Some estimates for fractional and Carleson operators and sparse domination of uncentered variational truncations

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I remember his words:

Te mando um presente atachado....

(*I send you an attached gift....*)

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Zusammenfassung in deutscher Sprache

Nach den einführenden Kapiteln 1 und 2 untersuchen wir im ersten Teil dieser Arbeit, bestehend aus den Kapiteln 3, 4, 5, 6, gewichtete Ungleichungen für die dyadische Version sogenannter nicht homogener, bilinearer und linearer, fraktionaler oder Carleson Operatoren.

In Kapitel 3 untersuchen wir auch schwache gewichtete Abschätzungen für dyadische bilineare Operatoren mit Summation über dünnbesetzte Mengen dyadischer Würfel. In diesem Fall entfernen wir die Abhängigkeit der Abschätzung von der in [Zor16] eingeführten multilinearen Fujii–Wilson A_∞-Charakteristik.

In Kapitel 4 verallgemeinern wir gewichtete Spurungleichungen von Verbitsky [Ver99] auf Operatoren mit nicht notwendigerweise homogenem Kern.

Die Hauptresultate des ersten Teils dieser Arbeit befinden sich in Kapiteln 5 und 6. Hier charakterisieren wir gewichtete $L^{p_1} \times L^{p_2} \to L^q$ starke Abschätzungen für dyadische bilineare Operatoren. Diese Ergebnisse erweitern [HV17, Theorem 1.2], wo die entsprechende Charakterisierung für lineare Operatoren bewiesen wurde. In beiden Kapiteln betrachten wir $p_1, p_2 > 1$.

In Kapitel 5 betrachten wir den Fall $0 < q < 1$. Wir zeigen eine explizitere, aber nicht direkt vergleichbare Charakterisierung, indem wir den bilinearen Fall auf den linearen Fall zurückführen und den Faktorisierungssatz auf lineare Operatoren anwenden.

In Kapitel 6 betrachten wir den Fall $0 < q < r$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} \leq 1$. Dafür benutzen wir eine kleine Verfeinerung des multilinearen Maurey-Faktorisierungssatzes aus [Sch84]. Damit erhalten wir eine stetige und eine diskrete Charakterisierung starker bilinearer Abschätzungen. Diese Charakterisierungen zeigen die Äquivalenz der Beschränktheit des bilinearen *Operators* und einer dazugehörigen *Bilinearform*.

Im zweiten Teil der Arbeit, Kapitel 7, betrachten wir eine Verschärfung maximaler Abschätzungen für abgeschnittene Calderón–Zygmund-Operatoren. Für einen Calderón–Zygmund-Kern K ist die punktweise r-Variation der zugehörigen abgeschnittenen Operatoren gegeben durch

$$\mathcal{T}^r f(x) := \sup_{s \leq t_1 < \cdots < t_J \leq t} \Big(\sum_{j=1}^{J-1} \Big| \int_{t_j < |x-y| < t_{j+1}} K(x,y) f(y) dy \Big|^r \Big)^{1/r}, \quad 2 < r < \infty.$$

Wir folgern aus bekannten L^p-Abschätzungen für diesen Operator eine neue Abschätzung durch dünnbesetzte Operatoren, die wiederum gewichtete Abschätzungen impliziert.

Contents

Chapter 1

Introduction

This thesis is motivated by the inequalities of the form

$$\|u\|_{L^q(\Omega,\sigma)} \leq C\|u\|_{W^{1,p}(\Omega,\mu)}. \tag{1.0.1}$$

In the most basic case when $\Omega = \mathbb{R}^n$ and both σ and μ are the Lebesgue measure, these are classical Sobolev inequalities, see works of Sobolev,Gagliardo and Nirenberg.They extend to domains Ω satisfying a so called cone condition. Another interesting example of σ is the $(n-1)$-dimensional Hausdorff measure on a hypersurface in Ω. These classical results can be found, for example, in [Maz03] and [AF03].

For general domains Ω and measures σ and a homogeneous version of the Sobolev norm, Maz'ya [Maz60], [Maz61], [Maz62a], [Maz62b], [Maz63], [Maz64] showed that certain trace inequalities hold if and only if certain isoperimetric and isocapacitary inequalities hold on Ω. In the case $p = 1$, $\sigma = \mu$ the Lebesgue measure, these isoperimetric inequalities concern the volume $\mathcal{H}^n(E)$ and the area of the interior part of the boundary $\mathcal{H}^{n-1}(\partial E \cap \Omega)$ of an arbitrary subset E of the domain Ω, see [Maz03, Theorem 3.5].

Sobolev spaces $W^{\alpha,p}$ are related to Bessel potential spaces $L^{\alpha,p}$. For $0 < \alpha < p$, the inhomogeneous Bessel kernel K_α is characterized by $\widehat{K_\alpha}(\xi) = (1+|\xi|^2)^{-\alpha/2}$. The functions in $L^{\alpha,p}$ are those having the form $P_\alpha f := K_\alpha * f$ with $f \in L^p$, and the norm is given by $\|K_\alpha * f\|_{L^{\alpha,p}} = \|f\|_{L^p}$. By Plancherel's theorem, $W^{\alpha,p} = L^{\alpha,p}$ with equivalence of norms if $p = 2$ and α is a positive integer. In [Cal61] this equivalence was extended to $1 < p < \infty$. The L^p norms are simpler to deal with than the $W^{\alpha,p}$ norms, so the identification between $W^{\alpha,p}$ and $L^{\alpha,p}$ is convenient. For more details about Bessel potential space see also [AH96].

In (1.0.1) we will refer to σ as a *trace measure*. A measure σ is called a trace measure for $L^{\alpha,p}$ if and only if the Bessel potential operator $K_\alpha * f$ is bounded from L^p to $L^p(\sigma)$. In [Maz62b], [Maz63] and [Maz64] trace measures for Bessel spaces were characterized via capacity. In the cases when Bessel spaces coincide with Sobolev spaces this also provides a characterization of measures for inequalities as (1.0.1).The Bessel capacity is defined on compact $K \subset \mathbb{R}^n, 1 < p < \frac{n}{\alpha}$, as

$$C_{\alpha,p}(K) := \inf\{||f||_{L^p}^p; f \in L^p, P_\alpha f \geq 1 \text{ on } K\}.$$

The $(\alpha, p)-$capacity can be extended to any $E \subset \mathbb{R}^n$ (see [AH96, Section 2]). A necessary and sufficient condition for the embedding of $L^{\alpha,p}$ in $L^p(\sigma)$ with $1 < p < \frac{n}{\alpha}$ is that there must be a constant $C(\sigma) > 0$ such that for any $E \subset \mathbb{R}^n$:

$$\sigma(E) \leq C(\sigma)C_{\alpha,p}(E). \tag{1.0.2}$$

This kind of inequality is called an *isocapacitary inequality* by Maz'ya and it is proved by the following capacity strong type inequality also proved by Maz'ya in [Maz73]:

$$\int_0^\infty C_{\alpha,p}(\{K_\alpha * f > t\})t^{p-1}dt \leq C||f||_{L^p}^p.$$

Another known inequality in potential theory is Wolff 's inequality

$$||K_\alpha * \sigma||_{L^{p'}}^{p'} \leq C \int_{\mathbb{R}^n} W_{\alpha,p}^\sigma d\sigma,$$

where for $\alpha p < n$ the function

$$W_{\alpha,p}^\sigma(x) := \int_0^\infty [r^{\alpha p - n}\sigma(B_r(x))^{p'-1}]\frac{dr}{r}$$

is called a *Wolff potential.* The boundedness of potential $W_{\alpha,p}^\sigma$ is a sufficient but not necessary condition for (1.0.2). Indeed, for any compact set $K \subset \mathbb{R}^n$ if σ is a Borel measure supported on K, we have by Hölder inequality that for $f \in L^{\alpha,p}$ with $P_\alpha f \geq 1_K$

$$\begin{aligned}\sigma(K) &\leq \int_{\mathbb{R}^n}(K_\alpha * f)d\sigma \leq \|f\|_{L^p}\|K_\alpha * \sigma\|_{L^{p'}}.\\ &\leq C\|f\|_{L^p}\|W_{\alpha,p}^\sigma\|_{L^\infty}^{\frac{1}{p'}}\sigma(K)^{\frac{1}{p'}}.\end{aligned}$$

So if we suppose that $\|W_{\alpha,p}^\sigma\|_\infty \leq \infty$, then (1.0.2) holds. On the other hand it is clear that (1.0.2) does not imply the boundedness of $W_{\alpha,p}^\sigma$.

Kerman and Sawyer [KS86] found a characterization of trace measures for more general potential operators by testing on balls (or equivalently on the dyadic cubes) in $\mathbb{R}^n$. They studied the trace inequality conditions for potential operators T_k defined as convolution operators with kernel k on L^p functions. The kernel k is assumed to be a locally integrable function on $\mathbb{R}^n$, nonnegative and radially decreasing. The inhomogeneous Bessel potential $K_\alpha * f$ defined above and the homogeneous Riesz potential $I_\alpha f$ whose kernel K_α is characterized by $\widehat{K_\alpha}(\xi) = |\xi|^{-\alpha}$ (i.e., $I_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)dy, 0 < \alpha < n$) are included in this family of potential operators. In these potential spaces a Borel measure σ is a trace measure if $T_k : L^p \to L^p(\sigma)$ is bounded, similarly to the case of Bessel spaces. [KS86, Theorem 2.3] states that for a positive locally finite Borel measure σ, a sufficient and necessary condition for the boundedness of T_k is that there must be $C(\sigma) > 0$ such that for any dyadic cube (or any ball) $Q \subset \mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \left[\int_Q k(x-y)d\sigma(y) \right]^{p'} dx \leq C(\sigma)\sigma(Q) \tag{1.0.3}$$

where p' is such that $pp' = p + p'$. The inequality (1.0.3) means that the dual operator T_k^* is bounded on the characteristic functions of dyadic cubes. The operator $T_k^* : L^{p'}(\sigma) \to L^{p'}, T_k^*(f)(x) = \int_{\mathbb{R}^n} k(x-y)f(y)d\sigma(y)$ is bounded if for any $f \in L^{p'}(\sigma)$:

$$\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} k(x-y)f(y)d\sigma(y) \right]^{p'} dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p'} d\sigma(x).$$

So for $f = 1_Q$ we find (1.0.3). Conditions on when f is replaced by the are characteristic function known in the literature as *test conditions*.

Using the Kerman-Sawyer condition (see condition (2.9) in [Ver99, Proposition 2.4]) for an auxiliary measure Verbitsky [Ver99] (in the case $1 < p < \infty$, $0 < q < p < \infty$, $0 < \alpha < n$) proved that the trace inequality

$$\|I_\alpha f\|_{L^q(d\sigma)} \leq C\|f\|_{L^p(dx)}, \forall f \in L^p(\mathbb{R}^n) \tag{1.0.4}$$

holds for a given positive Borel measure σ on $\mathbb{R}^n$ if and only if

$$W^\sigma_{\alpha,p} \in L^{\frac{q(p-1)}{p-q}}(d\sigma).$$

The proof proceeds via the inequality $\|I_\alpha f\|_{L^p(d\vartheta)} \leq C\|f\|_{L^p(dx)}$ with $d\vartheta = [W^\sigma_{\alpha,p}(x)]^{1-p} d\sigma$ that implies (1.0.4) by Hölder. Our initial goal is to give a result similar to the result of Verbitsky for a certain operator $T_\lambda(\overrightarrow{f})$ nonhomogeneous defined in (1.0.9) (in its linear version). The rest of the work concerns a bilinear version of $T_\lambda(\overrightarrow{f})$. We will work with kernels that are not necessarily radially decreasing or translation invariant.

A family of bilinear operators related to Riesz potentials are the bilinear fractional integral operators

$$\mathcal{I}_\alpha(f_1, f_2)(x) = \int_{\mathbb{R}^n} \frac{f_1(x-t)f_2(x+t)}{|t|^{n-\alpha}} dt, 0 < \alpha < n.$$

Such operators have a long history and were studied for example in [Gra92] [KS99] and [GK01]. The operators have attracted interest because of their similarity to the bilinear Hilbert transform

$$\mathcal{H}(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} \frac{f_1(x-t)f_2(x+t)}{t} dt.$$

Estimates for the bilinear Hilbert transform can be found, for example, in [LT97], [LT98] and [LT99]. If $\mathcal{I}_\alpha$ satisfies an $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ estimates, then a scaling argument shows that p_1, p_2, and q must satisfy the relationship

$$\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} = \frac{1}{p} - \frac{\alpha}{n}. \tag{1.0.5}$$

Conversely, if $1 < p_1, p_2 < \infty$ and q is defined by equation (1.0.5), then

$$\mathcal{I}_\alpha : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^q(\mathbb{R}^n),$$

see [Gra92; KS99; GK01]. In the case $q > 1$ this follows from linear bounds. Indeed, for any pair of conjugate exponents $\frac{1}{r} + \frac{1}{s} = 1$ Hölder's inequality yields

$$\mathcal{I}_\alpha(f_1, f_2) \le I_\alpha(f_1^r)^{\frac{1}{r}} I_\alpha(f_2^s)^{\frac{1}{s}}, \tag{1.0.6}$$

where $I_\alpha f$ is the linear Riesz operator defined above. If we let $r = \frac{p_1}{p}$ and $s = \frac{p_2}{p}$ with $\frac{1}{r} + \frac{1}{s} = 1$ for a suitable p, then $r, s > 1$ and $I_\alpha : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$. Hence

$$\begin{aligned}
\|\mathcal{I}_\alpha(f_1, f_2)\|_{L^q(\mathbb{R}^n)} &\le \left(\int_{\mathbb{R}^n} I_\alpha(f_1^r)^{\frac{q}{r}} I_\alpha(f_2^s)^{\frac{q}{s}} dx \right)^{\frac{1}{q}} \\
&\le \left(\int_{\mathbb{R}^n} I_\alpha(f_1^r)^q dx \right)^{\frac{1}{qr}} \left(\int_{\mathbb{R}^n} I_\alpha(f_2^s)^q dx \right)^{\frac{1}{qs}} \\
&\le C \left(\int_{\mathbb{R}^n} f_1^{p_1} dx \right)^{\frac{1}{p_1}} \left(\int_{\mathbb{R}^n} f_2^{p_2} dx \right)^{\frac{1}{p_2}} = \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2}(\mathbb{R}^n)}.
\end{aligned}$$

The same argument gives weighted estimates of the form

$$\|\mathcal{I}_\alpha(f_1, f_2) w_1 w_2\|_{L^q(\mathbb{R}^n)} \le C \|f_1 w_1\|_{L^{p_1}(\mathbb{R}^n)} \|f_2 w_2\|_{L^{p_2}(\mathbb{R}^n)} \tag{1.0.7}$$

in the case $q > 1$ and w_1, w_2 weight (See [IKS10].) Indeed, Muckenhoupt and Wheeden [MW71] showed that for $1/q = 1/p - \alpha/n$

$$\|(I_\alpha f)w\|_{L^q(\mathbb{R}^n)} \le C\|fw\|_{L^p(\mathbb{R}^n)}$$

holds if and only if $w \in A_{p,q}$, i.e.,

$$[w]_{A_{p,q}} := \sup \left(\frac{1}{|Q|}\int_Q w^q dx\right)^{\frac{1}{q}} \left(\frac{1}{|Q|}\int_Q w^{-p'} dx\right)^{\frac{1}{p'}} < \infty.$$

Using inequality (1.0.6), it was observed in [Ber+14] that for $1 < p_1, p_2, p, q < \infty$ satisfying (1.0.5), then (1.0.7) holds when $w_1, w_2 \in A_{p,q}$. Using inequality (1.0.6) and the linear theory once again, one may derive a variety of two weight (really three weight) inequalities of the form

$$\|\mathcal{I}_\alpha(f_1, f_2)u\|_{L^q(\mathbb{R}^n)} \le C\|f_1 v_1\|_{L^{p_1}(\mathbb{R}^n)}\|f_2 v_2\|_{L^{p_2}(\mathbb{R}^n)},$$

for example, if the pairs of weights (u, v_1) and (u, v_2) individually satisfy Sawyer's testing conditions or certain bump conditions, see [Pra10] and [Pér94].

In the case $q \le 1$ the estimate (1.0.6) is not useful. A different argument giving weighted estimates in this case can be found in [Moe14].

Now we describe our setting. Let μ and μ_i for $i = 1, ..., N$ be nonnegative measures on $\mathbb{R}^n$. Let $\mathcal{D}$ be the family of all dyadic cubes $Q = 2^{-k}(m + [0,1)^n), k \in \mathbb{Z}, m \in \mathbb{Z}^n$. We denote by $w_1, \ldots, w_{N+1}$ measurable functions on $\mathbb{R}^n$, by $\overrightarrow{f} = (f_1, \ldots, f_N)$ an N-tuple of functions on $\mathbb{R}^n$ and by λ_Q, $Q \in \mathcal{D}$, a family of nonnegative numbers. We are concerned with inequalities of the type

$$\|T_\lambda(\overrightarrow{fw})\|_{L^q(w_{N+1}d\mu_{N+1})} \lesssim \prod_{i=1}^{N} \|f_i\|_{L^{p_i}(w_i d\mu_i)}, \tag{1.0.8}$$

where the notation $A \lesssim B$ means that $A \le CB$ with a constant C that does not depend on the functions f and

$$T_\lambda(\overrightarrow{f})(x) := T^{\mathcal{D}}_{N,\lambda,\mu,\mu_i,Q}(\overrightarrow{f})(x) = \sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\mu(Q)^N} \prod_{i=1}^{N} \left(\int_Q f_i d\mu_i\right) 1_Q(x), \tag{1.0.9}$$

that is, in the case that $0 < q \le r, 1 < p_1, \ldots, p_N < \infty$, with a focus on the case $N = 2$. One method for the linear case $N = 1$ will also be presented. In Chapter 2 and Chapter 3 we consider $r = \left(\frac{1}{p_1} + \frac{1}{p_2}\right)^{-1}$ such that $q = r < 1$. In Chapter 4,we conside the case $N = 1$, $r = p$ and $0 < q < p < \infty, p > 1$. In Chapter 4 we work with $w_1, w_2 = 1$, $\mu_2 = \sigma$

an arbitrary nonnegative measure on $\mathbb{R}^n$ and $\mu_1 = \mu$ Lebesgue measure. In Chapter 5 we consider the case $0 < q < 1$. In Chapter 6, we consider r such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} \le 1$ and $0 < q < r$.

Initially, as a preliminary in theory, we present a strong (in Chapter 2) and we give a weak (in Chapter 3) $A_p - A_\infty$ weight estimate for $T_\lambda(f_1 w_1, f_2 w_2)$, in which the sum is taken over a sparse collection $S \subset \mathcal{D}$ (see Definition 7.2.12). In Chapter 2 the constant found depends of a Fujii–Wilson A_∞-Characteristic [Zor16]. In Chapter 3 we remove this dependency.

In Chapter 4 we extend the results of [Ver99] in the case $\lambda_Q = \mu(Q)^{\frac{\alpha}{n}}$ to more general sequences (λ_Q), namely those satisfying

$$\sum_{Q' \subseteq Q} \lambda_{Q'} w_1(Q') \sim \lambda_Q w_1(Q)$$

for a certain measure w_1 defined in Chapter 4 and

$$\sup_{x,P:x\in P} \sum_{Q:Q\subsetneq P, x\in Q} \lambda_Q \left[\sum_{S:P\subseteq S} \lambda_S^{1-p} \right]^{p'-1} < \infty. \tag{1.0.10}$$

(See Theorem 4.3.4.)

The method is similar to the method of [Ver99]. Note, however, that in the proof of our Proposition 4.2.3 we use properties only of the dyadic operator. Maz'ya and Verbitsky use particular properties of the continuous operator $I_\alpha(fw)$ (See Theorem 1.7 and Lemma 2.1 in [Ver99]).

Let $1 < p < \infty$, $0 < q < p$ and $0 < s < p$, and w, μ be two nonnegative Borel measures on $\mathbb{R}^n$. Suppose that that the pair $((\lambda_Q^s)_Q, \mu)$ satisfies the dyadic logarithmic bounded oscillation (DBLO) condition

$$\sup_{x\in Q} \frac{1}{\mu(Q)} \sum_{Q'\subset Q} \lambda_{Q'} 1_{Q'}(x) \lesssim \inf_{x\in Q} \frac{1}{\mu(Q)} \sum_{Q'\subset Q} \lambda_{Q'} 1_{Q'}(x) \tag{1.0.11}$$

for all dyadic cubes Q (this DBLO condition was introduced in [COV06]). In this case Cascante and Ortega [CO09, Theorem 2.8] proved that the inequality

$$\left\| \left(\sum_{Q\in\mathcal{D}} \lambda_Q^s \rho_Q^s 1_Q \right)^{\frac{1}{s}} \right\|_{L^q(dw)} \le C \left\| \sup_Q (\rho_Q 1_Q) \right\|_{L^p(dx)}$$

holds for all sequences of nonnegative numbers $(\rho_Q)_Q$ if and only if

$$\sum_{Q\in\mathcal{D}} \lambda_Q^s \left[\inf_{x\in Q} \sum_{Q'\subset Q} \lambda_{Q'}^s 1_{Q'}(x) \right]^{(\frac{p}{s})'-1} \left(\frac{w(Q)}{\mu(Q)} \right)^{(\frac{p}{s})'-1} 1_Q \in L^{\frac{q(p-s)}{s(p-q)}}(dw).$$

Our Theorem 4.3.4 extends the above result by Cascante and Ortega, but without using the DBLO condition (See Corollary 4.3.10).

We observe that Carleson sequences do not in general satisfy the above conditions (1.0.10) and (1.0.11). We are looking for a method to cover such sequences as well. As an initial illustration of a method covering Carleson sequences we present in Chapter 2 the theory of [Zor16] in the particular case $\lambda_Q = 1_{Q\in\mathcal{S}}$, $\mathcal{S}$ sparse. The constant C obtained this method depends on with a *Fujii–Wilson characteristic* introduced in [Zor16] (which in the linear case coincides with the A_∞ characteristic originating in [Fuj78; Wil87]).

[HL18, Theorem 1.2] suggests that the dependence on the Fujii–Wilson characteristic in the corresponding weak type estimate can be removed, analogously to the estimate for the multilinear maximal operator. This question was left open in [Zor16]. In Chapter 3 we make partial progress on this question.

Li and Sun [LS16] characterized the boundedness

$$T(\cdot w_1, \cdot w_2) : L^{p_1}(w_1) \times L^{p_2}(w_2) \to L^{p_3'}(w_3)$$

for exponents $p_1, p_2, p_3 \in (1,\infty)$ satisfying $\frac{1}{p_i} + \frac{1}{p_j} \geq 1$ for $i \neq j$, and Tanaka [Tan15] under the weaker restriction $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 1$. The idea is to reduce the bilinear case to the linear case by fixing one of the arguments: from the bilinear operator $T(\cdot w_1, \cdot w_2)$ we obtain the localized linear operator

$$f_2 \mapsto T_R\Bigg(\frac{1_R}{w_1(R)^{\frac{1}{p_1}}} w_1, f_2 w_2\Bigg) := \frac{1_R}{w_1(R)^{\frac{1}{p_1}}} \sum_{Q\in\mathcal{D}: Q\subseteq R} \frac{\lambda_Q}{\mu(Q)^2} w_1(Q) \int_Q f_2 dw_2 1_Q.$$

Boundedness of the linear operator is then characterized by the Sawyer testing conditions or the discrete Wolff potential depending on the exponents ([Hän15, Theorem 4.6] for $1 < p \leq q < \infty$ and [LSU09, Theorem 1.3] for $1 < q < p < \infty$).

In the case

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1$$

boundedness of bilinear positive dyadic operators was characterized by sequential testing conditions in [HHL16, Theorem 1.16]. In Chapter 5 we consider two weight $L^{p_1} \times L^{p_2} \to L^q$ bounds for bilinear positive dyadic operators in case $0 < q < 1, p_1, p_2 > 1$. Using parallel stopping cubes, characterization of boundedness of vector valued operators terms of discrete multipliers (see Lemma 5.1.1), and equivalence between sparse and Carleson conditions (see proof of Theorem 5.1.4), we show that boundedness of the bilinear operator is equivalent to the estimate

$$\Big\| \sum_{\substack{F_i \in \mathcal{F}_i \\ F_i \subseteq F_j}} \Lambda^j_{F_i} \frac{1}{w_i(F_i)} \Big(\int_{F_i} f_i dw_i \Big) 1_{F_i} \Big\|_{L^q(w_j)} \lesssim w_j(F_j)^{\frac{1}{q}} \|f_i\|_{L^{p_i}(w_i)}, i, j = 1, 2, i \neq j,$$

where $\Lambda^j_{F_i}$ depends only on $\lambda_Q, w_i, w_j, w_3(Q), \mu$ and on the stopping times F_i, F_j and stopping parents π_i, π_j. Using the above reduction and the ideas in the proof of Theorem [HV17, Theorem 1.2] we obtain Theorem 5.2.1.

In Chapter 6 we extend [HV17, Theorem 1.2] (which is based on Maurey's factorization theorem) to bilinear operators presenting a quantitative version of the factorization result from [Sch84]. (See Theorem 6.3.1.)

In Chapter 7 we formulate Lacey's sparse domination technique in terms of a variational refinement of nontangential maximal functions. As an application we obtain sparse bounds for sharp variational truncations of singular integral operators and a variational version of the Hardy–Littlewood maximal operator. This chapter previously appeared as [dZ16].

Chapter 2

Strong type L^p estimates for bilinear sparse operators with an explicit constant

In this chapter we give strong type estimates for a family of bilinear and linear sparse operators, where we consider coefficients $\lambda_Q = 1_{Q\in\mathcal{S}}$, $\mathcal{S}$ sparse, as presented in [Zor16]. We elaborate on the proof. The constant in our estimate is a nice combination of A_p and A_∞ type constants of the weights. Operators from this family are known for example to relate to bilinear Hilbert transforms and bilinear Calderón–Zygmund operators. We use a version of the *Fujii-Wilson* A_∞ condition introduced in [Zor16].

Similar weighted estimates of the type that we are interested in have been first obtained in [HP13]. The problem of optimal dependence of constants in weighted inequalities on characteristics of the weights has been studied in [Muc72]. Similar questions for singular integral operators were studied by a number of authors, we refer to [HPR12] and [Hyt14]. In [Ler13] the problem of proving weighted estimates is reduced to sparse operators. So we concentrate on weighted estimates for sparse operators.

We give here a particular case of [Zor16, Theorem 1.12] with $m = 3, r_i = 1, \rho_i = 0$ and $1 < t_i < \infty$, where we take $\alpha = q^{-1}$ and $t_i = p_i$. (Note that with these assumptions [Zor16, Theorem 1.12] makes no sense in the linear case).

Theorem 2.0.1. *Let S be a sparse collection of cubes, Definition 7.2.12. Let $Q_0 \in \mathcal{S}$. Let $w_i, i = 1, 2, 3$ be weights, μ nonnegative measure , $f_i, i = 1, 2, 3$ positive measurable functions with support in Q_0. Let $1 < p_i < \infty$ and* *consider*

$$q := \left(\sum_{i=1}^{2} \frac{1}{p_i}\right)^{-1} < 1.$$

Then

$$\left\| \sum_{Q\in\mathcal{S}} \frac{1}{\mu(Q)^2} \prod_{i=1}^{2} \left(\int_Q f_i w_i\right) 1_Q \right\|_{L^q(w_3)} \lesssim C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)}$$

with

$$C = \sup_Q (w_1)_Q^{1/p_1'} (w_2)_Q^{1/p_2'} (w_3)_Q^{1/q} \Bigg[\sup_Q \left(\int_Q \sup_{x\in Q', Q'\in\mathcal{D}} \mu(Q')^{-1} \int_{Q'} 1_Q w_1 \right)^{1/p_1}$$
$$\left(\int_Q \sup_{x\in Q', Q'\in\mathcal{D}} \mu(Q')^{-1} \int_{Q'} 1_Q w_2 \right)^{1/p_2} \left(\int_Q w_1\right)^{-1/p_1} \left(\int_Q w_2\right)^{-1/p_2} \Bigg].$$

We give here also another special case of [Zor16] that is essentially [Zor16, Theorem 1.11] in case $m = 2, r_i = 1, \rho_i = 0, t_i = p_i$ and $J_r = \emptyset$.

Theorem 2.0.2. *Let $Q_0 \in \mathcal{S}$, μ be non-negative measure, $w_i, i = 1, 2$, be weights, $f_i, i = 1, 2$, positive measurable functions with support in Q_0. Consider $\sum_{i=1}^{2} 1/p_i = 1$ with $1 < p_i < \infty$. Then*

$$\sum_{Q\in\mathcal{S}} \mu(Q)^{-1} \prod_{i=1}^{2} \left(\int_Q f_i w_i\right) \lesssim C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)},$$

where

$$C = \left(\sup_Q (w_1)_Q^{1/p_1'} (w_2)_Q^{1/p_2'} \right) \sup_Q \left(\int_Q \sup_{x\in Q', Q'\in\mathcal{D}} \mu(Q')^{-1} \int_{Q'} 1_Q w_1 \right)^{\frac{1}{p_1}} \left(\int_Q w_1\right)^{-\frac{1}{p_1}}.$$

2.1 Preliminaries

We need the following basic inequality.

Theorem 2.1.1. *If $s \geq 1$ then*

$$\left(\sum_{i=1}^{N} a_i\right)^{s} \leq s \sum_{i=1}^{N} a_i \left(\sum_{j\geq i}^{N} a_j\right)^{s-1} \tag{2.1.2}$$

for every summable sequence $\{a_i\}_{\in\mathbb{Z}}$ of nonnegative reals.

Proof. We induct on N. The case $N = 1$ is obvious. Assume that the claim holds for $N = k$. For $N = k + 1$ note that

$$(a+b)^s - b^s = \int_b^{a+b} sx^{s-1}dx \leq \int_b^{a+b} s(a+b)^{s-1}dx = sa(b+a)^{s-1},$$

i.e.,

$$(a+b)^s \leq sa(a+b)^{s-1} + b^s,$$

which implies

$$\left(\sum_{i=1}^{k+1} a_i\right)^s = \left(a_1 + \sum_{i=1}^{k} a_{i+1}\right)^s \leq sa_1\left(\sum_{i=1}^{k+1} a_i\right)^{s-1} + \left(\sum_{i=1}^{k} a_{i+1}\right)^s. \qquad \square$$

We will use the following result about L^s norms that is part of [COV04, Proposition 2.2] and [Tan14, Lemma 2.1].

Lemma 2.1.3. *For every $1 < s < \infty$ there exists $C_s > 0$ such that for every positive locally finite measure σ on $\mathbb{R}^n$ and any positive numbers $\lambda_Q, Q \in \mathcal{D}$, we have*

$$\int \left(\sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} 1_Q(x)\right)^s d\sigma(x) \lesssim_s \sum_{Q\in\mathcal{D}} \lambda_Q \left(\sigma(Q)^{-1} \sum_{Q'\subseteq Q} \lambda_{Q'}\right)^{s-1}.$$

Proof. We use Theorem 2.1.1. First, we verify the case $1 < s \leq 2$. It follows from (2.1.2) that

$$\begin{aligned}
\int \left(\sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} 1_Q\right)^s d\sigma &\leq s \sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} \int_Q \left(\sum_{Q'\subset Q} \frac{\lambda_{Q'}}{\sigma(Q')} 1_{Q'}\right)^{s-1} d\sigma \\
&\leq s \sum_{Q\in\mathcal{D}} \lambda_Q \left(\frac{1}{\sigma(Q)} \int_Q \left(\sum_{Q'\subset Q} \frac{\lambda_{Q'}}{\sigma(Q')} 1_{Q'}\right)^{(s-1)\frac{1}{s-1}} d\sigma\right)^{s-1} \left(\frac{1}{\sigma(Q)} \int_Q d\sigma\right)^{\frac{1}{\left(\frac{1}{s-1}\right)'}} \\
&\leq s \sum_{Q\in\mathcal{D}} \lambda_Q \left(\frac{1}{\sigma(Q)} \int_Q \left(\sum_{Q'\subset Q} \frac{\lambda_{Q'}}{\sigma(Q')} 1_{Q'}\right) d\sigma\right)^{s-1} \\
&= s \sum_{Q'\subset Q} \lambda_Q \left(\frac{1}{\sigma(Q)} \sum_{Q'\subset Q} \lambda_{Q'}\right)^{s-1},
\end{aligned}$$

where we have used $s - 1 \leq 1$, Hölder's inequality and $\int 1_{Q'} = \sigma(Q')$.

In the case $s > 2$ we use induction. For integer $k \geq 2$ we assume that the conclusion of the theorem holds for any $k - 1 < s \leq k$ and have to show that it also holds for $k < s \leq k + 1$. By (2.1.2)

$$\int \left(\sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} 1_Q\right)^s d\sigma \leq s \sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} \int_Q \left(\sum_{Q'\subset Q} \frac{\lambda_{Q'}}{\sigma(Q')} 1_{Q'}\right)^{s-1} d\sigma.$$

Applying the induction hypothesis for $k-1 < s-1 \leq k$, with the measure $1_Q\sigma$ and the set $(\lambda_{Q'})_{Q'}$ where $\lambda_{Q'} = 0$ for cubes $Q' \not\subset Q$, we obtain:

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} \int_Q \Big(\sum_{Q'\subset Q} \frac{\lambda_{Q'}}{\sigma(Q')} 1_{Q'} \Big)^{s-1} d\sigma \\
&\qquad \leq C_1(s-1) \sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} \Big[\sum_{Q'\subset Q} \lambda_{Q'} \Big(\frac{1}{\sigma(Q')} \sum_{Q''\subset Q'} \lambda_{Q''} \Big)^{s-2} \Big] \\
&\qquad = C_1(s-1) \Big[\sum_{Q'} \lambda_{Q'} \Big(\frac{1}{\sigma(Q')} \sum_{Q''\subset Q'} \lambda_{Q''} \Big)^{s-2} \Big] \sum_{Q'\subset Q} \frac{\lambda_Q}{\sigma(Q)} \\
&\qquad \leq C_1(s-1) \Big(\sum_{Q'} \lambda_{Q'} \Big(\frac{1}{\sigma(Q')} \sum_{Q''\subset Q} \lambda_{Q''} \Big)^{s-1} \Big)^{\frac{s-2}{s-1}} \cdot \\
&\qquad\qquad \cdot \Big(\sum_{Q'} \lambda_{Q'} \Big(\sum_{Q'\subset Q} \frac{\lambda_Q}{\sigma(Q)} \Big)^{s-1} \Big)^{\frac{1}{s-1}},
\end{aligned}$$

where in the last inequality we use Hölder's inequality for sums with exponents $\frac{1}{s-1}$ and $\frac{s-1}{s-2}$. Since

$$\sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} \int_Q \Big(\sum_{Q'\subset Q} \frac{\lambda_{Q'}}{\sigma(Q')} 1_{Q'} \Big)^{s-1} d\sigma = \sum_{Q'} \lambda_{Q'} \Big(\sum_{Q'\subset Q} \frac{\lambda_Q}{\sigma(Q)} \Big)^{s-1}$$

we obtain

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} \int_Q \Big(\sum_{Q'\subset Q} \frac{\lambda_{Q'}}{\sigma(Q')} 1_{Q'} \Big)^{s-1} d\sigma \\
&\qquad\qquad \leq (C_1(s-1))^{\frac{s-1}{s-2}} \sum_{Q'} \lambda_{Q'} \Big(\frac{1}{\sigma(Q')} \sum_{Q''\subset Q} \lambda_{Q''} \Big)^{s-1}
\end{aligned}$$

for $k < s \leq k+1$, which shows the conclusion for this case. $\square$

Lemma 2.1.4. *Let $0 \leq b_1, b_2$ be such that $b = b_1 + b_2 < 1$. Then for every sparse collection $\mathcal{S}$, every cube Q, all non-negative measures μ and all positive functions w_1, w_2 we have*

$$\sum_{Q'\subseteq Q, Q'\in\mathcal{S}} \mu(Q') \prod_{i=1}^{2} (w_i)_{Q'}^{b_i} \lesssim \mu(Q) \prod_{i=1}^{2} (w_i)_{Q'}^{b_i}.$$

Proof. By definition of sparseness,

$$\text{LHS} \;\le \sum_{Q'\subseteq Q, Q'\in\mathcal{S}} \mu(E(Q'))\prod_{i=1}^{2}(w_i)_{Q'}^{b_i} \;=\; \sum_{Q'\subseteq Q, Q'\in\mathcal{S}} \int_Q 1_{E_{Q'}} \prod_{i=1}^{2}(w_i 1_Q)_{Q'}^{b_i}.$$

Recall that the Hardy–Littlewood maximal function is defined by

$$M(f) = \sup_{x\in Q}(|f|)_Q = \sup_{x\in Q}\frac{1}{\mu(Q)}\int |f|.$$

By Fubini and $\sum_{Q'} 1_{E_{Q'}} \le 1$ we have

$$\begin{aligned}\text{LHS} \le \int_Q \sum_{Q'}\prod_{i=1}^{2}(w_i 1_Q)_{Q'}^{b_i} 1_{E_{Q'}} &\le \int_Q \sum_{Q'}\prod_{i=1}^{2} M(w_i 1_Q)(x)^{b_i} 1_{E_{Q'}} \\ &\le \int_Q \prod_{i=1}^{2} M(w_i 1_Q)(x)^{b_i} \sum_{Q'} 1_{E_{Q'}} \le \int_Q \prod_{i=1}^{2} M(w_i 1_Q)(x)^{b_i}.\end{aligned}$$

By Hölder's inequality

$$\text{LHS} \le \prod_{i=1}^{2}\left(\int_Q (M(w_i 1_Q))^b\right)^{b_i/b}.$$

In each factor we have the estimate

$$\begin{aligned}\int_Q (M(w_i 1_Q))^b &\lesssim \sum_{k\in\mathbb{Z}} 2^{kb}\mu(Q\cap\{M(w_i 1_Q)^b > 2^{kb}\}) \\ &\lesssim \mu(Q)\sum_{k\in\mathbb{Z}} 2^{kb}\min(1, 2^{-k}(w_i)_Q) \\ &\lesssim \max(1/b, 1/(1-b))\mu(Q)(w_i)_Q^b,\end{aligned}$$

where the first inequality follows by the weak type $(1,1)$ inequality for the maximal function and by the property

$$f \ge 0 \Rightarrow \int_Q f = \int_0^\infty \mu(\{f>\lambda\}\subseteq Q)d\lambda \le \sum_k 2^{kb}\mu(\{f > 2^{kb}\}).$$

$\square$

An important result here is the the following lemma.

Lemma 2.1.5. *Let $p > 1$. For all $0 < \beta < \infty$*

$$\int \left(\sum_{Q \in \mathcal{S}} (w_1)_Q^{p\beta} 1_Q \right)^{1/\beta} dw_2 \lesssim \left(\sup_Q (w_1)_Q^{p-1} \sup_Q (w_2)_Q \right) \sum_{Q \in \mathcal{S}} \mu(Q)(w_1)_Q$$

for all non-negative measures μ and all positive functions w_1, w_2.

Proof. For sufficiently small β there exists an ϵ such that

$$\beta = \frac{\beta p}{p} < \epsilon \leq \min\left(\frac{1}{1/\beta - 1}, \frac{\beta p}{p-1}, 1/1 \right) = \min\left(\frac{\beta}{1-\beta}, \beta \frac{p}{p-1}, 1 \right).$$

Then $\beta < 1$. Consider $\beta = 1/p$. So

$$1/p < \epsilon = \min\left(\frac{1}{p-1}, \frac{1}{p-1}, 1 \right) = \begin{cases} 1, p \geq 2 \\ \frac{1}{p-1}, p < 2. \end{cases}$$

By Lemma 2.1.3,

$$\begin{aligned} &\int \left(\sum_{Q \in \mathcal{S}} 1_Q \frac{(w_1)_Q w_2(Q)}{w_2(Q)} \right)^p dw_2 \\ &\qquad \lesssim \sum_{Q \in \mathcal{S}} (w_1)_Q w_2(Q) \left(w_2(Q)^{-1} \sum_{Q' \subseteq Q} (w_1)_{Q'} \mu(Q')(w_2)_{Q'} \right)^{p-1} \\ &\qquad\qquad \leq \left(\sup_Q (w_1)_Q^{\epsilon(p-1)(p-1)} (w_2)_Q^{\epsilon(p-1)} \right) \cdot \\ &\qquad \cdot \sum_Q \mu(Q)(w_1)_Q(w_2)_Q \left(w_2(Q)^{-1} \sum_{Q' \subseteq Q} \mu(Q')(w_1)_{Q'}^{1-\epsilon(p-1)} (w_2)_{Q'}^{1-\epsilon} \right)^{p-1}. \end{aligned}$$

By construction we have

$$1 - \epsilon(p_1 - 1) \geq 0, 1 - \epsilon \geq 0$$

and

$$1 - \epsilon(p-1) + 1 - \epsilon = 2 - \epsilon p < 2 - \frac{1}{p} p = 1.$$

Hence by Lemma 2.1.4 we obtain

$$\begin{aligned}
&\int \left(\sum_{Q\in\mathcal{D}} 1_Q (w_1)_Q \right)^p dw_2 \\
&\qquad\qquad \lesssim \left(\sup_Q (w_1)_Q^{\epsilon(p-1)(p-1)} (w_2)_Q^{\epsilon(p-1)} \right) \cdot \\
&\qquad \cdot \sum_Q \mu(Q)(w_1)_Q (w_2)_Q \left(w_2(Q)^{-1}\mu(Q)(w_1)_Q^{1-\epsilon(p-1)}(w_2)_Q^{1-\epsilon} \right)^{p-1} \\
&= \left(\sup_Q (w_1)_Q^{\epsilon(p-1)(p-1)} (w_2)_Q^{\epsilon(p-1)} \right) \sum_Q \mu(Q)(w_1)_Q^{1+(p-1)(1-\epsilon(p-1))}(w_2)_Q^{1+(p-1)(-\epsilon)} \\
&\qquad\qquad\qquad \le \sup_Q (w_1)_Q^{p-1}(w_2)_Q \sum_Q \mu(Q)(w_1)_Q.
\end{aligned}$$

□

Given a weight w, define the weighted dyadic *Hardy–Littlewood maximal operator* $M_w^{\mathcal{D}}$ by

$$M_w^{\mathcal{D}} f(x) := \sup_{Q\in\mathcal{D}:Q\ni x} \frac{1}{w(Q)} \int_Q |f| w.$$

The following result is known as the Hardy- Littlewood dyadic maximal theorem. It's can be found in [LN15].

Theorem 2.1.6. *The maximal operator $M_w^{\mathcal{D}}$ satisfies*

$$\|M_w^{\mathcal{D}} f\|_{L^p(w)} \le \frac{p}{p-1} \|f\|_{L^p(w)} \quad (1 < p \le \infty).$$

Proof. Let $\mathcal{F} \subset \mathcal{D}$ be any finite family of cubes. By the Monotone Convergence Theorem it suffices to consider the restricted maximal function

$$M_w^{\mathcal{F}} f = \begin{cases} \max \frac{1}{w(Q)} \int_Q |f| w, & \text{if } Q \in \mathcal{F}, Q \ni x, x \in \bigcup_{Q\in\mathcal{F}} Q \\ 0, & \text{otherwise.} \end{cases}$$

For $\lambda > 0$, let

$$\Omega_\lambda = \{x \in \mathbb{R}^n : M_w^{\mathcal{F}} f(x) > \lambda\}.$$

Then Ω_λ is just the union of the maximal cubes $Q_j \in \mathcal{F}$ with the property that

$$\int_{Q_j} |f| w > \lambda w(Q_j).$$

Since Q_j are disjoint, we get

$$w(\Omega_\lambda) = \sum_j w(Q_j) \leq \frac{1}{\lambda} \sum_j \int_{Q_j} |f|w = \frac{1}{\lambda} \int_{\Omega_\lambda} |f|w.$$

This implies the weak type bound for $M_w^{\mathcal{F}}$:

$$w\{x \in \mathbb{R}^n : M_w^{\mathcal{D}} f(x) > \lambda\} \leq \frac{1}{\lambda} \|f\|_{L^1(w)} \quad (\lambda > 0). \tag{2.1.7}$$

To get the $L^p(w)-$bound for $1 < p < \infty$ (the remaining case $p = \infty$ is obvious), just integrate (2.1.7) with the weight $p\lambda^{p-1}$:

$$\begin{aligned}
\|M_w^{\mathcal{D}} f\|_{L^p(w)}^p &= p \int_0^\infty \lambda^{p-1} w(\Omega_\lambda) d\lambda \\
&\leq p \int_0^\infty \lambda^{p-2} \Big(\int_{\Omega_\lambda} |f|w \Big) d\lambda \\
&= p \int \int_{\{(\lambda,x):0<\lambda<M_w^{\mathcal{D}} f(x)\}} \lambda^{p-2} |f(x)| w(x) dx d\lambda \\
&= \frac{p}{p-1} \int_{\mathbb{R}^n} (M_w^{\mathcal{D}} f)^{p-1} |f| w \\
&\leq \frac{p}{p-1} \Big(\int_{\mathbb{R}^n} (M_w^{\mathcal{D}} f)^p w \Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^n} |f|^p w \Big)^{1/p}.
\end{aligned}$$

Assuming that f is bounded and compactly supported (so all integrals in the last inequality are finite),we conclude that

$$\Big(\int_{\mathbb{R}^n} (M_w^{\mathcal{D}} f)^p w \Big)^{1/p} \leq \frac{p}{p-1} \Big(\int_{\mathbb{R}^n} |f|^p w \Big)^{1/p}.$$

For an arbitrary function $f \in L^p(w)$, consider the truncated functions

$$f_t(x) = \begin{cases} f(x), & \text{if } |x| < t, |f(x)| < t \\ 0 & \text{otherwise} \end{cases}$$

and use the monotone convergence theorem with $t \to \infty$. $\square$

Denote

$$M_{\rho,w}^{\mathcal{D}} f(x) := \sup_{Q \in \mathcal{D}} \frac{1}{w(Q)^{1-\rho}} \int_Q |f| w 1_Q(x), 0 \leq \rho < 1.$$

The boundedness $M^{\mathcal{D}}_{\rho,w}$ from $L^p(w)$ to $L^q(w)$ was proved in [Moe12, Theorem 2.3] with constant

$$\left(1+\frac{p'}{q}\right)^{1-\rho} = (p')^{1-\rho}\left(1-\rho\right)^{1-\rho}.$$

When $\rho = 0$ we denote $M^{\mathcal{D}}_w := M^{\mathcal{D}}_{0,w}$ and we get the well known sharp bound

$$\|M^{\mathcal{D}}_w f\|_{L^p(w)} \le p'\|f\|_{L^p(w)}.$$

See below for the boundedness of $M^{\mathcal{D}}_{\rho,w}$ from $L^p(w)$ to $L^q(w)$. The reasoning is identical to what was done in [Moe12, Theorem 2.3].

Theorem 2.1.8. *Let $0 \le \rho < 1, 1 < p < 1/\rho$ and $1/q = 1/p - \rho$. Then for every dyadic grid $\mathcal{D}$, and every measure w we have*

$$\left\|\sup_Q \frac{1}{w(Q)^{1-\rho}}\int_Q |f| w 1_Q\right\|_{L^q(w)} \lesssim_{\rho,p} \|f\|_{L^p(w)}$$

and the constant does not depend on the measure w and the dyadic grid $\mathcal{D}$.

Proof. By the standard properties of dyadic cubes we get the inequality

$$w(\{x : M^{\mathcal{D}}_{\rho,w} f(x) > \lambda\}) \le \left(\frac{1}{\lambda}\int_{\{M^{\mathcal{D}}_{\rho,w} f>\lambda\}} |f(x)|dw(x)\right)^{\frac{1}{1-\rho}}.$$

Let $q_o = \frac{1}{1-\rho}$, then $q > q_0$. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} M^{\mathcal{D}}_{\rho,w} f(x)^q dw &= q\int_0^\infty \lambda^{q-1} w(\{x : M^{\mathcal{D}}_{\rho,w} f(x) > \lambda\})d\lambda \\
&\le q\int_0^\infty \lambda^{q-1}\left(\frac{1}{\lambda}\int_{\{M^{\mathcal{D}}_{\rho,w} f(x)>\lambda\}} |f(x)|dw(x)\right)^{q_0} d\lambda \\
&\le q\left(\int_{\mathbb{R}^n} |f(x)|\left(\int_0^{M^{\mathcal{D}}_{\rho,w} f(x)} \lambda^{q-q_0-1}\right)^{1/q_0} dw(x)\right)^{q_0} \\
= \frac{q}{q-q_0}\left(\int_{\mathbb{R}^n} |f(x)| M^{\mathcal{D}}_{\rho,w} f(x)^{\frac{q}{p'}} dw(x)\right)^{q_0} &\le \frac{q}{q-q_0}\|f\|^{q_0}_{L^p(w)}\|M^{\mathcal{D}}_{\rho,w} f\|^{\frac{qq_0}{p'}}_{L^q(w)},
\end{aligned}$$

where in the second inequality we used Minkowski's integral inequality and Hölder's inequality in the last. Note only that $\frac{q}{q-q_0} = 1 + \frac{p'}{q}$. □

2.2 Proof of Theorem 2.0.2

Proof. We define the collection F_i of cubes for the pair $(f_i, w_i), i = 1, 2$, i.e.,

$$\mathcal{F}_i = \bigcup_{k=0} \mathcal{F}_i^k$$

where $\mathcal{F}_i^0 := \{Q_0\}, Q_0 \in \mathcal{S}$

$$\mathcal{F}_i^{k+1} := \bigcup_{F \in \mathcal{F}_i^k} ch(F)$$

and

$$ch(F) = \max\left\{Q \subset F : w_i(F)^{-1} \int_F f_i w_i < \frac{1}{2} w_i(Q)^{-1} \int_Q f_i w_i\right\}.$$

We define, for $Q \in \mathcal{S}$,

$$\begin{cases} \pi_1(Q) := \min\{F_1 \supseteq Q : F_1 \in \mathcal{F}_1\} \\ \pi_2(Q) := \min\{F_2 \supseteq Q : F_2 \in \mathcal{F}_2\}. \end{cases}$$

Then we can rewrite the series

$$\sum_{Q \subset Q_0} = \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_2 \in \mathcal{F}_2}} \sum_{\substack{Q : \pi_1(Q) = F_1 \\ \pi_2(Q) = F_2}} \leq \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_2 \subseteq F_1}} \sum_{\substack{Q : \pi_1(Q) = F_1 \\ \pi_2(Q) = F_2}} + \sum_{\substack{F_2 \in \mathcal{F}_2 \\ F_1 \subsetneq F_2}} \sum_{\substack{Q : \pi_1(Q) = F_1 \\ \pi_2(Q) = F_2}},$$

where we observed that if the inner sum over $Q : \pi(Q) = (F_1, F_2)$ is not empty, then there is some $Q \subseteq F_1 \cap F_2$, hence $F_1 \cap F_2 \neq \emptyset$, and thus $F_2 \subseteq F_1$ or $F_1 \subsetneq F_2$. Since the proof can be done in completely symmetric way, we shall concentrate ourselves on the first case. Consider Q with $\pi_1(Q) \subset \pi_2(Q)$. Since $Q \subseteq F_2 \subseteq F_1$, $\int_{F_2} w_2 = \int_{F_2} 1_{F_2} w_2$ $\int_Q w_2 = \int_Q 1_Q w_2$, $(w_1)_Q := \frac{1}{\mu(Q)} w_1(Q)$ and by definition of $ch(F_2)$, we have

$$
\begin{aligned}
(i) &:= \sum_{\substack{F_1\in\mathcal{F}_1\\ F_2\subseteq F_1}} \sum_{\substack{Q:\pi_1(Q)=F_1\\ \pi_2(Q)=F_2}} \mu(Q)^{-1}\prod_{i=1}^{2}\Big(\int_Q f_i w_i\Big)\\
&= \sum_{\substack{F_1\in\mathcal{F}_1\\ F_2\subseteq F_1}} \sum_{\substack{Q:\pi_1(Q)=F_1\\ \pi_2(Q)=F_2}} \mu(Q)\prod_{i=1}^{2}\Big(w_i(Q)^{-1}\int_Q f_i w_i\Big)(w_i)_Q\\
&\le 2\int\sum_{F_1}\Big(\sum_{F_2:\pi_1(F_2)=F_1} 1_{F_2} w_2(F_2)^{-1}\int_{F_2} f_2 w_2\Big)\cdot\\
&\qquad\cdot\sum_{\substack{Q:\pi_1(Q)=F_1\\ \pi_2(Q)=F_2}} \mu(Q)\Big(w_1(Q)^{-1}\int_Q f_1 w_1\Big)(w_1)_Q(w_2)_Q.
\end{aligned}
$$

By definition $ch(F_1)$,

$$
\begin{aligned}
(i) \le 4\int\sum_{F_1}\Big(\sum_{F_2:\pi_1(F_2)=F_1} 1_{F_2} w_2(F_2)^{-1}\int_{F_2} f_2 w_2\Big)\Big(1_{F_1} w_1(F_1)^{-1}\int_{F_1} f_1 w_1\Big)\cdot\\
\cdot\sum_{Q:\pi_1(Q)=F_1} 1_Q (w_1)_Q\, dw_2.
\end{aligned}
$$

Applying Hölder with the pair of conjugate exponents p_2 and p_1 we obtain

$$
\begin{aligned}
(i) &\lesssim \Big\{\int\sum_{F_1}\Big[\sum_{F_2:\pi_1(F_2)=F_1} 1_{F_2}(w_2(F_2)^{-1}\int_{F_2} f_2 w_2)\Big]^{p_2} dw_2\Big\}^{1/p_2}\\
&\cdot\Big\{\int\sum_{F_1}\Big(1_{F_1} w_1(F_1)^{-1}\int_{F_1} f_1 w_1\Big)^{p_1}\Big(\sum_{Q:\pi_1(Q)=F_1} 1_Q (w_1)_Q\Big)^{p_1} dw_2\Big\}^{1/p_1}\\
&\qquad := (I)\cdot(II).
\end{aligned}
$$

For (II) note that

$$
(II) \le \Big\{\sum_{F_1}\Big(1_{F_1} w_1(F_1)^{-1}\int_{F_1} f_1 w_1\Big)^{p_1}\int\Big(\sum_{Q:\pi_1(Q)=F_1} 1_Q (w_1)_Q\Big)^{p_1} dw_2\Big\}^{1/p_1}.
$$

By Lemma 2.1.5

$$
\int\Big(\sum_{Q:\pi_1(Q)=F_1} 1_Q (w_1)_Q\Big)^{p_1} dw_2 \lesssim \sup_Q (w_1)_Q^{p_1-1}\sup_Q (w_2)_Q \sum_{Q:\pi_1(Q)=F_1}\mu(Q)(w_1)_Q.
$$

So we obtain

$$
\begin{aligned}
&(II) \\
&\le \left\{ \sum_{F_1} \left(1_{F_1} w_1(F_1)^{-1} \int_{F_1} f_1 w_1 \right)^{p_1} \sup_Q (w_1)_Q^{p_1-1} \sup_Q (w_2)_Q \sum_{Q:\pi_1(Q)=F_1} \mu(Q)(w_1)_Q \right\}^{\frac{1}{p_1}} \\
&\le \left(\sup_Q (w_1)_Q^{\frac{p_1-1}{p_1}} \sup_Q (w_2)_Q^{1/p_1} \right) \left\{ \sum_{F_1} \left(1_{F_1} w_1(F_1)^{-1} \int_{F_1} f_1 w_1 \right)^{p_1} [w_1]_{FW} \int_{F_1} w_1 \right\}^{\frac{1}{p_1}} \\
&= \left(\sup_Q (w_1)_Q^{p_1-1/p_1} \sup_Q (w_2)_Q^{1/p_1} \right) [w_1]_{FW}^{1/p_1} \left(\int \sum_{F_1} \left(1_{F_1} w_1(F_1)^{-1} \int_{F_1} f_1 w_1 \right)^{p_1} w_1 \right)^{\frac{1}{p_1}},
\end{aligned}
$$

where

$$
[w_1]_{FW}^{1/p_1} = \sup_Q \left(\int_Q \sup_{x\in Q', Q'\in\mathcal{D}} \mu(Q')^{-1} \int_{Q'} 1_Q w_1 \right)^{1/p_1} \left(\int_Q w_1 \right)^{-1/p_1}
$$

and we use in the last inequality that

$$
\sum_{Q:\pi_1(Q)=F_1} \mu(Q)(w_1)_Q \lesssim \sum_{Q\subseteq F_1} \mu(E_Q)(w_1)_Q \le \int_{F_1} M w_1 \le [w_1]_{FW} \int_{F_1} w_1.
$$

Therefore, we have

$$
\begin{aligned}
(i) &\lesssim (I)\cdot(II) \\
&\lesssim \left\| \sup_{Q\in\mathcal{D}} \left(w_2(Q)^{-1} \int_Q f_2 w_2 \right) \right\|_{L^{p_2}(w_2)} \cdot \left(\sup_Q (w_1)_Q^{1/p_1'} \sup_Q (w_2)_Q^{1/p_2'} \right) \\
&\quad \cdot \sup_Q \left(\int_Q \sup_{x\in Q', Q'\in\mathcal{D}} \mu(Q')^{-1} \int_{Q'} 1_Q w_1 \right)^{1/p_1} \left(\int_Q w_1 \right)^{-1/p_1} \\
&\quad \cdot \left(\int \sum_{F_1} \left(1_{F_1} w_1(F_1)^{-1} \int_{F_1} f_1 w_1 \right)^{p_1} w_1 \right)^{1/p_1}.
\end{aligned}
$$

Moreover, choose $\lambda_{F_1} := w_1(Q)^{-1} \int_Q f_1 w_1$ and note that

$$
\begin{aligned}
\sum_{F_1} \lambda_{F_1}^{p_1} 1_{F_1}(x) = \sum_{Q_1\supseteq\cdots\supseteq Q_N\ni x} \lambda_{Q_i}^{p_1} &\le \sum_i 2^{(i-N)p_1} \lambda_{Q_N}^{p_1} \le C\lambda_{Q_N}^{p_1} \\
&\lesssim \sup_{F_1\ni x} \lambda_F^{p_1} =: (M\lambda(x))^{p_1},
\end{aligned}
$$

where in the first inequality we use $\lambda_{Q_{j+l}} \geq 2^l \lambda_{Q_j}$, $j+l=N$ and $j=i$. Then

$$\sum_{F_1} \left(1_{F_1} w_1(F_1)^{-1} \int_{F_1} f_1 w_1 \right)^{p_1} \lesssim (M w_1 f_1)^{p_1}.$$

By Theorem 6.1.3 we conclude the proof. □

2.3 Proof of Theorem 2.0.1

Proof. The left-hand side of the conclusion can be estimated by

$$\left[\int \left(\sum_{Q \in S} \frac{1}{\mu(Q)^2} \left(\prod_{i=1}^{2} \int f_i w_i \right) 1_Q \right)^q dw_3 \right]^{\frac{1}{q}}.$$

We will use the notation of the previous theorem. Since

$$\sum_Q \approx \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_2 \in \mathcal{F}_2}} \sum_{\substack{Q : \pi_1(Q) = F_1 \\ \pi_2(Q) = F_2}} \tag{2.3.1}$$

and by definition of $ch(F_2)$ and $ch(F_1)$ we obtain that (2.3.1) is estimated by

$$\left\{ \int \left[\sum_{F_1, F_2} \lambda_{2,F_2} \lambda_{1,F_1} \sum_{\substack{Q : \pi_1(Q) = F_1 \\ \pi_2(Q) = F_2}} (w_1)_Q (w_2)_Q 1_Q \right]^q dw_3 \right\}^{\frac{1}{q}},$$

with

$$\lambda_{i,F_i} = 1_{F_i} w_i(F_i)^{-1} \int_{F_i} f_i w_i.$$

By the subadditivity of the function $x \to x^{1/\alpha}$ (i.e., $(x+y)^q \leq x^q + y^q$) this is bounded by

$$\left\{ \sum_{F_1, F_2} \lambda_{2,F_2}^q \lambda_{1,F_1}^q \int \left(\sum_{\substack{Q : \pi_1(Q) = F_1 \\ \pi_2(Q) = F_2}} (w_1)_Q (w_2)_Q 1_Q \right)^q dw_3 \right\}^{\frac{1}{q}}. \tag{2.3.2}$$

Using [Zor16, Lemma 2.4] with

$$\tilde{s_1} = \tilde{s_2} = q, \tilde{s_3} = 0, \tilde{q_1} = q/p_1', \tilde{q_2} = q/p_2', \tilde{q_3} = 1$$

it can be proved that

$$\int \left(\sum_Q 1_Q (w_1)_Q (w_2)_Q \right)^q dw_3 \lesssim \sup_Q (w_1)_Q^{\frac{q}{p_1'}} (w_2)_Q^{\frac{q}{p_2'}} (w_3)_Q \sum_Q \mu(Q) (w_1)_Q^{\frac{q}{p_1}} (w_2)_Q^{\frac{q}{p_2}}.$$

So (2.3.2) is estimated by

$$\left\{ \sum_{F_1,F_2} \lambda_{2,F_2}^q \lambda_{1,F_1}^q [\overrightarrow{w}]^{\tilde{q}} \sum_Q \mu(Q)(w_1)_Q^{\frac{q}{p_1}} (w_2)_Q^{\frac{q}{p_2}} \right\}^{\frac{1}{q}}, \tag{2.3.3}$$

where

$$[\overrightarrow{w}]^{\tilde{q}} = \sup_Q (w_1)_Q^{\frac{q}{p_1'}} (w_2)_Q^{\frac{q}{p_2'}} (w_3)_Q.$$

So by definition of sup, (2.3.3) gives the estimate

$$\le [\overrightarrow{w}]^{\tilde{q}} \Bigg(\sum_{F_1,F_2} \lambda_{1,F_1}^q \lambda_{2,F_2}^q [\overrightarrow{w}]_{FW}^{(1/p_i)_{i\neq 3}} \int_{F_1\cap F_2} \prod_{i=1}^2 w_i^{\frac{q}{p_i}} \Bigg)^{\frac{1}{q}}$$

$$= [\overrightarrow{w}]^{\tilde{q}} [\overrightarrow{w}]_{FW}^{(1/p_i)_{i\neq 3}} \Bigg(\int \prod_{i=1}^2 \sum_{F_i} 1_{F_i} \lambda_{i,F_i}^q w_i^{q/p_i} \Bigg)^{\frac{1}{q}}$$

with

$$[\overrightarrow{w}]_{FW}^{(1/p_i)_{i\neq 3}} = \left[\sup_Q \left(\int_Q \sup_{x\in Q', Q'\in\mathcal{D}} \mu(Q')^{-1} \int_{Q'} 1_Q w_1 \right)^{1/p_1} \left(\int_Q w_1 \right)^{-1/p_1} \right] \cdot \left[\sup_Q \left(\int_Q \sup_{x\in Q', Q'\in\mathcal{D}} \mu(Q')^{-1} \int_{Q'} 1_Q w_2 \right)^{1/p_2} \left(\int_Q w_2 \right)^{-1/p_2} \right].$$

By Hölder's inequality we obtain the estimate

$$\le [\overrightarrow{w}]^{\tilde{q}} [\overrightarrow{w}]_{FW}^{(1/p_i)_{i\neq 3}} \prod_{i=1}^2 \Bigg(\int \sum_{F_i} 1_{F_i} \lambda_{i,F_i}^{p_i} w_i \Bigg)^{1/p_i}.$$

By definition of stopping times this is bounded by

$$[\overrightarrow{w}]^{\tilde{q}} [\overrightarrow{w}]_{FW}^{(1/p_i)_{i\neq 3}} \prod_{i=1}^2 ||M\lambda_i||_{L^{p_i}(w_i)}$$

and the result follows by Theorem 6.1.3. $\square$

Chapter 3

Weak type A_p estimates

It is conjectured in [Zor16] that a weak type version of Theorem 2.0.1 holds with a smaller constant, namely the multilinear A_p characteristic. Here we show that the Fujii–Wilson characteristic can be replaced by the multilinear A_p characteristic by a (larger) product of two linear A_p characteristics.

In this sense, here we give a weak $L^{p_1} \times L^{p_2} \to L^{q,\infty}$ estimate for a bilinear dyadic fractional sparse operator, where $q < 1$ and $1 < p_i < \infty$ with $q^{-1} = \sum_i 1/p_i$ and $i = 1, 2$. The method is inspired by the proof of Theorem 1.2 in [HL18].

We obtain the following new result.

Theorem 3.0.1. *Let* $q = \left(\sum_{i=1}^{2} \frac{1}{p_i} \right)^{-1} < 1, 1 < p_i < \infty, i = 1, 2, w_1, w_2, w_3$ *be weights,* μ *nonnegative measure,* $f_i, i = 1, 2,$ *positive measurable functions and* S *sparse. Then*

$$\left\| \sum_{Q \in \mathcal{S}} \frac{1}{\mu(Q)^2} \prod_{i=1}^{2} \left(\int f_i w_i \right) 1_Q \right\|_{L^{q,\infty}(w_3)} \lesssim C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)},$$

where $C = \left[\sup_Q (w_3)_Q (w_1)_Q^{(p_1 - 1)} \right]^{\frac{1}{p_1}} \left[\sup_P (w_3)_P (w_2)_P^{(p_2 - 1)} \right]^{\frac{1}{p_2}}.$

3.1 Notation and tools

Let $0 < p < \infty$ and f in the weak Lebesgue space

$$L^{p,\infty}(w) = \{ f : \mathbb{R}^n \to \mathbb{R}; \|f\|_{L^{p,\infty}(w)} := \sup_{\lambda > 0} \lambda w(\{x : |f(x)| \geq \lambda\})^{1/p} < \infty \}.$$

We have the following dual expression.

Lemma 3.1.1.

$$\sup_{\lambda>0} \lambda w(\{x : |f(x)| \geq \lambda\})^{1/p} \sim \sup_{0<w(E)<\infty} \inf_{E'\subset E: w(E')\geq \frac{1}{2}w(E)} w(E)^{-1/p'} \int_{E'} |f| dw.$$

Proof. First we show $\lesssim$. Let A be the value of the right-hand side. Let $E \subset \{|f| \geq \lambda\}$ be a finite measure subset. We have to show $\lambda w(E)^{1/p} \lesssim A$. By definition of A for every $\epsilon > 0$ there exists $E' \subset E$ with $w(E') \geq \frac{1}{2}w(E)$ and $w(E)^{-1/p'} \int_{E'} |f| dw \leq A + \epsilon$. Hence

$$w(E) \leq 2w(E') = 2\lambda^{-1} \int_{E'} \lambda dw \leq 2\lambda^{-1} \int_{E'} |f| dw \leq 2\lambda^{-1} w(E)^{1/p'} (A + \epsilon).$$

Dividing both sides by $w(E')^{1/p'}$ we obtain

$$w(E)^{1/p} \lesssim \lambda^{-1}(A + \epsilon).$$

Note that this inequality does not involve E'. Taking infimum over ϵ we obtain

$$w(E)^{1/p} \lesssim \lambda^{-1} A,$$

and this concludes the proof of the inequality $\lesssim$.

Now we show $\gtrsim$. Let A be the value of the left-hand side. Let E be a measurable set with $0 < w(E) < \infty$. We have to show that there exists a subset $E' \subset E$ such that $w(E') \geq \frac{1}{2}w(E)$ and $w(E)^{-1/p'} \int_{E'} |f| dw \lesssim A$. Let $\lambda > 0$ be chosen later and let

$$E'' := \{|f| > \lambda\}, \quad E' := E \setminus E''.$$

Then by definition of A we have

$$w(E'') \leq (A/\lambda)^p.$$

Choosing $\lambda = A(w(E)/2)^{-1/p}$ we ensure $w(E') \geq \frac{1}{2}w(E)$. Moreover,

$$w(E)^{-1/p'} \int_{E'} |f| dw \leq w(E)^{-1/p'} \int_{E'} \lambda dw \leq w(E)^{-1/p'} \lambda w(E') \leq \lambda w(E)^{1/p} \lesssim A.$$

This concludes the proof of $\gtrsim$. □

Let now $1 < p_1, p_2, p_3 < \infty$.

Consider

$$T(f_1 w_1, f_2 w_2)(x) = \sum_{Q\in S} \frac{1}{\mu(Q)^2} \prod_{i=1}^{2} (\int_Q f_i w_i) 1_Q(x).$$

By Lemma 3.1.1 we see that the following (I) and (II) are equivalent :

(I) for all $E : \mu(E) < \infty$ there exists $E' \subseteq E : w_3(E') \geq \frac{1}{2}w_3(E)$ such that

$$\int |T(f_1w_1, f_2w_2)1_{E'}|dw_3 \lesssim \|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}w_3(E)^{1/p_3}$$

(II)

$$\|T(f_1w_1, f_2w_2)\|_{L^{p_3',\infty}(w_3)} \lesssim \prod_{i=1,2}\|f_i\|_{L^{p_i}(w_i)}.$$

To facilitate comparison with other chapters we note that (I) can be written as

$$|\sum_{Q\in S}\frac{1}{\mu(Q)^2}\prod_{i=1}^{3}\int_Q f_iw_i| \leq C\prod_{i=1}^{3}\|f_i\|_{L^{p_i}(w_i)}$$

with $f_3 = 1_{E'}$ and $\|f_3\|_{L^{p_3}(w_3)} = w_3(E')^{1/p_3}$.

3.2 Proof of Theorem 3.0.1

We want to prove that

$$\sup_{\lambda>0}\lambda w_3\Big\{\sum_{Q\in\mathcal{S}}\frac{1}{\mu(Q)^2}\prod_{i=1}^{2}\Big(\int f_iw_i\Big)1_Q > \lambda\Big\}^{\frac{1}{q}}$$
$$\lesssim \Big[\sup_Q(w_3)_Q(w_1)_Q^{(p_1-1)}\Big]^{\frac{1}{p_1}}\Big[\sup_P(w_3)_P(w_2)_P^{(p_2-1)}\Big]^{\frac{1}{p_2}}\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(w_i)}$$

for every $\lambda > 0$. By homogeneity it suffices to consider

$$\lambda = \sum_{l,k\geq 0}2^{-\epsilon(l+k)}, \quad \epsilon = \frac{2-q}{2}, \tag{3.2.1}$$

so that $\epsilon - 1 + q < 0$. Let

$$S_{l,k} = \Big\{Q\in\mathcal{S} : \frac{B_1}{2} < \frac{(f_1w_1)_Q}{2^{-k}} \leq B_1, \frac{B_2}{2} < \frac{(f_2w_2)_Q}{2^{-l}} \leq B_2\Big\}, \quad k,l > 0,$$

and

$$S^j_{-B_j} : \{Q\in\mathcal{S} : (f_jw_j)_Q > B_j\}, j = 1,2,$$

with

$$(f_jw_j)_Q := \frac{1}{\mu(Q)}\int_Q f_jw_j,$$

and B_j constants to be chosen later. We have

$$w_3\Big\{ \sum_{Q\in S} \frac{1}{\mu(Q)^2} \prod_{i=1}^{2} \Big(\int f_i w_i \Big) 1_Q > \lambda \Big\} = w_3\Big\{ \sum_{Q\in S} (f_1 w_1)_Q (f_2 w_2)_Q 1_Q > \lambda \Big\}$$
$$\leq w_3\Big\{ \sum_{l,k\geq 0} \sum_{Q\in S_{l,k}} (f_1 w_1)_Q (f_2 w_2)_Q 1_Q > \lambda \Big\} + w_3\Big\{ \bigcup_{Q\in S^1_{-B_1}} Q \Big\} + w_3\Big\{ \bigcup_{Q\in S^2_{-B_2}} Q \Big\}.$$

Note that

$$w_3\Big\{ \sum_{l,k\geq 0} \sum_{Q\in S_{l,k}} (f_1 w_1)_Q (f_2 w_2)_Q 1_Q > \lambda \Big\}$$
$$= w_3\Big\{ \sum_{l,k\geq 0} \sum_{Q\in S_{l,k}} (f_1 w_1)_Q (f_2 w_2)_Q 1_Q > \sum_{l,k\geq 0} 2^{-\epsilon(l+k)} \Big\}$$
$$\leq \sum_{l,k\geq 0} w_3\Big\{ \sum_{Q\in S_{l,k}} 2^{-l} 2^{-k} B_1 B_2 1_Q > 2^{-\epsilon(l+k)} \Big\}$$

$$= \sum_{l,k\geq 0} w_3\Big\{ \sum_{Q\in S_{l,k}} 1_Q > (B_1 B_2)^{-1} 2^{(l+k)} 2^{-\epsilon(l+k)} \Big\}. \tag{3.2.2}$$

Considering $\mathcal{S}$ such that

$$\mu\left(\bigcup_{\substack{Q'\subset Q \\ Q',Q\in\mathcal{S}}} Q' \right) \leq \frac{1}{4}\mu(Q),$$

and

$$E_S(Q) = Q \setminus \bigcup_{Q'\in ch_S(Q)} Q'$$

with

$$ch_S(Q) = \{Q' \in \mathcal{S} : Q' \subset Q,\ \nexists Q'' \in \mathcal{S} : Q' \subset Q'' \subset Q\}$$

and since $1_{E(Q)} = 1_Q - \sum_{Q' \in ch(Q)}$, we have

$$
\begin{aligned}
(f_1 w_1 1_{E_{S_{l,k}}(Q)})_Q &= \frac{1}{\mu(Q)} \int_Q f_1 w_1 - \frac{1}{\mu(Q)} \sum_{Q' \in ch_{S_{l,k}}(Q)} \int_{Q'} f_1 w_1 \\
&= \frac{1}{\mu(Q)} \int_Q f_1 w_1 - \sum_{Q' \in ch_{S_{l,k}}(Q)} \frac{\mu(Q')}{\mu(Q)} \frac{1}{\mu(Q')} \int_{Q'} f_1 w_1 \\
&\geq \frac{1}{\mu(Q)} \int_Q f_1 w_1 - \frac{1}{4} 2^{-k} \geq \frac{1}{2} (f_1 w_1)_Q .
\end{aligned}
$$

Moreover , since $1_{E(Q)} \leq 1$, then

$$f_1 w_1 1_{E(Q)} \leq f_1 w_1 \quad \text{and} \quad (f_1 w_1 1_{E(Q)})_Q \leq (f_1 w_1)_Q .$$

So $(f_1 w_1 1_{E(Q)})_Q \sim (f_1 w_1)_Q$. Analogously $(f_2 w_2 1_{E(Q)})_Q \sim (f_2 w_2)_Q$. Then

$$(f_1 w_1 1_{E(Q)})_Q^q (f_2 w_2 1_{E(Q)})_Q^q \approx 2^{-(l+k)q} (B_1 B_2)^q . \tag{3.2.3}$$

Using the property that $w_3\{f > a\} \leq a^{-1} \int |f| w_3$ and (3.2.3) we have

$$
\begin{aligned}
&\sum_{l,k \geq 0} w_3 \Big\{ \sum_{Q \in S_{l,k}} 1_Q > (B_1 B_2)^{-1} 2^{(l+k)} 2^{-\epsilon(l+k)} \Big\} \\
&\leq \sum_{l,k \geq 0} B_1 B_2 2^{(l+k)(\epsilon-1)} (B_1 B_2)^{-q} 2^{(l+k)q} \int \sum_{Q \in S_{l,k}} 1_Q (f_1 w_1 1_{E_{S_{l,k}}(Q)})_Q^q (f_2 w_2 1_{E_{S_{l,k}}(Q)})^q w_3 .
\end{aligned}
$$

Note that

$$\frac{q}{p_1} + \frac{q}{p_2} = 1 .$$

So by Hölder inequality with exponents $p_1 q^{-1}$ and $p_2 q^{-1}$ we have

$$
\begin{aligned}
&\sum_{Q \in S_{l,k}} w_3(Q) (f_1 w_1 1_{E(Q)})_Q^q (f_2 w_2 1_{E(Q)})_Q^q \\
&\qquad \leq \Big(\sum_{Q \in S_{l,k}} w_3(Q) (f_1 w_1 1_{E(Q)})_Q^{p_1} \Big)^{\frac{q}{p_1}} \Big(\sum_{Q \in S_{l,k}} w_3(Q) (f_2 w_2 1_{E(Q)})_Q^{p_2} \Big)^{\frac{q}{p_2}} .
\end{aligned}
$$

Moreover, also by Hölder inequality

$$
\begin{aligned}
(f_i w_i 1_{E(Q)})_Q^{p_i} &\leq \Big(\frac{1}{\mu(Q)} \int_Q f_i^{p_i} w_i 1_{E(Q)} \Big) \Big(\frac{1}{\mu(Q)} \int_Q 1^{p_i'} w_i \Big)^{p_i/p_i'} \\
&= \Big(\frac{1}{\mu(Q)} \int_Q f_i^{p_i} w_i 1_{E(Q)} \Big) (w_i)_Q^{p_i - 1}, i = 1, 2,
\end{aligned}
$$

and by $\int 1_Q w_3 = w_3(Q)$, we have

$$\int \sum_{Q\in S_{l,k}} 1_Q (f_1 w_1 1_{E(Q)})_Q^q (f_2 w_2 1_{E(Q)})_Q^q w_3 = \sum_{Q\in S_{l,k}} w_3(Q)(f_1 w_1 1_{E(Q)})_Q^q (f_2 w_2 1_{E(Q)})_Q^q$$
$$\leq \Big[\sum_{Q\in S_{l,k}} \frac{w_3(Q)(w_1)_Q^{(p_1-1)}}{\mu(Q)}(\int_Q f_1^{p_1} w_1 1_{E(Q)})\Big]^{\frac{q}{p_1}}\cdot$$
$$\cdot\Big[\sum_{Q\in S_{l,k}} \frac{w_3(Q)(w_2)_Q^{(p_2-1)}}{\mu(Q)}(\int_Q f_2^{p_2} w_2 1_{E(Q)})\Big]^{\frac{q}{p_2}}.$$

So, since $\int_Q f_i^{p_i} w_i 1_{E(Q)} = \int_{E(Q)} f_i^{p_i} w_i$,

$$\sum_{l,k\geq 0} w_3\Big\{\sum_{S_{l,k}} 1_Q > (B_1B_2)^{-1}2^{(l+k)}2^{-\epsilon(l+k)}\Big\}$$
$$\leq \sum_{l,k\geq 0}(B_1B_2)^{1-q}2^{(l+k)(\epsilon-1)}2^{(l+k)q}\Big[\sum_{Q\in S_{l,k}}\frac{w_3(Q)}{\mu(Q)}(w_1)_Q^{(p_1-1)}(\int_{E(Q)} f_1^{p_1}w_1)\Big]^{\frac{q}{p_1}}$$
$$\cdot\Big[\sum_{Q\in S_{l,k}}\frac{w_3(Q)}{\mu(Q)}(w_2)_Q^{(p_2-1)}(\int_{E(Q)} f_2^{p_2}w_2)\Big]^{\frac{q}{p_2}}.$$

Then using (3.2.2) we ontain

$$w_3\Big\{\sum_{l,k\geq 0}\sum_{Q\in S_{l,k}}(f_1w_1)_Q(f_2w_2)_Q 1_Q > \lambda\Big\} \leq (B_1B_2)^{1-q}\sum_{l,k\geq 0}2^{(l+k)(\epsilon-1+q)}\cdot$$
$$\cdot\Big[\sup_Q (w_3)_Q(w_1)_Q^{(p_1-1)}\Big]^{\frac{q}{p_1}}\Big[\sup_Q (w_3)_Q(w_2)_Q^{(p_2-1)}\Big]^{\frac{q}{p_2}}\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(w_i)}^q. \tag{3.2.4}$$

Moreover, since $Mfw > B \Leftrightarrow M\frac{fw}{B} > 1$ and by [Zor16, Theorem 1.7], we have

$$w_3\Big\{\bigcup_{Q\in S_{-1}^j} Q\Big\} \leq w_3\Big(M(f_iw_i) > B_i\Big) \lesssim \Big[\sup_Q (w_3)_Q(w_i)_Q^{p_i-B_j}\Big]\frac{\|f_i\|_{L^{p_i}(w_i)}^{p_i}}{B_i^{p_i}}. \tag{3.2.5}$$

From (3.2.4) and (3.2.5) we obtain

$$w_3\Big\{\sum_{Q\in S}\frac{1}{\mu(Q)^2}\prod_{i=1}^{2}\Big(\int f_iw_i\Big)1_Q > \lambda\Big\} \lesssim C_1^{\frac{q}{p_1}}C_2^{\frac{q}{p_2}}(B_1B_2)^{1-q}+C_1B_1^{-p_1}+C_2B_2^{-p_2}$$

with

$$C_i = \sup_Q (w_3)_Q (w_i)_Q^{(p_i-1)} \|f_i\|_{L^{p_i}(w_i)}^{p_i}, i = 1, 2.$$

Choose $B_1^{-p_1} = C_1^{\frac{q}{p_1}} C_2^{\frac{q}{p_2}} C_1^{-1}$ and $B_2^{-p_2} = C_1^{\frac{q}{p_1}} C_2^{\frac{q}{p_2}} C_2^{-1}$ then

$$B_1^{-1} B_2^{-1} = C_1^{-\frac{1}{p_1}} C_2^{-\frac{1}{p_2}} C_1^{\frac{q}{p_1 p_2}} C_1^{\frac{q}{p_1 p_1}} C_2^{\frac{q}{p_2 p_2}} C_2^{\frac{q}{p_1 p_2}} = 1$$

and

$$C_1^{\frac{q}{p_1}} C_2^{\frac{q}{p_2}} (B_1 B_2)^{1-q} = C_1 B_1^{-p_1} = C_2 B_2^{-p_2} = C_1^{\frac{q}{p_1}} C_2^{\frac{q}{p_2}}.$$

So,

$$\begin{aligned} &\lambda^q w_3 \Big\{ \sum_Q \frac{1}{\mu(Q)^2} \prod_{i=1}^2 \int \Big(f_i w_i \Big) 1_Q > \lambda \Big\} \\ \lesssim &\left[\sup_Q (w_3)_Q (w_1)_Q^{(p_1-1)} \right]^{\frac{q}{p_1}} \left[\sup_P (w_3)_P (w_2)_P^{(p_2-1)} \right]^{\frac{q}{p_2}} \prod_{i=1}^2 \|f_i\|_{L^{p_i}(w_i)}^q \end{aligned}$$

for λ given by (3.2.2). By homogeneity

$$\begin{aligned} \Big\| \sum_{Q \in \mathcal{S}} \frac{1}{\mu(Q)^2} \prod_{i=1}^2 \Big(\int f_i w_i \Big) 1_Q \Big\|_{L^{q,\infty}(w_3)} &= \sup_{\lambda > 0} \lambda w_3 \Big\{ \sum_{Q \in \mathcal{S}} \frac{1}{\mu(Q)^2} \prod_{i=1}^2 \Big(\int f_i w_i \Big) 1_Q > \lambda \Big\}^{\frac{1}{q}} \\ &\lesssim \left[\sup_Q (w_3)_Q (w_1)_Q^{(p_1-1)} \right]^{\frac{1}{p_1}} \left[\sup_P (w_3)_P (w_2)_P^{(p_2-1)} \right]^{\frac{1}{p_2}} \prod_{i=1}^2 \|f_i\|_{L^{p_i}(w_i)}. \end{aligned}$$

Chapter 4

Trace inequalities via an auxiliary measure

In this chapter we give an introduction to weighted strong $L^p \to L^q$ estimates for non homogeneous discrete linear operators, in case $0 < q < 1$ and $p > 1$. The main results of this section are Theorems 4.3.4 and 4.3.14. The method is inspired by the ideas of the homogeneous case given in [Ver99]. We give a Wolf type and a Sawyer type conditions for the specific case in which we deal only with the dyadic operator (See Lemma 4.1.1, proof of Lemma 4.2.1 and additional hypothesis (4.2.7) in the Proposition 4.2.3).

We consider

$$T(f)(x) = \sum_{Q \in \mathcal{D}} \frac{\lambda_Q}{\mu(Q)} \Big(\int_Q f(y) d\mu(y) \Big) 1_Q(x),$$

where λ_Q, $Q \in \mathcal{D}$ are nonnegative numbers.

Note here that we work with (1.0.9) where $w_1, w_2 = 1$, $\mu_2 = \sigma$ an arbitrary nonnegative measure on $\mathbb{R}^n$ and $\mu_1 = \mu$ Lebesgue measure and we denote $T_\lambda := T$.

The term trace was first used by Elias Stein in relation to traces of Bessel potentials in hyperplanes, with respect to the corresponding Lebesgue measure on the hyperplane (see, for example, [KS86]).

4.1 Wolff type inequality

Theorem 4.1.1. *Let σ be a positive Borel measure on $\mathbb{R}^n$ and $1 < p < \infty$ and $(\lambda_Q)_{Q \in \mathcal{D}}$ be a sequence of nonnegative numbers associated with dyadic* *cubes Q. Assume that*

$$\sum_{Q'\subseteq Q} \lambda_{Q'}\sigma(Q') \sim \lambda_Q \sigma(Q) \tag{4.1.2}$$

for every Q. Then there is $A > 0$ so that

$$\int \left(\sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\mu(Q)} \sigma(Q) 1_Q(x) \right)^{p'} d\mu(x) = \int_{\mathbb{R}^n} (T\sigma)^{p'} d\mu(x) \leq A \sum_Q \left[\frac{\sigma(Q)}{\mu(Q)} \lambda_Q \right]^{p'} \mu(Q).$$

Proof. By Lemma 2.1.3 we have

$$\int_{\mathbb{R}^n} (T\sigma)^{p'} d\mu(x) \lesssim \sum_Q \lambda_Q \sigma(Q) \Big(\mu(Q)^{-1} \sum_{Q'\subseteq Q} \lambda_{Q'}\sigma(Q') \Big)^{p'-1}$$
$$\leq \sum_Q \lambda_Q \sigma(Q) \big(\mu(Q)^{-1} \lambda_Q \sigma(Q) \big)^{p'-1} \leq \sum_Q \sigma(Q)^{p'} \lambda_Q^{p'} \mu(Q)^{1-p'}.$$

□

4.2 Kerman–Sawyer type theorem

We define

$$T_Q(\sigma) = \sum_{P\subseteq Q} \frac{\lambda_P}{\mu(P)} \sigma(P) 1_P \quad \text{and} \quad T_Q^{'}(\sigma) = \sum_{P\nsubseteq Q} \frac{\lambda_P}{\mu(P)} \sigma(P) 1_P$$

and also

$$\tilde{T}(g d\sigma)(x) := \sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\mu(Q)} \Big(\int_Q g(y) d\sigma(y) \Big) 1_Q(x), g \geq 0.$$

Lemma 4.2.1. *Let $1 \leq p < \infty$, σ be a positive Borel measure on $\mathbb{R}^n$ and $0 \leq g \in L^1_{loc}(d\sigma)$. Suppose that*

$$\tilde{T}(g d\sigma)(x) < \infty.$$

Then

$$[\tilde{T}(g d\sigma)(x)]^p \leq C \tilde{T}[g(\tilde{T}(g d\sigma))^{p-1} d\sigma](x). \tag{4.2.2}$$

Proof. We have

$$\begin{aligned}
\tilde{T}(gd\sigma)(x)^p &= \Big(\sum_{x\in Q\in\mathcal{D}} \frac{\lambda_Q}{\mu(Q)}(\int_Q g(y)d\sigma(y))\Big)^p \\
&\le p\sum_{x\in Q} \frac{\lambda_Q}{\mu(Q)}(\int_Q g(y)d\sigma(y))\Big(\sum_{Q'\supseteq Q} \frac{\lambda_{Q'}}{\mu(Q')}(\int_{Q'} g(y)d\sigma(y))\Big)^{p-1} \\
&\le p\sum_{x\in Q} \frac{\lambda_Q}{\mu(Q)}(\int_Q g(y)d\sigma(y))\Big(\inf_Q \tilde{T}(gd\sigma)(x)\Big)^{p-1} \\
&\le p\sum_{x\in Q} \frac{\lambda_Q}{\mu(Q)}(\int_Q g(y)(\tilde{T}(gd\sigma))^{p-1}d\sigma(y)) \\
&= p\tilde{T}[g(\tilde{T}(gd\sigma))^{p-1}d\sigma](x), \qquad \square
\end{aligned}$$

where the first inequality follows by (2.1.2).

Proposition 4.2.3. *Let $1 < p < \infty$, σ be a positive Borel measure on $\mathbb{R}^n$ and v be defined by*

$$dv = (T\sigma)^{p'}dx.$$

Then the following conditions are equivalent.

1.
$$T[(T\sigma)^{p'}](x) \le cT\sigma(x) < \infty \quad a.e. \tag{4.2.4}$$

2.
$$\|Tf\|_{L^p(dv)} \le Cc\|f\|_{L^p}, \forall f \in L^p(\mathbb{R}^n). \tag{4.2.5}$$

3.
$$\|Tf\|_{L^p(d\sigma)} \le Cc^{1/p'}\|f\|_{L^p}, \forall f \in L^p(\mathbb{R}^n), \tag{4.2.6}$$

If we additionally assume

$$\sup_{x,P:x\in P} \sum_{Q:Q\subsetneq P, x\in Q} \lambda_Q \Big[\sum_{S:P\subseteq S} \lambda_S^{1-p}\Big]^{p'-1} < \infty, \tag{4.2.7}$$

then the above conditions are also equivalent to

4. *DFS-condition*

$$\int_Q [T_Q(\sigma)]^{p'} d\mu(x) \le C\sigma(Q), \quad Q \in \mathcal{D}, \tag{4.2.8}$$

Proof of (4.2.5) $\implies$ (4.2.6). By Lemma 4.2.1 with $d\sigma = dx$, Fubini, and Hölder we conclude

$$\|Tf\|^p_{L^p(d\sigma)} \le C \int_{\mathbb{R}^n} T[f(Tf)^{p-1}]d\sigma = C \int_{\mathbb{R}^n} f(Tf)^{p-1}(T\sigma)dx \le C\|f\|_{L^p}\|Tf\|^{p-1}_{L^p(dv)}. \quad (4.2.9)$$

□

Proof of (4.2.4) $\implies$ (4.2.5). Repeating the above argument with v in place of σ, we obtain

$$\|Tf\|^p_{L^p(dv)} \le C\|f\|_{L^p}\|Tf\|^{p-1}_{L^p(dv_1)}, \quad (4.2.10)$$

where by (4.2.4)

$$dv_1 = (Tv)^{p'}dx = [T(T\sigma)^{p'}]^{p'}dx \le c^{p'}dv.$$

Here c is the constant in (4.2.2). Hence by (4.2.10) and the preceding estimate

$$\|Tf\|^p_{L^p(dv)} \le Cc^{p'(p-1)/p}\|f\|_{L^p}\|Tf\|^{p-1}_{L^p(dv)}.$$

Assuming that $\|Tf\|_{L^p(dv)} < \infty$, we get

$$\|Tf\|_{L^p(dv)} \le Cc\|f\|_{L^p}$$

which proves (4.2.5). □

Proof of (4.2.6) $\implies$ (4.2.8). By duality (4.2.6) is equivalent to the inequality

$$\|T(gd\sigma)\|_{L^{p'}} \le C\|g\|_{L^{p'}(d\sigma)}, \forall g \in L^{p'}(d\sigma).$$

Letting $g = 1_Q$, we see that (4.2.8) holds. □

Proof of (4.2.8) $\implies$ (4.2.4). Note that

$$T_S(\sigma) = \sum_{S:P\subset S} \frac{\sigma(S)\lambda_S 1_S}{\mu(S)} \ge \frac{\sigma(S)\lambda_S 1_S}{\mu(S)}$$

and hence it follows from (4.2.8) that

$$\sigma(S) \gtrsim \int_S T_S(\sigma)^{p'}d\mu(x) \ge \int_S \sigma(S)^{p'}\lambda_S^{p'}\mu(S)^{-p'}1_S d\mu(x) = \mu(S)^{1-p'}\lambda_S^{p'}\sigma(S)^{p'}.$$

So

$$\sigma(S) \lesssim \lambda_S^{-p}\mu(S). \tag{4.2.11}$$

Moreover by (4.2.8)

$$\begin{aligned} v(Q) = \int_Q (T\sigma)^{p'} d\mu(y) &\leq C \int (T_Q\sigma)^{p'} d\mu(y) + C \int_Q (T'_Q\sigma)^{p'} d\mu(y) \\ &\lesssim \sigma(Q) + r(Q), \end{aligned}$$

with $r(Q) = \mu(Q)\big(\sum_{P:Q\subsetneq P} \frac{\lambda_P}{\mu(P)}\sigma(P)\big)^{p'}$. Hence

$$Tv \lesssim T\sigma + \sum_Q \frac{\lambda_Q}{\mu(Q)} 1_Q r(Q).$$

Using Lemma 2.1.3 and (4.2.11)

$$\begin{aligned} \sum_Q \frac{\lambda_Q}{\mu(Q)} 1_Q r(Q) &\lesssim C \sum_Q \lambda_Q 1_Q \Big[\sum_{P:Q\subsetneq P} \frac{\sigma(P)\lambda_P}{\mu(P)}\Big]^{p'} \\ &\lesssim \sum_Q \lambda_Q 1_Q \sum_{P:Q\subsetneq P} \frac{\sigma(P)\lambda_P}{\mu(P)} \Big[\sum_{S:P\subseteq S} \frac{\sigma(S)\lambda_S}{\mu(S)}\Big]^{p'-1} \\ &\lesssim \sum_Q \lambda_Q 1_Q \sum_{P:Q\subsetneq P} \frac{\sigma(P)\lambda_P}{\mu(P)} \Big[\sum_{S:P\subseteq S} \lambda_S^{1-p}\Big]^{p'-1} \\ &= \sum_P \frac{\sigma(P)L_P}{\mu(P)} \sum_{Q:Q\subsetneq P} \lambda_Q 1_Q \Big[\sum_{S:P\subseteq S} \lambda_S^{1-p}\Big]^{p'-1}. \end{aligned}$$

Since

$$\sup_{x,P:x\in P} \sum_{Q:Q\subsetneq P, x\in Q} \lambda_Q \Big[\sum_{S:P\subseteq S} \lambda_S^{1-p}\Big]^{p'-1} < \infty$$

we obtain (4.2.4). □

4.3 Tools and estimates

Lemma 4.3.1. *Let $1 < p < \infty, \sigma$ be a Borel measure on $\mathbb{R}^n$, $(\lambda_Q)_Q$ be a sequence of nonnegative numbers associated with dyadic cubes Q and $(L_Q)_Q$ be a sequence of nonnegative numbers. Define a measure σ_1 by*

$$d\sigma_1(x) = \frac{1}{\Big[\sum_{Q\in\mathcal{D}} L_Q 1_Q(x)\Big]^{p-1}} d\sigma.$$

Then for every dyadic cube P we have

$$\sum_{Q\subseteq P} L_Q \sigma(Q)^{1-p'} \sigma_1(Q)^{p'} \le \sigma_1(P).$$

Proof. By definition of $\sigma_1(Q)$, Hölder's inequality, and the definition of σ_1 again, we obtain

$$\begin{aligned}
\sum_{Q\subseteq P} L_Q \sigma(Q)^{1-p'} \sigma_1(Q)^{p'} &= \sum_{Q\subseteq P} L_Q \sigma(Q)^{1-p'} \left\{ \int_Q \frac{d\sigma}{\left[\sum_{R\in\mathcal{D}} L_R 1_R(x)\right]^{p-1}} \right\}^{p'} \\
&\le \sum_{Q\subseteq P} L_Q \int_Q \frac{d\sigma}{\left[\sum_{R\in\mathcal{D}} L_R 1_R(x)\right]^{p}} \\
&= \sum_{Q\subseteq P} L_Q \int_Q \frac{d\sigma_1(x)}{\sum_{R\in\mathcal{D}} L_R 1_R(x)} \\
&= \int_P \frac{(\sum_{Q\subseteq P} L_Q 1_Q(x)) d\sigma_1(x)}{\sum_{R\in\mathcal{D}} L_R 1_R(x)} \\
&\le \sigma_1(P).
\end{aligned}$$

□

Theorem 4.3.2. *Let $1 < p < \infty, \sigma$ be a Borel measure on $\mathbb{R}^n$. Let $(\lambda_Q)_{Q\in\mathcal{D}}$ be a sequence of nonnegative numbers. Consider*

$$L_Q := \left[\frac{\sigma(Q)}{\mu(Q)} \lambda_Q^p\right]^{p'-1}.$$

Define a measure σ_1 by

$$d\sigma_1(x) = \frac{1}{\left[\sum_{Q\in\mathcal{D}} L_Q 1_Q(x)\right]^{p-1}} d\sigma.$$

Assume that (4.1.2) *holds for the measure σ_1 and also that* (4.2.7) *holds. Then we have the trace inequality*

$$\|Tf\|_{L^p(d\sigma_1)} \le C\|f\|_{L^p(d\mu(x))}, \forall f \in L^p(\mathbb{R}^n). \tag{4.3.3}$$

Proof. By Theorem 4.1.1 and Lemma 4.3.1

$$\int_P [T_P(\sigma_1)^{p'}] \lesssim \sum_{Q\subseteq P} \left[\frac{\sigma_1(Q)}{\mu(Q)} \lambda_Q\right]^{p'} \mu(Q) = \sum_{Q\subseteq P} L_Q \sigma(Q)^{1-p'} \sigma_1(Q)^{p'} \le \sigma_1(P)$$

for every dyadic cube P. Hence by Proposition 4.2.3

$$\|Tf\|_{L^p(d\sigma_1)} \leq C\|f\|_{L^p(dx)}, \forall f \in L^p.$$

□

Theorem 4.3.4. *Let* $0 < q < p < \infty$ *and* $p > 1$. *Let* σ *be a positive Borel measure and* $(\lambda_Q)_Q$ *be a sequence of nonnegative numbers. Assume that* (4.1.2) *holds for the measure* σ_1 *and also that* (4.2.7) *holds. Then the trace inequality*

$$\|Tf\|_{L^q(d\sigma)} \leq C\|f\|_{L^p(d\mu(x))} \forall f \in L^p(\mathbb{R}^n) \tag{4.3.5}$$

holds if only if

$$\sum_{Q\in\mathcal{D}} \left[\frac{\sigma(Q)}{\mu(Q)}\lambda_Q^p\right]^{p'-1} 1_Q \in L^{\frac{q(p-1)}{p-q}}(d\sigma). \tag{4.3.6}$$

Proof. Suppose that (4.3.6) holds. Then by Theorem 4.3.2

$$\|Tf\|_{L^p(d\sigma_1)} \leq C\|f\|_{L^p(d\mu(x))}, \forall f \in L^p(\mathbb{R}^n). \tag{4.3.7}$$

Let

$$V = W^{\frac{1}{p'}}; \quad W := \sum_{Q\in\mathcal{D}} \left[\frac{\sigma(Q)}{\mu(Q)}\lambda_Q^p\right]^{p'-1} 1_Q.$$

Using the preceding estimate, Hölder's inequality, and the fact that $\frac{p-q}{q(p-1)} = \frac{qp}{p'(p-q)}$, we obtain

$$\begin{aligned}
\|T_\lambda(f)\|_{L^q(d\sigma)} &= \|T_\lambda(f)V^{-1}V\|_{L^q(d\sigma)} \\
&\leq \|T_\lambda(f)V^{-1}\|_{L^p(d\sigma)}\|V\|_{L^{\frac{1}{\frac{1}{q}-\frac{1}{p}}}(d\sigma)} \\
&= \|T_\lambda(f)\|_{L^p(d\sigma_1)}\|W\|^{\frac{1}{p'}}_{L^{\frac{q(p-1)}{p-q}}(d\sigma)} \\
&\leq C\|f\|_{L^p(d\mu(x))},
\end{aligned}$$

which proves (4.3.5). Conversely, suppose that (4.3.5) holds. Let $\{\rho_Q\}_{Q\in\mathcal{D}}$ be an arbitrary sequence of real numbers such that

$$\sum_Q \mu(\rho_Q)^p < \infty$$

and set

$$f(x) = \sup_Q\{\mu(Q)^{-\frac{1}{p}}\mu(\rho_Q)1_Q(x)\}.$$

Then
$$\|f\|^p_{L^p(dx)} \le \sum_Q \mu(\rho_Q)^p$$
and
$$\sum_{Q\in\mathcal{D}} \lambda_Q \frac{1}{\mu(Q)}\Big(\int_Q f(y)d\mu(y)\Big)1_Q(x) \ge C\sum_Q \mu(\rho_Q)\lambda_Q\mu(Q)^{-\frac{1}{p}}1_Q(x).$$
So, for all $\{\rho_Q\}\in l^p$, we obtain the inequality
$$\Big\|\sum_Q \mu(\rho_Q)\lambda_Q\mu(Q)^{-\frac{1}{p}}1_Q(x)\Big\|_{L^q(d\sigma)} \le C\Big(\sum_Q \rho_Q^p\Big)^{\frac{1}{p}}.$$
Applying [Ver96, Theorem 3 (c)] we conclude that
$$\sum_{Q\in\mathcal{D}}\left[\frac{\sigma(Q)}{\mu(Q)}\lambda_Q^p\right]^{p'-1}1_Q \in L^{\frac{q(p-1)}{p-q}}(d\sigma).$$
$\square$

Corollary 4.3.8. *Let $0<q<p<\infty, p>1, 0<\alpha<n$ and σ a positive Borel measure. Then the trace inequality*
$$\Big\|\sum_{Q\in\mathcal{D}}\mu(Q)^{\frac{\alpha}{n}-1}\Big(\int_Q f(y)d\mu(y)\Big)1_Q\Big\|_{L^q(d\sigma)} \le C\|f\|_{L^p(d\mu(x))}, \forall f\in L^p(\mathbb{R}^n) \tag{4.3.9}$$
holds if only if
$$\sum_{Q\in\mathcal{D}}\left[\sigma(Q)\mu(Q)^{\frac{\alpha p}{n}-1}\right]^{p'-1}1_Q \in L^{\frac{q(p-1)}{p-q}}(d\sigma).$$

Proof. Consider $\lambda_Q := \mu(Q)^{\frac{\alpha}{n}}$. Note that the condition (4.1.2) holds. Indeed,
$$\begin{aligned}\sum_{Q'\subseteq Q}\lambda_{Q'}\sigma(Q') &= \sum_{n=0}^{\infty}\sum_{Q'\subseteq Q,\ell(Q)=2^{-n}\ell(Q)}\lambda_{Q'}\sigma(Q')\\ &= \lambda_Q\sum_{n=0}^{\infty}2^{-nda}\sum_{Q'\subseteq Q,\ell(Q)=2^{-n}\ell(Q)}\sigma(Q') = \lambda_Q\sum_{n=0}^{\infty}2^{-nda}\sigma(Q)\sim\lambda_Q\sigma(Q).\end{aligned}$$

Moreover, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^r}$ converges when $r > 1$,

$$\sum_{Q:Q\subsetneq P, x\in Q} \mu(Q)^{\frac{\alpha}{n}} \Bigg[\sum_{S:P\subseteq S} \mu(S)^{\frac{\alpha}{n}(1-p)} \Bigg]^{p'-1} = \sum_{Q:Q\subsetneq P, x\in Q} \mu(Q)^{\frac{\alpha}{n}} \Bigg[\sum_{S:P\subseteq S} \frac{1}{\mu(S)^{\frac{\alpha}{n}(p-1)}} \Bigg]^{p'-1}$$
$$= C_1^{p'-1} C_2$$

and the condition (4.2.7) is also satisfied for this λ_Q. So the result follows by Theorem 4.3.4. □

The following corollary characterizes the inequality (4.3.11) similarly to Cascate and Ortega (see [CO09, Theorem 2.8]), but now without DBL0 condition.

Corollary 4.3.10. *Let $0 < q, s < \infty$, $1 < p < \infty, q < p$ and $s < p$, and let σ positive Borel measure on $\mathbb{R}^n$ and $(\lambda_Q)_Q$ be a sequence of nonnegative numbers associated with dyadic cubes Q. Assume that (4.1.2) holds for the measure σ_1 and also that (4.2.7) holds. Then the inequality*

$$\Bigg\| \Bigg(\sum_{Q\in\mathcal{D}} \lambda_Q^s \rho_Q^s 1_Q \Bigg)^{\frac{1}{s}} \Bigg\|_{L^q(d\sigma)} \le C \Big\| \sup_Q (\rho_Q 1_Q) \Big\|_{L^p(dx)} \tag{4.3.11}$$

holds for arbitrary sequences of nonnegative numbers $(\rho_Q)_Q$ if only if

$$\sum_{Q\in\mathcal{D}} \left[\frac{\sigma(Q)}{\mu(Q)} \lambda_Q^{ps} \right]^{\frac{p}{s}'-1} 1_Q \in L^{\frac{q(p-s)}{s(p-q)}}(d\sigma).$$

Proof. If in (4.3.11) we substitute ρ_Q^s by ρ_Q, put $\tilde{p} = \frac{p}{s}$ and $\tilde{q} = \frac{q}{s}$, we see that this estimate can be rewritten as

$$\left(\int_{\mathbb{R}^n} \Bigg(\sum_{Q\in\mathcal{D}} \rho_Q \lambda_Q^s 1_Q \Bigg)^{\tilde{q}} d\sigma \right)^{\frac{1}{\tilde{q}}} \le C \Big\| \sup_Q (\rho_Q 1_Q) \Big\|_{L^{\tilde{p}}(dx)},$$

where now $0 < \tilde{q} < \tilde{p}$ and $\tilde{p} > 1$. Applying the Lemma 5.1.1 we have that above is equivalent to

$$\left(\int_{\mathbb{R}^n} \Bigg(\sum_{Q\in\mathcal{D}} \frac{1}{\mu(Q)} \Big(\int_Q f d\mu \Big) \lambda_Q^s 1_Q \Bigg)^{\tilde{q}} d\sigma \right)^{\frac{1}{\tilde{q}}} \le C \|f\|_{L^{\tilde{p}}(d\mu(x))}.$$

The result follows by Theorem 4.3.4. □

Theorem 4.3.12 (S. Treil, [Tre15])**.** *Let* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$*. Then* (4.2.6) *holds if only if for all* $Q_0 \in \mathcal{D}$

$$\begin{aligned} \int_{Q_0 \in \mathcal{D}} \left(\sum_{Q \in \mathcal{D}: Q \subset Q_0} \lambda_Q 1_Q \right)^p d\sigma \leq C^p \mu(Q_0), \\ \int_{Q_0 \in \mathcal{D}} \left(\sum_{Q \in \mathcal{D}: Q \subset Q_0} \frac{\lambda_Q}{\mu(Q)} \sigma(Q) 1_Q \right)^p d\sigma \leq C^{p'} \sigma(Q_0). \end{aligned} \tag{4.3.13}$$

With what has been seen above we get easily by Theorema 4.3.12 the following.

Theorem 4.3.14. *Let* $0 < q < p < \infty$ *and* $p > 1$*. Let* σ *be a positive Borel measure and* $(\lambda_Q)_Q$ *be a sequence of nonnegative numbers. Define a measure* σ_1 *by*

$$d\sigma_1(x) = \frac{d\sigma}{\left[\sum_{Q \in \mathcal{D}} \sigma(Q)^{p'-1} \mu(Q)^{1-p'} \lambda_Q^{p'} 1_Q(x) \right]^{p-1}} \quad .$$

Assume that the condition (4.3.13) *holds for the measure* σ_1*. Then the trace inequality*

$$\|Tf\|_{L^q(d\sigma)} \leq C \|f\|_{L^p(d\mu(x))} \forall f \in L^p(\mathbb{R}^n) \tag{4.3.15}$$

holds if only if

$$\sum_{Q \in \mathcal{D}} \left[\frac{\sigma(Q)}{\mu(Q)} \lambda_Q^p \right]^{p'-1} 1_Q \in L^{\frac{q(p-1)}{p-q}}(d\sigma). \tag{4.3.16}$$

Proof. By 4.3.12 we have

$$\|Tf\|_{L^p(d\sigma_1)} \leq C \|f\|_{L^p(d\mu(x))}, \forall f \in L^p(\mathbb{R}^n).$$

Then the result follows using the reasoning of Theorem 4.3.4. $\square$

Chapter 5

Weighted $L^{p_1} \times L^{p_2} \to L^q$ bounds for bilinear positive dyadic operators in case $0 < q < 1$

In this chapter we characterize weighted $L^{p_1} \times L^{p_2} \to L^q$ strong type estimates for bilinear dyadic operators in case $0 < q < 1 < p_i, i = 1, 2$. Interesting examples of such bilinear operators are dyadic versions of bilinear fractional integrals (with $\lambda_Q = \mu(Q)^{\frac{\alpha}{n}}, 0 < \alpha < 2n$) or sparse operators (with λ_Q a Carleson sequence).

The main result here is Theorem 5.2.1. In the proof of this we use parallel stopping cubes, characterization of boundedness of vector valued operators in terms of discrete multipliers (see Lemma 5.1.1), and equivalence between sparse and Carleson conditions (see proof of Theorem 5.1.4) to reduce from the bilinear to the linear case. After this reduction, we follow the reasoning in the proof of [HV17, Theorem 1.2].

5.1 Preliminaries

Below we give some auxiliary results. The following theorem is based on [CO17, Lemma 2.1] and [CO09, Lemma 2.1].

Lemma 5.1.1. *Given $0 < q < \infty, 1 < p < \infty, 1 \leq s \leq \infty$ and $(\lambda_Q)_{Q \in \mathcal{D}}$ a sequence of nonnegative real numbers. Let w_1, w_2 be positive Borel measures on $\mathbb{R}^n$ and f a nonnegative function. The estimate*

$$\left\| \left(\sum_{Q \in \mathcal{D}} \lambda_Q^s \left(\int_Q f dw_1 \right)^s 1_Q \right)^{\frac{1}{s}} \right\|_{L^q(w_2)} \leq C \|f\|_{L^p(w_1)} \tag{5.1.2}$$

holds if and only if there exists C such that for every seuqence $(\rho_Q)_Q$ of non-negative numbers we have

$$\left\| \left(\sum_{Q \in \mathcal{D}} \lambda_Q^s (w_1(Q))^s \rho_Q^s 1_Q \right)^{\frac{1}{s}} \right\|_{L^q(w_2)} \le C \| \sup_{Q \in \mathcal{D}} (\rho_Q 1_Q) \|_{L^p(w_1)}. \tag{5.1.3}$$

Proof. Let

$$f = \sup_Q (\rho_Q 1_Q).$$

We have

$$\begin{aligned} \frac{1}{w_1(Q)} \int_Q f dw_1 = \frac{1}{w_1(Q)} \int_Q \sup_{Q' \in \mathcal{D}} (\rho_{Q'} 1_{Q'}) dw_1 \ge \frac{1}{w_1(Q)} \int_Q \rho_Q 1_Q dw_1 \\ = \rho_Q. \end{aligned}$$

Then if (5.1.2) holds we obtain

$$\begin{aligned} &\left\| \left(\sum_{Q \in \mathcal{D}} \lambda_Q^s (w_1(Q))^s \rho_Q^s 1_Q \right)^{\frac{1}{s}} \right\|_{L^q(w_2)} \\ \le &\left\| \left(\sum_{Q \in \mathcal{D}} \lambda_Q^s (w_1(Q))^s \left(\frac{1}{w_1(Q)} \int_Q f dw_1 \right)^s 1_Q \right)^{\frac{1}{s}} \right\|_{L^q(w_2)} \\ = &\left\| \left(\sum_{Q \in \mathcal{D}} \lambda_Q^s \left(\int_Q f dw_1 \right)^s 1_Q \right)^{\frac{1}{s}} \right\|_{L^q(w_2)} \\ \le &\, C \|f\|_{L^p(w_1)} = C \| \sup_{Q \in \mathcal{D}} (\rho_Q 1_Q) \|_{L^p(w_1)}. \end{aligned}$$

Conversely, let

$$\rho_Q = \frac{1}{w_1(Q)} \int_Q f dw_1.$$

Since $p > 1$ we know that the dyadic maximal operator with respect to $w_1, M_{w_1}^{\mathcal{D}}$ given by

$$M_{w_1}^{\mathcal{D}} f(x) = \sup_Q \frac{1}{w_1(Q)} \int_Q f dw_1$$

is strong type (p,p) with respect to w_1. Then if (5.1.3) holds we have

$$
\begin{aligned}
&\left\|\Big(\sum_{Q\in\mathcal{D}}\lambda_Q^s\Big(\int_Q f dw_1\Big)^s 1_Q\Big)^{\frac{1}{s}}\right\|_{L^q(w_2)}\\
=&\left\|\Big(\sum_{Q\in\mathcal{D}}\lambda_Q^s(w_1(Q))^s\Big(\frac{1}{w_1(Q)}\int_Q f dw_1\Big)^s 1_Q\Big)^{\frac{1}{s}}\right\|_{L^q(w_2)}\\
&\leq C\|\sup_{Q\in\mathcal{D}}(\rho_Q 1_Q)\|_{L^p(w_1)}\\
&= C\|M_{w_1}^{\mathcal{D}} f\|_{L^p(w_1)}\\
&\leq C\|f\|_{L^p(w_1)}.
\end{aligned}
$$

□

Lemma 5.1.4 ([CO17, Lemma 4.6]). *Let $(b_Q)_Q$ be a sequence of non-negative real numbers. Let $0 < q < \infty$ and $q \leq s \leq \infty$. Let μ be a positive locally finite Borel measure with no point mass*[1]*. Define $\tilde{s} := s/q$. Then*

$$
\left\|\Big(\sum_Q b_Q^s 1_Q\Big)^{\frac{1}{s}}\right\|_{L^q(\mu)} \sim_{q,s} \sup_{E(Q)}\Big(\sum_Q b_Q^q \mu(E_Q)^{\frac{1}{\tilde{s}'}}\mu(Q)^{\frac{1}{\tilde{s}}}\Big)^{\frac{1}{q}},
$$

where the supremum is taken over all collections $(E_Q)_Q$ of pairwise disjoint sets with $E_Q \subset Q$. The implicits constants do not depend on the sequence $(b_Q)_Q$.

Lemma 5.1.5 (see proof Theorem 1.2-[HV17]). *Let $\mathcal{Q} := \{Q \in \mathcal{D} : \lambda_Q > 0, \sigma(Q) > 0 \ \text{ and } \ w(Q) > 0\}$, where λ_Q is nonnegative reals numbers. The following assertions are equivalent:*

1. *There exists a function ξ, with $\xi > 0$ $dw-a.e.$ on every cube $Q \in \mathcal{Q}$, that satisfies the pair of condictions*

$$
\int \xi dw \lesssim_q 1, \tag{5.1.6}
$$

$$
\Big(\int\Big(\sum_{Q\in\mathcal{Q}}\lambda_Q(\xi^{-(\frac{1-q}{q})})_Q^w\frac{w(Q)}{\sigma(Q)}1_Q\Big)^{p'}d\sigma\Big)^{\frac{1}{p'}} \lesssim_q C. \tag{5.1.7}
$$

[1] The measure μ has no point masses if for each measurable set A and for every $m \in [0,\mu(A)]$, there exists a measurable subset $H \subset A$ such that $\mu(H) = m$.

2. *There exists a family $\{a_Q\}_{Q\in\mathcal{Q}}$ of positive reals that satisfies the pair of conditions*
$$\int \Big(\sup_{Q\in\mathcal{Q}} a_Q 1_Q\Big)^{\frac{q}{1-q}} dw \lesssim_1 1, \tag{5.1.8}$$
$$\Big(\int \Big(\sum_{Q\in\mathcal{Q}} \lambda_Q a_Q^{-1} \frac{w(Q)}{\sigma(Q)} 1_Q\Big)^{p'} d\sigma\Big)^{\frac{1}{p'}} \lesssim_q C. \tag{5.1.9}$$

Proof. First note that the continuous conditions imply the discret ones. We set $a_Q^{-1} := (\xi^{-(\frac{1-q}{q})})_Q^w$ for every cube $Q \in \mathcal{Q}$. Thus, condition (5.1.9) becomes condition (5.1.7). By Jensen's inequality together with the convexity of the function $t \to t^{-q}$, and the Hardy Littlewood maximal inequality, condition (5.1.6) implies condition (5.1.8) through
$$\begin{aligned}\int \Big(\sup_{Q\in\mathcal{Q}} a_Q 1_Q\Big)^{\frac{q}{1-q}} dw &= \int \sup_{Q\in\mathcal{Q}} \Big(((\xi^{-(\frac{1-q}{q})})_Q^w)^{-q}\Big)^{\frac{1}{1-q}} dw \\ &\le \int \Big(\sup_{Q\in\mathcal{Q}} (\xi^{1-q})_Q^w 1_Q\Big)^{\frac{1}{1-q}} dw \lesssim_q \int \xi dw.\end{aligned}$$
Next, we prove that the discret conditions imply the continous ones. We set
$$\xi := \Big(\sup_{Q\in\mathcal{Q}} a_Q 1_Q\Big)^{\frac{q}{1-q}}.$$
Thus, condition (5.1.6) becomes condition (5.1.8). By estimating the supremum from below by omitting all but one cube from the indexation, we see that condition (5.1.9) implies condition (5.1.7). □

5.1.1 Factorization through weak L_1

Theorem 5.1.10 (Pisier, [Pis86])**.** *Let $0 < q < 1$ and $\{f_i\}_{i\in I}$ be a family of measurable functions. The following assertions are equivalent.*

1. *There is a constant C_1 and a function $\phi \in L_1(\mu), \phi \ge 0, \int \phi d\mu \le 1$ such that, for all measurable subsets $E \subset \mathbb{R}^n$,*
$$\|1_E f_i\|_{L_q(\mu)} \le C_1 \Big(\int_E \phi d\mu\Big)^{\frac{1}{q}-1}. \tag{5.1.11}$$

2. *There is a constant C_2 and a function $\phi \in L_1(\mu), \phi \ge 0, \int \phi d\mu \le 1$ such that for every $i \in I$ we have $\{\phi = 0\} \subset \{f_i = 0\}$ and*
$$\|\phi^{-\frac{1}{q}} f_i\|_{L_{1,\infty}(\phi\mu)} \le C_2. \tag{5.1.12}$$

3. *There is a constant C such that for every finitely supported family* [2] $(\alpha_i)_{i\in I}$ *of real numbers we have*

$$\|\sup_{i\in I}|\alpha_i f_i|\|_{L^q(\mu)} \le C(\sum|\alpha_i|). \tag{5.1.13}$$

Proof of (5.1.11) $\implies$ (5.1.12) . Let $v = \phi\mu$. Fix $i \in I, t > 0$, and let

$$E = \{|f_i| > t\phi^{\frac{1}{q}}\}.$$

Then (5.1.11) implies

$$\begin{aligned} tv(E)^{1/q} = t\Big(\int_E \phi d\mu\Big)^{1/q} \le t\Big(\int_E |f_i|^q/t^q d\mu\Big)^{1/q} = \Big(\int_E |f_i|^q d\mu\Big)^{1/q} \\ \le C_1\Big(\int_E \phi d\mu\Big)^{1/q-1} = C_1 v(E)^{1/q-1}, \end{aligned}$$

hence

$$tv(E) \le C_1,$$

so that (5.1.12) holds. □

Proof of (5.1.12) $\implies$ (5.1.13). If $v = \phi\mu$ we have

$$\begin{aligned} \big\|\sup_i|\alpha_i f_i|\big\|_{L^q(\mu)} &= \big\|\sup_i \phi^{-\frac{1}{q}}|\alpha_i f_i|\big\|_{L^q(v)} \\ &\le C_3(q,1)\big\|\sup_i \phi^{-\frac{1}{q}}|\alpha_i f_i|\big\|_{L_{1,\infty}(v)} \\ &\le C_3(q,1)\sum_i \big\|\phi^{-\frac{1}{q}}\alpha_i f_i\big\|_{L_{1,\infty}(v)} \\ = C_3(q,1)\sum_i \alpha_i\big\|\phi^{-\frac{1}{q}} f_i\big\|_{L_{1,\infty}(v)} &\le C_3(q,1)C_2\sum_i \alpha_i. \end{aligned}$$

□

Proof of (5.1.13) $\implies$ (5.1.11) . Let n be a fixed integer and

$$C_n = \sup\{\big\|\sup_{i\in J}|\alpha_i f_i|\big\|_{L^q(\mu)} \mid J \subset I, \mathrm{card}J = n, \sum_{i\in J}|\alpha_i| \le 1\}.$$

Since C_n is bounded we may assume without loss of generality that $C_n \uparrow C$ and $C_n \ne 0$. Let $\delta_n > 1$ be a sequence such that $\delta_n \to 1$ when $n \to \infty$. By

[2] $(\alpha_i)_{i\in I}$ is a *finitely supported family* if $\{i : \alpha_i \ne 0\}$ is finite.

definition of C_n we can find, for each fixed n, a subset $J_n \subset I$ with cardinality n and scalars $(\alpha_i)_{i\in J_n}$ such that

$$\sum_{i\in J}|\alpha_i| \leq C_n^{-1}\delta_n \quad \text{and} \quad \left\|\sup_{i\in J_n}|\alpha_i f_i|\right\|_{L^q(\mu)} = 1.$$

Let

$$\phi_n = \left(\sup_{i\in J_n}|\alpha_i f_i|\right)^q$$

and let $i \in I$ be arbitrary. By definition of C_{n+1}, we have for all $\epsilon > 0$

$$\left\|\phi_n^{\frac{1}{q}} \vee (\epsilon C_n^{-1} f_i)\right\|_q \leq \frac{C_{n+1}}{C_n}(\delta_n + \epsilon). \tag{5.1.14}$$

Let $\beta_n = \frac{C_{n+1}}{C_n}(\delta_n + \epsilon)$. Note that $\beta_n \to (1+\epsilon)$ when $n \to \infty$. Let $E \subset \mathbb{R}^n$ be any measurable set. We deduce from (5.1.14) that

$$\int_{E^c}\phi_n d\mu + \epsilon^q C_n^{-q}\int_E |f_i|^q d\mu \leq \beta_n^q,$$

hence since $\int \phi_n d\mu = 1$

$$\epsilon^q C_n^{-q}\int_E |f_i|^q d\mu \leq \int_E \phi_n d\mu + \beta_n^q - 1, \tag{5.1.15}$$

and this holds for all $E, i \in I, \epsilon > 0$ and every n. Since $\frac{1}{q} > 1$ and assuming that there is a constant K such that for all finite sequences of scalars (α_n) we have

$$\|\sup|\alpha_n\phi_n|\|_1 \leq C(\sum|\alpha_n|^{\frac{1}{q}})^q,$$

then the sequence $\{\phi_n\}$ is uniformly integrable. Indeed, note that if (A_n) is any sequence of mutually disjoint sets and $q + \frac{1}{\beta} = 1$, then the last inequality implies

$$\left(\sum\int_{A_n}|\phi_n|^\beta\right)^{\frac{1}{\beta}} \leq C.$$

Then $\{\phi_n\}$ is a bounded sequence in L_1 and

$$\int_{A_n}|\phi_n| \to 0 \quad \text{when} \quad n \to \infty.$$

Therefore $\{\phi_n\}$ is uniformly integrable. Let now ϕ be a cluster point of $\{\phi_n\}$ for the weak topology $\sigma(L^1, L^\infty)$ We have $\phi \geq 0$ and $\int \phi d\mu = 1$. Passing to the limit in (5.1.15) we obtain

$$\forall i \in I, \epsilon^q \int_E |f_i|^q d\mu \leq C^q\left(\int_E \phi d\mu + q\epsilon\right). \tag{5.1.16}$$

Dividing by ϵ^q and minimizing over $\epsilon > 0$, we find

$$\Big(\int_E |f_i|^q d\mu\Big)^{\frac{1}{q}} \leq C\Big(1-q\Big)^{1-\frac{1}{q}}\Big(\int_E \phi d\mu\Big)^{\frac{1}{q}-1}$$

which yields (5.1.11) with $C_1 = C\Big(1-q\Big)^{1-\frac{1}{q}}$. □

The following factorization theorem can be proved by a similar argument.

Theorem 5.1.17 (Maurey factorization, [Mau73]). *Let $0 < q < p < \infty$, $\{f_i\}_{i\in I}$ a family of measurable functions, and $C < \infty$. Then the following assertions are equivalent.*

1. *There is a function $\phi > 0$ in $L_1(\mu)$ with $\int \phi d\mu \leq 1$ such that*

$$\|\phi^{-\frac{1}{q}} f_i\|_{L_1(\phi\mu)} \leq C, \forall i \in I.$$

2. *For all finitely supported families $(\alpha_i)_{i\in I}$ of real numbers*

$$\|\sum_i |\alpha_i f_i|\|_{L^q(\mu)} \leq C\sum_i |\alpha_i|.$$

Moreover 1.⇔ 2. with the same constant C (contrary to the situation of Theorem 5.1.10).

Proof. The implication 1. ⇒ 2. is elementary. For the converse, we can adapt the preceding argument. Let

$$B_n = \sup\{\|\sum_{i\in J} |\alpha_i f_i|\|_{L^q(\mu)} \mid J \subset I, \mathrm{card} J = n, \sum |\alpha_i| \leq 1\}.$$

Let $C = \sup B_n$. Let $\delta_n > 1$ be a sequence such that $\delta_n \to 1$. By definition of B_n we can find, for each fixed n, a subset $J_n \subset I$ with cardinality n and scalars $(\alpha_i)_{i\in J_n}$ such that

$$\sum_{i\in J} |\alpha_i| \leq B_n^{-1}\delta_n \ \text{ and } \ \Big\|\sup_{i\in J_n} |\alpha_i f_i|\Big\|_{L^q(\mu)} = 1.$$

Let

$$\phi_n = \Big(\sup_{i\in J_n} |\alpha_i f_i|\Big)^q$$

and let $i \in I$ be arbitrary. By definition of B_{n+1}, we have for all $\epsilon > 0$ and all $i \in I$,

$$\int (\phi_n^{\frac{1}{q}} + C^{-1}\epsilon |f_i|)^q d\mu \le (\delta_n + \epsilon)^q.$$

Reasoning as above, we find that ϕ_n is uniformly integrable. Let ϕ be a cluster point for $\sigma(L^1, L^\infty)$. We have

$$\int (\phi_n^{\frac{1}{q}} + C^{-1}\epsilon |f_i|)^q d\mu \le (1 + \epsilon)^q$$

hence

$$\int (1 + \epsilon C^{-1} |f_i| \phi^{-\frac{1}{q}})^q \phi d\mu \le (1 + \epsilon)^q.$$

Since $\int \phi d\mu = 1$, letting $\epsilon \to 0$ we see that this implies

$$C^{-1} \int |f_i| \phi^{\frac{1}{q}} \phi d\mu \le 1. \qquad \square$$

Theorem 5.1.18. *Let $0 < q < 1$. Let E be a Banach (or merely quasi-Banach) space. The following properties of a bounded operator $T : E \to L^q$ are equivalent.*

1. *There is a constant C such that, for all finite sequences (f_i) in E, we have*
$$\big\|\sup_i |T(f_i)|\big\|_q \le C\Big(\sum_i \|f_i\|\Big). \tag{5.1.19}$$

2. *There is a constant C_1 such that there is a $\phi \in L_1(\mu), \phi \ge 0$ and $\int \phi d\mu \le 1$ satisfying for all $f \in E$ and for all measurable E*
$$\|T(f) 1_E\|_{L^q(\mu)} \le C_1 \|f\| \Big(\int_E \phi d\mu\Big)^{\frac{1}{q} - 1}. \tag{5.1.20}$$

3. *There is a constant C_2 and a function $\phi \in L_1(\mu), \phi \ge 0$ and $\int \phi d\mu \le 1$ such that $\{\phi = 0\} \subset \{|T(f)| = 0\}$ for all f and*
$$\|\phi^{-\frac{1}{q}} T(f)\|_{L_{1,\infty}(\phi\mu)} \le C_2 \|f\| \quad \forall f \in E. \tag{5.1.21}$$

4. *The operator T admits a factorization of the form*
$$E \xrightarrow{\tilde{T}} L_{1,\infty}(\phi\mu) \xrightarrow{M} L_q(\mu), \tag{5.1.22}$$
where $\phi \in L_1(\mu), \phi \ge 0$ and $\int \phi d\mu \le 1$, where M is the (bounded) operator of multiplication by $\phi^{\frac{1}{q}}$ and where $\tilde{T}$ is bounded (Note that necessary $\tilde{T} = M^{-1}T$).

Proof. The equivalences (5.1.19) ⇔ (5.1.20) ⇔ (5.1.21) follow immediately from Theorem 5.1.10. Moreover (5.1.22) is nothing but a restatement of (5.1.21). □

5.2 Estimates

Let w_1, w_2, w_3 be Borel measures on $\mathbb{R}^n$ and $f_i \in L^{p_i}(w_i), i = 1, 2, 3$. We define the collections $\mathcal{F}_i$ of cubes for the pairs $(f_i, w_i), i = 1, 2, 3$. Namely,

$$\mathcal{F}_i = \bigcup_{k=0} \mathcal{F}_i^k$$

where $\mathcal{F}_i^0 := \{Q_0\}$, Q_0 large enough fixed,

$$\mathcal{F}_i^{k+1} := \bigcup_{F \in \mathcal{F}_i^k} ch(F)$$

where

$$ch(F) := max\left\{ Q \subset F : w_i(F)^{-1} \int_F f_i w_i < \frac{1}{2} w_i(Q)^{-1} \int_Q f_i w_i \right\}.$$

Observe that

$$\begin{aligned} \sum_{F' \in ch(F)} w_i(F') &\le \left(\frac{2}{w_i(F)} \int_F f_i w_i \right)^{-1} \sum_{F' \in ch(F)} \int_{F'} f_i w_i \\ &\le \left(\frac{2}{w_i(F)} \int_F f_i w_i \right)^{-1} \int_F f_i w_i = \frac{w_i(F)}{2} \end{aligned}$$

and hence

$$w_i(E_{\mathcal{F}_i}(F)) := w_i \left(F \setminus \bigcup_{F' \in ch(F)} F' \right) \ge \frac{w_i(F)}{2},$$

where the set $E_{\mathcal{F}_i}(F), F \in \mathcal{F}_i$, are pairwise disjoint.

We define for $Q \in \mathcal{D}$

$$\begin{aligned} \pi_1(Q) &:= \min\{F_1 \supseteq Q : F_1 \in \mathcal{F}_1\} \\ \pi_2(Q) &:= \min\{F_2 \supseteq Q : F_2 \in \mathcal{F}_2\} \end{aligned}$$

and denote

$$(w_i)_Q = \frac{1}{\mu(Q)} w_i(Q).$$

We say that w is in A_∞ if

$$\sup_{R\in\mathcal{D}} \frac{1}{w(R)} \int_R M_R^\mu(w)d\mu < \infty,$$

where, for each $R \in \mathcal{D}$, the localized Hardy- Littlewood Maximal operator M_R^μ is defined by

$$M_R^\mu(w) := \sup_{Q\in\mathcal{D}, Q\subseteq R} \frac{w(Q)}{\mu(Q)} 1_Q.$$

Below we will prove the main theorem of this section.

Theorem 5.2.1. *Let $\mathcal{F}_i$ be the collection and $\pi_1(Q), \pi_2(Q)$ defined as above, w_1, w_2, w_3 be Borel measures on $\mathbb{R}^n$, μ a nonnegative measure fixed on $\mathbb{R}^n$, and f_i positive functions in $L^{p_i}(w_i), i = 1, 2, 3$. Let $(\lambda_Q)_Q$ be a sequence of non-negative real numbers. Assume that, in addition, the measures w_1, w_2 have no point masses and are in A_∞.Consider $0 < q < 1 < p_i, i = 1, 2$. Let*

$$\Lambda^j_{F_i} := \Lambda^{q\ Q\ \lambda_Q\ \pi_i\pi_j w_i w_j w_3\ \mu}_{F_iF_j} = \Bigg(\sum_{\substack{Q:\pi_i(Q)=F_i \\ \pi_j(Q)=F_j}} (w_i)_Q^q (w_j)_Q^q w_3(Q)\lambda_Q^q \Bigg)^{\frac{1}{q}} w_j(F_i)^{-\frac{1}{q}}$$

for $i, j = 1, 2, i \neq j$. Define the collection $\mathcal{G}^i$ of the remaining cubes

$$\mathcal{G}^i = \{F_i \in \mathcal{F}_i, F_i \subseteq F_j : \Lambda^j_{F_i} > 0, w_i(F_i) > 0 \quad \text{and} \quad w_j(F_i) > 0\},$$

$i, j = 1, 2, i \neq j$. Let

$$\sup_{\mathcal{F}_i} \inf_{\Lambda^j_{F_i} = h^j_{F_i} c^j_{F_i}} w_j(F_j)^{-\frac{1}{q}} \Bigg(\int \Bigg(\sup_{F_i\in\mathcal{G}^i, F_i\subseteq F_j} h^j_{F_i} 1_{F_i} \Bigg)^{\frac{q}{1-q}} dw_j \Bigg)^{\frac{1-q}{q}} \cdot$$
$$\cdot \Bigg(\int \Bigg(\sum_{F_i\in\mathcal{G}^i, F_i\subseteq F_j} c^j_{F_i} \frac{w_j(Q)}{w_i(Q)} \Bigg)^{p'} dw_i \Bigg)^{\frac{1}{p'}} = A_i, \quad (5.2.2)$$

$i, j = 1, 2, i \neq j$. Let B be the best constant in

$$\Bigg\| \sum_{Q\in\mathcal{D}} \lambda_Q \frac{1}{\mu(Q)^2} \Bigg(\prod_{i=1}^2 \int_Q f_i w_i \Bigg) 1_Q \Bigg\|_{L^q(w_3)} \leq B \prod_{i=1}^2 \|f_i\|_{L^{p_i}(w_i)}. \quad (5.2.3)$$

Then $B \lesssim A_1 + A_2$, $A_1 \lesssim B$, and $A_2 \lesssim B$.

Proof. We begin with the inequality $B \lesssim A_1 + A_2$. We can rewrite the series

$$\sum_{Q\subset Q_0} = \sum_{\substack{F_1\in\mathcal{F}_1\\ F_2\in\mathcal{F}_2}} \sum_{\substack{Q:\pi_1(Q)=F_1\\ \pi_2(Q)=F_2}} \leq \sum_{\substack{F_1\in\mathcal{F}_1\\ F_2\subseteq F_1}} \sum_{\substack{Q:\pi_1(Q)=F_1\\ \pi_2(Q)=F_2}} + \sum_{\substack{F_2\in\mathcal{F}_2\\ F_1\subseteq F_2}} \sum_{\substack{Q:\pi_1(Q)=F_1\\ \pi_2(Q)=F_2}}$$

where we observed that if the inner sum over $Q : (\pi_1(Q), \pi_2(Q)) = (F_1, F_2)$ is not empty, then there is some $Q \subseteq F_1 \cap F_2$, hence $F_1 \cap F_2 \neq \emptyset$, and thus $F_2 \subseteq F_1$ or $F_1 \subseteq F_2$. Replacing the sum over Q by the second term on the right-hand side we will show $B \lesssim A_1$, the first term is symmetric.

Consider Q with $F_1 = \pi_1(Q) \subseteq \pi_2(Q) = F_2$. If $F' \in ch(F_2)$ satisfies $F' \subseteq Q$, then by definition of π_2 we must have

$$\pi_2(\pi_1(F')) = \begin{cases} F_2 & \text{if } F^{'} \notin \mathcal{F}_1 \\ F^{'} & \text{if } F^{'} \in \mathcal{F}_1. \end{cases} \tag{5.2.4}$$

In fact, by definition of π_1 and hypothesis we have $\pi_1(F') \subseteq F_1 \subseteq F_2$, then by defiition π_2 we have $\pi_2(\pi_1(F')) \subseteq F_2$. On the other hand, if $F' \notin \mathcal{F}_1$, by definition π_1, $\pi_1(F') \supsetneq F'$, then by definition π_2, $\pi_2(\pi_1(F')) \supsetneq F'$. Since, $F' \in ch(F_2)$ then $F' \subset F_2$. So $F_2 \subseteq \pi_2(\pi_1(F'))$. Moreover, if $F' \in \mathcal{F}_1$ by definition π_1, $\pi_1(F') = F' \in \mathcal{F}_2$, so by definition π_2 we have $\pi_2(\pi_1(F')) = F'$.

By this observation we define

$$ch^*(F_2) := \{F' \in ch(F_2) : F^{'} \text{ satisfies } (5.2.4)\}.$$

We further observe that if $\pi_1(Q) \subseteq \pi_2(Q) = F_2$ and $F^{'} \in ch^*(F_2)$, then $Q \cap F' \in \{F', \emptyset\}$, so we can regard f_1 as a constant on $F^{'}$ in the integral over Q, that is, $\int_Q f_1 w_1 = \int_Q f_1^{F_2} w_1$ with

$$f_1^{F_2} = f_1 1_{E(F_2)} + \sum_{F'\in ch^*(F_2)} 1_{F'} w_1(F')^{-1} \int_{F'} f_1 w_1$$

with

$$E(F_2) = F_2 \backslash \bigcup_{F'\in ch(F_2)} F'.$$

Indeed, since

$$1_{E(F_2)} = 1_{F_2} - \sum_{F'\in ch(F_2)} 1_{F'}$$

and $Q \subseteq F_2$ we have

$$\int_Q f_1 w_1 = \int_Q 1_{F_2} f_1 w_1 = \int_Q 1_{E(F_2)} f_1 w_1 + \int_Q \sum_{F'\in ch(F_2)} 1_{F'} f_1 w_1.$$

If $Q' \cap F' \neq \emptyset$, then either $F' \subsetneq Q$ or $Q \subseteq F'$. But the latter is not possible, since it would imply that $\pi_2(Q) \subseteq F' \subsetneq F_2$, contracting $\pi_2(Q) = F_2$. Thus for the nonzero terms in

$$\sum_{F' \in ch(F_2)} \int_{F' \cap Q} f_1 w_1$$

we must have $F' \subsetneq Q \subseteq F_2$.

So we may restrict this summation to $ch^*(F_2)$. Then we have

$$\begin{aligned}
\int_Q f_1 w_1 &= \int_Q f_1 1_{E(F_2)} w_1 + \sum_{F' \in ch(F_2)} \int_{F' \cap Q} f_1 d_1 \\
&= \int_Q f_1 1_{E(F_2)} w_1 + \sum_{F' \in ch^*(F_2), F' \subsetneq Q} \int_{F'} f_1 w_1 \\
&= \int_Q f_1 1_{E(F_2)} w_1 + \int_Q \Big[\sum_{F' \in ch^*(F_2)} w_1(F^{'})^{-1} \Big(\int_{F'} f_1 w_1 \Big) 1_{F'} \Big] w_1 \\
&= \int_Q f_1^{F_2} dw_1.
\end{aligned}$$

We denote

$$\begin{aligned}
(f_i)_{Q,w_i} &= w_i(Q)^{-1} \int_Q f_i dw_i \\
(f_i w_i)_Q &= \frac{1}{\mu(Q)} \int_Q f_i dw_i.
\end{aligned}$$

Note that

$$\begin{aligned}
&\Big\| \sum_{Q \in \mathcal{D}} \lambda_Q \frac{1}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) 1_Q \Big\|_{L^q(w_3)}^q \\
&\leq \int \sum_{Q \in \mathcal{D}} \lambda_Q^q (f_1 w_1)_Q^q (f_2 w_2)_Q^q 1_Q dw_3 \\
&\approx \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_2 \in \mathcal{F}_2}} \sum_{\substack{Q : \pi_1(Q) = F_1 \\ \pi_2(Q) = F_2}} (f_1 w_1)_Q^q (f_2 w_2)_Q^q w_3(Q) \lambda_Q^q \\
&\leq \sum_{F_2 \in \mathcal{F}_2} \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_1 \subseteq F_2}} \sum_{\substack{Q : \pi_1(Q) = F_1 \\ \pi_2(Q) = F_2}} (f_1 w_1)_Q^q (f_2 w_2)_Q^q w_3(Q) \lambda_Q^q \\
&+ \sum_{F_1 \in \mathcal{F}_1} \sum_{\substack{F_2 \in \mathcal{F}_2 \\ F_2 \subseteq F_1}} \sum_{\substack{Q : \pi_1(Q) = F_1 \\ \pi_2(Q) = F_2}} (f_1 w_1)_Q^q (f_2 w_2)_Q^q w_3(Q) \lambda_Q^q.
\end{aligned}$$

We concentrate on

$$(**) := \sum_{F_2 \in \mathcal{F}_2} \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_1 \subseteq F_2}} \sum_{\substack{Q : \pi_1(Q) = F_1 \\ \pi_2(Q) = F_2}} (f_1 w_1)_Q^q (f_2 w_2)_Q^q w_3(Q) \lambda_Q^q.$$

We want to show

$$(**) \lesssim A_1^q \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)}^q.$$

Since

$$\begin{aligned}
(f_2 w_2)_Q^q &= (f_2)_{Q,w_2}^q (w_2)_Q^q \\
(f_1 w_1)_Q^q &\simeq (f_1^{F_2} w_1)_Q^q \\
(f_2)_{Q,w_2}^q &\leq (f_2)_{F_2,w_2}^q \\
(f_1^{F_2} w_1)_Q &= (f_1^{F_2})_{Q,w_1} (w_1)_Q \\
(f_1^{F_2})_{Q,w_1}^q &\leq (f_1^{F_2})_{F_1,w_1}^q
\end{aligned}$$

we have

$$\begin{aligned}
(**) &= \sum_{F_2 \in \mathcal{F}_2} \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_1 \subseteq F_2}} \sum_{\substack{Q : \pi_1(Q) = F_1 \\ \pi_2(Q) = F_2}} (f_2)_{Q,w_2}^q (w_2)_Q^q (f_1)_{Q,w_1}^q (w_1)_Q^q w_3(Q) \lambda_Q^q \\
&\lesssim \sum_{F_2 \in \mathcal{F}_2} (f_2)_{F_2,w_2}^q \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_1 \subseteq F_2}} (f_1)_{F_1,w_1}^q \sum_{\substack{Q : \pi_1(Q) = F_1 \\ \pi_2(Q) = F_2}} (w_2)_Q^q (w_1)_Q^q w_3(Q) \lambda_Q^q \\
&\lesssim \sum_{F_2 \in \mathcal{F}_2} (f_2)_{F_2,w_2}^q w_2(F_2) w_2(F_2)^{-1} \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_1 \subseteq F_2}} (f_1^{F_2})_{F_1,w_1}^q \sum_{\substack{Q : \pi_1(Q) = F_1 \\ \pi_2(Q) = F_2}} (w_1)_Q^q (w_2)_Q^q w_3(Q) \lambda_Q^q.
\end{aligned}$$

Now

$$w_2(F_2)^{-1} \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_1 \subseteq F_2}} (f_1^{F_2})_{F_1,w_1}^q \sum_{\substack{Q : \pi_1(Q) = F_1 \\ \pi_2(Q) = F_2}} (w_1)_Q^q (w_2)_Q^q w_3(Q) \lambda_Q^q$$

$$= w_2(F_2)^{-1} \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_1 \subseteq F_2}} \left(w_1(F_1)^{-1} \left(\int_{F_1} f_1^{F_2} dw_1 \right) \Lambda_{F_1}^2 w_2(F_1)^{1/q} \right)^q. \tag{5.2.5}$$

We want to estimate this by $A_1^q \|f_1^{F_2}\|_{L^{p_1}(w_1)}^q$. With $\tilde{s}' = \frac{1}{1-q}$ the claimed estimate is equivalent to

$$w_2(F_2)^{-\frac{1}{q}}\Bigg(\sum_{\substack{F_1\in\mathcal{F}_1\\F_1\subseteq F_2}} w_1(F_1)^{-q}(\Lambda^2_{F_1})^q w_2(F_1)\bigg(\int_{F_1} f_1 dw_1\bigg)^q w_2(E_{\mathcal{F}_1 F_1})^{-\frac{1}{\tilde{s}'}}\cdot$$

$$\cdot w_2(F_1)^{-\frac{1}{\tilde{s}}} w_2(E_{\mathcal{F}_1 F_1})^{\frac{1}{\tilde{s}'}} w_2(F_1)^{\frac{1}{\tilde{s}}}\Bigg)^{\frac{1}{q}} \lesssim A_1\|f_1\|_{L^{p_1}(w_1)}$$

with $E_{\mathcal{F}_1 F_1} := E_{\mathcal{F}_1}(F_1) = F_1\setminus\bigcup_{F\in ch_{\mathcal{F}_1}(F_1)} F'$. Applying Lemma 5.1.4 with $s=1, \tilde{s}=\frac{1}{q}$ and

$$b_{F_1} = w_1(F_1)^{-1}w_2(F_1)^{-1}\Lambda^2_{F_1} w_2(F_1)^{1/q}(\int_{F_1} f_1 dw_1) w_2(E_{F_1})^{1-\frac{1}{q}}$$

the above inequality follows from

$$\bigg\|\bigg(\sum_{\substack{F_1\in\mathcal{F}_1\\F_1\subseteq F_2}} b_{F_1} 1_{F_1}\bigg)\bigg\|_{L^q(w_2)} \le A_1 w_2(F_2)^{\frac{1}{q}}\|f_1\|_{L^{p_1}(w_1)},$$

i.e.,

$$\Bigg\|\sum_{\substack{F_1\in\mathcal{F}_1\\F_1\subseteq F_2}} w_1(F_1)^{-1}w_2(F_1)^{-1}\Lambda^2_{F_1} w_2(F_1)^{1/q} w_2(E_{F_1})^{1-\frac{1}{q}}\cdot$$

$$\cdot(\int_{F_1} f_1 dw_1) 1_{F_1}\Bigg\|_{L^q(w_2)} \le A_1 w_2(F_2)^{\frac{1}{q}}\|f_1\|_{L^{p_1}(w_1)}.$$

Since $\mathcal{F}_1$ is w_1- sparse and w_2 is A_∞, then $\mathcal{F}_1$ is also w_2-sparse. Then $w_2(E_{F_1}) \approx w_2(F_1)$ and the last inequality is equivalent to

$$\Bigg\|\sum_{\substack{F_1\in\mathcal{F}_1\\F_1\subseteq F_2}} \Lambda^2_{F_1}\frac{1}{w_1(F_1)}\bigg(\int_{F_1} f_1 dw_1\bigg) 1_{F_1}\Bigg\|_{L^q(w_2)} \lesssim A_1 w_2(F_2)^{\frac{1}{q}}\|f_1\|_{L^{p_1}(w_1)}. \quad (5.2.6)$$

By Maurey's factorization we can see that (5.2.6) is equivalent to the existence of a Borel measurable function $\xi \ge 0$ such that

$$\int \xi dw_2 \le 1$$

and

$$\int \bigg(\sum_{\substack{F_1\in\mathcal{F}_1\\F_1\subseteq F_2}} \Lambda^2_{F_1}(f)^{w_1}_{F_1} 1_{F_1}\bigg)\xi^{-(\frac{1-q}{q})} dw_2 \lesssim A_1 w_2(F_2)^{\frac{1}{q}}\|f_1\|_{L^{p_1}(w_1)}. \quad (5.2.7)$$

Furthermore, we have

$$\{\xi = 0\} \supseteq \bigcap_{f_1 \in L^{p_1}(w_1)} \Big\{ \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_1 \subseteq F_2}} \Lambda^2_{F_1} \frac{1}{w_1(F_1)} \Big(\int_{F_1} f_1 dw_1 \Big) 1_{F_1} = 0 \Big\},$$

which means

$$\text{if } \ \Lambda^2_{F_1} > 0 \ \text{ and } \ w_2(F_1) > 0, \ \text{ then } \ \xi > 0 \ \ dw_2 - \text{a.e. } \ \text{on } \ F_1. \tag{5.2.8}$$

This condition guarantees that no division by zero occurs, as we may assume that the cubes F_1 with $\Lambda^2_{F_1} = 0$ or $w_2(F_1) = 0$ (or $w_1(Q) = 0$) are omitted from the summation because such cubes do not contribute to inequality (5.2.6). From now on we restrict the indexation to be over the collection $\mathcal{G}^1$ of the remaining cubes

$$\mathcal{G}^1 = \{F_1 \in \mathcal{F}_1, F_1 \subseteq F_2 : \Lambda^2_{F_1} > 0, w_1(F_1) > 0, \ \text{ and } \ w_2(F_1) > 0\}.$$

By interchanging the order of integration and summation in (5.2.7) and using duality between $L^{p_1}(w_1)$ and $L^{p'_1}(w_1)$, we see that (5.2.7) is equivalent to

$$\Big(\int \Big(\sum_{\substack{F_1 \in \mathcal{G}^1 \\ F_1 \subseteq F_2}} \Lambda^2_{F_1} (\xi^{-(\frac{1-q}{q})})^{w_2}_{F_1} \frac{w_2(F_1)}{w_1(F_1)} 1_{F_1} \Big)^{p'_1} dw_1 \Big)^{\frac{1}{p'_1}} \lesssim A_1 w_2(F_2)^{\frac{1}{q}}.$$

Then we can say that (5.2.6) is equivalent to the existence of a function ξ with $\xi > 0$ dw_2- a.e. on every cube $F_1 \in \mathcal{G}^1$, that satisfies the conditions

$$\int \xi dw_2 \lesssim_q 1,$$

$$\Big(\int \Big(\sum_{\substack{F_1 \in \mathcal{G}^1 \\ F_1 \subseteq F_2}} \Lambda^2_{F_1} (\xi^{-(\frac{1-q}{q})})^{w_2}_{F_1} \frac{w_2(F_1)}{w_1(F_1)} 1_{F_1} \Big)^{p'_1} dw_1 \Big)^{\frac{1}{p'_1}} \lesssim_q A_1 w_2(F_2)^{\frac{1}{q}}.$$

Discretizing this is equivalent to the existence of a family $\{a_{F_1}\}_{\substack{F_1 \in \mathcal{G}^1 \\ F_1 \subseteq F_2}}$ of positive reals that satisfies the pair of conditions

$$\begin{aligned} \int \Big(\sup_{\substack{F_1 \in \mathcal{G}^1 \\ F_1 \subseteq F_2}} a_{F_1} 1_{F_1} \Big)^{\frac{q}{1-q}} dw_2 &\lesssim_1 1, \\ \Big(\int \Big(\sum_{\substack{F_1 \in \mathcal{G}^1 \\ F_1 \subseteq F_2}} \Lambda^2_{F_1} a^{-1}_{F_1} \frac{w_2(F_1)}{w_1(F_1)} 1_{F_1} \Big)^{p'} dw_1 \Big)^{\frac{1}{p'_1}} &\lesssim_q A_1 w_2(F_2)^{\frac{1}{q}} \end{aligned} \tag{5.2.9}$$

(see Lemma 5.1.5). This holds by hypothesis (5.2.2). □

Proof. Now we show $A_1, A_2 \lesssim B$.

By Lemma 5.1.4 with $s = 1$ (which implies $\frac{1}{\tilde{s}} = q$ and $\frac{1}{\tilde{s}'} = 1 - q$) we have that (5.2.3) is equivalent to

$$\left(\sum_{Q\in\mathcal{D}}\left(\lambda_Q \frac{1}{\mu(Q)^2}\left(\prod_{i=1}^{2}\int_Q M_{w_i} f_i w_i\right)\right)^q w_3(E_Q)^{1-q} w_3(Q)^q\right)^{\frac{1}{q}} \lesssim B\prod_{i=1}^{2}\|M_{w_i} f_i\|_{L^{p_i}(w_i)}$$
$$\lesssim B\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(w_i)}.$$

Since

$$\sum_{Q\in\mathcal{D}} \approx \sum_{Q\subset Q_0} = \sum_{\substack{F_1\in\mathcal{F}_1\\F_2\in\mathcal{F}_2}}\ \sum_{\substack{Q:\pi_1(Q)=F_1\\\pi_2(Q)=F_2}}$$

and $w_3(E_Q) \approx w_3(Q)$ the inequality above is equivalent to

$$\left(\sum_{\substack{F_1\in\mathcal{F}_1\\F_2\in\mathcal{F}_2}}\ \sum_{\substack{Q:\pi_1(Q)=F_1\\\pi_2(Q)=F_2}} \lambda_Q^q\left(\frac{1}{\mu(Q)^2}\right)^q\left(\prod_{i=1}^{2}\int_Q M_{w_i} f_i w_i\right)^q w_3(Q)\right)^{\frac{1}{q}} \lesssim B\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(w_i)}.$$

Restricting the sum to $F_1 \subseteq F_2$ we obtain the estimate

$$\left(\sum_{F_2\in\mathcal{F}_2}\ \sum_{\substack{F_1\in\mathcal{F}_1\\F_1\subseteq F_2}}\ \sum_{\substack{Q:\pi_1(Q)=F_1\\\pi_2(Q)=F_2}} (M_{w_1} f_1 w_1)_Q^q (M_{w_2} f_2 w_2)_Q^q w_3(Q)\lambda_Q^q\right)^{\frac{1}{q}} \le B\prod_{i=1}^{2}\|f_i\|_{L_i^p(w_i)}$$

which is equivalent to

$$\left(\sum_{F_2\in\mathcal{F}_2}\ \sum_{\substack{F_1\in\mathcal{F}_1\\F_1\subseteq F_2}}\ \sum_{\substack{Q:\pi_1(Q)=F_1\\\pi_2(Q)=F_2}} (M_{w_2} f_2)_{Q,w_2}^q (w_2)_Q^q (M_{w_1} f_1)_{Q,w_1}^q (w_1)_Q^q w_3(Q)\lambda_Q^q\right)^{\frac{1}{q}}$$
$$\le B\prod_{i=1}^{2}\|f_i\|_{L_i^p(w_i)}. \quad (5.2.10)$$

This is equivalent to

$$\left(\sum_{F_2\in\mathcal{F}_2}\ \sum_{\substack{F_1\in\mathcal{F}_1\\F_1\subseteq F_2}}\left(\sum_{\substack{Q:\pi_1(Q)=F_1\\\pi_2(Q)=F_2}} (M_{w_1} f_1)_{Q,w_1}^q (M_{w_2} f_2)_{Q,w_2}^q (w_2)_Q^q (w_1)_Q^q w_3(Q)\lambda_Q^q\right)\cdot\right.$$
$$\left.\cdot w_2(E_{F_1})^{-\frac{1}{\tilde{s}'}} w_2(F_1)^{-\frac{1}{\tilde{s}}} w_2(E_{F_1})^{\frac{1}{\tilde{s}'}} w_2(F_1)^{\frac{1}{\tilde{s}}}\right)^{\frac{1}{q}} \le B\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(w_i)}.$$

Applying Lemma 5.1.4 with

$$b_{F_1} = w_2(F_1)^{-1} w_2(E_{F_1})^{1-\frac{1}{q}} \Bigg(\sum_{\substack{Q:\pi_1(Q)=F_1 \\ \pi_2(Q)=F_2}} (M_{w_1} f_1)^q_{Q,w_1} (M_{w_2} f_2)^q_{Q,w_2} (w_2)^q_Q (w_1)^q_Q w_3(Q) \lambda^q_Q \Bigg)^{\frac{1}{q}}$$

we obtain

$$\Bigg\| \sum_{F_2 \in \mathcal{F}_2} \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_1 \subseteq F_2}} w_2(F_1)^{-1} w_2(E_{F_1})^{1-\frac{1}{q}} \cdot$$

$$\cdot \Bigg(\sum_{\substack{Q:\pi_1(Q)=F_1 \\ \pi_2(Q)=F_2}} (M_{w_1} f_1)^q_{Q,w_1} (M_{w_2} f_2)^q_{Q,w_2} (w_2)^q_Q (w_1)^q_Q w_3(Q) \lambda^q_Q \Bigg)^{\frac{1}{q}} 1_{F_1} \Bigg\|_{L^q(w_2)}$$

$$\leq B \prod_{i=1}^{2} \|f_i\|_{L^p_i(w_i)}.$$

We can write the above as

$$\Bigg\| \sum_{F_2 \in \mathcal{F}_2} \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_1 \subseteq F_2}} \left(\int_{F_1} f_1 w_1 \right) \left(\int_{F_1} f_1 w_1 \right)^{-1} w_2(F_1)^{-1} w_2(E_{F_1})^{1-\frac{1}{q}} \cdot$$

$$\cdot \Bigg(\sum_{\substack{Q:\pi_1(Q)=F_1 \\ \pi_2(Q)=F_2}} (M_{w_1} f_1)^q_{Q,w_1} (M_{w_2} f_2)^q_{Q,w_2} (w_2)^q_Q (w_1)^q_Q w_3(Q) \lambda^q_Q \Bigg)^{\frac{1}{q}} 1_{F_1} \Bigg\|_{L^q(w_2)}$$

$$\leq B \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)}.$$

But by Lemma 5.1.1 with $s = 1$ this estimate holds if and only if there exists $C > 0$ such that for any sequence $(\rho_{F_1})_{F_1}$ of nonnegative numbers

$$\Bigg\| \sum_{F_2 \in \mathcal{F}_2} \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_1 \subseteq F_2}} \Gamma^2_{F_1} \rho_{F_1} 1_{F_1} \Bigg\|_{L^q(w_2)} \leq C \|\sup_{F_1} (\rho_{F_1} 1_{F_1})\|_{L^{p_1}(w_1)} \tag{5.2.11}$$

with

$$\Gamma^2_{F_1} = w_1(F_1)\bigg(\int_{F_1} f_1 w_1\bigg)^{-1} w_2(F_1)^{-1} w_2(E_{F_1})^{1-\frac{1}{q}} \cdot$$
$$\cdot \bigg(\sum_{\substack{Q:\pi_1(Q)=F_1\\ \pi_2(Q)=F_2}} (M_{w_1}f_1)^q_{Q,w_1}(M_{w_2}f_2)^q_{Q,w_2}(w_1)^q_Q(w_2)^q_Q w_3(Q)\lambda^q_Q\bigg)^{\frac{1}{q}}$$
$$\approx \bigg(\sum_{\substack{Q:\pi_1(Q)=F_1\\ \pi_2(Q)=F_2}} (M_{w_1}f_1)^q_{Q,w_1}(M_{w_2}f_2)^q_{Q,w_2}(w_1)^q_Q(w_2)^q_Q w_3(Q)\lambda^q_Q\bigg)^{\frac{1}{q}} w_2(F_1)^{-\frac{1}{q}} w_1(F_1)\bigg(\int_{F_1} f_1 w_1\bigg)^{-1}$$

Considering $f_2 = 1_{F_2}$ we obtain

$$\bigg\|(f_2)_{F_2,w_2}\sum_{\substack{F_1\in\mathcal{F}_1\\ F_1\subseteq F_2}} \Gamma^2_{F_1}\rho_{F_1}1_{F_1}\bigg\|_{L^q(w_2)} \le C\|\sup_{F_1}(\rho_{F_1}1_{F_1})\|_{L^{p_1}(w_1)}$$

with

$$\Gamma^2_{F_1} = w_2(F_1)^{-1}w_2(E_{F_1})^{1-\frac{1}{q}}\bigg(\sum_{\substack{Q:\pi_1(Q)=F_1\\ \pi_2(Q)=F_2}} (w_1)^q_Q(w_2)^q_Q w_3(Q)\lambda^q_Q\bigg)^{\frac{1}{q}}$$
$$\approx \bigg(\sum_{\substack{Q:\pi_1(Q)=F_1\\ \pi_2(Q)=F_2}} (w_1)^q_Q(w_2)^q_Q w_3(Q)\lambda^q_Q\bigg)^{\frac{1}{q}} w_2(F_1)^{-\frac{1}{q}}.$$

The last inequality is equivalent to

$$\bigg\|\sum_{\substack{F_1\in\mathcal{F}_1\\ F_1\subseteq F_2}} \Gamma^2_{F_1}\frac{1}{w_1(F_1)}\bigg(\int_{F_1} f_1 dw_1\bigg)1_{F_1}\bigg\|_{L^q(w_2)} \lesssim \tilde{A}_1\|f_1\|_{L^{p_1}(w_1)} \tag{5.2.12}$$

with $\tilde{A} = B\|f_2\|_{L^p(w_2)}$. By Maurey's factorization we can see that (5.2.12) is equivalent to the existence of a Borel measurable function $\xi \ge 0$ such that

$$\int \xi dw_2 \le 1$$

and

$$\int \bigg(\sum_{\substack{F_1\in\mathcal{F}_1\\ F_1\subseteq F_2}} \Gamma^2_{F_1}(f)^{w_1}_{F_1}1_{F_1}\bigg)\xi^{-(\frac{1-q}{q})}dw_2 \lesssim \tilde{A}_1\|f_1\|_{L^{p_1}(w_1)}. \tag{5.2.13}$$

With the same reasoning previously used and by interchanging the order of integration and summation in (5.2.13) and using the duality between $L^{p_1}(w_1)$ and $L^{p'_1}(w_1)$, we see that (5.2.13) is equivalent to

$$\Big(\int\Big(\sum_{\substack{F_1\in\mathcal{G}^1\\F_1\subseteq F_2}}\Gamma^2_{F_1}(\xi^{-(\frac{1-q}{q})})^{w_2}_{F_1}\frac{w_2(F_1)}{w_1(F_1)}1_{F_1}\Big)^{p'_1}dw_1\Big)^{\frac{1}{p'_1}}\lesssim\tilde{A}_1.$$

Then we can say that (5.2.12) is equivalent to the existence of a function ξ with $\xi>0$ dw_2- a.e. on every cube $F_1\in\mathcal{G}^1$, that satisfies the pair of conditions

$$\int\xi dw_2\lesssim_q 1,$$
$$\Big(\int\Big(\sum_{\substack{F_1\in\mathcal{G}^1\\F_1\subseteq F_2}}\Gamma^2_{F_1}(\xi^{-(\frac{1-q}{q})})^{w_2}_{F_1}\frac{w_2(F_1)}{w_1(F_1)}1_{F_1}\Big)^{p'_1}dw_1\Big)^{\frac{1}{p'_1}}\lesssim_q\tilde{A}_1.$$

Discretizing this is equivalent to the existence of a family $\{a_{F_1}\}_{\substack{F_1\in\mathcal{G}^1\\F_1\subseteq F_2}}$ of positive reals that satisfies the pair of conditions

$$\begin{aligned}&\int\Big(\sup_{\substack{F_1\in\mathcal{G}^1\\F_1\subseteq F_2}}a_{F_1}1_{F_1}\Big)^{\frac{q}{1-q}}dw_2\lesssim_1 1,\\&\Big(\int\Big(\sum_{\substack{F_1\in\mathcal{G}^1\\F_1\subseteq F_2}}\Gamma^2_{F_1}a^{-1}_{F_1}\frac{w_2(F_1)}{w_1(F_1)}1_{F_1}\Big)^{p'}dw_1\Big)^{\frac{1}{p'_1}}\lesssim_q\tilde{A}_1\end{aligned}\tag{5.2.14}$$

(see Lemma 5.1.5). This holds by hypothesis (5.2.2). □

Chapter 6

Weighted $L^{p_1} \times L^{p_2} \to L^q$ bounds for bilinear positive dyadic operators in case $0 < q < r$ and $\dfrac{1}{p_1} + \dfrac{1}{p_2} = \dfrac{1}{r} \leq 1$ with some function dependent bound

Still extending [HV17, Theorem 1.2], we characterize here weighted $L^{p_1} \times L^{p_2} \to L^q$ estimates for a dyadic version of a so-called non homogeneous bilinear fractional integral operator, as in the Chapter 5, but now in case $0 < q < r, p_1, p_2 > 1$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} \leq 1$.

Theorem 6.1.6 deals with the case $r = 1$. We use an argument analogous to the proof of [HV17, Theorem 1.2] and a bilinear version of Maurey's factorization theorem [Sch84]. In the linear case this argument reduces boundedness of a linear *operator* to boundedness of a linear *form*. Since bounded linear forms on L^p are exactly members of $L^{p'}$, this gives a very short characterization. In the bilinear case we obtain a bilinear form, and there does not seem to be a short description of its boundedness.

Theorem 6.3.1 deals with the general case $r \geq 1$. Here we use a slightly more general multilinear version of Maurey's factorization theorem.

6.1 Preliminaries and Tools

Let f be a real-valued function defined on the product set $X \times Y$ of two arbitrary sets X, Y. The function f is said to be convex on X if for any two elements $x_1, x_2 \in X$ and two numbers $\xi_1, \xi_2 \geq 0$ with $\xi_1 + \xi_2 = 1$, there exists an element $x_0 \in X$ such that

$$f(x_0, y) \leq \xi_1 f(x, y) + \xi_2 f(x_2, y)$$

for all $y \in Y$. Similarly f is said be concave on Y if for any two elements $y_1, y_2 \in Y$ and two numbers $\eta_1, \eta_2 \geq 0$ with $\eta_1 + \eta_2 = 1$, there exists an $y_0 \in Y$ such that

$$f(x, y_0) \geq \eta_1 f(x, y_1) + \eta_2 f(x, y_2)$$

for all $x \in X$. We say that $f(x, y)$ is lower (resp. upper) semi-continuous on X (resp. Y) if

$$f(x_0, y) \leq \lim_{x \to x_0} \inf f(x, y) \quad \left(\text{resp.} f(x_0, y) \geq \lim_{y \to y_0} \sup f(x, y_0)\right).$$

Theorem 6.1.1. *(Ky Fan's minimax theorem,[Fan53], Theorem 1) Let X, Y be two compact Hausdorff spaces and f a real-valued function defined on $X \times Y$. Suppose that, for every $y \in Y, f(x, y)$ is lower semi-continuous on X and, for every $x \in X, f(x, y)$ is upper semi-continuous on Y. Then:*

(i) The equality

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y)$$

holds if and only if the following condition is satisfied: For any two finite sets

$$\{x_1, x_2, \cdots, x_n\} \subset X \quad \textit{and} \quad \{y_1, y_2, \cdots, y_n\} \subset Y,$$

there exist $x_0 \in X$ and $y_o \in Y$ such that

$$f(x_0, y_k) \leq f(x_1, y_0), 1 \leq i \leq n, 1 \leq k \leq m.$$

(ii) In particular, if f is convex on X and concave on Y, then (i) holds.

Lemma 6.1.2. *(Lemma 1,[Mau74]) Let $1 < p < \infty$ and $0 < q < \infty$. Denote*

$$K_p := \{f; f \quad \textit{is convex}, f \geq 0, \int f^p d\mu \leq 1\}.$$

Then $f \to \int f^{-q} d\mu$ is convex s.c.i.(that is, growing and continuing to the left) on K_p in the topology $\sigma(L^p, L^{p'})$.

The following result is know as the Hardy- Littlewood dyadic maximal theorem. It's can be found in [Moe12].

Theorem 6.1.3. *(Hardy Littlewood maximal inequality) Let $1 < p \leq \infty$. Then for all measures w we have*

$$\left\| \sup_{Q \in \mathcal{D}} \frac{1}{w(Q)} \int_Q |f| w 1_Q \right\|_{L^p(w)} \lesssim_p \|f\|_{L^p(w)}$$

and the constant does not depend on w and on the family of dyadic cubes $\mathcal{D}$.

The following theorem is given by Schep [Sch84], the proof is partially inspired by Maurey's work. We put the proof below to make it easier to understand the text.

Theorem 6.1.4. *(Maurey [cf. Schep]) Let $A \subseteq L^q$ be a convex set of non-negative functions such that $\int f^q w_3 \leq 1$ for all $f \in A$. Assume $0 < q < 1$. Then there exists $\phi \geq 0$ in L_r with $\|\phi\|_r \leq 1$ and $r^{-1} = q^{-1} - 1$ such that $\int \frac{f}{\phi} w_3 \leq 1$ for all $f \in A$.*

Proof. Let $s = (1-q)^{-1}$ and let U_s be the positive unit ball of L_s. Then U_s is weakly compact since $1 < s < \infty$. Define

$$F : U_s \times A \to \mathbb{R}_+ \cup \{\infty\}$$

by

$$F(h, f) = \int \frac{f}{h^{\frac{1}{q}}} w_3,$$

where we employ $0/0 = 0$ as a convention. Then for every $f \in A$, $F(h, f)$ is convex and lower semicontinuous with respect to the weak topology of L_s (see Lemma 6.1.2). Moreover, for every $h \in U_s, F(h, s)$ is concave on A. Apply *Ky Fan's minimax theorem* (see Theorem 6.1.1) to obtain

$$\min_{h \in U_s} \max_{f \in A} F(h, f) = \max_{f \in A} \min_{h \in U_s} F(h, f).$$

Since $F(h, f) \leq 1$ for $h = f^{q(1-q)}$, it follows that there exists $h_0 \in U_s$ such that

$$F(h_0, f) = \int \frac{f}{h_0^{\frac{1}{q}}} w_3 \leq 1$$

for all $f \in A$. So there exists $\phi = h_0^{\frac{1}{q}}, \phi \geq 0$ in L_r with $\|\phi\|_r \leq 1$ and $r^{-1} = q^{-1} - 1$ such that

$$\int \frac{f}{\phi} w_3 \leq 1 \forall f \in A.$$

$\square$

We need also of the following lemma.

Lemma 6.1.5. *[cf. Schep] Let $X_1, \dots, X_m$ be measure spaces and*

$$E := \{u : \prod_{j=1}^{m} X_j \to \mathbb{R} \text{ measurable}\}.$$

Let $p_j \in [1, \infty]$ with $\sum_{j=1}^{m} 1/p_j \leq 1$. Then the functional

$$\rho_{p_1,\dots,p_m}(u) := \inf\left(\prod_{j=1}^{m} \|f_j\|_{p_j} : |u| \leq \otimes_{j=1}^{m} |f_j|\right),$$

where $f_j : X_j \to \mathbb{R}$, is subadditive on E.

Proof. Without loss of generality we may assume $\sum_{j=1}^{m} 1/p_j = 1$, otherwise we can add a $m+1$ factor X_{m+1} consisting of one point and $1/p'_{m+1} = \sum_{j=1}^{m} 1/p_j$ and observe

$$\rho_{p_1,\dots,p_m}(u) = \rho_{p_1,\dots,p_m,p_{m+1}}(u \otimes 1).$$

Let $u, v \in E$ and $\epsilon > 0$. Without loss of generality we assume that $\rho(u) < \infty$ and $\rho(v) < \infty$. Then there are $f_j, g_j : X_j \to \mathbb{R}$ with

$$|u(x_1, \dots, x_m)| \leq |f_1(x_1)| \cdots |f_m(x_m)|, \quad |v(x_1, \dots, x_m)| \leq |g_1(x_1)| \cdots |g_m(x_m)|$$

and

$$\rho(u) \leq \prod_{j=1}^{m} \|f_j\|_{p_j} - \epsilon, \quad \rho(v) \leq \prod_{j=1}^{m} \|g_j\|_{p_j} - \epsilon.$$

Without loss of generality we may assume $f_j \not\equiv 0$ for all j. Replacing each f_j by $(\prod_k \|f_k\|_{p_k})^{1/p_j} f_j / \|f_j\|_{p_j}$ we assume

$$\|f_1\|_{p_1}^{p_1} = \cdots = \|f_m\|_{p_m}^{p_m} \leq \rho(u) + \epsilon$$

and similarly

$$\|g_1\|_{p_1}^{p_1} = \cdots = \|g_m\|_{p_m}^{p_m} \leq \rho(v) + \epsilon.$$

By Hölder's inequality for the sum over 2 points we get

$$\begin{aligned} |u(x_1, \dots, x_m) + v(x_1, \dots, x_m)| &\leq |f_1(x_1)| \cdots |f_m(x_m)| + |g_1(x_1)| \cdots |g_m(x_m)| \\ &\leq (|f_1(x_1)|^{p_1} + |g_1(x_1)|^{p_1})^{\frac{1}{p_1}} \cdots (|f_m(x_m)|^{p_m} + |g_m(x_m)|^{p_m})^{\frac{1}{p_2}}. \end{aligned}$$

Hence

$$\begin{aligned}\tilde{\rho}(u+v) &\leq \prod_{j=1}^{m} \|(|f_j|^{p_j}+|g_j|^{p_j})^{1/p_j}\|_{p_j}\\ &= \prod_{j=1}^{m}(\|f_j\|_{p_j}^{p_j}+\|g_j\|_{p_j}^{p_j})^{\frac{1}{p_j}}\\ &\leq \tilde{\rho}(u)+\tilde{\rho}(v)+2\epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we obtain the claimed subadditivity. □

Note that, comparing with what was presented in the characterization via factorization by [HV17] in the linear case, we see for bilinear case the following.

Theorem 6.1.6. *Let w_1, w_2, w_3 measurable functions on $\mathbb{R}^n$, μ a nonnegative measure, $\sum_i p_i^{-1} = r^{-1} = 1, i = 1,2$ and $(\lambda_Q)_Q$ be a sequence of non-negative real numbers. Consider $0 < q < 1 < p_i, i = 1,2$. Denote by $\mathcal{D}$ the collection of dyadic cubes and $\mathcal{Q} := \{Q \in \mathcal{D}; \lambda_Q > 0, w_1(Q) > 0 \quad and \quad w_2(Q) > 0\}$. Denote*

$$L^{p_1'}(w_1, L^{p_2'}(w_2)) := \left\{ f : \left(\int \left(\int |f(x,y)|^{p_2'} w_2 dy \right)^{\frac{1}{p_2'}p_1'} w_1 dx \right)^{\frac{1}{p_1'}} < \infty \right\}.$$

Let

1. *There exists a family $\{a_Q\}_{Q\in\mathcal{Q}}$ of positive reals that satisfies the pair of conditions*

$$\int \left(\sup_{Q\in\mathcal{Q}} a_Q 1_Q \right)^{\frac{q}{1-q}} w_3 \lesssim 1, \tag{6.1.7}$$

$$\left\| \sum_{Q\in\mathcal{Q}} \lambda_Q a_Q^{-1} \frac{w_3(Q)}{\mu(Q)^2} 1_Q(y) 1_Q(z) \right\|_{L^{p_1'}(w_1, L^{p_2'}(w_2))} \lesssim C. \tag{6.1.8}$$

2. *There exists a function Φ with $\Phi > 0$ $dw_3-a.e.$ on every cube $Q \in \mathcal{Q}$, that satisfies the pair of conditions*

$$\int \Phi w_3 \leq 1, \tag{6.1.9}$$

$$\left\| \int \sum_{Q\in\mathcal{Q}} \frac{\lambda_Q}{\mu(Q)^2} 1_Q(x) 1_Q(y) 1_Q(z) \Phi(x)^{\frac{-(1-q)}{q}} w_3(x) \right\|_{L^{p_1'}(w_1, L^{p_2'}(w_2))} \leq C. \tag{6.1.10}$$

3. There exists $\Phi \geq 0$ *in* $L_1(w_3)$ *satisfying*

$$\int \Phi w_3 \leq 1$$

$$\exists C, \forall f_i, \int \sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\mu(Q)^2}\Big(\prod_{i=1}^{2}\int f_i w_i\Big) 1_Q \Phi^{-\frac{(1-q)}{q}} w_3 \leq C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)}. \tag{6.1.11}$$

4.

$$\exists B, \forall f_i, \Big\|\sum_{Q\in\mathcal{D}} \lambda_Q \frac{1}{\mu(Q)^2}\Big(\prod_{i=1}^{2}\int_Q f_i w_i\Big) 1_Q\Big\|_{L^q(w_3)} \leq B \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)}. \tag{6.1.12}$$

We have
i) 1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4.
ii)4 $\Rightarrow$ 3 $\not\Rightarrow$ 2 $\Rightarrow$ 1.

Proof of $1 \Rightarrow 2$. Put

$$\Phi := \Big(\sup_{Q\in\mathcal{Q}} a_Q 1_Q\Big)^{\frac{q}{1-q}}.$$

Clearly

$$\int \Phi w_3 = \int \Big(\sup_{Q\in\mathcal{Q}} a_Q 1_Q\Big)^{\frac{1}{1-q}} w_3 \leq 1.$$

Moreover, using that

$$\int 1_Q(x) dw_3(x) = w_3(Q) \quad \text{and} \quad \Big(\sup_{Q\in\mathcal{Q}} a_Q 1_Q\Big)^{\frac{q}{1-q}} \geq a_Q^{\frac{q}{1-q}} 1_Q$$

we have

$$\Big\|\int \sum_{Q\in\mathcal{Q}} \frac{\lambda_Q}{\mu(Q)^2} 1_Q(x) 1_Q(y) 1_Q(z) \Phi(x)^{\frac{-(1-q)}{q}} w_3(x)\Big\|_{L^{p'_1}(w_1, L^{p'_2}(w_2))}$$

$$= \Big\|\int \sum_{Q\in\mathcal{Q}} \frac{\lambda_Q}{\mu(Q)^2} 1_Q(x) 1_Q(y) 1_Q(z) \Big(\sup_{Q\in\mathcal{Q}} a_Q 1_Q(x)\Big)^{\frac{q}{1-q}\left(-\frac{(1-q)}{q}\right)} w_3(x)\Big\|_{L^{p'_1}(w_1, L^{p'_2}(w_2))}$$

$$\leq \Big\|\sum_{Q\in\mathcal{Q}} \lambda_Q a_Q^{-1} \frac{w_3(Q)}{\mu(Q)^2} 1_Q(y) 1_Q(z)\Big\|_{L^{p'_1}(w_1, L^{p'_2}(w_2))} \lesssim C.$$

$\square$

Proof of $2 \Rightarrow 3$. The proof follows by Fubini. Indeed,

$$\int \sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\mu(Q)^2}\Big(\prod_{i=1}^{2}\int f_i w_i\Big) 1_Q \Phi^{-\frac{(1-q)}{q}} w_3$$

$$\simeq \int_Q \int_Q \sum_{Q\in\mathcal{Q}} \frac{\lambda_Q}{\mu(Q)^2} 1_Q(x)1_Q(y)f_1w_1(y)1_Q(z)f_2w_2(z)\Phi^{-\frac{(1-q)}{q}}(x)w_3(x)dydz$$

$$\leq \Big\| \int \sum_{Q\in\mathcal{Q}} \frac{\lambda_Q}{\mu(Q)^2} 1_Q(x)1_Q(y)1_Q(z)\Phi(x)^{\frac{-(1-q)}{q}} w_3(x)\Big\|_{L^{p_1'}(w_1,L^{p_2'}(w_2))}$$

$$\leq C \leq C\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(w_i)}.$$

□

Proof of $3 \Rightarrow 4$. By Hölder inequality with expoents $\frac{1}{q}$ and $\frac{1}{1-q}$ we obtain

$$\Big\|\sum_{Q\in\mathcal{D}} \lambda_Q \frac{1}{\mu(Q)^2}\Big(\prod_{i=1}^{2}\int_Q f_i w_i\Big)1_Q\Big\|_{L^q(w_3)}$$

$$= \Big[\int\Big(\sum_{Q\in\mathcal{D}} \lambda_Q \frac{1}{\mu(Q)^2}\Big(\prod_{i=1}^{2}\int_Q f_i w_i\Big)1_Q\Big)^q \Phi^{-(1-q)}\Phi^{1-q}w_3\Big]^{\frac{1}{q}}$$

$$\leq \Big[\int\Big(\sum_{Q\in\mathcal{D}} \lambda_Q \frac{1}{\mu(Q)^2}\Big(\prod_{i=1}^{2}\int_Q f_i w_i\Big)1_Q\Big)\Phi^{-\frac{(1-q)}{q}}w_3\Big]\Big(\int \Phi w_3\Big)^{\frac{1-q}{q}}$$

$$\leq C\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(w_i)}.$$

□

Proof of $4 \Rightarrow 3$. Let

$$L_{p_i} := L_{p_i}(X_i,\mu_i) \text{ for some } \ p_i \geq 1$$

and

$$E := \{u : X_1 \times X_2 \to \mathbb{R}; u \ \text{ measurable}\}.$$

Define

$$\rho(u) := \inf\Big(\|f_1w_1\|_{p_1}\|f_2w_2\|_{p_2}, |u(x_1,x_2)| \leq (f_1w_1)(x_1)\cdot(f_2w_2)(x_2)\Big)$$

$\forall x_1 \in X_1, x_2 \in X_2$. By Lemma 6.1.5 ρ is subadditive in E provided $\sum_i 1/p_i = 1$. Denote

$$T_\lambda(f_1w_1, f_2w_2) = \sum_Q \frac{\lambda_Q}{\mu(Q)^2}\Big(\prod_{i=1}^{2}\int f_iw_i\Big)1_Q.$$

Now , since T_λ a bilinear operator, there is only one linear operator $T_{\lambda L}$ given by

$$T_{\lambda_L}(u) = \sum_Q \frac{\lambda_Q}{\mu(Q)^2}\Big(\int_{Q\times Q} u\Big)1_Q; u : X_1 \times X_2 \to \mathbb{R},$$

such that

$$T_\lambda(f_1w_1, f_2w_2) = T_{\lambda L}(f_1w_1 \otimes f_2w_2),$$

where

$$(f_1w_1 \otimes f_2w_2)(x_1, x_2) = f_1(x_1) \cdot f_2(x_2).$$

Let $u \geq 0, u \in E, f_1 \otimes f_2 \geq u$. Since $T_{\lambda L}$ is positive and by (6.1.12) we have

$$\|T_{\lambda L}(u)\|_q \leq \|T_{\lambda L}\big(f_1w_1 \otimes f_2w_2\big)\|_q = \|T_\lambda(f_1w_1, f_2w_2)\|_q \leq \|T_\lambda\| \prod_i \|f_iw_i\|_{p_i}.$$

Taking the infimum over f_iw_i, we obtain

$$\|T_{\lambda L}(u)\|_q \leq \|T_\lambda\|\rho(u).$$

Define

$$A := \frac{1}{\|T_\lambda\|}T_{\lambda L}(B_E) \subseteq L^q.$$

Note that A is convex. Let

$$\int f^q w_3 \leq 1, \forall f \in A.$$

By Theorem 6.1.4 there exists $\phi \geq 0$ in L_r with $\|\phi\|_r \leq 1$ and $r^{-1} = q^{-1} - 1$ such that

$$\int \frac{f}{\phi} w_3 \leq 1, \forall f \in A.$$

Considering $f = \frac{1}{\|T_\lambda\|}T_\lambda(f_1w_1, f_2w_2)$ we obtain that exists a Borel measurable function $\phi \geq 0$ in L_r with

$$\Big(\int \phi^r w_3\Big)^{\frac{1}{r}} \leq 1, r^{-1} = q^{-1} - 1$$

such that

$$\int \frac{1}{\|T_\lambda\|} T_\lambda(f_1 w_1, f_2 w_2)\phi^{-1} w_3 \leq B \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)}.$$

So, the inequality (6.1.12) implies the existence of $\Phi = \phi^{\frac{q}{1-q}} \geq 0$ in $L_1(w_3)$ with

$$\int \Phi w_3 \leq 1$$

and

$$\int T_\lambda(f_1 w_1, f_2 w_2)\Phi^{-\frac{(1-q)}{q}} w_3 \leq C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)}. \tag{6.1.13}$$

□

Proof of $3 \not\Rightarrow 2$. Consider $p_1 = p_2 = 2, w_1 = w_2 = 1, q = 1$ and

$$\lambda_Q = \begin{cases} \frac{\mu(Q)^2}{w_3(Q)}, l(Q) = 1 \\ 0, \quad \text{otherwise.} \end{cases}$$

We obtain by Cauchy Schwarz

$$\begin{aligned} \int \sum_Q \frac{\lambda_Q}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i \Big) 1_Q \phi^{\frac{-(1-q)}{q}} w_3 &= \sum_Q \Big(\prod_{i=1}^{2} \int f_i \Big) \\ &= \sum_{x,y} f_1(x) f_2(y) \\ &\leq ||f_1||_2 ||f_2||_2. \end{aligned}$$

But,

$$\begin{aligned} &\Big\| \int \sum_Q \frac{\lambda_Q}{\mu(Q)^2} 1_Q(x) 1_Q(y) 1_Q(z) \Phi(x)^{\frac{-(1-q)}{q}} w_3(x) \Big\|_{L^2(w_1, L^2(w_2))} \\ &\qquad = \Big\| \sum_Q 1_Q(x) 1_Q(y) 1_Q(z) \Big\|_{L^2(1, L^2(1))} \\ &= \Big(\int \Big(\int \Big| \sum_Q 1_Q(x) 1_Q(y) 1_Q(z) \Big|^2 dy \Big) dx \Big)^{\frac{1}{2}} = \infty. \end{aligned}$$

□

Proof of $2 \Rightarrow 1$. We set

$$a_Q^{-1} = \frac{1}{w_3(Q)} \int \Phi^{-\frac{(1-q)}{q}} w_3$$

for every cube $Q \in \mathcal{Q}$. Thus, condition (6.1.8) becomes condition (6.1.10). By Jensen's inequality together with the convexity of the function $t \to t^{-q}$, and the Hardy Littlewood maximal inequality, condition (6.1.9) implies (6.1.7) through

$$\begin{aligned}
\int \left(\sup_{Q \in \mathcal{D}} a_Q 1_Q \right)^{\frac{q}{1-q}} w_3 &= \int \left(\sup_{Q \in \mathcal{D}} \left(\frac{1}{w_3(Q)} \int \Phi^{-\frac{(1-q)}{q}} w_3 \right)^{-1} 1_Q \right)^{\frac{q}{1-q}} w_3 \\
&= \int \sup_{Q \in \mathcal{D}} \left(\frac{1}{w_3(Q)} \int \Phi^{-\frac{(1-q)}{q}} dw_3 \right)^{-q} 1_Q \Bigg)^{\frac{1}{1-q}} w_3 \\
&\leq \int \left(\sup_{Q \in \mathcal{D}} \left(\frac{1}{w_3(Q)} \int \Phi^{(1-q)} w_3 \right) 1_Q \right)^{\frac{1}{1-q}} w_3 \\
&\leq \int \Phi w_3.
\end{aligned}$$

□

6.2 Estimates for $r = 1$

Now, we obtain our first main result of this chapter.

Theorem 6.2.1. *Let $w_1, w_2.w_3$ be measurable functions on $\mathbb{R}^n, \mu$ a nonnegative measure function, $\sum_i p_i^{-1} = r^{-1} = 1, i = 1, 2$ and $(\lambda_Q)_Q$ be a sequence of non-negative real numbers. Consider $0 < q < 1 < p_i, i = 1, 2$. Denote $\mathcal{D}$ to be the collection of dyadic cubes and $\mathcal{Q} := \{Q \in \mathcal{D}; \lambda_Q > 0, w_1(Q) > 0$ and $w_2(Q) > 0)\}$. Let*

$3'$. There exists a family $\{a_Q\}_{Q \in \mathcal{Q}}$ of positive reals that satisfies the pair of conditions

$$\int \left(\sup_{Q \in \mathcal{Q}} a_Q 1_Q \right)^{\frac{q}{1-q}} w_3 \leq 1 \tag{6.2.2}$$

$$\sup_{\|f_i\|_{L^{p_i}(w_i)} \leq 1} \sum_{Q \in \mathcal{Q}} \frac{\lambda_Q}{\mu(Q)^2} \left(\prod_{i=1}^{2} \int_Q f_i w_i \right) a_Q^{-1} w_3(Q) \leq C \tag{6.2.3}$$

and 3. and 4. as in the previous theorem. Then we have

$$3'. \Leftrightarrow 3. \Leftrightarrow 4.$$

Proof of $3 \Rightarrow 3'$. Consider

$$a_Q^{-1} = \frac{1}{w_3(Q)} \int \Phi^{\frac{-(1-q)}{q}} w_3.$$

We have

$$\begin{aligned}
&\sum_{Q \in \mathcal{Q}} \frac{\lambda_Q}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) a_Q^{-1} w_3(Q) \\
&= \sum_{Q \in \mathcal{Q}} \frac{\lambda_Q}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) \Big(\frac{1}{w_3(Q)} \int \Phi^{-\frac{(1-q)}{q}} w_3 \Big) w_3(Q) \\
&= \int \sum_{Q \in \mathcal{Q}} \frac{\lambda_Q}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) 1_Q \Phi^{-\frac{(1-q)}{q}} w_3 \\
&\le C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}}(w_i).
\end{aligned}$$

The rest of the proof follows as in the proof 2. $\Rightarrow$ 1. in the previous theorem.

□

Proof of $3' \Rightarrow 3$. Consider

$$\Phi := \Big(\sup_{Q \in \mathcal{Q}} a_Q 1_Q \Big)^{\frac{q}{1-q}}.$$

Then

$$\int \Phi w_3 = \int \Big(\sup_{Q \in \mathcal{Q}} a_Q 1_Q \Big)^{\frac{q}{1-q}} w_3 \lesssim 1.$$

Moreover,

$$\begin{aligned}
&\int \sum_{Q \in \mathcal{Q}} \frac{\lambda_Q}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) 1_Q \Phi^{\frac{-(1-q)}{q}} w_3 \\
&= \sum_{Q \in \mathcal{Q}} \frac{\lambda_Q}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) \Big(\sup_{Q \in \mathcal{Q}} a_Q 1_Q \Big)^{\frac{q}{1-q}(-\frac{(1-q)}{q})} w_3(Q) \\
&\le \sum_{Q \in \mathcal{Q}} \frac{\lambda_Q}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) \phi_Q^{-1} 1_Q w_3(Q) \le C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}}(w_i).
\end{aligned}$$

□

6.3 Estimates for $r \geq 1$

The second main result of this chapter is given here.

Theorem 6.3.1. *Let w_1, w_2, w_3 be measurable functions, μ a nonnegative measure and λ_Q non-negative real numbers. Let $1 < p_1, p_2 < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} \leq 1$. Consider $0 < q < r$. Denote $\mathcal{D}$ to be the collection of dyadic cubes and $\mathcal{Q} := \{Q \in \mathcal{D}; \lambda_Q > 0, w_1(Q) > 0 \quad and \quad w_2(Q) > 0)\}$. The following assertions are equivalent:*

1.

$$\exists B, \forall f_1, f_2, \left\| \sum_{Q \in \mathcal{D}} \lambda_Q \frac{1}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int_Q f_i w_i \Big) 1_Q \right\|_{L^q(w_3)} \leq B \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)}. \tag{6.3.2}$$

2. *There exists a $0 \leq \Phi, \|\Phi\|_{L^s(w_3)} \leq 1$ with $s^{-1} = q^{-1} - r^{-1}$ such that*

$$\exists C, \forall f_1, f_2, \left\| \Big(\sum_{Q \in \mathcal{D}} \lambda_Q \frac{1}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int_Q f_i w_i \Big) 1_Q \Big) \Phi^{-1} \right\|_{L^r(w_3)} \leq C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)}.$$

3. *There exists a family $\{a_Q\}_{Q \in \mathcal{Q}}$ of positive reals satisfying the pair of conditions*

$$\int \Big(\sup_{Q \in \mathcal{Q}} a_Q^{\frac{1}{r}} 1_Q \Big)^s w_3 \leq 1, \tag{6.3.3}$$

$$\sup_{\|f_i\|_{L^{p_i}(w_i)} \leq 1} \int \Big(\sum_{Q \in \mathcal{Q}} 1_Q \frac{\lambda_Q a_Q^{-\frac{1}{r}}}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int_Q f_i w_i \Big) \Big)^r w_3 \leq C. \tag{6.3.4}$$

Proof of $2 \Rightarrow 1$. Since $\|fg\|_q \leq \|f\|_s \|g\|_r$ with $\frac{1}{q} = \frac{1}{s} + \frac{1}{r}$, we have

$$\begin{aligned} &\left\| \sum_{Q \in \mathcal{D}} \lambda_Q \frac{1}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) 1_Q \right\|_{L^q(w_3)} \\ \leq &\left\| \sum_{Q \in \mathcal{D}} \lambda_Q \frac{1}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) 1_Q \Phi^{-1} \right\|_{L^r(w_3)} \left\| \Phi \right\|_{L^s(w_3)} \\ &\leq C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)}. \end{aligned}$$

$\square$

Proof of $1 \Rightarrow 2$. If $q = r$ then $\Phi \equiv 1$ satisfies the condition of the theorem. Assume now $q < r$. Define

$$\begin{aligned} \tilde{T} : L_{p_1} \times L_{p_2} \times L_{r'} &\to L_0 \\ (f_1 w_1, f_2 w_2, f_3) &\mapsto \tilde{T}(f_1 w_1, f_2 w_2, f_3) = f_3 T(f_1 w_1, f_2 w_2) \end{aligned}$$

with

$$T(f_1 w_1, f_2 w_2) := \sum_{Q \in \mathcal{D}} \lambda_Q \frac{1}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int_Q f_i w_i \Big) 1_Q.$$

Define also $\tilde{q}$ such that $\frac{1}{\tilde{q}} = \frac{1}{r'} + \frac{1}{q} > 1$. By Hölder inequality with exponent r' and q , we have

$$\begin{aligned} \int |\tilde{T}(f_1 w_1, f_2 w_2, f_3)|^{\tilde{q}} w_3 d\mu &= \int |f_3 T(f_1 w_1, f_2 w_2)|^{\tilde{q}} w_3 d\mu \\ \leq \Big(\int |f_3|^{r'} w_3 d\mu \Big)^{\frac{\tilde{q}}{r'}} &\Big(\int |T(f_1 w_1, f_2 w_2)|^q w_3 d\mu \Big)^{\frac{\tilde{q}}{q}} < \infty. \end{aligned}$$

So $\tilde{T}$ maps into $L_{\tilde{q}}(w_3 d\mu)$. There is a positive linear operator

$$\tilde{T}_L : L_{p_1} \otimes L_{p_2} \otimes L_{r'} \to L_{\tilde{q}}$$

where

$$\tilde{T}_L(u)(x) := \Big[\sum_Q \frac{\lambda_Q}{\mu(Q)^2} \Big(\int_{Q \times Q} u(x_1, x_2, x) dx_1 dx_2 \Big) 1_Q(x) \Big]$$

and $u : X_1 \times X_2 \times X \to \mathbb{R}$, such that

$$\tilde{T}(f_1 w_1, f_2 w_2, f_3) = \tilde{T}_L(f_1 w_1 \otimes f_2 w_2 \otimes f_3).$$

Define

$$\tilde{\rho}_{p_1, p_2, r'}(u) := \inf \Big[\Big(\prod_{i=1}^{2} ||f_i w_i||_{p_i} \Big) ||f_3||_{r'} ; |u(x_1, x_2)| \leq \Big(\prod_{i=1}^{2} (f_i w_i)(x_i) \Big) f_3 \Big]$$

for all $x_1 \in X_1, x_2 \in X_2$. By Lemma 6.1.5 $\tilde{\rho}_{p_1, p_2, r'}$ is subadditive in E provided $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{r'} = 1$. Let $u \geq 0, u \in E, f_1 w_1 \otimes f_2 w_2 \otimes f_3 \geq u$. Since $\tilde{T}_L$ is positive and by (6.3.2) we have

$$\begin{aligned} \|\tilde{T}_L(u)\|_{\tilde{q}} \leq \|\tilde{T}_L\big(f_1 w_1 \otimes f_2 w_2 \otimes f_3\big)\|_{\tilde{q}} &= \|f_3 \tilde{T}(f_1 w_1, f_2 w_2)\|_{\tilde{q}} \\ &\leq \|\tilde{T}\| \prod_i \|f_i w_i\|_{p_i} \|f_3\|_{r'}. \end{aligned}$$

Taking the infimum over $f_i w_i$, we obtain

$$\|\tilde{T}_L(u)\|_{\tilde{q}} \leq \|\tilde{T}\|\rho_{p_1,p_2,r'}(u).$$

Define

$$\tilde{A} := \frac{1}{\|\tilde{T}\|}\tilde{T}_L(B_E) \subseteq L^{\tilde{q}}.$$

This is a convex set and

$$\int \tilde{f}^{\tilde{q}} w_3 \leq 1 \forall \tilde{f} \in \tilde{A}.$$

Since

$$\tilde{q}^{-1} = (r')^{-1} + \frac{1}{q} = 1 + \left(\frac{1}{q} - r^{-1}\right) > 1$$

and $1/s = 1/\tilde{q} - 1$ by the Maurey factorization theorem (Theorem 6.1.4) there exists $\Phi \geq 0$ in $L_s(w_3)$ with $\|\Phi\|_{L^s(w_3)} \leq 1$ and

$$\int \frac{\tilde{f}}{\Phi} w_3 \leq 1 \forall \tilde{f} \in \tilde{A}.$$

Substituting $\tilde{f} = \frac{1}{\|\tilde{T}\|\|f_1\|\|f_2\|\|f_3\|}\tilde{T}_L(f_1 w_1, f_2 w_2, f_3)$ we obtain

$$\int T(f_1 w_1, f_2 w_2) f_3 \Phi^{-1} w_3 \leq B \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)} \|f_3\|_{r'}.$$

Since this holds for every f_3 and by duality this implies

$$\|T(f_1 w_1, f_2 w_2)\Phi^{-1}\|_{L^r(w_3)} \leq B \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)}.$$

$\square$

Proof of $3 \Rightarrow 2$. Consider $\Phi = \sup\limits_Q 1_Q a_Q^{\frac{1}{r}}$. We have

$$\int \Phi^s w_3 = \int \left(\sup_Q 1_Q a_Q^{\frac{1}{r}}\right)^s w_3 \leq 1.$$

Moreover

$$
\begin{aligned}
&\int \Big(\sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) 1_Q \Big)^r \Phi^{-r} w_3 \\
&= \int \Big(\sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) 1_Q \Big)^r \Big(\sup_Q 1_Q a_Q^{\frac{1}{r}} \Big)^{-r} w_3 \\
&= \int \Big(\sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) 1_Q \Big)^r \Big(\sup_Q a_Q 1_Q \Big)^{-1} w_3 \\
&= \int \Big(\sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) 1_Q \Big)^r \Big(\inf_Q a_Q^{-1} 1_Q \Big) w_3 \\
&\leq \int \Big[\sum_{Q\in\mathcal{D}} 1_Q \frac{\lambda_Q a_Q^{-\frac{1}{r}}}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int_Q f_i w_i \Big) \Big]^r w_3 \leq C \prod_{i}^{2} \|f_i\|_{L^{p_i}(w_i)}.
\end{aligned}
$$

$\square$

Proof of $2 \Rightarrow 3$. Consider

$$
a_Q^{-1} = \frac{1}{w_3(Q)} \int \Phi^{-r} w_3.
$$

By Jensen inequality for $t \to t^{-\frac{q}{t}}$ and Hardy Littlewood maximal inequality,

$$
\begin{aligned}
\int \Big(\sup_Q a_Q^{\frac{1}{r}} 1_Q \Big)^s w_3 &= \int \Big(\sup_{Q\in\mathcal{D}} \Big(\frac{1}{w_3(Q)} \int \Phi^{-r} w_3 \Big)^{-\frac{1}{r}} 1_Q \Big)^s w_3 \\
&= \int \sup_{Q\in\mathcal{D}} \Big(\frac{1}{w_3(Q)} \int \Phi^{-r} w_3 \Big)^{-\frac{s}{r}} 1_Q w_3 \\
&\leq \int \sup_{Q\in\mathcal{D}} \Big(\frac{1}{w_3(Q)} \int \Phi^{q} w_3 \Big)^{\frac{s}{q}} w_3 \leq \int \Phi^s w_3.
\end{aligned}
$$

Moreover by Hardy Littlewood maximal inequality,

$$
\begin{aligned}
&\int \Big(\sum_{Q\in\mathcal{D}} 1_Q \frac{\lambda_Q a_Q^{-\frac{1}{r}}}{\mu(Q)^2} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) \Big)^r w_3 \\
&= \int \Big(\sum_{Q\in\mathcal{D}} 1_Q \frac{\lambda_Q}{\mu(Q)^2} \Big(\frac{1}{w_3(Q)} \int \Phi^{-r} w_3 \Big)^{\frac{1}{r}} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) \Big)^r w_3 \\
&\leq \int \Big(\sum_{Q\in\mathcal{D}} 1_Q \frac{\lambda_Q}{\mu(Q)^2} \Phi^{-1} \Big(\prod_{i=1}^{2} \int f_i w_i \Big) \Big)^r w_3 \leq C^r \prod_{I=1}^{2} \|f_i\|_{L^{p_i}(w_i)}^r = C.
\end{aligned}
$$

□

Chapter 7

Sparse domination of uncentered variational truncations (joint with [dZ16])

In this chapter we provide a versatile formulation of Lacey's recent sparse pointwise domination technique with a local weak type estimate on a nontangential maximal function as the only hypothesis. We verify this hypothesis for sharp variational truncations of singular integrals in the case when unweighted L^2 estimates are available. This extends previously known sharp weighted estimates for smooth variational truncations to the case of sharp variational truncations. We also include a sparse domination result for iterated commutators of multilinear operators with BMO functions. This chapter is taken from the paper paper [dZ16].

7.1 Introduction

Sparse domination has been introduced by Lerner [Ler13] in order to simplify the proof of the A_2 theorem for Calderón–Zygmund (CZ) operators (see [Hyt14] and [Hyt12] for a comprehensive history of this result). A new approach to sparse domination via weak type endpoint estimates has been recently discovered by Lacey [Lac15, Theorem 4.2], quantitatively refined by Hytönen, Roncal, and Tapiola [HRT15, Theorem 2.4], and streamlined by Lerner [Ler15]. In a short period of time since 2015 this idea has been applied in many settings which go beyond CZ theory, and we are not going to survey these developments. In the CZ setting it is by now well understood that sparse domination follows from suitable localized non-tangentional endpoint estimates; several abstract results formalizing this principle appeared in

[Ler16; dZ16; Con+17]. These techniques have been applied to r-variational estimates for truncated singular integrals in [HLP13] (smooth truncations) and [dZ16] (sharp truncations).

We want to extend these r-variational estimates to a class of non-convolution type singular integrals. We formulate our results on classes of spaces of homogeneous type that include homogeneous nilpotent Lie groups. In this setting we also obtain some sharp weighted inequalities for square functions and r-variation of averages. Our first result is an abstract implementation of Lacey's argument that can be applied as a black box in a number of situations, for instance to multilinear operators (recovering the sparse domination result in [DHL15a]), to intrinsic square functions (see [Zor17], where the second author uses Theorem 7.1.1 to extend some results in [LL15]), and also to variational truncations of singular integrals that will be the second topic of this chapter.

We will use the following version of the nontangential maximal function. Let (X, ρ, μ) be a space of homogeneous type (see Section 7.2 for definitions) and let F be a function on the set

$$\mathcal{X} := \{(x, s, t) \in X \times (0, \infty) \times (0, \infty) : s \le t\}.$$

We define the non-tangentional maximal operator (of aperture $a \ge 0$) localized to a set $Q \subset X$ by

$$(\mathcal{N}_{a,Q}F)(x) := 1_Q(x) \sup_{y \in X, \rho(x,y) < as < at \le \mathrm{dist}(y, X \setminus Q)} F(y, s, t).$$

We will omit Q from the notation if $Q = X$ and we will also omit a if $a = 1$.

We consider the only vertical line inside the tent over Q.

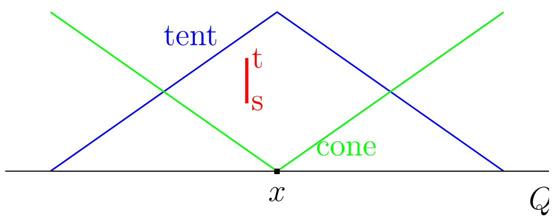

Figure 7.1: non-tangential maximal operator ($a = 1$) localized to $Q \subset X$.

Theorem 7.1.1. *For every space of homogeneous type (X, ρ, μ) and every choice of adjacent systems of dyadic cubes $\mathcal{D}^\alpha$ there exist $\epsilon, \eta > 0$ such that the following holds. Let $F : \mathcal{X} \to [0, \infty]$ be a function that is monotonic in the sense that*

$$s \le s' \le t' \le t \implies F(x, s', t') \le F(x, s, t)$$

and subadditive in the sense that

$$s \le s' \le t \implies F(x,s,t) \le F(x,s,s') + F(x,s',t).$$

Suppose that for every dyadic cube Q there exists $c_Q \ge 0$ such that

$$\mu\{\mathcal{N}_Q F > c_Q\} \le \epsilon \mu(Q). \tag{7.1.2}$$

Then there exist η-sparse collections $\mathcal{S}^{\alpha,k_0} \subset \mathcal{D}^\alpha$ of cubes such that

$$\mathcal{N} F \le \liminf_{k_0 \to -\infty} \sum_\alpha \sum_{Q \in \mathcal{S}^{\alpha,k_0}} 1_Q c_Q \tag{7.1.3}$$

holds pointwise almost everywhere.

One situation in which Theorem 7.1.1 does not apply as a black box is that of commutators of (multi)linear operators with BMO functions, and we provide the necessary modifications to the argument in Section 7.6, where a multilinear extension of [LOR16, Theorem 1.1] is proved.

Now, we return to the space $X = \mathbb{R}^d$ with the Euclidean distance and the Lebesgue measure. Let K be an ω-Calderón–Zygmund (CZ) kernel (see Section 7.2 for definitions) and consider the corresponding truncation operator given by

$$\mathcal{T} f(x,s,t) := \int_{s<|x-y|<t} K(x,y) f(y) \mathrm{d}y. \tag{7.1.4}$$

For $1 \le r < \infty$ we define the homogeneous [1] variation operator, acting on functions on $\mathcal{X}$, by

$$(\dot{\mathcal{V}}^r F)(x,s,t) := \sup_{s \le t_1 < \cdots < t_J \le t} \Big(\sum_{j=1}^{J-1} |F(x,t_j,t_{j+1})|^r \Big)^{1/r},$$

and analogously for $r = \infty$ with the ℓ^∞ norm in place of the ℓ^r norm.

It is known that, if the kernel K is of convolution type, i.e. $K(x,y) = k(x-y)$, satisfies the cancellation condition

$$\int_{\partial B(0,t)} k(x) \mathrm{d}x = 0, \qquad t > 0,$$

and satisfies one of the following additional conditions:

---

[1] The notation "$\dot{V}^r$" is not standard and is motivated by the embeddings $\dot{B}_r^{1/r,1} \to \dot{V}^r \to \dot{B}_r^{1/r,\infty}$ between the spaces of bounded homogeneous variation and homogeneous Besov spaces [BP74].

1. the kernel k is homogeneous of degree $-d$, that is, $k(tx) = t^{-d}k(x)$ for $t > 0$, or
2. the kernel k satisfies the smoothness condition $|k'(y)| \lesssim |y|^{-d-1}$,

then, for $r > 2$, the operator $\mathcal{N}_0 \circ \dot{\mathcal{V}}^r \circ \mathcal{T}$ is bounded on $L^p(\mathbb{R}^d)$ and has weak type (1,1). The strong type bounds in the case 1 have been proved in [Cam+03, Theorem A] (see also [JSW08] and [DHL15b]) and in the case 2 in [MST15, Theorem A.1]. In both cases the L^p bounds imply the weak type $(1,1)$ bound by [Cam+03, Theorem B] (note that the Hörmander condition assumption (1.8) assumed in this article follows from the Dini condition).

Our second main result is that these bounds remain true with $\mathcal{N}_0$ replaced by $\mathcal{N}_a$, $a > 0$.

Theorem 7.1.5. *Let K be an ω-CZ kernel on $\mathbb{R}^d$, $r > 2$, and assume that $\mathcal{N}_0 \circ \dot{\mathcal{V}}^r \circ \mathcal{T}$ has weak type (1,1). Then also $\mathcal{N}_a \circ \dot{\mathcal{V}}^r \circ \mathcal{T}$ has weak type (1,1) for every $a > 0$.*

The novelty of this result are the sharp truncations in (7.1.4). An analogous result with $1_{(s,t)}$ replaced by appropriately scaled smooth truncations is implicitly contained in [HLP13].

The appearance of cones with positive aperture in Theorem 7.1.5 allows us to apply Theorem 7.1.1 to variational truncations of singular integrals. Indeed, the localized operator $\mathcal{N}_Q \circ \dot{\mathcal{V}}^r \circ \mathcal{T}$ is dominated by the global operator $\mathcal{N} \circ \dot{\mathcal{V}}^r \circ \mathcal{T}$, and therefore has weak type (1,1) uniformly in Q. On the other hand, the localized operator depends only on the values of f on Q, and therefore (7.1.2) is satisfied for the function $F = \dot{\mathcal{V}}^r \mathcal{T} f$ with $c_Q = \frac{C}{\epsilon}\mu(Q)^{-1}\int_Q |f|$. Therefore, $\mathcal{N} \circ \dot{\mathcal{V}}^r \circ \mathcal{T} f$ can be estimated by sparse operators (7.2.13).

Sparse operators are known to satisfy very good weighted estimates, the currently best results can be found in [HL15] (L^p bounds with $p > 1$) and [DLR16] (the weak type (1,1) endpoint). Consequently, we obtain sharp weighted estimates for the variationally truncated operators $\mathcal{N} \circ \dot{\mathcal{V}}^r \circ \mathcal{T}$, unifying the previous results for sharp truncations with unspecified dependence on the characteristic of the weight [MTX15b; MTX15a] and for smooth truncations with sharp dependence on the characteristic of the weight [HLP13].

7.2 Notation and tools

7.2.1 Spaces of homogeneous type

Definition 7.2.1. A *quasi-metric* on a set X is a function $\rho : X \times X \to [0, \infty)$ such that $\rho(x, y) = 0 \iff x = y$ that is symmetric and satisfies the quasitriangle inequality

$$\rho(x,y) \leq A_0(\rho(x,z) + \rho(z,y)) \quad \text{for all} \quad x,y,z \in X$$

with some $A_0 < \infty$ independent of x, y, z.

A measure μ on a quasi-metric space (X, ρ) is called *doubling* if there exists $A_1 < \infty$ such that

$$\mu(B(x,2r)) \leq A_1 \mu(B(x,r)) \quad \text{for all} \quad x \in X, r > 0,$$

where $B(x,r) = \{y \in X : \rho(x,y) < r\}$ are the quasimetric balls of radius r. These balls need not be open, but can be made open by passing to an equivalent quasi-metric [MS79]. A tuple (X, ρ, μ) consisting of a set X, a quasi-metric ρ, and a doubling measure μ is called a *space of homogeneous type*.

A space of homogeneous type (X, ρ, μ) is called (Ahlfors–David) *d-regular*, $d > 0$, if there exist $0 < c, C < \infty$ such that for all $x \in X$ and $r > 0$ we have

$$cr^d \leq \mu(B(x,r)) \leq Cr^d.$$

We say that a family $\mathcal{D}$ of subsets of X has the *small boundary property* if there exist $\eta > 0$ and $C_3 < \infty$ such that for every $Q \in \mathcal{D}$ and every $0 < \tau \leq 1$

$$\mu(\partial_{\tau \operatorname{diam}(Q)} Q) \leq C_3 \tau^\eta \mu(Q), \tag{7.2.2}$$

where

$$\partial_\tau(Q) = \{x \in Q : \operatorname{dist}(x, X \setminus Q) \leq \tau\} \cup \{x \in X \setminus Q : \operatorname{dist}(x, Q) \leq \tau\}. \tag{7.2.3}$$

We say that (X, ρ, μ) has the small boundary property if the collection of all metric balls has the small boundary property.

We denote the measure of a set Q by $\mu(Q)$ and the average of a function f over Q by $\langle f \rangle_Q = \mu(Q)^{-1} \int_Q f \mathrm{d}\mu$.

7.2.2 Adjacent systems of dyadic cubes

Filtrations on spaces of homogeneous type that closely resemble dyadic filtrations on $\mathbb{R}^d$ have been first constructed by Christ [Chr90] and are now commonly known as *Christ cubes*. We recall their properties.

Definition 7.2.4. Let (X, ρ, μ) be a space of homogeneous type. A *system of dyadic cubes* $\mathcal{D}$ with constants $\kappa > 1$, $a_0 > 0$, $C_1 < \infty$ consists of collections $\mathcal{D}_k$, $k \in \mathbb{Z}$, of open subsets of X such that and constants $\kappa > 1$, $a_0, \eta > 0$, $C_1, C_2 < \infty$ with the following properties.

1. $\forall k \in \mathbb{Z} \quad \mu(X \setminus \cup_{Q \in \mathcal{D}_k} Q) = 0$,
2. If $l \geq k$, $Q \in \mathcal{D}_l$, $Q' \in \mathcal{D}_k$, then either $Q' \subseteq Q$ or $Q' \cap Q = \emptyset$,
3. For every $l \geq k$ and $Q' \in \mathcal{D}_k$ there exists a unique $Q \in \mathcal{D}_l$ such that $Q \supseteq Q'$,
4. $\forall k \in \mathbb{Z}, Q \in \mathcal{D}_k \quad \exists c_Q \in X : B(c_Q, a_0\kappa^k) \subseteq Q \subseteq B(c_Q, C_1\kappa^k)$.

We use $\mathcal{D}$ to denote the disjoint union of $\mathcal{D}_k$.

If in addition the collection $\mathcal{D}$ has the small boundary property (7.2.2), then we call $\mathcal{D}$ a *Christ* system of dyadic cubes.

Theorem 7.2.5 ([Chr90])**.** *Every space of homogeneous type admits a system of Christ dyadic cubes.*

For our purposes we do not need the small boundary property enjoyed by the Christ cubes, but we do need adjacent systems of cubes that have covering properties similar to those of shifted dyadic cubes in $\mathbb{R}^d$. Such systems have been constructed in [HK12].

Definition 7.2.6. Let (X, μ) be a measure space. A *system of dyadic sets* $\mathcal{D}$ consists of a sequence $(\mathcal{D}_k)_{k\in\mathbb{Z}}$ of collections of measurable subsets of X such that for all $l \leq k$, $l, k \in \mathbb{Z}$, 1. and 2. of definition 7.2.4 holds.

By an abuse of notation the sets Q remember their generation $k(Q)$ (the *unique* number such that $Q \in \mathcal{D}_{k(Q)}$), even though it is allowed that the same Q (viewed as a set) may occur in different generations $\mathcal{D}_k$. The relation $Q' \subseteq Q$ implies the inequality $k(Q') \geq k(Q)$ and the relation $Q' = Q$ implies $k(Q') = k(Q)$.

Definition 7.2.7. Let (X, ρ, μ) be a quasi-metric measure space and assume that the measure μ has full support. A *system of dyadic cubes* is a system of dyadic sets $\mathcal{D}$ such that for some $0 < \delta < 1$, $0 < c_1 \leq C_1 < \infty$ and all $k \in \mathbb{Z}$ and $Q = Q^k_\alpha \in \mathcal{D}_k$ there exists $z = z(Q) = z^k_\alpha \in X$ such that $B(z, a_0\delta^k) \subseteq Q \subseteq B(z, C_1\delta^k)$.

Definition 7.2.8. Let (X, ρ, μ) be a quasi-metric measure space and assume that the measure μ has full support. Systems of dyadic cubes $\mathcal{D}^\alpha$, $\alpha \in A$, are said to be *adjacent* if there exists $C_3 < \infty$ such that for every $z \in X$ and $r > 0$ there exist $\alpha \in A$, $k \in \mathbb{Z}$, and $Q \in \mathcal{D}^\alpha_k$ such that $B(z, r) \subset Q \subset B(z, C_3 r)$.

Theorem 7.2.9 (Theorem 4.1, [HK12])**.** *Every space of homogeneous type admits a finite collection of adjacent systems of dyadic cubes.*

Example 7.2.10. Let $X = \mathbb{R}^d$ with the Euclidean distance and the Lebesgue measure. For each $\alpha \in \{0,1,2\}^d$ the corresponding *shifted system of dyadic cubes* is given by

$$\mathcal{D}^\alpha = \{2^{-k}([0,1)^d + m + (-1)^k \frac{1}{3}\alpha), k \in \mathbb{Z}, m \in \mathbb{Z}^d\}.$$

Then the systems $\mathcal{D}^\alpha$, $\alpha \in \{0,1,2\}^d$, are adjacent. In fact, on $\mathbb{R}^d$ one can construct $d+1$ shifted systems of dyadic cubes that are adjacent [Mei03].

Example 7.2.11. Let (X,μ) be a measure space and let $\mathcal{D}$ be a system of dyadic sets. Define a metric on X by

$$\rho(x,x') := \inf\{2^{-k} : \exists Q \in \mathcal{D}_k : x, x' \in Q\}.$$

Then the system $\mathcal{D}$ is a system of dyadic cubes with respect to this metric, and this system is adjacent. For instance, the standard dyadic cubes in $\mathbb{R}^d$ are an adjacent system of dyadic cubes with respect to the dyadic metric. This does not preclude one from considering CZ operators on $\mathbb{R}^d$ with respect to the Euclidean metric and allows one to recover Lerner's version [Ler15] of the pointwise sparse domination theorem from Theorem 7.1.1.

7.2.3 Sparse and Carleson collections

Definition 7.2.12. Let $\mathcal{D}$ be a system of dyadic sets on a measure space (X,μ). A collection $\mathcal{S} \subset \mathcal{D}$ is called

1. (η,μ)-*sparse* (sparse with respect to measure μ for a fixed constant $\eta > 0$) if there exist pairwise disjoint subsets $E(Q) \subset Q \in \mathcal{S}$ with $\mu(E(Q)) \geq \eta\mu(Q)$
2. Λ-*Carleson* if one has $\sum_{Q' \subset Q, Q' \in \mathcal{S}} \mu(Q') \leq \Lambda\mu(Q)$ for all $Q \in \mathcal{D}$.

When η and μ are evident we write only η-*sparse* oder simply *sparse*.

It is known that a collection is η-sparse if and only if it is $1/\eta$-Carleson [LN15, §6.1]. The corresponding *sparse operator* is given by

$$A_{\mathcal{S}} f = \sum_{Q \in \mathcal{S}} 1_Q \langle |f| \rangle_{CQ} . \tag{7.2.13}$$

The sparse operators (7.2.13)

can be dominated by finite linear combinations of similar sparse operators/square functions with respect to adjacent dyadic grids in which the averages of f are taken over Q instead of CQ, cf. [Ler16, Remark 4.3]. Hence the usual estimates for sparse operators [HL15; DLR16] apply to (7.2.13).

We say that an operator T is pointwise controlled by a sparse operator with constant $C < \infty$ if for every function f there exist 1/2-sparse collections $\mathcal{S}^n \subset \mathcal{D}$, $n \in \mathbb{N}$, such that

$$|Tf| \leq C \liminf_{n \to \infty} A_{\mathcal{S}^n} f,$$

holds pointwise almost everywhere.

7.2.4 ω-Calderón–Zygmund kernels

An ω-Calderón–Zygmund (CZ) kernel is a function $K : \mathbb{R}^d \times \mathbb{R}^d \setminus (\text{diagonal}) \to \mathbb{C}$ that satisfies the size estimate

$$|K(x,y)| \leq \frac{C_K}{|x-y|^d} \tag{7.2.14}$$

and the smoothness estimate

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq \omega\Big(\frac{|x-x'|}{|x-y|}\Big) \frac{1}{|x-y|^d} \tag{7.2.15}$$

for $|x-y| > 2|x-x'| > 0$ with some *modulus of continuity* $\omega : [0,\infty) \to [0,\infty)$ (that is, a subadditive function: $\omega(t+s) \leq \omega(t) + \omega(s)$ for all $s,t \geq 0$) that satisfies the *Dini condition*

$$\|\omega\|_{\mathrm{Dini}} := \int_0^1 \omega(t) \frac{\mathrm{d}t}{t} < \infty. \tag{7.2.16}$$

7.3 Uncentered variational estimates

Consider the averaging operator

$$\mathcal{A}f(x,s,t) := A_t f(x) - A_s f(x), \quad A_t f(x) := \mu(\{|x-y| < t\})^{-1} \int_{|x-y|<t} f(x+y) \mathrm{d}y. \tag{7.3.1}$$

It satisfies the following uncentered variational estimates.

Lemma 7.3.2. *Let $r > 2$ and $a \geq 0$. Then $\mathcal{N}_a \circ \dot{\mathcal{V}}^r \circ \mathcal{A}$ is bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$, and has weak type $(1,1)$.*

Sketch of proof. We have

$$\mathcal{A}f(x,s,t) = A_t f(x) - A_s f(x) - \big(E_{k(t)}f(x) + E_{k(s)}f(x)\big) \\ +\big(E_{k(t)}f(x) + E_{k(s)}f(x)\big),$$

where E_k denotes the conditional expectation onto the $\sigma-$ algebra generated by dyadic cubes $\mathcal{Q}_k$ with lenght 2^k. Then

$$\mathcal{N}_a\big(\dot{\mathcal{V}}^r(\mathcal{A}f(x,s,t))\big) \\ \leq \mathcal{N}_a\Bigg(\dot{\mathcal{V}}^r\bigg(\big(A_t f(x) - A_s f(x)\big) - \big(E_{k(t)}f(x) + E_{k(s)}f(x)\big)\bigg)\Bigg) \\ +\mathcal{N}_a\big(\dot{\mathcal{V}}^r\big(E_{k(t)}f(x) + E_{k(s)}f(x)\big)\big).$$

Moreover, since $r > 2 \Rightarrow l^r \supseteq l^2$,

$$\dot{\mathcal{V}}^r\bigg(\big(A_t f(x) - A_s f(x)\big) - \big(E_{k(t)}f(x) + E_{k(s)}f(x)\big)\bigg) \\ \leq \dot{\mathcal{V}}^2\bigg(\big(A_t f(x) - A_s f(x)\big) - \big(E_{k(t)}f(x) + E_{k(s)}f(x)\big)\bigg) \lesssim \bigg(\sum_{k=k(t_0)}^{\infty}\big(S_k f(x)\big)^2\bigg)^{\frac{1}{2}},$$

where

$$S_k f := \sup_R |\tilde{F}_{t,s,k(t),k(s),f,A,E}| + \sup_{s<t_1<\cdots<t_j<t}\bigg(\sum_{j=1}^{j-1}|\tilde{F}_{t_j,t_{j+1},k(t_j),k(t_{j+1}),f,A,E}|\bigg)$$

with

$$\tilde{F}_{a,b,k(a),k(b),f,A,E} = (A_a f - A_b f) - (E_{k(a)} + E_{k(b)}).$$

□

Then the L^p bound, $1 < p < \infty$, for the dyadic version of this operator is a direct consequence of Lépingle's inequality for martingales. The real version can be compared with the dyadic version using the uncentered square function from [KZ15, Theorem 1.4]. Finally, the weak type $(1,1)$ bound follows by [KZ15, Proposition 5.1].

Note that the results cited from [KZ15] continue to hold with $3\mathcal{Q}_k$ replaced by $C\mathcal{Q}_k$ in the definitions of $\tilde{S}_k$ and $\tilde{R}_k$ for an arbitrary C; in our case we can take e.g. $C = 100(a+1)$.

Alternatively, note that $\mathcal{N}_a$ can be seen as the usual nontangential maximal operator of aperture a applied to the function $(x,s) \mapsto \sup_{t>s} F(x,s,t)$.

Hence the operator $\mathcal{N}_a \circ \dot{\mathcal{V}}^r \circ \mathcal{A}$ has weak type $(1,1)$/strong type (p,p) for all $a > 0$ provided that this holds for some $a > 0$, see e.g. [Ste93, §II.2.5.1].

The next lemma compares variational truncations of ω-CZ kernels at nearby points. The case $r = \infty$ of this lemma appeared in [HRT15, Lemma 2.3].

Lemma 7.3.3. *Let $r > 1$, $x, x' \in \mathbb{R}^d$, $0 < \epsilon \leq \delta \leq \infty$, and suppose $|x - x'| \leq \epsilon/2$. Let also K be an ω-CZ kernel. Then*

$$
\begin{aligned}
|\dot{\mathcal{V}}^r \mathcal{T} f(x, \epsilon, \delta) - \dot{\mathcal{V}}^r \mathcal{T} f(x', \epsilon, \delta)| \lesssim_d (\|\omega\|_{\mathrm{Dini}} + r' C_K) \sup_{\epsilon \leq t \leq \delta} A_t |f|(x) \\
+ C_K (\dot{\mathcal{V}}^r \mathcal{A} |f|(x, \epsilon, \delta) + \dot{\mathcal{V}}^r \mathcal{A} |f|(x', \epsilon, \delta)).
\end{aligned}
$$

Theorem 7.1.5 is an immediate consequence of Lemma 7.3.3, Lemma 7.3.2, and the Hardy–Littlewood maximal inequality (See subsection 7.5).

Proof of Lemma 7.3.3. By the triangle inequality on ℓ^r the left-hand side of the conclusion is bounded by

$$
\sup_{\epsilon \leq t_1 < \cdots < t_J \leq \delta} \Big(\sum_{j=1}^{J-1} \Big| \int_{t_j <} x - y| < t_{j+1} | K(x, y) f(y) - \int_{t_j <} x' - y| < t_{j+1} | K(x', y) f(y) \Big|^r \Big)^{1/r}.
$$

For a fixed sequence $t_1 < \cdots < t_J$ we estimate this by

$$
\begin{aligned}
&\Big(\sum_{j=1}^{J-1} \Big| \int_{t_j <} x - y| < t_{j+1} | (K(x, y) - K(x', y)) f(y) \Big|^r \Big)^{1/r} \\
+ &\Big(\sum_{j=1}^{J-1} \Big| \Big(\int_{t_j <} x - y| < t_{j+1} | - \int_{t_j <} x' - y| < t_{j+1} | \Big) K(x', y) f(y) \Big|^r \Big)^{1/r} =: I + II.
\end{aligned}
$$

In the first term we estimate the ℓ^r norm by the ℓ^1 norm and proceed as in [HRT15, Lemma 2.3]:

$$
\begin{aligned}
I &\leq \sum_{j=1}^{J-1} \int_{t_j < |x-y| < t_{j+1}} |K(x, y) - K(x', y)| |f(y)| \\
&\leq \int_{\epsilon < |x-y| < \delta} \omega\Big(\frac{|x - x'|}{|x - y|} \Big) \frac{|f(y)|}{|x - y|^d} \\
&\leq \sum_{k=0}^{\infty} \omega\Big(\frac{\epsilon/2}{2^k \epsilon} \Big) \int_{2^k \epsilon < |x-y| < \min(2^{k+1}\epsilon, \delta)} \frac{|f(y)|}{|x - y|^d} \\
&\lesssim_d \sum_{k=0}^{\infty} \omega(2^{-k-1}) \sup_{\epsilon < t < \delta} A_t |f|(x) \\
&\lesssim \|\omega\|_{\mathrm{Dini}} \sup_{\epsilon < t < \delta} A_t |f|(x).
\end{aligned}
$$

In order to estimate the second term we use an idea from [MTX15a]. If $t_{j+1} - t_j \le 2|x - x'|$, then we estimate

$$|1_{t_j<|x-y|<t_{j+1}} - 1_{t_j<|x'-y|<t_{j+1}}| \le 1_{t_j<|x-y|<t_{j+1}} + 1_{t_j<|x'-y|<t_{j+1}}.$$

Otherwise we estimate

$$\begin{aligned}
&|1_{t_j<|x-y|<t_{j+1}} - 1_{t_j<|x'-y|<t_{j+1}}| \\
&\qquad \le |1_{t_j<|x-y|} - 1_{t_j<|x'-y|}| + |1_{|x-y|<t_{j+1}} - 1_{|x'-y|<t_{j+1}}| \\
&\qquad\quad \le 1_{t_j<|x-y|<t_j+|x-x'|} + 1_{t_j<|x'-y|<t_j+|x-x'|} \\
&\qquad\qquad + 1_{t_{j+1}-|x-x'|<|x-y|<t_{j+1}} + 1_{t_{j+1}-|x-x'|<|x'-y|<t_{j+1}}.
\end{aligned}$$

Thus we may estimate II by a sum of two terms of the form

$$\Big(\sum_{j=1}^{J'-1} (\int_{s_j<|x_0-y|<s_{j+1}} |K(x',y)||f(y)|)^r\Big)^{1/r},$$

where $x_0 = x, x'$ and the sequence $\epsilon \le s_1 < \cdots < s_{J'} \le \delta$ has bounded differences: $|s_{j+1} - s_j| \le 2|x - x'|$. Using the hypothesis that $|x - x'| < \epsilon/2$ and the kernel estimate we can bound the above by a dimensional constant times

$$C_K\Big(\sum_{j=1}^{J'-1} (s_{j+1}^{-d} \int_{s_j<|x_0-y|<s_{j+1}} |f(y)|)^r\Big)^{1/r}.$$

The above ℓ^r norm can be written as

$$\begin{aligned}
&\Big(\sum_{j=1}^{J'-1} \Big(s_{j+1}^{-d}\big(\int_{|x_0-y|<s_{j+1}} |f(y)| - \int_{|x_0-y|<s_j} |f(y)|\big)\Big)^r\Big)^{1/r} \\
&\le \Big(\sum_{j=1}^{J'-1} \big(s_{j+1}^{-d}\int_{|x_0-y|<s_{j+1}} |f(y)| - s_j^{-d}\int_{|x_0-y|<s_j} |f(y)|)\big)^r\Big)^{1/r} + \Big(\sum_{j=1}^{J'-1} ((s_j^{-d} - s_{j+1}^{-d})\int_{|x_0-y|<s_j} |f(y)|)^r\Big)^{1/r} \\
&\lesssim_d \dot{V}^r(A_s|f|(x_0) : \epsilon < s < \delta) + \sup_{\epsilon<s<\delta} A_s|f|(x_0)\Big(\sum_{j=1}^{J'-1} ((s_j^{-d} - s_{j+1}^{-d})/s_j^{-d})^r\Big)^{1/r}.
\end{aligned}$$

It remains to obtain a uniform bound on the last bracket. By homogeneity we may assume $1 < s_1 < s_2 < \dots$ and $s_{j+1} - s_j \leq 1$. Then

$$\begin{aligned}
\Big(\sum_j ((s_j^{-d} - s_{j+1}^{-d})/s_j^{-d})^r\Big)^{1/r} &= \Big(\sum_j (1-(s_j/s_{j+1})^d)^r\Big)^{1/r} \\
&\leq d\Big(\sum_j (1-s_j/s_{j+1})^r\Big)^{1/r} = d\Big(\sum_{n\in\mathbb{N}} \sum_{s_j\in[n,n+1)} (\frac{s_{j+1}-s_j}{s_{j+1}})^r\Big)^{1/r} \\
&\leq d\Big(\sum_{n\in\mathbb{N}} (\sum_{s_j\in[n,n+1)} \frac{s_{j+1}-s_j}{n})^r\Big)^{1/r} \leq d\Big(\sum_{n\in\mathbb{N}} (\frac{2}{n})^r\Big)^{1/r} \lesssim \frac{d}{r-1}. \quad \square
\end{aligned}$$

The proof of Lemma 7.3.3 in fact shows that the homogeneous r-variation in its conclusion can be restricted to the "short variation" that can be controlled (for $r \geq 2$) by the uncentered square function in [KZ15, Theorem 1.4]. Thus the application of Lépingle's inequality (through the use of Lemma 7.3.2) to estimate the error term in the above proof is not strictly necessary (but helps us to avoid additional notation).

7.4 Sparse domination

The main ingredient in the proof of Theorem 7.1.1 is the cube selection rule in Lacey's recursion lemma [Lac15, Lemma 4.7] and its quantitative refinement [HRT15, Lemma 2.8]. It can be formulated in terms of the localized non-tangentional maximal operator as follows.

Let F be a subadditive monotonic function on $\mathcal{X}$. Let $Q_0 \in \mathcal{D}_0$ be a dyadic cube $\lambda : Q_0 \to [0,\infty]$ any function defined on Q_0. Let

$$\sigma(y) := \inf\{\tau > 0 : F(y,\tau,\operatorname{dist}(y,\complement Q_0)) \leq \lambda(y)\}, \quad y \in Q_0,$$

and let

$$Y := \{y \in Q_0 : \sigma(y) > 0\}.$$

For each $y \in Y$ choose a dyadic cube $Q_y \subset Q_0$ that contains $B(y, 2\sigma(y))$ and diameter $Q_y \lesssim \sigma(y), Q_y \subseteq B(y, C\sigma(y))$ for some $C < \infty$

(such a cube exists by definition of adjacent systems). Let $\mathcal{Q} = \mathcal{Q}_\lambda(F, Q_0)$ be the collection of the maximal cubes among the Q_y's. Then for every $y \in Y$ we have

$$F(y,\operatorname{dist}(y,\complement Q),\operatorname{dist}(y,\complement Q_0)) \leq \lambda(y) \tag{7.4.1}$$

for some $Q \in \mathcal{Q}$, since this holds with Q replaced by Q_y (indeed, if the left-hand side is non-zero, then $\sigma(y) < \operatorname{dist}(y,\complement Q_0)$ with strict inequality, so that by construction $\operatorname{dist}(y, \complement Q) > \sigma(y)$ holds also with strict inequality). In particular, by subadditivity of F we obtain

$$\mathcal{N}_{0,Q_0}F \le 1_{Q_0}(\lambda + \sup_{Q\in\mathcal{Q}} \mathcal{N}_{0,Q}F).$$

Lemma 7.4.2. *Suppose that the function $\lambda(x)$ equals a constant λ. Then the collection $\mathcal{Q} = \mathcal{Q}_\lambda(F, Q_0)$ of dyadic cubes $Q \subset Q_0$ constructed above satisfies*

$$\sum_{Q\in\mathcal{Q}} \mu(Q) \lesssim \mu(\{\mathcal{N}_{Q_0}F > \lambda\}) \tag{7.4.3}$$

and for every subadditive function $\tilde{F} \le F$ we have

$$\mathcal{N}_{Q_0}\tilde{F} \le 1_{Q_0}(\lambda + \sup_{Q\in\mathcal{Q}} \mathcal{N}_Q\tilde{F}). \tag{7.4.4}$$

Proof. We write the left-hand side of (7.4.3) as

$$\sum_\alpha \sum_{Q\in\mathcal{Q}\cap\mathcal{D}^\alpha} \mu(Q)$$

and fix α. Since the cubes in $\mathcal{Q}\cap\mathcal{D}^\alpha$ are disjoint and each of them contains $B(y,\sigma(y))$ for some $y \in Y$ and has side length $\lesssim \sigma(y)$, the inner sum is bounded by a constant (depending on the doubling constant) times the measure of

$$\bigcup_{y\in Y}\{x : |x-y| < \sigma(y)\} \subset \{x \in Q_0 : \mathcal{N}_{Q_0}F(x) > \lambda\}.$$

So we have

$$\begin{aligned}
\sum_{\alpha\in\mathcal{Q}} \mu(Q) &= \sum_\alpha \sum_{Q\in\mathcal{Q}\cap\mathcal{Q}^\alpha} \mu(Q) \\
&\lesssim \sum_\alpha \sum_{Q\in\mathcal{Q}\cap\mathcal{Q}^\alpha} \mu(B(y_Q, \sigma(y_Q))) \\
&\le \sum_\alpha \mu\Big(\bigcup_{Q\in\mathcal{Q}\cap\mathcal{Q}^\alpha} B(y_Q,\sigma(y_Q))\Big) \\
&\le \sum_\alpha \mu\Big(\bigcup_{y\in Y}\{x : |x-y| < \sigma(y)\}\Big) \\
&\le \sum_\alpha \mu\Big(\{x \in Q_0 : \mathcal{N}_{Q_0}F(x) > \lambda\}\Big).
\end{aligned}$$

It remains to prove (7.4.4). If $\mathcal{N}_{Q_0}\tilde{F}(x) > \lambda$, then the supremum in the definition of $\mathcal{N}_{Q_0}\tilde{F}(x)$ can be restricted to Y. Indeed, if $y \notin Y$, then $\sigma(y) \leq 0$, (by definition of Y), this implies by definition of $\sigma(y)$ that

$$\forall r > 0, \tilde{F}(y, r, \operatorname{dist}(y, \complement Q_0)) \leq F(y, r, \operatorname{dist}(y, \complement Q_0)) \leq \lambda.$$

Therefore $y \in Y$. We obtain

$$\begin{aligned}\mathcal{N}_{Q_0}\tilde{F}(x) &= \sup_{y\in Y} \tilde{F}(y, |x-y|, \operatorname{dist}(y, \complement Q_0))\\ &\leq \sup_{y\in Y} \inf_{Q\in\mathcal{Q}} \Big(\tilde{F}(y, \operatorname{dist}(y, \complement Q), \operatorname{dist}(y, \complement Q_0)) + \tilde{F}(y, |x-y|, \operatorname{dist}(y, \complement Q))\Big)\\ &\leq \lambda + \sup_{y\in Y}\sup_{Q\in\mathcal{Q}} \tilde{F}(y, |x-y|, \operatorname{dist}(y, \complement Q))\end{aligned}$$

by monotonicity and subadditivity of $\tilde{F}$, the assumption $\tilde{F} \leq F$, and (7.4.1). The last summand can be non-zero only if $|x-y| < \operatorname{dist}(y, \complement Q)$, so that $x \in Q$, so it can be estimated by $\mathcal{N}_Q\tilde{F}(x)$. □

7.4.1 Proof of Theorem 7.1.1

For a cube Q denote by $\mathcal{Q}(Q)$ the family provided by Lemma 7.4.2 applied Q with $\lambda = c_Q$, so that

$$\mu(Q)^{-1} \sum_{Q'\in\mathcal{Q}(Q)} \mu(Q') \leq C_{(7.4.3)}\epsilon. \tag{7.4.5}$$

Therefore, in view of the doubling hypothesis,

$$k(Q') > k(Q) \quad \text{for all} \quad Q' \in \mathcal{Q}(Q)$$

provided that ϵ is small enough.

Indeed, suppose by contradiction that $k(Q') \leq k(Q)$; then by definition (7.2.7) and the triangular inequality we have

$$\begin{aligned}Q \subseteq B(z_Q, C_1\delta^{k(Q)}) &\subseteq B(y, 2A_0C_1\delta^{k(Q)})\\ &\subseteq B(z_{Q'}, A_0(2A_0C_1\delta^{k(Q)} + C_1\delta k^1(Q')))\\ &\subseteq B(z_{Q'}, (2A_0^2+1)C_1\delta^{k(Q')}), \quad k = k(Q).\end{aligned}$$

Then by the doubling hypothesis and (7.4.3) we obtain

$$\begin{aligned}\mu(Q) &\leq \mu\big(B(z_{Q'}, (2A_0^2+1)C_1\delta^{k(Q')})\big)\\ &\lesssim \mu\big(B(z_{Q'}, a_0\delta^{k(Q')})\big) \leq \mu(Q') \lesssim \epsilon\mu(Q).\end{aligned}$$

Following the proof of [Lac15, Theorem 4.2], start with

$$\mathcal{P}_{k_0} := \bigcup_{\alpha} \mathcal{D}_{k_0}^{\alpha}$$

and define inductively

$$\begin{aligned} \mathcal{P}_k^* &:= \mathcal{P}_k \cap \cup_\alpha \mathcal{D}_k^\alpha, \\ \mathcal{P}_{k+1} &:= \text{maximal cubes in } (\mathcal{P}_k \setminus \mathcal{P}_k^*) \cup \bigcup_{P \in \mathcal{P}_k^*} \mathcal{Q}(P). \end{aligned}$$

The sparse collection in the conclusion of the theorem will be given by

$$\mathcal{S}^\alpha := \mathcal{S} \cap \mathcal{D}^\alpha, \quad \mathcal{S} := \bigcup_{k \geq k_0} \mathcal{P}_k^*.$$

Let us first verify the Carleson property for the collections $\mathcal{S}^\alpha$. We call the cubes $Q \in \mathcal{Q}(P)$, $P \in \mathcal{P}_k^*$, the $\mathcal{Q}$-*children* of P. Note that a cube can have many $\mathcal{Q}$-parents. We claim that all $\mathcal{Q}$-descendants of any cube P are contained in a ball $B(z(P), C\delta^{k(P)})$, where C is a constant that depends only on the quasimetric constant and δ. Indeed, if $(z_0, z_1, \dots)$ is a sequence of points with $\rho(z_n, z_{n+1}) \leq C\delta^n$, then by the triangular inequality

$$\rho(z_{2^m n}, z_{2^m(n+1)}) \leq A_0^m C\sigma^n \quad \text{with} \quad \sigma = \delta^{2^m}.$$

Choosing m so large that $\sigma A_0 < 1$, we can estimate

$$\begin{aligned} \rho(z_0, z_{2^m n}) &\leq A_0(\rho(z_{2^m 0}, z_{2^m 1}) + A_0(\rho(z_{2^m 1}, z_{2^m 2}) + \dots))) \\ &\leq A_0^m C \sum_{l=0}^{\infty} (A_0\sigma)^l \leq \frac{A_0^m}{1 - A_0\sigma} C, \end{aligned}$$

and the claim follows.

Now let $Q, Q' \in \mathcal{S}^\alpha$ with $Q' \subsetneq Q$, so that in particular $k(Q') > k(Q)$. Then by construction $Q' \notin \mathcal{P}_{k(Q)}$ (Note here that $Q \in \mathcal{P}_{k(Q)}^* \subset \mathcal{P}_{k(Q)}$ and use the property $A, B \in \mathcal{P}_k, A \neq B \Rightarrow A \subsetneq B$ and $B \subsetneq A$). On the other hand, by

i) $Q' \in \mathcal{P}_{k(Q')} \Rightarrow \mathcal{Q}$-ancestor P in $\mathcal{P}_{k(Q)}$,

ii) since by the above argument Q' is contained in a ball of radius $C\delta^{k(P)}$ with center in P, the cube P must in turn be contained in $B(z(Q), C\delta^{k(Q)})$ for some larger constant C.

iii) since the elements of $\mathcal{P}_{k(Q)} \cap \mathcal{D}^\alpha$ are maximal and therefore disjoint, the family $\mathcal{P}_{k(Q)}$ has bounded overlap,

iv) doubling property of our measure space

follows that

v) the total measure of all possible ancestors in $\mathcal{P}_{k(Q)}$ is bounded by a multiple of $\mu(Q)$.

vi) if $\epsilon < 1/C_{(7.4.3)}$, then the total mass of all $\mathcal{Q}$-descendants of each P is bounded by a constant times the measure of P.

which completes the verification of the Carleson condition.

Proof of **v)**. By the doubling property

$$\sum_\alpha \sum_{P \in \mathcal{P}_{k(Q)} \cap \mathcal{D}^\alpha, P \subseteq B(z(Q), C\delta^{k(Q)})} \mu(P)$$

$$\leq \sum_\alpha \mu(B(z(Q), C\delta^{k(Q)})) \lesssim \mu(B(z(Q), C\delta^{k(Q)})) \lesssim \mu(B(z(Q), \delta^{k(Q)})) \leq \mu(Q).$$

□

Proof of **vi)**. By (7.4.5) and since

$$\sum_{n=1}^{N+1} \sum_{Q \in \mathcal{D}^n(P)} \mu(Q) = \sum_{Q \in \mathcal{D}^{N+1}(P)} \mu(Q) = \sum_{Q \in \mathcal{D}^N} \sum_{Q' \in \mathcal{D}(Q)} \mu(Q')$$

we have

$$\sum_{Q, Q-descendant(P)} \mu(Q) = \lim_{N \to \infty} \sum_{n=1}^{N+1} \sum_{Q \in \mathcal{D}^n(P)} \mu(Q) \leq \frac{1}{1 - \epsilon C_{(7.4.3)}} \mu(P).$$

□

It remains to show (7.1.3). Consider the family of truncations of the function F given by

$$F_\tau(x, t, s) := F(x, \max(t, \tau), \max(s, \tau)).$$

By induction on $K \geq k_0$ we obtain

$$\max_{Q_0 \in \mathcal{P}_{k_0}} \mathcal{N}_{Q_0} F_\tau \leq \sum_{k=k_0}^{K-1} \sum_{Q \in \mathcal{P}_k^*} c_Q 1_Q + \max_{Q \in \mathcal{P}_K} \mathcal{N}_Q F_\tau \tag{7.4.6}$$

for each $\tau > 0$. Indeed, the base case $K = k_0$ holds trivially, and in the inductive step we can apply (7.4.4) and obtain

$$\begin{aligned}\max_{Q\in\mathcal{P}_K} \mathcal{N}_Q F_\tau &= \max\Big\{\max_{Q\in\mathcal{P}_K\setminus\mathcal{P}_K^*} \mathcal{N}_Q F_\tau, \max_{Q\in\mathcal{P}_K^*} \mathcal{N}_Q F_\tau\Big\} \\ &\le \max\Big\{\max_{Q\in\mathcal{P}_K\setminus\mathcal{P}_K^*} \mathcal{N}_Q F_\tau, \max_{Q\in\mathcal{P}_K^*}(c_Q 1_Q + \max_{Q'\in\mathcal{Q}(Q)} \mathcal{N}_{Q'} F_\tau)\Big\} \\ &\le \max\Bigg\{\max_{Q\in\mathcal{P}_K\setminus\mathcal{P}_K^*} \mathcal{N}_Q F_\tau, \max_{Q\in\mathcal{P}_K^*}\max_{Q'\in\mathcal{Q}(Q)} \mathcal{N}_{Q'} F_\tau\Bigg\} + \max_{Q\in\mathcal{P}_K^*} c_Q 1_Q \\ &\le \max_{Q\in\mathcal{P}_{K+1}} \mathcal{N}_Q F_\tau + \sum_{Q\in\mathcal{P}_K^*} c_Q 1_Q.\end{aligned}$$

The second summand on the right-hand side of (7.4.6) vanishes identically for each fixed $\tau > 0$ and K that are so large that $\delta^K \ll \tau$. Thus we have obtained

$$\max_{Q_0\in\mathcal{P}_{k_0}} \mathcal{N}_{Q_0} F_\tau \le \sum_\alpha \sum_{Q\in\mathcal{S}^\alpha} 1_Q c_Q,$$

and the left-hand side converges to $\mathcal{N}F$ pointwise as $\tau \to 0$ and $k_0 \to -\infty$.

7.5 Proof of Theorem 7.1.5

We have

$$\mathcal{N}_a F(x) \le \sup_{\substack{x\in X\\ s<t\le\infty}} F(x,s,t) + \sup_{\substack{y\in X\\ s<t\le\infty\\ \rho(x,y)\le as}} (F(y,s,t) - F(x,s,t)).$$

Consider $F = (\dot{\mathcal{V}}^r \circ \mathcal{T})(f)$. Since $\|f+g\|_{L^{1,\infty}} \le 2\|f\|_{L^{1,\infty}} + 2\|g\|_{L^{1,\infty}}$, we obtain

$$\|\mathcal{N}_a \circ (\dot{\mathcal{V}}^r \circ \mathcal{T})(f)\|_{L^{1,\infty}} \le 2\|\mathcal{N}_0(\dot{\mathcal{V}}^r \circ \mathcal{T})(f)\|_{L^{1,\infty}} + 2\cdot$$
$$\cdot\Bigg\| \sup_{y\in X, s<t\le\infty, \rho(x,y)\le as} \big((\dot{\mathcal{V}}^r\circ\mathcal{T})(f)(y,s,t) - (\dot{\mathcal{V}}^r\circ\mathcal{T})(f)(x,s,t)\big)\Bigg\|_{L^{1,\infty}_X}.$$

By Lemma 7.3.3 and $f \le g \Rightarrow \|f\|_{1,\infty} \le \|g\|_{1,\infty}$,

$$\Bigg\| \sup_{y\in X, s<t\le\infty, \rho(x,y)\le as} \big((\dot{\mathcal{V}}^r\circ\mathcal{T})(f)(y,s,t) - (\dot{\mathcal{V}}^r\circ\mathcal{T})(f)(x,s,t)\big)\Bigg\|_{L^{1,\infty}_X}$$

$$\leq C\|Mf\|_{L^{1,\infty}} + C\|\mathcal{N}\dot{\mathcal{V}}^r\mathcal{A}f\|_{L^{1,\infty}}.$$

Then the result follows by Hardy- Litlewood maximal inequality and Lemma 7.3.2.

7.6 Commutators of BMO functions and CZ operators

In this section we prove a sparse domination theorem for iterated commutators of BMO functions with multilinear operators that extends [LOR16, Theorem 1.1]. An m-linear operator $\mathcal{T}$ taking an m-tuple $\vec{f} = (f_1, \dots, f_m)$ of functions defined on X to a function defined on $\mathcal{X}$ is called *local* if $\mathcal{T}(\vec{f})(x, s, t)$ depends only on the restrictions of the functions f_j to the ball $B(x, t)$. The main case of interest are truncations of multilinear CZ operators.

Let B be an index set and $\jmath : B \times \{0, 1\} \to \{0, \dots, m\}$. For a tuple of functions $\vec{b} = (b_\beta)_{\beta \in B}$, $j \in \{0, \dots, m\}$, and an index $a \in \{0, 1\}^B$ let $b_{a,j} := \prod_{\beta:\jmath(\beta,a(\beta))=j}(-1)^{a(\beta)}b_\beta$. The (iterated) $\jmath$-commutator of $\vec{b}$ with an m-linear operator $\mathcal{T}$ is defined by

$$[\vec{b}, \mathcal{T}]_\jmath(\vec{f})(x, s, t) := \sum_{a \in \{0,1\}^B} b_{a,0}(x)\mathcal{T}(\overrightarrow{fb_a})(x, s, t),$$

where $\overrightarrow{fb_a}$ is the vector $(f_1 b_{a,1}, \dots, f_m b_{a,m})$. Multilinear operators of this type have been studied in [Ler+09].

The next result extends [LOR16, Theorem 1.1]. Note that it holds for spaces of homogeneous type; this allows one to recover a number of results in that setting, see e.g. [AD14].

Theorem 7.6.1. *For every space of homogeneous type (X, ρ, μ) and every choice of adjacent systems of dyadic cubes $\mathcal{D}^\alpha$ there exists $0 < \eta < 1$ such that the following holds. Let $1 \leq r \leq \infty$ and let $\mathcal{T}$ be an m-linear local operator such that*

$$C_{\mathcal{T}} := \|\mathcal{N} \circ \dot{\mathcal{V}}^r \circ \mathcal{T}\|_{L^1 \times \dots \times L^1 \to L^{1/m,\infty}} < \infty. \tag{7.6.2}$$

Let $B, \jmath, \vec{b}$ be as above and let $c_{\beta,Q}$ for $\beta \in B$ and $Q \in \cup_\alpha \mathcal{D}^\alpha$ be arbitrary numbers. Let also Q_0 be an initial dyadic cube and $f_1, \dots, f_m \in L^\infty(Q_0)$. Then there exist η-sparse collections $\mathcal{S}^{\alpha,k_0} \subset \mathcal{D}^\alpha$ such that we have

$$\mathcal{N}_0\dot{\mathcal{V}}^r[\vec{b}, \mathcal{T}]_\jmath\vec{f} \lesssim C_{\mathcal{T}} \liminf_{k_0 \to -\infty} \sum_\alpha \sum_{Q \in \mathcal{S}^{\alpha,k_0}} \{\vec{b}, \vec{f}\}_{\jmath,Q}$$

pointwise almost everywhere, where

$$\{\vec{b},\vec{f}\}_{\jmath,Q}(x) := 1_Q(x) \sum_{a\in\{0,1\}^B} |b_{a,0,Q}(x)| \prod_{j=1}^m \langle |b_{a,j,Q} f_j| \rangle_Q$$

and

$$b_{a,j,Q} := \prod_{\beta:\jmath(\beta,a(\beta))=j} (-1)^{a(\beta)}(b_\beta - c_{\beta,Q}).$$

In absence of commutators ($B=\emptyset$) this follows directly from Theorem 7.1.1, and in fact the centered operator $\mathcal{N}_0$ can be replaced by the uncentered operator $\mathcal{N}$ in the conclusion. In presence of commutators the most interesting choice of constants is of course $c_{\beta,Q} = \langle b_\beta \rangle_Q$.

Proof of Theorem 7.6.1. The only difference from Theorem 7.1.1 is that we need a suitable substitute for (7.4.3) when

$$F = \dot{\mathcal{V}}^r [\vec{b}, \mathcal{T}]_\jmath \vec{f}$$

and

$$\lambda(x) = \epsilon^{-1} C_B \{\vec{b},\vec{f}\}_{Q_0}(x).$$

Note that, by multilinearity of $\mathcal{T}$, the function F does not change when replacing b_β by $b_\beta - c_{\beta,Q_0}$. For each $y\in Y$ we have

$$\lambda(y) < F(y, \frac{1}{2}\sigma(y), \mathrm{dist}(y, \complement Q_0)).$$

By the triangle inequality for the ℓ^r norm this implies

$$\epsilon^{-1} C_B |b_{a,0,Q_0}(y)| \prod_{j=1}^m \langle |b_{a,j,Q_0} f_j| \rangle_{Q_0} \\ < |b_{a,0,Q_0}(y)| \dot{\mathcal{V}}^r \mathcal{T}(\overrightarrow{f b_{a,Q_0}})(y, \frac{1}{2}\sigma(y), \mathrm{dist}(y, \complement Q_0))$$

for some $a\in\{0,1\}^B$. Since this inequality is strict, the factor $|b_{a,0,Q_0}(y)|$ cannot be zero and can be canceled. It follows that

$$\bigcup_{y\in Y} B(y, \sigma_y/4) \subset \bigcup_{a\in\{0,1\}^B} \Big\{ \dot{\mathcal{V}}^r \mathcal{T}(\overrightarrow{f b_{a,Q_0}}) > \epsilon^{-1} C_B \prod_{j=1}^m \langle |b_{a,j,Q_0} f_j| \rangle_{Q_0} \Big\},$$

and the measures of the latter sets are bounded by $\epsilon^{1/m}|Q_0|$ by definition of C_B and locality of $\mathcal{T}$. This provides the estimate $\sum_{Q\in\mathcal{Q}} |Q| \lesssim \epsilon^{1/m}|Q_0|$. $\square$

The above domination theorem requires as input an endpoint weak type estimate (7.6.2) for $\mathcal{N} \circ \mathcal{V}^r \circ \mathcal{T}$. In the multilinear case such bounds are known only for $r = \infty$ (that is, for maximal truncations) and can be found in [DHL15a] (where they are stated for $X = \mathbb{R}^d$). More precisely, the weak type estimate for $\mathcal{N}_0 \circ \mathcal{V}^\infty \circ \mathcal{T}$ is proved in [DHL15a, §6] and the weak type estimate for $\mathcal{N} \circ \mathcal{V}^\infty \circ \mathcal{T}$ is effectively proved in [DHL15a, §3.1]. The main difference from the linear case is the need to use the multilinear maximal function from [Ler+09, Theorem 3.3].

In the linear case one can obtain the hypothesis (7.6.2) with $2 < r < \infty$ for a certain class of CZ operators from Theorem 7.1.5. Using the results of [LOR16, §4] this implies weighted estimates for variational truncations of commutators of CZ operators with BMO functions.

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