# Quotient Presentations of Mori Dream Spaces

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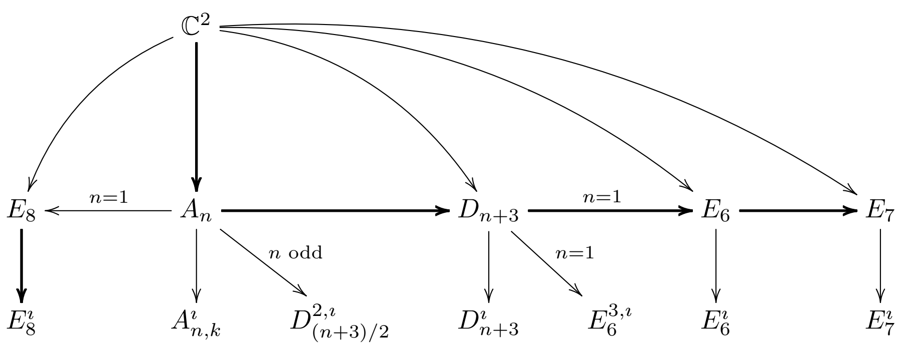
# Contents

# Introduction

The purpose of the present thesis is to investigate different quotient presentations of *Mori Dream Spaces* and in particular of *varieties of Fano type*. The approach is twofold: on the one hand, we iterate the construction of *Cox rings* in order to express varieties of Fano type as GIT-quotients of factorial canonical quasicones by solvable reductive groups. On the other hand, we aim to construct such factorial canonical quasicones as (iterated) quotients of affine space by the group $\mathrm{SL}_n(\mathbb{C})$. In order to do so, we develop techniques to effectively compute invariant rings of $\mathrm{SL}_n(\mathbb{C})$.

Let us begin with taking a look at the following classical diagram, that can be obtained for example from the classification of Brieskorn [**23**]. It contains all log terminal surface singularities, where arrows symbolize quotients by finite subgroups of $\mathrm{GL}_2(\mathbb{C})$ and the *Gorenstein* ADE-singularities in the middle row serve as index one covers for those of higher Gorenstein index in the bottom row:




In fact, almost all concepts developed throughout this thesis are reflected in the diagram: the singularities here arise as vertices of *quasicones*, affine varieties with a $\mathbb{C}^*$-action such that all orbit closures meet in one point - the vertex. All of them are *Mori Dream Spaces* - normal varieties with a finitely generated divisor class group $\mathrm{Cl}(X)$ and a finitely generated $\mathrm{Cl}(X)$-graded multisection ring

$$\mathcal{R}(X) := \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)),$$

called the *Cox ring* of $X$. Examples of Mori Dream Spaces (MDS) include toric [**30**], spherical [**26**], and Fano type varieties [**16**]. They have been introduced by Hu and Keel in [**57**] and behave very nicely with respect to the minimal model program: a $D$-MMP can be run for any divisor $D$ and any sequence of $D$-flips terminates.

Another nice property of MDS is that they can be presented as a GIT-quotient of the *total coordinate space* $\overline{X} := \operatorname{Spec} \mathcal{R}(X)$ by the *characteristic quasitorus*

$H_X := \operatorname{Spec}\mathbb{C}[\operatorname{Cl}(X)]$, generalizing the well known representation of projective space $\mathbb{P}_{n-1} = (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*$. To be exact, $X$ is a good quotient by $H_X$ of the set of semistable points $\widehat{X} := \overline{X}^{\mathrm{ss}} \subseteq \overline{X}$ with complement of codimension at least two, called the *characteristic space*.

In the above diagram, the bold arrows symbolize such quotients by characteristic quasitori. As mentioned before, the arrows from the middle to the bottom row represent *index one covers* and in fact, the concatenation of such an arrow with a precedent bold one gives a characteristic quasitorus quotient as well. Thus all total coordinate spaces in our diagram are Gorenstein in the sense that the canonical class $K_{\overline{X}}$ is Cartier, and one might be tempted to hope that this is a general phenomenon. Our first result says that this is in fact true:

THEOREM 1. *Let $X$ be an* MDS*. Then $\overline{X}$ is Gorenstein.*

Another feature of the diagram is the representation of singularities as vertices of quasicones, which from the perspective of Cox ring theory is a nice property. Firstly, because quasicones have trivial Picard group, and thus all information on their characteristic quasitorus is concentrated in the local class group of the vertex, so the Cox ring truly reflects properties of this very point. Secondly, because the total coordinate space $\overline{X}$ inherits the quasicone property from $X$. In particular, if $X$ is a quasicone or complete MDS, then $\overline{X}$ is always a quasicone, as we show in Section 2.3. So roughly said, even for complete varieties, the information of the Cox ring is concentrated in one point - though it is not truly a local ring. This is reflected by the characterization from [**43, 28, 62**], stating that a projective variety is of Fano type if and only if it has finitely generated log terminal Cox ring. Again, our diagram from above suggests a similar statement for log terminal quasicones. One might say this is no surprise since all morphisms in the diagram are finite and it is well known that finite morphisms preserve log terminality. Nevertheless, the statement generalizes to all *Kawamata log terminal* (klt) quasicones and we arrive at the following refined characterization:

THEOREM 2. *Let $X$ be projective (affine). Then it is of Fano type (a* klt *quasicone) if and only if $\overline{X}$ is a Gorenstein canonical quasicone.*

Till now, we did not talk about the curved arrows going out from $\mathbb{C}^2$. They arise as concatenations of characteristic quasitorus quotients and are in fact quotients by *solvable* groups (The arrow pointing to $E_8$ is an exception, we will come to that later). By the *derived normal series* of such a group, one retrieves the respective chain of abelian quasitorus quotients.

This phenomenon can be generalized as well: we speak of *iteration of Cox rings*, when the total coordinate space $\mathcal{X}^{(1)} := \overline{X}$ of an MDS $X$ is an MDS as well and has total coordinate space $\mathcal{X}^{(2)} := \overline{\overline{X}}$ - and so on. Denoting the respective characteristic spaces by $X^{(i)} \subseteq \mathcal{X}^{(i)}$, we eventually arrive at a chain of quotients $\cdots \to X^{(2)} \to X^{(1)} \to X$, where one has three possibilities: either in every step $X^{(i)}$ is an MDS with nontrivial class group, then $X$ has *infinite iteration of Cox rings*. Or at some point $m \in \mathbb{N}$, the iteration stops and $X$ has *finite iteration of Cox rings*, which can have two reasons: either $X^{(m)}$ is factorial, so $\operatorname{Cl}(X^{(m)})$ is trivial, or $X^{(m)}$ is non-MDS. In both of the latter cases, we call $\mathbb{C}[X^{(m)}]$ the *master Cox ring* of $X$.

Now observe that for all quasicones in our diagram, iteration of Cox rings is finite and we have two types of master Cox rings: $\mathbb{C}^2$ and $E_8$, both factorial. This observation generalizes to varieties of Fano type and klt quasicones as well, which makes our next result:

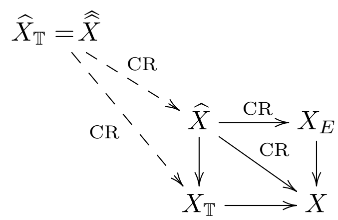
THEOREM 3. *Let $X$ be of Fano type or a* klt *quasicone. Then $X$ has finite iteration of Cox rings with factorial master Cox ring.*

Note that Theorem 2 already guarantees MDS-ness of all $X^{(i)}$ in the Cox ring iteration of Fano type varieties and klt quasicones. *Finiteness* of the iteration follows from the next lemma, which is of interest in its own right. When we speak of a *downgrading* of a graded ring, we mean the same ring, but graded (in a compatible way) by a subgroup of the former grading group. Downgradings lead to a factorization of the respective quotients, and Lemma 1 shows how characteristic quasitorus quotients behave and are preserved in such a situation.

LEMMA 1. *Let $X$ be an* MDS *and $H_X = E \times \mathbb{T}$ its characteristic quasitorus with torsion and torus part $E$ and $\mathbb{T}$. Then:*

(i) *The Cox ring of the geometric quotient $X_E := \widehat{X}/E$ is a downgrading of the Cox ring of $X$.*
(ii) *The characteristic space $\widehat{X}$ is an* MDS *if and only if $X_{\mathbb{T}} := \widehat{X} /\!\!/ \mathbb{T}$ is an* MDS *and in that case, the Cox ring $\mathcal{R}(\widehat{X})$ is a downgrading of $\mathcal{R}(X_{\mathbb{T}})$.*

*In particular, we have the following commutative diagram of* GIT*-quotients, where* CR *denotes characteristic quasitorus quotients:*



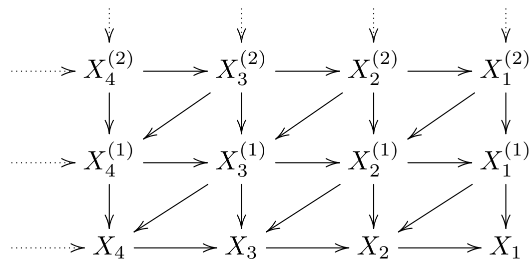

Roughly speaking, the first assertion of Lemma 1 allows to reduce to the case of finite characteristic quasitori in the proof of Theorem 1 - which factor through index one covers and preserve Gorensteinness. By the second assertion of Lemma 1, one can construct a certain chain of *finite Galois covers* from the Cox ring iteration and such a chain must be finite by [**45**, Thm. 1.1], which then proves Theorem 3.

The key observation leading to Lemma 1 is that if an MDS $X$ has a *quotient presentation* $X = Y /\!\!/ H$ with a variety $Y$ and a quasitorus $H$ with certain good properties, see [**6**, Def. 4.2.1.1], then the characteristic space $\widehat{X} \to X$ factors through $Y \to X$, see [**6**, Thm. 4.2.1.4] and Proposition 2.1.2.

Now consider Okawa's Theorem [**73**, Thm. 1.1], which states that for a surjective morphism $f\colon X \to Y$ between projective varieties, if $X$ is an MDS then $Y$ is as well. We can in fact say more if $f$ is a quotient presentation, namely if $Y$ and its characteristic space $\widehat{Y}$ are MDS, then $X$ is so, see Proposition 2.1.2. In terms of iteration of Cox rings, this generalizes to the following:

THEOREM 4. *Consider a chain of quotient presentations $\cdots \to X_2 \to X_1$. Denote the $i$-th iterated characteristic space of $X_j$ by $X_j^{(i)}$ if it exists, i.e. $X_j^{(1)} := \widehat{X}_j$ etc. Then if one of the $X_i$ has infinite iteration of Cox rings, the others have as well. If one has factorial master Cox ring, the others have as well and all master*

*Cox rings coincide. In both of these cases, we get a commutative* web of Cox ring iterations*:*




Note that if for example $X_1$ has finite iteration of Cox rings with non-MDS master Cox ring $\mathbb{C}[X_1^{(3)}]$, we get MDS-ness only for $X_2, X_2^{(1)}, X_3$. So in such case, the maximal number of steps in the Cox ring iteration becomes important.

Due to [**6**, Thm. 4.2.1.4], the characteristic space $\widehat{X} \to X$ is a universal reductive abelian (possibly nondiscrete) 'cover' of $X$ in the sense that it factors through any quotient presentation $Y \to X$. This can be generalized for MDS with finite iteration of Cox rings and factorial master Cox ring in the following way.

Corollary 1. *Let $X$ be an* MDS *with finite iteration of Cox rings and factorial master Cox ring $\mathbb{C}[X^{(m)}]$. Let also $X = Y /\!\!/ G$ be a quotient of a normal variety $Y$ by a solvable reductive group $G$, such that $Y$ has only constant invertible $G$-invariant functions and $G$ acts freely on a subset with complement of codimension at least two in $Y$. Then the quotient $\left(X^{(m)}\right)^{\mathrm{ss}} \to X$, where $\left(X^{(m)}\right)^{\mathrm{ss}}$ denotes the set of semistable points of $X^{(m)}$, factors through $Y \to X$. We call $\left(X^{(m)}\right)^{\mathrm{ss}} \to X$ the* universal solvable quotient presentation *of $X$.*

Note that this does not mean in general that a solvable quotient $Y \to X$ factors through $\widehat{X} \to X$, compare [**7**, §3] for the affine case. Moreover, if $X$ has non-MDS master Cox ring, difficulties as with Theorem 4 arise.

Speaking about Cox ring iteration, the question comes up if one can effectively compute the iteration chain for a certain variety $X$. In general, this is a difficult task, but it is feasible for certain classes of varieties. The Cox ring iteration of toric varieties for example is trivial, since their Cox ring is a polynomial ring [**30**], while spherical varieties have factorial master Cox ring and the iteration has at most two steps [**42**]. In the joint work [**5**] with Arzhantsev, Hausen, and Wrobel, we computed the iteration chain explicitly for log terminal quasicones with a torus action *of complexity one*, i.e. where the dimension of the torus is one less than that of the variety itself. It turns out that the iteration in any dimension is reflected by our initial diagram.

But first let us have a look at the Cox rings of quasicones and complete varieties with a torus action of complexity one. They are (isomorphic to) suitably graded rings given by trinomial relations

$$T_0^{l_0} + T_1^{l_1} + T_2^{l_2}, \quad \lambda_1 T_1^{l_1} + T_2^{l_2} + T_3^{l_3}, \ldots, \quad \lambda_{r-2} T_{r-2}^{l_{r-2}} + T_{r-1}^{l_{r-1}} + T_r^{l_r},$$

where the $1 \neq \lambda_i \in \mathbb{C}^*$ are pairwise different coefficients, the $l_i$ are vectors in $\mathbb{N}^{n_i}$, and the $T_i^{l_i}$ are of the form $T_i^{l_i} = T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}$.

For example, if $r = 2$, all $n_i$ equal one, and the exponents $(l_0, l_1, l_2)$ form one of the *platonic triples* $(5,3,2)$, $(4,3,2)$, $(3,3,2)$, $(x,2,2)$, $(x,y,1)$, then the relations

$$T_0^5 + T_1^3 + T_2^2, \quad T_0^4 + T_1^3 + T_2^2, \quad T_0^3 + T_1^3 + T_2^2, \quad T_0^n + T_1^2 + T_2^2, \quad T_0^n + T_1^m + T_2$$

are those of the total coordinate spaces $E_8$, $E_6$, $D_4$, $A_n$ and $\mathbb{C}^2$ from our diagram. This generalizes to the following complexity one version of Theorem 2:

THEOREM 5. *Let $X$ be a projective (affine) rational variety with a torus action of complexity one. Then $X$ is of Fano type (a* klt *quasicone) if and only if after decreasingly ordering the maximal exponents $\ell_i := \max(l_{i1}, \dots, l_{in_i})$ in the relations of $\mathcal{R}(X)$, we have $\ell_i = 1$ for $i \geq 3$ and $(\ell_0, \ell_1, \ell_2)$ form a platonic triple.*

Moreover, the spectrum $\overline{X} = \operatorname{Spec} \mathcal{R}(X)$ of a Cox ring of such form is factorial if and only if the greatest common divisors $\mathfrak{l}_i = \gcd(l_{i1}, \dots, l_{in_i})$ of the exponents are pairwise coprime. Finally, we can explicitly list the exponents of the Cox ring of non-factorial $\overline{X}$ in terms of the ordered exponents of $\mathcal{R}(X)$, see Corollary 2.6.8:

| $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ | exponent vectors of $\mathcal{R}(\overline{X})$ |
|---|---|
| $(4,3,2)$ | $l_1,\, l_1,\, \frac{l_0}{2},\, \frac{l_2}{2},\, l_3,\, l_3,\, \dots,\, l_r,\, l_r$ |
| $(3,3,2)$ | $\frac{l_0}{3},\, \frac{l_1}{3},\, l_2,\, l_2,\, l_2,\, \dots,\, l_r,\, l_r,\, l_r$ |
| $(2k,2,2)$ | $\frac{l_0}{2},\, \frac{l_0}{2},\, \frac{l_1}{2},\, \frac{l_1}{2},\, \frac{l_2}{2},\, \frac{l_2}{2},\, l_3,\, l_3,\, l_3,\, l_3,\, \dots,\, l_r,\, l_r,\, l_r,\, l_r$ |
| $(2k+1,2,2)$ | $l_0,\, l_0,\, \frac{l_1}{2},\, \frac{l_2}{2},\, l_3,\, l_3,\, \dots,\, l_r,\, l_r$ |
| $(\mathfrak{l}_0, \mathfrak{l}_1, 1)$ | $\frac{l_0}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)},\, \frac{l_1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)},\, \underbrace{l_2, \dots, l_2}_{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)},\, \dots,\, \underbrace{l_r, \dots, l_r}_{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)}$ |

With this at hand, we will in the last part of the thesis determine the *Cox ring iteration tree* of canonical and compound Du Val threefold singularities with a two-torus action.

But first let us have a look at the left side of our initial diagram. The two arrows pointing to $E_8$ are of a different kind than the others. The one going out from $A_1$ reflects a quotient by the (simple) alternating group $\mathcal{A}_5$ and the curved one going out from $\mathbb{C}^2$ a quotient by the (perfect) binary icosahedral group, which is an extension of $\mathcal{A}_5$ by the cyclic group of order two, again reflected by the composition $\mathbb{C}^2 \to A_1 \to E_8$ in our diagram.

This observation motivates the second part of the present thesis. The aim is to find factorial quasicones that can serve as master Cox rings for MDS. The idea is to construct them explicitly as quotients of simple linear groups, generalizing the $\mathcal{A}_5$-quotient representation of $E_8$. The approach is via *classical invariant theory*: given a representation $\varphi : G \to \mathrm{GL}(W)$ of a simple linear group $G$, we aim to compute invariants and relations explicitly.

From now on, we restrict to the case $G = \mathrm{SL}_n(\mathbb{C})$ and denote by $V$ the standard representation, by $\Lambda^k$, $S^k$, Ad, and $S_\lambda$ the $k$-th exterior power, $k$-th symmetric power, adjoint representation, and Schur module for a vector $\lambda$ of natural numbers. Reducible representations are denoted by sums and multiples of these symbols and the dual by a starred version.

Already among those representations of $\mathrm{SL}_2$ of which the rings of invariants are classically known, there are some that in fact give factorial quasicones with a torus action of complexity one, for example the relations of the invariant rings of $4V$, $2V+S^2$, $V+S^3$, which are:

$$T_{01}T_{02}+T_{11}T_{12}+T_{21}T_{22}, \quad T_{01}^2+T_{11}^2T_{12}+T_{21}T_{22}, \quad T_{01}^3+T_{11}^2T_{12}+T_{21}^2.$$

In the classical invariant theory of the nineteenth century, *brackets* have been used to effectively compute invariants for $\mathrm{SL}_2$. The idea behind is roughly the following: all invariant polynomial functions $W \to \mathbb{C}$ of a representation $W$ of $\mathrm{SL}_n$ come from the *determinant*. Irreducible subrepresentations of $W$ can be taken as *column entries* for the determinant. So for example on $W = mV$, one has $\binom{m}{n}$ nontrivial choices of combinations for the $n$ columns of the determinant. One of these choices is then denoted by a bracket $[i_1 \cdots i_n]$, where $1 \leq i_1 < \ldots < i_n \leq m$. Between such brackets the well known *Plücker relations* hold.

But what if $V$ is not the only irreducible subrepresentation of $W$? A $k$-th power of $V$ requires $k$ columns of the determinant, in the bracket we denote this by $k$ copies of the same letter corresponding to the representation. It also may happen now that such $k$-th power is distributed over multiple determinants in a *product*, as for example in $[add][dbc]$, representing an invariant of the $\mathrm{SL}_3$-representation $W = 3V + S^3$, where $a, b, c, d$ stand for $V, V, V, S^3$ respectively. Setting $\det := x_1 \wedge x_2 \wedge x_3$, the invariant given by the above bracket expression can be seen as the concatenation of $\varphi\colon W \to T_6^6(V)$, given by

$$(t_1, t_2, t_3, s) \longmapsto t_1 \otimes s \otimes t_2 \otimes t_3 \otimes \det \otimes \det,$$

with the tensor contraction $C_{1,2,3,4,5,6}^{1,2,3,4,5,6}\colon T_6^6(V) \to \mathbb{C}$. The bracket here specifies which indices have to be contracted - namely those of a copy of det and a tensor represented by a letter standing in the bracket corresponding to det.

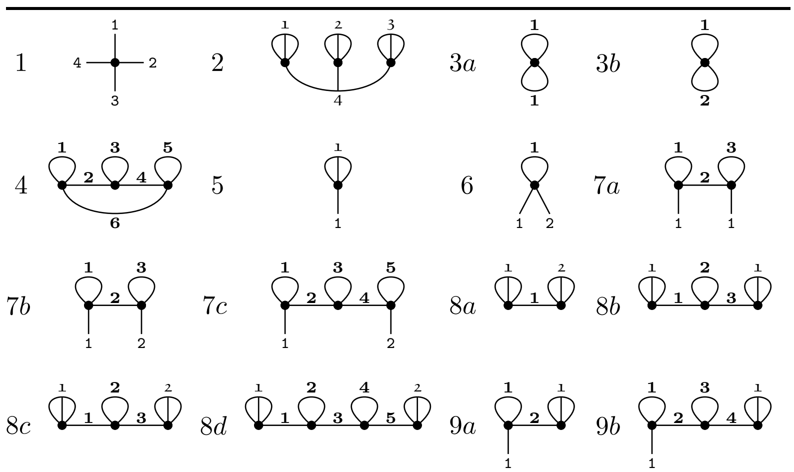
For such generalized brackets, Plücker relations hold as well, and they can be used to investigate if a bracket expression reduces to a sum of products of 'minimal' ones, which then represent generators of the ring of invariants. Letters corresponding to symmetric and exterior powers behave differently with respect to the Plücker relations and it is in general a challenging task to determine such generators, which is why with the exception of a few serial cases, in the nineteenth century only a limited number of particular invariant rings of $\mathrm{SL}_2$-representations have been determined. Moreover, it took untill 1987 that the bracket notation for *combinations of symmetric and exterior powers* has found a rigorous foundation in terms of *superalgebras*, which is due to Grosshans, Rota and Stein [**46**]. But as soon as the degree $n$ of $\mathrm{SL}_n$ becomes bigger and the structure of the representation $W$ more complex, the brackets and the impacts of the Plücker relations on them become more and more involved.

In Chapter 3, we develop a graphical method to represent the invariants, making it much easier to see the effects of Plücker relations and to compute generators for the invariant ring. In addition, this method enables one to apply graph theoretical results. The idea is to associate to a bracket expression a *hypergraph*, containing a vertex for each bracket and an *$i$-edge* for each letter with $i$ occurences (i.e. corresponding to a $i$-th power of $V$) in the bracket expression. By coloring the edges (with natural numbers), we can distinguish between different $i$-th powers of $V$ in $W$. Now consider $W$ to be a sum of fundamental representations of $\mathrm{SL}_n$, i.e.

$$W := \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^{n_i} V_{i,j},$$

where $V_{i,j} := \Lambda^i V$. We first investigate the case $\mathrm{SL}_4$. Colors $1, 2, \ldots, N$ of $i$-edges stand representatively for arbitrary but ascending colors $1 \leq j_1, j_2, \ldots, j_N \leq n_i$.

THEOREM 6. *The following graphs yield a minimal set of generators for the algebra of invariants* $\mathbb{C}[W]^{\mathrm{SL}_4}$.




We will clarify the meaning of these graphs by the example of No. 9a, comprising edges of all types. We have one invariant of this type for each choice of subrepresentations $V_{1,j}$, $V_{2,k_1}$, $V_{2,k_2}$, $V_{3,l} \subseteq W$, where $1 \leq j \leq n_1$, $1 \leq k_1 < k_2 \leq n_2$, and $1 \leq l \leq n_3$, since we have one 1-edge, two 2-edges, and one 3-edge.

Now we build up a map from $V_{1,j} \oplus V_{2,k_1} \oplus V_{2,k_2} \oplus V_{3,l} \subseteq W$ to the tensor algebra. The image is a tensor product of *one element of each subrepresentation* - since the graph comprises *one edge of each corresponding color* - and *two tensors* det - since the graph has *two vertices.* We get the map

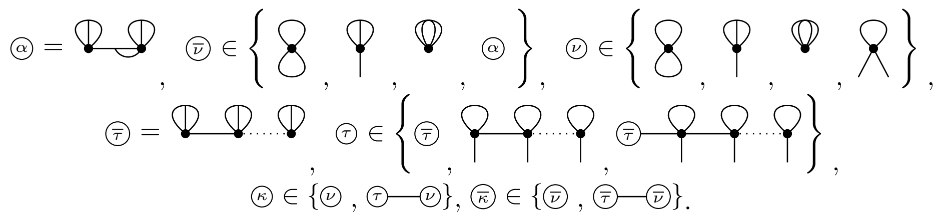
$$(t_{1,j}, t_{2,k_1}, t_{2,k_2}, t_{3,l}) \mapsto t_{1,j} \otimes t_{2,k_1} \otimes t_{2,k_2} \otimes t_{3,l} \otimes \det \otimes \det .$$

Finally we have to contract indices according to the graph. Suppose the first vertex corresponds to the first det, then all indices of $t_{1,j}$ and $t_{2,k_1}$ and the first index of $t_{2,k_2}$ have to be contracted with the indices of the first det, since the corresponding edges are connected to the first vertex. The remaining index of $t_{2,k_2}$ and all of $t_{3,l}$ have to be contracted with those of the second det. This yields the invariant corresponding to the graph.

In Theorem 6, one can observe some general phenomena: first, each vertex has a *looping edge* and thus their *virtual degree* (i.e. the number of connections going out to other vertices) is strictly smaller than their true degree. We can always achieve this for a system of generators, to be exact, we can even achieve the biggest $i$-edges to be looping. Note that for example if $n_1 = n_3 = 0$, i.e. we have only 2-edges, this already yields that graphs in a minimal generating set must be cycles, which was observed in [**58**] by using brackets. With our graphical method, we can see that such cycles are in a minimal generating set only if they have one or three vertices, cf. graphs 3 and 4. For details and more examples on the benefit of the graphical method, we refer to Chapter 3.

In Section 3.2 - still in the case of $\mathrm{SL}_4$ - we then give relations holding between the invariants. Moreover, from the invariants we deduce a minimal generating set of *covariants* - i.e. generators of $\mathbb{C}[W]^U$ for a maximal unipotent subgroup $U$ of $\mathrm{SL}_4$ - and give a geometric interpretation in terms of linear subspaces of $\mathbb{P}^3$. Covariants correspond to *graphs with dummy edges* and become important in the classification of complete intersection invariant rings later.

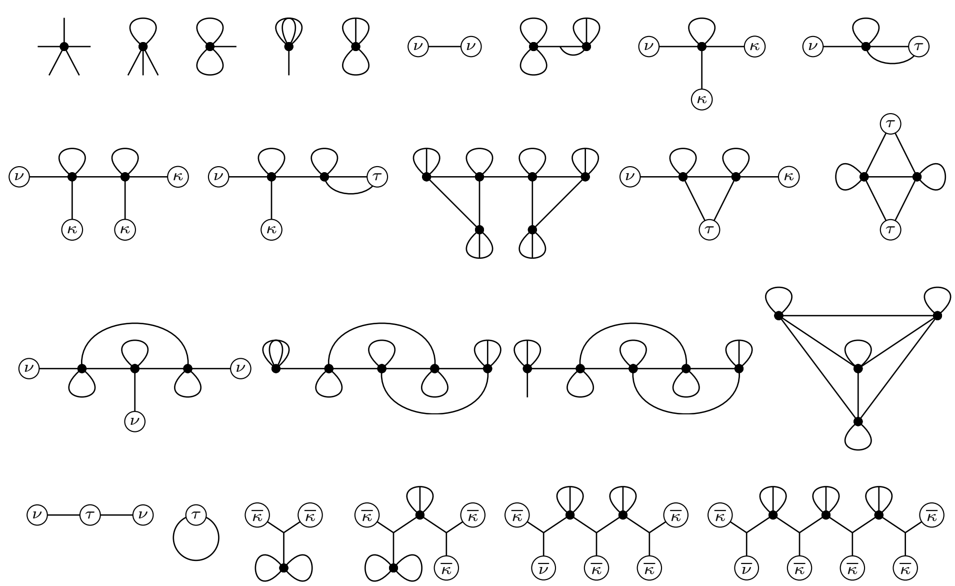
The case of $\mathrm{SL}_5$ is much more involved. Thus in the following theorem, we do not give all different colorings of each graph. We did this exemplarily in the special case $n_i = 0$ for $i \neq 2$, see Proposition 3.2.8. Moreover, by the duality of $\Lambda^i V$ and $\Lambda^{n-i} V$, we have *mirror invariants* and also mirror graphs, where $i$-edges of the one correspond to $(n-i)$-edges of the other. For example, the first two graphs from Theorem 6 are mirrors of each other, while graphs three to six are their own mirror each. Thus we consider only one graph of each mirror pair in the following. Moreover, there are the following types of 'building blocks' that can be attached to some of the graphs for $\mathrm{SL}_5$:

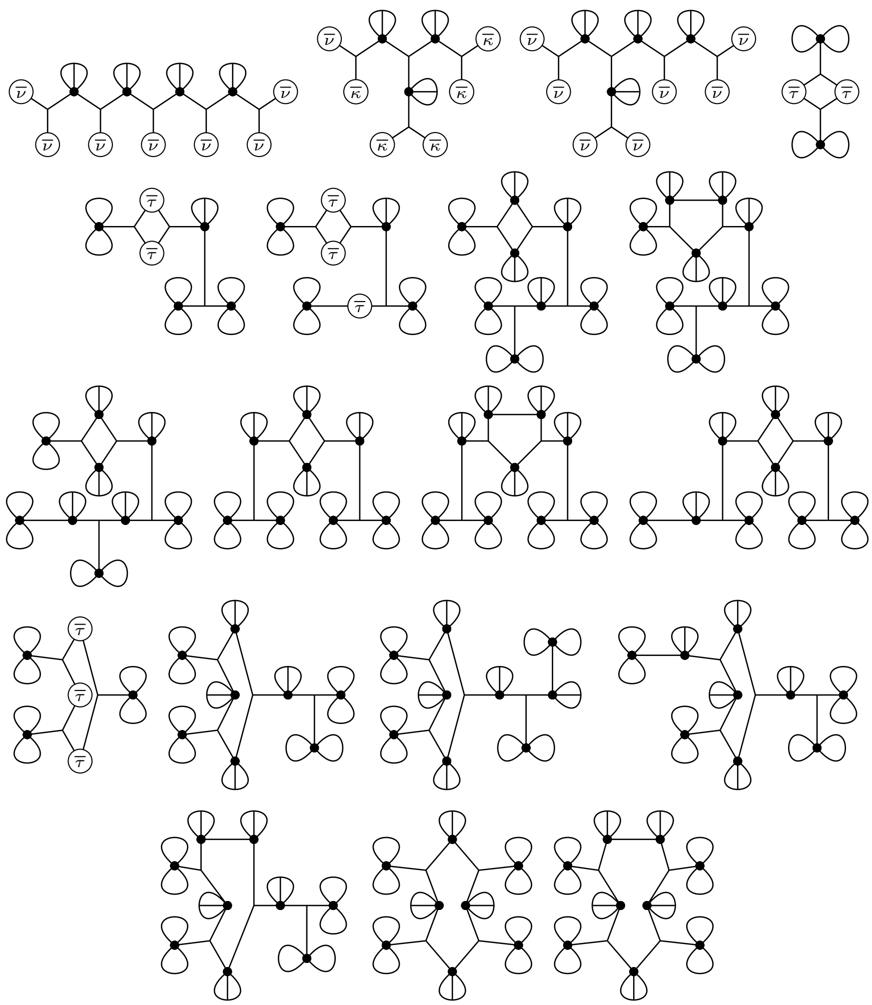


THEOREM 7. *The following graphs with all possible combinations of building blocks attached and all possible colorings with respect to the conditions:*

- *the number of vertices with at least one looping* 2*-edge plus the number of non-looping two edges is less than or equal to nine,*
- *the number of blocks* ⓐ *plus the number of* 3*-edges that are not part of a block* ⓐ *is less than or equal to nine,*
- *the number of* 3*-edges is less than or equal to the number of* 2*-edges,*

*together with their mirrors constitute a generating set of* $\mathbb{C}[W]^{\mathrm{SL}_5}$.





Though it seems reasonable to compute generating sets also for some $n \geq 6$, we now address ourselves to another task. Since all Cox rings of varieties with a torus action of complexity one are complete intersections, it is interesting from our perspective which invariant rings of $\mathrm{SL}_n$ are of such kind. In fact, it is a long-standing task in invariant theory to determine representations with a reasonably simple - meaning regular, hypersurface or complete intersection - ring of invariants. Representations of connected simple groups with regular ring of invariants have been classified in [**61, 2, 86**], while *irreducible* representations with complete intersection invariant rings can be found in [**71**].

*Reducible* representations of $\mathrm{SL}_2$ with complete intersection invariant ring are classically known, see [**44**]. Those for arbitrary $\mathrm{SL}_n$ have been classified by Shmelkin in [**91**], while three single and three serial cases have been left open.

In Section 3.5 we settle these cases and thus in combination with Shmelkin's results obtain the full list of representations of $\mathrm{SL}_n$ with complete intersection invariant ring:

Theorem 8. *A representation of* $\mathrm{SL}_n$ *has a non-regular but complete intersection invariant ring if and only if it or its dual is contained in the following table. If the invariant ring is not a hypersurface, we note its homological dimension by a boldface subscript.*

| $\mathrm{SL}_2$ | $\mathrm{SL}_3$ | $\mathrm{SL}_4$ | $\mathrm{SL}_5$ | $\mathrm{SL}_6$ | $\mathrm{SL}_7$ |
|---|---|---|---|---|---|
| $4V$ | $2S^2+(S^2)^*$ | $S^3$ | $V+3\Lambda^2$ | $V+V^*+\Lambda^2+\Lambda^3$ | $3V+\Lambda^3$ |
| $3S^2$ | $S^2+\mathrm{Ad}$ | $6\Lambda^2$ | $3V^*+2\Lambda^2$ | $2V^*+\Lambda^2+\Lambda^3$ | $2V+2V^*+\Lambda^3$ |
| $V+2S^2$ | $V+S^3$ | $V+4\Lambda^2$ | $V+2\Lambda^2+\Lambda^3$ | $2V+\Lambda^2+{\Lambda^3}_{\mathbf{2}}$ | $V+3V^*+\Lambda^3$ |
| $2V+S^2$ | $V^*+S^3$ | $2V+3\Lambda^2$ | $V^*+3{\Lambda^2}_{\mathbf{2}}$ | $3V+3V^*+{\Lambda^3}_{\mathbf{2}}$ | $4V^*+\Lambda^3$ |
| $V+S^3$ | $3S^2$ | $V+V^*+3\Lambda^2$ | $V^*+2\Lambda^2+{\Lambda^3}_{\mathbf{3}}$ | $4V+V^*+{\Lambda^3}_{\mathbf{2}}$ | $\Lambda^2+\Lambda^3$ |
| $S^2+S^3$ | $2\mathrm{Ad}$ | $V+S_{(2,2)}$ | $4V^*+2{\Lambda^2}_{\mathbf{4}}$ | $4V+2V^*+{\Lambda^3}_{\mathbf{4}}$ | $\Lambda^3+\Lambda^5$ |
| $V+S^4$ | | $\Lambda^2+S_{(2,2)}$ | | | $3V+V^*+{\Lambda^3}_{\mathbf{2}}$ |
| $S^2+S^4$ | | $3\Lambda^2+{S^2}_{\mathbf{2}}$ | | | |
| $2S^4$ | | $2\Lambda^2+\mathrm{Ad}_{\mathbf{3}}$ | | | $\mathrm{SL}_8$ |
| $S^5$ | | $2V+2V^*+2{\Lambda^2}_{\mathbf{2}}$ | | | $2V+\Lambda^3$ |
| $S^6$ | | $3V+V^*+2{\Lambda^2}_{\mathbf{2}}$ | | | $V+V^*+\Lambda^3$ |
| $2{S^3}_{\mathbf{2}}$ | | $4V+2{\Lambda^2}_{\mathbf{3}}$ | | | $2V^*+\Lambda^3$ |

| $\mathrm{SL}_n$ | | |
|---|---|---|
| $n \geq 3$ | $n \geq 4$ | $n \geq 5$ |
| $nV+nV^*$ | $nV^*+\Lambda^2, n$ even | $kV+lV^*+2{\Lambda^2}_{\mathbf{max}(\lfloor \frac{n}{2} \rfloor, n-l-1)}$, |
| $nV^*+S^2$ | $V+nV^*+\Lambda^2$ | $k+l=4, l \leq n-2.$ |
| $V+(n-1)V^*+S^2$ | $3V+(n-1)V^*+\Lambda^2$ | $4V+\Lambda^2+{(\Lambda^2)^*}_{\frac{n+2}{2}}, n$ even |
| $2V+rV^*+S^2, r \leq n-3$ | $4V+rV^*+\Lambda^2, r \leq n-3$ | $3V+V^*+\Lambda^2+{(\Lambda^2)^*}_{\lceil \frac{n}{2} \rceil}$ |
| $V+2S^2$ | $2V+nV^*+{\Lambda^2}_{\mathbf{2}}$ | $2V+2V^*+\Lambda^2+{(\Lambda^2)^*}_{\lfloor \frac{n}{2} \rfloor}$ |
| $V^*+2S^2$ | $4V+(n-2)V^*+{\Lambda^2}_{\mathbf{2}}$ | $2V+S^2+{(\Lambda^2)^*}_{\frac{n+2}{2}}, n$ even |
| $V+S^2+(S^2)^*$ | $2V+\Lambda^2+{S^2}_{n-1}$ | $V+V^*+S^2+{(\Lambda^2)^*}_{\lceil \frac{n}{2} \rceil}$ |
| $V+V^*+\mathrm{Ad}$ | $V+V^*+\Lambda^2+{S^2}_{n-2}$ | $2V^*+S^2+{(\Lambda^2)^*}_{\lfloor \frac{n}{2} \rfloor}$ |
| $2V+(n-2)V^*+{S^2}_{\mathbf{2}}$ | $2V^*+\Lambda^2+{S^2}_{\mathbf{max}(2,n-3)}$ | $V+S^2+(\Lambda^2)^*, n$ odd |
| $(n+1)V+r{V^*}_{r}, r \leq n$ | | $3V+\Lambda^2+(\Lambda^2)^*, n$ odd |

Apart from our graphical methods, two key techniques are used in the proof: firstly, (a slight modification of) an algorithm by Xin [**107, 108**] for computations of Hilbert series, see Section 3.4. Secondly, the *Crosshair-Sieve algorithm*, developed in Section 3.3 in order to prove that a certain ideal basis is a Gröbner basis. This algorithm aims to construct an optimal monomial ordering with respect to given monomials, on the one hand avoiding difficulties arising from standard monomial orderings, on the other hand without the computational cost of Gröbner fans.

Our final task is to combine all techniques developed throughout the thesis to redraw our initial diagram for *compound Du Val* (cDV) and *canonical* threefold singularities with a torus action of complexity one. Miles Reid introduced cDV singularities in [**79**]. They are the Gorenstein canonical threefold singularities with a general hyperplane section being canonical (and thus an ADE-singularity). Terminal threefold singularities are cyclic quotients of isolated cDV singularities. For more important properties see [**64**, Sec. 5.3].

We first have to classify all such singularities. Note that terminal threefold singularities have been classified by Mori [**69**] and *toric* canonical threefold singularities by Ishida and Iwashita [**59**]. Toric cDV singularities are known due to Dais [**33**].

The case of cDV singularities with a torus action of complexity one is treated in Section 4.1. Our classification yields the following, where the cDV-type of a singularity $x$ is denoted by $S(x_1), \dots, S(x_r) \to cS(x)$, where $S(x_i)$ stands for the ADE-type of singular curves meeting in $x$ and $S(x)$ for that of a general hyperplane section through $x$.

Theorem 9. *The following table provides the equations of the compound Du Val singularities ( in form of the vertex of a quasicone) which are toric (nos. 1 – 3) or non-toric with a torus action of complexity one (nos. 4 – 18).*

| *No.* | *cDV-type* | *equation in* $\mathbb{C}^4$ |
|---|---|---|
| *1* | $A_l \times \mathbb{C}$ | $T_1T_2 + T_3^{l+1}$ |
| *2* | $A_{l_1-1}, A_{l_2-1} \to cA_{l_1+l_2-1}$ | $T_1T_2 + T_3^{l_1}T_4^{l_2}$ |
| *3* | $A_1, A_1, A_1 \to cD_4$ | $T_1^2 + T_2T_3T_4$ |
| *4* | $D_{l+3} \times \mathbb{C}$ | $T_1^2 + T_2^2T_3 + T_3^{l+2}$ |
| *5* | $A_1, A_{l-1} \to cD_{l+4}$ | $T_1^2 + T_2^2T_3 + T_3T_4^{l+2}$ |
| *6* | $E_6 \times \mathbb{C}$ | $T_1^2 + T_2^3 + T_3^4$ |
| *7* | $E_7 \times \mathbb{C}$ | $T_1^2 + T_2^3 + T_2T_3^3$ |
| *8* | $E_8 \times \mathbb{C}$ | $T_1^2 + T_2^3 + T_3^5$ |
| *9a* | $A_{l-1} \to cA_L$ | $T_1T_2 + \left(T_3^{L_1+1} + T_4^{L_2+1}\right)^l$, <br> $L = \min(L_1+1, L_2+1)l - 1$ |
| *9b* | $A_{l_j-1} \to cA_L$ | $T_1T_2 + \prod_{j=1}^{r-1} \left(jT_3^{L_1+1} + (2j-1)T_4^{L_2+1}\right)^{l_j}$, <br> $L = \min(L_i+1) \sum l_j - 1$ |
| *9c* | $A_{L_3-1}, A_{l_j-1} \to cA_L$ | $T_1T_2 + T_3^{L_3} \prod_{j=1}^{r-1} \left(jT_3^{L_1} + (2j-1)T_4^{L_2+1}\right)^{l_j}$, <br> $L = \min(L_3 + L_1 \sum l_j - 1, L_2 \sum l_j - 1)$ |
| *9d* | $A_{L_3-1}, A_{L_4-1}, A_{l_j-1} \to cA_L$ | $T_1T_2 + T_3^{L_3}T_4^{L_4} \prod_{j=1}^{r-1} \left(jT_3^{L_1} + (2j-1)T_4^{L_2}\right)^{l_j}$, <br> $L = \min_{k=3,4}(L_k + l_{k-2} \sum l_j - 1)$ |

| | | |
|---|---|---|
| *10* | $A_{l+1} \to cD_{l+3}$ | $T_1^2 + T_2^2T_3 + T_4^{l+2}$ |
| *11* | $A_{2l+1} \to cD_{2l+2}$ | $T_1^2 + T_2^2T_3 + T_2T_4^{l+1}$ |
| *12* | $A_{l_2-1}, D_{l_1+2} \to cD_{l_1+l_2+2}$ | $T_1^2 + T_2^2T_3 + T_3^{l_1+1}T_4^{l_2}$ |
| *13* | $A_1, A_1 \to cD_{l+3}$ | $T_1^2 + T_2T_3T_4 + T_4^{l+2}$ |
| *14* | $A_1, A_1, A_2 \to cE_6$ | $T_1^2 + T_2^3 + T_3^2T_4^2$ |
| *15* | $D_4 \to cE_6, cE_7$ | $T_1^2 + T_2^3 + T_3^3T_4$ |
| *16* | $A_1, D_4 \to cE_7$ | $T_1^2 + T_2^3 + T_2T_3T_4^2$ |
| *17* | $A_2, D_4 \to cE_8$ | $T_1^2 + T_2^3 + T_3^2T_4^3$ |
| *18* | $E_6 \to cE_8$ | $T_1^2 + T_2^3 + T_3T_4^4$ |

*Here, parameters are integers greater than zero with the exponents containing $L_1$, $L_2$ in nos. 9a to 9d being coprime, $A_0$ means that there is no singularity and $D_l \cong A_l$ for $l \leq 3$.*

In the case of canonical threefold singularities, we restrict to those of Gorenstein index $\imath \geq 2$, since for $\imath = 1$, canonicity is equivalent to log terminality, which makes this class too big for a reasonable classification. For $\imath \geq 2$, integer point freeness of rational polytopes gives manageable combinatorial restrictions. For a first subclass of singularities, we give their Cox rings with respective grading matrix $Q$, representing them as the quotient $\overline{X} /\!\!/ H_X$. Their Gorenstein index $\imath$ and another invariant, the *canonical multiplicity* $\zeta$, are given as well. The canonical multiplicity appears in our diagram from the beginning as well - as first superscript of the exceptional singularities $D^{2,\imath}_{(n+3)/2}$ and $E_6^{3,\imath}$ - see also the original classification of surface singularities by Brieskorn [**23**, Satz 2.9]. We refer to Section 1.4 for details on this invariant.

Theorem 10. *Let $X$ be a canonical threefold singularity of Gorenstein index $\imath \geq 2$ admitting a two-torus action. The following table lists those $X$ that are either sporadic or belong to a series of singularities with up to three parameters.*

| *No.* | $\mathcal{R}(X)$ | $\mathrm{Cl}(X)$ | $Q$ | $\imath$ | $\zeta$ |
|---|---|---|---|---|---|
| *1* | $\mathbb{C}[T_1, \ldots, T_4]$ | $\mathbb{Z} \times \mathbb{Z}/2\mathfrak{d}\mathbb{Z}$, $m, n \in \mathbb{Z}_{\geq 1}$ $\mathfrak{d} = \gcd(2m, m+n)$ | $\begin{bmatrix} -\frac{m+n}{\mathfrak{d}} & \frac{2n}{\mathfrak{d}} & -\frac{m+n}{\mathfrak{d}} & \frac{2m}{\mathfrak{d}} \\ \overline{\alpha_1+\mathfrak{d}} & \overline{-(2\alpha_1+\alpha_2)} & \overline{\alpha_1} & \overline{\alpha_2} \end{bmatrix}$ with $2m\alpha_1 + (m+n)\alpha_2 = \mathfrak{d}$ | 2 | 1 |
| *2* | $\mathbb{C}[T_1, \ldots, T_3]$ | $\mathbb{Z}/\imath m\mathbb{Z}$, $m, n \in \mathbb{Z}_{\geq 1}$, $\gcd(n, \imath) = 1$ | $\begin{bmatrix} \overline{1} & \overline{m\alpha_1} & \overline{-1} \end{bmatrix}$ with $n\alpha_1 \equiv 1 \mod \imath$ | $\geq 2$ | 1 |
| *3* | $\mathbb{C}[T_1, \ldots, T_3]$ | $\mathbb{Z}/4m\mathbb{Z}$, $m \in \mathbb{Z}_{\geq 2}$ | $\begin{bmatrix} \overline{2} & \overline{2m-1} & \overline{-1} \end{bmatrix}$ | 2 | 1 |
| *4* | $\mathbb{C}[T_1, \ldots, T_3]$ | $\mathbb{Z}/10\mathbb{Z}$ | $\begin{bmatrix} \overline{1} & \overline{1} & \overline{3} \end{bmatrix}$ | 2 | 1 |
| *5* | $\mathbb{C}[T_1, \ldots, T_3]$ | $\mathbb{Z}/9\mathbb{Z}$ | $\begin{bmatrix} \overline{1} & \overline{4} & \overline{7} \end{bmatrix}$ | 3 | 1 |
| *6* | $\frac{\mathbb{C}[T_1, \ldots, T_4]}{\langle T_1^3T_2 + T_3^3 + T_4^2 \rangle}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} \overline{1} & \overline{1} & \overline{0} & \overline{1} \end{bmatrix}$ | 2 | 1 |

| | | | | | |
|---|---|---|---|---|---|
| *7* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^3T_2+T_3^3T_4+T_5^2\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} 3 & -1 & 3 & -1 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}$ | 2 | 1 |
| *8* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^3+T_2^2T_3+T_4^2\rangle}$ | $\mathbb{Z}/3\mathbb{Z}$ | $\begin{bmatrix} \bar{2} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$ | 3 | 1 |
| *9* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^3+T_2^2+T_3T_4\rangle}$ | $\mathbb{Z}/10\mathbb{Z}$ | $\begin{bmatrix} \bar{2} & \bar{3} & \bar{9} & \bar{7} \end{bmatrix}$ | 2 | 1 |
| *10* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^4+T_2^3+T_3^2\rangle}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} \bar{1} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ | 2 | 2 |
| *11* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^4+T_2^3T_3+T_4^2\rangle}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}$ | 2 | 2 |
| *12* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^4+T_2^3T_3+T_4^2\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} 5 & 7 & -1 & 10 & -1 \\ \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ | 2 | 2 |
| *13* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^4+T_2^3+T_3^2\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} -6 & -8 & -12 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$ | 2 | 2 |
| *14* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^4+T_2^3T_3T_4+T_5^2\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} 4 & 6 & -1 & -1 & 8 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ | 2 | 2 |
| *15* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^3+T_2^3+T_3^2\rangle}$ | $\mathbb{Z}/3\mathbb{Z}$ | $\begin{bmatrix} \bar{1} & \bar{2} & \bar{0} & \bar{2} \end{bmatrix}$ | 3 | 3 |
| *16* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^3+T_2^3+T_3^2T_4\rangle}$ | $\mathbb{Z}/3\mathbb{Z}$ | $\begin{bmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{2} \end{bmatrix}$ | 3 | 3 |
| *17* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^4+T_2^2+T_3^2T_4\rangle}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\begin{bmatrix} \bar{1} & \bar{0} & \bar{3} & \bar{2} \end{bmatrix}$ | 2 | 2 |
| *18* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^4+T_2^2+T_3^2T_4T_5\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$ | $\begin{bmatrix} 1 & 2 & 3 & -1 & -1 \\ \bar{0} & \bar{2} & \bar{0} & \bar{3} & \bar{1} \end{bmatrix}$ | 2 | 2 |
| *19* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^4+T_2^2+T_3^2T_4\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} 3 & 6 & 7 & -2 & -2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$ | 2 | 2 |
| *20* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^4+T_2^4+T_3T_4\rangle}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\begin{bmatrix} \bar{5} & \bar{7} & \bar{1} & \bar{3} \end{bmatrix}$ | 2 | 2 |
| *21* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^4+T_2^2+T_3T_4\rangle}$ | $\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix} \bar{2} & \bar{1} & \bar{3} & \bar{5} \end{bmatrix}$ | 2 | 2 |
| *22* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^8+T_2^2+T_3T_4\rangle}$ | $\mathbb{Z}/10\mathbb{Z}$ | $\begin{bmatrix} \bar{6} & \bar{9} & \bar{7} & \bar{1} \end{bmatrix}$ | 2 | 2 |
| *23* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^2+T_2^2+T_3T_4\rangle}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\begin{bmatrix} \bar{5} & \bar{1} & \bar{7} & \bar{3} \end{bmatrix}$ | 4 | 2 |
| *24* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^3+T_2^3+T_3T_4\rangle}$ | $\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix} \bar{3} & \bar{1} & \bar{4} & \bar{5} \end{bmatrix}$ | 3 | 3 |
| *25* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^5+T_2^4+T_3T_4\rangle}$ | $\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix} \bar{4} & \bar{5} & \bar{1} & \bar{1} \end{bmatrix}$ | 2 | 3 |
| *26* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^{k-1}+T_2T_3+T_4T_5\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/2k\mathbb{Z}$ | $\begin{bmatrix} 0 & 1 & -1 & -1 & 1 \\ \bar{1} & \bar{k} & \overline{-1} & \overline{k-1} & \bar{0} \end{bmatrix}$ | 2 | 1 |
| *27* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^{2k}+T_2^2+T_3^2\rangle}$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | $\begin{bmatrix} \bar{1} & \overline{k+1} & \bar{k} & \bar{1} \\ \bar{1} & \bar{k} & \overline{k+1} & \bar{0} \end{bmatrix}$ | 2 | 2 |
| *28* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^{2k}+T_2^2+T_3^2\rangle}$ | $\mathbb{Z}\times(\mathbb{Z}/2\mathbb{Z})^2$ | $\begin{bmatrix} 2 & 2k & 2k & -1 & -1 \\ \bar{1} & \overline{k+1} & \bar{k} & \bar{0} & \bar{1} \\ \bar{1} & \bar{k} & \overline{k+1} & \bar{0} & \bar{0} \end{bmatrix}$ | 2 | 2 |
| *29* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^{2k+1}+T_2^2+T_3^2\rangle}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\begin{bmatrix} \bar{2} & \overline{1-2k} & \overline{-1-2k} & \bar{1} \end{bmatrix}$ | 4 | 2 |
| *30* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^{k\zeta-2}+T_2^2+T_3T_4\rangle}$ | $\mathbb{Z}/2k\mathbb{Z}$ | $\begin{bmatrix} \bar{1} & \overline{k(1-\zeta/2)-1} & \overline{k\zeta-1} & \bar{1} \end{bmatrix}$ | 2 | $\in 4\mathbb{Z}$ |
| *31* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^{5\zeta-2}+T_2^2+T_3T_4\rangle}$ | $\mathbb{Z}/10\mathbb{Z}$ | $\begin{bmatrix} \bar{4} & \bar{1} & \bar{3} & \bar{9} \end{bmatrix}$ | 2 | $\geq 3$ |
| *32* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^{3\zeta-4}+T_2^4+T_3T_4\rangle}$ | $\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix} \bar{2} & \bar{1} & \bar{5} & \bar{5} \end{bmatrix}$ | 2 | $\in 2\mathbb{Z}+1$ |
| *33* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^{3\zeta-2}+T_2^2+T_3T_4\rangle}$ | $\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix} \bar{2} & \bar{1} & \bar{3} & \bar{5} \end{bmatrix}$ | 2 | $\geq 3$ |
| *34* | $\frac{\mathbb{C}[T_1,\dots,T_6]}{\left\langle \begin{matrix} T_1^{3\zeta-2}+T_2^2+T_3T_4 \\ 2T_2^2+T_3T_4+T_5T_6 \end{matrix} \right\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix} 0 & 0 & 1 & -1 & -1 & 1 \\ \bar{2} & \bar{1} & \bar{3} & \bar{5} & \bar{2} & \bar{0} \end{bmatrix}$ | 2 | $\geq 3$ |

| | | | | | |
|---|---|---|---|---|---|
| *35* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^{3\zeta-2}+T_2^2+T_3T_4\rangle}$ | $\mathbb{Z}/12\mathbb{Z}$ | $\begin{bmatrix}\overline{4} & \overline{2} & \overline{5} & \overline{11}\end{bmatrix}$ | 2 | $\geq 3$ |
| *36* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle (T_1T_2)^{3\zeta-2}+T_3^2+T_4T_5\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}-1 & 1 & 0 & 1 & -1\\ \overline{2} & \overline{0} & \overline{1} & \overline{3} & \overline{5}\end{bmatrix}$ | 2 | $\geq 3$ |
| *37* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^{3\zeta-2}T_2^{\zeta-2}+T_3^2+T_4T_5\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix}-1 & 3 & -2 & 3 & -7\\ \overline{1} & \overline{1} & \overline{1} & \overline{0} & \overline{0}\end{bmatrix}$ | 2 | $\geq 3$ |
| *38* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^{2\zeta-2}+T_2^2+T_3T_4\rangle}$ | $\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}\overline{5} & \overline{5} & \overline{3} & \overline{1}\end{bmatrix}$ | 3 | $\in 6\mathbb{Z}-1$ |
| *39* | $\frac{\mathbb{C}[T_1,\dots,T_6]}{\left\langle \begin{smallmatrix} T_1^{2\zeta-2}+T_2^2+T_3T_4 \\ 2T_2^2+T_3T_4+T_5T_6\end{smallmatrix}\right\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}0 & 0 & 1 & -1 & -1 & 1\\ \overline{5} & \overline{5} & \overline{3} & \overline{1} & \overline{4} & \overline{0}\end{bmatrix}$ | 3 | $\in 6\mathbb{Z}-1$ |
| *40* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^{2\zeta-2}+T_2^2+T_3T_4\rangle}$ | $\mathbb{Z}/12\mathbb{Z}$ | $\begin{bmatrix}\overline{10} & \overline{10} & \overline{7} & \overline{1}\end{bmatrix}$ | 3 | $\in 6\mathbb{Z}-1$ |
| *41* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^{2\zeta-2}+T_2^2+T_3T_4\rangle}$ | $\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}\overline{5} & \overline{5} & \overline{2} & \overline{2}\end{bmatrix}$ | 3 | $\in 6\mathbb{Z}-1$ |
| *42* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle (T_1T_2)^{2\zeta-2}+T_3^2+T_4T_5\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}1 & -1 & 0 & -1 & 1\\ \overline{0} & \overline{5} & \overline{5} & \overline{2} & \overline{2}\end{bmatrix}$ | 3 | $\in 6\mathbb{Z}-1$ |
| *43* | $\frac{\mathbb{C}[T_1,\dots,T_6]}{\langle (T_1T_2)^{2\zeta-2}+(T_3T_4)^2+T_5T_6\rangle}$ | $\mathbb{Z}^2\times\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}1 & -1 & 0 & 0 & -1 & 1\\ 0 & 0 & -1 & 1 & 1 & -1\\ \overline{0} & \overline{5} & \overline{5} & \overline{0} & \overline{2} & \overline{2}\end{bmatrix}$ | 3 | $\in 6\mathbb{Z}-1$ |
| *44* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle (T_1T_2)^{2\zeta-2}+T_3^2+T_4T_5\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}1 & -1 & 0 & -2 & 2\\ \overline{0} & \overline{5} & \overline{5} & \overline{1} & \overline{3}\end{bmatrix}$ | 3 | $\in 6\mathbb{Z}-1$ |
| *45* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^{2\zeta-2}+(T_2T_3)^2+T_4T_5\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}0 & -1 & 1 & 1 & -1\\ \overline{5} & \overline{5} & \overline{0} & \overline{3} & \overline{1}\end{bmatrix}$ | 3 | $\in 6\mathbb{Z}-1$ |
| *46* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^{2\zeta-2}+(T_2T_3)^2+T_4T_5\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}0 & -1 & 1 & 2 & -2\\ \overline{5} & \overline{5} & \overline{0} & \overline{3} & \overline{1}\end{bmatrix}$ | 3 | $\in 6\mathbb{Z}-1$ |
| *47* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^{2\zeta-2}T_2^{\zeta-2}+T_3^2+T_4T_5\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$ | $\begin{bmatrix}-1 & 2 & -1 & 1 & -3\\ \overline{0} & \overline{2} & \overline{0} & \overline{1} & \overline{2}\end{bmatrix}$ | 3 | $\in 6\mathbb{Z}-1$ |
| *48* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^{2\zeta-3}+T_2^3+T_3T_4\rangle}$ | $\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}\overline{3} & \overline{5} & \overline{2} & \overline{1}\end{bmatrix}$ | 3 | $\geq 4$ |
| *49* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^{2\zeta-3}+(T_2T_3)^3+T_4T_5\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}0 & -1 & 1 & 1 & -1\\ \overline{3} & \overline{5} & \overline{0} & \overline{2} & \overline{1}\end{bmatrix}$ | 3 | $\geq 4$ |
| *50* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^{2\zeta-3}+T_2^3+T_3T_4\rangle}$ | $\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}\overline{1} & \overline{1} & \overline{4} & \overline{5}\end{bmatrix}$ | 3 | $\in 9\mathbb{Z}$, $9\mathbb{Z}-3$ |
| *51* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^{2\zeta-3}+(T_2T_3)^3+T_4T_5\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}0 & -1 & 1 & 1 & -1\\ \overline{1} & \overline{1} & \overline{0} & \overline{4} & \overline{5}\end{bmatrix}$ | 3 | $\in 9\mathbb{Z}$, $9\mathbb{Z}-3$ |
| *52* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle (T_1T_2)^{2\zeta-3}+T_3^3+T_4T_5\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}-1 & 1 & 0 & 1 & -1\\ \overline{1} & \overline{0} & \overline{1} & \overline{4} & \overline{5}\end{bmatrix}$ | 3 | $\in 9\mathbb{Z}$, $9\mathbb{Z}-3$ |
| *53* | $\frac{\mathbb{C}[T_1,\dots,T_4]}{\langle T_1^{2\zeta-4}+T_2^4+T_3T_4\rangle}$ | $\mathbb{Z}/6\mathbb{Z}$ | $\begin{bmatrix}\overline{1} & \overline{1} & \overline{5} & \overline{5}\end{bmatrix}$ | 3 | $\in 12\mathbb{Z}+1$, $12\mathbb{Z}-5$ |
| *54* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^{k\zeta-1}+T_2T_3+T_4T_5\rangle}$ | $\mathbb{Z}\times\mathbb{Z}/2k\mathbb{Z}$ | $\begin{bmatrix}0 & 1 & -1 & -1 & 1\\ \overline{1} & \overline{k} & \overline{-1} & \overline{k-1} & \overline{0}\end{bmatrix}$ | 2 | $\in 2\mathbb{Z}+1$ |
| *55* | $\frac{\mathbb{C}[T_1,\dots,T_5]}{\langle T_1^{\zeta-2}+T_2^2+T_3T_4\rangle}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\begin{bmatrix}\overline{2} & \overline{\zeta-2} & \overline{\zeta} & \overline{\zeta}\end{bmatrix}$ | 4 | $\in 2\mathbb{Z}+1$ |

The remaining canonical threefold singularities with a two-torus action come in large series. We present them in the form of a *defining matrix* $P$, of which the columns represent ray generators of a canonical ambient toric variety $Z$, having the same divisor class group as $X$. Thus for a certain $P$, one can compute $\mathrm{Cl}(X)$ explicitly. The relations of the Cox ring of $X$ in $\mathcal{R}(Z)$ are given by the trinomials from page 4 with exponents given by the entries in the first $r$ rows of $P$. For details we refer to Chapter 1.

THEOREM 11. *Let $X$ be a canonical threefold singularity of Gorenstein index $\imath \geq 2$ admitting a two-torus action. The following table lists those $X$ that belong to a series with more than three parameters. It holds $\imath = 2$ for all but No. 59, where we have $\imath \geq 2$. For the canonical multiplicity, we have $\zeta = 4$ for No. 56, $\zeta = 2$ for No. 57 and $\zeta > \imath$ for Nos. 58 and 59. Moreover, in all cases we have $r \geq 2$, $\mathbf{k} \in \mathbb{Z}^t_{\geq 1}$, $\mathbf{d_0} \in \mathbb{Z}^t$ for some $t \in \mathbb{Z}_{\geq 1}$.*

$$56a : \begin{bmatrix} -2\mathbf{k}-1 & 2 & 0 & 0 & 0 & \dots & 0 & 0 \\ -2\mathbf{k}-1 & 0 & 2 & 0 & 0 & \dots & 0 & 0 \\ -2\mathbf{k}-1 & 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2\mathbf{k}-1 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ \mathbf{d_0} & 1 & 1 & 0 & d_3 & \dots & 0 & d_r \\ \mathbf{k} & 0 & 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \qquad 56b : \begin{bmatrix} -2\mathbf{k}-1 & 2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ -2\mathbf{k}-1 & 0 & 2 & 2 & 0 & 0 & \dots & 0 & 0 \\ -2\mathbf{k}-1 & 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2\mathbf{k}-1 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ \mathbf{d_0} & 1 & 1 & d_2 & 0 & d_3 & \dots & 0 & d_r \\ \mathbf{k} & 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$57a : \begin{bmatrix} -\mathbf{k} & 2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -\mathbf{k} & 0 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -\mathbf{k} & 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -\mathbf{k} & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0 \\ \mathbf{d_0} & 1 & 1 & 0 & d_3 & \dots & 0 & d_r & d'_1 & d'_2 \\ 1-\mathbf{k} & 0 & 2 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \end{bmatrix} \qquad 57b : \begin{bmatrix} -2 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -2 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & d_2 & \dots & 0 & d_r & d'_1 & d'_2 \\ 0 & 2 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$58a : \begin{bmatrix} 2(\zeta-k) & 2k & 0 & 0 & \dots & 0 & 0 \\ 2(\zeta-k) & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(\zeta-k) & 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 1 & 0 & d_2 & \dots & 0 & d_r \\ 2\left(\frac{k\mu+1}{\zeta}-\mu\right) & 2\frac{k\mu+1}{\zeta} & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \qquad 58b : \begin{bmatrix} 2(\zeta-k) & 2k-\zeta & 2k & 0 & 0 & \dots & 0 & 0 \\ 2(\zeta-k) & 2k-\zeta & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(\zeta-k) & 2k-\zeta & 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 1 & d_{11} & 0 & d_2 & \dots & 0 & d_r \\ 2\left(\frac{k\mu+1}{\zeta}-\mu\right) & 2\frac{k\mu+1}{\zeta}-\mu & 2\frac{k\mu+1}{\zeta} & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$58c : \begin{bmatrix} 2(\zeta-k) & 2k-\zeta & 2k-\zeta & 2k & 0 & 0 & \dots & 0 & 0 \\ 2(\zeta-k) & 2k-\zeta & 2k-\zeta & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(\zeta-k) & 2k-\zeta & 2k-\zeta & 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & d_{11} & d_{12} & 1 & 0 & d_2 & \dots & 0 & d_r \\ 2\left(\frac{k\mu+1}{\zeta}-\mu\right) & 2\frac{k\mu+1}{\zeta}-\mu & 2\frac{k\mu+1}{\zeta}-\mu & 2\frac{k\mu+1}{\zeta} & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$59a : \begin{bmatrix} -k\imath & \mathfrak{l} & 0 & 0 & \cdots & 0 & 0 \\ -k\imath & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -k\imath & 0 & 0 & 0 & \cdots & 1 & 1 \\ d & d & 0 & d_2 & \cdots & 0 & d_r \\ \imath\frac{1-\mu k}{\mathfrak{l}+k\imath} & \frac{\imath+\mu\mathfrak{l}}{\mathfrak{l}+k\imath} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad 59b : \begin{bmatrix} -k\imath & \mathfrak{l} & \mathfrak{l} & 0 & 0 & \cdots & 0 & 0 \\ -k\imath & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -k\imath & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ d & d & d+d_1 & 0 & d_2 & \cdots & 0 & d_r \\ \imath\frac{1-\mu k}{\mathfrak{l}+k\imath} & \frac{\imath+\mu\mathfrak{l}}{\mathfrak{l}+k\imath} & \frac{\imath+\mu\mathfrak{l}}{\mathfrak{l}+k\imath} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

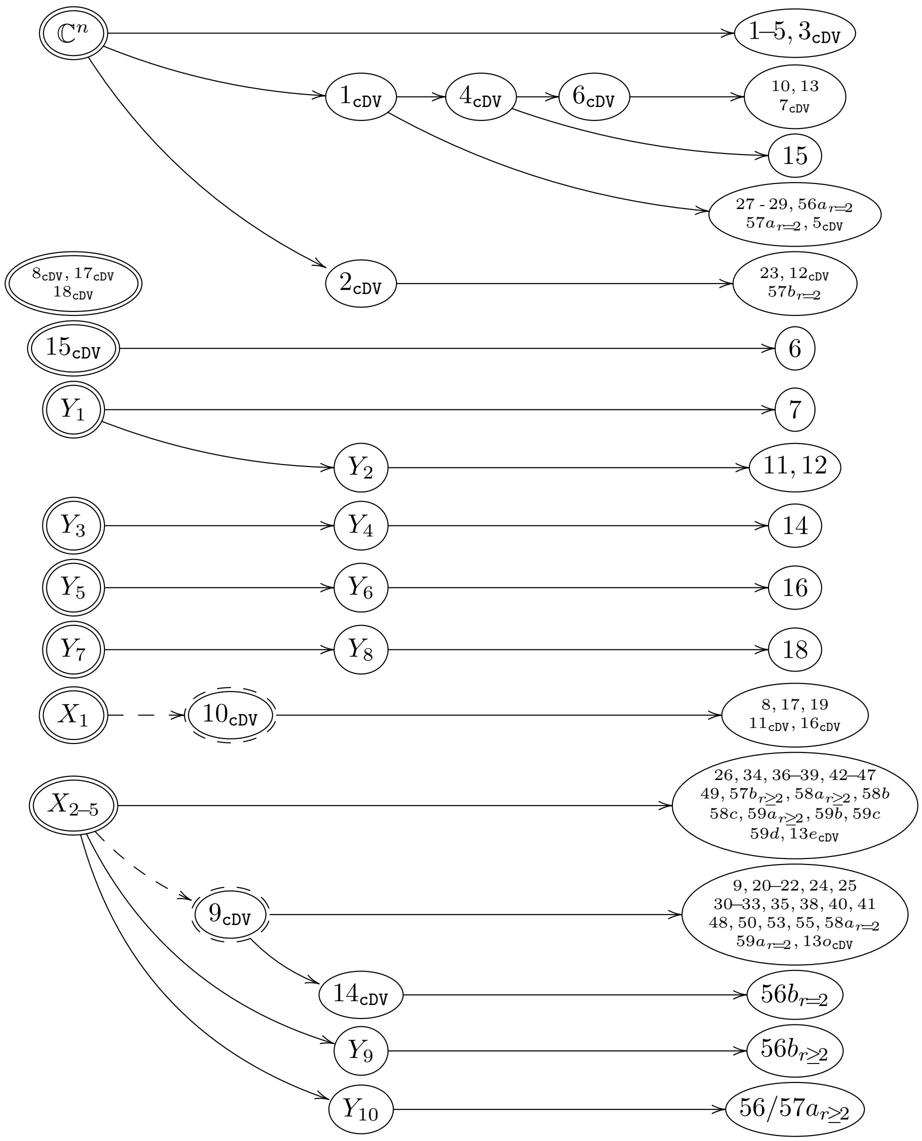
$$59c : \begin{bmatrix} -k\imath & -k\imath & \mathfrak{l} & 0 & 0 & \cdots & 0 & 0 \\ -k\imath & -k\imath & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -k\imath & -k\imath & 0 & 0 & 0 & \cdots & 1 & 1 \\ d & d+d_0 & d & 0 & d_2 & \cdots & 0 & d_r \\ \imath\frac{1-\mu k}{\mathfrak{l}+k\imath} & \imath\frac{1-\mu k}{\mathfrak{l}+k\imath} & \frac{\imath+\mu\mathfrak{l}}{\mathfrak{l}+k\imath} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad 59d : \begin{bmatrix} -k\imath & -k\imath & \mathfrak{l} & \mathfrak{l} & 0 & 0 & \cdots & 0 & 0 \\ -k\imath & -k\imath & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -k\imath & -k\imath & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ d & d+d_0 & d & d+d_1 & 0 & d_2 & \cdots & 0 & d_r \\ \imath\frac{1-\mu k}{\mathfrak{l}+k\imath} & \imath\frac{1-\mu k}{\mathfrak{l}+k\imath} & \frac{\imath+\mu\mathfrak{l}}{\mathfrak{l}+k\imath} & \frac{\imath+\mu\mathfrak{l}}{\mathfrak{l}+k\imath} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

The terminal ones among these singularities are identified by the following:

COROLLARY 2. *Let $X$ be a* terminal *threefold singularity of Gorenstein index $\imath \geq 2$ admitting a two-torus action. Then $X$ either is No. 2 with $m = 1$ or one of Nos. 59a-d with $d_i = 1$ for all $i$.*

Now we are ready to finally present the joint *Cox ring iteration tree* for canonical and compound Du Val threefold singularities.

THEOREM 12. *Denote by* 1−59 *the canonical singularities from Theorems 10, 11 and by* $1_{\mathtt{cDV}}$−$18_{\mathtt{cDV}}$ *the compound Du Val singularities from Theorem 9. Let the canonical singularities $Y_1$−$Y_{10}$ and $X_1$−$X_5$ with torus action of complexity one be given as in Definition 4.4.10 on page 175. We have the following complete tree* *of Cox ring iterations for these singularities, where arrows indicate total coordinate spaces and double frames factorial singularities. For dashed double frames and arrows, factoriality depends on parameters.*




COROLLARY 3. *All roots of the tree from Theorem 12 are generalized compound Du Val in the sense that they have a hyperplane section with at most canonical singularities. In particular, every three-dimensional singularity of complexity one that is either canonical of Gorenstein index $\imath \geq 2$ or compound Du Val is a quotient of a factorial generalized compound Du Val singularity of complexity one.*

CHAPTER 1

# Preliminaries

In this preliminary chapter, we discuss Mori Dream Spaces and their Cox rings, in particular rational varieties with an effective torus action of complexity one - i.e. where the dimension of the acting torus is one less than the one of the variety. We will call them *$T$-varieties of complexity one* in the following. Unless stated otherwise, we will work over the field $\mathbb{C}$ of complex numbers.

While in the first section, we investigate MDS and Cox rings in general, we will focus on Cox rings of $T$-varieties of complexity one in the second section. A standard reference for these first two sections is [**6**]. Section 1.3 then investigates properties of singularities of $T$-varieties of complexity one, mainly those that are reflected in the Cox ring. Section 1.4 follows up on this task, in particular by introducing a new invariant to the canonical divisor of a variety of complexity one, the *canonical multiplicity* $\zeta$, which provides more structure for the Cox rings of such varieties. These last two sections 1.3 and 1.4 contain new results that have been published in the joint work [**5**] with Arzhantsev, Hausen and Wrobel.

## 1.1. Mori Dream Spaces and Cox rings

Let $X$ be a normal algebraic variety over the field $\mathbb{C}$ of complex numbers. A *prime divisor* is a 1-codimensional irreducible subvariety $D \subseteq X$. By $\mathrm{WDiv}(X)$, we denote the free abelian group generated by the prime divisors, with elements called *Weil divisors*. To a nonzero rational function $f \in \mathbb{C}(X)^*$, we associate the Weil divisor

$$\mathrm{div}(f) := \sum_{D \text{ prime}} \nu_D(f) \cdot D,$$

where the coefficients $\nu_D(f)$ are the vanishing or pole orders of $f$ along $D$. We call a Weil divisor $D$ *principal*, if $D = \mathrm{div}(f)$ for some $f$. If $D$ is locally principal, we call it *Cartier*. Cartier and principal divisors generate subgroups of $\mathrm{WDiv}(X)$, which we denote by $\mathrm{CaDiv}(X)$ and $\mathrm{PDiv}(X)$ respectively. We then have the quotient groups $\mathrm{Pic}(X) = \mathrm{CaDiv}(X)/\mathrm{PDiv}(X)$ and $\mathrm{Cl}(X) = \mathrm{WDiv}(X)/\mathrm{PDiv}(X)$, the *Picard group* and *divisor class group*. We use the following notions: if $\mathrm{Cl}(X)$ is trivial (torsion), we say $X$ is factorial (almost factorial). If $\mathrm{Cl}(X)/\mathrm{Pic}(X)$ is trivial (torsion), we say that $X$ is locally factorial ($\mathbb{Q}$-factorial).

For a Weil divisor $D$, we can define its divisorial sheaf $\mathcal{O}_X(D)$ by setting

$$\Gamma(U, \mathcal{O}_X(D)) := \{f \in \mathbb{C}(X)^* \mid (\mathrm{div}(f) + D)|_U \geq 0\} \cup \{0\}$$

for open $U \subseteq X$. Since $f_1 \cdot f_2 \in \Gamma(U, \mathcal{O}_X(D_1 + D_2))$ for $f_i \in \Gamma(U, \mathcal{O}_X(D_i))$, to a subgroup $K \subseteq \mathrm{WDiv}(X)$ we can associate its *sheaf of divisorial algebras*:

$$\mathcal{S} := \bigoplus_{D \in K} \mathcal{O}_X(D).$$

Now in principle, Cox sheaves are such sheaves of $K$-graded $\mathcal{O}_X$-algebras for certain subgroups $K$ of $\mathrm{WDiv}(X)$ and Cox rings are their rings of global sections. In the original definition of [**57**], a free Picard group was assumed and $K$ chosen so that the canonical map sending a divisor $D$ to its class $[D]$ in $\mathrm{Cl}(X)$ would yield an isomorphism $K \to \mathrm{Pic}(X)$. In [**13**], the freeness assumption on $\mathrm{Pic}(X)$ was dropped. Then in [**38**] and [**14**], a free class group with an isomorphism $K \to \mathrm{Cl}(X)$ was studied. Finally in [**49**], Hausen considered the most general case of $\mathrm{Cl}(X)$ being allowed to have torsion. Here, we take this most general viewpoint from [**49**] that is also adopted in the comprehensive work [**6**]. Note that apart from this, even if $\mathrm{Cl}(X)$ has torsion, $K$ is sometimes taken to be isomorphic to the free part of $\mathrm{Cl}(X)$, e.g. in [**43**].

From now on assume $X$ is a normal algebraic variety with only constant global invertible functions and finitely generated divisor class group. Following [**6**, Sec. 1.4.2], we fix a subgroup $K \subseteq \mathrm{WDiv}(X)$ such that the canonical map $K \to \mathrm{Cl}(X)$ is surjective. Denote by $K^0$ its kernel and let $\chi : K^0 \to \mathbb{C}(X)^*$ be a character, such that $\mathrm{div}(\chi(D)) = D$ for $D \in K^0$. Let $\mathcal{S}$ be the sheaf of divisorial algebras defined as above associated to $K$ and let $\mathcal{I}$ be the sheaf of ideals of $\mathcal{S}$ generated by sections $1 - \chi(D)$ for $D \in K^0$. The quotient sheaf $\mathcal{R} := \mathcal{S}/\mathcal{I}$ is $\mathrm{Cl}(X)$-graded with homogeneous parts

$$\mathcal{R}_{[D]} := \pi\left(\bigoplus_{D' \in [D]} \mathcal{S}_{D'}\right),$$

with $\pi\colon \mathcal{S} \to \mathcal{R}$ denoting the projection. We call this sheaf the *Cox sheaf* of $X$. The *Cox ring* is its ring of global sections

$$\mathcal{R}(X) := \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{R}_{[D]}(X),$$

with $\mathrm{Cl}(X)$-homogeneous parts $\mathcal{R}_{[D]}(X) := \Gamma(X, \mathcal{R}_{[D]})$.

If $\mathcal{R}(X)$ is a finitely generated $\mathbb{C}$-algebra, $X$ is called a Mori Dream Space (MDS) due to [**57**]. It was shown in [**57**] that this property is equivalent to the possibility of running a $D$-MMP for any divisor $D$ and that any sequence of $D$-flips terminates [**57**, Def. 1.10, Prop. 2.9]. Their definition was restricted to projective varieties, while here we will call any normal algebraic variety with finitely generated divisor class group and Cox ring an MDS. From now on, $X$ is assumed to be an MDS if not stated otherwise.

Cox rings enjoy many nice properties. We focus on two of them, that are of special importance for us in the following. If $\mathrm{Cl}(X)$ is free, then $\mathcal{R}(X)$ is factorial (i.e. a unique factorization domain), see [**13, 38, 4**]. In [**4**], this was deduced from the more general property of being *factorially graded*, which also holds for $\mathcal{R}(X)$ if $\mathrm{Cl}(X)$ has torsion.

Definition 1.1.1. Let $K$ be a finitely generated abelian group and $R$ be a $K$-graded integral $\mathbb{C}$-algebra.

- A homogeneous nonzero non-unit $f \in R$ is called *$K$-prime*, if for any homogeneous elements $h, g$ with $f | g \cdot h$, we either have $f|g$ or $f|h$.
- We say that $R$ is *factorially $K$-graded*, if any homogeneous nonzero non-unit has a unique (up to association) representation as a product of $K$-primes.

An important observation from [**9**, Thm. 1.5] is that being factorially $K$-graded does not depend on the free part of $K$:

THEOREM 1.1.2. *Let $R$ be a $K \oplus \mathbb{Z}^m$-graded integral $\mathbb{C}$-algebra. Then $R$ is factorially $K \oplus \mathbb{Z}^m$-graded if and only if it is factorially $K$-graded. Moreover, the product representations of homogeneous elements are identical with respect to both gradings.*

Note that factoriality of $\mathcal{R}(X)$ for free $\mathrm{Cl}(X)$ follows directly from Theorem 1.1.2. We come to another important property of $\mathcal{R}(X)$. Recall that a *quasitorus* is a diagonalizable affine algebraic group and can be seen as a direct product of a torus $\mathbb{T} = (\mathbb{C}^*)^n$ and a finite abelian group. Finitely generated abelian groups and quasitori are in a 1-to-1-correspondence by associating to a quasitorus $H$ its group of characters $\mathbb{X}(H)$ (i.e. homomorphisms $H \to \mathbb{C}^*$) and to a finitely generated abelian group $K$ the quasitorus $H := \mathrm{Spec}\,\mathbb{C}[K]$, see [**6**, Thm. 1.2.1.4].

Moreover, $K$-gradings of affine algebras and quasitorus actions on affine varieties are in a 1-to-1-correspondence as well, see [**6**, 1.2.2.4].

For the action of an affine algebraic reductive group $G$ on an affine variety $X = \mathrm{Spec} R$, classical invariant theory defines the ring of invariants

$$R^G := \{f \in R | f(g \cdot x) = f(x) \forall x \in X, g \in G\} \subseteq R,$$

and the induced morphism $\pi\colon X \to Y := \mathrm{Spec}\, R^G$ has the properties of a *good quotient*, i.e. it is affine, $G$-invariant and $\mathcal{O}_Y \cong (\pi_* \mathcal{O}_X)^G$. Good quotients of non-affine varieties are locally of the same form. We denote good quotients by $X /\!\!/ G$. If in addition all fibers are $G$-orbits, we call $X /\!\!/ G$ a geometric quotient and write $X/G$. Finite quotients are geometric.

Now we call $\overline{X} := \mathrm{Spec}\,\mathcal{R}(X)$ the *total coordinate space* of $X$. The $\mathrm{Cl}(X)$-grading induces an action of the *characteristic quasitorus* $H_X := \mathrm{Spec}\,\mathbb{C}[\mathrm{Cl}(X)]$ on $\overline{X}$. Moreover, there is an $H_X$-invariant open subvariety $\widehat{X} \subseteq \overline{X}$ with complement of codimension at least two, the *characteristic space* of $X$, so that there is a good quotient

$$p\colon \widehat{X} \to \widehat{X} /\!\!/ H_X$$

and $X$ is isomorphic to $\widehat{X} /\!\!/ H_X$. All varieties arising from *small* birational modifications of $X$ can be represented as quotients of such subsets with complement of codimension at least two by $H_X$. If $X$ is affine, we have $\widehat{X} = \overline{X}$. The action of $H_X$ is free over the smooth locus of $X$, see [**6**, Prop. 1.6.1.6]. In particular, it is of the following form:

DEFINITION 1.1.3. We say that the action of an affine algebraic group $G$ on a variety $X$ is *strongly stable*, if there is an open $G$-invariant subset $X' \subseteq X$ so that

- $X \setminus X'$ is of codimension at least two.
- $G$ acts freely on $X'$.
- for any $x \in X'$, the orbit $G \cdot x$ is closed in $X'$.

The two properties of factoriality of a $K$-grading and strong stability of the action of the corresponding quasitorus $H$ together with only constant invertible $H$-homogeneous elements are essential for Cox rings, see [**6**, Thm. 1.6.4.3].

On the other hand, if one drops the requirement of factoriality of such grading, one gets so called *quotient presentations* of $X$. We make this notion precise in the following definition, see [**6**, Sec. 4.2.1].

Definition 1.1.4. Let $X$ be a normal algebraic variety with only constant global invertible functions and $\mathrm{Cl}(X)$ finitely generated. A *quotient presentation* of $X$ is a good quotient $\pi\colon \widetilde{X} \xrightarrow{/\!\!/ H} X$, where $H$ acts on $\widetilde{X}$ strongly stably with only constant $H$-homogeneous invertible functions on $\widetilde{X}$.

If $X$ is an MDS, then by [**6**, Thm. 1.2.1.4] we have a 1-to-1-correspondence between quotient presentations of $X$ and subgroups of $\mathrm{Cl}(X)$. In particular, if for a subgroup $L \subseteq \mathrm{Cl}(X)$ we set $\tilde{L} := \mathrm{Cl}(X)/L$ and $H_L := \mathrm{Spec}\,\mathbb{C}[L]$, $H_{\tilde{L}} := \mathrm{Spec}\,\mathbb{C}[\tilde{L}]$, by [**6**, Constr. 4.2.1.2] we have a commutative diagram of quotient presentations

$$\widehat{X} \xrightarrow{/\!\!/ H_{\tilde{L}}} X /\!\!/ H_{\tilde{L}} \xrightarrow{/\!\!/ H_L} X \,, \qquad \widehat{X} \xrightarrow{/\!\!/ H_X} X .$$

Now recall the representation of *toric varieties* $Z$ - i.e. normal algebraic varieties acted on by a torus $\mathbb{T}$ of the same dimension - by polytopal fans $\Sigma$, see [**31**]. The rays of these fans correspond to torus invariant divisors, and these divisors generate the class group $\mathrm{Cl}(Z)$. In view of Cox ring theory, we say that the projection $Q : \mathrm{WDiv}(Z)^{\mathbb{T}} \to \mathrm{Cl}(Z)$ maps every torus invariant divisor to its *degree*. The matrix $P$ with the ray generators of $\Sigma$ standing in its columns maps the ray generators of the $m$-dimensional positive orthant $\Delta$ (which is the cone of the toric variety $\mathbb{C}^m$ - the total coordinate space of $Z$) to the rays of $\Sigma$. Now the characteristic space $\widehat{Z} \subseteq \overline{Z} = \mathbb{C}^m$ is given by the subfan $\overline{\Sigma}$ of $\Delta$ having as cones the preimages of all the cones of $\Sigma$.

Thus the toric variety $Z$ can be represented for example by its Cox ring $\mathbb{C}^m$ with the grading given by $Q$ and the fan $\overline{\Sigma}$. It is also possible to encode $\overline{\Sigma}$ by a combinatorial object in $\mathrm{Cl}(Z)$ called a *bunch of cones*, see [**14, 49**] and also [**6**, Ch. 2, 3], but we do not use this approach directly in the present thesis. In principle, both a bunch of cones in some finitely generated abelian group and the fan $\Sigma$ combinatorially encode the same data, and depending on the goal, it is more useful to work with the one or the other tool, if not with both of them.

Now apart from toric varieties, the key observation is that an MDS that is an $\mathrm{A}_2$-variety - i.e. a variety $X$ such that any two points of $X$ admit a common affine neighbourhood - always admits a *neat* embedding $\imath : X \to Z$ into a toric variety $Z$. Here neat means that preimages of torus invariant prime divisors of $Z$ are prime divisors of $X$ and the induced pullback $\imath^* : \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$ is an isomorphism, see [**6**, Def. 3.2.5.2]. There is a minimal such neat embedding, given by a minimal fan $\Sigma$, where $X$ intersects nontrivially every toric orbit given by a cone of $\Sigma$. We call this the *canonical toric embedding* of $X$.

In this situation, $X$ comes with a decomposition into locally closed subvarieties corresponding to the intersections with the toric orbits of $Z$. In particular, $\mathrm{Cl}(X)$ is generated by the intersections with torus invariant prime divisors of $Z$ and we have closed embeddings $\overline{X} \subseteq \overline{Z}$ and $\widehat{X} \subseteq \widehat{Z}$ of characteristic and total coordinate spaces as well. The Cox ring $\mathcal{R}(X)$ is a quotient of $\mathcal{R}(Z) = \mathbb{C}[x_1, \ldots, x_m]$ by the vanishing ideal $\mathcal{I}(\overline{X})$ and inherits the grading given by $Q$. For more details we refer to [**6**, Sec. 3.1–3.3]. In the following section, we discuss more properties of the canonical toric embedding in the case of rational varieties acted on by a torus of codimension one.

## 1.2. Cox rings of $T$-varieties of complexity one

We consider rational *$T$-varieties of complexity one*, i.e. varieties admitting an effective action of an algebraic torus with general orbits of codimension one. These varieties have finitely generated class group and Cox ring and admit a manageable description in terms of the canonical toric embedding from the last section. We will focus on complete varieties and quasicones and refer to [**52, 50, 53, 6, 5**].

The following construction provides rings $R$ with coarsest gradings by finitely generated abelian groups $K_0$. Downgrading to subgroups $K \subseteq K_0$ will then provide Cox rings for varieties $X$ with Cox ring $R$, characteristic quasitorus $H = \operatorname{Spec} \mathbb{C}[K]$ and an action of the torus $T = H_0/H$ with $H_0 = \operatorname{Spec} \mathbb{C}[K_0]$ of complexity one.

CONSTRUCTION 1.2.1. Fix integers $r, n > 0$, $m \geq 0$ and a partition $n = n_0 + \ldots + n_r$. For each $0 \leq i \leq r$, fix a tuple $l_i \in \mathbb{Z}_{>0}^{n_i}$ and define a monomial

$$T_i^{l_i} \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \ \in \ \mathbb{C}[T_{ij}, S_k;\ 0 \leq i \leq r,\ 1 \leq j \leq n_i,\ 1 \leq k \leq m].$$

We will shortly write $\mathbb{C}[T_{ij}, S_k]$ for the above polynomial ring. Let $A := (a_0, \ldots, a_r)$ be a $2 \times (r+1)$-matrix with pairwise linearly independent columns $a_i \in \mathbb{C}^2$. Set $I := \{0, \ldots, r-2\}$ and for every $i \in I$ define

$$g_i \ := \ \det \begin{bmatrix} T_i^{l_i} & T_{i+1}^{l_{i+1}} & T_{i+2}^{l_{i+2}} \\ a_i & a_{i+1} & a_{i+2} \end{bmatrix} \ \in \ \mathbb{C}[T_{ij}, S_k].$$

We build up an $r \times (n+m)$ matrix from the exponent vectors $l_0, \ldots, l_r$ of these polynomials:

$$P_0 \ := \ \begin{bmatrix} -l_0 & l_1 & & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ -l_0 & 0 & & l_r & 0 & \ldots & 0 \end{bmatrix}.$$

Denote by $P_0^*$ the transpose of $P_0$ and consider the projection

$$Q \colon \mathbb{Z}^{n+m} \ \to \ K_0 \ := \ \mathbb{Z}^{n+m} / \operatorname{im}(P_0^*).$$

Denote by $e_{ij}, e_k \in \mathbb{Z}^{n+m}$ the canonical basis vectors corresponding to the variables $T_{ij}$, $S_k$. Define a $K_0$-grading on $\mathbb{C}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) \ := \ Q(e_{ij}) \ \in \ K_0, \qquad \deg(S_k) \ := \ Q(e_k) \ \in \ K_0.$$

This is the coarsest possible grading of $\mathbb{C}[T_{ij}, S_k]$ leaving the variables and the $g_i$ homogeneous. In particular, we have a $K_0$-graded factor algebra

$$R(A, P_0) \ := \ \mathbb{C}[T_{ij}, S_k] / \langle g_i;\ i \in I \rangle.$$

The $\mathbb{C}$-algebra $R(A, P_0)$ is a complete intersection and admits only constant invertible $K_0$-homogeneous elements. It is also $K_0$-factorial. The generic quotient of the action of $H_0$ on $\operatorname{Spec} R(A, P_0)$ is the projective line. As $X$ is rational if and only if this generic quotient is rational, the two possibilities for the quotient are $\mathbb{P}^1$ and $\mathbb{C}^1$, see [**53**]. But quasicones and complete varieties, the two types we focus on in this thesis, both have generic quotient $\mathbb{P}^1$. The downgradings of $K_0$ leading to Cox rings come from enhancements of the matrix $P_0$ as the following construction shows:

CONSTRUCTION 1.2.2. Let integers $r$, $n = n_0 + \ldots + n_r$, $m$ and data $A$ and $P_0$ as in Construction 1.2.1. Fix $1 \le s \le n + m - r$, choose an integral $s \times (n + m)$ matrix $d$ and build the $(r + s) \times (n + m)$ stack matrix

$$P := \begin{bmatrix} P_0 \\ d \end{bmatrix}.$$

We require the columns of $P$ to be pairwise different primitive vectors generating $\mathbb{Q}^{r+s}$ as a vector space. Let $P^*$ denote the transpose of $P$ and consider the projection

$$Q \colon \mathbb{Z}^{n+m} \to K := \mathbb{Z}^{n+m}/\mathrm{im}(P^*).$$

Denoting as before by $e_{ij}, e_k \in \mathbb{Z}^{n+m}$ the canonical basis vectors corresponding to the variables $T_{ij}$ and $S_k$, we obtain a $K$-grading on $\mathbb{C}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) := Q(e_{ij}) \in K, \qquad \deg(S_k) := Q(e_k) \in K.$$

This $K$-grading coarsens the $K_0$-grading of $\mathbb{C}[T_{ij}, S_k]$ given in Construction 1.2.1. In particular, we have the $K$-graded factor algebra

$$R(A, P) := \mathbb{C}[T_{ij}, S_k]/\langle g_i;\ i \in I\rangle.$$

If the columns of $P$ are the ray generators of a pointed polyhedral cone, then there is a unique affine variety

$$X(A, P) := \mathrm{Spec}\, R(A, P) /\!\!/ \mathrm{Spec}\, \mathbb{C}[K],$$

having $R(A, P)$ as its Cox ring.

So $R(A, P_0)$ and $R(A, P)$ are the same algebra, but with a different grading. Nevertheless, $R(A, P)$ is $K$-factorial as well. Also note that the relations of $R(A, P)$ are slightly more general than the ones from page 4. By appropriately scaling the variables $T_{ij}$, we arrive at the ones from the introduction for any matrix $A$.

REMARK 1.2.3. Consider the defining matrix $P$ of a $K$-graded ring $R(A, P)$ as in Construction 1.2.2. Write $v_{ij} = P(e_{ij})$ and $v_k = P(e_k)$ for the columns of $P$. The $i$-th column block of $P$ is $(v_{i1}, \ldots, v_{in_i})$ and by the data of this block we mean $l_i$ and the $s \times n_i$ block $d_i$ of $d$. We introduce *admissible operations* on $P$:

(i) swap two columns inside a block $v_{i1}, \ldots, v_{in_i}$,
(ii) exchange the data $l_{i_1}, d_{i_1}$ and $l_{i_2}, d_{i_2}$ of two column blocks,
(iii) add multiples of the upper $r$ rows to one of the last $s$ rows,
(iv) any elementary row operation among the last $s$ rows,
(v) swapping among the last $m$ columns.

The operations of type (iii) and (iv) do not change the associated ring $R(A, P)$, while the types (i), (ii), (v) correspond to certain renumberings of the variables of $R(A, P)$ keeping the (graded) isomorphy type.

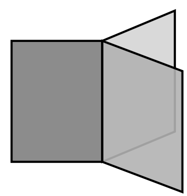
REMARK 1.2.4. If $R(A, P)$ is not a polynomial ring, then we can always assume that $P$ is *irredundant* in the sense that $l_{i1} + \ldots + l_{in_i} > 1$ holds for $i = 0, \ldots, r$.

All Cox rings of rational $T$-varieties $X$ of complexity one that are either complete or quasicones are isomorphic to rings of the form $R(A, P)$. Moreover, if such $X$ is complete (a quasicone) then the columns of $P$ generate $\mathbb{Q}^{r+s}$ as a cone (are the ray generators of a pointed convex polyhedral cone). The possible open subsets $\widehat{X} \subseteq \overline{X}$ now can be encoded by bunches of cones, see [**6**] for more details, but here we do not want to go this way. As discussed in the previous section, there is a canonical ambient toric variety $X \subseteq Z$, and in the present case, the columns of $P$ are the ray generators of the fan $\Sigma$ of $Z$. In case $X$ is complete, there are in general various choices for $\Sigma$, but if $X$ is affine, there is only one choice, see [**5**, Rem. 2.7], since $\widehat{X} = \overline{X}$. Thus for data $A, P$ with columns of $P$ being ray generators of a pointed polyhedral cone, there is a unique affine variety $X = X(A, P)$ with a torus action of complexity one such that $R(A, P)$ is the Cox ring of $X(A, P)$ and basically all varieties $X$ of this form arise from data $A, P$ [**53**, Thm. 1.8]. By Tevelev [**98**], the tropical variety of $X$ can be used to determine the toric orbits which $X$ intersects nontrivially:

Construction 1.2.5. Let $X = X(A, P)$ be the unique affine variety with Cox ring $R(A, P)$, where the columns of $P$ are the ray generators of a pointed polyhedral cone as in Construction 1.2.2. The *tropical variety* of $X$ is the fan $\mathrm{trop}(X)$ in $\mathbb{Q}^{r+s}$ consisting of the cones

$$\lambda_i \ := \ \mathrm{cone}(v_{i1}) + \mathrm{lin}(e_{r+1}, \ldots, e_{r+s}) \text{ for } i \ = \ 0, \ldots, r, \qquad \lambda \ := \ \lambda_\iota \cap \ldots \cap \lambda_r,$$

where $v_{ij} \in \mathbb{Z}^{r+s}$ denote the first $n$ columns of $P$ and $e_k \in \mathbb{Z}^{r+s}$ the $k$-th canonical basis vector; we call $\lambda_i$ a *leaf* and $\lambda$ the *lineality part* of $\mathrm{trop}(X)$.



Construction 1.2.6. Let $X = X(A, P)$ be as in Construction 1.2.5. For a face $\delta_0 \preceq \delta$ of the orthant $\delta \subseteq \mathbb{Q}^{n+m}$, let $\delta_0^* \preceq \delta$ denote the complementary face and call $\delta_0$ *relevant* if the relative interior of $P(\delta_0)$ intersects $\mathrm{trop}(X)$. Then we obtain fans $\widehat{\Sigma}$ in $\mathbb{Z}^{n+m}$ and $\Sigma$ in $\mathbb{Z}^{r+s}$ of pointed cones by setting

$$\widehat{\Sigma} \ := \ \{\delta_1 \preceq \delta_0; \ \delta_0 \preceq \delta \text{ relevant}\}, \qquad \Sigma \ := \ \{\sigma \preceq P(\delta_0); \ \delta_0 \preceq \delta \text{ relevant}\}.$$

The toric varieties $\widehat{Z}$ and $Z$ associated with $\widehat{\Sigma}$ and $\Sigma$, respectively, and $\overline{Z} = \mathbb{C}^{n+m}$ fit into a commutative diagramm of characteristic spaces and total coordinate spaces

$$\begin{array}{ccc} \overline{X}(A,P) & \subseteq & \overline{Z} \\ \cup\!\!\mid & & \cup\!\!\mid \\ \widehat{X}(A,P) & \subseteq & \widehat{Z} \\ \big\downarrow {\scriptstyle /\!\!/ H} & & \big\downarrow {\scriptstyle /\!\!/ H} \\ X(A,P) & \subseteq & Z \end{array}$$

The horizontal inclusions are $T$-equivariant closed embeddings, where $T$ acts on $Z$ as the subtorus of the $(r + s)$-torus corresponding to $0 \times \mathbb{Z}^s \subseteq \mathbb{Z}^{r+s}$. Moreover, $X(A, P)$ intersects every closed toric orbit of $Z$.

Remark 1.2.7. As there is a unique affine $X$ with Cox ring $R(A, P)$ as in Construction 1.2.6, we do not need to use bunches of cones and the tropical variety of $X$ fully determines the *minimal toric ambient variety* $Z$ of $X$, thus Construction 1.2.6 in this case is equivalent to [**5**, Constr. 2.9].

The minimal toric ambient variety can be used for the resolution of singularities of $X$. The following two-step construction for resolving singularities is valid for complete varieties as well [**6**, Thm. 3.4.4.9], but here we will restrict to the affine case.

Construction 1.2.8. Let $X = X(A, P)$ be obtained from Construction 1.2.2 and consider the canonical toric embedding $X \subseteq Z$ and the defining fan $\Sigma$ of $Z$.

- Let $\Sigma' = \Sigma \sqcap \mathrm{trop}(X)$ be the coarsest common refinement.
- Let $\Sigma''$ be any regular subdivision of the fan $\Sigma'$.

Then $\Sigma'' \to \Sigma$ defines a proper toric morphism $Z'' \to Z$ and with the proper transform $X'' \subseteq Z''$ of $X \subseteq Z$, the morphism $X'' \to X$ is a resolution of singularities.

Remark 1.2.9. In the setting of Construction 1.2.8, the variety $X''$ has again a torus action of complexity one and thus is of the form $X'' = X(A'', P'', \Phi'')$ with $\Phi''$ a bunch of cones, which is the Gale dual of $\Sigma''$, see [**6**, Ch. 2]. When resolving singularities in Chapter 4, we will work with the fan $\Sigma''$ instead. It encodes the same information as $\Phi''$, but in a more immediate way from the viewpoint of resolution of singularities. Thus we will encode such varieties by the data $X = X(A, P, \Sigma)$ in general. Moreover, we have $A'' = A$ and $P''$ is obtained from $P$ by inserting the primitive generators of $\Sigma''$ as new columns.

### 1.3. Singularities of $T$-varieties of complexity one

We start the investigation of singularities by recalling the singularity types arising in the minimal model program. Let $X$ be a $\mathbb{Q}$-Gorenstein variety, i.e., some non-zero multiple of a canonical divisor $K_X$ on $X$ is an integral Cartier divisor. Then, for any resolution of singularities $\varphi\colon X' \to X$ and appropriate choices of canonical divisors, one has the ramification formula

$$K_{X'} - \varphi^*(K_X) \ = \ \sum a_i E_i,$$

where the $E_i$ are the prime components of the exceptional divisors and the coefficients $a_i \in \mathbb{Q}$ are the discrepancies of the resolution. The variety $X$ is said to have at most *log terminal (canonical, terminal)* singularities, if for every resolution of singularities the discrepancies $a_i$ satisfy $a_i > -1$ ($a_i \geq 0$, $a_i > 0$).

In [**10**], the "anticanonical complex" has been introduced for Fano varieties and served as a tool to study singularities of the above type. The purpose here is to extend this approach and to generalize results from [**10**] to the non-complete and non-$\mathbb{Q}$-factorial cases. As an application, we characterize log terminality in Theorem 1.3.11 via platonic triples occuring in the Cox ring. This leads to the proof of Theorem 5, which will be given in Chapter 2.

Now, let $X = X(A, P, \Sigma)$ be a rational $T$-variety of complexity one arising from Construction 1.2.2. Consider the embedding $X \subseteq Z$ into the minimal toric ambient variety. Then $X$ and $Z$ share the same divisor class group

$$K \ = \ \mathrm{Cl}(X) \ = \ \mathrm{Cl}(Z)$$

and the same degree map $Q\colon \mathbb{Z}^{n+m} \to K$ for their Cox rings. Let $e_Z \in \mathbb{Z}^{n+m}$ denote the sum over the canonical basis vectors $e_{ij}$ and $e_k$ of $\mathbb{Z}^{n+m}$. Then, with the defining relations $g_\iota, \ldots, g_{r-2}$ of the Cox ring $R(A, P)$, the canonical divisor classes of $Z$ and $X$ are given as

$$\mathcal{K}_Z \ = \ -Q(e_Z) \ \in \ K, \qquad\qquad \mathcal{K}_X \ = \ \sum_{i=\iota}^{r-2+\iota} \deg(g_i) + \mathcal{K}_Z \ \in \ K,$$

see [**6**, Prop. 3.3.3.2]. Define a (rational) polyhedron

$$B(-\mathcal{K}_X) \ := \ Q^{-1}(-\mathcal{K}_X) \cap \mathbb{Q}^{n+m}_{\geq 0} \ \subseteq \ \mathbb{Q}^{n+m}$$

and let $B := B(g_\iota) + \ldots + B(g_{r-2+\iota}) \subseteq \mathbb{Q}^{n+m}$ denote the Minkowski sum of the Newton polytopes $B(g_i)$ of the relations $g_\iota, \ldots, g_{r-2+\iota}$ of $R(A, P)$.

Definition 1.3.1. Let $X = X(A, P, \Sigma)$ such that $-\mathcal{K}_X$ is ample.

(i) The *anticanonical polyhedron* of $X$ is the dual polyhedron $A_X \subseteq \mathbb{Q}^{r+s}$ of the polyhedron

$$B_X \ := \ (P^*)^{-1}(B(-\mathcal{K}_X) + B - e_\Sigma) \ \subseteq \ \mathbb{Q}^{r+s}.$$

(ii) The *anticanonical complex* of $X$ is the coarsest common refinement of polyhedral complexes

$$A_X^c \ := \ \mathrm{faces}(A_X) \sqcap \Sigma \sqcap \mathrm{trop}(X).$$

(iii) The *relative interior* of $A_X^c$ is the interior of its support with respect to the intersection $\mathrm{Supp}(\Sigma) \cap \mathrm{trop}(X)$.

(iv) The *relative boundary* $\partial A_X^c$ is the complement of the relative interior of $A_X^c$ in $A_X^c$.

A first statement expresses the discrepancies of a given resolution of singularities via the anticanonical complex; the proof is a straightforward generalization of the one given in [**10**] for the Fano case, it is given in [**105**].

Proposition 1.3.2. *Let $X = X(A, P, \Sigma)$ such that $-\mathcal{K}_X$ is ample and $X'' \to X$ a resolution of singularities as in Construction 1.2.8. For any ray $\varrho \in \Sigma''$, let $v_\varrho$ be its primitive generator, $v'_\varrho$ its leaving point of $A_X^c$ provided $\varrho \not\subseteq A_X^c$ and $D_\varrho$ the prime divisor on $X''$ obtained by intersecting $X''$ with the toric divisor of $Z''$ corresponding to $\varrho$. Then the discrepancy $a_\varrho$ along $D_\varrho$ satisfies*

$$a_\varrho \ = \ -1 + \frac{||v_\varrho||}{||v'_\varrho||} \quad \textit{if } \varrho \not\subseteq A_X^c, \qquad\qquad a_\varrho \leq -1 \quad \textit{if } \varrho \subseteq A_X^c.$$

The next result characterizes the existence of at most log terminal (canonical, terminal) singularities in terms of the anticanonical complex; this generalizes [**10**, Thm. 1.4], and the proof goes along the lines of the respective proof, where one replaces [**10**, Prop. 2.3] by the above Proposition 1.3.2.

Theorem 1.3.3. *Let $X = X(A, P, \Sigma)$ be such that $-\mathcal{K}_X$ is ample. Then the following statements hold.*

(i) *$A_X^c$ contains the origin in its relative interior and all primitive generators of the fan $\Sigma$ are vertices of $A_X^c$.*

(ii) *$X$ has at most log terminal singularities if and only if the anticanonical complex $A_X^c$ is bounded.*

(iii) *$X$ has at most canonical singularities if and only if $0$ is the only lattice point in the relative interior of $A_X^c$.*

(iv) *$X$ has at most terminal singularities if and only if $0$ and the primitive generators $v_\varrho$ for $\varrho \in \Sigma^{(1)}$ are the only lattice points of $A_X^c$.*

We describe the structure of the anticanonical complex in more detail, which generalizes in particular statements on the $\mathbb{Q}$-factorial Fano case obtained in [**10**].

CONSTRUCTION 1.3.4. Let $X = X(A, P, \Sigma)$. Write $v_{ij} := P(e_{ij})$ and $v_k := P(e_k)$ for the columns of $P$. Consider a pointed cone of the form

$$\tau \ = \ \mathrm{cone}(v_{0j_0}, \ldots, v_{rj_r}) \ \subseteq \ \mathbb{Q}^{r+s},$$

that means that $\tau$ contains exactly one $v_{ij}$ for every $i = 0, \ldots, r$. If $\tau \in \Sigma$, we call it a *$P$-elementary cone*, if not, we call it a *fake elementary cone*. We associate the following numbers with a ($P$- or fake) elementary cone $\tau$:

$$\ell_{\tau,i} \ := \ \frac{l_{0j_0} \cdots l_{rj_r}}{l_{ij_i}} \text{ for } i = 0, \ldots, r, \qquad \ell_\tau \ := \ (1-r) l_{0j_0} \cdots l_{rj_r} + \sum_{i=0}^{r} l_{\tau,i}.$$

Moreover, we set

$$v(\tau) \ := \ \ell_{\tau,0} v_{0j_0} + \ldots + \ell_{\tau,r} v_{rj_r} \ \in \ \mathbb{Z}^{r+s}, \qquad \varrho(\tau) \ := \ \mathbb{Q}_{\geq 0} \cdot v(\tau) \ \in \ \mathbb{Q}^{r+s}.$$

We denote by $\mathrm{T}(A, P, \Sigma)$ the set of all $P$-elementary cones $\tau \in \Sigma$. For a given $\sigma \in \Sigma$, we denote by $\mathrm{T}(\sigma)$ the set of all $P$-elementary faces of $\sigma$.

REMARK 1.3.5. Let $X = X(A, P, \Sigma)$ and $\lambda_0, \ldots, \lambda_r \subseteq \mathrm{trop}(X)$ the leaves of the tropical variety of $X$. As in [**10**, Def. 4.1], we say that

(i) a cone $\sigma \in \Sigma$ is a *leaf cone* if $\sigma \subseteq \lambda_i$ holds for some $i = 0, \ldots, r$,
(ii) a cone $\sigma \in \Sigma$ is called *big* if $\sigma \cap \lambda_i^\circ \neq \emptyset$ holds for all $i = 0, \ldots, r$.

Observe that a given cone $\sigma \in \Sigma$ is big if and only if $\sigma$ contains some $P$-elementary cone as a subset.

PROPOSITION 1.3.6. *Let $X = X(A, P, \Sigma)$ be of Type 2 such that $-\mathcal{K}_X$ is ample. Denote by $\lambda_0, \ldots, \lambda_r$ the leaves of $\mathrm{trop}(X)$ and by $\lambda = \lambda_0 \cap \ldots \cap \lambda_r$ its lineality part.*

(i) *The fan $\Sigma \sqcap \mathrm{trop}(X)$ consists of the cones $\sigma \cap \lambda$ and $\sigma \cap \lambda_i$, where $\sigma \in \Sigma$ and $i = 0, \ldots, r$. Here, one always has $\sigma \cap \lambda \preceq \sigma \cap \lambda_i$.*
(ii) *The fan $\Sigma \sqcap \mathrm{trop}(X)$ is a subfan of the normal fan of the polyhedron $B_X$. In particular, for every cone $\sigma \cap \lambda_i$, there is a vertex $u_{\sigma,i} \in B_X$ with*

$$\partial A_X^{\mathrm{c}} \cap \sigma \cap \lambda_i \ = \ \{v \in \sigma \cap \lambda_i; \ \langle u_{\sigma,i}, v \rangle = -1\}.$$

(iii) *If a $P$-elementary cone $\tau$ is contained in some $\sigma \in \Sigma$, then $\tau$ is simplicial, $v(\tau) \in \tau^\circ$ holds, $\varrho(\tau)$ is a ray, $\varrho(\tau) = \tau \cap \lambda$ holds as well as $\mathbb{Q}\varrho(\tau) = \mathbb{Q}\tau \cap \lambda$.*
(iv) *Let $\sigma \in \Sigma$ be any cone. Then, for every $i = 0, \ldots, r$, the set of extremal rays of $\sigma \cap \lambda_i \in \Sigma \sqcap \mathrm{trop}(X)$ is given by*

$$(\sigma \cap \lambda_i)^{(1)} \ = \ \{\varrho(\sigma_0); \ \sigma_0 \in \mathrm{T}(\sigma)\} \cup \{\varrho \in \sigma^{(1)}; \ \varrho \subseteq \lambda_i\}.$$

(v) *The set of rays of $\Sigma \sqcap \mathrm{trop}(X)$ consists of the rays of $\Sigma$ and the rays $\varrho(\sigma_0)$, where $\sigma_0 \in \mathrm{T}(A, P, \Sigma)$.*
(vi) *If a $P$-elementary cone $\tau$ is contained in some $\sigma \in \Sigma$, then the minimum value among all $\langle u, v(\tau) \rangle$, where $u \in B_X$, equals $-\ell_\tau$.*
(vii) *Let the $P$-elementary cone $\tau$ be contained in $\sigma \in \Sigma$. Then $\varrho(\tau) \not\subseteq A_X^{\mathrm{c}}$ holds if and only if $\ell_\tau > 0$ holds; in this case, $\varrho(\tau)$ leaves $A_X^{\mathrm{c}}$ at $v(\tau)' = \ell_\tau^{-1} v(\tau)$.*

(viii) *The vertices of* $A_X^c$ *are the primitive generators of* $\Sigma$*, i.e. the columns of* $P$*, and the points* $v(\sigma_0)' = \ell_{\sigma_0}^{-1} v(\sigma_0)$*, where* $\sigma_0 \in \mathrm{T}(A, P, \Sigma)$ *and* $\ell_{\sigma_0} > 0$.

Proof. Assertion (i) holds more generally. Indeed, the coarsest common refinement $\Sigma_1 \sqcap \Sigma_2$ of any two quasifans $\Sigma_i$ in a common vector space consists of the intersections $\sigma_1 \cap \sigma_2$, where $\sigma_i \in \Sigma_i$. Moreover, the faces of a given cone $\sigma_1 \cap \sigma_2$ of $\Sigma_1 \sqcap \Sigma_2$ are precisely the cones $\sigma_1' \cap \sigma_2'$, where $\sigma_i' \preceq \sigma_i$.

We show (ii). Let $\Sigma'$ be the complete fan in $\mathbb{Q}^{r+s}$ defined by the class $-\mathcal{K}_X \in K$. Since $-\mathcal{K}_X$ is ample, the fan $\Sigma$ is a subfan of $\Sigma'$. The preimage $P^{-1}(\Sigma')$ consists the cones $P^{-1}(\sigma')$, where $\sigma' \in \Sigma'$, and is the normal fan of $B(-\mathcal{K}_X) \subseteq \mathbb{Q}^{n+m}$. Moreover, $P^{-1}(\mathrm{trop}(X))$ turns out to be a subfan of the normal fan of $B \subseteq \mathbb{Q}^{n+m}$. It follows that $P^{-1}(\Sigma') \sqcap P^{-1}(\mathrm{trop}(X))$ is a subfan of the normal fan of $B(-\mathcal{K}_X)+B$. Projecting the involved fans via $P$ to $\mathbb{Q}^{r+s}$ gives the assertion.

To obtain (iii), consider first any $P$-elememtary $\tau = \mathrm{cone}(v_{0j_0}, \ldots, v_{rj_r})$. Then $v_{0j_0}, \ldots, v_{rj_r}$ is linearly dependent if and only if $v(\tau) = 0$ holds. The latter is equivalent to 0 being an inner point of $\tau$. Thus, if $\tau$ is contained in some $\sigma \in \Sigma$, then $\tau$ is pointed and thus must be simplicial. The remaining part is then obvious; recall that the lineality part of $\mathrm{trop}(X)$ equals the vector subspace $0 \times \mathbb{Q}^s \subseteq \mathbb{Q}^{r+s}$.

We turn to (iv). First, we claim that if $\sigma_0 \in \Sigma$ is big and $\varrho(\tau) = \varrho(\tau')$ holds for any two $P$-elementary cones $\tau, \tau' \subseteq \sigma$, then $\sigma_0$ is $P$-elementary. Assume that $\sigma_0$ is not $P$-elementary. Then we find some $1 \le t \le r$ and cones

$$\tau \;=\; \mathrm{cone}(v_{0j_0}, \ldots, v_{tj_{t-1}}, v_{tj_t}, v_{tj_{t+1}}, \ldots, v_{rj_r}) \;\subseteq\; \sigma_0,$$

$$\tau' \;=\; \mathrm{cone}(v_{0j_0}, \ldots, v_{tj_{t-1}}, v_{tj_t'}, v_{tj_{t+1}}, \ldots, v_{rj_r}) \;\subseteq\; \sigma_0$$

with $j_t \neq j_t'$ and thus $\tau \neq \tau'$. Here, we may assume that $c_\tau^{-1} l_{tj_t} \ge c_{\tau'}^{-1} l_{tj_t'}$ holds with the greatest common divisors $c_\tau$ and $c_{\tau'}$ of the entries of $v(\tau)$ and $v(\tau')$ respectively. Then even $c_\tau^{-1}\ell_{\tau,i} \ge c_{\tau'}^{-1}\ell_{\tau',i}$ must hold for all $1 \le i \le r$. Since, the rays $\varrho(\tau)$ and $\varrho(\tau')$ coincide, also their primitive generators $c_{\tau'}^{-1} v(\tau')$ and $c_\tau^{-1} v(\tau)$ coincide. By the definition of $v(\tau)$ and $v(\tau')$, this implies

$$c_{\tau'}^{-1}\ell_{\tau',t} v_{tj_t'} \;=\; c_\tau^{-1}\ell_{\tau,k} v_{tj_t} + \sum_{i \neq t} (c_\tau^{-1}\ell_{\tau,i} - c_{\tau'}^{-1}\ell_{\tau',i}) v_{ij_i}.$$

We conclude $v_{tj_t'} \in \tau$. Since $v_{tj_t'}$ is an extremal ray of $\sigma_0$ and $\tau' \subseteq \sigma_0$ holds, $v_{tj_t'}$ generates an extremal ray of $\tau$. This contradicts to the choice of $j_t'$ and the claim is verified.

Now, consider the equation of (iv). To verify "$\subseteq$", let $\varrho$ be an extremal ray of $\sigma \cap \lambda_i$. We have to show that $\varrho = \varrho(\sigma_0)$ holds for some $\sigma_0 \in \mathrm{T}(\sigma)$ or that $\varrho$ is a ray of $\sigma$ with $\varrho \subseteq \lambda_i$. According to (ii), there is a face $\sigma_\varrho \preceq \sigma$ such that $\varrho = \sigma_\varrho \cap \lambda$ or $\varrho = \sigma_\varrho \cap \lambda_i$ holds. We choose $\sigma_\varrho$ minimal with respect to this property, that means that we have $\varrho^\circ \subseteq \sigma_\varrho^\circ$. We distinguish the following cases.

*Case 1.* We have $\varrho = \sigma_\varrho \cap \lambda$. If $\sigma_\varrho \subseteq \lambda$ holds, then we obtain $\varrho = \sigma_\varrho$ and thus $\varrho \subseteq \lambda_i$ is an extremal ray of $\sigma$. So, assume that $\sigma_\varrho$ is not contained in $\lambda$. Then, because of $\sigma_\varrho^\circ \cap \lambda \neq \emptyset$, there is a $P$-elementary cone $\tau \subseteq \sigma_\varrho$. Using (i), we obtain

$$\varrho(\tau) \;=\; \tau \cap \lambda \;\subseteq\; \sigma_\varrho \cap \lambda \;=\; \varrho$$

and thus $\varrho = \varrho(\tau)$. As this does not depend on the particular choice of the $P$-elementary cone $\tau \subseteq \sigma_\varrho$, the above claim yields $\sigma_0 := \sigma_\varrho \in \mathrm{T}(\sigma)$ and $\varrho = \varrho(\sigma_0)$.

*Case 2.* We don't have $\varrho = \sigma_\varrho \cap \lambda$. Then $\varrho = \sigma_\varrho \cap \lambda_i$ and $\varrho^\circ \subseteq \lambda_i^\circ$ hold. If $\sigma_\varrho \subseteq \lambda_i$ holds, then we obtain $\varrho = \sigma_\varrho$ and thus $\varrho \subseteq \lambda_i$ is an extremal ray of $\sigma$. So, assume that $\sigma_\varrho$ is not contained in $\lambda_i$. Then $\sigma_\varrho \cap \lambda_j^\circ$ is non-empty for all $j = 0, \ldots, r$. Thus, there is a $P$-elementary cone $\tau \subseteq \sigma_\varrho$. Using (i), we obtain

$$\varrho(\tau) \ = \ \tau \cap \lambda \ \subseteq \ \sigma_\varrho \cap \lambda \ = \ \varrho$$

and thus $\varrho = \varrho(\tau)$. As this does not depend on the particular choice of the $P$-elementary cone $\tau \subseteq \sigma_\varrho$, the above claim yields $\sigma_0 := \sigma_\varrho \in \mathrm{T}(\sigma)$ and $\varrho = \varrho(\sigma_0)$.

We verify the inclusion "$\supseteq$". Consider a face $\sigma_0 \in \mathrm{T}(\sigma)$. As seen just before, the extremal rays of $\sigma_0 \cap \lambda_i$ are $\varrho(\sigma_0)$ and the rays of $\sigma_0$ that lie in $\lambda_i$. Since $\sigma_0 \cap \lambda_i$ is a face of $\sigma \cap \lambda_i$, the ray $\varrho(\sigma_0)$ is an extremal ray of $\sigma \cap \lambda_i$. Finally, consider an extremal ray $\varrho \preceq \sigma$ with $\varrho \subseteq \lambda_i$. Then $\varrho = \varrho \cap \lambda_i$ is a face of $\sigma \cap \lambda_i$.

The proof of Assertion (iv) is complete now. Assertion (v) is a direct consequence of (iv).

We turn to Assertions (vi), (vii) and (viii). Let $\widehat{\tau} \preceq \widehat{\sigma} \preceq \mathbb{Q}^{n+m}$ be the faces with $P(\widehat{\tau}) = \tau$ and $P(\widehat{\sigma}) = \sigma$. Moreover, let $e_\tau \in \widehat{\tau}$ be the (unique) point with $P(e_\tau) = v(\tau)$. The minimum value $\langle u, v(\tau)\rangle$ is attained at some vertex $u \in B_X$. For this $u$, we find vertices $e_\sigma \in B(-\mathcal{K}_X)$ and $e_B \in B$ with

$$u \ = \ (P^*)^{-1}(e_\sigma + e_B - e_Z).$$

Here, $e_\sigma$ is any vertex of $B(-\mathcal{K}_X)$ such that $\widehat{\sigma}$ is contained in the cone of the normal fan of $B(-\mathcal{K}_X)$ associated with $e_\sigma$; such $e_\sigma$ exists due to ampleness of $-\mathcal{K}_X$ and $e_\sigma$ vanishes along $\widehat{\sigma}$. Together we have

$$e_\tau \ = \ \sum_{i=0}^{r} l_{ij_i} e_{ij_i}, \qquad\qquad \langle u, v(\tau)\rangle \ = \ \langle e_\sigma + e_B - e_Z, e_\tau\rangle.$$

As mentioned, $\langle e_\sigma, e_\tau\rangle = 0$ holds. Moreover, $\langle e, e_\tau\rangle = (r-1) l_{0j_0} \cdots l_{rj_r}$ holds for every $e \in B$. We conclude $\langle u, v(\tau)\rangle = -\ell_\tau$ and Assertion (vi). Moreover, Assertions (vii) and (viii) are direct consequences of (vi) and (ii). □

Example 1.3.7. Consider the $E_6$-singular affine surface $X = V(z_1^4 + z_2^3 + z_3^2) \subseteq \mathbb{C}^3$. It inherits a $\mathbb{C}^*$-action from the action

$$t \cdot (z_1, z_2, z_3) \ = \ (t^3 z_1, t^4 z_2, t^6 z_3)$$

on $\mathbb{C}^3$. The divisor class group and the Cox ring of the surface $X$ are explicitly given by

$$\mathrm{Cl}(X) \ = \ \mathbb{Z}/3\mathbb{Z}, \qquad\qquad \mathcal{R}(X) \ = \ \mathbb{C}[T_1, T_2, T_3]/\langle T_1^3 + T_2^3 + T_3^2\rangle,$$

where the $\mathrm{Cl}(X)$-degrees of $T_1$, $T_2$, and $T_3$ are $\bar{1}$, $\bar{2}$ and $\bar{0}$. The minimal toric ambient variety is affine and corresponds to the cone

$$\sigma \ = \ \mathrm{cone}((-3,-3,-2), (3,0,1), (0,2,1)).$$

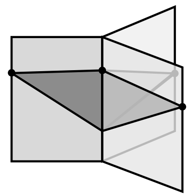
Denoting by $e_i \in \mathbb{Q}^3$ the $i$-th canonical basis vector, the tropical variety $\mathrm{trop}(X)$ in $\mathbb{Q}^3$ is given as

$$\mathrm{trop}(X) \ = \ \mathrm{cone}(e_1, \pm e_3) \cup \mathrm{cone}(e_2, \pm e_3) \cup \mathrm{cone}(-e_1 - e_2, \pm e_3).$$

The anticanonical polyhedron $A_X \subseteq \mathbb{Q}^3$ is non bounded with recession cone generated by $(-1,-1,-1)$, $(1,0,0)$, $(0,1,0)$. The vertices of $A_X$ are

$$(-3,-3,-2), \ (3,0,1), \ (0,2,1), \ (0,0,1).$$

The anticanonical complex $A_X^c = A_X \sqcap \Sigma \sqcap \mathrm{trop}(X)$ lives inside $\mathrm{trop}(X)$ and looks as follows.



COROLLARY 1.3.8. *Let $X = X(A, P, \Sigma)$, such that $-\mathcal{K}_X$ is ample. Let $\tau$ be a $P$-elementary cone contained in some $\sigma \in \Sigma$. Assume $\varrho(\tau) \not\subseteq A_X^c$ and denote by $c_\tau$ the greatest common divisor of the entries of $v(\tau)$. Then, for any resolution of singularities $\varphi\colon X'' \to X$ provided by 1.2.8, the discrepancy along the prime divisor of $X''$ corresponding to $\varrho(\tau)$ equals $c_\tau^{-1}\ell_\tau - 1$.*

COROLLARY 1.3.9. *Let $X = X(A, P, \Sigma)$ such that $-\mathcal{K}_X$ is ample and let $\tau = \mathrm{cone}(v_{0j_0}, \ldots, v_{rj_r})$ be contained in some $\sigma \in \Sigma$.*

(i) *If $X$ has at most log terminal singularities, then $l_{0j_0}^{-1} + \ldots + l_{rj_r}^{-1} > r - 1$ holds.*
(ii) *If $X$ has at most canonical singularities, then $l_{0j_0}^{-1} + \ldots + l_{rj_r}^{-1} \geq r - 1 + c_\tau l_{0j_0}^{-1} \cdots l_{rj_r}^{-1}$ holds.*
(iii) *If $X$ has at most terminal singularities, then $l_{0j_0}^{-1} + \ldots + l_{rj_r}^{-1} > r - 1 + c_\tau l_{0j_0}^{-1} \cdots l_{rj_r}^{-1}$ holds.*

REMARK 1.3.10. Let $a_0, \ldots, a_r$ be positive integers. Then $a_0^{-1} + \ldots + a_r^{-1} > r - 1$ holds if and only if $(a_0, \ldots, a_r)$ is a platonic tuple.

THEOREM 1.3.11. *Let $X = X(A, P, \Sigma)$ such that $-\mathcal{K}_X$ is ample. Then the following statements are equivalent.*

(i) *The variety $X$ has at most log terminal singularities.*
(ii) *For every $P$-elementary $\tau = \mathrm{cone}(v_{0j_0}, \ldots, v_{rj_r})$ contained in a cone of $\Sigma$, the exponents $l_{0j_0}, \ldots, l_{rj_r}$ form a platonic tuple.*

PROOF. Assume that $X = X(A, P, \Sigma)$ is log terminal. Then Corollary 1.3.9 (i) tells us that for every $P$-elementary $\tau = \mathrm{cone}(v_{0j_0}, \ldots, v_{rj_r})$ contained in a cone of $\Sigma$, the corresponding exponents $l_{0j_0}, \ldots, l_{rj_r}$ form a platonic tuple.

Now assume that (ii) holds. Then every $(l_{0j_0}, \ldots, l_{rj_r})$ is a platonic tuple. Consequently, we have $\ell_\tau > 0$ for every $P$-elementary cone $\tau$. Proposition 1.3.6 shows that $A_X^c$ is bounded for $X = X(A, P, \Sigma)$. Theorem 1.3.3 (ii) tells us that $X$ is log terminal. □

## 1.4. The canonical multiplicity of a $T$-variety of complexity one

For a $\mathbb{Q}$-Gorenstein variety $X$, the well known *Gorenstein index* is the smallest positive integer $\imath_X$ such that $\imath_X \mathcal{K}_X$ is Cartier. In this section, we attach another invariant to the canonical divisor of $X$. It will be crucial for the classifications in Chapter 4.

REMARK 1.4.1. Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein, affine $T$-variety arising from Construction 1.2.2. We consider canonical divisors $K_X$ on $X$ that are of the following form, cf. [**6**, Prop. 3.3.3.2]:

$$(1.4.1.1) \qquad -\sum_{i,j} D_{ij} - \sum_k E_k + \sum_{\alpha=1}^{r-1} \sum_{j=0}^{n_{i_\alpha}} l_{i_\alpha j} D_{i_\alpha j}, \qquad 0 \le i_\alpha \le r.$$

Then $\imath_X K_X$ is the divisor of a $T$-homogeneous rational function. Any two $\imath_X K_X$ with $K_X$ of shape (1.4.1.1) differ by the divisor of a $T$-invariant rational function, and thus, all the functions with divsors $\imath_X K_X$, where $K_X$ as in (1.4.1.1), are homogeneous with respect to the same weight $\eta_X \in \mathbb{X}(T)$.

Definition 1.4.2. Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein, affine $T$-variety arising from Construction 1.2.2. We call $\eta_X \in \mathbb{X}(T)$ of Remark 1.4.1 the *canonical weight* of $X$. The *canonical multiplicity* of $X$ is the minimal non-negative integer $\zeta_X$ such that $\eta_X = \zeta_X \cdot \eta'_X$ holds with a primitive element $\eta'_X \in \mathbb{X}(T)$.

Proposition 1.4.3. *Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein, affine $T$-variety with at most log terminal singularities. Then $\zeta_X > 0$ holds. Moreover, for any positive integer $\imath$, the following statements are equivalent.*

(i) *The variety $X$ is of Gorenstein index $\imath$.*
(ii) *There exist integers $\mu_1, \ldots, \mu_r$ with $\gcd(\mu_1, \ldots, \mu_r, \zeta_X, \imath) = 1$ such that with $\mu_0 := \imath(r-1) - \mu_1 - \ldots - \mu_r$ we obtain integral vectors*

$$\nu_i := (\nu_{i1}, \ldots, \nu_{in_i}) \text{ with } \nu_{ij} := \frac{\imath - \mu_i l_{ij}}{\zeta_X},$$
$$\nu' := (\nu'_1, \ldots, \nu'_m) \text{ with } \nu'_k := \frac{\imath}{\zeta_X}$$

*and by suitable elementary row operations on the $(d, d')$-block, the matrix $P$ gains $(\nu_0, \ldots, \nu_r, \nu')$ as its last row, i.e., turns into the shape*

$$\tilde{P} = \begin{pmatrix} -l_0 & l_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -l_0 & 0 & \ldots & l_r & 0 \\ * & * & \ldots & * & * \\ \nu_0 & \nu_1 & \ldots & \nu_r & \nu' \end{pmatrix}.$$

Proof. We work with an anticanonical divisor $K_X$ on $X$ such that $-K_X$ is of the form (1.4.1.1):

$$-K_X := \sum_{i,j} D_{ij} + \sum_k E_k - (r-1) \sum_{j=1}^{n_0} l_{0j} D_{0j}.$$

The Picard group of $X$ is trivial since $X$ is a quasicone, see [**70**]. Thus, $\imath_X K_X$ is the divisor of some toric character $\chi^u$, where

$$u = (\mu_1, \ldots, \mu_r, \eta_1, \ldots, \eta_s) \in \mathbb{Z}^{r+s}.$$

Note that $-(\eta_1, \ldots, \eta_s) \in \mathbb{Z}^s = \mathbb{X}(T)$ is the canonical weight $\eta_X$ of $X$. Moreover, the divisor $\imath_X K_X = \operatorname{div}(\chi^u)$ corresponds to the vector $P^* \cdot u \in \mathbb{Z}^{m+n}$ under the identification of toric divisors with lattice points via $D_{ij} \mapsto e_{ij}$ and $E_k \mapsto e_k$.

We claim that $\eta_X$ is non-trivial. Otherwise, $\eta_1 = \ldots = \eta_s = 0$ holds. As noted, the $ij$-th and $k$-th components of the vector $P^* \cdot u$ are the multiplicities of $D_{ij}$ and $D_k$ in $\imath_X K_X$, respectively. More explicitly, this leads to the conditions

$$m = 0, \qquad \imath_X((r-1)l_{0j} - 1) = (\mu_1 + \ldots + \mu_r) l_{0j}, \qquad \imath_X = \mu_i l_{ij}$$

for all $i$ and $j$. Plugging the third into the second one, we obtain that $l_{0j_0}^{-1}+\ldots+l_{rj_r}^{-1}$ equals $r-1$ for any choice of $1 \le j_i \le n_i$. According to Corollary 1.3.9 (i), this contradicts to log terminality of $X$. Knowing that $\eta_X$ is non-zero, we obtain that $\zeta_X$ is non-zero.

Now, assume that (i) holds, i.e., we have $\imath = \imath_X$. Let $u \in \mathbb{Z}^{r+s}$ as above. Then we have $\zeta_X = \gcd(\eta_1, \ldots, \eta_s)$ and $\mathrm{div}(\chi^u) = \imath K_X$ implies $\gcd(\mu_1, \ldots, \mu_r, \zeta_X, \imath) = 1$. Next, choose a unimodular $s \times s$ matrix $\mathcal{B}$ with $\mathcal{B}^{-1} \cdot (\eta_1, \ldots, \eta_s) = (0, \ldots, 0, \zeta_X)$. Consider $\tilde{P} := \mathrm{diag}(E_r, \mathcal{B}^*) \cdot P$ and

$$\tilde{u} \ = \ (\mu_1, \ldots, \mu_r, 0, \ldots, 0, \zeta_X) \ \in \ \mathbb{Z}^{r+s}.$$

Observe that we have $P^* \cdot u = \tilde{P}^* \cdot \tilde{u}$. Comparing the entries of $\tilde{P}^* \cdot \tilde{u}$ with the multiplicities of the prime divisors $D_{ij}$ and $D_k$ in $\imath K_X$ shows that the last row of $\tilde{P}$ is as claimed.

Conversely, if (ii) holds, consider $u := (\mu_1, \ldots, \mu_r, 0, \ldots, 0, \zeta_X)$. Then we obtain $\imath K_X = \mathrm{div}(\chi^u)$. Using $\gcd(\mu_1, \ldots, \mu_r, \zeta_X, \imath) = 1$, we conclude that $\imath$ is the Gorenstein index of $X$. □

Remark 1.4.4. Let $X = X(A,P)$ be a $\mathbb{Q}$-Gorenstein, affine $T$-variety and $K_X$ a canonical divisor on $X$ as in (1.4.1.1). Then $\imath_X K_X$ is the divisor of some toric character $\chi^u$, where

$$u \ = \ (\mu_1, \ldots, \mu_r, \eta_1, \ldots, \eta_s) \ \in \ \mathbb{Z}^{r+s}.$$

In this situation, we have $\eta_X = (\eta_1, \ldots, \eta_s) \in \mathbb{X}(T)$ for the canonical weight of $X$ and the canonical multiplicity of $X$ is given by $\zeta_X = \gcd(\eta_1, \ldots, \eta_s)$. If $P$ is in the shape of Proposition 1.4.3, then $\eta_X = (0, \ldots, 0, \zeta_X)$ holds and $-\mu_1, \ldots, -\mu_r$ satisfy the conditions of 1.4.3 (ii).

Remark 1.4.5. The defining matrix $P$ of a given $\mathbb{Q}$-Gorenstein, affine $T$-variety $X = X(A,P)$ is in the shape of Proposition 1.4.3 if and only if for every $i = 0, \ldots, r$, the numbers $\mu_i := (\imath_X - \zeta_X \nu_{i1}) l_{i1}^{-1}$ satisfy

(i) $\zeta_X \nu_{ij} + \mu_i l_{ij} = \imath_X$ for $i = 1, \ldots, r$ and $j = 1, \ldots, n_i$,
(ii) $\zeta_X \nu_{0j} + \mu_0 l_{0j} = \imath_X$, for $\mu_0 := \imath_X(r-1) - \mu_1 - \ldots - \mu_r$ and $j = 1, \ldots, n_0$,
(iii) $\gcd(\mu_1, \ldots, \mu_r, \zeta_X, \imath_X) = 1$,
(iv) $\zeta_X \nu'_k = \imath_X$ for $k = 1, \ldots, m$.

Corollary 1.4.6. *Let $X = X(A,P)$ be a $\mathbb{Q}$-Gorenstein, affine $T$-variety with at most log terminal singularities. Then, for every $\imath \in \mathbb{Z}_{\ge 1}$, the following statements are equivalent.*

(i) *The variety $X$ is of Gorenstein index $\imath$ and of canonical multiplicity one.*
(ii) *One can choose the defining matrix $P$ to be of the shape*

$$\begin{pmatrix} -l_0 & l_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -l_0 & 0 & \ldots & l_r & 0 \\ * & * & \ldots & * & * \\ \boldsymbol{\imath} - \imath(r-1)l_0 & \boldsymbol{\imath} & \ldots & \boldsymbol{\imath} & \boldsymbol{\imath} \end{pmatrix},$$

*where $\boldsymbol{\imath}$ stands for a vector $(\imath, \ldots, \imath)$ of suitable length.*

Proof. If (i) holds, then we may assume $P$ to be as $\tilde{P}$ in Proposition 1.4.3. Adding the $\mu_i$-fold of the $i$-th row to the last row brings $P$ into the desired form.

If (ii) holds, take $u = (0, \dots, 0, -1) \in \mathbb{Z}^{r+s}$. Then $P^* \cdot u \in \mathbb{Z}^{n+m}$ defines a divisor $\imath K_X$ with $K_X$ a canonical divisor of shape (1.4.1.1) and we see $\zeta_X = 1$. □

PROPOSITION 1.4.7. *Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein affine $T$-variety with at most log terminal singularities and canonical multiplicity $\zeta_X > 1$. Then we can choose $P$ of shape 1.4.3 (ii) such that $l_{ij} = 1$ and $\nu_{ij} = 0$ holds for $i = 3, \dots, r$ and $j = 1, \dots, n_i$ and, moreover, $P$ satisfies one of the following cases:*

| *Case* | $(l_{01}, l_{11}, l_{21})$ | $(\nu_0, \nu_1, \nu_2)$ | $\zeta_X$ | $\imath_X$ |
|---|---|---|---|---|
| *(i)* | $(4, 3, 2)$ | $\frac{1}{2}(\boldsymbol{\imath_X} + l_0,\, \boldsymbol{\imath_X} - \imath_X l_1,\, \boldsymbol{\imath_X} - l_2)$ | 2 | $0 \mod 2$ |
| *(ii)* | $(3, 3, 2)$ | $\frac{1}{3}(\boldsymbol{\imath_X} - l_0,\, \boldsymbol{\imath_X} + l_1,\, \boldsymbol{\imath_X} - \imath_X l_2)$ | 3 | $0 \mod 3$ |
| *(iii)* | $(2k+1, 2, 2)$ | $\frac{1}{4}(\boldsymbol{\imath_X} - \imath_X l_0,\, \boldsymbol{\imath_X} - l_1,\, \boldsymbol{\imath_X} + l_2)$ | 4 | $2 \mod 4$ |
| *(iv)* | $(2k, 2, 2)$ | $\frac{1}{2}(\boldsymbol{\imath_X} - l_0,\, \boldsymbol{\imath_X} + l_1,\, \boldsymbol{\imath_X} - \imath_X l_2)$ | 2 | $0 \mod 2$ |
| *(v)* | $(k, 2, 2)$ | $\frac{1}{2}(\boldsymbol{\imath_X} - \imath_X l_0,\, \boldsymbol{\imath_X} - l_1,\, \boldsymbol{\imath_X} + l_2)$ | 2 | $0 \mod 2$ |
| *(vi)* | $(k_0, k_1, 1)$ | $(\nu_0,\, \nu_1,\, \zeta_X^{-1}(\boldsymbol{\imath_X} - \imath_X l_2))$ | | |

*where $\boldsymbol{\imath_X}$ stands for a vector $(\imath_X, \dots, \imath_X)$ of suitable length, and in Case (vi), all the numbers $(\imath_X - \nu_{0j_0}\zeta_X)/l_{0j_0}$ and $(\nu_{1j_1}\zeta_X - \imath_X)/l_{1j_1}$ are integral and coincide.*

PROOF. Since $X = X(A, P)$ has at most log terminal singularities, Theorem 5 guarantees that the Cox ring $\mathcal{R}(X) = R(A, P)$ is platonic. Thus, suitably exchanging data of column blocks, we achieve $l_{ij} = 1$ for all $i \geq 3$. Next, we bring $P$ into the form of Proposition 1.4.3 (ii). Finally, subtracting the $\nu_{ij}$-fold of the $i$-th row from the last one, we achieve $\nu_{ij} = 0$ for $i = 3, \dots, r$.

Observe that our new matrix $P$ still satisfies the conditions of Remark 1.4.5. For the integers $\mu_i$ defined there, we have

$$\mu_0 + \mu_1 + \mu_2 \;=\; \mu_3 \;=\; \dots \;=\; \mu_r \;=\; \imath_X. \tag{1.4.7.1}$$

Moreover, for $i = 0, 1, 2$ set $\ell_i := l_{01}l_{11}l_{21}/l_{i1}$. Then, because of $\imath_X + \mu_i l_{ij} = \nu_{ij}\zeta_X$, we obtain

$$\gcd(\ell_0, \ell_1, \ell_2)^{-1} \sum_{i=0}^{2} \ell_i(\imath_X - \mu_i l_{ij}) \;=\; \alpha\zeta_X \quad \text{for some } \alpha \in \mathbb{Z}. \tag{1.4.7.2}$$

Finally, Remark 1.4.5 ensures

$$1 \;=\; \gcd(\mu_1, \dots, \mu_r, \zeta_X, \imath_X) \;=\; \gcd(\mu_1, \mu_2, \zeta_X, \imath_X). \tag{1.4.7.3}$$

We will now apply these conditions to establish the table of the assertion. Since $(l_{01}, l_{11}, l_{21})$ is a platonic triple, we have to discuss the following cases.

*Case 1*: $(l_{01}, l_{11}, l_{21})$ equals $(5, 3, 2)$. Our task is to rule out this case. Using (1.4.7.1) and (1.4.7.2), we see that $\zeta_X$ divides

$$\imath_X \;=\; 31\imath_X - 30(\mu_0 + \mu_1 + \mu_2) \;=\; 6(\imath_X - 5\mu_0) + 10(\imath_X - 3\mu_1) + 15(\imath_X - 2\mu_2).$$

Consequently, (1.4.7.3) becomes $\gcd(\mu_1, \mu_2, \zeta_X) = 1$ and from $\imath_X - \mu_i l_{ij} = \nu_{ij}\zeta_X$ we infer that $\zeta_X$ divides $5\mu_0$, $3\mu_1$ and $2\mu_2$. This leaves us with the three possibilities $\zeta_X = 2, 3, 6$.

If $\zeta_X = 2$ holds, then $\zeta_X$ divides $\mu_0$ and $\mu_1$ but not $\mu_2$; if $\zeta_X = 3$ holds, then $\zeta_X$ divides $\mu_0$ and $\mu_2$ but not $\mu_1$. Both contradict to the fact that $\zeta_X$ divides $\imath_X = \mu_0 + \mu_1 + \mu_2$. Thus, only $\zeta_X = 6$ is left. In that case, $\zeta_X$ must divide $\mu_0$. Since $\zeta_X$ divides $\imath_X = \mu_0 + \mu_1 + \mu_2$, we see that $\zeta_X$ divides $\mu_1 + \mu_2$. Moreover, $\zeta_X \mid 3\mu_1$ gives $\mu_1 = 2\mu_1'$ and $\zeta_X \mid 2\mu_2$ gives $\mu_2 = 3\mu_2'$ with integers $\mu_1', \mu_2'$. Now, as $\zeta_X = 6$ divides $2\mu_1' + 3\mu_2'$, we obtain that $\mu_2'$ and hence $\mu_2$ are even. This contradicts $\gcd(\mu_1, \mu_2, \zeta_X) = 1$.

*Case 2*: $(l_{01}, l_{11}, l_{21})$ equals $(4, 3, 2)$. Similarly as in the preceding case, we apply (1.4.7.1) and (1.4.7.2) to see that $\zeta_X$ divides

$$\imath_X \;=\; 13\imath_X - 12(\mu_0 + \mu_1 + \mu_2) \;=\; \frac{1}{2}\big(6(\imath_X - 4\mu_0) + 8(\imath_X - 3\mu_1) + 12(\imath_X - 2\mu_2)\big).$$

As before, we conclude $\gcd(\mu_1, \mu_2, \zeta_X) = 1$ and obtain that $\zeta_X$ divides $4\mu_0$, $3\mu_1$ and $2\mu_2$. This reduces to $\zeta_X = 2, 3, 6$.

If $\zeta_X = 3$ holds, then $\zeta_X$ divides $\mu_0$ and $\mu_2$ but not $\mu_1$, contradicting the fact that $\zeta_X$ divides $\imath_X = \mu_0 + \mu_1 + \mu_2$. If $\zeta_X = 6$ holds, then we obtain $\mu_0 = 3\mu_0'$, $\mu_1 = 2\mu_1'$ and $\mu_2 = 3\mu_2'$ with suitable integers $\mu_i'$. Since $\zeta_X$ divides $\imath_X = \mu_0 + \mu_1 + \mu_2$, we obtain that $\mu_2$ is divisible by 3, contradicting $\gcd(\mu_1, \mu_2, \zeta_X) = 1$.

Thus, the only possibility left is $\zeta_X = 2$. We show that this leads to Case (i) of the assertion. Observe that $\mu_1$ is even, $\mu_2$ is odd because of $\gcd(\mu_1, \mu_2, \zeta_X) = 1$ and $\mu_2$ is odd because $\imath_X = \mu_0 + \mu_1 + \mu_2$ is even. Recall that the vectors $\nu_i$ in the last row of $P$ are given as

$$\nu_i \;=\; \frac{1}{\zeta_X}(\boldsymbol{\imath_X} - \mu_i l_i) \;=\; \frac{1}{2}\boldsymbol{\imath_X} - \frac{\mu_i}{2} l_i.$$

Thus, adding the $(-\mu_0 - \mu_2)/2$-fold of the first row and the $(\mu_2 - 1)/2$-fold of the second row to the last row brings $P$ into the shape of Case (i).

*Case 3*: $(l_{01}, l_{11}, l_{21})$ equals $(3, 3, 2)$. As in the two preceding cases, we infer from (1.4.7.1) and (1.4.7.2) that $\zeta_X$ divides

$$\imath_X \;=\; 7\imath_X - 6(\mu_0 + \mu_1 + \mu_2) \;=\; \frac{1}{3}\big(6(\imath_X - 3\mu_0) + 6(\imath_X - 3\mu_1) + 9(\imath_X - 2\mu_2)\big).$$

Since $\gcd(\mu_1, \mu_2, \zeta_X) = 1$ and $\zeta_X$ divides $3\mu_0$, $3\mu_1$, $2\mu_2$, we are left with $\zeta_X = 2, 3, 6$. If $\zeta_X = 2$ or $\zeta_X = 6$ holds, then $\mu_0$, $\mu_1$ and $\imath_X = \mu_0 + \mu_1 + \mu_2$ must be even. Thus also $\mu_2$ must be even, contradicting $\gcd(\mu_1, \mu_2, \zeta_X) = 1$.

Let $\zeta_X = 3$. We show that this leads to Case (ii) of the assertion. First, 3 divides $\mu_2$ and $\imath_X = \mu_0 + \mu_1 + \mu_2$, hence also $\mu_0 + \mu_1$. Moreover, 3 divides neither $\mu_0$ nor $\mu_1$ because of $\gcd(\mu_1, \mu_2, \zeta_X) = 1$. Interchanging, if necessary, the data of the column blocks no. 0 and 1, we achieve that 3 divides $\mu_0 - 1$ and $\mu_1 + 1$. So, at the moment, the $\nu_i$ in the last row of $P$ are of the form

$$\nu_i \;=\; \frac{1}{\zeta_X}(\boldsymbol{\imath_X} - \mu_i l_i) \;=\; \frac{1}{3}\boldsymbol{\imath_X} - \frac{\mu_i}{3} l_i.$$

Adding the $(\mu_1 + 1)/3$-fold of the first and the $(-\mu_0 - \mu_1)/3$-fold of the second to the last row of $P$, we arrive at Case (ii).

*Case 4*: $(l_{01}, l_{11}, l_{21})$ equals $(k, 2, 2)$ with $k \geq 3$ odd. Then (1.4.7.1) and (1.4.7.2) show that $\zeta_X$ divides

$$2\imath_X \;=\; (2+2k)\imath_X - 2k(\mu_0 + \mu_1 + \mu_2) \;=\; \frac{1}{2}(4(\imath_X - k\mu_0) + 2k(\imath_X - 2\mu_1) + 2k(\imath_X - 2\mu_2)).$$

*Case 4.1*: $\zeta_X$ doesn't divide $\imath_X$. Then we have $2\imath_X = \alpha\zeta_X$ with $\alpha \in \mathbb{Z}$ odd. Thus, $\zeta_X$ is even and $2\mu_i = \imath_X - \nu_{ij}\zeta_X$ implies that $4\mu_i$ is an odd multiple of $\zeta_X$ for $i = 1, 2$. In particular, 4 divides $\zeta_X$. Moreover, (1.4.7.3) implies $\gcd(\mu_1, \mu_2, \zeta_X/2) = 1$ and we obtain $\zeta_X = 4$. That means $\imath_X \equiv 2 \mod 4$. Since $\zeta_X = 4$ divides $\imath_X - k\mu_0$ and $k$ is odd, we conclude $\mu_0 \equiv 2 \mod 4$. Then $\mu_0 + \mu_1 + \mu_2 = \imath_X \equiv 2 \mod 4$ implies that 4 divides $\mu_1 + \mu_2$. Interchanging, if necessary, the data of the column blocks no. 1 and 2, we can assume $\mu_1 \equiv -\mu_2 \equiv 1 \mod 4$. Then, adding the $(\mu_1 - 1)/4$-fold of the first and the $(\mu_2 + 1)/4$-fold of the second to the last row of $P$, we arrive at Case (iii) of the assertion.

*Case 4.2*: $\zeta_X$ divides $\imath_X$. Then (1.4.7.3) becomes $\gcd(\mu_1, \mu_2, \zeta_X) = 1$. Since $\zeta_X$ divides $2\mu_1$ and $2\mu_2$, we see that $\zeta = 2$ holds and $\mu_1$, $\mu_2$ are odd. Adding the $(\mu_1 - 1)/2$-fold of the first and the $(\mu_2 + 1)/2$-fold of the second to the last row of $P$ leads to Case (v) of the assertion.

*Case 5*: $(l_{01}, l_{11}, l_{21})$ equals $(k, 2, 2)$ with $k \geq 2$ even. Then (1.4.7.1) and (1.4.7.2) show that $\zeta_X$ divides

$$\imath_X \;=\; (k+1)\imath_X - k(\mu_0 + \mu_1 + \mu_2) \;=\; \frac{1}{4}(4(\imath_X - k\mu_0) + 2k(\imath_X - 2\mu_1) + 2k(\imath_X - 2\mu_2)).$$

As earlier, we conclude that $\zeta_X | 2\mu_i$ for $i = 1, 2$ and $\zeta_X = 2$. Since $\gcd(\mu_1, \mu_2, 2) = 1$ holds and $\mu_0 + \mu_1 + \mu_2 = \imath_X$ is even, two of the $\mu_i$ are be odd and one is even. If $\mu_1$ and $\mu_2$ are odd, then adding the $(\mu_1 - 1)/2$-fold of the first and the $(\mu_2 + 1)/2$-fold of the second to the last row of $P$ leads to Case (v). Now, let $\mu_0$ be odd. Interchanging, if necessary, the data of the column blocks no. 1 and 2, we achieve that $\mu_1$ is odd. Then we add the $(\mu_1 + 1)/2$-fold of the first and the $(-\mu_0 - \mu_1)/2$-fold of the second to the last row of $P$ and arrive at Case (iv) of the assertion.

*Case 6.* $(l_{01}, l_{11}, l_{21})$ equals $(k_0, k_1, 1)$, where $k_0, k_1 \in \mathbb{Z}_{>0}$. We subtract the $\nu_{21}$-fold of the second row of $P$ from the last one. Since $\nu_{21} = (\imath_X - \mu_2)/\zeta_X$ holds, we obtain $\nu_2 = \zeta_X^{-1}(\boldsymbol{\imath_X} - \imath_X l_2)$. Moreover, (1.4.7.1) becomes $\mu_0 + \mu_1 = 0$. We arrive at Case (vi) of the assertion by observing

$$(\imath_X - \nu_{0j_0}\zeta_X)/l_{0j_0} \;=\; \mu_0 \;=\; -\mu_1 \;=\; (\nu_{1j_1}\zeta_X - \imath_X)/l_{1j_1}.$$

□

Example 1.4.8. We discuss the rational affine $\mathbb{C}^*$-surfaces $X$ with at most log terminal singularities. First, the affine toric surfaces $X = \mathbb{C}^2/C_k$ show up here, where $C_k$ is the cyclic group of order $k$ acting diagonally. In terms of toric geometry, these surfaces are given as

$$X \;=\; \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2], \qquad \sigma = \operatorname{cone}((k, \imath),\, (\imath, k + m)),$$

where $k, m \in \mathbb{Z}_{>0}$ with $\gcd(k, \imath) = \gcd(k + m, \imath) = 1$ and $\imath$ is the Gorenstein index of $X$; see [**31**, Chap. 10] for more background. Now consider a quasicone non-toric $\mathbb{C}^*$-surface $X = X(A, P)$. As a quotient of $\mathbb{C}^2$ by a finite group, $X$ has finite divisor class group [**23**] and thus $P$ is a $3 \times 3$ matrix of the shape

$$P \;=\; \begin{bmatrix} -l_{01} & l_{11} & 0 \\ -l_{01} & 0 & l_{21} \\ d_{01} & d_{11} & d_{21} \end{bmatrix}.$$

Theorem 5 says that $(l_{01}, l_{11}, l_{21})$ is a platonic triple. Moreover, Corollary 1.4.6 and Proposition 1.4.7 provide us with constraints on the $d_{i1}$. Having in mind that $P$ is of rank three with primitive columns, one directly arrives at the following possibilities, where $\zeta = \zeta_X$ is the canonical multiplicity and $\imath = \imath_X$ the Gorenstein index:

| Type | $P$ | $\zeta$ | $\imath$ |
|---|---|---|---|
| $D_n^{1,\imath}$ | $\begin{bmatrix} -n+2 & 2 & 0 \\ -n+2 & 0 & 2 \\ -n\imath+3\imath & \imath & \imath \end{bmatrix}$ | 1 | $\gcd(\imath, 2n) = 1$ |
| $D_{2n+1}^{2,\imath}$ | $\begin{bmatrix} -2n+1 & 2 & 0 \\ -2n+1 & 0 & 2 \\ (1-n)\imath & \imath/2+1 & \imath/2-1 \end{bmatrix}$ | 2 | $\gcd(\imath, 8n-4) = 4$ |
| $E_6^{1,\imath}$ | $\begin{bmatrix} -3 & 3 & 0 \\ -3 & 0 & 2 \\ -2\imath & \imath & \imath \end{bmatrix}$ | 1 | $\gcd(\imath, 6) = 1$ |
| $E_6^{3,\imath}$ | $\begin{bmatrix} -3 & 3 & 0 \\ -3 & 0 & 2 \\ \imath/3-1 & \imath/3+1 & -\imath/3 \end{bmatrix}$ | 3 | $\gcd(\imath, 18) = 9$ |
| $E_7^{1,\imath}$ | $\begin{bmatrix} -4 & 3 & 0 \\ -4 & 0 & 2 \\ -3\imath & \imath & \imath \end{bmatrix}$ | 1 | $\gcd(\imath, 6) = 1$ |
| $E_8^{1,\imath}$ | $\begin{bmatrix} -5 & 3 & 0 \\ -5 & 0 & 2 \\ -4\imath & \imath & \imath \end{bmatrix}$ | 1 | $\gcd(\imath, 30) = 1$ |

For geometric details on these surfaces, we refer to the work of Brieskorn [**23**], and, in the context of the McKay Correspondence, Wunram [**106**] and Wemyss [**100**].

CHAPTER 2

# Gorensteinness and iteration of Cox rings

In this chapter, we prove that finitely generated Cox rings are Gorenstein and that iteration of Cox rings is finite with factorial master Cox ring for Fano type varieties and klt quasicones. Moreover, we determine explicitly how Cox rings iterate in the case of $T$-varieties of complexity one.

The results of Sections 2.1–2.5, where we prove Theorems 1–5 and Lemma 1, are contained in the preprint [**20**]. Those of Section 2.6 have been published in the joint work [**5**] with Arzhantsev, Hausen and Wrobel.

## 2.1. Quotient presentations

In [**48**], Hashimoto introduced the notion of *almost principal fiber bundles.* The definition is as follows.

Definition 2.1.1. Let $Y$ and $X$ be normal varieties, moreover let $G$ be an affine algebraic group, acting on $Y$ such that $\varphi\colon Y \to X =: Y /\!\!/ G$ is a good quotient. Then we call $\varphi$ an almost principal $G$-bundle if there exist open subsets $U \subseteq Y$ and $V \subseteq X$ with complement of codimension at least two so that

$$\varphi|_U : U \to V$$

is a principal $G$-bundle.

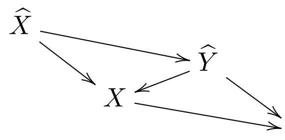
This notion naturally comes into play in the setting of Cox rings, as the representation of an MDS $X$ as the quotient $\widehat{X} /\!\!/ H_X$ is of such kind, see [**6**, Prop. 1.6.1.6]. In fact, $V$ can always be chosen to be the regular locus of $X$. Moreover, the quotient presentations from Definition 1.1.4 having the property that the Cox construction factors through them are *almost principal quasitorus bundles* with the additional assumption that all invertible functions homogeneous with respect to the grading group are constant. We will in the following use the notions quotient presentation and almost principal quasitorus bundle interchangeably, where we assume only constant invertible homogeneous functions and that $V$ from Definition 2.1.1 can be chosen as $X_{\mathrm{reg}}$.

Okawa has shown in [**73**] that if $f\colon X \to Y$ is a surjective morphism of projective varieties and $X$ is an MDS, then $Y$ is as well. Bäker showed in [**8**] that the same holds if $f\colon X \to Y = X /\!\!/ G$ is a quotient presentation with not necessarily projective normal $X$ and a reductive affine algebraic group $G$. We can in fact say more if $G$ is a quasitorus:

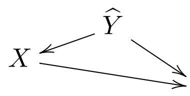
Proposition 2.1.2. *Let $\varphi\colon X \to Y$ be an almost principal $H$-bundle of normal varieties with $H$ a quasitorus. We have the following:*

(i) *If $H$ is a torus, then $X$ is an* MDS *if and only if $Y$ is so and in this case $\mathcal{R}(X) = \mathcal{R}(Y)$.*

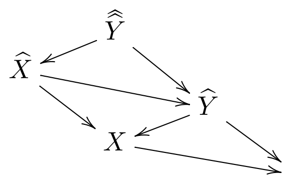
(ii) *If $X$ is an* MDS*, then $Y$ is so and we have a commutative diagram of almost principal quasitorus bundles:*




(iii) *If $Y$ is an* MDS*, then its characteristic space is an almost principal quasitorus bundle over $X$, i.e. we have at least the following commutative subdiagram:*




*Moreover, if $\widehat{Y}$ is an* MDS*, then $X$ is so and we get the enhanced diagram*




Remark 2.1.3. One can observe that here iteration of Cox rings comes into play, that is, if $Y$ has iteration of Cox rings with at least two steps, then $X$ is an MDS. For further generalizations see Proposition 2.4.5 and of course Theorem 4.

Before proving Proposition 2.1.2, we state a corollary that is crucial for both Gorensteinness of Cox rings and finiteness of Cox ring iteration. It shows that among the quotient presentations of $X$, the two exceptional ones $\widehat{X}/E \to X$ and $\widehat{X} /\!\!/ \mathbb{T} \to X$ - with $E$ and $\mathbb{T}$ being the torsion and torus part of $H_X$ - have very special properties.

Corollary 2.1.4 (Lemma 1). *Let $X$ be an* MDS *and $H_X = E \times \mathbb{T}$ its characteristic quasitorus with torsion and torus part $E$ and $\mathbb{T}$ respectively. We have the following:*

(i) *The Cox rings of $X$ and of the geometric quotient $X_E := \widehat{X}/E$ coincide. In particular, $X_E$ is a quasiaffine* MDS *and almost factorial.*
(ii) *The characteristic space $\widehat{X}$ is an* MDS *if and only if $X_{\mathbb{T}} := \widehat{X} /\!\!/ \mathbb{T}$ is an* MDS *and in that case, their Cox rings coincide.*

Proof. The assertions follow directly if one of $E$ or $\mathbb{T}$ is trivial - note that $\widehat{X}$ is factorial in the first case. So assume that none of them is trivial.

We begin with (i). By Proposition 2.1.2, since $X_E \to X$ is an almost principal torus bundle, both share the same Cox ring. Thus $E$ is the characteristic quasitorus of $X_E$ and since it is finite and $X_E$ is open in $\overline{X}/E$, which is affine, it is quasiaffine and almost factorial. Also (ii) follows directly from (i) of Proposition 2.1.2. □

Remark 2.1.5. In fact in Corollary 2.1.4 (i), we can replace $E$ by $E \times T$ for any subtorus $T \subseteq \mathbb{T}$ and still $X_{E\times T}$ has Cox ring $\mathcal{R}(X)$, while quasiaffineness may fail.

Proof of Proposition 2.1.2. We consider the first assertion. If $X$ is an MDS, then $\mathcal{R}(X)$ is factorially $\mathrm{Cl}(X)$-graded. Thus by Theorem 1.1.2, it is also factorially $\mathrm{Cl}(X) \times \mathbb{X}(H)$-graded, since the character group $\mathbb{X}(H)$ of $H$ is torsion free. All involved actions are free over smooth loci. By [**6**, Thm. 1.6.4.3], we get that $Y$ is an MDS and its Cox ring coincides with $\mathcal{R}(X)$.

Now assume $Y$ is an MDS. Then by [**6**, Thm. 4.2.1.4], we have a commutative diagram of quotient presentations

$$\widehat{Y} \xrightarrow{/\!\!/(H_Y/H)} X \xrightarrow{/\!\!/H} Y, \qquad \widehat{Y} \xrightarrow{/\!\!/H_Y} Y .$$

Since $\mathcal{R}(Y)$ is factorially $\mathrm{Cl}(Y)$-graded and $\mathrm{Cl}(Y) = \mathbb{X}(H_Y/H) \times \mathbb{X}(H)$, it is factorially $\mathbb{X}(H_Y/H)$-graded by Theorem 1.1.2. By [**6**, Thm. 1.6.4.3], the Cox ring of $X$ is $\mathcal{R}(Y)$.

We come to assertion (ii). If $X$ is an MDS, then $Y$ is so by [**8**, Thm. 1.1] and by [**6**, Thm. 4.2.1.4], we get a quotient presentation $\widehat{Y} \to X$. Applying [**6**, Thm. 4.2.1.4] again, we get a quotient presentation $\widehat{X} \to \widehat{Y}$ and thus the desired diagram. For (iii), apply [**6**, Thm. 4.2.1.4] several times as above. □

The following proposition uses reduction to positive characteristic. We refer to [**56, 47, 89**] for definitions and details, with an emphasis on Cox ring related problems in [**43, 1**].

The main purpose of [**48**] is to show how properties are preserved by almost principal fiber bundles. In the same spirit, varying [**64**, Prop. 5.20], we show the following on preservation of log-terminality.

Proposition 2.1.6. *Let $\varphi\colon X \to Y := X /\!\!/ G$ be an almost principal $G$-bundle with $G$ an affine algebraic group. Let $X$ and $Y$ be $\mathbb{Q}$-Gorenstein. Then $X$ is log-terminal if and only if $Y$ is so.*

Proof. By [**97**, Cor. 3.4], a pair $(Z, \Delta_Z)$ is Kawamata log-terminal if and only if it is of locally $F$-regular type in the sense that all local rings are of strongly $F$-regular type. For reduction modulo $p$, see [**43**, Ch. 2] and [**56**, Ch. 2]. So the assertion follows from [**48**, Thm. 13.14]. □

Remark 2.1.7. If [**48**, Thm. 13.14] is valid for *pairs* as well, then we can state Proposition 2.1.6 also for pairs $(X, \Delta_X)$, $(Y, \Delta_Y)$. Also note that there is a different definition of being of locally $F$-regular type than the one used above. Namely the a priori stronger one that $Z$ is covered by affine globally $F$-regular subsets, see [**89**, Def. 3.1]. One can show that these two notions are in fact equivalent, see [**88**] with slightly different definitions.

The statement of Proposition 2.1.6 has a different, more local nature than the corresponding ones [**43**, Prop. 4.6] and [**1**, Prop. 7.17]. These guarantee not only log-terminality but the global property of being of Fano type for a variety $X$ if the Cox ring is log-terminal. Combining these two viewpoints, we get the following.

Corollary 2.1.8. *Let $X$ be a log-terminal* MDS *that is not of Fano type. Then the complement of $\widehat{X}$ in $\overline{X}$ contains all non log-terminal singularities of $\overline{X}$.*

## 2.2. Cox rings are Gorenstein

The purpose of this section is to prove Theorem 1. For convenience, we state it here again in the following form:

THEOREM 2.2.1 (Theorem 1). *Let $X$ be an* MDS. *Then $K_{\overline{X}}$ is Cartier. In particular, if $\overline{X}$ is Cohen Macaulay, it is Gorenstein. If $\overline{X}$ is rational, it is Gorenstein canonical.*

REMARK 2.2.2. In general, one cannot expect a finitely generated Cox ring to be Cohen Macaulay or $\mathbb{Q}$-factorial. For a non Cohen Macaulay example by Gongyo, see the paper [**28**] by Brown. For a collection of non $\mathbb{Q}$-factorial examples, see [**5**, Thm. 1.9], where the toric one given by $xy + z^a w^b$ is probably the easiest one. It is the Cox ring of the affine threefold with $(\mathbb{C}^*)^2$-action given by

$$x^2 + y^2 z + z^a w^b.$$

PROOF OF THEOREM 2.2.1. If $H_X$ is torsion free, then $\overline{X}$ is factorial and thus numerically Gorenstein. Otherwise, consider $Y := \widehat{X}/E$, where $E$ is the torsion part of $H_X$. Then $Y$ is almost factorial by Corollary 2.1.4. Then a multiple of $K_X$ is principal, say $rK_X = 0$ in $\mathrm{Cl}(Y)$. This means we have its *global* canonical or index-1-cover $\rho\colon \mathcal{Y} \to Y$, see [**64**, 5.19]. Since $\rho$ is étale over $Y_{\mathrm{reg}}$, $\mathbb{Z}/r\mathbb{Z}$ acts strongly stably on $\mathcal{Y}$ so that $\rho$ becomes a quotient presentation of $Y$. Now [**6**, Thm. 4.2.1.4] gives us the commutative diagram of quotient presentations

$$\widehat{X} \xrightarrow{/\!\!/(H_Y/(\mathbb{Z}/r\mathbb{Z}))} \mathcal{Y} \xrightarrow{/\!\!/(\mathbb{Z}/r\mathbb{Z})} Y, \qquad \widehat{X} \xrightarrow{/\!\!/H_Y} Y.$$

Note that even if $\mathcal{R}(\mathcal{Y})$ exists, it may differ from $\mathcal{R}(X)$. Now since $K_{\mathcal{Y}}$ is Cartier and $\widehat{X} \to \mathcal{Y}$ is unramified over $\mathcal{Y}_{\mathrm{reg}}$, also $K_{\widehat{X}}$ is Cartier, see [**64**, Prop. 5.20]. So $\mathcal{R}(X)$ is numerically Gorenstein. The additions follow directly. ☐

In general, one cannot expect $K_{\widehat{X}}$ to be trivial, though $K_{\mathcal{Y}}$ is. An important class of varieties where this is true are the quasicones from Section 2.3.

REMARK 2.2.3. The proof of Theorem 2.2.1 suggests the notion of a - non-finite but étale in codimension one - index-1-cover of non-$\mathbb{Q}$-factorial singularities with finitely generated divisor class group. It is clear that $\mathcal{Y}$ coincides with the usual index-1-cover of $X$ if $X$ is $\mathbb{Q}$-factorial, while for non $\mathbb{Q}$-factorial but $\mathbb{Q}$-Gorenstein $X$, both differ from each other. A true generalization would require that there is a unique minimal subtorus $T \subseteq \mathbb{T}$ such that $X_{H_X/T}$ is $\mathbb{Q}$-Gorenstein. Then one could define the usual index-1-cover of $X_{H_X/T}$ to be the *generalized index-1-cover* of $X$.

## 2.3. Quasicone Cox rings

In [**16**], it was shown that varieties of Fano type are MDS. So in a way, the Fano type property *guarantees* MDS-ness, while non Fano type varieties may or may not be MDS. In this section, we give a criterion (namely being a *klt* quasicone) for affine varieties that guarantees MDS-ness and, moreover, guarantees *preservation* of MDS-ness by taking Cox rings. This is exactly what we need for iteration of Cox rings. Since we also show that Cox rings of Fano varieties are such quasicones, one can say that also the Fano type property not only guarantees MDS-ness but also its preservation.

Definition 2.3.1. An affine variety $X = \operatorname{Spec}(A)$ is called a *quasicone*, if one of the following equivalent conditions holds:

(i) $X$ allows a $\mathbb{C}^*$-action and the closures of all $\mathbb{C}^*$-orbits meet in one common point.
(ii) $A$ is $\mathbb{Z}_{\geq 0}$ graded and $A_0 = \mathbb{C}$.
(iii) $A$ is $\mathbb{Z}_{\geq 0}$ graded and has homogeneous generators $a_1, \ldots, a_n$ of strictly positive degree.

For a set of homogeneous generators of $A$ as in (iii) and the corresponding embedding of $X$ into $\mathbb{C}^n$, the origin is the common point of the orbit closures from (i), it is called the *vertex* of $X$. We refer to [**34, 37**] for detailed treatments of quasicones.

Lemma 2.3.2. *Let $X$ be a normal quasicone. Then* $\operatorname{Pic}(X) = 0$.

Proof. This is [**70**, Le. 5.1], compare also [**40**, Cor. 10.3] and [**5**, Cor. 2.14]. □

Remark 2.3.3. If $X$ is not normal, $\operatorname{Pic}(X)$ can be nontrivial. Consider for example $A = \mathbb{C}[x, y]/(x^3 - y^2)$ with $\operatorname{Pic}(A) = \mathbb{C}_+$.

Proposition 2.3.4. *Let $X$ be an* MDS *that is either a quasicone or has only constant global regular functions, e.g. $X$ is complete. Then $\overline{X}$ is a* quasicone*. If $x$ is the vertex of $\overline{X}$,* $\operatorname{Cl}(\overline{X}, x) \cong \operatorname{Cl}(\overline{X})$ *and every irreducible component of* $\operatorname{Sing}(\overline{X})$ *intersects $x$ nontrivially.*

Proof. First assume $X$ has only constant global regular functions. We have a decomposition $H_X = E \times \mathbb{T}$ of the characteristic quasitorus of $X$ with nontrivial torus part $\mathbb{T}$, since otherwise $X = X_E$ would be quasiaffine with Proposition 2.1.4. Then the finite morphism $X_{\mathbb{T}} \to X$ is proper, hence $X_{\mathbb{T}}$ has only constant global regular functions. Consider the $\mathbb{Z}^k$-grading of $A := \mathcal{R}(X)$ corresponding to the quotient $\widehat{X} \to X_{\mathbb{T}}$, observe $A_0 = \mathbb{C}$ and set $\mathfrak{m} = A \setminus A_0$. Since the weight cone contains no line, by appropriately projecting onto $\mathbb{Z}$, we can assume $k = 1$ and $\overline{X}$ is a quasicone.

Now let $X$ be a quasicone. Here the coordinate ring $A := \mathbb{C}[X]$ is $\mathbb{Z}_{\geq 0}$-graded and generators *and relations* of $A$ are of strictly positive degree. The grading of $A$ can be lifted to $\mathcal{R}(X)$ and thus $\mathcal{R}(X)$ is $\mathbb{Z}^{\dim(H)+1}$ graded, where $H$ is the characteristic quasitorus of $X$. As we have $\mathcal{R}(X)_{0_H} = A_0 = \mathbb{C}$, the $\mathbb{Z}^{\dim(H)+1}$-weight cone of $\mathcal{R}(X)$ again contains no lines and by appropriately projecting onto $\mathbb{Z}$, we see that $\overline{X}$ is a quasicone.

The remaining assertions follow directly from $X$ being a quasicone. □

Remark 2.3.5. If an affine MDS $X$ is a quasicone, the fan of the canonical ambient toric variety $Z$, see Chapter 1, is in fact (almost) *a cone*, since the torus action of $X$ comes from the torus action on $Z$ and thus the vertex of $X$ is the torus fixed point of $Z$. In general, the fan of the canonical ambient toric variety of affine MDS that are not quasicones is only a subfan of the fan of faces of a cone, possibly not containing the big cone itself.

Proof of Theorem 2. If $X$ is projective, then the assertion follows from [**43**, Thm. 1.1] and Theorem 2.2.1, Proposition 2.3.4. If $X$ is affine, then let $(X, \Delta)$ be klt. In particular, since $\operatorname{Pic}(X)$ is trivial, an integer multiple of $K_X + \Delta$ is trivial. By [**16**, Cor. 1.4.3], there is a *small* birational contraction $\varphi\colon Y \to X$, which is a log terminal model. There is a $\mathbb{Q}$-divisor $\Delta' \geq 0$ on $Y$ such that $K_Y + \Delta' = \varphi^*(K_x + \Delta)$. It may occur that $K_Y + \Delta'$ has no trivial $\mathbb{N}$-multiple any more, but it is still $\mathbb{Q}$-factorial and nef. By [**16**, Cor. 3.9.2], $K_Y + \Delta'$ is semiample and thus by the proof of [**16**, Cor. 1.1.9], see also [**16**, Sec. 1.3], $\mathcal{R}(Y)$ and thus $\mathcal{R}(X)$ is finitely generated. Since $X$ is klt, it is of globally or equivalently locally $F$-regular type, see [**97**, Cor. 3.4] and for the equivalence [**89**]. On the other hand, since $\mathcal{R}(X)$ is a Gorenstein quasicone by Theorem 2.2.1 and Proposition 2.3.4, it is of $F$-regular type if and only if it is canonical. Now we follow exactly the lines of [**43**, Proof of Thm. 4.7] and see that $\mathcal{R}(X)$ is canonical. If $X$ is affine and $\mathcal{R}(X)$ is a Gorenstein canonical quasicone, we still have to show that $X$ is a quasicone, i.e. the converse of Proposition 2.3.4 in the affine case. But this is clear since if $T_X \subseteq \mathbb{T}$ is the torus part of $H_X$ inside the maximal torus $\mathbb{T}$ acting on $\overline{X}$, then with respect to the corresponding $\mathbb{Z}^{\dim(T_X)}$-grading of $\mathcal{R}(X)$, we have $\mathbb{C} \neq \mathbb{C}[X] \subseteq \mathcal{R}(X)_0$, i.e. $\dim(\mathbb{T}/T_X) \geq 0$ and $\mathbb{C}[X]_0 = \mathbb{C}$ with respect to the corresponding $\mathbb{Z}^{\dim(\mathbb{T}/T_X)}$-grading. By projecting to $\mathbb{Z}$ as in the proof of Proposition 2.3.4, we see that $X$ is a quasicone. ☐

We can also directly infer Theorem 5 from Theorem 2:

Proof of Theorem 5. By Theorem 2 a projective (affine) variety $X$ is of Fano type (a klt quasicone) if and only if $\mathcal{R}(X)$ is a Gorenstein canonical quasicone. If now $X$ is a $T$-variety of complexity one, we see that $R(A, P)$ is (with a different grading) also the Cox ring of a $\mathbb{Q}$-factorial quasicone $Y$ by appropriately enhancing the corresponding matrix $P_0$ to a matrix $P$, see Constructions 1.2.1, 1.2.2. Observe that for example $Y$ can be taken as $X_E$ from Lemma 1. Then by Theorem 2 again $Y$ is log terminal if and only if $\mathcal{R}(X)$ is a Gorenstein canonical quasicone. Now due to 1.3.11, $Y$ is log terminal if and only if the maximal exponents $\ell_i = \max(l_{i1}, \ldots, l_{in_i})$ in the relations of $\mathcal{R}(X)$ form a platonic tuple. Altogether, we arrive at the statement of Theorem 5. ☐

## 2.4. Iteration of Cox rings

In this section, we introduce iteration of Cox rings and prove some related results.

Definition 2.4.1. Let $X$ be an MDS. If the characteristic space $X^{(1)} := \widehat{X}$ or equivalently the total coordinate space $\mathcal{X}^{(1)} := \overline{X}$ is an MDS as well, we can consider their characteristic and total coordinate spaces $X^{(2)} := \widehat{\widehat{X}}$ and $\mathcal{X}^{(2)} := \overline{\overline{X}}$. By iterating this procedure, we get a commutative diagram

(2.4.1.1)
$$
\begin{array}{ccccccccc}
\cdots\!\!\to & \mathcal{X}^{(3)} & \xrightarrow{/\!\!/ H_2} & \mathcal{X}^{(2)} & \xrightarrow{/\!\!/ H_1} & \mathcal{X}^{(1)} & & \\
 & \uparrow & & \uparrow & & \uparrow & & \\
\cdots\!\!\to & X^{(3)} & \xrightarrow{/\!\!/ H_2} & X^{(2)} & \xrightarrow{/\!\!/ H_1} & X^{(1)} & \xrightarrow{/\!\!/ H_0} & X
\end{array}
$$

of 1-Gorenstein varieties (with the possible exception of $X$). Here horizontal arrows stand for quotients by characteristic quasitori and vertical arrows stand for the inclusions of the $X^{(i)}$ as open subsets with complement of codimension at least two in $\mathcal{X}^{(i)}$. We call this the *iteration of Cox rings* of $X$. If $X$ is affine, then so are the $X^{(i)} = \mathcal{X}^{(i)}$.

REMARK 2.4.2. In the *iteration of Cox rings* of an MDS $X$, we have three possibilities:

- For each $i \in \mathbb{N}$, $X^{(i)}$ is an MDS with nontrivial divisor class group $\mathrm{Cl}(X^{(i)})$. We say that $X$ has *infinite iteration of Cox rings* and set $N := \infty$.
- For some $N \in \mathbb{N}$, either $X^{(N)}$ is not an MDS or it is factorial. We call $\mathfrak{R}(X) := \mathbb{K}[X^{(N)}]$ the *master Cox ring* of $X$. In this case the diagram 2.4.1.1 is finite and we say that $X$ has *finite iteration of Cox rings*.
  - If $X^{(N)}$ is factorial, we say that $X$ has factorial master Cox ring.
  - If $X^{(N)}$ is not an MDS, we say that $X$ has non-MDS master Cox ring.

PROPOSITION 2.4.3. *For each natural number $i \leq N$, we can represent $X$ as a quotient of $X^{(i)}$ by a solvable reductive group $G_i$ acting strongly stably and the chain of abelian quotients*

$$X^{(i)} \xrightarrow{/\!\!/ H_{i-1}} \cdots \xrightarrow{/\!\!/ H_2} X^{(2)} \xrightarrow{/\!\!/ H_1} X^{(1)} \xrightarrow{/\!\!/ H_0} X$$

*can be retrieved by setting $H_j := G_i^{(j-1)}/G_i^{(j)}$ and $X^{(j)} := X^{(i)} /\!\!/ G_i^{(j-1)}$ for the $k$-th derived subgroups $G_i^{(k)}$ of $G_i$.*

PROOF. Following [**5**, Proof of Thm. 3], we construct solvable linear algebraic groups $G_j \subseteq \mathrm{Aut}(X_j)$ acting algebraically on $X_j$ such that the unit component $G_j^0 \subseteq G_j$ is a torus, $G_j$ contains $H_{j-1}$ as a normal subgroup, $G_{j-1} = G_j/H_{j-1}$ holds and we have $G_1 = H_0$.

Start with $G_1 := H_0$, acting on $X^{(1)}$. According to [**6**, Thm. 2.4.3.2], there exists an (effective) action of a torus $\mathcal{G}_1$ on $X^{(2)}$ lifting the action of $G_1^0$ on $X^{(1)}$ and commuting with the action of $H_1$ on $X^{(2)}$. Moreover, [**7**, Thm. 5.1] provides us with an exact sequence of groups

$$1 \longrightarrow H_1 \longrightarrow \mathrm{Aut}(X^{(2)}, H_1) \xrightarrow{\pi} \mathrm{Aut}(X^{(1)}) \longrightarrow 1 ,$$

where $\mathrm{Aut}(X^{(2)}, H_1)$ denotes the group of automorphisms of $X^{(2)}$ normalizing the quasitorus $H_1$. Set $G_2 := \pi^{-1}(G_1)$. Then $H_1^0\mathcal{G}_1$, as a factor group of the torus $H_1^0 \times \mathcal{G}_1$ by a closed subgroup, is an algebraic torus and it is of finite index in $G_2$. Thus, $G_2$ is an affine algebraic group with $G_2^0 = H_1^0\mathcal{G}_1$ being a torus. By construction, $H_1 \subseteq G_2$ is the kernel of $\alpha_1 := \pi|_{G_2}$ and hence a normal subgroup. Moreover, $G_2$ is solvable and acts algebraically on $X^{(2)}$. Iterating this procedure gives a sequence

$$G_j \xrightarrow{\alpha_{j-1}} G_{j-1} \xrightarrow{\alpha_{j-2}} \quad \ldots \quad \xrightarrow{\alpha_2} G_2 \xrightarrow{\alpha_1} G_1 \xrightarrow{\alpha_0} 1$$

of group epimorphisms, where, as wanted, $G_j$ is a solvable reductive group acting algebraically on $X^{(j)}$ such that $H_j = \ker(\alpha_j)$ is the characteristic quasitorus of $X^{(j)}$.

From [**6**, Prop. 1.6.1.6], we infer that $G_1 = H_1$ acts freely on the preimage $U_2 \subseteq X_2$ of the set of smooth points $U_1 \subseteq X_1$ and moreover, the complement $X_2 \setminus U_2$ is of codimension at least two in $X_2$. Let $U_3 \subseteq X_3$ be the preimage of $U_2 \subseteq X_2$. Again, the complement of $U_3$ is of codimension at least two in $X_3$ and, as $U_2$ consists of smooth points of $X_2$, the quasitorus $H_2$ acts freely on $U_3$. Because of $G_2/H_2 = G_1$, we conclude that $U_3$ is $G_2$-invariant and $G_2$ acts freely on $U_2$. Repeating this procedure, we end up with an open set $U_i \subseteq X_i$ having complement of codimension at least two such that $G_i$ acts freely on $U_i$. Thus, $G_i$ acts strongly stably on $X_i$. Now set $\mathcal{D}_j := G_i/G_j$ for $j \leq i$. Then $X^{(j)} = X^{(i)} /\!\!/ \mathcal{D}_j$. and $H_j = \mathcal{D}_{j-1}/\mathcal{D}_j$. Moreover for each $\mathcal{D}_j$, its action on $X_i$ is strongly stable, as remarked before.

Let us check that each $\mathcal{D}_k$-stable divisor on $X^{(i)}$ is principal. But this holds since $X^{(i)} \to X^{(i)} /\!\!/ \mathcal{D}_k$ factors through the characteristic spaces

$$X^{(i)} \to X^{(i-1)} \to \ldots \to X^{(k)} = X^{(i)} /\!\!/ \mathcal{D}_k.$$

Now by [**7**, Prop. 3.5], we obtain a commutative diagram

$$\begin{array}{ccc} X^{(i)} /\!\!/ [\mathcal{D}_j, \mathcal{D}_j] & \xrightarrow{\quad \beta \quad} & X^{(i)} /\!\!/ \mathcal{D}_{j+1} \\ & {\scriptstyle /\!\!/ \mathcal{D}_j/[\mathcal{D}_j,\mathcal{D}_j]} \searrow \quad \swarrow {\scriptstyle /\!\!/ \mathcal{D}_j/\mathcal{D}_{j+1}} & \\ & X^{(i)} /\!\!/ \mathcal{D}_j & \end{array}$$

where the left downward map is a total coordinate space. As $\mathcal{D}_j/\mathcal{D}_{j+1} = H_{j+1}$ is abelian, $[\mathcal{D}_j, \mathcal{D}_j]$ is contained in $\mathcal{D}_{j+1}$ and we have the horizontal morphism $\beta$. Since the right hand side is a total coordinate space as well, we infer from [**6**, Sec. 1.6.4] that $\beta$ is an isomorphism. This finally implies $\mathcal{D}_{i+1} = [\mathcal{D}_i, \mathcal{D}_i]$. □

Example 2.4.4. It is clear that toric varieties trivially have finite iteration of Cox rings with polynomial master Cox ring. On the other hand, in [**42**], it was shown that *spherical* varieties have finite iteration of Cox rings with factorial master Cox ring and at most two iteration steps.

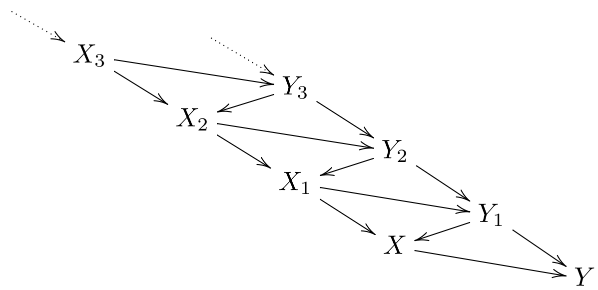
Originally the notion of Cox ring iteration was introduced in [**5**] by Arzhantsev, Hausen, Wrobel and the author, where it was also shown that for log terminal singularities with a torus action of complexity one, Cox ring iteration is finite with factorial master Cox ring. Moreover, the possible chains for these singularities have been calculated explicitly in [**5**, Rem. 6.7]. These results can be found in Section 2.6. In dimension two, one retrieves exactly the representation of the log-terminal singularities as finite quotients of $\mathbb{C}^2$, i.e. the Cox ring iteration chains of those singularities form a tree with the single root $\mathbb{C}^2$, as was explained in the introduction. Recall that in Theorem 12 in the introduction, we give the Cox ring iteration tree for *compound du Val* and canonical threefold singularities with a two-torus action. It turns out that all master Cox rings here are compound du Val.

In [**54**], Hausen and Wrobel gave criteria for affine varieties with a torus action of complexity one to have finite iteration of Cox rings with factorial master Cox ring. In fact, one can directly verify from their computations that all varieties with a torus action of complexity one have finite iteration of Cox rings, though with possibly non-MDS master Cox ring.

The following *relative* version of Cox ring iteration is the first generalization of Proposition 2.1.2, from which then directly follows the most general version, the *web of Cox ring iterations* from Theorem 4.

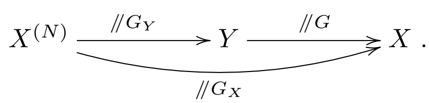
Proposition 2.4.5 (Relative iteration of Cox rings). *Let the quasitorus $H$ act on $X$, such that the good quotient $Y := X /\!\!/ H$ exists and is an almost principal $H$-bundle. Then $X$ has finite iteration of Cox rings if and only if $Y$ has so. Moreover, $X$ has factorial master Cox ring if and only if $Y$ has so and in this case,* $\mathfrak{R}(X) = \mathfrak{R}(Y)$.

Proof. If $H$ is torsion free, by Proposition 2.1.2 (i), $X$ and $Y$ have the same Cox ring, i.e. the same iteration of Cox rings from the second step on. So assume $H$ has torsion. From Proposition 2.1.2 (iii), we get a Cox ring iteration ladder:



So if $X_i$ is an MDS, then so is $Y_i$ and if $Y_{j+1}$ is an MDS, then so is $X_j$. Thus $X$ has finite iteration of Cox rings if and only if $Y$ has so. If $\mathfrak{R}(X)$ is the factorial master Cox ring of $X$, then $X_N$ is an almost principal quasitorus bundle over $Y_N$ and $\mathfrak{R}(X)$ is factorially graded with respect to the associated character group, since it is factorial. So $\mathfrak{R}(X)$ is the Cox ring of $Y_N$ and thus the master Cox ring of $Y$. If on the other hand $\mathfrak{R}(Y)$ is the factorial master Cox ring of $Y$, then $Y_N$ is an almost principal quasitorus bundle over $X_{N-1}$ and we can apply the same argument. □

Proof of Corollary 1. Let $Y \to X$ be a quotient by the solvable group $G$ as in the corollary. Then by any normal series of $G$ we get a chain of quotient presentations $Y \to \cdots \to X_3 \to X_2 \to X$. Thus by Theorem 2.2.1, $Y$ is an MDS with factorial master Cox ring $\mathfrak{R}(X)$ and the diagram



is commutative, which proves the assertion. □

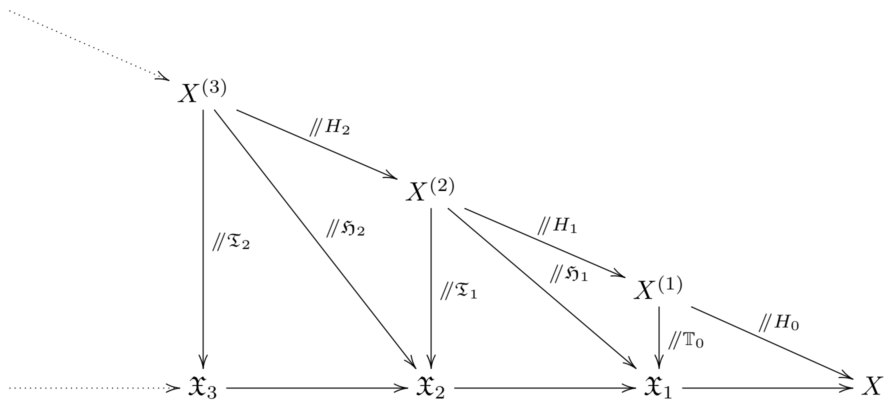
## 2.5. Iteration of Cox rings for Fano type varieties and klt quasicones

In this section, we prove Theorem 3 using results from previous sections. We restate the theorem for convenience:

Theorem 2.5.1 (Theorem 3). *Let $X$ be of Fano type or a* klt *quasicone. Then $X$ has finite iteration of Cox rings with factorial master Cox ring.*

Proof. Let $X$ be a klt quasicone and assume it has infinite iteration of Cox rings. If $H_i$ is a torus for some $i \in \mathbb{N}$, then $X_{i+1}$ is factorial by [**6**, Prop. 1.4.1.5]. So each $H_i$ has a nontrivial torsion part $E_i$ and a possibly trivial torus part $\mathbb{T}_i$.

Denote $\mathfrak{X}_1 := X_{\mathbb{T}} = X^{(1)} /\!\!/ \mathbb{T}_0$. By Corollary 2.1.4, $X^{(2)}$ is the characteristic space of $\mathfrak{X}_1$. Denote by $\mathfrak{H}_1$ the characteristic quasitorus of $\mathfrak{X}_1$ and by $\mathfrak{T}_1$ its torus part. Define $\mathfrak{X}_2 := X^{(2)} /\!\!/ \mathfrak{T}_1$. By iteration of Cox rings, we get the following infinite commutative diagram, where all diagonal arrows are characteristic spaces:




The bottom of this diagram gives a sequence of finite Galois morphisms between klt quasicones étale in codimension one such that all compositions are Galois as well, compare the lifting of the abelian group actions from Proposition 2.4.3. By [**45**, Thm. 1.1], all but finitely many of the $\mathfrak{X}_{i+1} \to \mathfrak{X}_i$ must be étale. But none of them is, since they are all ramified over the vertex, see [**6**, Prop. 1.6.2.5]. So $X$ must have finite iteration of Cox rings with master Cox ring $\mathfrak{R}(X) = \mathbb{C}[X^{(N)}]$ for some $N \in \mathbb{N}$.

Since all $X^{(i)}$ in the iteration of Cox rings of $X$ are Gorenstein canonical quasicones and thus MDS by Theorem 2, $X^{(N)}$ is an MDS and thus factorial. The assertion for varieties of Fano type follows directly. □

### 2.6. Iteration of Cox rings for klt quasicones of complexity one

In this section, for log terminal quasicones with a torus action of complexity one with coordinate ring $R(A, P_0)$ from Construction 1.2.1, we explicitly determine the exponents $l'_{ij}$ in the relations of $\mathcal{R}(\operatorname{Spec} R(A, P_0))$. This enables us to compute Cox ring iteration chains for such varieties.

The underlying observation is the following: the (relations of) Cox rings $R(A, P)$ of klt quasicones of complexity one have platonic exponent triples. We call such rings *platonic*. The maximal torus $H_0^0$ acting on $R(A, P)$ has character group $K_0^0 := K_0/K_0^{\mathrm{tors}}$, where $K_0 = \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*)$ and $P_0$ is the submatricx of $P$ given by Construction 1.2.2. We know that $\overline{X} = \operatorname{Spec} R(A, P_0)$ has finitely generated class group and Cox ring by Theorem 3. Thus the generic quotient of $\overline{X}$ by $H_0^0$ must be rational, see [**6**, Sec. 4.4.1]. We can then use [**6**, Thm. 4.4.1.6] to compute $\mathcal{R}(\overline{X})$.

Remark 2.6.1. Let $R(A, P)$ be a ring as in Construction 1.2.2 or 1.2.1. Applying suitable admissible operations, one achieves that $P$ is *ordered* in the sense that $l_{i1} \geq \ldots \geq l_{in_i}$ for all $i = 0, \ldots, r$ and $l_{01} \geq \ldots \geq l_{r1}$ hold. For an ordered $P$, the ring $R(A, P)$ is platonic if and only if $(l_{01}, l_{11}, l_{21})$ is a platonic triple and $l_{i1} = 1$ holds for $i \geq 3$.

Definition 2.6.2. The *leading platonic triple* of a ring $R(A, P)$ is the triple $(l_{01}, l_{11}, l_{21})$ obtained after ordering $P$. Set $\mathfrak{l}_i := \gcd(l_{i1}, \ldots, l_{n_i})$ and $\mathfrak{l} := \gcd(\mathfrak{l}_0, \ldots, \mathfrak{l}_r)$

LEMMA 2.6.3. *Consider a matrix $P_0$ with $m = 0$ and $r = 2$ as in Construction 1.2.1:*

$$P_0 \;=\; \begin{bmatrix} -l_0 & l_1 & 0 \\ -l_0 & 0 & l_2 \end{bmatrix}.$$

*Then, with $\mathfrak{l}_{ij} = \gcd(\mathfrak{l}_i, \mathfrak{l}_j)$ , we obtain*

$$K_0^{\mathrm{tors}} \;=\; (\mathbb{Z}^n/\mathrm{im}(P_0^*))^{\mathrm{tors}} \;\cong\; \mathbb{Z}/\mathfrak{l}\mathbb{Z} \times \mathbb{Z}/(\mathfrak{l}\mathfrak{l}_{01}\mathfrak{l}_{02}\mathfrak{l}_{12})\mathbb{Z}.$$

PROOF. Suitable elementary column operations on $P_0$ transform the entries $l_i$ to $(\mathfrak{l}_i, 0, \ldots, 0)$. Thus, $K_0^{\mathrm{tors}} \cong (\mathbb{Z}^3/\mathrm{im}(P_1^*))^{\mathrm{tors}}$ holds with the $2 \times 3$ matrix

$$P_1 \;:=\; \begin{bmatrix} -\mathfrak{l}_0 & \mathfrak{l}_1 & 0 \\ -\mathfrak{l}_0 & 0 & \mathfrak{l}_2 \end{bmatrix}.$$

The determinantal divisors of $P_0$ are $\gcd(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ and $\gcd(\mathfrak{l}_0\mathfrak{l}_1, \mathfrak{l}_0\mathfrak{l}_2, \mathfrak{l}_1\mathfrak{l}_2)$. Thus, the invariant factors of $P_0$ are $\mathfrak{l}$ and $\mathfrak{l}\mathfrak{l}_{01}\mathfrak{l}_{02}\mathfrak{l}_{12}$; see [**72**]. □

Note that if $K_0^{tors}$ is trivial, $R(A, P_0)$ is factorial by [**4**]. Thus $R(A, P_0)$ is factorial if the $\mathfrak{l}_i$ are pairwise coprime.

LEMMA 2.6.4. *Let $R(A, P_0)$ be as in Construction 1.2.1 and platonic such that $l_{i1} \geq \ldots \geq l_{in_i}$ holds for all $i$ and $l_{i1} = 1$ for $i \geq 3$. Assume* $\gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \mathfrak{l}$. *Then, with $K_0 = \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*)$, the kernel of $\mathbb{Z}^{n+m} \to K_0/K_0^{\mathrm{tors}}$ is generated by the rows of the matrix*

$$P_1 \;:=\; \begin{bmatrix} \frac{-1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)}l_0 & \frac{1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)}l_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \frac{-1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}l_0 & 0 & \frac{1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}l_2 & 0 & 0 & & & \\ -l_0 & 0 & & \mathbf{1} & 0 & \vdots & & \vdots \\ \vdots & & & \vdots & \ddots & \vdots & & \\ -l_0 & 0 & \ldots & 0 & \mathbf{1} & 0 & \ldots & 0 \end{bmatrix},$$

*where the symbols $\mathbf{1}$ indicate vectors of length $n_i$ with all entries equal to one.*

PROOF. Observe that the rows of $P_0$ generate a sublattice of finite index in the row lattice $P_1$. Thus, we have a commutative diagram

$$\begin{array}{ccc} K_0 & \longrightarrow & K_0/K_0^{\mathrm{tors}} \\ & \searrow \quad \nearrow & \\ & \mathbb{Z}^{n+m}/\mathrm{im}(P_1^*) & \end{array}$$

It suffices to show that $\mathbb{Z}^{n+m}/\mathrm{im}(P_1^*)$ is torsion free. Applying suitable elementary column operations to $P_1$ reduces the problem to showing that for the $2 \times 3$ matrix

$$\begin{bmatrix} \frac{\mathfrak{l}_0}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} & \frac{\mathfrak{l}_1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} & 0 \\ \frac{\mathfrak{l}_0}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)} & 0 & \frac{\mathfrak{l}_2}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} \end{bmatrix},$$

all determinantal divisors equal one. The entries of the above matrix are coprime and its $2 \times 2$ minors are

$$\frac{\mathfrak{l}_0\mathfrak{l}_2}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}, \qquad \frac{\mathfrak{l}_1\mathfrak{l}_2}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}, \qquad \frac{\mathfrak{l}_0\mathfrak{l}_1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}.$$

up to sign. By assumption, we have $\gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \mathfrak{l}$. Consequently, we obtain

$$\gcd(\mathfrak{l}_0\mathfrak{l}_2, \mathfrak{l}_0\mathfrak{l}_1, \mathfrak{l}_1\mathfrak{l}_2) \;=\; \gcd(\mathfrak{l}_0\mathfrak{l}, \mathfrak{l}_1\mathfrak{l}_2) \;=\; \gcd(\mathfrak{l}_0, \mathfrak{l}_1)\gcd(\mathfrak{l}_0, \mathfrak{l}_2)$$

and therefore the second determinantal divisor equals one. The first one equals one as well due to coprime entries and the assertion follows. ☐

LEMMA 2.6.5. *Let $R(A, P_0)$ be as in Construction 1.2.1 and $\overline{X} = \operatorname{Spec} R(A, P_0)$. Then, for any generator $T_{01}$ of $R(A, P_0)$, we have*

$$V(\overline{X}, T_{01}) \ \cong\ V(T_{01}) \cap V(T_1^{l_1} - T_i^{l_i};\ i = 2, \ldots, r) \ \subseteq\ \mathbb{C}^{n+m}.$$

*In particular, the number of irreducible components of $V(\overline{X}, T_{01})$ equals the product of the invariant factors of the matrix*

$$\begin{bmatrix} -\mathfrak{l}_1 & \mathfrak{l}_2 & & 0 \\ \vdots & & \ddots & \\ -\mathfrak{l}_1 & 0 & & \mathfrak{l}_r \end{bmatrix}.$$

PROOF. First observe that the ideal $\langle T_{01}, g_0, \ldots, g_{r-2} \rangle \subseteq \mathbb{C}[T_{ij}, S_k]$ is generated by binomials which can be brought into the above form by scaling the variables appropriately. Now consider the homomorphism of tori

$$\pi\colon \mathbb{T}^{n_1+\ldots+n_r} \ \to\ \mathbb{T}^{r-1}, \qquad (t_1, \ldots, t_r) \ \mapsto\ \left( \frac{t_2^{l_2}}{t_1^{l_1}}, \ldots, \frac{t_r^{l_r}}{t_1^{l_1}} \right).$$

Then the number of connected components of $\ker(\pi)$ equals the product of the invariant factors of the above matrix. Moreover, $\mathbb{T}^{n_0-1} \times \ker(\pi) \times \mathbb{T}^m$ is isomorphic to $V(\overline{X}, T_{01}) \cap \mathbb{T}^{n+m}$. Finally, one directly checks that $V(\overline{X}, T_{01})$ has no further irreducible components outside $\mathbb{T}^{n+m}$. ☐

LEMMA 2.6.6. *Let $R(A, P_0)$ be as in Construction 1.2.1 with platonic exponent tuple. Assume that $P_0$ is ordered. Then the number $c(i)$ of irreducible components of $V(\overline{X}, T_{ij})$ is given as*

| $i$ | $0$ | $1$ | $2$ | $\geq 3$ |
|---|---|---|---|---|
| $c(i)$ | $\gcd(\mathfrak{l}_1, \mathfrak{l}_2)$ | $\gcd(\mathfrak{l}_0, \mathfrak{l}_2)$ | $\gcd(\mathfrak{l}_0, \mathfrak{l}_1)$ | $\mathfrak{l}^2 \gcd(\mathfrak{l}_0, \mathfrak{l}_1) \gcd(\mathfrak{l}_0, \mathfrak{l}_2) \gcd(\mathfrak{l}_1, \mathfrak{l}_2)$ |

PROOF. Suitable admissible operations turn $T_{ij}$ to $T_{01}$. Then the number of components is computed via Lemma 2.6.5. ☐

PROPOSITION 2.6.7. *Let $R(A, P_0)$ be as in Construction 1.2.1 with platonic exponent tuple. Assume that $P_0$ is ordered and let $P_1$ be as in Lemma 2.6.4. Set*

$$n_{i,1}, \ldots, n_{i,c(i)} \ :=\ n_i, \qquad l_{ij,1}, \ldots, l_{ij,c(i)} \ :=\ \gcd((P_1)_{1,ij}, \ldots, (P_1)_{r,ij}).$$

*The $l_{i,\alpha} := (l_{i1,\alpha}, \ldots, l_{in_i,\alpha}) \in \mathbb{Z}^{n_{i,\alpha}}$ build up an $(r'+s) \times (n'+m)$ matrix $P_0'$, where $n' := c(0)n_0 + \ldots + c(r)n_r$. With a suitable matrix $A'$, the following holds.*

(i) *The affine variety $\operatorname{Spec} R(A', P_0')$ is the total coordinate space of the affine variety $\operatorname{Spec} R(A, P_0)$,*

(ii) *The leading platonic triple (l.p.t.) of $R(A', P')$ can be expressed in terms of that of $R(A, P)$ as*

| *l.p.t. of $R(A, P)$* | *l.p.t. of $R(A', P')$* |
|---|---|
| $(4, 3, 2)$ | $(3, 3, 2)$ |
| $(3, 3, 2)$ | $(2, 2, 2)$ |
| $(y, 2, 2)$ | $(z, z, 1)$ *or* $\left(\frac{y}{2}, 2, 2\right)$ |
| $(x, y, 1)$ | $\left(\frac{x}{\gcd(\mathfrak{l}_0, \mathfrak{l}_1)}, \frac{y}{\gcd(\mathfrak{l}_0, \mathfrak{l}_1)}, 1\right)$ |

*Proof.* We compute the Cox ring of $\overline{X} = \operatorname{Spec} R(A, P_0)$ according to [**6**, Thm. 4.4.1.6]. By [**53**, Cor. 1.9], the statement given there is valid also in the affine case. That means that we have to figure out which invariant divisors are identified under the rational map onto the curve $Y$ with function field $\mathbb{C}(\overline{X})^{H_0^0}$, where $H_0^0$ is the unit component of $H_0 = \operatorname{Spec} \mathbb{C}[K_0]$, and we have to determine the orders of isotropy groups of invariant divisors.

Let $P_1$ be as in Lemma 2.6.4. Then the torus $H_0^0$ acts diagonally on $\mathbb{C}^{n+m}$ with weights provided by the projection $Q_1 \colon \mathbb{Z}^{n+m} \to K_0^0$, where $K_0^0 = \mathbb{Z}^{n+m}/\operatorname{im}(P_1^*)$ equals the character group of $H_0^0$. Consider the commutative diagram

$$\begin{array}{ccc} \overline{X}_0 & \subseteq & \mathbb{C}_0^{n+m} \\ \downarrow & & \downarrow \\ \overline{X}_0/H_0^0 & \subseteq & \mathbb{C}_0^{n+m}/H_0^0 \\ \downarrow & & \downarrow \\ Y & \subseteq & \mathbb{P} \end{array}$$

where $\overline{X}_0 \subseteq \overline{X}$ and $\mathbb{C}_0^{n+m} \subseteq \mathbb{C}^{n+m}$ denote the open $H_0^0$-invariant subsets obtained by removing all coordinate hyperplanes $V(S_k)$ and all intersections $V(T_{i_1j_1}, T_{i_2j_2})$ with $(i_1, j_1) \neq (i_2, j_2)$ from $\mathbb{C}^{n+m}$. Moreover, the geometric quotient spaces in the middle row are possibly non-separated and the maps to the lower row are separation morphisms.

We determine the orders of isotropy groups. Every point in $\mathbb{T}^{n+m}$ has trivial $H_0^0$-isotropy. Thus, we only have to look what happens on the sets $V(T_{ij}) \cap \mathbb{C}_0^{n+m}$. According to [**6**, Prop. 2.1.4.2], the order of isotropy group of $H_0^0$ at any point $x \in V(T_{ij}) \cap \mathbb{C}_0^{n+m}$ equals the greatest common divisor of the entries of the $ij$-th column of $P_1$:

$$|H_{0,x}^0| \ = \ l'_{ij} \ := \ \gcd((P_1)_{1,ij}, \ldots, (P_1)_{r,ij}) \quad \text{for all } x \in V(T_{ij}) \cap \mathbb{C}_0^{n+m}.$$

Now we figure out which $H_0^0$-invariant divisors of $\overline{X}_0$ are identified under the map $\overline{X}_0 \to Y$. Lemma 2.6.6 provides us explicit numbers $c(0), \ldots, c(r)$ such that for fixed $i$ and $j = 1, \ldots, n_i$, we have the decomposition into prime divisors

$$V(\overline{X}, T_{ij}) \ = \ D_{ij,1} \cup \ldots \cup D_{ij,c(i)},$$

in particular, the number $c(i)$ does not depend on the choice of $j$. The components $D_{ij,1}, \ldots, D_{ij,c(i)}$ lie in the common affine chart $W_0 \subseteq \overline{X}_0$ obtained by localizing at all $T_{i'j'}$ different from $T_{ij}$. Their images thus lie in the affine chart $W_0/H_0^0 \subseteq \overline{X}_0/H_0^0$. Consequently, the $D_{ij,1}, \ldots, D_{ij,c(i)}$ have pairwise disjoint images under the composition $\overline{X}_0 \to \overline{X}_0/H_0^0 \to Y$.

On the other hand, $V(\overline{X}, T_{ij})$ and $V(\overline{X}, T_{ij'})$ are identified isomorphically under the separation map $\overline{X}_0/H_0^0 \to Y$ Thus, suitably numbering, we obtain for every $i$, and $\alpha = 1, \ldots, c(i)$ a chain

$$D_{i1,\alpha}, \ldots, D_{in_i,\alpha},$$

of divisors identified under the morphism $\overline{X}_0/H_0^0 \to Y$. The order of isotropy for any $x \in D_{ij,\alpha}$ equals $l'_{ij}$. Now, using [**6**, Thm. 4.4.1.6], we can compute the defining relations of the Cox ring of $\overline{X}$, which establishes the two assertions. $\square$

Corollary 2.6.8. *Let $R(A, P_0)$ be a non factorial platonic ring with ordered $P_0$ and let as usual $\mathfrak{l}_i = \gcd(l_{i1}, \dots, l_{in_i})$ and $\overline{X} := \operatorname{Spec} R(A, P_0)$. Then the exponents of the defining relations of $\mathcal{R}(\overline{X})$ are listed in the following table.*

| $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ | *exponent vectors of* $\mathcal{R}(\overline{X})$ |
|---:|---|
| $(4, 3, 2)$ | $l_1,\ l_1,\ \frac{l_0}{2},\ \frac{l_2}{2},\ l_3,\ l_3,\ \dots,\ l_r,\ l_r$ |
| $(3, 3, 2)$ | $\frac{l_0}{3},\ \frac{l_1}{3},\ l_2,\ l_2,\ l_2,\ \dots,\ l_r,\ l_r,\ l_r$ |
| $(2k, 2, 2)$ | $\frac{l_0}{2},\ \frac{l_0}{2},\ \frac{l_1}{2},\ \frac{l_1}{2},\ \frac{l_2}{2},\ \frac{l_2}{2},\ l_3,\ l_3,\ l_3,\ l_3,\ \dots,\ l_r,\ l_r,\ l_r,\ l_r$ |
| $(2k + 1, 2, 2)$ | $l_0,\ l_0,\ \frac{l_1}{2},\ \frac{l_2}{2},\ l_3,\ l_3,\ \dots,\ l_r,\ l_r$ |
| $(\mathfrak{l}_0, \mathfrak{l}_1, 1)$ | $\frac{l_0}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)},\ \frac{l_1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)},\ \underbrace{l_2, \dots, l_2}_{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)},\ \dots,\ \underbrace{l_r, \dots, l_r}_{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)}$ |

CHAPTER 3

# Invariant rings of $\mathrm{SL}_n$

This chapter is devoted to the computation of invariant rings of special linear groups $\mathrm{SL}_n$ over the complex numbers. In the first section, we develop the graphical method to describe invariants of $\mathrm{SL}_n$ as sketched in the introduction. A related method for invariants of $\mathrm{SL}_2$ was developed by Olver and Shakiban in [**74**]. We apply our method in Section 3.2 to compute (minimal generating sets of) invariants, covariants - i.e. invariants of maximal unipotent subgroups - and relations of $\mathrm{SL}_4$ as well as invariants of $\mathrm{SL}_5$ for *sums of fundamental representations* - i.e. arbitrary reducible representations $W$ with irreducible subrepresentations being fundamental ones $\Lambda^k V$.

The third and fourth section develop algorithms used in the fifth one, where we complete the classification of $\mathrm{SL}_n$-representations with complete intersection invariant ring by Shmelkin [**91**]. The *Crosshair-Sieve algorithm* from Section 3.3 allows efficient computation of Gröbner bases in certain cases, for example quasicones. In Section 3.4, which is devoted to Hilbert series computations, on the one hand, we refine an algorithm by Xin [**107**] for *MacMahon partition analysis* and on the other hand we describe how to modify a standard algorithm for *univariate* Hilbert series to the *multivariate* case.

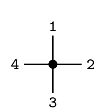
These algorithms together with the graphical method from Section 3.1 are then used to prove that six certain representations of $\mathrm{SL}_n$ that have been left open by Shmelkin in [**91**] have complete intersection invariant rings. The results of the present chapter are contained in the preprints [**18, 19, 21**].

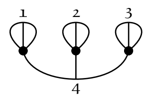
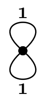
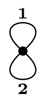
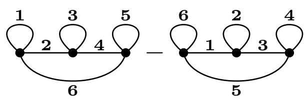
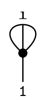
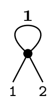
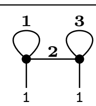
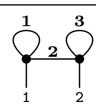
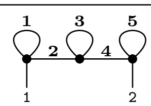
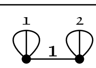
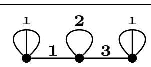
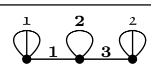
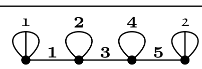
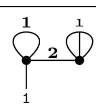
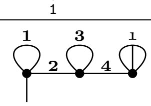
We now restate Theorem 6 on invariants of $\mathrm{SL}_4$ with more details, including the explicit forms of some invariants, and then present relations and covariants for $\mathrm{SL}_4$. Let $W$ be a sum of fundamental representations of $\mathrm{SL}_n$, i.e.

$$W := \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^{n_i} V_{i,j},$$

where $V_{i,j} := \Lambda^i V$. In the following, colors $1, 2, \ldots, N$ of $i$-edges stand representatively for arbitrary but ascending colors $1 \leq k_1, k_2, \ldots, k_N \leq n_i$.

Theorem 3.1. *Let $\mathrm{SL}_4$ act on an arbitrary sum of fundamental representations $W$. Then a minimal generating set of $\mathbb{C}[W]^{\mathrm{SL}_4}$ is the following, where if the respective invariant is too big to display, only the number of monomials is given.*

| *Graph* | *Invariant* | *Symbol* |
|---|---|---|
|  | $\begin{vmatrix} x_{11} \cdots x_{41} \\ \vdots \ \ddots \ \vdots \\ x_{14} \cdots x_{44} \end{vmatrix}$ | $\lvert 1234 \rvert$ |

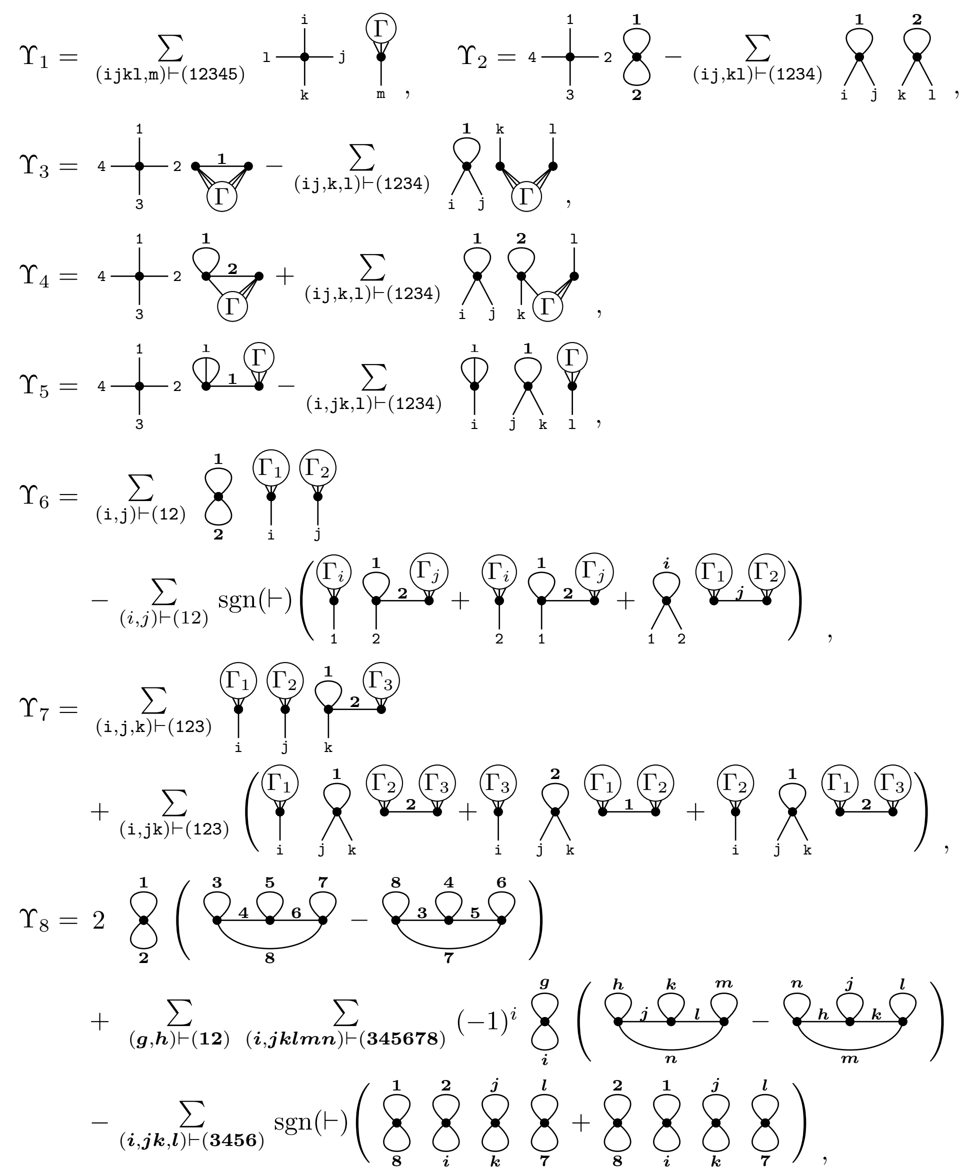
| | | |
|---|---|---|
|  | $\begin{vmatrix} z_{1234} \cdots z_{4234} \\ \vdots \ \ddots \ \vdots \\ z_{1123} \cdots z_{4123} \end{vmatrix}$ | $\left\|{}^{1234}\right\|$ |
|  | $2(y_{\mathbf{1}12}y_{\mathbf{1}34} - y_{\mathbf{1}13}y_{\mathbf{1}24} + y_{\mathbf{1}14}y_{\mathbf{1}23})$ | $\left\|\mathbf{11}\right\|$ |
|  | $y_{\mathbf{1}12}y_{\mathbf{2}34} - y_{\mathbf{1}13}y_{\mathbf{2}24} + y_{\mathbf{1}14}y_{\mathbf{2}23}$<br>$+y_{\mathbf{1}34}y_{\mathbf{2}12} - y_{\mathbf{1}24}y_{\mathbf{2}13} + y_{\mathbf{1}23}y_{\mathbf{2}14}$ | $\left\|\mathbf{12}\right\|$ |
|  | $\begin{vmatrix} y_{\mathbf{1}12} \cdots y_{\mathbf{6}12} \\ \vdots \ \ddots \ \vdots \\ y_{\mathbf{1}34} \cdots y_{\mathbf{6}34} \end{vmatrix}$ | $\left\|\mathbf{123456}\right\|$ |
|  | $x_{11}z_{1234} - x_{12}z_{1134} + x_{13}z_{1124} - x_{14}z_{1123}$ | $\left\|{}_1{}^1\right\|$ |
|  | $y_{\mathbf{1}12}(x_{13}x_{24}-x_{14}x_{23}) + y_{\mathbf{1}13}(x_{14}x_{22}-x_{12}x_{24})$<br>$+y_{\mathbf{1}14}(x_{12}x_{23}-x_{13}x_{22}) + y_{\mathbf{1}23}(x_{11}x_{24}-x_{14}x_{21})$<br>$+y_{\mathbf{1}24}(x_{13}x_{21}-x_{11}x_{23}) + y_{\mathbf{1}34}(x_{11}x_{22}-x_{12}x_{21})$ | $\left\|{}_1\mathbf{1}_2\right\|$ |
|  | 96 | $\left\|{}_1\mathbf{123}_1\right\|$ |
|  | 108 | $\left\|{}_1\mathbf{123}_2\right\|$ |
|  | 972 | $\left\|{}_1\mathbf{12345}_2\right\|$ |
|  | 12 | $\left\|{}^1\mathbf{1}^2\right\|$ |
|  | 96 | $\left\|{}^1\mathbf{123}^1\right\|$ |
|  | 108 | $\left\|{}^1\mathbf{123}^2\right\|$ |
|  | 972 | $\left\|{}^1\mathbf{12345}^2\right\|$ |
|  | 36 | $\left\|{}_1\mathbf{12}^1\right\|$ |
|  | 324 | $\left\|{}_1\mathbf{1234}^1\right\|$ |

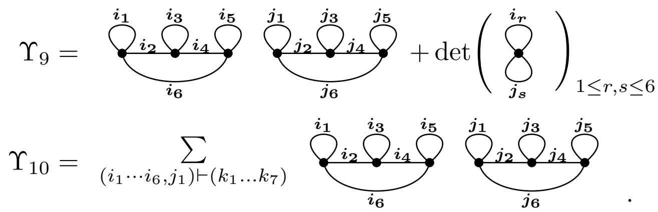
Now we present relations holding between the invariants from above. Most of them can be generalized to $n \geq 5$. The notation

$$(u_1, \ldots, u_M) \vdash (u)$$

means that we sum over all partitions of the word $u$ in $M$ subwords $u_k = l_{k,1} \cdots l_{k,N_k}$ of the lengths $N_k$. If we have an ordering on the letters of $u$, then we require $u$ and all subwords to be ordered and for every summand define $\mathrm{sgn}(\vdash)$ to be the sign of the underlying permutation of letters $u \mapsto u_1 \cdots u_M$.

THEOREM 3.2. *The following sums of graphs correspond to polynomials in the ideal of relations of* $\mathbb{C}[W]^{\mathrm{SL}_4}$*, where the not necessarily connected (sub-)graphs* $\Gamma$ *and* $\Gamma_i$ *have to be chosen such that all involved graphs are out of the minimal generating set from Theorem 3.1.*





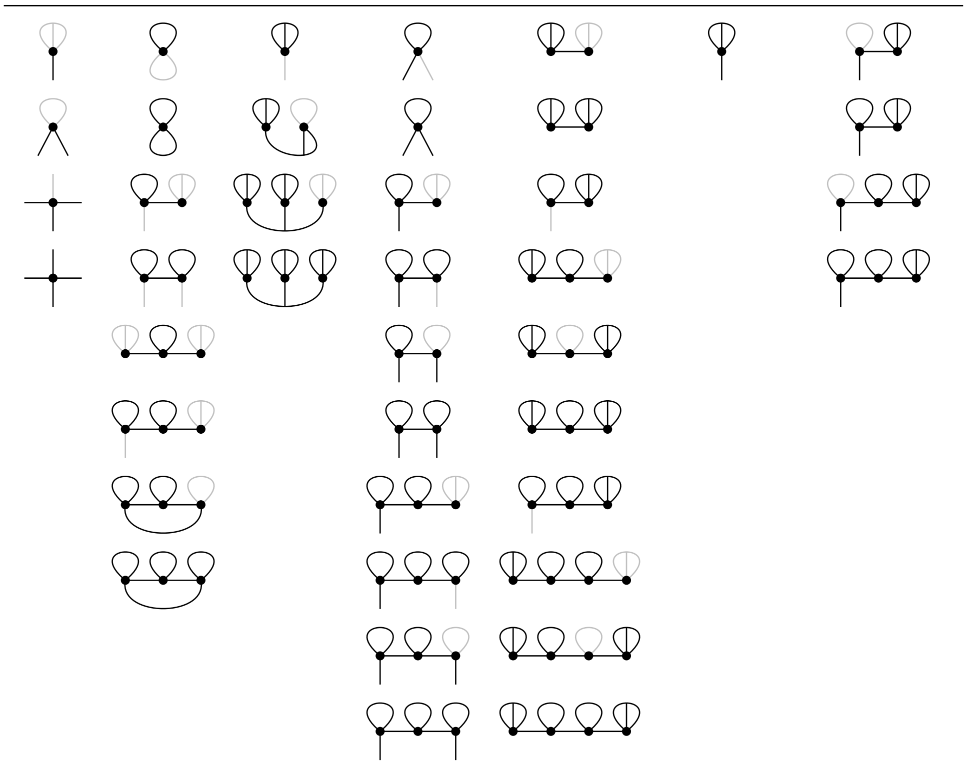

These relations emerge from somewhat natural principles that are discussed in the proof of Theorem 3.2 in Section 3.2.2. This fact together with calculations of the Hilbert series for small values of $n_i$ using [**35**, §4.6] and the algorithm from Section 3.4 leads us to the following:

Conjecture 3.3. *The graphsums from Theorem 3.2 generate the ideal of relations of* $\mathbb{C}[W]^{\mathrm{SL}_4}$.

The next theorem presents graphs for the covariants of $\mathrm{SL}_4$, where the gray 'dummy' edges can be seen as corresponding to dummy representations $W$ of $\mathrm{SL}_4$, i.e. a covariant is an invariant of the action of $\mathrm{SL}_4$ on $W \oplus W$. So it is a map from $W$ not to $\mathbb{C}$, but to $\mathbb{C}[W]$.

Theorem 3.4. *A minimal generating set for the algebra of covariants for the action of* $\mathrm{SL}_4$ *on a sum of fundamental representations* $W$ *is given by appropriate colorings of the (non-dummy) edges of the following graphs.*

| 1-*edges* | 2-*edges* | 3-*edges* | $(1,2)$-*edges* | $(2,3)$-*edges* | $(1,3)$-*edges* | $(1,2,3)$-*edges* |
|---|---|---|---|---|---|---|



## 3.1. Invariants of $\mathrm{SL}_n$ and colored hypergraphs

**3.1.1. Brackets and invariants.** This section is merely a summary of the parts of [**46**] that are relevant for antisymmetric tensors. All notation is as close as possible to the one from [**46**]. Fix a natural number $n$, a complex vector space $V$ of dimension $n$ and a basis $e_1, \ldots, e_n$ of $V$. Now let the special linear group $\mathrm{SL}_n$ act on $V$ by multiplication from the left. This induces an action of $\mathrm{SL}_n$ on $\Lambda^i V$ for every $1 \leq i \leq n-1$. Fix some integer $n_i \geq 0$ for every such $i$ and set $V_{i,j} := \Lambda^i V$ for every $1 \leq i \leq n-1$ and $1 \leq j \leq n_i$. Then the action of $\mathrm{SL}_n$ on $V$ finally induces an action on

$$W := W_{(n_1,\ldots,n_{n-1})} = \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^{n_i} V_{i,j} = \bigoplus_{i=1}^{n-1} \left( \bigwedge^i V \right)^{n_i}.$$

We can identify the ring of polynomial functions on $W$ with $\mathbb{C}\left[T_{i,j,\iota_1\cdots\iota_i}\right]$, where $1 \leq j \leq n_i$, $\{\iota_1 < \ldots < \iota_i\} \subseteq \{1, \ldots, n\}$. We do this by linearly mapping $T_{i,j,\iota_1\cdots\iota_i}$ onto a function $f$ so that for an element

$$t = \sum_{i=1}^{n-1} \sum_{j=1}^{n_i} \sum_{\iota_1 < \ldots < \iota_{\nu_i}} t_{i,j,\iota_1\cdots\iota_i} e_{\iota_1} \wedge \ldots \wedge e_{\iota_i}$$

of $W$ we have $f(t) = t_{i,j\cdots\iota_i}$. Now following [**46**] we introduce an ordered alphabet $P = \{1, \ldots, n\}$ with $1 < \ldots < n$ of so called *places* and the algebra $\mathrm{Ext}(P)$, which is the exterior algebra generated by the places. We denote multiplication in $\mathrm{Ext}(P)$ by juxtaposition. Moreover, for every $V_{i,j}$, we introduce an infinite number of so called *letters* $a_{i,j,k}$ for all $k \in \mathbb{N}$ forming the alphabet $L$. We set $a_{i_1,j_1,k_1} < a_{i_2,j_2,k_2}$ if either $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$ or $i_1 = i_2$ and $j_1 = j_2$ and $k_1 < k_2$.

Definition 3.1.1. Let $A$ be an alphabet, then the *divided powers algebra* $\mathrm{Div}(A)$ is the commutative algebra generated by symbols $a^{(i)}$, where $a \in A$ and $i \in \mathbb{N}$. We denote multiplication in $\mathrm{Div}(A)$ by juxtaposition. Moreover, we set $a^{(0)} = 1$, $a^{(1)} = a$, and impose the identity

$$a^{(i)} a^{(j)} = \binom{i+j}{j} a^{(i+j)}.$$

We define the length of the word $a^{(i)}$ to be $|a^{(i)}| = i$.

Now we proceed with the divided powers algebra $\mathrm{Div}(L)$ generated by the alphabet $L$ of letters and define a third alphabet $[L|P]$, the *letterplace alphabet* having as elements pairs $(x|\alpha)$, where $x \in L$, $\alpha \in P$. The algebra $\mathrm{Ext}([L|P])$ is called the *fourfold algebra*.

Definition 3.1.2. We define a bilinear form

$$(*|*) : \mathrm{Div}(L) \times \mathrm{Ext}(P) \to \mathrm{Ext}([L|P]),$$

called the *biproduct*, by the following:

(i) $(w|v) = 0$ if $w$ and $v$ are words of different length.
(ii) $(w|v) = (x|\alpha)$ if $w = x$ is a letter and $v = \alpha$ is a place, thus the image is a single letterplace.
(iii) $(1|1) = 1$.
(iv) $(w|vu) = \sum_{w_1 w_2 = w} (w_1|v)\,(w_2|u)$, where the sum ranges over all pairs $w_1, w_2$ of subwords of $w$ such that $w_1 w_2 = w$.

(v) $(vu|w) = \sum_{w_1 w_2 = \pm w} (-1)^{\delta(w_1,w_2)} (v|w_1)(u|w_2)$, where $\delta(w_1,w_2)$ is the number of transpositions needed to obtain the word $w$ from the word $w_1 w_2$.

We give some examples to clarify these rules.

EXAMPLE 3.1.3. Let $a \in L$. We have a look at the image of $(a^{(2)}, 12)$ under the biproduct. First we want to use Rule $(iv)$ with $v = 1$, $u = 2$. Due to Rule $(i)$ we only have to consider pairs of subwords of length one. We have $a^{(2)} = \frac{1}{2}aa$ and we have *two* pairs of possible subwords of length one, since we have to distinguish the two $a$'s. Thus we get

$$(a^{(2)}|12) = \frac{1}{2}(aa|12) = \frac{1}{2}((a|1)(a|2) + (a|1)(a|2)) = (a|1)(a|2).$$

Using Rule $(v)$ instead, we compute

$$(a^{(2)}|12) = \frac{1}{2}(aa|12) = \frac{1}{2}((a|1)(a|2) - (a|2)(a|1)) = (a|1)(a|2).$$

More generally, for arbitrary $i, j, k, l$, we get

$$\left(a_{i,j,k}^{(l)}|i_1 \dots i_l\right) = (a_{i,j,k}|i_1) \cdots (a_{i,j,k}|i_l)\,.$$

For different letters $a, b \in L$, we compute

$$(ab|12) = (a|1)(b|2) + (b|1)(a|2) = (ba|12).$$

Now for letters $a_1, \dots, a_n$, we define the *bracket* in $a_1, \dots, a_n$ to be the element

$$[a_1 \dots a_n] := (a_1 \cdots a_n | 1 \cdots n)$$

of $\mathrm{Ext}([L|P])$. A *bracket monomial* is a product of brackets and a *bracket polynomial* is a linear combination of bracket monomials. We denote the subalgebra of all bracket polynomials of $\mathrm{Ext}([L|P])$ by $\mathrm{Br}(L)$.

LEMMA 3.1.4 ([**46**], 'Exchange Lemma', p. 60). *Let $u, v, w$ be words in* $\mathrm{Div}(L)$. *Then*

$$\sum_{u_1 u_2 = u} [u_1 v]\,[u_2 w] \;=\; (-1)^{n-|w|} \sum_{v_1 v_2 = v} [v_1 u]\,[v_2 w]\,.$$

PROPOSITION 3.1.5. *All identities among bracket polynomials can be deduced from the identity of Lemma 3.1.4 with $|u| = 2$, $|w| = n - 1$.*

PROOF. The fact that all identities can be deduced from Lemma 3.1.4 follows directly from Theorem 8 of [**46**]. Thus - as was stated in [**46**] on page xv - it can be used for an abstract definition of (skew) brackets. The fact that all those identities stem from the ones with $\mathrm{Length}(u) = 2$ is clear. □

REMARK 3.1.6. In Proposition 3.1.5, one can replace '$|u| = 2$, $|w| = n-1$' with '$|u| = n+1$, $|w| = 0$'.

Finally, we bring together brackets and invariants of the action of $\mathrm{SL}_n$ on $W$ by the following linear map.

DEFINITION 3.1.7. Let the linear *umbral operator*

$$\begin{aligned} U : \mathrm{Ext}([L|P]) &\to \mathbb{C}[W] \\ f &\mapsto \langle U, f \rangle \end{aligned}$$

be defined by the following:

(i) $\left\langle U, \left(a_{i,j,k}^{(i)}|\iota_1 \ldots \iota_i\right)\right\rangle = T_{i,j,\iota_1,\ldots,\iota_i}$,

(ii) $\left\langle U, \left(a_{i,j,k}^{(l)}|\iota_1 \ldots \iota_l\right)\right\rangle = 0$ if $l \neq i$,

(iii) $\left\langle U, \prod_{i,j,k} \left(a_{i,j,k}^{(l_{i,j,k})}|\iota_1 \ldots \iota_{l_{i,j,k}}\right)\right\rangle = \prod_{i,j,k} \left\langle U, \left(a_{i,j,k}^{(l_{i,j,k})}|\iota_1 \ldots \iota_{l_{i,j,k}}\right)\right\rangle$,

where in (iii), the order of the letterplaces in the word $\prod_{i,j,k}(a_{i,j,k}^{(l_{i,j,k})}|\iota_1 \ldots \iota_{l_{i,j,k}})$ must be according to the order of the letters $a_{i,j,k}$.

Example 3.1.8. For arbitrary $n$ and any permutation $\sigma \in S_n$, we have

$$\langle U, [a_{1,1,1} \cdots a_{1,n,1}]\rangle = \left\langle U, \left[a_{1,\sigma(1),1} \cdots a_{1,\sigma(n),1}\right]\right\rangle = \begin{vmatrix} T_{1,1,1} & \cdots & T_{1,n,1} \\ \vdots & \ddots & \vdots \\ T_{1,1,n} & \cdots & T_{1,n,n} \end{vmatrix}$$

as $a_{1,1,1} < \ldots < a_{1,n,1}$.

Theorem 3.1.9 ([**46**], Thm. 18). *The umbral operator* $U : \mathrm{Ext}([L|P]) \to \mathbb{C}[W]$ *is surjective and its restriction to the bracket polynomials* $\mathrm{Br}(L)$ *is onto* $\mathbb{C}[W]^{\mathrm{SL}_n}$.

We are only interested in bracket polynomials that are not in the kernel of $U$. Thus in the following, we consider the subalgebra Bra(L) of *appropriate* bracket polynomials, where if a letter $a_{i,j,k}$ turns up in an appropriate bracket monomial, it does so exactly $i$ times. Of course, the restriction of $U$ to Bra(L) is still onto $\mathbb{C}[W]^{\mathrm{SL}_n}$.

**3.1.2. Brackets and graphs.** In this section, we develop the basis of our method: bracket polynomials are associated with formal sums of colored hypergraphs.

Definition 3.1.10. Let $X$ be a set. Then we denote by $\mathcal{M}(X)$ the set of nonempty multisets composed of elements of $X$.

Definition 3.1.11. Let $m$ be a positive integer. An undirected $n$-regular *colored hypergraph* $\Gamma$ with $m$ vertices is a pair $\Gamma = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1 < \ldots < v_m\}$ is the ordered set of *vertices* and $\mathcal{E} \in \mathcal{P}(\mathcal{M}(\mathcal{V}) \times \mathbb{N} \times \mathbb{N})$ is the set of colored *hyperedges* $e = (e_1, e_2, e_3)$, and for all vertices $v$, we have

$$\sum_{e \in E} \#_v(e_1) = n,$$

where $\#_v(e_1)$ is the *number of connections* of $e$ to $v$, i.e. the number of occurences of the element $v$ in the multiset $e_1$. By the *virtual degree* of a vertex $v$, we mean the number

$$\mathrm{vdeg}(v) = n - \sum_{(\{v,\ldots,v\},e_2,e_3) \in E} |\{v, \ldots, v\}|.$$

By the *effective graph* $\Gamma_{\text{eff}}$ of $\Gamma = (\mathcal{V}, \mathcal{E})$, we denote the subgraph

$$\Gamma_{\text{eff}} = (\mathcal{V}, \mathcal{E} \setminus \{e|e_1 = \{v, \ldots, v\}, v \in \mathcal{V}\}).$$

If $e = (e_1, e_2, e_3)$ is a hyperedge, then we call $k = |e_1|$ the *size*, $e_2 \in \mathbb{N}$ the *color* and $e_3 \in \mathbb{N}$ the *shading* of $e$. We call $e$ a $k$-edge. We say that $e$ is connected to $v$, if $v \in e_1$. If $e$ is connected to only one $v$, then we call it a *looping edge*.

Observe that multiple edges and loops are allowed in this definition of a hypergraph, so it is truly a *pseudo-hypergraph.*

DEFINITION 3.1.12. Now let $G$ be the $\mathbb{C}$-vector space of formal sums of $\mathbb{C}$-multiples of $n$-regular colored hypergraphs. On $G$, we define a (non-abelian) multiplication as follows. For $\Gamma_1 = (\{v_1 < \ldots < v_m\}, \mathcal{E}_1)$ and $\Gamma_2 = (\{w_1 < \ldots < w_m\}, \mathcal{E}_2)$ in $G$, we set

$$\Gamma_1\Gamma_2 := \Gamma_1 \cdot \Gamma_2 := (\{v_1 < \ldots < v_m < w_1 < \ldots < w_m\}, \mathcal{E}_1 \cup \mathcal{E}_2)$$

and extend to formal sums of graphs in the obvious way. This makes $G$ a $\mathbb{C}$-algebra. We call elements $\Upsilon = \sum a_i\Gamma_i \in G$ *graphsums.*

To a hypergraph $\Gamma = (\{v_1 < \ldots < v_m\}, \mathcal{E}_1)$, we associate a bracket monomial $p_\Gamma = b_1 \cdots b_m$ with brackets $b_1, \ldots, b_m$ defined by:

$$b_i := \left[\prod_{e\in\mathcal{E}} a_{|e_1|,e_2,e_3}^{(\#_{v_i}(e_1))}\right].$$

This gives a linear surjective map $\gamma : G \to \mathrm{Bra(L)}$ by setting

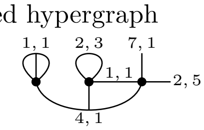
$$\gamma : \sum a_i\Gamma_i \mapsto \sum a_i p_{\Gamma_i}.$$

Now we set $\mathcal{G} := G/\ker(\gamma)$ and by $\gamma' : \mathcal{G} \to \mathrm{Bra}(L)$ denote the induced isomorphism.

EXAMPLE 3.1.13. We associate the bracket monomial

$$\left[a_{3,1,1}^{(3)}a_{3,4,1}\right]\left[a_{2,2,3}^{(2)}a_{3,4,1}a_{2,1,1}\right]\left[a_{3,4,1}a_{2,1,1}a_{1,7,1}a_{1,2,5}\right]$$

to the color-and-shading-labeled hypergraph




CONVENTION 3.1.14. We will often speak only of *graphs*, when we mean colored hypergraphs. Moreover, we will not number vertices of graphs, but will assume that they are ordered ascending from left to right. We also ignore shading of edges, assuming that all $k$-edges of the same color have different shading.

DEFINITION 3.1.15. We say that two graphsums $\Upsilon_1$ and $\Upsilon_2$ are *equivalent*, writing $\Upsilon_1 \simeq \Upsilon_2$ , if $\Upsilon_1 - \Upsilon_2 \in \ker(U \circ \gamma)$. We call a graphsum $\Upsilon$ *reducible*, if it is equivalent to zero or to some $\sum a_i\Gamma_i$ with all $\Gamma_i$ disconnected. A graphsum is irreducible if it is not reducible.

Several graphsums $\Upsilon_1, \ldots, \Upsilon_N$ are called *reducibly independent*, if the only reducible linear combination $\sum a_i\Upsilon_i$ is the trivial one. If for two graphsums $\Upsilon_1$, $\Upsilon_2$ the linear combination $\Upsilon_1 - \Upsilon_2$ is reducible, we call them *reducibly equivalent* and write $\Upsilon_1 \simeq_r \Upsilon_2$. We say that a set of reducibly independent irreducible graphs has property $(RI)$.

REMARK 3.1.16. The Exchange Lemma 3.1.4 leads to equivalencies between graphsums. If two graphsums are equivalent in this way, for any pair of graphs occuring in the two graphsums, there is a color-preserving one-to-one-correspondence between the edges.

THEOREM 3.1.17. *Let $M$ be a maximal set with property $(RI)$. Then $M$ is in one-to-one-correspondence to a minimal generating set of $\mathbb{C}[W]^{\mathrm{SL}_n}$ by $\Gamma \mapsto U\circ\gamma(\Gamma)$.*

PROOF. Let $M$ be a maximal set of reducibly independent irreducible graphs in $G$. Let $F := U \circ \gamma(M)$. Since $U \circ \gamma$ is surjective, for any element $f$ of $\mathbb{C}[W]^{SL_n}$, we have a graphsum $\Gamma$ with $U \circ \gamma(\Gamma) = f$. If $\Gamma$ is irreducible, then either $\Gamma \in M$ and thus $f \in F$, or if $\Gamma \notin M$, due to maximality of $M$, there is a reducible nontrivial linear combination

$$\Gamma + \sum_{\Gamma' \in M} a_{\Gamma'} \Gamma'.$$

Due to linearity of $U \circ \gamma$, we can proceed with reducible $\Gamma$. Either $\Gamma = 0$, then $f = 0$, or $\Gamma$ is equivalent to a graphsum of disconnected graphs. We can assume all connected subgraphs are irreducible and thus proceed with an irreducible graph, where the number of vertices is strictly less than that of $\Gamma$. Since the number of vertices of graphs is bounded from below, this procedure comes to an end. So $F$ generates $\mathbb{C}[W]^{SL_n}$. The minimality of $F$ follows immediately from $M$ being reducibly independent. □

LEMMA 3.1.18. *Let the graph $\Gamma = (\mathcal{V}, \mathcal{E})$ have a hyperedge $e' = (e'_1, e'_2, e'_3)$ with $v \in e'_1$. Then $\Gamma$ is equivalent to a graphsum $\sum_i (\mathcal{V}, \mathcal{E}_i)$ where $\mathcal{E}_i \setminus \{e | v \in e_1\} \subseteq \mathcal{E} \setminus \{e | v \in e_1\}$ and $(\{v, ..., v\}, e'_2, e'_3) \in \mathcal{E}_i$ for all $i$.*

PROOF. We can assume that $v$ is the smallest element of $\mathcal{V}$. If $\#_v(e'_1) = |e'_1|$, we are done. Thus take $\#_v(e'_1) = \kappa < |e'_1|$. We can assume that for the second smallest element $v'$ of $\mathcal{V}$ we have $\#'_v(e'_1) = \nu \geq 1$. So

$$\gamma(\Gamma) = \left[a^{(\kappa)}_{|e'_1|, e'_2, e'_3} w_1\right] \left[a^{(\nu)}_{|e'_1|, e'_2, e'_3} w_2\right] \mathfrak{m}$$

for some $w_1, w_2 \in L$ and $\mathfrak{m}$ a bracket monomial. Now applying Lemma 3.1.4 with $u = a^{(\kappa+\nu)}_{|e'_1|, e'_2, e'_3}$, we see that $\Gamma$ is equivalent to a graphsum $\sum_i (\mathcal{V}, \mathcal{E}_i)$ where $\mathcal{E}_i \setminus \{e | v \in e_1\} \subseteq \mathcal{E} \setminus \{e | v \in e_1\}$ and $(e''_1, e'_2, e'_3) \in \mathcal{E}_i$ with $\#_v(e''_1) = \kappa + \nu$. Iterating this gives the desired result. □

REMARK 3.1.19. When searching for a maximal set of reducibly independent irreducible graphs, Lemma 3.1.18 can be used to simplify the graphs effectively by *making the biggest edges looping.*

PROPOSITION 3.1.20. *For each $1 \leq k \leq n$, if $n_k \geq \binom{n}{k}$ and $J \subseteq \{1, \ldots, n_k\}$ has cardinality $\binom{n}{k}$, then $\det : \oplus_{j \in J} V_{k,j} \to \mathbb{C}$ equals $U \circ \gamma(\Upsilon_{\det(k,J)})$ for some irreducible graphsum $\Upsilon_{\det(k,J)}$.*

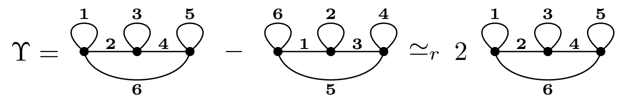
*Moreover, let $\Gamma$ be an irreducible graph with arbitrary coloring which is not reducibly equivalent to $c\Upsilon_{\det(k,J)}$ for any $1 \leq k \leq n$, $J \subseteq \{1, \ldots, n_k\}$ and $c \in \mathbb{C}^*$. Let $M$ be the set of graphs of the same form as $\Gamma$ but with $k$-edges of only $\binom{n}{k} - 1$ different colors. Then at least one element of $M$ is irreducible.*

PROOF. This follows directly from Theorem 9.2 of [**77**]. □

## 3.2. Invariants of sums of fundamental representations of $SL_4$ and $SL_5$

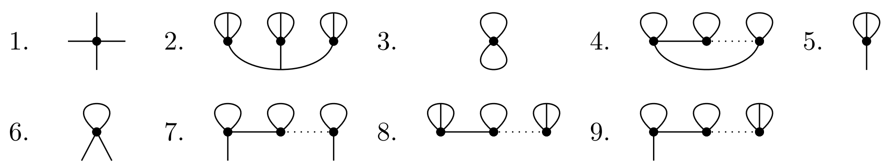
**3.2.1. Invariants of $SL_4$.** In the case of $SL_4$, it turns out that elementary graph theoretical and combinatorial considerations suffice to determine a minimal generating set of $\mathbb{C}[W]^{SL_4}$. This means that the present subsection is almost totally self-contained. We prove Theorem 6 and the slightly more general version Theorem 3.1 in the following.

Remark 3.2.1. According to Proposition 3.1.20, some multiple of graph no. 4 from Theorem 6 must be reducibly equivalent to some graphsum $\Upsilon$ with $U \circ \gamma(\Upsilon) = \det$. In fact,




from Theorem 3.1 has this property, as it is alternating in the colors, i.e. $\Upsilon$ is not only reducibly, but truly equivalent to $\mathrm{sgn}(\sigma)\Upsilon_\sigma$, where $\sigma$ is a permutation of the colors. The more pleasant display has of course graph no. 4, while for some purposes like for example finding relations, $\Upsilon$ will do better. The corresponding bracket polynomial, together with those of graphs no. 3a and 3b, are also given in [**67, 83, 101**], while no attempt is made there to show that these provide a minimal generating set of $\mathbb{C}[W_{(0,n_2,0,0)}]^{\mathrm{SL}_4}$.

Lemma 3.2.2. *Every irreducible graph for the action of* $\mathrm{SL}_4$ *on* $W$ *is reducibly equivalent to a graphsum* $\sum a_i\Gamma_i$*, where all* $\Gamma_i$ *are of the same form. This form is one of the following:*




Proof. Let $\Gamma$ be an irreducible graph. First assume $\Gamma$ has only 1-edges. Then $\Gamma$ is connected and thus irreducible only if it has one vertex. We are in Case 1.

Now assume $\Gamma$ has only 3-edges. Then with Lemma 3.1.18, it is reducibly equivalent to a sum of graphs $\Gamma_i$ having only vertices of virtual degree one. Now take one vertex $v_1$ of an arbitrary $\Gamma_i$. There must be a non-looping 3-edge $e$ with one connection to $v_1$. Since all vertices have virtual degree one, there must be two vertices $v_2$ and $v_3$ with $e = \{v_1, v_2, v_3\}$. Thus $v_1$, $v_2$ and $v_3$ together with $e$ and their respective looping 3-edges are a connected component of $\Gamma_i$ and since $\Gamma_i$ is connected, we are in Case 2.

If now $\Gamma$ has only 2-edges and it has only one vertex, we are in Case 3. Assume it has more than one vertex. Again by Lemma 3.1.18, $\Gamma$ is reducibly equivalent to a sum of connected graphs $\Gamma_i$, where for each of them every vertex has one (and only one) looping edge and is thus of virtual degree two. The effective graph of an arbitrary $\Gamma_i$ must be a *connected simple 2-regular graph*, i.e. a cycle, and we are in Case 4.

We come to the cases where $\Gamma$ has hyperedges of two different sizes. Let us begin assuming there are 1- and 3-edges. Again using Lemma 3.1.18, we can move on to some $\Gamma_i$ with vertices $v_1, \dots, v_N$ each with a looping 3-edge. If there is an additional vertex $v_{N+1}$ with a connection to a 3-edge, we can move on to a graph having $N+1$ vertices with a looping 3-edge. If there is an additional vertex with no connection to a 3-edge, it constitutes a connected component like in Case 1. Thus we can assume all vertices of $\Gamma_i$ have a looping 3-edge. Now if any vertex despite for its looping 3-edge has a connection to another 3-edge, we have a connected component like in Case 2. Thus each vertex must have a connection to a 1-edge. Since $\Gamma_i$ is connected we are in Case 5.

Now let $\Gamma$ have 1- and 2-edges. We use Lemma 3.1.18, move on to some $\Gamma_i$ and can with the same argumentation as in the previous case assume that every vertex has a looping 2-edge. If there is a vertex connected to two 1-edges, we are in Case 6. If not, there are vertices of virtual degree one and two. Thus the effective graph of $\Gamma_i$ must be a *chain* and we are in Case 7.

One can follow the same argumentation if $\Gamma$ has 2- and 3-edges. By Lemma 3.1.18, we have vertices of virtual degree one and two and the effective graph of $\Gamma_i$ must be a chain, giving Case 8.

Finally let $\Gamma$ have 1-, 2-, and 3-edges. Once more, we have vertices of virtual degree one and two by Lemma 3.1.18 and the effective graph of $\Gamma_i$ must be a chain, leading to Case 9. □

When we say '$\Gamma$ is of the form $n$' in the following, we mean that the graph $\Gamma$ falls under Case $n$ of Lemma 3.2.2, while we use the term 'graph no. n' for the colored graphs from Theorem 6.

Lemma 3.2.3. *Let $\Gamma$ be one of the graphs from Lemma 3.2.2 with two or more vertices and arbitrary coloring, let $\sigma$ be a permutation of the colors of the 2-edges and $\Gamma_\sigma$ be the graph with permuted colors. Then $\Gamma \simeq_r \operatorname{sgn}(\sigma)\Gamma_\sigma$.*

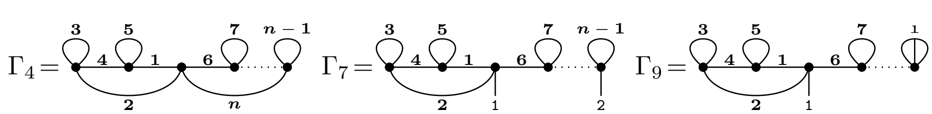
Proof. We show that $\Gamma \simeq_r -\Gamma_\sigma$, where $\sigma$ swaps the colors of a looping and a non-looping 2-edge connected with the same vertex. From this the assertion follows immediately.

So let $v \neq v'$ be vertices of $\Gamma$ and $(\{v,v\}, j_1, k_1)$, $(\{v,v'\}, j_2, k_2)$ the respective two 2-edges. We use Lemma 3.1.4 with $u = a^{(2)}_{2,j_1,k_1} a^{(2)}_{2,j_2,k_2}$ and get $\Gamma + \Gamma_\sigma \simeq -\Gamma'$, where $\Gamma'$ has the looping 2-edges $(\{v,v\}, j_1, k_1)$, $(\{v,v\}, j_2, k_2)$ and is thus reducible. So $\Gamma \simeq_r -\Gamma_\sigma$. □

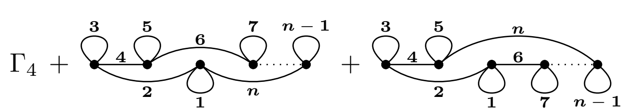
Proof of Theorem 6. For graphs $\Gamma$ of the forms 1, 2 and 5, the corresponding invariants are in classical terms 'invariants of systems of vectors and linear forms', see for example [**77**, p. 254]. Graphs of the form 3 either give the 'Pfaffian' if both edges have the same color, or a variation of it. Graphs $\Gamma$ of the form 6 neither are disconnected nor $U \circ \gamma(\Gamma) = 0$ unless both 1-edges have the same color.

The remaining types to check for irreducibility are 4, 7, 8, 9. By the duality of $V$ and $\Lambda^{n-1}V$, we can reduce form 8 to form 7. Thus 4, 7, 9 remain. In all these cases we require all 2-edges to be of pairwise different color, otherwise Lemma 3.2.3 with $\sigma$ swapping two edges of the same color would result in $\Gamma \simeq_r (\Gamma + \Gamma_\sigma)/2$, which is reducible.

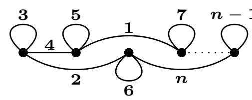
Now first we show that if a graph of one of these types has four or more vertices, it is reducible. We consider graphs $\Gamma_i$ of the forms:

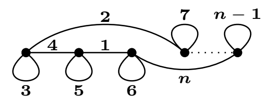



We proceed exemplarily with $\Gamma_4$. Applying Lemma 3.1.4 with $u$ the word corresponding to the 2-edge of color **1**, we get that



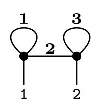
is reducible. Let us call the second and third graph in this sum $\Gamma_{4,1}$ and $\Gamma_{4,2}$ respectively. Since graphs with permuted vertices are equivalent, by swapping colors 1 and 5 as well as 2 and 4, we get $\Gamma_{4,2} \simeq_r \Gamma_{4,1}$, so $\Gamma_4 \simeq_r -2\Gamma_{4,1}$. Applying Lemma 3.1.4 again, now with $u$ the word corresponding to the 2-edge of color 6, we get that

$\Gamma_4 +$ 

$+$ 


is reducible. Calling the second and third graph in this sum $\Gamma_{4,3}$ and $\Gamma_{4,4}$ respectively, by symmetry reasons, we have $\Gamma_{4,3} \simeq_r \Gamma_{4,4}$ and by swapping the edges of colors 1 and 6 in $\Gamma_{4,3}$ we get $\Gamma_{4,3} \simeq_r -\Gamma_{4,1}$, thus $2\Gamma_{4,1} \simeq_r -2\Gamma_{4,1}$ and $\Gamma_{4,1}$ must be reducible. Exactly the same procedure, namely two times applying Lemma 3.1.4, one time on the edge of color 1, one time on that of color 6, leads to reducibility of graphs of the forms 4, 7, and 9 with four or more vertices. We consider these with three or less vertices in the following.

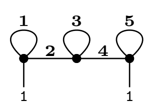
**Case 1: $\Gamma$ is of the form 7.** Here $\Gamma$ has either two or three vertices. We first check these with two vertices. All of the graphs with 2-edges of pairwise different colors **1**, **2**, **3**, and 1-edges of possibly non-different colors 1 and 2 are either reducible or of the form $\Gamma_\sigma$, where $\sigma$ permutes colors of 2-edges and

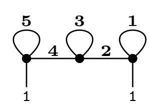
$\Gamma =$ 

.

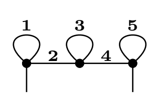
We define a map $\phi : G \to \mathbb{C}$ by setting $\phi(c\Gamma_\sigma) = \mathrm{sgn}(\sigma)c$ and $\phi(\Gamma') = 0$ for graphs $\Gamma'$ of other forms. Since $\Gamma \notin \ker(U \circ \gamma)$ and all relations from Lemma 3.1.4 with $|u| = 2$ involving $\Gamma$ are compatible with $\phi$ in the sense that $\phi(\Upsilon_1) = \phi(\Upsilon_2)$ if $\gamma(\Upsilon_1) = \gamma(\Upsilon_2)$ for two graphsums $\Upsilon_1 = \Gamma + \Upsilon_1'$ and $\Upsilon_2$, $\phi$ induces a well-defined map $\phi' : G/\ker(U \circ \gamma) \to \mathbb{C}$ and $\Gamma$ is irreducible.

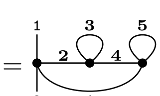
Now if $\Gamma$ is of type 7 with three vertices and the two 1-edges are of the same color, we have $U \circ \gamma(\Gamma) = -U \circ \gamma(\Gamma_\tau)$, where $\tau$ interchanges the two 1-edges. On the other hand, we have



$\simeq_r$ 


by swapping colors **1** and **5** as well as **2** and **4** of 2-edges and see that $\Gamma$ is reducible. Now all graphs with three vertices and with 2-edges of pairwise different colors **1-5** and 1-edges of different colors 1 and **2** are either reducible or of the form $\Gamma_\sigma$, $\Gamma'_\sigma$, where $\sigma$ permutes 2-edges and

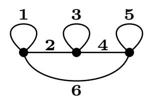
$\Gamma =$ 

, $\Gamma' =$ 

.

By applying Lemma 3.1.4 on the 2-edge of $\Gamma'$ of color **1**, we see that $\Gamma' \simeq_r -2\Gamma$. Similarly as we did before, we define a map $\phi : G \to \mathbb{C}$ by setting $\phi(c\Gamma_\sigma) = \mathrm{sgn}(\sigma)c$, $\phi(c\Gamma'_\sigma) = -\mathrm{sgn}(\sigma)2c$ and $\phi(\Gamma^*) = 0$ for graphs $\Gamma^*$ of other forms. As before, $\phi$ induces a well-defined map $\phi' : G/\ker(U \circ \gamma) \to \mathbb{C}$ and $\Gamma$ thus is irreducible.

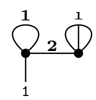
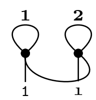
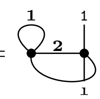
**Case 2: $\Gamma$ is of the form 4.** If such a graph has two vertices, it has two edges $(\{v_1, v_2\}, j_1, k_1)$ and $(\{v_1, v_2\}, j_2, k_2)$. Then by Lemma 3.1.4 with $u = a^{(2)}_{2,j_1,k_1} a^{(2)}_{2,j_2,k_2}$, it is reducible. The case of three vertices remains. All graphs with three vertices and six 2-edges of different colors **1-6** are either reducible or of the form $\Gamma_\sigma$, where

$\Gamma =$ 

.

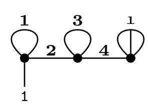
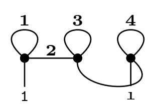
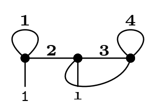
Again the map $\phi : G \to \mathbb{C}$ defined by $\phi(c\Gamma_\sigma) = \mathrm{sgn}(\sigma)c$ and $\phi(\Gamma^*) = 0$ for graphs $\Gamma^*$ of other forms induces a well-defined map $\phi' : G/\ker(U \circ \gamma) \to \mathbb{C}$ and $\Gamma$ thus is irreducible.

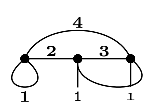
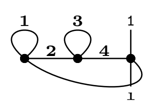
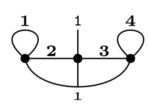
**Case 3: $\Gamma$ is of the form 9.** Here $\Gamma$ can have two or three vertices. We begin with the case of two. All graphs with one 1-edge of color 1, two 2-edges of colors **1** and **2**, and one 3-edge of color 1 are either reducible or of the form $(\Gamma_i)_\sigma$ with

$\Gamma_1 =$  , $\Gamma_2 =$  , $\Gamma_3 =$  .

Applying Lemma 3.1.4 with $u$ the word corresponding to the 3-edge, we see $\Gamma_3 \simeq_r \Gamma_2 \simeq_r \Gamma_1$. The map $\phi : G \to \mathbb{C}$ defined by $\phi(c(\Gamma_i)_\sigma) = \mathrm{sgn}(\sigma)c$ and $\phi(\Gamma^*) = 0$ for graphs $\Gamma'$ of other forms induces a well-defined map $\phi' : G/\ker(U \circ \gamma) \to \mathbb{C}$ and $\Gamma_1$ thus is irreducible.

We come to those graphs with three vertices. All graphs with one 1-edge of color 1, four 2-edges of pairwise different colors **1-4**, and one 3-edge of color 1 are either reducible or of the form $(\Gamma_i)_\sigma$ with

$\Gamma_1 =$  , $\Gamma_2 =$  , $\Gamma_3 =$  ,

$\Gamma_4 =$  , $\Gamma_5 =$  , $\Gamma_6 =$  .

We apply Lemma 3.1.4 with $u$ the word corresponding to the 3-edge (to two ends of the 3-edge in the case of $\Gamma_6$) and get $\Gamma_1 \simeq_r \Gamma_3 \simeq_r -\Gamma_2 \simeq_r -\Gamma_5$ and $\Gamma_6 \simeq_r \Gamma_4 \simeq_r -2\Gamma_1$. The map $\phi : G \to \mathbb{C}$ defined in the usual form induces a well-defined map $\phi' : G/\ker(U \circ \gamma) \to \mathbb{C}$ and $\Gamma_1$ thus is irreducible. The proof is complete.

□

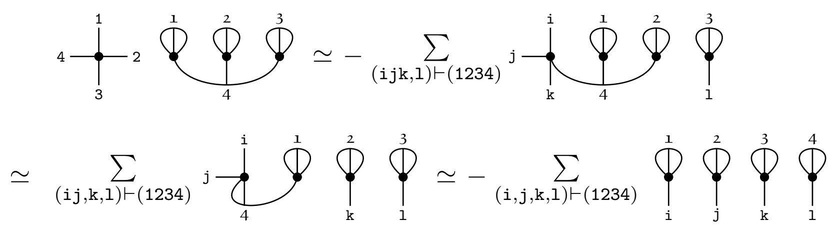
Remark 3.2.4. To show that the graphs with four (five in the case of those of form 8 respectively) or more vertices are reducible, we also could use Proposition 3.1.20 together with the fact that graphs with two 2-edges of the same color are reducible. We preferred the more self-contained version here, because it provides more insight *why* this is so from our 'graphical' viewpoint. We want to stress that this reducibility is almost impossible to see without the graph notation, which might explain why Huang in [**58**] could not reduce her generating set of cycles to a minimal one.

Moreover, to show the irreducibility of the remaining graphs, one can also compute the Hilbert series for small but sufficiently large values of respective $n_i$'s using [**35**, §4.6] and Xin's algorithm [**107**] for MacMahon partition analysis. In fact, Xin's algorithm performs very good for such small values.

PROOF OF THEOREM 3.1. Theorem 6 provides us with a maximal set of reducibly independent irreducible graphs. The corresponding invariants can be computed according to the rules from Definition 3.1.7. The author used the *DifferentialGeometry* package of MAPLE for these computations. □

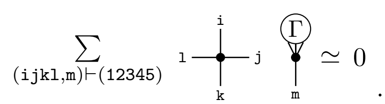
**3.2.2. Relations of $\mathrm{SL}_4$.**

EXAMPLE 3.2.5. Consider graphs no. 1 and 2 from Theorem 6. By applying the Plücker relation from Lemma 3.1.4 three times, we can *pull over* the 3-edge of color 4 and get the well known - see [**77**, p. 255] - relation:

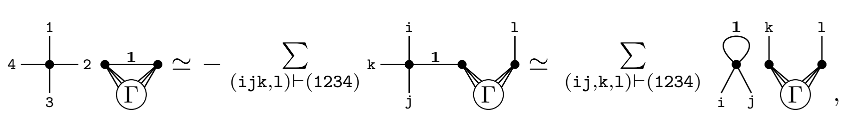



PROOF OF THEOREM 3.2. We identify four somewhat *natural principles* generating relations between invariants. All relations from Theorem 3.2 stem from these principles.

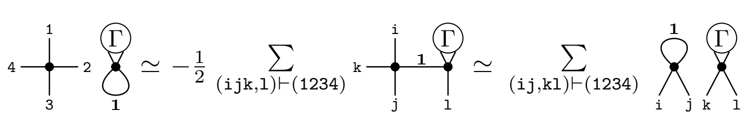
**Principle 1a: permuting five 1-edges.** This comes from applying the Plücker relation from Lemma 3.1.4 on five 1-edges *once*. Let Γ be an arbitrary graph, connected to a 1-edge, then



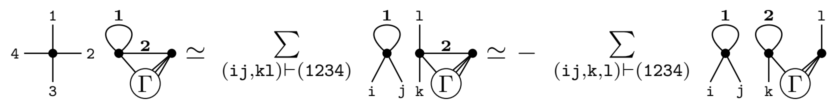

**Principle 1b: pulling over a $k$-edge.** If we have a product of graph no. 1 and a second graph with a 2- or 3-edge, we can *pull this $k$-edge over to graph no.* 1 *and distribute a total of $k$ 1-edges to the second graph.* There are at most $k$ applications of Lemma 3.1.4 necessary. In the case of a non-looping 2-edge, we get



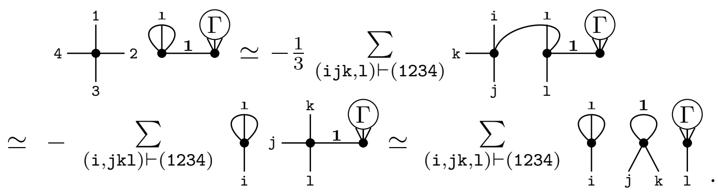

where Γ does not have to be connected. In the case of a looping 2-edge, we have




which is fine if Γ is a looping 2-edge. If not, then the vertex connected to the 2-edge of color **1** is connected to another, but non-looping, 2-edge - say of color **2** - and

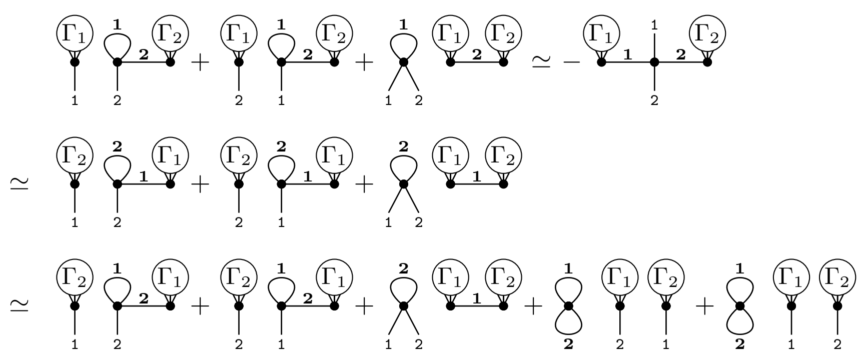


holds. Now in the case of a 3-edge *pulled over* to graph no. 1, Example 3.2.5 shows what happens for the non-looping 3-edge of graph no. 2. Since we can interchange the edges of graph no. 2 as we want by a simple application of Lemma 3.1.4, we get the same relation for one of the looping 3-edges. Thus we can exclude graph no. 2 in the following, which means that we only have to consider graphs with a looping 3-edge on a vertex connected to a 2-edge (of color **1**). Here we get

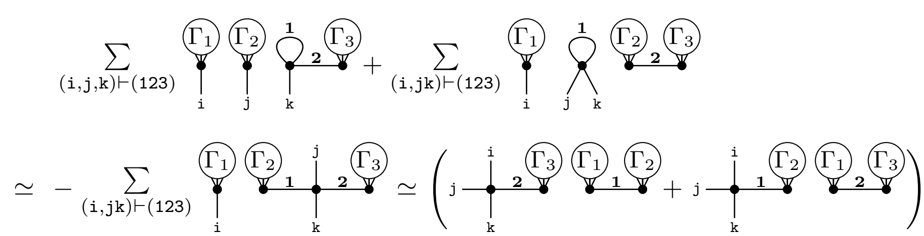


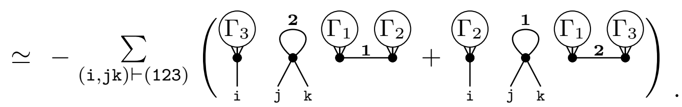
Observe that Principle 1a can be seen as a special case of Principle 1b, namely pulling over a 1-edge.

**Principle 2: bringing together two 1-edges (of two different graphs).** This only works if one of the 1-edges is connected to a vertex with one looping 2-edge and one non-looping 2-edge. So any combination of graphs no. 1, 5, and 6 gives no or no new relation. If one of the graphs is graph no. 1, we can reduce to Principle 1 or 2. So we exclude this as well and get

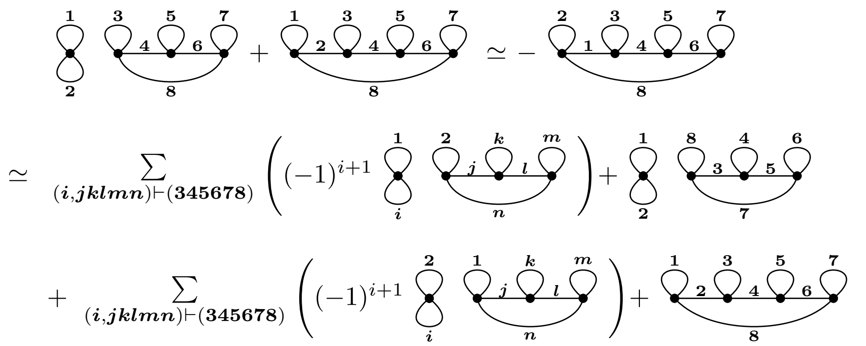


**Principle 3: bringing together three 1-edges.** Here as well one of the 1-edges must be connected to a vertex with a looping and a non-looping 2-edge and none of the graphs must be no. 1. We have

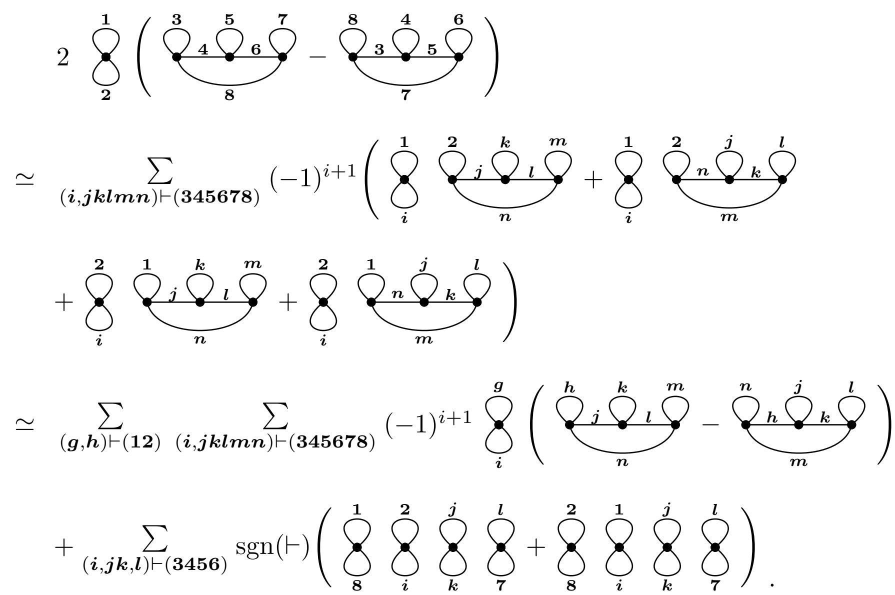




**Principle 4a: going around circular graphs.** We get the second identity in the following by going around the circular graph with four vertices first with the 2-edge of color **1** and then with the 2-edge of color **2**:



Thus we get



**Principle 4b: determinantal relations.** Consider the matrices

$$B := \begin{pmatrix} y_{\boldsymbol{j_1}34} & y_{\boldsymbol{j_2}34} & y_{\boldsymbol{j_3}34} & y_{\boldsymbol{j_4}34} & y_{\boldsymbol{j_5}34} & y_{\boldsymbol{j_6}34} \\ -y_{\boldsymbol{j_1}24} & -y_{\boldsymbol{j_2}24} & -y_{\boldsymbol{j_3}24} & -y_{\boldsymbol{j_4}24} & -y_{\boldsymbol{j_5}24} & -y_{\boldsymbol{j_6}24} \\ y_{\boldsymbol{j_1}23} & y_{\boldsymbol{j_2}23} & y_{\boldsymbol{j_3}23} & y_{\boldsymbol{j_4}23} & y_{\boldsymbol{j_5}23} & y_{\boldsymbol{j_6}23} \\ y_{\boldsymbol{j_1}14} & y_{\boldsymbol{j_2}14} & y_{\boldsymbol{j_3}14} & y_{\boldsymbol{j_4}14} & y_{\boldsymbol{j_5}14} & y_{\boldsymbol{j_6}14} \\ -y_{\boldsymbol{j_1}13} & -y_{\boldsymbol{j_2}13} & -y_{\boldsymbol{j_3}13} & -y_{\boldsymbol{j_4}13} & -y_{\boldsymbol{j_5}13} & -y_{\boldsymbol{j_6}13} \\ y_{\boldsymbol{j_1}12} & y_{\boldsymbol{j_2}12} & y_{\boldsymbol{j_3}12} & y_{\boldsymbol{j_4}12} & y_{\boldsymbol{j_5}12} & y_{\boldsymbol{j_6}12} \end{pmatrix}$$

$$A := \begin{pmatrix} y_{\boldsymbol{i_1}12} & y_{\boldsymbol{i_2}12} & y_{\boldsymbol{i_3}12} & y_{\boldsymbol{i_4}12} & y_{\boldsymbol{i_5}12} & y_{\boldsymbol{i_6}12} \\ y_{\boldsymbol{i_1}13} & y_{\boldsymbol{i_2}13} & y_{\boldsymbol{i_3}13} & y_{\boldsymbol{i_4}13} & y_{\boldsymbol{i_5}13} & y_{\boldsymbol{i_6}13} \\ y_{\boldsymbol{i_1}14} & y_{\boldsymbol{i_2}14} & y_{\boldsymbol{i_3}14} & y_{\boldsymbol{i_4}14} & y_{\boldsymbol{i_5}14} & y_{\boldsymbol{i_6}14} \\ y_{\boldsymbol{i_1}23} & y_{\boldsymbol{i_2}23} & y_{\boldsymbol{i_3}23} & y_{\boldsymbol{i_4}23} & y_{\boldsymbol{i_5}23} & y_{\boldsymbol{i_6}23} \\ y_{\boldsymbol{i_1}24} & y_{\boldsymbol{i_2}24} & y_{\boldsymbol{i_3}24} & y_{\boldsymbol{i_4}24} & y_{\boldsymbol{i_5}24} & y_{\boldsymbol{i_6}24} \\ y_{\boldsymbol{i_1}34} & y_{\boldsymbol{i_2}34} & y_{\boldsymbol{i_3}34} & y_{\boldsymbol{i_4}34} & y_{\boldsymbol{i_5}34} & y_{\boldsymbol{i_6}34} \end{pmatrix} \quad C := \begin{pmatrix} |\boldsymbol{i_1 j_1}| & \cdots & |\boldsymbol{i_1 j_6}| \\ \vdots & \ddots & \vdots \\ |\boldsymbol{i_6 j_1}| & \cdots & |\boldsymbol{i_6 j_6}| \end{pmatrix}.$$

We have $\det(A) = |\boldsymbol{i_1 \cdots i_6}|$, $\det(B) = -|\boldsymbol{j_1 \cdots j_6}|$ and $A^{\mathrm{T}}B = C$. Thus

$$\det(C) + |\boldsymbol{i_1 \cdots i_6}||\boldsymbol{i_1 \cdots i_6}| = 0$$

holds. Moreover, we have the standard Plücker identity for determinants of matrices of the form $A$:

$$\sum_{(i_1 \cdots i_6, j_1) \vdash (k_1 \ldots k_7)} |\boldsymbol{i_1 \cdots i_6}||\boldsymbol{j_1 \cdots j_6}| = 0.$$

Both of the above identities could also be achieved via 'going around circular graphs', which turns out to be a lot harder than this approach. Since we found all relations from Theorem 3.2, the proof is complete. □

**3.2.3. Covariants of $\mathrm{SL}_4$.** The purpose of this section is to prove Theorem 3.4 and to give a geometric interpretation of the covariants therein. We will in fact start from this geometric viewpoint and then relate it to the covariants of $\mathrm{SL}_4$.

For linear subspaces of the complex projective space $\mathbb{P}^3(\mathbb{C})$ , one can formulate incidence relations such as 'point $\mathtt{x}$ lies on line $\boldsymbol{l}$' or 'plane $\mathfrak{E}$ and line $\boldsymbol{l}$ intersect in one point' et cetera. We can translate these relations into the vanishing (or nonvanishing) of covariants of $\mathrm{SL}_4$ in the following way.

Consider the exterior algebra $\Lambda(\mathbb{C}^4)$ and let us here denote the exterior product inside this algebra by $\vee$ instead of $\wedge$ for geometrical reasons. Then a $k$-dimensional linear subspace $A$ of $\mathbb{C}^4$ with basis $x_1, \ldots, x_k$ is identified - by slight abuse of notation - with $A := x_1 \vee \ldots \vee x_k \in \Lambda^k(\mathbb{C}^4)$. We call such elements of $\Lambda(\mathbb{C}^4)$ extensors (of step $k$). Moreover, we call the exterior product $A \vee B$ of two extensors the *join*, as it is an extensor being nonzero if and only if $\dim(A \cap B) = 0$ and in this case it corresponds to the *span* of $A$ and $B$. We identify extensors of step four with scalars and denote them by a bracket $[x_1 \cdots x_4] := x_1 \vee \ldots \vee x_4$ of the determinant. To represent the *intersection* of subspaces $A = a_1 \vee \ldots \vee a_j$ and $B = b_1 \vee \ldots \vee b_k$, we define the operation $A \wedge B$, called the *meet*, by

$$A \wedge B := \sum_{(a', a'') \vdash (a_1 \cdots a_j)} \mathrm{sgn}(\vdash)[a'_1 \cdots a'_{4-k} b_1 \cdots b_k] \cdot a''_1 \vee \ldots \vee a''_{j+k-4},$$

where $(a', a'') \vdash (a_1 \cdots a_j)$ means that we sum over all decompositions of $a_1 \cdots a_j$ in two ordered subwords $a', a''$ and $\mathrm{sgn}(\vdash)$ is the sign of the underlying permutation $a \mapsto a'a''$. Then $A \wedge B$ is nonzero if and only if $A$ and $B$ span $\mathbb{C}^4$ and in this case it corresponds to the intersection of the subspaces $A$ and $B$. We call $\Lambda(\mathbb{C}^4)$ the

*Grassmann-Cayley-Algebra*. For more details we refer to [**96, 93**]. The Grassmann-Cayley-Algebra is a fundament for quantum logic, see [**60, 99, 12**].

Now identify $k$-dimensional linear subspaces of $\mathbb{C}^4$ with $(k-1)$-dimensional linear-projective subspaces of $\mathbb{P}^3(\mathbb{C})$. The incidence relations from above stay untouched under the natural action of $\mathrm{SL}_4$ on $\mathbb{C}^4$, they are *invariant*. This is to say, they must be given by polynomials in their homogeneous coordinates - invariant under the action of $\mathrm{SL}_4$. The *invariants* are polynomial expressions in the brackets from above, but there are invariant functions, the *covariants*, that are not scalar-valued but tensor valued. One can see them as 'incomplete invariants' (like the meet $A \wedge B$ from above) that can be completed to bracket expressions by *generic* subspaces of $\mathbb{P}^3(\mathbb{C})$. An example of an invariant is the determinant $[x_1x_2x_3x_4]$ that is zero if and only if the points $x_1, \dots, x_4$ lie in a common plane. An example of a covariant would be $[x_1x_2 **] := x_1 \vee x_2$, which is zero if and only if $x_1 = x_2$, being equivalent to the condition that for a generic 'dummy' line $l$ the dimension of the span $x_1 \vee x_2 \vee l$ is smaller than three, i.e. the bracket $[x_1x_2l]$ vanishes.

Incidence relations and covariants of $\mathrm{SL}_n$ are on the one hand important in *classical* mechanics, rigidity of joint-bar-body-hinge-frameworks [**32, 102, 103**], with applications in such fields as computer aided geometric design [**39**], machine learning [**94**], robotics [**11**] and biology [**104, 85**]. On the other hand, they as well come into play in *quantum* mechanics, entanglement and quantum information theory [**12, 36**].

From the mathematical perspective, all geometric incidence relations can be formulated by means of covariants [**96**, Eq. 3.3.11] and due to Hilbert [**55**] these form a finitely generated algebra. Thus the classical invariant theoretic task of finding a *minimal generating set* for the algebra of invariants or covariants becomes important from the geometric point of view as it equals a minimal set of geometric relations that *suffices to describe all geometric relations*. As we already have determined a minimal generating set for the algebra of invariants in Subsection 3.2.1, it remains to derive such set for the algebra of covariants - i.e. to prove Theorem 3.4. Afterwards, we discuss the geometric meaning of these covariants.

According to [**82**, Thm. 3], in our graphical language the covariants correspond to subgraphs of the invariant graphs or equivalently to the invariant graphs with some $k$-edges made 'dummy', corresponding to generic $k$-extensors. Moreover, we identify those graphs whose subgraphs of non-dummy edges coincide and of course those identified by a Plücker relation. So a covariant will not map to $\mathbb{C}$, but to the polynomial ring $\mathbb{C}[W]$, where $W \cap W = \emptyset$ is the vector space of the dummy edges. Thus a covariant is zero if and only if all its coefficients are zero. There are several equivalent definitions due to isomorphisms, including that the algebra of covariants $\mathrm{Cov}(W)^{\mathrm{SL}_n}$ is isomorphic to the algebra of invariants $\mathbb{C}[W]^U$ for a maximal unipotent subgroup $U$ of $\mathrm{SL}_n$, see [**92, 91**]. This isomorphism is induced by evaluating the graph at $e_1 \vee \ldots \vee e_k$ for *each* dummy $k$-edge. We will use this isomorphism in Section 3.5 to compute *contractions* of invariant rings.

Proof of Theorem 3.4. We will refer to the entry in the $i$-th row and $j$-th column of the table from Theorem 3.4 as $\mathcal{I}_{ij}$. A minimal generating set of the algebra of invariants is given in Theorem 6. These are exactly the graphs from Theorem 3.4 without dummy edges. Moreover, all graphs with one dummy edge and one vertex must be included in a minimal generating set of $\mathrm{Cov}(W)^{\mathrm{SL}_n}$. By duality of $V$ and $\Lambda^3 V \cong V^*$, we have to include $\mathcal{I}_{15}$ - which is dual to $\mathcal{I}_{14}$ - as well.

Now to extract a minimal generating set from all possible remaining subgraphs, we use the already mentioned fact that we can evaluate the graph at $e_1 \vee \ldots \vee e_k$ for *each* dummy $k$-edge. The resulting polynomials are exactly the members of a minimal generating set of the algebra of invariants $\mathbb{C}[W]^U$ for a maximal unipotent subgroup $U$ due to [**92**].

So because the cycle $\mathcal{I}_{82}$ can be assumed antisymmetric in the edges, the only covariant coming from a subgraph of $\mathcal{I}_{82}$ to include in a minimal generating set is $\mathcal{I}_{72}$ with one dummy edge. By the same reasoning, from the subgraphs of $\mathcal{I}_{(10)4}$ we have to include those that either have one dummy 1-edge or one dummy 2-edge. Of each type, we have exactly one generator due to antisymmetry of the edges. By duality of $V$ and $\Lambda^3 V \cong V^*$, we have the same for subgraphs of $\mathcal{I}_{(10)5}$ with the role of 1- and 3-edges interchanged.

Now consider subgraphs of $\mathcal{I}_{64}$. This graph is symmetric in the 1-edges and antisymmetric in the 2-edges. So we have to include those subgraphs that either have one dummy 2-edge or one or two dummy 1-edges - three in total. We have the analogous statement for subgraphs of the dual graph $\mathcal{I}_{65}$.

Only subgraphs of $\mathcal{I}_{27}$ and $\mathcal{I}_{47}$ remain. These graphs are antisymmetric in the 2-edges, so we have to include subgraphs with one dummy 1-, 2-, or 3-edge and those with a dummy 1- *and* a dummy 3-edge. The resulting set of covariants is generating due to the arguments above. It is minimal since the set of invariants from Theorem 6 is minimal: if one of the covariant graphs could be expressed as a polynomial in the others by Plücker relations, the same would hold for the respective invariant graphs with dummy edges replaced by real edges. □

Now we give geometric interpretations of vanishing of the covariants from Theorem 3.4. We show how Grassmann-Cayley-Algebra expressions can be translated directly into brackets and graphs as in Section 3.1.1 and how this is the more immediate way compared to e.g. the approach from [**96**] using $k$ *different* letters corresponding to a basis of a $k$-extensor. To illustrate this, we begin with the following example:

EXAMPLE 3.2.6. Consider the case of lines in $\mathbb{P}^2(\mathbb{C})$. In [**96**, Ex. 3.3.3], the Grassmann-Cayley-Algebra expression for the condition that three pairwise different lines $\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}_3$ in $\mathbb{P}^2(\mathbb{C})$ meet in one point, which is

$$(\boldsymbol{l}_1 \wedge \boldsymbol{l}_2) \wedge \boldsymbol{l}_3 = 0,$$

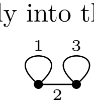
is translated into a bracket algebra expression in the following way: assume each $\boldsymbol{l}_i$ is spanned by two ponts $\mathtt{x}_i, \mathtt{y}_i$, then the left hand side of the above condition becomes

$$((\mathtt{x}_1 \vee \mathtt{y}_1) \wedge (\mathtt{x}_2 \vee \mathtt{y}_2)) \wedge (\mathtt{x}_3 \vee \mathtt{y}_3).$$

By the definition of join and meet, this can be translated into the bracket expression

$$[x_1 y_1 x_2][y_2 x_3 y_3] - [x_1 y_1 y_2][x_2 x_3 y_3].$$

This is at least unsatisfying as far as we had to choose a basis for the lines. But we can do better. Associate letters $a_i$ to the $\boldsymbol{l}_i$ as proposed in Section 3.1.1. Now using the definition of the (skew) bracket, the first expression in the $\boldsymbol{l}_i$ is translated into $[a_1 a_1 a_2][a_2 a_3 a_3]$ or equivalently into the graph



This graph as well as the bracket corresponds to the determinant in the coordinates of the $\boldsymbol{l}_i$, which is exactly what we want and in fact reflects the underlying geometry: the lines $\boldsymbol{l}_i$ meet in one point if and only if the corresponding planes in $\mathbb{C}^3$ meet in one line. This is the case if and only if the normal vectors of these planes, which are nothing else than the coordinate vectors of the extensors $\boldsymbol{l}_i$, are linearly dependent, i.e. if their determinant vanishes.

We move back to $\mathbb{P}^3(\mathbb{C})$. The translation from Grassmann-Cayley-Algebra expressions into bracket expressions and graphs can be done by the definitions of meet and join, adapted to the definitions of skew brackets from Section 3.1.1. Let $C$ be an expression in the Grassmann-Cayley-Algebra involving only operations $\vee$ and $\wedge$.

Then for $A \vee B$ in $C$, where $A$ and $B$ are extensors of step $k_A$ and $k_B$, either $k_A + k_B > 4$, which means that the expression is zero by definition, or replace $A \vee B$ by the bracket $[a_A \cdots a_A a_B \cdots a_B * \cdots *]$ containing $k_A$ and $k_B$ times the letters $a_A$ and $a_B$ respectively. On the other hand, for $A \wedge B$, either $k_A + k_B < 4$ - then the expression is zero by definition - or replace it by

$$\binom{k_B}{4-k_A}[a_A \cdots a_A a_B \cdots a_B][a_B \cdots a_B * \cdots *],$$

with $k_A$ and $4 - k_A$ times the letters $a_A$ and $a_B$ in the first and $k_A + k_B - 4$ times the letter $a_B$ in the second bracket. After replacing all elementary expressions by brackets, we have now expressions of types

$$[w_1] \cdots [w_n][w * \cdots *] \vee [v_1] \cdots [v_m][v * \cdots *],$$
$$[w_1] \cdots [w_n][w * \cdots *] \wedge [v_1] \cdots [v_m][v * \cdots *].$$

In the first case, either $|w| + |v| > 4$, then again the expression is equal to zero (a special case of this is one of $|w|$ and $|v|$ being equal to four, i.e. one of the bracket monomials is not a covariant but an invariant), or we can replace it by

$$[w_1] \cdots [w_n][v_1] \cdots [v_m][wv * \cdots *].$$

In the second case, either $|w| + |v| < 4$ - the expression is equal to zero - or replace it by

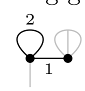
$$\sum_{(v',v'') \vdash v} \mathrm{shuff}_{v',v''} \cdot [w_1] \cdots [w_n][v_1] \cdots [v_m][wv'][v'' * \cdots *],$$

where the prefactor $\mathrm{shuff}_{v',v''}$ counts the number of shuffles resulting in the same decomposition $(v', v'')$ of $v$. Thus if $v$ only consists of one letter, it becomes the binomial prefactor from above. This procedure leads to a bracket polynomial, from which we get a graph. The reverse procedure is more involved and there is no general algorithm available, see [**96**, Sec. 3.5], but in most cases of the covariants from Theorem 3.4, the corresponding Grassmann-Cayley-Algebra expressions are obvious. We now give an example of how generic subspaces come into play. From now on we let letters in brackets corresponding to extensors be the same as the naming letter of the respective subspace, but in standard math-italic font.

Example 3.2.7. We want to find a Grassmann-Cayley-Algebra expression, a bracket polynomial and a graph that detect if two lines $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ in $\mathbb{P}^3(\mathbb{C})$ are identical. This property is equivalent to saying their span has dimension one. But we have no immediate criterion for this in the Grassmann-Cayley-Algebra. The operations therein more or less can only 'detect' if two lines intersect or not. But we have a criterion for a point $\mathtt{x}$ lying on a line $\boldsymbol{l}$, namely $\mathtt{x} \vee \boldsymbol{l} = 0$ or $[xll*] = 0$ equivalently. So we can reformulate the condition '$\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are identical' by 'a generic point of $\boldsymbol{l}_1$ lies on $\boldsymbol{l}_2$'. We can create generic points on $\boldsymbol{l}_1$ by the meet of $\boldsymbol{l}_1$ with a generic plane $\mathfrak{E}$, thus our condition equals

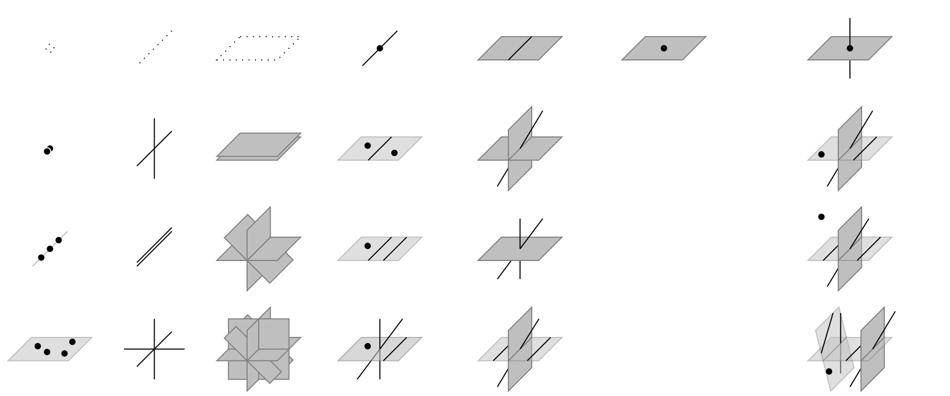
$$0 = (\mathfrak{E} \wedge \boldsymbol{l}_1) \vee \boldsymbol{l}_2 = [EEEl_1][l_1 l_2 l_2 *]$$

If we now replace the star in the second bracket by a generic point $\mathtt{x}$, we still have the same condition and the corresponding graph is entry $\mathcal{I}_{32}$ from Theorem 3.4:




In the table below, we list configurations for which the covariants from the first four rows of Theorem 3.4 vanish. The light gray transparent planes are auxiliary planes, while all other points, lines and planes represent one of the edges of the covariant graph. The dashing in the fist three entries means that the respective covariant vanishes if and only if the extensor is equal to zero, i.e. nonexistent. Double points, dots and planes in entries $(2, 1)$, $(2, 3)$ and $(3, 2)$ mean identity of extensors. Note that we always list the most general configuration; for example $\mathcal{I}_{24}$ vanishes if the two points lie on the line, but the most general configuration is the two points and the line lying on a common plane.

| points | lines | planes | pts. & lns. | lns. & pns. | pts. & pns. | pts. & lns. & pns. |
|---|---|---|---|---|---|---|



One sees that naturally the more edges the graphs have, the more complicated the configurations become. For example in the case of $\mathcal{I}_{47}$ we have a point $\mathtt{x}$, four lines $\boldsymbol{l}_1, \ldots, \boldsymbol{l}_4$ and a plane $\mathfrak{E}$. The configuration for vanishing $\mathcal{I}_{47}$ described in words would be that (1) the intersection point of the line $\boldsymbol{l}_4$ and the plane $\mathfrak{E}$, (2) the intersection point of the line $\boldsymbol{l}_2$ and the plane through the line $\boldsymbol{l}_1$ and the point $\mathtt{x}$, and (3) the line $\boldsymbol{l}_3$ - all three lie on a common plane.

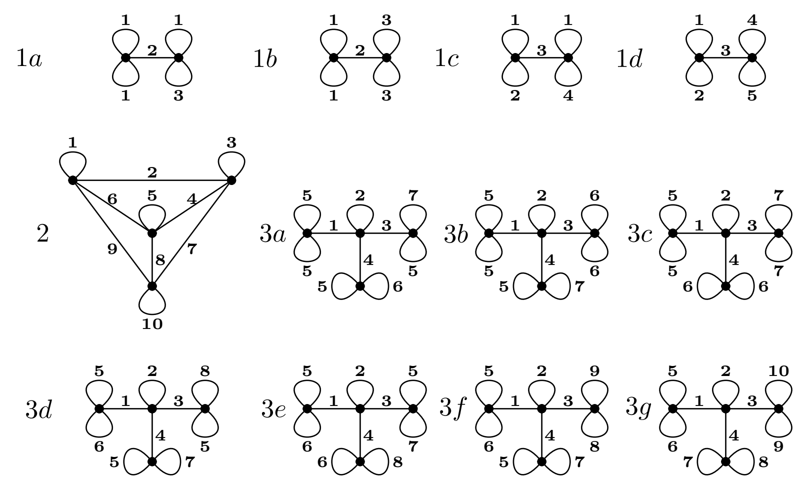
For this reason, we do not list configurations for all covariants from Theorem 3.4, but in all cases but for the cycles $\mathcal{I}_{72}$ and $\mathcal{I}_{82}$, it is possible to derive a Grassmann-Cayley-Algebra expression by starting at an 'end-vertex' and reversing the process from above, i.e. replacing subsets of brackets with elementary $\vee$- and $\wedge$-operations. In the case of $\mathcal{I}_{82}$, the reference [**83**] provides a geometric interpretation: the invariant $\mathcal{I}_{82}$ vanishes, if the six lines form a *line complex*, which means they either have a common transversal or are reciprocal to a common screw.

Then $\mathcal{I}_{72}$ vanishes if the five lines and a generic sixth line either all have a common transversal (which only happens when they intersect in one point) or are reciprocal to a common screw.

We finish with a comment on [**96**, Ex. 3.4.5, 3.4.6]. In these examples, bracket expressions for four and five lines having a common transversal are derived. These expressions are invariants, and since the only generators for the algebra of *invariants* for five or less lines are those of type $\mathcal{I}_{22}$ - with one invariant of this type for each pair of lines - they can be described only by means of $\mathcal{I}_{22}$. This can be seen directly in the formula from [**96**, Thm. 3.4.7] but is not so obvious in the case of the formula for four lines on page 106 of [**96**].

**3.2.4. Invariants of $\mathrm{SL}_5$.** In order to prove Theorem 7, we need some more techniques than the ones we developed for $\mathrm{SL}_4$. The duality of $\Lambda^k V$ and $\Lambda^{n-k} V$ becomes very important and we introduce a new reducibility notion that is essential (and probably would be so even more in higher dimensions). As Theorem 7 does not provide any colorings, we give these exemplarily for $W_{(0,n_2,0,0)}$ in the following.

PROPOSITION 3.2.8. *The following graphs constitute a maximal set of reducibly independent irreducible graphs for the action of* $\mathrm{SL}_5$ *on* $W_{(0,n_2,0,0)}$, *where in each case* $1 \leq \mathbf{1} < \mathbf{2} < \ldots \leq n_2$ *are pairwise different colors of the* 2*-edges.*




DEFINITION 3.2.9. We say that a vertex is of type $\mathcal{V}_{k_1 \cdots k_r}^{l_1 \cdots l_s}$, if it has a looping $k_i$-edge for every $1 \leq i \leq r$ and has one connection to a $l_j$-edge for every $1 \leq j \leq s$.

DEFINITION 3.2.10. The *virtual degree type* $d(\Gamma)$ of a graph $\Gamma$ with $k$ vertices is the descending sequence $(d_1, \ldots, d_k)$ of virtual degrees of vertices of $\Gamma$. We define a partial order on the set of graphs for the action of $\mathrm{SL}_n$ with $k$ vertices by setting

$$\begin{aligned} \Gamma < \Gamma' &:\Leftrightarrow (d_1, \ldots, d_k) = d(\Gamma) < d(\Gamma') = (d'_1, \ldots, d'_k) \\ &:\Leftrightarrow d_1 \leq d'_1, \ldots, d_{k-1} \leq d'_{k-1}, d_k < d'_k. \end{aligned}$$

We call a graphsum $\sum \Gamma_i$ *degree-reducible*, if it is reducibly equivalent either to 0 or to a graphsum $\sum \Gamma'_j$ with $d(\Gamma'_j) < d(\Gamma_i)$ for all $i$, $j$. Moreover, in analogy to Definition 3.1.15, we say that graphsums $\Upsilon_1, \ldots, \Upsilon_N$ are *degree-reducibly independent*, if a linear combination $\sum a_i \Upsilon_i$ is degree-reducible only if all $a_i$ are equal to zero. If for two graphsums $\Upsilon_1$, $\Upsilon_2$ the linear combination $\Upsilon_1 - \Upsilon_2$ is degree-reducible, we call them *degree-reducibly equivalent* and write $\Upsilon_1 \simeq_d \Upsilon_2$. We say that a set of degree-reducibly independent degree-irreducible graphs has property $(DI)$.

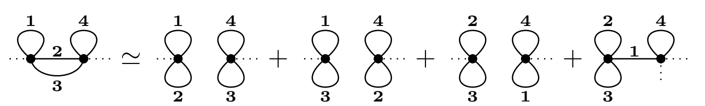
Lemma 3.2.11. *A maximal set with property $(DI)$ is also a maximal set with property $(RI)$, i.e. a set of reducibly independent irreducible graphs.*

Proof. Let $M$ be a maximal set with property $(DI)$. Of course, $M$ has property $(RI)$. Assume $M$ is not maximal with that property. Then there is a graph $\Gamma$, so that $M' = M \cup \{\Gamma\}$ still has property $(RI)$, but not $(DI)$. Thus $\Gamma + \sum a_i \Gamma_i$ is degree-reducible for some $\Gamma_i \in M$. So $\Gamma + \sum a_i \Gamma_i \simeq_r \sum b_j \Gamma'_j$ with reducibly independent irreducible $\Gamma'_j$ so that $d(\Gamma'_j) < d(\Gamma)$ for all $j$. Since $M'$ has property $(RI)$, not all $\Gamma'_j$ can be elements of $M'$. Take $\Gamma'_k \notin M'$ and assume $M \cup \{\Gamma'_k\}$ does not have property $(RI)$. Since $\Gamma'_k$ is irreducible, there must be a reducible sum $\sum c_i \Gamma_i + \Gamma'_k$ with not all $c_i$ equal to zero. So we find

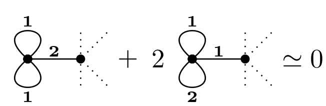
$$\Gamma + \sum (a_i + b_i c_i)\Gamma_i \simeq_r \sum_{j \neq k} b_j \Gamma'_j$$

Thus there must be some $\Gamma'_l$ so that $M'' = M \cup \{\Gamma'_k\}$ has property $(RI)$ and $d(\Gamma'_k) < d(\Gamma)$. Since $(0, \ldots, 0)$ is a lower bound for the virtual degree type, iterating this procedure gives a contradiction. □

Proof of Proposition 3.2.8. Due to Lemma 3.2.11, we only have to consider degree-irreducible graphs. Due to Lemma 3.1.18, such a graph can be assumed to have two types of vertices: such with one - type $\mathcal{V}_2$ - and such with two looping 2-edges - type $\mathcal{V}_{22}$ - , being of virtual degree three and one respectively. A graph with a double edge is not necessarily reducible but degree-reducible, due to

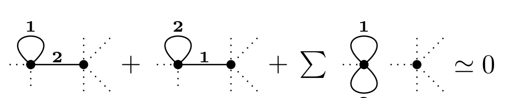


So we can exclude such graphs as well. We call vertices with two looping edges *black holes*, because they 'absorb colors' in the sense that we can not interchange the colors of the two looping edges with other edges' colors in the way we are used to from the $\mathrm{SL}_4$-case. Colors can only be extracted if two of the adjacent edges have the same color:



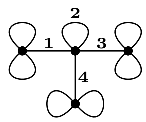
On the other hand, if all three adjacent edges have the same color, the graph evaluates to zero under $U \circ \gamma$. So we exclude this case as well and first let $\Gamma$ have two vertices, then it clearly is of the form 1 from the proposition and we get the relevant colorings by evaluating all other non-equivalent colorings to zero.

Let now $\Gamma$ have four or more vertices and $\Gamma_2$ be the subgraph consisting of vertices of type $\mathcal{V}_2$ and all edges with a connection to one of these vertices. Let $\Gamma_{\sigma,2}$ be the graph $\Gamma$ with the colors inside $\Gamma_2$ permuted by $\sigma$. Then similar as in the $\mathrm{SL}_4$-case, but now with degree-reducibly equivalence, we get $\Gamma \simeq_d \mathrm{sgn}(\sigma)\Gamma_{\sigma,2}$, due to

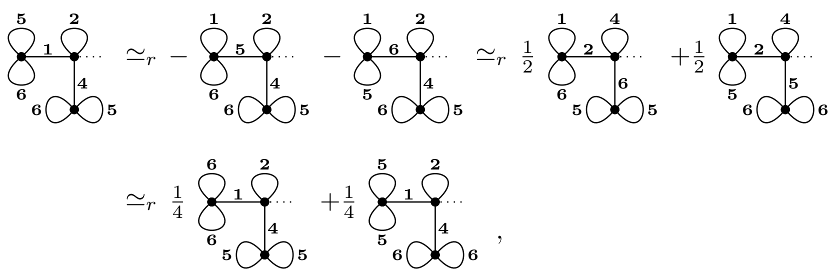


Thus we can use Proposition 3.1.20 to conclude that either $\Gamma$ has four vertices and $\Gamma_2$ ten edges or $\Gamma_2$ has at most $9 = \binom{5}{2} - 1$ edges. In the first case, $\Gamma_{\text{eff}}$ is the simple cubic connected graph $K_4$ and we find Graph 2 from the proposition. In the second case, we distinguish between the number of vertices of $\Gamma_2$:

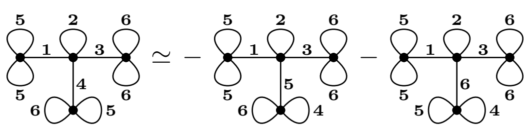
**Case 1: $\Gamma_2$ has one vertex.** Here $\Gamma$ must be of the form:




If this graph is reducible for any coloring, it must be reducible for a coloring where the remaining looping edges are of colors $\mathbf{1}, \ldots, \mathbf{9}$. If the two looping edges of one black hole are $\mathbf{1}, \ldots, \mathbf{4}$-colored, the graph is reducible. If one looping edge of a black hole is $\mathbf{1}, \ldots, \mathbf{4}$-colored, say $\mathbf{1}$, and the other $\mathbf{5}, \ldots, \mathbf{9}$-colored, by moving the $\mathbf{1}$-colored edge of $\Gamma_2$ to the black hole, this graph is reducibly equivalent to the one with two looping edges of color $\mathbf{1}$ at the black hole and one edge of color $\mathbf{5}$ in $\Gamma_2$. So by swapping colors $\mathbf{1}$ and $\mathbf{5}$, we can assume that the looping edges of black holes are $\mathbf{5}, \ldots, \mathbf{9}$-colored. If the four looping edges of two black holes are colored with only one color, the graph is reducible by moving one of the colored edges of the first to the second black hole. If two black holes each have colored their looping edges with the same two colors, say $\mathbf{5}$ and $\mathbf{6}$, then we get

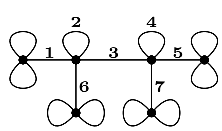


The only remaining possible form for a $\mathbf{1}, \ldots, \mathbf{6}$-colored graph is thus reducible:



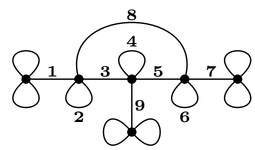
For graphs with more than six colors, we get the reducibly independent possibilities $3a$-$3g$.

**Case 2: $\Gamma_2$ has two vertices.** Then $\Gamma$ must have six vertices and is of the form




Due to the considerations from the previous case, this graph must be irreducible for some coloring of the remaining edges with colors $\mathbf{8}, \mathbf{9}$. But any such graph is reducible.

**Case 3: $\Gamma_2$ has three vertices.** Here $\Gamma_2$ either contains more than nine edges, or $\Gamma$ must be of the form




.

But this graph is reducible for all the remaining edges colored with colors $\mathbf{1}, \ldots, \mathbf{9}$. So it is reducible for any coloring.

**Case 4: $\Gamma_2$ has four or more vertices.** Then it contains more than nine edges, which is a contradiction. The proof is complete. □

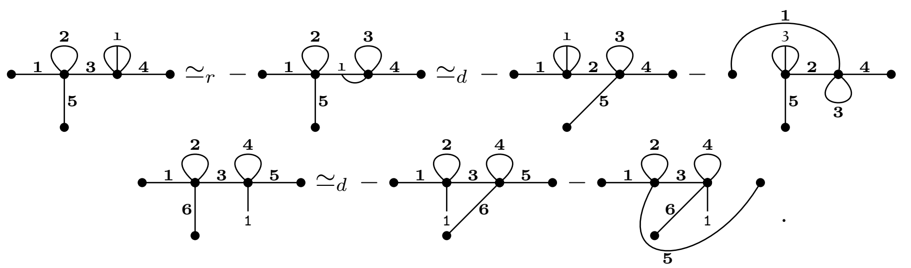
Proof of Theorem 7. We only consider degree-irreducible graphs and do not list explicit colorings. We also can assume that the number of 2-edges is greater or equal to the number of 3-edges due to the duality of $\Lambda^k V$ and $\Lambda^{n-k} V$. First of all, we have vertices of the type $\mathcal{V}_{11111}$, these constitute a connected component, the standard determinant. Besides, there are vertices of virtual degree one of the types $\mathcal{V}_4$, $\mathcal{V}_{22}$, $\mathcal{V}_{13}$, and $\mathcal{V}_{112}$. We have vertices of virtual degree two of the types $\mathcal{V}_3$ and $\mathcal{V}_{21}$. Lastly, we have vertices of virtual degree three of the type $\mathcal{V}_2$. We can assume that there are no vertices of virtual degree four, since such vertex would have a looping 1-edge and connections to $k$-edges with $k > 1$. But we can make such $k$-edge looping at the vertex which would yield a higher virtual degree. We have no double 2-edges due to degree-irreducibility, see the previous proof of Proposition 3.2.8.

All possible graphs with one vertex are irreducible for a suitable choice of colors. Those with two vertices either have a non-looping 2-edge and any combination of vertices of types $\mathcal{V}_4$, $\mathcal{V}_{22}$, $\mathcal{V}_{13}$, and $\mathcal{V}_{112}$, or they have two 2-edges and two 3-edges and due to degree-irreducibility, for any combination of colors, there is one unique such graph.

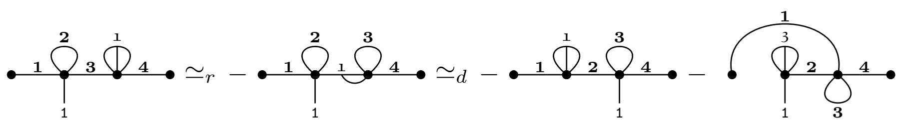
So let $\Gamma$ have three or more vertices. We can assume that there is no non-looping 4-edge, since if there is one and it is only connected to vertices with looping 4-edges, this constitues a connected component with mirror the graph with one vertex of the type $\mathcal{V}_{11111}$, on the other hand, if it is connected to a vertex without a looping 4-edge, we can pull it over to this vertex.

If there is a vertex of type $\mathcal{V}_2$, $\mathcal{V}_{12}$ or $\mathcal{V}_{112}$, then there is no non-looping 3-edge. This is due to degree-irreducibility: by pulling over the non-looping 3-edge to the vertex, we achieve that (after reordering descendingly if necessary) all virtual degrees are smaller or equal than before and at least one is strictly smaller. All 2-edges but the looping ones of black holes $\mathcal{V}_{22}$ can be permuted and we have $\Gamma_\sigma \simeq_d \operatorname{sgn}(\sigma)\Gamma$ as usual.

**Case 1: $\Gamma$ has a vertex of type $\mathcal{V}_2$.** Here we have no non-looping 3-edge and thus in principle, all graphs stem from those from the proof of Proposition 3.2.8, with three possible modifications. Firstly, vertices of type $\mathcal{V}_{22}$ can be replaced by such of types $\mathcal{V}_4$, $\mathcal{V}_{13}$ or $\mathcal{V}_{112}$. Secondly, arms or cycles can be prolonged by inserting vertices of type $\mathcal{V}_3$ and $\mathcal{V}_{12}$, and lastly, two arms can be connected to a cycle by replacing the two 'end-vertices' with one vertex of type $\mathcal{V}_3$ or $\mathcal{V}_{12}$. The number of 2-edges here is always bounded by Proposition 3.1.20. We can assume that vertices of types $\mathcal{V}_3$ and $\mathcal{V}_{12}$ only are on two of three sides of a vertex of type $\mathcal{V}_2$ by the following:



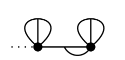
Moreover, a graph with two vertices of types $\mathcal{V}_3$ and $\mathcal{V}_{12}$ joined by a 2-edge and the graph with these two vertices swapped differ degree-reducibly by a graph with an additional vertex of type $\mathcal{V}_2$:



This graph is either reducible or is considered in the list of generators as well, so we can in fact swap two such vertices. This directly leads to the graphs from the Theorem.

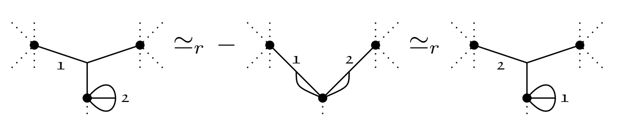
**Case 2: $\Gamma$ has no vertex of type $\mathcal{V}_2$ and no non-looping 3-edge.** In this case, the only non-looping edges are still those of size two. But now, we only have vertices of virtual degree one and two. Thus we have two types: chains and cycles.

**Case 3: $\Gamma$ has a non-looping 3-edge.** We have no vertices of types $\mathcal{V}_2$, $\mathcal{V}_{12}$ or $\mathcal{V}_{112}$. Thus we only have vertices of types $\mathcal{V}_4$, $\mathcal{V}_{22}$, $\mathcal{V}_{13}$, $\mathcal{V}_3$. Assume a non-looping 3-edge of $\Gamma$ has two connections to one vertex, then this vertex has a looping 3-edge. At the second vertex connected to the non-looping 3-edge, there must be a looping 3-edge as well, because otherwise, these two vertices constitute a connected component and we have assumed that $\Gamma$ has more than two vertices. Thus this part of the graph must be of the form

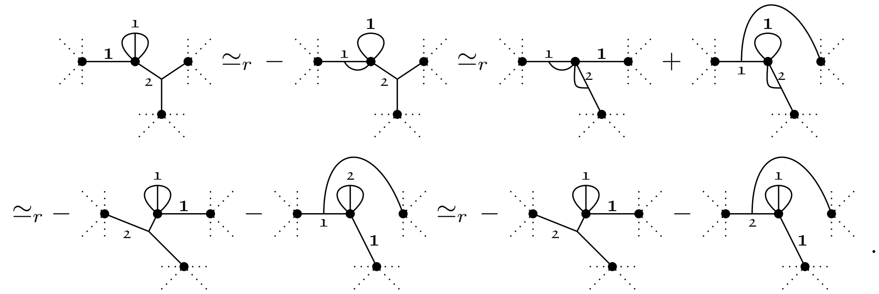


,

with two additional edges connected to the left vertex. Now the essential constraint comes from the number of cycles of $\Gamma$. If $\Gamma$ has more than one cycle, the number of 3-edges exceeds the number of 2-edges, since all vertices in the cycles must be of type $\mathcal{V}_3$. By



we can swap 3-edges adjacent to a vertex. In fact, we see that if both have the same color, the graph evaluates to zero since changing shadings of two $k$-edges for odd $k$ results in reversed sign, as can be seen directly from the corresponding brackets, cf. Section 3.1. Moreover, we can assume non-looping 2-edges to be only on two sides of a non-looping 3-edge due to



This is the mirrored version of the first equation from Case 1. The mirrored version of vertices $\mathcal{V}_{22}$ are the subgraphs from the beginning of Case 3 without the left looping 3-edge. They appear as blocks ⓐ in Theorem 7. We conclude that the number of blocks ⓐ plus the number of 3-edges that are not part of a block ⓐ is less than or equal to nine.

So first assume $\Gamma$ has no cycle. Due to the restriction on the number of 3-edges, only for one non-looping 3-edge of such graph there can be non-looping 3-edges on three sides, i.e. we have a 'star'. Here only three or four non-looping 3-edges are possible. We get the second and third graph in the first row on page 9. If we have a 'chain', up to five non-looping 3-edges are possible. The last four graphs on page 8 and the first on page 9 are of this type.

Now assume $\Gamma$ has a cycle (of non-looping 3-edges). Then all vertices of virtual degree one must be of type $\mathcal{V}_{22}$, otherwise the number of 3-edges would exceed the number of 2-edges. The cycle can be made up by two, three, or four 3-edges, where the total number of non-looping 3-edges is smaller or equal to four. We get the remaining graphs from the theorem.

Finally consider the restrictions on numbers of edges. The last one - namely that the number of 3-edges is less than or equal to the number of 2-edges - can be made since in the theorem we are considering the mirror graphs as well. For all graphs with no non-looping 3-edges, the first two restrictions are due to the previous observations in Cases 1 and 2, and due to the last restriction as well as the form of the graphs. For graphs with looping 3-edges, the same arguments hold with 2- and 3-edges interchanged. □

## 3.3. The Crosshair-Sieve algorithm for Gröbner bases

The Crosshair-sieve-algorithm 3.5.11 is developed in Subsection 3.5.2 of Section 3.5 in order to show that a certain ideal basis is a Gröbner basis. We give a general but very rough account of this algorithm here. It lies between standard Gröbner basis computations with usually a limited amount of available monomial orders and the computation of the Gröbner fan, introduced in [**68**]. In particular, if we want to find a Gröbner basis with certain good properties (for instance a particular set of leading monomials), then it is in general not very likely that the standard monomial orders such as lexicographic, total degree reverse lexicographic et cetera produce such basis. This problem is adressed by Gröbner fan computations, see [**41**]. On the other hand, the computation of the whole Gröbner fan encoding all possible Gröbner bases of one ideal or a universal - i.e. with respect to all monomial orders - Gröbner basis, see [**95**], can be too complex, see [**65**] for such problems in applied Gröbner basis theory. This is in particular the case if some parameters are involved as in the three series of invariant rings we will encounter in Section 3.5. Moreover, even the computation of a single Gröbner basis for a badly suited monomial order can be very complex in such cases.

So for a polynomial $f$ denote by $\mathrm{lm}(f)$ the leading monomial of $f$ wih respect to a given monomial order. Assume we are in the following situation: we have a finite set of polynomials $f_i$ with monomials $m_i$ appearing in the $f_i$ and we want to find a monomial order such that the $f_i$ have leading monomials $\mathrm{lm}(f_i) = m_i$. If for example the $m_i$ pairwise have no variable in common, then the $f_i$ are already a Gröbner basis of the ideal generated by themselves. This follows directly from the Buchberger algorithm [**29**, Thm. 3.3] as all $S$-polynomials reduce to zero.

Any monomial order on $s$ variables can be expressed as a matrix order, see [**81, 63**], with a matrix $\mathcal{M} \in \mathrm{GL}_s(\mathbb{R})$ where degrees are given by the first row of $\mathcal{M}$ with ties broken by the second row et cetera. In fact, we only require $\mathcal{M}$ to have $s$ columns, an arbitrary number of rows and rank $s$. What we want to do is building up a matrix $\mathcal{M}$ such that the induced monomial order provides the required leading terms.

Algorithm 3.3.1 (*General Crosshair-sieve*). In order to do so, let $d$ be the maximal total degree of all $f_i$ and denote the variables by $x_1, \ldots, x_s$. Set $x_0 := 1$. To each monomial $x_1^{a_1} \cdots x_s^{a_s}$ associate a degree vector $\sum a_j e_j \in \mathbb{R}^s$. Denote the rows of $\mathcal{M}$ by sums $\mathcal{M}_\nu = \sum b_j e_j^*$.

To each row $\mathcal{M}_\nu$ of $\mathcal{M}$ associate a symmetric tensor $S_\nu \in S^d(\mathbb{R}^{s+1})$, the so called *sieve*, collecting the degrees associated by $\mathcal{M}_\nu$ to all monomials in the $x_i$ of degree $\leq d$. In the entry $(S_\nu)_{i_1,\ldots,i_d}$ stands the degree of $x_{i_1} \cdots x_{i_d}$. The degrees of all monomials of one $f_i$ are a collection of entries $(S_\nu)_{i_1,\ldots,i_d}$.

Now start with $\nu = 1$ and $\mathcal{M}_\nu = 0$. We 'target' a monomial $m_i$ by increasing the coefficient $b_j$ in $\mathcal{M}_\nu$ for at least one $x_j$ occuring in $m_i$. We do this for every $m_i$ *in such way that* $\deg_\nu(m_i) \geq \deg_\nu(m_i')$ *for all monomials* $m_i'$ *of* $f_i$. It is necessary for the algorithm to work (and for the $m_i$ to be a possible set of leading monomials) that such way exists. This is the case for example if all $f_i$ are homogeneous and we set $b_j = 1$ for all $x_j$ occuring in some $m_i$. But this condition is not sufficient of course. At least for one single $i$, we require $\deg_\nu(m_i)$ to be truly greater than $\deg_\nu(m_i')$ for all other monomials $m_i'$ of $f_i$. We say $m_i$ is *filtered out* by $S_\nu$. Which means we do not have to target this very $m_i$ in $S_{\nu+1}$ any more and thus get less *unwanted interferences of* $b_j$ leading to high degrees of unwanted monomials. It is clear that this algorithm terminates if and only if at each step - for each row of $\mathcal{M}$ - at least one $m_i$ is filtered out.

Remark 3.3.2. The success of this approach of course relies heavily on the structure of the $f_i$, i.e. how the degrees of their monomials are distributed over $S_\nu$. In practice, for example in invariant theory, the $f_i$ will most likely inherit some symmetry like weighted homogeneity and a symmetric distribution over $S_\nu$.

Example 3.3.3. We give a short account of how such properties fit together very well in the serial cases considered in Section 3.5. For details we refer to Algorithm 3.5.11. Since the polynomials considered there are *weighted homogeneous*, even if the $m_i$ are not of maximal total degree among all monomials in the $f_i$, there will be some variables occuring in the $m_i$ but not in the $m'_i$ of higher total degree. Targeting these variables at first will filter out monomials of the same total degree as the $m_i$. Now we can arrange the variables in such way that degrees of one $f_i$ lie on counterdiagonals in the 2-tensors $S_\nu$ (seen as symmetric matrices). Moreover, as we have some freedom in the choice of the $m_i$, we can arrange them on diagonals. Now targeting them by setting $b_j = 1$ for each $x_j$ in some $m_i$ that has not yet been filtered out, we see that at least the right- and lowermost entry of $S_\nu$ will be filtered out at each step. The picture of $S_\nu$ with one $m_i$ targeted in this way is exactly that of a *crosshair*.

## 3.4. Two algorithms for Hilbert series computations

We compute Hilbert series of the invariant rings of the three single $\mathrm{SL}_n$ representations in Subsection 3.5.1 and of low dimensional representatives of the serial cases in Subsection 3.5.2 of Section 3.5 by the following method.

According to [**35**, Sec. 4.6] the (univariate) Hilbert series of the invariant ring of a representation of a reductive algebraic group can be expressed as the *constant term* in $l$ variables $z_1, \dots, z_l$ of a rational function of the form

$$\frac{f(z_1, \dots, z_l)}{(1 - f_1(z_1, \dots, z_l)t) \cdots (1 - f_k(z_1, \dots, z_l)t)},$$

with monomials $f_i$. It is one of the main objectives of *MacMahon partition analysis* to compute constant terms of so called *Elliot-rational* functions, with various applications and implementations, see [**3, 107, 108**]. Constant terms in more than one variable $z_1, \dots, z_l$ are treated as iterated constant terms in one variable $z_i$. The constant term can be directly read off if one has a partial fraction decomposition of the rational function. This is the approach of the algorithm `Ell` developed in [**107**] with an implementation in Maple. But there are two bottlenecks: firstly, if one of the denominator factors is not *linear* in $z_i$ and secondly, if two of the denominator factors are not relatively prime. These problems lead to a very significant increase in runtime, which makes it practically impossible to address complicated problems. In theory, the first problem can be solved by introducing roots of unity, the second one by introducing 'slack variables' $s_i$, one for each denominator factor, so that the function will change to

$$\frac{f(z_1, \dots, z_l)}{(1 - f_1(z_1, \dots, z_l)ts_1) \cdots (1 - f_k(z_1, \dots, z_l)ts_k)}.$$

This is the approach of [**108**]. But the output of such algorithm is a large sum of rational functions - most of them with poles at some $s_i = 1$ - which must be simplified to a single rational function where we can set $s_i = 1$. So the problem of computational complexity is only shifted. In [**108**], MacMahon partition analysis then is used again to compute terms constant in $v_i = s_i - 1$ for each of the output functions, which may by far be the part of the computation with the highest complexity, see [**108**, Sec. 5.3].

In the Hilbert series computations needed in Subsection 3.5.1, we did not succeed in reasonable time with any of the mentioned algorithms. So we propose the following modified version: assume that any pair of denominator factors that is not relatively prime is identical. Instead of introducing a slack variable for each factor, we introduce *one* slack variable $s$ and take different powers of $s$ for each member of a collection of identical denominator factors. A function of the form

$$\frac{f}{(1-f_1t)^{n_1}\cdots(1-f_kt)^{n_k}}$$

will thus become

$$\frac{f}{(1-f_1t)\cdots(1-f_1ts^{n_1-1})\cdots(1-f_kt)\cdots(1-f_kts^{n_k-1})}.$$

The output of the algorithm from [**107**] applied to such function will be a large sum of rational functions with no chance to be simplified by e.g. standard Maple simplification. This is where the second aspect of our modification comes into play: most of the rational functions will have a pole in $s=1$, but of different *order*. Let $c\in\mathbb{N}$ be the highest pole order. Since we know that the resulting function has no pole at $s=1$, the sum of all functions with a pole of order $c$ must have a pole of order $c-1$ at most. Since it is likely that such sum consists of not nearly as much terms as the whole sum, it might be possible to compute it and thus reduce the highest overall pole order. Iterating this procedure will of course finally result in the sum of all rational functions, our desired output.

The Maple-worksheets for the Hilbert series computations with the above methods applied to the $\mathrm{SL}_5$ representations $2\Lambda^2+\Lambda^3+V^*$, $3\Lambda^2+V^*$ and $2\Lambda^2+4V^*$ are available as supplementary material to our preprint [**19**]. We claim that the described modification is useful for at least some more problems of certain complexity.

Remark 3.4.1. The above algorithm may analogically be used to compute multivariate Hilbert series, i.e. not with respect to total degree but a certain multigrading. As such Hilbert series contain more information, compare Subsection 3.5.2, they may be more useful than the univariate ones in some situations, for example if one is searching for generators and relations. But then on the other hand, one has to compute multivariate Hilbert series of ideals given by generators and multidegrees. To our knowledge, there is no such implementation available, though it is analogue to computation of univariate Hilbert series by means of Gröbner bases. A basic implementation in Maple, applied to computing the Hilbert series of the algebra $A_3$ from Lemma 3.5.16, is provided as supplementary material to [**19**] as well.

### 3.5. Representations of $\mathrm{SL}_n$ with complete intersection invariant ring

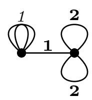
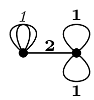
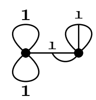
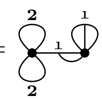
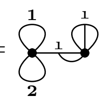
The aim of this section is to prove Theorem 8. As we mentioned in the introduction, the task is to decide if the six cases left open in Shmelkin's classification [**91**] have complete intersection invariant ring or not. These six cases are:

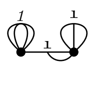
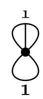
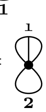
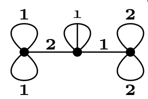
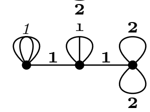
$$(\mathrm{SL}_5, 2\Lambda^2+\Lambda^3+V^*),\quad (\mathrm{SL}_5, 3\Lambda^2+V^*),\quad (\mathrm{SL}_7, 3V+\Lambda^3+V^*),$$
$$(\mathrm{SL}_n, 2\Lambda^2+4V^*),\quad (\mathrm{SL}_n, V+2\Lambda^2+3V^*),\quad (\mathrm{SL}_n, S^2+\Lambda^2+2V^*),\quad 5\le n\in 2\mathbb{Z}+1.$$

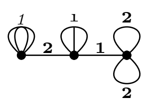
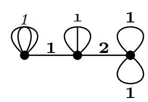
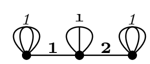
We will use the graph method from Section 3.1 here for combinations of different types of tensors - antisymmetric *and* symmetric. Graphically, we will distinguish between antisymmetric and symmetric tensors by using jagged edges for the latter ones. Their 'behaviour' with respect to Plücker relations will be different, so the 'Exchange Lemma' 3.1.4 is not valid in that form for symmetric tensors. We refer to [**46**] for details, but anyway will not apply Plücker relations to symmetric tensors in the following. In the serial cases, we will depict $(n-1)$-edges by 'solid' versions and (in the rare cases when they appear) $(n-k)$-edges by thinner solid versions depending on $k$. Compare for example Lemma 3.5.7, where the gray solid edge in the graph $c_1^\circ$ is an $(n-1)$-edge, while the one in $c_2^\circ$ is an $(n-3)$-edge.

**3.5.1. The single cases.** Consider the $\mathrm{SL}_5$-representations $U_1 := 2\Lambda^2 + \Lambda^3 + \Lambda^4$ , $U_2 := 3\Lambda^2 + \Lambda^4$ and the $\mathrm{SL}_7$-representation $U_3 := 3V + \Lambda^3 + \Lambda^6$. These are the three single cases that Shmelkin left open in [**91**]. We prove that all three are complete intersections in the following.

PROPOSITION 3.5.1. *The ring of invariants of $U_1$ is a complete intersection of homological dimension two minimally generated by*

$g_{122} =$ , $g_{211} =$ , $g_{11} =$ , $g_{22} =$ , $g_{12} =$ ,

$g =$ , $g_1 =$ , $g_2 =$ , $g_{112122} =$ , $g_{1122} =$ ,

$g_{2122} =$ , $g_{1211} =$ , $g_{21} =$  .

PROOF. The list of generators can be extracted from Theorem 7. Since the Krull dimension of the ring of invariants is 11, the homological dimension is two and it is a complete intersection due to [**76**, Rem. to Prop. 1.5]. We also computed the Hilbert series of $\mathbb{C}[U_1]^{\mathrm{SL}_5}$, it is

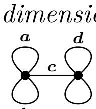
$$\frac{(1-t^{10})(1-t^{12})}{(1-t^2)^2(1-t^3)(1-t^4)^5(1-t^5)(1-t^6)^3(1-t^7)}.$$

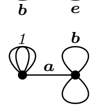
The following relations hold between the respective invariants. One gets them for example by linear dependencies between the possible monomials of the degrees given by the Hilbert series. As can be seen by comparison with the above Hilbert series, they are generators of the ideal of relations.

$$16g_{1122}g_2g_{211} - 6g_2^2g_{211}^2 - 3g_1g_{1122}g_{122} - 8g_{11}g_{122}^2 - 8g_{12}g_{122}g_{211} - 3g_{112122}g_{21}$$
$$-16g_{1122}^2 - 16g_{1211}g_{2122},$$
$$2gg_{112122} - 4g_{12}g_{1122} - 2g_{2122}g_{11} - 2g_{1211}g_{22} - g_{122}g_2g_{11} + g_{211}g_1g_{22} - 2g_{211}g_2g_{12}.$$

$\square$

PROPOSITION 3.5.2. *The ring of invariants of $U_2$ is a complete intersection of homological dimension three minimally generated by*

$f_{abcde} =$ , $abcde \in \{12311, 12322, 13233, 11233, 11322, 22133\}$;

$f_{abc} =$ , $abc \in \{122, 322, 233, 133, 123, 213\}$.

PROOF. The list of generators can be extracted from Theorem 7. Since the Krull dimension of the ring of invariants is 11, the homological dimension is three.

We need to show that the ideal of relations is generated by three polynomials. The Hilbert series of $\mathbb{C}[W]^{\mathrm{SL}_5}$ is

$$\frac{(1-t^9)^3}{(1-t^4)^8(1-t^5)^6}$$

and the following three relations hold between the invariants:

$$-2f_{122}f_{13233} - 2f_{322}f_{11233} + f_{233}f_{11322} - f_{133}f_{12322} + 2f_{123}f_{22133},$$

$$2f_{211}f_{13233} - 2f_{311}f_{22133} + f_{133}f_{11322} - f_{233}f_{12311} + 2f_{213}f_{11233},$$

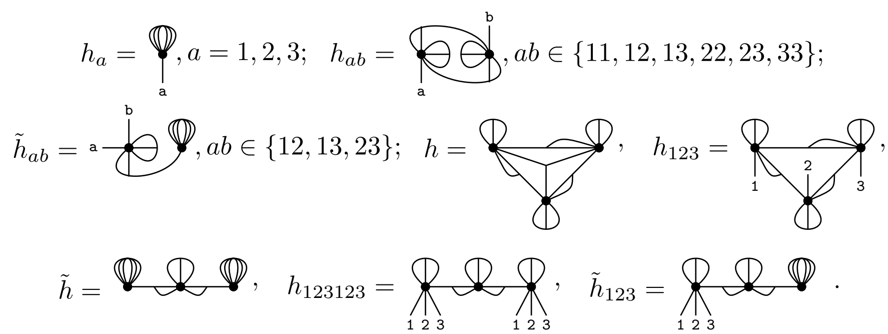
$$2f_{311}f_{12322} - 2f_{211}f_{22133} + f_{122}f_{11233} + f_{322}f_{12311} - 2(f_{213} + f_{123})f_{11322}.$$

The Hilbert series of $\mathbb{C}[f_{abcde}, f_{abc}]/I$ with the ideal $I$ generated by these three polynomials and the Hilbert series of $\mathbb{C}[W]^{\mathrm{SL}_5}$ from above coincide by a computation with Maple, thus the assertion follows.

□

Let in the following be all 3- and 6-edges of color one.

Proposition 3.5.3. *The ring of invariants of $U_3$ is a complete intersection of homological dimension two minimally generated by*




Proof. The Krull dimension of the ring of invariants is 15. The first 15 invariants can be extracted from [**91**, Table 7]. Also from there, we know that an irreducible graph with five 3-edges and six 1-edges - two of each color $1, 2, 3$ - must exist. Since such a graph can not be equivalent to a sum of disconnected graphs due to the lack of appropriate graphs with fewer vertices, any graph with such edges either evaluates to zero or it is irreducible and the irreducible ones are pairwise reducibly equivalent. The penultimate one from the above list is such an irreducible graph. Remaining graphs from a maximal set of reducibly independent irreducible ones must now contain at least one 3-, one 6- and one 1-edge of each possible color, since otherwise, they would appear already for some subrepresentation for which the generators of the ring of invariants are known. The last one from the above list is of such kind.

Now we follow the outline of Shmelkin [**91**, pp. 221, 227]. Let $z = e_1 \wedge e_2 \wedge e_5 + e_3 \wedge e_4 \wedge e_6 + e_1 \wedge e_3 \wedge e_7 + e_2 \wedge e_4 \wedge e_7$. Then $\mathrm{SL}_7 z$ is dense in $V(h)$ and the isotropy group of $z$ is $G_2$. Denote by $g_i$ the restrictions of all generators of $\mathbb{C}[W]^{\mathrm{SL}_5}$ but $h$ to $3V + z + \Lambda^6$. Consider the obvious $\mathbb{Z}^5$ and $\mathbb{Z}^4$-gradings of $\mathbb{C}[W]$ and $\mathbb{C}\left[4\mathbb{C}^7\right] = \mathbb{C}\left[3V + \Lambda^6\right]$ respectively. Then the proof of [**90**, Prop. 4.5] says that the Hilbert series of the algebras $\mathbb{C}\left[4\mathbb{C}^7\right]^{G_2}$ and $\mathbb{C}\left[g_i\right]$ are identical. Now due to [**87**], the multivariate Hilbert series of the first algebra is

$$\frac{(1-t_1^2t_2^2t_3^2t_4^2)}{\prod_{1\le i\le j\le 4}(1-t_it_j)\prod_{1\le k\le 4}\left(1-t_1t_2t_3t_4t_k^{-1}\right)(1-t_1t_2t_3t_4)}.$$

Now consider all invariants from the proposition but $h$. Let $e_i = e_1\wedge ...\wedge e_{i-1}\wedge e_{i+1}...\wedge e_7$. We can restrict to $2V+\mathbb{C}e_7+z+(e_1^*+\mathbb{C}e_3^*+\mathbb{C}e_7^*)$ by [**91**, Ex. on p. 221]. If we further restrict to $2V+e_7+z+(e_1^*+e_3^*+\mathbb{C}e_7^*)$, then $\tilde{h}$ and $h_{33}$ become constants. The restrictions - denoted by Fraktur letters - of the remaining invariants satisfy the relations

$$\mathfrak{h}_{13}^2\mathfrak{h}_{22}-2\mathfrak{h}_{12}\mathfrak{h}_{13}\mathfrak{h}_{23}+\mathfrak{h}_{11}\mathfrak{h}_{23}^2+2\mathfrak{h}_{12}^2-2\mathfrak{h}_{11}\mathfrak{h}_{22}-8\mathfrak{h}_{123}^2,$$

$$\begin{gathered}\mathfrak{h}_3^2\mathfrak{h}_{12}^2-2\mathfrak{h}_2\mathfrak{h}_3\mathfrak{h}_{12}\mathfrak{h}_{13}+\mathfrak{h}_2^2\mathfrak{h}_{13}^2-\mathfrak{h}_3^2\mathfrak{h}_{11}\mathfrak{h}_{22}+2\mathfrak{h}_1\mathfrak{h}_3\mathfrak{h}_{13}\mathfrak{h}_{22}+2\mathfrak{h}_2\mathfrak{h}_3\mathfrak{h}_{11}\mathfrak{h}_{23}-2\mathfrak{h}_1\mathfrak{h}_3\mathfrak{h}_{12}\mathfrak{h}_{23}\\-2\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_{13}\mathfrak{h}_{23}+\mathfrak{h}_1^2\mathfrak{h}_{23}^2-2\mathfrak{h}_2^2\mathfrak{h}_{11}+4\mathfrak{h}_1\mathfrak{h}_2\mathfrak{h}_{12}-2\mathfrak{h}_1^2\mathfrak{h}_{22}-4\mathfrak{h}_{23}\tilde{\mathfrak{h}}_{12}\tilde{\mathfrak{h}}_{13}+2\mathfrak{h}_{22}\tilde{\mathfrak{h}}_{13}^2\\+4\mathfrak{h}_{13}\tilde{\mathfrak{h}}_{12}\tilde{\mathfrak{h}}_{23}-4\mathfrak{h}_{12}\tilde{\mathfrak{h}}_{13}\tilde{\mathfrak{h}}_{23}+2\mathfrak{h}_{11}\tilde{\mathfrak{h}}_{23}^2-8\mathfrak{h}_3\tilde{\mathfrak{h}}_{12}\mathfrak{h}_{123}+8\mathfrak{h}_2\tilde{\mathfrak{h}}_{13}\mathfrak{h}_{123}-8\mathfrak{h}_1\tilde{\mathfrak{h}}_{23}\mathfrak{h}_{123}+4\tilde{\mathfrak{h}}_{12}^2\\-16\tilde{\mathfrak{h}}_{123}^2+32\mathfrak{h}_{123123}.\end{gathered}$$

The first one corresponds to the syzygy of $(\mathrm{SL}_7, 3V+\Lambda^3V)$ from [**91**, Table 7]. The second one is new. By suitably multiplying with $\tilde{h}$ and $h_{33}$ and adding $hh_{123123}$, we get the syzygies

$$h_{13}^2h_{22}-2h_{12}h_{13}h_{23}+h_{11}h_{23}^2+h_{33}h_{12}^2-h_{11}h_{22}h_{33}-8h_{123}^2+hh_{123123},$$

$$\begin{gathered}h_3^2h_{12}^2-2h_2h_3h_{12}h_{13}+h_2^2h_{13}^2-h_3^2h_{11}h_{22}+2h_1h_3h_{13}h_{22}+2h_2h_3h_{11}h_{23}-2h_1h_3h_{12}h_{23}\\-2h_1h_2h_{13}h_{23}+h_1^2h_{23}^2-2h_2^2h_{11}h_{33}+4h_1h_2h_{12}h_{33}-2h_1^2h_{22}h_{33}-4h_{23}\tilde{h}_{12}\tilde{h}_{13}+2h_{22}\tilde{h}_{13}^2\\+4h_{13}\tilde{h}_{12}\tilde{h}_{23}-4h_{12}\tilde{h}_{13}\tilde{h}_{23}+2h_{11}\tilde{h}_{23}^2-8h_3\tilde{h}_{12}h_{123}+8h_2\tilde{h}_{13}h_{123}-8h_1\tilde{h}_{23}h_{123}+4\tilde{h}_{12}^2h_{33}\\-16\tilde{h}_{123}^2+32h_{123123}\tilde{h}.\end{gathered}$$

The Hilbert series of the algebra with these two syzygies is

$$\frac{(1-t_1^2t_2^2t_3^2t_4^2)(1-t_1^2t_2^2t_3^2)}{\prod_{1\le i\le j\le 4}(1-t_it_j)\prod_{1\le k\le 4}\left(1-t_1t_2t_3t_4t_k^{-1}\right)(1-t_1t_2t_3t_4)(1-t_1^2t_2^2t_3^2)}$$

and coincides with the one of $\mathbb{C}\left[4\mathbb{C}^7\right]^{G_2}$, which proves the assertion. □

REMARK 3.5.4. An alternative way to prove Proposition 3.5.3 would be to decompose

$$U_3=\left(\Lambda^3V\right)+\left(3V+\Lambda^6V\right),$$

where for the first $\mathrm{SL}_7$-module, we know a set of generators of the algebra of covariants due to [**82**] and for the second one, such set is easy to determine with the same techniques we apply in Section 3.5.2. Following up the approach of Subsection 3.5.2, from these covariant algebras one can explicitly compute a minimal set of generators coinciding with the one from the proposition, find relations between them of suitable multidegree and show that these cut out a complete intersection of the right dimension.

**3.5.2. The serial cases.** In this section, let always $n = 2p+1$ for $p \in \mathbb{N}_{\geq 2}$. We consider the three $\mathrm{SL}_n$-representations $V_1 := \Lambda^2 V + 2\Lambda^{2p} V$, $V_2 := V + \Lambda^2 V + \Lambda^{2p} V$, $V_3 := S^2 V$. The serial cases that were left open in [**91**] are

$$W_1 := 2V_1, \quad W_2 := V_1 + V_2, \quad W_3 := V_1 + V_3.$$

Our approach is similar in all cases and involves three steps, which we describe in the following. For the first step, let us denote the maximal unipotent subgroup of $\mathrm{SL}_n$ of lower triangular matrices by $U$, the *opposite* maximal unipotent subgroup of upper triangular matrices by $U^o$ and the normalizing maximal torus by $T$. By Theorem 0.2 of [**75**], for affine varieties $X$ and $Y$, the algebra $\mathbb{C}[X+Y]^{\mathrm{SL}_n}$ is a deformation of $(\mathbb{C}[X]^U \otimes \mathbb{C}[Y]^{U^o})^T$ and both algebras share the same Hilbert series with respect to a common $\mathrm{SL}_n$-stable grading. We explicitly compute $A_i := (\mathbb{C}[V_1]^U \otimes \mathbb{C}[V_i]^{U^o})^T$ and its Hilbert series for $i = 1, 2, 3$. In order to do this, in Lemmata 3.5.10, 3.5.14, 3.5.16 we apply our graph method to algebras of covariants.

Now the explicit form of the $A_i$ not only provides us degrees of potential syzygies for $\mathbb{C}[W_i]^{\mathrm{SL}_n}$, but also *important parts* of these syzygies, because the syzygies of $A_i$ turn out to be - as one would expect - 'contractions' of syzygies holding in $\mathbb{C}[W_i]^{\mathrm{SL}_n}$. Here, the 'contraction' can be seen as dropping certain monomials from a polynomial. Not all of them deform to *generators* of the ideal of syzygies of $\mathbb{C}[W_i]^{\mathrm{SL}_n}$ - those that do not are responsible for $A_i$ not being a complete intersection. But those that do so represent *important parts*, because they have no variables in common and suggest a way to prove the ci. property.

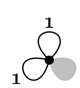
In step two, we find explicit forms of syzygies for $\mathbb{C}[W_i]^{\mathrm{SL}_n}$, which is made possible by our graph theoretic method from Section 3.1.

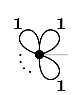
Finally in step three, we prove that the syzygies we found in step two generate the ideal of syzygies. In all three cases, we find a suitable monomial matrix order such that these syzygies are a Gröbner basis with respect to this order and derive the ci. property. The respective leading monomials also occur in the contracted versions of the syzygies in $A_i$. A posteriori, we see that $A_i$ is just not the *best* contraction of $\mathbb{C}[W_i]^{\mathrm{SL}_n}$ for proving the ci. property: we can deform $A_i$ to some algebra $B_i$ that on the one hand lacks all the superfluous equations and generators of $A_i$ and on the other hand keeps the important parts of the syzygies of $\mathbb{C}[W_i]^{\mathrm{SL}_n}$.

3.5.2.1. *Rings of Covariants.* In the following, we compute rings of covariants of the representations $V_1, V_2, V_3$. Here again dummy edges as in Subsection 3.2.3 of Section 3.2 have to be considered. When we speak of *applying the Plücker relation to one or several edges*, we mean that we consider the relation coming from the Exchange Lemma 3.1.4 with $u$ the word corresponding to the aforementioned edges.

LEMMA 3.5.5. *The algebra of covariants* $\mathrm{Cov}\,(\mathrm{SL}_{2p+1}, V_1)$ *for* $p \geq 2$ *is minimally generated by the* $4p+3$ *covariants*

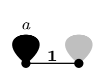
$c_1 =$ 

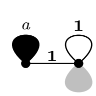
, $c_2 =$ 

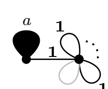
$, \ldots,$ $c_p =$ 

,

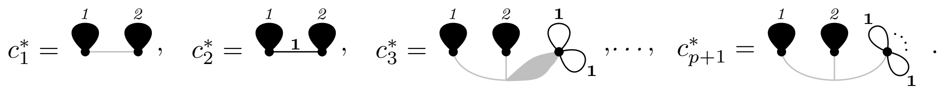
$c_1^{(a)} =$ 

, $c_2^{(a)} =$ 

, $c_3^{(a)} =$ 

$, \ldots,$ $c_{p+1}^{(a)} =$ 

$, a = 1, 2;$

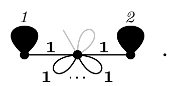



*Moreover, the ideal of relations is generated by*

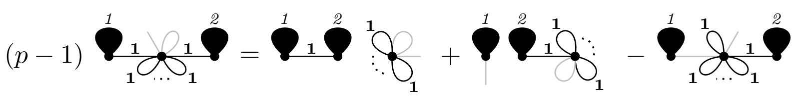
$$c_p\, c_1^* - p\, c_1^{(1)}\, c_{p+1}^{(2)} + p\, c_1^{(2)}\, c_{p+1}^{(1)}.$$

PROOF. Due to [**82**, Thm. 3], covariant graphs are similar to invariant graphs but can in addition contain looping dummy $k$-edges, behaving as if they correspond to additional copies of $\Lambda^k V$. In fact, the algebra of covariants is isomorphic to the algebra of invariants $\mathbb{C}[W]^{U(G)}$ for a maximal unipotent subgroup $U(G)$ of $G$, see [**92**]. The isomorphism is given - if we choose the upper triangular matrices for $U(G)$ - by evaluation at $e_1 \wedge \ldots \wedge e_k$ for each dummy $k$-edge. The multidegrees of the covariants can be deduced from [**25**, Table 1]. The explicit forms of the graphs for these covariants are then obvious.

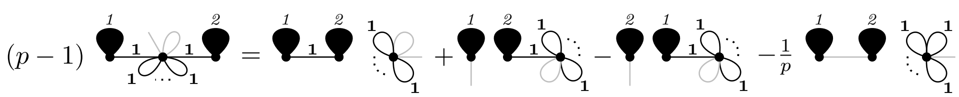
Since the Krull dimension of $\mathbb{C}[W]^{U(G)}$ is $4p+2$, see [**25**], we have one syzygy. We obtain this syzygy by considering the graph



Applying the Plücker relation to the left non-looping 2-edge gives:




Now in the last graph on the right, we apply the Plücker relation to the non-looping 2-edge of color one and get:

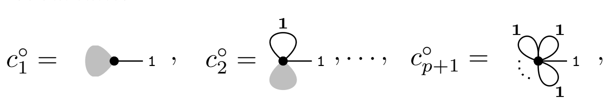


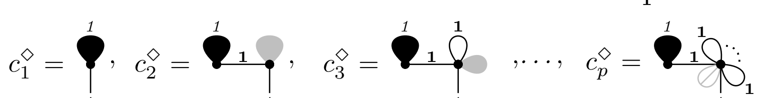

But due to the isomorphism from above, we can evaluate at $e_1$ and $e_1 \wedge e_2$ for the dummy 1- and 2-edge respectively and see that the two leftmost graphs disappear. We arrive at the desired relation.

∎

REMARK 3.5.6. Observe that in Lemma 3.5.5, we did not always choose the graph with a *looping* dummy edge for the invariant, but it is easy to deduce one version from another by applying the Plücker relation to the dummy edge.

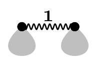
LEMMA 3.5.7. *The algebra of covariants* $\mathrm{Cov}(\mathrm{SL}_{2p+1}, V_2)$ *for* $p \geq 2$ *is polynomial, generated by the* $c_1, \ldots, c_p$ *and* $c_1^{(1)}, \ldots, c_{p+1}^{(1)}$ *from Lemma 3.5.5 and in addition by the covariants*

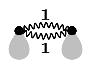


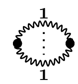



PROOF. Due to the equivalence with the algebra of invariants of $U(\mathrm{SL}_{2p+1})$, we get multidegrees of covariants from [**25**, Table 1], which in addition gives polynomiality. It is straightforward to find the only possible covariants of the matching multidegrees. $\square$

LEMMA 3.5.8. *The algebra of covariants* $\mathrm{Cov}(\mathrm{SL}_n, V_3)$ *is polynomial, generated by the* $n$ *covariants*

$c_1^{\triangle} =$ 

, $c_2^{\triangle} =$ 

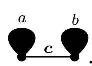
$, \dots,$ $c_n^{\triangle} =$ 

.

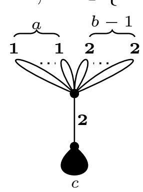
PROOF. Here, we get polynomiality and multidegrees of covariants from [**24**, Thm. 3] and proceed as in Lemma 3.5.7. $\square$

3.5.2.2. *The case* $(\mathrm{SL}_{2p+1}, W_1)$.

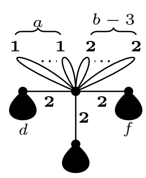
PROPOSITION 3.5.9. *The ring of invariants* $\mathbb{C}[W_1]^{\mathrm{SL}_{2p+1}}$ *is a complete intersection. Its homological dimension is four if* $p = 2$ *and* $2p-4$ *if* $p \geq 3$. *It is minimally generated by*

$\mathfrak{i}_{abc} =$ 

$, ab \in \{12, 13, 14, 23, 24, 34\}, c \in \{1, 2\};$

$\mathfrak{j}_{abc} =$ 

$, a, b \geq 1, c \in \{1, 2, 3, 4\};$

$\mathfrak{k}_{abc} =$ 

$, a, b \geq 3, def \in \{234, 134, 124, 123\}, c = \{1, 2, 3, 4\} \setminus \{d, e, f\}.$

LEMMA 3.5.10. *Denote by* $x$ *the images of the covariants* $c$ *from Lemma 3.5.5 in* $\mathbb{C}[V_1]^U$ *and by* $y$ *those in* $\mathbb{C}[V_1]^{U^o}$. *The algebra* $A_1 = (\mathbb{C}[V_1]^U \otimes \mathbb{C}[V_1]^{U^o})^T$ *is minimally generated by* $x_2^*, y_2^*$ *and the entries of the matrices*

$$f^{(1)} := \begin{bmatrix} x_2^{(1)} \\ x_2^{(2)} \end{bmatrix} \cdot \begin{bmatrix} y_1^{(1)} & y_1^{(2)} & y_p \end{bmatrix}, \quad f^{(2)} := \begin{bmatrix} x_1 \\ x_3^* \end{bmatrix} \cdot \begin{bmatrix} y_{p+1}^{(1)} & y_{p+1}^{(2)} & y_1^* \end{bmatrix},$$

$$g^{(1)} := \begin{bmatrix} y_2^{(1)} \\ y_2^{(2)} \end{bmatrix} \cdot \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & x_p \end{bmatrix}, \quad g^{(2)} := \begin{bmatrix} y_1 \\ y_3^* \end{bmatrix} \cdot \begin{bmatrix} x_{p+1}^{(1)} & x_{p+1}^{(2)} & x_1^* \end{bmatrix}$$

$$h^{(k)} := \begin{bmatrix} x_{2+k} \\ x_{4+k}^* \end{bmatrix} \cdot \begin{bmatrix} y_{p-k}^{(1)} & y_{p-k}^{(2)} \end{bmatrix}, \quad i^{(k)} := \begin{bmatrix} y_{2+k} \\ y_{4+k}^* \end{bmatrix} \cdot \begin{bmatrix} x_{p-k}^{(1)} & x_{p-k}^{(2)} \end{bmatrix}, \quad k = 0, \dots, p-3.$$

*The ideal of syzygies is generated by all* $2 \times 2$*-minors of the matrices* $f^{(j)}$, $g^{(j)}$, $h^{(k)}$, $i^{(k)}$ *and*

$$f_{a3}^{(1)} f_{b3}^{(2)} - p f_{a1}^{(1)} f_{b2}^{(2)} + p f_{a2}^{(1)} f_{b1}^{(2)}, \; g_{a3}^{(1)} g_{b3}^{(2)} - p g_{a1}^{(1)} g_{b2}^{(2)} + p g_{a2}^{(1)} g_{b1}^{(2)}, \quad a, b \in \{1, 2\}.$$

Proof. Under the isomorphism between $\mathrm{Cov}(\mathrm{SL}_n, V_1)$ and $\mathbb{C}[V_1]^U$, the images of the covariants from Lemma 3.5.5 have weights according to [**25**, Table 1]. From this, we directly deduce generators of the ring of invariants and the ideal of syzygies, similar as in [**5**, Proof of Thm. 1.8].

□

Proof of Prop. 3.5.9. The list of generators can be extracted from [**91**, Le. 2.5], it is minimal due to [**91**, Proof of Thm. 0.2]. We come to the syzygies. Let us first have a look at the Hilbert series of $A_1$ with respect to the $\mathbb{Z}^6$-grading of $W_1$, where variables $t_i$ correspond to copies of $\Lambda^2 V$ and $s_i$ to $\Lambda^{n-1}V$ respectively. Lemma 3.5.10 provides us with generators and syzygies. We have

$$\begin{aligned} A_1 = \mathbb{C}[x_2^*, y_2^*] &\otimes \mathbb{C}[x_1, x_3^*, x_2^{(1)}, x_2^{(2)}, y_p, y_1^*, y_1^{(1)}, y_1^{(2)}, y_{p+1}^{(1)}, y_{p+1}^{(2)}]^T \\ &\otimes \mathbb{C}[y_1, y_3^*, y_2^{(1)}, y_2^{(2)}, x_p, x_1^*, x_1^{(1)}, x_1^{(2)}, x_{p+1}^{(1)}, x_{p+1}^{(2)}]^T \\ &\otimes \bigotimes_{k=0}^{p-3} \mathbb{C}[x_{2+k}, x_{4+k}^*, y_{p-k}^{(1)}, y_{p-k}^{(2)}]^T \\ &\otimes \bigotimes_{k=0}^{p-3} \mathbb{C}[y_{2+k}, y_{4+k}^*, x_{p-k}^{(1)}, x_{p-k}^{(2)}]^T. \end{aligned}$$

The first three algebras have the Hilbert series

$$\frac{1}{(1-t_1s_1s_2)(1-t_2s_3s_4)},$$

$$\frac{(1-t_1^2t_2^ps_1s_3s_4)(1-t_1^2t_2^ps_2s_3s_4)}{(1-t_1s_1s_3)(1-t_1s_1s_4)(1-t_1s_2s_3)(1-t_1s_2s_4)(1-t_1t_2^ps_1)(1-t_1t_2^ps_2)(1-t_1t_2^ps_3)(1-t_1t_2^ps_4)(1-t_1s_3s_4)},$$

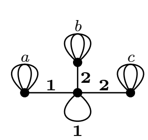
$$\frac{(1-t_1^pt_2^2s_1s_2s_3)(1-t_1^pt_2^2s_1s_2s_4)}{(1-t_2s_1s_3)(1-t_2s_2s_3)(1-t_2s_1s_4)(1-t_2s_2s_4)(1-t_1^pt_2s_3)(1-t_1^pt_2s_4)(1-t_1^pt_2s_1)(1-t_1^pt_2s_2)(1-t_2s_1s_2)},$$

while the $k$-th factors of the fourth and fifth algebra have the respective Hilbert series

$$\frac{(1-t_1^{5+2k}t_2^{2p-2k-2}s_1s_2s_3s_4)}{(1-t_1^{2+k}t_2^{p-k-1}s_3)(1-t_1^{2+k}t_2^{p-k-1}s_4)(1-t_1^{3+k}t_2^{p-k-1}s_1s_2s_3)(1-t_1^{3+k}t_2^{p-k-1}s_1s_2s_4)},$$

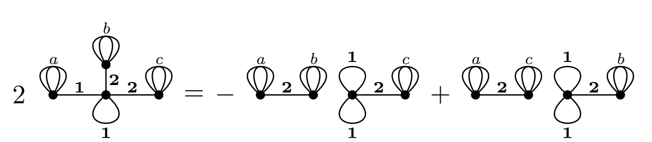
$$\frac{(1-t_2^{5+2k}t_1^{2p-2k-2}s_1s_2s_3s_4)}{(1-t_2^{2+k}t_1^{p-k-1}s_1)(1-t_2^{2+k}t_1^{p-k-1}s_2)(1-t_2^{3+k}t_1^{p-k-1}s_1s_3s_4)(1-t_2^{3+k}t_1^{p-k-1}s_2s_3s_4)}.$$

Now we distinguish between the cases $p = 2, 3, \geq 4$ as they behave differently. For $p = 2$, the Hilbert series of $\mathbb{C}[W_1]^{\mathrm{SL}_5}$ is the product of those of the first three algebras from above. Each factor in the denominator corresponds to one generator. Now consider the graphs

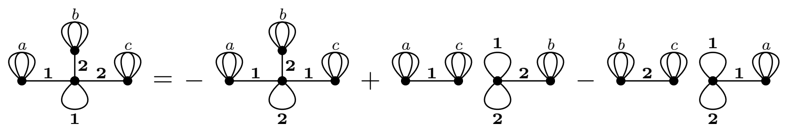


$, abc \in \{234, 134, 124, 123\}.$

Applying the Plücker relation to the two edges of color 1, we get



On the other hand, if we apply the Plücker relation to the right edge of color 2, we get

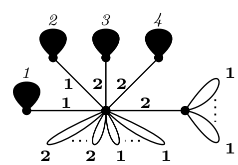



Finally applying the Plücker relation to the vertical edge of color 2 in the first graph on the right and on the two edges of same color in the other ones, we get the syzygy

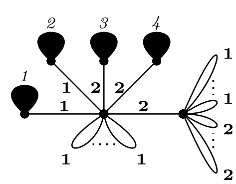
$$\mathfrak{i}_{ab1}\mathfrak{j}_{12c} - \mathfrak{i}_{ac1}\mathfrak{j}_{12b} + \mathfrak{i}_{bc1}\mathfrak{j}_{12a} + \mathfrak{i}_{ab2}\mathfrak{j}_{21c} - \mathfrak{i}_{ac2}\mathfrak{j}_{21b} + \mathfrak{i}_{bc2}\mathfrak{j}_{21a}.$$

The Hilbert series of the ideal generated by the resulting four syzygies for all combinations of $a, b, c$ coincides with that of $A_1$, so we are done with the case $p = 2$.

In the case $p = 3$, since the homological dimension is two, the c.i. property follows by [**91**, Le. 5.1], explicit syzygies can be obtained in the same way as for $p \geq 4$ in the following. So consider $p \geq 4$. Here in the decomposition of $A_1$ from above, apart from the first three algebras, we have one factor for each $k = 0, \ldots, p-3$ in the fourth and fifth one. In the resulting Hilbert series, the factors $(1 - t_i^p t_j^2 s_a s_b s_c)$ in the denominator and numerator cancel out and in the reduced fraction, again all denominator factors correspond to generators. So we need syzygies corresponding to $(1 - t_1^k t_2^{2p+3-k} s_1 s_2 s_3 s_4)$, $k = 4, \ldots, 2p-1$. One of $k$ and $2p + 3 - k$ is always greater than $p + 1$. Let us assume that $k > p + 1$ in the following. Consider the graphs




These graphs are decomposable in *two ways*. The first one is applying the Plücker relation on all two-edges of color 2 but those two going up. In the resulting sum, all graphs are either disconnected or evaluate to zero, since they have a vertex with $p$ looping two-edges of color 1 and one additional non-looping two-edge of color 1. For the second one, we observe that we can move a looping two-edge of color 1 from the rightmost to the central vertex by first applying the Plücker relation to all edges connected to the rightmost vertex - now the central and rightmost vertex are connected by a two-edge of color 1 - and then applying it to this connecting edge and all looping edges of color 1 at the central vertex. Iterating this procedure gives us a sum of disconnected graphs plus the graph




Now finally applying in this graph the Plücker relation to all 2-edges of color 1 looping at the central vertex and the one that connects to the vertex with the looping $(n-1)$-edge of color 1, we see that it is decomposable as well. The resulting relation has the form

$$\mathfrak{f}_k := \left( \sum_{i=1}^{4} \sum_{\substack{a+c=k \\ b+d=2p+3-k}} \mathfrak{j}_{abi}\mathfrak{k}_{cdi} \right) + \left( \sum_{(ij,l,m)\vdash(1234)} \sum_{\substack{e=1,2 \\ a+c=k-\delta_{e1} \\ b+d=2p+3-k-\delta_{e2}}} \mathfrak{i}_{ije}\mathfrak{j}_{abl}\mathfrak{j}_{cdm} \right),$$

where we ignore coefficients as they are not important for what follows. The notation $(ij,k,l) \vdash (1234)$ means that we sum over all subdivisions of the word 1234 in two words of length one and one of length two, i.e. we have twelve summands. The same procedure applies for $2p+3-k > p+1$ by interchanging 2-edges of colors 1 and 2, yielding respective syzygies in this case as well.

What remains is to show that the $\mathfrak{f}_k$ are a Gröbner basis for the ideal of syzygies of $\mathbb{C}[W_1]^{\mathrm{SL}_n}$. Consider the $\mathbb{Z}^{4n}$ grading of $\mathbb{C}[W_1]^{\mathrm{SL}_n}$ given by sending $\mathfrak{i}_{abc}$, $\mathfrak{j}_{abc}$, $\mathfrak{k}_{abc}$ to respective basis elements $e^{\mathfrak{i}}_{abc}$, $e^{\mathfrak{j}}_{abc}$, $e^{\mathfrak{k}}_{abc}$. Denote the dual basis of $(\mathbb{Z}^{4n})^*$ by $x^{\mathfrak{i}}_{abc}$, $x^{\mathfrak{j}}_{abc}$, $x^{\mathfrak{k}}_{abc}$. Now we need to find a monomial grading so that the leading monomials of each $\mathfrak{f}_k$ have no variable in common. Let us construct a matrix $\mathcal{M}$ with $4n$ columns and of rank $4n$, so that the associated grading meets our requirements. Our first goal is that the monomials $\mathfrak{j}_{abi}\mathfrak{k}_{cdi}$ are greater than the $\mathfrak{i}_{ije}\mathfrak{j}_{abk}\mathfrak{j}_{cdl}$. This is achieved by setting the first row $\mathcal{M}_1$ of $\mathcal{M}$ to

$$\mathcal{M}_1 := \sum x^{\mathfrak{i}}_{abc} + \sum x^{\mathfrak{j}}_{abc} + \sum 3x^{\mathfrak{k}}_{abc}.$$

After this preparation, we want to construct rows $\mathcal{M}_2, \mathcal{M}_3, \ldots$ for the desired grading matrix. In order to do so, we present the Crosshair-sieve Algorithm 3.3.1 from Section 3.3 adapted to the present situation.

---

ALGORITHM 3.5.11 (*Crosshair-sieve*). We see each row of the matrix $\mathcal{M}$ as a sieve (of increasing fineness), that filters out some of the monomials we want to be the leading ones of the $\mathfrak{f}_k$.

Now for some row $\mathcal{M}_\nu$ consider a $(p-1)\times(p-4)\times 4$-matrix (or tensor if you want) $\mathcal{S}_\nu$, where in the $(r,s,t)$-th entry stands the degree of $\mathfrak{j}_{r(p+1-r)t}\mathfrak{k}_{(s+2)(p-s)t}$. We say that the first entry gives the *row*, the second one the *column* and the third one the *level* of the *sieve* $\mathcal{S}_\nu$.

So entries in the same level and row correspond to monomials sharing some $\mathfrak{j}_{r(p+1-r)t}$ and entries in the same level and column some $\mathfrak{k}_{(s+2)(p-s)t}$ respectively. Observe that the degrees of all monomials of some $\mathfrak{f}_k$ stand in the counterdiagonal of $\mathcal{S}_\nu$ given by $r+s+2=k$ (remember $k=4,\ldots,2p-1$).

To construct a row of $\mathcal{M}$, we begin with $\mathcal{M}_\nu = 0$. Now if we want a monomial $\mathfrak{j}_{abi}\mathfrak{k}_{cdi}$ to be filtered out (i.e. to be the one of highest degree of some $\mathfrak{f}_{a+c}$), we target this monomial setting

$$\mathcal{M}_\nu := \mathcal{M}_\nu + x^{\mathfrak{j}}_{abi} + x^{\mathfrak{k}}_{cdi}.$$

In fact, at the $i$-th level of $\mathcal{S}_\nu$, this looks like a crosshair targeting our monomial. Exemplarily in the case $p=9$, we have the following picture:

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 2 | 1 |
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 | 0 |

Now $\mathfrak{j}_{abi}\mathfrak{k}_{cdi}$ is the leading monomial - with degree two - of $\mathfrak{f}_{a+c}$, but we have many leading monomials (of degree one) of other $\mathfrak{f}_k$'s that share a variable, which we do not want. So we have to target all other desired leading monomials in the same way. If this happens at another level of $\mathcal{S}_\nu$, all is fine, but if two desired leading monomials are at the same level - and this must happen for $p \geq 3$, since we only have four levels but need $2p-4$ leading monomials - we get unwanted crossings, as for example the italic ones in:

| | | | | |
|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 |
| *2* | 1 | 1 | 2 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 2 | 1 | 1 | *2* | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 |

So it may happen that in an $\mathfrak{f}_k$ some unwanted monomials are of the same degree as the desired leading polynomials with respect to $\mathcal{M}_\nu$, so the sieve $\mathcal{S}_\nu$ is *too coarse*. But as long as it filters out at least one desired leading monomial, we do not have to target this one in the following sieve $\mathcal{S}_{\nu+1}$, so we get lesser unwanted crossings and may be able to filter out another desired leading monomial. *So if at each stage we can filter out at least one, the algorithm will terminate with the desired set of leading monomials.* If the resulting matrix is not of full rank, we add rows to achieve full rank and found the desired grading.

---

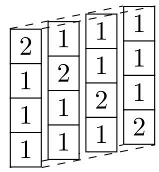
The termination of the algorithm obviously depends on the choice of the desired leading monomials, which results in a choice of arrangements of crosshairs eventually leading to a sieve that does not filter any desired leading monomial.

So we need to find a set of monomials for which the algorithm terminates. We do this by induction in $p$. Let $\mathcal{M}^p$ be the grading matrix for respective $p$ with first row $\mathcal{M}_1^p := \sum x_{abc}^{\mathfrak{i}} + \sum x_{abc}^{\mathfrak{j}} + \sum 3x_{abc}^{\mathfrak{k}}$ as defined above. Let $\mathcal{S}^p$ be the associated collection of sieves with $\mathcal{S}_i^p$ corresponding to $\mathcal{M}_i^p$. For $p = 4$, apart from permuting the last index, we only have one choice for the leading monomials lm:

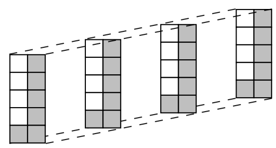
$$\mathrm{lm}(\mathfrak{f}_4) = \mathfrak{j}_{1p1}\mathfrak{k}_{3(p-1)1}, \quad \mathrm{lm}(\mathfrak{f}_5) = \mathfrak{j}_{2(p-1)2}\mathfrak{k}_{3(p-1)2}, \quad \mathrm{lm}(\mathfrak{f}_6) = \mathfrak{j}_{3(p-2)3}\mathfrak{k}_{3(p-1)3},$$

$$\mathrm{lm}(\mathfrak{f}_7) = \mathfrak{j}_{4(p-3)4}\mathfrak{k}_{3(p-1)4}.$$

Applying Algorithm 3.5.11 to this choice, the sieve $\mathcal{S}_2^4$ becomes



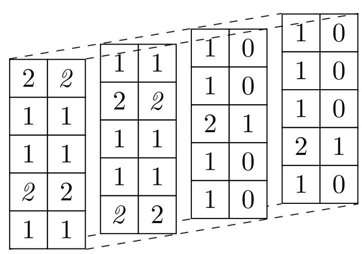

This sieve already filters out our desired monomials. We describe the first iteration step in detail: for $p = 5$, the sieves have one additional column and row, so they are of the form



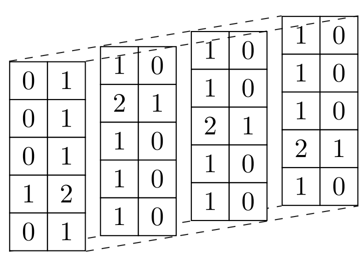
Observe that all entries on the counterdiagonal planes correspond to monomials of one $\mathfrak{f}_k$. For $\mathfrak{f}_4, \dots, \mathfrak{f}_7$, we keep the choice of leading monomials, while for $\mathfrak{f}_8$ and $\mathfrak{f}_9$ we choose

$$\mathrm{lm}(\mathfrak{f}_8) = \mathfrak{j}_{4(p-3)1}\mathfrak{k}_{4(p-2)1}, \quad \mathrm{lm}(\mathfrak{f}_9) = \mathfrak{j}_{5(p-4)2}\mathfrak{k}_{4(p-2)2},$$

which results in $\mathcal{S}_2^5$ (with italic unwanted highest degrees) becoming




We see that $\mathrm{lm}(\mathfrak{f}_9) = \mathfrak{j}_{5(p-4)2}\mathfrak{k}_{4(p-2)2}$ and $\mathrm{lm}(\mathfrak{f}_4) = \mathfrak{j}_{1p1}\mathfrak{k}_{3(p-1)1}$ are filtered out and so in the next sieve $\mathcal{S}_3^5$ we do not have to target them, which results in



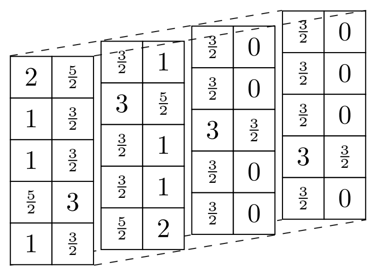
So $\mathcal{S}_3^5$ filters out the remaining leading monomials and we are done. Now for general $p$, choose the leading monomials for $\mathfrak{f}_4, \dots, \mathfrak{f}_{2p-3}$ as for $p-1$ and

$$\mathrm{lm}(\mathfrak{f}_{2p-2}) = \mathfrak{j}_{(p-1)21}\mathfrak{k}_{(p-1)31}, \quad \mathrm{lm}(\mathfrak{f}_{2p-1}) = \mathfrak{j}_{p12}\mathfrak{k}_{(p-1)32}.$$

With this choice, the desired leading monomial of $\mathfrak{f}_{2p-1}$ (among others, but this is just a bonus) is obviously filtered out by $\mathcal{S}_2^p$. In $\mathcal{S}_3^p$, it must not be targeted any more, and since the corresponding entry is the only one right and below the one of $\mathrm{lm}(\mathfrak{f}_{2p-2})$, at least this will now be filtered out by $\mathcal{S}_3^p$. So every $\mathcal{S}_\nu^p$ will filter out at least $\mathrm{lm}(\mathfrak{f}_{2p-\nu+1})$, which means that we are done. The $\mathfrak{f}_k$ with respective leading monomials are a Gröbner basis of the ideal of relations of $\mathbb{C}[W_1]^{\mathrm{SL}_n}$ and thus it is a complete intersection.

□

REMARK 3.5.12. In fact, we only need one row of $\mathcal{M}$ to achieve our desired leading monomials by subsequently scaling the $\mathcal{M}_\nu$ with highest index $\nu$ by a factor smaller than one and adding it to $\mathcal{M}_{\nu-1}$. With factor $\frac{1}{2}$, in the case $p = 5$ from above, the one remaining sieve then becomes



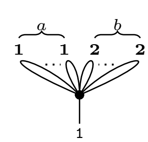

The grading matrix may then be filled up arbitrarily. Of course this gives a different grading but the leading monomials stay the same.

3.5.2.3. *The case* $(\mathrm{SL}_{2p+1}, W_2)$.

PROPOSITION 3.5.13. *The ring of invariants* $\mathbb{C}[W_2]^{\mathrm{SL}_{2p+1}}$ *is a complete intersection. Its homological dimension is two for* $p = 2$ *and* $2p - 3$ *for* $p \geq 3$. *It is minimally generated by the*

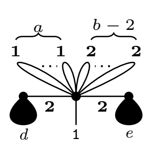
$$\mathfrak{i}_{abc}, ab \in \{12, 13, 23\}, c \in \{1, 2\}; \quad \mathfrak{j}_{abc}, a, b \geq 1, c \in \{1, 2, 3\}; \quad \mathfrak{k}_{ab4}, a, b \geq 3.$$

*from Proposition 3.5.9 and in addition*

$\mathfrak{h}_a =$  $a \in \{1, 2, 3\}$; $\mathfrak{l}_{ab} =$  $, a, b \geq 0;$





$\mathfrak{m}_{abc} =$  $, a, b \geq 2, de \in \{23, 13, 12\}, c = \{1, 2, 3\} \setminus \{d, e\}.$



LEMMA 3.5.14. *Denote by* $x$ *the images of the* $c$ *in* $\mathbb{C}[V_1]^U$ *and by* $y$ *those in* $\mathbb{C}[V_2]^{U^o}$. *The algebra* $A_2 = (\mathbb{C}[V_1]^U \otimes \mathbb{C}[V_2]^{U^o})^T$ *is minimally generated by*

*$x_2^*, y_{p+1}^\circ, y_1^\diamond$ and the entries of the matrices*

$$d := \begin{bmatrix} y_p^\circ \\ y_{p+1}^{(1)} \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_3^* \end{bmatrix}, \quad e := \begin{bmatrix} y_p \\ y_1^{(1)} \end{bmatrix} \cdot \begin{bmatrix} x_2^{(1)} & x_2^{(2)} \end{bmatrix},$$

$$f := \begin{bmatrix} y_1^\circ \\ y_2^{(1)} \end{bmatrix} \cdot \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & x_p \end{bmatrix}, \quad g := \begin{bmatrix} y_1 \\ y_2^\diamond \end{bmatrix} \cdot \begin{bmatrix} x_{p+1}^{(1)} & x_{p+1}^{(2)} & x_1^* \end{bmatrix},$$

$$h^{(k)} := \begin{bmatrix} x_{2+k} \\ x_{4+k}^* \end{bmatrix} \cdot \begin{bmatrix} y_{p-k+1}^\circ & y_{p-k}^{(1)} \end{bmatrix}, \quad i^{(k)} := \begin{bmatrix} x_{p-k}^{(1)} \\ x_{p-k}^{(2)} \end{bmatrix} \cdot \begin{bmatrix} y_{2+k} & y_{3+k}^\diamond \end{bmatrix}, \quad k = 0, \dots, p-3.$$

*The ideal of syzygies is generated by all $2 \times 2$-minors of the matrices $d$, $e$, $f$, $g$, $h^{(k)}$, $i^{(k)}$ and*

$$f_{a3}g_{b3} - pf_{a1}g_{b2} + pf_{a2}g_{b1}, \quad a, b \in \{1, 2\}.$$

PROOF. We proceed exactly as in Lemma 3.5.10. □

PROOF OF PROP. 3.5.13. The list of generators can be extracted from [**91**, Le. 2.5], it is minimal due to [**91**, Proof of Thm. 0.2]. We come to the syzygies. Let us first have a look at the Hilbert series of $A_2$ with respect to the $\mathbb{Z}^6$-grading of $W_2$, where variables $t_i$, $s_i$ and $r$ correspond to copies of $\Lambda^2 V$, $\Lambda^{n-1} V$ and $V$ respectively. Lemma 3.5.14 provides us with generators and syzygies. We have

$$\begin{aligned} A_2 = \mathbb{C}[x_2^*, y_{p+1}^\circ, y_1^\diamond] &\otimes \mathbb{C}[y_{p+1}^{(1)}, y_p^\circ, y_1^*, x_1, x_3^*]^T \otimes \mathbb{C}[y_1^{(1)}, y_p, x_2^{(1)}, x_2^{(2)}]^T \\ &\otimes \mathbb{C}[y_2^{(1)}, y_1^\circ, x_1^{(1)}, x_1^{(2)}, x_p, y_1, y_2^\diamond, x_{p+1}^{(1)}, x_{p+1}^{(2)}, x_1^*]^T \\ &\otimes \bigotimes_{k=0}^{p-3} \mathbb{C}[x_{2+k}, x_{4+k}^*, y_{p-k+1}^\circ, y_{p-k}^{(1)}]^T \\ &\otimes \bigotimes_{k=0}^{p-3} \mathbb{C}[y_{2+k}, y_{3+k}^\diamond, x_{p-k}^{(1)}, x_{p-k}^{(2)}]^T. \end{aligned}$$

The first four algebras have the Hilbert series

$$\frac{1}{(1-t_1s_1s_2)(1-t_2^pr)(1-s_3r)}, \quad \frac{(1-t_1^3t_2^{2p-1}s_1s_2s_3r)}{(1-t_1t_2^{p-1}r)(1-t_1t_2^ps_3)(1-t_1^2t_2^{p-1}s_1s_2r)(1-t_1^2t_2^ps_1s_2s_3)},$$

$$\frac{(1-t_1^2t_2^ps_1s_2s_3)}{(1-t_1t_2^ps_1)(1-t_1t_2^ps_2)(1-t_1s_1s_3)(1-t_1s_2s_3)},$$

$$\frac{(1-t_1^pt_2^2s_1s_2s_3)(1-t_1^pt_2s_1s_2r)}{(1-s_1r)(1-s_2r)(1-t_1^pr)(1-t_2s_1s_3)(1-t_2s_2s_3)(1-t_1^pt_2s_3)(1-t_1^pt_2s_1)(1-t_1^pt_2s_2)(1-t_2s_1s_2)},$$

whereas the ones of the $k$-th factors of the fifth and sixth algebra are

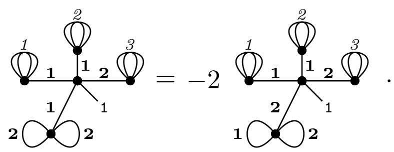
$$\frac{(1-t_1^{2k+5}t_2^{2p-2k-3}s_1s_2s_3r)}{(1-t_1^{2+k}t_2^{p-k-1}s_3)(1-t_1^{3+k}t_2^{p-k-1}s_1s_2s_3)(1-t_1^{2+k}t_2^{p-k-2}r)(1-t_1^{3+k}t_2^{p-k-2}s_1s_2r)},$$

$$\frac{(1-t_1^{2p-2k-2}t_2^{2k+4}s_1s_2s_3r)}{(1-t_1^{p-k-1}t_2^{2+k}s_1)(1-t_1^{p-k-1}t_2^{2+k}s_2)(1-t_1^{p-k-1}t_2^{2+k}s_1s_3r)(1-t_1^{p-k-1}t_2^{2+k}s_2s_3r)}.$$

As in the proof of Proposition 3.5.9, we distinguish between $p = 2, 3, \geq 4$. For $p = 2$, we have homological dimension two and according to the Hilbert series and the given 18 generators, there are two syzygies corresponding to the factors $(1 - t_1^2t_2^2s_1s_2s_3)$ and $(1 - t_1^3t_2^3s_1s_2s_3r)$. The first one already occured in $\mathbb{C}[W_1]^{\mathrm{SL}_5}$, it is

$$\mathfrak{i}_{121}\mathfrak{j}_{123} - \mathfrak{i}_{131}\mathfrak{j}_{122} + \mathfrak{i}_{231}\mathfrak{j}_{121} + \mathfrak{i}_{122}\mathfrak{j}_{213} - \mathfrak{i}_{132}\mathfrak{j}_{212} + \mathfrak{i}_{232}\mathfrak{j}_{211}.$$

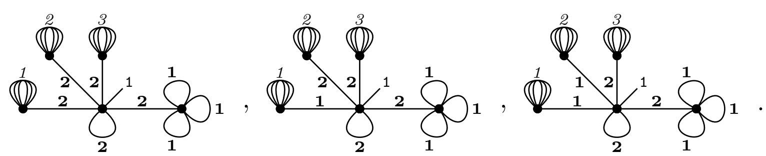
For an explicit form of the second one, consider the graphs



Applying the Plücker relation to the diagonal edges in each of both graphs gives a relation, which we can write - ignoring coefficients - as

$$\sum_{(i,j,k)\vdash(123)} \mathfrak{h}_i\mathfrak{j}_{12j}\mathfrak{j}_{21k} + \sum_{(ij,k)\vdash(123)} \left(\mathfrak{i}_{ij1}(\mathfrak{j}_{12k}\mathfrak{l}_{11}+\mathfrak{j}_{21k}\mathfrak{l}_{02}) + \mathfrak{i}_{ij2}(\mathfrak{j}_{12k}\mathfrak{l}_{20}+\mathfrak{j}_{21k}\mathfrak{l}_{11})\right).$$

Now consider the case $p = 3$. Here as with $W_1$, we encounter all types of invariants but $\mathfrak{k}_{ab4}$. We need three relations corresponding to $(t_1^k t_2^{8-k} s_1 s_2 s_3 r)$ for $k = 3, 4, 5$. Consider the graphs



Applying Plücker relations to them results - for $k = 3, 4, 5$ - in

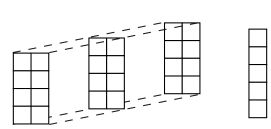
$$\mathfrak{g}_k = \sum_{(i,j,l)\vdash(123)} \sum \mathfrak{h}_i\mathfrak{j}_{abj}\mathfrak{j}_{cdl} + \sum_{(ij,l)\vdash(123)} \left(\sum \mathfrak{j}_{abl}\mathfrak{m}_{cdl} + \sum \mathfrak{i}_{ije}\mathfrak{j}_{abl}\mathfrak{l}_{cd}\right),$$

where in the second and fourth sum, we sum over $a + c = k, b + d = 2p + 2 - k$ and in the last one over $e = 1, 2$ and $a + c = k - \delta_{e1}, b + d = 2p + 2 - k - \delta_{e2}$. In the terms $\mathfrak{j}_{abl}\mathfrak{m}_{cdl}$, we observe that $a = b = 2$ must hold, so $\mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_5$ contain terms $\mathfrak{j}_{22l}\mathfrak{m}_{13l}, \mathfrak{j}_{22l}\mathfrak{m}_{22l}, \mathfrak{j}_{22l}\mathfrak{m}_{31l}$ respectively. We apply the Crosshair-sieve-algorithm 3.5.11. First we make the $\mathfrak{m}$ the variables of highest degree and second, we filter out $\mathfrak{j}_{221}\mathfrak{m}_{131}, \mathfrak{j}_{222}\mathfrak{m}_{222}, \mathfrak{j}_{223}\mathfrak{m}_{313}$ as leading monomials of $\mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_5$, which shows that they form a Gröbner basis and that $\mathbb{C}[W_2]^{\mathrm{SL}_7}$ is a complete intersection.

Now for $p \geq 4$, the situation is similar, but the relations $\mathfrak{g}_k$ now have additional terms:

$$\mathfrak{g}_k = \sum \mathfrak{k}_{ab4}\mathfrak{l}_{cd} + \sum_{(i,j,l)\vdash(123)} \mathfrak{h}_i\mathfrak{j}_{abj}\mathfrak{j}_{cdl} + \sum_{(ij,l)\vdash(123)} \left(\sum \mathfrak{j}_{abl}\mathfrak{m}_{cdl} + \sum \mathfrak{i}_{ije}\mathfrak{j}_{abl}\mathfrak{l}_{cd}\right).$$

We apply the Crosshair-sieve-algorithm 3.5.11 - in a preliminary step $\mathcal{M}_1$ making variables $\mathfrak{m}$ and $\mathfrak{k}$ the ones of highest degree - with the following sieves: $\mathcal{S}_\nu$ now comprises two parts - a $p\times(p-2)\times 3$-matrix $\mathcal{S}_{\nu,1}$ and a $(p+1)\times(p-3)$-matrix $\mathcal{S}_{\nu,2}$, where in the $(r, s, t)$-th entry of $\mathcal{S}_{\nu,1}$ stands the degree of $\mathfrak{j}_{r(p+1-r)t}\mathfrak{m}_{(s+1)(p-s)t}$ and in the $(r, s)$-th entry of $\mathcal{S}_{\nu,2}$ the degree of $\mathfrak{k}_{(s+2)(p-s)4}\mathfrak{l}_{(r-1)(p+1-r)}$. So for $p = 4$, the sieves are of the form



Again, monomials occuring in one $\mathfrak{g}_k$ lie on the counterdiagonals. For $\mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_5$, we choose the same leading monomials as in the case $p = 3$, while for $\mathfrak{g}_6, \mathfrak{g}_7$, we take

$$\mathrm{lm}(\mathfrak{g}_6) = \mathfrak{l}_{31}\mathfrak{k}_{334}, \quad \mathrm{lm}(\mathfrak{g}_7) = \mathfrak{j}_{411}\mathfrak{m}_{321}.$$

With this choice $\mathcal{S}_2$ filters out all leading monomials but the ones of $\mathfrak{g}_4$ and $\mathfrak{g}_6$, which is done by $\mathcal{S}_3$. If for general $p$, we choose the same leading monomials as for $p-1$ and for the two additional relations we take

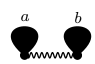
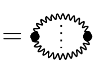
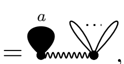
$$\mathrm{lm}(\mathfrak{g}_{2p}) = \mathfrak{j}_{(p-1)22}\mathfrak{m}_{(p-3)42}, \quad \mathrm{lm}(\mathfrak{g}_{2p-1}) = \mathfrak{j}_{p33}\mathfrak{m}_{(p-3)43},$$

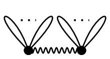
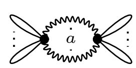
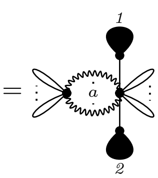
by the same argument as in the proof of Proposition 3.5.9, each $\mathcal{S}_\nu^p$ filters out at least the leading monomial of the remaining $\mathfrak{g}_k$ with highest index, so the Crosshair-sieve-algorithm terminates, the $\mathfrak{g}_k$ are a Gröbner basis and $\mathbb{C}[W_2]^{\mathrm{SL}_n}$ is a complete intersection.

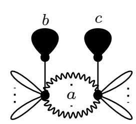
□

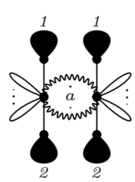
3.5.2.4. *The case* $(\mathrm{SL}_{2p+1}, W_3)$. Let from now on all 2-edges be of color one.

Proposition 3.5.15. *The ring of invariants* $\mathbb{C}[W_3]^{\mathrm{SL}_{2p+1}}$ *is a complete intersection. Its homological dimension is* $2p-2$. *It is minimally generated by* $\mathfrak{i}_{121}$ *from Proposition 3.5.9 and*

$\mathfrak{n}_{ab} =$  $, ab \in \{11, 12, 22\},\ \mathfrak{o} =$  $,\ \mathfrak{p}_a =$  $, a \in \{1, 2\},$

$\mathfrak{q}_1 =$  $,\ \mathfrak{q}_a =$  $,\ \mathfrak{r}_a =$  $, 3 \leq a \leq 2p-1 \in 2\mathbb{Z}+1,$

$\mathfrak{s}_{abc} =$  $, 4 \leq a \leq 2p \in 2\mathbb{Z}, bc \in \{11, 12, 22\},$

$\mathfrak{t}_a =$  $, 5 \leq a \leq 2p-1 \in 2\mathbb{Z}+1.$

Lemma 3.5.16. *Denote by* $x$ *the images of the* $c$ *in* $\mathbb{C}[V_1]^U$ *and by* $y$ *those in* $\mathbb{C}[V_3]^{U^o}$. *The algebra* $A_3 = (\mathbb{C}[V_1]^U \otimes \mathbb{C}[V_3]^{U^o})^T$ *is minimally generated by* $x_2^*, y_{2p+1}^\triangle$ *and the entries on and above the diagonal of the matrices*

$$f := y_1^\triangle \cdot \begin{bmatrix} {x_1^{(1)}}^2 & x_1^{(1)}x_1^{(2)} & x_1^{(1)}x_p \\ x_1^{(1)}x_1^{(2)} & {x_1^{(2)}}^2 & x_1^{(2)}x_p \\ x_1^{(1)}x_p & x_1^{(2)}x_p & x_p^2 \end{bmatrix},\ g := y_2^\triangle \cdot \begin{bmatrix} {x_{p+1}^{(1)}}^2 & x_{p+1}^{(1)}x_{p+1}^{(2)} & x_{p+1}^{(1)}x_1^* \\ x_{p+1}^{(1)}x_{p+1}^{(2)} & {x_{p+1}^{(2)}}^2 & x_{p+1}^{(2)}x_1^* \\ x_{p+1}^{(1)}x_1^* & x_{p+1}^{(2)}x_1^* & {x_1^*}^2 \end{bmatrix},$$

$$h^{(k)} := y_{2k+1}^\triangle \cdot \begin{bmatrix} x_{p-k}^2 & x_{p-k}x_{p+2-k}^* \\ x_{p-k}x_{p+2-k}^* & {x_{p+2-k}^*}^2 \end{bmatrix}, k = 1, \ldots, p-1,$$

$$i^{(k)} := y_{2k+2}^\triangle \cdot \begin{bmatrix} {x_{p+1-k}^{(1)}}^2 & x_{p+1-k}^{(1)}x_{p+1-k}^{(2)} \\ x_{p+1-k}^{(1)}x_{p+1-k}^{(2)} & {x_{p+1-k}^{(2)}}^2 \end{bmatrix}, k = 1, \ldots, p-1.$$

*The ideal of relations is generated by all $2 \times 2$-minors of the matrices $f, g, h^{(k)}, i^{(k)}$ with at most one entry below the diagonal and in addition*

$$f_{a3}g_{b3} - pf_{a2}g_{1b} + pf_{1a}g_{2b}, \quad a, b \in \{1, 2, 3\}.$$

Proof. We proceed exactly as in Lemmata 3.5.10 and 3.5.14. □

Proof of Prop. 3.5.15. We have

$$A_3 = \mathbb{C}[x_2^*, y_{2p+1}^{\triangle}] \otimes \mathbb{C}[y_1^{\triangle}, y_2^{\triangle}, x_1^{(1)}, x_1^{(2)}, x_p, x_{p+1}^{(1)}, x_{p+1}^{(2)}, x_1^*]^T \otimes \bigotimes_{k=1}^{p-1} \mathbb{C}[y_{2k+1}^{\triangle}, x_{p-k}, x_{p+2-k}^*]^T \otimes \bigotimes_{k=1}^{p-1} \mathbb{C}[y_{2k+2}^{\triangle}, x_{p+1-k}^{(1)}, x_{p+1-k}^{(2)}]^T.$$

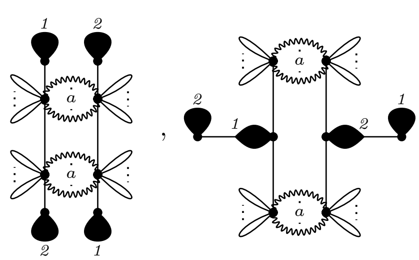
With variables $t$, $s$ and $r_i$ corresponding to copies of $\Lambda^2 V$, $S^2 V$ and $\Lambda^{n-1} V$ respectively, the first two algebras and the $k$-th factors of the third and fourth algebra have respective Hilbert series

$$\frac{1}{(1-tr_1r_2)(1-s^{2p+1})}, \frac{1-s^3t^{2p}r_1^2r_2^2}{(1-st^{2p})(1-sr_2t^p)(1-sr_2^2)(1-sr_1t^p)(1-sr_1r_2)(1-sr_1^2)}$$

$$\frac{1-s^{4k+2}t^{4p-4k+2}r_1^2r_2^2}{(1-s^{2k+1}t^{2p-2k})(1-s^{2k+1}t^{2p-2k+1}r_1r_2)(1-s^{2k+1}t^{2p-2k+2}r_1^2r_2^2)},$$

$$\frac{1-s^{4k+4}t^{4p-4k}r_1^2r_2^2}{(1-s^{2k+2}t^{2p-2k}r_1^2)(1-s^{2k+2}t^{2p-2k}r_1r_2)(1-s^{2k+2}t^{2p-2k}r_2^2)}$$

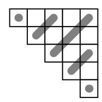
By [**91**, Proof of Thm. 0.4], a minimal system of generators has the same set of multidegrees as the system of generators of $A_3$ from Lemma 3.5.16, where those generators, for which $A_3$ has a syzygy of smaller or equal multidegree, must be checked for reducibility and omitted if reducible. This is the case for all entries of $g$ and for $h_{2,2}^{(1)}$, where invariants having the same multidegree can be shown to be reducible by applying Plücker relations to 2-edges. We arrive at the proposed minimal system of generators of $\mathbb{C}[W_3]^{\mathrm{SL}_{2p+1}}$. Concerning the syzygies, the Hilbert series tells us to search for ones corresponding to $(1 - s^{4k+2}t^{4p-4k+2}r_1^2r_2^2)$ and $(1 - s^{4k+4}t^{4p-4k}r_1^2r_2^2)$. Consider the reducible graphs



for $a = 2k + 1$ in the left and $a = 2k + 2$ in the right one. For both types, there are *two ways* to transform them into a sum of disconnected ones, so we get a syzygy for each of them. These two ways are the following: for the left one, consider the Plücker relation applied to the 2-edges connecting the upper and lower part, but for the first way pulling both to the upper vertices and for the second way pulling one to the upper and one to the lower vertices. This results in two equivalent sums of reducible graphs, containing $\mathfrak{r}_a^2$ and $\mathfrak{q}_a\mathfrak{t}_a$ respectively. We do not list the other summands but remark that this is exactly the part of the relation that reflects in the algebra $A_3$ as the determinant of $h^{(k)}$.

For the right graph, consider for the first way the Plücker relation applied to the 2-edges connecting the upper and middle part, and for the second way, which is more important for us, the Plücker relation applied to the horizontal non-looping $2p$-edges, both pulled towards the middle. The resulting syzygy contains - resulting from the second way - $\mathfrak{s}_{a12}^2$ and $\mathfrak{s}_{a11}\mathfrak{s}_{a22}$, which again are exactly the part of the relation reflecting in $A_3$, but now as the determinant of $i^{(k)}$.

Finally, we apply the Crosshair-sieve-algorithm 3.5.11 to show that $\mathbb{C}[W_3]^{\mathrm{SL}_n}$ is a complete intersection. This is done by choosing the $\mathfrak{r}_a^2$ and $\mathfrak{s}_{a12}^2$ as leading monomials. The sieves here can be viewed as part-matrices up and above the diagonal, containing in the $(i,j)$-th entry the degree of $\mathfrak{r}_{2+i}\mathfrak{r}_{2+j}$, $\mathfrak{r}_{2+i}\mathfrak{s}_{(2+j)12}$, $\mathfrak{s}_{(2+i)12}\mathfrak{r}_{2+j}$ and $\mathfrak{s}_{(2+i)12}\mathfrak{s}_{(2+j)12}$ for $(i,j) \in \{(2\mathbb{Z}+1)^2, (2\mathbb{Z}+1)\times 2\mathbb{Z}, 2\mathbb{Z}\times(2\mathbb{Z}+1), 2\mathbb{Z}^2\}$ respectively. By the multidegrees of the syzygies, monomials that may occur in one relation lie on a counterdiagonal containing an entry on the diagonal (corresponding to a square monomial), as is shown in the following example picture:



Now as the targeted monomials lie on the diagonal, $\mathcal{S}_1$ filters out $\mathfrak{r}_3^2$ and $\mathfrak{s}_{(2p)12}^2$, then $\mathcal{S}_2$ filters out $\mathfrak{s}_{412}^2$ and $\mathfrak{r}_{(2p-1)}^2$ et cetera. So we found a Gröbner basis for the ideal of relations of $\mathbb{C}[W_3]^{\mathrm{SL}_n}$ and it is a complete intersection.

□

CHAPTER 4

# Classification of compound Du Val and canonical threefold singularities of complexity one

This chapter is devoted to the classification of compound Du Val and canonical threefold singularities of complexity one. In particular, we prove Theorems 9, 10 and 11.

In Section 4.1, we provide first results concerning compound Du Val singularities. The classification follows in Section 4.2. The results of these two sections have been published in the joint work [**5**] with Arzhantsev, Hausen and Wrobel. In Section 4.3.1, we shortly discuss $k$-empty polytopes, i.e. convex rational polytopes $\mathsf{P} \subseteq \mathbb{Q}^n$ such that the only points of $k\mathbb{Z}^n$ lying in $\mathsf{P}$ are its vertices. We relate such polytopes to certain subpolytopes of the anticanonical complex of a canonical threefold singularity of complexity one. This enables us to classify such singularities in the last section, Section 4.4. These last two sections will be published in the joint work [**22**] with Hättig.

## 4.1. Compound Du Val singularities

Between the Gorenstein terminal and canonical threefold singularities lie the compound Du Val singularities, introduced by Miles Reid in [**79**], see also [**80, 66, 64**]. We discuss compound Du Val singularities in the context of $T$-varieties of complexity one and provide first constraints on the defining data for affine threefolds, preparing the proof of our classification results.

Definition 4.1.1. [**79**, Def. 2.1], [**64**, Thm. 5.34, Cor. 2.3.2]. A normal, canonical, Gorenstein threefold singularity $x \in X$ is called *compound Du Val*, if one of the following equivalent criteria is satisfied:

(i) For a general hypersurface $Y \subseteq X$ with $x \in Y$, the point $x$ is a Du Val surface singularity of $Y$.

(ii) Near $x$, the threefold $X$ is analytically isomorphic to a hypersurface of the following shape

$$V(f(T_1,T_2,T_3) + g(T_1,T_2,T_3,T_4)\,T_4) \;\subseteq\; \mathbb{C}^4,$$

where $f$ is a defining polynomial for a Du Val surface singularity in $\mathbb{C}^3$ and $g$ is any polynomial in $T_1,T_2,T_3,T_4$.

(iii) For every resolution $\varphi\colon X' \to X$ of singularities and every irreducible exceptional divisor $E \subseteq \varphi^{-1}(x)$, the discrepancy of $E$ is greater than zero.

(iv) There is a resolution $\varphi\colon X' \to X$ of singularities such that every irreducible exceptional divisor $E \subseteq \varphi^{-1}(x)$ is of discrepancy greater than zero.

For an affine toric threefold $X$, Condition 4.1.1 (iv) means the following: $X$ is defined by the cone over $\triangle \times \{1\}$ with a hollow lattice polytope $\triangle \subseteq \mathbb{Q}^2$, where hollow means that $\triangle$ has no lattice points in its interior. Based on this characterization, one obtains the list of toric compound Du Val singularities provided in [**33**]:

Proposition 4.1.2. *Let $X$ be an affine toric variety with a compound Du Val singularity. Then $X \cong X(\sigma)$ holds with a cone $\sigma \subseteq \mathbb{Q}^3$ generated by the columns of one of the following matrices*

$$(1)\ \begin{bmatrix} 0 & 0 & k \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},\ k \in \mathbb{Z}_{\geq 2}, \qquad (2)\ \begin{bmatrix} 0 & 0 & k_1 & k_2 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},\ k_1, k_2 \in \mathbb{Z}_{\geq 1}, \qquad (3)\ \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Proof. After removing the third row from the matrices, we find in their columns the vertices of the hollow polytopes $\triangle \subseteq \mathbb{Q}^2$; see [**78**]. □

We turn to non-toric singularities. The basic tool is the anticanonical complex $A_X^c$, described in Proposition 1.3.6. The following statement specifies a bit more.

Proposition 4.1.3. *Let $X = X(A, P)$ be a Gorenstein canonical threefold quasicone such that $P$ is in the form of Proposition 1.4.3. Consider the intersections*

$$\partial A_X^c(\lambda) := \partial A_X^c \cap \lambda, \quad \partial A_X^c(\lambda_i) := \partial A_X^c \cap \lambda_i, \quad \partial A_X^c(\lambda_i, \tau) := \partial A_X^c(\lambda_i) \cap \tau,$$

*where $\partial A_X^c$ is the relative boundary of the anticanonical complex, $\lambda \subseteq \mathrm{trop}(X)$ the lineality part, $\lambda_0, \ldots, \lambda_r \subseteq \mathrm{trop}(X)$ are the leaves and $\tau$ is any $P$-elementary cone.*

(i) *Let $x_1, \ldots, x_{r+2}$ be the standard coordinates on the column space $\mathbb{Q}^{r+2}$ of $P$ and set $x_0 := -x_1 - \ldots - x_r$. Then $x_i, x_{r+1}, x_{r+2}$ are linear coordinates on the three-dimensional vector space $\mathrm{Lin}_{\mathbb{Q}}(\lambda_i)$ and we have*

$$\partial A_X^c(\lambda_i) \ = \ A_X^c \cap \lambda_i \cap \mathcal{H}_i \ \subseteq \ \mathrm{Lin}_{\mathbb{Q}}(\lambda_i)$$

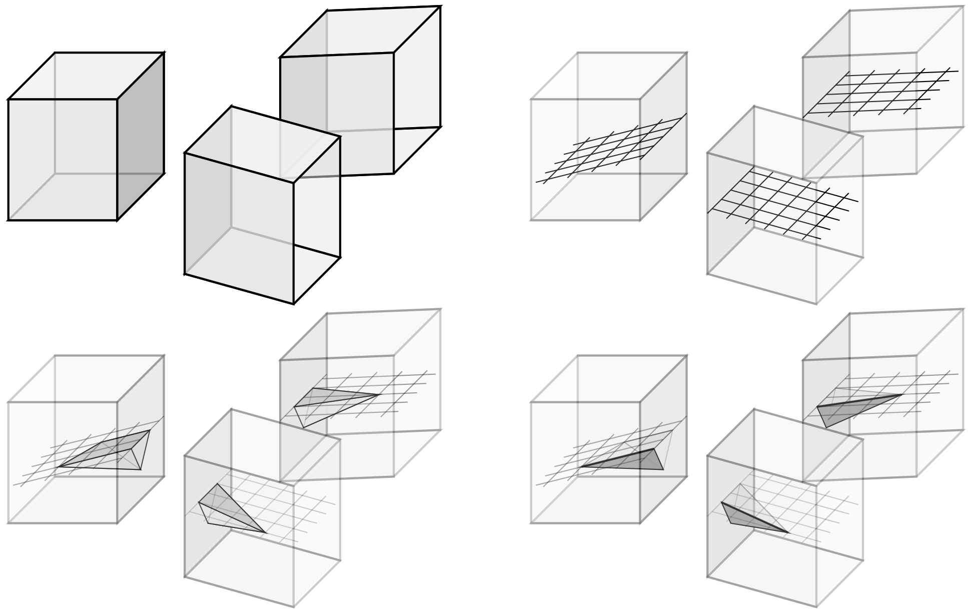
*with the plane $\mathcal{H}_i := V(\zeta_X x_{r+2} + \mu_i x_i - \imath_X) \subseteq \mathrm{Lin}_{\mathbb{Q}}(\lambda_i)$, where $\mu_i$ is the integer defined in Remark 1.4.5. In particular, for fixed $i$, the columns $v_{ij}$ of $P$ lie on the half plane $\lambda_i \cap \mathcal{H}_i$.*

(ii) *The set $A_X^c \cap \tau$ is a two-dimensional and $\partial A_X^c \cap \tau$ a one-dimensional polyhedral complex respectively. Furthermore, $\partial A_X^c(\lambda_i, \tau)$ is a line segment.*

Proof. We show (i). Let $\sigma \subseteq \mathbb{Q}^{r+2}$ be the cone over the columns of $P$. Then the set $\partial A_X^c(\lambda_i)$ equals $\partial A_X^c \cap \sigma \cap \lambda_i$. By the assumption on $P$, the equation from Proposition 1.3.6 (ii) gives the assertion.

For (iii), write $\tau = \mathrm{cone}(w_0, \ldots, w_r)$ with $w_i \in \lambda_i$. Observe that $A_X^c \cap \tau \cap \lambda_i$ has the vertices $0$, $w_i$, $v(\tau)'$ and is thus two-dimensional. Only $w_i$ and $v(\tau)'$ satisfy the equation $\zeta_X x_{r+2} + \mu_i x_i = \imath_X$. Thus $A_X^c \cap \tau$ is two-dimensional and $\partial A_X^c \cap \tau$ as well as $\partial A_X^c(\lambda_i, \tau)$ are one-dimensional. □

The following figures visualize the situation of Proposition 4.1.3 for the case $r = 2$. The first one shows the leaves $\lambda_i$, the second one the half planes $\lambda_i \cap \mathcal{H}_i$, the third one all $A_X^c(\lambda_i)$ and the last one all $A_X^c(\lambda_i, \tau)$ for a given $P$-elementary cone $\tau$.



The following statement shows that the relative boundary $\partial A_X^c$ of the anti-canonical complex replaces the lattice polytope $\triangle$ from the toric setting discussed before.

PROPOSITION 4.1.4. *Let $X = X(A,P)$ be a Gorenstein canonical threefold quasicone. Then $X$ has at most compound Du Val singularities if and only if there are no integral points in the relative interior of $\partial A_X^c$.*

LEMMA 4.1.5. *Let $X = X(A,P,\Sigma)$ and let $\sigma \in \Sigma$ be a big cone, cf. Remark 1.3.5.*

- (i) *The toric orbit $T_Z \cdot z_\sigma \subseteq Z$ corresponding to the cone $\sigma \in \Sigma$ is contained in $X \subseteq Z$.*
- (ii) *If $T_Z \cdot z_\sigma \subseteq X$ contains a singular point of $X$, then every point of $T_Z \cdot z_\sigma$ is singular in $X$.*

PROOF. We show (i). By the structure of the defining relations $g_i$, the corresponding statement holds for $\overline{X} \subseteq \overline{Z} = \mathbb{C}^{n+m}$. Passing to the quotient by the characteristic quasitorus $H$ gives the assertion.

We turn to (ii). Let $z \in \widehat{X}$ be a point mapping to $T_Z \cdot z_\sigma$. Using once more the specific shape of the defining relations $g_i$, we see that if the point $z \in \widehat{X}$ is singular in $\overline{X}$, then every point of $\mathbb{T}^{n+m} \cdot z$ is singular in $\overline{X}$. Thus, the assertion follows from [**6**, Cor. 3.3.1.12]. $\square$

PROOF OF PROPOSITION 4.1.4. Let $Z$ be the minimal toric ambient variety of $X$. Recall that $Z$ is the affine toric variety defined by the cone $\sigma$ over the columns of $P$ and that the toric fixed point $x \in Z$ belongs to $X$. For any point $x' \in X$ different from $x$, we infer from Lemma 4.1.5 and [**6**, 3.4.4.6] that, if $x'$ is singular in $X$, it belongs to a curve consisting of singular points of $X$. According to [**64**, Cor. 5.4], the point $x'$ is at most a compound Du Val singularity. Thus, $X$ has at most compound Du Val singularities if and only if every prime divisor $E \subseteq \varphi^{-1}(x)$ has positive discrepancy; use Condition 4.1.1 (iv). By Proposition 1.3.2, the latter holds if and only if there are no integral points in $\partial A_X^c \cap \sigma^\circ$, which in turn is the relative interior of $\partial A_X^c$. $\square$

Definition 4.1.6. Let the matrix $P$ be ordered in the sense of Remark 2.6.1. By the *leading block* of $P$, we mean the matrix $[v_{01}, \ldots, v_{r1}]$.

Lemma 4.1.7. *Let $X = X(A, P)$ be a Gorenstein canonical threefold quasicone of canonical multiplicity one.*

(i) *By admissible operations one achieves that $P$ is ordered in the sense of Remark 2.6.1, in the form of Corollary 1.4.6 and the entry $\mathfrak{d}_i$ sitting in column $v_{i1}$ and row number $r+1$ of $P$ satisfies $\mathfrak{d}_i = 0$ whenever $i \geq 3$.*
(ii) *In the situation of (i), the leading block of the matrix $P$ is fully determined by the data $(l_{01}, l_{11}, l_{21}; \mathfrak{d}_0, \mathfrak{d}_1, \mathfrak{d}_2)$.*

Proof. The leading block contains the leading platonic triple $(l_{01}, l_{11}, l_{21})$. All other $l_{i1}$ must be equal to one. Due to Corollary 1.4.6, the last row of $P$ is determined by these data. Subtracting the $\mathfrak{d}_i$-fold of the $i$-th from the $r+1$-th row, we obtain $\mathfrak{d}_i = 0$ for $i \geq 3$. Thus apart from $l_{01}, l_{11}, l_{21}$, the only free parameters in the leading block are $\mathfrak{d}_0, \mathfrak{d}_1, \mathfrak{d}_2$. $\square$

Definition 4.1.8. In the situation of Lemma 4.1.7 (i), we call $(l_{01}, l_{11}, l_{21}; \mathfrak{d}_0, \mathfrak{d}_1, \mathfrak{d}_2)$ the *leading block data* of $P$.

Proposition 4.1.9. *Let $X = X(A, P)$ be a Gorenstein canonical threefold quasicone of canonical multiplicity one in the form of Lemma 4.1.7. By admissible operations, keeping the form of Lemma 4.1.7, we achieve that the leading block has one of the following data:*

| | | |
|---|---|---|
| *(i)* $(5, 3, 2; 0, 0, 0)$ | *(ii)* $(4, 3, 2; 0, 0, 0)$ | *(iii)* $(4, 3, 2; 1, 0, 0)$ |
| *(iv)* $(3, 3, 2; 0, 0, 0)$ | *(v)* $(3, 3, 2; 1, 0, 0)$ | *(vi)* $(l_{01}, 2, 2; 0, 0, 0)$ |
| *(vii)* $(l_{01}, 2, 2; 1, 0, 0)$ | *(viii)* $(l_{01}, 2, 2; 0, 1, 0)$ | *(ix)* $(l_{01}, l_{11}, 1; \mathfrak{d}_0, 0, 0)$ |

Proof. We go through all possible leading platonic triples and explicitly list the admissible operations on $P$ that produce the desired leading block data. First, we modify $P$ by subtracting the $i$-th row from the last for $i \geq 3$. Then we have

$$\nu_{01} \;=\; 1 - l_{01}, \qquad \nu_{11} \;=\; \nu_{21} \;=\; 1, \qquad \nu_{i1} \;=\; \mathfrak{d}_i \;=\; 0, \; i = 3, \ldots, r.$$

In the sequel, by "applying $a = (a_1, a_2, a_3)$" we mean performing the following sequence of admissible operations on $P$: add the $a_1$-fold of the first, the $a_2$-fold of the second and the $a_3$-fold of the last to the penultimate row of $P$.

*Case 1*: The leading platonic triple is $(5, 3, 2)$. We arrive at Case (i) by applying

$$a \;=\; (2\mathfrak{d}_0 + 3\mathfrak{d}_1 + 5\mathfrak{d}_2,\; 3\mathfrak{d}_0 + 5\mathfrak{d}_1 + 7\mathfrak{d}_2,\; -6\mathfrak{d}_0 - 10\mathfrak{d}_1 - 15\mathfrak{d}_2).$$

*Case 2*: The leading platonic triple is $(4, 3, 2)$. If $\mathfrak{d}_0 \equiv \mathfrak{d}_2 \mod 2$ holds, then we arrive at Case (ii) by applying

$$a \;=\; \left(\mathfrak{d}_0 + \mathfrak{d}_1 + 2\mathfrak{d}_2,\; \frac{3}{2}\mathfrak{d}_0 + 2\mathfrak{d}_1 + \frac{5}{2}\mathfrak{d}_2,\; -3\mathfrak{d}_0 - 4\mathfrak{d}_1 - 6\mathfrak{d}_2\right).$$

If $\mathfrak{d}_0 \equiv \mathfrak{d}_2 + 1 \mod 2$ holds, then we arrive at Case (iii) by applying

$$a \;=\; \left(\mathfrak{d}_0 + \mathfrak{d}_1 + 2\mathfrak{d}_2 - 1,\; \frac{3}{2}\mathfrak{d}_0 + 2\mathfrak{d}_1 + \frac{5}{2}\mathfrak{d}_2 - \frac{3}{2},\; -3\mathfrak{d}_0 - 4\mathfrak{d}_1 - 6\mathfrak{d}_2 + 3\right).$$

*Case 3*: The leading platonic triple is $(3,3,2)$. We distinguish the cases $\mathfrak{d}_0 \equiv \mathfrak{d}_1 \mod 3$ and $\mathfrak{d}_0 \equiv \mathfrak{d}_1 + 1 \mod 3$ (if $\mathfrak{d}_0 \equiv \mathfrak{d}_1 - 1 \mod 3$, then exchange the data of the blocks 0 and 1 of $P$). We arrive at Cases (iv) and (v) by applying respectively

$$a \;=\; \left(\frac{2}{3}\mathfrak{d}_0 + \frac{1}{3}\mathfrak{d}_1 + \mathfrak{d}_2,\ \mathfrak{d}_0 + \mathfrak{d}_1 + \mathfrak{d}_2,\ -2\mathfrak{d}_0 - 2\mathfrak{d}_1 - 3\mathfrak{d}_2\right),$$

$$a \;=\; \left(\frac{2}{3}\mathfrak{d}_0 + \frac{1}{3}\mathfrak{d}_1 + \mathfrak{d}_2 - \frac{2}{3},\ \mathfrak{d}_0 + \mathfrak{d}_1 + \mathfrak{d}_2 - 1,\ -2\mathfrak{d}_0 - 2\mathfrak{d}_1 - 3\mathfrak{d}_2 + 2\right).$$

*Case 4:* The leading platonic triple is $(l_{01},2,2)$. We distinguish several subcases and will work with

$$a \;=\; \left(\frac{1}{2}\mathfrak{d}_0 + \frac{l_{01}-2}{4}\mathfrak{d}_1 + \frac{l_{0j_0}}{4}\mathfrak{d}_2,\ \frac{1}{2}\mathfrak{d}_0 + \frac{l_{01}}{4}\mathfrak{d}_1 + \frac{l_{01}-2}{4}\mathfrak{d}_2,\ -\mathfrak{d}_0 - \frac{l_{01}}{2}(\mathfrak{d}_1 + \mathfrak{d}_2)\right).$$

*4.1:* We have $l_{01} \equiv 1 \mod 4$.

*4.1.1:* $\mathfrak{d}_1 \equiv \mathfrak{d}_2 \mod 4$. If $\mathfrak{d}_0$ is even, then applying $a$, we arrive at Case (vi). If $\mathfrak{d}_0$ is odd, then applying $a + (-1/2,\ -1/2,\ 1)$, we arrive at Case (vii).

*4.1.2:* $\mathfrak{d}_1 \equiv \mathfrak{d}_2 + 1 \mod 4$. If $\mathfrak{d}_0$ is even, then applying $a + (1/4,\ -1/4,\ 1/2)$ leads to Case (viii). If $\mathfrak{d}_0$ is odd, then applying $a + (-1/4,\ 1/4,\ 1/2)$ and exchanging the data of column blocks 1 and 2 leads to Case (viii).

*4.1.3:* $\mathfrak{d}_1 \equiv \mathfrak{d}_2 - 1 \mod 4$. Exchanging the data of column blocks 1 and 2, we are in 4.1.2 and thus arrive at Case (viii).

*4.1.4:* $\mathfrak{d}_1 \equiv \mathfrak{d}_2 + 2 \mod 4$. If $\mathfrak{d}_0$ is odd, then applying $a$, we arrive at Case (vi). If $\mathfrak{d}_0$ is even, then applying $a + (-1/2,\ -1/2,\ 1)$ leads to Case (vii).

*4.2:* We have $l_{01} \equiv 2 \mod 4$.

*4.2.1:* $\mathfrak{d}_0 \equiv \mathfrak{d}_1 \equiv \mathfrak{d}_2 \mod 2$. Applying $a$, we arrive at Case (vi).

*4.2.2:* $\mathfrak{d}_0 \equiv \mathfrak{d}_1 \not\equiv \mathfrak{d}_2 \mod 2$. Applying $a + (0,\ -1/2,\ 1)$, we arrive at Case (viii).

*4.2.3:* $\mathfrak{d}_0 \equiv \mathfrak{d}_2 \not\equiv \mathfrak{d}_1 \mod 2$. Exchanging the data of column blocks 1 and 2, we are in 4.2.2 and thus arrive at Case (viii).

*4.2.4:* $\mathfrak{d}_0 \not\equiv \mathfrak{d}_1 \equiv \mathfrak{d}_2 \mod 2$. Applying $a + (-1/2,\ -1/2,\ 1)$, we arrive at Case (vii).

*4.3 and 4.4:* $l_{01} \equiv 3 \mod 4$ or $l_{01} \equiv 3 \mod 4$, respectively. These cases are settled by similar arguments as 4.1 and 4.2. That means that the same admissible operations are applied after, if necessary exchanging the data of column blocks 1 and 2.

*Case 5:* The leading platonic triple is $(l_{01}, l_{11}, 1)$. Applying $(0,\ \mathfrak{d}_1 - \mathfrak{d}_2,\ -\mathfrak{d}_1)$, we arrive at Case (ix).

Finally, in each of the cases (i) to (ix), we modify the matrix $P$ obtained so far by adding the $i$-th row to the last one for $i = 3, \dots, r$. This brings $P$ again into the form of Lemma 4.1.7 (i). □

## 4.2. Proof of Theorem 9

In Propositions 4.2.1, 4.2.3 and 4.2.4, we classify the compound Du Val singularities admitting a torus action of complexity one and list their defining matrices $P$, numerated according to their appearance in Theorem 9. We begin with the case of $\mathbb{Q}$-factorial non-toric threefolds of canonical multiplicity one.

PROPOSITION 4.2.1. *Let $X$ be a non-toric threefold quasicone. Assume that $X$ is $\mathbb{Q}$-factorial, of canonical multiplicity one and has at most compound Du Val singularities. Then $X$, for suitable $A$, is isomorphic to $X(A,P)$, where $P$ is one of the following matrices:*

$$(8)\ \begin{bmatrix} -5 & 3 & 0 & 0 \\ -5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 1 & 1 & 1 \end{bmatrix} \qquad (7)\ \begin{bmatrix} -4 & 3 & 0 & 0 \\ -4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 1 & 1 & 1 \end{bmatrix} \qquad (18)\ \begin{bmatrix} -4 & -1 & 3 & 0 \\ -4 & -1 & 0 & 2 \\ 1 & 3 & 0 & 0 \\ -3 & 0 & 1 & 1 \end{bmatrix}$$

$$(6)\ \begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 1 & 1 \end{bmatrix} \qquad (15)\ \begin{bmatrix} -3 & -1 & 3 & 0 \\ -3 & -1 & 0 & 2 \\ 1 & 2 & 0 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix} \qquad (17)\ \begin{bmatrix} -3 & -2 & 3 & 0 \\ -3 & -2 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ -2 & -1 & 1 & 1 \end{bmatrix}$$

$$(4)\ \begin{bmatrix} -k & 2 & 0 & 0 \\ -k & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 1-k & 1 & 1 & 1 \end{bmatrix} \qquad (12\text{-e-e})\ \begin{bmatrix} -k_1 & -k_2 & 2 & 0 \\ -k_1 & -k_2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1-k_1 & 1-k_2 & 1 & 1 \end{bmatrix} \qquad (5\text{-o})\ \begin{bmatrix} -k & 2 & 0 & 0 \\ -k & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 1-k & 1 & 1 & 1 \end{bmatrix}$$

$$(11)\ \begin{bmatrix} -k & 2 & 1 & 0 \\ -k & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 1-k & 1 & 1 & 1 \end{bmatrix} \qquad (12\text{-o-e/o})\ \begin{bmatrix} -2k_1 & -2k_2-1 & 2 & 0 \\ -2k_1 & -2k_2-1 & 0 & 2 \\ 0 & k_1-k_2 & 1 & 0 \\ 1-2k_1 & -2k_2 & 1 & 1 \end{bmatrix} \qquad (16)\ \begin{bmatrix} -4 & 2 & 1 & 0 \\ -4 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ -3 & 1 & 1 & 1 \end{bmatrix}$$

$$(5\text{-e})\ \begin{bmatrix} -2k-1 & 2 & 0 & 0 \\ -2k-1 & 0 & 2 & 0 \\ 0 & 1 & 0 & k+1 \\ -2k & 1 & 1 & 1 \end{bmatrix} \qquad (10\text{-o})\ \begin{bmatrix} -2k-1 & 2 & 1 & 0 \\ -2k-1 & 0 & 0 & 2 \\ 0 & 1 & \left\lceil \frac{2k+1}{4} \right\rceil & 0 \\ -2k & 1 & 1 & 1 \end{bmatrix}$$

*where the parameters $k, k_1, k_2$ are positive integers and in (4), (5-o) and (11), we have $k \geq 2$. Moreover, (12-e-e) indicates that the two exponents in the defining equation of Theorem 9 (12) are even, in (5-o) the exponent is odd etc..*

PROOF. We may assume that $P$ is irredundant and in the form of Proposition 4.1.9. As $X$ is $\mathbb{Q}$-factorial, the matrix $P$ has precisely $r+2$ columns, i.e., is a square matrix. Since we assume $P$ to be irredundant and $l_{ij} = 1$ holds for $i \geq 3$, we must have $n_i \geq 2$ for $i \geq 3$. This forces $r \leq 3$. The strategy is now to compute suitable parts of $\partial A_X^c$ explicitly according to Proposition 1.3.6 and to use the fact that they do not contain interior lattice points, as guaranteed by Proposition 4.1.4.

Consider the case $r = 3$. Here, we have $n_0 = n_1 = n_2 = 1$ and $n_3 = 2$. Moreover, $(l_{01}, l_{11}, l_{21})$ is a platonic triple with $l_{21} > 1$ and $l_3 = (1,1)$ holds. The column apart from the leading block of $P$ is $v_{32} = (0,0,1,t,0)$, where we may assume that $t$ is a positive integer. The vertices of $\partial A_X^c(\lambda)$ thus are

$$\left(0,0,0,\frac{\alpha}{\beta},1\right), \qquad \left(0,0,0,\frac{\alpha + t l_{01} l_{11} l_{21}}{\beta},1\right),$$

where

$$\alpha := \mathfrak{d}_0 l_{11} l_{21} + \mathfrak{d}_1 l_{01} l_{21} + \mathfrak{d}_2 l_{01} l_{11}, \quad \beta := l_{11} l_{21} + l_{01} l_{21} + l_{01} l_{11} - l_{01} l_{11} l_{21}$$

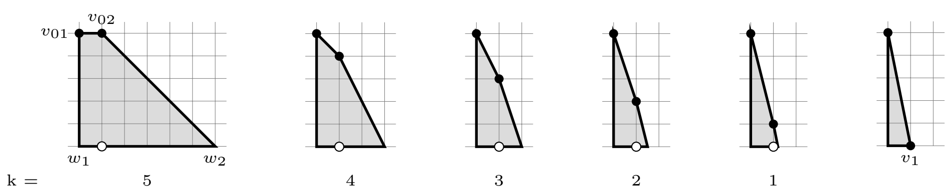
Since $l_{01}, l_{11}, l_{21}$ all differ from one, $t l_{01} l_{11} l_{21}/\beta \geq 2$ holds and thus $\partial A_X^c(\lambda)$ contains an integral point in its relative interior. Consequently $r = 3$ is impossible.

We are left with the case $r = 2$. Here, $P$ is a $4 \times 4$ matrix, the leading block columns are $v_{01}, v_{11}, v_{21}$ and the column $v$ of $P$ apart from these three is one of

$$v_{02} = (-k,-k,t,1-k), \quad v_{12} = (k,0,0,t,1), \quad v_{22} = (0,k,t,1), \quad v_1 = (0,0,t,1).$$

We now go through the list of all possible leading block data provided by Proposition 4.1.9. We will often compute the line segment $\partial A_X^c(\lambda) \subseteq \mathbb{Q}^4$ from Proposition 4.1.3 explicitly. According to Proposition 1.3.6, the $P$-elementary cone spanned by the columns of the leading block produces the first vertex $w_1$ of $\partial A_X^c(\lambda)$ and the second vertex $w_2$ either arises from a (unique) second $P$-elementary cone or one has $w_2 = v = v_1$.

Let $P$ have the leading block data $(5,3,2;0,0,0)$. Then the first vertex of $\partial A_X^c(\lambda)$ is $w_1 = (0,0,0,1)$. Consider the case that the additional column $v$ lies in the relative interior $\lambda_0^\circ \subseteq \lambda_0$. Then $v = v_{02} = (-k,-k,t,1-k)$ with $1 \le k \le 5$, where we may assume $t > 0$. We compute $w_2 = (0,0,6t/(6-k),1)$. The following figures show $\partial A_X^c(\lambda_0) \subseteq \mathcal{H}_0$ with the lower edge being $\partial A_X^c(\lambda)$, where the plane $\mathcal{H}_0$ is defined as in Proposition 4.1.3:

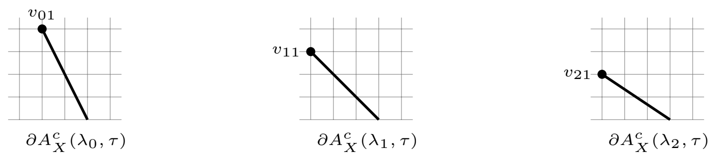



where the last figure indicates the case of the additional column lying in $\lambda$, treated below. Now, because of $6t/(6-k) \ge 6/5$, we find the point $(0,0,1,1)$ in the relative interior of $\partial A_X^c(\lambda)$ and hence in the relative interior of $\partial A_X^c$. According to Proposition 4.1.4, we leave the compound Du Val case here.

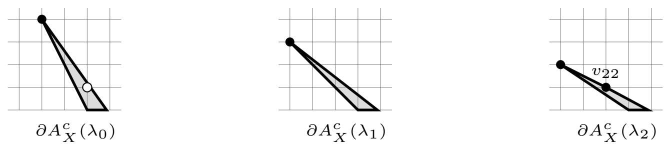
We proceed in a more condensed way. Assume $v \in \lambda_1^\circ$. Then $v = v_{12} = (k,0,t,1)$ with $1 \le k \le 3$ and we can assume $t > 1$. We obtain $w_2 = (0,0,10t/(10-3k),1)$. We find again $(0,0,1,1)$ in $\partial A_X^c(\lambda)^\circ$ and thus leave the compound Du Val case. Assume $v \in \lambda_2^\circ$. Then $v = v_{22} = (0,k,t,1)$ with $1 \le k \le 2$ and we can assume $t > 1$. We obtain $w_2 = (0,0,15t/(15-7k,1)$. Once more, $(0,0,1,1)$ shows up in $\partial A_X^c(\lambda)^\circ$ and we leave the compound Du Val case. Finally, assume $v = v_1 = (0,0,t,1)$. We may assume $t > 0$. Only for $t = 1$ there are no lattice points in $\partial A_X^c(\lambda)^\circ$. Moreover, if $t = 1$, then all $\partial A_X^c(\lambda_i)$ are hollow polytopes of the first type of Proposition 4.1.2 and we arrive at matrix (8) from the assertion defining the compound Du Val singularity $E_8 \times \mathbb{C}$.

Let $P$ have the leading block data $(4,3,2;0,0,0)$. Also here, the first vertex of $\partial A_X^c(\lambda)$ is $w_1 = (0,0,0,1)$. Assume $v \in \lambda_0^\circ$. Then $v = v_{02} = (-k,-k,t,1-k)$ with $1 \le k \le 4$, where we may assume $t > 0$. We obtain $w_2 = (0,0,6t/(6-k),1)$. Thus, $(0,0,1,1)$ lies in $\partial A_X^c(\lambda)^\circ$ and we leave the compound Du Val case. Assume $v \in \lambda_1^\circ$. Then $v = v_{12} = (k,0,t,1)$, where $1 \le k \le 3$ and we can assume $t > 0$. We obtain $w_2 = (0,0,4t/(4-k),1)$ and find $(0,0,1,1)$ in the relative interior of $\partial A_X^c(\lambda)$ and thus leave the compound Du Val case. Assume $v \in \lambda_2^\circ$. Then $v = v_{22} = (0,k,t,1)$ with $k = 1,2$ and we can assume $t > 0$. We obtain $w_2 = (0,0,12t/(12-5k),1)$ and see that $(0,0,1,1)$ lies in $\partial A_X^c(\lambda)^\circ$. Thus, we leave the compound Du Val case. Finally, assume $v = v_1 = (0,0,t,1)$. For $t > 1$, we find $(0,0,1,1)$ in $\partial A_X^c(\lambda)^\circ$. The case $t = 1$ gives matrix (7), defining the compound Du Val singularity $E_7 \times \mathbb{C}$.

Let $P$ have the leading block data $(4,3,2;1,0,0)$. Here, the first vertex of $\partial A_X^c(\lambda)$ is $w_1 = (0,0,3,1)$. To visualize the setting, consider the $P$-elementary cone $\tau \subseteq \mathbb{Q}^4$ generated by the columns $v_{01}, v_{11}, v_{21}$ of the leading block and the line segments $\partial A_X^c(\lambda_i, \tau) \subseteq \mathcal{H}_i$, where $i = 0,1,2$, from Proposition 4.1.3:




Note that the additional column $v$ is represented in the above figures by a lattice point not contained in $\partial A_X^c(\lambda_i,\tau)$, indicated by the black line. Going through the cases, we will also have to look at the polytopes $\partial A_X^c(\lambda_i)$ and will encounter the following situations:

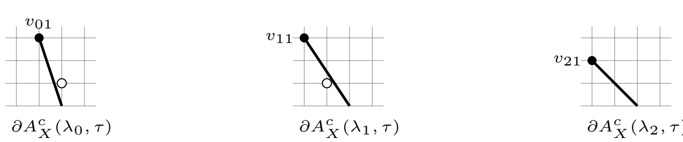



Assume $v \in \lambda_0^\circ$. Then $v = v_{02} = (-k,-k,t,1-k)$ with $1 \le k \le 4$. The second vertex of $\partial A_X^c(\lambda)$ is $w_2 = (0,0,6t/(6-k),1)$. We find one of the points $(0,0,4,1)$ or $(0,0,2,1)$ in $\partial A_X^c(\lambda)^\circ$ for $k=2,4$. Moreover, for $k=3$, we find $(-1,-1,3,0)$ in $\partial A_X^c(\lambda_0)^\circ$. Thus, we end up with non compound du Val singularities for $k=2,3,4$. In the case $k=1$, we may assume $t>2$. Only for $t=3$, no lattice points are inside $\partial A_X^{c\,\circ}$. For $t>3$, the point $(-1,-1,3,0)$ lies in $\partial A_X^c(\lambda_0)^\circ$. So with $t=3$, we obtain matrix (18), defining a compound Du Val singularity.

We show that the remaining possible locations of $v$ all lead to non compound Du Val singularities. Assume $v \in \lambda_1^\circ$. Then $v = v_{12} = (k,0,t,1) \in \lambda_1^\circ$ with $1 \le k \le 3$. The second vertex of $\partial A_X^c(\lambda)$ is $w_2 = (0,0,(k+4t)/(4-k),1)$. Thus, either $(0,0,2,1)$ or $(0,0,4,1)$ lies in $\partial A_X^c(\lambda)^\circ$. Assume $v \in \lambda_2^\circ$. Then $v = v_{22} = (0,k,t,1)$ with $k=1,2$, where we can assume $t>1$ or $t>0$ accordingly. The second vertex of $\partial A_X^c(\lambda)$ is $w_2 = (0,0,(3k+12t)/(12-5k),1)$. For $k=2$, we find $(0,0,4,1)$ in $\partial A_X^c(\lambda)^\circ$. For $k=1$, the segment $\partial A_X^c(\lambda)$ is of length $(12t-18)/7$. Thus, for $t \ge 3$, we find a lattice point in $\partial A_X^c(\lambda)^\circ$. For $t=2$, we look at $\partial A_X^c(\lambda_0)^\circ$ and see that it contains $(-1,-1,3,0)$; see the figure above. Finally, if $v = v_1 \in \lambda$, one finds $(-1,-1,3,0)$ in $\partial A_X^c(\lambda_0)^\circ$.

Let $P$ have the leading block data $(3,3,2;0,0,0)$. Then the first vertex of $\partial A_X^c(\lambda)$ is $w_1 = (0,0,0,1)$. Assume $v = (-k,-k,t,1-k) \in \lambda_0^\circ$ or $v = (k,0,t,1) \in \lambda_1^\circ$ with $k=1,2,3$. Then we can assume $t>0$. We obtain $w_2 = (0,0,6t/(6-k),1)$, find $(0,0,1,1)$ in $\partial A_X^c(\lambda)^\circ$ and thus leave the compound Du Val case. If $v = (0,k,t,1) \in \lambda_2^\circ$, with $k=1,2$, we can assume $t>0$. We obtain $w_2 = (0,0,3t/(3-k),1)$ and find $(0,0,1,1)$ in $\partial A_X^c(\lambda)^\circ$. Thus also here, we leave the compound Du Val case. Finally, if $v = (0,0,t,1) \in \lambda$, then we end up with $t=1$ and the matrix (6), defining the compound du Val singularity $E_6 \times \mathbb{C}$.

Let $P$ have the leading block data $(3,3,2;1,0,0)$. Then the first vertex of $\partial A_X^c(\lambda)$ is $w_1 = (0,0,2,1)$. We will take a look at the leaves:



Assume $v \in \lambda_0^\circ$. Then $v = v_{01} = (-k, -k, t, 1-k)$ with $k = 1, 2, 3$. We obtain $w_2 = (0, 0, 6t/(6-k), 1)$. In the case $k = 3$ as well as in the case $k = 2$ with $t \neq 1$, we find one of $(0, 0, 1, 1)$ and $(0, 0, 3, 1)$ in $\partial A_X^c(\lambda)^\circ$ and leave the compound Du Val case. For $k = 2$ and $t = 1$, there are no lattice points in $\partial A_X^c$ and the resulting matrix is (15), defining a compound Du Val singularity. If $k = 1$ and $t \neq 2$, we find $(0, 0, 1, 1)$ or $(0, 0, 3, 1)$ in $\partial A_X^c(\lambda)^\circ$. The case $t = 2$ leads to the matrix (7), defining a compound Du Val singularity. The case of $v \in \lambda_1^\circ$ can be reduced by means of admissible operations to the previous case. We show that for the remaining possible locations of $v$, we leave the compound du Val case. If $v = (0, k, t, 1) \in \lambda_2^\circ$, then $w_2 = (0, 0, (3t+k)/(3-k), 1)$ and we find $(0, 0, 1, 1)$ or $(0, 0, 3, 1)$ in $\partial A_X^c(\lambda)^\circ$. If $v = (0, 0, t, 1) \in \lambda$, then $(-1, -1, 2, 0)$ or $(1, 0, 1, 1)$ lies in $\partial A_X^c(\lambda)^\circ$.

Let $P$ have the leading block data $(l_{01}, 2, 2; 0, 0, 0)$. Then the first vertex of $\partial A_X^c(\lambda)$ is $(0, 0, 0, 1)$. Assume $v \in \lambda_0^\circ$. Then $v = v_{02} = (-k, -k, t, 1-k)$ with $1 \leq k \leq l_{01}$, where we can assume $t > 0$. We have $w_2 = (0, 0, t, 1)$. For $t > 1$, we obtain $(0, 0, 1, 1) \in \partial A_X^c(\lambda)^\circ$ and thus leave the compound Du Val case. For $t = 1$, the resulting singularity is compound Du Val for every $k$ and has defining matrix (12-e-e) with $k_1 \geq k_2$. Assume $v \in \lambda_1^\circ$. Then $v = v_{12} = (k, 0, t, 1)$ with $k = 1, 2$. We can assume $l_{01} > 2$ and $t > 0$. For $k = 1$ we have $w_2 = (0, 0, 2tl_{01}/(2 + l_{01}), 1)$ and for $k = 2$, we have $w_2 = (0, 0, tl_{01}/2, 1)$. In both cases, $\partial A_X^c(\lambda)^\circ$ contains $(0, 0, 1, 1)$ and we obtain a non compound Du Val singularity. The case of $v \in \lambda_2^\circ$ can be transformed via exchanging the data of blocks 1 and 2 into the previous one. Finally, if $v = (0, 0, t, 1) \in \lambda$, then we must have $t = 1$ and this gives the compound Du Val singularity $D_{l_{01}+2} \times \mathbb{C}$, defined by the matrix (4).

Let $P$ have the leading block data $(l_{01}, 2, 2; 1, 0, 0)$. Then the first vertex of $\partial A_X^c(\lambda)$ is $(0, 0, 1, 1)$. Assume $v \in \lambda_0^\circ$. Then $v = v_{02} = (-k, -k, t, 1-k)$ with $1 \leq k \leq l_{01}$. We can assume $t < 1$. For $t < 0$, we have $(0, 0, 0, 1) \in \partial A_X^c(\lambda)^\circ$. For $t = 0$, we obtain a matrix (12-e-e) as in the case of leading block data $(l_{01}, 2, 2; 0, 0, 0)$, now with $k_1 \leq k_2$. Assume $v \in \lambda_1^\circ$. The case $l_{01} = 2$ can be transformed via admissible operations into the case of leading block data $(l_{01}, 2, 2; 0, 1, 0)$ and an additional column in $\lambda_0^\circ$, which is discussed below. So, let $l_{01} > 2$. Then $v = (k, 0, t, 1)$, where $k = 1, 2$. For $k = 2$, we can assume $t > 0$. We obtain $w_2 = (0, 0, 1 + tl_{01}/2, 1)$ and $(0, 0, 2, 1) \in \partial A_X^c(\lambda)^\circ$ and thus leave the compound Du Val case. Now let $k = 1$. Here, $t$ may be any integer and we obtain $w_2 = (0, 0, 2(1 + l_{01}t)/(2 + l_{01}), 1)$. Only for $t = 0, 1$ there are no lattice points in $\partial A_X^c(\lambda)^\circ$. Both cases lead by admissible operations to the compound Du Val singularity with defining matrix (11). The case of $v \in \lambda_2^\circ$ can be transformed to the previous one by exchanging the data of column blocks 1 and 2. Finally, if $v \in \lambda$, then it equals either $(0, 0, 0, 1)$ or $(0, 0, 2, 1)$. Both cases lead to the compound Du Val singularity with defining matrix (5o).

Let $P$ have leading block data $(l_{01}, 2, 2; 0, 1, 0)$. Then the first vertex of $\partial A_X^c(\lambda)$ is $w_1 = (0, 0, l_{01}/2, 1)$.

*Case 1:* The exponent $l_{01}$ is even. Assume $v \in \lambda$. Then $v = v_1 = w_2 = (0, 0, t, 1)$. Exchanging the data of blocks 0 and 1 transforms the case $l_{01} = 2$ into the corresponding case with leading block data $(l_{01}, 2, 2; 1, 0, 0)$ treated before. So, let $l_{01} > 2$. Having no lattice points in $\partial A_X^c(\lambda)^\circ$ implies $t = l_{01}/2 \pm 1$. But then, there are integer points in $\partial A_X^c(\lambda_0)^\circ$: for $t = l_{01}/2 + 1$ we find

$$\left(-1, -1, \frac{l_{01}}{2}, 0\right) = \frac{1}{l_{01}} v_{01} + \frac{1}{2} w_1 + \left(\frac{1}{2} - \frac{1}{l_{01}}\right) w_2$$

and for $t = l_{01}/2 - 1$ we find

$$\left(-1, -1, \frac{l_{01}}{2} - 1, 0\right) \;=\; \frac{1}{l_{01}} v_{01} + \left(\frac{1}{2} - \frac{2}{l_{01}}\right) w_1 + \left(\frac{1}{l_{01}} + \frac{1}{2}\right) w_2.$$

Assume $v \in \lambda_0^\circ$. Then $v = v_{02} = (-k, -k, t, 1-k)$ with $1 \le k \le l_{01}$. The second vertex of $\partial A_X^c(\lambda)$ is $w_2 = (0, 0, t + k/2, 1)$. If $k$ is even, then having no lattice points in $\partial A_X^c(\lambda)^\circ$ implies $t = (l_{01} - k)/2 \pm 1$. Again there are integer points in $\partial A_X^c(\lambda_0)^\circ$: for $t = (l_{01} - k)/2 + 1$ we find

$$\left(-1, -1, \frac{l_{01}}{2}, 0\right) \;=\; \frac{1}{k} v_{02} + \frac{1}{2} w_1 + \left(\frac{1}{2} - \frac{1}{k}\right) w_2$$

and for $t = (l_{01} - k)/2 - 1$ we find

$$\left(-1, -1, \frac{l_{01}}{2} - 1, 0\right) \;=\; \frac{1}{k} v_{02} + \frac{1}{2} w_1 + \left(\frac{1}{2} - \frac{1}{k}\right) w_2.$$

If $k$ is odd, then having no lattice points in $\partial A_X^c(\lambda)^\circ$ implies $t = (l_{01} - k \pm 1)/2$. For both choices of $t$, this setting produces a compound Du Val singularity with matrix (12-o-e/o) and parameters $k_1 \ge k_2$.

Before entering the discussion of the cases $v \in \lambda_i^\circ$ with $i = 1, 2$, the parameter $k$ occurring in $v$ might be $k = 1, 2$ and the vertex $w_2$ is given by

$$w_2 \;=\; \begin{cases} \left(0, 0, \frac{2tl_{01}}{2l_{01} + 2k - kl_{01}}, 1\right), & v = (k, 0, t, 1) \in \lambda_1^\circ, \\ \left(0, 0, \frac{2tl_{01} + kl_{01}}{2l_{01} + 2k - kl_{01}}, 1\right), & v = (0, k, t, 1) \in \lambda_2^\circ. \end{cases}$$

*Case 1.1:* We have $l_{01} \equiv 0 \mod 4$. If $v \in \lambda_2^\circ$ or $v = (2, 0, t, 1) \in \lambda_1^\circ$, then we find one of $(0, 0, l_{01}/2 \pm 1, 1)$ in $\partial A_X^c(\lambda)^\circ$. Thus, we are left with $v \in \lambda_1^\circ$ and $k = 1$. For any $t \ne l_{01}/4 + 1$, we find the lattice point $(1, 0, l_{01}/4 + 1, 1)$ in $\partial A_X^c(\lambda_1)^\circ$. Thus, we end up with

$$v \;=\; (k, 0, t, 1) \;=\; (1, 0, l_{01}/4 + 1, 1), \qquad w_2 \;=\; (0, 0, l_{01}(l_{01} + 4)/(2l_{01} + 4), 1).$$

Note that the segment $\partial A_X^c(\lambda)$ contains no lattice points, because its length equals $l_{01}/(l_{01} + 2) < 1$. Taking a look at $\lambda_0$, we observe

$$(-1, -1, l_{01}/2, 0) \in \partial A_X^c(\lambda_0)^\circ \iff \frac{l_{01}}{l_{01} + 2} > \frac{l_{01}}{2(l_{01} - 1)} \iff l_{01} > 4.$$

Thus, to obtain compound Du Val singularities, we must have $l_{01} \le 4$. As $l_{01} \equiv 0 \mod 4$ holds, only $l_{01} = 4$ is left and, indeed, this leads to the compound Du Val singularity with defining matrix (16).

*Case 1.2:* We have $l_{01} \equiv 2 \mod 4$. If $v \in \lambda_1^\circ$ or $v = (0, 2, t, 1) \in \lambda_2^\circ$, then we find one of $(0, 0, l_{01}/2 \pm 1, 1)$ in $\partial A_X^c(\lambda)^\circ$. Thus, we are left with $v \in \lambda_2^\circ$ and $k = 1$. For any $t \ne l_{01}/4 + 1/2$, we find the lattice point $(0, 1, l_{01}/4 + 1/2, 1)$ in $\partial A_X^c(\lambda_2)^\circ$. We end up with

$$v \;=\; (0, k, t, 1) \;=\; (0, 1, l_{01}/4 + 1/2, 1) \;\in\; \lambda_2^\circ.$$

Similar to Case 1.1, we obtain that $(-1, -1, l_{01}/2, 0) \in \partial A_X^c(\lambda_0)^\circ$ as soon as $l_{01} > 4$. Thus, only $l_{01} = 2$ might lead to a compound Du Val singularity. In this case, we exchange the data of blocks 0 and 2 and land in case of leading block data $(l_{01}, 2, 2; 0, 1, 0)$ and an additional column in $\lambda_0^\circ$.

*Case 2:* The exponent $l_{01}$ is odd. If $v \in \lambda$, then $v = v_1 = w_2 = (0, 0, (l_{01} + 1)/2, 1)$ holds and we arrive at the compound Du Val singularity with defining matrix (5e).

If $v \in \lambda_0^\circ$ holds, then the arguing runs similar as in Case 1. Only for $k$ odd and $v = v_{02} = (-k, -k, (l_{01} - k + 1)/2, 1 - k)$, there are no lattice points in the relative interior of $\partial A_X^c(\lambda)^\circ$ and we end up with the matrix (12-o-e/o) as in Case 1, but now with parameters $k_1 \le k_2$.

Assume $v \in \lambda_1^\circ$. Then $v = v_{12} = (k, 0, t, 1)$ with $k = 1, 2$. The case $k = 2$ gives $w_2 = (0, 0, tl_{01}/2, 1)$, the point $(0, 0, (l_{01} \pm 1)/2, 1)$ lies $\partial A_X^c(\lambda)^\circ$ and thus we leave the compound Du Val case. So, let $k = 1$. Then we have $v = (1, 0, t, 1) \in \lambda_1^\circ$. Moreover, $w_1 = (0, 0, l_{01}/2, 1)$ and $w_2 = (0, 0, 2tl_{01}/(2 + l_{01}), 1)$. Now, as $l_{01}$ is odd, we see that $\partial A_X^c(\lambda)$ to have no lattice points in the relative interior means

$$\frac{1}{2} \;\ge\; \left| \frac{2tl_{01}}{2 + l_{01}} - \frac{l_{01}}{2} \right|.$$

If $l_{01} \equiv 1 \mod 4$, this is only fulfilled for $t = (l_{01} + 3)/4$. If $l_{01} \equiv 3 \mod 4$, it is only fulfilled for $t = (l_{01} + 1)/4$. Altogether, it is fulfilled for $t = \lceil l_{01}/4 \rceil$. This leads to the compound Du Val singularity with defining matrix (10o).

The case $v \in \lambda_2^\circ$ can be transformed by suitable admissible operations to the case $v \in \lambda_1^\circ$ just discussed.

Let $P$ have leading block data $(l_{01}, l_{11}, 1; \mathfrak{d}_0, 0, 0)$. As $P$ is irredundant, the additional column is forced to be $(0, 1, t, 1) \in \lambda_2^\circ$ and we have $l_{01}, l_{11} \ge 2$. The vertices of $\partial A_X^c(\lambda)$ turn out to be

$$w_1 \;=\; \left(0, 0, \frac{\mathfrak{d}_0 l_{11}}{l_{01} + l_{11}}, 1\right), \qquad w_2 \;=\; \left(0, 0, \frac{\mathfrak{d}_0 l_{11} + tl_{01} l_{11}}{l_{01} + l_{11}}, 1\right).$$

We have $0 < tl_{01}l_{11}/(l_{01} + l_{11}) \le 1$ only for $t = 1$ and $l_{01} = l_{11} = 2$. In this case, the second inequality becomes an equality and thus $w_1$ is integral which implies $\mathfrak{d}_0 = 0$. We arrive at the compound Du Val singularity with matrix (12-e-e) and parameters $k_1 = k_2 = 1$. □

We turn to the non-toric non-$\mathbb{Q}$-factorial threefolds, still of canonical multiplicity one. The following observation provides the link to the $\mathbb{Q}$-factorial case. Given defining data $A, P$ for a ring $R(A, P)$, we will have to deal with quadratic submatrices $P'$ of $P$, obtained by erasing columns and rows from $P$. The corresponding submatrix $A'$ of $A$ gathers all columns $a_i$ of $A$ such that at least one column $v_{ij}$ is not erased from $P$ when passing to $P'$.

Lemma 4.2.2. *Let $X = X(A, P)$ be a compound Du Val threefold quasicone of canonical multiplicity $\zeta_X$ with $P$ irredundant in the form of Proposition 1.4.3 and ordered in the sense of Remark 2.6.1.*

- (i) *Let $P'$ be an $(r+2) \times (r+2)$ submatrix of $P$ such that for any $i = 0, \dots, r$ at least one $v_{ij}$ is not erased from $P$.*
  - *(a) $A' = A$ and $P'$ are defining data of Type 2 in the sense of Construction 1.2.2; moreover, $P'$ is in the form of Proposition 1.4.3.*
  - *(b) $X' = X(A', P')$ is a $\mathbb{Q}$-factorial threefold with at most compound Du Val singularities of canonical multiplicity $\zeta_{X'} = \zeta_X$.*

  *Moreover, one always finds a submatrix $P'$ as above being ordered and having the same leading block as $P$.*
- (ii) *Every $P'$ as in (i) admits a $4 \times 4$ submatrix $P''$ with the same leading block as $P'$ such that*
  - *(a) $A''$ and $P''$ are defining data of Type 2 in the sense of Construction 1.2.2, the matrix $P''$ is ordered and the form of Proposition 1.4.3.*

   (b) *The varieties $X' = X(A', P')$ and $X'' = X(A'', P'')$ are equivariantly isomorphic to each other.*

(iii) *If the leading platonic triple of $P$ is different from $(x, y, 1)$, then $r = 2$ holds.*

(iv) *One always finds $P'$ and $P''$ as in (ii) with the same leading block as $P$ such that*
   (a) *in case of the leading platonic triple of $P$ differing from $(x, y, 1)$, up to admissible operations, $P''$ is one of the matrices from Proposition 4.2.1.*
   (b) *in case of the leading platonic triple of $P$ being equal to $(x, y, 1)$, we have $n_2'' = 2$ for $P''$.*

Proof. We verify (i). Note that each column of $P'$ is as well a column of $P$. The columns of $P$ generate the extremal rays of a full dimensional cone $\sigma \subseteq \mathbb{Q}^{r+2}$. Thus, also the columns of $P'$ generate the extremal rays of a cone $\sigma' \subseteq \mathbb{Q}^{r+2}$. We show that $\sigma'$ is full dimensional. If $P'$ has a column $v_1 \in \lambda$, then, using Proposition 1.3.6 (iii) we see that the remaining $r + 1$ columns of $P'$ are linearly independent and $v_1$ does not lie in their linear span. If $P'$ has no column inside $\lambda$, then we can form two different $P$-elementary cones $\tau_1$ and $\tau_2$ out of columns of $P'$. The corresponding $v_{\tau_i} \in \tau_i^\circ$ generate the pointed two-dimensional cone $\sigma' \cap \lambda$ and we see that the columns of $P$ generate $\mathbb{Q}^{r+2}$. Thus, we can conclude that $P'$ satisfies the conditions of Construction 1.2.2 and, together with $A' = A$ gives defining data. Observe that $X' = X(A', P')$ is $\mathbb{Q}$-factorial by construction. Using Remark 1.4.4, we obtain $\zeta_{X'} = \zeta_X$ and see that $P'$ still is in the form of Proposition 1.4.3. Using Remark 1.4.5, we conclude $\imath_{X'} = \imath_X = 1$. Moreover, according to Proposition 1.3.6, the anticanonical complex $\mathcal{A}^c_{X'}$ is a subcomplex of $\mathcal{A}^c_X$ and the same holds for $\partial \mathcal{A}^c_{X'}$ and $\partial \mathcal{A}^c_X$. Thus, Proposition 4.1.4 shows that $X'$ inherits from $X$ the property of having at most compound Du Val singularities. The supplement is obvious.

We prove (ii). For $r = 2$, there is nothing to show. So, assume $r \geq 3$. If $P'$ has a column $v_k \in \lambda$, then we have $n_i = l_{i1} = 1$ for $i \geq 3$ and Remark 1.2.4, applied $r - 2$ times, yields the desired $4 \times 4$ matrix $P''$. We turn to the case that $P'$ has no column in $\lambda$. Then $n_k = 2$ for some $0 \leq k \leq r$ and all other $n_i$ equal one. If $k \leq 2$ holds, then we have $n_i = l_{i1} = 1$ for $i \geq 3$ and proceed as before to obtain $P''$. We discuss $k = 3$. First assume that the leading platonic triple of $P'$ equals $(x, y, 1)$. Then, exchanging the data of column blocks 3 and 2 of $P'$, we are in the case $k \leq 2$ just treated. If the leading platonic triple of $P'$ differs from $(x, y, 1)$ then, applying $r - 3$ times Remark 1.2.4, we arrive at an irredundant $5 \times 5$ matrix $P''$ defining a variety $X'' = X(A'', P'')$ isomorphic to $X' = X(A', P')$; a contradiction to Proposition 4.2.1. Finally, if $k \geq 4$, then we exchange the data of column blocks $k$ and 3 of $P'$ and are in the case $k = 3$. This proves (ii).

We turn to (iii). Assume $r \geq 3$. Since $P$ is irredundant and ordered in the sense of Remark 2.6.1, we have $n_i \geq 2$ and $l_{ij} = 1$ for $i \geq 3$. Consider the submatrices

$$P' := [v_{01}, v_{11}, v_{21}, v_{31}, v_{32}, v_{41}, \ldots, v_{r1}], \qquad P^\sim := [v_{01}, v_{11}, v_{21}, v_{31}, v_{32}].$$

Let $P''$ be the matrix obtained by erasing from $P^\sim$ erasing all but the first three and the last two rows. Then $P''$ is an irredundant $5 \times 5$ matrix and $X'' = X(A'', P'')$ is isomorphic to $X' = X(A', P')$; a contradiction to Proposition 4.2.1.

Finally, we show (iv). For (a), observe that because of $\imath_{X''} = \imath_X = 1$, Proposition 1.4.7 gives $\zeta_{X''} = \zeta_X = 1$. Thus $X''$ is $\mathbb{Q}$-factorial compound Du Val and $P''$

must, up to admissible operations, be one of the matrices from Proposition 4.2.1. We turn to (b). For any $i \geq 2$, we have $n_i \geq 2$, because $P$ is irredundant. Consider the submatrices

$$P' \ := \ [v_{01}, v_{11}, v_{21}, v_{22}, v_{31}, \ldots, v_{r1}], \qquad P^{\sim} \ := \ [v_{01}, v_{11}, v_{21}, v_{22}].$$

Then we obtain the desired $P''$ from $P^{\sim}$ by erasing all but the first two and the last two rows. □

Proposition 4.2.3. *Let $X = X(A, P)$ be a non-toric affine threefold. Assume that $X$ is not $\mathbb{Q}$-factorial, of canonical multiplicity one and has at most compound Du Val singularities. Then $P$ can be assumed to be the matrix*

$$\text{(10-e)} \quad \begin{bmatrix} -k & 2 & 1 & 0 & 0 \\ -k & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1-k & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad k \in \ \mathbb{Z}_{\geq 2}.$$

Proof. The strategy is to look first for not necessarily irredundant matrices $P''$ with $r'' = 2$ defining a $\mathbb{Q}$-factorial $X'' = X(A'', P'')$ of canonical multiplicity one with at most compound Du Val singularities. Then we obtain, up to admissible operations, all matrices $P$ with $X(A, P)$ satisfying the assumptions of the proposition by enlarging the $P''$ in the sense of Lemma 4.2.2. We organize the subsequent discussion according to the possible leading block data, as listed in Proposition 4.1.9, and treat pairs $P'', P$ sharing the same leading block data. Note that we have $r = 2$ for $P$ whenever the leading platonic triple differs from $(x, y, 1)$.

Consider the leading block data $(5, 3, 2; 0, 0, 0)$. Proposition 4.2.1 tells us that after suitable admissible operations, we have

$$P'' \ = \ \begin{bmatrix} -5 & 3 & 0 & 0 \\ -5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 1 & 1 & 1 \end{bmatrix}.$$

After performing the corresponding admissible operations on $P$, we find $P''$ as a submatrix of $P$. Moreover, $P$ has at least one further column and thus a submatrix

$$P''' \ = \ \begin{bmatrix} -5 & 3 & 0 & * \\ -5 & 0 & 2 & * \\ 0 & 0 & 0 & * \\ -4 & 1 & 1 & * \end{bmatrix}.$$

Lemma 4.2.2 (i) says that $X''' = X(A''', P''')$ is $\mathbb{Q}$-factorial, of canonical multiplicity one and with at most compound Du Val singularities. Thus, up to admissible operations, $P'''$ occurs in the list of Proposition 4.2.1. So, the last column must be one of

$$(0, 0, 1, 1), \qquad (0, 0, -1, 1)$$

The first case is impossible, because the columns of the defining matrix $P$ are pairwise different. For $(0, 0, -1, 1)$ as last column, the point $(0, 0, 0, 1)$ lies in $\partial A_X^c(\lambda)^\circ$; a contradiction to Proposition 4.1.4.

The case of leading block data $(4, 3, 2; 0, 0, 0)$ is treated by exactly the same arguments as the preceding case.

Consider the leading block data $(4,3,2;1,0,0)$. Again, Proposition 4.2.1 tells us that, up to admissible operations, we have

$$P'' = \begin{bmatrix} -4 & -1 & 3 & 0 \\ -4 & -1 & 0 & 2 \\ 1 & 3 & 0 & 0 \\ -3 & 0 & 1 & 1 \end{bmatrix}.$$

Adapting $P$ by admissible operations, it comprises $P''$ as a submatrix. As before, we obtain a matrix $P'''$ by enhancing the leading block with a further column of $P$, which this time must be one of

$$(-1,-1,3,0), \qquad (-1,-1,2,0).$$

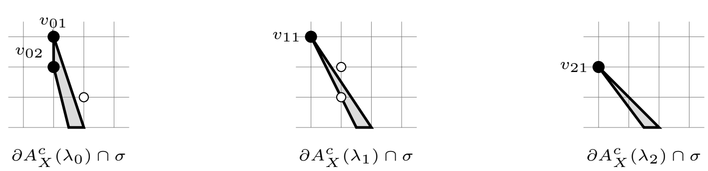
The first leads to two identical columns of $P$ and this is excluded. For the second we find $(0,0,3,1)$ inside $\partial A_X^c(\lambda)^\circ$ and leave the compound Du Val case.

The case of leading block data $(3,3,2;0,0,0)$ runs exactly as the case of $(5,3,2;0,0,0)$.

Consider the leading block data $(3,3,2;1,0,0)$. Here Proposition 4.2.1 leaves us with two possibilities for the submatrix $P''$ of the accordingly adapted $P$. The first possibility is

$$P'' = \begin{bmatrix} -3 & -2 & 3 & 0 \\ -3 & -2 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ -2 & -1 & 1 & 1 \end{bmatrix} \tag{4.2.3.1}$$

with columns $v_{01}, v_{02}, v_{11}, v_{21}$. Using as above Proposition 4.2.1, we arrive at three possibilities for submatrices $P''' = [v_{01}, v_{11}, v_{21}, *]$; with $\sigma = \mathrm{cone}(v_{01}, v_{02}, v_{11}, v_{21})$, we find the following situation in the polytopes $\partial A_X^c(\lambda_i) \cap \sigma$:




where the circles indicate the prospective columns $*$ of $P'''$ leading to compound Du Val singularities $X(A''', P''')$ of canonical multiplicity one. They are

$$(-1,-1,2,0) \in \lambda_0, \quad (1,0,1,1),\ (2,0,1,1) \in \lambda_1.$$

The lower one in the middle picture is contained in $\sigma$ which is not possible. The other two force $(0,0,2,1)$ to lie in $\partial A_X^c(\lambda)^\circ$ which is as well impossible. So, (4.2.3.1) does not occur as a submatrix of $P$. The second possibility is

$$P'' = \begin{bmatrix} -3 & -1 & 3 & 0 \\ -3 & -1 & 0 & 2 \\ 1 & 2 & 0 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix}.$$

Here we proceed analogously as with (4.2.3.1) and see the only possible additional column in $P$ is $(1,0,1,1)$. In this case again $(0,0,2,1)$ lies in $\partial A_X^c(\lambda)^\circ$ and we leave the compound Du Val case.

Consider the leading block data $(l_{01}, 2, 2; 0, 0, 0)$. Here Proposition 4.2.1 tells us that the submatrix $P''$ of the accordingly adapted $P$ is

$$P'' = \begin{bmatrix} -k_1 & -k_2 & 2 & 0 \\ -k_1 & -k_2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1-k_1 & 1-k_2 & 1 & 1 \end{bmatrix},$$

where we allow $k_2 = 0$ here and in this case change the second and fourth column to have a proper defining matrix. A possible further column for $P'''$ must have the form $(-k_3, -k_3, t, 1-k_3)$ with $t = \pm 1$. For $t = 1$, one of $(-k_2, -k_2, 1, 1-k_2)$ or $(-k_3, -k_3, 1, 1-k_3)$ does not give an extremal ray of the cone spanned by the columns of $P$. For $t = -1$, the point $(0, 0, 0, 1)$ lies in $\partial A_X^c(\lambda)^\circ$ and we leave the compound Du Val case.

Consider the leading block data $(l_{01}, 2, 2; 1, 0, 0)$. Proposition 4.2.1 allows two choices for the submatrix $P''$ of the accordingly adapted $P$. The first one is

$$P'' = \begin{bmatrix} -k & 2 & 0 & 0 \\ -k & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 1-k & 1 & 1 & 1 \end{bmatrix}.$$

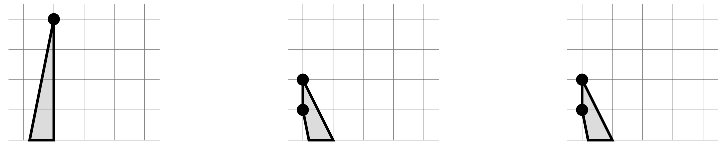
We check the possible further columns of $P$. A column in $\lambda$ would lead to $(0, 0, 1, 1) \in \partial A_X^c(\lambda)^\circ$ and this is impossible. For any $P'''$ sharing the first three columns with $P''$, the additional column, due to Proposition 4.2.1, must be $(1, 0, t, 1)$ or $(0, 1, t, 1)$, where $t = 0, 1$. For $t = 0$, such column would not generate an extremal ray of the cone spanned by the columns of $P$. For $t = 1$, we obtain $(0, 0, 1, 1) \in \partial A_X^c(\lambda)^\circ$ and we leave the compound Du Val case. The second choice is

$$P'' = \begin{bmatrix} -k & 2 & 1 & 0 \\ -k & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 1-k & 1 & 1 & 1 \end{bmatrix}.$$

Proposition 4.2.1 tells us that $(1, 0, t, 1)$ or $(0, 1, t, 1)$ with $t = 0, 1$ are the only possible further columns of $P$. But $(1, 0, 0, 1)$ is impossible, since this column already exists in $P$ and for $(1, 0, 1, 1)$, we obtain $(0, 0, 1, 1) \in \partial A_X^c(\lambda)^\circ$. The same holds for $(0, 1, 1, 1)$. For $(0, 1, 0, 1)$, the line segment $\partial A_X^c(\lambda)$ has, in addition to $w_1 = (0, 0, 1, 1)$, the vertex

$$w_2 = \left(0, 0, \frac{1}{1+k}, 1\right).$$

If we have a look at the leaves, we see that we get a compound Du Val singularity with defining matrix (10-e):



$\partial A_X^c(\lambda_0)$ $\partial A_X^c(\lambda_1)$ $\partial A_X^c(\lambda_2)$

Consider the leading block data $(l_{01}, 2, 2; 0, 1, 0)$. Proposition 4.2.1 allows four possible submatrices $P''$ of the suitably adapted $P$. We distinguish the following cases.

*Case 1:* The exponent $l_{01}$ is odd. First assume $P$ has after suitable admissible operations a submatrix

$$P'' = \begin{bmatrix} -2k_1 - 1 & -2k_2 & 2 & 0 \\ -2k_1 - 1 & -2k_2 & 0 & 2 \\ 0 & \frac{k_1 - k_2 + 1}{2} & 1 & 0 \\ -2k_1 & 1 - 2k_2 & 1 & 1 \end{bmatrix}.$$

Assume the matrix $P$ has a further column $(-k, -k, t, 1 - k)$ in $\lambda_0$. We regard the submatrix containing this further column as well as the last two columns of $P''$ and either the first (if $k$ odd) or the second (if $k$ even) of $P''$. This matrix does not show up in Proposition 4.2.1 and we leave the compound Du Val case. So $P$ can have no further column $(-k, -k, t, 1 - k)$.

Also an additional column $(0, 0, t, 1)$ in the lineality part is impossible, because due to Proposition 4.2.1, the only possibilities are $t = k_1$ and $t = k_1 + 1$. But these would either not give an extremal ray of the cone spanned by the columns of $P$ (for $t = k_1 + 1$) or $(-1, -1, k_1, 0)$ would show up in $\partial A_X^c(\lambda_0)^\circ$. Now the last possibility is an additional column $(1, 0, t, 1)$ in $\lambda_1$ or $(0, 1, t, 1)$ in $\lambda_2$. But the possible values of $t$, i.e. those giving a compound Du Val submatrix of type (10-o) from Proposition 4.2.1, either generate no extremal ray of the cone spanned by the columns of $P$ or $(-1, -1, k_1, 0)$ is an interior point of $\partial A_X^c(\lambda_0)$. Thus assume $P$ has, after suitable admissible operations, no submatrix of the above form and one

$$P'' = \begin{bmatrix} -2k - 1 & 2 & 1 & 0 \\ -2k - 1 & 0 & 0 & 2 \\ 0 & 1 & \lceil \frac{2k+1}{4} \rceil & 0 \\ -2k & 1 & 1 & 1 \end{bmatrix}.$$

Now, the submatrix of $P$ given by the first, second and third column of this submatrix and one further column must as well be of this form after suitable admissible operations. So the only possible additional column is $(0, 1, \lceil (2k + 1)/4 \rceil - 1, 1)$ in $\lambda_2$, but then $(-1, -1, k_1, 0)$ is an inner point of $\partial A_X^c(\lambda_0)$ and we leave the compound Du Val case.

*Case 2:* The exponent $l_{01}$ equals 4. After suitable admissible operations, the matrix $P$ has a submatrix

$$P'' = \begin{bmatrix} -4 & 2 & 1 & 0 \\ -4 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ -3 & 1 & 1 & 1 \end{bmatrix}.$$

A further column must, together with the first two and the last row of $P''$, give a compound Du Val submatrix $P'''$ of $P$ as well. So due to Proposition 4.2.1, the only possible further column is $(1, 0, 1, 1)$. But with this, the point $(0, 0, 2, 1)$ is an inner point of $\partial A_X^c(\lambda)$ and we leave the compound Du Val case.

Consider the leading block data $(l_{01}, l_{11}, 1; \mathfrak{d}_0, 0, 0)$. Note that here, we also have to take care about redundant matrices $P''$. Proposition 4.2.1 provides us with one irredundant matrix

$$P'' \;=\; \begin{bmatrix} -2 & 2 & 0 & 0 \\ -2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}.$$

The only possible further columns of $P$ are of the form $(-2,-2,t_0,-1)$, $(2,0,t_1,1)$ or $(0,1,t_2,0)$. Each of them would stretch the segment $\partial A_X^c(\lambda)$ which already has the vertices $(0,0,0,1)$ and $(0,0,1,1)$.

Now we treat the redundant $P''$, which means to deal with $l_{11}=1$. Due to Lemma 4.2.2 (iv) (b), after suitable admissible operations, the matrix $P$ has a submatrix

$$P'' = \begin{bmatrix} -l_{01} & 1 & 0 & 0 \\ -l_{01} & 0 & 1 & 1 \\ \mathfrak{d}_0 & 0 & 0 & t_2 \\ 1-l_{01} & 1 & 1 & 1 \end{bmatrix}.$$

But since $P$ is irredundant, it must have a further submatrix

$$P''' = \begin{bmatrix} -l_{01} & 1 & 1 & 0 & 0 \\ -l_{01} & 0 & 0 & 1 & 1 \\ \mathfrak{d}_0 & 0 & t_1 & 0 & t_2 \\ 1-l_{01} & 1 & 1 & 1 & 1 \end{bmatrix}$$

comprising $P''$ and one further column in $\lambda_1$. For this matrix and the vertices of the respective $\partial A_{X'''}^c(\lambda)$, we have

$$w_1 \;=\; \left(0,0,\frac{\mathfrak{d}_0}{l_{01}+1},1\right), \qquad w_2 \;=\; \left(0,0,\frac{\mathfrak{d}_0+(t_1+t_2)l_{01}}{l_{01}+1},1\right),$$

But $(t_1+t_2)l_{01}/(l_{01}+1)\le 1$ only for $t_1=t_2=l_{01}=1$. But as $P$ is irredundant, it must have a sixth column $(-1,-1,\mathfrak{d}_0+t_0,0)$ in $P$. The distance between the vertices of $\partial A_X^c(\lambda)$ becomes

$$\frac{t_0+t_1+t_2}{2} \;\ge\; \frac{3}{2}.$$

Thus, $\partial A_X^c(\lambda)^\circ$ contains an integral point. So we obtain no compound Du Val singularity in this case. □

Finally, we have to deal with the non-toric threefolds of canonical multiplicity greater than one.

Proposition 4.2.4. *Let $X = X(A,P)$ be a non-toric affine threefold. Assume that $X$ is of canonical multiplicity greater than one and has at most compound Du Val singularities. Then one may assume $P$ to be one of the following matrices:*

$$\text{(9)}\quad \begin{bmatrix} -k & -k & \zeta_X-k & \zeta_X-k & 0 & 0 & \cdots & 0 & 0 \\ -k & -k & 0 & 0 & 1 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & & \\ -k & -k & 0 & 0 & 0 & 0 & & 1 & 1 \\ 0 & \mathfrak{d}_0 & 0 & \mathfrak{d}_1 & 0 & \mathfrak{d}_2 & \cdots & 0 & \mathfrak{d}_r \\ \frac{1-\mu k}{\zeta_X} & \frac{1-\mu k}{\zeta_X} & \frac{1-\mu k}{\zeta_X}+\mu & \frac{1-\mu k}{\zeta_X}+\mu & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\text{(13-e)}\begin{bmatrix} -2\zeta_X+1 & 1 & 1 & 0 & 0 \\ -2\zeta_X+1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(13-o)}\begin{bmatrix} -2\zeta_X+2 & 2 & 0 & 0 \\ -2\zeta_X+2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ \zeta_X & -1 & 0 & 0 \end{bmatrix} \quad \text{(14)}\begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 \end{bmatrix}.$$

*In (9), $r \ge 2$ holds, the integers $\zeta_X \ge 2$ and $k \ge 1$ are coprime and $\mu$ is the unique integer $1 \le \mu < \zeta_X$ with $\zeta_X \mid (1-\mu k)$. Moreover $\mathfrak{d}_i \in \mathbb{Z}_{\ge 1}$ holds for $i \ge 0$ and if*

*$k \ge 2$ ($\zeta_X - k \ge 2$), then one may erase the second (fourth) column of the matrix. In (13-e), we have $\zeta_X \ge 2$. In (13-o), we have $\zeta_X \ge 3$ odd. In (14), we have $\zeta_X = 2$.*

Proof. The strategy is similar to that of the proof of Proposition 4.2.3. We look first for not necessarily irredundant matrices $P''$ with $r = 2$ and $n_2'' = 2$ defining a $\mathbb{Q}$-factorial $X'' = X(A'', P'')$ with at most compound Du Val singularities and of canonical multiplicity bigger than one. Lemma 4.2.2 then ensures that for $X = X(A, P)$ satisfying the assumptions of the proposition, the matrix $P$ contains, after suitable admissible operations, one of our $P''$ as a submatrix with the same leading platonic triple as $P$. In other words, we can construct the possible $P$ by suitably enlarging $P''$.

The matrix $P''$ we are looking for is $4 \times 4$. Since $\zeta_{X''} > 1$ holds, we are in the setting of Proposition 1.4.7 and because of $\imath_{X''} = 1$, we end up in Case 1.4.7 (vi). In addition to the leading block, we have the extra column $v_{22}$ in $P''$. Moreover, the integer $\mu := (1 - \nu_{01}\zeta_{X''})/l_{01}$ as well as $l_{01}$ and $l_{11}$ must all be coprime to $\zeta_{X''}$, since we have the integer entries $\nu_{01} = (1 - \mu l_{01})/\zeta_{X''}$ and $\nu_{11} = (1 + \mu l_{11})/\zeta_{X''}$. We also see that $\zeta_{X''}$ divides $l_{01} + l_{11}$ by subtracting $\nu_{01}$ and $\nu_{11}$ from each other. Now let

$$k_0 := \lfloor l_{01}/\zeta_{X''} \rfloor, \qquad k_1 := \lceil l_{11}/\zeta_{X''} \rceil, \qquad \delta := l_{01} - k_0\zeta_{X''}.$$

Furthermore, let in this proof $\mathfrak{d}_{ij}$ be the third entry of the column $v_{ij}$ of $P''$. With these definitions, our matrix has the following shape

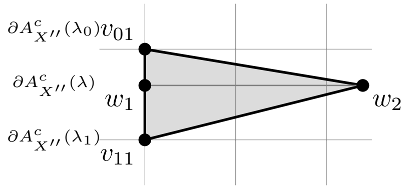
$$P'' = \begin{bmatrix} -(k_0\zeta_{X''} + \delta) & k_1\zeta_{X''} - \delta & 0 & 0 \\ -(k_0\zeta_{X''} + \delta) & 0 & 1 & 1 \\ \mathfrak{d}_{01} & \mathfrak{d}_{11} & 0 & \mathfrak{d}_{22} \\ \frac{1-\mu\delta}{\zeta_{X''}} - \mu k_0 & \frac{1-\mu\delta}{\zeta_{X''}} + \mu k_1 & 0 & 0 \end{bmatrix}, \tag{4.2.4.1}$$

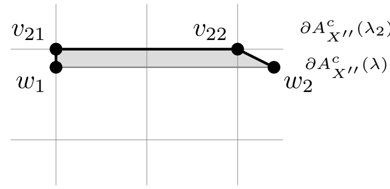
where we achieve $1 \le \mu < \zeta_{X''}$ by subtracting the $\lfloor \mu/\zeta_{X''} \rfloor$-fold of the first from the last row, simultaneously. Moreover, we achieve $\mathfrak{d}_{01} = 0$ by subtracting the $\mathfrak{d}_{01}\zeta_{X''}$-fold of the last and the $\mathfrak{d}_{01}\mu$-fold of the first from the penultimate row. Exchanging, if necessary, the data of column blocks 0 and 1, we achieve $k_1 > k_0 \ge 0$. We now figure out those $P''$ defining a compound Du Val singularity. For this, we consider several constellations of $k_0$ and $k_1$.

*Case 1:* We have $k_0 = 0$ and $k_1 = 1$. Here we can also achieve $\mathfrak{d}_{11} = 0$ by subtracting the $\mathfrak{d}_{11}(1 - \mu\delta)/\zeta_{X''}$-fold of the first and the $\mathfrak{d}_{11}\delta$-fold of the last from the penultimate row. The vertices of $\partial A^c_{X''}(\lambda)$ are

$$w_1 = \left(0, 0, 0, \frac{1}{\zeta_{X''}}\right), \qquad w_2 = \left(0, 0, \frac{\mathfrak{d}_{22}\delta(\zeta_{X''} - \delta)}{\zeta_{X''}}, \frac{1}{\zeta_{X''}}\right).$$

We illustrate the situation for the case $\delta = 2$, $\zeta_{X''} = 5$, $\mathfrak{d}_{22} = 2$ below; observe that the lineality part $\lambda$ contains no integer points and the union of the $\lambda_i \cap \mathcal{H}_i \cap \mathbb{Z}^4$ for $i = 0, 1$ is a sublattice



The polytope $\partial A^{c}_{X''}(\lambda_0)$ does not contain integer points $(-k,-k,t,(1-\mu k)/\zeta_{X''})$ in its relative interior as for such integer points $k<\delta$ and $(1-\mu k)/\zeta_{X''}$ integral must hold, but $\delta$ is minimal with the second property. The same holds for $\partial A^{c}_{X''}(\lambda_1)$ and $\partial A^{c}_{X''}(\lambda_2)$ respectively. All points in $\partial A^{c}_{X''}(\lambda)$ have $1/\zeta_{X''}$ as last coordinate, thus are not integral. So, there is no integral point in the relative interior of $\partial A^{c}_{X''}$. Thus $P''$ defines a $\mathbb{Q}$-factorial compound Du Val singularity and meanwhile looks as follows:

$$\begin{bmatrix} -\delta & \zeta_{X''}-\delta & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ 0 & 0 & 0 & \mathfrak{d}_{22} \\ \frac{1-\mu\delta}{\zeta_{X''}} & \frac{1-\mu\delta}{\zeta_{X''}}+\mu & 0 & 0 \end{bmatrix}, \qquad \gcd(\delta,\zeta_{X''})=1, \quad \mathfrak{d}_{22}\in\mathbb{Z}_{>0}. \tag{4.2.4.2}$$

Now we check the possibilities of enlarging $P''$ in the sense of Lemma 4.2.2 to a matrix $P$ defining a non-$\mathbb{Q}$-factorial $X(A,P)$ as in the proposition. As further columns we can insert one or both of

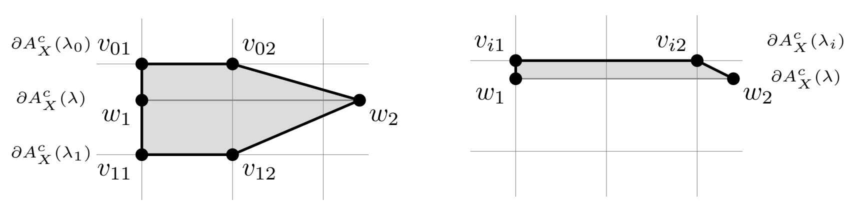
$$v_{02}=\left(-\delta,-\delta,\mathfrak{d}_{02},\frac{1-\mu\delta}{\zeta_{X''}}\right), \qquad v_{12}=\left(\zeta_{X''}-\delta,0,\mathfrak{d}_{12},\frac{1-\mu\delta}{\zeta_{X''}}+\mu\right),$$

with $\mathfrak{d}_{i2}\in\mathbb{Z}_{>0}$ arbitrary. We can not add other columns $(-k,-k,0,(1-\mu k)/\zeta_{X''})$ in $\lambda_0$. This is because first, $k\leq\delta$ must hold since $(\delta,\zeta_{X''}-\delta,1)$ is the leading platonic triple. Second, $k=k'\zeta_{X''}+\delta$ with $k'\geq 0$ must hold. So we get $k=\delta$. But then one of the columns

$$\left(-\delta,-\delta,\mathfrak{d}_{01},\frac{1-\mu\delta}{\zeta_{X''}}\right), \qquad \left(-\delta,-\delta,\mathfrak{d}_{02},\frac{1-\mu\delta}{\zeta_{X''}}\right), \qquad \left(-\delta,-\delta,\mathfrak{d}_{03},\frac{1-\mu\delta}{\zeta_{X''}}\right)$$

lies in the cone spanned by the other two. It can give no extremal ray of the cone spanned by the columns of $P$; a contradiction. Exactly the same argument shows that no more columns can be added in $\lambda_1$ and $\lambda_2$.

Moreover, we can increase $r$ from two to arbitrary to get $P$ from $P''$. The leaves $\lambda_0,\dots,\lambda_2$ stay untouched, we add new columns in leaves $\lambda_3,\dots,\lambda_r$. First we have $l_{ij}=1$, $n_i\geq 2$ for $i\geq 3$ due to log-terminality and irredundancy. Second, by the same argument as above for $\lambda_0,\dots,\lambda_2$, we have $n_i\leq 2$. Thus $n_i=2$ holds for $i\geq 3$. So $\lambda_i$ for $i\geq 3$ must have the same structure as $\lambda_2$ with two columns $e_i$ and $e_i+\mathfrak{d}_{i2}e_{r+1}$. Here $\mathfrak{d}_{i2}\in\mathbb{Z}_{>0}$ arbitrary and $e_j$ denotes the $j$-th basis vector. The distances $\mathfrak{d}_{i2}$ between $v_{i1}$ and $v_{i2}$ for $0\leq i\leq r$ and in consequence between $w_1$ and $w_2$ may vary. Nevertheless, all polytopes $\partial A^{c}_{X}(\lambda_i)$ are subsets of polytopes of the second type of Proposition 4.1.2 as also the following exemplary picture shows:




$i\geq 2$

So for any $P$ of this form, there are no integral points in the relative interior of $\partial A^{c}_{X}$. Furthermore, as we have seen above, no more columns can be added in any leaf. In total, we get the series (9) of defining matrices $P$ of compound Du Val singularities.

*Case 2:* We have $k_1 \geq 2$. Recall that we have $P''$ of shape (4.2.4.1) with $\mathfrak{d}_{01} = 0$. Let $x_1, \ldots, x_4$ be the standard coordinates on the column space $\mathbb{Q}^4$ of $P''$. Consider the line segments $\partial A^c_{X''}(\lambda)$ and

$$L_{0,X''} := \partial A^c_{X''}(\lambda_0) \cap \{x_1 = x_2 = -\delta\},\ L_{1,X''} := \partial A^c_{X''}(\lambda_1) \cap \{x_1 = \zeta_X - \delta, x_2 = 0\},$$

Let $w_1$, $w_2$ denote the vertices of $\partial A^c_{X''}(\lambda)$. Moreover, let $\omega_{01}$, $\omega_{02}$ be the vertices of $L_{0,X''}$ and $\omega_{11}$, $\omega_{12}$ the vertices of $L_{1,X''}$. Then we have

$$\begin{aligned}
w_1 &= \left(0, 0, \frac{\mathfrak{d}_{11}(k_0\zeta_{X''} + \delta)}{\zeta_{X''}(k_1 + k_0)}, \frac{1}{\zeta_{X''}}\right), \\
w_2 &= w_1 + \mathfrak{d}_{22}\frac{(k_0\zeta_{X''} + \delta)(k_1\zeta_{X''} - \delta)}{\zeta_{X''}(k_1 + k_0)} e_3, \\
\omega_{01} &= \left(-\delta, -\delta, \frac{\mathfrak{d}_{11}k_0\zeta_{X''}}{\zeta_{X''}(k_1 + k_0)}, \frac{1 - \mu\delta}{\zeta_{X''}}\right), \\
\omega_{02} &= \omega_{01} + \mathfrak{d}_{22}\frac{(k_1\zeta_{X''} - \delta)k_0}{k_1 + k_0} e_3, \\
\omega_{11} &= \left(\zeta_{X''} - \delta, 0, \mathfrak{d}_{11}\frac{\zeta_{X''}k_1 - \delta k_0 + \zeta_{X''}k_1k_0 - \delta}{(k_1\zeta_{X''} - \delta)(k_1 + k_0)}, \frac{1 - \mu\delta}{\zeta_{X''}} + \mu\right), \\
\omega_{12} &= \omega_{11} + \mathfrak{d}_{22}\frac{(k_0\zeta_{X''} + \delta)(k_1 - 1)}{k_1 + k_0} e_3.
\end{aligned}$$

Since there must be no integral point in the relative interior of the line segments $L_{0,X''}$ and $L_{1,X''}$, we at least require

$$\mathfrak{d}_{22}\frac{(k_1\zeta_{X''} - \delta)k_0}{k_1 + k_0} \leq 1, \qquad \mathfrak{d}_{22}\frac{(k_0\zeta_{X''} + \delta)(k_1 - 1)}{k_1 + k_0} \leq 1. \tag{4.2.4.3}$$

These inequalities will be observed in the following different cases.

*Case 2.1:* We have $k_0 = 0$. Here, the inequalities (4.2.4.3) ease to $\mathfrak{d}_{22}\delta(k_1 - 1)/k_1 \leq 1$. We distinguish between $\delta = 1$ and $\delta > 1$.

*Case 2.1.1:* We have $\delta = 1$. Here the matrix $P''$ is redundant. So any matrix $P$ with such submatrix must have an additional column in $\lambda_0$. We move on to a matrix $P$ also containing this additional column. Such matrix is of the form

$$P = \begin{bmatrix} -1 & -1 & k_1\zeta_X - 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 \\ 0 & \mathfrak{d}_{02} & \mathfrak{d}_{11} & 0 & \mathfrak{d}_{22} \\ 0 & 0 & k_1 & 0 & 0 \end{bmatrix},$$

where we can assume $\mathfrak{d}_{02} > 0$. But here the length of the line segment $L_{1,X}$ is

$$(\mathfrak{d}_2 + \mathfrak{d}_{02})(k_1 - 1)/k_1,$$

which is less or equal to one - which must hold if it does not contain an integral point - only for $\mathfrak{d}_{02} = \mathfrak{d}_{22} = 1$ and $k_1 = 2$. Thus by adding multiples of the last to the penultimate row, we can assume that $\mathfrak{d}_{11}$ equals one or zero. If $\mathfrak{d}_{11} = 1$, then the line segment $L_{1,X}$ has the vertices

$$\left(\zeta_X - 1, 0, \frac{2\zeta_X + 1}{4\zeta_X - 2}, 1\right), \qquad \left(\zeta_X - 1, 0, \frac{2\zeta_X + 1}{4\zeta_X - 2} + 1, 1\right).$$

So it contains an integer point in its relative interior, since $(2\zeta_X + 1)/(4\zeta_X - 2)$ is not integral. If $\mathfrak{d}_{11} = 0$, then $L_{1,X}$ has the vertices

$$(\zeta_X - 1, 0, 0, 1), \qquad (\zeta_X - 1, 0, 1, 1)$$

and thus contains no integer points. Since $L_{1,X}^{\circ}$ is the only subset of $\partial A_X^{c\circ}$ that may contain integer points, we get the series of defining matrices $(13e)$ with arbitrary $\zeta_X$ from this.

Such $P$ cannot again be the submatrix of a non-$\mathbb{Q}$-factorial matrix with possibly larger $r$. This is because for any additional column in $\lambda_0, \ldots, \lambda_2$, the line segment $L_{1,X}$ would be stretched and then contains one of the points $(\zeta_X - 1, 0, 0, 1)$ or $(\zeta_X - 1, 0, 1, 1)$ in its relative interior. The same holds for additional leaves, which by irredundancy must contain at least two columns and also would lead to a stretching of $L_{1,X}$.

*Case 2.1.2:* We have $\delta > 1$. Here $\mathfrak{d}_{22}\delta(k_1-1)/k_1 \geq 2(k_1-1)/k_1$ holds. Thus (4.2.4.3) is fulfilled only for $k_1 = \delta = 2$ and $\mathfrak{d}_{22} = 1$. Moreover $\zeta_{X''}$ must be odd since $l_{01} = 2$ is even. Also $\mu = (\zeta_{X''} + 1)/2$ holds, i.e. we have the matrix

$$P'' = \begin{bmatrix} -2 & 2\zeta_{X''} - 2 & 0 & 0 \\ -2 & 0 & 1 & 1 \\ 0 & \mathfrak{d}_{11} & 0 & 1 \\ -1 & \zeta_{X''} & 0 & 0 \end{bmatrix}.$$

By admissible operations, again $\mathfrak{d}_{11}$ can be assumed to be equal to zero or one. For $\mathfrak{d}_{11} = 1$, the line segment $L_{1,X''}$ has the vertices

$$\left(\zeta_{X''} - 2, 0, \frac{\zeta_{X''} - 1}{\zeta_{X''} - 2}, \frac{\zeta_{X''} - 1}{2}\right), \qquad \left(\zeta_{X''} - 2, 0, \frac{\zeta_{X''} - 1}{\zeta_{X''} - 2} + 1, \frac{\zeta_{X''} - 1}{2}\right),$$

which have an integer point inbetween due to $(\zeta_{X''}-1)/(\zeta_{X''}-2)$ not being integral. In case $\mathfrak{d}_{11}$ equals zero, the segment $L_{1,X''}$ has the vertices

$$\left(\zeta_{X''} - 2, 0, 0, \frac{\zeta_{X''} - 1}{2}\right), \qquad \left(\zeta_{X''} - 2, 0, 1, \frac{\zeta_{X''} - 1}{2}\right).$$

Since again $L_{1,X''}^{\circ}$ is the only subset of $\partial A_{X''}^{c\circ}$ that may contain integer points, we get a compound Du Val series with defining matrices (13-o) and odd $\zeta_{X''}$. With exactly the same argument as in Case 2.1.1, these matrices cannot serve as submatrices for other compound Du Val defining matrices.

*Case 2.2:* We have $k_0 \geq 1$. Here, the first inequality of (4.2.4.3) leads to

$$1 \leq k_0 \leq \frac{k_1}{k_1\zeta_{X''} - \delta - 1} \;\Rightarrow\; 0 \geq k_1(\zeta_{X''} - 1) - \delta - 1. \tag{4.2.4.4}$$

*Case 2.2.1:* We have $k_1 \geq 3$. Remembering $\delta < \zeta_{X''}$, we in total require $\delta < \zeta_{X''} \leq (\delta + 4)/3$ from the above inequality (4.2.4.4), leading to $1 = \delta < \zeta \leq 5/3$. This gives a contradiction, since $\zeta_{X''}$ is integral.

*Case 2.2.2:* We have $k_1 = 2$. The inequality (4.2.4.4) gives $\delta < \zeta_{X''} \leq (\delta + 3)/2$ here, leading to $\delta < 3$. While $\delta = 2$ leads to $\zeta_{X''} \leq 5/2$, which contradicts $\delta < \zeta_{X''}$, the case $\delta = 1$ allows $\zeta_{X''} = 2$. The first inequality of (4.2.4.3) can only be fulfilled for $\mathfrak{d}_2 = k_0 = 1$ here. Furthermore, $\mu = 1$ must hold and inserting everything in (4.2.4.1), we get a defining matrix

$$P'' = \begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 1 & 1 \\ 0 & \mathfrak{d}_{11} & 0 & 1 \\ -1 & 2 & 0 & 0 \end{bmatrix}.$$

Here in a first step, by admissible operations we can assume $\mathfrak{d}_{11} \in \{0, 1, 2\}$. In a second step, the vertices

$$\omega_{11} = \left(1, 0, \mathfrak{d}_{11}\frac{2}{3}, 1\right), \qquad \omega_{12} = \left(1, 0, \mathfrak{d}_{11}\frac{2}{3} + 1, 1\right)$$

of the line segment $L_{1,X''}$ are integer only for $\mathfrak{d}_{11} = 0$. Exactly the same holds for $L_{0,X''}$. So in this case, $P''$ itself gives the compound Du Val defining matrix (14). By the same arguments as in Case 2.1.1, these matrix cannot serve as submatrix for other compound Du Val defining matrices. □

We now provide the necessary input for establishing the defining equation in $\mathbb{C}^4$ of our compound Du Val singularities. Recall that the Cox ring $\mathcal{R}(X)$ of $X = X(A, P)$ is determined by the defining data, where generators and relations are read off directly and the degree matrix $Q$ of $\mathcal{R}(X)$, listing the generator deegres in $\mathrm{Cl}(X) = K$, needs to be computed.

PROPOSITION 4.2.5. *Consider $X = X(A, P)$ as in Case (9) of Proposition 4.2.4 with the defining matrix $P$ and the parameters therein. As indicated there, we have four subcases:*

$$\textit{(9a)} \quad P = \begin{bmatrix} k & \zeta - k & 0 & 0 \\ k & 0 & 1 & 1 \\ 0 & 0 & 0 & \mathfrak{d} \\ \frac{1-\mu k}{\zeta} & \frac{1-\mu k}{\zeta} + \mu & 0 & 0 \end{bmatrix}, \qquad \textit{(9b)} \quad P = \begin{bmatrix} -k & \zeta_X - k & 0 & 0 & \cdots & 0 & 0 \\ -k & 0 & 1 & 1 & & 0 & 0 \\ \vdots & \vdots & & & \ddots & & \\ -k & 0 & 0 & 0 & & 1 & 1 \\ 0 & 0 & 0 & \mathfrak{d}_2 & \cdots & 0 & \mathfrak{d}_r \\ \frac{1-\mu k}{\zeta_X} & \frac{1-\mu k}{\zeta_X} + \mu & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\textit{(9c)} \quad P = \begin{bmatrix} -k & \zeta_X - k & \zeta_X - k & 0 & 0 & \cdots & 0 & 0 \\ -k & 0 & 0 & 1 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & & & \ddots & & \\ -k & 0 & 0 & 0 & 0 & & 1 & 1 \\ 0 & 0 & \mathfrak{d}_1 & 0 & \mathfrak{d}_2 & \cdots & 0 & \mathfrak{d}_r \\ \frac{1-\mu k}{\zeta_X} & \frac{1-\mu k}{\zeta_X} + \mu & \frac{1-\mu k}{\zeta_X} + \mu & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\textit{(9d)} \quad P = \begin{bmatrix} -k & -k & \zeta_X - k & \zeta_X - k & 0 & 0 & \cdots & 0 & 0 \\ -k & -k & 0 & 0 & 1 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & & \\ -k & -k & 0 & 0 & 0 & 0 & & 1 & 1 \\ 0 & \mathfrak{d}_0 & 0 & \mathfrak{d}_1 & 0 & \mathfrak{d}_2 & \cdots & 0 & \mathfrak{d}_r \\ \frac{1-\mu k}{\zeta_X} & \frac{1-\mu k}{\zeta_X} & \frac{1-\mu k}{\zeta_X} + \mu & \frac{1-\mu k}{\zeta_X} + \mu & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

*According to these subcases, the divisor class group $\mathrm{Cl}(X)$ and the degree matrix $Q$ of the Cox ring $\mathcal{R}(X)$ are given as follows:*

*(9a)* *one has $\mathrm{Cl}(X) = \mathbb{Z}/\mathfrak{d}\mathbb{Z}$ and $Q = \begin{bmatrix} \overline{0} & \overline{0} & \overline{1} & -\overline{1} \end{bmatrix}$,*

*(9b)* *with $\mathfrak{d} := \gcd(\mathfrak{d}_2, \ldots, \mathfrak{d}_r)$ and integers $\alpha_i$ such that $\alpha_2\mathfrak{d}_2 + \ldots + \alpha_r\mathfrak{d}_r = \mathfrak{d}$ holds, one has $\mathrm{Cl}(X) = \mathbb{Z}^{r-2} \times \mathbb{Z}/\mathfrak{d}\mathbb{Z}$ and*

$$Q = \begin{bmatrix} 0 & 0 & -\mathfrak{d}_3 & \mathfrak{d}_3 & \mathfrak{d}_2 & -\mathfrak{d}_2 & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & & \\ & & -\mathfrak{d}_r & \mathfrak{d}_r & 0 & 0 & & \mathfrak{d}_2 & -\mathfrak{d}_2 \\ 0 & 0 & -\overline{\alpha_2} & \overline{\alpha_2} & -\overline{\alpha_3} & \overline{\alpha_3} & \cdots & -\overline{\alpha_r} & \overline{\alpha_r} \end{bmatrix},$$

(9c) *with* $\mathfrak{d} := \gcd(\mathfrak{d}_1, \ldots, \mathfrak{d}_r)$ *and integers* $\alpha_i$ *such that* $\alpha_1\mathfrak{d}_1 + \ldots + \alpha_r\mathfrak{d}_r = \mathfrak{d}$ *holds, one has* $\mathrm{Cl}(X) = \mathbb{Z}^{r-1} \times \mathbb{Z}/\mathfrak{d}\mathbb{Z}$ *and*

$$Q = \begin{bmatrix} 0 & \mathfrak{d}_2 & -\mathfrak{d}_2 & -\mathfrak{d}_1 & \mathfrak{d}_1 & & & & & \\ & & & -\mathfrak{d}_3 & \mathfrak{d}_3 & \mathfrak{d}_2 & -\mathfrak{d}_2 & & 0 & 0 \\ \vdots & & & \vdots & \vdots & & & \ddots & & \\ & & & -\mathfrak{d}_r & \mathfrak{d}_r & 0 & 0 & & \mathfrak{d}_2 & -\mathfrak{d}_2 \\ 0 & -\overline{\alpha_1} & \overline{\alpha_1} & -\overline{\alpha_2} & \overline{\alpha_2} & -\overline{\alpha_3} & \overline{\alpha_3} & \cdots & -\overline{\alpha_r} & \overline{\alpha_r} \end{bmatrix},$$

(9d) *with* $\mathfrak{d} := \gcd(\mathfrak{d}_0, \ldots, \mathfrak{d}_r)$ *and integers* $\alpha_i$ *such that* $\alpha_0\mathfrak{d}_0 + \ldots + \alpha_r\mathfrak{d}_r = \mathfrak{d}$ *holds, one has* $\mathrm{Cl}(X) = \mathbb{Z}^{r} \times \mathbb{Z}/\mathfrak{d}\mathbb{Z}$ *and*

$$Q = \begin{bmatrix} \mathfrak{d}_2 & -\mathfrak{d}_2 & 0 & 0 & -\mathfrak{d}_0 & \mathfrak{d}_0 & & & & & & \\ 0 & 0 & \mathfrak{d}_2 & -\mathfrak{d}_2 & -\mathfrak{d}_1 & \mathfrak{d}_1 & & & & & & \\ & & & & -\mathfrak{d}_3 & \mathfrak{d}_3 & \mathfrak{d}_2 & -\mathfrak{d}_2 & & 0 & 0 \\ & & & & \vdots & \vdots & & & \ddots & & \\ & & & & -\mathfrak{d}_r & \mathfrak{d}_r & 0 & 0 & & \mathfrak{d}_2 & -\mathfrak{d}_2 \\ -\overline{\alpha_0} & \overline{\alpha_0} & -\overline{\alpha_1} & \overline{\alpha_1} & -\overline{\alpha_2} & \overline{\alpha_2} & -\overline{\alpha_3} & \overline{\alpha_3} & \cdots & -\overline{\alpha_r} & \overline{\alpha_r} \end{bmatrix}.$$

PROOF. Let $P^*$ be the transpose of $P$. Then we have $\mathrm{Cl}(X) \cong \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$ and $Q\colon \mathbb{Z}^{n+m} \to \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$ is the projection, see Construction 1.2.2. To describe $\mathrm{Cl}(X)$ and $Q$ explicitly, choose unimodular matrices $V$ and $W$ such that $S := V \cdot P^* \cdot W$ is in Smith Normal Form, let $\beta_1, \ldots, \beta_\nu$ denote the elementary divisors and $\beta$ the number of zero rows of $S$. Then

$$\mathrm{Cl}(X) \;\cong\; \mathbb{Z}^\beta \oplus \mathbb{Z}/\beta_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/\beta_\nu\mathbb{Z}.$$

Moreover, the matrix $Q$ is basically the stack of the last $\nu + \beta$ rows of $V$. Now, we elaborate this explicitly for Case (9d). Set $\kappa_1 := (k\mu - 1)/\zeta_X - \mu$ and $\kappa_2 := \zeta_X - k$. Then

$$V := \begin{bmatrix} \kappa_1 & 0 & \frac{1-k\mu}{\zeta_X} & 0 & k\kappa_1 & 0 & k\kappa_1 & 0 & \cdots & k\kappa_1 & 0 \\ & & & & 1 & 0 & & & & & \\ & & & & & & 1 & 0 & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & 1 & 0 \\ \kappa_2 & 0 & k & 0 & k\kappa_2 & 0 & k\kappa_2 & 0 & \cdots & k\kappa_2 & 0 \\ \mathfrak{d}_2 & -\mathfrak{d}_2 & 0 & 0 & -\mathfrak{d}_0 & \mathfrak{d}_0 & & & & & \\ 0 & 0 & \mathfrak{d}_2 & -\mathfrak{d}_2 & -\mathfrak{d}_1 & \mathfrak{d}_1 & & & & & \\ & & & & -\mathfrak{d}_3 & \mathfrak{d}_3 & \mathfrak{d}_2 & -\mathfrak{d}_2 & & 0 & 0 \\ & & & & \vdots & \vdots & & & \ddots & & \\ & & & & -\mathfrak{d}_r & \mathfrak{d}_r & 0 & 0 & & \mathfrak{d}_2 & -\mathfrak{d}_2 \\ -\alpha_0 & \alpha_0 & -\alpha_1 & \alpha_1 & -\alpha_2 & \alpha_2 & -\alpha_3 & \alpha_3 & \cdots & -\alpha_r & \alpha_r \end{bmatrix}, \qquad W := \begin{bmatrix} E_r & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

are both unimodular matrices and turn the matrix $P^*$ into Smith Normal Form: we have

$$V \cdot P^* \cdot W = \begin{bmatrix} E_{r+1} & 0 \\ 0 & \mathfrak{d} \\ 0 & 0 \end{bmatrix}.$$

This proves the assertion for Case 9(d). The other cases run similarly. □

PROPOSITION 4.2.6. *Consider* $X = X(A, P)$ *with the defining matrix* $P$ *and the parameters therein as in Propositions 4.1.2, 4.2.1, 4.2.3 and 4.2.4, except Case 4.2.4 (9). Then the divisor class group* $\mathrm{Cl}(X)$ *and the degree matrix* $Q$ *of the Cox ring* $\mathcal{R}(X)$ *are given as follows.*

| $P$ | $\mathrm{Cl}(X)$ | $Q$ |
|---|---|---|
| *4.1.2 (1)* | $\mathbb{Z}/k\mathbb{Z}$ | $[\overline{1}\ \overline{0}\ \overline{k-1}]$ |

| | | |
|---|---|---|
| *4.1.2 (2)* | $\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$<br>$k := \gcd(k_1, k_2)$ | $\begin{bmatrix} k_2 & -k_1 & -k_2 & k_1 \\ -\overline{\alpha_1} & -\overline{\alpha_2} & \overline{\alpha_1} & \overline{\alpha_2} \end{bmatrix}$<br>$\alpha_1 k_1 + \alpha_2 k_2 = k$ |
| *4.1.2 (3)* | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} \bar{1} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ |
| *4.2.1 (4)*<br>*k even* | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$ |
| *4.2.1 (4)*<br>*k odd* | $\mathbb{Z}/4\mathbb{Z}$ | $\begin{bmatrix} \bar{2} & \bar{1} & \bar{3} & \bar{0} \end{bmatrix}$ |
| *4.2.1 (5-e)* | $\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} \bar{0} & \overline{k+1} & \bar{k} & \bar{1} \end{bmatrix}$ |
| *4.2.1 (5-o)* | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} \bar{0} & \bar{1} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$ |
| *4.2.1 (6)* | $\mathbb{Z}/3\mathbb{Z}$ | $\begin{bmatrix} \bar{1} & \bar{2} & \bar{0} & \bar{0} \end{bmatrix}$ |
| *4.2.1 (7)* | $\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$ |
| *4.2.1 (8)* | $\{0\}$ | — |
| *4.2.3 (10-e)* | $\mathbb{Z}$ | $\begin{bmatrix} 0 & 1 & -2 & -1 & 2 \end{bmatrix}$ |
| *4.2.1 (10-o)* | $\{0\}$ | — |
| *4.2.1 (11)* | $\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$ |
| *4.2.1 (12-e-e)*<br>*$k_1$ even* | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$ |
| *4.2.1 (12-e-e)*<br>*$k_1$ odd* | $\mathbb{Z}/4\mathbb{Z}$ | $\begin{bmatrix} \bar{2} & \bar{0} & \bar{3} & \bar{1} \end{bmatrix}$ |
| *4.2.1 (12-o-e/o)* | $\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$ |
| *4.2.4 (13-e)* | $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} 0 & 1 & -1 & -1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$ |
| *4.2.4 (13-o)* | $\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$ |
| *4.2.4 (14)* | $\mathbb{Z}/3\mathbb{Z}$ | $\begin{bmatrix} \bar{1} & \bar{2} & \bar{0} & \bar{0} \end{bmatrix}$ |
| *4.2.1 (15, 17, 18)* | $\{0\}$ | — |
| *4.2.1 (16)* | $\mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$ |

PROOF. The arguing is the same as for Proposition 4.2.5. Note that the cases without parameters can easily be done by computer, e.g. using [**51**]. □

PROOF OF THEOREM 9. Propositions 4.1.2, 4.2.1, 4.2.3 and 4.2.4 provide us with the defining matrices $P$ of the compound Du Val threefold singularities $X$ of complexity one. This gives in particular their Cox rings $\mathcal{R}(X) = R(A,P)$. The grading of the Cox ring by $\mathrm{Cl}(X) = K$ is given by the degree matrices $Q$ provided in Propositions 4.2.5 and 4.2.6. We have $X = \operatorname{Spec} R(A,P)_0$ and will obtain the describing equation for $X \subseteq \mathbb{C}^4$ from a suitable presentation of the degree zero part $R(A,P)_0$ of $R(A,P)$ by generators and relations.

We exemplarily carry this procedure out for the case (9d) from Proposition 4.2.5. A glance at the degree matrix $Q$ given in Proposition 4.2.5 (9d) shows that the following monomials are of $K$-degree zero:

$$x_i \ := \ T_{i1}T_{i2}, \ i = 0, \ldots, r, \qquad x_{r+1} \ := \ T_{01}^{\mathfrak{d}_0} \cdots T_{r1}^{\mathfrak{d}_r}, \qquad x_{r+2} \ := \ T_{02}^{\mathfrak{d}_0} \cdots T_{r2}^{\mathfrak{d}_r}.$$

Obviously, any monomial $h$ in the $T_{ij}$ is a product of powers of $x_0, \ldots, x_{r+2}$ and a monomial $h'$ depending of at most one variable $T_{ij}$ per $i$ and at most on $r-1$ variables in total. By the shape of $Q$, such a monomial $h'$ is of degree zero if and only if it is constant. We conclude that $x_0, \ldots, x_{r+2}$ generate $R(A,P)_0$. Now consider the morphism

$$\pi\colon \mathbb{C}^n \ \to\ \mathbb{C}^{r+3}, \qquad z \ \mapsto\ (x_0(z), \ldots, x_{r+2}(z)).$$

Then $X = \operatorname{Spec} R(A,P)_0$ is the image of $\overline{X} = \operatorname{Spec} R(A,P)$ under $\pi$. We claim that $X = \pi(\overline{X}) \subseteq \mathbb{C}^{r+3}$ is contained in the zero set of the polynomials

$$\begin{gathered} x_{r+1}x_{r+2} - x_0^{\mathfrak{d}_0} \cdots x_r^{\mathfrak{d}_r}, \\ x_0^k + x_1^{\zeta-k} + x_2, \quad x_1^{\zeta-k} + 2x_2 + x_3, \\ x_2 + 3x_3 + x_4, \quad \ldots, \quad x_{r-2} + (r-1)x_{r-1} + x_r. \end{gathered}$$

Indeed, the first polynomial is an obvious relation between the $x_i$ and the remaining ones pull back via $\pi$ to the Cox ring relations given by the matrix $P$. The above relations allow elimination of variables $x_3, \ldots, x_r$: starting with the last relation, we successively plug these into the first one and arrive at

$$x_{r+1}x_{r+2} - x_0^{\mathfrak{d}_0}x_1^{\mathfrak{d}_1}\prod_{i=2}^{r}(a_ix_0^k + b_ix_1^{\zeta-k})^{\mathfrak{d}_i},$$

where we can by a suitable coordinate change achieve that $a_i = i-1$ and $b_i = 2i-3$ hold for all $i = 2, \ldots, r$. Thus, $X$ can be realized as a closed subset inside the hypersurface $X' \subseteq \mathbb{C}^4$ defined by the above polynomial. As the latter is irreducible, we conclude $X = X'$, which proves the assertion in case (9d).

For the other non-factorial compound Du Val singularities, one argues analogously. In the cases (9a), (9b) and (9c), we obtain the following invariants $x_i$ and relations among them:

| No. | Invariant monomials $x_0, \ldots, x_\nu$ | Relations among $x_i$ |
|---|---|---|
| 9a | $T_{21}^{\mathfrak{d}}, T_{22}^{\mathfrak{d}}, T_{21}T_{22}, T_{01}, T_{11}$ | $x_0x_1 - x_2^{\mathfrak{d}}$<br>$x_3^k + x_4^{\zeta-k} + x_2$ |
| 9b | $T_{01}, T_{11}, T_{21}T_{22}, \ldots, T_{r1}T_{r2}$,<br>$T_{21}^{\mathfrak{d}_2} \cdots T_{r1}^{\mathfrak{d}_r}, T_{22}^{\mathfrak{d}_2} \cdots T_{r2}^{\mathfrak{d}_r}$ | $x_{r+1}x_{r+2} - x_2^{\mathfrak{d}_2} \cdots x_r^{\mathfrak{d}_r}$<br>$x_0^k + x_1^{\zeta-k} + x_2$<br>$\vdots$<br>$x_{r-2} + (r-1)x_{r-1} + x_r$ |
| 9c | $T_{01}, T_{11}T_{12}, T_{21}T_{22}, \ldots, T_{r1}T_{r2}$,<br>$T_{11}^{\mathfrak{d}_1} \cdots T_{r1}^{\mathfrak{d}_r}, T_{12}^{\mathfrak{d}_1} \cdots T_{r2}^{\mathfrak{d}_r}$ | $x_{r+1}x_{r+2} - x_1^{\mathfrak{d}_1} \cdots x_r^{\mathfrak{d}_r}$<br>$x_0^k + x_1^{\zeta-k} + x_2$<br>$\vdots$<br>$x_{r-2} + (r-1)x_{r-1} + x_r$ |

For the cases different from 9, we list the relevant data in the following table; note that the cases without parameters can be settled by computer, e.g., using [**51**]:

| No. | Relations in $\mathcal{R}(X)$ | Invariant monomials $x_1, \ldots, x_\nu$ | Relations among $x_i$ |
|---|---|---|---|
| 1 | | $T_1^k, T_3^k, T_1T_3, T_2$ | $x_1x_2 - x_3^k$ |

| | | | |
|---|---|---|---|
| 2 | | $T_1^{k_1}T_2^{k_2}, T_3^{k_1}T_4^{k_2}$, $T_1T_3, T_2T_4$ | $x_1x_2 - x_3^{k_1}x_4^{k_2}$ |
| 3 | | $T_1T_2T_3, T_1^2, T_2^2, T_3^2$ | $x_1^2 - x_2x_3x_4$ |
| 4 | $T_1^k + T_2^2 + T_3^2$ <br> $k$ even | $T_1T_2T_3, T_1^2, T_2^2, T_3^2, T_4$ | $x_1^2 - x_2x_3x_4$ <br> $x_2^{k/2} + x_3 + x_4$ |
| 4 | $T_1^k + T_2^2 + T_3^2$ <br> $k$ odd | $T_2^4, T_3^4, T_1T_2^2, T_1T_3^2$, $T_2T_3, T_1^2, T_4$ | $x_1x_2 - x_5^4$ <br> $x_6^{\frac{k-1}{2}}x_3 + x_2 + x_5^2$ <br> $x_6^{\frac{k-1}{2}}x_4 + x_1 + x_4^2$ <br> $x_6^{\frac{k+1}{2}} + x_3 + x_4$ |
| 5-e | $T_1^{2k+1} + T_2^2 + T_3^2$ | $T_4^2, T_2^2, T_2T_4, T_1, T_3$ | $x_1x_2 - x_3^2$ <br> $x_4^{2k+1} + x_5^2 + x_2$ |
| 5-o | $T_1^k + T_2^2 + T_3^2$ | $T_2T_3T_4, T_2^2, T_3^2, T_4^2, T_1$ | $x_1^2 - x_2x_3x_4$ <br> $x_5^k + x_2 + x_3$ |
| 6 | $T_1^3 + T_2^3 + T_3^2$ | $T_1^3, T_2^3, T_1T_2, T_3, T_4$ | $x_1x_2 - x_3^3$ <br> $x_1 + x_2 + x_4^2$ |
| 7 | $T_1^4 + T_2^3 + T_3^2$ | $T_1^2, T_3^2, T_1T_3, T_2, T_4$ | $x_1x_2 - x_3^2$ <br> $x_1^2 + x_4^3 + x_2$ |
| 10-e | $T_1^k + T_2^2T_3 + T_4^2T_5$ | $T_2^2T_3, T_4^2T_5$, $T_2T_4, T_3T_5, T_1$ | $x_1x_2 - x_3^2x_4$ <br> $x_5^k + x_1 + x_2$ |
| 11 | $T_1^k + T_2^2T_3 + T_4^2$ | $T_2^2, T_4^2, T_2T_4, T_1, T_3$ | $x_1x_2 - x_3^2$ <br> $x_4^k + x_1x_5 + x_2$ |
| 12-e-e | $T_1^{k_1}T_2^{k_2} + T_3^2 + T_4^2$ <br> $k_1$ even | $T_1T_3T_4, T_1^2, T_3^2, T_4^2, T_2$ | $x_1^2 - x_2x_3x_4$ <br> $x_2^{\frac{k_1}{2}}x_5^{k_2} + x_3 + x_4$ |
| 12-e-e | $T_1^{k_1}T_2^{k_2} + T_3^2 + T_4^2$ <br> $k_1$ odd | $T_3^4, T_4^4, T_3^2T_1, T_4^2T_1$, $T_3T_4, T_1^2, T_2$ | $x_1x_2 - x_5^4$ <br> $x_6^{\frac{k-1}{2}}x_7^{k_2}x_3 + x_2 + x_5^2$ <br> $x_6^{\frac{k-1}{2}}x_7^{k_2}x_4 + x_1 + x_4^2$ <br> $x_6^{\frac{k+1}{2}}x_7^{k_2} + x_3 + x_4$ |
| 12-o-e/o | $T_1^{2k_1}T_2^{2k_2+1} + T_3^2 + T_4^2$ | $T_1^2, T_4^2, T_1T_4, T_2, T_3$ | $x_1x_2 - x_3^2$ <br> $x_1^{k_1}x_4^{2k_2+1} + x_5^2 + x_2$ |
| 13-e | $T_1^{2\zeta-1} + T_2T_3 + T_4T_5$ | $T_2^2T_3^2, T_4^2T_5^2, T_1T_2T_3$, $T_1T_4T_5, T_1^2, T_2T_4, T_3T_5$ | $x_3x_4 - x_5x_6x_7$ <br> $x_1x_2 - x_6^2x_7^2$ <br> $x_3^2 - x_5x_1$ <br> $x_4^2 - x_5x_2$ <br> $x_5^\zeta + x_3 + x_4$ <br> $x_5^{\zeta-1}x_3 + x_1 + x_6x_7$ <br> $x_5^{\zeta-1}x_4 + x_2 + x_6x_7$ |
| 13-o | $T_1^{2\zeta-2} + T_2^2 + T_3T_4$ | $T_1^2, T_2^2, T_1T_2, T_3, T_4$ | $x_1x_2 - x_3^2$ <br> $x_1^{\zeta-1} + x_2 + x_4x_5$ |

| | | | |
|---|---|---|---|
| 14 | $T_1^3 + T_2^3 + T_3T_4$ | $T_1^3, T_2^3, T_1T_2, T_3, T_4$ | $x_1x_2 - x_3^3$ <br> $x_4x_5 + x_1 + x_2$ |
| 16 | $T_1^4 + T_2^2T_3 + T_4^2$ | $T_1^2, T_4^2, T_1T_4, T_2, T_3$ | $x_1x_2 - x_3^2$ <br> $x_1^2 + x_4^2x_5 + x_2$ |

□

## 4.3. $k$-empty and canonical polytopes

**4.3.1. $k$-empty polytopes.** In the following we give a description of a standard form of $k$-empty lattice triangles. Then, Farey sequences are used to classify such triangles. We set $\Delta := \{(x, y) \in \mathbb{Z}^2;\ 0 \le y < x\}$. Members of $k\mathbb{Z}^n$ for $k \in \mathbb{Z}_{\ge 1}$ are called *$k$-fold lattice points*.

Definition 4.3.1. Let $n, k \in \mathbb{Z}_{\ge 1}$ and consider a convex rational polytope $\mathsf{P} \subseteq \mathbb{Q}^n$. The set of vertices of $\mathsf{P}$ is denoted by $\mathcal{V}(\mathsf{P})$, the relative interior by $\mathsf{P}^\circ$ and the boundary by $\partial\mathsf{P}$. We call $\mathsf{P}$

(i) a *lattice polytope*, if $\mathcal{V}(\mathsf{P}) \subseteq \mathbb{Z}^n$.
(ii) a *lattice polygon*, if $\mathsf{P}$ is a lattice polytope and $n = 2$.
(iii) *$k$-empty*, if $\mathsf{P} \cap k\mathbb{Z}^n \subseteq \mathcal{V}(\mathsf{P})$.

Definition 4.3.2. The group $\mathrm{Aff}_k^n(\mathbb{Z})$ of *$k$-affine unimodular transformations in* $\mathbb{Q}^n$ is defined by

$$\mathrm{Aff}_k^n(\mathbb{Z}) := \{T \colon \mathbb{Q}^n \to \mathbb{Q}^n;\ T(v) = Av + w,\ A \in \mathrm{GL}_n(\mathbb{Z}), w \in k\mathbb{Z}^n\}.$$

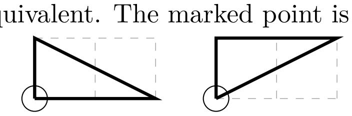
It naturally acts on the set of lattice polytopes in $\mathbb{Q}^n$. Lattice polytopes $\mathsf{P}_1$ and $\mathsf{P}_2$ are called *$k$-equivalent*, if $\mathsf{P}_2 \in \mathrm{Aff}_k(\mathbb{Z}) \cdot \mathsf{P}_1$. Additionally, we call 1-equivalent polytopes *lattice equivalent*.

Remark 4.3.3. Let $T \in \mathrm{Aff}_k^n(\mathbb{Z})$ and $\mathsf{P}$ be a lattice polytope. Then the following hold.

(i) $T(\mathbb{Z}^n) = \mathbb{Z}^n$.
(ii) $T(k\mathbb{Z}^n) = k\mathbb{Z}^n$.
(iii) $T(\mathcal{V}(\mathsf{P})) = \mathcal{V}(T(\mathsf{P}))$.
(iv) $T(\partial\mathsf{P}) = \partial T(\mathsf{P})$.
(v) $T(\mathsf{P}^\circ) = T(\mathsf{P})^\circ$.
(vi) $\mathrm{vol}(\mathsf{P}) = \mathrm{vol}(T(\mathsf{P}))$.

Therefore, the number of vertices, the number of (interior) lattice points and the number of (interior) $k$-fold lattice points are invariant under the action of $\mathrm{Aff}_k^n(\mathbb{Z})$.

Note that $k$-equivalence of lattice polytopes indeed depends on the specific value of $k$. Consider for example the following pair of lattice polygons. They are 1-equivalent but not 2- equivalent. The marked point is the origin.



Definition 4.3.4. Let $k \in \mathbb{Z}_{\ge 1}$ and $\mathsf{P}$ be a lattice polygon. We set

$$a_{\mathsf{P}} := \min\{\text{number of lattice points in the relative interior of } \mathsf{E}\} + 1$$

where $\mathsf{E}$ runs through the edges of $\mathsf{P}$ which have a vertex in $k\mathbb{Z}^2$.

DEFINITION 4.3.5 (Standard form of $k$-empty lattice triangles). Let $k \in \mathbb{Z}_{\geq 1}$ and $\mathsf{S}$ be a $k$-empty lattice polygon with exactly three vertices such that one of them is in $k\mathbb{Z}^2$. We refer to $\mathsf{S}$ as in *standard form*, if the following conditions are satisfied.

(i) $\mathsf{S}$ has the vertices $(0,0), (0, a_\mathsf{S})$ and $(x,y)$ where $(x,y) \in \Delta$.
(ii) If $(x,y) \notin k\mathbb{Z}^2$ and $\gcd(x,y) = a_\mathsf{S}$ then for each $z = 1, \ldots, y-1$ we have $a_\mathsf{S} \nmid z$ or $a_\mathsf{S} x \nmid a_\mathsf{S}^2 - zy$.
(iii) If $(x,y) \in k\mathbb{Z}^2$ and $\gcd(x,y) = a_\mathsf{S}$ then for each $z = 1, \ldots, y-1$ we have $a_\mathsf{S} \nmid z$ or $a_\mathsf{S} x \nmid a_\mathsf{S}(z+y) - zy$.

If $\mathsf{S}$ is in standard form, we write $\mathsf{S} = \Delta(a_\mathsf{S}, x, y)$. The simplex $\mathsf{S}$ is called *minimal*, if $a_\mathsf{S} = 1$.

REMARK 4.3.6. The last two conditions of Definition 4.3.5 ensure that the second coordinate of the vertex $(x,y) \in \Delta$ is minimal. To illustrate this, consider the 2-equivalent 2-empty polytopes

$$\begin{aligned} \mathsf{P}_1 &= \operatorname{conv}((0,0),(0,1),(5,3)), \\ \mathsf{P}_2 &= \operatorname{conv}((0,0),(0,1),(5,2)). \end{aligned}$$

We can see that $\mathsf{P}_1$ does not fulfill condition (ii) so it is not in standard form whereas $\mathsf{P}_2$ is. Therefore, these conditions make sure that the *right* vertex lies on the vertical axis. This is relevant in case that there are several edges which attain the minimum of Definition 4.3.4.

PROPOSITION 4.3.7. *Let $\mathsf{S}$ be a $k$-empty lattice triangle with a vertex $z \in k\mathbb{Z}^2$. Then there is a unique lattice triangle $\mathsf{S}'$ in standard form that is $k$-equivalent to $\mathsf{S}$.*

PROOF. There are three cases depending on the number of vertices of $\mathsf{S}$ in $k\mathbb{Z}^2$.
*Case* 1*: There is exactly one vertex $z \in k\mathbb{Z}^2$.* Let $v_1$ and $v_2$ be the other two vertices and $d_i$ the number of lattice points in the relative interior of the edge of $\mathsf{S}$ with vertices $z$ and $v_i$. We have $a_\mathsf{S} = \min\{d_1, d_2\} + 1$.
*Case* 1.1*: $d_1 \neq d_2$.* We can assume, without loss of generality, that $d_1 < d_2$. So we have $a_\mathsf{S} = d_1 + 1$. Consider the $k$-affine unimodular transformation $T_1$ given by $T_1(v) = v - z$. The coordinates of $T_1(v_1)$ have the greatest common divisor $a_\mathsf{S}$. Thus, there is a $k$-affine unimodular transformation $T_2$ that leaves the origin fixed and takes $T_1(v_1)$ to $(0, a_\mathsf{S})$. A third transformation $T_3$ sends $T_2(T_1(v_2))$ to a point $(x,y) \in \Delta$ without changing the coordinates of $T_2(T_1(v_1))$ and $T_2(T_1(z))$. The $k$-empty lattice triangle $\mathsf{S}' := T_3 \circ T_2 \circ T_1(S)$ satisfies the conditions of Definition 4.3.5. Note that $\gcd(x,y) \neq a_\mathsf{S}$ since $d_1 \neq d_2$. The representation is obviously unique in this case.
*Case* 1.2*: $d_1 = d_2$.* As before, let $T_1$ be given by $T_1(v) = v - z$. Then, let $T_2$ be the $k$-affine unimodular transformation that leaves the origin fixed and sends $T_1(v_1)$ to $(0, a_\mathsf{S})$. We choose a transformation $T_3$ that takes $T_2(T_1(v_2))$ to a point $(x,y) \in \Delta$ without changing the coordinates of $(0, a_\mathsf{S})$ and $(0,0)$. Analogously, let $T_2'$ be the $k$-affine unimodular transformation that leaves the origin fixed and sends $T_1(v_2)$ to $(0, a_\mathsf{S})$. We choose a transformation $T_3'$ that takes $T_2'(T_1(v_1))$ to a point $(x,y') \in \Delta$ without changing the coordinates of $(0, a_\mathsf{S})$ and $(0,0)$. Without loss of generality we have $y < y'$. Set $\mathsf{S}' := T_3 \circ T_2 \circ T_1(S)$.

The lattice triangle $\mathsf{S}'$ fulfills condition (i) of Definition 4.3.5. Suppose that it does not satisfy condition (ii). That means there is a $z$ with $1 \leq z \leq y-1$ such that $a_{\mathsf{S}}|z$ and $a_{\mathsf{S}}x|a_{\mathsf{S}}^2 - zy$. Consider the transformation $T$ given by the matrix

$$A = \begin{pmatrix} -\frac{y}{a_{\mathsf{S}}} & \frac{x}{a_{\mathsf{S}}} \\ \frac{a_{\mathsf{S}}^2 - zy}{a_{\mathsf{S}}x} & \frac{z}{a_{\mathsf{S}}} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$$

and apply it to $\mathsf{S}'$. Since $(x,z) \in \Delta$ is a vertex of of $T(\mathsf{S}')$, we have $z = y'$. So $y' = z < y < y'$ which is a contradiction. Thus $\mathsf{S}'$ is in standard form and by construction unique.

*Case* 2*: There are exactly two vertices $z_1, z_2 \in k\mathbb{Z}^2$.* Let $v$ be the third vertex and $d_i$ the number of lattice points in the relative interior of the edge of $\mathsf{S}$ with vertices $z_i$ and $v$. As in Case 1 we have $a_{\mathsf{S}} = \min\{d_1, d_2\} + 1$.

*Case* 2.1*: $d_1 \neq d_2$.* Without loss of generality $d_1 < d_2$. So we have $a_{\mathsf{S}} = d_1 + 1$. Consider the $k$-affine unimodular transformation $T_1$ which is given by $T_1(v') = v' - z_1$. Let $T_2$ be such that $T_2(0) = 0$ and $T_2(v - z_1) = (0, a_{\mathsf{S}})$. Then, choose a third transformation $T_3$ which fixes $(0,0$ and $(0, a_{\mathsf{S}})$ and sends $T_2(T_1(z_2))$ to a point $(x,y) \in \Delta$. The lattice triangle $\mathsf{S}' := T_3 \circ T_2 \circ T_1(S)$ satisfies the conditions of Definition 4.3.5. As before, note that $\gcd(x,y) \neq a_{\mathsf{S}}$ since $d_1 \neq d_2$. The representation is unique.

*Case* 2.2*: $d_1 = d_2$.* Let $T_1$ be given by $T_1(v') = v' - z_1$ and let $T_2$ be the transformation fixing the origin such that $T_2(T_1(v)) = (0, a_{\mathsf{S}})$. Choose additionally $T_3$ such that the origin and $(0, a_{\mathsf{S}})$ are fixed and $(x,y) := T_3(T_2(T_1(z_2))) \in \Delta$. Accordingly, let $T_1'$ be defined by $T_1'(v') = v' - z_2$ and $T_2'$ fixing $(0,0)$ and $T_2'(T_1'(v)) = (0, a_{\mathsf{S}})$. Finally, choose a transformation $T_3'$ which fixes $(0,0)$ and $(0, a_{\mathsf{S}})$ and $(x, y') := T_3'(T_2'(T_1'(z_2))) \in \Delta$. Again, without loss of generality, we have $y < y'$. We set $\mathsf{S}' := T_3 \circ T_2 \circ T_1(S)$.

Assume that Condition (iii) of Definition 4.3.5 is not satisfied. Then there is a $z$ with $1 \leq z \leq y - 1$ such that $a_{\mathsf{S}}|z$ and $a_{\mathsf{S}}x|a_{\mathsf{S}}(z+y) - zy$. Consider the transformation $T$ given by the matrix

$$A = \begin{pmatrix} 1 - \frac{y}{a_{\mathsf{S}}} & \frac{x}{a_{\mathsf{S}}} \\ \frac{a_{\mathsf{S}}(z+y) - zy}{a_{\mathsf{S}}x} & \frac{z}{a_{\mathsf{S}}} - 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$$

and apply it to the lattice triangle

$$\mathrm{conv}\left((0,0), (x, y - a_{\mathsf{S}})), (x,y)\right)$$

which is $k$-equivalent to $\mathsf{S}'$. The lattice triangle $T(\mathsf{S}')$ has the vertex $(x,z)$ and so we have $z = y'$. That means $y' = z < y < y'$ which is a contradiction. Therefore $\mathsf{S}'$ is in standard form and the uniqueness is clear by construction.

*Case* 3*: There are three vertices $z_1, z_2, z_3 \in k\mathbb{Z}^2$.* Since the number of lattice points in the relative interior of each edge of $\mathsf{S}$ is $k-1$, the only possible standard form is the lattice triangle $\mathrm{conv}\left((0,0), (0,k), (k,0)\right)$. □

Definition 4.3.8. Let $k \in \mathbb{Z}_{\geq 1}$. We define the *$k$-th Farey sequence* to be

$$F_k := \left(\frac{f_1}{f_2};\ f_1, f_2 \in \mathbb{Z},\ 0 \leq f_1 < f_2 \leq k,\ \gcd(f_1, f_2) = 1\right).$$

Members of $F_k$ are called *$k$-th Farey numbers*.

Let $f = \frac{f_1}{f_2}$ be a $k$-th Farey number. We define the *$k$-th Farey strip* corresponding to $f$ to be the polyhedron

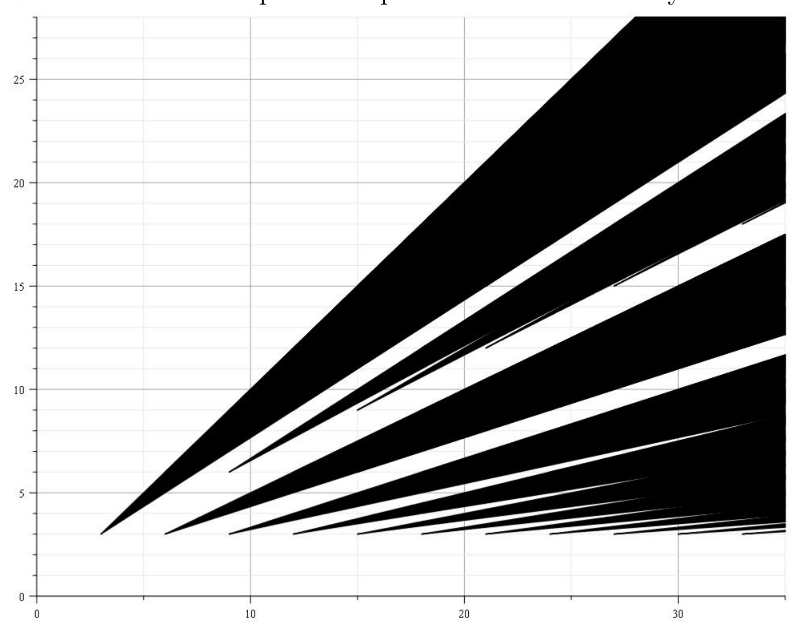
$$F_{k,f} \;:=\; \begin{cases} \left\{ \begin{pmatrix} x \\ y \end{pmatrix};\ 0 \;<\; \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} -f_1 \\ f_2 \end{pmatrix} \;<\; k \right\}, & \text{if } f_2 = k, \\ \left\{ \begin{pmatrix} x \\ y \end{pmatrix};\ 0 \;<\; \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} -f_1 \\ f_2 \end{pmatrix} \;\leq\; k \right\}, & \text{if } f_2 \neq k. \end{cases}$$

REMARK 4.3.9. By definition, the number of $k$-th Farey strips equals the length of the $k$-th Farey sequence. That is, there are $\varphi(1) + \cdots + \varphi(k)$ many $k$-th Farey strips where $\varphi$ is the Euler totient function.

DEFINITION 4.3.10. A *spike* attached to a $k$-th Farey strip $F_{k,f}$ is a 2-dimensional convex polytope $\mathsf{S}$ with exactly three rational vertices satisfying the following conditions.

(i) Two of the vertices of $\mathsf{S}$ are on the line $y = \frac{f_1}{f_2}x + \frac{f_2}{k}$.
(ii) One vertex of $\mathsf{S}$ is above the line $y = \frac{f_1}{f_2}x + \frac{f_2}{k}$.
(iii) If $(x, y)$ is a lattice point in $\mathsf{S}$, then $\operatorname{conv}((0,0),(0,1),(x,y))$ is $k$-empty.

The following picture shows the $k$-th Farey strips and spikes attached to them for $k = 3$. There are four strips where spikes are attached to only two of them.




PROPOSITION 4.3.11. *Let $k \in \mathbb{Z}_{\geq 1}$ and $f = \frac{f_1}{f_2}$ be a $k$-th Farey number. If the lattice triangle $\mathsf{S} = \operatorname{conv}((0,0),(0,1),(x,y))$ is contained in the $k$- th Farey strip $F_{k,f}$, it is $k$-empty.*

PROOF. Assume that $\mathsf{S}$ is not $k$-empty. Then we find $a, b \in \mathbb{Z}_{\geq 1}$ such that $(ka, kb) \in \mathsf{S} \subseteq F_{k,f}$. By definition we have $0 < f_2kb - f_1ka \leq k$ and therefore $0 < f_2b - f_1a \leq 1$. So $f_2b - f_1a = 1$. This means that $(ka, kb)$ cannot lie in the interior of $F_{k,f}$. If $f_2 = k$, this is already a contradiction. If on the other hand $f_2 \neq k$, the point $(ka, kb)$ must be a vertex of $\mathsf{S}$ which is again a contradiction. $\square$

Definition 4.3.12. Let $k \in \mathbb{Z}_{\geq 1}$. A $k$-empty lattice triangle in standard form which is not contained in a $k$-th Farey strip is called *sporadic*.

Proposition 4.3.13. *Let $k \in \mathbb{Z}_{\geq 1}$. The number of minimal sporadic $k$-empty lattice triangles in standard form is finite. The first coordinate of the vertex in $\Delta$ is bounded by $(k^2-1)k-1$.*

Proof. Let $(x,y) \in \mathbb{Z}^2$ such that $\operatorname{conv}((0,0,),(0,1),(x,y))$ is a $k$-empty triangle in normal form which is not completely contained in a $k$-th Farey strip. Then there is a $k$-th Farey strip $F_{k,f}$ and a spike $\mathsf{S}$ attached to it, such that $(x,y)$ is in the interior of it.

We have $f_2 \neq k$. Otherwise $\mathsf{S}$ is empty. For some $i > k - f_2$ the spike has the vertices

$$\left(ik, \frac{f_1}{f_2}ik + \frac{k}{f_2}\right), \; \left((i+f_2)k, \frac{f_1}{f_2}(i+f_2)k + \frac{k}{f_2}\right),$$
$$\left(\frac{(i+f_2)ik}{i-k+f_2}, \frac{\frac{f_1}{f_2}(i+f_2)ik + i\frac{k}{f_2}}{i-k+f_2}\right).$$

Its area is therefore

$$A(\mathsf{S}) \; := \; \frac{1}{2}\left(\frac{ik^2}{i-k+f_2} - k^2\right).$$

Let $I(\mathsf{S})$ be the number of interior integral points of $\mathsf{S}$ and $B(\mathsf{S})$ the number of integral points on the boundary of $\mathsf{S}$. By Pick's theorem we have

$$A(\mathsf{S}) = I(\mathsf{S}) + \frac{B(\mathsf{S})}{2} - 1.$$

Furthermore, it is $A(\mathsf{S}) < 1$ if and only if

$$i \; > \; \frac{k^3 - f_2k^2 + 2k - 2f_2}{2}.$$

Since $B(\mathsf{S}) \geq 1$, if $A(\mathsf{S}) < 1$ then $I(\mathsf{S}) < 1$. So $x$ is bounded from above by $(k^2-1)k-1$. □

Corollary 4.3.14. *Let $k \in \mathbb{Z}_{\geq 1}$. The number of $k$-empty lattice triangles in standard form which are not contained in a $k$-th Farey strip is finite.*

Proof. Let $\mathsf{S} = \Delta(a,x,y)$ be a $k$-empty lattice triangle in standard form. Then, there is a minimal $k$-empty lattice triangle $\mathsf{S}' = \Delta(1,x,y)$ contained in $\mathsf{S}$. □

Remark 4.3.15. The proof of Proposition 4.3.13 shows that we can list the sporadic minimal $k$-empty lattice triangles in standard form explicitly for a given $k \in \mathbb{Z}_{\geq 1}$. The following table lists the number of those simplices for low values of $k$.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| # | 0 | 2 | 7 | 32 | 96 | 279 |

### 4.3.2. Canonical polytopes.

DEFINITION 4.3.16. Let $d \in \mathbb{Z}_{\geq 2}, r \in \mathbb{Z}_{\geq d}$ and $v_0 := 0, v_1, \ldots, v_r \in \mathbb{Q}^d$ be pairwise different so that the polytope $\mathsf{P} := \mathrm{conv}(v_0, \ldots, v_r)$ is full-dimensional and the $v_i$ are the vertices of $\mathsf{P}$. We say that $\mathsf{P}$ is *$\mathbb{Q}$-Gorenstein of index $\imath \in \mathbb{Z}_{\geq 1}$* if there exist $\alpha_1, \ldots, \alpha_d \in \mathbb{Z}$ with

$$\sum_{j=1}^{d} \alpha_j v_{ij} = \imath$$

for all $i = 1, \ldots, r$ and $\imath$ is minimal with that property.

For a $\mathbb{Q}$-Gorenstein polytope $\mathsf{P} = \mathrm{conv}(v_0, \ldots, v_r)$ of index $\imath$ and $\alpha_1, \ldots, \alpha_d \in \mathbb{Z}$ with $\sum_{j=1}^{d} \alpha_j v_{ij} = \imath$, we say that $\mathsf{P}$ is *canonical* (*terminal*) if for each lattice point $0 \neq p \in \mathsf{P}$ we have $\sum_{j=1}^{d} \alpha_j p_j = \imath$ (if $\mathsf{P} \cap \mathbb{Z}^d \subseteq \{v_0, \ldots, v_r\}$).

REMARK 4.3.17. Let $v_1, \ldots, v_r \in \mathbb{Z}^d$ be primitive and $X$ be the affine toric variety defined by the full-dimensional cone $\mathrm{cone}(v_1, \ldots, v_r)$. Then $X$ is $\mathbb{Q}$-Gorenstein if and only if the polytope $\mathsf{P} := \mathrm{conv}(0, v_1, \ldots, v_r)$ is $\mathbb{Q}$-Gorenstein and in this case it is canonical (terminal) if and only if $\mathsf{P}$ is canonical (terminal).

It is well known that the only canonical lattice polygons are those of index one. From this, we can deduce the following statement for canonical polytopes with rational vertices.

LEMMA 4.3.18. *Let $\mathsf{P} \subseteq \mathbb{Q}^2$ be $\mathbb{Q}$-Gorenstein of index $\imath \geq 2$. Then if $\mathsf{P}$ contains two lattice points different from the origin, it can not be canonical.*

PROOF. Let $\mathsf{P} := \mathrm{conv}(v_0, \ldots, v_r)$ contain the two lattice points $p_1, p_2$ and choose $\alpha_1, \ldots, \alpha_d \in \mathbb{Z}$ satisfying $\sum_{j=1}^{d} \alpha_j v_{ij} = \imath$. Then if $\sum \alpha_j p_{ij} = \imath$ for $i = 1, 2$, the polygon $\mathsf{P}$ contains the lattice polygon $\mathsf{P}' = \mathrm{conv}(0, p_1, p_2)$ of index $\imath$. But $\mathsf{P}'$ and thus also $\mathsf{P}$ cannot be canonical. If in turn $\sum_{j=1}^{d} \alpha_j p_{ij} < \imath$ for some $i \in \{1, 2\}$, then $\mathsf{P}$ is not canonical by definition. $\square$

In dimension three, we have a classification of the canonical lattice polytopes by Ishida and Iwashita, which we recall in the following theorem:

THEOREM 4.3.19. [**59**, Theorem 3.6]. *Let $\mathsf{P} \subseteq \mathbb{Q}^3$ be a $\mathbb{Q}$-Gorenstein canonical lattice polytope of index $\imath$ with vertices $0, v_1, \ldots, v_r$. Then, up to lattice equivalence, the polytope $\mathsf{P}$ belongs to one of the following cases.*

| *Case* | $r$ | $\imath$ | $v_1, \ldots, v_r$ | *Conditions* |
|---|---|---|---|---|
| *(i)* | *any* | 1 | *any* | |
| *(ii)* | 4 | 2 | $(1,0,2), (1+m,1,2), (1,2,2), (1-n,1,2)$ | $n, m \in \mathbb{Z}_{\geq 1}$ |
| *(iii)* | 3 | $\geq 2$ | $(1,n,\imath), (0,n,\imath), (1,n+m,\imath)$ | $n, m \in \mathbb{Z}_{\geq 1}$ |
| *(iv)* | 3 | 2 | $(1,1,2), (0,1,2), (2,1+2m,2)$ | $m \in \mathbb{Z}_{\geq 2}$ |
| *(v)* | 3 | 2 | $(1,-2,2), (-1,-1,2), (2,1,2)$ | |
| *(vi)* | 3 | 3 | $(1,-1,3), (0,-1,3), (2,2,3)$ | |

*Additionally,* $\gcd(n, \imath) = 1$ *holds in Case* (*iii*) *and* $\mathsf{P}$ *is terminal if and only if it falls under Case* (*iii*) *with* $m = 1$.

There is another very useful assertion from [**59**]:

Corollary 4.3.20. [**59**, Lemma 2.2]. *Let* $a, b, \imath \in \mathbb{Z}$. *The three-dimensional polytope*

$$\mathsf{P} := \operatorname{conv}(0, (a, b, \imath), (a+1, b, \imath), (a, b+1, \imath), (a+1, b+1, \imath))$$

*is only canonical for* $\imath = 1$.

Using the data from Theorem 4.3.19, we compute the divisor class group and Cox ring of the toric threefold singularities in Section 4.4. But for now, we go one step further and have a look at certain canonical polytopes that have not only integer but also rational vertices.

Lemma 4.3.21. *The rational polygon* $\operatorname{conv}((0,0), (k, \imath), (k + 1/q, \imath))$ *with* $k \in \mathbb{Z}$, $\imath, q \in \mathbb{Z}_{>1}$ *is canonical if and only if there is an integral* $0 < c < q$ *with*

$$kc \equiv -1 \mod \imath.$$

Proof. Let $\mathsf{P} := \operatorname{conv}((0,0), (k, \imath), (k + 1/q, \imath))$ with $k \in \mathbb{Z}$ and $2 \leq q$. If $\gcd(k, \imath) \neq 1$, then $\mathsf{P}$ cannot be canonical. Thus we have $\gcd(k, \imath) = 1$. The polygon $\mathsf{P}$ contains no lattice points but $(0,0)$ and $(k, \imath)$ if and only if the lattice polygon

$$\mathsf{P}_1 := \operatorname{conv}((0,0), (qk, q\imath), (qk+1, q\imath))$$

contains no $q$-fold points but $(0,0)$ and $(qk, q\imath)$. But $\mathsf{P}_1$ is $q$-equivalent to the polygon

$$\mathsf{P}_2 := \operatorname{conv}((0,0), (0, q), (\imath, q - c))$$

with integer $0 < q - c < \imath$ and $kc \equiv \imath - 1 \mod \imath$. The $q$-affine unimodular transformation is given by the matrix

$$\begin{pmatrix} \imath & -k \\ -c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}),$$

where appropriate $c$ and $d$ exist since $k$ and $\imath$ are coprime.

Now if $q - \imath < -c \leq 0$, then the $q$-fold point $(q, q)$ lies inside $\mathsf{P}_2$ while for $0 < c < q$, no $q$-fold point lies inside $\mathsf{P}_2$. The assertion follows. □

Corollary 4.3.22. *The rational polygon* $\operatorname{conv}((0,0), (k, \imath), (k + 1/2, \imath))$ *with* $k \in \mathbb{Z}$, $\imath \in \mathbb{Z}_{>1}$ *is canonical if and only if* $k + 1$ *is a multiple of* $\imath$, *i.e. it is lattice equivalent to* $\operatorname{conv}((0,0), (1/2, \imath), (1, \imath))$.

Corollary 4.3.23. *The rational polygon* $\operatorname{conv}((0,0), (k - 1/2, \imath), (k + 1/2, \imath))$ *with* $k \in \mathbb{Z}$, $\imath \in \mathbb{Z}_{>1}$ *is canonical if and only if* $\imath = 2$ *and* $k$ *is odd.*

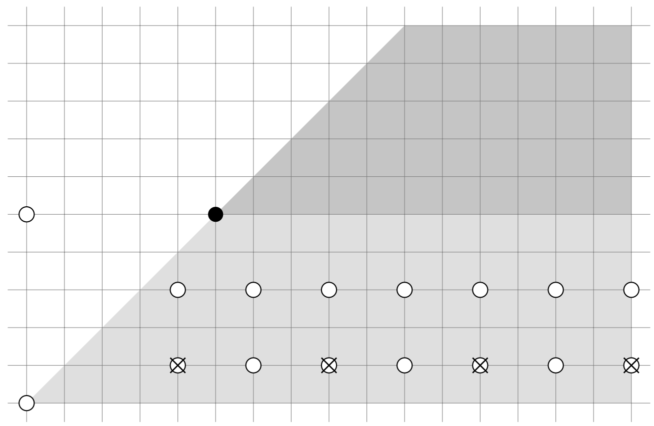
Corollary 4.3.24. *The rational polygon* $\operatorname{conv}((0,0), (k, \imath), (k + 1/3, \imath))$ *with* $k \in \mathbb{Z}$, $\imath \in \mathbb{Z}_{>1}$ *is canonical if and only if either* $k + 1$ *is a multiple of* $\imath$ *or* $\imath$ *is odd and* $k \equiv \frac{\imath - 1}{2} \mod \imath$.

Corollary 4.3.25. *The rational polygon* $\operatorname{conv}((0,0), (k - 1/3, \imath), (k + 1/3, \imath))$ *with* $k \in \mathbb{Z}$, $\imath \in \mathbb{Z}_{>1}$ *is canonical if and only if* $\imath = 2$ *and* $k$ *is odd or* $\imath = 3$ *and* $k \equiv 1, 2 \mod 3$.

Remark 4.3.26. We compare polytopes $\mathsf{Q} := \operatorname{conv}((0,0),(k,\imath),(k+2/5,\imath))$ with the polytopes $\operatorname{conv}((0,0),(k,\imath),(k+1/3,\imath))$ from Corollary 4.3.24. The polytope $\mathsf{Q}$ is canonical if and only if the polytope

$$\operatorname{conv}((0,0),(0,5),(2\imath,5-2c))$$

with $0<5-2c<2\imath$ and $kc \equiv -1 \mod \imath$ is 5-empty. More specifically, if it contains no 5-fold points but $(0,0)$ and $(0,5)$. Hence, it cannot contain the 5-fold point $(5,5)$ and so $(2\imath,5-2c)$ cannot lie in the dark gray area, but must be in the light gray area. The points of the form $(2\imath,5-2c)$ and $(0,0)$, $(0,5)$ are marked with circles. The ones which violate the condition that $c$ and $\imath$ are coprime are crossed out.



Altogether, we see that $c=1$ or $c=2$, if $\imath$ is odd. Thus $\mathsf{Q}$ is canonical if and only if $\operatorname{conv}((0,0),(k,\imath),(k+1/3,\imath))$ is canonical.

Lemma 4.3.27. *The rational polygon* $\operatorname{conv}((0,0),(k+1/2,\imath),(k+4/5,\imath))$ *with* $k\in\mathbb{Z}$, $\imath\in\mathbb{Z}_{>1}$ *contains no lattice points but* $(0,0)$ *if and only if it is lattice equivalent to one of the polytopes*

$$\operatorname{conv}((0,0),(-1/2,\imath),(-1/5,\imath)),$$
$$\operatorname{conv}((0,0),(1/2,\imath),(4/5,\imath)),$$
$$\operatorname{conv}\left((0,0),\left(\frac{\imath}{2}-\frac{1}{2},\imath\right),\left(\frac{\imath}{2}-\frac{1}{5},\imath\right)\right) \qquad \text{only if } 2|\imath.$$

Proof. Let $\mathsf{P} := \operatorname{conv}((0,0),(k+1/2,\imath),(k+4/5,\imath))$ with $k\in\mathbb{Z}$ and $\imath\in\mathbb{Z}_{>1}$. The polygon $\mathsf{P}$ contains no lattice points but $(0,0)$ if and only if the lattice polygon

$$\mathsf{P}_1 := \operatorname{conv}((0,0),(10k+5,10\imath),(10k+8,10\imath))$$

contains no 10-fold points but $(0,0)$. Consider the 10-affine unimodular transformation given by

$$\begin{pmatrix} -2\imath & 2k+1 \\ c & d \end{pmatrix} \in \operatorname{GL}_2(\mathbb{Z})$$

which yields a 10-equivalent lattice polygon

$$\mathsf{P}_2 := \operatorname{conv}((0,0),(0,g),(6\imath,g+3c))$$

where $g := c\cdot(10k+5)+d\cdot 10\imath = \gcd(10k+5,10\imath) \geq 5$. Note that $g\leq 9$. Since $g$ can't be $6,7,8$ or $9$ we have $g=5$. We also assume that $0<5+3c<6\imath$. The lattice polygon $\mathsf{P}_2$ contains no 10-fold points but $(0,0)$ if and only if one of the following cases occurs.

*Case* 1*: The lattice point* $(6\imath, 5+3c)$ *lies in the relative interior of the Farey strip* $F_{10,\frac{1}{2}}$. Then we have $5+3c = 3\imath+2$, so $c = \imath - 1$.
*Case* 2*: The lattice point* $(6\imath, 5+3c)$ *lies in the relative interior of the Farey strip* $F_{10,0}$. In this case we have $c = \pm 1$.
*Case* 3*: The lattice point* $(6\imath, 5+3c)$ *lies in the relative interior of a spike attached to* $F_{10,0}$. The spikes are of the form

$$\operatorname{conv}\left((10m, 10), (10(m+1), 10), \left(\frac{10m(m+1)}{m-1}, \frac{10m}{m-1}\right)\right)$$

where $2 \le m \le 11$. This leaves the possibilities $(\imath, c) \in \{(5,2), (7,2), (9,2)\}$. But those cannot occur since $c \equiv \pm 1 \mod 2\imath$. □

Lemma 4.3.28. *The rational polygon* $\operatorname{conv}((0,0), (i, k+1/2), (j, k+1/2))$ *with* $i \neq j \in \mathbb{Z}$*,* $k \in \mathbb{Z}_{\ge 0}$ *contains no lattice points but* $(0,0)$ *if and only if* $k = 0$.

Proof. Let $\mathsf{P} := \operatorname{conv}((0,0), (i, k+1/2), (j, k+1/2))$ with $i, j \in \mathbb{Z}, k \in \mathbb{Z}_{\ge 0}$ and without loss of generality $i < j$. The polygon $\mathsf{P}$ contains no lattice points but $(0,0)$ if and only if the lattice polygon

$$\mathsf{P}_1 := \operatorname{conv}((0,0), (2i, 2k+1), (2j, 2k+1))$$

contains no 2-fold points but $(0,0)$. Consider the 2-affine unimodular transformation given by

$$\begin{pmatrix} -(2k+1) & 2i \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$$

which yields a 2-equivalent lattice polygon

$$\mathsf{P}_2 := \operatorname{conv}((0,0), (0,g), (2\cdot(2k+1)(j-i), g + c(j-i)))$$

where $g := c \cdot 2i + d \cdot (2k+1) = \gcd(2i, 2k+1) \ge 1$. Since there are no 2-fold points but $(0,0)$, we have $g = 1$. Moreover, we can assume that $0 < 1 + c(j-i) < 2\cdot(2k+1)(j-i)$. Now, the lattice polygon $\mathsf{P}_2$ contains no 2-fold points but $(0,0)$ if and only if one of the following cases occurs.
*Case* 1*: The lattice point* $(2\cdot(2k+1)(j-i), 1+c(j-i))$ *lies in the relative interior of the Farey strip* $F_{2,\frac{1}{2}}$. Then the lattice point lies on the line $y = \frac{1}{2}x + 1$ and we have $2k+1 = c$. Since the matrix above has determinant $\pm 1$, we get $2k+1 = \pm 1$ and so $k = 0$.
*Case* 2*: The lattice point* $(2\cdot(2k+1)(j-i), 1+c(j-i))$ *lies in the relative interior of the Farey strip* $F_{2,0}$. Then we have $c = 0$ and by considering the determinant again we see $k = 0$.
*Case* 3*: The lattice point* $(2\cdot(2k+1)(j-i), 1+c(j-i))$ *lies in the relative interior of a spike attached to* $F_{2,0}$. The spikes are of the form

$$\operatorname{conv}\left((2m, 2), (2(m+1), 2), \left(\frac{2m(m+1)}{m-1}, \frac{2m}{m-1}\right)\right)$$

where $2 \le m \le 4$. These interiors do not contain any lattice points, so there are no additional possibilities. □

## 4.4. Canonical singularities

**4.4.1. Toric singularities.** This subsection deals with the *toric* canonical threefold singularities. From the data provided by Theorem 4.3.19, we compute the class groups and the Cox rings.

PROPOSITION 4.4.1. *If $X$ is a toric canonical threefold singularity, then it is either - Case $(i)$ - Gorenstein or belongs to one of the following other cases with respective total coordinate space, class group and grading:*

| *Case* | $\overline{X}$ | $\mathrm{Cl}(X)$ | $Q$ |
|---|---|---|---|
| $(ii)$ | $\mathbb{C}^4$ | $\mathbb{Z} \times \mathbb{Z}/2\mathfrak{d}\mathbb{Z}$ <br> $\mathfrak{d} = \gcd(2m, m+n)$ | $\begin{bmatrix} -\frac{m+n}{\mathfrak{d}} & \frac{2n}{\mathfrak{d}} & -\frac{m+n}{\mathfrak{d}} & \frac{2m}{\mathfrak{d}} \\ \overline{\alpha_1 + \mathfrak{d}} & \overline{-(2\alpha_1 + \alpha_2)} & \overline{\alpha_1} & \overline{\alpha_2} \end{bmatrix}$ <br> with $2m\alpha_1 + (m+n)\alpha_2 = \mathfrak{d}$ |
| $(iii)$ | $\mathbb{C}^3$ | $\mathbb{Z}/\imath m\mathbb{Z}$ | $\begin{bmatrix} \overline{1} & \overline{m\alpha_1} & \overline{-1} \end{bmatrix}$ with $n\alpha_1 \equiv 1 \mod \imath$ |
| $(iv)$ | $\mathbb{C}^3$ | $\mathbb{Z}/4m\mathbb{Z}$ | $[\overline{2}, \overline{2m-1}, \overline{-1}]$ |
| $(v)$ | $\mathbb{C}^3$ | $\mathbb{Z}/10\mathbb{Z}$ | $[\overline{1}, \overline{1}, \overline{3}]$ |
| $(vi)$ | $\mathbb{C}^3$ | $\mathbb{Z}/9\mathbb{Z}$ | $[\overline{1}, \overline{4}, \overline{7}]$ |

*The singularity $X$ is terminal if and only if it falls under Case $(iii)$ with $m = 1$.*

PROOF. By writing the vertices of the canonical polytopes from Theorem 4.3.19 as columns, we get a matrix $P$ representing the canonical toric threefold singularities $X$ as affine threefolds with a two-torus action. We only observe the Cases $(ii)$ to $(vi)$, since Case $(i)$ consists of all Gorenstein toric threefold singularities. In any case, we determine unimodular matrices $V$ and $W$, so that $S := V \cdot P^* \cdot W$ is in Smith normal form. Then if $\beta_1, \ldots, \beta_\nu$ are the elementary divisors and $\beta$ the number of zero rows of $S$, we have that

$$\mathrm{Cl}(X) \cong \mathbb{Z}^\beta \oplus \mathbb{Z}/\beta_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/\beta_\nu\mathbb{Z}.$$

Furthermore, the degree matrix $Q$ is the stack of the last $\nu + \beta$ rows of $V$. We go through the Cases $(ii)$ to $(vi)$ in the following. For Case $(ii)$, let $\mathfrak{d} := \gcd(2m, m+n)$ and $\alpha_1, \alpha_2 \in \mathbb{Z}$ with $2m\alpha_1 + (m+n)\alpha_2 = \mathfrak{d}$. Then

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -(m+1) & 1 & 0 & 0 \\ \alpha_1 + \mathfrak{d} & -(2\alpha_1 + \alpha_2) & \alpha_1 & \alpha_2 \\ -\frac{(m+n)}{\mathfrak{d}} & \frac{2n}{\mathfrak{d}} & -\frac{(m+n)}{\mathfrak{d}} & \frac{2m}{\mathfrak{d}} \end{bmatrix} \cdot P^* \cdot \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2m \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2\mathfrak{d} \\ 0 & 0 & 0 \end{bmatrix}.$$

So $\mathrm{Cl}(X) = \mathbb{Z}/2\mathfrak{d}\mathbb{Z}$ and

$$Q = \begin{bmatrix} \overline{\alpha_1 + \mathfrak{d}} & \overline{-(2\alpha_1 + \alpha_2)} & \overline{\alpha_1} & \overline{\alpha_2} \\ -\frac{m+n}{\mathfrak{d}} & \frac{2n}{\mathfrak{d}} & -\frac{m+n}{\mathfrak{d}} & \frac{2m}{\mathfrak{d}} \end{bmatrix}.$$

In Case $(iii)$, with $\alpha_1, \alpha_2$ meeting $\alpha_1 n + \alpha_2 m = 1$, the equality $S = V \cdot P^* \cdot W$ becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m\imath \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & -m\alpha_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & n & \imath \\ 0 & n & \imath \\ 1 & n+m & \imath \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha_1 & \imath \\ 0 & \alpha_2 & -n \end{bmatrix},$$

leading to the respective class group and degree matrix. The same equation has the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4m \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -2 & 1-2m & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 1+2m & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

in Case $(iv)$, while in the cases $(v)$ and $(vi)$, the data $\mathrm{Cl}(X)$ and $Q$ can easily be computed using the Maple package [**51**]. $\square$

**4.4.2. Matrices $P$ for non-toric singularities.** In the following, we classify the matrices $P$ defining non-toric canonical threefold singularities of complexity one. The classification is divided into the case $\zeta_X = 1$ and the subcases of Proposition 1.4.7. We begin with those singularities of canonical multiplicity one:

PROPOSITION 4.4.2. *Let $X$ be a non-toric threefold singularity of complexity one. Assume that $X$ is of canonical multiplicity one, Gorenstein index $\imath \geq 2$ and is at most canonical. Then for suitable $A$, we have $X \cong X(A, P_i)$, where $P_i$ is one of the following matrices:*

$$P_1 = \begin{bmatrix} -3 & -1 & 3 & 0 \\ -3 & -1 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ -4 & 0 & 2 & 2 \end{bmatrix} \quad P_2 = \begin{bmatrix} -3 & -1 & 3 & 1 & 0 \\ -3 & -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & -1 & -1 \\ -4 & 0 & 2 & 2 & 2 \end{bmatrix} \quad P_3 = \begin{bmatrix} -3 & 2 & 1 & 0 \\ -3 & 0 & 0 & 2 \\ 2 & 0 & 2 & 1 \\ -6 & 3 & 3 & 3 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} -3 & 2 & 0 & 0 \\ -3 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -4 & 2 & 2 & 2 \end{bmatrix} \quad P_5 = \begin{bmatrix} -k & 1 & 1 & 0 & 0 \\ -k & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 2-2k & 2 & 2 & 2 & 2 \end{bmatrix}$$

PROOF. Let $P$ be a matrix so that $X(A, P)$ is a canonical threefold with $\zeta_X = 1$ admitting a two-torus action. We may assume that $P$ is in the normal form of Corollary 1.4.6. Since $l_{ij} = 1$ holds for $i \geq 3$, we have $n_i \geq 2$ for $i \geq 3$. We now go through all possible different leading platonic tuples. Let $e_1, \ldots, e_{r+2}$ be the standard basis of the column space of $P$. Set $e_0 := -e_1 - \ldots - e_r$.

Let in the following $\tau_1$ be the (fake or $P$-) elementary cone generated by the columns of the leading block and by $\tau_i$ for $i \geq 2$ denote other (fake or $P$-) elementary cones. For such cones, recall that $v(\tau_i)' = \partial A_X^c(\lambda) \cap \tau_i$ denotes the point where $\lambda \cap \tau_i$ leaves $A_X^c$.

*Case 1: leading platonic tuple* $(5, 3, 2)$. The matrix $P$ has leading block

$$\begin{bmatrix} -5 & 3 & 0 & 0 & \ldots & 0 \\ -5 & 0 & 2 & 0 & \ldots & 0 \\ -5 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -5 & 0 & 0 & 0 & \ldots & 1 \\ d_{011} & d_{111} & d_{211} & 0 & \ldots & 0 \\ 6\imath - 5\imath r & \imath & \imath & \imath & \ldots & \imath \end{bmatrix}$$

due to Corollary 1.4.6. If $r \geq 3$, due to irredundancy we have at least another column $v_{32} = e_3 + d_{321}e_{r+1} + \imath e_{r+2}$ with $d_{321} \neq 0$. We thus have an elementary cone $\tau_2$ generated by $v_{32}$ and all columns of the leading block but $v_{31}$. So $A_X^c(\lambda)$ contains the integer points

$$\begin{aligned} v(\tau_1)' &= (0, \ldots, 0, 6d_{011} + 10d_{111} + 15d_{211}, \imath), \\ v(\tau_2)' &= (0, \ldots, 0, 6d_{011} + 10d_{111} + 15d_{211} + 30d_{321}, \imath) \end{aligned}$$

and thus the corresponding variety can not be canonical with Lemma 4.3.18. So $r = 2$ remains. We have leading block

$$\begin{bmatrix} -5 & 3 & 0 \\ -5 & 0 & 2 \\ d_{011} & d_{111} & d_{211} \\ -4\imath & \imath & \imath \end{bmatrix}$$

here. So we get $v(\tau_1)' = (0, 0, 6d_{011} + 10d_{111} + 15d_{211}, \imath) \in \mathbb{Z}^4$. Thus $\partial A_X^c(\lambda_i, \tau_1)$ contains the integer points

$$\begin{aligned}
p_{0,t} &:= (-t, -t, (6-t)d_{011} + 2(5-t)d_{111} + 3(5-t)d_{211}, (1-t)\imath) && t = 0, 1, \ldots, 5\\
p_{1,t} &:= (t, 0, 2(3-t)d_{011} + (10-3t)d_{111} + 5(3-t)d_{211}, \imath) && t = 0, 1, 2, 3\\
p_{2,t} &:= (0, t, 3(2-t)d_{011} + 5(2-t)d_{111} + (15-7t)d_{211}, \imath) && t = 0, 1, 2
\end{aligned}$$

These points are not allowed to be columns of $P$, as they would lie inside $\tau_1$. Any column of $P$ not contained in the leading block must now be of the form $v_{i2} = p_{i,t} + se_3$ for $0 \neq s \in \mathbb{Z}$ and $p_{i,t}$ one of the points above. Let now $\tau_2$ be the elementary cone generated by this additional column and the two leading block columns from the other two leaves. Depending on $i = 0, 1, 2$, we have $v(\tau_2)' = v(\tau_1)' + \delta_i e_3$, with

$$\delta_0 = 6s/(6-t), \quad \delta_1 = 10s/(10-3t), \quad \delta_2 = 15s/(15-7t).$$

Now since $|ks/(k-x)| \geq 1$ for $0 \neq s \in \mathbb{Z}$ and $0 \leq x < k$, between the integer point $v(\tau_1)'$ and $v(\tau_2)'$ lies another integer point in $A_X^c(\lambda)$. Lemma 4.3.18 tells us that the corresponding variety cannot be canonical.

*Case 2: leading platonic tuple* $(4, 3, 2)$. The matrix $P$ has leading block

$$\begin{bmatrix}
-4 & 3 & 0 & 0 & \ldots & 0\\
-4 & 0 & 2 & 0 & \ldots & 0\\
-4 & 0 & 0 & 1 & \ldots & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
-4 & 0 & 0 & 0 & \ldots & 1\\
d_{011} & d_{111} & d_{211} & 0 & \ldots & 0\\
5\imath - 4\imath r & \imath & \imath & \imath & \ldots & \imath
\end{bmatrix}$$

due to Corollary 1.4.6. If $r \geq 3$, due to irredundancy we have at least another column $v_{32} = e_3 + d_{321}e_{r+1} + \imath e_{r+2}$ with $d_{321} \neq 0$. So $A_X^c(\lambda)$ contains the integer points

$$\begin{aligned}
v(\tau_1)' &= (0, \ldots, 0, 3d_{011} + 4d_{111} + 6d_{211}, \imath),\\
v(\tau_2)' &= (0, \ldots, 0, 3d_{011} + 4d_{111} + 6d_{211} + 12d_{321}, \imath)
\end{aligned}$$

and as in Case 1, the corresponding variety can not be canonical. So we have $r = 2$ again and the leading block

$$\begin{bmatrix}
-4 & 3 & 0\\
-4 & 0 & 2\\
d_{011} & d_{111} & d_{211}\\
-3\imath & \imath & \imath
\end{bmatrix}$$

with $v(\tau_1)' = (0, 0, 3d_{011} + 4d_{111} + 6d_{211}, \imath) \in \mathbb{Z}^4$. As in Case 1, any column of $P$ not contained in the leading block is of the form $v_{i2} = p_{i,t} + se_3$, now with

$$\begin{aligned}
p_{0,t} &:= (-t, -t, (3-t/2)d_{011} + (4-t)d_{111} + (6-3t/2)d_{211}, (1-t)\imath) && t = 0, 1, \ldots, 4\\
p_{1,t} &:= (t, 0, (3-t)d_{011} + (4-t)d_{111} + 2(3-t)d_{211}, \imath) && t = 0, 1, 2, 3\\
p_{2,t} &:= (0, t, (3-3t/2)d_{011} + 2(2-t)d_{111} + (6-5t/2)d_{211}, \imath) && t = 0, 1, 2
\end{aligned}$$

For $i = 0, 1, 2$, we have $v(\tau_2)' = v(\tau_1)' + \delta_i e_3$, with

$$\delta_0 = 6s/(6-t), \quad \delta_1 = 8s/(8-2t), \quad \delta_2 = 12s/(12-5t).$$

The third coordinate of the points $p_{i,t}$ is in $\mathbb{Z}$ for $i = 1$ and also for $i = 0, 2$ if additionally $t \in 2\mathbb{Z}$. It is as well in $\mathbb{Z}$ if $d_{011} - d_{211} \in 2\mathbb{Z}$. In these cases we use $|ks/(k-x)| \geq 1$ for $0 \leq x < k$ to obtain another (in addition to $v(\tau_1)'$) integer point in $A_X^c(\lambda)$. In the remaining cases, the third coordinate of the points $p_{i,t}$ is in $\mathbb{Z} + 1/2$ and we have $s \in \mathbb{Z} + 1/2$. We first examine $i = 0$ and $t = 3$, here the distance $6s/(6-t)$ between $v(\tau_1)'$ and $v(\tau_2)'$ becomes $2s$, which means that again there must be an integer point inbetween.

Only $t = 1$ and $i = 0, 2$ remain. Since $d_{011} - d_{211} \in 2\mathbb{Z} + 1$, the Gorenstein index $\imath$ must be odd, otherwise the first or the third column of the leading block would not be primitive. By admissible operations, we achieve $d_{011} = 1$, $d_{211} = 0$.

We first examine the case $i = 0$. This leads to $v(\tau_1)' = (0, 0, 3 + 4d_{111}, \imath)$ and $v(\tau_2)' = (0, 0, 3 + 4d_{111} + 6s/5, \imath)$. Only for $s = \pm 1/2$, there is no integer point inbetween. In these two cases, the polytope

$$\mathrm{conv}(0_{\mathbb{Z}_4}, (0, 0, 3 + 4d_{111}, \imath), (0, 0, 3 + 4d_{111} \pm 1/2, \imath)$$

is contained in $A_X^c(\lambda)$. In the first case $s = 1/2$, Corollary 4.3.22 tells us that $4 + 4d_{111}$ is a multiple of $\imath$. Our leading block together with the column $v_{02}$ has the form

$$\begin{bmatrix} -4 & -1 & 3 & 0 \\ -4 & -1 & 0 & 2 \\ 1 & 3d_{111} + 2 & d_{111} & 0 \\ -3\imath & 0 & \imath & \imath \end{bmatrix}.$$

By the admissible operations of adding the $(1+d_{111})$-fold of the first, the $(2+2d_{111})$-fold of the second and the $-4(1+d_{111})/\imath$-fold of the fourth all to the third row, we obtain

$$\begin{bmatrix} -4 & -1 & 3 & 0 \\ -4 & -1 & 0 & 2 \\ 1 & -1 & -1 & 0 \\ -3\imath & 0 & \imath & \imath \end{bmatrix}.$$

But $A_X^c(\lambda_0)$ contains the point $(-1, -1, 0, -1)$ and thus the corresponding variety can not be canonical. In the second case $s = -1/2$, Corollary 4.3.22 tells us that $2+4d_{111}$ is a multiple of $\imath$. By the admissible operations of adding the $(1+d_{111})$-fold of the first, the $(1 + 2d_{111})$-fold of the second and the $-2(1 + 2d_{111})/\imath$-fold of the fourth all to the third row, we obtain the exact same matrix as above and are done with this case as well. The final case $i = 2$ is analogue to the case $i = 0$.

*Case 3: leading platonic tuple* $(3, 3, 2)$. The matrix $P$ has leading block

$$\begin{bmatrix} -3 & 3 & 0 & 0 & \dots & 0 \\ -3 & 0 & 2 & 0 & \dots & 0 \\ -3 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -3 & 0 & 0 & 0 & \dots & 1 \\ d_{011} & d_{111} & d_{211} & 0 & \dots & 0 \\ 4\imath - 3\imath r & \imath & \imath & \imath & \dots & \imath \end{bmatrix}$$

due to Corollary 1.4.6. If $r \geq 3$, due to irredundancy we have at least another column $v_{32} = e_3 + d_{321}e_{r+1} + \imath e_{r+2}$ in $P$ with $d_{321} \neq 0$. So $A_X^c(\lambda)$ contains the integer points

$$\begin{aligned} v(\tau_1)' &= (0, \dots, 0, 2(d_{011} + d_{111}) + 3d_{211}, \imath), \\ v(\tau_2)' &= (0, \dots, 0, 2(d_{011} + d_{111}) + 3d_{211} + 6d_{321}, \imath) \end{aligned}$$

and with Lemma 4.3.18, the corresponding variety can not be canonical. Thus only $r = 2$ is possible and the leading block is

$$\begin{bmatrix} -3 & 3 & 0 \\ -3 & 0 & 2 \\ d_{011} & d_{111} & d_{211} \\ -2\imath & \imath & \imath \end{bmatrix}$$

with $v(\tau_1)' = (0, 0, 2(d_{011} + d_{111}) + 3d_{211}, \imath) \in \mathbb{Z}^4$. The points $p_{i,t}$ in the intersection of $\partial A_X^c(\lambda_i, \tau_1)$ with the hyperplanes $\{x_i = t\}$ have the forms:

$$p_{0,t} := (-t, -t, (2 - t/3)d_{011} + (2 - 2t/3)d_{111} + (3 - t)d_{211}, (1 - t)\imath) \quad t = 0, 1, 2, 3$$
$$p_{1,t} := (t, 0, (2 - 2t/3)d_{011} + (2 - t/3)d_{111} + (3 - t)d_{211}, \imath) \quad t = 0, 1, 2, 3$$
$$p_{2,t} := (0, t, (2 - t)d_{011} + (2 - t)d_{111} + (3 - t)d_{211}, \imath) \quad t = 0, 1, 2$$

Now any column of $P$ not contained in the leading block must be of the form $v_{i2} = p_{i,t} + se_3$ with $0 \neq s \in \mathbb{Q}$. This is because it must firstly be contained in $\partial A_X^c$, which fixes the last coordinate, secondly it is not in the leading block, restricting its first two coordinates and lastly it must not be contained in $\tau_1$, so in fact $s \neq 0$. Now for $i = 0, 1, 2$ let $\tau_2$ be the elementary cone generated by $v_{i2}$ and $v_{j1}$ for $j \neq i$. For $i = 0, 1, 2$, we have $v(\tau_2)' = v(\tau_1)' + \delta_i e_3$, with

$$\delta_0 = 6s/(6 - t), \quad \delta_1 = 6s/(6 - t), \quad \delta_2 = 3s/(3 - t).$$

The third coordinate of the points $p_{i,t}$ is in $\mathbb{Z}$ for $i = 2$ and also for $i = 0, 1$ if additionally $d_{011} - d_{111} \in 3\mathbb{Z}$ or $t = 0, 3$. Since $v_{i2}$ must be integer, in these cases also $s$ must be integer. Then we use $|ks/(k-t)| \geq 1$ for $0 \leq t < k$ and integer $s \neq 0$ to obtain another (in addition to $v(\tau_1)'$) integer point in $A_X^c(\lambda)$. Lemma 4.3.18 shows that such variety can not be canonical.

It remains $i = 0, 1$ with $t = 1, 2$ and $d_{011} - d_{111} \notin 3\mathbb{Z}$. Here $\imath \notin 3\mathbb{Z}$, because otherwise neither of $d_{011}$ and $d_{111}$ would be in $3\mathbb{Z}$ due to primitivity of the columns of $P$. But then $d_{011} + d_{111} \in 3\mathbb{Z}$ and $v(\tau_1)'$ would not be primitive. The cases $i = 0$ and $i = 1$ are equivalent, since the leaves $\lambda_1$ and $\lambda_0$ till now are interchangeable. So we only treat $i = 0$ in the following. Moreover, by admissible operations, we can achieve $d_{011} = 0$ and $d_{111} = -1$.

This works as follows: We have $d_{011} - d_{111} \notin 3\mathbb{Z}$. Thus $2d_{011} + d_{111} \pm 1 \in 3\mathbb{Z}$. We have that $\imath$ and 3 are coprime. Thus there always exists an $a \in \mathbb{Z}$ so that adding $-a + (2d_{011} + d_{111} \pm 1)/3$ times the first, $-a + d_{011} + d_{111} \pm 1$ times the second and $(3a - 2d_{011} - 2d_{111} \pm 2)/\imath$ the fourth to the third row is an admissible operation. Then we have $d_{011} = 0$ and $d_{111} = \pm 1$, where a possible negation of the third row if required leads to the desired form. Now we examine the two remaining cases $t = 1, 2$ with $P$ in this form.

*Case 3.1:* $t = 1$. In order not to have two integer points in $A_X^c(\lambda)$, we require $s = -2/3$ or $s = 1/3$. For $s = -2/3$, Corollary 4.3.22 applied to the polytope

$$\operatorname{conv}(0_{\mathbb{Z}_4}, v(\tau_1)', v(\tau_2)')$$

tells us that $3d_{211} - 3$ is a multiple of $\imath$ and then adding the $(d_{211} - 1)$-fold of the first and second and the $3(1 - d_{211})/\imath$-fold of the last to the third row, the matrix $P$ can be brought into the form:

$$\begin{bmatrix} -3 & -1 & 3 & 0 \\ -3 & -1 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ -2\imath & 0 & \imath & \imath \end{bmatrix}.$$

But the first two columns of $P$ here build up a two-dimensional polytope in $A_X^c(\lambda_0)$ that can not be canonical due to Lemma 4.3.18. For $s = 1/3$, the leading block together with the column $v_{02}$ has the form:

$$\begin{bmatrix} -3 & -1 & 3 & 0 \\ -3 & -1 & 0 & 2 \\ 0 & 2d_{211} - 1 & -1 & d_{211} \\ -2\imath & 0 & \imath & \imath \end{bmatrix}.$$

The polytope $\mathrm{conv}(0_{\mathbb{Z}_4}, (0, 0, 3d_{211} - 2, \imath), (0, 0, 3d_{211} - 2 + 2/5, \imath))$ is contained in $A_X^c(\lambda)$. According to Remark 4.3.26, we have two possible cases:

*Case 3.1.1:* $3d_{211} - 1$ *is a multiple of* $\imath$. Then the admissible operation of adding the $d_{211}$-fold of the first, the $(d_{211} - 1)$-fold of the second and the $(1 - 3d_{211})/\imath$-fold of the last to the third row brings the matrix into the form:

$$P_{1,\imath} := \begin{bmatrix} -3 & -1 & 3 & 0 \\ -3 & -1 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ -2\imath & 0 & \imath & \imath \end{bmatrix}.$$

We have a look at

$$\begin{aligned} & \mathrm{conv}(0_{\mathbb{Z}_4}, v(\tau_1)', v_{01}, v_{02}) \\ = & \mathrm{conv}(0_{\mathbb{Z}_4}, (0, 0, -1, \imath), (-3, -3, 1, -2\imath), (-1, -1, 0, 0)) \subseteq A_X^c(\lambda_0). \end{aligned}$$

There is a "critical" point that can lie inside this polytope, namely

$$(-1, -1, 0, -1) = \frac{1}{\imath} v(\tau_1)' + \frac{1}{\imath} v_{01} + \frac{\imath - 3}{\imath} v_{02}.$$

Only for $\imath = 2$, it does not and we get our first canonical singularity with defining matrix $P_1 := P_{1,2}$.

*Case 3.1.2:* $\imath$ *is odd and* $6d_{211} - 3$ *is a multiple of* $\imath$. Since $\imath$ and 3 are coprime, we have that $2d_{211} - 1$ is a multiple of $\imath$ and moreover, with admissible operations, we can achieve $d_{211} = (\imath + 1)/2$. A simple computation as above shows that $(1, 0, \imath - 2, \imath - 1)$ lies inside $A_X^c(\lambda_1)$ for any odd $\imath$ coprime to three. We come to:

*Case 3.2:* $t = 2$. In order not to have two integer points in $A_X^c(\lambda)$, we require $s = -1/3$. Then Corollary 4.3.22 tells us that $3d_{211} - 3$ is a multiple of $\imath$ and the same admissible operation as in Case 3.1 for $s = -2/3$ brings the matrix into the form

$$\begin{bmatrix} -3 & -2 & 3 & 0 \\ -3 & -2 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ -2\imath & 0 & \imath & \imath \end{bmatrix}.$$

and again as in Case 3.1 the first two columns of $P$ together with $0_{\mathbb{Z}_4}$ build up a non-canonical two-dimensional polytope.

Now as we have seen, the only possible matrix with an additional column in $\lambda_0$ has leading block together with the additional column

$$\begin{bmatrix} -3 & -1 & 3 & 0 \\ -3 & -1 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ -4 & 0 & 2 & 2 \end{bmatrix}.$$

As we have seen above, the blocks $\lambda_0$ and $\lambda_1$ are interchangeable, so there must be a column in $\lambda_1$ that together with the leading block gives an equivalent matrix. This column is $(1, 0, -1, 2)$. This is because the admissible operations of subtracting the first from the third and adding the last to the third row, then negating the third row and finally changing the data of the leaves $\lambda_0$ and $\lambda_1$ brings the matrix from above in the form:

$$\begin{bmatrix} -3 & 3 & 1 & 0 \\ -3 & 0 & 0 & 2 \\ 1 & 0 & -1 & -1 \\ -4 & 2 & 2 & 2 \end{bmatrix}.$$

Thus it is possible that a defining matrix $P$ contains *both* of these columns and in fact all subpolytopes of the anticanonical complex of

$$P_2 := \begin{bmatrix} -3 & -1 & 3 & 1 & 0 \\ -3 & -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & -1 & -1 \\ -4 & 0 & 2 & 2 & 2 \end{bmatrix}$$

are canonical. Thus we get our first non-$\mathbb{Q}$-factorial variety with defining matrix $P_2$. As there are no other possible additional columns, we are done with Case 3.

*Case 4: leading platonic tuple* $(l_{01}, 2, 2)$. The matrix $P$ has leading block

$$\begin{bmatrix} -l_{01} & 2 & 0 & 0 & \dots & 0 \\ -l_{01} & 0 & 2 & 0 & \dots & 0 \\ -l_{01} & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -l_{01} & 0 & 0 & 0 & \dots & 1 \\ d_{011} & d_{111} & d_{211} & 0 & \dots & 0 \\ \imath(1 - l_{01}(r-1)) & \imath & \imath & \imath & \dots & \imath \end{bmatrix}$$

with $l_{01} \geq 2$ due to Corollary 1.4.6. If $r \geq 3$, due to irredundancy we have at least another column $v_{32} = e_3 + d_{321}e_{r+1} + \imath e_{r+2}$ in $P$ with $d_{321} \neq 0$. So $A_X^c(\lambda)$ contains the points

$$v(\tau_1)' = \left(0, \dots, 0, d_{011} + \frac{l_{01}(d_{111} + d_{211})}{2}, \imath\right),$$
$$v(\tau_2)' = \left(0, \dots, 0, d_{011} + \frac{l_{01}(d_{111} + d_{211})}{2} + l_{01}d_{321}, \imath\right)$$

and since $|l_{01}d_{321}| \geq 2$, with Lemma 4.3.18, the corresponding variety can not be canonical. Thus only $r = 2$ is possible and the leading block is

$$\begin{bmatrix} -l_{01} & 2 & 0 \\ -l_{01} & 0 & 2 \\ d_{011} & d_{111} & d_{211} \\ (1 - l_{01})\imath & \imath & \imath \end{bmatrix}$$

with $v(\tau_1)' = \left(0, \dots, 0, d_{011} + \frac{l_{01}(d_{111}+d_{211})}{2}, \imath\right) \in \mathbb{Z}^4$. We distinguish two cases in the following:

*Case 4.1: $\imath$ is even.* Since the last two columns of the leading block must be primitive, $d_{111}$ and $d_{211}$ must be odd and thus by adding appropriate multiples of the first two rows to the third, we can assume $d_{111} = d_{211} = 1$. So $v(\tau_1)' = (0, 0, d_{011} + l_{01}, \imath)$ is integer. The points $p_{i,t}$, defined as in Case 3, here have the form:

$$\begin{aligned} p_{0,t} &:= (-t, -t, d_{011} + l_{01} - t, (1-t)\imath) && t = 0, \dots, l_{01} \\ p_{1,t} &:= (t, 0, d_{011} + l_{01} + t(1 - d_{011} - l_{01})/2, \imath) && t = 0, 1, 2 \\ p_{2,t} &:= (0, t, d_{011} + l_{01} + t(1 - d_{011} - l_{01})/2, \imath) && t = 0, 1, 2 \end{aligned}$$

Again, additional columns must be of the form $v_{i2} = p_{i,t} + se_3$. For $i = 0, 1, 2$, we have $v(\tau_2)' = v(\tau_1)' + \delta_i e_3$, with

$$\delta_0 = s, \quad \delta_1 = \delta_2 = \frac{2l_{01}s}{2l_{01} - (l_{01} - 2)t}.$$

In the first case $i = 0$, we have integer $s$ and thus $A_X^c$ contains two integer points, we get no canonical variety. The cases $i = 1, 2$ are again equivalent. We examine

$i = 1$. If $t = 0, 2$ or $d_{011} + l_{01}$ is odd, then $s$ is integer. In all these cases, since $l_{01} \geq 2$, we have that

$$0 \neq \frac{2l_{01}s}{2l_{01} - (l_{01} - 2)t} \in \mathbb{Z},$$

so $A_X^c(\lambda)$ contains two integer points and we are done with these cases. The case $t = 1$ and $d_{011} + l_{01}$ even remains. Here $s \in \mathbb{Z} + 1/2$. Only for $s = \pm 1/2$, we have

$$\left| \frac{2l_{01}s}{2l_{01} - (l_{01} - 2)t} \right| < 1,$$

otherwise we have two integer points in $A_X^c(\lambda)$. Now with $s = \pm 1/2$, from Corollary 4.3.22 applied to the polytope

$$\operatorname{conv}\left(0_{\mathbb{Z}_4}, (0, 0, d_{011} + l_{01}, \imath), \left(0, 0, d_{011} + l_{01} \pm \frac{l_{01}}{l_{01} + 2}, \imath\right)\right) \subseteq A_X^c(\lambda)$$

follows that $d_{011} + l_{01} \pm 1$ is a multiple of $\imath$. But this is a contradiction since $\imath$ is even and $d_{011} + l_{01}$ is even as well.

*Case 4.2: $\imath$ is odd.* If $d_{111} \equiv d_{211} \mod 2$, then by if necessary adding the last row to the third and then adding appropriate multiples of the first and second row to the third, we achieve $d_{111} = d_{211} = 0$. If $d_{111} \equiv d_{211} + 1 \mod 2$ , we achieve $d_{111} = 0$, $d_{211} = 1$. We distinguish both cases in the following. Observe that in the first as well as in the second case, we do not have to distinguish between additional columns in the first or in the second leaf due to admissible operations.

*Case 4.2.1: $d_{111} = d_{211} = 0$.* So $v(\tau_1)' = (0, 0, d_{011}, \imath)$. The points $p_{i,t}$, defined as in Case 4.1, here have the form:

$$\begin{aligned} p_{0,t} &:= (-t, -t, d_{011}, (1-t)\imath) && t = 0, \dots, l_{01} \\ p_{1,t} &:= (t, 0, d_{011}(1 - t/2), \imath) && t = 0, 1, 2 \\ p_{2,t} &:= (0, t, d_{011}(1 - t/2), \imath) && t = 0, 1, 2 \end{aligned}$$

Additional columns must be of the form $v_{i2} = p_{i,t} + se_3$. We have $v(\tau_2)' = v(\tau_1)' + \delta_i e_3$ with $\delta_i$ as in Case 4.1. Thus as in Case 4.1, the only possibility for a canonical singularity is $i = 1$ ($i = 2$ is equivalent), $t = 1$, odd $d_{011}$, $s = \pm 1/2$, since otherwise $A_X^c(\lambda)$ would contain two integer points. Moreover, from Corollary 4.3.22 applied on the polytope

$$\operatorname{conv}\left(0_{\mathbb{Z}_4}, (0, 0, d_{011}, \imath), \left(0, 0, d_{011} \pm \frac{l_{01}}{l_{01} + 2}, \imath\right)\right) \subseteq A_X^c(\lambda)$$

follows that $d_{011} \pm 1$ is a multiple of $\imath$. By the admissible operations of adding the $(d_{011} \pm 1)/2$-fold of the first and second to the third row and subtracting the $(d_{011} \pm 1)/\imath$-fold of the last from the third row and optionally negating the third row, we achieve the following form for our matrix:

$$\begin{bmatrix} -l_{01} & 2 & 1 & 0 \\ -l_{01} & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ (1 - l_{01})\imath & \imath & \imath & \imath \end{bmatrix}.$$

But $A_X^c(\lambda_1)$ contains a two-dimensional polytope with vertices $0_{\mathbb{Z}_4}$ and the second and third column of $P$, which can not be canonical. We are done with Case 4.2.1.

*Case 4.2.2:* $d_{111} = 0$, $d_{211} = 1$. So $v(\tau_1)' = (0, 0, d_{011} + l_{01}/2, \imath)$ is integer. The points $p_{i,t}$, defined as in Case 3, here have the form:

$$\begin{aligned} p_{0,t} &:= (-t, -t, d_{011} + (l_{01} - t)/2, (1-t)\imath) && t = 0, \dots, l_{01} \\ p_{1,t} &:= (t, 0, (1 - t/2)(d_{011} + l_{01}/2), \imath) && t = 0, 1, 2 \\ p_{2,t} &:= (0, t, (1 - t/2)(d_{011} + l_{01}/2) + t/2, \imath) && t = 0, 1, 2 \end{aligned}$$

Again, additional columns must be of the form $v_{i2} = p_{i,t} + se_3$. Again we have $v(\tau_2)' = v(\tau_1)' + \delta_i e_3$ with $\delta_i$ as in Case 4.1.

*Case 4.2.2.1:* $i = 0$. If $l_{01}$ is even and $t$ as well, then $s$ is integer and $A_X^c(\lambda)$ contains the two integer points $v(\tau_1)'$ and $v(\tau_2)'$. If both are odd, then Corollary 4.3.23 applied to $\operatorname{conv}(0_{\mathbb{Z}_4}, v(\tau_1)', v(\tau_2)')$ forces $\imath = 2$, which is a contradiction, since $\imath$ is odd. So it remains $l_{01} + t$ odd and $s = \pm 1/2$.

*Case 4.2.2.1.1: $l_{01}$ even, $t$ odd.* Corollary 4.3.22 forces $d_{011} + l_{01}/2 \pm 1$ to be a multiple of $\imath$. The polytope

$$\begin{aligned} \mathsf{B}_1 := \operatorname{conv}(0_{\mathbb{Z}_4}, &(0, 0, d_{011} + l_{01}/2, \imath), (0, 0, d_{011} + l_{01}/2 \pm 1/2, \imath), \\ &(-2, -2, d_{011} + (l_{01} - 2)/2, -\imath), (-1, -1, d_{011} + (l_{01} - 1)/2 - \pm 1/2, 0)) \end{aligned}$$

is contained in $A_X^c(\lambda_0)$. Its union $\mathsf{B}_{12} := \mathsf{B}_1 \cup \mathsf{B}_2$ with

$$\begin{aligned} \mathsf{B}_2 := \operatorname{conv}(0_{\mathbb{Z}_4}, &(0, 0, d_{011} + l_{01}/2, \imath), (0, 0, d_{011} + l_{01}/2 \pm 1/2, \imath), \\ &(1, 1, d_{011} + (l_{01} + 1)/2, 2\imath)) \end{aligned}$$

is lattice equivalent to the polytope from Corollary 4.3.20 and can thus not be canonical. But since $\mathsf{B}_2$ is canonical, $\mathsf{B}_1$ can not be.

*Case 4.2.2.1.2: $l_{01}$ odd, $t$ even.* Corollary 4.3.22 forces $d_{011} + l_{01}/2 \pm 1/2$ to be a multiple of $\imath$. The argument is the same as before, now with

$$\begin{aligned} \mathsf{B}_1 := \operatorname{conv}(0_{\mathbb{Z}_4}, &(0, 0, d_{011} + l_{01}/2, \imath), (0, 0, d_{011} + l_{01}/2 \pm 1/2, \imath), \\ &(-2, -2, d_{011} + (l_{01} - 2)/2 \pm 1/2, -\imath), (-1, -1, d_{011} + (l_{01} - 1)/2, 0)) \\ \mathsf{B}_2 := \operatorname{conv}(0_{\mathbb{Z}_4}, &(0, 0, d_{011} + l_{01}/2, \imath), (0, 0, d_{011} + l_{01}/2 \pm 1/2, \imath), \\ &(1, 1, d_{011} + (l_{01} + 1)/2 \pm 1/2, 2\imath)). \end{aligned}$$

*Case 4.2.2.2:* $i = 1$. If $l_{01}$ and $t$ are even, then $s$ is integer and $A_X^c(\lambda)$ contains two integer points. If $l_{01}$ is even and $t = 1$, then if $(d_{011} + l_{01}/2)$ is even, $s$ is integer and $A_X^c(\lambda)$ contains two integer points. If $d_{011} + l_{01}/2$ is odd, we require $s = \pm 1/2$ and Corollary 4.3.22 forces $d_{011} + l_{01}/2 \pm 1$ to be a multiple of $\imath$. By the admissible operations of adding the $(d_{011} + l_{01}/2 \pm 1)/2$-fold of the first and second to the third row and subtracting the $(d_{011} + l_{01}/2 \pm 1)/\imath$-fold of the last from the third row, we achieve the following form for our matrix:

$$\begin{bmatrix} -l_{01} & 2 & 1 & 0 \\ -l_{01} & 0 & 0 & 2 \\ -(l_{01}/2 \pm 1) & 0 & 0 & 1 \\ (1 - l_{01})\imath & \imath & \imath & \imath \end{bmatrix}.$$

But as in Case 4.2.1, $A_X^c(\lambda_1)$ contains a two-dimensional polytope with vertices $0_{\mathbb{Z}_4}$ and the second and third column of $P$, which can not be canonical. Now the case $l_{01}$ odd remains. We distinguish $t = 0, 1, 2$.

*Case 4.2.2.2.1:* $t = 0$. We need $s = \pm 1/2$ here and again Corollary 4.3.22 forces $d_{011} + l_{01}/2 \pm 1/2$ to be a multiple of $\imath$. Similar to the last case, by admissible operations we achieve the form

$$\begin{bmatrix} -l_{01} & 2 & 0 & 0 \\ -l_{01} & 0 & 2 & 0 \\ -(l_{01}/2 \pm 1/2) & 0 & 1 & 0 \\ (1-l_{01})\imath & \imath & \imath & \imath \end{bmatrix}$$

for our matrix and again $A_X^c(\lambda_1)$ contains a non-canonical two-dimensional polytope with vertices $0_{\mathbb{Z}_4}$ and the second and fourth column of $P$.

*Case 4.2.2.2.2: $t = 1$.* We require $d_{011}/2+l_{01}/4+s \in \mathbb{Z}$. By if necessary subtracting the second from the third row and afterwards negating the third row, we can achieve that $d_{011}$ is even. Now we distinguish two subcases:

*Case 4.2.2.2.2.1: $l_{01} \equiv 1 \mod 4$.* So $s \in \mathbb{Z} + 3/4$. We have a look at the leaving points of the two elementary cones:

$$\begin{aligned} v(\tau_1)' &= (0, 0, d_{011} + l_{01}/2, \imath)\,, \\ v(\tau_2)' &= (0, 0, d_{011} + l_{01}/2 + 2l_{01}s/(l_{01} + 2), \imath)\,. \end{aligned}$$

For $|s| \geq 3/4$, we have $|l_{01}/2 + 2l_{01}s/(l_{01} + 2)| \geq 3l_{01}/(2l_{01} + 4) > 1$ since $l_{01} > 4$. In this case applying Corollary 4.3.23 to $A_X^c(\lambda)$, we get $\imath = 2$, which is a contradiction. Thus the only possibility is $s = -1/4$. Lemma 4.3.27 tells us that either $d_{011} + (l_{01} - 1)/2$ or $d_{011} + (l_{01} + 1)/2$ is a multiple of $\imath$.

*Case 4.2.2.2.2.1.1: $d_{011} + (l_{01} - 1)/2$ is a multiple of $\imath$.* Since $l_{01} \equiv 1 \mod 4$, it is even as well. Thus adding the $(d_{011} + (l_{01} - 1)/2)/2$-fold of the first and the second to the third and subtracting the $(d_{011} + (l_{01} - 1)/2)/\imath$-fold of the last from the third row, we bring our matrix into the form

$$\begin{bmatrix} -l_{01} & 2 & 1 & 0 \\ -l_{01} & 0 & 0 & 2 \\ (1-l_{01})/2 & 0 & 0 & 1 \\ (1-l_{01})\imath & \imath & \imath & \imath \end{bmatrix}.$$

Again $A_X^c(\lambda_1)$ contains a non-canonical two-dimensional polytope with vertices $0_{\mathbb{Z}_4}$ and the second and third column of $P$.

*Case 4.2.2.2.2.1.2: $d_{011} + (l_{01} + 1)/2$ is a multiple of $\imath$.* It is odd as well. So we can write $d_{011} = k\imath - (l_{01} + 1)/2$ with odd $k \in \mathbb{Z}$. Adding the $(k-1)\imath/2$-fold of the first and second and the $(1-k)$-fold of the last to the third row, we bring our matrix into the form

$$\begin{bmatrix} -l_{01} & 2 & 1 & 0 \\ -l_{01} & 0 & 0 & 2 \\ \imath-(1+l_{01})/2 & 0 & (\imath-1)/2 & 1 \\ (1-l_{01})\imath & \imath & \imath & \imath \end{bmatrix}.$$

We have a look at the polytope

$$\begin{aligned} \operatorname{conv}\left(0_{\mathbb{Z}_4}, \left(-l_{01}, -l_{01}, \imath - \frac{1+l_{01}}{2}, (1-l_{01})\imath\right), \left(0, 0, \imath - \frac{1}{2}, \imath\right), \left(0, 0, \frac{3l_{01}+2}{2(l_{01}+2)}\right)\right) \\ = \operatorname{conv}(0_{\mathbb{Z}_4}, v_{01}, v(\tau_1)', v(\tau_2)') \subseteq A_X^c(\lambda_0) \end{aligned}$$

and the point

$$(-1, -1, \imath - 2, -1) = \frac{1}{l_{01}} v_{01} + \left(1 - \frac{\imath + 2(1+l_{01})}{\imath l_{01}}\right) v(\tau_1)' + \frac{l_{01}+2}{\imath l_{01}} v(\tau_2)',$$

which is in the relative interior since $\imath \geq 3$ and $l_{01} \geq 5$. Thus the corresponding variety is not canonical.

*Case 4.2.2.2.2.2: $l_{01} \equiv 3 \mod 4$.* Here $s \in \mathbb{Z} + 1/4$ and with the same argument as in Case 4.2.2.2.2.1, we have $s = 1/4$. Lemma 4.3.27 tells us that either $d_{011} + (l_{01} -$

1)/2 or $d_{011} + (l_{01}+1)/2$ is a multiple of $\imath$. So we distinguish the same subcases as in Case 4.2.2.2.2.1.:

*Case 4.2.2.2.2.2.1:* $d_{011} + (l_{01}-1)/2$ *is a multiple of* $\imath$. Since $l_{01} \equiv 3 \mod 4$, it is now odd. With the same admissible operations as in Case 4.2.2.2.2.1.2, we arrive at the matrix

$$\begin{bmatrix} -l_{01} & 2 & 1 & 0 \\ -l_{01} & 0 & 0 & 2 \\ \imath - (l_{01}-1)/2 & 0 & (\imath+1)/2 & 1 \\ (1-l_{01})\imath & \imath & \imath & \imath \end{bmatrix}.$$

Now the point

$$(-1,-1,\imath-1,-1) = \frac{1}{l_{01}} v_{01} + \left(1 - \frac{\imath + 2(1+l_{01})}{\imath l_{01}}\right) v(\tau_1)' + \frac{l_{01}+2}{\imath l_{01}} v(\tau_2)'$$

is again in the relative interior of $A_X^c(\lambda_0)$ for $\imath \geq 3$ and $l_{01} \geq 7$. But for $l_{01} = 3$, this is the case only for $\imath \geq 4$. So for $\imath = 3$, we get a canonical singularity from the matrix

$$P_3 := \begin{bmatrix} -3 & 2 & 1 & 0 \\ -3 & 0 & 0 & 2 \\ 2 & 0 & 2 & 1 \\ -6 & 3 & 3 & 3 \end{bmatrix}.$$

*Case 4.2.2.2.2.2.2:* $d_{011} + (l_{01}+1)/2$ *is a multiple of* $\imath$. Since $l_{01} \equiv 3 \mod 4$, $d_{011} + (l_{01}+1)/2$ is now even. With the same admissible operations as in Case 4.2.2.2.2.1.1, we arrive at the matrix

$$\begin{bmatrix} -l_{01} & 2 & 1 & 0 \\ -l_{01} & 0 & 0 & 2 \\ (1+l_{01})/2 & 0 & 0 & 1 \\ (1-l_{01})\imath & \imath & \imath & \imath \end{bmatrix}.$$

Again $A_X^c(\lambda_1)$ contains a non-canonical two-dimensional polytope with vertices $0_{\mathbb{Z}_4}$ and the second and third column of $P$.

*Case 4.2.2.2.3:* $t = 2$. Here $s$ is integer. Applying Corollary 4.3.23 to $A_X^c(\lambda)$, we get $\imath = 2$, which is a contradiction.

So all in all we found one $\mathbb{Q}$-factorial matrix in Case 4, namely

$$P_3 = \begin{bmatrix} -3 & 2 & 1 & 0 \\ -3 & 0 & 0 & 2 \\ 2 & 0 & 2 & 1 \\ -6 & 3 & 3 & 3 \end{bmatrix}.$$

Thus we have to check if an additional column $(0,1,d_{211},3)$ in $\lambda_2$ is possible. But for such a column, we need $|s| = 1/4$ and have the same situation as in Case 4.2.2.2.2.2.1, so we get no canonical singularity. We are done with Case 4.

*Case 5: leading platonic tuple* $(l_{01}, l_{11}, 1)$. The matrix $P$ has leading block

$$\begin{bmatrix} -l_{01} & l_{11} & 0 & 0 & \dots & 0 \\ -l_{01} & 0 & 1 & 0 & \dots & 0 \\ -l_{01} & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -l_{01} & 0 & 0 & 0 & \dots & 1 \\ d_{011} & d_{111} & 0 & 0 & \dots & 0 \\ \imath(1-l_{01}(r-1)) & \imath & \imath & \imath & \dots & \imath \end{bmatrix}$$

with $l_{01} \geq l_{11} \geq 1$ due to Corollary 1.4.6. If $r \geq 3$, then due to irredundancy we have at least *two* more columns $v_{22} = e_2 + d_{221}e_{r+1}$ and $v_{32} = e_3 + d_{321}e_{r+1}$, where we can assume $d_{221} \geq 1$ by if this is not the case adding the $-d_{221}$ of the second to the penultimate row. The same holds for $d_{321}$. So the intersections of $\partial A_X^c$ with the elementary cones $\tau_1$ set up by the leading block and $\tau_2$ set up by the leading block with columns $v_{21}$ and $v_{31}$ replaced by $v_{22}$ and $v_{32}$ are the leaving points

$$v(\tau_1)' = \left(0, \ldots, 0, \frac{l_{11}d_{011} + l_{01}d_{111}}{l_{11} + l_{01}}, \imath\right),$$

$$v(\tau_2)' = \left(0, \ldots, 0, \frac{l_{11}d_{011} + l_{01}d_{111} + (d_{221} + d_{321})l_{01}l_{11}}{l_{11} + l_{01}}, \imath\right).$$

Now we distinguish three subcases.

*Case 5.1:* $l_{11} \geq 2$. Here if $r \geq 3$, we have $|v(\tau_1)' - v(\tau_2)'| \geq \frac{2l_{01}l_{11}}{l_{11}+l_{01}} \geq 2$ and thus $A_X^c(\lambda)$ contains two integer points besides $0_{\mathbb{Z}_{r+2}}$. Lemma 4.3.18 tells us that the corresponding variety can not be canonical. So we have $r = 2$. The necessity of an additional column $v_{21}$ remains and the leading block together with it as well as the two leaving points are

$$\begin{bmatrix} -l_{01} & l_{11} & 0 & 0 \\ -l_{01} & 0 & 1 & 1 \\ d_{011} & d_{111} & 0 & d_{221} \\ \imath(1-l_{01}) & \imath & \imath & \imath \end{bmatrix},$$

$$v(\tau_1)' = \left(0, 0, \frac{l_{11}d_{011} + l_{01}d_{111}}{l_{11} + l_{01}}, \imath\right), v(\tau_2)' = v(\tau_1)' + \frac{d_{221}l_{01}l_{11}}{l_{11} + l_{01}}e_3$$

So now $|v(\tau_1)' - v(\tau_2)'| \geq \frac{l_{01}l_{11}}{l_{11}+l_{01}} \geq 2$ if $l_{11} \geq 4$ or if $l_{11} = 3$ and $l_{01} \geq 6$. As above, we can exclude these cases with Lemma 4.3.18. We examine the remaining cases with $l_{11} = 3$. If $l_{01} = 3$, then even for $d_{211} = 1$ one of $v(\tau_1)'$ and $v(\tau_2)'$ is integer and their distance is $3/2$, Lemma 4.3.18 rules out this case. If $l_{01} = 4$, then $d_{211} = 1$ must hold and we have

$$v(\tau_1)' = \left(0, 0, \frac{3d_{011} + 4d_{111}}{7}, \imath\right), \quad v(\tau_2)' = \left(0, 0, \frac{3d_{011} + 4d_{111} + 12}{7}, \imath\right).$$

So $A_X^c(\lambda)$ might not contain two integer points only if $3d_{011} + 4d_{111} \equiv 1 \mod 7$. Corollary 4.3.23 forces $\imath = 2$. By admissible operations, we achieve $d_{111} = 0$. So $d_{011} = 5 + 7k$ and with Corollary 4.3.23, we have odd $k$. By adding the $k$-fold of the first, the $3k$-fold of the second and the $-3k/2$-fold of the last to the third row, we achieve $d_{011} = 5$. But then $(-1, -1, 2, -1)$ lies inside $A_X^c(\lambda_0)$ and the corresponding variety is not canonical. It remains $l_{01} = 5$. Again $d_{211} = 1$ must hold and we have

$$v(\tau_1)' = \left(0, 0, \frac{3d_{011} + 5d_{111}}{8}, \imath\right), \quad v(\tau_2)' = \left(0, 0, \frac{3d_{011} + 5d_{111} + 15}{8}, \imath\right),$$

so we have two integer points in $A_X^c(\lambda)$.

We examine the cases with $l_{11} = 2$. First from $|v(\tau_1)' - v(\tau_2)'| = \frac{2l_{01}d_{221}}{2+l_{01}} \geq 2$ if $d_{221} \geq 2$ follows that $d_{221} = 1$ must hold. We have

$$v(\tau_1)' = \left(0, 0, \frac{2d_{011} + l_{01}d_{111}}{l_{01} + 2}, \imath\right), \quad v(\tau_2)' = \left(0, 0, \frac{2d_{011} + l_{01}d_{111} + 2l_{01}}{l_{01} + 2}, \imath\right).$$

Now for $l_{01} \geq 6$, we have $|v(\tau_1)' - v(\tau_2)'| \geq 3/2$ and therefore $A_X^c(\lambda)$ either contains two integer points or a polytope like in Corollary 4.3.23, which forces $\imath = 2$. For $l_{01} = 5$, the same holds, since $|v(\tau_1)' - v(\tau_2)'| = 10/7$ and for $2d_{011} + 5d_{111} \equiv 0, 4, 5, 6 \mod 7$, the polytope $A_X^c(\lambda)$ contains two integer points while for $2d_{011} + 5d_{111} \equiv 1, 2, 3 \mod 7$, it contains a polytope like in Corollary 4.3.23. For $l_{01} = 4$, we have $|v(\tau_1)' - v(\tau_2)'| = 4/3$ and for $d_{011} + 2d_{111} \equiv 0, 2 \mod 3$ we have two integer points while for $d_{011} + 2d_{111} \equiv 1 \mod 3$, we have a polytope like in Corollary 4.3.23. For

$l_{01} = 3$, we have $|v(\tau_1)' - v(\tau_2)'| = 6/5$, so for $2d_{011} + 3d_{111} \equiv 0, 4 \mod 5$, we have two integer points and for $2d_{011} + 3d_{111} \equiv 2 \mod 5$, we have a polytope like in Corollary 4.3.23. For $l_{01} = 2$, we have $|v(\tau_1)' - v(\tau_2)'| = 1$, so for $d_{011} + d_{111}$ even, we have two integer points and for $d_{011} + d_{111}$ odd, we have a polytope like in Corollary 4.3.23. This means that $\imath = 2$ must hold in all cases but $l_{01} = 3$, $2d_{011} + 3d_{111} \equiv 1, 3 \mod 5$. So first let $l_{01} = 3$ and $2d_{011} + 3d_{111} \equiv 1, 3 \mod 5$. But using Corollary 4.3.25, we either get $\imath = 2$ or $\imath = 3$. In the case $\imath = 3$, by admissible operations we achieve $d_{111} = 0$ and our matrix has the form:

$$\begin{bmatrix} -3 & 2 & 0 & 0 \\ -3 & 0 & 1 & 1 \\ d_{011} & 0 & 0 & 1 \\ -6 & 3 & 3 & 3 \end{bmatrix}.$$

Since $d_{111} = 0$, we either have $2d_{011}/5 = 3k + 1/5$ or $2d_{011}/5 = 3k + 8/5$ for some $k \in \mathbb{Z}$. We can add multiples of 15 to $d_{011}$ due to admissible operations. Since in the first case we have odd $k$ and in the second one, we have even $k$, we get $d_{011} \in \{4, 8\}$. These two cases are equivalent by admissible operations as well. But for $d_{011} = 4$, the point $(-1, -1, 2, 0)$ lies inside $A_X^c(\lambda_0)$ and the corresponding variety is not canonical.

So only the cases with $l_{11} = \imath = 2$ are left. Here by primitivity of the second column and admissible operations, we achieve $d_{111} = 1$ and have the matrix

$$\begin{bmatrix} -l_{01} & 2 & 0 & 0 \\ -l_{01} & 0 & 1 & 1 \\ d_{011} & 1 & 0 & 1 \\ 2(1 - l_{01}) & 2 & 2 & 2 \end{bmatrix}.$$

By adding the $k(l_{01} + 2)$-fold of the first, the $2k(l_{01} + 2)$-fold of the second and the $-k(l_{01} + 2)$-fold of the last to the third row for $k = \lfloor d_{011}/(l_{01} + 2) \rfloor$, we can assume $0 \leq d_{011} \leq l_{01} + 1$. Since

$$v(\tau_1)' = \left(0, 0, \frac{2d_{011} + l_{01}}{l_{01} + 2}, 2\right), \quad v(\tau_2)' = \left(0, 0, \frac{2d_{011} + l_{01} + 2l_{01}}{l_{01} + 2}, 2\right),$$

we need $2d_{011} + l_{01} = i(l_{01} + 2) + j$ with $i = 0, 2$ and $j \in \{1, 2, 3\}$ in order not to have $(0, 0, 1, 1)$ or $(0, 0, 2, 1)$ inside $A_X^c(\lambda)$. First consider $i = 0$. Since $l_{01} \geq 2$, this forces $d_{011} = 0$ and $l_{01} = 3$ due to primitivity of the first column. This results in the matrix

$$P_4 := \begin{bmatrix} -3 & 2 & 0 & 0 \\ -3 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -4 & 2 & 2 & 2 \end{bmatrix},$$

giving a canonical singularity. Now consider $i = 2$. This leads to $d_{011} = (l_{01} + j')/2$ with $j' \in \{5, 6, 7\}$.

*Case a:* $j' = 5$. Then $l_{01}$ is odd. The point

$$(-1, -1, 2, -1) = \frac{1}{l_{01}} v_{01} + \frac{l_{01} - 4}{4l_{01}} v(\tau_1)' + \frac{1}{4} v(\tau_2)'$$

lies inside $A_X^c(\lambda_0)$ for $l_{01} \geq 5$. But for $l_{01} = 3$, the resulting matrix is equivalent to the canonical one from above by adding the first, three times the second and $-2$ times the last to the third row and afterwards negating the third row. So we get no new canonical singularity in this case.

*Case b:* $j' = 6$. Then $l_{01}$ is even. Due to primitivity of the first column, we require $l_{01} \equiv 0 \mod 4$. The point

$$(-1, -1, 2, -1) = \frac{1}{l_{01}} v_{01} + \frac{l_{01} - 2}{4l_{01}} v(\tau_1)' + \frac{l_{01} - 2}{4l_{01}} v(\tau_2)'$$

lies in $A_X^c(\lambda_0)$ and so the corresponding singularity can not be canonical.

*Case c:* $j' = 7$. Then $l_{01}$ is odd. The point

$$(-1,-1,2,-1) = \frac{1}{l_{01}} v_{01} + \frac{1}{4} v(\tau_1)' + \frac{l_{01}-4}{4l_{01}} v(\tau_2)'$$

lies inside $A_X^c(\lambda_0)$ for $l_{01} \geq 5$. For $l_{01} = 3$, the resulting matrix again is equivalent to the canonical one from above.

The final step in Case 5.1 is now to check if to the canonical matrix

$$\begin{bmatrix} -3 & 2 & 0 & 0 \\ -3 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -4 & 2 & 2 & 2 \end{bmatrix}$$

columns in $\lambda_0$ or $\lambda_1$ can be added, as no additional columns in $\lambda_3$ are possible due to convexity.

*Case a: additional column* $v_{02} = (-t,-t,s,2(1-t))$ *with* $t = 1,2,3$. Then for $\tau_3 := \mathrm{cone}(v_{02}, v_{11}, v_{21})$ and $\tau_4 := \mathrm{cone}(v_{02}, v_{11}, v_{22})$, we get

$$v(\tau_3)' = \left(0, 0, \frac{2s+t}{2+t}, 2\right), \quad v(\tau_4)' = \left(0, 0, \frac{2s+3t}{2+t}, 2\right).$$

Now $s = 0$ is impossible, since in this case $\mathrm{conv}(0_{\mathbb{Z}^4}, v_{01}, v_{02})$ is a non-canonical two-dimensional cone. For $s \leq -1$, the point $(-1,-1,0,-1)$ lies inside

$$\mathrm{conv}(0_{\mathbb{Z}^4}, v_{01}, v_{02}, v(\tau_1)').$$

For $s \geq 2$ and $s = 1$, $t = 2,3$, the point $(0,0,1,1)$ lies inside the polytope $\mathrm{conv}(0_{\mathbb{Z}^4}, v(\tau_1)', v(\tau_4)')$. But $s = 1$, $t = 1$ is impossible, since the corresponding point $(-1,-1,1,0)$ already lies inside $\mathrm{conv}(v_{01}, v(\tau_1)', v(\tau_2)')$.

*Case b: additional column* $v_{12} = (t,0,s,2)$ *with* $t = 1,2$. Here $s = 1$ is impossible since the corresponding points already lie inside $\mathrm{conv}(v_{11}, v(\tau_1)', v(\tau_2)')$. Then for $\tau_3 := \mathrm{cone}(v_{01}, v_{12}, v_{21})$ and $\tau_4 := \mathrm{cone}(v_{01}, v_{12}, v_{22})$, we get

$$v(\tau_3)' = \left(0, 0, \frac{3s}{3+t}, 2\right), \quad v(\tau_4)' = \left(0, 0, \frac{3(s+t)}{3+t}, 2\right).$$

For $s \leq 0$, the point $(0,0,0,1)$ lies inside $\mathrm{conv}(0_{\mathbb{Z}^4}, v(\tau_1)', v(\tau_3)')$ and for $s \geq 2$, the point $(0,0,1,1)$ lies inside $\mathrm{conv}(0_{\mathbb{Z}^4}, v(\tau_2)', v(\tau_4)')$. We are done with Case 5.1.

*Case 5.2:* $l_{11} = 1$, $l_{01} \geq 2$. We can assume $d_{111} = 0$ and due to irredundancy, we need an additional column $v_{12} = (1, 0, d_{121}, \imath)$ in $\lambda_1$, where as before, we can assume $d_{121} \geq 1$. If $r \geq 3$, with $\tau_2 := \mathrm{cone}(v_{01}, v_{12}, v_{32}, v_{41}, \ldots, v_{r1})$ we have

$$v(\tau_1)' = \left(0, \ldots, 0, \frac{d_{011}}{1+l_{01}}, \imath\right),$$
$$v(\tau_2)' = \left(0, \ldots, 0, \frac{d_{011} + (d_{121} + d_{221} + d_{321})l_{01}}{1+l_{01}}, \imath\right).$$

But $|v(\tau_1)' - v(\tau_2)'| \geq \frac{3l_{01}}{1+l_{01}} \geq 2$, since $l_{01} \geq 2$, so a corresponding singularity cannot be canonical. Thus we have $r = 2$ and our matrix has the form:

$$\begin{bmatrix} -l_{01} & 1 & 1 & 0 & 0 \\ -l_{01} & 0 & 0 & 1 & 1 \\ d_{011} & 0 & d_{121} & 0 & d_{221} \\ \imath(1-l_{01}) & \imath & \imath & \imath & \imath \end{bmatrix}.$$

With $\tau_2 := \mathrm{cone}(v_{01}, v_{12}, v_{32})$, we have

$$v(\tau_1)' = \left(0, 0, \frac{d_{011}}{1+l_{01}}, \imath\right), \quad v(\tau_2)' = \left(0, 0, \frac{d_{011} + (d_{121} + d_{221})l_{01}}{1+l_{01}}, \imath\right).$$

So $|v(\tau_1)' - v(\tau_2)'| = \frac{(d_{121}+d_{221})l_{01}}{1+l_{01}} \leq 2$ only if $d_{121} = d_{221} = 1$. But $A_X^c(\lambda)$ contains two integer points but for $d_{011} \equiv 1 \mod 1+l_{01}$. In this case it contains a polytope like in Corollary 4.3.23. This forces $\imath = 2$ and $d_{011} = 1 + 2k(1+l_{01})$ for some $k \in \mathbb{Z}$. But by adding $2k$ times the first and second and $-k$ times the last to the third row, we arrive at the matrix

$$P_5 := \begin{bmatrix} -l_{01} & 1 & 1 & 0 & 0 \\ -l_{01} & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 2(1-l_{01}) & 2 & 2 & 2 & 2 \end{bmatrix},$$

giving a canonical singularity for arbitrary $l_{01} \geq 2$.

*Case 5.3:* $l_{11} = l_{01} = 1$. We can assume $d_{111} = 0$ and due to irredundancy, we need additional columns $v_{02} = (-1, -1, d_{021}, \imath)$ in $\lambda_0$ and $v_{12} = (1, 0, d_{121}, \imath)$ in $\lambda_1$, where as before, we can assume $d_{121} \geq 1$ and moreover $d_{021} > d_{011}$ . With $\tau_2 := \mathrm{cone}(v_{02}, v_{12}, v_{32}, v_{41}, \ldots, v_{r1})$, we have

$$v(\tau_1)' = \left(0, 0, \frac{d_{011}}{2}, \imath\right), \quad v(\tau_2)' = \left(0, 0, \frac{d_{021} + d_{121} + d_{221}}{2}, \imath\right),$$

thus $|v(\tau_1)' - v(\tau_2)'| \geq 3/2$ and one of $v(\tau_1)'$ and $v(\tau_2)'$ is integer, so the corresponding singularity cannot be canonical. We are done with all cases.

□

Now we go through the cases of Proposition 1.4.7 with $\zeta_X > 1$, beginning with Case $(i)$ in the following proposition:

Proposition 4.4.3. *Let $X$ be a non-toric threefold singularity of complexity one. Assume that $X$ is of canonical multiplicity two, even Gorenstein index $\imath \geq 2$, the leading platonic triple is $(4, 3, 2)$ and $X$ is at most canonical. Then for suitable $A$, $X$ is isomorphic to $X(A, P_i)$, where $P_i$ is one of the following matrices:*

$$P_6 = \begin{bmatrix} -4 & 3 & 0 & 0 \\ -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & 5 \\ 3 & -2 & 0 & 1 \end{bmatrix} \quad P_7 = \begin{bmatrix} -4 & 3 & 1 & 0 \\ -4 & 0 & 0 & 2 \\ 0 & 0 & 3 & 1 \\ 3 & -2 & 0 & 0 \end{bmatrix} \quad P_8 = \begin{bmatrix} -4 & 3 & 1 & 0 & 0 \\ -4 & 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 1 & 7 \\ 3 & -2 & 0 & 0 & 1 \end{bmatrix}$$

$$P_9 = \begin{bmatrix} -4 & 3 & 0 & 0 & 0 \\ -4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 5 & 7 \\ 3 & -2 & 0 & 1 & 1 \end{bmatrix} \quad P_{10} = \begin{bmatrix} -4 & 3 & 1 & 1 & 0 \\ -4 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 5 & 1 \\ 3 & -2 & 0 & 0 & 0 \end{bmatrix}$$

Proof. The matrix $P$ falls under Case $(i)$ of Proposition 1.4.7, thus $\imath$ is even and the leading block has the shape

$$\begin{bmatrix} -4 & 3 & 0 & 0 & \ldots & 0 \\ -4 & 0 & 2 & 0 & \ldots & 0 \\ -4 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -4 & 0 & 0 & 0 & \ldots & 1 \\ d_{011} & d_{111} & d_{211} & 0 & \ldots & 0 \\ \frac{\imath}{2}+2 & -\imath & \frac{\imath}{2}-1 & 0 & \ldots & 0 \end{bmatrix}.$$

If $r \geq 3$, due to irredundancy we have at least another column $v_{32} = e_3 + d_{321}e_{r+1}$ in $P$ with $d_{321} \geq 1$. So $A_X^c(\lambda)$ contains the integer points

$$v(\tau_1)' = (0, \ldots, 0, 3d_{011} + 4d_{111} + 6d_{211}, \imath/2),$$
$$v(\tau_2)' = (0, \ldots, 0, 3d_{011} + 4d_{111} + 6d_{211} + 12d_{321}, \imath/2)$$

and with Lemma 4.3.18, we get $\imath = 2$. By primitivity of the third column, we can write $d_{211} = 2k+1$ for some $k \in \mathbb{Z}$, and by subtracting the $8k + 2d_{011} + 3d_{111}$-fold of the first, the $k$-fold of the second and the $12k + 3d_{011} + 4d_{111}$-fold of the last from the penultimate row, we achieve $d_{011} = d_{111} = 0$ and $d_{211} = 1$. so we get

$$v(\tau_1)' = (0, \dots, 0, 6, 1), \quad v(\tau_2)' = (0, \dots, 0, 6 + 12d_{321}, 1).$$

We have a look at $A_X^c(\lambda_0)$. The point

$$(-1, -1, 2, 1) = \frac{1}{4}v_{01} + \frac{6d_{321} - 1}{24d_{321}}v(\tau_1)' + \frac{1}{24d_{321}}v(\tau_2)'$$

lies inside, so we have no canonical singularity. Thus only $r = 2$ is possible. Now we have to check which additional columns are possible.

*Case 1: additional column* $v_1 = (0, 0, d_1, \imath/2)$ *in* $\lambda$. Since $v(\tau_1)'$ is integer, an additional column in the lineality part means two integer points in $A_X^c(\lambda)$, which forces $\imath = 2$ with Lemma 4.3.18. With the same arguments as above, the leading block together with the additional column has the form:

$$\begin{bmatrix} -4 & 3 & 0 & 0 \\ -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & d_1 \\ 3 & -2 & 0 & 1 \end{bmatrix}.$$

Now the point

$$(-1, -1, 1, 1) = \frac{1}{4}v_{01} + \frac{d_1 - 4}{4d_1 - 24}v(\tau_1)' + \frac{1}{12 - 2d_1}v_1'$$

lies inside $A_X^c(\lambda_0)$ for $d_1 \leq 4$, and the point

$$(-1, -1, 2, 1) = \frac{1}{4}v_{01} + \frac{d_1 - 8}{4d_1 - 24}v(\tau_1)' + \frac{1}{2d_1 - 12}v_1'$$

lies inside $A_X^c(\lambda_0)$ for $d_1 \geq 8$. So the cases $d_1 = 5, 7$ remain as $d_1 = 6$ is not possible due to $v(\tau_1)' = (0, 0, 6, 1)$. But by subtracting the second, eight times the first and 12 times the last from the third row and negating the third row afterwards, we see that both cases are equivalent and give a canonical singularity with matrix

$$P_6 := \begin{bmatrix} -4 & 3 & 0 & 0 \\ -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & 5 \\ 3 & -2 & 0 & 1 \end{bmatrix}.$$

*Case 2: additional column* $v_{02} = (-t, -t, d_{021}, (\imath + t)/2)$ *in* $\lambda_0$. Since $v_{02}$ is integer, we have $t = 4, 2$.

*Case 2.1:* $t = 4$. We can assume $d_{021} > d_{011}$ and have the two integer points $v(\tau_i)' = (0, 0, 3d_{0i1} + 4d_{111} + 6d_{211}, \imath/2)$ inside $A_X^c(\lambda)$, forcing $\imath = 2$ and by the same admissible operations as in Case 1, we get $d_{011} = d_{111} = 0$, $d_{211} = 1$, $v(\tau_1)' = (0, 0, 6, 1)$ and $v(\tau_2)' = (0, 0, 6 + 3d_{021}, 1)$. As we have seen in Case 1, the point $(-1, -1, 2, 1)$ lies inside $A_X^c(\lambda_0)$.

*Case 2.2:* $t = 2$. Here $v(\tau_1)'$ is integer and $v(\tau_2)' = (0, 0, 3(d_{021} + d_{211})/2 + d_{111}, \imath/2)$. We distinguish two subcases in the following.

*Case 2.2.1:* $\imath \equiv 2 \mod 4$. Here due to primitivity of $v_{02}$ and $v_{21}$, we require both $d_{021}$ and $d_{211}$ to be odd. Thus $v(\tau_2)'$ is integer, which forces $\imath = 2$. But then either $d_{021} \leq 1$ or $d_{021} \geq 5$ and either $(-1, -1, 1, 1)$ or $(-1, -1, 2, 1)$ lies inside $A_X^c(\lambda_0)$.

*Case 2.2.2:* $\imath \equiv 0 \mod 4$. Here $d_{021} + d_{211}$ must be odd due to Lemma 4.3.18. By if necessary adding the last row to the third, we can achieve $d_{021}$ odd and $d_{211}$ even. Then by adding appropriate multiples of the first two rows to the third, we get $d_{021} = 1$ and $d_{211} = 0$. We have

$$v(\tau_1)' = (0, 0, 3d_{011} + 4d_{111}, \imath/2), \quad v(\tau_2)' = (0, 0, 3/2 + d_{111}, \imath/2).$$

Since $A_X^c(\lambda)$ must not contain two integer points due to Lemma 4.3.18, we require $3(d_{011} + d_{111}) \in \{1, 2\}$, which is not possible.

*Case 3: additional column* $v_{12} = (t, 0, d_{121}, \imath(1-t)/2)$ *in* $\lambda_1$. Here we have the possibilities $t = 3, 2, 1$.

*Case 3.1:* $t = 3$. We can assume $d_{121} > d_{111}$ and have the two integer points $v(\tau_i)' = (0, 0, 3d_{011} + 4d_{1i1} + 6d_{211}, \imath/2)$ inside $A_X^c(\lambda)$, forcing $\imath = 2$ and by the same admissible operations as in Case 1, we get $d_{011} = d_{111} = 0$, $d_{211} = 1$, $v(\tau_1)' = (0, 0, 6, 1)$ and $v(\tau_2)' = (0, 0, 6 + 4d_{121}, 1)$. As we have seen in Case 1, the point $(-1, -1, 2, 1)$ lies inside $A_X^c(\lambda_0)$.

*Case 3.2:* $t = 2$. Here both $v(\tau_1)'$ and $v(\tau_2)' = (0, 0, d_{011} + 2(d_{121} + d_{211}), \imath/2)$ are integer, forcing $\imath = 2$, and by the same admissible operations as in Case 1, we get $d_{011} = d_{111} = 0$, $d_{211} = 1$, $v(\tau_1)' = (0, 0, 6, 1)$ and $v(\tau_2)' = (0, 0, 2(1 + d_{121}), 1)$. As we have seen in Case 1, one of the points $(-1, -1, 1, 1)$ and $(-1, -1, 2, 1)$ lies inside $A_X^c(\lambda_0)$, since either $2(1 + d_{121}) \leq 4$ or $2(1 + d_{121}) \geq 8$.

*Case 3.3:* $t = 1$. By subtracting $d_{121}$ times the first from the third row, we achieve $d_{121} = 0$. Furthermore, either $\imath \equiv 2 \mod 4$, then due to primitivity of $v_{21}$, the entry $d_{211}$ is odd and by adding appropriate multiples of the second to the third row, we achieve $d_{211} = 1$, or $\imath \equiv 0 \mod 4$, then $\imath/2 - 1$ is odd and by if necessary adding the last and then adding appropriate multiples of the second to the third row, we achieve $d_{211} = 1$ as well. So $v(\tau_1)' = (0, 0, 3d_{011} + 4d_{111} + 6, \imath/2)$ and $v(\tau_2)' = (0, 0, (d_{011} + 2)/3, \imath/2)$.

*Case 3.3.1:* $\imath > 2$. Here Lemma 4.3.18 forces

$$3d_{011} + 4d_{111} + 6 \in \{d_{011}/3, (d_{011} + 1)/3, d_{011}/3 + 1, (d_{011} + 4)/3\}.$$

But if $3d_{011} + 4d_{111} + 6 = d_{011}/3$, then $d_{011} = 3k$ for some $k \in \mathbb{Z}$ and $4k + 2d_{111} + 3 = 0$ follows, a contradiction. If $3d_{011} + 4d_{111} + 6 = d_{011}/3 + 1$, then $8k + 4d_{111} + 5 = 0$ follows, a contradiction as well. For $3d_{011} + 4d_{111} + 6 = (d_{011} + 1)/3$, we have $d_{011} = 3k - 1$ for some $k \in \mathbb{Z}$ and $8k + 4d_{111} + 3 = 0$ follows, while for $3d_{011} + 4d_{111} + 6 = (d_{011} + 4)/3$, we have $d_{011} = 3k - 1$ for some $k \in \mathbb{Z}$ as well and $4k + 2d_{111} + 1 = 0$ follows, both contradictions.

*Case 3.3.2:* $\imath = 2$. Here, our matrix has the form

$$\begin{bmatrix} -4 & 3 & 1 & 0 \\ -4 & 0 & 0 & 2 \\ 0 & 0 & d_{121} & 1 \\ 3 & -2 & 0 & 0 \end{bmatrix}$$

and we have $v(\tau_1)' = (0, 0, 6, 1)$ and $v(\tau_2)' = (0, 0, (2 + 4d_{121})/3, 1)$. As we have seen in Case 1, in order not to have $(-1, -1, 1, 1)$ or $(-1, -1, 2, 1)$ contained in $A_X^c(\lambda_0)$, we require $4 < (2 + 4d_{121})/3 < 8$, which leads to $d_{121} \in \{3, 5\}$, as $d_{121} = 4$ is impossible. But with the same admissible operations as in Case 1, these two are equivalent and give a canonical singularity with matrix

$$P_7 := \begin{bmatrix} -4 & 3 & 1 & 0 \\ -4 & 0 & 0 & 2 \\ 0 & 0 & 3 & 1 \\ 3 & -2 & 0 & 0 \end{bmatrix}.$$

*Case 4: additional column* $v_{22} = (0, t, d_{221}, (\imath - t)/2)$ *in* $\lambda_2$. We can assume $d_{021} > d_{011}$ and have the two integer points $v(\tau_i)' = (0, 0, 3d_{011} + 4d_{111} + 6d_{2i1}, \imath/2)$ inside $A_X^c(\lambda)$, forcing $\imath = 2$ and by the same admissible operations as in Case 1, we get $d_{011} = d_{111} = 0$, $d_{211} = 1$, $v(\tau_1)' = (0, 0, 6, 1)$ and $v(\tau_2)' = (0, 0, 6 + 6d_{221}, 1)$. As we have seen in Case 1, the point $(-1, -1, 2, 1)$ lies inside $A_X^c(\lambda_0)$.

We finally have to check if the possible additional columns can be combined. Such a matrix must be of one of the following three forms.

*Case a:*

$$\begin{bmatrix} -4 & 3 & 1 & 0 & 0 \\ -4 & 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 1 & d_1 \\ 3 & -2 & 0 & 0 & 1 \end{bmatrix},$$

with $d_1 \in \{5, 7\}$. But for $d_1 = 5$, the column $v_1$ would lie inside $\operatorname{cone}(v_{01}, v_{12}, v_{21})$. The Case $d_1 = 7$ gives a canonical singularity with matrix $P_8$.

*Case b:*

$$\begin{bmatrix} -4 & 3 & 0 & 0 & 0 \\ -4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 5 & 7 \\ 3 & -2 & 0 & 1 & 1 \end{bmatrix},$$

which gives a canonical singularity with matrix $P_9$.

*Case c:*

$$\begin{bmatrix} -4 & 3 & 1 & 1 & 0 \\ -4 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 5 & 1 \\ 3 & -2 & 0 & 0 & 0 \end{bmatrix},$$

which gives the last canonical singularity with matrix $P_{10}$.

□

We come to Case (*ii*) of Proposition 1.4.7.

PROPOSITION 4.4.4. *Let $X$ be a non-toric threefold singularity of complexity one. Assume that $X$ is of canonical multiplicity three, Gorenstein index $\imath \geq 2$, the leading platonic triple is $(3, 3, 2)$ and $X$ is at most canonical. Then for suitable $A$, $X$ is isomorphic to $X(A, P_i)$, where $P_i$ is one of the following matrices:*

$$P_{11} = \begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & 1 \end{bmatrix} \quad P_{12} = \begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

PROOF. Our matrix $P$ falls under Case (*ii*) of Proposition 1.4.7, thus $\imath$ is a multiple of 3 and the leading block has the shape

$$\begin{bmatrix} -3 & 3 & 0 & 0 & \dots & 0 \\ -3 & 0 & 2 & 0 & \dots & 0 \\ -3 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -3 & 0 & 0 & 0 & \dots & 1 \\ d_{011} & d_{111} & d_{211} & 0 & \dots & 0 \\ \frac{\imath}{3} - 1 & \frac{\imath}{3} + 1 & -\frac{\imath}{3} & 0 & \dots & 0 \end{bmatrix}.$$

If $r \geq 3$, due to irredundancy we have at least another column $v_{32} = e_3 + d_{321}e_{r+1}$ in $P$ with $d_{321} \geq 1$. So $A_X^c(\lambda)$ contains the integer points

$$v(\tau_1)' = (0, \dots, 0, 2(d_{011} + d_{111}) + 3d_{211}, \imath/3),$$
$$v(\tau_2)' = (0, \dots, 0, 2(d_{011} + d_{111}) + 3d_{211} + 6d_{321}, \imath/3)$$

and with Lemma 4.3.18, only $\imath = 3$ is possible. By primitivity of the first column and by if necessary negating the penultimate row, we can assume $d_{011} = 3k + 1$ for some $k \in \mathbb{Z}$. By adding the $(2d_{211}+d_{111}+4k)$-fold of the first to the penultimate and subtracting the $(2d_{211} + d_{111} + 3k)$-fold of the second and the $(3d_{211} + 2d_{111} + 6k)$-fold of the last to the penultimate row, we achieve $d_{011} = 1$, $d_{111} = d_{211} = 0$ and $v(\tau_1)' = (0, \dots, 0, 2, 1)$ and $v(\tau_2)' = (0, \dots, 0, 2 + 6d_{321}, \imath/3)$. But for $d_{321} \leq 0$, the point $(1, 0, 0, 1)$ and for $d_{321} \geq 2$, the point $(1, 0, 1, 1)$ lies inside $A_X^c(\lambda_1)$, while $d_{321} = 1$ is not possible, since then $v_{32}$ would lie inside $\tau_1$. Thus only $r = 2$ is possible and the leading block is

$$\begin{bmatrix} -3 & 3 & 0 \\ -3 & 0 & 2 \\ d_{011} & d_{111} & d_{211} \\ \frac{\imath}{3} - 1 & \frac{\imath}{3} + 1 & -\frac{\imath}{3} \end{bmatrix}.$$

Now we have to check which additional columns are possible.

*Case 1: additional column* $v_1 = (0, 0, d_1, \imath/3)$ *in* $\lambda$. Since $v(\tau_1)'$ is integer, an additional column in the lineality part means two integer points in $A_X^c(\lambda)$, which forces $\imath = 3$. The leading block together with the additional column has the form

$$\begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ d_{011} & d_{111} & d_{211} & d_1 \\ 0 & 2 & -1 & 1 \end{bmatrix}.$$

By primitivity of the first column and by if necessary negating the third row, we can assume $d_{011} = 3k + 1$ for some $k \in \mathbb{Z}$. By adding the $(2d_{211} + d_{111} + 4k)$-fold of the first to the third and subtracting the $(2d_{211} + d_{111} + 3k)$-fold of the second and the $(3d_{211} + 2d_{111} + 6k)$-fold of the last to the third row, we achieve $d_{011} = 1$, $d_{111} = d_{211} = 0$ and $v(\tau_1)' = (0, 0, 2, 1)$. Thus $d_1 = 2$ is impossible and the following cases remain.

*Case 1.1:* $d_1 \leq 0$. Then inside $A_X^c(\lambda_1)$ lies the point

$$(1, 0, 0, 1) = \frac{1}{3}v_{01} + \frac{d_1}{3(d_1 - 2)}v(\tau_1)' - \frac{2}{3(d_1 - 2)}v_1.$$

*Case 1.2:* $d_1 \geq 3$. Then inside $A_X^c(\lambda_1)$ lies the point

$$(1, 0, 1, 1) = \frac{1}{3}v_{01} + \frac{(d_1 - 3)}{3(d_1 - 2)}v(\tau_1)' + \frac{1}{3(d_1 - 2)}v_1.$$

*Case 1.3:* $d_1 = 1$. Since $A_X^c$ is canonical in this case, we get a canonical singularity from the matrix

$$P_{11} := \begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & 1 \end{bmatrix}.$$

*Case 2: additional column* $v_{02} = (-t, -t, d_{021}, (\imath - t)/3)$ *in* $\lambda_0$. Since $v_{02}$ must be integer, we have $t = 3$ and can assume $d_{021} > d_{011}$. Thus $A_X^c(\lambda)$ contains the two integer points $v(\tau_i)' = (0, 0, 2(d_{0i1} + d_{111}) + 3d_{211}, \imath/3)$ with $i = 1, 2$. As in Case 1, this forces $\imath = 3$. By the same admissible operations as in Case 1, we can now achieve $d_{011} = 1$, $d_{111} = d_{211} = 0$, $v(\tau_1)' = (0,0,2,1)$ and $v(\tau_2)' = (0,0,2d_{021},1)$. But for $d_{021} \le 0$, the point $(1,0,0,1)$ and for $d_{021} \ge 2$, the point $(1,0,1,1)$ lies inside $A_X^c(\lambda_1)$, as Case 1 shows.

*Case 3: additional column* $v_{12} = (t, 0, d_{121}, (\imath + t)/3)$ *in* $\lambda_1$. The proceeding is exactly the same as in Case 2.

*Case 4: additional column* $v_{22} = (0, t, d_{221}, \imath(1-t)/3)$ *in* $\lambda_2$. Here we have the possibilities $t = 1, 2$.

*Case 4.1:* $t = 2$. Here we can assume $d_{221} > d_{211}$ and $A_X^c(\lambda)$ contains the two integer points $v(\tau_i)' = (0,0,2(d_{011}+d_{111}) + 3d_{2i1}, \imath/3)$. This again forces $\imath = 3$ and the proceeding is again the same as in the previous cases.

*Case 4.2:* $t = 1$. We have the two integer points $v(\tau_1)' = (0,0,2(d_{011}+d_{111}) + 3d_{211}, \imath/3)$ and $v(\tau_2)' = (0,0,d_{011}+d_{111}+3d_{221}, \imath/3)$ inside $A_X^c(\lambda)$, again forcing $\imath = 3$ and we thus achieve $d_{011} = 1$, $d_{111} = d_{211} = 0$, $v(\tau_1)' = (0,0,2,1)$ and $v(\tau_2)' = (0,0,1+3d_{021},1)$. So regarding $A_X^c(\lambda_1)$, we see that the only possibility is $d_{021} = 0$, giving a canonical singularity with matrix

$$P_{12} := \begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}.$$

The two matrices we found cannot be combined, since $(0,0,1,1)$ lies inside

$$\operatorname{cone}((-3,-3,1,0),(3,0,0,2),(0,1,0,0)).$$

□

We proceed with Case (*iii*) of Proposition 1.4.7.

PROPOSITION 4.4.5. *Let $X$ be a non-toric threefold singularity of complexity one. Assume that $X$ is of canonical multiplicity four, Gorenstein index $\imath \ge 2$, the leading platonic triple is $(l_{01}, 2, 2)$ with $l_{01}$ odd and $X$ is at most canonical. Then for suitable $A$, $X$ is isomorphic to $X(A, P_i)$, where $P_i$ is one of the matrices*

$$P_{13} = \begin{bmatrix} -(2\mathbf{k}+1) & 2 & 0 & 0 & 0 & \dots & 0 & 0 \\ -(2\mathbf{k}+1) & 0 & 2 & 0 & 0 & \dots & 0 & 0 \\ -(2\mathbf{k}+1) & 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -(2\mathbf{k}+1) & 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ \mathbf{d_0} & 1 & 1 & 0 & d_3 & \dots & 0 & d_r \\ \mathbf{k} & 0 & 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \qquad P_{14} = \begin{bmatrix} -(2\mathbf{k}+1) & 2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ -(2\mathbf{k}+1) & 0 & 2 & 2 & 0 & 0 & \dots & 0 & 0 \\ -(2\mathbf{k}+1) & 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -(2\mathbf{k}+1) & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ \mathbf{d_0} & 1 & 1 & d_2 & 0 & d_3 & \dots & 0 & d_r \\ \mathbf{k} & 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

*In both cases $r \ge 2$ and $\mathbf{k}, \mathbf{d_0} \in \mathbb{Z}^t$ for some $t \in \mathbb{Z}_{\ge 1}$.*

PROOF. Our matrix $P$ falls under Case (*iii*) of Proposition 1.4.7, thus $\imath \equiv 2 \mod 4$ and the leading block has the shape

$$\begin{bmatrix} -(2k+1) & 2 & 0 & 0 & \dots & 0 \\ -(2k+1) & 0 & 2 & 0 & \dots & 0 \\ -(2k+1) & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(2k+1) & 0 & 0 & 0 & \dots & 1 \\ d_{011} & d_{111} & d_{211} & 0 & \dots & 0 \\ -k\imath/2 & (\imath-2)/4 & (\imath+2)/4 & 0 & \dots & 0 \end{bmatrix}.$$

One of $(\imath-2)/4$ and $(\imath+2)/4$ is even, while the other is odd. So if both $d_{111}$ and $d_{211}$ are odd, adding appropriate multiples of the first two rows to the penultimate gives $d_{111} = d_{211} = 1$. We achieve the same if one is odd and the other even by before adding the last to the penultimate row. So we proceed with $d_{111} = d_{211} = 1$.

*Case 1:* $r \geq 3$. Due to irredundancy we have at least another column $v_{32} = e_3 + d_{321}e_{r+1}$ in $P$ with $d_{321} > 0$. So $A_X^c(\lambda)$ contains the points

$$v(\tau_1)' = (0, \ldots, 0, d_{011} + 2k + 1, \imath/4)\,,$$
$$v(\tau_2)' = (0, \ldots, 0, d_{011} + 2k + 1 + (2k+1)d_{321}, \imath/4)\,.$$

Applying Lemma 4.3.28, we get $\imath = 2$. In this case, all $A_X^c(\lambda_i)$ but for $i = 1, 2$ have height one. But the only possible integer points in the relative interiors of $A_X^c(\lambda_1)$, $A_X^c(\lambda_2)$ are those of the form $(1, 0, \ldots, 0, k, 0)$. Thus in every leaf but $A_X^c(\lambda_1)$ arbitrary columns can be added to the leading block to get a canonical singularity - if the resulting matrix is an appropriate defining matrix $P$. We write this as the two series $P_{13}$ and $P_{14}$ from above.

*Case 2:* $r = 2$. If $\imath = 2$, all $\partial A_X^c(\lambda_i)$ have height one as above, we are in the same situation as in Case 1 and the resulting matrix is the case $r = 2$ of the series $P_{13}$ and $P_{14}$. So we can assume $\imath \geq 6$. Our leading block is

$$\begin{bmatrix} -(2k+1) & 2 & 0 \\ -(2k+1) & 0 & 2 \\ d_{011} & 1 & 1 \\ -k\imath/2 & (\imath-2)/4 & (\imath+2)/4 \end{bmatrix}$$

and $v(\tau_1)' = (0, 0, d_{011} + 2k + 1, \imath/4)$. But for any allowed additional column, we get $v(\tau_2)' = (0, 0, \kappa, \imath/4)$ with $\kappa \in \mathbb{Z}$. Applying Lemma 4.3.28 again, we get $\imath = 2$, a contradiction. □

The next Proposition deals with Case $(iv)$ of Proposition 1.4.7.

PROPOSITION 4.4.6. *Let $X$ be a non-toric threefold singularity of complexity one. Assume that $X$ is of canonical multiplicity two, even Gorenstein index, the leading platonic triple is $(2k, 2, 2)$ and $X$ is at most canonical. Then for suitable $A$, $X$ is isomorphic to $X(A, P_i)$, where $P_i$ is one of the matrices*

$$P_{15} = \begin{bmatrix} -2k & 2 & 0 & 0 \\ -2k & 0 & 2 & 0 \\ 1 & 1 & 1 & 2k+2 \\ 1-k & 2 & -1 & 1 \end{bmatrix} \quad P_{16} = \begin{bmatrix} -2k & 2 & 0 & 0 & 0 \\ -2k & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 2k & 2k+2 \\ 1-k & 2 & -1 & 1 & 1 \end{bmatrix}$$

$$P_{17} = \begin{bmatrix} -4 & 2 & 0 & 0 \\ -4 & 0 & 2 & 1 \\ 1 & 1 & 1 & 4 \\ -1 & 2 & -1 & 0 \end{bmatrix} \quad P_{18} = \begin{bmatrix} -4 & 2 & 0 & 0 & 0 \\ -4 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 4 \\ -1 & 2 & -1 & 0 & 0 \end{bmatrix} \quad P_{19} = \begin{bmatrix} -4 & 2 & 0 & 0 & 0 \\ -4 & 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 4 & 4 \\ -1 & 2 & -1 & 0 & 1 \end{bmatrix}$$

PROOF. Our matrix $P$ falls under Case $(iv)$ of Proposition 1.4.7, thus $\imath$ is even and the leading block has the shape

$$\begin{bmatrix} -2k & 2 & 0 & 0 & \ldots & 0 \\ -2k & 0 & 2 & 0 & \ldots & 0 \\ -2k & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2k & 0 & 0 & 0 & \ldots & 1 \\ d_{011} & d_{111} & d_{211} & 0 & \ldots & 0 \\ \imath/2-k & \imath/2+1 & -\imath/2 & 0 & \ldots & 0 \end{bmatrix}$$

with $k \geq 2$ and we have $v(\tau_1)' = (0, \ldots, 0, d_{011} + k(d_{111} + d_{211}), \imath/2)$. If now $r \geq 3$, we have at least another column $v_{321}$ so that $v(\tau_2)' = v(\tau_1)' + 2kd_{321}e_{r+1}$ and we can assume $d_{321} \geq 1$ as before. Since both $v(\tau_i)'$ are integer, this forces $\imath = 2$. Now $d_{111}$ must be odd, so by if necessary adding the last to the penultimate and appropriate multiples of the first two to the penultimate row we achieve $d_{111} = d_{211} = 1$. Now by adding $-2\lfloor d_{011}/2\rfloor$ times the first, $\lfloor d_{011}/2\rfloor$ times the second and $2\lfloor d_{011}/2\rfloor$ times the last to the penultimate row, we achieve $d_{011} \in \{0,1\}$. Assume $d_{011} = 0$, then

$$(-1,-1,k-1,0) = \frac{1}{2k}v_{01} + \frac{k-1}{2k}v(\tau_1)'$$

lies inside $A_X^c(\lambda_0)$. But if $d_{011} = 1$, then since $k \geq 2$ inside $A_X^c(\lambda_0)$ lies

$$(-1,-1,k,0) = \frac{1}{2k}v_{01} + \frac{2d_{321}k + 2d_{321} - 1}{4kd_{321}}v(\tau_1)' + \frac{1}{4kd_{321}}v(\tau_2)'.$$

So we can assume $r = 2$. We achieve $d_{111} = d_{211} = 1$ as before. Additional columns in $\lambda$, $\lambda_0$, $\lambda_1$ as well as in $\lambda_2$ with $l_{22} = 2$ lead to two integer points in $A_X^c(\lambda)$ and thus to $\imath = 2$. We treat this in the following first case while in the second case we consider the case of an additional column in $\lambda_2$ with $l_{22} = 1$.

*Case 1.* We have $\imath = 2$ and achieve $d_{011} = 1$. So $v(\tau_1)' = (0,0,1+2k,1)$. Now if we have $i = 1,2$ and an additional column $v_{i2}$ in $\lambda_i$ with $l_{i2} = 2$, then we can assume $d_{i21} \geq 2$ and thus $v(\tau_2)' = v(\tau_1)' + k(d_{i21}-1)e_3$. The point

$$(-1,-1,k,0) = \frac{1}{2k}v_{01} + \frac{k(k-1)(d_{i21}-1) - k}{2k^2(d_{i21}-1)}v(\tau_1)' + \frac{1}{2k(d_{i21}-1)}v(\tau_2)'$$

lies inside $A_X^c(\lambda_0)$. For an additional column $v_{02} = (-2k', -2k', d_{021}, 1-k')$ in $\lambda_0$, where we can assume $d_{021}$ odd and $d_{021} + 2k' \geq 3 + 2k$, again $(-1,-1,k,0)$ lies inside. Now assume an additional column $v_1 = (0,0,d_1,1)$ in $\lambda$. We can assume $d_1 > 1+2k$ for if not, i.e. $d_1 \leq 1+2k$, adding the $1+4k$-fold of the first, the $-2(k+1)$-fold of the second and the $-2(2k+1)$-fold of the last to the third row and negating the third row afterwards gives $d_1 > 1+2k$. Then the point

$$(-1,-1,k,0) = \frac{1}{2k}v_{01} + \frac{d_1k - 2k^2 - d_1 + 1}{2k(d_1-2k-1)}v(\tau_1)' + \frac{1}{2(d_1-2k-1)}v_1$$

lies inside $A_X^c(\lambda_0)$ for $d_1 \geq 3+2k$. Thus we require $d_1 = 2+2k$ and get the matrix

$$P_{15} := \begin{bmatrix} -2k & 2 & 0 & 0 \\ -2k & 0 & 2 & 0 \\ 1 & 1 & 1 & 2k+2 \\ 1-k & 2 & -1 & 1 \end{bmatrix},$$

giving a canonical singularity.

*Case 2: additional column* $v_{22} = (0,1,d_{221},0)$ *in* $\lambda_2$. The leading block together with the additional column has the form

$$\begin{bmatrix} -2k & 2 & 0 & 0 \\ -2k & 0 & 2 & 1 \\ d_{011} & 1 & 1 & d_{221} \\ \imath/2 - k & \imath/2+1 & -\imath/2 & 0 \end{bmatrix}$$

and we have $v(\tau_1)' = (0,0,d_{011}+2k,\imath/2)$. We write $d_{221} = (d_{011}+2k+1)/2 + s$ for suitable $s$ and get

$$v(\tau_2)' = (0,0,d_{011}+2k+2ks/(k+1),\imath/2)\,.$$

Then $|v(\tau_1)' - v(\tau_2)'| < 1$ if $|s| < 1/2 + 1/(2k)$. Since $k \geq 2$, this is only possible for $d_{011}$ even, $s = \pm 1/2$. We examine these cases and the case $|v(\tau_1)' - v(\tau_2)'| \geq 1$ afterwards.

*Case 2.1:* $d_{011}$ *even,* $s = 1/2$. Here we have $|v(\tau_1)' - v(\tau_2)'| = k/(k+1) \geq 1/2$. With Corollary 4.3.22, this forces $(\imath/2)|(d_{011}+2k+1)$ and thus $\imath \equiv 2 \mod 4$ and $k$ even. We achieve $d_{011} = \imath/2 - 2k - 1$ by adding the $(\imath+2)(2d_{011}+4k+2-\imath)/(4\imath)$-fold of the first, the $(\imath - 2d_{011} - 4k - 2)/4$-fold of the second and the $(\imath - 2d_{011} - 4k - 2)/\imath$-fold of the last to the third row. But then inside $A_X^c(\lambda_0)$ lies the point

$$(-1, -1, (\imath - 6)/4, (\imath - 2)/4) = \frac{1}{2k} v_{01} + \frac{k-1}{2k} v(\tau_1)'.$$

*Case 2.2: $d_{011}$ even, $s = -1/2$.* Here we have $|v(\tau_1)' - v(\tau_2)'| = k/(k+1) \geq 1/2$. With Corollary 4.3.22, this forces $(\imath/2)|(d_{011} + 2k - 1)$ and thus $\imath \equiv 2 \mod 4$ and $k$ even. We achieve $d_{011} = \imath/2 - 2k + 1$ by adding the $(\imath+2)(2d_{011} + 4k - 2 - \imath)/(4\imath)$-fold of the first, the $(\imath - 2d_{011} - 4k + 2)/4$-fold of the second and the $(\imath - 2d_{011} - 4k + 2)/\imath$-fold of the last to the third row. But then the point

$$(-1, -1, (\imath - 2)/4, (\imath - 2)/4) = \frac{1}{2k} v_{01} + \frac{k-1}{2k} v(\tau_1)'$$

lies inside $A_X^c(\lambda_0)$.

*Case 2.3:* $|v(\tau_1)' - v(\tau_2)'| \geq 1$. This leads to $\imath = 2$ and we can assume $d_{011} = 1$ as we have seen above. Now $s$ is integer and we can assume $s \geq 1$ by the same admissible operation as in Case 1. Then $|v(\tau_1)' - v(\tau_2)'| = 2sk/(k+1) \geq 2$ if $s \geq 2$ and this gives the interior point $(-1, -1, k, 0)$ in $A_X^c(\lambda_0)$, as we also have seen in Case 1. So $s = 1$ remains. The point

$$(-1, -1, k, 0) = \frac{1}{2k} v_{01} + \frac{k-3}{4k} v(\tau_1)' + \frac{k+1}{4k} v(\tau_2)'$$

lies inside $A_X^c(\lambda_0)$ for $k \geq 3$. So we get a canonical singularity from the matrix

$$P_{17} := \begin{bmatrix} -4 & 2 & 0 & 0 \\ -4 & 0 & 2 & 1 \\ 1 & 1 & 1 & 4 \\ -1 & 2 & -1 & 0 \end{bmatrix}.$$

Finally, we have to check which of the additional columns can be combined. On the one hand, we have the matrix $P_{15}$ where in the last column, we can also write $2k$ instead of $2k + 2$, see Case 1. Thus we can also add *both* columns to the leading block and get another canonical singularity with matrix $P_{16}$. On the other hand, we have the matrix $P_{17}$ where we can take $(0, 1, 2, 0)^\top$ instead of the last column and get the same singularity. Thus here as well we can take both of these columns in addition to the leading block and get the matrix $P_{18}$. Now as the leading block of $P_{17}$ is the special case $k = 2$ of the leading block of $P_{15}$, we also have to check if we can add columns $(0, 0, 4, 1)^\top$ and $(0, 0, 6, 1)^\top$. But $(0, 0, 6, 1)^\top$ lies inside $\operatorname{cone}(v_{01}, v_{11}, v_{22})$, while $(0, 0, 4, 1)^\top$ can be added to achieve the matrix

$$P_{19} := \begin{bmatrix} -4 & 2 & 0 & 0 & 0 \\ -4 & 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 4 & 4 \\ -1 & 2 & -1 & 0 & 1 \end{bmatrix},$$

giving a canonical singularity. To this matrix, the column $(0, 1, 2, 0)^\top$ can not be added, since then $(0, 0, 4, 1)^\top$ would lie inside $\operatorname{cone}(v_{01}, v_{11}, (0, 1, 2, 0)^\top)$.

□

We come to Case $(v)$ of Proposition 1.4.7.

PROPOSITION 4.4.7. *Let $X$ be a non-toric threefold singularity of complexity one. Assume that $X$ is of canonical multiplicity two, even Gorenstein index, the* *leading platonic triple is $(k,2,2)$ and $X$ is at most canonical. Then for suitable $A$, $X$ is isomorphic to $X(A,P_i)$, where $P_i$ is one of the matrices*

$$P_{20} = \begin{bmatrix} -\mathbf{k} & 2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -\mathbf{k} & 0 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -\mathbf{k} & 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -\mathbf{k} & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0 \\ \mathbf{d_0} & 1 & 1 & 0 & d_3 & \dots & 0 & d_r & d_1' & d_2' \\ 1-\mathbf{k} & 0 & 2 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \end{bmatrix} \qquad P_{21} = \begin{bmatrix} -(2k+1) & 2 & 0 & 0 \\ -(2k+1) & 0 & 2 & 0 \\ -k & 0 & 1 & 1 \\ -4k & 1 & 3 & 2 \end{bmatrix}$$

*One or both of the last two columns of $P_{20}$ may be omitted, if the requirements on the matrix from Construction 1.2.2 are still fulfilled.*

Proof. Our matrix $P$ falls under Case $(v)$ of Proposition 1.4.7, thus $\imath$ is even and the leading block has the shape

$$\begin{bmatrix} -l_{01} & 2 & 0 & 0 & \dots & 0 \\ -l_{01} & 0 & 2 & 0 & \dots & 0 \\ -l_{01} & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -l_{01} & 0 & 0 & 0 & \dots & 1 \\ d_{011} & d_{111} & d_{211} & 0 & \dots & 0 \\ \imath/2(1-l_{01}) & \imath/2-1 & \imath/2+1 & 0 & \dots & 0 \end{bmatrix}$$

with $l_{01} \in \mathbb{Z}_{\geq 2}$. Now we distinguish the case $\imath = 2$ and $\imath \geq 4$.

*Case 1:* $\imath = 2$. Here we can assume $d_{111} = d_{211} = 1$ by adding appropriate multiples of the first two rows to the penultimate and also $d_{011} = 0$ by afterwards adding the $d_{011}$-fold of the second and the $-d_{011}$ fold of the last to the penultimate row. Now we see that $A_X^c(\lambda_0)$ and the $A_X^c(\lambda_i)$ for $i = 3, \dots, r$ have height one and are thus canonical for any allowed additional column. In $A_X^c(\lambda_1)$, $A_X^c(\lambda_2)$, the points that may be inside are of the forms $(1,0,\dots,0,k,0)$ and $(0,1,0,\dots,0,k,0)$. So in these leaves, no additional columns are allowed. We denote the resulting series by

$$\begin{bmatrix} -\mathbf{k} & 2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -\mathbf{k} & 0 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -\mathbf{k} & 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -\mathbf{k} & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0 \\ \mathbf{d_0} & 1 & 1 & 0 & d_3 & \dots & 0 & d_r & d_1' & d_2' \\ 1-\mathbf{k} & 0 & 2 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \end{bmatrix}.$$

*Case 2:* $\imath \geq 4$. Here $r \geq 3$ is impossible, since then we would have at least an additional column $v_{321}$ and would get $|v(\tau_1)' - v(\tau_2)'| \geq 2$, so that $A_X^c(\lambda)$ contains two integer points forcing $\imath = 2$, a contradiction. So we have $r = 2$. We have

$$v(\tau_1)' = (0, 0, d_{011} + l_{01}(d_{111} + d_{211})/2, \imath/2)$$

and distinguish three subcases in the following.

*Case 2.1:* $\imath \equiv 2 \mod 4$. Here both $d_{111}$ and $d_{211}$ must be odd and adding appropriate multiples of the first two rows to the third, we achieve $d_{111} = d_{211} = 1$. Thus $v(\tau_1)'$ is integer and no second integer point is allowed in $A_X^c(\lambda)$. Since for any allowed additional column such a second integer point will appear, we get no canonical singularity in this case.

*Case 2.2:* $\imath \equiv 0 \mod 4$, $d_{111} \equiv d_{211} \mod 2$. By if necessary adding the last to the third and afterwards adding appropriate multiples of the first two rows to the third, we achieve $d_{111} = d_{211} = 0$, i.e. a leading block of the form

$$\begin{bmatrix} -l_{01} & 2 & 0 \\ -l_{01} & 0 & 2 \\ d_{011} & 0 & 0 \\ \imath/2(1-l_{01}) & \imath/2-1 & \imath/2+1 \end{bmatrix}.$$

Again $v(\tau_1)'$ is integer and the only additional columns that do not give a second integer point in $A_X^c(\lambda)$ are those of the form $(2, 0, d_{121}, \imath/2-1)$ and $(0, 2, d_{221}, \imath/2-1)$ in $\lambda_1$ and $\lambda_2$ respectively with $d_{i21}$ odd. Both leaves are equivalent by adding the first and subtracting the second from the last row, so we consider additional columns $(2, 0, d_{121}, \imath/2-1)$. For such a column, we have

$$v(\tau_2)' = (0, 0, d_{011} + l_{01}d_{121}/2, \imath/2),$$

so we require $l_{01}d_{121}/2 = \pm 1/2$, which is impossible due to $l_{01} \geq 2$.

*Case 2.3: $\imath \equiv 0 \mod 4$, $d_{111} \equiv d_{211} + 1 \mod 2$.* By the same proceeding as in the previous case, we get $d_{111} = 0$, $d_{211} = 1$ and $v(\tau_1)' = (0, 0, d_{011} + l_{01}/2, \imath/2)$. Now in the following subcases, we check which additional columns are possible.

*Case 2.3.1: additional column $v_1 = (0, 0, d_1, \imath/2)$ in $\lambda$.* With Lemma 4.3.18, we get $l_{01} = 2k + 1$ odd for some $k \in \mathbb{Z}_{\geq 1}$ and $d_1 = d_{011} + (l_{01} \pm 1)/2$. From this we get that $\imath/2$ divides $d_{011} + (l_{01} \pm (-1))/2$.

*Case 2.3.1.1: $d_1 = d_{011} + (l_{01} + 1)/2$.* Here $\imath/2$ divides $d_{011} + k$. By adding the $1 - \imath/2$-fold of the first, the $-(1 + \imath/2)$-fold of the second and two times the last row to the third, we can add multiples of $\imath$ to $d_{011}$, so we can achieve one of the following cases:

*Case 2.3.1.1.1: $d_{011} = -k$.* Here if $\imath \geq 8$, the point

$$(-1, -1, 0, -1) = \frac{1}{2k+1}v_{01} + \frac{2(\imath k - 4k - 2)}{\imath(2k+1)}v(\tau_1)' + \frac{2}{\imath}v_1$$

lies inside $A_X^c(\lambda_0)$, so we require $\imath = 4$. We get the canonical matrix

$$P_{21} := \begin{bmatrix} -(2k+1) & 2 & 0 & 0 \\ -(2k+1) & 0 & 2 & 0 \\ -k & 0 & 1 & 1 \\ -4k & 1 & 3 & 2 \end{bmatrix}.$$

*Case 2.3.1.1.2: $d_{011} = \imath/2 - k$.* This is equivalent to Case 2.3.1.1.1 by adding the $\imath/4$-fold of the first two and the $-1$-fold of the last to the third row.

*Case 2.3.1.2: $d_1 = d_{011} + (l_{01} - 1)/2$.* Here $\imath/2$ divides $d_{011} + k + 1$. By the same admissible operation as in Case 2.3.1.1, we can assume $d_{011} = -k - 1$. But now this is equivalent to the Case 2.3.1.1 by subtracting the second from the third row and afterwards negating the third row.

*Case 2.3.2: additional column $v_{02} = (-l_{02}, -l_{02}, d_{021}, \imath/2(1 - l_{02}))$ in $\lambda_0$.* For such a column, we have $v(\tau_2)' = (0, 0, d_{021} + l_{02}/2, \imath/2)$, which of course must not be equal to $v(\tau_1)'$. Thus we distinguish several subcases.

*Case 2.3.2.1: $l_{01} \equiv l_{02} \equiv 0 \mod 2$.* In this case, both $v(\tau_i)'$ are integer, which is a contradiction to $\imath \geq 4$ with Lemma 4.3.18.

*Case 2.3.2.2: $l_{01} \equiv l_{02} \equiv 1 \mod 2$.* Here, the third entry of both $v(\tau_i)'$ is in $\mathbb{Z} + 1/2$ and so, $A_X^c(\lambda)$ contains a polytope like in Corollary 4.3.23, forcing $\imath = 4$ and - writing $l_{01} = 2k_1 + 1$, $l_{02} = 2k_2 + 1$ - we achieve $d_{011} = -k_1$ as in the previous case, forcing $d_{021} = 1 - k_2$. But for $k_1 \geq 2$, inside $A_X^c(\lambda_0)$ lies the point

$$(-1, -1, 0, -1) = \frac{1}{2k_1+1}v_{01} + \frac{2k_1 - 3}{4(2k_1+1)}v(\tau_1)' + \frac{1}{4}v(\tau_2)',$$

while for $k_1 = 1$, we have $k_2 \in \{0, 1\}$, and thus inside $A_X^c(\lambda_0)$ lies

$$(-1,-1,0,-1) = \frac{3-2k_2}{12} v_{01} + \frac{1}{4} v_{02} + \frac{k_2}{6} v(\tau_1)'.$$

*Case 2.3.2.3:* $l_{01} \equiv l_{02}+1 \equiv 1 \mod 2$. Here we drop the condition $l_{01} \geq l_{02}$ coming from $(v_{01}, v_{11}, v_{21})$ being the leading block so that we do not have to examine a separate subcase. From Lemma 4.3.18 and Corollary 4.3.22, we get that (by if necessary negating the third row) for the third entries of the $v(\tau_i)'$

$$d_{021} + l_{02}/2 = d_{011} + l_{01}/2 + 1/2 = k\imath/2 + 1$$

By the same admissible operations as in Case 2.3.1, we get - writing $l_{01} = 2k_1 + 1$, $l_{02} = 2k_2$ - that $d_{011} = -k_1$, $d_{021} = 1 - k_2$ holds. But as we also have seen in Case 2.3.1.1.1, this forces $\imath = 4$ with $v_1$ replaced with $v(\tau_2)'$. But here, the point

$$(-1,-1,0,-1) = \frac{1}{2k_2} v_{02} + \frac{k_2 - 1}{2k_2} v(\tau_2)'$$

again lies inside $A_X^c(\lambda_0)$, so we get no canonical singularity.

*Case 2.3.3: additional column* $v_{12} = (-2, -2, d_{121}, \imath/2 - 1)$ *in* $\lambda_1$. This leads to $v(\tau_2)' = (0, 0, d_{011} + l_{01}(1 + d_{121})/2, \imath/2)$, where we can assume $d_{121} \geq 1$. We have $|v(\tau_1)' - v(\tau_2)'| = \frac{l_{01} d_{121}}{2} \geq 3/2$ if $d_{121} \geq 2$ or $l_{01} \geq 3$, which is not possible due to Lemma 4.3.18. So we require $d_{121} = 1$ and $l_{01} = 2$. But here both $v(\tau_i)'$ are integer, so this is not possible as well. The proof is complete. □

Finally, we deal with the last Case $(vi)$ of Proposition 1.4.7.

PROPOSITION 4.4.8. *Let $X$ be a non-toric threefold singularity of complexity one. Assume that $X$ is of arbitrary canonical multiplicity $\zeta > 1$, the leading platonic triple is $(l_{01}, l_{11}, 1)$ and $X$ is at most canonical. Then for suitable $A$, $X$ is isomorphic to $X(A, P_i)$, where $P_i$ is one of the matrices*

$$P_{22} = \begin{bmatrix} -4 & 4 & 0 & 0 \\ -4 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & 3 & 0 & 0 \end{bmatrix} \quad P_{23} = \begin{bmatrix} -4 & 2 & 0 & 0 \\ -4 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{bmatrix} \quad P_{24} = \begin{bmatrix} -8 & 2 & 0 & 0 \\ -8 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 5 & 0 & 0 & 0 \end{bmatrix} \quad P_{25} = \begin{bmatrix} -2 & 2 & 0 & 0 \\ -2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix}$$

$$P_{26} = \begin{bmatrix} -2 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -2 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & d_2 & \dots & 0 & d_r & d'_1 & d'_2 \\ 0 & 2 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \end{bmatrix} \quad P_{27} = \begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix} \quad P_{28} = \begin{bmatrix} -5 & 4 & 0 & 0 \\ -5 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 2 & 0 & 0 \end{bmatrix}$$

$$P_{29} = \begin{bmatrix} 2 - m_0\zeta & 2 & 0 & 0 \\ 2 - m_0\zeta & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 + \frac{m_0(2-\zeta)}{2} & 1 & 0 & 0 \end{bmatrix}, \zeta \in 4\mathbb{Z}_{\geq 1} \qquad P_{30} = \begin{bmatrix} 2 - 5\zeta & 2 & 0 & 0 \\ 2 - 5\zeta & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 5 & 0 & 0 & 0 \end{bmatrix}, \zeta \geq 3$$

$$P_{31} = \begin{bmatrix} 4 - 3\zeta & 4 & 0 & 0 \\ 4 - 3\zeta & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ \frac{7-3\zeta}{2} & 2 & 0 & 0 \end{bmatrix}, \zeta \in 2\mathbb{Z}_{\geq 2} + 1 \qquad P_{32} = \begin{bmatrix} 2 - 3\zeta & 2 & 0 & 0 & 0 & 0 \\ 2 - 3\zeta & 0 & 1 & 1 & 0 & 0 \\ 2 - 3\zeta & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{33} = \begin{bmatrix} 2 - 3\zeta & 2 & 0 & 0 \\ 2 - 3\zeta & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 0 & 0 \end{bmatrix} \quad P_{34} = \begin{bmatrix} 2 - 3\zeta & 2 & 0 & 0 \\ 2 - 3\zeta & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{bmatrix} \quad P_{35} = \begin{bmatrix} 2 - 3\zeta & 2 - 3\zeta & 2 & 0 & 0 \\ 2 - 3\zeta & 2 - 3\zeta & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 3 & 3 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{36} = \begin{bmatrix} 2 - 3\zeta & 2 - \zeta & 2 & 0 & 0 \\ 2 - 3\zeta & 2 - \zeta & 0 & 1 & 1 \\ 0 & 3 & 1 & 0 & 1 \\ 3 & 1 & 0 & 0 & 0 \end{bmatrix}$$

*For* $P_{37} - P_{39}$ *it holds* $k\mu + 1 \equiv 0 \mod \zeta$, $d_{11} \leq 0$, $d_{12} \geq 2 + k\sum_{i=2}^{r} d_i$.

$$P_{37} = \begin{bmatrix} 2(\zeta-k) & 2k & 0 & 0 & \dots & 0 & 0 \\ 2(\zeta-k) & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(\zeta-k) & 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 1 & 0 & d_2 & \dots & 0 & d_r \\ 2\left(\frac{k\mu+1}{\zeta} - \mu\right) & 2\frac{k\mu+1}{\zeta} & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad P_{38} = \begin{bmatrix} 2(\zeta-k) & 2k-\zeta & 2k & 0 & 0 & \dots & 0 & 0 \\ 2(\zeta-k) & 2k-\zeta & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(\zeta-k) & 2k-\zeta & 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 1 & d_{11} & 0 & d_2 & \dots & 0 & d_r \\ 2\left(\frac{k\mu+1}{\zeta} - \mu\right) & 2\frac{k\mu+1}{\zeta} - \mu & 2\frac{k\mu+1}{\zeta} & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$P_{39} = \begin{bmatrix} 2(\zeta-k) & 2k-\zeta & 2k-\zeta & 2k & 0 & 0 & \dots & 0 & 0 \\ 2(\zeta-k) & 2k-\zeta & 2k-\zeta & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(\zeta-k) & 2k-\zeta & 2k-\zeta & 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & d_{11} & d_{12} & 1 & 0 & d_2 & \dots & 0 & d_r \\ 2\left(\frac{k\mu+1}{\zeta} - \mu\right) & 2\frac{k\mu+1}{\zeta} - \mu & 2\frac{k\mu+1}{\zeta} - \mu & 2\frac{k\mu+1}{\zeta} & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

*For* $P_{40} - P_{49}$ *it holds* $\zeta \equiv 5 \mod 6$.

$$P_{40} = \begin{bmatrix} 2-2\zeta & 2 & 0 & 0 & 0 & 0 \\ 2-2\zeta & 0 & 1 & 1 & 0 & 0 \\ 2-2\zeta & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 4-\zeta & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad P_{41} = \begin{bmatrix} 2-2\zeta & 2 & 0 & 0 \\ 2-2\zeta & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 4-\zeta & 1 & 0 & 0 \end{bmatrix} \quad P_{42} = \begin{bmatrix} 2-2\zeta & 2 & 0 & 0 \\ 2-2\zeta & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 4-\zeta & 1 & 0 & 0 \end{bmatrix}$$

$$P_{43} = \begin{bmatrix} 2-2\zeta & 2 & 0 & 0 \\ 2-2\zeta & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 \\ 4-\zeta & 1 & 0 & 0 \end{bmatrix} P_{44} = \begin{bmatrix} 2-2\zeta & 2-2\zeta & 2 & 0 & 0 \\ 2-2\zeta & 2-2\zeta & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 4-\zeta & 4-\zeta & 1 & 0 & 0 \end{bmatrix} P_{45} = \begin{bmatrix} 2-2\zeta & 2-2\zeta & 2 & 2 & 0 & 0 \\ 2-2\zeta & 2-2\zeta & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ 4-\zeta & 4-\zeta & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$P_{46} = \begin{bmatrix} 2-2\zeta & 2-2\zeta & 2 & 0 & 0 \\ 2-2\zeta & 2-2\zeta & 0 & 1 & 1 \\ 1 & 3 & 0 & 0 & 1 \\ 4-\zeta & 4-\zeta & 1 & 0 & 0 \end{bmatrix} \quad P_{47} = \begin{bmatrix} 2-2\zeta & 2 & 2 & 0 & 0 \\ 2-2\zeta & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 4-\zeta & 1 & 1 & 0 & 0 \end{bmatrix} \quad P_{48} = \begin{bmatrix} 2-2\zeta & 2 & 2 & 0 & 0 \\ 2-2\zeta & 0 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 4-\zeta & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$P_{49} = \begin{bmatrix} 2-2\zeta & 2-\zeta & 2 & 0 & 0 \\ 2-2\zeta & 2-\zeta & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 4-\zeta & \frac{5-\zeta}{2} & 1 & 0 & 0 \end{bmatrix} \quad P_{50} = \begin{bmatrix} 3-2\zeta & 3 & 0 & 0 \\ 3-2\zeta & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix} \quad P_{51} = \begin{bmatrix} 3-2\zeta & 3 & 3 & 0 & 0 \\ 3-2\zeta & 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

*For* $P_{52} - P_{54}$ *it holds* $\zeta \equiv 6 \mod 9$.

$$P_{52} = \begin{bmatrix} 3-2\zeta & 3 & 0 & 0 \\ 3-2\zeta & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ \frac{9-2\zeta}{3} & 1 & 0 & 0 \end{bmatrix} \quad P_{53} = \begin{bmatrix} 3-2\zeta & 3 & 3 & 0 & 0 \\ 3-2\zeta & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ \frac{9-2\zeta}{3} & 1 & 1 & 0 & 0 \end{bmatrix} \quad P_{54} = \begin{bmatrix} 3-2\zeta & 3-2\zeta & 3 & 0 & 0 \\ 3-2\zeta & 3-2\zeta & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ \frac{9-2\zeta}{3} & \frac{9-2\zeta}{3} & 1 & 0 & 0 \end{bmatrix}$$

*For* $P_{55} - P_{57}$ *it holds* $\zeta \equiv 0 \mod 9$.

$$P_{55} = \begin{bmatrix} 3-2\zeta & 3 & 0 & 0 \\ 3-2\zeta & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ \frac{12-4\zeta}{3} & 2 & 0 & 0 \end{bmatrix} \quad P_{56} = \begin{bmatrix} 3-2\zeta & 3 & 3 & 0 & 0 \\ 3-2\zeta & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ \frac{12-4\zeta}{3} & 2 & 2 & 0 & 0 \end{bmatrix} \quad P_{57} = \begin{bmatrix} 3-2\zeta & 3-2\zeta & 3 & 0 & 0 \\ 3-2\zeta & 3-2\zeta & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ \frac{12-4\zeta}{3} & \frac{9-2\zeta}{3} & 1 & 0 & 0 \end{bmatrix}$$

$$P_{58} = \begin{bmatrix} 4-2\zeta & 4 & 0 & 0 \\ 4-2\zeta & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ \frac{5-\zeta}{2} & 1 & 0 & 0 \end{bmatrix}, \zeta \equiv 7 \mod 12 \qquad P_{59} = \begin{bmatrix} 4-2\zeta & 4 & 0 & 0 \\ 4-2\zeta & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ \frac{9-3\zeta}{2} & 3 & 0 & 0 \end{bmatrix}, \zeta \equiv 1 \mod 12$$

*For* $P_{60}$, $P_{61}$ *it holds* $\zeta \in 2\mathbb{Z}_{\geq 2} + 1$.

$$P_{60} = \begin{bmatrix} 1-m_0\zeta & 1 & 1 & 0 & 0 \\ 1-m_0\zeta & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 2m_0 & 0 & 0 & 0 & 0 \end{bmatrix}, m_0 \geq 2 \qquad P_{61} = \begin{bmatrix} 2-\zeta & 2 & 0 & 0 \\ 2-\zeta & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

*For* $P_{62} - P_{65}$ *it holds* $\zeta = k\imath + \mathfrak{l}$, $\gcd(k, \mathfrak{l}) = \gcd(\mathfrak{d}, \imath) = 1$ *for all* $\mathfrak{d} \in \mathbb{Z} \cap [d, d + d_0]$.

$$P_{62} = \begin{bmatrix} -k\imath & \mathfrak{l} & 0 & 0 & \cdots & 0 & 0 \\ -k\imath & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -k\imath & 0 & 0 & 0 & \cdots & 1 & 1 \\ d & d & 0 & d_2 & \cdots & 0 & d_r \\ \imath\frac{1-\mu k}{\mathfrak{l}+k\imath} & \frac{\imath+\mu\mathfrak{l}}{\mathfrak{l}+k\imath} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad P_{63} = \begin{bmatrix} -k\imath & \mathfrak{l} & \mathfrak{l} & 0 & 0 & \cdots & 0 & 0 \\ -k\imath & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -k\imath & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ d & 0 & d+d_1 & 0 & d_2 & \cdots & 0 & d_r \\ \imath\frac{1-\mu k}{\mathfrak{l}+k\imath} & \frac{\imath+\mu\mathfrak{l}}{\mathfrak{l}+k\imath} & \frac{\imath+\mu\mathfrak{l}}{\mathfrak{l}+k\imath} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$P_{64}=\begin{bmatrix} -k\imath & -k\imath & \mathfrak{l} & 0 & 0 & \cdots & 0 & 0 \\ -k\imath & -k\imath & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -k\imath & -k\imath & 0 & 0 & 0 & \cdots & 1 & 1 \\ d & d+d_0 & 0 & 0 & d_2 & \cdots & 0 & d_r \\ \imath\frac{1-\mu k}{\mathfrak{l}+k\imath} & \imath\frac{1-\mu k}{\mathfrak{l}+k\imath} & \frac{\imath+\mu\mathfrak{l}}{\mathfrak{l}+k\imath} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad P_{65}=\begin{bmatrix} -k\imath & -k\imath & \mathfrak{l} & \mathfrak{l} & 0 & 0 & \cdots & 0 & 0 \\ -k\imath & -k\imath & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -k\imath & -k\imath & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ d & d+d_0 & 0 & d+d_1 & 0 & d_2 & \cdots & 0 & d_r \\ \imath\frac{1-\mu k}{\mathfrak{l}+k\imath} & \imath\frac{1-\mu k}{\mathfrak{l}+k\imath} & \frac{\imath+\mu\mathfrak{l}}{\mathfrak{l}+k\imath} & \frac{\imath+\mu\mathfrak{l}}{\mathfrak{l}+k\imath} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

PROOF. According to (the proof of) Proposition 1.4.7, there exists an integer $\mu$ with $\gcd(\mu,\zeta,\imath)=1$, so that the matrix $P$ has leading block

$$\begin{bmatrix} -l_{01} & l_{11} & 0 & 0 & \dots & 0 \\ -l_{01} & 0 & 1 & 0 & \dots & 0 \\ -l_{01} & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -l_{01} & 0 & 0 & 0 & \dots & 1 \\ d_{011} & d_{111} & 0 & 0 & \dots & 0 \\ \frac{\imath-\mu l_{01}}{\zeta} & \frac{\imath+\mu l_{11}}{\zeta} & 0 & 0 & \dots & 0 \end{bmatrix}$$

with $l_{01}\geq l_{11}\geq 1$. Moreover we require $\gcd(\mu,\zeta)=1$, otherwise since $(\imath-\mu l_{01})/\zeta$ is integer, $\gcd(\imath,\mu,\zeta)\neq 1$ would follow. Moreover, subtracting the last entry of $v_{01}$ from the last entry of $v_{11}$, we see that $\mu(l_{01}+l_{11})/\zeta$ is integer, leading to $l_{01}+l_{11}=m\zeta$ for some $m\in\mathbb{Z}_{\geq 1}$. We distinguish three cases in the following.

*Case 1:* $\zeta|\imath$. Here we require $\gcd(\mu,\zeta)=1$, so $\zeta|l_{01},l_{11}$. Set $\mathfrak{l}_i=l_{i1}/\zeta$ for $i=0,1$. Due to irredundancy we have at least another column $v_{22}=e_2+d_{221}e_{r+1}$, where we can assume $d_{221}\geq 1$ by if this is not the case adding the $-d_{221}$ of the second to the penultimate row. So the intersections of $\partial A_X^c$ with the elementary cones $\tau_1$ set up by the leading block and $\tau_2$ set up by the leading block with column $v_{21}$ replaced by $v_{22}$ are the leaving points

$$v(\tau_1)'=\left(0,\dots,0,\frac{\mathfrak{l}_1d_{011}+\mathfrak{l}_0d_{111}}{\mathfrak{l}_0+\mathfrak{l}_1},\imath/\zeta\right),$$
$$v(\tau_2)'=\left(0,\dots,0,\frac{\mathfrak{l}_1d_{011}+\mathfrak{l}_0d_{111}+d_{221}\mathfrak{l}_0\mathfrak{l}_1\zeta}{\mathfrak{l}_0+\mathfrak{l}_1},\imath/\zeta\right).$$

We distinguish three subcases in the following.

*Case 1.1:* $\zeta=2$. We achieve $\mu=1$ by admissible operations and distinguish some more subcases.

*Case 1.1.1:* $\mathfrak{l}_1\geq 2$. Then $|v(\tau_1)'-v(\tau_2)'|\geq 2$, forcing $\imath=2$. Then $A_X^c(\lambda_0)\cap\{x_{r+2}=0\}$ is the convex hull of $0_{\mathbb{Z}_{r+2}}$ and the two points

$$w_1=\left(-2,-2,\frac{\mathfrak{l}_1d_{011}+\mathfrak{l}_0d_{111}+d_{011}-d_{111}}{\mathfrak{l}_0+\mathfrak{l}_1},0\right),$$
$$w_2=\left(-2,-2,\frac{\mathfrak{l}_1d_{011}+\mathfrak{l}_0d_{111}+d_{011}-d_{111}+2d_{221}\mathfrak{l}_1(\mathfrak{l}_0-1)}{\mathfrak{l}_0+\mathfrak{l}_1},0\right).$$

Thus if $\mathfrak{l}_1\geq 3$ or $\mathfrak{l}_1=2$ and $\mathfrak{l}_0\geq 4$, we have $|w_1-w_2|\geq 2$ and due to Lemma 4.3.18, $A_X^c(\lambda_0)\cap\{x_{r+2}=0\}$ can not be canonical. So we have $\mathfrak{l}_1=2$ and $\mathfrak{l}_0\in\{2,3\}$ and can assume $d_{111}=0$.

*Case 1.1.1.1:* $\mathfrak{l}_0=3$. In this case,

$$w_1=\left(-2,-2,\frac{3}{5}d_{011},0\right),\ w_2=\left(-2,-2,\frac{3}{5}d_{011}+\frac{8}{5}d_{221},0\right),$$

which forces $d_{221}=1$, $r=2$ and $d_{011}\equiv 7 \mod 10$ due to Corollary 4.3.23. By admissible operations, we achieve $d_{011}=-3$. But then, $A_X^c(\lambda_1)$ contains the point $(1,0,0,1)$.

*Case 1.1.1.2:* $\mathfrak{l}_0 = 2$. In this case,

$$w_1 = \left(-2, -2, \frac{3}{4}d_{011}, 0\right), w_2 = \left(-2, -2, \frac{3}{4}d_{011} + d_{221}, 0\right),$$

which forces $d_{221} = 1$, $r = 2$ and $d_{011} \equiv 1, 3, 6 \mod 8$ due to Corollary 4.3.23. By admissible operations, we get $d_{011} \in \{1, 6\}$. For $d_{011} = 6$, we have the point $(1, 0, 1, 1)$ in $A_X^c(\lambda_1)$, while for $d_{011} = 1$, the resulting matrix

$$P_{22} := \begin{bmatrix} -4 & 4 & 0 & 0 \\ -4 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & 3 & 0 & 0 \end{bmatrix}$$

gives a canonical singularity. It is easy to check that no column can be added to this matrix.

*Case 1.1.2:* $\mathfrak{l}_1 = 1, \mathfrak{l}_0 \geq 2$. We have $|v(\tau_1)' - v(\tau_2)'| \geq 3/2$ if $\mathfrak{l}_0 \geq 2$ and for $\mathfrak{l}_0 = 2$, the polytope $A_X^c(\lambda)$ either contains two points or a cone like in Corollary 4.3.23 and thus, due to this corollary and Lemma 4.3.18, we get $\imath \in \{2, 4\}$.

*Case 1.1.2.1:* $\imath = 4$. Here we require $d_{211} = 1$ and $r = 2$, since otherwise $|v(\tau_1)' - v(\tau_2)'| \geq 3$. Moreover, we achieve $d_{111} = 0$ by admissible operations. Recall the leaving points

$$v(\tau_1)' = \left(0, \ldots, 0, \frac{d_{011}}{\mathfrak{l}_0 + 1}, 2\right), v(\tau_2)' = \left(0, \ldots, 0, \frac{d_{011} + 2\mathfrak{l}_0}{\mathfrak{l}_0 + 1}, 2\right).$$

We require $d_{011} \equiv 1 \mod 2(\mathfrak{l}_0 + 1)$ due to Corollary 4.3.23 and Lemma 4.3.18 and by admissible operations, we achieve $d_{011} = 1$. But then inside $A_X^c(\lambda_0)$ lies the point

$$(-1, -1, 1, 1) = \frac{1}{2\mathfrak{l}_0}v_{01} + \frac{2\mathfrak{l}_0 - 3}{8\mathfrak{l}_0}v(\tau_1)' + \frac{4\mathfrak{l}_0 - 1}{8\mathfrak{l}_0}v(\tau_2)'.$$

*Case 1.1.2.2:* $\imath = 2$. Here $d_{111}$ must be odd and by adding appropriate multiples of the first to the penultimate row, we achieve $d_{111} = 1$. As in Case 1.1.1, we investigate $A_X^c(\lambda_0) \cap \{x_{r+2} = 0\}$, which here is the convex hull of $0_{\mathbb{Z}_{r+2}}$ and the two points

$$w_1 = \left(-2, -2, \frac{2d_{011} + \mathfrak{l}_0 - 1}{\mathfrak{l}_0 + 1}, 0\right), w_2 = \left(-2, -2, \frac{2d_{011} + \mathfrak{l}_0 - 1 + 2d_{211}(\mathfrak{l}_0 - 1)}{\mathfrak{l}_0 + 1}, 0\right).$$

Lemma 4.3.18 forces $d_{211} = 1$ and $r = 2$ again and together with Corollary 4.3.23, it leads to $2d_{011} \equiv \mathfrak{l}_0 + 4, \mathfrak{l}_0 + 5, \mathfrak{l}_0 + 6 \mod 2\mathfrak{l}_0 + 2$. Since by admissible operations we can add multiples of $\mathfrak{l}_0 + 1$ to $d_{011}$, we achieve $2d_{011} \in \{\mathfrak{l}_0 + 4, \mathfrak{l}_0 + 5, \mathfrak{l}_0 + 6\}$.

*Case 1.1.2.2.1:* $\mathfrak{l}_0$ *odd*. Here we get $d_{011} = (\mathfrak{l}_0 + 5)/2$. Since $v_{01}$ must be primitive, we require $\mathfrak{l}_0 + 5 \equiv 2 \mod 4$ in addition. So $\mathfrak{l}_0 \geq 5$. Then $A_X^c(\lambda_0)$ contains

$$w_1 = \left(-6, -6, \frac{3\mathfrak{l}_0 + 7}{\mathfrak{l}_0 + 1}, -2\right), w_2 = \left(-6, -6, \frac{5\mathfrak{l}_0 + 1}{\mathfrak{l}_0 + 1}, -2\right)$$

and thus the point $(-3, -3, 2, -1)$. It can not be canonical.

*Case 1.1.2.2.2:* $\mathfrak{l}_0 = 2$. By admissible operations, the two possible cases $d_{011} = 3, 4$ are equivalent to $d_{011} = 0$ and give a canonical singularity with defining matrix $P_{23}$.

*Case 1.1.2.2.3:* $\mathfrak{l}_0 \geq 4$ *even.* By admissible operations, the two possibilities for $d_{011}$ are equivalent, so we let $d_{011} = (\mathfrak{l}_0 + 4)/2$. Here $A_X^c(\lambda_0)$ contains the points

$$w_1 = \left(-6, -6, \frac{3\mathfrak{l}_0 + 5}{\mathfrak{l}_0 + 1}, -2\right), w_2 = \left(-6, -6, \frac{5\mathfrak{l}_0 - 1}{\mathfrak{l}_0 + 1}, -2\right)$$

and thus the point $(-3, -3, 2, -1)$ if $\mathfrak{l}_0 > 4$. If $\mathfrak{l}_0 = 4$, we get a canonical singularity with matrix $P_{24}$.

*Case 1.1.3:* $\mathfrak{l}_1 = \mathfrak{l}_0 = 1$. We distinguish two subcases.

*Case 1.1.3.1:* $\imath \geq 4$. Lemma 4.3.18 forces $d_{211} = 1$ and $r = 2$. Together with Corollary 4.3.23, it leads to $\imath = 4$, $d_{111} = 0$ and $d_{011} \equiv 1 \mod 4$. By admissible operations, we achieve $d_{011} = 1$. The resulting matrix

$$P_{25} := \begin{bmatrix} -2 & 2 & 0 & 0 \\ -2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix}$$

gives a canonical singularity. It is easy to check that no columns can be added.

*Case 1.1.3.2:* $\imath = 2$. We achieve $d_{011} = d_{111} = 1$ here. It is clear that $r$ and the $d_{i11}$ can be arbitrary and canonicity is preserved. In $\lambda_0$ and $\lambda_1$, no columns can be added, but up to two in the lineality part. We denote this series by

$$P_{26} := \begin{bmatrix} -2 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -2 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & d_2 & \dots & 0 & d_r & d_1' & d_2' \\ 0 & 2 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \end{bmatrix}.$$

*Case 1.2:* $\zeta = 3$. Here if $\mathfrak{l}_0 \geq 2$, then $|v(\tau_1)' - v(\tau_2)'| = |3d_{221}\mathfrak{l}_0\mathfrak{l}_1/(\mathfrak{l}_0 + \mathfrak{l}_1)| \geq 2$. Moreover, if $\mathfrak{l}_0 = \mathfrak{l}_1 = 1$, then $A_X^c(\lambda)$ contains two integer points as well. Thus $\imath = 3$ due to Lemma 4.3.18 and Corollary 4.3.23. By adding appropriate multiples of the first to the last row, we achieve $\mu \in \{-1, 1\}$.

*Case 1.2.1:* $\mathfrak{l}_0 = \mathfrak{l}_1 = 1$. Here by interchanging the data of the first two leaves, we achieve $\mu = -1$. We achieve $d_{111} = 1$ and $d_{011} \in \{0, 1\}$ by admissible operations. The polytope $A_X^c(\lambda_0)$ contains the points

$$w_1 = \left(-1, -1, \frac{3d_{011} + 1}{6}, 1\right), w_2 = \left(-1, -1, \frac{3d_{011} + 1 + 3d_{211}}{6}, 1\right).$$

Because otherwise there would be an integer point inbetween $w_1$ and $w_2$, which violates canonicity, we require $d_{211} = 1$, $r = 2$ and $d_{011} = 0$. This gives the next canonical singularity with matrix $P_{27}$. It is easy to check that no additional columns are possible.

*Case 1.2.2:* $\mathfrak{l}_0 \geq 2$, $\mathfrak{l}_1 = 1$. We distinguish two subcases.

*Case 1.2.2.1:* $\mu = -1$. We achieve $d_{111} = 1$ as above. The polytope $A_X^c(\lambda_0)$ contains the points

$$w_1 = \left(-1, -1, \frac{3d_{011} - 1 + 2\mathfrak{l}_0}{3(\mathfrak{l}_0 + 1)}, 1\right), w_2 = \left(-1, -1, \frac{3d_{011} - 1 + 2\mathfrak{l}_0 + 3(2d_{211}\mathfrak{l}_0 - 1)}{3(\mathfrak{l}_0 + 1)}, 1\right)$$

and thus, there is an integer point inbetween, violating canonicity.

*Case 1.2.2.2:* $\mu = 1$. We achieve $d_{111} = 0$ by admissible operations. The polytope $A_X^c(\lambda_0)$ contains the points

$$w_1 = \left(-3, -3, \frac{2d_{011}}{\mathfrak{l}_0 + 1}, 0\right), w_2 = \left(-3, -3, \frac{2d_{011} + 3d_{211}(\mathfrak{l}_0 - 1)}{\mathfrak{l}_0 + 1}, 0\right),$$

so for $\mathfrak{l}_0 \geq 3$, due to Corollary 4.3.23 and Lemma 4.3.18, the corresponding singularity cannot be canonical. For $\mathfrak{l}_0 = 2$, we get $d_{211} = 1$ and $r = 2$ as above, moreover $d_{011} \equiv 5, 7 \mod 9$. Thus by admissible operations, we achieve $d_{011} = -2$. But then the point $(1, 0, 0, 1)$ is contained in $A_X^c(\lambda_1)$.

*Case 1.2.3:* $\mathfrak{l}_0, \mathfrak{l}_1 \geq 2$. By if necessary exchanging the data of the first two leaves, we achieve $\mu = 1$. Bear in mind that by doing this, we cannot assume $\mathfrak{l}_0 \geq \mathfrak{l}_1$ anymore, i.e. the matrix possibly is not in standard form. As in Case 1.2.2.2, there are two points $w_1, w_2$ in $A_X^c(\lambda_0) \cap \{x_1 = x_2 = -1\}$, but now their distance is $3d_{211}\mathfrak{l}_1(\mathfrak{l}_0 - 1)/(\mathfrak{l}_0 + \mathfrak{l}_1) \geq 3/2$, so that the corresponding singularity cannot be canonical.

*Case 1.3:* $\zeta \geq 4$. Here $|v(\tau_1)' - v(\tau_2)'| \geq 2$. Thus $\imath = \zeta$ due to Lemma 4.3.18. If now $A_X^c(\lambda)$ contains two integer points $(0, 0, k, 1)$ and $(0, 0, k+2, 1)$, then Theorem 4.3.19 forces $\mathfrak{l}_0 = \mathfrak{l}_1 = 1$. But for $\mathfrak{l}_0, \mathfrak{l}_1 \geq 2$, we have $|v(\tau_1)' - v(\tau_2)'| \geq 3$, so $A_X^c(\lambda)$ contains two such points. If $\mathfrak{l}_0 = 2$, $\mathfrak{l}_1 = 1$, then $|v(\tau_1)' - v(\tau_2)'| = 8/3$ and the penultimate entry of $v(\tau_1)'$ is a multiple of $1/3$. Therefore also in this case, $A_X^c(\lambda)$ contains two such points. The case $\mathfrak{l}_0 = \mathfrak{l}_1 = 1$ is left. Now if $\zeta \geq 5$, then $A_X^c(\lambda)$ contains two such points as well. Applying Theorem 4.3.19 to $A_X^c(\lambda_0) \cup A_X^c(\lambda_1)$, we get $\zeta = \imath = 2$, a contradiction. Thus only $\zeta = 4$ is left. We require $d_{221} = 1$, $r = 2$, since otherwise again two such integer points would be inside $A_X^c(\lambda)$. Moreover, we have $\mu \equiv 1, 3 \mod 4$. By exchanging the data of the first two leaves if necessary and adding appropriate multiples of the first row to the last, we achieve $\mu = 1$, resulting in $d_{011} = d_{111} = 1$. But then, the point $(1, 0, 1, 1)$ lies inside $A_X^c(\lambda_1)$.

*Case 2:* $\zeta \nmid \imath$, $\zeta < \imath$. We can write $\imath = k\zeta + i$ for integer $k \geq 1$ and $1 \leq i \leq \zeta - 1$. Moreover, $\zeta \nmid l_{01}, l_{11}$ due to integrality of the last entries of $v_{01}, v_{11}$ and $\zeta \nmid \imath$. Thus as in Case 1, the polytope $A_X^c(\lambda)$ contains the points

$$v(\tau_1)' = \left(0, \ldots, 0, \frac{l_{11}d_{011} + l_{01}d_{111}}{l_{11} + l_{01}}, \imath/\zeta\right),$$
$$v(\tau_2)' = \left(0, \ldots, 0, \frac{l_{11}d_{011} + l_{01}d_{111} + d_{221}l_{01}l_{11}}{l_{11} + l_{01}}, \imath/\zeta\right).$$

Remember that $l_{01} = m\zeta - l_{11}$ for some $m \in \mathbb{Z}_{\geq 1}$. If $A_X^c(\lambda)$ contains an integer point with last coordinate $k$, then it is not canonical. But the line segment $A_X^c(\lambda) \cap \{x_{r+2} = k\}$ contains the points

$$w_1 = \left(0, \ldots, 0, k\frac{l_{11}d_{011} + (m\zeta - l_{11})d_{111}}{m(k\zeta + i)}, k\right),$$
$$w_2 = \left(0, \ldots, 0, k\frac{l_{11}d_{011} + (m\zeta - l_{11})d_{111} + d_{221}(m\zeta - l_{11})l_{11}}{m(k\zeta + i)}, k\right).$$

Since for $l_{11} \geq 4$, $|w_1 - w_2| \geq 1$, we have $l_{11} \leq 3$. Moreover, we require

$$d_{221}k(m\zeta - l_{11})l_{11} \leq m(k\zeta + i) - 2$$

in order to have no integer point in $A_X^c(\lambda) \cap \{x_{r+2} = k\}$. So $d_{211} = 1$, $r = 2$.

*Case 2.1:* $l_{11} = 3$. The inequality from above becomes

$$m(2k\zeta - i) \leq 9k - 2$$

here. If $m \geq 2$, at least $k(2\zeta - 9/2) \leq i - 1$ must be fulfilled. Since $i \leq \zeta - 1$, this forces $k = i = 1$, $m = 2$, $\zeta = 2$ and thus $l_{01} = 1$, a contradiction. If $m = 1$, at least $k(2\zeta - 9) \leq i - 2$ must hold. For $k \geq 2$, this leads to $\zeta \leq 5$ and thus $l_{01} \leq 2$, a contradiction. For $k = 1$, it leads to $\zeta = 6$, $i = 5$ and $l_{01} = 3$. But since $\frac{11+3\mu}{6}$ cannot be integral, this is not possible.

*Case 2.2:* $l_{11} = 2$. The inequality from above becomes

$$m(k\zeta - i) \leq 4k - 2$$

here and we have $\zeta \geq 3$. The case $m \geq 3$ can be excluded.

*Case 2.2.1:* $m = 2$. The fraction in the penultimate entries of $w_1$ and $w_2$ can be reduced by two, so even the inequality $k(\zeta - 2) \leq i - 2$ must be fulfilled. But this is impossible.

*Case 2.2.2:* $m = 1$. We require $\zeta \geq 4$ here. The inequality from above becomes $k(\zeta - 4) \leq i - 2$. This leads to $i \in \{\zeta - 1, \zeta - 2\}$.

*Case 2.2.2.1:* $\zeta = 4$. Here $i = 2$ is impossible, since again, the fraction in the penultimate entries of $w_1$ and $w_2$ can be reduced by two. The case $i = 3$ remains, but the last entry of $v_{01}$, which is $(4k + 3 - 2\mu)/4$, must be integer, which is impossible.

*Case 2.2.2.2:* $\zeta \geq 5$. Here $k \geq 3$ is impossible and $k = 2$ leads to $\zeta = 5$.

*Case 2.2.2.2.1:* $k = 2$. We require $i = 4$ here. But the fraction in the penultimate entries of $w_1$ and $w_2$ can be reduced by two, which makes this case impossible.

*Case 2.2.2.2.2:* $k = 1$. If $i = \zeta - 1$, then $\zeta$ must be odd and we achieve $\mu = (1 - \zeta)/2$ and $d_{111} = 0$ as well as $d_{011} = \zeta$. But then the point

$$(-1, -1, 3, 2) = \frac{1}{\zeta - 2} v_{01} + \frac{(\zeta - 3)\zeta}{(\zeta - 2)(2\zeta - 1)} v(\tau_1)' + \frac{(\zeta - 3)\zeta}{4\zeta^2 - 10\zeta + 4} v(\tau_1)'$$

lies in $A_X^c(\lambda_0)$. If $i = \zeta - 2$, then $\zeta$ must be odd and we achieve $\mu = 1 - \zeta$ and $d_{111} = 1$ as well as $d_{011} = (3 - \zeta)/2$. But then in $A_X^c(\lambda_0)$ lies the point

$$(-1, -1, 0, 2) = \frac{1}{\zeta - 2} v_{01} + \frac{(\zeta - 3)\zeta}{4(\zeta - 2)(\zeta - 1)} v(\tau_1)' + \frac{(\zeta - 3)\zeta}{4\zeta^2 - 12\zeta + 8} v(\tau_1)'.$$

*Case 2.3:* $l_{11} = 1$, $l_{01} \geq 2$. We need an additional column $v_{12} = (1, 0, \ldots, 0, d_{121}, 0)$ in $\lambda_1$ here. The polytope $A_X^c(\lambda)$ contains the points

$$v(\tau_1)' = \left(0, \ldots, 0, \frac{d_{011} + l_{01} d_{111}}{1 + l_{01}}, \imath/\zeta\right),$$
$$v(\tau_3)' = \left(0, \ldots, 0, \frac{d_{011} + l_{01} d_{111} + (d_{121} + d_{221}) l_{01}}{1 + l_{01}}, \imath/\zeta\right).$$

Thus the line segment $A_X^c(\lambda) \cap \{x_{r+2} = k\}$ contains the points

$$w_1 = \left(0, \ldots, 0, k\frac{d_{011} + (m\zeta - 1) d_{111}}{m(k\zeta + i)}, k\right),$$
$$w_3 = \left(0, \ldots, 0, k\frac{d_{011} + (m\zeta - 1) d_{111} + (d_{121} + d_{221})(m\zeta - 1)}{m(k\zeta + i)}, k\right).$$

So $k(d_{121} + d_{221})(m\zeta - 1) \leq m(k\zeta + i) - 2$ must hold. But this is impossible.

*Case 2.4:* $l_{01} = l_{11} = 1$. This forces $\zeta = 2$ and $i = 1$. We need another column $v_{02} = (-1, \dots, -1, d_{021}, 0)$ in $\lambda_0$ here. The polytope $A_X^c(\lambda)$ contains the points

$$v(\tau_1)' = \left(0, \dots, 0, \frac{d_{011} + d_{111}}{2}, \imath/2\right),$$
$$v(\tau_4)' = \left(0, \dots, 0, \frac{d_{011} + d_{111} + (d_{021} + d_{121} + d_{221})l_{01}}{2}, \imath/2\right).$$

Thus the line segment $A_X^c(\lambda) \cap \{x_{r+2} = k\}$ contains the points

$$w_1 = \left(0, \dots, 0, k\frac{d_{011} + d_{111}}{2k+1}, k\right),$$
$$w_3 = \left(0, \dots, 0, k\frac{d_{011} + d_{111} + (d_{021} + d_{121} + d_{221})}{2k+1}, k\right).$$

But $|w_1 - w_3| \geq 1$, so $A_X^c(\lambda) \cap \{x_{r+2} = k\}$ contains an integer point.

*Case 3:* $\zeta > \imath \geq 2$. Recall that $l_{11} + l_{01} = m\zeta$. We now write $l_{11} = m_1\zeta + \mathfrak{l}$ and $l_{01} = m_0\zeta - \mathfrak{l}$ with $m_0 + m_1 = m$. Now the leading block is of the form

$$\begin{bmatrix} -(m_0\zeta - \mathfrak{l}) & m_1\zeta + \mathfrak{l} & 0 & 0 & \dots & 0 \\ -(m_0\zeta - \mathfrak{l}) & 0 & 1 & 0 & \dots & 0 \\ -(m_0\zeta - \mathfrak{l}) & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(m_0\zeta - \mathfrak{l}) & 0 & 0 & 0 & \dots & 1 \\ d_{011} & d_{111} & 0 & 0 & \dots & 0 \\ \frac{\imath + \mu\mathfrak{l}}{\zeta} - \mu m_0 & \frac{\imath + \mu\mathfrak{l}}{\zeta} + \mu m_1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Since $A_X^c(\lambda)$ provides no restrictions in this case any more, our approach is different than in the previous cases. Observe that all $v_{0i}$ and $v_{1i}$ lie on one affine hyperplane and moreover $v(\tau_1)'$ lies on the line segment with end points $v_{01}$ and $v_{11}$. Thus $A_X^c(\lambda_0) \cup A_X^c(\lambda_1)$ contains a polytope of the form $\mathsf{Q} := \operatorname{conv}(\nu_0, \nu_1, \nu_\tau)$ with

$$\nu_0 := \left(\mathfrak{l} - m_0\zeta, d_{011}, \frac{\imath + \mu\mathfrak{l}}{\zeta} - \mu m_0\right),$$
$$\nu_1 := \left(m_1\zeta + \mathfrak{l}, d_{111}, \frac{\imath + \mu\mathfrak{l}}{\zeta} + \mu m_1\right),$$
$$\nu_\tau := \left(0, \frac{(m_0\zeta - \mathfrak{l})d_{111} + (m_1\zeta + \mathfrak{l})d_{011} + \left(\sum_{i=2}^r d_{i21}\right)(m_0\zeta - \mathfrak{l})(m_1\zeta + \mathfrak{l})}{(m_0 + m_1)\zeta}, \frac{\imath}{\zeta}\right).$$

Now we search for integer points inside this polytope. If we find a line segment inside that contains three or more integer points and neither $\nu_0$ nor $\nu_1$, then Theorem 4.3.19 inter alia forces $\imath = 2$. If we find two parallel line segments inside that both contain two integer points, Theorem 4.3.19 forces $\imath = 2$ as well. If one of those parallel line segments even contains three integer points, the corresponding singularity cannot be canonical. We distinguish some subcases.

*Case 3.1:* $m_1 \geq 1$. We have $m_0 \geq 2$ here, due to $l_{01} \geq l_{11}$. We have a look at the line segment $\mathsf{Q}_1 = \mathsf{Q} \cap \{x_1 = \mathfrak{l}\}$, it has the endpoints

$$w_1 := \left(\mathfrak{l}, \frac{d_{111}m_0 + d_{011}m_1}{m_0 + m_1}, \frac{\imath + \mu\mathfrak{l}}{\zeta}\right)$$
$$w_2 := w_1 + \frac{\left(\sum_{i=2}^r d_{i21}\right) m_1(m_0\zeta - \mathfrak{l})}{m_0 + m_1} e_2.$$

So it contains at least $k$ integer points if

$$\left(\sum_{i=2}^{r} d_{i21}\right) m_1(m_0\zeta - \mathfrak{l}) - k(m_0 + m_1) + 1 \geq 0.$$

The line segment $\mathsf{Q}_0 = \mathsf{Q} \cap \{x_1 = \mathfrak{l} - \zeta\}$ on the other hand contains at least $k$ integer points if

$$\left(\sum_{i=2}^{r} d_{i21}\right) m_0(m_1\zeta + \mathfrak{l}) - k(m_0 + m_1) + 1 \geq 0.$$

*Case 3.1.1:* $m_1 \geq 2$. We have $m_0 \geq 3$ here. Both line segments contain three or more integer points.

*Case 3.1.2:* $m_1 = 1$. We have $m_0 \geq 2$ here.

*Case 3.1.2.1:* $\zeta = 3$. This forces $\imath = r = 2$, $d_{221} = 1$.

*Case 3.1.2.1.1:* $\mathfrak{l} = 1$. We achieve $\mu = d_{111} = 1$. Now both line segments contain two or more points if $m_0 \geq 3$. If $m_0 \geq 5$, then $\mathsf{Q}_0$ contains three or more points. We have a look at the remaining cases.

*Case 3.1.2.1.1.1:* $m_0 = 4$. Here $\mathsf{Q}_1$ contains three or more integer points if not $d_{011} \equiv 2, 3, 4 \mod 5$ and $\mathsf{Q}_0$ if not $d_{011} \equiv 2, 4 \mod 5$. We achieve $d_{011} \in \{2, 4\}$ by admissible operations. If $d_{011} = 2$, then $(-1, -1, 1, 0)$ is an inner point of $A_X^c(\lambda_0)$. If $d_{011} = 4$, then $(-1, -1, 2, 0)$ is an inner point of $A_X^c(\lambda_0)$.

*Case 3.1.2.1.1.2:* $m_0 = 3$. Here $\mathsf{Q}_1$ or $\mathsf{Q}_0$ contains three or more integer points if not $d_{011} \equiv 0, 2 \mod 4$. But this is not possible due to primitivity of $v_{01}$.

*Case 3.1.2.1.1.3:* $m_0 = 2$. We achieve $d_{011} \in \{0, 1, 2\}$ here. But for $d_{011} = 1, 2$, the point $(-1, -1, 1, 0)$ lies inside $A_X^c(\lambda_0)$, while for $d_{011} = 0$, we get a canonical singularity from the matrix

$$P_{28} := \begin{bmatrix} -5 & 4 & 0 & 0 \\ -5 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 2 & 0 & 0 \end{bmatrix}.$$

It is easy to check that no column can be added.

*Case 3.1.2.1.2:* $\mathfrak{l} = 2$. We achieve $\mu = -1$, $d_{111} = 0$ and can assume $m_0 \geq 3$ here. But $\mathsf{Q}_1$ contains at least two and $\mathsf{Q}_0$ at least three integer points, so we get no canonical singularity.

*Case 3.1.2.2:* $\zeta = 4$. Both $\mathsf{Q}_1$ and $\mathsf{Q}_0$ contain at least two integer points. This forces $\imath = 2$. Since $\gcd(\zeta, \imath, \mu) = 1$, we have $\mu \in \{-1, 1\}$ and consequently $\mathfrak{l} = 2$. Then $Q_0$ contains three or more points if not $m_0 = 2$. We achieve $\mu = -1$ and $d_{011} = 1$. But then $A_X^c(\lambda_1)$ contains the point $(1, 0, 1, 0)$.

*Case 3.1.2.3:* $\zeta \geq 5$. At least one of $\mathsf{Q}_1$ and $\mathsf{Q}_0$ contains three or more integer points while the other contains two or more. So we get no canonical singularity from this case.

*Case 3.2:* $m_1 = 0, m_0 \geq 2$.

*Case 3.2.1:* $\mathfrak{l} = 1$. This leads to $\mu = -\imath$ and we require an additional column $v_{12} = (1, 0, \ldots, 0, d_{121}, 0)$, while we can assume $d_{111} = 0$. Thus $A_X^c(\lambda_0) \cup A_X^c(\lambda_1)$

now contains a polytope of the form $\mathsf{Q}^* := \operatorname{conv}(\nu_0, \nu_1, \nu_{1,2}, \nu_{\tau,2})$ with

$$\nu_{1,2} := (1, d_{121}, 0)\,,$$
$$\nu_{\tau,2} := \left(0, \frac{d_{011} + \left(\sum_{i=1}^r d_{i21}\right)(m_0\zeta - 1)}{m_0\zeta}, \frac{\imath}{\zeta}\right).$$

Now the line segment $\mathsf{Q}_0^* = \mathsf{Q}^* \cap \{x_1 = 1 - \zeta\}$ has the endpoints

$$w_1^* := \left(1 - \zeta, \frac{d_{011}}{m_0}, \imath\right)$$
$$w_2^* := w_1^* + \frac{\left(\sum_{i=1}^r d_{i21}\right)(m_0 - 1)}{m_0} e_2.$$

If this line segment contains two integer points, then with the two integer points in $\mathsf{Q}^* \cap \{x_1 = 1\}$ and applying Corollary 4.3.20, we get a contradiction. This forces $r = 2$ and $d_{121} = d_{221} = 1$ as well as $d_{011} \equiv 1 \mod m_0$. Our matrix is of the form

$$\begin{bmatrix} 1 - m_0\zeta & 1 & 1 & 0 & 0 \\ 1 - m_0\zeta & 0 & 0 & 1 & 1 \\ d_{011} & 0 & 1 & 0 & 1 \\ m_0\imath & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now $\mathsf{Q}^*$ lies inside a polytope $\mathsf{Q}^+ = \operatorname{conv}(\nu_0, \nu_1, (1, 2, 0))$, while $\mathsf{Q}^+ \setminus \mathsf{Q}^*$ can not contain any integer point. Since $\mathsf{Q}^+$ is of type *(ii)* from Theorem 4.3.19, we get $\imath = 2$, odd $\zeta \geq 3$ and $d_{011} = 1$. This gives the series $P_{60}$ of canonical singularities.

*Case 3.2.2:* $\mathfrak{l} \geq 2$. Now the line segment $\mathsf{Q}_0 = \mathsf{Q} \cap \{x_1 = \mathfrak{l} - \zeta\}$ has the endpoints

$$w_1 := \left(\mathfrak{l} - \zeta, \frac{d_{111}(m_0 - 1) + d_{011}}{m_0}, \frac{\imath + \mu(\mathfrak{l} - \zeta)}{\zeta}\right)$$
$$w_2 := w_1 + \frac{\left(\sum_{i=2}^r d_{i21}\right)\mathfrak{l}(m_0 - 1)}{m_0} e_2.$$

We have a second line segment $\mathsf{Q}_0' = \mathsf{Q} \cap \{x_1 = \mathfrak{l} - 2\zeta\}$ with the endpoints

$$w_1' := \left(\mathfrak{l} - 2\zeta, \frac{d_{111}(m_0 - 2) + 2d_{011}}{m_0}, \frac{\imath + \mu(\mathfrak{l} - 2\zeta)}{\zeta}\right),$$
$$w_2' := w_1 + \frac{\left(\sum_{i=2}^r d_{i21}\right)\mathfrak{l}(m_0 - 2)}{m_0} e_2.$$

If both of these line segments have two or more integer points, the singularity cannot be canonical.

*Case 3.2.2.1:* $m_0 \geq 4$. Here, both line segments contain two integer points if $\mathfrak{l} \geq 3$. This forces $\mathfrak{l} = r = 2$, $d_{221} = 1$ and we achieve $d_{111} = 1$. Now we have $A_X^c(\lambda_0) = \mathsf{Q}^- \cap \{x_1 \leq 0\}$, where $\mathsf{Q}^-$ is the polytope

$$\left(\left(2 - m_0\zeta, 2 - m_0\zeta, d_{011}, \frac{\imath + 2\mu}{\zeta} - \mu m_0\right), \left(2, 2, 1, \frac{\imath + 2\mu}{\zeta}\right), \left(2, 2, 3, \frac{\imath + 2\mu}{\zeta}\right)\right).$$

We have two possibilities for $\mu$ now.

*Case 3.2.2.1.1:* $\mu = (\zeta - \imath)/2$. We achieve $d_{111} = 0$. Since $\mathsf{Q}^- \cap \{x_1 > 0\}$ can not contain any integer point, $A_X^c(\lambda_0)$ is canonical if and only if $\mathsf{Q}^-$ is. Then Theorem 4.3.19 forces $\imath = 2$ and $d_{011} = 1$. Since $\gcd(\mu, \zeta, \imath) = 1$ must hold, we require $\zeta \equiv 0 \mod 4$. We get a canonical singularity from the corresponding matrix

$$P_{29} := \begin{bmatrix} 2 - m_0\zeta & 2 & 0 & 0 \\ 2 - m_0\zeta & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 + \frac{m_0(2-\zeta)}{2} & 1 & 0 & 0 \end{bmatrix}.$$

*Case 3.2.2.1.2:* $\mu = -\imath/2$. Here $\imath$ must be even. We achieve $d_{111} = 1$. First assume $d_{011} = km_0+2$ with $k \in \mathbb{Z}$. But then the point $(1-\zeta, 1-\zeta, k+1, \imath/2)$ lies in $A_X^c(\lambda_0)$. If $d_{011} = km_0$, then $\mathsf{Q}_0$ and $\mathsf{Q}_0'$ contain two integer points. This is impossible. Now let $d_{011} = km_0+j$ with $j \neq 0, 2$. Only for $2d_{011} \equiv 3, 4, 5 \mod m_0$, the line segment $\mathsf{Q}_0'$ contains one integer point. But since this point is in the relative interior of $\mathsf{Q}_0'$ and since $\mathsf{Q}_0$ has two integer points in its relative interior, Theorem 4.3.19 forces $\imath = 2$.

*Case 3.2.2.1.2.1: $m_0$ even.* Here we achieve $d_{011} = (4 - m_0)/2$. But $A_X^c(\lambda_0)$ contains the point $(1 - 2\zeta, 1 - 2\zeta, 0, 2)$.

*Case 3.2.2.1.2.2: $m_0$ odd.* Here we achieve $d_{011} = (3 - m_0)/2$. If $m_0 \geq 7$, then the point $(1 - 2\zeta, 1 - 2\zeta, 0, 2)$ again lies in $A_X^c(\lambda_0)$. For $m_0 = 5$, we get a canonical singularity with defining matrix $P_{30}$.

*Case 3.2.2.2:* $m_0 = 3$. Here for $\mathfrak{l} \geq 5$, both $\mathsf{Q}_0$ and $\mathsf{Q}_0'$ contain two or more integer points.

*Case 3.2.2.2.1:* $\mathfrak{l} = 4$. Here we require $r = 2$, $d_{221} = 1$ and since $\mathsf{Q}_0$ contains three or more integer points, also $\imath = 2$.

*Case 3.2.2.2.1.1: $\zeta$ odd.* We get $\mu = (\zeta - 1)/2$ here and achieve $d_{111} = 1$, $d_{011} = 0$. The corresponding singularity is canonical with matrix $P_{31}$.

*Case 3.2.2.2.1.2: $\zeta$ even.* We get $\mu \in \{(\zeta - 2)/4, (3\zeta - 2)/4\}$. If $\mu = (\zeta - 2)/4$, we achieve $d_{111} = 0$, $\zeta = 8k + 2$, since otherwise $\mathsf{Q}_0'$ would contain a 2-fold point. But then $\mathsf{Q}_0$ contains a 2-fold point. If $\mu = (3\zeta - 2)/4$, we achieve $d_{111} = 0$, $\zeta = 8k + 6$, since otherwise $\mathsf{Q}_0'$ would contain a 2-fold point. But then $\gcd(\mu, \imath, \zeta) = 2 \neq 1$, a contradiction.

*Case 3.2.2.2.2:* $\mathfrak{l} = 3$. Here we require $r = 2$, $d_{221} = 1$ and since $\mathsf{Q}_0$ contains two points that do not lie on the line between $\nu_0$ and $\nu_1$ and $\mathsf{Q}_0'$ contains at least one integer point, $\imath = 2$ is forced. Moreover $\zeta$ must be odd, since otherwise due to integrality of $(3\mu + 2)/\zeta$, we would have even $\mu$ as well, violating $\gcd(\mu, \zeta, \imath) = 1$. We have the possibilities $\mu \in \{(\zeta-2)/3, 2(\zeta-1)/3\}$. But in both cases $\mathsf{Q}_0$ contains a 2-fold point.

*Case 3.2.2.2.3:* $\mathfrak{l} = 2$. Here $\sum_{i=2}^r d_{i21} \geq 3$ is impossible.

*Case 3.2.2.2.3.1:* $\sum_{i=2}^r d_{i21} = 2$. Here $\mathsf{Q}_0$ contains three or more integer points and $\mathsf{Q}_0'$ at least one, which forces $\imath = 2$. We have the possibilities $\mu \in \{-1, (\zeta - 2)/2\}$. If $\mu = -1$, we achieve $d_{111} = 1$ and $d_{011} = 0$. Since $\sum_{i=2}^r d_{i21} = 2$ is possible for either $r = 2$ and $d_{221} = 2$ or $r = 3$ and $d_{321} = d_{221} = 1$, we get two canonical singularities with defining matrices $P_{32}$ and $P_{33}$ from this. It is easy to check that no column can be added. If $\mu = (\zeta - 2)/2$, then due to $\gcd(\zeta, \mu, \imath) = 1$, we have $\zeta \in 4\mathbb{Z}$. But then $\mathsf{Q}_0$ contains a 2-fold point.

*Case 3.2.2.2.3.2:* $\sum_{i=2}^r d_{i21} = 1$. This requires $r = 2$, $d_{221} = 1$. We have the possibilities $\mu \in \{-\imath/2, (\zeta - \imath)/2\}$.

*Case 3.2.2.2.3.2.1:* $\mu = -\imath/2$. We need $\imath$ even and get $d_{111} = 1$ here.

*Case 3.2.2.2.3.2.1.1:* $d_{011} \equiv 0 \mod 3$. This forces $\imath = 2$ since $\mathsf{Q}_0$ contains two integer points one of which lies inside the polytope spanned by the other and $\nu_0$ and $\nu_1$. We achieve $d_{011} = 0$. This gives a canonical singularity. We see that two columns can be added: $(2 - 3\zeta, 2 - 3\zeta, 1, 3)$ and $(2 - \zeta, 2 - \zeta, 3, 0)$. Both together can not be added since then $(1 - \zeta, 1 - \zeta, 1, 1)$ would lie inside $A_X^c(\lambda_0)$. We get the defining matrices $P_{34}$, $P_{35}$, $P_{36}$.

*Case 3.2.2.2.3.2.1.2:* $d_{011} \equiv 1 \mod 3$. For $\imath = 2$, this is equivalent to $d_{011} \equiv 0 \mod 3$. So we can assume $\imath > 2$. We write $d_{011} = 3k + 1$. Here $\mathsf{Q}$ contains the polytope

$$\mathsf{Q}' = \operatorname{conv}\left((2 - 3\zeta, 3k + 1, 3\imath/2), (2, 1, 0), (2 - \zeta, k + 2, \imath/2)\right)$$

with integer vertices. This polytope must fall under Case (*iii*) of Theorem 4.3.19 if canonical. Let $\alpha, \beta \in \mathbb{Z}$ with $-\alpha\zeta + \beta\imath/2 = 1$. This is possible due to $1 = \gcd(\zeta, \imath, \mu) = \gcd(\zeta, \imath/2)$. Then multiplication from the left with the unimodular matrix

$$\begin{bmatrix} \alpha(1+k) & -1 & \beta(1+k) \\ -\alpha k & 1 & -\beta k \\ \imath/2 & 0 & \zeta \end{bmatrix}$$

transforms $\mathsf{Q}'$ to the polytope

$$\begin{aligned} \operatorname{conv}\big( & (2\alpha(k+1) + 2, -2\alpha k + 1, \imath), (2\alpha(k+1) - 1, -2\alpha k + 2, \imath), \\ & (2\alpha(k+1) - 1, -2\alpha k + 1, \imath)\big). \end{aligned}$$

Then due to Theorem 4.3.19, we get $k \equiv -\zeta \mod \imath/2$. By admissible operations, we achieve $k = -\zeta$. Then by subtracting the first from the third row, we achieve $d_{011} = d_{111} = -1$. We have a look at $\mathsf{Q} \cap \{x_2 = 0\}$, this polytope contains the points

$$(2 - 3\zeta/2, 0, 3\imath/4), \quad (2 - \zeta, 0, 2\imath/2), \quad \left(\frac{3\zeta - 4}{3\zeta - 2}, 0, \frac{3\imath}{6\zeta - 4}\right).$$

But since the first two differ by $\zeta/2$ and the last two by $\frac{3\zeta-5}{3\zeta-2} \geq 1/3$ in the last coordinate, by Corollary 4.3.22 we get $\imath = 2$, which we excluded.

*Case 3.2.2.2.3.2.1.3:* $d_{011} \equiv 2 \mod 3$. We write $d_{011} = 3k - 1$. Here $\mathsf{Q}_0'$ contains the 2-fold point $(2 - 2\zeta, 2 - 2\zeta, 2k, \imath)$.

*Case 3.2.2.2.3.2.2:* $(\zeta - \imath)/2$. We get $d_{111} = 1$ as well. Since $\gcd(\mu, \zeta, \imath) = 1$, we require $\zeta - \imath \equiv 2 \mod 4$.

*Case 3.2.2.2.3.2.2.1:* $d_{011} \equiv 2 \mod 3$. Since $\mathsf{Q}$ contains a polytope of Type (*iii*) from Theorem 4.3.19, we get that $\mathsf{Q}$ must be lattice equivalent to a polytope $\mathsf{Q}_s$ with vertices

$$(0, k, \imath), (1, k + 2, \imath), \left(\frac{6\zeta - 2}{3\zeta}, k + \frac{8 - 6\zeta}{3\zeta}, \imath\right),$$

where we can assume $0 \leq k \leq \imath - 1$. Now the point $(1, k, \imath - 1)$ lies inside $\operatorname{conv}(\mathsf{Q}_s, 0_{\mathbb{Z}_3})$, if $2/(k + 4) \geq 1/(\imath - 1)$. This is the case for any possible $k$ if $\imath \geq 5$. For $\imath = 4$, it is not the case only for $k = 3$, but this is equivalent to $k = -1$, and then the point $(-1, 1, 3)$ lies inside. For $\imath = 3$ and $k = 0, 1$, the point $(1, 0, 2)$ lies inside, while for $k = 2$, the point $(1, 1, 2)$ lies inside. Now $\imath = 2$ remains, from which follows $\zeta \in 4\mathbb{Z}$. Since $d_{011}$ must be odd, we achieve $d_{011} = -1$. We get a canonical singularity, for which it is easy to check, that no columns can be added. The matrix appears as the case $m_0 = 3$ of $P_{29}$.

*Case 3.2.2.2.3.2.2.2:* $d_{011} \equiv 0 \mod 3$. As we have seen above in the respective case for $\mu - \imath/2$, this forces $\imath = 2$. But then $\mathsf{Q}_0$ contains a 2-fold point.

*Case 3.2.2.2.3.2.2.3:* $d_{011} \equiv 1 \mod 3$. If $\imath$ is even, $\mathsf{Q}_0$ contains a 2-fold point. So $\imath$ must be odd. But as we have seen in the respective case for $\mu = -\imath/2$, we require $\imath = 2$, a contradiction.

*Case 3.2.2.3:* $m_0 = 2$.

*Case 3.2.2.3.1:* $\sum_{i=2}^{r} d_{i21} \geq 3$. This forces $\imath = 2$. Since $\mathsf{Q}_0$ contains a 2-fold point otherwise, we require that $\mathfrak{l}$ is even. This determines $\mu$ if we suppose $1 \leq \mu \leq \zeta - 1$. By admissible operations, we achieve $d_{011} = d_{111} = 1$. We get a series of canonical singularities, where optionally one or two columns of the form $(\mathfrak{l} - \zeta, \mathfrak{l} - \zeta, d_{0i1}, (\mathfrak{l}\mu + 2)/\zeta - \mu)$ can be added. This gives the defining matrices $P_{37}$, $P_{38}$, $P_{39}$.

*Case 3.2.2.3.2:* $\sum_{i=2}^{r} d_{i21} = 2$. The polytope $\mathsf{Q}$ in any case contains a lattice polytope and at least one integer point in the relative interior, leading to $\imath = 2, 3$ with Theorem 4.3.19.

*Case 3.2.2.3.2.1:* $\imath = 3$. This forces $\mathfrak{l} = 2$. By admissible operations, we achieve $d_{111} = 0$, since $(2\mu + 3)/\zeta$ is odd. We achieve $\mu = (\zeta - 3)/2$. Moreover $d_{011}$ must be odd, so we achieve $d_{011} \in \{-1, 1\}$. Due to $\gcd(\mu, \zeta, \imath) = 1$ and since $\zeta$ must be odd, we achieve $\zeta \equiv 1, -1 \mod 6$.

First let $d_{011} = -1$. Then $\zeta \equiv -1 \mod 6$ is not allowed since $\mathsf{Q}_0$ would contain a 3-fold point. If $\zeta \equiv 1 \mod 6$, then $A_X^{\mathfrak{c}}(\lambda_0)$ contains the point $((4 - \zeta)/3, (4 - \zeta)/3, 0, (7 - \zeta)/6)$.

Now let $d_{011} = 1$. If $\zeta \equiv 1 \mod 6$, then $A_X^{\mathfrak{c}}(\lambda_0)$ contains the point $((4 - \zeta)/3, (4 - \zeta)/3, 1, (7 - \zeta)/6)$. For $\zeta \equiv 5 \mod 6$, we get a canonical singularity. Since for $\sum_{i=2}^{r} d_{i21} = 2$, we have the two possibilities $r = 2, d_{221} = 2$ and $r = 3, d_{221} = d_{321} = 1$, we get two different singularities with defining matrices $P_{40}$, $P_{41}$.

*Case 3.2.2.3.2.2:* $\imath = 2$. As we have seen in Case 3.2.2.3.1, we require $\mathfrak{l}$ even. The resulting singularities can be seen as part of the series from Case 3.2.2.3.1.

*Case 3.2.2.3.3:* $\sum_{i=2}^{r} d_{i21} = 1$. This forces $r = 2$, $d_{221} = 1$.

*Case 3.2.2.3.3.1:* $\imath = 2$. The resulting singularities must be part of the series from Case 3.2.2.3.1, if $\mathfrak{l}$ is even. So we assume that $\mathfrak{l}$ is odd. But then $\mathsf{Q}_0$ contains a 2-fold point, which is not possible.

*Case 3.2.2.3.3.2:* $\imath = 3$. We have $\mathfrak{l} \leq 4$.

*Case 3.2.2.3.3.2.1:* $\mathfrak{l} = 2$. As in Case 3.2.2.3.2.1, we achieve $d_{111} = 0$, $\mu = (\zeta - 3)/2$ and $\zeta \equiv 5 \mod 6$. The possibilities for $d_{011}$ are $1, 2$. These two can also be combined, yielding one additional column. Another column of the form $(2, 0, 1, 1)$ can be added to this combination. Now let $d_{011} = 1$ not in combination with $d_{021} = 2$. Then in addition $d_{021} = 3$ is possible or a column of one of the forms $(2, 0, 1, 1), (2, 0, 2, 1), (2 - \zeta, 2 - \zeta, 2, (5 - \zeta)/2)$. Any combination of possible columns with $d_{011} = 2$ and without $d_{021} = 1$ is isomorphic to one of the previous ones. We get the defining matrices $P_{42} - P_{49}$.

*Case 3.2.2.3.3.2.2:* $\mathfrak{l} = 3$. We achieve $d_{111} = 1$. If $3 \nmid \zeta$, then we achieve $\mu = -1$. We achieve $d_{011} \in \{0, 1\}$. If $d_{011} = 1$, then the point $(2 - \zeta, 2 - \zeta, 1, 1)$ lies inside $A_X^{\mathfrak{c}}(\lambda_0)$. If $d_{011} = 0$, the resulting singularity is canonical. We can also add the column $(3, 0, 2, 0)$ and get the defining matrices $P_{50}$, $P_{51}$. These singularities are also possible for $3|\zeta$, $\mu = -1$. So we assume $3|\zeta$, $\mu \in \{\zeta/3 - 1, 2\zeta/3 - 1\}$ now. We achieve $d_{111} = 0$.

First assume $\mu = \zeta/3 - 1$, then we require $\zeta/3 \equiv 0, 2 \mod 3$. If $\zeta/3 \equiv 0 \mod 3$, then we achieve $d_{011} \in \{1, 2, 4, 5\}$. But then the point $(2 - \zeta/3, 2 - \zeta/3, 1, 1 - \zeta/9)$ lies inside $A_X^{\mathfrak{c}}(\lambda_0)$. If $\zeta/3 \equiv 2 \mod 3$, then we achieve $d_{011} \in \{1, 2\}$, both giving isomorphic canonical singularities. For $d_{011} = 1$, we can also add the column $(3, 0, 1, 1)$. We get the defining matrices $P_{52} - P_{54}$.

Now finally assume $\mu = 2\zeta/3 - 1$, then we require $\zeta/3 \equiv 0, 1 \mod 3$. If $\zeta/3 \equiv 1 \mod 3$, then we achieve $d_{011} \in \{1, 2, 4, 5\}$. But then the point $(2-\zeta/3, 2-\zeta/3, 1, (15-2\zeta)/9)$ lies inside $A_X^c(\lambda_0)$. If $\zeta/3 \equiv 0 \mod 3$, we achieve $d_{011} \in \{1, 2\}$, both giving isomorphic canonical singularities. For $d_{011} = 1$, we can also add the column $(3, 0, 1, 2)$. We get the defining matrices $P_{55} - P_{57}$.

*Case 3.2.2.3.3.2.3:* $\mathfrak{l} = 4$. We have the possibilities $\mu \in \{(\zeta-3)/4, 3(\zeta-1)/4\}$ and can assume $d_{111} = 0$. We have $\zeta \equiv 1, 2 \mod 3$, otherwise $\gcd(\mu, \zeta, \imath) = 3 \neq 1$.

*Case 3.2.2.3.3.2.3.1:* $\mu = (\zeta - 3)/4$. This leads to $\zeta \equiv 7, -1 \mod 12$. For $\zeta \equiv 7 \mod 12$, we get $d_{011} = 1$ and get a canonical singularity with defining matrix $P_{58}$. For $\zeta \equiv -1 \mod 12$, we achieve $d_{011} \in \{1, 2, 4, 5\}$. But then the point $((8-\zeta)/3, (8-\zeta)/3, 1, (11-\zeta)/8)$ lies inside $A_X^c(\lambda_0)$.

*Case 3.2.2.3.3.2.3.2:* $\mu = 3(\zeta - 1)/4$. This leads to $\zeta \equiv 1, 5 \mod 12$. For $\zeta \equiv 5 \mod 12$, we achieve $d_{011} \in \{1, 2, 4, 5\}$. But then for $\zeta \geq 17$, the point $((8-\zeta)/3, (8-\zeta)/3, 1, (9-\zeta)/4)$ lies inside $A_X^c(\lambda_0)$. For $\zeta = 5$, the point $(1, 0, 1, 1)$ lies inside $A_X^c(\lambda_1)$. For $\zeta \equiv 1 \mod 12$, we get $d_{011} = 1$ and get a canonical singularity with defining matrix $P_{59}$.

*Case 3.2.2.3.3.3:* $\imath \geq 4$. Here $\mathfrak{l} \geq 4$ is impossible due to Theorem 4.3.19.

*Case 3.2.2.3.3.3.1:* $\mathfrak{l} = 3$. We require $d_{011} \equiv d_{111} \mod 2$. Here $\mathsf{Q}$ contains the polytope

$$\mathsf{Q}' = \operatorname{conv}(\nu_0, \nu_1, (3-\zeta, (d_{011} + d_{111})/2 + 1, (\imath + 3\mu)/\zeta - \mu)).$$

This polytope must fall under Case (*iii*) of Theorem 4.3.19 if canonical. But then $\mathsf{Q}$ contains a polytope of the form

$$\operatorname{conv}\left((k + 1/3, \imath), (k + (5-2\zeta)/(3-2\zeta), \imath)\right),$$

which can not be canonical for $\imath \geq 4$ due to Corollary 4.3.25.

*Case 3.2.2.3.3.3.2:* $\mathfrak{l} = 2$.

*Case 3.2.2.3.3.3.2.1:* $\imath$ *odd.* This leads to odd $\zeta$ and $\mu = (\zeta - \imath)/2$. We achieve $d_{111} = 0$. First assume that $d_{011}$ is odd. But then $\mathsf{Q}$ contains a polytope of the form

$$\operatorname{conv}\left((k + 1/2, \imath), (k + (3-2\zeta)/(2-2\zeta), \imath)\right),$$

which can not be canonical for $\imath \geq 4$ due to Corollary 4.3.25. Now assume that $d_{011}$ is even. Then applying Corollary 4.3.22, we get that $\imath | 2$, a contradiction.

*Case 3.2.2.3.3.3.2.2:* $\imath$ *even.* Here $\zeta$ can be odd or even.

*Case 3.2.2.3.3.3.2.2.1:* $\zeta$ *odd.* This forces $\mu = -\imath/2$ and we achieve $d_{111} = 1$ and $d_{011}$ odd. Again applying Corollary 4.3.22, we get that $\imath | 2$, a contradiction since $\imath \geq 4$.

*Case 3.2.2.3.3.3.2.2.2:* $\zeta$ *even.* For $\mu = -\imath/2$, we have the same conclusion as in the previous case. So $\mu = (\zeta - \imath)/2$ remains. The proceeding is exactly the same as in Case 3.2.2.3.3.3.2.1.

*Case 3.3:* $m_1 = 0, m_0 = 1$.

*Case 3.3.1:* $\mathfrak{l} = 1$. Here as in Case 3.2.1, the polytope $A_X^c(\lambda_0) \cup A_X^c(\lambda_1)$ now contains a polytope of the form $\mathsf{Q}^* := \operatorname{conv}(\nu_0, \nu_1, \nu_{1,2}, \nu_{\tau,2})$ with

$$\nu_{1,2} := (1, d_{121}, 0),$$
$$\nu_{\tau,2} := \left(0, \frac{d_{011} + \left(\sum_{i=1}^r d_{i21}\right)(m_0\zeta - 1)}{m_0\zeta}, \frac{\imath}{\zeta}\right).$$

Moreover, $\mathsf{Q}^*$ lies inside a polytope $\mathsf{Q}^+ = \operatorname{conv}(\nu_0, \nu_1, (1, \sum_{i=1}^r d_{i21}, 0))$, while $\mathsf{Q}^+ \setminus \mathsf{Q}^*$ can not contain any integer point. Since $\sum_{i=1}^r d_{i21} \geq 2$ and $\mathsf{Q}^+$ is of type $(iii)$ from Theorem 4.3.19, we get $\zeta = k\imath + 1$ with $k \in \mathbb{Z}_{\geq 1}$ and $\gcd(d_{011}, \imath) = 1$.

*Case 3.3.2:* $\mathfrak{l} \geq 2$. We first assume $\sum_{i=2}^r d_{i21} = 1$, i.e. $r = 2$ and $d_{i21} = 1$. For greater values of $\sum_{i=2}^r d_{i21}$, the polytope $\mathsf{Q}$ contains a respective polytope $\mathsf{Q}'$ of a singularity with $\sum_{i=2}^r d_{i21} = 1$. Thus only if for a configuration of $\imath, \zeta, \mathfrak{l}, \mu$ we find a canonical singularity with $\sum_{i=2}^r d_{i21} = 1$, it is possible to find one with $\sum_{i=2}^r d_{i21} > 1$. Our matrix $P$ has the following form:

$$\begin{bmatrix} \mathfrak{l} - \zeta & \mathfrak{l} & 0 & 0 \\ \mathfrak{l} - \zeta & 0 & 1 & 1 \\ d_{011} & d_{111} & 0 & 1 \\ \frac{\imath + \mu\mathfrak{l}}{\zeta} - \mu & \frac{\imath + \mu\mathfrak{l}}{\zeta} & 0 & 0 \end{bmatrix}.$$

Now by admissible operations, multiples of $\imath$ can be added to $d_{011}$ without changing $d_{111}$ and multiples of $\zeta$ and $\mu$ can be added to *the difference between* $d_{011}$ *and* $d_{111}$. Since $\gcd(\mu, \zeta, \imath) = 1$, we achieve $d_{011} = d_{111} := d \in \mathbb{Z}$. We bring $P^*$ in Smith normal form. Since $\gcd(\mu, \zeta) = 1$, we can choose $\alpha, \beta \in \mathbb{Z}$ with $\alpha\zeta + \beta\mu = 1$. Then

$$S := \begin{bmatrix} -1 & 1 & \mathfrak{l} - \zeta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ \frac{\beta\imath + \mathfrak{l}}{\zeta} & 1 - \frac{\beta\imath + \mathfrak{l}}{\zeta} & d + \mathfrak{l}\left(1 - \frac{\beta\imath + \mathfrak{l}}{\zeta}\right) & -d \end{bmatrix}, \quad T := \begin{bmatrix} \alpha & 0 & 0 & -\mu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta & 0 & 0 & \zeta \end{bmatrix}$$

are unimodular and $S \cdot P^* \cdot T = \operatorname{Diag}(1, 1, 1, \imath)$. This means that the class group is $\mathbb{Z}/\imath\mathbb{Z}$ and the total coordinate space given by the equation $T_0^{\mathfrak{l} - \zeta} + T_1^{\mathfrak{l}} + T_2T_3$ is the index one cover, while

$$Q = \begin{bmatrix} \frac{\beta\imath + \mathfrak{l}}{\zeta} & 1 - \frac{\beta\imath + \mathfrak{l}}{\zeta} & d + \mathfrak{l}\left(1 - \frac{\beta\imath + \mathfrak{l}}{\zeta}\right) & -d \end{bmatrix} \in (\mathbb{Z}/\imath\mathbb{Z})^4$$

is the grading matrix. Moreover the only integer points in $\partial A_X^c$ are the columns of the matrix $P$. This means that the corresponding singularity is canonical if and only if it is terminal. We thus can use the classification of Mori [**69**] to determine the canonical ones of this type. The relevant theorem of [**69**] is Theorem 12 due to the form of the equation of the index one cover. There are three possible cases.

*Case 3.3.2.1: Case (1) of* [**69**, Thm. 12] *holds.* Since we can exchange the data of the first two leaves, we can assume $\mathfrak{l} - \zeta = -k\imath$ for some $k \in \mathbb{Z} \geq 1$. Thus from [**69**, Thm. 12] we get that $\gcd(d, \imath) = 1$ must hold and $\frac{\beta\imath + \mathfrak{l}}{\zeta} \equiv 1 \mod \imath$, so $\frac{1 - \mu k}{\zeta}$ must be integer. So $1 = \gcd(k, \zeta) = \gcd(k, k\imath + \mathfrak{l}) = \gcd(k, \mathfrak{l})$ must hold. Our matrix now has the form

$$\begin{bmatrix} -k\imath & \mathfrak{l} & 0 & 0 \\ -k\imath & 0 & 1 & 1 \\ d & d & 0 & 1 \\ \imath\frac{1 - \mu k}{\mathfrak{l} + k\imath} & \frac{\imath + \mu\mathfrak{l}}{\mathfrak{l} + k\imath} & 0 & 0 \end{bmatrix}.$$

It is now clear that $\sum_{i=2}^r d_{i21}$ can be augmented in this case in any way perhaps losing terminality but keeping canonicity. Also in $\lambda_1$ columns of the form $(-k\imath, \dots, -k\imath, d, \frac{\imath + \mu\mathfrak{l}}{\mathfrak{l} + k\imath})$ can be added as well as columns of the form

$$(\mathfrak{l}, 0, \dots, 0, d_{021}, \imath\frac{1 - \mu k}{\mathfrak{l} + k\imath})$$

in $\lambda_0$, as long as for all $\delta \in \mathbb{Z} \cap [d, d_{021}]$, we have $\gcd(\delta, \imath) = 1$. We get four series $P_{62}$-$P_{65}$ of canonical singularities.

*Case 3.3.2.2: Case (2) of* [**69**, Thm. 12] *holds.* Here we have $\imath = 4$, require $d = -1$ and a look at $Q$ tells us that $\mathfrak{l} = 2$ must hold and $2|(2\mu + 4)/\zeta$, so we achieve $\mu = -2$. Thus $\zeta$ must be odd. It is easy to check that no column can be added to the resulting matrix $P_{61}$.

*Case 3.3.2.3: Case (3) of* [**69**, Thm. 12] *holds.* Let without restriction $\mathfrak{l} - \zeta = -2k$ for some $k \in \mathbb{Z}_{\geq 1}$ and $(1 - \mu k)/\zeta$ integer. So $1 = \gcd(k, \zeta) = \gcd(k, 2k + \mathfrak{l}) = \gcd(k, l)$ follows. That means we are in Case (1) of [**69**, Thm. 12]. The determination of defining matrices $P$ for canonical threefold singularities with two-torus action is complete. □

**4.4.3. Proof of Theorems 10, 11, 12 and Corollaries 2, 3.** Propositions 4.4.2-4.4.8 provide the defining matrices $P$ of the non-toric canonical threefold singularities with two-torus action, while Proposition 4.4.1 provides the Cox rings and class groups of the toric ones. Thus the remaining task for proving Theorems 10 and 11 is to determine the Cox rings and class groups of those non-toric ones that do not belong to a "many parameter series".

Proof of Theorems 10 and 11. The Cox ring (without the grading) can be read off directly from the defining matrix $P$, see Construction 1.2.1. As already stated in the proof of Proposition 4.4.1, to determine class group and grading matrix $Q$, we need to find unimodular matrices $V$ and $W$, so that $V \cdot P^* \cdot W = S$ is in Smith normal form. The cases without parameters can be done by computer, e.g. using [**51**].

The few parameter series can be done by computer as well - with a little more effort. Bear in mind that we have to find Smith normal forms over $\mathbb{Z}[x_1, \ldots, x_{\mathfrak{n}}]$ with $\mathfrak{n} \in \{1, 2, 3\}$ here. But this is not a principal ideal domain, so it is not guaranteed that a Smith normal form exists. In other words, it is not clear a priori that for a defining matrix $P$ with parameters, we get polynomial formulae for the class group and the grading matrix $Q$. But we can compute the Smith normal form for several integer values of the parameters and interpolate the entries of respective $V$, $W$, $S$ by polynomials. We then take the interpolating polynomials as entries of matrices $\mathbf{V}$, $\mathbf{W}$, $\mathbf{S}$ and check a posteriori if these matrices are well defined.

This works for all few parameter series and we are done with the computation of the class groups and grading matrices. □

Remark 4.4.9. The following list provides information about which matrices from Propositions 4.4.2-4.4.8 correspond to the singularities from Theorems 10 and 11. Bear in mind that the toric singularities with Nos. 1-5 from Theorem 10 correspond to the cones given by the Cases $(ii) - (vi)$ of Theorem 4.3.19.

| $P_{1-4}$ | $P_5$ | $P_{6-12}$ | $P_{13}$ | $P_{14}$ | $P_{15,16}$ | $P_{17-19}$ | $P_{20}$ | $P_{21}$ |
|---|---|---|---|---|---|---|---|---|
| 6-9 | 26 | 10-16 | 56a | 56b | 27,28 | 17-19 | 57a | 29 |

| $P_{22-25}$ | $P_{27,28}$ | $P_{29-31}$ | $P_{32,33}$ | $P_{34}$ | $P_{35,36}$ | $P_{37-39}$ | $P_{40,41}$ | $P_{42}$ |
|---|---|---|---|---|---|---|---|---|
| 20-23 | 24,25 | 30-32 | 34,35 | 33 | 36,37 | 58a-c | 39,40 | 38 |

| $P_{43}$ | $P_{44-51}$ | $P_{52-54}$ | $P_{55-57}$ | $P_{58,59}$ | $P_{60,61}$ | $P_{62-65}$ | | |
|---|---|---|---|---|---|---|---|---|
| 41 | 42-49 | 50-52$_{\zeta \in 9\mathbb{Z}-3}$ | 50-52$_{\zeta \in 9\mathbb{Z}}$ | 53 | 54,55 | 59a-d | | |

Proof of Corollary 2. To check which of the canonical singularities are terminal, we have a look at $\partial A_X^c$ in each case. If there is an integer point different from the columns of the defining matrix $P$, the singularity $X(P)$ is not terminal. □

Definition 4.4.10. In order to prove Theorem 12, we define the following canonical quasicones of complexity one:

$$X_1 = V(T_1^2T_2 + T_3^2T_4 + T_5^{\frac{l+2}{2}}), \qquad X_2 = V(T_1T_2 + T_3T_4 + T_5^{l-1}),$$

$$X_5 = V(T_1^{L_1}T_2^{L_1} + T_3^{L_2}T_4^{L_2} + T_5T_6,\ T_3^{L_2}T_4^{L_2} + 2T_5T_6 + T_7T_8,$$
$$T_5T_6 + 3T_7T_8 + T_9T_{10},\ \ldots,\ T_{2r-3}T_{2r-2} + (r-1)T_{2r-1}T_{2r} + T_{2r+1}T_{2r+2}),$$

$$Y_1 = V(T_1^3T_2 + T_3^3T_4 + T_5^2),\ Y_2 = V(T_1^4 + T_2^3T_3 + T_4^2),\ Y_3 = V(T_1^3T_2T_3 + T_4^3T_5T_6 + T_7^2),$$
$$Y_4 = V(T_1^4 + T_2^3T_3T_4 + T_5^2),\ Y_5 = V(T_1^2T_2 + T_3^2T_4 + T_5^2T_6),\ Y_6 = V(T_1^3 + T_2^3 + T_3^2T_4),$$
$$Y_7 = V(T_1^2T_2T_3 + T_4^2T_5T_6 + T_7^2),\ Y_8 = V(T_1^4 + T_2^2 + T_3^2T_4T_5),$$
$$Y_9 = V(T_0^k + T_1^2 + T_2^2, \ldots, (r-1)T_{(r-2)1}T_{(r-2)2} + T_{(r-1)1}T_{(r-1)2} + T_{r1}T_{r2}),$$
$$Y_{10} = V(T_0^k + T_1^2 + T_{21}^2T_{22}^2, \ldots, (r-1)T_{(r-2)1}T_{(r-2)2} + T_{(r-1)1}T_{(r-1)2} + T_{r1}T_{r2}).$$

Moreover, to obtain $X_4$, set $T_4 = 1$ and for $X_3$ in addition $T_2 = 1$ in the equations of $X_5$.

Proof of Theorem 12. For the canonical and compound Du Val threefold singularities, the Cox rings can be read off directly from the defining matrix $P$. The Cox rings thereof can be computed using Corollary 2.6.8, yielding the tree from the theorem with quasicones $X_1, \ldots, X_5$ and $Y_1, \ldots, Y_{10}$ from Definition 4.4.10. □

Proof of Corollary 3. The generalized compound Du Val property for the singularities $Y_1, Y_3, Y_5, Y_7$ follows from taking the hyperplane section given by $T_1 = T_3$. For $X_1, \ldots, X_5$, it follows from taking the section $T_3 = T_4$. The second assertion follows directly. □

CHAPTER 5

# Iteration of $\mathrm{SL}_n$-quotients: an outlook

In this section, we study properties of *iteration of* $\mathrm{SL}_n$*-quotients*. While invariant rings of certain representations of $\mathrm{SL}_n$ provide factorial canonical Cox rings, we aim to iterate such quotients as well.

First, observe that if $X$ is a factorial variety, then a good quotient $X /\!\!/ G$ by a *semisimple* group $G$ acting on $X$ is still factorial, see [**15**]. Moreover, if $X$ has rational singularities, then $X /\!\!/ G$ has so by [**17**]. In our situation - i.e. a factorial and thus Gorenstein variety $X$ - this already yields canonicity of the quotient if $X$ is canonical. In [**84**] it was shown that if $X$ is log-terminal then so is the quotient (provided it is $\mathbb{Q}$-Gorenstein). So iterating $\mathrm{SL}_n$-quotients will preserve factoriality and canonicity and thus will provide more candidates for master Cox rings of Fano type varieties. The question now is how to construct in a systematic way such iterations. By [**27**], we have the following:

Construction 5.1. *Let $G$ be a reductive algebraic group. Let $W$ be a $G$-module and let $\mathbb{C}[W]^G$ be generated by $g_1, \ldots, g_r \in \mathbb{C}[W]$. Let the ideal of syzygies be generated by $h_1, \ldots, h_s \in \mathbb{C}[x_1, \ldots, x_r]$. So we have the closed embedding*

$$\begin{array}{rcl} W /\!\!/ G & \stackrel{\cong}{\longrightarrow} & V(h_1, \ldots, h_s) \subseteq \mathbb{C}^r. \\ x & \mapsto & (g_1(x), \ldots, g_r(x)). \end{array}$$

*Now let $f_1, \ldots, f_t \in \mathbb{C}[W]^G$. Then $X := V(f_1, \ldots, f_t) \subseteq W$ is a closed $G$-invariant affine variety. The quotient $X /\!\!/ G \subseteq W /\!\!/ G$ is given by*

$$X /\!\!/ G \stackrel{\cong}{\longrightarrow} V(h_1, \ldots, h_s, f_1^*, \ldots, f_t^*) \subseteq \mathbb{C}^r,$$

*where $f_i^*$ is the image of $f_i$ under the $\mathbb{C}$-algebra isomorphism*

$$\varphi\colon \mathbb{C}[W]^G \to \mathbb{C}[x_1, \ldots, x_r] / \langle h_1, \ldots, h_s \rangle .$$

In the following we give examples of different techniques - using the above construction - to obtain factorial Cox rings of varieties with a torus action of complexity one. Recall that all such rings are of the form $R(A, P_0)$ from Construction 1.2.1. We can assume that such a ring is given by relations

$$T_0^{l_0} + T_1^{l_1} + T_2^{l_2}, \quad T_1^{l_1} + 2T_2^{l_2} + T_3^{l_3}, \ldots, \quad T_{r-2}^{l_{r-2}} + (r-1)T_{r-1}^{l_{r-1}} + T_r^{l_r},$$

where due to canonicity the maximal exponents $\ell_i := \max(l_{i1}, \ldots, l_{in_i})$ are equal to one for $i \geq 3$ and $(\ell_0, \ell_1, \ell_2)$ form a platonic tuple. Moreover, due to factoriality, the gcd-exponents $\mathfrak{l}_i := \gcd(l_{i1}, \ldots, l_{in_i})$ are pairwise coprime.

Example 5.2. *There are some invariant rings of* $\mathrm{SL}_n$*-representations that are of this form. We will denote the quotient variety of a representation* $(\mathrm{SL}_n, W)$ *by*

*$W /\!\!/ \mathrm{SL}_n$. We have the following:*

$$\begin{aligned} 4V /\!\!/ \mathrm{SL}_2 &= V(T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}), \\ 2V + S^2 /\!\!/ \mathrm{SL}_2 &= V(T_{01}^2 + T_{11}^2 T_{12} + T_{21}T_{22}), \\ V + S^3 /\!\!/ \mathrm{SL}_2 &= V(T_{01}^2 + T_{11}^2 T_{12} + T_{21}^3). \end{aligned}$$

*From now on, we will often name generators of the ring of invariants by their image under the $\mathbb{C}$-algebra homomorphism $\varphi$.*

The above Cox rings arise directly from invariant rings of $\mathrm{SL}_n$. We go one step beyond.

EXAMPLE 5.3. *Consider the $\mathrm{SL}_2$-representation $V + 2S^2$. Let $x_0, x_1$, $y_0, y_1, y_2$ and $u_0, u_1, u_2$ be coordinates on $V$ and the first and second copy of $S^2$ respectively. The ring of invariants is generated by*

$$\begin{aligned} T_{01} &= -2\,u_0x_0x_1y_2 + 2\,u_0x_1^2y_1 + 2\,u_1x_0^2y_2 - 2\,x_1^2y_0u_1 - 2\,u_2x_0^2y_1 + 2\,u_2x_0x_1y_0, \\ T_{11} &:= -x_0^2y_2 + 2\,x_0x_1y_1 - x_1^2y_0, \\ T_{12} &:= 2\,u_0u_2 - 2\,u_1^2, \\ T_{21} &:= -u_0x_1^2 + 2\,u_1x_0x_1 - u_2x_0^2, \\ T_{22} &:= 2\,y_0y_2 - 2\,y_1^2, \\ S &:= y_2u_0 - 2\,y_1u_1 + y_0u_2. \end{aligned}$$

*After suitably scaling, the ideal of relations is generated by*

$$f := T_{01}^2 + T_{11}^2T_{12} + T_{21}^2T_{22} + T_{11}T_{21}S.$$

*Thus the quotient $X /\!\!/ \mathrm{SL}_2$ of $X := V(y_2u_0 - 2\,y_1u_1 + y_0u_2) \subseteq V + 2S^2$ is*

$$X /\!\!/ \mathrm{SL}_2 \cong V(f, S) \cong V(T_{01}^2 + T_{11}^2T_{12} + T_{21}^2T_{22}) \subset \mathbb{C}^5,$$

*which is a factorial quasicone of complexity one. On the other hand, as an affine variety, $X$ is isomorphic to $(4V /\!\!/ \mathrm{SL}_2) \times \mathbb{C}^2$ from Example 5.2. Thus we have an $\mathrm{SL}_2$-iteration chain of factorial quasicones of complexity one:*

$$\mathbb{C}^{10} \xrightarrow{/\!\!/(\mathrm{SL}_2, 4V)} V(T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}) \xrightarrow{/\!\!/(\mathrm{SL}_2, V+2S^2)} V(T_{01}^2 + T_{11}^2T_{12} + T_{21}^2T_{22}).$$

EXAMPLE 5.4. *Once more, consider the $\mathrm{SL}_2$-representation $4V$. Denote by $x_{k1}, x_{k2}$ the coordinates on the $k$-th copy of $V$. Then the invariants are the minors*

$$S_{ij} := x_{i1}x_{j2} - x_{j1}x_{i2}$$

*and the ideal of relations is generated by*

$$f := S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23}.$$

*So by suitably scaling, the invariant ring is isomorphic to the factorial quasicone $V(T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22})$ of complexity one. Now for $N \geq 3$, consider the action of $\mathrm{SL}_2$ on $W := 4V \times \mathbb{C}^{N-1}$, where $\mathbb{C}^{N-1}$ has coordinates $T_{21} \cdots T_{2(N-1)}$. Then the quotient $X /\!\!/ \mathrm{SL}_2$ of $X := V(x_{21}x_{32} - x_{31}x_{22} - T_{21} \cdots T_{2(N-1)}) \subseteq W$ is*

$$X /\!\!/ \mathrm{SL}_2 \cong V(f, S_{23} - T_{21} \cdots T_{2(N-1)}) \cong V(T_{01}T_{02} + T_{11}T_{12} + T_{21} \cdots T_{2N})$$

*By iterating this procedure, we get a chain of factorial quasicones of complexity one for arbitrary* $N$ *with* $n := \frac{N(N+1)}{2}$*:*

$$\begin{array}{c} \mathbb{C}^n \\ \big\downarrow /\!\!/ (\mathrm{SL}_2, 4V) \\ V(T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}) \\ \big\downarrow /\!\!/ (\mathrm{SL}_2, 4V) \\ \vdots \\ \big\downarrow /\!\!/ (\mathrm{SL}_2, 4V) \\ V(T_{01}T_{02} + T_{11}T_{12} + T_{21} \cdots T_{2N}). \end{array}$$

The next example shows that the $\mathrm{SL}_n$-iteration chains for a factorial Cox ring do not have to be unique.

EXAMPLE 5.5. *Again consider the* $\mathrm{SL}_2$*-representation* $4V$*, with the same notation as in Example 5.4. Now define the affine subvariety*

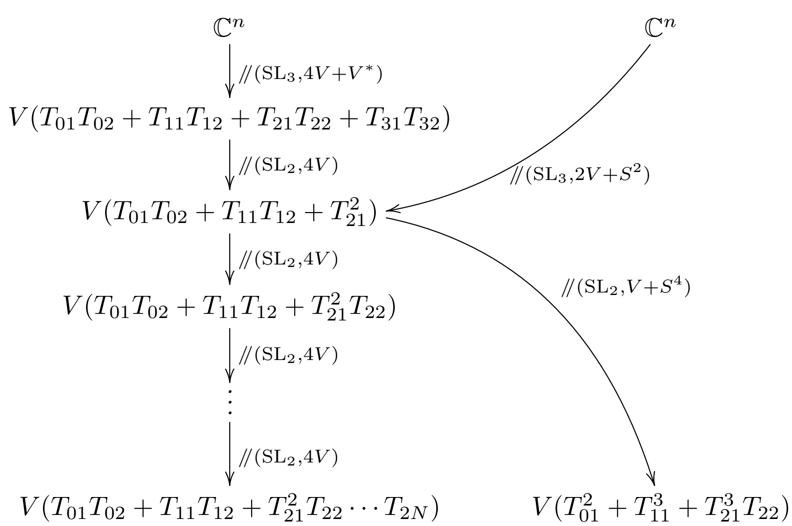
$$X := V(x_{21}x_{32} - x_{31}x_{22} - x_{11}x_{42} + x_{41}x_{12}) \subseteq 4V.$$

*The quotient of* $X$ *by* $\mathrm{SL}_2$ *is*

$$Y := X /\!\!/ \mathrm{SL}_2 \cong V(f, S_{23} - S_{14}) \cong V(T_{01}T_{02} + T_{11}T_{12} + T_{21}^2).$$

*On the other hand,* $X$ *is isomorphic to the maximal spectrum of the invariant ring of the* $SL_n$*-representation* $4V + \Lambda^2$ *for arbitrary* $n \geq 3$*, see e.g.* [**91**, Table 1]*. Moreover, the invariant ring of the* $SL_n$*-representation* $2V + S^2$ *for arbitrary* $n \geq 3$ *has maximal spectrum isomorphic to* $Y$ *as well. So* $Y$ *can be represented as an iterated* $\mathrm{SL}_n$*-quotient in numerous ways. The question arises if there is one canonical way among them.*

*Now in a similar way as in Example 5.4, considering subvarieties of the* $\mathrm{SL}_2$*-representations* $4V$ *and* $V + S^4$ *isomorphic to* $Y$*, we get the following picture of iteration chains:*



We conclude our series of examples with a rather involved one:

EXAMPLE 5.6. *Consider the* $\mathrm{SL}_3$*-representation* $4V + 2V^*$*. With the notation from Example 5.4, its ring of invariants has maximal spectrum isomorphic to*

$$X := V(x_{11}x_{22} - x_{12}x_{21} - y_{11}y_{22} - y_{12}y_{21}, x_{11}x_{32} - x_{12}x_{31} - y_{11}y_{32} - y_{12}y_{31}),$$

*see for example* [**77**, p. 255]*. Now consider the* $(\mathrm{SL}_2 \times \mathrm{SL}_2)$*-representation* $4V \times 4V$*, where the first copy of* $\mathrm{SL}_2$ *acts as usual on the first copy of* $4V$*, trivial on the second and vice versa. Let* $x_{ij}$ *and* $y_{ij}$ *be coordinates on the first and second copy of* $4V$ *respectively. Then* $X \subseteq 4V \times 4V$ *is* $(\mathrm{SL}_2 \times \mathrm{SL}_2)$*-stable and since the quotient* $(4V \times 4V) /\!\!/ (\mathrm{SL}_2 \times \mathrm{SL}_2)$ *is given by relations*

$$f := S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23}, \quad g := R_{12}R_{34} - R_{13}R_{24} + R_{14}R_{23},$$

*the quotient* $X /\!\!/ (\mathrm{SL}_2 \times \mathrm{SL}_2)$ *is isomorphic to*

$$V(f, g, S_{12} - R_{12}, S_{13} - R_{13}) \cong V(z_{21}z_{42} - z_{22}z_{41} - T_{12}T_{13}, z_{21}z_{32} - z_{22}z_{31} - T_{22}T_{23}),$$

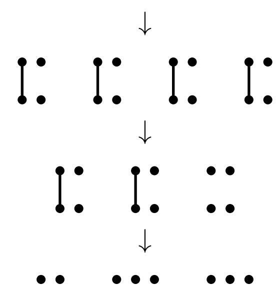
*which we denote by* $Y$*. Finally let* $\mathrm{SL}_2$ *act on* $W := 4V \times \mathbb{C}^4$*, with coordinates* $z_{ij}$ *and* $T_{12}, T_{13}, T_{22}, T_{23}$ *on* $4V$ *and* $\mathbb{C}^4$ *respectively. Then* $Y \subseteq W$ *is* $\mathrm{SL}_2$*-stable and* $Y /\!\!/ \mathrm{SL}_2$ *is isomorphic to* $V(T_{01}T_{02} + T_{11}T_{12}T_{13} + T_{21}T_{22}T_{23})$*. Altogether, we have the following chain of* $\mathrm{SL}_2$*-quotients:*

$$\begin{array}{c}
\mathbb{C}^{22} \\
\big\downarrow {\scriptstyle /\!\!/(\mathrm{SL}_3, 4V+2V^*)} \\
V\begin{pmatrix} T_{01}S_{02} + T_{11}S_{12} + T_{21}S_{22} + T_{31}S_{32}, \\ T_{01}R_{02} + T_{11}R_{12} + T_{21}R_{22} + T_{31}R_{32} \end{pmatrix} \\
\big\downarrow {\scriptstyle /\!\!/(\mathrm{SL}_2, 4V) \times (\mathrm{SL}_2, 4V)} \\
V\begin{pmatrix} T_{01}S_{02} + T_{11}S_{12} + S_{21}S_{22}, \\ T_{01}R_{02} + T_{11}R_{12} + R_{21}R_{22} \end{pmatrix} \\
\big\downarrow {\scriptstyle /\!\!/(\mathrm{SL}_2, 4V)} \\
V(T_{01}T_{02} + T_{11}T_{12}T_{13} + T_{21}T_{22}T_{23})
\end{array}$$

The above examples show that apart from sporadic ones, we can systematically produce series of factorial master Cox rings of complexity one by iterating $\mathrm{SL}_n$-quotients. Nevertheless, this becomes more and more involved as the relations become more complicated. Thus new techniques are needed to obtain more (or *all*) master Cox rings as iterated $\mathrm{SL}_n$-quotients. As an outlook, we ask for two possible directions:

QUESTION 5.7. *Is there a way to handle more complicated iterations of* $\mathrm{SL}_n$ *contructively?*

We shortly discuss one idea that may help to answer this question. Both Cox rings of $T$-varieties of complexity one and invariant rings of $\mathrm{SL}_n$ have the following building plan: they are given by certain weighted homogeneous relations $f_1, \dots, f_r$, where all or at least some subset $f_{i_1}, \dots, f_{i_s}$ are of the same form, but containing different variables. Some of the variables in the $f_i$ coincide. We can graphically depict one $f_i$ by groups of vertices in one row, a monomial $m_{ij}$ of $f_i$ is represented by a group of vertices, where each vertex represents a variable occuring in $m_{ij}$. If now two variables (in one or two $f_i$, i.e. in one or two rows) coincide, we connect them with an edge. In the course of an iteration of $\mathrm{SL}_n$-quotients, we get a chain of graphs. Considering e.g. Example 5.6, with the empty graph standing for $\mathbb{C}^n$, we have:



It seems natural to ask if certain $\mathrm{SL}_n$-actions can be translated into certain transformations of graphs of this kind. We leave this open here and conclude with a second possible direction:

QUESTION 5.8. *Let $X$ be a factorial affine variety (e.g. a factorial quasicone). Is there a way to obtain from $X$ a pair $(G, Y)$ - unique with respect to some properties - with a reductive algebraic group $G$ and an affine variety $Y$ - so that $X := Y /\!\!/ G$ - in a similar way as e.g. the pair $(\mathrm{Cl}(X), \overline{X})$ in the Cox construction or the pair $(\pi_1(X), \tilde{X})$ with fundamental group $\pi_1(X)$ and universal cover $\tilde{X}$?*

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# Deutsche Zusammenfassung

Das Thema der vorliegenden Arbeit sind *Mori Dream Spaces* und deren Darstellungen als Quotienten nach reduktiven algebraischen Gruppen. Mori Dream Spaces wurden in der Arbeit [**57**] von Hu und Keel eingeführt. In der ursprünglichen Definition, die nur projektive Varietäten einschließt, werden sie durch ein optimales Verhalten im Hinblick auf das *minimal model program* gekennzeichnet. In [**57**, Prop. 2.9] wird dann gezeigt, dass dies äquivalent ist dazu, dass sowohl die Divisorenklassengruppe $\mathrm{Cl}(X)$ als auch der Ring der globalen Schnitte

$$\mathcal{R}(X) := \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)),$$

der sogenannte *Coxring*, endlich erzeugt ist, wobei wiederum in der ursprünglichen Definition statt $\mathrm{Cl}(X)$ lediglich die *Picardgruppe* $\mathrm{Pic}(X)$ Verwendung fand, diese als torsionsfrei und $X$ als $\mathbb{Q}$-faktoriell angenommen wurde. In den folgenden Jahren wurden diese Definitionen - insbesondere in [**13, 38, 14, 49**] - zunehmend verallgemeinert auf nicht notwendig $\mathbb{Q}$-faktorielle Prävarietäten mit endlich erzeugter, nicht notwendig torsionsfreier Divisorenklassengruppe.

Nicht nur aufgrund ihrer guten Eigenschaften im Hinblick auf das minimal model program, sondern auch aufgrund der Tatsache, dass viele geometrische Aspekte, z.B. Eigenschaften von Singularitäten, anhand ihrer Coxringe verstanden werden können, haben Mori Dream Spaces in den vergangenen Jahren große Aufmerksamkeit erfahren. So wurde gezeigt, dass abgesehen von torischen Varietäten auch sphärische Varietäten und solche vom Fano Typ, d.h. mit einer amplen kanonischen Klasse und höchstens log-terminalen Singularitäten, Mori Dream Spaces sind.

Daraufhin konnte mit verschiedenen Methoden in [**43, 28, 62**] gezeigt werden, dass eine projektive Varietät genau dann vom Fano Typ ist, wenn ihr Coxring log-terminal ist (eine analoge Charakterisierung von Mori Dream Spaces vom *Calabi-Yau Typ* mittels eines log-kanonischen Coxrings findet sich ebenfalls in [**43, 62**], wobei zu beachten ist, dass Calabi-Yau Varietäten nicht zwingend Mori Dream Spaces sind). Zuvor schon wurde ebenfalls mithilfe unterschiedlicher Ansätze gezeigt, dass Coxringe von Varietäten mit *torsionsfreier* Divisorenklassengruppe faktoriell sind [**13, 38, 4**], was insbesondere impliziert, dass in diesem Fall $\mathcal{R}(X)$ Gorenstein ist. Faktorialität gilt im Allgemeinen - also für nicht torsionsfreie Klassengruppe - nicht, aber ein Hauptergebnis der vorliegenden Arbeit ist, dass endlich erzeugte Coxringe immer Gorenstein sind.

Torische Varietäten als 'Ursprung' der Coxring-Theorie besitzen nicht nur alle einen endlich erzeugten Coxring, dieser ist sogar polynomiell, und diese Eigenschaft charakterisiert wiederum torische Varietäten eindeutig. Im Gegensatz dazu wurde für Varietäten, die lediglich die Wirkung eines niedrigerdimensionalen Torus $\mathbb{T}$ erlauben, in [**52**] gezeigt, dass der Coxring genau dann endlich erzeugt ist, wenn dies für den geometrischen Quotienten $X_0/\mathbb{T}$ gilt, wobei $X_0$ die Menge der Punkte von $X$ mit endlicher Isotropiegruppe bezeichne. Das erlaubt im Falle eines in $X$ lediglich eins-kodimensionalen Torus $\mathbb{T}$ - wir nennen solche Varietäten im Folgenden 'Komplexität eins $\mathbb{T}$-Varietäten' - eine kombinatorische Beschreibung ähnlich der von torischen Varietäten.

Diese Tatsache ist einer allgemeineren Beobachtung geschuldet: $A_2$-Varietäten $X$ - d.h. solche mit der Eigenschaft, dass zwei beliebige Punkte eine gemeinsame affine Umgebung besitzen - mit endlich erzeugtem Coxring können immer kanonisch so in eine torische Varietät $Z$ eingebettet werden, dass sie mit ihr die selbe Divisorenklassengruppe teilen und das Maximalspektrum des Coxrings $\overline{X} := \operatorname{Spec} \mathcal{R}(X)$ als abgeschlossene affine Varietät in $\overline{Z}$ realisiert werden kann. Wir nennen $\overline{X}$ auch den totalen Koordinatenraum von $X$. Man beachte, dass $\overline{Z}$ als $\mathbb{C}^n$ angenommen werden kann. Eine aus der torischen Geometrie bekannte Konstruktion erlaubt es nun, die torische Varietät $Z$ als GIT-Quotient (d.h. als Quotient einer offenen Untervarität mit mindestens zwei-kodimensionalem Komplement) von $\overline{Z}$ nach dem *charakteristischen Quasitorus* $H_Z := \operatorname{Spec} \mathbb{C}[\operatorname{Cl}(Z)]$ darzustellen. Diese Konstruktion kann auf $X$ übertragen werden, da nicht nur die Divisorenklassengruppen übereinstimmen, sondern auch $\overline{X}$ invariant unter der Wirkung von $H_X = H_Z$ auf $\overline{Z}$ ist. Wir erhalten eine Darstellung $X = \widehat{X} /\!\!/ H_X$, wobei der charakteristische Raum $\widehat{X} \subseteq \overline{X}$ gerade offen mit mindestens zwei-kodimensionalem Komplement und eindeutig durch den charakteristischen Raum $\widehat{Z}$ bestimmt ist.

Ziel dieser Arbeit ist es nun, sowohl Eigenschaften von Coxringen - insbesondere solche, die sich in der Varietät $X$ niederschlagen - zu untersuchen, als auch die oben genannte Quotientenkonstruktion zu verallgemeinern. Dies auf zwei verschiedene Arten: erstens durch Iteration ebendieser Konstruktion, d.h. durch Betrachtung des Coxrings des totalen Koordinatenraums $\overline{X}$ etc., und zweitens im Falle einer faktoriellen Varietät $X$, wo diese Konstruktion aufgrund von $\operatorname{Cl}(X) = 0$ trivial ist, durch Berechnung von Invariantenringen der speziellen linearen Gruppe $\operatorname{SL}_n(\mathbb{C})$. Diese Invariantenringe liefern aufgrund der Einfachheit der Gruppe $\operatorname{SL}_n(\mathbb{C})$ gerade faktorielle Kandidaten für den Koordinatenring (und damit den Coxring) solcher Varietäten.

Wir stellen im Folgenden in der Reihenfolge der Kapitel die zentralen Ergebnisse der vorliegenden Arbeit vor.

In den ersten beiden Abschnitten des einführenden Kapitels 1 werden die notwendigen Grundlagen besprochen - zum einen in Hinblick auf Mori Dream Spaces im Allgemeinen, zum anderen für Komplexität eins $\mathbb{T}$-Varietäten im Besonderen. Die letzten beiden Abschnitte von Kapitel 1 enthalten bereits neue Ergebnisse, diese wurden in der Arbeit [**5**] mit Arzhantsev, Hausen und Wrobel veröffentlicht. Wir zeigen dort, dass affine Komplexität eins $\mathbb{T}$-Varietäten $X$, die Singularitäten $x$ repräsentieren in dem Sinne, dass $x$ im Abschluss jedes Orbits eines eindimensionalen Untertorus von $\mathbb{T}$ liegt - sogenannte *Quasikegel* $X$ mit Spitze $x$ - genau dann log-terminal sind, wenn die Exponententupel in den beschreibenden Gleichungen von $\mathcal{R}(X)$ eine der folgenden Formen annehmen:

$$(5, 3, 2, 1, \ldots, 1), \quad (4, 3, 2, 1, \ldots, 1), \quad (3, 3, 2, 1, \ldots, 1),$$

$$(x, 2, 2, 1, \ldots, 1), \quad (x, y, 1, \ldots, 1).$$

Wir nennen die führenden Tripel solcher Tupel *platonisch*. Weiter führen wir im letzten Abschnitt von Kapitel 1 eine neue Invariante von Komplexität eins $\mathbb{T}$-Varietäten ein, die sogenannte *kanonische Multiplizität*, die sich für Klassifikationen als nützlich erweist. In Beispiel 1.4.8 erhalten wir so mit einfachen Mitteln exemplarisch die auf Brieskorn [**23**] zurückgehende Klassifikation log-terminaler Flächensingularitäten wieder.

In Kapitel 2 wird zum einen die schon erwähnte Aussage bewiesen, dass endlich erzeugte Coxringe Gorenstein sind, zum anderen beschäftigen wir uns mit der ebenfalls schon erwähnten *Iteration von Coxringen*. Eine für beide Untersuchungen grundlegende Beobachtung, die hier Erwähnung finden soll, ist die folgende (vgl. Korollar 2.1.4): teilen wir den charakteristischen Quasitorus $H_X$ eines Mori Dream Spaces $X$ in Torus- und Torsionsanteil $\mathbb{T}$ und $E$ auf und betrachten die Quotienten $\overline{X} /\!\!/ \mathbb{T}$ und $\overline{X} /\!\!/ E$, so ist zum einen $\overline{X} /\!\!/ E$ Mori Dream Space und besitzt denselben Coxring wie $X$, zum anderen ist $\overline{X} /\!\!/ \mathbb{T}$ genau dann Mori Dream Space, wenn $\overline{X}$ es ist und beide teilen sich in diesem Fall denselben Coxring.

Diese Beobachtung erlaubt es uns im zweiten Abschnitt von Kapitel 2, den Beweis der Gorenstein-Eigenschaft von $\mathcal{R}(X)$ auf eine *endliche* Divisorenklassengruppe zurückzuführen. Wir zeigen anschließend, dass die Cox-Konstruktion durch die Index-1-Überdeckung faktorisiert, was zusammen mit der Tatsache, dass die Gorenstein-Eigenschaft von endlichen Quotienten erhalten wird, den Beweis komplettiert.

Im dritten Abschnitt zeigen wir, dass die schon erwähnten Quasikegel nicht nur im Falle, dass sie log-terminal sind, die affinen Gegenstücke zu Varietäten vom Fano Typ darstellen, also ebenfalls einen endlich erzeugten Coxring mit log-terminalem Maximalspektrum haben, sondern sich die Quasikegel-Eigenschaft auch auf den Coxring überträgt. Das führt zusammen mit der Gorenstein-Eigenschaft von Coxringen zu folgender Charakterisierung, siehe Theorem 2, die die erwähnte Charakterisierung von Varietäten vom Fano Typ aus [**43, 28, 62**] verfeinert und verallgemeinert:

THEOREM. *Sei $X$ eine projektive (affine) Varietät. Dann ist $X$ genau dann vom Fano Typ (ein Kawamata log-terminaler Quasikegel), wenn $\mathcal{R}(X)$ ein kanonischer Gorenstein Quasikegel ist.*

Wir zeigen dann, dass dieses Kriterium für Komplexität eins $\mathbb{T}$-Varietäten dazu äquivalent ist, dass die maximalen Exponenten in den Gleichungen des Coxrings die schon erwähnten platonischen Tripel bilden.

Der vierte Abschnitt von Kapitel 2 führt die Iteration von Coxringen ein, indem - falls für einen Mori Dream Space $X$ der totale Koordinatenraum $\overline{X}$ ebenfalls Mori Dream Space ist - wiederum dessen Coxring $\mathcal{R}(\overline{X})$ und totaler Koordinatenraum $X^{(2)} := \overline{\overline{X}}$ betrachtet werden, dann im Falle dass $X^{(2)}$ Mori Dream Space ist, dessen totaler Koordinatenraum $X^{(3)}$ usw. Hierbei sind drei Szenarien möglich: erstens ist in jedem Schritt $X^{(i)}$ Mori Dream Space mit nichttrivialer Klassengruppe, dann hat $X$ *unendliche Coxring-Iteration*, oder für ein $N \in \mathbb{N}$ ist entweder $X^{(N)}$ kein Mori Dream Space, oder faktoriell - hat also triviale Klassengruppe. Wir sagen in diesen Fällen, dass $X$ *endliche Coxring-Iteration* hat, mit entweder nicht-MDS oder faktoriellem *Master-Coxring* $\mathfrak{R}(X) := \mathbb{C}[X^{(N)}]$. Wir zeigen ebenfalls, dass für jedes $i \leq N$ die Varietät $X$ dargestellt werden kann als Quotient von $X^{(i)}$ nach einer reduktiven auflösbaren Gruppe, durch deren Normalreihe wir im Gegenzug die Sequenz der charakteristischen Quasitori $H_{X^{(i)}}$ zurückerhalten.

Schließlich zeigen wir im fünften Abschnitt von Kapitel 2, dass Varietäten vom Fano Typ und log-terminale Quasikegel endliche Coxring-Iteration mit faktoriellem Master-Coxring haben. Man beachte hier, dass in diesem Fall nach obigem Theorem bereits nur die zwei Möglichkeiten einer unendlichen oder einer endlichen Coxring-Iteration mit faktoriellem Master-Coxring übrigbleiben. Es bleibt also zu zeigen, dass die Coxring-Iteration tatsächlich endlich ist. Dies erhalten wir, indem wir mithilfe des oben beschriebenem Korollars 2.1.4 wiederum auf den Fall von endlichen Quotienten zurückführen. Eine Sequenz von endlichen Quotienten (mit bestimmten hier vorliegenden Eigenschaften) muss aber nach einem Theorem von Greb, Kebekus und Peternell [**45**] endlich sein, woraus auch die Endlichkeit der Sequenz von Quasitorusquotienten folgt.

Im sechsten und letzten Abschnitt von Kapitel 2 berechnen wir im konkreten Fall von Komplexität eins $\mathbb{T}$-Varietäten explizit die Gleichungen von iterierten Coxringen, vgl. Korollar 2.6.8.

In Kapitel 3 beschäftigen wir uns mit Invariantenringen der speziellen linearen Gruppe $\mathrm{SL}_n(\mathbb{C})$. Diese sind im Zusammenhang der vorliegenden Arbeit deshalb von Interesse, da sie gerade Beispiele von faktoriellen quasihomogenen Ringen liefern, also Kandidaten für die *Endpunkte* der Coxring-Iteration, die faktoriellen Master-Coxringe sind. In der klassischen Invariantentheorie des neunzehnten Jahrhunderts wurden vor allem Invariantenringe der Gruppe $\mathrm{SL}_2$ untersucht und explizit berechnet. Ein Ansatz zur Berechnung war die *symbolische Methode*, wobei *brackets* (also Klammern) zur Darstellung von Invarianten dienen. Dies folgendermaßen: eine unmittelbar ersichtliche unter der Wirkung von $\mathrm{SL}_n$ invariante Polynomfunktion ist die Determinante auf $\mathrm{Mat}(n, n, \mathbb{C}) = (\mathbb{C}^n)^n$. Unter einem gewissen Gesichtspunkt ist sie sogar die einzige. Betrachten wir die Determinante nämlich als kovarianten Tensor $\det = e_1 \wedge \ldots \wedge e_n$, wobei $e_i$ die $i$-te Standardkoordinatenfunktion auf $\mathbb{C}^n$ sei, so kann jede Invariante einer Darstellung von $\mathrm{SL}_n$ ( also eines $\mathrm{SL}_n$-Moduls $W$) dargestellt werden als lineare Abbildung von $W$ in die Tensoralgebra $T(\mathbb{C}^n)$ gefolgt von einer vollständigen Kontraktion des Tensorprodukts des Bildes mit einer passenden Anzahl von kovarianten Tensoren det.

Eine zu einem Tensor det korrespondierende *bracket*, die genau $n$ Buchstaben enthält, welche wiederum zu irreduziblen Unterdarstellungen von $W$ korrespondieren, kodiert nun, welche Indizes kontrahiert werden. Betrachten wir zum Beispiel die $\mathrm{SL}_2$-Darstellung $W := \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus S^2(\mathbb{C}^2)$, wobei $S^k(V)$ das $k$-fache symmetrische Produkt von $V$ bezeichne, und repräsentieren wir die beiden Kopien von $\mathbb{C}^2$ sowie $S^2(\mathbb{C}^2)$ mit Buchstaben $a_1, a_2$ und $b$, so repräsentieren die brackets

$$[a_1 a_2], \quad [a_1 a_1], \quad [bb], \quad [a_1 b][b a_1]$$

jeweils die Invarianten

$$\begin{array}{rcl} W & \to & \mathbb{C} \\ (t_1, t_2, s) & \mapsto & \left\{ \begin{array}{rcl} C_{12}^{12}\, t_1 \otimes t_2 \otimes \det & = & t_{11}t_{22} - t_{12}t_{21} \\ C_{12}^{12}\, t_1 \otimes t_1 \otimes \det & = & 0 \\ C_{12}^{12}\, s \otimes \det & = & s_{11}s_{22} - s_{12}^2 \\ C_{1234}^{1234}\, t_1 \otimes s \otimes t_2 \otimes \det \otimes \det & = & 2t_{11}t_{12}s_{12} - t_{11}^2 s_{22} - t_{22}^2 s_{11}, \end{array} \right. \end{array}$$

wobei $t_{i1}, t_{i2}$ jeweils die Koordinaten auf der $i$-ten Kopie von $\mathbb{C}^2$ sowie $s_{11}, s_{22}, s_{12}$ diejenigen auf $S^2(\mathbb{C}^2)$ seien. Zusammen mit den zugehörigen Invarianten der brackets $[a_1 b][b a_2]$ und $[a_2 b][b a_2]$ bilden diese insgesamt fünf Invarianten ein minimales Erzeugendensystem des Invariantenrings $\mathbb{C}[W]^{\mathrm{SL}_2}$. Insbesondere sind die aus zwei brackets bestehenden Invarianten $[a_i b][ba_j]$ also nicht darstellbar als Polynom in jenen mit nur einer bracket. Dennoch erfüllen die Invarianten die Relation

$$[bb][a_1 a_2]^2 - 2[a_1 b][ba_1][a_2 b][ba_2] + 2[a_1 b][ba_2]^2,$$

welche das Relationenideal sogar erzeugt. Diese Relation kann aus den bekannten *Plückerrelationen* abgeleitet werden, wobei sich hier Buchstaben, welche zu Kopien von $\mathbb{C}^2$ korrespondieren, wie gewohnt verhalten, d.h. auf $W := (\mathbb{C}^2)^4$ erhalten wir beispielsweise die Relation

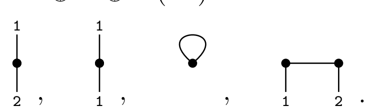
$$[a_1 a_2][a_3 a_4] - [a_1 a_3][a_2 a_4] + [a_1 a_4][a_2 a_3].$$

Buchstaben, die für andere Darstellungen von $\mathrm{SL}_n$ stehen, zeigen ein davon abweichendes 'Verhalten', was zu anderen Relationen wie beispielsweise der ersteren führt. Die symbolische Methode für kombinierte Summen symmetrischer und antisymmetrischer Produkte von $\mathbb{C}^n$ wurde von Grosshans, Rota und Stein in [**46**] eingeführt. Darauf aufbauend zeigte beispielsweise Huang in [**58**], dass der Invariantenring $\mathbb{C}[\bigoplus_{i=1}^m \Lambda^2(\mathbb{C}^4)]^{\mathrm{SL}_4}$ für beliebiges $m$ von brackets der Form

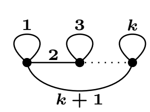
$$[b_{k+1} b_1 b_1 b_2][b_2 b_3 b_3 b_4] \cdots [b_{k-1} b_k b_k b_{k+1}]$$

erzeugt wird. Bei zunehmender Dimension $n$ und zunehmender Anzahl an irreduziblen Unterdarstellungen von $W$ werden jedoch die symbolische Methode und insbesondere die Auswirkungen der Plückerrelationen schnell unübersichtlich.

An diesem Punkt setzt Kapitel 3 an, wobei eine graphische Methode entwickelt wird, die zumindest teilweise diese Probleme umgehen kann. Statt brackets betrachten wir *gefärbte Hypergraphen*, wobei jeder Knoten einer bracket entspricht und eine $i$-Hyperkante (also eine Kante mit $i$ Verbindungen zu Knoten) einem zu einem $i$-fachen Produkt von $\mathbb{C}^n$ gehörenden Buchstaben. Für jedes Vorkommen des zugehörigen Buchstabens besitzt die Kante eine Verbindung zum entsprechenden Knoten. Insbesondere wenn $W$ mehrere $i$-fache Produkte von $\mathbb{C}^n$ enthält, so unterscheiden wir $i$-Hyperkanten durch entsprechende Einfärbung. Die vier Invarianten der $\mathrm{SL}_2$-Darstellung $W := \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus S^2(\mathbb{C}^2)$ von oben entsprechen also den Graphen




Die Darstellung mittels Graphen erlaubt in vielen Fällen eine tiefere Einsicht in die Auswirkungen der Plückerrelationen und darüberhinaus die Anwendung graphentheoretischer Aussagen. Beispielsweise kann so gezeigt werden, dass von den zyklischen Graphen




die gerade den oben erwähnten Invarianten von $\mathbb{C}[\bigoplus_{i=1}^m \Lambda^2(\mathbb{C}^4)]^{\mathrm{SL}_4}$ entsprechen, genau solche mit *einem oder drei Knoten* ein *minimales* Erzeugendensystem bilden. Im zweiten Abschnitt von Kapitel 3 bestimmen wir mithilfe unserer graphischen Methode für *Summen von fundamentalen Darstellungen* $W$, also für $\mathrm{SL}_n$-Moduln der Form

$$W := \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^{n_i} V_{i,j},$$

mit $V_{i,j} := \Lambda^i V$, minimale Erzeugendensysteme der Invariantenringe im Falle $n = 4, 5$, sowie im Falle $n = 4$ ein minimales Erzeugendensystem für den *Kovariantenring* und das Relationenideal.

In den folgenden drei Abschnitten von Kapitel 3 beschäftigen wir uns mit Darstellungen von $\mathrm{SL}_n$ mit einem Invariantenring, der ein vollständiger Durchschnitt ist. Solche Darstellungen wurden von Shmelkin in [**91**] klassifiziert, wobei für drei einzelne und drei Serien von Darstellungen keine Entscheidung getroffen werden konnte. Wir zeigen in Abschnitt 3.5, dass der Invariantenring aller dieser übriggebliebener Darstellungen ein vollständiger Durchschnitt ist. Dazu verwenden wir - abgesehen von der bereits vorgestellten graphischen Methode - einen in Abschnitt 3.3 entwickelten Algorithmus, welcher es ermöglicht, zu zeigen, dass gewisse Polynome eine Gröbnerbasis bilden, indem eine bestimmte geeignete Monomordnung gefunden wird. Darüberhinaus stellen wir in Abschnitt 3.4 dar, wie ein Algorithmus von Xin [**107**] durch leichte Modifikation dazu genutzt werden kann, Hilbertreihen von Invariantenringen zu berechnen. Diese Methoden werden schließlich im letzten Abschnitt zusammen mit Berechnungen von Kovariantenringen - ebenfalls mithilfe der graphischen Methode - dazu genutzt, die Klassifikation der $\mathrm{SL}_n$-Darstellungen mit einem Invariantenring, der ein vollständiger Durchschnitt ist, abzuschließen.

Das folgende Kapitel 4 beschäftigt sich mit zwei Unterklassen von log-terminalen dreidimensionalen Singularitäten mit Toruswirkung der Komplexität eins, genauer gesagt deren Klassifikation. Die ersten beiden Abschnitte behandeln sogenannte *compound Du Val* Singularitäten, kanonische Gorenstein Punkte $x$ mit der Eigenschaft, dass ein allgemeiner Hyperebenenschnitt durch $x$ ebenfalls nur kanonische Singularitäten besitzt. Nach dem einführenden Abschnitt 4.1 werden in Abschnitt 4.2 solche compound Du Val Singularitäten mit einer Toruswirkung der Komplexität eins vollständig klassifiziert. Der Abschnitt 4.3 enthält vorbereitende Arbeiten, hauptsächlich betreffend bestimmte Polytope ohne innere ganzzahlige Punkte, für die in Abschnitt 4.4 folgende Klassifikation von kanonischen dreidimensionalen Singularitäten mit einer Toruswirkung der Komplexität eins vom Gorensteinindex $\imath \geq 2$. Abschließend bestimmen wir den gemeinsamen Coxring-Iterations-Baum dieser Singularitäten, wobei sich herausstellt, dass alle Wurzeln - also alle faktoriellen Master-Coxringe - das compound Du Val Kriterium erfüllen.

Das letzte Kapitel 5 hat den Charakter eines Ausblicks: anhand von mehreren Beispielen zeigen wir, wie durch Iteration von Quotienten nach $\mathrm{SL}_n$ neue faktorielle Master-Coxringe gewonnen werden können. Insbesondere im Fall von solchen mit Toruswirkung der Komplexität eins können wir so ganze Serien von Beispielen generieren. Dies legt die Hoffnung nahe, dass entweder mit verfeinerten Methoden noch allgemeinere Serien von Maximalspektra faktorieller Master-Coxringe als iterierte Quotienten nach $\mathrm{SL}_n$ *konstruktiv* dargestellt werden können oder wir - ausgehend von einer faktoriellen Varietät $X$, solch eine Quotientendarstellung als *Invariante* von $X$ erhalten, analog zu $\mathcal{R}(X)$ oder auch der universellen Überlagerung.

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