High-Frequency Trading: Insights from Analytical Models and Simulated Agent-Based Models

Dissertation
zur Erlangung des Doktorgrades
der Wirtschafts- und Sozialwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen

vorgelegt von
Lina Kalimullina (geb. Kazantseva), M.Sc.
aus Menzelinsk

Tübingen
2018

Tag der mündlichen Prüfung:	20.02.2019
Dekan:	Professor Dr. rer. soc. Josef Schmid
1. Gutachter:	Professor Dr.-Ing. Rainer Schöbel
2. Gutachter:	Professor Dr. rer. pol. Werner Neus

Acknowledgments

This dissertation is the result of my work at the University of Tübingen between September 2013 and April 2018. I would like to express my sincere gratitude to the people whose continuous support, encouragement, and patience allowed me to complete this thesis.

The key interim results of this research were presented in The Fifth German Network on Economic Dynamics (GENED) Meeting (Karlsruhe, Germany) in October 2017. While elaborating on the topic of this dissertation, I participated in the conferences "Computing in Economics and Finance" (Oslo, Norway), "Artificial Economics" (Barcelona, Spain), The Second GENED Meeting (Darmstadt, Germany) in 2014, and in The Third GENED Meeting (Bochum, Germany) in 2015. I would like to thank all the participants of the aforementioned conferences and meetings for their helpful feedback and interesting discussions.

I wish to express my sincere appreciation to my supervisor Prof. Dr.-Ing. Rainer Schöbel for being open to my research interests, for sharing his expertise with me and giving me useful advice, for continuously inspiring and encouraging me. I would also like to thank my second supervisor, Prof. Dr. Werner Neus, for his helpful comments and valuable feedback.

Moreover, I am very grateful to my chair colleagues - Vera Klöckner, Dr. Ingrid Kudlik, Dr. Stefan Rostek, and Raphael Sturn - for their support and a productive and comfortable working atmosphere. Many thanks to my colleagues from the Economic department and friends from the Master's program who continuously inspired me.

Last but not least, I would like to express my heartfelt gratitude to my family for their belief in me. I am thankful to my parents Nina and Alexander, as well as to my brother Andrei for their love and support. I wish to express my deepest gratitude to my husband Anton for his encouragement, patience and weekends unconditionally spent with our daughter Elani and thus providing me enough time for my research; without him, the completion of this dissertation would not have been possible.

Contents

List of Tables

List of Figures

List of Acronyms

Symbol	**Meaning**
ABM(s)	agent-based model(s)
Eq.	equilibrium
FT(s)	fast trader(s), fast trading
HFT(s)	high-frequency trader(s), high-frequency trading
IST(s)	informed slow trader(s), informed slow trading
IT(s)	informed trader(s), informed trading
LO(s)	limit order(s)
LOB	limit order book
MEP(s)	maximizer(s) of expected profit from limit orders
MO(s)	market order(s)
RT(s)	random trader(s), random trading
ST(s)	slow trader(s), slow trading

List of Symbols

Symbol	Meaning
$\hat{}, \check{}$	diactrics denoting different types of cutoff prices
$\rightarrow$	convergence sign
$\mathbf{0}$	zero matrix
A	ask order
AR	acceptance ratio
av	superscript of average values
α	probability that an agent is an FT or an IST
B	bid order
b	subscript used for buyers
$BestAsk$	best ask
$BestBid$	best bid
$border$	superscript denoting borderlines between equilibria
D	dividend
Δ	change in metrics
δ	normally distributed random variable
E	expectation
EHS	effective half spread
EPV	expected present value
ES	effective spread
$ESPO$	earned spread at passive orders
ε	discrete innovation of the fundamental value
$\mathbf{\Phi}$	limiting matrix of $\mathbf{P}^n$
φ	probability of an equilibrium event
$\boldsymbol{\varphi}$	stationary probabilities vector
γ	probability for an agent to be a buyer
h	order, equilibrium, or agent type
$\mathbf{I}$	identity matrix
ID	identifier
i	subscript differentiating an agent side, or a type, or an identifier, or random variables
j	information level / subscript differentiating random variables
k	order, equilibrium, or agent type (used to show the difference from h)
L	absolute value of the reservation value component y
m	midquote/superscript referring to the market value
MT	maker-taker ratio/maker-taker rate
$multi$	subscript for multiperiod values
$N(0, x)$	normal distribution with mean 0 and variance x
N	number / amount (e.g. of transactions, limit orders, etc.)

O	order
OIB	order imbalance
OT	order-to-trade ratio
$\mathbf{P}$	transition matrix
P	price
p	probability of ST's order to be satisfied next step
$Part$	participation
PE	pricing error
π	risk of being picked-off
PQS	proportional quoted spread
Pr	probability
$PSAO$	paid spread at aggressive orders
q	probability of the FT's or IST's order to be executed / dummy for a transaction type
QS	quoted spread
R	private reservation value / return
r_e	risk-adjusted interest rate
r_f	risk-free interest rate
RAD	relative absolute deviation
RET	excess return
RHO	correlation between prices and fundamental value
RHS	realized half spread
RPP	realized positioning profit
RR	relative return
S	asset endowment
s	subscript used for sellers / time step within a period
sim	superscript denoting market metrics obtained from the simulated data
σ	absolute value of the fundamental value innovation
T	number of simulated periods
t	time period
TC	trading costs
TP	trading profit
TR	trading rate
tr	transaction
ts	uninterrupted series of time steps where a certain activity happens (e.g., transactions)
τ	costs of immediacy
$U(0,1)$	uniform distribution
u	uniformly distributed random variable
V	expected profit from an order
v	fundamental value
W	welfare / wealth
w	probability of some order to be sent at t and to be executed at $t+1$

X	random variable
x	variance of the normal distribution
y	reservation value component that determines the market side of an agent
Z	random variable, found as a difference of X_i and X_j

1 Introduction

High-frequency trading (HFT) has been the most important market structure development in the past decade. HFT is a type of algorithmic trading which involves rapid trading of securities. HFT is "likely to affect nearly all aspects of its (market's) performance"[1], since it accounts for more than 50% of the trading volume of U.S.-listed equities. According to Huh (2014), HFT amounts to $50 - 85\%$ of the daily trading volume.

In this chapter, we introduce the topic and research methods, provide a literature review and comment on the organization of the dissertation. As it will become evident, HFT is a multidimensional activity, which leads to vagueness in its description; the definitions of this phenomenon evolve over time and embrace more and more characteristics and facts. There are three HFT research methods discussed: analytical models, empirical research, and agent-based models. Potential advantages and disadvantages of each of these methods are pointed out. Moreover, we provide a literature review with respect to the HFT influence on the market.

1.1 High-frequency trading: definition and influence on the market

Initially, the U.S. Securities and Exchange Commission (see SEC (2014)) distinguished only two main characteristics of high-frequency traders (HFTs):

(1) Professionalism of traders acting within a proprietary firm (which can be variously organized and (not) registered as a broker-dealer);

(2) A high scale of trading activity on a daily basis.

Later on, SEC added five further characteristics which are nowadays usually attributed to HFT:

(3) Use of high speed technologies and sophisticated programs for the purpose of generating, routing, and executing orders;

(4) Use of co-location and individual data feeds to reduce latency of the system;

(5) Short time needed for the creation and liquidation of positions;

(6) Submission of many orders, most of which are liquidated, almost immediately;

(7) Closing almost all positions at the end of a trading day: avoidance of unhedged overnight positions[2].

Gomber, Arndt, Lutat, and Uhle (2011) add the following properties to this list:

(8) Profit from buying/selling due to the middleman functions;

[1] See SEC (2014), p.4.

[2] This characteristic is discussed in detail in Easley, Lopez de Prado, and O'Hara (2011). The authors mention that the inventories of HFTs are very low: they are turned approximately five times a day.

(9) Extracting extremely low margins per trade;

(10) Focus on highly liquid assets and instruments.

German High Frequency Trading Act (2013) classifies a firm as an HFT, if it:

(1) Utilizes infrastructures to minimize latency;

(2) Uses decision systems to create, transmit, or execute orders automatically without human intervention;

(3) Has a high intra-day trading messages volume caused by the submissions or cancellations of quotes.

The multidimensional character of HFT explains certain vagueness of its definition, which hinders the complete and thorough investigation of the phenomenon. Any attempt to classify firms and activities as HFT or non-HFT would be limited: non-HFT activities might be classified as HFT and non-HFT based on various criteria.
Proprietary firms may engage in different activities, some of them may bring benefits to the market, while others may harm it. Therefore, when investigating the HFT influence, it is more reasonable to concentrate not on a single standardized definition of HFT, but rather focus on separate HFT characteristics, tools, or strategies. SEC (2014) describes the following strategies of HFT firms:

(1) Passive market-making involves posting limit orders that rest for some time in a limit order book; this strategy provides liquidity to a marketplace. A market-maker bears the risk of losing to more informed traders, therefore she cancels or updates orders frequently. Such trading aims to profit from a bid-ask spread and from liquidity rebates (or as Jones (2013) put it: the "maker fees").

(2) Arbitrage involves trading on temporary discrepancies between the prices of the same product on different markets or of the related products. Statistical arbitrage is based on deviations from stable theoretical relationships. Event arbitrage depends on certain recurring events which might lead to predictable short-term responses.

(3) Structural strategies are meant to profit from structural differences on the market, e.g. latency difference between agents. Low-latency strategies are considered to be naive HFT strategies: computer scientists try to trade on very short-term price discrepancies in different markets by gaining access to information faster than the majority of (low-latency) traders.

(4) Directional trading is intended to create an asset position depending on the anticipated future price change. These strategies apply a textual analysis and trade on inferred news (see Jones (2013), p.9):

- Order anticipation includes the process of testing whether a large buyer or a seller exists. This strategy takes into account the advantage of an anticipated big transaction. It is usually executed through ticker tape trading or filter trading, when algorithmic trading catches trading announcements, news, and other events. This is also called news-based trading.
- Momentum ignition involves a series of quotes and trades with the purpose to ignite a rapid price movement to a certain direction.

HFTs are perceived to be competing with other HFTs, as human traders are not able to operate in such a high-frequency environment. HFT opponents claim that HFT strategies are very risky compared to the traditional buy-and-hold strategies. HFT is usually believed to have contributed to the volatility increase that led to the Flash Crash in 2012. When index funds rebalance their portfolios, HFTs "strip" profits from investors by determining big orders or further order flow and acting respectively. Clark (2012) reports that:

(1) Risk controls are poorer in HFT due to the time pressure;

(2) There are no stringent processes to develop, test, and deploy a code used in trading mechanisms;

(3) There are a lot of erroneous or out-of-control algorithms.

The advocates of HFT argue that it:

(1) Improves market liquidity in terms of higher volume of limit orders (e.g. Biais, Declerck, and Moinas (2016));

(2) Reduces spreads (e.g. Menkveld (2013));

(3) Lowers trading costs: HFT activity smooths out the price effects of order imbalances (e.g. Subrahmanyam and Zheng (2016));

(4) Makes prices more efficient and increases the informativeness of quotes and prices (e.g Brogaard, Hendershott, and Riordan (2013)).

However, in the current state it is not clear how HFT influences the volatility of the market (there is evidence of both increased and decreased volatility due to HFT activities): whether HFT provides or abstains from providing liquidity on the volatile markets. It is also open to discussion whether increased liquidity benefits the market, as the liquidity provided by HFTs can be described as "phantom" liquidity (other traders cannot benefit from this additional liquidity).

Jones (2013), Gomber, Arndt, Lutat, and Uhle (2011), and Menkveld (2016) summarize the previous research related to the HFT influence on the market. Jones (2013) concludes that HFT leads to a better price discovery, but there is no evidence of its adverse effect on the average results. Gomber, Arndt, Lutat, and Uhle (2011) claim that HFT improves the forecastability of the average trading prices, as they become more concentrated around their mean. Moreover, in general HFT improves the market quality: out of eight sources considered, only one suggests that HFT might have a negative influence on the market quality. Menkveld (2016) states that HFTs are good for the market in the role of better informed agents but bad in the role of faster acting agents. When informational and speed properties are combined, theoretical studies predict mixed results. The assessment of the final influence is non-trivial and depends on the initial market state. Model calibrations and event studies find a moderately positive effect of HFT entering the market; it is a popular view to treat the HFT activity as making the financial system more fragile (e.g., Gerig (2012)).

1.2 Analytical models and empirical research vs ABM as research instruments

In the recent research literature on financial markets modeling, two separate and barely connected approaches can be found: one line of research relies on analytical models[3] (e.g. Foucault (1999)), while the proponents of the other approach construct artificial stock markets through agent-based models (ABMs) and use extensive simulation techniques (e.g. Tóth and Scalas (2008)). Since these methods are advocated by different researchers, assumptions and set-ups often vary a lot, so that a direct comparison of the two types of models is not possible. The analytical models have the advantage of ending up with closed-form solutions and exact values of market metrics. However, for tractability reasons, these models should be quite simple and based on restrictive assumptions. The simulated markets are more powerful in this respect, since assumptions might be less strict, and a better representation of real systems might be achieved. ABMs end up with possible distributions of market metrics; exact (scalar) solutions are not possible. ABMs can be represented as a complex system of procedures and algorithms. Usually, these procedures are blackboxes for users, which makes it difficult to inspect and adjust them, if necessary. Moreover, ABMs are also based on a high number of assumptions (e.g., agents are assumed to act in a certain way, follow a certain algorithm, use certain information, etc.).

Law (2015) summarizes some concepts and definitions of agent-based modeling, even though there is a lot of disagreement among researchers on these grounds. He defines an agent as "an autonomous "entity" that can sense its environment, including other agents, and use this information in making decisions"[4]. ABM can be considered as bottom-up modeling: it emphasizes agents' individual behavior and their interactions. Agents have some attributes and a set of rules which determine their decision-making process and their behavior; they could also have learning or adaptive mechanisms determining their behavior. However, agents are the main

[3] The analytical models are usually dynamic limit order book models and may be based on the game theory. In these models, equilibrium can be reached.

[4] See Law (2015), p.694.

but not the only element of an ABM. In ABMs of financial markets, a marketplace should also be modeled. Some ABMs can be analytically tractable, therefore they can be classified as analytical models as well. The majority of ABMs are based on extensive heterogeneity, which does not allow for analytical tractability. Some researchers classify ABM as a special case of discrete-event simulation[5].

Simulations help to check how the system functions under various initial parameters or alternative system configurations. As compared to the real-world or laboratory experiments, simulation studies offer a higher degree of control.

Nevertheless, ABMs are not the mainstream of the current financial markets research, which is partially explained by their disadvantages. First, many simulation runs are required to receive a reliable estimator; this causes additional computational time and memory requirements. Second, computational ABMs are usually criticized for their complexity, amount of parameters, and difficulty of interpretation and generalization.

However, computational ABMs serve as a very important complement to the analytical models, as research based on the representative investor concept is overly simplified. ABMs add a certain complexity that allows to see how real-life heterogeneity affects markets. ABMs bring analytical research one step closer to the real systems, whereas the complexity of these models can be controlled. With the help of ABMs, the effects of HFT entry and HFT activities can be evaluated from different viewpoints, under various scenarios and initial market states. Interactions between trader groups can be analyzed in detail. Moreover, an ABM provides an opportunity to calculate multiple market quality metrics and assess various policy instruments.

The empirical research of HFT, which is the current mainstream, is prone to a number of difficulties. The multidimensional nature of HFT leads to dissimilar classifications used by various financial databases. Usually, researchers do not have an opportunity to inspect database groupings because all entries are anonymous. Some researchers try to come up with their own classification of traders or activities based on statistical characteristics, but it does not guarantee the correctness of the classification. For example, Hendershott and Riordan (2013) utilize the exchange classification to distinguish algotraiding from human trading; Brogaard, Hendershott, and Riordan (2013) use data flags from NASDAQ data showing whether a trading activity involves HFT; while Baron, Brogaard, and Kirilenko (2012) (self-)classify traders into different categories, including passive and aggressive HFT.

Given these difficulties in the empirical research of HFT, agent-based models could serve as an additional efficient tool for the investigation of a possible HFT influence on the market quality. Compared to the empirical methods for HFT research, a simulation of ABMs offers the following advantages:

(1) A researcher explicitly empowers some agents with certain HFT characteristics, so that the problem of a potential incorrect classification disappears.

[5]These simulation models develop a system at some discrete (countable) points in time, see Law (2015), p.6.

(2) It is possible to construct a benchmark market that is not affected by HFT: such benchmarks are usually unavailable in the real markets.

(3) A researcher has more control over the system: it is possible to include only one HFT characteristic into the market, but it can also be made sophisticated by several HFT properties.

(4) It is possible to run the market game several times with various random input parameters (e.g., fundamental value) and base sound conclusions on the average measures. Empirical research is limited by one historical scenario of the real market development; the main driving force behind the final result cannot be identified clearly: determining whether a change in the final results is brought about by an interchange of random parameters or an entry of HFT becomes challenging.

(5) Considering point (4), simulations might be run under various market configurations. The agent-based models serve as a testbed for possible regulations, policies, or other changes of the trading system.

1.3 Related agent-based and analytical models

In this section, we outline the agent-based and analytical models that describe the HFT influence on the market.

Leal, Napoletano, Roventini, and Fagiolo (2016) developed an ABM with low- and high-frequency traders. Low-frequency traders can switch between fundamentalist and chartist strategies, while an HFT uses directional strategies. The trading rules of a low-frequency trader are based on chronological time, whereas an HFT has event-driven rules; therefore price fluctuations may cause an HFT to enter the trading game. In the next version of the model, Leal and Napoletano (2017) examined HFT targeted regulatory policies such as minimum resting times, circuit breakers, cancellation fees, and transactions taxes. The authors' finding was a trade-off between market stability and resilience: policies that control market volatility slow down the market recovery likewise. The reason for this is a twofold role of HFTs: on the one hand, they can cause flash crashes, on the other hand, they play a key role in the post-crash recovery.

Arifovic, Chiarella, He, and Wei (2016) proposed an ABM with learning and high- and low-frequency traders with asymmetric information. They investigated the main sources of the HFT profit as well as its influence on market liquidity and efficiency.

The ABM by Yim, Oh, and Kim (2015) shows that the HFT influence on the market depends on whether an agent can update her type or not. Izumi, Toriumi, and Matsui (2009) analyze whether algorithmic trading influences financial stability by using the volatility and average trading volume measures.

Bernales (2014) as well as Rojcek and Ziegler (2016) rely on the dynamic limit order book model which cannot be solved analytically. Rather, the authors achieve an equilibrium numerically using simulations. Rojcek and Ziegler (2016) find that HFT improves the market quality under asymmetric information but only if competition among HFTs is strong enough. Bernales (2014) shows that market performance can be improved only under specific conditions.

Biais, Foucault, and Moinas (2011) construct a model with fast traders having some private information and a higher probability to trade. The authors focus on the Pareto-dominant equilibrium. Overall, algorithmic trading leads to negative externalities, as it creates an adverse selection problem for slow traders. The equilibrium level of investment in algorithmic trading is analyzed. Informational asymmetries are endogenous in the market with HFTs, since large investments in fixed costs related to HFT can be made only by large institutions, while small slow traders bear higher trading costs.

Jovanovic and Menkveld (2016) model an HFT as a middleman between early coming and late coming agents. In this model, HFT entry leads to higher liquidity and the improvement of the adverse selection problem. However, welfare effects are very modest. The authors find that changing the trading mechanism to a continuous double auction is better than HFT entry.

Foucault, Hombert, and Rosu (2016) conceptualize a fast trader as an informed speculator and compare this agent with a slow speculator. They conclude that speed matters, since HFT is responsible for a higher share of the trading volume. However, a fast trader receives a higher share of her trading profits from price changes in the long-run, which suggests that fundamental information should be interesting at least to some HFTs.

Boco, Germain, and Rousseau (2017) come up with a model where agents have a different speed of trading. They find an equilibrium in the model where an HFT competes with a slow trader and then generalize it to the case when several HFTs compete against each other and against several slow traders. The researchers end up with an optimal level of speed for the HFT activity.

The model by Hoffmann (2014) is the baseline analytical model in this dissertation: it presupposes a dynamic limit order market with fast traders (FTs) and slow traders (STs). A FT has an informational advantage over a ST due to FT's speed advantage. This model results in equilibrium order-setting rules dependent on the current market state. The model leads to important conclusions about the HFTs' influence on the market and its dynamics and offers a testbed for the policies' analysis.

1.4 Literature review: HFT influence on the market

The bulk of research related to HFT is empirical research. As summarized by Jones (2013), the main conclusion in all the empirical papers is that HFT and algorithmic trading in general improve the market quality. Some researchers concentrate on certain events or changes on different markets. For example, Hendershott, Jones, and Menkveld (2011) studied the automation of quote dissemination on NYSE, the factor that lead to an increase in automated trading, they also investigated whether algorithmic trading improves market liquidity. They largely admit that the results are sensitive to the stock size, but in general liquidity and informational efficiency improve. Hendershott and Moulton (2011) studied the introduction of the Hybrid market on NYSE in 2006 that increased automation and reduced the execution time for market orders from 10 seconds to less than one second. Menkveld (2013) analyzed the HFT market-maker entry to the Dutch stock market. Due to the trade reporting requirements, the trades of this new market-maker are observable and distinguishable, therefore this paper presents important insights behind the HFT market-making activities. The market is compared to the Belgian stocks, which were restricted from trading on Chi-X, and are therefore considered to be unaffected by the HFT market-maker entry. Riordan and Storkenmaier (2012) examined an important upgrade of the trading system on the Deutsche Börse: an introduction of a new Xetra release in 2007, which decreased latency from 50 ms to 10 ms.

Another group of researchers tried to distinguish the HFT activities from the low-latency (human) trading. Bellia, Pelizzon, Subrahmanyam, Uno, and Yuferova (2017) found no speed advantage of HFTs compared to the other traders. HFTs help the market to improve its quality without any privileges, since the speed advantage turns out to be non-crucial. According to Brogaard (2010) and Chaboud, Chiquoine, Hjalmarsson, and Vega (2014), most of the time HFTs follow the price reversal strategy driven by order imbalances, and their trading strategies are more correlated with each other than with the strategies of non-HFTs. It means that HFTs rely on less diversified trading strategies and can exaggerate market movements. Kirilenko, Kyle, Samadi, and Tuzun (2017) found that HFTs mainly transact with other HFTs. On the opposite, Groth (2011) claims that algorithmic traders follow strategies that are as diverse as human strategies. Huh (2014) studied how liquidity-taking HFTs affect liquidity-providing HFTs.

Table 1 sums up the conclusions from the relevant literature about the negative and positive effects of HFT. Here different types of research are distinguished (analytical, empirical, or ABM), and various market metrics are described. Table 1 concentrates on the main research conclusions only.

1.5 Research goals and organization of the dissertation

The overall goal of this dissertation is to identify and analyze specific channels by which HFT influences market quality, its efficiency, as well as the market participants' behavior and welfare[6]. In conducting this research, we intend to answer the following questions:

- What makes HFT a profitable activity?
- Does HFTs' entry improve the other market participants' welfare?
- Does HFT improve price efficiency, liquidity, or some other market quality metrics?

[6] In our previous research, see Kalimullina and Schöbel (2015), with the help of the agent-based model we investigated how transaction taxes influence the results of the market game among heterogeneously informed agents.

Table 1: HFT influence on market quality: literature review

Analytical models	Empirical research	Agent-based models
	Liquidity	
Boco, Germain, and Rousseau (2017): HFT is beneficial to market liquidity Jovanovic and Menkveld (2016): HFT entry coincides with 17% increase in trade frequency Foucault, Hombert, and Rosu (2016): FT is responsible for a higher share of the trading volume Biais, Foucault, and Moinas (2011): HFTs increase the trading volume due to their construction but an adverse selection reduces it. A large proportion of HFTs can decrease the trading volume compared to the market with STs only Hoffmann (2014): HFT increases the trading volume except for the combination of a low volatility and low share of FTs	Hendershott, Jones, and Menkveld (2011): algorithmic trading improves liquidity Biais, Declerck, and Moinas (2016): FTs provide liquidity by leaving limit orders in a limit order book Hasbrouck and Saar (2013): HFT leads to improved liquidity Subrahmanyam and Zheng (2016): HFTs increase liquidity provision when the market is volatile Chaboud, Chiquoine, Hjalmarsson, and Vega (2014): during the periods of market stress, algorithmic trading really provides liquidity Groth (2011): during periods of high volatility, algorithmic traders do not withdraw liquidity from the market Carrion (2013): HFTs provide liquidity when it is deficient (when spreads are wide) and take liquidity when it is abundant (when spreads are tight) Hendershott and Riordan (2009): liquidity is consumed by HFTs when it is cheap and is provided when it is expensive Hendershott and Riordan (2013): algorithmic trading consumes liquidity when bid-ask spreads are narrow, while supplying liquidity when spreads are broad Huh (2014): HFTs submit market orders when they have some information, i.e. they take liquidity; market-making HFTs increase market liquidity, but they provide less liquidity replenishment when markets are volatile Brogaard (2010): HFTs' trading level changes only moderately as volatility increases Clapham, Haferkorn, and Zimmermann (2017): recovery of a limit order book in terms of depth happens solely through the activities of human traders and takes significantly longer time than the reduction of spreads Bellia, Pelizzon, Subrahmanyam, Uno, and Yuferova (2017): HFTs do not affect liquidity	Arifovic, Chiarella, He, and Wei (2016): HFT increases both supply and demand of liquidity Yim, Oh, and Kim (2015): if the agent type is non-updating, FTs increase liquidity Bernales (2014): when the majority are less skilled traders, algorithmic traders act as liquidity takers. Algorithmic traders improve liquidity when the participation of traditional traders is smaller than that of FTs
	Cancellation of orders	
	Huh (2014): HFTs overflow the system by posting and canceling orders that do not benefit anyone Subrahmanyam and Zheng (2016): HFTs' cancellation frequency is not higher than that of non-HFTs. The cancellation option is used strategically better by HFTs	Arifovic, Chiarella, He, and Wei (2016): HFT increases order cancellations Leal, Napoletano, Roventini, and Fagiolo (2016): effects of order cancellation are not only negative. Higher cancellation rates increase the probability of flash crashes but decrease their duration and accelerate price recovery

Spreads		
	Huh (2014), Menkveld (2013): market-making HFTs reduce spreads Hendershott, Jones, and Menkveld (2011): algorithmic trading reduces spreads for large stocks Riordan and Storkenmaier (2012): latency decrease reduces spreads for small to medium stocks Clapham, Haferkorn, and Zimmermann (2017): HFTs are solely responsible for spread reduction within the first seconds after liquidity shocks Hendershott and Moulton (2011): HFT increases spreads	Leal, Napoletano, Roventini, and Fagiolo (2016), Arifovic, Chiarella, He, and Wei (2016): HFT increases spreads Bernales (2014): when the share of FTs is predominant, spreads are reduced
Price efficiency		
Boco, Germain, and Rousseau (2017): HFT benefits price efficiency Hoffmann (2014): HFT brings quotes closer to fundamental values if volatility is high	Bellia, Pelizzon, Subrahmanyam, Uno, and Yuferova (2017): HFTs lead price discovery Brogaard (2010): HFTs add a lot to the price discovery process Brogaard, Hendershott, and Riordan (2013): HFT positively influences price efficiency Riordan and Storkenmaier (2012): with latency decrease prices become more efficient Hendershott and Moulton (2011): HFT reduces noise in prices, prices become more efficient Hendershott and Riordan (2009): algorithmic trading contributes to the price discovery process more than human trading Carrion (2013): prices are more efficient on the days with a high HFTs' participation Hendershott, Jones, and Menkveld (2011): algorithmic trading improves information content in quotes. For large stocks, it improves trade-related price discovery Brogaard, Hendershott, and Riordan (2013): HFTs' liquidity demanding orders increase price efficiency by trading in the direction of permanent price changes and in the opposite direction of transitory price errors Gerig (2012): if safeguards are not implemented, during the times of stress, price errors spread fast through the financial system	Arifovic, Chiarella, He, and Wei (2016): HFT increases efficiency of informational dissemination Bernales (2014): FTs' entry improves price efficiency (reduces microstructure noise) Yim, Oh, and Kim (2015): if the agent type is non-updating, FTs increase market efficiency
Adverse selection		
Hoffmann (2014): FTs impose the adverse selection risk (risk of being picked-off) on STs Biais, Foucault, and Moinas (2011): HFTs' ability to process information before STs creates adverse selection Jovanovic and Menkveld (2016): HFT entry coincides with 23% drop in adverse selection costs for price quotes	Hendershott and Moulton (2011): HFT increases adverse selection Huh (2014): liquidity-taking HFTs induce information asymmetry Menkveld (2013): market-making HFT reduces adverse selection Riordan and Storkenmaier (2012): with latency decrease, adverse selection costs drop dramatically Gerig (2012): HFT activity reduces transaction costs (adverse selection) Riordan and Storkenmaier (2012), Hendershott and Moulton (2011): algorithmic trading leads to a decrease in adverse selection costs Hendershott, Jones, and Menkveld (2011): algorithmic trading improves the adverse selection problem for large stocks Brogaard, Hendershott, and Riordan (2013): HFTs' liquidity supplying orders are adversely selected	Bernales (2014): adverse selection is amplified with a higher share of FTs

Welfare, profits or benefits of agents		
Biais, Foucault, and Moinas (2011): HFT helps to raise gains from trade Jovanovic and Menkveld (2016): HFT entry coincides with a modest (one percentage point) welfare increase. If HFTs are the only agents equipped with hard information, they can reduce welfare. If late coming agents also see hard information with some probability, HFTs can increase welfare Boco, Germain, and Rousseau (2017): STs are worse-off on the market with HFTs. The speed differential brings profit to HFTs and benefits a liquidity trader Hoffmann (2014): efficiency increase due to the HFT activity does not benefit low-frequency traders Foucault, Hombert, and Rosu (2016): FTs' trades are more correlated with price changes in the short run, but they receive a higher share of the trading profit from price changes in the long run	Gerig (2012): Informed traders make less profit when there is an HFT Brogaard (2010): HFT activity does not damage non-HFTs Menkveld (2013): HFTs are predominantly market-makers; on average, an HFT experiences losses on her net position, and since her orders are mostly passive, she earns spreads Carrion (2013): HFTs' market-making activity is profitable even without liquidity rebates, while aggressive trading is unprofitable even before accounting for taker-fees occurs Baron, Brogaard, and Kirilenko (2012): HFTs as intermediaries are highly profitable and generate unusually high average Sharpe ratio. Aggressive HFTs are much more profitable than passive or mixed HFTs. Profits are mainly derived from STs who can be opportunistic traders, fundamental (institutional) traders, or small (retail) traders Moosa and Ramiah (2015): when defining HFTs in terms of their holding period and frequency of trading only, it turns to be not as profitable as it is believed to be: holding an asset for a longer period and less frequent trading may be more profitable Bellia, Pelizzon, Subrahmanyam, Uno, and Yuferova (2017): HFTs make profits by coming to the market earlier, but the speed advantage turns out to be non-crucial Biais, Declerck, and Moinas (2016): only proprietary FTs are able to provide liquidity to the market without experiencing losses	Rojcek and Ziegler (2016): HFT activities do not influence investors' welfare, whereas slow speculators' welfare is negatively affected Arifovic, Chiarella, He, and Wei (2016): HFTs make significant profits, the main sources of which are the informational advantage and learning rather than the speed advantage Bernales (2014): algorithmic traders with the informational advantage only increase global welfare, while algorithmic traders with the speed advantage only reduce it. When both informational and speed advantage are experienced by algorithmic traders, this results in a positive synergy, which increases economy welfare by more than under the condition of the single informational advantage. Algorithmic trading damages traditional traders' (STs') profits
Volatility		
Aldridge (2016): if HFT is restricted, the downward movement of prices is much more extreme than with HFTs	Kirilenko, Kyle, Samadi, and Tuzun (2017): HFT activities contribute to increased volatility but do not cause flash crashes Groth (2011): algorithmic traders' activities lead to as high an increase in volatility as that caused by human traders, and this increase is insignificant Hasbrouck and Saar (2013): HFT leads to decreased short-term volatility Menkveld (2013): market-making HFTs do not affect market volatility Chaboud, Chiquoine, Hjalmarsson, and Vega (2014): causality between algorithmic trading and volatility increase is not found	Leal, Napoletano, Roventini, and Fagiolo (2016), Arifovic, Chiarella, He, and Wei (2016): HFT increases volatility
Order imbalances		
	Subrahmanyam and Zheng (2016): liquidity provision from the HFT side is less affected by order imbalance shocks, HFTs stabilize the market	Leal, Napoletano, Roventini, and Fagiolo (2016): HFTs synchronize on the selling side of the market generating flash crashes

- Does HFT have a more crucial impact on the market compared to a simple informed trading or random trading?

In answering these questions, we utilize analytical models as well as ABMs. We try to bridge the gap between analytical models and ABMs for financial markets, since they are presently too divergent.
We set up a simplified analytical dynamic limit order book model and the agent-based version of it: the two models have to converge if assumptions are harmonized. The interim research results discussing the matter in question were summarized in the working paper "From Analytical Markets to Artificial Stock Markets: A New Type of Agent-Based Models", see Kalimullina (2017).
Furthermore, using the simplified analytical model, we want to study the symmetry issue between the two market sides (buy side and sell side). What does the symmetry mean? Are asymmetric equilibria possible for the equal shares of the market sides? How does the market sentiment influence equilibrium order-setting rules and market results?
As the next step, we revisit the model by Hoffmann (2014) to check whether equilibria may be impacted, if both market sides are considered. We recalculate the market metrics from the original paper and represent them graphically to provide better insights into the model results. Moreover, we disentangle fast traders' informational advantage from their speed advantage, the latter additionally including the possibility to revise quotes before the next slow trader comes. To do so, we build the modified analytical model with informed slow traders (ISTs), who have complete informational advantage but no speed advantage. We compare the fast traders' and informed slow traders' influence on the market and their trading partners and draw important conclusions.
All the analytical models and ABMs built in this research are provided in Table 2.
The dissertation is organized as follows:
In Chapter 2, a simplified equilibrium dynamic model with only slow traders, who can take either the buying or selling market side, is constructed. In Chapter 3, this model is extended by adding fast traders: by doing so, we revise the model by Hoffmann (2014). In Chapter 4, this model is modified by introducing informed slow traders instead of fast traders to investigate how the market game changes, if the speed advantage is substituted by the unconditional informational advantage. Chapter 5 compares the analytical results of the two previous models: the question of whether the market is better-off with fast traders or with informed slow traders is addressed here. Chapter 6 is devoted to the simple ABM of a stock market. As the first step, we show that the ABM results converge to the analytical model results, if the ABM is based on the analytical model specifications from Chapter 2. As the second step, we extend the analytically-based ABM by increasing the heterogeneity of agents: the ABM results with HFTs and with informed slow traders participating in trade are investigated. Chapter 7 extends the simple ABM of Chapter 6 by allowing for more realistic assumptions. A more realistic ABM makes it possible to investigate the HFTs' influence on the market quality, its efficiency, as

well as on the other market participants' behavior and final results. Chapter 8 concludes by summarizing the main results and outcomes of this dissertation.

Table 2: Models constructed in this research

Analytical Models		
1. Simplified model	2. Hoffmann (2014) model with FTs	3. Modified model with ISTs
Only STs, shares of buyers and sellers may be non-equal	STs and FTs: • FTs have the speed advantage (which leads to conditional informational advantage, i.e. with respect to other STs only) • FTs can adjust their orders to STs' level correctly, but they still run the risk of being picked-off by next FTs	STs and ISTs • ISTs cannot revise or adjust orders, but also do not have the risk of being picked-off
Simple ABMs		
Two set-ups: (i) two profit maximizers, convergence to the analytical model is proved, (ii) one profit maximizer and one random trader	One profit maximizer and one FT	One profit maximizer and one IST
A more realistic computational ABM		
Random traders, fundamentalists, and HFTs; various configurations		

Note: FT stays for a fast trader, ST is a slow trader, and IST is an informed slow trader

2 Simplified Analytical Model with Slow Traders

Foucault (1999) developed a simple analytical model which allowed for a mix of limit orders and market orders in an equilibrium framework. He found that volatility plays a major role in determining the composition of orders. Volatility increases the risk of being picked-off, therefore limit orders providers ask for a higher compensation, which makes trading more costly, and more agents decide to execute their trade through limit orders. However, the fill-rate decreases, and spreads are damaged. The average trading costs for buy (sell) market orders rise (fall) following the ratio of buy-to-sell orders. In this model, in each time step there is a risk of market termination.

The model proposed by Hoffmann (2014) is based on Foucault (1999), but it assumes an infinite market session and enriches the original model by adding an HFT. In this chapter, the model by Hoffmann (2014) will be simplified by populating the market with slow traders (STs) only; such a revised model would serve as a benchmark for the concept developed by Hoffmann (2014), which is necessary to assess the HFT influence.

In Hoffmann (2014), traders participate equally likely on both sides of the market: there are comparable chances for an agent to be a buyer or a seller. This assumption simplifies the analysis considerably, as both sides of the market are treated as symmetric. However, the buyers' and sellers' market shares are not generally equal. A market sentiment could be bearish (more sellers, which results in lower prices) as well as bullish (more buyers and consequently higher prices). A sentiment is not always caused by a change in the fundamentals, rather it is a result of behavioral factors such as the herding effect.

The limit order market is presented as a sequential bargaining game where agents can either accept an existing limit order and make a transaction (post a market order) or make an offer to the next coming agent by posting a new limit order. An agent's bargaining power is determined by her outside option, which is endogenous and equals the expected profit from posting a limit order. Depending on the share of buyers and sellers, these groups can experience a higher or lower market power.

There is a number of limitations of this model stemming from its simplicity: (i) since orders allow to trade one unit of asset only, implications about the limit order book depth are impossible to make, (ii) since there might be only one limit order in a limit order book per one time step, comprehensive conclusions about the order flow and spreads are unachievable.

The goal of this chapter is to investigate how the model equilibrium and market features are affected by the shares of the market sides. Do the equal shares of the market sides always lead to a symmetry in the order-setting process: can non-symmetric order-setting rules potentially lead to an equilibrium? What does "symmetry" exactly mean: does symmetry in the shares of the market sides also lead to a symmetry in order values only or all the other market measures likewise? Are there any equilibrium types that are preferred to others?

For the purpose of analytical and agent-based models synchronization, we adjust the assumptions of the analytical model. Moreover, we provide market measures for both the modified and original research set-ups (under the assumptions presented in the literature). The key results discussed in this chapter were presented in The Fifth German Network on Economic Dynamics (GENED) Meeting (Karlsruhe, Germany) in October 2017, see Kalimullina (2017).

2.1 Model set-up

There is a single risky asset traded in the market; its fundamental value follows a random walk:

$$v_t = v_{t-1} + \varepsilon_t,$$

where ε_t is i.i.d. (independent and identically distributed) discrete innovation which is equally likely to be positive or negative: $|\varepsilon_t| = \sigma$.

The traders arrive at the market sequentially, i.e. only one agent acts in the market per time period. The traders are risk-neutral, and γ is the probability that an agent is a buyer[7,8].

In spite of the fact that the traders are of the same type, i.e. STs, the buyers and sellers have different private reservation values:

$$R_{i,t} = v_t + y_i,$$

where $y_i = \pm L$, t refers to the time subscript, and i differentiates a buyer from a seller: $i = b$ for a buyer and $i = s$ for a seller. The buyer's (seller's) private value of an asset is higher (lower) than its current fundamental value by $|y_i| = L$. Therefore, she is willing to invest in an asset (strives to sell an asset). A difference in private values is essential for trade existence; the market would be idle without this contrast. Apart from this symmetric difference in private values and possible different shares, the buy side is assumed to be identical to the sell side. In this simplified model, no agents possess superior information.

All the traders potentially face a risk of being picked-off once a limit order is sent, since the STs have no opportunity to cancel or change a limit order. All the traders are assumed to be homogeneous in terms of information they receive about the next realization of the fundamental value and about the market in general.

At the very first step of the market game, an acting agent can only fill a limit order book with a limit order, while each next step an agent can either accept the existing order and make a transaction, or she can post a new limit order (if a limit order book is empty, if the existing order is on the wrong side, or if the price seems to be unreasonable for an acting agent). If a limit order from the previous step is accepted, its value serves as the current market order value and as the transaction price: $O_t = O^m_{t+1}$. Both limit orders and market orders are posted for one unit of a risky asset as maximum. Limit orders stay in a limit order book for one period only and then automatically expire[9] for tractability reasons. Therefore, at each time point t, a

[7]The probability that an agent is a seller is $1 - \gamma$.

[8]Hoffmann (2014) assumption of $\gamma = \frac{1}{2}$ is modified in our analysis: $\gamma \in (0, 1)$.

[9]The agent acting at $t + 1$ exclusively has a chance to transact on O_t.

limit order book is either empty or includes a single limit order (either bid or ask) for one unit of a risky asset.

The temporal flow of the market game in Figure 1 shows the agents' possible actions at the very first step $t = 0$[10], at $t + 1$, and at $t + 2$. Further actions follow the same scheme as at $t + 2$. The exogenous fundamental value of a risky asset becomes known to an acting agent once she enters the market. Based on this new knowledge, an agent decides whether to satisfy the existing order or to post her own limit order. However, the exact fundamental value for the next period is not known. The type of the agent coming next is not predetermined either. A new limit order is based therefore on the expected fundamental value; the fundamental value is changed first when the next agent enters the market. The current market state is described by the share of buyers γ, by the volatility of the fundamental value σ, by the value of the individual reservation value component L, as well as by the current order on the market, if it exists, O_t: (γ, σ, L) and O_t.

Definition 2.1. An agent has two possibilities in the market: either to satisfy an existing limit order by making a transaction, or to post a new limit order. Each of these two options has its own value for an agent, and they serve as outside options to each other. $V_{i,t}^{MO}$ and $V_{i,t}^{LO}$ denote the outside option values (or expected profits) for an agent i at any point of time t for (from) a limit order or a market order, respectively.

Proposition 2.1. *In equilibrium,* $0 \leqslant V_{i,t}^{LO} < 2L$.

Proof. In the equilibrium situation, the expected profit from posting a limit order cannot be negative, otherwise the other outside option has to be used[11]. If two agents from the opposite market sides trade, they share the surplus of $2L$. However, given that there is a limit order in a limit order book, a transaction happens with the probability of $\max(\gamma, 1-\gamma)$. Therefore, the maximum expected gain from trade per period is $\max(\gamma, 1-\gamma) \cdot 2L < 2L$, given that $0 < \gamma, 1-\gamma < 1$. □

A buyer (seller) posting a bid (ask) faces a trade-off between its aggressiveness, which increases its probability to be executed in the next period and its expected profitability conditional on execution. The optimality condition forces the agents to choose the lowest (highest) value at which the buy (sell) transaction is possible given that the probability of execution is unchanged. This value corresponds to the sell (buy) cutoff price[12]. Such a choice maximizes the potential profit from a transaction for the posting agent, while it has arbitrarily the same probability of execution as bids (asks) on a slightly higher (lower) level.

Definition 2.2. The cutoff price is the limit order value at which an agent is irrelevant between executing this order and posting a new limit order. When posting a limit order at t, an agent sets its value to the possible cutoff price of the next coming agent at $t + 1$.

[10] A limit order book is empty at the starting point of the game.
[11] If an outside option is not available, an agent has to stay idle in this step.
[12] As in Foucault (1999), cutoff prices are similar to reservation values, but they are endogenous and depend on the expected gain from limit orders.

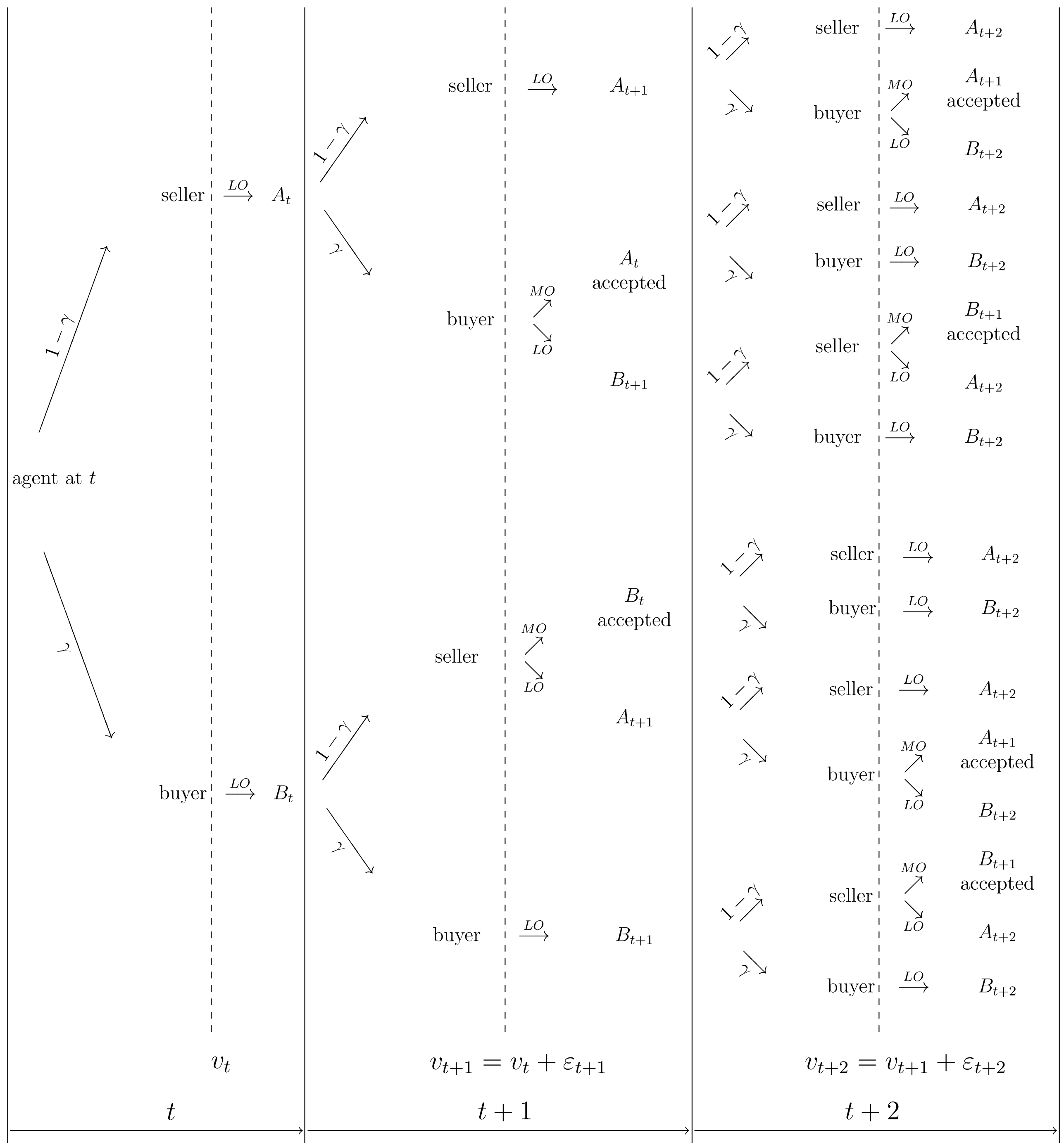


Figure 1: Temporal flow for the simplified analytical model

Solid vertical lines and horizontal arrows indicate the borders of a single period. Dashed vertical lines mark time points when new fundamental information becomes available (in the later models, some agents may have access to information earlier); the realized fundamental value is displayed at the bottom of the line.

The cutoff price is determined by comparing the payoff from a possible transaction with its outside option value. An acting agent at $t+1$ accepts $O^m_{t+1} = O_t$ only if the profit from a transaction exceeds her outside option value; otherwise this agent is better-off by choosing the outside option (by posting her own limit order).

An agent has to choose from a set of several possible cutoff prices: an order can be based either on the positive or negative innovation of the fundamental value and can correspond to different agent types. Due to randomness and incomplete information, this choice is risky but can be optimized. In the following, four types of cutoff prices will be defined: even if an agent is a buyer, at a certain price she can decide to turn her side and to sell; similar is true for a seller. To find the seller's sell $\hat{B}_t^{v+\varepsilon}$ (buyer's buy $\hat{A}_t^{v+\varepsilon}$) cutoff price, the profit from a selling (buying) transaction on the left-hand side is compared with its outside option on the right-hand side: $B_{t+1}^m - R_{s,t+1} \geq V_{s,t+1}^{LO}$ ($R_{b,t+1} - A_{t+1}^m \geq V_{b,t+1}^{LO}$), the resulting optimal order values at t are:

$$\begin{aligned} B_{t+1}^m &= B_t = \hat{B}_t^{v+\varepsilon} = (v_t + \varepsilon_t - L) + V_{s,t+1}^{LO}, \\ A_{t+1}^m &= A_t = \hat{A}_t^{v+\varepsilon} = (v_t + \varepsilon_t + L) - V_{b,t+1}^{LO}. \end{aligned}$$

One can make the same analysis for the buyer's sell $\check{B}_t^{v+\varepsilon}$ (seller's buy $\check{A}_t^{v+\varepsilon}$) cutoff price: $B_{t+1}^m - R_{b,t+1} \geq V_{b,t+1}^{LO}$ ($R_{s,t+1} - A_{t+1}^m \geq V_{s,t+1}^{LO}$), which results in:

$$\begin{aligned} B_{t+1}^m &= B_t = \check{B}_t^{v+\varepsilon} = (v_t + \varepsilon_t + L) + V_{b,t+1}^{LO}, \\ A_{t+1}^m &= A_t = \check{A}_t^{v+\varepsilon} = (v_t + \varepsilon_t - L) - V_{s,t+1}^{LO}. \end{aligned}$$

The potential values for a bid order are: $\hat{B}_t^{v-\sigma}, \hat{B}_t^{v+\sigma}, \check{B}_t^{v-\sigma}, \check{B}_t^{v+\sigma}$, while for an ask order they are: $\hat{A}_t^{v-\sigma}, \hat{A}_t^{v+\sigma}, \check{A}_t^{v-\sigma}, \check{A}_t^{v+\sigma}$.

Following Hoffmann (2014) and Foucault (1999), our study incorporates stationary solutions which result from the Markov-perfect equilibrium. In this type of equilibria, agents act based on some current set of market variables and do not look at the (entire) history of the game[13]. "Markov" refers to the fact that an equilibrium depends on the current and not on the past state of nature. "Perfect" indicates that the equilibrium conditions are correct for any time step during the game. In the modeled market set-up, the agents are memoryless and post orders based on the current market conditions. Regardless of the arrival date, the order placement strategy depends on that of the next coming agent. Since this order placement is endogenous, and given that the game is infinite, there is no point of time at which one can start solving for equilibrium recursively. Instead of this, stationary solutions which do not depend on time apply[14]. The value of the outside option is assumed to be unchanged during the game. The model leads to general order-setting rules rather than to individual solutions for each time step. The time subscripts can therefore be omitted.

Lemma 2.1. *In equilibrium, a buyer never sells, while a seller never buys.*

Proof. Equilibrium is defined as a set of the most profitable order-sending strategies for both sides of the market, so that no incentive to deviate exists. Therefore, it is necessary to check whether buying and selling transactions are profitable for a buyer and a seller, given all potential order values.

[13] See Maskin and Tirole (2001).
[14] See Foucault (1999), p.107.

A buyer (seller) chooses a bid (ask) value from the available sell (buy) cutoff prices.
First, it is verified whether a posting buyer gets a profit from using the possible sell cutoff prices if an order is satisfied during the next period; for this purpose the difference $R_{b,t+1} - B^m_{t+1}$ is calculated:

- If a buyer uses the seller's sell cutoff price $\hat{B}^{v+\varepsilon}_t$:
  - based on the positive fundamental value change[15] $\hat{B}^{v+\sigma}_t = (v_t + \sigma - L) + V^{LO}_s$:
    - If the real $\varepsilon_t = \sigma$, the profit from buying is $v_t + \sigma + L - \hat{B}^{v+\sigma}_t = 2L - V^{LO}_s > 0$,
    - If the real $\varepsilon_t = -\sigma$, the profit from buying is $v_t - \sigma + L - \hat{B}^{v+\sigma}_t = 2L - 2\sigma - V^{LO}_s$, which can be positive for certain parameters.
  - based on the negative fundamental value change $\hat{B}^{v-\sigma}_t = (v_t - \sigma - L) + V^{LO}_s$:
    - If the real $\varepsilon_t = \sigma$, the profit from buying is $v_t + \sigma + L - \hat{B}^{v-\sigma}_t = 2L + 2\sigma - V^{LO}_s > 0$,
    - If the real $\varepsilon_t = -\sigma$, the profit from buying is $v_t - \sigma + L - \hat{B}^{v-\sigma}_t = 2L - V^{LO}_s > 0$.
- If a buyer uses the buyer's sell cutoff price $B^m_{t+1} = \check{B}^{v+\varepsilon}_t$:
  - based on the positive fundamental value change $\check{B}^{v+\sigma}_t = (v_t + \sigma + L) + V^{LO}_b$:
    - If the real $\varepsilon_t = \sigma$, the profit from buying is $v_t + \sigma + L - \check{B}^{v+\sigma}_t = -V^{LO}_b < 0$,
    - If the real $\varepsilon_t = -\sigma$, the profit from buying is $v_t - \sigma + L - \check{B}^{v+\sigma}_t = -2\sigma - V^{LO}_b < 0$.
  - based on the negative fundamental value change $\check{B}^{v-\sigma}_t = (v_t - \sigma + L) + V^{LO}_b$:
    - If the real $\varepsilon_t = \sigma$, the profit from buying is $v_t + \sigma + L - \check{B}^{v-\sigma}_t = 2\sigma - V^{LO}_b$, which can be positive for certain parameters,
    - If the real $\varepsilon_t = -\sigma$, the profit from buying is $v_t - \sigma + L - \check{B}^{v-\sigma}_t = -V^{LO}_b < 0$.

From the previous analysis it follows that a posting buyer uses the seller's sell cutoff prices, since all such bid orders have a potential to bring a positive payoff, whereas there is only one buyer's sell cutoff price which can potentially lead to a profit. For this bid to be executed, a selling agent at $t+1$ should have an incentive to take it. Therefore, it stands to test whether a selling agent's profit at $t+1$ turns out to be greater than her outside option: only under this condition a transaction satisfies the equilibrium requirements. Only those bid values that appear to be possible from a posting agent's point of view are accounted for, the other bids are already excluded from the universe of the equilibrium orders.

- If a buyer uses the seller's sell cutoff price $B^m_{t+1} = \hat{B}^{v+\varepsilon}_t$:
  - based on the positive fundamental value change $\hat{B}^{v+\sigma}_t = (v_t + \sigma - L) + V^{LO}_s$:
    - If the real $\varepsilon_t = \sigma$, the profit from selling is $\hat{B}^{v+\sigma}_t - (v_t + \sigma - L) = V^{LO}_s$,
    - If the real $\varepsilon_t = -\sigma$, the profit from selling is $\hat{B}^{v+\sigma}_t - (v_t - \sigma - L) = 2\sigma + V^{LO}_s > V^{LO}_s$.

[15] In the following, $V^{LO}_{i,t+1}$ is referred to as V^{LO}_i.

- based on the negative fundamental value change $\hat{B}_t^{v-\sigma} = (v_t - \sigma - L) + V_s^{LO}$:
  * If the real $\varepsilon_t = \sigma$, the profit from selling is $\hat{B}_t^{v-\sigma} - (v_t + \sigma - L) = V_s^{LO} - 2\sigma < V_s^{LO}$,
  * If the real $\varepsilon_t = -\sigma$, the profit from selling is $\hat{B}_t^{v-\sigma} - (v_t - \sigma - L) = V_s^{LO}$.

- If a buyer uses the buyer's sell cutoff price, we check only the case based on the negative fundamental value change $B_{t+1}^m = \check{B}_t^{v-\sigma} = (v_t - \sigma + L) + V_b^{LO}$ and only if the real $\varepsilon_t = \sigma$:
  - profit from selling is $\check{B}_t^{v-\sigma} - (v_t + \sigma + L) = V_b^{LO} - 2\sigma < V_b^{LO}$.

In case of using the seller's sell cutoff price, a selling agent at $t+1$ has an incentive to execute the market bid almost in all cases but one. The only possible scenario of using the buyer's sell cutoff price by a posting agent turns out to be non-optimal for a selling buyer at $t+1$. Therefore, transactions at the buyer's sell cutoff prices never happen in equilibrium markets. Moreover, buyers never sell in equilibrium.

Second, it is verified whether a posting seller gets a profit from using the buy cutoff prices; for this purpose the difference $A_{t+1}^m - R_{s,t+1}$ is calculated:

- If a seller uses the buyer's buy cutoff price $A_{t+1}^m = \hat{A}_t^{v+\varepsilon}$:
  - based on the positive fundamental value change $\hat{A}_t^{v+\sigma} = (v_t + \sigma + L) - V_b^{LO}$:
    * If the real $\varepsilon_t = \sigma$, the profit from selling is $\hat{A}_t^{v+\sigma} - (v_t + \sigma - L) = 2L - V_b^{LO} > 0$,
    * If the real $\varepsilon_t = -\sigma$, the profit from selling is $\hat{A}_t^{v+\sigma} - (v_t - \sigma - L) = 2L + 2\sigma - V_b^{LO} > 0$.
  - based on the negative fundamental value change $\hat{A}_t^{v-\sigma} = (v_t - \sigma + L) - V_b^{LO}$:
    * If the real $\varepsilon_t = \sigma$, the profit from selling is $\hat{A}_t^{v-\sigma} - (v_t + \sigma - L) = 2L - 2\sigma - V_b^{LO}$, which can be positive for certain parameters,
    * If the real $\varepsilon_t = -\sigma$, the profit from selling is $\hat{A}_t^{v-\sigma} - (v_t - \sigma - L) = 2L - V_b^{LO} > 0$.
- If a seller uses the seller's buy cutoff price $A_{t+1}^m = \check{A}_t^{v+\varepsilon}$:
  - based on the positive fundamental value change $\check{A}_t^{v+\sigma} = (v_t + \sigma - L) - V_s^{LO}$:
    * If the real $\varepsilon_t = \sigma$, the profit from selling is $\check{A}_t^{v+\sigma} - (v_t + \sigma - L) = -V_s^{LO} < 0$,
    * If the real $\varepsilon_t = -\sigma$, the profit from selling is $\check{A}_t^{v+\sigma} - (v_t - \sigma - L) = 2\sigma - V_s^{LO}$, which can be positive for certain parameters.
  - based on the negative fundamental value change $\check{A}_t^{v-\sigma} = (v_t - \sigma - L) - V_s^{LO}$:
    * If the real $\varepsilon_t = \sigma$, the profit from selling is $\check{A}_t^{v-\sigma} - (v_t + \sigma - L) = -2\sigma - V_s^{LO} < 0$,
    * If the real $\varepsilon_t = -\sigma$, the profit from selling is $\check{A}_t^{v-\sigma} - (v_t - \sigma - L) = -V_s^{LO} < 0$.

A posting seller should use the buyer's buy cutoff prices, because all such ask orders have a potential to bring a positive payoff, whereas there is only one seller's buy cutoff price which might result in a profit for a posting seller. For this ask to be executed, a buying agent at $t+1$ should have an incentive to accept it. Therefore, additionally it is checked whether a buying agent's profit at $t+1$ turns out to be greater than her outside option: only under this condition a transaction satisfies the equilibrium requirements. Only those ask values that appear to be possible from a posting agent's point of view are tested; the other asks are already excluded from the universe of the equilibrium orders.

- If a seller uses the buyer's buy cutoff price $A^m_{t+1} = \hat{A}^{v+\varepsilon}_t$:
  - based on the positive fundamental value change $\hat{A}^{v+\sigma}_t = (v_t + \sigma + L) - V^{LO}_b$:
    * If the real $\varepsilon_t = \sigma$, the profit from buying is $(v_t + \sigma + L) - \hat{A}^{v+\sigma}_t = V^{LO}_b$,
    * If the real $\varepsilon_t = -\sigma$, the profit from buying is $(v_t - \sigma + L) - \hat{A}^{v+\sigma}_t = -2\sigma + V^{LO}_b < V^{LO}_b$.
  - based on the negative fundamental value change $\hat{A}^{v-\sigma}_t = (v_t - \sigma + L) - V^{LO}_b$:
    * If the real $\varepsilon_t = \sigma$, the profit from buying is $(v_t + \sigma + L) - \hat{A}^{v-\sigma}_t = V^{LO}_b + 2\sigma > V^{LO}_b$,
    * If the real $\varepsilon_t = -\sigma$, the profit from buying is $(v_t - \sigma + L) - \hat{A}^{v-\sigma}_t = V^{LO}_b$.
- If a seller uses the seller's buy cutoff price, only the case based on the positive fundamental value change $A^m_{t+1} = \check{A}^{v+\sigma}_t = (v_t + \sigma - L) - V^{LO}_s$ and only for the real $\varepsilon_t = -\sigma$ is verified:
  - the profit from buying is $(v_t - \sigma - L) - \check{A}^{v+\sigma}_t = V^{LO}_s - 2\sigma < V^{LO}_s$.

In case of using the buyer's buy cutoff price, a buying agent at $t+1$ has an incentive to satisfy this ask almost in all cases but one. The only possible scenario of using the seller's buy cutoff price by a posting agent turns out to be non-optimal for a buying seller at $t+1$. Therefore, transactions at the sellers' buy cutoff prices never happen in equilibrium markets. Moreover, sellers never buy in equilibrium. □

Corollary 2.1. *A buyer (seller) bases her limit orders on the seller's sell $\hat{B}^{v+\varepsilon}_t$ (buyer's buy $\hat{A}^{v+\varepsilon}_t$) cutoff prices only.*

Proof. Under the buyer's sell cutoff prices, this conclusion follows directly from the non-profitability of a buying transaction for a posting buyer under the three possible order-state combinations and from the non-profitability of a bid's order execution by a selling buyer at $t+1$ in the fourth possible combination. The same is true for the seller's buy cutoff prices: for three cases of order-state combinations, such an ask is not profitable for a posting seller, if a transaction happens, whereas in the last case the satisfaction of such an ask is not profitable for a buying seller coming at $t+1$. □

Corollary 2.2. *$\hat{B}^{v-\sigma}_t$ ($\hat{A}^{v+\sigma}_t$) is satisfied only under a negative (positive) fundamental value change at $t+1$, while $\hat{B}^{v+\sigma}_t$ ($\hat{A}^{v-\sigma}_t$) is satisfied in both scenarios.*

Proof. This follows directly from the (non-)profitability of a transaction from the executing agent's point of view (see the comparison of the transaction profit with the outside option of an agent acting at $t+1$). □

Corollary 2.3. *In equilibrium, $\hat{B}_t^{v-\sigma} \leqslant \hat{B}_t^{v+\sigma}$ and $\hat{A}_t^{v-\sigma} \leqslant \hat{A}_t^{v+\sigma}$.*

Proof. The two inequalities follow directly from the representation of the cutoff prices as the sum of their elements:

$$\hat{B}_t^{v-\sigma} = v - \sigma - L + V_s^{LO} \leqslant v + \sigma - L + V_s^{LO} = \hat{B}_t^{v+\sigma},$$
$$\hat{A}_t^{v-\sigma} = v - \sigma + L - V_b^{LO} \leqslant v + \sigma + L - V_b^{LO} = \hat{A}_t^{v+\sigma}.$$

□

Having outlined the general set-up of the model and the market game, it is essential to describe the equilibrium conditions, which is the main aim of the next section.

2.2 General equilibrium conditions for any shares of market sides

According to Foucault (1999), the order placement decision consists of two components: (i) the choice of the order type (a limit order or a market order), (ii) the choice of the value for a limit order (if an agent chooses to place a limit order). Corollary 2.1 accounts for possible optimal orders (in this model, O_k, k is an order from the universe of possible equilibrium orders). Each order O_k placed at t can be characterized by the probability of its execution at $t+1$, $p(O_k)$. Due to the discreteness of innovations, this probability function has an increasing step-form. The probability of execution moves to the next level (value) once an order reaches the next cutoff price. Recalling that an agent faces a trade-off between the aggressiveness of her order and its profitability given that it is executed during the next step, the order-setting problem should account for an expected profit from a limit order. Thus, the equilibrium objective function includes not just the order profit conditional on execution, but also the probability of its execution. A buyer's and seller's objective functions can be presented as follows:

$$V_b^{LO} = \max_{B_k} \left[p(B_k)\big((v+\varepsilon+L) - B_k\big) \right],$$
$$V_s^{LO} = \max_{A_k} \left[p(A_k)\big(A_k - (v+\varepsilon-L)\big) \right],$$

where ε is a possible change of the fundamental value after an order O_k is placed. An agent becomes aware of her type (buyer or seller) after she enters the market. Therefore, the equilibrium rules should account for the profitability of both market sides expressed by the following inequality conditions:

$$E(V_b^{LO}|B_k) \geq E(V_b^{LO}|B_{h\neq k}), \tag{2.1}$$
$$E(V_s^{LO}|A_k) \geq E(V_s^{LO}|A_{h\neq k}). \tag{2.2}$$

Equilibrium is determined by simultaneously solving the previous two inequalities for a certain order choice k: in this way, one identifies the region of the market state parameters where profitable deviations from the chosen combination of orders do not occur. The equilibrium rules allow an agent to end up with the highest possible expected profit given the market state (γ, σ, L).

Proposition 2.2. *For the fixed parameters (γ, σ, L), a Markov-perfect equilibrium exists in the limit order market. Table 4 shows combinations of the most profitable order setting strategies for certain market states, as well as the outside option values for each of these cases.*

Table 4: Equilibrium conditions for (γ, σ, L) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4
B_k	$\hat{B}^{v-\sigma}$	$\hat{B}^{v-\sigma}$	$\hat{B}^{v+\sigma}$	$\hat{B}^{v+\sigma}$
A_k	$\hat{A}^{v-\sigma}$	$\hat{A}^{v+\sigma}$	$\hat{A}^{v-\sigma}$	$\hat{A}^{v+\sigma}$
V_b^{LO}	$\frac{(1-\gamma)\left(2L(1-\gamma)+\gamma\sigma\right)}{2-\gamma+\gamma^2}$	$\frac{2L(2-\gamma)(1-\gamma)}{4-\gamma+\gamma^2}$	$\frac{(1-\gamma)^2(2L-\sigma)}{1-\gamma+\gamma^2}$	$\frac{2(1-\gamma)\left(L(2-\gamma)-\sigma\right)}{2-\gamma+\gamma^2}$
V_s^{LO}	$\frac{2\gamma\left(L(1+\gamma)-\sigma\right)}{2-\gamma+\gamma^2}$	$\frac{2L\gamma(1+\gamma)}{4-\gamma+\gamma^2}$	$\frac{\gamma^2(2L-\sigma)}{1-\gamma+\gamma^2}$	$\frac{\gamma\left(2L\gamma+(1-\gamma)\sigma\right)}{2-\gamma+\gamma^2}$
Condition for σ	$\frac{(1-\gamma)2L}{2-2\gamma+\gamma^2} \leq \sigma \leq \frac{(1+\gamma)2L}{4-\gamma+\gamma^2}$	$\begin{cases}\sigma \geq \frac{(1+\gamma)2L}{4-\gamma+\gamma^2} \\ \sigma \geq \frac{(2-\gamma)2L}{4-\gamma+\gamma^2}\end{cases}$	$\begin{cases}\sigma \leq \frac{(1-\gamma)2L}{2-2\gamma+\gamma^2} \\ \sigma \leq \frac{\gamma 2L}{\gamma^2+1}\end{cases}$	$\frac{\gamma 2L}{\gamma^2+1} \leq \sigma \leq \frac{(2-\gamma)2L}{4-\gamma+\gamma^2}$

Proof. Since there are only two possible orders per market side available to be placed, we need to check four combinations for the possibility of equilibrium:

- Eq.1: $B_k = \hat{B}^{v-\sigma}$, $A_k = \hat{A}^{v-\sigma}$
  Recalling Corollary 2.2, it is easy to end up with probabilities of limit orders' execution. Knowing these probabilities[16], one can calculate the values of the outside option for these orders:

$$V_b^{LO} = \frac{1-\gamma}{2}(v - \sigma + L - \hat{B}^{v-\sigma}) = \frac{1-\gamma}{2}(2L - V_s^{LO}),$$
$$V_s^{LO} = \frac{\gamma}{2}(\hat{A}^{v-\sigma} - (v - \sigma - L)) + \frac{\gamma}{2}(\hat{A}^{v-\sigma} - (v + \sigma - L)) = \frac{\gamma}{2}(2L - V_b^{LO}) + \frac{\gamma}{2}(2L - 2\sigma - V_b^{LO}).$$

  By simultaneously solving the two previous equations, the values of the outside options for this equilibrium can be determined:

$$V_b^{LO} = \frac{(1-\gamma)(2L(1-\gamma)+\gamma\sigma)}{2-\gamma+\gamma^2},$$
$$V_s^{LO} = \frac{2\gamma(L(1+\gamma)-\sigma)}{2-\gamma+\gamma^2}.$$

[16]These probabilities consist of the probability to meet an agent from the other market side during the next period and the probability of a certain value innovation. Some of the orders are satisfied only if the value changes into a certain direction.

To estimate the set of parameters under which this equilibrium potentially holds, a similar procedure to the one described in Equation 2.1 and Equation 2.2 is applied:

$$\frac{1-\gamma}{2}(v-\sigma+L-\hat{B}^{v-\sigma}) \geq \frac{1-\gamma}{2}(v-\sigma+L-\hat{B}^{v+\sigma})+\frac{1-\gamma}{2}(v+\sigma+L-\hat{B}^{v+\sigma}),$$
$$\frac{\gamma}{2}(\hat{A}^{v-\sigma}-(v-\sigma-L))+\frac{\gamma}{2}(\hat{A}^{v-\sigma}-(v+\sigma-L)) \geq \frac{\gamma}{2}(\hat{A}^{v+\sigma}-(v+\sigma-L)),$$

which transform to the following system of inequalities:

$$\frac{1-\gamma}{2}(2L-2\sigma-V_s^{LO}) \leq 0,$$
$$\frac{\gamma}{2}(2L-2\sigma-V_b^{LO}) \geq 0.$$

By plugging in the calculated values of V_b^{LO} and V_s^{LO}, the previous system of inequalities is solved for σ, and the following equilibrium area is determined (Eq.1):

$$\frac{(1-\gamma)2L}{2-2\gamma+\gamma^2} \leq \sigma \leq \frac{(1+\gamma)2L}{4-\gamma+\gamma^2}.$$

- Eq.2: $B_k = \hat{B}^{v-\sigma}, A_k = \hat{A}^{v+\sigma}$
  The outside option values for these orders:

$$V_b^{LO} = \frac{1-\gamma}{2}(v-\sigma+L-\hat{B}^{v-\sigma}) = \frac{1-\gamma}{2}(2L-V_s^{LO}),$$
$$V_s^{LO} = \frac{\gamma}{2}(\hat{A}^{v+\sigma}-(v+\sigma-L)) = \frac{\gamma}{2}(2L-V_b^{LO}),$$

both of which lead to:

$$V_b^{LO} = \frac{2L(2-\gamma)(1-\gamma)}{4-\gamma+\gamma^2},$$
$$V_s^{LO} = \frac{2L\gamma(1+\gamma)}{4-\gamma+\gamma^2}.$$

The following inequalities help to identify the set of parameters under which this equilibrium potentially holds:

$$\frac{1-\gamma}{2}(v-\sigma+L-\hat{B}^{v-\sigma}) \geq \frac{1-\gamma}{2}(v-\sigma+L-\hat{B}^{v+\sigma})+\frac{1-\gamma}{2}(v+\sigma+L-\hat{B}^{v+\sigma}),$$
$$\frac{\gamma}{2}(\hat{A}^{v+\sigma}-(v+\sigma-L)) \geq \frac{\gamma}{2}(\hat{A}^{v-\sigma}-(v-\sigma-L))+\frac{\gamma}{2}(\hat{A}^{v-\sigma}-(v+\sigma-L)),$$

both of which transform to:

$$\frac{1-\gamma}{2}(2L-2\sigma-V_s^{LO}) \leq 0,$$
$$\frac{\gamma}{2}(2L-2\sigma-V_b^{LO}) \leq 0.$$

The following system of inequalities is applied for describing the equilibrium area (Eq.2):

$$\begin{cases} \sigma \geq \frac{(1+\gamma)2L}{4-\gamma+\gamma^2} \\ \sigma \geq \frac{(2-\gamma)2L}{4-\gamma+\gamma^2} \end{cases}.$$

- Eq.3: $B_k = \hat{B}^{v+\sigma}, A_k = \hat{A}^{v-\sigma}$
  The outside option values for these orders:

$$\begin{aligned} V_b^{LO} &= \frac{1-\gamma}{2}(v-\sigma+L-\hat{B}^{v+\sigma}) + \frac{1-\gamma}{2}(v+\sigma+L-\hat{B}^{v+\sigma}) \\ &= \frac{1-\gamma}{2}(2L-2\sigma-V_s^{LO}) + \frac{1-\gamma}{2}(2L-V_s^{LO}), \\ V_s^{LO} &= \frac{\gamma}{2}(\hat{A}^{v-\sigma}-(v-\sigma-L)) + \frac{\gamma}{2}(\hat{A}^{v-\sigma}-(v+\sigma-L)) \\ &= \frac{\gamma}{2}(2L-V_b^{LO}) + \frac{\gamma}{2}(2L-2\sigma-V_b^{LO}), \end{aligned}$$

both of which lead to:

$$\begin{aligned} V_b^{LO} &= \frac{(1-\gamma)^2(2L-\sigma)}{1-\gamma+\gamma^2}, \\ V_s^{LO} &= \frac{\gamma^2(2L-\sigma)}{1-\gamma+\gamma^2}. \end{aligned}$$

To find the set of parameters under which this equilibrium potentially holds, the following inequalities should be solved:

$$\begin{aligned} \frac{1-\gamma}{2}(v-\sigma+L-\hat{B}^{v+\sigma}) + \frac{1-\gamma}{2}(v+\sigma+L-\hat{B}^{v+\sigma}) &\geq \frac{1-\gamma}{2}(v-\sigma+L-\hat{B}^{v-\sigma}), \\ \frac{\gamma}{2}(\hat{A}^{v-\sigma}-(v-\sigma-L)) + \frac{\gamma}{2}(\hat{A}^{v-\sigma}-(v+\sigma-L)) &\geq \frac{\gamma}{2}(\hat{A}^{v+\sigma}-(v+\sigma-L)), \end{aligned}$$

both of which transform to:

$$\begin{aligned} \frac{1-\gamma}{2}(2L-2\sigma-V_s^{LO}) &\geq 0, \\ \frac{\gamma}{2}(2L-2\sigma-V_b^{LO}) &\geq 0. \end{aligned}$$

The following system of inequalities is used for describing the equilibrium area (Eq.3):

$$\begin{cases} \sigma \leq \frac{(1-\gamma)2L}{2-2\gamma+\gamma^2} \\ \sigma \leq \frac{\gamma 2L}{\gamma^2+1}. \end{cases}$$

- Eq.4: $B_k = \hat{B}^{v+\sigma}, A_k = \hat{A}^{v+\sigma}$

The outside option values for these orders:

$$\begin{aligned}
V_b^{LO} &= \frac{1-\gamma}{2}(v-\sigma+L-\hat{B}^{v+\sigma}) + \frac{1-\gamma}{2}(v+\sigma+L-\hat{B}^{v+\sigma}) \\
&= \frac{1-\gamma}{2}(2L-2\sigma-V_s^{LO}) + \frac{1-\gamma}{2}(2L-V_s^{LO}), \\
V_s^{LO} &= \frac{\gamma}{2}(\hat{A}^{v+\sigma}-(v+\sigma-L)) = \frac{\gamma}{2}(2L-V_b^{LO}),
\end{aligned}$$

both of which lead to:

$$\begin{aligned}
V_b^{LO} &= \frac{2(1-\gamma)\big(L(2-\gamma)-\sigma\big)}{2-\gamma+\gamma^2}, \\
V_s^{LO} &= \frac{\gamma\big(2L\gamma+(1-\gamma)\sigma\big)}{2-\gamma+\gamma^2}.
\end{aligned}$$

Finding the set of parameters under which this equilibrium potentially holds, means solving the following inequalities:

$$\begin{aligned}
\frac{1-\gamma}{2}(v-\sigma+L-\hat{B}^{v+\sigma}) + \frac{1-\gamma}{2}(v+\sigma+L-\hat{B}^{v+\sigma}) &\geq \frac{1-\gamma}{2}(v-\sigma+L-\hat{B}^{v-\sigma}), \\
\frac{\gamma}{2}(\hat{A}^{v+\sigma}-(v+\sigma-L)) &\geq \frac{\gamma}{2}(\hat{A}^{v-\sigma}-(v-\sigma-L)) + \frac{\gamma}{2}(\hat{A}^{v-\sigma}-(v+\sigma-L)),
\end{aligned}$$

both of which transform to:

$$\begin{aligned}
&\frac{1-\gamma}{2}(2L-2\sigma-V_s^{LO}) \geq 0, \\
&\frac{\gamma}{2}(2L-2\sigma-V_b^{LO}) \leq 0.
\end{aligned}$$

As a result of the calculations, the following inequalities describing the equilibrium area are derived (Eq.4):

$$\frac{\gamma 2L}{\gamma^2+1} \leq \sigma \leq \frac{(2-\gamma)2L}{4-\gamma+\gamma^2}.$$

□

In Figure 2, the equilibrium map for the simplified model is illustrated.

In general, all four achieved equilibria are not symmetric: the bargaining powers of both sides are different, $V_b^{LO} \neq V_s^{LO}$, and the orders of both market sides are differently distant from the current fundamental value. In Eq.2, which renders a high variance of a fundamental value change, both sides of the market prefer low fill-rate orders, while in Eq.3, encompassing lower variance, both sides choose high fill-rate orders[17]. In Eq.1 and Eq.4, choices are mixed in terms of fill-rates.

[17]Orders' classification in terms of fill-rates is presented in Hoffmann (2014), p.159: low fill-rate orders are satisfied only under one state of the fundamental value change ($\hat{B}^{v-\sigma}$ only if $\varepsilon_{t+1} = -\sigma$, whereas $\hat{A}^{v+\sigma}$ only if $\varepsilon_{t+1} = \sigma$); high fill-rate orders are satisfied in both possible states ($\hat{B}^{v+\sigma}$ and $\hat{A}^{v-\sigma}$).

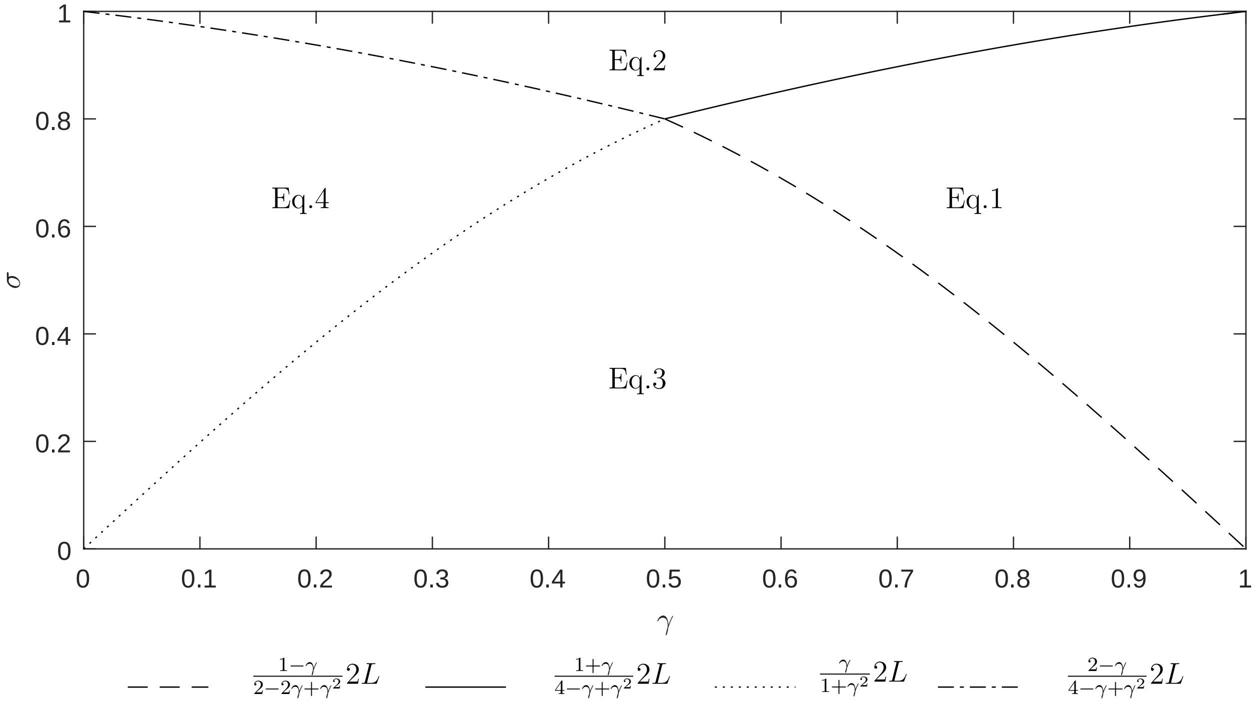


Figure 2: Equilibrium map for the simplified analytical model ($L = 1$)

Proposition 2.3. *In equilibrium, high (low) volatility forces the agents to post low (high) fill-rate orders. A large (small) share of a certain market side causes the agents on this side to post orders with low (high) fill-rates.*

Proposition 2.3 describes a situation typical of real financial markets. First, the high volatility of fundamental values increases the severeness of adverse selection risk, and the agents try to protect themselves by using low fill-rate orders, which will be profitable in any case, if executed[18]. Second, a large share of buyers (sellers) makes the bargaining power (the value of the outside option) of the opposite side significant. This makes trade unprofitable for a posting agent from the numerous side, if a high fill-rate order is used when the guess is wrong, and the real fundamental value change turns out to be the opposite[19]. In such a case, it is safer for an agent from the numerous group to use low fill-rate orders.

Corollary 2.4. *If there are more sellers (buyers) on the market, $\gamma < 0.5$ ($\gamma > 0.5$), the bargaining power of the opposite market side is higher: buyers (sellers) have a greater expected profit from sending a limit order. This makes the equilibria asymmetric.*

Proof. This can be proven by comparing the pairs V_b^{LO} and V_s^{LO} for each equilibrium and determining the parameter values γ, under which the described inequalities hold. □

Figure 3 shows 3D-graphs of the outside option values under different equilibrium conditions. The maximum values of the outside options are reached by buyers and sellers on the opposite

[18]This result is in line with Hoffmann (2014), p.160.

[19]The probability of such a transaction to happen is called the risk of being picked-off, see Hoffmann (2014) and Section 2.5.5.

edges of the equilibrium map: the more buyers (sellers) there are and the smaller volatility σ is, which corresponds partially to Eq.1 and Eq.3 (Eq.2 and Eq.4), the greater is the outside option value for a seller (buyer), while the bargaining power of a buyer (seller) stays on the positive but minimum level. The scarce market side exercises a greater bargaining power. This power is magnified, if volatility is low, because it decreases the risk of adverse selection.

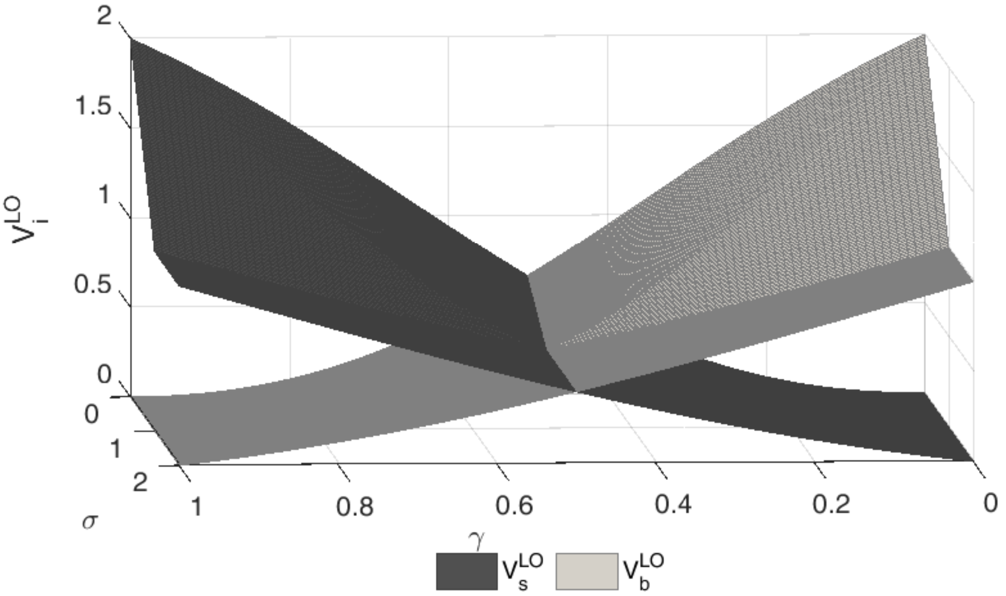


Figure 3: Bargaining power of the market sides for the simplified analytical model ($L = 1$)

The bargaining power and expected profit from posting a limit order, V_i^{LO}, can be described as the "imaginary" profit. This profit is based on the future reservation value $R_{i,t+1}$, and one part of this reservation value, y_i, is never realized, since the fundamental value changes only by the absolute value of σ. Due to this fact, the attempt to check how the real expected profit from posting a limit order differs from the imaginary one becomes an interesting research task.

Corollary 2.5. *While the imaginary expected profit from posting a limit order is positive for both sides of the market in all the four equilibria, the real expected profit may become negative, see Table 5.*

Table 5: The real expected profit from limit orders for (γ, σ, L) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4
$V_{b,\text{ real}}^{LO}$	$\frac{1-\gamma}{2} \cdot \frac{2\gamma\sigma+L\cdot(2-3\gamma-\gamma^2)}{2-\gamma+\gamma^2})$	$\frac{1-\gamma}{2} \cdot L \cdot \frac{(\gamma+4)(1-\gamma)}{4-\gamma+\gamma^2}$	$\frac{(1-\gamma)(L(1-\gamma-\gamma^2)-\sigma(1-\gamma))}{1-\gamma+\gamma^2}$	$\frac{(1-\gamma)(L(2-\gamma-\gamma^2)-2\sigma)}{2-\gamma+\gamma^2}$
$V_{s,\text{ real}}^{LO}$	$\frac{\gamma(L\gamma(3-\gamma)-2\sigma)}{2-\gamma+\gamma^2}$	$\frac{\gamma}{2} \cdot L \cdot \frac{\gamma(5-\gamma)}{4-\gamma+\gamma^2}$	$-\frac{\gamma(L(1-3\gamma+\gamma^2)+\gamma\sigma)}{1-\gamma+\gamma^2}$	$\frac{\gamma}{2} \cdot \frac{L(-2+5\gamma-\gamma^2)+2(1-\gamma)\sigma}{2-\gamma+\gamma^2}$

Proof. Based on the combinations of possible orders for each equilibrium and their probabilities of execution, the real expected profit is verified by comparing a potential order with a possible fundamental value instead of the reservation value:

- Eq.1:

$$
\begin{aligned}
V^{LO}_{b,\text{ real}} &= \frac{1-\gamma}{2}(v-\sigma-\hat{B}^{v-\sigma}) = \frac{1-\gamma}{2}(L-V^{LO}_s)\\
&= \frac{1-\gamma}{2}(L-\frac{2\gamma(L(1+\gamma)-\sigma)}{2-\gamma+\gamma^2}) = \frac{1-\gamma}{2}(2\gamma\sigma+L\cdot\frac{2-3\gamma-\gamma^2}{2-\gamma+\gamma^2}),\\
V^{LO}_{s,\text{ real}} &= \frac{\gamma}{2}(\hat{A}^{v-\sigma}-(v-\sigma))+\frac{\gamma}{2}(\hat{A}^{v-\sigma}-(v+\sigma)) = \frac{\gamma}{2}(L-V^{LO}_b)+\frac{\gamma}{2}(L-2\sigma-V^{LO}_b)\\
&= \frac{\gamma}{2}(L-\frac{(1-\gamma)(2L(1-\gamma)+\gamma\sigma)}{2-\gamma+\gamma^2})+\frac{\gamma}{2}(L-2\sigma-\frac{(1-\gamma)(2L(1-\gamma)+\gamma\sigma)}{2-\gamma+\gamma^2})\\
&= \frac{\gamma(L\gamma(3-\gamma)-2\sigma)}{2-\gamma+\gamma^2}.
\end{aligned}
$$

- Eq.2:

$$
\begin{aligned}
V^{LO}_{b,\text{ real}} &= \frac{1-\gamma}{2}(v-\sigma-\hat{B}^{v-\sigma}) = \frac{1-\gamma}{2}(L-V^{LO}_s)\\
&= \frac{1-\gamma}{2}(L-\frac{2L\gamma(1+\gamma)}{4-\gamma+\gamma^2}) = \frac{1-\gamma}{2}\cdot L\cdot\frac{(\gamma+4)(1-\gamma)}{4-\gamma+\gamma^2},\\
V^{LO}_{s,\text{ real}} &= \frac{\gamma}{2}(\hat{A}^{v+\sigma}-(v+\sigma)) = \frac{\gamma}{2}(L-V^{LO}_b)\\
&= \frac{\gamma}{2}(L-\frac{2L(2-\gamma)(1-\gamma)}{4-\gamma+\gamma^2}) = \frac{\gamma}{2}\cdot L\cdot\frac{\gamma(5-\gamma)}{4-\gamma+\gamma^2}.
\end{aligned}
$$

- Eq.3:

$$
\begin{aligned}
V^{LO}_{b,\text{ real}} &= \frac{1-\gamma}{2}(v-\sigma-\hat{B}^{v+\sigma})+\frac{1-\gamma}{2}(v+\sigma-\hat{B}^{v+\sigma})\\
&= \frac{1-\gamma}{2}(L-2\sigma-V^{LO}_s)+\frac{1-\gamma}{2}(L-V^{LO}_s)\\
&= \frac{1-\gamma}{2}(L-2\sigma-\frac{\gamma^2(2L-\sigma)}{1-\gamma+\gamma^2})+\frac{1-\gamma}{2}(L-\frac{\gamma^2(2L-\sigma)}{1-\gamma+\gamma^2})\\
&= \frac{(1-\gamma)(L(1-\gamma-\gamma^2)-\sigma(1-\gamma))}{1-\gamma+\gamma^2},\\
V^{LO}_{s,\text{ real}} &= \frac{\gamma}{2}(\hat{A}^{v-\sigma}-(v-\sigma))+\frac{\gamma}{2}(\hat{A}^{v-\sigma}-(v+\sigma))\\
&= \frac{\gamma}{2}(L-V^{LO}_b)+\frac{\gamma}{2}(L-2\sigma-V^{LO}_b)\\
&= \frac{\gamma}{2}(L-\frac{(1-\gamma)^2(2L-\sigma)}{1-\gamma+\gamma^2})+\frac{\gamma}{2}(L-2\sigma-\frac{(1-\gamma)^2(2L-\sigma)}{1-\gamma+\gamma^2})\\
&= -\frac{\gamma(L(1-3\gamma+\gamma^2)+\gamma\sigma)}{1-\gamma+\gamma^2}.
\end{aligned}
$$

- Eq.4:

$$\begin{aligned}
V^{LO}_{b,\text{ real}} &= \frac{1-\gamma}{2}(v-\sigma-\hat{B}^{v+\sigma}) + \frac{1-\gamma}{2}(v+\sigma-\hat{B}^{v+\sigma}) \\
&= \frac{1-\gamma}{2}(L-2\sigma-V^{LO}_s) + \frac{1-\gamma}{2}(L-V^{LO}_s) \\
&= \frac{1-\gamma}{2}(L-2\sigma-\frac{\gamma\big(2L\gamma+(1-\gamma)\sigma\big)}{2-\gamma+\gamma^2}) + \frac{1-\gamma}{2}(L-\frac{\gamma\big(2L\gamma+(1-\gamma)\sigma\big)}{2-\gamma+\gamma^2}) \\
&= \frac{(1-\gamma)(L(2-\gamma-\gamma^2)-2\sigma)}{2-\gamma+\gamma^2}, \\
V^{LO}_{s,\text{ real}} &= \frac{\gamma}{2}(\hat{A}^{v+\sigma}-(v+\sigma)) = \frac{\gamma}{2}(L-V^{LO}_b) = \frac{\gamma}{2}(L-\frac{2(1-\gamma)\big(L(2-\gamma)-\sigma\big)}{2-\gamma+\gamma^2}) \\
&= \frac{\gamma}{2}\cdot\frac{L(-2+5\gamma-\gamma^2)+2(1-\gamma)\sigma}{2-\gamma+\gamma^2}.
\end{aligned}$$

From the equations above, it is possible to investigate for which parameters (γ, σ) these functions become negative (see the graphical solution in Figure 4 given $L = 1$). □

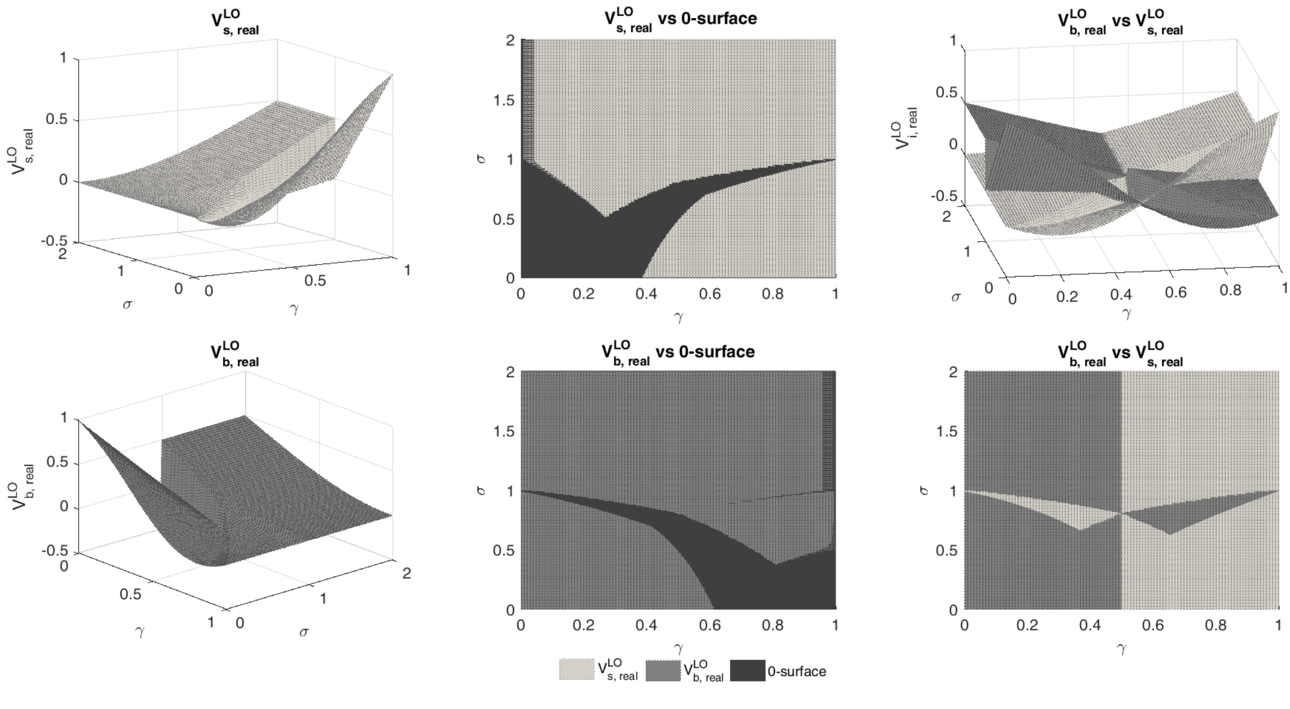


Figure 4: Real expected profit from limit orders for the simplified analytical model ($L = 1$)

Even if the equilibria are optimal, when based on V_i^{LO} maximization, they might turn out to be unprofitable, if $V^{LO}_{i,\text{ real}}$ is accounted for. The real expected profit is negative for a large share of the respective market side and small volatility, but might also be negative for some low share - moderate volatility states. The intuition behind this result is the following: if the share of a market side is large, this market side posts orders with smaller profitability. However, if volatility goes up, the profitability of those orders goes up, since agents try to protect themselves from the adverse selection risk. If the two real expected profits are compared, the results are similar to the imaginary profits: if the buyers' (sellers') share is larger, the sellers' (buyers') real expected profit is greater. However, for the real expected profit there is an exception region where this rule does not hold: for the moderate volatility values, the real expected profit from a limit order is not always greater for the scarce market side. When posting orders, the bargaining power of the opposite side might be overvalued, which distorts the order values and decreases their profitability.

Corollary 2.6. *The equilibria have the following characteristics:*

(a) On the borderline between two neighboring equilibria, an agent is indifferent towards the choice between the two.

(b) All equilibrium borders intersect at one point: $(\gamma = \frac{1}{2}, \sigma = \frac{4}{5}L)$. At this point, all the four equilibria exist and they are indifferently optimal based on the V_i^{LO}: the bargaining power is equal among the equilibria and market sides.

(c) More generally, if $\gamma = \frac{1}{2}$, $V_b^{LO} = V_s^{LO}$: if the two market sides are balanced, their outside options are equal[20]*, which makes these equilibria symmetric based on V_i^{LO}. However, only Eq.2 and Eq.3 are symmetric in terms of the orders distance from the current fundamental value, whereas Eq.1 and Eq.4 are not.*

Proof. Each point from this corollary is discussed below:

(a) The volatility σ is set to borderline values and the equality $V_{b,k}^{LO} = V_{b,h \neq k}^{LO}$ is checked for each of the four possible neighboring equilibrium pairs k and h.

(b) The proof follows from the pairwise equalization of all the four borderlines to each other and finding the intersection point (γ, σ). After plugging in the parameters $(\gamma = \frac{1}{2}, \sigma = \frac{4}{5}L, L = 1)$ into the V_i^{LO} formulae, it is received $V_b^{LO} = V_s^{LO} = \frac{2}{5}L$ for all the four equilibria.

(c) If the market shares of buyers and sellers are equal, Eq.1 and Eq.4 are possible only for one specific volatility value $(\sigma = \frac{4}{5}L)$. We already know that $V_b^{LO} = V_s^{LO} = \frac{2}{5}L$. Eq.2 and Eq.3 exist for the other parameters of volatility σ, therefore the general functional forms of the outside options are checked, given that $\gamma = \frac{1}{2}$. In Eq.2, $V_b^{LO} = V_s^{LO} = \frac{2}{5}L$, independently of σ. In Eq.3, $V_b^{LO} = V_s^{LO} = \frac{1}{3}(2L - \sigma)$. Thus, a conclusion can be drawn that all the four equilibria are symmetric in terms of the outside option value, if the shares of the market sides are equal. Further, the order-setting rules are applied to various equilibria. Again, Eq.1 and Eq.4 are verified only at one specific point, while Eq.2 and Eq.3 are checked in general for different σ with $\gamma = \frac{1}{2}$. Table 6 shows the respective order values. From Table 6 it follows that bid and ask orders are equidistant from v for Eq.2 and Eq.3, while asymmetric in Eq.1 (both orders are smaller than the current fundamental value) and Eq.4 (both orders are higher than the current fundamental value). □

[20] This case is used in the model by Hoffmann (2014).

Table 6: Order values for $(\gamma = \frac{1}{2}, \sigma, L)$ in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4
B_k	$v - \frac{7}{5}L$	$v - (\frac{3}{5}L + \sigma)$	$v - (\frac{1}{3}L - \frac{2}{3}\sigma)$	$v + \frac{1}{5}L$
A_k	$v - \frac{1}{5}L$	$v + (\frac{3}{5}L + \sigma)$	$v + (\frac{1}{3}L - \frac{2}{3}\sigma)$	$v + \frac{7}{5}L$

2.3 Equilibrium conditions for equal shares of market sides

The model proposed by Hoffmann (2014) is based on symmetric equilibria only (in terms of the bargaining power and the distance of orders from the fundamental value), given that the shares of the market sides are identical. In this section, the assumption that a market sentiment is neutral is followed: there is no sudden loss or gain anticipated in the fundamental value, there are no herding processes related to such expectations. For liquid assets in the long run the number of buyers and sellers tends to be the same (deviations and extreme cases are possible in the short term perspective). In this way, the simplified model is verified under the assumption incorporated in Hoffmann (2014). The results from Section 2.2 are tested to see how the equilibrium conditions are simplified under this assumption.

Proposition 2.4. *For the fixed parameters $(\gamma = \frac{1}{2}, \sigma, L)$, a Markov-perfect equilibrium exists in the limit order market. The equilibrium conditions are presented in Table 7.*

Table 7: Equilibrium conditions for $(\gamma = \frac{1}{2}, \sigma, L)$ in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4
B_k	$\hat{B}^{v-\sigma}$	$\hat{B}^{v-\sigma}$	$\hat{B}^{v+\sigma}$	$\hat{B}^{v+\sigma}$
A_k	$\hat{A}^{v-\sigma}$	$\hat{A}^{v+\sigma}$	$\hat{A}^{v-\sigma}$	$\hat{A}^{v+\sigma}$
V_b^{LO}	$\frac{2}{7}L + \frac{1}{7}\sigma$	$\frac{2}{5}L$	$\frac{2}{3}L - \frac{1}{3}\sigma$	$\frac{6}{7}L - \frac{4}{7}\sigma$
V_s^{LO}	$\frac{6}{7}L - \frac{4}{7}\sigma$	$\frac{2}{5}L$	$\frac{2}{3}L - \frac{1}{3}\sigma$	$\frac{2}{7}L + \frac{1}{7}\sigma$
$V_{b,\text{ real}}^{LO}$	$\frac{1}{28}L + \frac{1}{7}\sigma$	$\frac{3}{20}L$	$\frac{1}{6}L - \frac{1}{3}\sigma$	$\frac{5}{14}L - \frac{4}{7}\sigma$
$V_{s,\text{ real}}^{LO}$	$\frac{5}{14}L - \frac{4}{7}\sigma$	$\frac{3}{20}L$	$\frac{1}{6}L - \frac{1}{3}\sigma$	$\frac{1}{28}L + \frac{1}{7}\sigma$
Condition for σ	$\sigma = \frac{4}{5}L$	$\sigma \geq \frac{4}{5}L$	$\sigma \leq \frac{4}{5}L$	$\sigma = \frac{4}{5}L$

Proof. These results are achieved straight-forwardly by plugging in $\gamma = \frac{1}{2}$ into the equilibrium conditions of the more general set-up from Table 4. □

Figure 5 presents the results of transactions for both market sides, if $\gamma = \frac{1}{2}$, for the two cases of σ value: $\sigma \geq \frac{4}{5}L$ and $\sigma < \frac{4}{5}L$. For the first case, when $\sigma \geq \frac{4}{5}L$:

- The positive case bid is higher than the buyer's reservation value in the negative case $\hat{B}^{v+\sigma} > v - \sigma + L$, which brings a loss to a buyer, if this bid is executed;

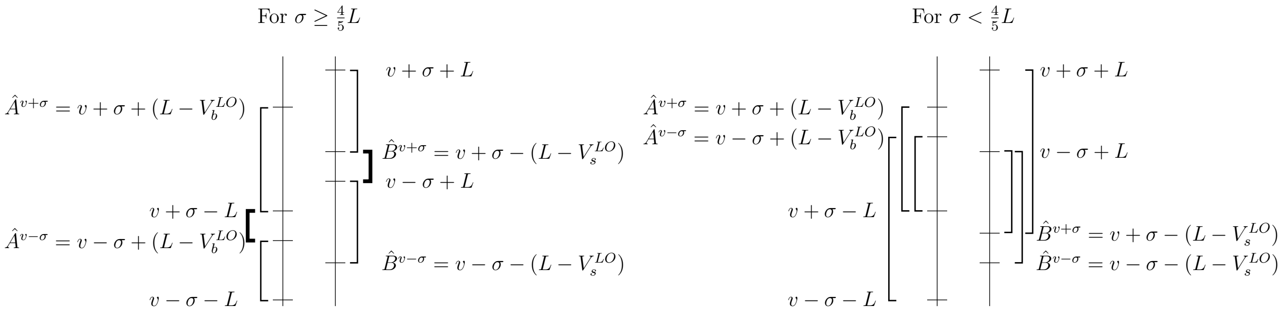


Figure 5: Possible transaction results on both market sides for $\gamma = \frac{1}{2}$ in the simplified analytical model

The absolute values of distances are presented arbitrary; relative positions of the illustrated values are important. Normal square brackets render a possible profit from a transaction, while bold square brackets demonstrate a possible loss.

- The negative case ask is smaller than the positive case of the sellers' reservation value $\hat{A}^{v-\sigma} > v + \sigma - L$, which brings a loss to a seller, if this ask is executed.

If $\sigma > \frac{4}{5}L$, it is more profitable for a buyer (seller) to post $\hat{B}^{v-\sigma}(\hat{A}^{v+\sigma})$, which helps to avoid losses, but has a smaller probability to be executed: for higher values of σ, a trader should use the low fill-rate orders.
If $\sigma = \frac{4}{5}L$, the negative case of the buyers' reservation value turns out to be equal to the positive case bid order: $v - \sigma + L = \hat{B}^{v+\sigma} = v + \sigma - \frac{3}{5}L$, and the negative case ask order is equal to the positive case of the sellers' reservation value: $\hat{A}^{v-\sigma} = v - \sigma + \frac{3}{5}L = v + \sigma - L$. This coincidence leads to the zero additional profit of high fill-rate orders in comparison to low fill-rate orders. Moreover, both orders have the same expected profit, if they are executed during the next period. For the buyer it is: $v - \sigma + L - \hat{B}^{v-\sigma} = v + \sigma + L - \hat{B}^{v+\sigma} = \frac{8}{5}L$, and for the seller it is: $\hat{A}^{v+\sigma} - (v + \sigma - L) = \hat{A}^{v-\sigma} - (v - \sigma - L) = \frac{8}{5}L$. Since the expected profit from both possible orders is equal, an agent is indifferent between them at the point $(\gamma = \frac{1}{2}, \sigma = \frac{4}{5}L)$. This leads to four possible equilibria at this point.
For the second case, if $\sigma < \frac{4}{5}L$, both realizations of the buyers' (sellers') reservation values are greater (smaller) than both possible bid (ask) orders, therefore both bids (asks) bring a positive payoff to a buyer (seller), if the order is executed. The expected profit is maximized, if $\hat{B}^{v+\sigma}(\hat{A}^{v-\sigma})$ is used, since it brings profit in both states of the fundamental value change.

2.4 Analytical vs agent-based model: harmonization of assumptions

Agent-based models (ABMs) are a powerful tool for financial markets research and an important addition to the analytical models. In Chapter 6, the gap between the two divergent types of models is bridged. This is done by harmonizing the assumptions of the two models. Hoffmann (2014) does not comment on the (im-)possibility of a certain agent making a transaction based on her own posted order. This assumption is not of great importance, if an infinite number of agents is assumed, since it almost nullifies the probability of the same agent acting in two consequent periods. However, this parameter makes a big difference, if there are only few agents, and chances for an agent to be selected for actions twice in a row are considerable. This assumption might be inevitable in practical implementations (in ABMs), where an infinite population of agents is unattainable, and the implementation of a high number of agents does not bring any additional value to the analysis but makes it unnecessarily complex. Moreover, ABMs enable tracking the performance of individual agents. For this, again, the amount of agents has to be limited. For simplicity, there are only two agents in our analysis, and they use an equilibrium map from the analytical model to make their decisions about order values. The agents can trade with each other only, an agent cannot satisfy her own limit order from the previous period[21].

For further investigation of the market characteristics, a neutral market sentiment is assumed and the opportunity to analyze all equilibria by setting up $\sigma = \frac{4}{5}L$ is used. The analysis is simplified by taking $L = 1$ without loss of generality[22].

2.5 Market metrics

To assess the results of this model, the market metrics presented in Hoffmann (2014) are analyzed. Besides, one more important variable - the real expected profit from posting a limit order - is introduced into the analysis. Below is the list and a short definition of all the market characteristics considered within the research:

- Expected profit from posting a limit order, V_i^{LO}, serves as the value of an outside option to sending a market order and measures an agent's bargaining power.
- Real expected profit from posting a limit order, $V_{i,\text{ real}}^{LO}$, is free from the imaginary part of the reservation value.
- Limit order execution probability, p_i, shows the probability of a posted limit order to be satisfied in the next market step.
- Probabilities of equilibrium events, φ_i^{LO} and φ_i^{MO}, show the likelihood to see a new limit order posting or transaction in the market in each time period.
- Risk of being picked-off, π_i, is the fraction of the high fill-rate order execution probability under the opposite fundamental value development and the full execution probability.
- Trading rate, TR, is the sum of probabilities to post a market order on both sides of the market.
- Costs of immediacy, $E(\tau_i)$, is the signed difference between a transaction price and the fundamental value of an asset.

[21] This is a significant change in assumptions compared to Hoffmann (2014); it influences market variables through the probabilities of execution.

[22] Conclusions from the comparative analysis depend on the relative values of σ and L only.

- Maker-taker ratio, MT_i, is the probability to trade via a limit order relative to the probability to trade via a market order.
- Pricing error, PE_i, is the market efficiency measure; it is determined as the absolute difference between the fundamental asset value and an order existing in the market.
- Agents' welfare, W_i, is the weighted average of an agent's profit from market orders and limit orders.

In the following, these market metrics are transferred to the simplified analytical model with slow traders for two set-ups: under the assumption of an infinite number of market participants and the assumption of two market participants. The former case is reported in parentheses and corresponds to the key assumptions of the analytical models presented in the literature. The latter case provides the results of the market set-up harmonized with the ABM from Chapter 6. In next sections, if market metrics are independent of some parameters, this parameter is referred to in an analytical form in a table's title (e.g., $(\gamma = \frac{1}{2}, \sigma, L)$ means that the considered measure is independent of σ and L, but is calculated for $\gamma = \frac{1}{2}$). The average values are based on the assumption that all the four equilibria are equally likely to appear during the market session.

2.5.1 Execution probabilities

Corollary 2.7. *For the fixed parameters $(\gamma = \frac{1}{2}, \sigma, L)$, the equilibrium probabilities of limit orders sent at t to be executed at $t+1$ are presented in Table 8.*

Table 8: Execution probabilities for $(\gamma = \frac{1}{2}, \sigma, L)$ in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4	Average
p_b	$\frac{1}{8}(\frac{1}{4})$	$\frac{1}{8}(\frac{1}{4})$	$\frac{1}{4}(\frac{1}{2})$	$\frac{1}{4}(\frac{1}{2})$	$\frac{3}{16}(\frac{3}{8})$
p_s	$\frac{1}{4}(\frac{1}{2})$	$\frac{1}{8}(\frac{1}{4})$	$\frac{1}{4}(\frac{1}{2})$	$\frac{1}{8}(\frac{1}{4})$	$\frac{3}{16}(\frac{3}{8})$

Proof. The execution probability of O_t sent by an agent at t can be expressed as a product of the probabilities that the next coming agent has another identifier (ID), that she is on the opposite side of the market, that a certain value change happens during the next period, and the conditional probability of this order to be executed at $t+1$ given all these three factors lead to a transaction:

$$
\begin{aligned}
p_{b,t} = {} & \Pr(\mathrm{ID}_{t+1} \neq \mathrm{ID}_t) \cdot \Pr(i_{t+1} = s) \cdot \Pr(\varepsilon_{t+1} = -\sigma) \cdot \Pr(B_t | \mathrm{ID}_{t+1} \neq \mathrm{ID}_t, i_{t+1} = s, \varepsilon_{t+1} = -\sigma) \\
& + \Pr(\mathrm{ID}_{t+1} \neq \mathrm{ID}_t) \cdot \Pr(i_{t+1} = s) \cdot \Pr(\varepsilon_{t+1} = \sigma) \cdot \Pr(B_t | \mathrm{ID}_{t+1} \neq \mathrm{ID}_t, i_{t+1} = s, \varepsilon_{t+1} = \sigma), \\
p_{s,t} = {} & \Pr(\mathrm{ID}_{t+1} \neq \mathrm{ID}_t) \cdot \Pr(i_{t+1} = b) \cdot \Pr(\varepsilon_{t+1} = -\sigma) \cdot \Pr(A_t | \mathrm{ID}_{t+1} \neq \mathrm{ID}_t, i_{t+1} = b, \varepsilon_{t+1} = -\sigma) \\
& + \Pr(\mathrm{ID}_{t+1} \neq \mathrm{ID}_t) \cdot \Pr(i_{t+1} = b) \cdot \Pr(\varepsilon_{t+1} = \sigma) \cdot \Pr(A_t | \mathrm{ID}_{t+1} \neq \mathrm{ID}_t, i_{t+1} = b, \varepsilon_{t+1} = \sigma).
\end{aligned}
$$

Given the assumptions of the model, it is possible to postulate the following:

- $\Pr(\text{ID}_{t+1} \neq \text{ID}_t) = \frac{1}{2}$, since there are only two agents participating in the market, and they have equal chances to be randomly selected in the market for an action;
- $\Pr(i_{t+1} = s) = \Pr(i_{t+1} = b) = \gamma = \frac{1}{2}$, since the share of buyers and sellers is symmetric;
- $\Pr(\varepsilon_{t+1} = \sigma) = \Pr(\varepsilon_{t+1} = -\sigma) = \frac{1}{2}$, since there are equal chances for the fundamental value to go up or down in the next period.

Only the conditional probabilities are different among the equilibria depending on whether the optimal order is a high or low fill-rate. Table 9 includes the conditional probabilities of execution given that the next coming agent has another identifier, she is on the other market side, and given a certain direction of the fundamental value development.

Table 9: Conditional execution probabilities for (γ, σ, L) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4
$\Pr(B_t \vert \text{ID}_{t+1} \neq \text{ID}_t, i_{t+1} = s, \varepsilon_{t+1} = -\sigma)$	1	1	1	1
$\Pr(B_t \vert \text{ID}_{t+1} \neq \text{ID}_t, i_{t+1} = s, \varepsilon_{t+1} = \sigma)$	0	0	1	1
$\Pr(A_t \vert \text{ID}_{t+1} \neq \text{ID}_t, i_{t+1} = b, \varepsilon_{t+1} = -\sigma)$	1	0	1	0
$\Pr(A_t \vert \text{ID}_{t+1} \neq \text{ID}_t, i_{t+1} = b, \varepsilon_{t+1} = \sigma)$	1	1	1	1

As the last step, one needs to use the formulae above to calculate the execution probabilities. □

Eq.2 and Eq.3 are symmetric for both sides of the market with respect to the probabilities of execution. The higher the volatility is, the lower is the probability, while for the low and moderate values of volatility, the probability of execution is higher. However, a high probability of execution does not allow to make any final conclusion about a higher profitability of one equilibrium over another, since the profits conditional on execution have to be taken into account. Eq.4 (Eq.1) is as good for a buyer (seller) as Eq.3 (Eq.2), but for a seller (buyer) the execution chances in the respective equilibrium decrease.

2.5.2 Probabilities of equilibrium events

To compute equilibrium outcomes in the future sections, one needs to know the equilibrium probability of a particular event. There are four mutually exclusive events possible at any time step in the market: (1) a buyer submits a bid limit order to a limit order book with the probability φ_b^{LO}, (2) a buyer executes an ask from the previous period (a buyer submitting a market order), φ_b^{MO}, (3) a seller submits an ask limit order, φ_s^{LO}, (4) a seller executes a bid from a limit order book (a seller submitting a market order), φ_s^{MO}.
To determine the probabilities of equilibrium events, we need to construct a transition matrix

which follows the Markov chain[23,24] with a set of states $\varphi = (\varphi_b^{LO}, \varphi_b^{MO}, \varphi_s^{LO}, \varphi_s^{MO})$; it shows the transition from one state at t to another at $t+1$. The game starts in one of these states and moves step by step to another state. Taking into consideration the share of each market side, it is necessary to additionally account for the following equalities: $\varphi_b^{LO} + \varphi_b^{MO} = \gamma$ and $\varphi_s^{LO} + \varphi_s^{MO} = 1 - \gamma$. The sum of probabilities of all the equilibrium events should be equal to one: $\varphi_b^{LO} + \varphi_b^{MO} + \varphi_s^{LO} + \varphi_s^{MO} = 1$. Transition probabilities are shown in Table 10.

Table 10: Transition probabilities in the simplified analytical model

Event$_{t+1}$ / Event$_t$	a buyer sending an LO	a buyer sending an MO	a seller sending an LO	a seller sending an MO
a buyer sending an LO	γ	0	$(1-\gamma)\cdot(1-p_{b\|s})$	$(1-\gamma)\cdot p_{b\|s}$
a buyer sending an MO	γ	0	$1-\gamma$	0
a seller sending an LO	$\gamma\cdot(1-p_{s\|b})$	$\gamma\cdot p_{s\|b}$	$1-\gamma$	0
a seller sending an MO	γ	0	$1-\gamma$	0

In Table 10, the probabilities of some transitions are zero: if a limit order book is empty at the end of t (if at t a market order took place), a market order at $t+1$ is not possible. If a limit order from the same type of agent is stored in a limit order book, the satisfaction of this limit order is not possible. In some of the cases, the probability of transition is equal to the share of the agent type active at $t+1$: if a limit order book is empty, the probability of posting a limit order equals the share of the next coming agent type. Or if an identical trader type comes during the next period to a non-empty limit order book, this agent has only a chance to post her own limit order. In other cases, transitions depend not only on the share of the next coming agent, but also on the (non-)execution probabilities conditional on the type of the next coming agent ($p_{i_t|i_{t+1}}$ for an execution and $1-p_{i_t|i_{t+1}}$ for a non-execution): if a limit order book is filled with an order of the opposite market side, the next coming agent decides either to satisfy this limit order or post her own limit order. The data in Table 10 leads to the following transition matrix $\mathbf{P}$:

$$\mathbf{P} = \begin{pmatrix} \gamma & 0 & (1-\gamma)\cdot(1-p_{b|s}) & (1-\gamma)\cdot p_{b|s} \\ \gamma & 0 & 1-\gamma & 0 \\ \gamma\cdot(1-p_{s|b}) & \gamma\cdot p_{s|b} & 1-\gamma & 0 \\ \gamma & 0 & 1-\gamma & 0 \end{pmatrix}.$$

[23]This is the type of the process where "the outcome of the given experiment can affect the outcome of the next experiment", see Grinstead and Snell (2003), p.405.

[24]Markov chain is characterized by "memorylessness": it has no memory of how the present state was reached, the next step depends exclusively on the present state.

This transition matrix is ergodic[25] and regular[26], therefore stationary probabilities can be determined as the fixed row vector $\boldsymbol{\varphi}$[27] computed as the left eigenvector of the transition matrix $\mathbf{P}$[28]: $\boldsymbol{\varphi}(\mathbf{P}-\mathbf{I}) = \mathbf{0}$[29]. Taking into account the probability relationships for different types of agents mentioned before and the following system of equations in the matrix form,

$$\begin{pmatrix}\varphi_b^{LO} & \varphi_b^{MO} & \varphi_s^{LO} & \varphi_s^{MO}\end{pmatrix}\begin{pmatrix} \gamma-1 & 0 & (1-\gamma)\cdot(1-p_{b|s}) & (1-\gamma)\cdot p_{b|s} \\ \gamma & -1 & 1-\gamma & 0 \\ \gamma\cdot(1-p_{s|b}) & \gamma\cdot p_{s|b} & -\gamma & 0 \\ \gamma & 0 & 1-\gamma & -1 \end{pmatrix}$$
$$= \begin{pmatrix}0 & 0 & 0 & 0\end{pmatrix},$$

six equations with four unknowns have to be solved:

$$\begin{aligned}
(\gamma-1)\varphi_b^{LO} + \gamma\varphi_b^{MO} + \gamma\cdot(1-p_{s|b})\varphi_s^{LO} + \gamma\varphi_s^{MO} &= 0\\
-\varphi_b^{MO} + \gamma\cdot p_{s|b}\varphi_s^{LO} &= 0\\
(1-\gamma)\cdot(1-p_{b|s})\varphi_b^{LO} + (1-\gamma)\varphi_b^{MO} - \gamma\varphi_s^{LO} + (1-\gamma)\varphi_s^{MO} &= 0\\
(1-\gamma)\cdot p_{b|s}\varphi_b^{LO} - \varphi_s^{MO} &= 0\\
\varphi_b^{LO} + \varphi_b^{MO} &= \gamma\\
\varphi_s^{LO} + \varphi_s^{MO} &= 1-\gamma.
\end{aligned}$$

This system of equations leads to the solution presented in Proposition 2.5.

Proposition 2.5. *For the fixed parameters (γ, σ, L), the stationary probabilities of equilibrium events are described by:*

$$\begin{aligned}
\varphi_b^{LO} &= \frac{\gamma(1-(1-\gamma)p_{s|b})}{1-\gamma(1-\gamma)p_{b|s}p_{s|b}},\\
\varphi_s^{LO} &= \frac{(1-\gamma)(1-\gamma p_{b|s})}{1-\gamma(1-\gamma)p_{b|s}p_{s|b}},\\
\varphi_b^{MO} &= \frac{\gamma(1-\gamma)p_{s|b}(1-\gamma p_{b|s})}{1-\gamma(1-\gamma)p_{b|s}p_{s|b}},\\
\varphi_s^{MO} &= \frac{\gamma(1-\gamma)p_{b|s}(1-(1-\gamma)p_{s|b})}{1-\gamma(1-\gamma)p_{b|s}p_{s|b}}.
\end{aligned}$$

Proof. It is more convenient to use the matrix form of the system of equations that has to be

[25] It is possible to move from a state to a state at least in a couple of steps, see Grinstead and Snell (2003), p.433.

[26] The second power of the matrix has only positive elements, see Grinstead and Snell (2003), p.433.

[27] This is the common row of the matrix $\mathbf{\Phi}$, a limiting matrix of $\mathbf{P}^n$. Moreover, this is the unique probability vector for the ergodic Markov chain, see Grinstead and Snell (2003), pp.434-435, p.438.

[28] See Grinstead and Snell (2003), p.437. Stationary probabilities are computed in the same manner in Colliard and Foucault (2012) and in Hoffmann (2014).

[29] $\mathbf{I}$ is an identity matrix.

solved:

$$
\begin{array}{c} (1)\\ (2)\\ (3)\\ (4)\\ (5)\\ (6) \end{array}
\left(\begin{array}{cccc|c}
\gamma-1 & \gamma & \gamma\cdot(1-p_{s|b}) & \gamma & 0\\
0 & -1 & \gamma\cdot p_{s|b} & 0 & 0\\
(1-\gamma)\cdot(1-p_{b|s}) & 1-\gamma & -\gamma & 1-\gamma & 0\\
(1-\gamma)\cdot p_{b|s} & 0 & 0 & -1 & 0\\
1 & 1 & 0 & 0 & \gamma\\
0 & 0 & 1 & 1 & 1-\gamma
\end{array}\right).
$$

One needs to perform a couple of modification steps to see that some equations are linear combinations of the others.

As the first step, the rows (1), (2), and (3) are summed up, and the result is written on the place of the row (3); the rest remains unchanged.

$$
\begin{array}{c} (1)\\ (2)\\ (3)\\ \cancel{(4)}\\ (5)\\ (6) \end{array}
\left(\begin{array}{cccc|c}
\gamma-1 & \gamma & \gamma\cdot(1-p_{s|b}) & \gamma & 0\\
0 & -1 & \gamma\cdot p_{s|b} & 0 & 0\\
(\gamma-1)\cdot p_{b|s} & 0 & 0 & 1 & 0\\
\cancel{(1-\gamma)\cdot p_{b|s}} & \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{0}\\
1 & 1 & 0 & 0 & \gamma\\
0 & 0 & 1 & 1 & 1-\gamma
\end{array}\right)
$$

Since the rows (3) and (4) are opposite, it is possible to neglect the row (4) and reduce the system of equations to five equations with four unknowns. As the second step, the following operation is performed: $\big((1)+(2)+(5)\big)/\gamma$; this result is written down in the row (1). Since the outcome of this operation gives absolutely the same result as $(5)+(6)$, the first row is the linear combination of the other two rows in the system, hence this row can be omitted.

$$
\begin{array}{c} \cancel{(1)}\\ (2)\\ (3)\\ \cancel{(4)}\\ (5)\\ (6) \end{array}
\left(\begin{array}{cccc|c}
\cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} & \cancel{1}\\
0 & -1 & \gamma\cdot p_{s|b} & 0 & 0\\
(\gamma-1)\cdot p_{b|s} & 0 & 0 & 1 & 0\\
\cancel{(1-\gamma)\cdot p_{b|s}} & \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{0}\\
1 & 1 & 0 & 0 & \gamma\\
0 & 0 & 1 & 1 & 1-\gamma
\end{array}\right)
$$

As a result, the system of equations is reduced to four equations with four unknowns:

$$
\left(\begin{array}{cccc|c}
0 & -1 & \gamma\cdot p_{s|b} & 0 & 0\\
(\gamma-1)\cdot p_{b|s} & 0 & 0 & 1 & 0\\
1 & 1 & 0 & 0 & \gamma\\
0 & 0 & 1 & 1 & 1-\gamma
\end{array}\right).
$$

This leads to the following four equations, the solution to which gives the probabilities of equilibrium events presented in the proposition:

$$\begin{aligned}
-\varphi_b^{MO} + \gamma \cdot p_{s|b}\varphi_s^{LO} &= 0 \\
(\gamma - 1) \cdot p_{b|s}\varphi_b^{LO} + \varphi_s^{MO} &= 0 \\
\varphi_b^{LO} + \varphi_b^{MO} &= \gamma \\
\varphi_s^{LO} + \varphi_s^{MO} &= 1 - \gamma.
\end{aligned}$$

□

Corollary 2.8. *For the fixed parameters ($\gamma = \frac{1}{2}, \sigma, L$), the stationary probabilities of equilibrium events are provided in Table 11.*

Table 11: Stationary probabilities of equilibrium events for ($\gamma = \frac{1}{2}, \sigma, L$) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4	Average
$p_{b\|s}$	$\frac{1}{4}(\frac{1}{2})$	$\frac{1}{4}(\frac{1}{2})$	$\frac{1}{2}(1)$	$\frac{1}{2}(1)$	$\frac{3}{8}(\frac{3}{4})$
$p_{s\|b}$	$\frac{1}{2}(1)$	$\frac{1}{4}(\frac{1}{2})$	$\frac{1}{2}(1)$	$\frac{1}{4}(\frac{1}{2})$	$\frac{3}{8}(\frac{3}{4})$
φ_b^{LO}	$\frac{12}{31}(\frac{2}{7})$	$\frac{4}{9}(\frac{2}{5})$	$\frac{2}{5}(\frac{1}{3})$	$\frac{14}{31}(\frac{3}{7})$	$\frac{587}{1395}(\frac{38}{105})$
φ_s^{LO}	$\frac{14}{31}(\frac{3}{7})$	$\frac{4}{9}(\frac{2}{5})$	$\frac{2}{5}(\frac{1}{3})$	$\frac{12}{31}(\frac{2}{7})$	$\frac{587}{1395}(\frac{38}{105})$
φ_b^{MO}	$\frac{7}{62}(\frac{3}{14})$	$\frac{1}{18}(\frac{1}{10})$	$\frac{1}{10}(\frac{1}{6})$	$\frac{3}{62}(\frac{1}{14})$	$\frac{221}{2790}(\frac{29}{420})$
φ_s^{MO}	$\frac{3}{62}(\frac{1}{14})$	$\frac{1}{18}(\frac{1}{10})$	$\frac{1}{10}(\frac{1}{6})$	$\frac{7}{62}(\frac{3}{14})$	$\frac{221}{2790}(\frac{29}{420})$

Proof. Follows directly from Proposition 2.5 and Table 8. The conditional probabilities are presented in Table 11. □

The probability of sending a limit order by an agent increases when she posts high fill-rate orders but the other side of the market prefers to use low fill-rate orders. A buyer (seller) has the highest probability to send a limit order in Eq.4 (Eq.1), where she uses a high fill-rate order and her counterparty uses a low fill-rate order; and the lowest probability is in Eq.1 (Eq.4), where she uses a low fill-rate order and her counterparty uses a high fill-rate order. The opposite relationship holds for the probability to send market orders: this probability increases, if an agent uses low fill-rate orders, and if the opposite side of the market prefers to use high fill-rate orders. A buyer (seller) would prefer Eq.1 (Eq.4), if she is impatient to send limit orders and wait until they are executed and prefers to use market orders. These results are intuitive, since the high fill-rate of the opposite side increases chances for an agent to take those orders, whereas if an agent posts low fill-rate orders, this enhances the relative probability of a market order as an equilibrium event even more. Neither Eq.1 nor Eq.4 are considered by Hoffmann (2014), but our results show that these two equilibria might be preferred by agents to symmetric-case equilibria, if agents want to have a higher probability to trade through market orders than wait until a limit order is satisfied.

2.5.3 Bargaining power and real expected profit from limit orders

As was discussed in the proof section to Corollary 2.6, the agents' bargaining power in all the four equilibria under the market state ($\gamma = \frac{1}{2}, \sigma = \frac{4}{5}, L = 1$) with an infinite number of participants equals $\frac{2}{5}L$ for both sides of the market. However, this value should be two times smaller if one assumes just two agents playing in the market, since the execution probabilities are halved. The same holds for the real expected profits from limit orders. Corollary 2.5 provides an analytical expression for the real expected profit, which is free from the imaginary part of the reservation value L; this part is not realized in a transaction, but serves as a personal expectation part.

Corollary 2.9. *For the fixed parameters* ($\gamma = \frac{1}{2}, \sigma = \frac{4}{5}, L = 1$)*, the imaginary and real expected profits from sending a limit order are provided in Table 12.*

Table 12: Imaginary and real expected profits from limit orders for ($\gamma = \frac{1}{2}, \sigma = \frac{4}{5}, L = 1$) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4	Average
V_b^{LO}	$\frac{1}{5}(\frac{2}{5})$	$\frac{1}{5}(\frac{2}{5})$	$\frac{1}{5}(\frac{2}{5})$	$\frac{1}{5}(\frac{2}{5})$	$\frac{1}{5}(\frac{2}{5})$
V_s^{LO}	$\frac{1}{5}(\frac{2}{5})$	$\frac{1}{5}(\frac{2}{5})$	$\frac{1}{5}(\frac{2}{5})$	$\frac{1}{5}(\frac{2}{5})$	$\frac{1}{5}(\frac{2}{5})$
$V_{b,\text{ real}}^{LO}$	$\frac{3}{40}(\frac{3}{20})$	$\frac{3}{40}(\frac{3}{20})$	$-\frac{1}{20}(-\frac{1}{10})$	$-\frac{1}{20}(-\frac{1}{10})$	$\frac{1}{80}(\frac{1}{40})$
$V_{s,\text{ real}}^{LO}$	$-\frac{1}{20}(-\frac{1}{10})$	$\frac{3}{40}(\frac{3}{20})$	$-\frac{1}{20}(-\frac{1}{10})$	$\frac{3}{40}(\frac{3}{20})$	$\frac{1}{80}(\frac{1}{40})$

Proof. Given the analytical expressions in Table 4 and Table 5, expected profits values under the parameters (γ, σ, L) are calculated. Moreover, profits are two times smaller with two market participants than with an infinite number of potential trading partners. □

The values of the imaginary and real expected profits discussed in Corollary 2.9 represent the profits from a limit order given that it is posted. However, a limit order is posted with the probability φ_i^{LO}: the average expected profit from posting a limit order in a multiperiod market game can be calculated as the product of the posting probability and the conditional profit:

$$V_{i,(\text{real}),\text{multi}}^{LO} = \varphi_i^{LO} \cdot V_{i,(\text{real})}^{LO}.$$

Corollary 2.10. *For the fixed parameters* ($\gamma = \frac{1}{2}, \sigma = \frac{4}{5}, L = 1$)*, the multiperiod imaginary and real expected profits from sending a limit order are provided in Table 13.*

Proof. Follows directly from Corollaries 2.8 and 2.9. □

As in Hoffmann (2014), both sides of the market are indifferent between the four equilibria if they base their decisions on the imaginary one-period expected profits from limit orders. However, once the multiperiod imaginary expected profit is considered, a buyer (seller) would prefer Eq.4 (Eq.1), which provides the highest multiperiod profitability from sending a limit

Table 13: Multiperiod imaginary and real expected profits for ($\gamma = \frac{1}{2}, \sigma = \frac{4}{5}, L = 1$) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4	Average
$V^{LO}_{b,multi}$	$\frac{12}{155}(\frac{4}{35})$	$\frac{4}{45}(\frac{4}{25})$	$\frac{2}{25}(\frac{2}{15})$	$\frac{14}{155}(\frac{6}{35})$	$\frac{587}{6975}(\frac{76}{525})$
$V^{LO}_{s,multi}$	$\frac{14}{155}(\frac{6}{35})$	$\frac{4}{45}(\frac{4}{25})$	$\frac{2}{25}(\frac{2}{15})$	$\frac{12}{155}(\frac{4}{35})$	$\frac{587}{6975}(\frac{76}{525})$
$V^{LO}_{b,\text{real},multi}$	$\frac{9}{310}(\frac{3}{70})$	$\frac{3}{90}(\frac{3}{50})$	$-\frac{1}{50}(-\frac{1}{30})$	$-\frac{7}{310}(-\frac{3}{70})$	$\frac{23}{4650}(\frac{1}{150})$
$V^{LO}_{s,\text{real},multi}$	$-\frac{7}{310}(-\frac{3}{70})$	$\frac{3}{90}(\frac{3}{50})$	$-\frac{1}{50}(-\frac{1}{30})$	$\frac{9}{310}(\frac{3}{70})$	$\frac{23}{4650}(\frac{1}{150})$

order; the low fill-rate equilibrium is preferred to the high fill-rate equilibrium.
If agents consider real expected profits from sending a limit order, they are no longer indifferent among the equilibria. The real profit is positive, if an agent uses low fill-rate orders and negative otherwise. Eq.1 and Eq.2 (Eq.2 and Eq.4) are equally likely preferred by buyers (sellers). Moreover, if the probability of sending a limit order is taken into account and the multiperiod real expected profits are considered, only Eq.2, which is the low fill-rate equilibrium, is preferred by both market sides.

2.5.4 Trading rate

The trading rate is defined as an unconditional probability to observe a transaction in a given period[30]; it is calculated as the sum of market order probabilities on both sides of the market:

$$TR = \varphi_b^{MO} + \varphi_s^{MO}.$$

Corollary 2.11. *For the fixed parameters ($\gamma = \frac{1}{2}, \sigma, L$), the trading rates are presented in Table 14.*

Table 14: Trading rates for ($\gamma = \frac{1}{2}, \sigma, L$) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4	Average
TR	$\frac{5}{31}(\frac{2}{7})$	$\frac{1}{9}(\frac{1}{5})$	$\frac{1}{5}(\frac{1}{3})$	$\frac{5}{31}(\frac{2}{7})$	$\frac{221}{1395}(\frac{29}{105})$

Proof. Follows directly from Corollary 2.8 and the formula above. □

The trading rate is a measure of the market quality and welfare, because it shows the probability to realize a profit from trade (i.e., how frequently profits from trade are realized). Four possible equilibria are not equal in terms of this metrics. The low fill-rate equilibrium has the smallest liquidity, the high fill-rate equilibrium has the highest one. Non-symmetric Eq.1 and Eq.4 are in between these two in terms of providing profits from trade.

[30]See Hoffmann (2014), p.161.

2.5.5 Risk of being picked-off

In the simplified model, all agents are assumed to be slow, i.e. they do not have an opportunity to change or cancel their limit orders once they are sent to a limit order book. In the real financial market, the risk of being picked-off exists, since agents do not monitor the market and their portfolio continuously. Agents who send a market order and use a limit order from the last step, can obtain "additional windfall profits on top of their outside option value"[31], if the price moves in the favorable direction from their perspective, but to the unfavorable direction from their trading partner's point of view. The risk of being picked-off exists only if high fill-rate orders are used, since only in this case the execution of an order in the unfavorable state could occur. If a high (low) bid (ask) is posted, but the fundamental value decreases (increases) during the next step, an agent might regret posting such a limit order, because the bid (ask) was too high (low). In case of a low fill-rate order, it is executed only in the same state the order is based on, which does not allow for the risk of being picked-off. The losses could have been avoided by a posting agent if there had been a chance of deleting or changing a limit order. The following formulae are used to find the risk of being picked-off for buyers and sellers:

$$\pi_b = \begin{cases} \frac{1}{2}\frac{p_{b|-\sigma}}{p_b} & \text{if } B = \hat{B}^{v+\sigma} \text{ (Eq.3 and Eq.4)} \\ 0 & \text{if } B = \hat{B}^{v-\sigma} \text{ (Eq.1 and Eq.2),} \end{cases} \quad \pi_s = \begin{cases} \frac{1}{2}\frac{p_{s|+\sigma}}{p_s} & \text{if } A = \hat{A}^{v-\sigma} \text{ (Eq.1 and Eq.3)} \\ 0 & \text{if } A = \hat{A}^{v+\sigma} \text{ (Eq.2 and Eq.4).} \end{cases}$$

Corollary 2.12. *For the fixed parameters ($\gamma = \frac{1}{2}, \sigma, L$), the risk of being picked-off is provided in Table 15.*

Table 15: Risk of being picked-off for ($\gamma = \frac{1}{2}, \sigma, L$) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4	Average
$p_{b\|-\sigma}$	$\frac{1}{4}(\frac{1}{2})$	$\frac{1}{4}(\frac{1}{2})$	$\frac{1}{4}(\frac{1}{2})$	$\frac{1}{4}(\frac{1}{2})$	$\frac{1}{4}(\frac{1}{2})$
$p_{s\|+\sigma}$	$\frac{1}{4}(\frac{1}{2})$	$\frac{1}{4}(\frac{1}{2})$	$\frac{1}{4}(\frac{1}{2})$	$\frac{1}{4}(\frac{1}{2})$	$\frac{1}{4}(\frac{1}{2})$
π_b	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
π_s	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{4}$

Proof. Follows directly from the formulae above and Table 8. □

As Table 15 shows, in Eq.2 neither of the market sides is subject to the risk of being picked-off, since both of them base their orders on the low fill-rate rules. In Eq.3, where both sides post their orders based on the high fill-rate strategies, they both have a symmetric probability of being picked-off. In mixed equilibria, only one side is subject to this risk, while the other side is safe in this respect. This makes Eq.1 and Eq.4 asymmetric in terms of the risk of being picked-off.

[31] See Hoffmann (2014), p.161.

2.5.6 Costs of immediacy

Following Foucault (1999) and Hoffmann (2014), the expected trading costs (costs of immediacy) $E(\tau_i)$ are defined as "the signed difference between the transaction price and the asset fundamental value". Equilibrium orders may be further away or closer to the fundamental value. The costs of immediacy depend on the direction of the fundamental value change ε as well as on the type of a posting agent: all these parameters determine the distance from the fundamental value. Given that $\tau^{\varepsilon}_{j,i}$ are the costs of immediacy of an i-type agent whose order is executed by a j-type agent conditional on the last move of the fundamental value ε, and $\omega^{\varepsilon}_{j,i}$ is the corresponding probability of this order appearance and execution in the equilibrium, the general formula for the expected costs of immediacy can be presented in the following way:

$$E(\tau_i) = \sum_{j,\varepsilon} \frac{\omega^{\varepsilon}_{j,i}}{\sum\limits_{j,\varepsilon} \omega^{\varepsilon}_{j,i}} \tau^{\varepsilon}_{j,i}.$$

Since an order can be satisfied only by a trader from the opposite market side, the general formula can be simplified to:

$$E(\tau_b) = \sum_{\varepsilon} \frac{\omega^{\varepsilon}_{s,b}}{\sum\limits_{\varepsilon} \omega^{\varepsilon}_{s,b}} \tau^{\varepsilon}_{s,b} = \frac{\omega^{-\sigma}_{s,b}\tau^{-\sigma}_{s,b} + \omega^{+\sigma}_{s,b}\tau^{+\sigma}_{s,b}}{\omega^{-\sigma}_{s,b} + \omega^{+\sigma}_{s,b}},$$

$$E(\tau_s) = \sum_{\varepsilon} \frac{\omega^{\varepsilon}_{b,s}}{\sum\limits_{\varepsilon} \omega^{\varepsilon}_{b,s}} \tau^{\varepsilon}_{b,s} = \frac{\omega^{-\sigma}_{b,s}\tau^{-\sigma}_{b,s} + \omega^{+\sigma}_{b,s}\tau^{+\sigma}_{b,s}}{\omega^{-\sigma}_{b,s} + \omega^{+\sigma}_{b,s}}.$$

Corollary 2.13. *For the fixed parameters ($\gamma = \frac{1}{2}, \sigma = \frac{4}{5}, L = 1$), the costs of immediacy are provided in Table 16.*

Table 16: Costs of immediacy for ($\gamma = \frac{1}{2}, \sigma = \frac{4}{5}, L = 1$) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4	Average
$E(\tau_b)$	$-\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$
$E(\tau_s)$	$\frac{3}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$

Proof. A seller's profit is calculated as the bid price which is received by a seller when a market order is executed minus the current fundamental value. A buyer receives an asset with the current fundamental value, but has to pay an ask price when her market order is executed. The opposite values have to be considered if one is interested in the costs of immediacy[32]:

$$\tau_{b,s} = v_t - B_t, \qquad \tau_{s,b} = A_t - v_t.$$

[32] In the following formulae, $v_t = v_{t-1} + \varepsilon$, and B_t as well as A_t are based on the fundamental value from the previous period, i.e. on v_{t-1}.

The probability of a limit order to be sent and executed, conditional on a certain fundamental value change, $\omega^{\varepsilon}_{j,i}$, is the product of two probabilities: the probability of a limit order to be posted at $t-1$, φ^{LO}_{j}, and the probability that this order is executed at t, conditional on the fundamental value development, $p_{j|\varepsilon}$:

$$\omega^{\varepsilon}_{j,i} = \varphi^{LO}_{j} \cdot p_{j|\varepsilon}.$$

Table 17 provides the intermediate values for computing the costs of immediacy as well as the final results for the model's set-up with infinitely many traders:

Table 17: Costs of immediacy for $(\gamma = \frac{1}{2}, \sigma, L)$ in the simplified analytical model with infinitely many traders

	Eq.1	Eq.2	Eq.3	Eq.4
$\tau^{-\sigma}_{b,s}$	$\frac{1}{7}L + \frac{4}{7}\sigma$	$\frac{3}{5}L$	$\frac{1}{3}L - \frac{5}{3}\sigma$	$\frac{5}{7}L - \frac{15}{7}\sigma$
$\tau^{+\sigma}_{b,s}$	$\frac{1}{7}L + \frac{18}{7}\sigma$	$\frac{3}{5}L + 2\sigma$	$\frac{1}{3}L + \frac{1}{3}\sigma$	$\frac{5}{7}L - \frac{1}{7}\sigma$
$\tau^{-\sigma}_{s,b}$	$\frac{5}{7}L - \frac{1}{7}\sigma$	$\frac{3}{5}L + 2\sigma$	$\frac{1}{3}L + \frac{1}{3}\sigma$	$\frac{1}{7}L + \frac{18}{7}\sigma$
$\tau^{+\sigma}_{s,b}$	$\frac{5}{7}L - \frac{15}{7}\sigma$	$\frac{3}{5}L$	$\frac{1}{3}L - \frac{5}{3}\sigma$	$\frac{1}{7}L + \frac{4}{7}\sigma$
$\omega^{-\sigma}_{b,s}$	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{3}{14}$
$\omega^{+\sigma}_{b,s}$	0	0	$\frac{1}{6}$	$\frac{3}{14}$
$\omega^{-\sigma}_{s,b}$	$\frac{3}{14}$	0	$\frac{1}{6}$	0
$\omega^{+\sigma}_{s,b}$	$\frac{3}{14}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
$E(\tau_b)$	$\frac{5}{7}L - \frac{8}{7}\sigma$	$\frac{3}{5}L$	$\frac{1}{3}L - \frac{2}{3}\sigma$	$\frac{1}{7}L + \frac{4}{7}\sigma$
$E(\tau_s)$	$\frac{1}{7}L + \frac{4}{7}\sigma$	$\frac{3}{5}L$	$\frac{1}{3}L - \frac{2}{3}\sigma$	$\frac{5}{7}L - \frac{8}{7}\sigma$

For the fixed parameters $(\gamma = \frac{1}{2}, \sigma = \frac{4}{5}, L = 1)$, the intermediate values and final results both for the model with an infinite number of agents and for two agents are presented in Table 18. As the final step, the intermediate values have to be plugged into the general formula for calculating the costs of immediacy. $\square$

In Eq.2, both sides face positive trading costs, since they base their orders on the low fill-rate rules. In Eq.3, where both sides post their orders based on the high fill-rate rules, they have symmetric negative costs of immediacy (profits of immediacy). In mixed equilibria, only one side experiences positive costs of immediacy, while the other has negative costs. This makes Eq.1 and Eq.4 asymmetric in terms of the costs of immediacy. For an agent it is better to post high fill-rate orders, since this leads to negative costs of immediacy.

Table 18: Individual costs of immediacy for ($\gamma = \frac{1}{2}, \sigma = \frac{4}{5}, L = 1$) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4
$\tau_{b,s}^{-\sigma}$	$\frac{3}{5}$	$\frac{3}{5}$	-1	-1
$\tau_{b,s}^{+\sigma}$	$\frac{11}{5}$	$\frac{11}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
$\tau_{s,b}^{-\sigma}$	$\frac{3}{5}$	$\frac{11}{5}$	$\frac{3}{5}$	$\frac{11}{5}$
$\tau_{s,b}^{+\sigma}$	-1	$\frac{3}{5}$	-1	$\frac{3}{5}$
$\omega_{b,s}^{-\sigma}$	$\frac{3}{31}(\frac{1}{7})$	$\frac{1}{9}(\frac{1}{5})$	$\frac{1}{10}(\frac{1}{6})$	$\frac{7}{62}(\frac{3}{14})$
$\omega_{b,s}^{+\sigma}$	$0(0)$	$0(0)$	$\frac{1}{10}(\frac{1}{6})$	$\frac{7}{62}(\frac{3}{14})$
$\omega_{s,b}^{-\sigma}$	$\frac{7}{62}(\frac{3}{14})$	$0(0)$	$\frac{1}{10}(\frac{1}{6})$	$0(0)$
$\omega_{s,b}^{+\sigma}$	$\frac{7}{62}(\frac{3}{14})$	$\frac{1}{9}(\frac{1}{5})$	$\frac{1}{10}(\frac{1}{6})$	$\frac{3}{31}(\frac{1}{7})$

2.5.7 Maker-taker ratio

To analyze whether the two types of agents are symmetric in terms of their market-making functions (sending limit orders to a limit order book) or market-taking functions (taking limit orders from a limit order book, or sending market orders), the maker-taker ratio, defined as "the probability of trading via a limit order divided by the probability of trading via a market order"[33], is determined:

$$MT_i = \frac{\omega_{i,j}}{\omega_{j,i}}.$$

Corollary 2.14. *For the fixed parameters ($\gamma = \frac{1}{2}, \sigma, L$), the maker-taker ratio is presented in Table 19.*

Table 19: Maker-taker ratio for ($\gamma = \frac{1}{2}, \sigma, L$) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4	Average
MT_b	$\frac{3}{7}(\frac{1}{3})$	$1(1)$	$1(1)$	$\frac{7}{3}(3)$	$\frac{25}{21}(\frac{4}{3})$
MT_s	$\frac{7}{3}(3)$	$1(1)$	$1(1)$	$\frac{3}{7}(\frac{1}{3})$	$\frac{25}{21}(\frac{4}{3})$

Proof. Follows directly from Table 18 and the above formula, but it can be found alternatively through the probabilities of equilibrium events and the conditional probabilities of execution:

$$\omega_{i,j} = \frac{1}{2}\omega_{i,j}^{-\sigma} + \frac{1}{2}\omega_{i,j}^{+\sigma} = \frac{1}{2}\varphi_i^{LO} \cdot p_{i|-\sigma} + \frac{1}{2}\varphi_i^{LO} \cdot p_{i|+\sigma} = \frac{1}{2}\varphi_i^{LO}(p_{i|-\sigma} + p_{i|+\sigma}) = \varphi_i^{LO} \cdot p_i.$$

□

Eq.2 and Eq.3 are symmetric with respect to making-taking functions of both sides of the market: buyers and sellers both provide as much liquidity as they take. Eq.1 and Eq.4 are

[33]See Hoffmann (2014), p.162.

asymmetric in this respect: in Eq.1, a buyer takes liquidity from the market, while a seller provides more limit orders than she takes, and the opposite is true in Eq.4. For this model set-up, the maker-taker ratio of a trader type is the multiplicative inverse of the other type's maker-taker ratio. If both market sides use the same fill-rate orders, they are also identical in terms of their degree of making the market. Using low fill-rate orders while the opposite side uses high fill-rate orders makes the first side a relative market-taker.

2.5.8 Pricing error

Even though there is no information asymmetry among the agents in the simplified model and no uncertainty about the current fundamental value, there are two other sources of potential price inefficiency. First, there is an uncertainty about the future fundamental value: the agents form limit orders under the condition of the lack of information about the direction of values change. Information arrival makes the posted limit orders stale, since the agents do not have an opportunity to change or cancel them. Second, the agents posting limit orders have a market power and, in their desire to get a profit, they move orders away from the fundamentals (when posting orders, traders consider their personal value element $\pm L$). It is important to verify whether prices are efficient and whether they are equally (in-)efficient on both sides of the market. For this purpose, the pricing error, defined as "the expectation of the absolute difference between the true asset value and the best available quote"[34], is used:

$$PE_b = E[|v - B^m|], \qquad PE_s = E[|A^m - v|].$$

In these formulae, both the fundamental value and an existing limit order are related to the same period. Keeping in mind that an existing limit order is based on the expected fundamental value, while v is the realized fundamental value, the following extended formulae for the pricing error can be achieved:

$$PE_b = \frac{1}{2}|v - \sigma - B^m| + \frac{1}{2}|v + \sigma - B^m|, \quad PE_s = \frac{1}{2}|A^m - (v - \sigma)| + \frac{1}{2}|A^m - (v + \sigma)|.$$

The pricing error of both market sides can be combined into an average measure that shows the price (in-)efficiency of the whole market:

$$PE = \frac{\varphi_b^{LO}}{\varphi_b^{LO} + \varphi_s^{LO}} PE_b + \frac{\varphi_s^{LO}}{\varphi_b^{LO} + \varphi_s^{LO}} PE_s.$$

Corollary 2.15. *For the fixed parameters ($\gamma = \frac{1}{2}, \sigma = \frac{4}{5}L, L = 1$), the equilibrium pricing errors are presented in Table 20.*

Proof. The pricing errors are independent of the probabilities of execution and probabilities of equilibrium events and are therefore the same for both model set-ups. For the aggregate

[34] See Hoffmann (2014), p.163.

Table 20: Pricing error for ($\gamma = \frac{1}{2}, \sigma = \frac{4}{5}L, L = 1$) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4	Average
PE_b	$\frac{7}{5}$	$\frac{7}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{11}{10}$
PE_s	$\frac{4}{5}$	$\frac{7}{5}$	$\frac{4}{5}$	$\frac{7}{5}$	$\frac{11}{10}$
PE	$\frac{14}{13}(\frac{26}{25})$	$\frac{7}{5}(\frac{7}{5})$	$\frac{4}{5}(\frac{4}{5})$	$\frac{14}{13}(\frac{26}{25})$	$\frac{283}{260}(\frac{107}{100})$

pricing error of both market sides, the respective probabilities of equilibrium events have to be applied.
For the parameters ($\gamma = \frac{1}{2}, \sigma, L$), the pricing error for the model with infinitely many market participants is presented in Table 21. □

Table 21: Pricing error for ($\gamma = \frac{1}{2}, \sigma, L$) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4
PE_b	$\frac{1}{7}L + \frac{11}{7}\sigma$	$\frac{3}{5}L + \sigma$	$\begin{cases}\sigma, & \text{if } \sigma > \frac{1}{5}L \\ \frac{1}{3}L - \frac{2}{3}\sigma, & \text{if } \sigma \leq \frac{1}{5}L\end{cases}$	$\begin{cases}\sigma, & \text{if } \sigma > \frac{1}{3}L \\ \frac{5}{7}L - \frac{8}{7}\sigma, & \text{if } \sigma \leq \frac{1}{3}L\end{cases}$
PE_s	$\begin{cases}\sigma, & \text{if } \sigma > \frac{1}{3}L \\ \frac{5}{7}L - \frac{8}{7}\sigma, & \text{if } \sigma \leq \frac{1}{3}L\end{cases}$	$\frac{3}{5}L + \sigma$	$\begin{cases}\sigma, & \text{if } \sigma > \frac{1}{5}L \\ \frac{1}{3}L - \frac{2}{3}\sigma, & \text{if } \sigma \leq \frac{1}{5}L\end{cases}$	$\frac{1}{7}L + \frac{11}{7}\sigma$
PE	$\begin{cases}\frac{2}{35}L + \frac{43}{35}\sigma, & \text{if } \sigma > \frac{1}{3}L \\ \frac{17}{35}L - \frac{2}{35}\sigma, & \text{if } \sigma \leq \frac{1}{3}L\end{cases}$	$\frac{3}{5}L + \sigma$	$\begin{cases}\sigma, & \text{if } \sigma > \frac{1}{5}L \\ \frac{1}{3}L - \frac{2}{3}\sigma, & \text{if } \sigma \leq \frac{1}{5}L\end{cases}$	$\begin{cases}\frac{2}{35}L + \frac{43}{35}\sigma, & \text{if } \sigma > \frac{1}{3}L \\ \frac{17}{35}L - \frac{2}{35}\sigma, & \text{if } \sigma \leq \frac{1}{3}L\end{cases}$

As follows from Table 21, Eq.2 and Eq.3 are symmetric with respect to the pricing error on both sides of the market: buyers and sellers are both subject to the same pricing error. Eq.1 and Eq.4 are asymmetric: in Eq.1, a buyer's pricing error is greater than that of the seller, and the opposite is true in Eq.4.
The low fill-rate equilibrium (Eq.2) has the highest pricing error, while the high fill-rate equilibrium (Eq.3) has the smallest one. From the perspective of the whole market, the high fill-rate equilibrium is better than mixed equilibria, and they in their turn are better than the low fill-rate equilibrium. An individual agent would choose an order with a higher fill-rate, since it minimizes the pricing error for her.

2.5.9 Welfare

Since each agent has an opportunity to post a limit order and a market order, an agent's welfare is the weighted average of expected profits of both of these options:

$$W_i = \frac{\varphi_i^{LO}}{\varphi_i^{LO} + \varphi_i^{MO}} V_i^{LO} + \frac{\varphi_i^{MO}}{\varphi_i^{LO} + \varphi_i^{MO}} V_i^{MO}, \tag{2.3}$$

where profit from a market order can be found as the difference between the transaction price and the subjective value. Since the difference between the fundamental value and transaction price corresponds to the trading costs, and the subjective value is different from the fundamental value by L, the profit from a market order is calculated as the difference between L and the transaction costs:

$$V_i^{MO} = L - E(\tau_i).$$

The aggregate market welfare is computed as the welfares of each agent type multiplied by their respective shares. An agent gains from trade only if a transaction happens (in itself, sending a limit order does not lead to any gain, if this order is not executed in the next step), and market orders happen with the probability equal to the trading rate. If a transaction happens, the aggregate gain from trade of both market sides is equal to $2L$. Therefore, there is an alternative way to determine the market welfare:

$$W = \gamma W_b + (1-\gamma)W_s = 2L \cdot TR.$$

Corollary 2.16. *For the fixed parameters ($\gamma = \frac{1}{2}, \sigma = \frac{4}{5}L, L = 1$), the market welfare is presented in Table 22.*

Table 22: Welfare for ($\gamma = \frac{1}{2}, \sigma = \frac{4}{5}L, L = 1$) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4	Average
W_b	$\frac{66}{155}(\frac{26}{35})$	$\frac{2}{9}(\frac{2}{5})$	$\frac{2}{5}(\frac{2}{3})$	$\frac{34}{155}(\frac{2}{5})$	$\frac{442}{1395}(\frac{58}{105})$
W_s	$\frac{34}{155}(\frac{2}{5})$	$\frac{2}{9}(\frac{2}{5})$	$\frac{2}{5}(\frac{2}{3})$	$\frac{66}{155}(\frac{26}{35})$	$\frac{442}{1395}(\frac{58}{105})$
W	$\frac{10}{31}(\frac{4}{7})$	$\frac{2}{9}(\frac{2}{5})$	$\frac{2}{5}(\frac{2}{3})$	$\frac{10}{31}(\frac{4}{7})$	$\frac{442}{1395}(\frac{58}{105})$

Proof. The values in Table 22 follow from the formulae above, the respective probabilities of equilibrium events from Corollary 2.8 and the costs of immediacy from Corollary 2.13.
For the model with an infinite number of market players and the fixed parameters ($\gamma = \frac{1}{2}, \sigma, L$), the analytical form of welfare is presented in Table 23.

Table 23: Welfare for ($\gamma = \frac{1}{2}, \sigma, L$) in the simplified analytical model

	Eq.1	Eq.2	Eq.3	Eq.4
W_b	$\frac{2}{7}L + \frac{4}{7}\sigma$	$\frac{2}{5}L$	$\frac{2}{3}L$	$\frac{6}{7}L - \frac{4}{7}\sigma$
W_s	$\frac{6}{7}L - \frac{4}{7}\sigma$	$\frac{2}{5}L$	$\frac{2}{3}L$	$\frac{2}{7}L + \frac{4}{7}\sigma$
W	$\frac{4}{7}L$	$\frac{2}{5}L$	$\frac{2}{3}L$	$\frac{4}{7}L$

To verify if the second part of the relationship, $W = 2L \cdot TR$, is true, the aggregate gains of the market, W, are compared with the doubled trading rate from Corollary 2.11. □

Symmetric cases discovered in Hoffmann (2014) are also symmetric in terms of the welfare of both market sides. The low fill-rate equilibrium brings about the smallest market welfare, while the high fill-rate equilibrium produces the highest market welfare. An individual agent receives the highest (lowest) welfare in asymmetric equilibria where she posts a low (high) fill-rate order but the opposite side uses high (low) fill-rate orders. Therefore, from the perspective of individual investors, mixed equilibria are better, but only those that involve their own low fill-rate orders. If a choice is to be made from the market perspective as a whole, the best option would be the high fill-rate equilibrium. Mixed strategies yield a moderate result only, while the low fill-rate equilibrium leads to the worst result.

2.5.10 Decisions on equilibrium order-setting rules based on market metrics

The four possible equilibria in the simplified model with STs, given the balanced shares of buyers and sellers, are symmetric only based on the one-period imaginary expected profit from sending a limit order. In terms of the other market metrics, the equilibria are quite divergent. The low fill-rate equilibrium provides no risk of being picked-off, it maximizes one-step and multiperiod real profits from sending a limit order for both sides of the market simultaneously. The high fill-rate equilibrium provides greater chances of execution, minimizes trading costs and pricing errors for both sides of the market simultaneously. Moreover, this equilibrium is the most optimal in terms of such aggregate market measures as trading rate, pricing error, and the total aggregate market welfare.

If an agent posts a low fill-rate order in the mixed equilibrium, this maximizes her probability of transaction through market orders, her one-period real expected profit from limit orders and her welfare, as well as minimizes her risk of being picked-off and trading costs. The same equilibrium maximizes the execution probability for the opposite side, the probability of sending a limit order and the multiperiod imaginary profit from limit orders, while minimizing the pricing error.

2.6 Summary

In this chapter, we designed a simplified infinite horizon dynamic limit order book model with slow traders following assumptions and notations by Hoffmann (2014). It was revealed that an agent makes transactions on the respective market side only, while transactions on the opposite market side are non-optimal either from her own perspective or from the perspective of her potential trading partner. Furthermore, we performed an analysis how the market sentiment influences equilibria. The numerous market side posts low fill-rate orders, while the high volatility on the market keeps the fill-rate of orders low, as the agents try to save themselves from the increased adverse selection. The equilibria are asymmetric with respect to the expected profit from sending a limit order: the scarce market side executes a higher bargaining power. The model is based on maximizing the imaginary expected profit from sending a limit order, while the real expected profit (free from the never-realized element L of the personal value) can become negative.

Following an implicit assumption about the neutrality of a market sentiment, popular in the research literature, we verified whether all possible equilibria for the equal shares of the market sides are truly symmetric. All of them turned out to be symmetric (and identical in the intersection point) in terms of the imaginary profit from sending a limit order. However, the two additional equilibria were mixed in terms of fill-rate. All the four possible equilibria were tested and the market metrics for the state of the market correspondent to the intersection point were computed. One of the essential outcomes of the analysis is that the equilibria are symmetric in terms of the fill-rates and imaginary profits are also symmetric in terms of all the other analyzed market metrics. However, the two mixed equilibria are asymmetric, even though they also yield symmetric imaginary profits from sending a limit order.

Additionally, we provided the values of the market metrics for the alternative market set-up with only two trading partners. An important remark at this point is that the basic set-up, consistent with the literature, assumes an infinite number of the market participants. With the help of this adjusted assumption, it became possible to harmonize the analytical model with the simple agent-based model discussed later in Chapter 6. The target market measures of the modified set-up will be referred to while discussing the convergence of the analytical model and its ABM alternative.

The choice of the equilibrium depends on the perspective (individual investor or the whole economy), as well as on a market metrics assumed to be critical for this choice. The high fill-rate equilibrium is the most optimal based on the aggregate trading rate, welfare, and trading error, but it brings about a high risk of being picked-off. Therefore, mixed equilibria or the low fill-rate equilibrium might be preferred by individual agents.

3 Analytical Model with High-Frequency Traders: Hoffmann (2014) Revisited

This chapter revisits the model by Hoffmann (2014). We add further details to the discussion of the model. We consider both sides of the market while determining equilibrium conditions[35]. Moreover, the equilibrium border lines and the intersection point will be studied in greater detail. We present the market measures graphically, which allows for additional model insights.

3.1 The model set-up

Hoffmann (2014) introduced a fast trader (FT) to the model by Foucault (1999). The main difference of an FT from a slow trader (ST) is that an FT can revise her limit orders after news arrival (i.e., after ε_{t+1} becomes known) but before the next ST comes to the marketplace. An FT has no advantage if the next agent is another FT: in this case, she does not have a chance to revise an order. Therefore, she still experiences a risk of being picked-off. In this situation, it is optimal for an FT to post orders aimed at FTs as the next coming agents: if an ST comes, an FT adjusts her order to the level acceptable for an ST. The order revision by an FT, even if it happens only before an ST, cannot be a disadvantage: the whole model is built on the assumption that the expected profit from posting a limit order by an FT (an FT's bargaining power) is not lower than that by an ST, $V_{FT}^{LO} \geq V_{ST}^{LO}$. An FT's speed (and therefore informational) advantage and her higher outside option creates inefficiency: an ST faces a trade-off between a higher probability of execution and a higher spread, if a limit order is executed.

The temporal flow of market events is presented in Figure 6[36].

FTs' cutoff prices are different from those of STs, since FTs have a higher value of the outside option:

$$\hat{B}_{FT}^{v+\varepsilon} = (v+\varepsilon-L) + V_{s,FT}^{LO} \geq \hat{B}_{ST}^{v+\varepsilon} = (v+\varepsilon-L) + V_{s,ST}^{LO},$$
$$\hat{A}_{FT}^{v+\varepsilon} = (v+\varepsilon+L) - V_{b,FT}^{LO} \leq \hat{A}_{ST}^{v+\varepsilon} = (v+\varepsilon+L) - V_{b,ST}^{LO}.$$

The model assumes equal shares of the market sides: as was shown in Section 2.3, the bargaining powers of the market sides under this assumption are symmetric, i.e. $V_{b,k}^{LO} = V_{s,k}^{LO}$[37]. However, not all of the possible equilibria are symmetric from the other perspectives either. Asymmetric equilibria appear only in one specific market state. Hoffmann (2014) presents an analysis of one market side only, claiming that the opposite side would experience identical results. The aim of our study is to check whether asymmetric equilibria are possible in this model by investigating the equilibrium conditions for both market sides.

[35] In Hoffmann (2014), they are assumed to be symmetric, and therefore only one market side is shown.
[36] This temporal flow is a modified version of the timing of events from Hoffmann (2014).
[37] Here, k =ST or k =FT.

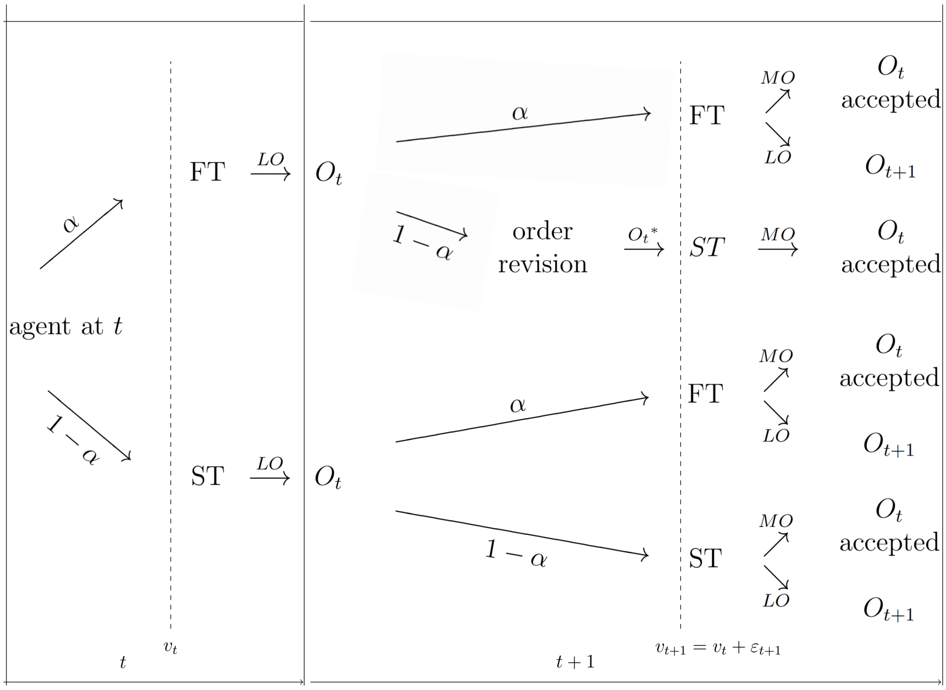


Figure 6: Temporal flow for the model with FTs

An acting agent strives to maximize her profit from sending a limit order, which defines equilibrium order-setting rules. The objective functions on the part of a slow buyer and a slow seller are defined as follows:

$$V^{LO}_{b,ST} = \max_{B_{ST}} [p(B_{ST})\big((v+\varepsilon+L) - B_{ST}\big)], \qquad V^{LO}_{s,ST} = \max_{A_{ST}} [p(A_{ST})\big(A_{ST} - (v+\varepsilon-L)\big)].$$

Since an FT can revise her order, if an ST comes next, her objective function differentiates between two cases: (i) if the next coming agent is an FT, which happens with the probability α, no revision of the initially posted order O_{FT} is possible, (ii) if an ST appears next, with the probability $1-\alpha$, an agent can revise her order based on the realized fundamental value change: $O^{+\sigma}_{FT}$ or $O^{-\sigma}_{FT}$.

$$\begin{aligned}
V^{LO}_{b,FT} = \max_{B_{FT}, B^{+\sigma}_{FT}, B^{-\sigma}_{FT}} &\big[\alpha q_{|FT}(B_{FT})\big((v+\varepsilon+L) - B_{FT}\big) \\
&+ \frac{1-\alpha}{2} q_{|ST,+\sigma}(B^{+\sigma}_{FT})\big((v+\sigma+L) - B^{+\sigma}_{FT}\big) \\
&+ \frac{1-\alpha}{2} q_{|ST,-\sigma}(B^{-\sigma}_{FT})\big((v-\sigma+L) - B^{-\sigma}_{FT}\big)\big], \\
V^{LO}_{s,FT} = \max_{A_{FT}, A^{+\sigma}_{FT}, A^{-\sigma}_{FT}} &\big[\alpha q_{|FT}(A_{FT})\big(A_{FT} - (v+\varepsilon-L)\big) \\
&+ \frac{1-\alpha}{2} q_{|ST,+\sigma}(A^{+\sigma}_{FT})\big(A^{+\sigma}_{FT} - (v+\sigma-L)\big) \\
&+ \frac{1-\alpha}{2} q_{|ST,-\sigma}(A^{-\sigma}_{FT})\big(A^{-\sigma}_{FT} - (v-\sigma-L)\big)\big],
\end{aligned}$$

where $q(O_{FT})$ is the execution probability of O_{FT} posted by an FT. As an FT forms the values $O_{FT}^{+\sigma}$ and $O_{FT}^{-\sigma}$ under certainty, i.e. knowing that the next agent is an ST and the fundamental value change, her maximization problem is limited to the choice of an order in case (i):

$$V_{b,FT}^{LO} = \max_{B_{FT}} \big[\alpha q_{|FT}(B_{FT})\big((v+\varepsilon+L) - B_{FT}\big)\big],$$
$$V_{s,FT}^{LO} = \max_{A_{FT}} \big[\alpha q_{|FT}(A_{FT})\big(A_{FT} - (v+\varepsilon-L)\big)\big].$$

The equilibrium strategies should take into account four inequality conditions: for each side of the market and for each agent type:

$$\begin{aligned} E(V_{b,ST}^{LO}|B_k) &\geq E(V_{b,ST}^{LO}|B_{h\neq k}) \\ E(V_{s,ST}^{LO}|A_k) &\geq E(V_{s,ST}^{LO}|A_{h\neq k}) \\ E(V_{b,FT}^{LO}|B_k) &\geq E(V_{b,FT}^{LO}|B_{h\neq k}) \\ E(V_{s,FT}^{LO}|A_k) &\geq E(V_{s,FT}^{LO}|A_{h\neq k}). \end{aligned} \tag{3.1}$$

3.2 Orders and probabilities of their execution

The universe of possible order values for a slow buyer includes $(\hat{B}_{ST}^{v-\sigma}, \hat{B}_{ST}^{v+\sigma}, \hat{B}_{FT}^{v-\sigma}, \hat{B}_{FT}^{v+\sigma})$, whereas a fast buyer has to decide between two possible cutoff prices $(\hat{B}_{FT}^{v-\sigma}, \hat{B}_{FT}^{v+\sigma})$. For a seller this choice is similar: for a slow seller it is $(\hat{A}_{ST}^{v-\sigma}, \hat{A}_{ST}^{v+\sigma}, \hat{A}_{FT}^{v-\sigma}, \hat{A}_{FT}^{v+\sigma})$ and for a fast seller it is $(\hat{A}_{FT}^{v-\sigma}, \hat{A}_{FT}^{v+\sigma})$. Hoffmann (2014) proved that for the sell cutoff prices it holds that $\hat{B}_{ST}^{v-\sigma} \leq \hat{B}_{FT}^{v-\sigma} \leq \hat{B}_{ST}^{v+\sigma} \leq \hat{B}_{FT}^{v+\sigma}$, while the buy cutoff prices should follow the symmetric inequality of: $\hat{A}_{FT}^{v-\sigma} \leq \hat{A}_{ST}^{v-\sigma} \leq \hat{A}_{FT}^{v+\sigma} \leq \hat{A}_{ST}^{v+\sigma}$. The smaller (higher) the bid (ask) is, the smaller is its execution probability.

Table 24 summarizes the states under which a certain order is executed and provides the probabilities of execution for each possible order value. Similar to Hoffmann (2014), this research introduces an infinite number of the market participants, and an order can be satisfied only if the next coming agent is on the opposite market side. Table 24 shows that the execution probability is a step function: for a certain range of order values the probability stays unchanged, while the profit can be improved if the minimum bid (maximum ask) is chosen. This explains why the choice of the other side cutoff prices as possible order values is optimal.

The equilibrium conditions can be determined by solving Inequalities 3.1 simultaneously for a certain k.

3.3 Equilibrium conditions

Hoffmann (2014) presents equilibrium orders for one side of the market only, claiming that the two sides are symmetric. The analysis of the simplified model with STs in Chapter 2 showed that even if the shares and bargaining powers are symmetric, in a certain market state some

Table 24: Execution probabilities for all the possible order values in the model with FTs

Order	Next agent, value state				Probabilities	
	$ST, -\sigma$	$ST, +\sigma$	$FT, -\sigma$	$FT, +\sigma$	$p(O)$	$q_{\vert FT}(O)$
$B < \hat{B}^{v-\sigma}_{ST}$	$-$	$-$	$-$	$-$	0	0
$\hat{B}^{v-\sigma}_{ST} \leq B < \hat{B}^{v-\sigma}_{FT}$	$+$	$-$	$-$	$-$	$\frac{1-\alpha}{4}$	0
$\hat{B}^{v-\sigma}_{FT} \leq B < \hat{B}^{v+\sigma}_{ST}$	$+$	$-$	$+$	$-$	$\frac{1}{4}$	$\frac{1}{4}$
$\hat{B}^{v+\sigma}_{ST} \leq B < \hat{B}^{v+\sigma}_{FT}$	$+$	$+$	$+$	$-$	$\frac{2-\alpha}{4}$	$\frac{1}{4}$
$B \geq \hat{B}^{v+\sigma}_{FT}$	$+$	$+$	$+$	$+$	$\frac{1}{2}$	$\frac{1}{2}$
$A > \hat{A}^{v+\sigma}_{ST}$	$-$	$-$	$-$	$-$	0	0
$\hat{A}^{v+\sigma}_{FT} < A \leq \hat{A}^{v+\sigma}_{ST}$	$-$	$+$	$-$	$-$	$\frac{1-\alpha}{4}$	0
$\hat{A}^{v-\sigma}_{ST} < A \leq \hat{A}^{v+\sigma}_{FT}$	$-$	$+$	$-$	$+$	$\frac{1}{4}$	$\frac{1}{4}$
$\hat{A}^{v-\sigma}_{FT} < A \leq \hat{A}^{v-\sigma}_{ST}$	$+$	$+$	$-$	$+$	$\frac{2-\alpha}{4}$	$\frac{1}{4}$
$A \leq \hat{A}^{v-\sigma}_{FT}$	$+$	$+$	$+$	$+$	$\frac{1}{2}$	$\frac{1}{2}$

"+" means that limit orders are executed in the certain state, while "-" means that orders are not executed in the certain state.

non-symmetric equilibrium combinations of the order-setting strategies exist. The analysis in question is extended in order to investigate whether asymmetric combinations of the order-setting rules lead to equilibria in the model with FTs.

Table 25 shows possible order combinations on the two market sides: by mixing each case from the buy-side (indicated by numbers) with the sell-side (specified by letters), one comes up with 64 possible combinations.

Table 25: Possible combinations of the order-setting rules in the model with FTs

Buy side								
B_{ST}	$\hat{B}^{v-\sigma}_{ST}$	$\hat{B}^{v+\sigma}_{ST}$	$\hat{B}^{v-\sigma}_{FT}$	$\hat{B}^{v+\sigma}_{FT}$	$\hat{B}^{v-\sigma}_{ST}$	$\hat{B}^{v+\sigma}_{ST}$	$\hat{B}^{v-\sigma}_{FT}$	$\hat{B}^{v+\sigma}_{FT}$
B_{FT}	$\hat{B}^{v-\sigma}_{FT}$	$\hat{B}^{v-\sigma}_{FT}$	$\hat{B}^{v-\sigma}_{FT}$	$\hat{B}^{v-\sigma}_{FT}$	$\hat{B}^{v+\sigma}_{FT}$	$\hat{B}^{v+\sigma}_{FT}$	$\hat{B}^{v+\sigma}_{FT}$	$\hat{B}^{v+\sigma}_{FT}$
Case ID	1	2	3	4	5	6	7	8
Equilibrium	Eq.1	Eq.3	Eq.2	–	–	Eq.4	–	Eq.5
Sell side								
A_{ST}	$\hat{A}^{v-\sigma}_{ST}$	$\hat{A}^{v+\sigma}_{ST}$	$\hat{A}^{v-\sigma}_{FT}$	$\hat{A}^{v+\sigma}_{FT}$	$\hat{A}^{v-\sigma}_{ST}$	$\hat{A}^{v+\sigma}_{ST}$	$\hat{A}^{v-\sigma}_{FT}$	$\hat{A}^{v+\sigma}_{FT}$
A_{FT}	$\hat{A}^{v-\sigma}_{FT}$	$\hat{A}^{v-\sigma}_{FT}$	$\hat{A}^{v-\sigma}_{FT}$	$\hat{A}^{v-\sigma}_{FT}$	$\hat{A}^{v+\sigma}_{FT}$	$\hat{A}^{v+\sigma}_{FT}$	$\hat{A}^{v+\sigma}_{FT}$	$\hat{A}^{v+\sigma}_{FT}$
Case ID	a	b	c	d	e	f	g	h
Equilibrium	Eq.4	–	Eq.5	–	Eq.3	Eq.1	–	Eq.2

The case ID is a combination of a number from the buy-side with a letter from the sell-side: there are 64 cases possible. The equilibrium identifier corresponds to that used by Hoffmann (2014). For example, case 1f describes Eq.1, case 3h is Eq.2 from Hoffmann (2014).

As follows from Table 25, not all of the combinations lead to equilibria. The equilibria indicated in Hoffmann (2014) are all area-equilibria, i.e. they are valid for a considerable area of parameters and not just for a point or a line. In the following, the possible symmetric equilibria are classified according to their fill-rate (high fill-rate, if they are executed in both states of the fundamental value change; low fill-rate, if execution happens only in one state)[38] and with respect to their specialization[39].

Proposition 3.1. *For the fixed parameters* (α, σ, L)*, a Markov-perfect equilibrium exists in the limit order market. Table 26 shows combinations of the most profitable order-setting strategies for certain market states, as well as the outside option values for each of these cases. The equilibrium map is presented in Figure 7.*

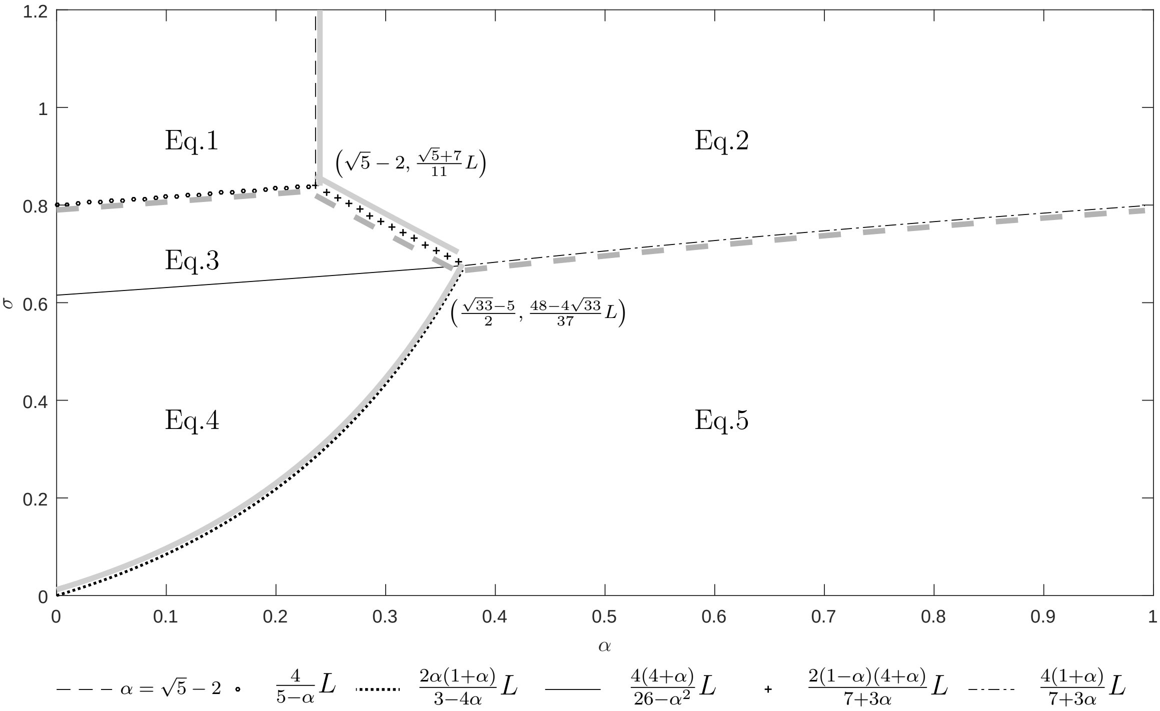


Figure 7: Equilibrium map for the model with FTs ($L = 1$)

The dashed gray line is σ_1^{border}, the border between the low fill-rate equilibria (above the line: Eq.1 and Eq.2) and the high fill-rate equilibria (below the line: Eq.3, Eq.4, and Eq.5) for an ST. The solid gray line is α_1^{border}, the border between the unspecialized equilibria (to the right from the line: Eq.2 and Eq.5) and the specialized equilibria (to the left from the line: Eq.1, Eq.3, and Eq.4) for an ST. These borderlines are presented in Equation 3.2.

The equilibrium map from Figure 7 is identical to the one presented in Hoffmann (2014). As follows from Table 26, all the equilibria provide identical bargaining power to both market sides, if market sentiment is neutral. The detailed investigation in Chapter 2 showed that on the border lines and at intersection points, there exist several neighboring equilibria (not

[38]This classification is identical to the one discussed in Section 2.2.

[39]See Hoffmann (2014), p.159: the strategy is called specialized, if the probability of execution by an ST at least in one state of the fundamental value change is higher than by an FT; otherwise the strategy is called unspecialized.

Table 26: Equilibrium conditions in the model with FTs

	Eq.1	Eq.2	Eq.3	Eq.4	Eq.5
B_{ST}	$\hat{B}_{ST}^{v-\sigma}$	$\hat{B}_{FT}^{v-\sigma}$	$\hat{B}_{ST}^{v+\sigma}$	$\hat{B}_{ST}^{v+\sigma}$	$\hat{B}_{FT}^{v+\sigma}$
B_{FT}	$\hat{B}_{FT}^{v-\sigma}$	$\hat{B}_{FT}^{v-\sigma}$	$\hat{B}_{FT}^{v-\sigma}$	$\hat{B}_{FT}^{v+\sigma}$	$\hat{B}_{FT}^{v+\sigma}$
A_{ST}	$\hat{A}_{ST}^{v+\sigma}$	$\hat{A}_{FT}^{v+\sigma}$	$\hat{A}_{ST}^{v-\sigma}$	$\hat{A}_{ST}^{v-\sigma}$	$\hat{A}_{FT}^{v-\sigma}$
A_{FT}	$\hat{A}_{FT}^{v+\sigma}$	$\hat{A}_{FT}^{v+\sigma}$	$\hat{A}_{FT}^{v+\sigma}$	$\hat{A}_{FT}^{v-\sigma}$	$\hat{A}_{FT}^{v-\sigma}$
$V_{ST,b/s}^{LO}$	$\frac{2(1-\alpha)}{5-\alpha}L$	$\frac{2(1+\alpha)}{7+3\alpha}L$	$\frac{2(2-\alpha)L-2\sigma}{6-\alpha}$	$\frac{2(2-\alpha)L-2\sigma}{6-\alpha}$	$\frac{2}{3}L-\frac{2}{3(1+\alpha)}\sigma$
$V_{FT,b/s}^{LO}$	$\frac{2(8-3\alpha-\alpha^2)}{(5-\alpha)(4+\alpha)}L$	$\frac{2(3-\alpha)}{7+3\alpha}L$	$\frac{2(2-\alpha)}{6-\alpha}L+\frac{4(1-\alpha)}{(6-\alpha)(4+\alpha)}\sigma$	$\frac{2(4+2\alpha-\alpha^2)}{(6-\alpha)(2+\alpha)}L+\frac{2-8\alpha+\alpha^2}{(6-\alpha)(2+\alpha)}\sigma$	$\frac{2}{3}L+\frac{1-3\alpha}{3(1+\alpha)}\sigma$
Area	$\begin{cases}\alpha\leq\sqrt{5}-2\\ \sigma\geq\frac{4}{5-\alpha}L\end{cases}$	$\begin{cases}\alpha\geq\sqrt{5}-2\\ \sigma\geq\frac{4(1+\alpha)}{7+3\alpha}L\\ \sigma\geq\frac{2(4+\alpha)(1-\alpha)}{7+3\alpha}L\end{cases}$	$\begin{cases}\sigma\leq\frac{4}{5-\alpha}L\\ \sigma\leq\frac{2(1-\alpha)(4+\alpha)}{7+3\alpha}L\\ \sigma\geq\frac{4(4+\alpha)}{26-\alpha^2}L\end{cases}$	$\begin{cases}\sigma\leq\frac{4(4+\alpha)}{26-\alpha^2}L\\ \sigma\geq\frac{2\alpha(1+\alpha)}{3-4\alpha}L\end{cases}$	$\begin{cases}\sigma\leq\frac{4(1+\alpha)}{7+3\alpha}L\\ \sigma\leq\frac{2\alpha(1+\alpha)}{3-4\alpha}L\end{cases}$

necessary symmetric) equivalent in terms of the bargaining power. Hoffmann (2014) mentions just five out of 64 possible combinations of the order-setting strategies. Even though these five cases only lead to area-equilibria, the other combinations may lead to line- or point-equilibria. All these additional equilibria are indicated in Table 27.

Table 27: Further line- and point-equilibria in the model with FTs

Line/Point	Further equilibrium cases
Border lines	
$\alpha = \sqrt{5} - 2, \sigma \in \left(\frac{\sqrt{5}+7}{11}L; +\infty\right)$	$1h, 3f$
$\alpha \in \left(0; \sqrt{5} - 2\right), \sigma = \frac{4}{5-\alpha}L$	$1e, 2f$
$\alpha \in \left(\sqrt{5} - 2; \frac{\sqrt{33}-5}{2}\right), \sigma = \frac{2(1-\alpha)(4+\alpha)}{7+3\alpha}L$	$2h, 3e$
$\alpha \in \left(0; \frac{\sqrt{33}-5}{2}\right), \sigma = \frac{4(4+\alpha)}{26-\alpha^2}L$	$2a, 6e$
$\in \left(\frac{\sqrt{33}-5}{2}; 1\right), \sigma = \frac{4(1+\alpha)}{7+3\alpha}L$	$3c, 3d, 3g, 4c, 4d, 4g, 4h,$
	$7c, 7d, 7g, 7h, 8d, 8g, 8h$
$\alpha \in \left(0; \frac{\sqrt{33}-5}{2}\right), \sigma = \frac{2\alpha(1+\alpha)}{3-4\alpha}L$	$6c, 8a$
Intersection points	
$\alpha = \sqrt{5} - 2, \sigma = \frac{\sqrt{5}+7}{11}L$	$1e, 1h, 2f, 2h, 3e, 3f$
$\alpha = \frac{\sqrt{33}-5}{2}, \sigma = \frac{48-4\sqrt{33}}{37}L$	$2a, 2c, 2d, 2g, 2h, 3a, 3c, 3d, 3e, 3g$
	$4a, 4c, 4d, 4e, 4g, 4h, 6c, 6d, 6e, 6g, 6h$
	$7a, 7c, 7d, 7e, 7g, 7h, 8a, 8d, 8e, 8g, 8h$

For a human as well as for an artificial agent it is very difficult to discern the market states (α, σ, L) under which line- and point-equilibria indicated in Table 27 take place. Due to this fact, the subsequent analysis concerns the area-equilibria only.

3.4 Market metrics

As the next step, the market metrics for the model with FTs (Hoffmann (2014)) are presented, following the investigation logic of Section 2.5. Proofs and further details could be found in Hoffmann (2014). Graphical representations of the market metrics will be used to make important conclusions and add further aspects to the discussion.

3.4.1 Execution probabilities

The execution probabilities of STs' orders p and of FTs' orders q are identical for both market sides, if buyers and sellers send symmetric orders: the probabilities in Table 28 may be interpreted as the execution probabilities on each market side. $q_{|ST}$ is identical in all the equilibria, because FTs adjust their quotes to the level of the next coming ST's cutoff price.

Table 28: Execution probabilities in the model with FTs

	Eq.1	Eq.2	Eq.3	Eq.4	Eq.5
$p_{\|ST,-\sigma}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$p_{\|ST,+\sigma}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$p_{\|ST}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$p_{\|FT,-\sigma}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$p_{\|FT,+\sigma}$	0	0	0	0	$\frac{1}{2}$
$p_{\|FT}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$p_{\|-\sigma}$	$\frac{1-\alpha}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$p_{\|+\sigma}$	0	0	$\frac{1-\alpha}{2}$	$\frac{1-\alpha}{2}$	$\frac{1}{2}$
p	$\frac{1-\alpha}{4}$	$\frac{1}{4}$	$\frac{2-\alpha}{4}$	$\frac{2-\alpha}{4}$	$\frac{1}{2}$

	Eq.1	Eq.2	Eq.3	Eq.4	Eq.5
$q_{\|ST,-\sigma}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$q_{\|ST,+\sigma}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$q_{\|ST}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$q_{\|FT,-\sigma}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$q_{\|FT,+\sigma}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
$q_{\|FT}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
$q_{\|-\sigma}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$q_{\|+\sigma}$	$\frac{1-\alpha}{2}$	$\frac{1-\alpha}{2}$	$\frac{1-\alpha}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
q	$\frac{2-\alpha}{4}$	$\frac{2-\alpha}{4}$	$\frac{2-\alpha}{4}$	$\frac{1}{2}$	$\frac{1}{2}$

Corollary 3.1. *An ST follows specialized strategies for a low share of FTs (high share of STs), when $\alpha \leq \alpha_1^{border}$; otherwise, an ST uses unspecialized strategies. An FT follows unspecialized strategies in all the equilibria, independent of the share and volatility parameters.*
An ST follows the high fill-rate strategies in case of low volatility, when $\sigma \leq \sigma_1^{border}$. An FT uses the high fill-rate equilibria in case of even lower volatility, when $\sigma \leq \sigma_2^{border}$. For high volatility, both STs and FTs use the low fill-rate equilibria.

Proof. First, the classification of equilibria with respect to their degree of fill-rate and specialization (see Table 29) based on the probabilities of execution (see Table 28) is important to perform.

Table 29: Classification of equilibria in the model with FTs

	Fill-rate type		Specialization type	
	ST	*FT*	*ST*	*FT*
Eq.1	low	low	specialized	unspecialized
Eq.2	low	low	unspecialized	unspecialized
Eq.3	high	low	specialized	unspecialized
Eq.4	high	high	specialized	unspecialized
Eq.5	high	high	unspecialized	unspecialized

Defining the following border lines, the corollary follows directly:

$$\alpha_1^{border} = \begin{cases} \sqrt(5) - 2 \\ \frac{2(1-\alpha)(4+\alpha)}{7+3\alpha} L \\ \frac{2\alpha(1+\alpha)}{3-4\alpha} L, \end{cases} \quad \sigma_1^{border} = \begin{cases} \frac{4}{5-\alpha} L \\ \frac{2(1-\alpha)(4+\alpha)}{7+3\alpha} L \\ \frac{4(1-\alpha)}{7+3\alpha} L, \end{cases} \quad \sigma_2^{border} = \begin{cases} \frac{4(4+\alpha)}{26-\alpha^2} L \\ \frac{4(1-\alpha)}{7+3\alpha} L. \end{cases} \quad \square \tag{3.2}$$

The high (low) volatility leads to posting limit orders with lower (higher) execution probabilities, since a more severe adverse selection with high volatility causes agents to use less aggressive limit orders in an attempt to protect themselves. If the share of FTs is low, it is not profitable for STs to target their limit orders at FTs, because it requires posting aggressive quotes with a slightly increased execution probability: STs follow the specialized (on STs) strategies. Only if the share of FTs is high enough, the unspecialized strategies are used. FTs always use the unspecialized strategies, but they revise quotes once they learn that an ST comes next, so that there is no risk for an FT, if the next to come is an ST.

The execution probabilities of STs' limit orders depend on the share only in the specialized equilibria (Eq.1, Eq.3, and Eq.4). The higher the share of FTs is, the smaller is the execution probability of STs' limit orders, since the quotes are specialized on STs, but their share becomes smaller. Only in the low fill-rate equilibria (Eq.1, Eq.2, and Eq.3) the execution probabilities of FTs' orders depend on the share: the higher the share of FTs is, the smaller is the execution probability of their orders. An FT adjusts her order before an ST comes, so that an order is executed with any fundamental value change; a smaller share of STs diminishes the execution probability, since orders are adjustable in fewer cases.

3.4.2 Probabilities of equilibrium events

In general, four equilibrium events are distinguished: (i) an ST sends a limit order, φ^{LO}_{ST}, (ii) an FT sends a limit order, φ^{LO}_{FT}, (iii) an ST makes a market order, φ^{MO}_{ST}, and (iv) an FT makes a market order, φ^{MO}_{FT}. In Hoffmann (2014), given that the market sides are symmetric, each of the probabilities of equilibrium events is the sum of the identical probabilities of those events on each market side. Knowing the market share of each agent type, it follows that $\varphi^{LO}_{FT}+\varphi^{MO}_{FT}=\alpha$ and $\varphi^{LO}_{ST}+\varphi^{MO}_{ST}=1-\alpha$. Relying on the same procedure as in Section 2.5.2, a transition matrix can be designed based on the data in Table 30.

Table 30: Transition probabilities in the model with FTs

Event$_t$ \ Event$_{t+1}$	ST sending an LO	ST sending an MO	FT sending an LO	FT sending an MO
ST sending an LO	$(1-\alpha)(1-p_{\|ST})$	$(1-\alpha)p_{\|ST}$	$\alpha(1-p_{\|FT})$	$\alpha p_{\|FT}$
ST sending an MO	$1-\alpha$	0	α	0
FT sending an LO	$(1-\alpha)(1-q_{\|ST})$	$(1-\alpha)q_{\|ST}$	$\alpha(1-q_{\|FT})$	$\alpha q_{\|FT}$
FT sending an MO	$1-\alpha$	0	α	0

Since $q_{|ST}=\frac{1}{2}$ for all the equilibria, this probability is substituted by its value in the following derivations. The transition matrix has the following form:

$$\mathbf{P} = \begin{pmatrix} (1-\alpha)(1-p_{|ST}) & (1-\alpha)p_{|ST} & \alpha(1-p_{|FT}) & \alpha p_{|FT} \\ 1-\alpha & 0 & \alpha & 0 \\ (1-\alpha)\frac{1}{2} & (1-\alpha)\frac{1}{2} & \alpha(1-q_{|FT}) & \alpha q_{|FT} \\ 1-\alpha & 0 & \alpha & 0 \end{pmatrix}$$

The left eigenvector of $\mathbf{P}$ associated with the unit modulus leads to the following stationary probability distribution $\varphi = (\varphi_{ST}^{LO}, \varphi_{ST}^{MO}, \varphi_{FT}^{LO}, \varphi_{FT}^{MO})$:

$$\begin{aligned} \varphi_{ST}^{LO} &= \frac{1-\alpha}{\chi}\big(1-\alpha(\frac{1}{2}-q_{|FT})\big), \\ \varphi_{ST}^{MO} &= \frac{1-\alpha}{\chi}\big(\chi-1+\alpha(\frac{1}{2}-q_{|FT})\big), \\ \varphi_{FT}^{LO} &= \frac{\alpha}{\chi}\big(1+(1-\alpha)(p_{|ST}-p_{|FT})\big), \\ \varphi_{FT}^{MO} &= \frac{\alpha}{\chi}\big(\chi-1-(1-\alpha)(p_{|ST}-p_{|FT})\big), \end{aligned}$$

where $\chi = (1+\alpha q_{|FT})(1+(1-\alpha)p_{|ST}) - \frac{\alpha(1-\alpha)}{2}p_{|FT}$.
These probabilities are used later in computing the market measures.

3.4.3 Bargaining power

Hoffmann (2014) shows that FTs' entry weakens STs' bargaining power, because $V_{ST}^{LO} < V_0^{LO} < V_{FT}^{LO}$. In this analysis, the choice of V_0^{LO} is critical. If FTs were not present at all, it would be logical to take the bargaining power from the simplified model with STs ($V_{ST}^{LO} = \frac{2}{5}$). Once the presence (even potential) of FTs is assumed by the market participants, one has to find the limiting values of the bargaining power when α converges to 0. As there are only three equilibria possible in the model with FTs when $\alpha \to 0$, the limiting values are given by:

$$\lim_{\alpha\to 0} V_{ST}^{LO} = \begin{cases} \frac{2}{5}L & \text{if } \sigma \geq \frac{4}{5}L \\ \frac{2}{3}L - \frac{1}{3}\sigma & \text{if } \sigma < \frac{4}{5}L. \end{cases}$$

Corollary 3.2. *Under the equilibrium conditions, the following holds:*

(a) The FTs' entry negatively affects STs' bargaining power: $V_0^{LO} \geq V_{ST}^{LO}$.

(b) The FTs' bargaining power is higher than that of STs both under the FTs' presence and absence: $V_{FT}^{LO} \geq V_0^{LO} \geq V_{ST}^{LO}$. *The higher the share of FTs is, the smaller is their bargaining power.*

Proof. These conclusions follow directly from the graphical representations of the respective functions in Figure 8 and Figure 9, but can also be proved analytically. □

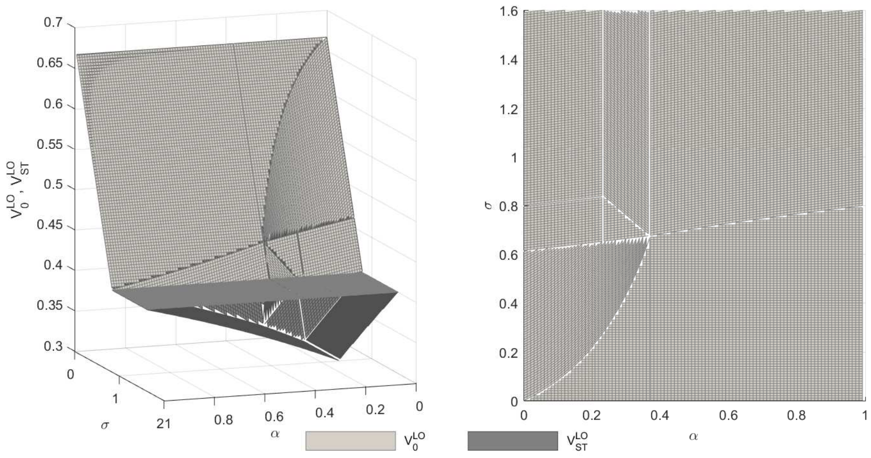


Figure 8: STs' bargaining power under the FTs' presence and absence

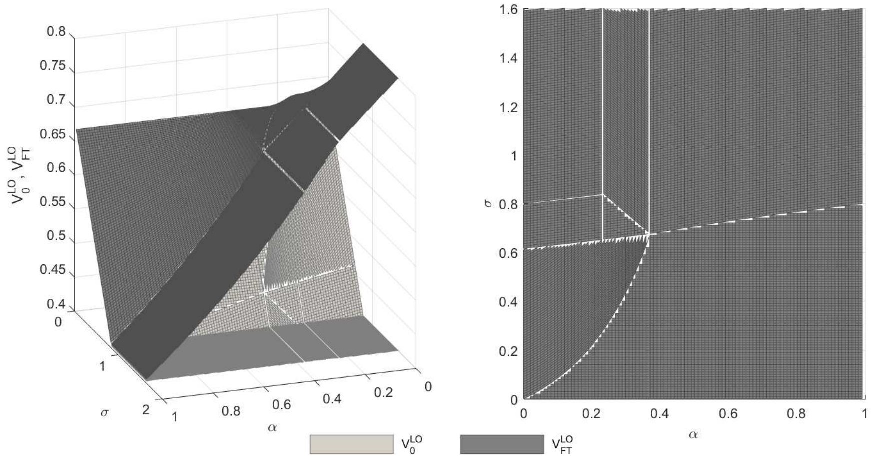


Figure 9: FTs' bargaining power and that of STs given the FTs' absence

3.4.4 Trading rate

To find the trading rate, the stationary probabilities of market orders as equilibrium events for both agent types are summed up:

$$TR = \varphi_{ST}^{MO} + \varphi_{FT}^{MO},$$

which results in the following:

$$TR = \begin{cases} \frac{4+\alpha-\alpha^2}{(4+\alpha)(5-\alpha)} & \text{for Eq.1} \\ \frac{4+3\alpha-3\alpha^2}{20-\alpha+\alpha^2} & \text{for Eq.2} \\ \frac{2-\alpha}{6-\alpha} & \text{for Eq.3} \\ \frac{4-\alpha+\alpha^2}{12+\alpha-\alpha^2} & \text{for Eq.4} \\ \frac{1}{3} & \text{for Eq.5.} \end{cases}$$

The limiting values of the trading rates if STs account for the presence of FTs but the share of the latter converges to 0 are equal to:

$$\lim_{\alpha \to 0} TR = \begin{cases} \frac{1}{5}L & \text{if } \sigma \geq \frac{4}{5}L \\ \frac{1}{3}L & \text{if } \sigma < \frac{4}{5}L. \end{cases}$$

The values of the trading rate are independent of volatility and influenced only by the shares of each agent type.

Corollary 3.3. *In equilibrium, the following holds:*

(a) The trading rates for the low fill-rate equilibria are lower than for the high fill-rate equilibria.

(b) For the high fill-rate unspecialized equilibrium (Eq.5, corresponds to the high share - low volatility states), the trading rate is independent of the share: a high share of FTs leads to a high number of transactions, and the risk of being picked-off is not so severe. For the low fill-rate equilibria and a small to moderate share of FTs, the trading rate increases with the share (Eq.1 and Eq.2). The trading rate deteriorates with the share increase for the low share - low volatility states (Eq.3 and Eq.4) and for the high share - high volatility states (Eq.2).

(c) The FTs' entry improves (diminishes) the trading rate if the volatility is high (low).

Proof. These conclusions follow directly from the graphical representations of the respective functions in Figure 10 and Figure 11, but can also be proved analytically. □

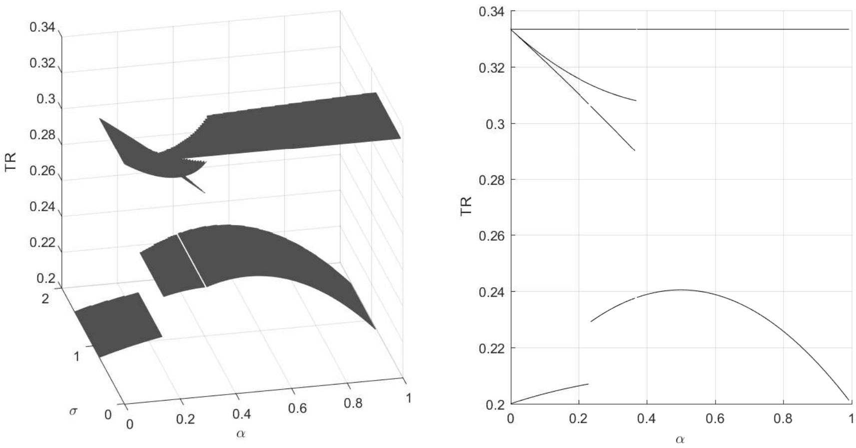


Figure 10: Trading rate for the model with FTs

The effect of FTs on the trading rate is two-fold. First, they reduce inefficiencies caused by high volatility, because the ability to revise orders allows FTs to be less cautious in posting limit orders. Second, they cause additional inefficiency, because STs receive a trade-off between the aggressiveness and profitability of their limit orders. The first effect is more pronounced than the second one for high volatility only.

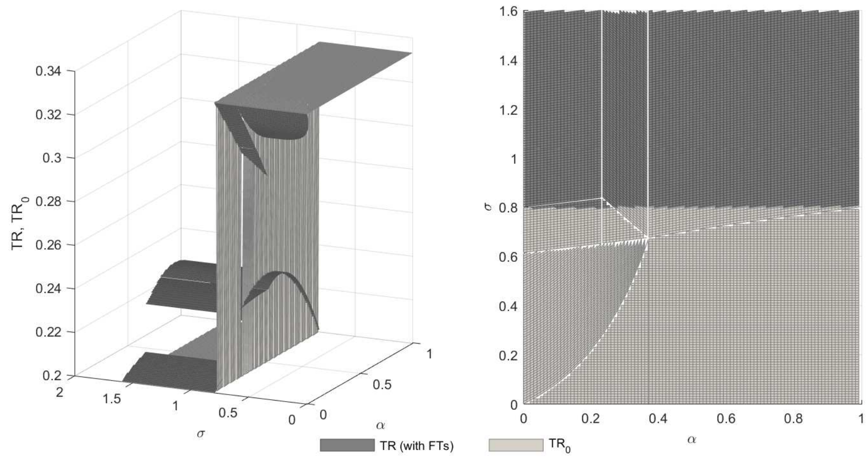


Figure 11: Trading rate for the model with FTs and with STs-only

3.4.5 Risk of being picked-off

As is the case in the simplified model with STs, the risk of being picked-off emerges when an agent uses a high fill-rate quote, but the realized fundamental value change turns out to be less profitable for an agent (i.e., when a high (low) bid (ask) is posted, but a low (high) fundamental value is realized). Given the symmetry of both market sides and equality of their order execution probabilities, the risk of being picked-off is calculated as follows:

$$\pi_{ST} = \begin{cases} \frac{\frac{1}{2}p_{|-\sigma}}{p} & \text{for Eq.3, Eq.4, and Eq.5} \\ 0 & \text{for Eq.1 and Eq.2,} \end{cases} \qquad \pi_{FT} = \begin{cases} \frac{\frac{1}{2}\alpha q_{|FT,-\sigma}}{q} & \text{for Eq.4 and Eq.5} \\ 0 & \text{for Eq.1, Eq.2, and Eq.3.} \end{cases}$$

As an FT adjusts her orders before STs, her risk of being picked-off triggers off negative scenarios only if the next coming agent is an FT. Plugging in the respective probabilities from Table 28, following results can be obtained:

$$\pi_{ST} = \begin{cases} \frac{1}{2} & \text{for Eq.3 and Eq.5} \\ \frac{1}{2-\alpha} & \text{for Eq.4} \\ 0 & \text{for Eq.1 and Eq.2,} \end{cases} \qquad \pi_{FT} = \begin{cases} \frac{\alpha}{2} & \text{for Eq.4 and Eq.5} \\ 0 & \text{for Eq.1, Eq.2, and Eq.3.} \end{cases}$$

If the share of FTs converges to 0, their risk of being picked-off disappears, while the maximum risk for STs is 50%:

$$\lim_{\alpha \to 0} \pi_{ST} = \begin{cases} \frac{1}{2} & \sigma < \frac{4}{5}L \\ 0 & \sigma \geq \frac{4}{5}L, \end{cases} \qquad \lim_{\alpha \to 0} \pi_{FT} = 0.$$

Corollary 3.4. *In equilibrium, the following holds:*

(a) Only in the specialized high fill-rate equilibrium (Eq.4, corresponds to the low share - low volatility states), the risk of being picked-off for the STs increases with the FTs' entry. In all the other market states, the risk of being picked-off for the STs does not change.

(b) *For the low volatility conditions, the risk of being picked-off for the FTs is lower than for the STs-only market. Moreover, the lower the share of the FTs is, the smaller is their risk of being picked-off.*

Proof. These conclusions follow directly from the graphical representations of the respective functions in Figure 12, but can also be proved analytically. □

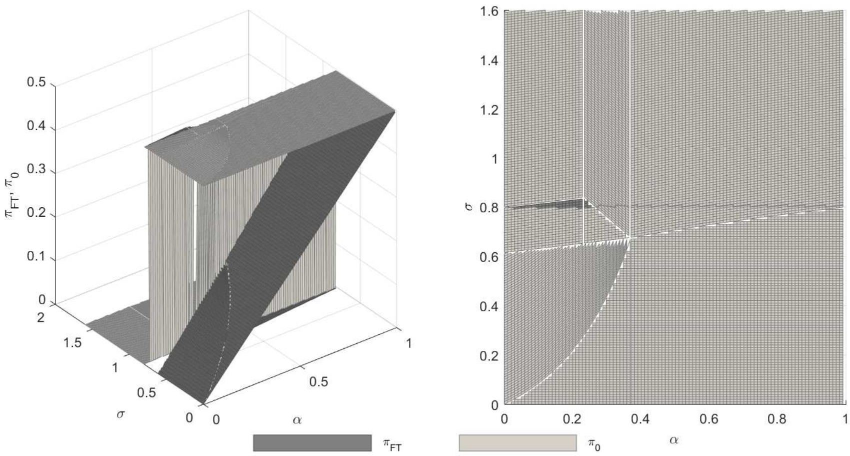


Figure 12: Risk of being picked-off for FTs and for STs with the FTs' absence

Since the FTs' bargaining power is higher, they accept only the most profitable limit orders. Therefore, the relative probabilities for STs to be picked-off increase with FTs' entry.

3.4.6 Costs of immediacy

The extended version of the costs of immediacy formula for an i-type trader takes up the following form:

$$E(\tau_i) = \frac{\omega^{-\sigma}_{ST,i}\tau^{-\sigma}_{ST,i} + \omega^{+\sigma}_{ST,i}\tau^{+\sigma}_{ST,i} + \omega^{-\sigma}_{FT,i}\tau^{-\sigma}_{FT,i} + \omega^{+\sigma}_{FT,i}\tau^{+\sigma}_{FT,i}}{\omega^{-\sigma}_{ST,i} + \omega^{+\sigma}_{ST,i} + \omega^{-\sigma}_{FT,i} + \omega^{+\sigma}_{FT,i}},$$

where $\omega^{\varepsilon}_{j,i}$ is constructed with three probabilities: (i) the probability of a j-type agent to post a limit order, φ^{LO}_j, (ii) the probability of this limit order to be executed conditional on ε and the next agent type i, $q_{|i,\varepsilon}$ or $p_{|i,\varepsilon}$, (iii) the probability that the next coming agent is an i-type agent, α or $1-\alpha$:

$$\omega^{\varepsilon}_{ST,ST} = \varphi^{LO}_{ST} \cdot p_{|ST,\varepsilon} \cdot (1-\alpha), \qquad \omega^{\varepsilon}_{FT,ST} = \varphi^{LO}_{FT} \cdot q_{|ST,\varepsilon} \cdot (1-\alpha),$$
$$\omega^{\varepsilon}_{ST,FT} = \varphi^{LO}_{ST} \cdot p_{|FT,\varepsilon} \cdot \alpha, \qquad \omega^{\varepsilon}_{FT,FT} = \varphi^{LO}_{FT} \cdot q_{|FT,\varepsilon} \cdot \alpha,$$

and τ are the costs of immediacy incurred by an i-type agent whose market order is executed by a j-type agent conditional on the latest fundamental value change ε. The costs of immediacy for an ST and an FT are the following:

$$E(\tau_{ST}) = \begin{cases} \frac{3+\alpha}{5-\alpha}L & \text{for Eq.1} \\ \frac{4+55\alpha-16\alpha^2+5\alpha^3}{(7+3\alpha)(4+3\alpha+\alpha^2)}L & \text{for Eq.2} \\ \frac{2+\alpha}{6-\alpha}L + \frac{\alpha^3-11\alpha^2+34\alpha-16}{4(6-\alpha)}\sigma & \text{for Eq.3} \\ \frac{2+\alpha}{6-\alpha}L + \frac{2(15\alpha-8-3\alpha^2)}{(6-\alpha)(4+\alpha-\alpha^2)}\sigma & \text{for Eq.4} \\ \frac{1}{3}L - \frac{4-6\alpha}{3(1+\alpha)}\sigma & \text{for Eq.5,} \end{cases}$$

$$E(\tau_{FT}) = \begin{cases} \frac{4+7\alpha+\alpha^2}{(5-\alpha)(4+\alpha)}L & \text{for Eq.1} \\ \frac{1+5\alpha}{7+3\alpha}L & \text{for Eq.2} \\ \frac{2+\alpha}{6-\alpha}L - \frac{(1-\alpha)(5-\alpha)(16+2\alpha-\alpha^2)}{2(6-\alpha)(4+\alpha)}\sigma & \text{for Eq.3} \\ \frac{8+20\alpha-10\alpha^2+3\alpha^3-\alpha^4}{(6-\alpha)(2+\alpha)(2+3\alpha-\alpha^2)}L - \frac{80+84\alpha-100\alpha^2+16\alpha^3}{2(6-\alpha)(2+\alpha)(2+3\alpha-\alpha^2)}\sigma & \text{for Eq.4} \\ \frac{1}{3}L - \frac{4}{3(1+\alpha)}\sigma & \text{for Eq.5.} \end{cases}$$

Corollary 3.5. *In equilibrium, the following holds:*

(a) The costs of immediacy for the STs are greater than those for the FTs: $E(\tau_{ST}) > E(\tau_{FT})$.

(b) With the FTs' entry, the costs of immediacy for the STs deteriorate (i.e. increase) for almost all the market states ($E(\tau_{ST}) > E(\tau_0)$)*; only for the moderate share - high volatility states, the costs of immediacy for the STs improve (i.e., decrease).*

(c) If volatility is high, the costs of immediacy for the FTs are better (i.e., smaller) than for the STs, if the FTs are not present on the market ($E(\tau_0) > E(\tau_{FT})$)*; if volatility is low, the costs of immediacy for the FTs are not better.*

Proof. These conclusions follow directly from graphical representations of the respective functions in Figure 13, Figure 14, and Figure 15, but can also be proved analytically. □

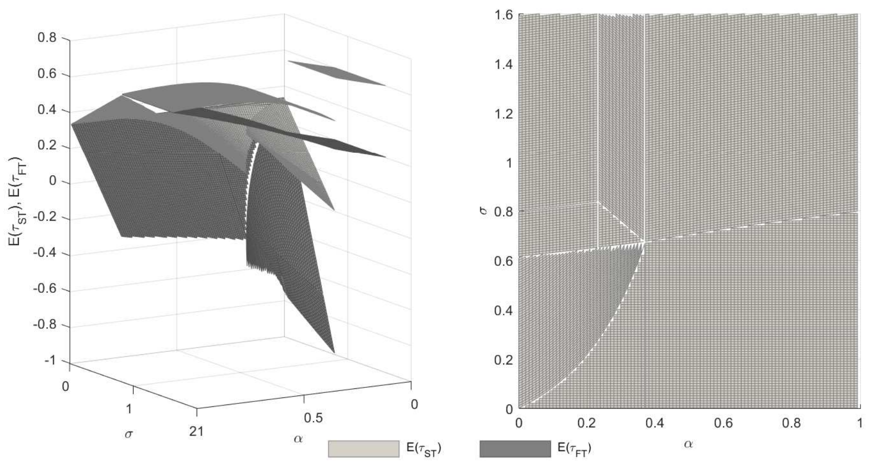


Figure 13: Costs of immediacy for FTs and STs

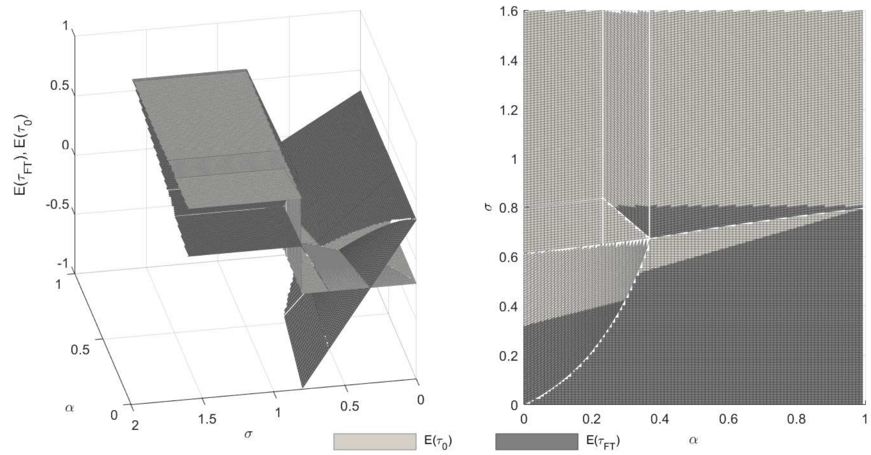


Figure 14: Costs of immediacy for FTs and for STs with the FTs' absence

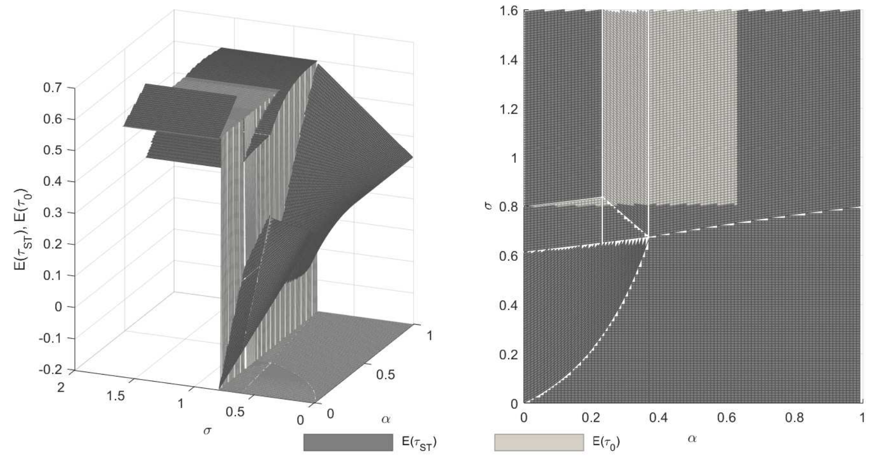


Figure 15: Costs of immediacy for STs with the FTs' absence and presence

Figure 13 is self-explanatory: the costs of immediacy for STs are greater than those for FTs, but the difference between them is not identical among the equilibria. As is evident from Figure 15, for the low fill-rate equilibria (Eq.1 and Eq.2), the costs of immediacy for STs are high and independent of volatility. For the low share - high volatility states, the costs of immediacy reach their maximum. For the high fill-rate equilibria (which correspond to lower volatility values), the costs of immediacy are smaller. This means that the costs of immediacy for STs increase for equilibria with higher volatility. The minimum costs of immediacy for STs are achieved in Eq.3. In Eq.3 and Eq.4, i.e. for the low share - moderate volatility states, they are negative, which is the most desirable scenario for an ST.

The costs of immediacy for FTs in the low fill-rate equilibria are independent of volatility. The maximum value is achieved in the high share states. The minimum is in Eq.3, for the low share - moderate volatility states. Under these states in Eq.3 and Eq.4, the obtained costs of immediacy are negative, which is optimal from the FTs' perspective.

3.4.7 Maker-taker ratio

The maker-taker ratio accounts for all the cases when an i-type agent serves a market-making function in the numerator and a market-taking function in the denominator:

$$MT_i = \frac{\omega_{i,ST} + \omega_{i,FT}}{\omega_{ST,i} + \omega_{FT,i}}.$$

Applying this formula, the following results are derived:

$$MT_{ST} = \begin{cases} \frac{4-5\alpha+\alpha^2}{4+5\alpha-\alpha^2} & \text{for Eq.1} \\ \frac{4-\alpha}{4+3\alpha+\alpha^2} & \text{for Eq.2} \\ \frac{(4-\alpha)(2-\alpha)}{8} & \text{for Eq.3} \\ \frac{2(2-\alpha)}{4+\alpha-\alpha^2} & \text{for Eq.4} \\ 1 & \text{for Eq.5,} \end{cases} \qquad MT_{FT} = \begin{cases} \frac{2-\alpha}{\alpha} & \text{for Eq.1} \\ \frac{4(2-\alpha)}{4-\alpha+\alpha^2} & \text{for Eq.2} \\ \frac{(5-\alpha)(2-\alpha)}{4} & \text{for Eq.3} \\ \frac{5-\alpha}{2+3\alpha-\alpha^2} & \text{for Eq.4} \\ 1 & \text{for Eq.5.} \end{cases}$$

The STs' maker-taker ratio with FTs' absence is a unit, $MT_0 = 1$, meaning that an ST is a balanced maker-taker: the probabilities of her trading through limit orders and market orders are equal.

Corollary 3.6. *In equilibrium, the following holds:*

(a) For all the market states, an FT is a market-maker, while an ST is a market-taker: $MT_{FT} \geq 1 \geq MT_{ST}$. In the unspecialized high fill-rate equilibrium (Eq.5), both agent types make the market to the same degree as they take it.

(b) The degree of market-making by the FTs increases with high volatility.

Proof. These conclusions could be made by considering the graphical representation of the respective functions in Figure 16 and using the following logic. Even though the maker-taker ratio is independent of volatility, one can check possible transitions through equilibria due to the volatility increase. If the volatility σ increases, while the parameters (α, L) stay unchanged, the following transitions among the equilibria are possible (first the longest path is shown, followed by shorter or partial paths):

(a) Eq.5 → Eq.4 → Eq.3 → Eq.1: Eq.5 → Eq.4, Eq.5 → Eq.3, Eq.5 → Eq.1, Eq.4 → Eq.3, Eq.4 → Eq.1, Eq.3 → Eq.1,

(b) Eq.5 → Eq.4 → Eq.3 → Eq.2, paths not mentioned before: Eq.5 → Eq.2, Eq.4 → Eq.2, Eq.3 → Eq.2.

As Figure 16 shows, the maker-taker ratio for STs decreases only if volatility goes up; the maker-taker ratio for FTs rises with increased volatility with one exception: for the transition Eq.3 → Eq.2, the maker-taker ratio for FTs reduces (for the states with the share a bit above the quarter level and high to medium volatility). □

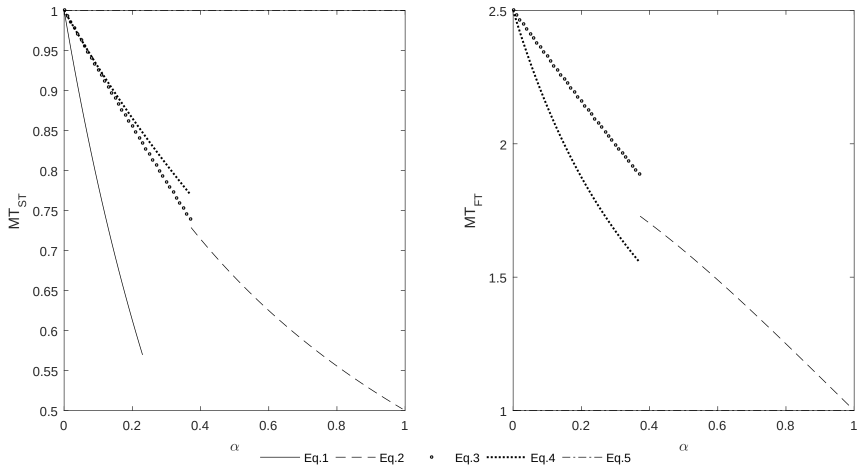


Figure 16: Maker-taker ratio for STs and FTs

Eq.1 is omitted from the right subplot: the maker-taker ratio for FTs in this equilibrium is the highest and converges to infinity.

The maker-taker ratio for FTs is always above one, since they have the maximum execution probability conditional on STs' entry due to their quote adjustment opportunities, and because they are less likely to submit market orders (accept existing limit orders) than STs due to their higher cutoff prices.

3.4.8 Pricing error

As the two market sides are symmetric and the probabilities of equilibrium events include probabilities for both of them, the pricing error is calculated as follows:

$$PE = \frac{\varphi_{ST}^{LO}}{\varphi_{ST}^{LO} + \varphi_{FT}^{LO}} PE_{ST} + \frac{\varphi_{FT}^{LO}}{\varphi_{ST}^{LO} + \varphi_{FT}^{LO}} PE_{FT},$$

where

$$PE_{ST} = \frac{1}{2}|v - \sigma - B_{ST}^{m}| + \frac{1}{2}|v + \sigma - B_{ST}^{m}|,$$

$$PE_{FT} = \frac{\alpha}{2}|v - \sigma - B_{FT}^{m}| + \frac{\alpha}{2}|v + \sigma - B_{FT}^{m}| + \frac{1-\alpha}{2}|v - \sigma - \hat{B}_{ST}^{v-\sigma}| + \frac{1-\alpha}{2}|v + \sigma - \hat{B}_{ST}^{v+\sigma}|.$$

The two last elements in the pricing error of an FT are identical for all the equilibria, as she adjusts her orders to the current change in the fundamental value and agent type, if the next coming agent is an ST. Here is the analytical form of the pricing error for the market as a whole[40]:

[40] The expression for Eq.2 does not exactly coincide with Hoffmann (2014): the discrepancy could be due to a typo in the original paper.

$$PE = \begin{cases} \frac{(3+\alpha)(4+\alpha)-2\alpha^2(5-\alpha)}{(5-\alpha)(4+\alpha)}L + \frac{4-\alpha(5-\alpha)(1-\alpha)}{4}\sigma & \text{for Eq.1} \\ \frac{4+35\alpha-36\alpha^2+21\alpha^3}{(7+3\alpha)(4-\alpha+\alpha^2)}L + \frac{4-5\alpha+5\alpha^2}{4-\alpha+\alpha^2}\sigma & \text{for Eq.2} \\ \frac{\alpha(5-\alpha)(2+\alpha)}{4(6-\alpha)}L + \frac{96-82\alpha+94\alpha^2-\alpha^3-8\alpha^4+\alpha^5}{4(6-\alpha)(4+\alpha)}\sigma & \text{for Eq.3} \\ \frac{\alpha(5-\alpha)(1-\alpha)(2+\alpha)}{(6-\alpha)(4+\alpha-\alpha^2)}L + \frac{24-18\alpha+22\alpha^2-9\alpha^3+\alpha^4}{(6-\alpha)(4+\alpha-\alpha^2)}\sigma & \text{for Eq.4 and } \sigma \geq \frac{2+\alpha}{2(5-\alpha)}L \\ \frac{(2+\alpha)(4+\alpha-6\alpha^2+\alpha^3)}{(6-\alpha)(4+\alpha-\alpha^2)}L + \frac{\alpha^4-9\alpha^3+14\alpha^2+30\alpha-16}{(6-\alpha)(4+\alpha-\alpha^2)}\sigma & \text{for Eq.4 and } \frac{4+\alpha^2}{26-\alpha^2}L \leq \sigma < \frac{2+\alpha}{2(5-\alpha)}L \\ \frac{16+20\alpha+4\alpha^2-23\alpha^3+3\alpha^4}{(6-\alpha)(2+\alpha)(4+\alpha-\alpha^2)}L - \frac{2(16-22\alpha+36\alpha^2-11\alpha^3+\alpha^4)}{(6-\alpha)(2+\alpha)(4+\alpha-\alpha^2)}\sigma & \text{for Eq.4 and } \sigma < \frac{4+\alpha^2}{26-\alpha^2}L \\ \frac{1}{3}L - \frac{2(2-3\alpha+3\alpha^2)}{3(1+\alpha)}\sigma & \text{for Eq.5 and } \sigma \leq \frac{1+\alpha}{7+3\alpha}L \\ \frac{\alpha(1-\alpha)}{3}L + \frac{3+2\alpha-2\alpha^2+3\alpha^3}{3(1+\alpha)}\sigma & \text{for Eq.5 and } \sigma > \frac{1+\alpha}{7+3\alpha}L, \end{cases}$$

Under the condition of FTs' absence, the pricing error has the following form:

$$\lim_{\alpha \to 0} PE = \begin{cases} \frac{3}{5}L + \sigma & \text{if } \sigma \geq \frac{4}{5}L \\ \sigma & \text{if } \frac{1}{5}L \leq \sigma < \frac{4}{5}L \\ \frac{1}{3}L - \frac{2}{3}\sigma & \text{if } \sigma < \frac{1}{5}L. \end{cases}$$

Corollary 3.7. *In equilibrium, the FTs' entry decreases (increases) the pricing error of the market quotes for high (low) volatility, $PE \leq PE_0$ ($PE \geq PE_0$).*

Proof. These conclusions follow directly from the graphical representations of the respective functions in Figure 17, but can also be proved analytically. □

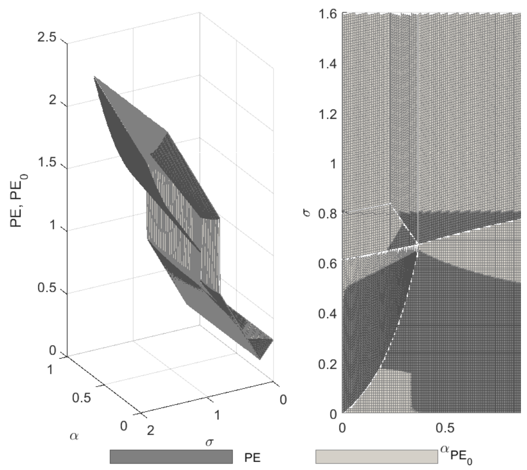


Figure 17: Pricing error in the model with FTs

FTs' influence on the pricing error is two-fold. On the one hand, their ability to revise orders brings quotes closer to the fundamental value. On the other hand, FTs' higher bargaining power can boost the pricing error. If volatility is high, the first effect is more pronounced, which improves the pricing error of the whole economy.

3.4.9 Welfare

To compute the welfare of each agent type, Formula 2.3 can be applied, where $i = ST$ or $i = FT$. The aggregate market welfare which includes the results for both agent types is calculated as follows:

$$W = \alpha W_{ST} + (1-\alpha)W_{FT} = 2L \cdot TR.$$

The analytical expressions for the welfare of the agent types are the following:

$$W_{ST} = \begin{cases} \frac{2(1-\alpha)}{5-\alpha}L & \text{for Eq.1} \\ \frac{2(28+\alpha+12\alpha^2-\alpha^3)}{(7+3\alpha)(20-\alpha+\alpha^2)}L & \text{for Eq.2} \\ \frac{2(2-\alpha)}{6-\alpha}L - \frac{\alpha(5-\alpha)}{2(6-\alpha)}\sigma & \text{for Eq.3} \\ \frac{2(2-\alpha)}{6-\alpha}L - \frac{6\alpha(5-\alpha)}{(4-\alpha)(3+\alpha)(6-\alpha)}\sigma & \text{for Eq.4} \\ \frac{2}{3}L - \frac{2\alpha}{3(1+\alpha)}\sigma & \text{for Eq.5,} \end{cases}$$

$$W_{FT} = \begin{cases} \frac{2(8-3\alpha-\alpha^2)}{(5-\alpha)(4+\alpha)}L & \text{for Eq.1} \\ \frac{2(3-\alpha)}{7+3\alpha}L & \text{for Eq.2} \\ \frac{2(2-\alpha)}{6-\alpha}L + \frac{1-\alpha}{2(6-\alpha)}\sigma & \text{for Eq.3} \\ \frac{2(48+24\alpha-10\alpha^2-3\alpha^3+\alpha^4)}{(4-\alpha)(3+\alpha)(6-\alpha)(2+\alpha)}L + \frac{6(\alpha-5)(\alpha-1)}{(4-\alpha)(3+\alpha)(6-\alpha)}\sigma & \text{for Eq.4} \\ \frac{2}{3}L + \frac{2(1-\alpha)}{3(1+\alpha)}\sigma & \text{for Eq.5.} \end{cases}$$

The limiting values of welfare are evident, if the share of FTs converges to 0 (W_0) and to 1 (W_1). In both cases, only those equilibria that are possible under each respective extreme share are considered. As it turns out, both limiting values are identical:

$$\lim_{\alpha \to 0} W = \lim_{\alpha \to 1} W = \begin{cases} \frac{2}{3}L & \sigma < \frac{4}{5}L \\ \frac{2}{5}L & \sigma \geq \frac{4}{5}L. \end{cases}$$

Corollary 3.8. *In equilibrium, the following holds:*

(a) The FTs' welfare is higher than that of the STs; the STs' welfare diminishes with the FTs' entry, while the FTs get an additional welfare: $W_{FT} > W_0 > W_{ST}$. The STs are better-off when there are no FTs on the market.

(b) Since $W_0 = W_1$, the FTs have a relative advantage only, which they can exercise if STs participate in the market.

(c) The aggregate market welfare increases with the FTs' entry under the condition of high volatility: $W > W_0$. In the low share - low volatility states, the aggregate welfare decreases:

$W < W_0$. In the moderate to high share - low volatility states, the aggregate welfare is not influenced by the FTs' entry.

Proof. These conclusions follow directly from the graphical representations of the respective functions in Figure 18, but can also be proved analytically. □

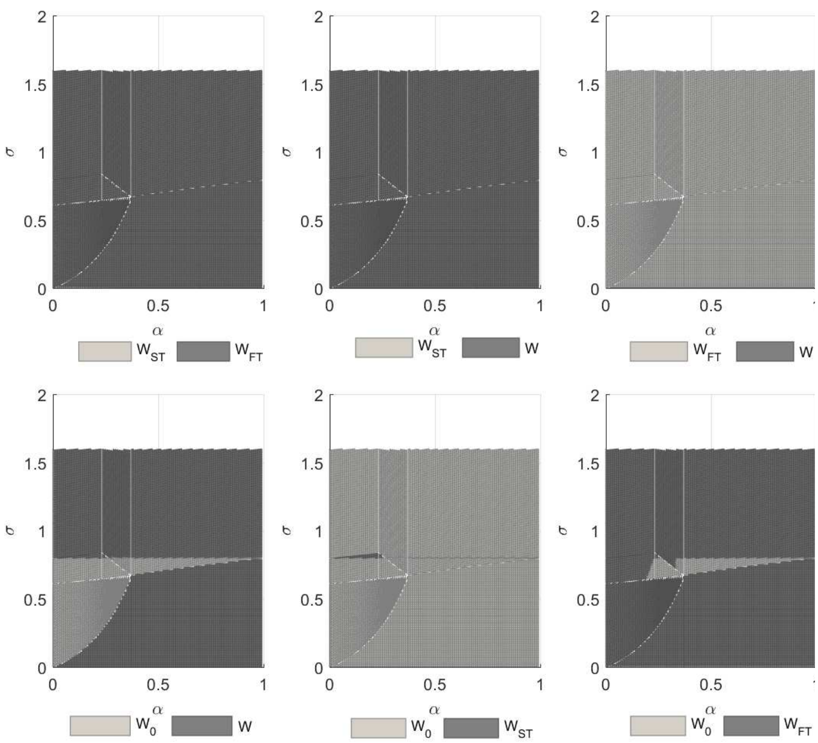


Figure 18: STs' and FTs' welfare, the aggregate market welfare, STs' welfare with the FTs' absence, and FTs' welfare with the STs' absence

FTs win the market game at the cost of STs. Only if volatility is high enough, the aggregate market welfare is better with FTs, while STs are still underprivileged.

3.5 Summary

In this chapter, we revisited the model with FTs by Hoffmann (2014) and extended it with an analysis of the equilibrium conditions from the perspective of both market sides: this considerably increased the amount of order-setting combinations necessary for investigation. It turned out that the only area-equilibria are those equilibria discussed in Hoffmann (2014). The other combinations of the order-setting rules led either to line- or to point-equilibria. Since the market states corresponding to the line- and point-equilibria were described by various combinations of the infinite decimal parameters (α, σ, L), it would be almost impossible for a human trader as well as for an algorithm to distinguish these states from the neighboring area-equilibria states. As a result, these equilibria were excluded from further consideration.

All the market measures from Hoffmann (2014) were verified, and the only discrepancy reported for the formula of the pricing error for a single equilibrium could be treated as a typo in the original paper. Therefore, the conclusions of our analysis are in accordance with the conclusions made by Hoffmann (2014). The most important outcomes are:

For a small share of FTs, an ST uses the specialized strategies, while an FT always follows the unspecialized strategies. With volatility being low, both agents use the high fill-rate strategies, whereas high volatility causes them to protect themselves from the increased adverse selection risk by turning to the low fill-rate strategies.

The FTs' entry diminishes STs' bargaining power and welfare in all states, increases their risk of being picked-off for the low share - low volatility cases, negatively affects their costs of immediacy (but for the moderate share - high volatility states, this measure may improve).

The FTs' entry improves the trading rate, the pricing error, and the aggregate market welfare if volatility is high. In the low share - low volatility states, the aggregate welfare deteriorates with FTs's entry; in the moderate to high share - low volatility states, the aggregate welfare is not influenced by FTs.

FTs win the market at the cost of STs. FTs have a higher bargaining power (it increases when the share of FTs goes down), lower costs of immediacy. If volatility is low, the risk of being picked-off for FTs is smaller and the costs of immediacy for FTs are better than in STs-only market. An FT serves a market-making function; her degree of market-making increases with volatility.

4 Modified Analytical Model with Informed Slow Traders

The model we introduce in this chapter is a modified version of the Hoffmann (2014) model. Instead of FTs, another type of agents is allowed to enter the marketplace.
The main difference of an FT from an ST in Hoffmann (2014) is that an FT receives information through her speed advantage, but it is only relative to STs: an FT has a possibility to revise a quote before an ST enters the market but after the fundamental value change becomes known. This advantage vanishes if the next coming agent is another FT. The main aim of this chapter is to investigate how the agents' welfare changes, if an agent does not have any speed advantage, but has the informational advantage only. The agents who possess the informational advantage but not the speed of reactions are designated as informed slow traders (ISTs). ISTs know with certainty the development of the fundamental price before the next agent enters the market, so she sets orders based on the correct predictions. However, an IST is not aware of the type of the next coming agent, and this creates uncertainty in her decision-making. Once an order is sent, it cannot be revised. For an IST, chances to get a profit are different from those of an FT in two ways: (i) an IST knows the exact fundamental value change independent of who comes next, while an FT can run in front of STs only, (ii) an IST cannot adjust her quotes up or down to the level of the next coming agent, while an FT adjusts her orders in such a way that they are executed by an ST for any fundamental value change.
After setting up the model and analyzing its results in the current chapter, we scrutinize the difference between the two agent types: a fast but conditionally informed agent with the possibility to revise her orders and a slow but unconditionally informed trader without the possibility to revise her quotes. "Conditionally" means that an agent can be considered informed only if the next coming agent is an ST. Unconditional informational advantage exists regardless of the agent type coming next. Moreover, it is essential to analyze which of the two informed agent types harms an ST most and which of them affects the whole economy most.

4.1 The model set-up

A temporal flow of the market events is presented in Figure 19.

Assumption 1. Given that an IST is more certain about the future fundamental value change and could set her quotes accordingly, her expected profit from posting a limit order is greater than that of an ST: $V^{LO}_{IST} \geq V^{LO}_{ST}$.

If there are two agent types and the shares of market sides are the same, the bounds for the bargaining value are described by Proposition 4.1.

Proposition 4.1. *In equilibrium,* $0 \leqslant V^{LO}_{k} < L$.

Proof. In equilibrium, the expected profit from posting a limit order cannot be negative, otherwise the other outside option is used. If two agents with different reservation values trade, they

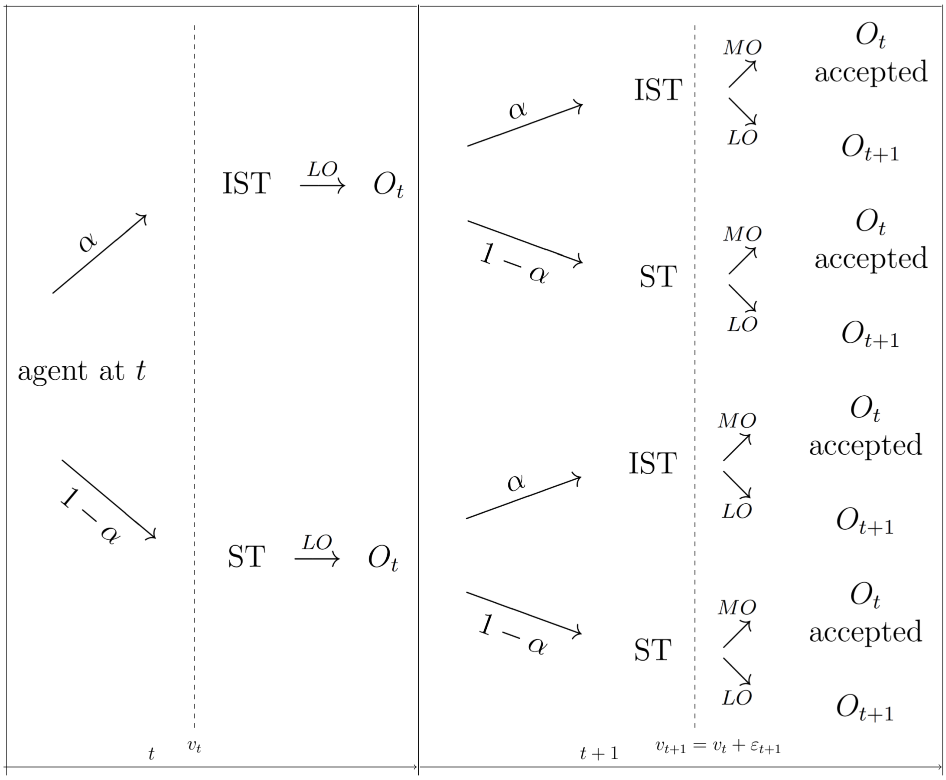


Figure 19: Temporal flow for the model with ISTs

share the surplus of $2L$. However, given that some limit order is posted, a transaction happens with the maximum probability of $\max(\alpha, 1-\alpha) \cdot \max(\gamma, 1-\gamma)$. Therefore, the maximum expected gain from trade per period is $\frac{1}{2} \cdot \max(\alpha, 1-\alpha) \cdot 2L < L$, given that $0 < \alpha, 1-\alpha < 1$ and $\gamma = \frac{1}{2}$. $\square$

ISTs have a higher value of the outside option, as follows from Assumption 1, and therefore the cutoff prices for two agent types differ:

$$\hat{B}_{IST}^{v+\varepsilon} = (v + \varepsilon - L) + V_{s,IST}^{LO} \geq \hat{B}_{ST}^{v+\varepsilon} = (v + \varepsilon - L) + V_{s,ST}^{LO},$$
$$\hat{A}_{IST}^{v+\varepsilon} = (v + \varepsilon + L) - V_{b,IST}^{LO} \leq \hat{A}_{ST}^{v+\varepsilon} = (v + \varepsilon + L) - V_{b,ST}^{LO}.$$

Lemma 4.1. *In equilibrium,* $\hat{B}_{ST}^{v-\sigma} \leq \hat{B}_{IST}^{v-\sigma} \leq \hat{B}_{ST}^{v+\sigma} \leq \hat{B}_{IST}^{v+\sigma}$ *holds for the sell cutoff prices, while* $\hat{A}_{IST}^{v-\sigma} \leq \hat{A}_{ST}^{v-\sigma} \leq \hat{A}_{IST}^{v+\sigma} \leq \hat{A}_{ST}^{v+\sigma}$ *holds for the buy cutoff prices.*

Proof. Knowing how the sell cutoff price is formed, the following inequality can be compiled:

$$\hat{B}_k^{v+\sigma} = v + \sigma - L + V_k^{LO} \leq v + \sigma.$$

Now it is possible to rewrite this inequality and add further entries on the right-hand side:

$$v + \sigma \geq v + \sigma - L + V_k^{LO} = \hat{B}_k^{v+\sigma} \geq v - \sigma - L + V_k^{LO} = \hat{B}_k^{v-\sigma}.$$

It follows that:

$$\hat{B}_{IST}^{v-\sigma} \leq \hat{B}_{IST}^{v+\sigma}, \qquad \hat{B}_{ST}^{v-\sigma} \leq \hat{B}_{ST}^{v+\sigma}.$$

The following inequality holds only if $\sigma \geq \frac{1}{2}L$:

$$\hat{B}_{ST}^{v+\sigma} \geq v+\sigma-L \geq v-\sigma \geq v-\sigma-L+V_{IST}^{LO} = \hat{B}_{IST}^{v-\sigma},$$

meaning that $\hat{B}_{ST}^{v+\sigma} \geq \hat{B}_{IST}^{v-\sigma}$ if $\sigma \geq \frac{1}{2}L$.
The next step is to prove that the same inequality holds also if $\sigma < \frac{1}{2}L$, i.e. for all values of σ. For this purpose, another strategy can be adopted. Let us consider the following combination of posted limit orders by different agent types:

- Case 1: An IST uses $\hat{B}_{IST}^{v+\varepsilon}$, while an ST posts $\hat{B}_{IST}^{v+\sigma}$
  Recalling that an IST knows the next fundamental value change, her expected profit from posting a limit order is the following:

$$\begin{aligned}
V_{IST}^{LO} &= \frac{\alpha}{2}\Big(\frac{1}{2}(v-\sigma+L-\hat{B}_{IST}^{v-\sigma}) + \frac{1}{2}(v+\sigma+L-\hat{B}_{IST}^{v+\sigma})\Big) \\
&+ \frac{1-\alpha}{2}\Big(\frac{1}{2}(v-\sigma+L-\hat{B}_{IST}^{v-\sigma}) + \frac{1}{2}(v+\sigma+L-\hat{B}_{IST}^{v+\sigma})\Big) \\
&= \frac{1}{2}(v+L) - \frac{1}{4}\hat{B}_{IST}^{v-\sigma} - \frac{1}{4}\hat{B}_{IST}^{v+\sigma}.
\end{aligned}$$

  As posting $\hat{B}_{IST}^{v+\sigma}$ by an ST is not necessarily her optimal strategy, the expected profit from posting a limit order should be greater than the one received using this order choice (this quote will be executed by any seller in the next step regardless of the fundamental value development):

$$\begin{aligned}
V_{ST}^{LO} &\geq \frac{\alpha}{2}\Big(\frac{1}{2}(v-\sigma+L-\hat{B}_{IST}^{v+\sigma}) + \frac{1}{2}(v+\sigma+L-\hat{B}_{IST}^{v+\sigma})\Big) \\
&+ \frac{1-\alpha}{2}\Big(\frac{1}{2}(v-\sigma+L-\hat{B}_{IST}^{v+\sigma}) + \frac{1}{2}(v+\sigma+L-\hat{B}_{IST}^{v+\sigma})\Big) \\
&= \frac{1}{2}(v+L-\hat{B}_{IST}^{v+\sigma}).
\end{aligned}$$

  An analysis of the difference of the expected profits from sending a limit order of the two agents, yields the following results:

$$\begin{aligned}
V_{IST}^{LO} - V_{ST}^{LO} &\leq \frac{1}{2}(v+L) - \frac{1}{4}\hat{B}_{IST}^{v-\sigma} - \frac{1}{4}\hat{B}_{IST}^{v+\sigma} - \frac{1}{2}(v+L-\hat{B}_{IST}^{v+\sigma}) \\
&= \frac{1}{4}\hat{B}_{IST}^{v+\sigma} - \frac{1}{4}\hat{B}_{IST}^{v-\sigma} = \frac{1}{4}(v+\sigma-L+V_{IST}^{LO}) - \frac{1}{4}(v-\sigma-L+V_{IST}^{LO}) \\
&= \frac{1}{2}\sigma \geq 0.
\end{aligned}$$

It follows that:

$$V_{IST}^{LO} - \sigma \le V_{IST}^{LO} \le V_{ST}^{LO} + \frac{1}{2}\sigma \le V_{ST}^{LO} + \sigma,$$
$$\hat{B}_{IST}^{v-\sigma} = V_{IST}^{LO} - \sigma + v - L \le V_{ST}^{LO} + \sigma + v - L = \hat{B}_{ST}^{v+\sigma}.$$

The last outcome leads to the conclusion that $\hat{B}_{IST}^{v-\sigma} \le \hat{B}_{ST}^{v+\sigma}$.

- Case 2: An IST uses $\hat{B}_{ST}^{v+\varepsilon}$ as her quote, while an ST posts $\hat{B}_{IST}^{v+\sigma}$
  ISTs' limit order will not be executed by another IST during the next step, it may be satisfied only by an ST. The analysis of the ST's quote is the same as in Case 1.

$$\begin{aligned} V_{IST}^{LO} &= \frac{1-\alpha}{2}\Big(\frac{1}{2}(v - \sigma + L - \hat{B}_{ST}^{v-\sigma}) + \frac{1}{2}(v + \sigma + L - \hat{B}_{ST}^{v+\sigma})\Big) \\ &= \frac{1-\alpha}{2}\big(v + L - \frac{1}{2}\hat{B}_{ST}^{v-\sigma} - \frac{1}{2}\hat{B}_{ST}^{v+\sigma}\big) \end{aligned}$$

The difference in the expected profits from sending a limit order for the two agent types:

$$\begin{aligned} V_{IST}^{LO} - V_{ST}^{LO} &\le \frac{1-\alpha}{2}\big(v + L - \frac{1}{2}\hat{B}_{ST}^{v-\sigma} - \frac{1}{2}\hat{B}_{ST}^{v+\sigma}\big) - \frac{1}{2}(v + L - \hat{B}_{IST}^{v+\sigma}) \\ &= \frac{1}{2}\sigma - \alpha L - \frac{1}{2}V_{ST}^{LO} + \frac{\alpha}{2}V_{ST}^{LO} + \frac{1}{2}V_{IST}^{LO}. \end{aligned}$$

This leads to the following result:

$$\begin{aligned} V_{IST}^{LO} - V_{ST}^{LO} &\le \frac{1}{2}\sigma - \alpha L - \frac{1}{2}V_{ST}^{LO} + \frac{\alpha}{2}V_{ST}^{LO} + \frac{1}{2}V_{IST}^{LO}, \\ \frac{1}{2}V_{IST}^{LO} &\le \frac{1}{2}\sigma - \alpha L + \frac{1}{2}V_{ST}^{LO}(1+\alpha), \\ V_{IST}^{LO} - \sigma &\le V_{ST}^{LO}(1+\alpha) - 2\alpha L \le V_{ST}^{LO} + \alpha(V_{ST}^{LO} - 2L) \le V_{ST}^{LO} \le V_{ST}^{LO} + \sigma. \end{aligned}$$

And finally:

$$\hat{B}_{IST}^{v-\sigma} = v - \sigma - L + V_{IST}^{LO} \le v + \sigma - L + V_{ST}^{LO} = \hat{B}_{ST}^{v+\sigma},$$
$$\hat{B}_{IST}^{v-\sigma} \le \hat{B}_{ST}^{v+\sigma}.$$

The inequality for the buy cutoff prices may be proved in a similar manner. □

4.2 Equilibrium conditions

In the market with ISTs, agents also aim at maximizing their expected profits from sending a limit order:

$$V_{b,k}^{LO} = \max_{B_k}\,[p(B_k)\big((v + \varepsilon + L) - B_k\big)], \qquad V_{s,k}^{LO} = \max_{A_k}\,[p(A_k)\big(A_k - (v + \varepsilon - L)\big)],$$

where $k = IST$ or $k = ST$. The possible universe of orders for an ST is $(\hat{B}^{v-\sigma}_{ST}, \hat{B}^{v+\sigma}_{ST}, \hat{B}^{v-\sigma}_{IST}, \hat{B}^{v+\sigma}_{IST})$ and $(\hat{A}^{v-\sigma}_{ST}, \hat{A}^{v+\sigma}_{ST}, \hat{A}^{v-\sigma}_{IST}, \hat{A}^{v+\sigma}_{IST})$, while an IST knows exactly the next fundamental value realization ε and therefore has to choose only between two orders per each market side: $(\hat{B}^{v+\varepsilon}_{ST}, \hat{B}^{v+\varepsilon}_{IST})$ and $(\hat{A}^{v+\varepsilon}_{ST}, \hat{A}^{v+\varepsilon}_{IST})$. The possible combinations of the order-setting rules and equilibrium combinations are stated in Table 31. It is assumed that the shares of the market sides are equal, and that only symmetric equilibria are possible (area equilibria); the line- and point-equilibria lie beyond the scope of this research, due to the decimal character of parameters corresponding to these equilibria.

Table 31: Possible combinations of order-setting rules in the model with ISTs

Buy side								
B_{ST}	$\hat{B}^{v-\sigma}_{ST}$	$\hat{B}^{v+\sigma}_{ST}$	$\hat{B}^{v-\sigma}_{IST}$	$\hat{B}^{v+\sigma}_{IST}$	$\hat{B}^{v-\sigma}_{ST}$	$\hat{B}^{v+\sigma}_{ST}$	$\hat{B}^{v-\sigma}_{IST}$	$\hat{B}^{v+\sigma}_{IST}$
B_{IST}	$\hat{B}^{v+\varepsilon}_{ST}$	$\hat{B}^{v+\varepsilon}_{ST}$	$\hat{B}^{v+\varepsilon}_{ST}$	$\hat{B}^{v+\varepsilon}_{ST}$	$\hat{B}^{v+\varepsilon}_{IST}$	$\hat{B}^{v+\varepsilon}_{IST}$	$\hat{B}^{v+\varepsilon}_{IST}$	$\hat{B}^{v+\varepsilon}_{IST}$
Case ID	1	2	3	4	5	6	7	8
Equilibrium	Eq.1	Eq.2	–	–	–	Eq.3	Eq.4	Eq.5
Sell side								
A_{ST}	$\hat{A}^{v-\sigma}_{ST}$	$\hat{A}^{v+\sigma}_{ST}$	$\hat{A}^{v-\sigma}_{IST}$	$\hat{A}^{v+\sigma}_{IST}$	$\hat{A}^{v-\sigma}_{ST}$	$\hat{A}^{v+\sigma}_{ST}$	$\hat{A}^{v-\sigma}_{IST}$	$\hat{A}^{v+\sigma}_{IST}$
A_{IST}	$\hat{A}^{v+\varepsilon}_{ST}$	$\hat{A}^{v+\varepsilon}_{ST}$	$\hat{A}^{v+\varepsilon}_{ST}$	$\hat{A}^{v+\varepsilon}_{ST}$	$\hat{A}^{v+\varepsilon}_{IST}$	$\hat{A}^{v+\varepsilon}_{IST}$	$\hat{A}^{v+\varepsilon}_{IST}$	$\hat{A}^{v+\varepsilon}_{IST}$
Case ID	a	b	c	d	e	f	g	h
Equilibrium	Eq.2	Eq.1	–	–	Eq.3	–	Eq.5	Eq.4

Proposition 4.2. *For the fixed parameters (α, σ, L), a unique Markov-perfect equilibrium exists in the limit order market. Table 32 presents the combinations of the most profitable order-setting strategies for certain parameters, as well as the values of the outside options in each of these cases. The equilibrium map is illustrated in Figure 20.*

Table 32: Equilibrium conditions in the model with ISTs

	Eq.1	Eq.2	Eq.3	Eq.4	Eq.5
B_{ST}	$\hat{B}^{v-\sigma}_{ST}$	$\hat{B}^{v+\sigma}_{ST}$	$\hat{B}^{v+\sigma}_{ST}$	$\hat{B}^{v-\sigma}_{IST}$	$\hat{B}^{v+\sigma}_{IST}$
B_{IST}	$\hat{B}^{v+\varepsilon}_{ST}$	$\hat{B}^{v+\varepsilon}_{ST}$	$\hat{B}^{v+\varepsilon}_{IST}$	$\hat{B}^{v+\varepsilon}_{IST}$	$\hat{B}^{v+\varepsilon}_{IST}$
V^{LO}_{ST}	$\frac{(1-\alpha)}{5-\alpha}2L$	$\frac{2-\alpha}{6-\alpha}2L - \frac{2}{6-\alpha}\sigma$	$\frac{2-\alpha}{6-\alpha}2L - \frac{2}{6-\alpha}\sigma$	$\frac{1}{3}L$	$\frac{2}{3}L - \frac{1}{2}\sigma$
V^{LO}_{IST}	$\frac{(1-\alpha)}{5-\alpha}4L$	$\frac{1-\alpha}{6-\alpha}4L + \frac{1-\alpha}{6-\alpha}\sigma$	$\frac{2}{3}L$	$\frac{2}{3}L$	$\frac{2}{3}L$
Area	$\begin{cases}\alpha \leq \frac{1}{5} \\ \sigma \geq \frac{4}{5-\alpha}L\end{cases}$	$\begin{cases}\sigma \geq \frac{10\alpha}{3(1-\alpha)}L \\ \sigma \leq \frac{4}{5-\alpha}L\end{cases}$	$\begin{cases}\sigma \leq \frac{10\alpha}{3(1-\alpha)}L \\ \sigma \leq \frac{6-5\alpha}{6}L \\ \sigma \geq \frac{8\alpha}{3(2-\alpha)}L\end{cases}$	$\begin{cases}\alpha \geq \frac{1}{5} \\ \sigma \geq \frac{6-5\alpha}{6}L \\ \sigma \geq \frac{2}{3}L\end{cases}$	$\begin{cases}\sigma \leq \frac{8\alpha}{3(2-\alpha)}L \\ \sigma \leq \frac{2}{3}L\end{cases}$

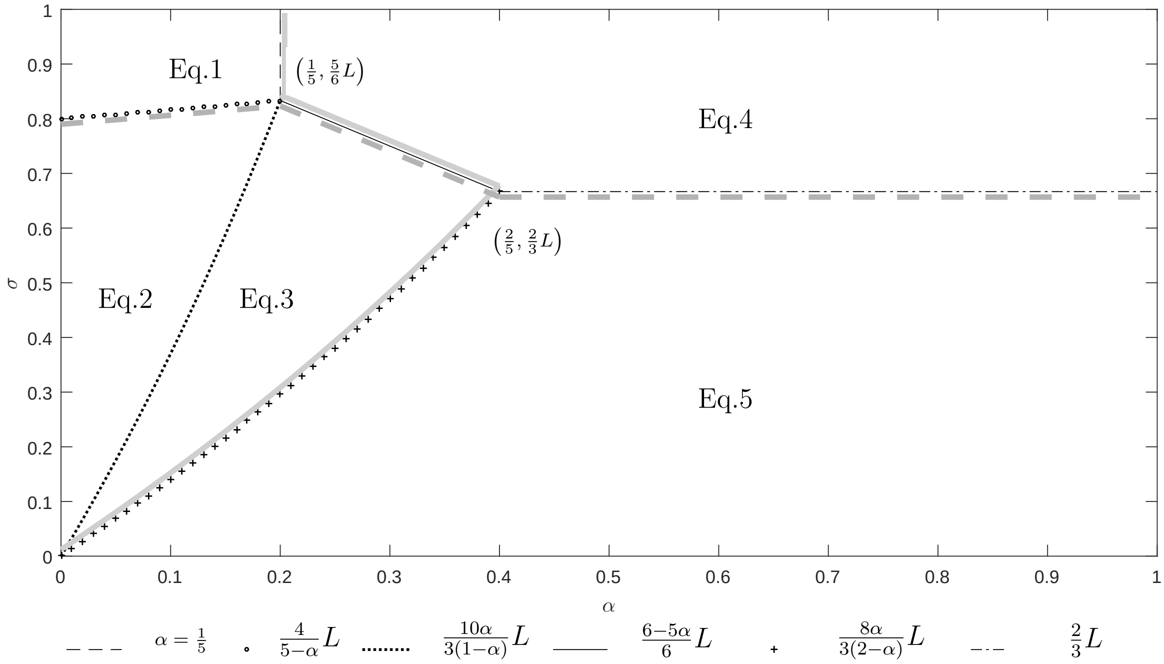


Figure 20: Equilibrium map for the model with ISTs ($L = 1$)

The dashed gray line is σ_3^{border}, the border between the low fill-rate equilibria (above the line: Eq.1 and Eq.4) and the high fill-rate equilibria (below the line: Eq.2, Eq.3, and Eq.5) for an ST. The solid gray line shows α_2^{border}, the border between the unspecialized equilibria (to the right of the line: Eq.4 and Eq.5) and the specialized equilibria (to the left of the line: Eq.1, Eq.2, and Eq.3) for an ST. These borderlines are presented in Equation 4.1.

Proof. In this proof, only those inequalities are considered that have to be solved to find area-equilibria; and only those cases of quote combinations are provided that actually lead to area-equilibria (the other cases from Table 31 can be checked in a similar manner). For this proof, Table 33 is used for reference.

- Eq.1: $B_{ST} = \hat{B}_{ST}^{v-\sigma}, B_{IST} = \hat{B}_{ST}^{v+\varepsilon}$

$$V_{ST}^{LO} = \frac{1-\alpha}{4}(v - \sigma + L - \hat{B}_{ST}^{v-\sigma})$$

$$V_{IST}^{LO} = \frac{1-\alpha}{4}(v - \sigma + L - \hat{B}_{ST}^{v-\sigma}) + \frac{1-\alpha}{4}(v + \sigma + L - \hat{B}_{ST}^{v+\sigma})$$

By simultaneously solving the two previous equations, the values of the outside options are determined if this equilibrium holds (see Table 32). To arrive at the set of parameters under which this equilibrium holds, one needs to solve the following equations, which is merely a comparison of the expected profit from sending a limit order for different orders.

An ST has no incentives to deviate from the equilibrium, if the following is true:

$$\frac{1-\alpha}{4}(v-\sigma+L-\hat{B}_{ST}^{v-\sigma}) \geq \frac{1}{4}(v-\sigma+L-\hat{B}_{IST}^{v-\sigma}),$$
$$\frac{1-\alpha}{4}(v-\sigma+L-\hat{B}_{ST}^{v-\sigma}) \geq \frac{1}{4}(v-\sigma+L-\hat{B}_{ST}^{v+\sigma}) + \frac{1-\alpha}{4}(v+\sigma+L-\hat{B}_{ST}^{v+\sigma}),$$
$$\frac{1-\alpha}{4}(v-\sigma+L-\hat{B}_{ST}^{v-\sigma}) \geq \frac{1}{4}(v-\sigma+L-\hat{B}_{IST}^{v+\sigma}) + \frac{1}{4}(v+\sigma+L-\hat{B}_{IST}^{v+\sigma}).$$

An IST has no incentives to deviate from her choice of a limit order, if the following holds:

$$\frac{1-\alpha}{4}(v-\sigma+L-\hat{B}_{ST}^{v-\sigma})+\frac{1-\alpha}{4}(v+\sigma+L-\hat{B}_{ST}^{v+\sigma}) \geq \frac{1}{4}(v-\sigma+L-\hat{B}_{IST}^{v-\sigma}) + \frac{1}{4}(v+\sigma+L-\hat{B}_{IST}^{v+\sigma}).$$

- Eq.2: $B_{ST} = \hat{B}_{ST}^{v+\sigma}, B_{IST} = \hat{B}_{ST}^{v+\varepsilon}$

$$V_{ST}^{LO} = \frac{1-\alpha}{4}(v-\sigma+L-\hat{B}_{ST}^{v+\sigma}) + \frac{1-\alpha}{4}(v+\sigma+L-\hat{B}_{ST}^{v+\sigma}) + \frac{\alpha}{4}(v-\sigma+L-\hat{B}_{ST}^{v+\sigma})$$
$$V_{IST}^{LO} = \frac{1-\alpha}{4}(v-\sigma+L-\hat{B}_{ST}^{v-\sigma}) + \frac{1-\alpha}{4}(v+\sigma+L-\hat{B}_{ST}^{v+\sigma})$$

An ST has no incentives to deviate from the equilibrium, if the following is true:

$$\frac{1}{4}(v-\sigma+L-\hat{B}_{ST}^{v+\sigma}) + \frac{1-\alpha}{4}(v+\sigma+L-\hat{B}_{ST}^{v+\sigma}) \geq \frac{1}{4}(v-\sigma+L-\hat{B}_{IST}^{v-\sigma}),$$
$$\frac{1}{4}(v-\sigma+L-\hat{B}_{ST}^{v+\sigma}) + \frac{1-\alpha}{4}(v+\sigma+L-\hat{B}_{ST}^{v+\sigma}) \geq \frac{1-\alpha}{4}(v-\sigma+L-\hat{B}_{ST}^{v-\sigma}),$$
$$\frac{1}{4}(v-\sigma+L-\hat{B}_{ST}^{v+\sigma}) + \frac{1-\alpha}{4}(v+\sigma+L-\hat{B}_{ST}^{v+\sigma}) \geq \frac{1}{4}(v-\sigma+L-\hat{B}_{IST}^{v+\sigma}) + \frac{1}{4}(v+\sigma+L-\hat{B}_{IST}^{v+\sigma}).$$

An IST has no incentives to deviate from her choice of a limit order, if the following applies:

$$\frac{1-\alpha}{4}(v-\sigma+L-\hat{B}_{ST}^{v-\sigma})+\frac{1-\alpha}{4}(v+\sigma+L-\hat{B}_{ST}^{v+\sigma}) \geq \frac{1}{4}(v-\sigma+L-\hat{B}_{IST}^{v-\sigma}) + \frac{1}{4}(v+\sigma+L-\hat{B}_{IST}^{v+\sigma}).$$

- Eq.3: $B_{ST} = \hat{B}_{ST}^{v+\sigma}, B_{IST} = \hat{B}_{IST}^{v+\varepsilon}$

$$V_{ST}^{LO} = \frac{1-\alpha}{4}(v-\sigma+L-\hat{B}_{ST}^{v+\sigma}) + \frac{1-\alpha}{4}(v+\sigma+L-\hat{B}_{ST}^{v+\sigma}) + \frac{\alpha}{4}(v-\sigma+L-\hat{B}_{ST}^{v+\sigma})$$
$$V_{IST}^{LO} = \frac{1}{4}(v-\sigma+L-\hat{B}_{IST}^{v-\sigma}) + \frac{1}{4}(v+\sigma+L-\hat{B}_{IST}^{v+\sigma})$$

An ST has no incentives to deviate from the equilibrium, if the following is true:

$$\frac{1}{4}(v-\sigma+L-\hat{B}^{v+\sigma}_{ST})+\frac{1-\alpha}{4}(v+\sigma+L-\hat{B}^{v+\sigma}_{ST})\geq\frac{1}{4}(v-\sigma+L-\hat{B}^{v-\sigma}_{IST}),$$
$$\frac{1}{4}(v-\sigma+L-\hat{B}^{v+\sigma}_{ST})+\frac{1-\alpha}{4}(v+\sigma+L-\hat{B}^{v+\sigma}_{ST})\geq\frac{1-\alpha}{4}(v-\sigma+L-\hat{B}^{v-\sigma}_{ST}),$$
$$\frac{1}{4}(v-\sigma+L-\hat{B}^{v+\sigma}_{ST})+\frac{1-\alpha}{4}(v+\sigma+L-\hat{B}^{v+\sigma}_{ST})$$
$$\geq\frac{1}{4}(v-\sigma+L-\hat{B}^{v+\sigma}_{IST})+\frac{1}{4}(v+\sigma+L-\hat{B}^{v+\sigma}_{IST}).$$

An IST has no incentives to deviate from her choice of a limit order, if the following applies:

$$\frac{1}{4}(v-\sigma+L-\hat{B}^{v-\sigma}_{IST})+\frac{1}{4}(v+\sigma+L-\hat{B}^{v+\sigma}_{IST})$$
$$\geq\frac{1-\alpha}{4}(v-\sigma+L-\hat{B}^{v-\sigma}_{ST})+\frac{1-\alpha}{4}(v+\sigma+L-\hat{B}^{v+\sigma}_{ST}).$$

- Eq.4: $B_{IST}=\hat{B}^{v-\sigma}_{IST}, B_{IST}=\hat{B}^{v+\varepsilon}_{IST}$

$$V^{LO}_{ST}=\frac{1-\alpha}{4}(v-\sigma+L-\hat{B}^{v-\sigma}_{IST})+\frac{\alpha}{4}(v-\sigma+L-\hat{B}^{v-\sigma}_{IST})$$
$$V^{LO}_{IST}=\frac{1}{4}(v-\sigma+L-\hat{B}^{v-\sigma}_{IST})+\frac{1}{4}(v+\sigma+L-\hat{B}^{v+\sigma}_{IST})$$

An ST has no incentives to deviate from the equilibrium, if the following is true:

$$\frac{1}{4}(v-\sigma+L-\hat{B}^{v-\sigma}_{IST})\geq\frac{1}{4}(v-\sigma+L-\hat{B}^{v+\sigma}_{ST})+\frac{1-\alpha}{4}(v+\sigma+L-\hat{B}^{v+\sigma}_{ST}),$$
$$\frac{1}{4}(v-\sigma+L-\hat{B}^{v-\sigma}_{IST})\geq\frac{1-\alpha}{4}(v-\sigma+L-\hat{B}^{v-\sigma}_{ST}),$$
$$\frac{1}{4}(v-\sigma+L-\hat{B}^{v-\sigma}_{IST})\geq\frac{1}{4}(v-\sigma+L-\hat{B}^{v+\sigma}_{IST})+\frac{1}{4}(v+\sigma+L-\hat{B}^{v+\sigma}_{IST}).$$

An IST has no incentives to deviate from her choice of a limit order, if the following applies:

$$\frac{1}{4}(v-\sigma+L-\hat{B}^{v-\sigma}_{IST})+\frac{1}{4}(v+\sigma+L-\hat{B}^{v+\sigma}_{IST})$$
$$\geq\frac{1-\alpha}{4}(v-\sigma+L-\hat{B}^{v-\sigma}_{ST})+\frac{1-\alpha}{4}(v+\sigma+L-\hat{B}^{v+\sigma}_{ST}).$$

- Eq.5: $B_{IST}=\hat{B}^{v+\sigma}_{IST}, B_{IST}=\hat{B}^{v+\varepsilon}_{IST}$

$$V^{LO}_{ST}=\frac{1}{4}(v-\sigma+L-\hat{B}^{v+\sigma}_{IST})+\frac{1}{4}(v-\sigma+L-\hat{B}^{v+\sigma}_{IST})$$
$$V^{LO}_{IST}=\frac{1}{4}(v-\sigma+L-\hat{B}^{v-\sigma}_{IST})+\frac{1}{4}(v+\sigma+L-\hat{B}^{v+\sigma}_{IST})$$

An ST has no incentives to deviate from the equilibrium, if the following is true:

$$\begin{aligned}
&\frac{1}{4}(v-\sigma+L-\hat{B}_{IST}^{v+\sigma})+\frac{1}{4}(v+\sigma+L-\hat{B}_{IST}^{v+\sigma})\\
&\qquad\geq\frac{1}{4}(v-\sigma+L-\hat{B}_{ST}^{v+\sigma})+\frac{1-\alpha}{4}(v+\sigma+L-\hat{B}_{ST}^{v+\sigma}),\\
&\frac{1}{4}(v-\sigma+L-\hat{B}_{IST}^{v+\sigma})+\frac{1}{4}(v+\sigma+L-\hat{B}_{IST}^{v+\sigma})\geq\frac{1-\alpha}{4}(v-\sigma+L-\hat{B}_{ST}^{v-\sigma}),\\
&\frac{1}{4}(v-\sigma+L-\hat{B}_{IST}^{v+\sigma})+\frac{1}{4}(v+\sigma+L-\hat{B}_{IST}^{v+\sigma})\geq\frac{1}{4}(v-\sigma+L-\hat{B}_{IST}^{v-\sigma}).
\end{aligned}$$

An IST has no incentives to deviate from her choice of a limit order, if the following applies:

$$\begin{aligned}
&\frac{1}{4}(v-\sigma+L-\hat{B}_{IST}^{v-\sigma})+\frac{1}{4}(v+\sigma+L-\hat{B}_{IST}^{v+\sigma})\\
&\qquad\geq\frac{1-\alpha}{4}(v-\sigma+L-\hat{B}_{ST}^{v-\sigma})+\frac{1-\alpha}{4}(v+\sigma+L-\hat{B}_{ST}^{v+\sigma}).\qquad\square
\end{aligned}$$

4.3 Market metrics

In the next step, the market metrics for the modified analytical model with ISTs are discussed following the procedure of Section 3.4. The proofs here are similar to those presented in Hoffmann (2014). Our analysis is concerned with graphical representations of the market measures.

4.3.1 Execution probabilities

Table 33 provides the execution probabilities of all possible order values for both agent types.

Table 33: Execution probabilities in the model with ISTs

	Eq.1	Eq.2	Eq.3	Eq.4	Eq.5
$p_{\|ST,-\sigma}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$p_{\|ST,+\sigma}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$
$p_{\|ST}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$
$p_{\|IST,-\sigma}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$p_{\|IST,+\sigma}$	0	0	0	0	$\frac{1}{2}$
$p_{\|IST}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$p_{\|-\sigma}$	$\frac{1-\alpha}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$p_{\|+\sigma}$	0	$\frac{1-\alpha}{2}$	$\frac{1-\alpha}{2}$	0	$\frac{1}{2}$
p	$\frac{1-\alpha}{4}$	$\frac{2-\alpha}{4}$	$\frac{2-\alpha}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

	Eq.1	Eq.2	Eq.3	Eq.4	Eq.5
$q_{\|ST,-\sigma}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$q_{\|ST,+\sigma}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$q_{\|ST}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$q_{\|IST,-\sigma}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$q_{\|IST,+\sigma}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$q_{\|IST}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$q_{\|-\sigma}$	$\frac{1-\alpha}{2}$	$\frac{1-\alpha}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$q_{\|+\sigma}$	$\frac{1-\alpha}{2}$	$\frac{1-\alpha}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
q	$\frac{1-\alpha}{2}$	$\frac{1-\alpha}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$q_{|ST}$ is the same for all the equilibria, since an IST knows the exact future fundamental value development.

Corollary 4.1. *An ST follows the specialized strategies if the share of the ISTs is small on the market, when $\alpha \leq \alpha_2^{border}$; an IST uses the specialized strategies for even a lower share, when $\alpha \leq \alpha_3^{border}$. For high shares, both the STs and ISTs turn to the unspecialized equilibria.*
An ST follows the high fill-rate strategies for low volatility, when $\sigma \leq \sigma_3^{border}$; otherwise an ST prefers the low fill-rate strategies. An IST follows the high fill-rate strategies regardless of the share and volatility values.

Proof. The starting point is a classification of the equilibria with respect to their degree of fill-rate and specialization (see Table 34) depending on the execution probabilities of quotes (see Table 33).

Table 34: Classification of the equilibria in the model with ISTs

	Fill-rate type		Specialization type	
	ST	*IST*	*ST*	*IST*
Eq.1	low	high	specialized	specialized
Eq.2	high	high	specialized	specialized
Eq.3	high	high	specialized	unspecialized
Eq.4	low	high	unspecialized	unspecialized
Eq.5	high	high	unspecialized	unspecialized

Defining the following borderlines, the corollary follows directly:

$$\alpha_2^{border} = \begin{cases} \frac{1}{5} \\ \frac{(6-5\alpha)}{6}L \\ \frac{8\alpha}{3(2-\alpha)}L, \end{cases} \qquad \alpha_3^{border} = \begin{cases} \frac{1}{5} \\ \frac{4}{5-\alpha}L, \end{cases} \qquad \sigma_3^{border} = \begin{cases} \frac{4}{5-\alpha}L \\ \frac{(6-5\alpha)}{6}L \\ \frac{2}{3}L. \end{cases} \qquad \square \tag{4.1}$$

For the model with ISTs, similar to Hoffmann (2014), more severe adverse selection under the condition of high volatility leads to less aggressive limit orders sent by STs, while low volatility allows STs to post more aggressive limit orders. ISTs always use the high fill-rate strategies, as they know the next precise fundamental value change. For a low ISTs' share, STs prefer not to target their quotes on ISTs, since a more aggressive order increases the execution probability only slightly. The unspecialized strategies are followed by STs only when the share of ISTs is high enough. ISTs, in their turn, post limit orders specialized on STs only, if the share of these agents is high enough (when the share of ISTs is low): even though a focus on STs brings an additional spread to ISTs, this decreases the execution probability, since the next IST does not satisfy such a specialized limit order.

4.3.2 Probabilities of equilibrium events

Given the four possible equilibrium events and the shares of the agent types, transition matrix $\mathbf{P}$ is constructed based on the data of Table 35:

Table 35: Transition probabilities in the model with ISTs

Event$_t$ \ Event$_{t+1}$	an ST sending an LO	an ST sending an MO	an IST sending an LO	an IST sending an MO
an ST sending an LO	$(1-\alpha)(1-p_{\|ST})$	$(1-\alpha)p_{\|ST}$	$\alpha(1-p_{\|IST})$	$\alpha p_{\|IST}$
an ST sending an MO	$1-\alpha$	0	α	0
an IST sending an LO	$(1-\alpha)(1-q_{\|ST})$	$(1-\alpha)q_{\|ST}$	$\alpha(1-q_{\|IST})$	$\alpha q_{\|IST}$
an IST sending an MO	$1-\alpha$	0	α	0

Since $q_{|ST} = \frac{1}{2}$ for all the equilibria, this probability is substituted by its value in the analytical derivations.

$$\mathbf{P} = \begin{pmatrix} (1-\alpha)(1-p_{|ST}) & (1-\alpha)p_{|ST} & \alpha(1-p_{|IST}) & \alpha p_{|IST} \\ 1-\alpha & 0 & \alpha & 0 \\ (1-\alpha)\frac{1}{2} & (1-\alpha)\frac{1}{2} & \alpha(1-q_{|IST}) & \alpha q_{|IST} \\ 1-\alpha & 0 & \alpha & 0 \end{pmatrix}$$

The left eigenvector of $\mathbf{P}$ associated with the unit modulus gives the following stationary probability distribution $\varphi = (\varphi_{ST}^{LO}, \varphi_{ST}^{MO}, \varphi_{IST}^{LO}, \varphi_{IST}^{MO})$:

$$\begin{aligned} \varphi_{ST}^{LO} &= \frac{1-\alpha}{\chi}\big(1-\alpha(\frac{1}{2}-q_{|IST})\big), \\ \varphi_{ST}^{MO} &= \frac{1-\alpha}{\chi}\big(\chi-1+\alpha(\frac{1}{2}-q_{|IST})\big), \\ \varphi_{IST}^{LO} &= \frac{\alpha}{\chi}\big(1+(1-\alpha)(p_{|ST}-p_{|IST})\big), \\ \varphi_{IST}^{MO} &= \frac{\alpha}{\chi}\big(\chi-1-(1-\alpha)(p_{|ST}-p_{|IST})\big), \end{aligned}$$

where $\chi = (1+\alpha q_{|IST})(1+(1-\alpha)p_{|ST}) - \frac{\alpha(1-\alpha)}{2}p_{|IST}$.

4.3.3 Bargaining power

In this section, the ISTs' bargaining power is compared with that of STs with ISTs' absence and presence.

Corollary 4.2. *Under the equilibrium conditions, the following holds true:*

(a) The ISTs' entry negatively affects the STs' bargaining power: $V_0^{LO} \geq V_{ST}^{LO}$.

(b) The ISTs' bargaining power is higher than the STs' bargaining power both with and without the ISTs' presence: $V_{IST}^{LO} \geq V_0^{LO} \geq V_{ST}^{LO}$. For small shares, the ISTs' bargaining power is a decreasing function of their share; for high shares, the ISTs' bargaining power is independent of the share.

Proof. These conclusions follow directly from the graphical representations of the respective functions in Figure 21 and Figure 22, but can also be proved analytically. □

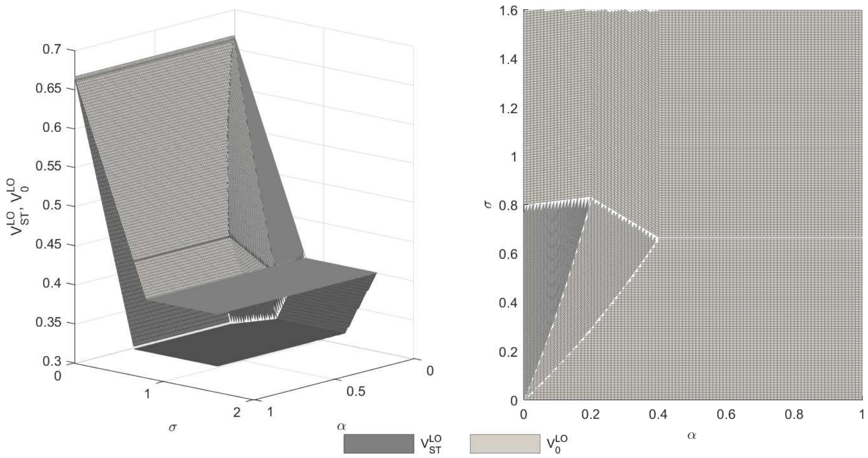


Figure 21: STs' bargaining power with the ISTs' presence and absence

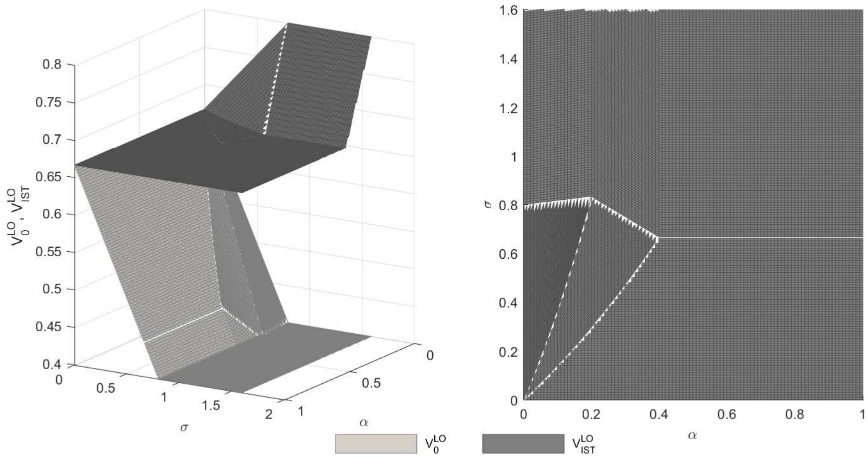


Figure 22: ISTs' bargaining power and that of STs with the ISTs' absence

4.3.4 Trading rate

The trading rates in the model with ISTs are determined in the same way as described in Section 3.4.4:

$$TR = \begin{cases} \frac{1-\alpha^2}{5-\alpha} & \text{for Eq.1} \\ \frac{(4+\alpha)(1-\alpha)}{4(3-\alpha)} & \text{for Eq.2} \\ \frac{4-\alpha+\alpha^2}{12+\alpha-\alpha^2} & \text{for Eq.3} \\ \frac{1+\alpha}{5+\alpha} & \text{for Eq.4} \\ \frac{1}{3} & \text{for Eq.5.} \end{cases}$$

Corollary 4.3. *In equilibrium, the following holds:*

(a) The trading rates for the low fill-rate equilibria are lower than those for the high fill-rate equilibria.

(b) For the high fill-rate unspecialized equilibrium (Eq.5), the trading rate is independent of the share. For the low fill-rate unspecialized equilibrium, the trading rate increases with the share (Eq.4, corresponds to the high share - high volatility states). The trading rate drops when the share goes up for the high fill-rate specialized equilibria (Eq.2 and Eq.3, corresponding to the low share - low volatility states).

(c) The ISTs' entry improves the trading rate for the majority of the market states. Only for the specialized high fill-rate equilibria (Eq.2 and Eq.3, corresponding to the low share - low volatility states), the trading rate falls.

Proof. These conclusions follow directly from the graphical representations of the respective functions in Figure 23 and Figure 24, but can also be proved analytically. □

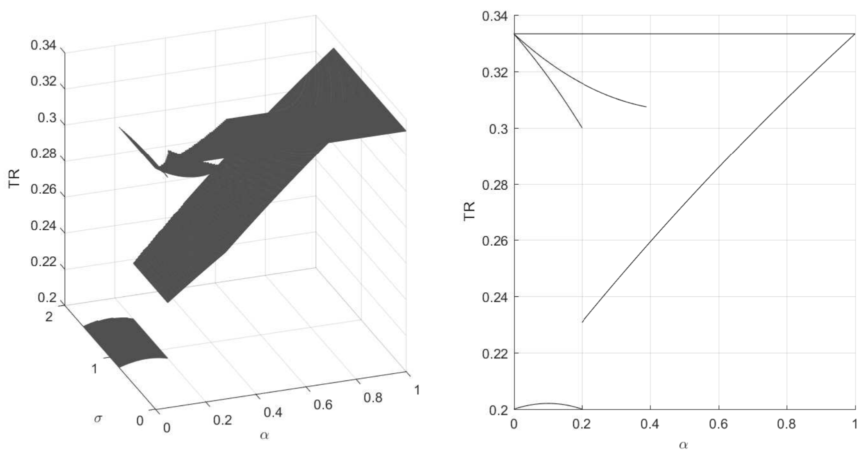


Figure 23: Trading rate for the model with ISTs

Just like FTs, ISTs have two-fold effects on the trading rate. On the one hand, ISTs' exact knowledge about the fundamental value change reduces the degree of caution with which ISTs

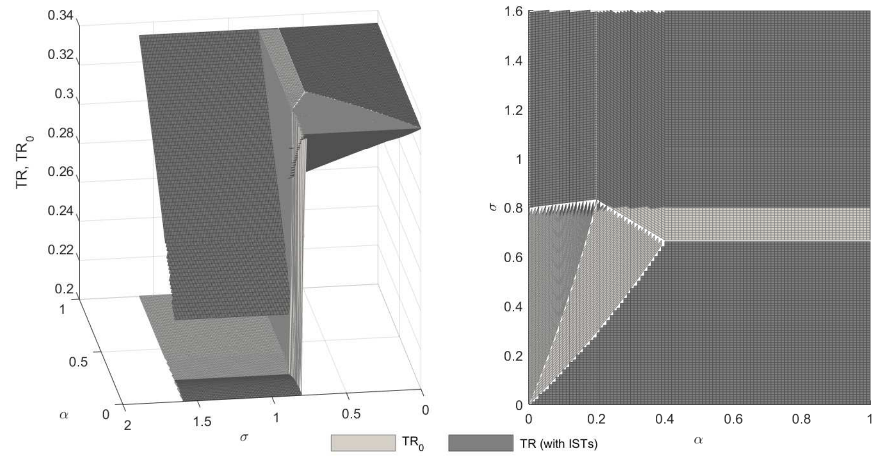


Figure 24: Trading rate for the model with ISTs and with STs-only

post limit orders, which in its turn increases the trading rate. On the other hand, STs receive a trade-off between the aggressiveness of quotes and their profitability. The second effect seems to be more pronounced than the first one for the low share - low volatility states, which reduces the trading rate. In all the other market states, the first effect is greater, therefore ISTs' entry influences the trading rate positively.

4.3.5 Risk of being picked-off

In the modified model with ISTs, the risk of being picked-off is experienced by STs only: ISTs always know the next fundamental price change and post their quotes based on the "correct" value:

$$\pi_{ST} = \begin{cases} \frac{\frac{1}{2}p_{|-\sigma}}{p} & \text{for Eq.2, Eq.3, and Eq.5} \\ 0 & \text{for Eq.1 and Eq.4,} \end{cases} \qquad \pi_{IST} = 0.$$

Plugging in the respective probabilities from Table 33 leads to the following results:

$$\pi_{ST} = \begin{cases} \frac{1}{2-\alpha} & \text{for Eq.2 and Eq.3} \\ \frac{1}{2} & \text{for Eq.5} \\ 0 & \text{for Eq.1 and Eq.4.} \end{cases}$$

Corollary 4.4. *In equilibrium, the risk of being picked-off for the STs does not increase with the ISTs' entry in most of the market states. Only for the specialized high fill-rate equilibria (Eq.2 and Eq.3, corresponding to the low share - low volatility states), the risk of being picked-off rises, while for the same equilibria with moderate volatility, the risk decreases.*

Proof. These conclusions follow directly from the functional form of the STs' risk of being picked-off. □

For STs, only in the specialized high fill-rate equilibria (Eq.2 and Eq.3), the risk of being picked-off depends on the share: the more ISTs there are, the higher is the risk of being picked-off for STs. The low share of ISTs allows them to pick off the STs' stale orders, but only as long as the volatility of the market is low.

4.3.6 Costs of immediacy

The costs of immediacy formula for the model with ISTs transforms to:

$$E(\tau_i) = \frac{\omega^{-\sigma}_{ST,i}\tau^{-\sigma}_{ST,i} + \omega^{+\sigma}_{ST,i}\tau^{+\sigma}_{ST,i} + \omega^{-\sigma}_{IST,i}\tau^{-\sigma}_{IST,i} + \omega^{+\sigma}_{IST,i}\tau^{+\sigma}_{IST,i}}{\omega^{-\sigma}_{ST,i} + \omega^{+\sigma}_{ST,i} + \omega^{-\sigma}_{IST,i} + \omega^{+\sigma}_{IST,i}},$$

where:

$$\begin{aligned} \omega^{\varepsilon}_{ST,ST} &= \varphi^{LO}_{ST} \cdot p_{|ST,\varepsilon} \cdot (1-\alpha), & \omega^{\varepsilon}_{IST,ST} &= \varphi^{LO}_{IST} \cdot q_{|ST,\varepsilon} \cdot (1-\alpha), \\ \omega^{\varepsilon}_{ST,IST} &= \varphi^{LO}_{ST} \cdot p_{|IST,\varepsilon} \cdot \alpha, & \omega^{\varepsilon}_{IST,IST} &= \varphi^{LO}_{IST} \cdot q_{|IST,\varepsilon} \cdot \alpha. \end{aligned}$$

An IST cannot adjust her quote to the coming agent's cutoff price level. Therefore, the costs of immediacy for an IST and an ST for executing the same quote are the same: $\tau^{\varepsilon}_{ST,ST} = \tau^{\varepsilon}_{ST,IST}$ and $\tau^{\varepsilon}_{IST,ST} = \tau^{\varepsilon}_{IST,IST}$. The costs of immediacy for each agent type are the following:

$$E(\tau_{ST}) = \begin{cases} \frac{3+\alpha}{5-\alpha}L & \text{for Eq.1} \\ \frac{2+\alpha}{6-\alpha}L + \frac{2(\alpha^3-8\alpha^2+19\alpha-8)}{(6-\alpha)(4-\alpha+\alpha^2)}\sigma & \text{for Eq.2} \\ \frac{\alpha^3-23\alpha^2+18\alpha+24}{3(6-\alpha)(4+\alpha-\alpha^2)}L - \frac{4(1-\alpha)(4-\alpha)}{(6-\alpha)(4+\alpha-\alpha^2)}\sigma & \text{for Eq.3} \\ \frac{1}{3}L & \text{for Eq.4} \\ \frac{1}{3}L - (1-\alpha)\sigma & \text{for Eq.5,} \end{cases}$$

$$E(\tau_{IST}) = \begin{cases} 0 & \text{for Eq.1} \\ \frac{2+\alpha}{6-\alpha}L - \frac{2(5-\alpha)}{6-\alpha}\sigma & \text{for Eq.2} \\ \frac{\alpha^3-17\alpha^2+24\alpha+12}{3(6-\alpha)(2+3\alpha-\alpha^2)}L - \frac{4(1-\alpha)(5-\alpha)}{(6-\alpha)(2+3\alpha-\alpha^2)}\sigma & \text{for Eq.3} \\ \frac{1}{3}L & \text{for Eq.4} \\ \frac{1}{3}L - (1-\alpha)\sigma & \text{for Eq.5.} \end{cases}$$

Table 36 lists all the values necessary for computing the costs of immediacy.

Corollary 4.5. *In equilibrium, the following holds:*

(a) The STs' costs of immediacy are greater than those of the ISTs: $E(\tau_{ST}) > E(\tau_{IST})$ for the specialized equilibria only (that correspond to the low share of the ISTs). In the unspecialized equilibria, the ISTs' and STs' costs of immediacy are the same.

(b) With the ISTs' entry, the STs' costs of immediacy increase for almost all the market

Table 36: Costs of immediacy in the model with ISTs: auxiliary values

	Eq.1	Eq.2	Eq.3	Eq.4	Eq.5
$\tau^{-\sigma}_{ST,ST/IST}$	$\frac{3+\alpha}{5-\alpha}L$	$\frac{2+\alpha}{6-\alpha}L-\frac{2(5-\alpha)}{6-\alpha}\sigma$	$\frac{2+\alpha}{6-\alpha}L-\frac{2(5-\alpha)}{6-\alpha}\sigma$	$\frac{1}{3}L$	$\frac{1}{3}L-2\sigma$
$\tau^{+\sigma}_{ST,ST/IST}$	$\frac{3+\alpha}{5-\alpha}L+2\sigma$	$\frac{2+\alpha}{6-\alpha}L+\frac{2}{6-\alpha}\sigma$	$\frac{2+\alpha}{6-\alpha}L+\frac{2}{6-\alpha}\sigma$	$\frac{1}{3}L+2\sigma$	$\frac{1}{3}L$
$\tau^{-\sigma}_{IST,ST/IST}$	$\frac{3+\alpha}{5-\alpha}L$	$\frac{2+\alpha}{6-\alpha}L+\frac{2}{6-\alpha}\sigma$	$\frac{1}{3}L$	$\frac{1}{3}L$	$\frac{1}{3}L$
$\tau^{+\sigma}_{IST,ST/IST}$	$\frac{3+\alpha}{5-\alpha}L$	$\frac{2+\alpha}{6-\alpha}L+\frac{2}{6-\alpha}\sigma$	$\frac{1}{3}L$	$\frac{1}{3}L$	$\frac{1}{3}L$
$\omega^{-\sigma}_{ST,ST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{ST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{ST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{ST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{ST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{ST}$
$\omega^{+\sigma}_{ST,ST}$	0	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{ST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{ST}$	0	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{ST}$
$\omega^{-\sigma}_{IST,ST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{IST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{IST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{IST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{IST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{IST}$
$\omega^{+\sigma}_{IST,ST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{IST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{IST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{IST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{IST}$	$\frac{1}{2}(1-\alpha)\varphi^{LO}_{IST}$
$\omega^{-\sigma}_{ST,IST}$	0	$\frac{1}{2}\alpha\varphi^{LO}_{ST}$	$\frac{1}{2}\alpha\varphi^{LO}_{ST}$	$\frac{1}{2}\alpha\varphi^{LO}_{ST}$	$\frac{1}{2}\alpha\varphi^{LO}_{ST}$
$\omega^{+\sigma}_{ST,IST}$	0	0	0	0	$\frac{1}{2}\alpha\varphi^{LO}_{ST}$
$\omega^{-\sigma}_{IST,IST}$	0	0	$\frac{1}{2}\alpha\varphi^{LO}_{IST}$	$\frac{1}{2}\alpha\varphi^{LO}_{IST}$	$\frac{1}{2}\alpha\varphi^{LO}_{IST}$
$\omega^{+\sigma}_{IST,IST}$	0	0	$\frac{1}{2}\alpha\varphi^{LO}_{IST}$	$\frac{1}{2}\alpha\varphi^{LO}_{IST}$	$\frac{1}{2}\alpha\varphi^{LO}_{IST}$
χ	$\frac{5-\alpha}{4}$	$\frac{12-5\alpha+\alpha^2}{8}$	$\frac{12+\alpha-\alpha^2}{8}$	$\frac{5+\alpha}{4}$	$\frac{3}{2}$
φ^{LO}_{ST}	$\frac{2(1-\alpha)(2-\alpha)}{5-\alpha}$	$\frac{4(1-\alpha)(2-\alpha)}{12-5\alpha+\alpha^2}$	$\frac{8(1-\alpha)}{12+\alpha-\alpha^2}$	$\frac{4(1-\alpha)}{5+\alpha}$	$\frac{2(1-\alpha)}{3}$
φ^{LO}_{IST}	α	$\frac{2\alpha(5-\alpha)}{12-5\alpha+\alpha^2}$	$\frac{2\alpha(5-\alpha)}{12+\alpha-\alpha^2}$	$\frac{4\alpha}{5+\alpha}$	$\frac{2\alpha}{3}$

states ($E(\tau_{ST}) > E(\tau_0)$); only for the moderate to high share - high volatility states, the STs' costs of immediacy improve.

(c) *If volatility is high, the ISTs' costs of immediacy are better (i.e. smaller) than those in the STs-only market ($E(\tau_0) > E(\tau_{IST})$); if volatility is low, the ISTs' costs of immediacy are not lower.*

Proof. These conclusions follow directly from the graphical representations of the respective functions in Figure 25, Figure 26, and Figure 27, but can also be proved analytically. □

For Eq.1, the ISTs' costs of immediacy is a non-defined value, since the zero numerator has to be divided by the zero denominator. An IST does not take limit orders from the market, regardless of who posts these orders. However, an IST still participates in trade through limit orders. The agents' costs of immediacy for the high share of ISTs (corresponding to the unspecialized equilibria Eq.4 and Eq.5) coincide. The ISTs' and STs' costs of immediacy are the same, if STs aim at limit orders execution by ISTs.

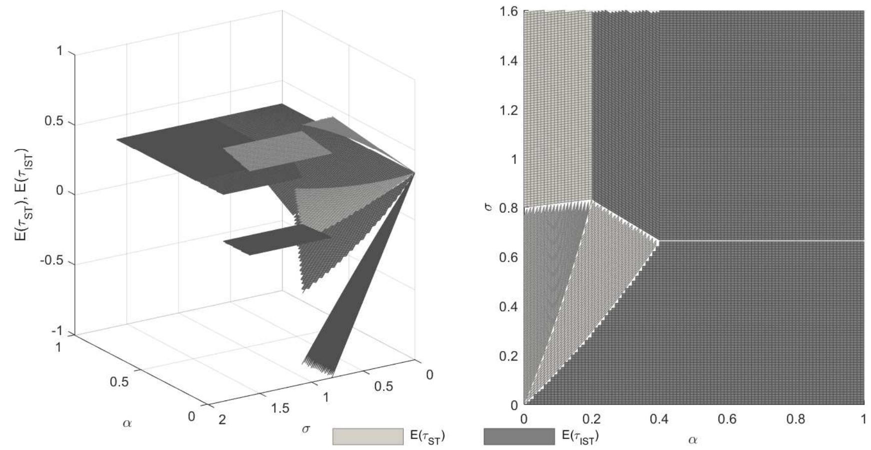


Figure 25: ISTs' and STs' costs of immediacy

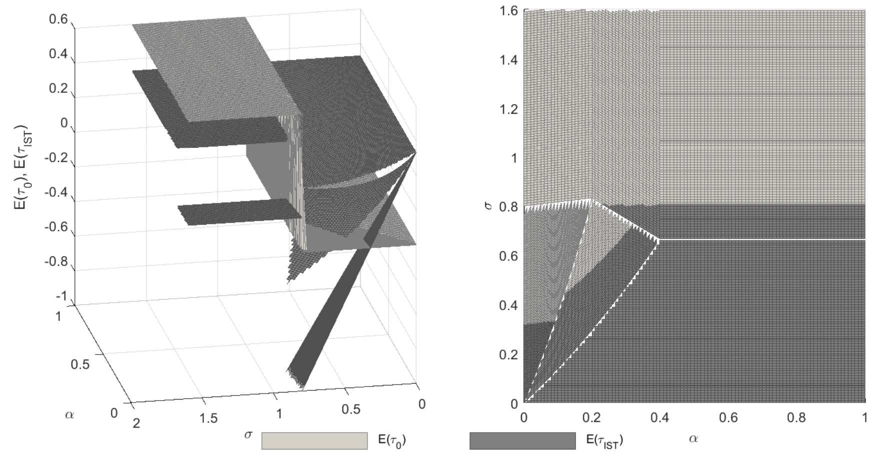


Figure 26: ISTs' costs of immediacy and those of STs with the ISTs' absence

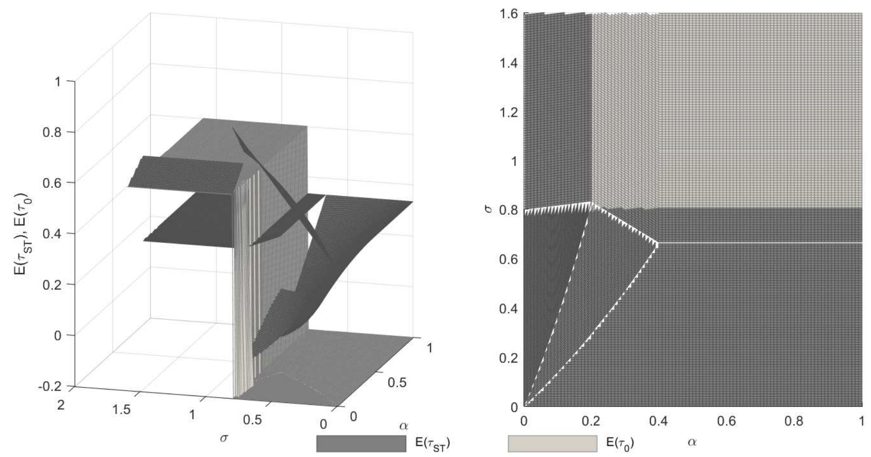


Figure 27: STs' costs of immediacy with the ISTs' presence and absence

4.3.7 Maker-taker ratio

The maker-taker ratio for the model with ISTs is calculated similarly to the procedure described in Section 3.4.7:

$$MT_{ST} = \begin{cases} \frac{(1-\alpha)(2-\alpha)}{2(1+\alpha)} & \text{for Eq.1} \\ \frac{(2-\alpha)^2}{4-\alpha+\alpha^2} & \text{for Eq.2} \\ \frac{2(2-\alpha)}{4+\alpha-\alpha^2} & \text{for Eq.3} \\ \frac{1}{1+\alpha} & \text{for Eq.4} \\ 1 & \text{for Eq.5,} \end{cases} \qquad MT_{IST} = \begin{cases} \infty(\text{only makes the market}) & \text{for Eq.1} \\ \frac{5-\alpha}{2-\alpha} & \text{for Eq.2} \\ \frac{5-\alpha}{2+3\alpha-\alpha^2} & \text{for Eq.3} \\ \frac{2}{1+\alpha} & \text{for Eq.4} \\ 1 & \text{for Eq.5.} \end{cases}$$

Corollary 4.6. *In equilibrium, the following holds:*

(a) For all the market states, an IST is a market-maker, while an ST is a market-taker: $MT_{IST} \geq 1 \geq MT_{ST}$. In the unspecialized high fill-rate equilibrium (Eq.5), both agent types make the market just as much as they take it.

(b) The degree of market-making by the ISTs increases with high volatility.

Proof. These conclusions are supported by the graphical representation of the respective functions in Figure 28 and the following logic. If the volatility σ increases and the parameters (α, L) stay unchanged, the following transitions between the equilibria are possible (first the longest path is shown, followed by shorter or partial paths):

(a) Eq.5 $\rightarrow$ Eq.3 $\rightarrow$ Eq.2 $\rightarrow$ Eq.1: Eq.5 $\rightarrow$ Eq.3, Eq.5 $\rightarrow$ Eq.2, Eq.5 $\rightarrow$ Eq.1, Eq.3 $\rightarrow$ Eq.2, Eq.3 $\rightarrow$ Eq.1, Eq.2 $\rightarrow$ Eq.1,

(b) Eq.5 $\rightarrow$ Eq.3 $\rightarrow$ Eq.4, paths not mentioned before: Eq.5 $\rightarrow$ Eq.4, Eq.3 $\rightarrow$ Eq.4.

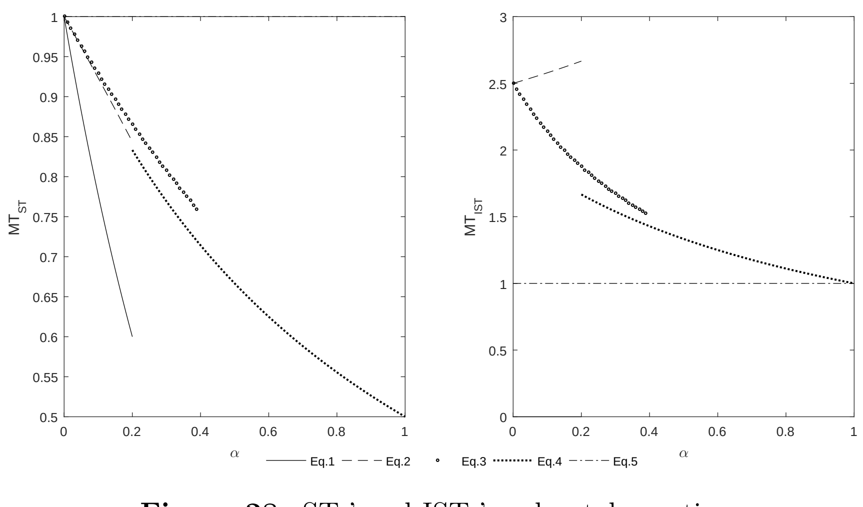


Figure 28: STs' and ISTs' maker-taker ratio

Eq.1 is omitted from the right subplot, ISTs' maker-taker ratio in this equilibrium is the highest and converges to infinity.

As Figure 28 shows, the STs' maker-taker ratio decreases, only if the volatility goes up; the ISTs' maker-taker ratio rises with increased volatility with one exception: for the transition Eq.3 $\rightarrow$ Eq.4, it drops (for the share of ISTs a bit above the quarter level and for high to medium volatility). $\square$

The ISTs' maker-taker ratio is always above or equal to one, since they know the exact next fundamental value change and they are less likely to submit market orders (accept existing limit orders) than STs because of their higher cutoff prices.

4.3.8 Pricing error

The pricing error in the model with ISTs is determined as:

$$PE = \frac{\varphi_{ST}^{LO}}{\varphi_{ST}^{LO} + \varphi_{IST}^{LO}} PE_{ST} + \frac{\varphi_{IST}^{LO}}{\varphi_{ST}^{LO} + \varphi_{IST}^{LO}} PE_{IST},$$

where

$$\begin{aligned} PE_{ST} &= \frac{1}{2}|v - \sigma - B_{ST}^{m}| + \frac{1}{2}|v + \sigma - B_{ST}^{m}|, \\ PE_{IST} &= \frac{1}{2}|v - \sigma - \hat{B}_{k}^{v-\sigma}| + \frac{1}{2}|v + \sigma - \hat{B}_{k}^{v-\sigma}|, \end{aligned}$$

and $k = \{ST, IST\}$ depending on the IST's quote choice. The resulting pricing error for the whole economy is calculated by applying the following formula:

$$PE = \begin{cases} \frac{(3+\alpha)}{(5-\alpha)}L + \frac{2(1-\alpha)(2-\alpha)}{4-\alpha+\alpha^2}\sigma & \text{for Eq.1} \\ \frac{(2+\alpha)}{(6-\alpha)}L + \frac{2(\alpha^3-8\alpha^2+19\alpha-8)}{(6-\alpha)(4-\alpha+\alpha^2)}\sigma & \text{for Eq.2 and } \sigma \leq \frac{2+\alpha}{2(5-\alpha)}L \\ \frac{\alpha(5-\alpha)(2+\alpha)}{(6-\alpha)(\alpha^2-\alpha+4)}L - \frac{2(\alpha^3-8\alpha^2+15\alpha-12)}{(6-\alpha)(4-\alpha+\alpha^2)}\sigma & \text{for Eq.2 and } \sigma \geq \frac{2+\alpha}{2(5-\alpha)}L \\ \frac{\alpha^3-23\alpha^2+18\alpha+24}{3(6-\alpha)(4+\alpha-\alpha^2)}L + \frac{4(1-\alpha)(\alpha-4)}{(6-\alpha)(4+\alpha-\alpha^2)}\sigma & \text{for Eq.3 and } \sigma \leq \frac{2+\alpha}{2(5-\alpha)}L \\ \frac{\alpha(5-\alpha)}{3(4+\alpha-\alpha^2)}L + \frac{4(1-\alpha)}{4+\alpha-\alpha^2}\sigma & \text{for Eq.3 and } \sigma \geq \frac{2+\alpha}{2(5-\alpha)}L \\ \frac{1}{3}L + (1-\alpha)\sigma & \text{for Eq.4} \\ \frac{1}{3}L - (1-\alpha)\sigma & \text{for Eq.5 and } \sigma \leq \frac{1}{6}L \\ \frac{\alpha}{3}L + (1-\alpha)\sigma & \text{for Eq.5 and } \sigma \geq \frac{1}{6}L. \end{cases}$$

Corollary 4.7. *In equilibrium, the ISTs' entry decreases (increases) the pricing error of the market quotes under the conditions of high (low) volatility, $PE \leq PE_0$ ($PE \geq PE_0$); there is one small exception region for the low share - low volatility states.*

Proof. These conclusions follow directly from the graphical representations of the respective functions in Figure 29, but can also be proved analytically. $\square$

The ISTs' influence on the pricing error is two-fold. On the one hand, their knowledge of the next fundamental value change brings orders closer to the fundamental value. On the other

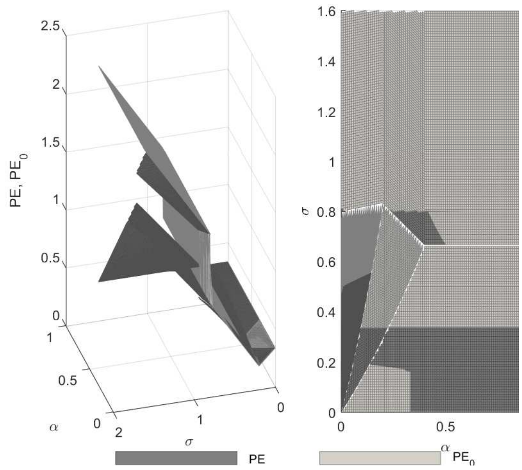


Figure 29: Pricing error in the model with ISTs

hand, the ISTs' higher bargaining power can increase the pricing error. If volatility is high, the first effect is more pronounced, which improves the pricing error of the whole economy. The same effect is also possible for the low share - low volatility states.

4.3.9 Welfare

To compute the welfare for each agent type, Formula 2.3 is applied, where $i = ST$ or $i = IST$. The aggregate market welfare is:

$$W = \alpha W_{ST} + (1-\alpha) W_{IST} = 2L \cdot TR.$$

The analytical expressions for the welfare of the agent types are:

$$W_{ST} = \begin{cases} \frac{2(1-\alpha)(5-3\alpha)}{(5-\alpha)^2} L & \text{for Eq.1} \\ \frac{2(24-14\alpha+7\alpha^2-\alpha^3)}{(6-\alpha)(12-5\alpha+\alpha^2)} L - \frac{2\alpha(5-\alpha)(3-\alpha)}{(6-\alpha)(12-5\alpha+\alpha^2)}\sigma & \text{for Eq.2} \\ \frac{2(72-30\alpha+\alpha^2+\alpha^3)}{3(6-\alpha)(12+\alpha-\alpha^2)} L - \frac{4\alpha(5-\alpha)}{(6-\alpha)(12+\alpha-\alpha^2)}\sigma & \text{for Eq.3} \\ \frac{2(3+\alpha)}{3(5+\alpha)} L & \text{for Eq.4} \\ \frac{2}{3}L - \frac{\alpha}{3}\sigma & \text{for Eq.5,} \end{cases}$$

$$W_{IST} = \begin{cases} \frac{4(1-\alpha)}{5-\alpha} L & \text{for Eq.1} \\ \frac{2(1-\alpha)(24-8\alpha+\alpha^2)}{(6-\alpha)(12-5\alpha+\alpha^2)} L + \frac{2(1-\alpha)(5-\alpha)(3-\alpha)}{(6-\alpha)(12-5\alpha+\alpha^2)}\sigma & \text{for Eq.2} \\ \frac{2(72-10\alpha-3\alpha^2+\alpha^3)}{3(6-\alpha)(12+\alpha-\alpha^2)} L + \frac{4(1-\alpha)(5-\alpha)}{(6-\alpha)(12+\alpha-\alpha^2)}\sigma & \text{for Eq.3} \\ \frac{2}{3}L & \text{for Eq.4} \\ \frac{2}{3}L + \frac{1-\alpha}{3}\sigma & \text{for Eq.5.} \end{cases}$$

Compared to the model with FTs, W_0 of the model with ISTs is unchanged, but the difference in the equilibrium conditions and the ISTs' welfare causes a different limiting welfare W_1:

$$\lim_{\alpha\to 0} W = \begin{cases} \frac{2}{3}L & \text{if } \sigma < \frac{4}{5}L \\ \frac{2}{5}L & \text{if } \sigma \geq \frac{4}{5}L, \end{cases} \qquad\qquad \lim_{\alpha\to 1} W = \frac{2}{3}L.$$

Corollary 4.8. *In equilibrium, the following holds:*

(a) The ISTs' welfare is higher than that of the STs, $W_{IST} > W_{ST}$; the ISTs win some additional welfare in comparison with the STs-only market, when ISTs are absent: $W_{IST} > W_0$. From the ISTs' point of view, it is better to act in the market with STs, $W_{IST} \geq W_1$.

(b) For the low volatility states and the low share - high volatility states, the STs' welfare deteriorates with the ISTs' entry: $W_{ST} < W_0$. In these states, the STs are better-off when there are no ISTs around. However, if the volatility is high and if there are enough ISTs, the STs gain from the ISTs' entry: $W_{ST} > W_0$.

(c) If the volatility is low, $W_0 = W_1$: the ISTs' presence may bring about an additional welfare only if the STs are also present; the ISTs have a relative advantage. With a high volatility, $W_0 < W_1$: the ISTs might be better-off also without the STs; this is an absolute advantage for the ISTs. For the whole economy, it is better if all the traders are ISTs and the volatility is high, $W_0 < W_1$ and $W < W_1$. If the volatility is low, the economy is indifferent between these two limiting cases, $W_0 = W_1$, since the advantage is only relative.

(d) For the low share of the ISTs, their entry decreases the aggregate market welfare, $W < W_0$. For all the other parameters, the aggregate market welfare improves with ISTs, $W > W_0$.

Proof. These conclusions follow directly from the graphical representations of the respective functions in Figure 30, but can also be proved analytically. □

ISTs win the market at the cost of STs. However, STs may also gain from the ISTs' entry. It depends on the initial market state whether the ISTs' entry brings about a positive or a negative payoff for the whole economy.

4.4 Summary

In this chapter, we designed and analyzed the modified analytical model. For this purpose, an FT was substituted by an IST: the FTs' informational advantage is detached from their speed advantage, which also includes the possibility to revise the quotes. An IST is assumed to be unconditionally informed but having no opportunity to revise her quotes, while an FT is only conditionally informed (only if the next to come is an ST), but she can adjust her quotes. The modified model is essential to investigate how an FT influences the market compared to a

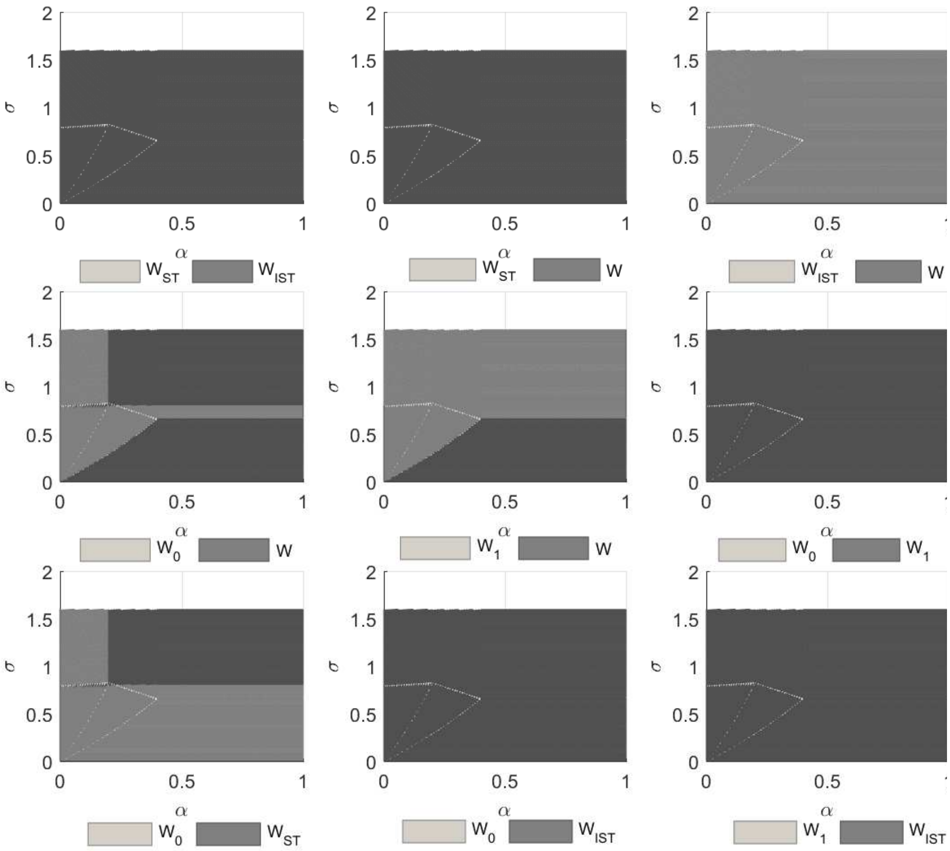


Figure 30: STs' and ISTs' welfare, the aggregate market welfare, STs' welfare with the ISTs' absence, and ISTs' welfare with the STs' absence

simple informed trader, which is discussed in the next chapter.

Both STs and ISTs use the specialized strategies when the share of ISTs on the market is low. An ST follows the high fill-rate equilibria for the low volatility states, while an IST always uses the high fill-rate strategies due to her knowledge of the fundamental value change.

The most important ways an ISTs influences the market are the following:

The ISTs' bargaining power and welfare is higher than those of STs; ISTs almost always play a role of market-makers; the degree of their market-making increases when volatility goes up.

The ISTs' entry generally negatively affects the STs' bargaining power and their costs of immediacy.

For the low share of ISTs, the ISTs' costs of immediacy are smaller than those of STs, and the ISTs' entry diminishes the aggregate market welfare and the STs' welfare.

For the low share - low volatility states, the ISTs' entry worsens the trading rate and pricing error, while also increasing the risk of being picked-off for STs.

With high volatility, the ISTs' entry improves the pricing error. Only for the high share - high volatility states, the STs' costs of immediacy and welfare may improve with the ISTs' entry.

5 Fast Trader vs Informed Slow Trader: Influence on Slow Traders and on the Market as a Whole

The aim of this chapter is to investigate how the FT's influence on the market and her trading partners differs from that of an IST, and which agent type enjoys better results. For this purpose, we compare the market measures for the two models described in Chapter 3 and Chapter 4.

5.1 Bargaining power

The next corollary sums up the similarities of and differences between the FTs' bargaining power with that of ISTs and their influence on the STs' bargaining power.

Corollary 5.1. *In equilibrium, the following holds:*

(a) The entry of both informed traders (FTs and ISTs) adversely affects the STs' bargaining power.

(b) For the specialized equilibria (corresponding to the low share states), the higher the share is, the greater is the deterioration of the STs' bargaining power.
For the unspecialized equilibria (corresponding to the high share states), an increase in the share of FTs (ISTs) leads to the improvement of (does not influence) the STs' bargaining power. However, it is still worse than under the condition of the informed traders absence.
For the low volatility states, the STs' bargaining power drops with an increase in volatility.
For the low (high) fill-rate equilibrium states, the STs' bargaining power is independent of volatility (decreases with volatility).

(c) In the high share states, the STs' bargaining power is higher with FTs than with ISTs. However, in these states an ST experiences a greater fall in the bargaining power from the IST's entry. In the low share states, both informed traders entry influences the STs' bargaining power in a similar way.

(d) The ISTs' (FTs') bargaining power does not depend on volatility in almost all states (the high volatility states). In the low share - low volatility states, the informed traders' bargaining power increases with rising volatility. In the unspecialized high fill-rate equilibrium states, the FTs' bargaining power decreases with rising volatility.

(e) In the low (high) share states, the FTs' (ISTs') bargaining power is greater than that of the ISTs (FTs): it is better to invest in becoming an FT (an IST) from the bargaining power perspective.

Proof. In (a), Corollary 3.2 and Corollary 4.2 are summed up. To prove (b), the function $V_0^{LO} - V_{ST}^{LO}$ in Figure 31 is analyzed, and part (c) follows from Figure 32 and Figure 33. Parts (d) and (e) are based on Figure 34. $\square$

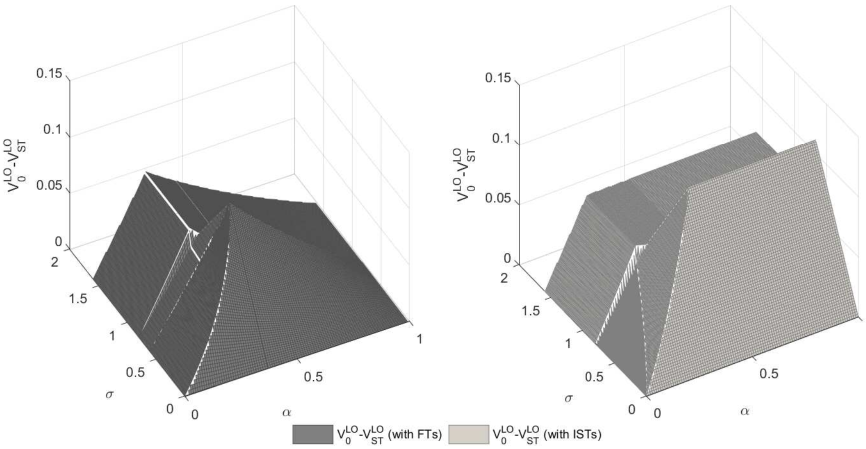


Figure 31: Difference between the STs' bargaining powers with and without the informed traders' presence: two models presented separately

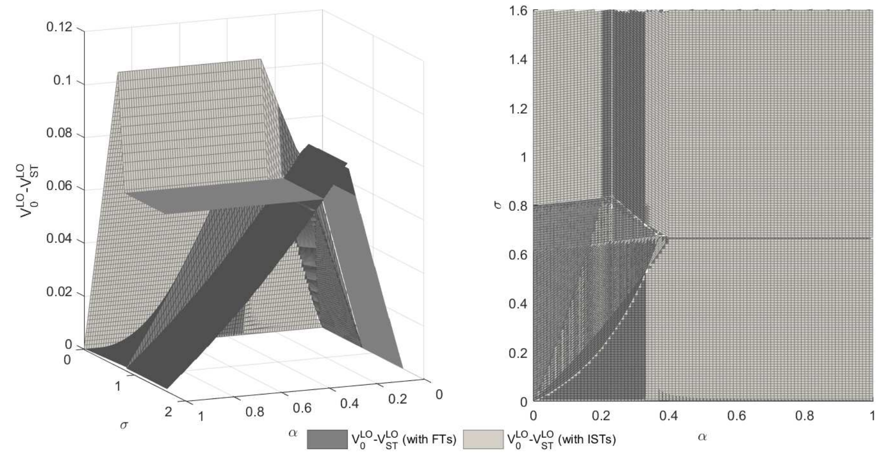


Figure 32: Difference between the STs' bargaining powers with and without the informed traders' presence: two models presented together

FTs adjust their quotes both to the level of the next coming agent and to the level of the fundamental value change: an FT pays a smaller (receives a higher) price than an IST but only if the next to come is an ST. ISTs know the exact fundamental value change, but they cannot adjust their quotes to the level of the next coming agent. In the low share states, the profits are mainly received through transactions with STs. An FT, due to her adjustment possibility, enjoys a higher amount of transactions and profits from sending a limit order (compared to an IST). In the high share states, transactions with the same trader type (an FT or an IST) becomes more important. In transactions with FTs, an FT cannot adjust her quotes and does not have the exact information about the fundamental value change, while an IST is at least sure about the value realization. Therefore, in the high share states, an IST has a higher bargaining power than an FT.

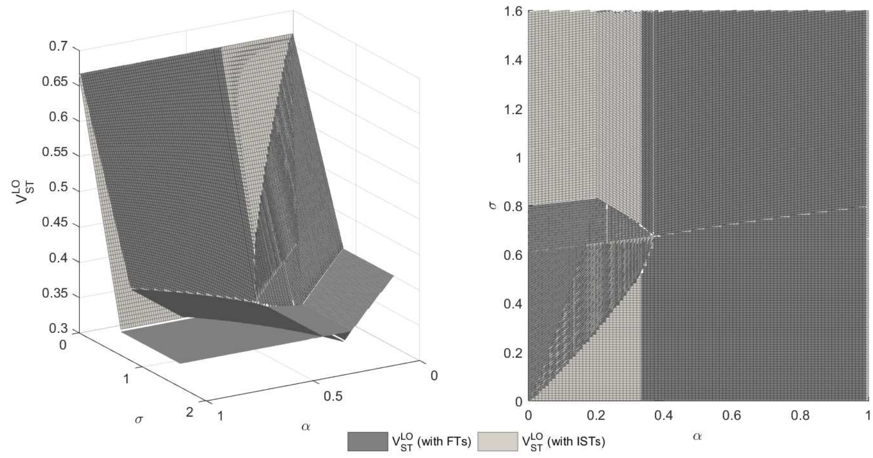


Figure 33: STs' bargaining power in the models with FTs and with ISTs

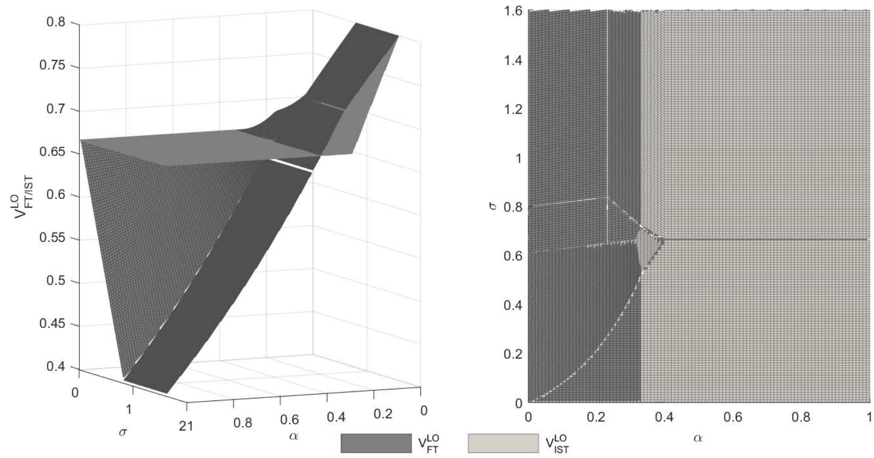


Figure 34: FTs' and ISTs' bargaining powers

5.2 Trading rate

The next corollary sums up the similarities of and differences between the trading rate in the markets with FTs and ISTs.

Corollary 5.2. *In equilibrium, the following holds:*

(a) For both models, the reaction of the trading rate to the share increase is similar.

(b) An FT brings more trade to the market compared to an IST if the share is low (but also for some moderate share states). However, in the high share states, an IST affects the trading rate more considerably in the positive direction.

(c) The FTs' entry improves the trading rate only in the high volatility states, while an IST boosts the trading rate also in the low volatility states with the exception of the specialized high fill-rate equilibrium: an IST promotes the trading rate for a broader range of the market states.

Proof. For part (a), the direction of change for both models described in Corollary 3.3 and Corollary 4.3 are summed up. For parts (b) and (c), Figure 35 is analyzed. □

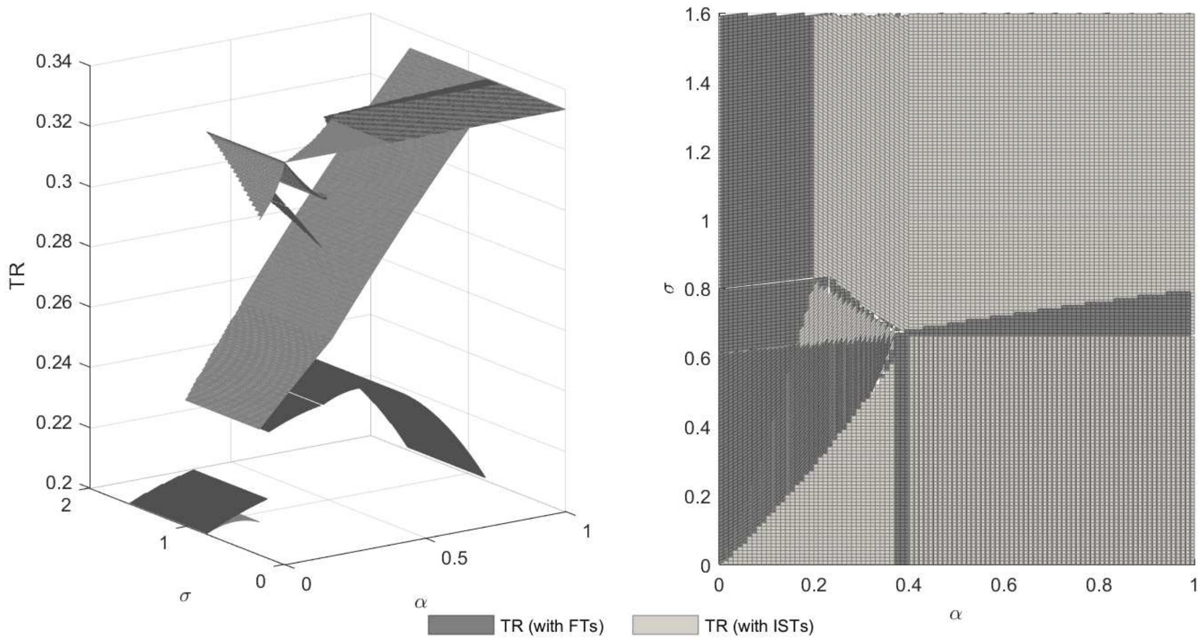


Figure 35: Trading rate for the models with FTs and with ISTs

FTs seem to have a similar influence on the trading rate as ISTs; however for a narrow range of the market states, her influence is more moderate. Nevertheless, if the share is low, FTs increase the trading rate more than ISTs.

5.3 Risk of being picked-off

In the next corollary, the risk of being picked-off in the markets with FTs and ISTs is described in terms of similarities and differences.

Corollary 5.3. *In equilibrium, the following holds:*

(a) The STs experience the same risk of being picked-off as the FTs, ISTs, and as the STs in the informed-traders-free market in the majority of the states. Only in the specialized high fill-rate equilibrium (corresponding to the low share - low volatility states), the risk of being picked-off for the STs is higher with informed traders.

(b) The risk of being picked-off for the ISTs is always zero, while an FT has a lower risk of being picked-off than the STs-only level. The lower the share is, the lower is the risk of being picked-off for the FTs.

Proof. This follows directly from Corollary 3.4 and Corollary 4.4. □

5.4 Costs of immediacy

The next corollary incorporates the similarities of and differences between the STs' costs of immediacy and those of informed traders in the markets with FTs and ISTs.

Corollary 5.4. *In equilibrium, the following holds:*

(a) For the majority of the market states, the STs' costs of immediacy are higher with the FTs than with the ISTs; only in the low share states, having an IST as a trading partner causes higher trading costs for an ST. The STs are better-off if there are a lot of ISTs instead of FTs and if there are few FTs instead of ISTs.

(b) In the low share states and in the high share - high volatility states, the FTs' costs of immediacy are higher than those of the ISTs. Otherwise, the ISTs' costs of immediacy are higher.

Proof. This follows directly from Figure 36 and Figure 37. □

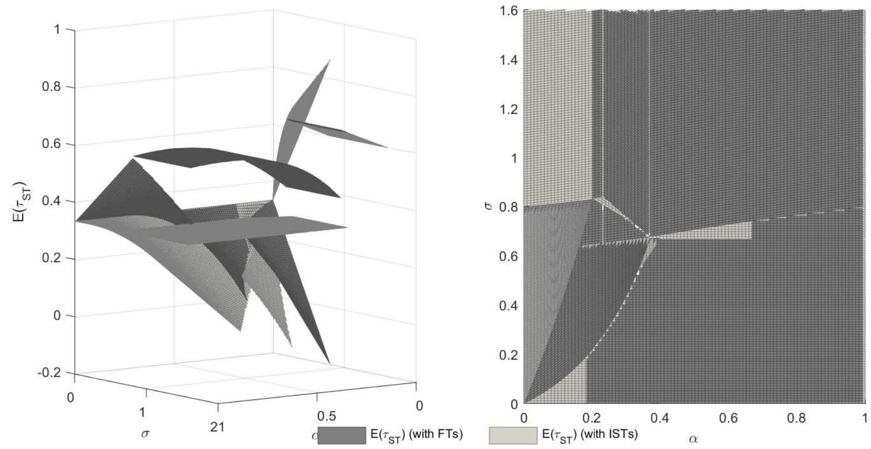


Figure 36: STs' costs of immediacy for the models with FTs and with ISTs

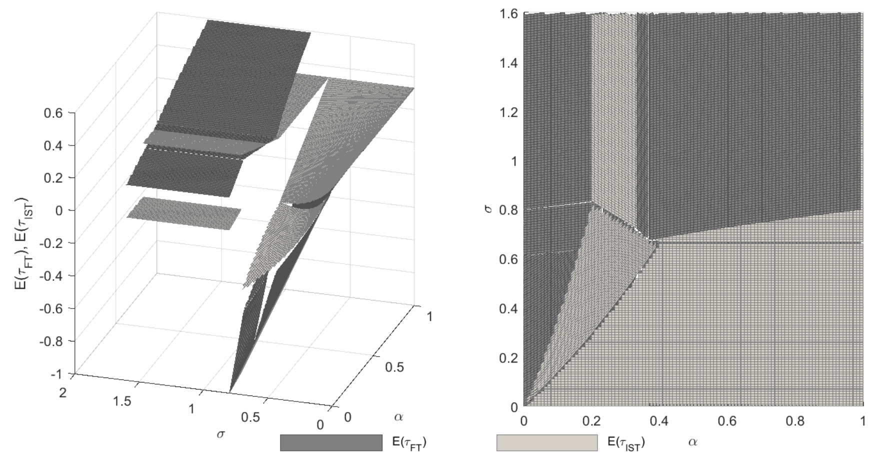


Figure 37: FTs' and ISTs' costs of immediacy

It depends on the initial market state whether an FT or an IST influences the STs' trading costs more positively. If there are a few informed traders, it is better for an ST if they are FTs and not ISTs. The informed traders' trading costs do not coincide, and they are lower for an FT in the high share - low volatility states. In the low share states and in the high share - high volatility states, being an FT causes higher costs of immediacy than being simply an IST.

5.5 Maker-taker ratio

The next corollary addresses the similarities of and differences between the STs' and informed traders' maker-taker ratios in the markets with FTs and ISTs.

Corollary 5.5. *In equilibrium, the following holds:*

(a) In the low share - high volatility states, an ST makes the market slightly more with the FTs around rather than with the ISTs. Otherwise, the STs' maker-taker ratio is the same in both models.

(b) In the high volatility states, the FTs make the market more than the ISTs. In the low share - low volatility states, the ISTs make the market more than the FTs. Otherwise, the maker-taker ratio of the two informed traders is the same.

Proof. Part (a) follows from Figure 38 and (b) from Figure 39. □

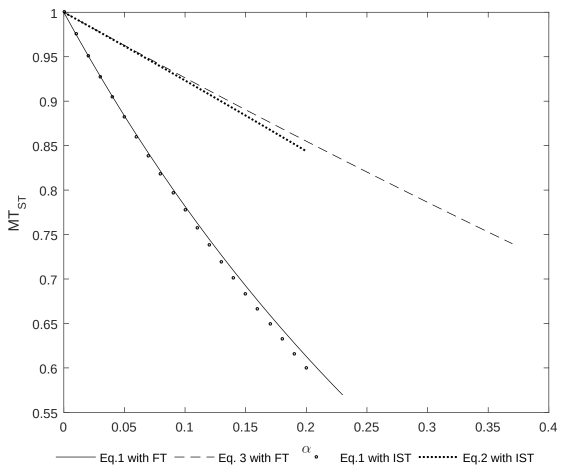


Figure 38: STs' maker-taker ratio for the models with FTs and with ISTs (only the high volatility equilibria)

Corollary 5.5 is in accordance with the common view that FTs prefer speculation games when the volatility is high, because they provide more liquidity in this case; ISTs prefer to wait until the volatility goes down.

5.6 Pricing error

The next corollary sums up the similarities of and differences between the pricing errors in the markets with FTs and ISTs.

Corollary 5.6. *In equilibrium, in the high (low) volatility states, the pricing error deteriorates more due to the FTs' (ISTs') entry than due to the ISTs' (FTs') entry.*

Proof. This follows directly from Figure 40. □

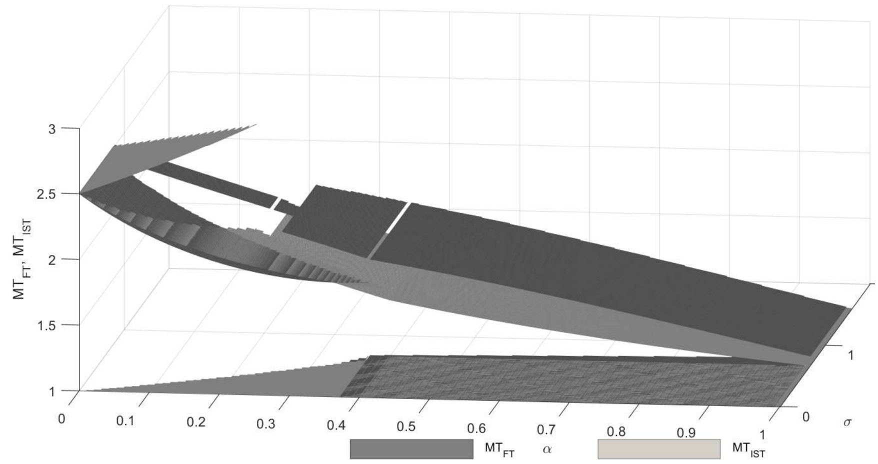


Figure 39: FTs' and ISTs' maker-taker ratios

Eq.1 is omitted, as it interferes the comparison of the other equilibria. MT_{IST} and MT_{FT} converge to positive infinite values for this equilibrium.

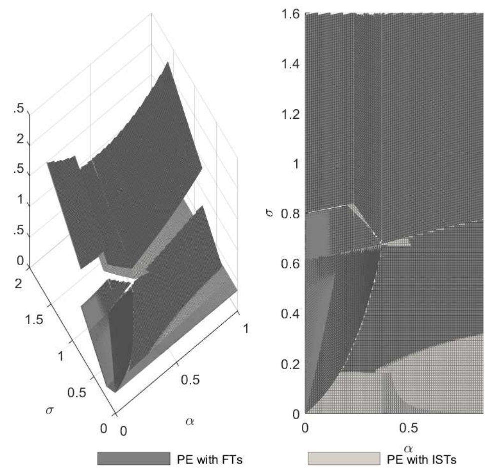


Figure 40: FTs' and ISTs' pricing errors

Corollary 5.6 suggests that the the pricing error is better (i.e. smaller) with FTs than with ISTs in the low volatility states only.

5.7 Welfare

The next corollary discusses the similarities of and differences between the STs' welfare and that of informed traders, as well as the aggregate market welfare in the markets with FTs and ISTs.

Corollary 5.7. *In equilibrium, the following holds:*

(a) An ST is better-off if she has an IST as a trading partner rather than an FT. Only in the low share - high volatility states, the STs would prefer the market with the FTs rather than with the ISTs.

(b) *An IST has a higher welfare than an FT in the non-low share - high volatility states. For all the other states, an FT is able to win more than an IST.*

(c) *Based on the welfare obtained by the informed traders, an ST invests in becoming an IST rather than an FT in the high volatility states, but in the low volatility states an ST is indifferent between investing in becoming an FT or an IST (given that all STs simultaneously decide to become informed traders).*

(d) *In the low volatility states, the aggregate market welfare is the same with both informed trader types. In the low share - high volatility states, the FTs' entry brings a higher welfare to the whole economy in comparison with the ISTs' entry. Otherwise, the aggregate market welfare is higher with the ISTs than with the FTs.*

(e) *In the low share - low volatility states, the economy is better-off with one agent type only: either an ST, an FT, or an IST. Based on the aggregate market welfare, the agents' mix does not bring any advantage to the economy. Only if the share is high enough, the mix may have a positive payoff for the economy as a whole. In the high share - low volatility states, the mix does not influence the aggregate market welfare.*
The FTs-only market is the same as the STs-only market, since speed is a relative advantage. The ISTs-only market is preferred to the STs-only market due to the informational advantage. In the high share - high volatility states, the ISTs-only market is better than a mixed market, which is in its turn better than the STs-only market.

Proof. These conclusions can be drawn by summing up Corollary 3.8 and Corollary 4.8, as well as by analyzing Figure 41. □

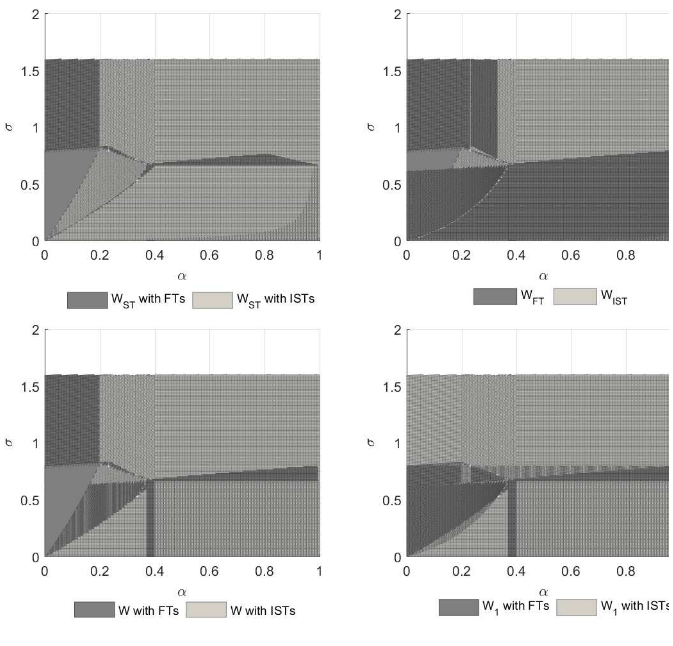


Figure 41: STs' and informed traders' welfare, the aggregate market welfare, and the informed traders' welfare in informed-traders-only markets with FTs and with ISTs

Corollary 5.7 suggests that FTs are preferred to ISTs by STs based on the aggregate market welfare in the low share - high volatility states only.

5.8 Fast trader vs informed slow trader: their influence compared

In this section, in order to compare and contrast FTs and ISTs, the equilibrium classifications according to their fill-rate and specialization are applied. Since the equilibria are valid for certain market states, implications about model preferences for the various states can be made. For example, the specialized high fill-rate equilibrium corresponds to the low share - low volatility states, while the unspecialized low fill-rate equilibrium corresponds to the high share - high volatility states. Even though the equilibrium areas of the two models do not coincide exactly, the above suggested classification may be generally applied to both models.

Table 37 sums up the results of the current chapter; it is used to justify the choice of the model for various market participants or for various equilibrium states in Table 38. As Table 38 suggests, there is no strict and simple answer whether the market is better with FTs or ISTs: the choice depends on the initial state and on the perspective, as well as on the market metrics which play the most critical role. The summed up results suggest, however, that for an ST, for a broader range of the states, it is better to have an IST as a trading partner, while for an informed trader it is better to be an FT rather than an IST for a broader range of the states. However, the economy as a whole is better-off with ISTs than with FTs for a broader area of states, but it may be the opposite. Few criteria are available for evaluating the two models from the whole market perspective.

If a higher welfare of each individual agent is to this individual agent's advantage, from the whole market perspective, a high welfare may be an evidence of a less efficient market. The market which operates efficiently cannot provide high average profits to all its participants. Therefore, from the whole market perspective, the preference is based on the minimum market welfare, while from the individual perspectives, the set-up providing a higher welfare is preferable.

5.9 Summary

In this chapter, we investigated differences between the FT's and ISTs' influence on the whole economy and their trading partners. For this purpose, the market measures for the two models were compared. The main conclusions were the following:

The STs' bargaining power is stronger with FTs in the high share states. The STs' costs of immediacy are lower with FTs only if the share is low. For low shares, the FTs' bargaining power is stronger, but her costs of immediacy (also in the high share - high volatility states) are higher. In the high volatility states, an FT provides more liquidity to the market than an IST. As for the trading rate, the market with FTs is more efficient when the share of FTs is low, while the market with ISTs is better for higher shares. In the high volatility environments, the market with ISTs is more informationally efficient than with FTs, whereas having an FT is better in

Table 37: Comparison of the models with FTs and with ISTs: auxiliary table

Market metrics	Equilibrium map areas											
	low α - low σ			low α - high σ			high α - high σ			high α - low σ		
Approximate equilibrium area classification												
Specialization type	specialized						unspecialized					
Fill-rate type	high fill-rate (i)			low fill-rate						high fill-rate (ii)		
Relationship between different characteristics of the two models												
Model type, sign if appropriate	with FTs	sign	with ISTs	with FTs	sign	with ISTs	with FTs	sign	with ISTs	with FTs	sign	with ISTs
Bargaining power												
V_{ST}		$\approx$			$\approx$		*	$>$		*	$>$	
$V_{ST} - V_0$ (ITs' entry influence)	neg		neg	neg		neg	neg		neg	neg		neg
V_{ST} when α goes up	decr		decr	decr		decr	incr *		indep	incr *		decr
V_{ST} when σ goes up	decr		decr	indep		indep	indep		indep	decr		decr
V_{IT}	**	$>$		**	$>$			$<$	**		$<$	**
$V_{IT} - V_0$	**	$>$		**	$>$			$<$	**		$<$	**
V_{IT} when α goes up	decr		decr	decr		decr	decr		indep	decr		indep
V_{IT} when σ goes up	incr		incr	indep		indep	indep		indep	decr		indep
Trading rate												
TR	***	$>$		***	$>$			$<$	***		$<$	***
$TR - TR_0$	0		0	pos		pos	pos		pos	0		pos ***
Costs of immediacy												
$E(\tau_{ST})$	*	$<$		*	$<$			$>$	*		$>$	*
$E(\tau_{IT})$		$>$	**		$>$	**	**	$<$		**	$<$	
Maker-taker rate												
MT_{ST}		$=$		*	$>$			$=$			$=$	
MT_{IT}		$<$	**	**	$>$		**	$>$			$=$	
Pricing error												
PE	***	$<$			$>$	***		$>$	***	***	$<$	
Welfare												
W_{ST}		$<$	*	*	$>$			$<$	*		$<$	*
W_{IT}	**	$>$		**	$>$			$<$	**	**	$>$	
$W_{IT} - W_0$		$=$			$<$	**		$<$	**		$=$	
W	***	$<$			$>$	***	***	$<$			$\approx$	

In this table, three types of comparison are provided: (i) Influence of informed traders' share on the market metrics ("incr" and "decr" mean increase and decrease, respectively; "indep" means independent); (ii) A sign of the difference between the market metrics on the market with informed traders and STs-only market ("pos" and "neg" mean positive and negative, respectively); (iii) A sign in the middle column describes an inequality sign between the market metrics in the two models (approximation sign means that either the two values are economically not significantly different from each other or that a small difference appears due to the non-matching equilibrium areas in the models). IT refers to informed trader.
The asterisks show whether an FT or an IST causes a more advantageous situation on the market: * denotes that this market configuration is better from a ST's perspective, ** means that this set-up is the best for an informed trader, while *** shows conditions when the market as a whole has the best quality.

Table 38: Comparison of the models with FTs and ISTs

Model superiority	with FTs			with ISTs		
Perspective	ST	FT	Whole market	ST	IST	Whole market
Equilibrium type	Choice per individual strategy					
Specialized high fill-rate	$E(\tau_{ST})$ $V_{FT} - V_0$	V_{FT} W_{FT}	PE TR W	W_{ST}	MT_{IST} $E(\tau_{IST})$	-
Specialized low fill-rate	MT_{ST} W_{ST} $E(\tau_{ST})$ $V_{FT} - V_0$	MT_{FT} V_{FT} W_{FT}	TR	$W_{IST} - W_0$	$E(\tau_{IST})$	W PE
Unspecialized low fill-rate	V_{ST}	$\mathrm{E}(\tau_{FT})$ MT_{FT}	W	$\mathrm{W}_{IST} - W_0$ $\mathrm{E}(\tau_{ST})$ W_{ST} $\mathrm{V}_{IST} - V_0$	MT_{IST} W_{IST} $\mathrm{E}(\tau_{IST})$ V_{IST}	PE TR
Unspecialized high fill-rate	V_{ST}	$\mathrm{E}(\tau_{FT})$ W_{FT}	PE	$\mathrm{E}(\tau_{ST})$ W_{ST} $\mathrm{V}_{IST} - V_0$	V_{IST}	TR-TR_0 TR
Fill-rate type	Choice generalized for fill-rate types					
High fill-rate		W_{FT}	PE	W_{ST}		
Low fill-rate		MT_{FT}		$\mathrm{W}_{IST} - W_0$	$\mathrm{E}(\tau_{IST})$	PE
Specialization type	Choice generalized for specialization types					
Specialized	$\mathrm{E}(\tau_{ST})$ $\mathrm{V}_{FT} - V_0$	V_{FT} W_{FT}	TR		$\mathrm{E}(\tau_{IST})$	
Unspecialized	V_{ST}	$\mathrm{E}(\tau_{FT})$		$\mathrm{E}(\tau_{ST})$ W_{ST} $\mathrm{V}_{IST} - V_0$	V_{IST}	TR

This table combines the choice of the market specification for various agent types and the market as a whole under various market states or equilibrium types (characterized by the share - volatility parameters). The market metrics in this table render the basis of a certain specification choice. If there are several market metrics inside one cell, a certain model is the most optimal based on all the metrics provided in this cell.

the low volatility states. With regards to the aggregate market welfare, the market with FTs is more efficient in the low share - low volatility states and in the high share - high volatility states. ISTs are more advantageous from the perspective of the aggregate market welfare in the low share - high volatility states. In the high volatility environment, ISTs' presence leads to a more efficient market in terms of the pricing error. In some other cases, FTs may also reduce the pricing error (e.g., if the volatility is low).

To decide whether FTs adversely affect the market results, it is essential to have a benchmark market to compare with. Is it correct to assume that all the agents are purely uninformed and slow (i.e., STs)? Most likely, there are some ISTs on the market. If the ISTs' presence is assumed, do FTs worsen the market results more than ISTs? The answer to this question is not that easy. One has to consider the initial market state (the share and volatility values), perspective of the analysis, as well as the most critical market measures. The decision map presented in Table 38 could potentially give an answer to each individual question.

6 Simple Agent-Based Model of Financial Markets

This chapter bridges the gap between the simple analytical market model from Chapter 2 and the computational agent-based model (ABM) for financial markets. Besides, since the two models converge (i.e. they lead to close results), we make some important conclusions based on their results. We extend the constructed ABM by increasing the heterogeneity of agents. Three further configurations are investigated: an ABM with random traders (RTs), fast traders (FTs), and informed slow traders (ISTs). As a result, important conclusions for each of the configurations are made concerning how each specific type of agents influences the market quality and trading partners. The key results of this chapter were presented in The Fifth German Network on Economic Dynamics (GENED) Meeting (Karlsruhe, Germany) in October 2017, see Kalimullina (2017).

6.1 Construction of the simple ABM

The ABM consists of several important building blocks, which are discussed in this section: starting with the trading mechanism, we proceed with the market environment, market participants, as well as the matching procedure and market session sequence.

6.1.1 Trading system

A trading system is the central part of this ABM: it sets rules for orders matching in fulfilling a transaction. Bauwens and Giot (2001) describe three main trading mechanisms used in analytical and agent-based research: (i) the Walrasian auction, which clears submitted supply and demand with a single equilibrium price, (ii) the quote-driven market, where agents trade with the market-maker, (iii) and the order-driven market without market-makers where agents trade directly with each other by posting orders to both sides of the market. A limit order book is an inherent part of the order-driven market: it stores all orders. The matching mechanism[41] checks whether a transaction is possible when a new order comes. According to Raberto, Cincotti, Dose, Focardi, and Marchesi (2005), the majority of stock exchanges are order-driven markets[42] where each transaction is made stepwise with an individual price; only this pricing mechanism leads to replication of stylized properties of financial markets such as fat tails of return distribution. Agents trading larger amounts of stock are more likely to do so in market-maker systems, while smaller individual investors trading smaller amounts are most likely to transact on order-driven markets due to their liquidity and cheapness[43].

In this dissertation, the order-driven market is conceptualized as the continuous double auction where agents come sequentially and have two possibilities: either to post a limit order, which is written down to a limit order book, or send a market order, which leads immediately to a

[41]See Section 6.1.3.
[42]Paris Bourse, the Toronto Stock Exchange, and majority of European exchanges are order-driven markets.
[43]See Bauwens and Giot (2001).

transaction[44]. To make assumptions in accordance with the analytical version of the model, a limit order book has to be either empty or include only one limit order per time step: if a limit order is not executed by the next coming agent, an order is removed from a limit order book. For the simple ABM, it is assumed that limit orders once sent, cannot be removed by a posting agent, but can be replaced or changed by a certain posting agent (an FT).

A limit order book consists of two functional parts: a buy (sell) side where bid (ask) orders are stored. All orders are sent only for one unit of stock. A bid order reflects the maximum price at which a buyer is ready to buy a stock, while an ask order is the minimum price at which a seller is ready to sell a stock. If an agent posts a market order, she is willing to buy or sell a stock at the best price currently available in the market; a transaction happens immediately at the best available limit order[45]. The transaction price is the price of a matching order from a limit order book: a market order value is not revealed by an agent, it is used only for her internal comparison in the decision-making process[46]. As our model is simplified by the fact that a limit order book has only one limit order per time step, there is no need to assume either the time priority rule or the price priority rule[47].

6.1.2 Economic environment

The simple ABM construction is based on Tóth and Scalas (2008) and Huber (2007). In this dissertation, we design the market as a multi-period game where heterogeneous artificial agents (computer algorithms) possess a portfolio of a risky asset (stock) and a risk-free asset (cash) and make their investment decisions based on the market information. Only one risky asset is available on the market; the risk-free rate equals zero. As a result, these agents face the question of investing in the risky asset only; the risk-free part of the portfolio stays in the form of cash. Short-selling is not allowed, and once endowment is used up, an agent is limited by a budget constraint and stays idle until her endowment changes and allows her to trade again.

The agents in this research are provided with very large endowments, so that they have a very small probability to face the budget constraint: at the beginning of a trading session, each trader possesses 10000 units of cash and 100 units of stock. The endowments of cash and stock are carried over from one period to another until the end of a market session. The market is populated with two agents only, while depending on the configuration these agents may be of various types[48].

The fundamental value follows a random walk as in the analytical model[49] with the starting

[44]See Cervone, Galavotti, and LiCalzi (2009).

[45]The best order is the highest bid for a selling transaction and the lowest ask for a buying transaction.

[46]Harris (2003) calls this pricing rule as the discriminatory rule. Some other pricing rules are described in the literature: in Lawrenz (2008), the price is the median of agents' expectations; Chiarella and Iori (2002) use the average of the best bid and the best ask to determine the transaction price; Hoffmann, Jager, and Von Eije (2007) use the average of market bid and ask weighted by the quantity of stocks offered or asked.

[47]For details, see Chiarella and Iori (2002) and Section 7.1.2.

[48]In the more realistic version of the ABM, more than two agents will participate in the market.

[49]See Section 2.1.

value v_0=40. The market is assumed to be a circulation market[50], i.e. the amount of stocks is unchanged during the whole market session, no new emissions or buyouts take place. This provides only two sources of the agents' wealth change: an asset price change and positioning profit (accumulation of wealth due to correct transactions in accordance with the realized fundamental value change). Other exogenous or endogenous sources of endowment change are not modeled. The information and transaction costs are set up to zero and not considered in this dissertation, however some extensions are possible[51].
To make the simple ABM comparable to the analytical model, each period should consist of one step only: only one agent can make a decision and act per period[52]. Each transaction is made maximum for one unit of stock[53]. There are 1000 periods per each market session.

6.1.3 Matching procedure

Once the value of an order is defined, the same set of rules are used for all types of agents to determine whether a transaction is possible under the current market conditions. This set of rules is called the matching procedure.
A transaction happens if the newly formed ask (bid) at t is below (above) the bid (ask) already existing in the market: $A_t < B_t^m$ ($B_t > A_t^m$). The other three conditions to be met are: (i) the potential transaction price is non-negative, (ii) a seller has at least one unit of stock in her endowment, and (iii) a buyer has enough cash in her endowment (it should be greater or equal to the potential transaction price).
A new limit order is posted to a limit order book, if an order is positive, and no transaction is possible in this step. To send a bid (ask), an agent should have enough cash (stock) for a potential buy (sell).

6.1.4 Market session sequence

In this section, the temporal flow of the market session is presented:

(1) The starting value of a stock is defined, and the whole time series of fundamental value changes ε_t is simulated.

(2) The initial endowment is distributed to agents.

(3) A continuous trading session happens:

  (a) A trader is chosen randomly to act in the market. There is only one agent acting per step, one step corresponds to one period.

[50]See Schredelseker (1984).
[51]See, e.g., Posada and Hernández (2010).
[52]In the analytical model, only one agent has a chance to act per one step when the new information hits the market. In reality, multiple transactions are possible until the next piece of information becomes available, and the periods between information reveals are not equal. The first issue will be modeled in a more realistic ABM in Chapter 7.
[53]The extension would be to allow transactions with a higher number of stocks; this number could be determined with some optimization rules.

(b) The random procedure decides which side of the market is active in this step.

(c) If the acting agent uses the analytical equilibrium map for her decision-making, the type of equilibrium is determined (or chosen randomly).

(d) An agent decides on the value of her order.

(e) A transaction happens, and the price is formed, if an order of an acting agent matches some existing order from a limit order book. Otherwise a new limit order is posted.

(f) The wealth bookkeeping mechanism changes endowments of agents participating in a transaction depending on their side (for a buyer, cash is decreased and the stock endowment is increased, the opposite is true for a seller). If a new limit order is posted, the endowment stays unchanged.

(4) At the end of a market session, the final wealth and the relative return of agents are calculated.

Figure 42 shows the temporal flow of the market session steps.

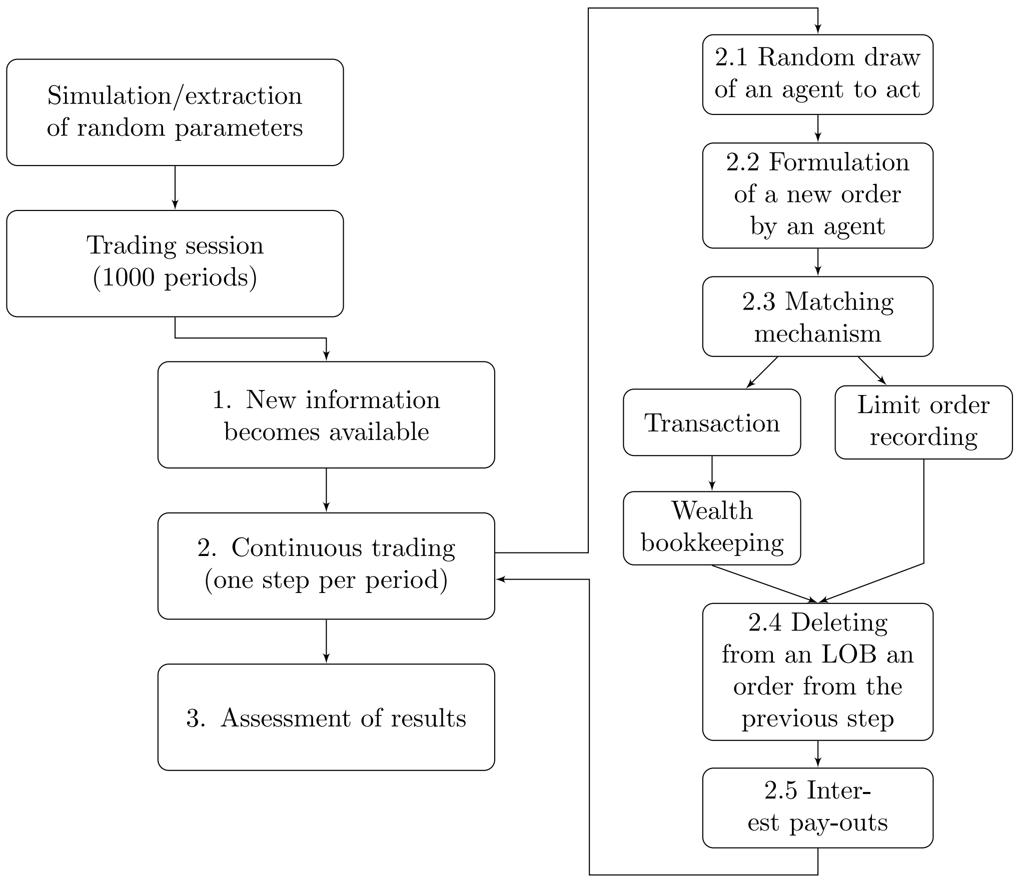


Figure 42: Temporal flow for the simple ABM
LOB refers to a limit order book

6.1.5 Market participants and their trading strategies

A distinctive feature of all computational ABMs[54] is that the role of human agents is played by artificial agents in the form of intelligence algorithms. These algorithms are the sets of exogenous rules which help to decide whether a market order or a limit order should be sent to the market, and if it is a limit order, what value it should have. Previous researchers implemented many possible types of artificial agents, whose decision-making rules are based on various market characteristics. For example, fundamentalists try to predict a future fundamental value and to maximize their final endowment by making correct transactions from a long-term perspective. Chartists do not spend their resources on the economic analysis and follow the technical analysis; their decision-making is based on the presence or absence of some trends in a price series. Random traders, as it follows from the name, decide randomly about the direction of a transaction and the value of a limit order, based on the minimal information about the last transaction price.

In the simple ABM, four types of artificial agents are modeled: a maximizer of an expected profit from limit orders (MEP), a random trader (RT), a high-frequency trader based on analytical rules (analytic HFT), and an informed slow trader (IST). MEPs, ISTs, and analytic HFTs use equilibrium maps from analytical models (previously discussed in this dissertation) as their decision rules.

In the following, decision-making rules of the mentioned agents are described. The decision rules use some input data to produce an order choice and its value, if necessary. The agents use various sources of information and different quantities of those. The input data may be classified as publicly available (e.g. an absolute value of the fundamental value change, a limit order book composition)[55] and private information (e.g. the exact direction of fundamental value change, or the type of the next coming agent)[56].

In the ABM, there is an opportunity to track the actions and performance of each separate investor by using individual identifiers. In order to avoid overcomplicating the market, only two individual traders are allowed to participate in it[57]. This special characteristic of the simple ABM is the main motivation to change the assumptions behind the analytical model, since only under these conditions, a comparison of the two models makes sense. An important point is the assumption that an agent trades only with another individual agent[58]. The two participating agents may have various types described in the following subsections.

6.1.5.1 Maximizer of expected profit from limit orders

The maximizer of the expected profit from limit orders (MEP) is a slow trader who aims at maximizing her expected profit from sending a limit order, which was defined in the analytical

[54]See LeBaron, Arthur, and Palmer (1999).

[55]The agents may or may not use this information for their decision-making.

[56]Private information is available only to some types of agents.

[57]This assumption will be changed in the more realistic ABM in Chapter 7.

[58]An agent cannot buy or sell from herself.

model[59] as V_i^{LO}. This agent uses the equilibrium map illustrated in Figure 2 and the equilibrium conditions from Table 4 as her decision rules: depending on the parameters (γ, σ, L)[60], an agent chooses her equilibrium order. If the fixed market parameters reach such values so that multiple equilibria are possible (on intersection lines or points), an agent is indifferent between them and makes her choice randomly. Allowing for this, the market can be analyzed more throughly under all possible equilibria[61].

An MEP turns out to be on the buy or the sell side randomly with the probability γ to be a buyer; she bases her order on the equilibrium bargaining power V_i^{LO} of the opposite market side. After setting an order on her side, an agent checks whether a limit order book is empty or not. If there is no opposite side limit orders, this order is posted as a limit order. If there is a limit order on the opposite side, an MEP compares her order with the existing one, and a transaction happens only if the conditions are profitable for her. If a potential transaction is not profitable, this order is posted as a new limit order.

Even though the analytical models and therefore equilibrium maps are based on the infinite number of market participants, MEPs[62] are assumed to believe there is an infinite number of possible counteragents and use this analytical map as their decision-making rule, while in the simple ABM an agent has only one possible contractor.

6.1.5.2 Fast trader: high-frequency trader based on analytical rules

Following the equilibrium map in Figure 7 and equilibrium conditions from Table 26, a fast trader (FT) chooses her orders based on the equilibrium for a set of parameters (α, σ, L). We call this FT a high-frequency trader based on analytical rules (analytic HFT) to distinguish her from a more realistic FT modeled in the extended ABM in Chapter 7. The probability for an FT to be a buyer is γ.

6.1.5.3 Informed slow trader

An informed slow trader (IST), in the same manner as an MEP and analytic HFT, bases her decisions on the equilibrium map on Figure 20 and equilibrium conditions from Table 32 as well as on the market set described by parameters (α, σ, L). The probability for an IST to be

[59]See Section 2.1.

[60]γ is a share of buyers, σ is the volatility of the fundamental value change, L is the private value element.

[61]The maximization of V_i^{LO} does not necessary lead to the optimization of other market metrics. Making a further analysis, we can distinguish the best equilibrium based on various market characteristics. However, implementation of the multiple-equilibrium environment makes sense only if the parameters are "catchable": the equilibrium state should be perceptible for a software as well as for a human trader. Non-terminating or long-terminating decimal parameter values are non-trivial to be distinguished by a software and a real human agent from the other close parameter sets. As a result, the agents end up with only one of the neighboring equilibrium parameter sets rather than directly on a border-case. Therefore, multiple borderline equilibria are discriminated for the easiest analytical model only (the model with only slow traders), and for more advanced models (models that include fast traders or informed slow traders), this analysis is not performed, since their border-case parameters are very difficult to catch. Instead of this, we analyze only one parameter set and one equilibrium.

[62]As well as the other agents who base their decisions on equilibrium maps, i.e. FTs and ISTs.

a buyer is γ.

6.1.5.4 Random trader

A random trader (RT) makes her decisions about the direction of a trade and the value of an order randomly. She does not use information about an absolute fundamental value change and constructs her order as a random deviation from the last transaction price. In explaining the nature of such behavior, we follow Black (1986)[63], who interprets it as an agents' desire to trade or belief that she trades on information. The strategy can also be described as the zero intelligence bidding strategy[64], since an agent does not strive to maximize her wealth, observe the market, remember past prices, gather experience, or learn.

The probability for an RT to be a buyer is γ. Once the market side is determined, an agent decides upon the value of an order: it deviates from the last transaction price by a normally distributed random variable $\delta = N(0, 4)$[65]. This order turns out to be either a limit order or a market order depending on the current limit order book composition, e.g. depending on the availability of a (suitable) opposite side limit order, if the endowment requirements are met.

6.2 Common random numbers: a variance reduction technique

To improve the reliability of statistical results and to reduce the variance of final estimators resulting from the usage of random variables, the common random numbers as the variance reduction technique is applied here.

Law (2015) describes the common random numbers technique as different from all the other variance reduction techniques, in that it is implemented for a comparison of different system configurations, while the other techniques are mainly intended for an analysis of a single configuration. Sloan and Unwin (1990) introduce the common random numbers method as the way to compare two or more alternative systems: if two simulated systems are compared under similar experimental conditions, any observed difference should be due to the difference in structures rather than in experimental conditions. Kleijnen, Ridder, and Rubinstein (2010) claim that the common random numbers method is often applied in practice, since it is natural to compare alternative systems under the same circumstances.

Brandimarte (2014) explains the common random numbers technique with the example of two random variables X_1 and X_2 stemming from the simulation of two different systems, even if these vary only in one parameter, and in general $E(X_1) \neq E(X_2)$. It is difficult to say whether the difference in estimators appears due to random noise or a change in a parameter. If an analysis of the difference between these two random variables is to be preformed, two sequences

[63]See Black (1986), p.531.

[64]See Ma and Leung (2008).

[65]The standard deviation of this random variable is 2 currency units, while the mean is 0. Even though the distribution is normal, negative orders are prevented by the matching procedure, see Section 6.1.3.

should be taken into account in the following way:

$$Z_j = X_{1,j} - X_{2,j}.$$

The expected value of the difference has a better quality, when the confidence interval of the estimator is narrow. This is achieved by decreasing the variance of the estimator:

$$Var(X_{1,j} - X_{2,j}) = Var(X_{1,j}) + Var(X_{2,j}) - 2Cov(X_{1,j}, X_{2,j}).$$

As the equation suggests, the variance of the difference can be reduced by generating a positive correlation between $X_{1,j}$ and $X_{2,j}$. Using the same random number streams when generating both X_1 and X_2, can lead to a positive correlation.

According to Law (2015), even though many simulation packages use by default the same streams of random numbers when simulating several configurations, for the common random numbers efficiency these streams have to be synchronized, which requires an additional effort from the researcher. Synchronization means matching up random numbers, or the usage of specific random numbers for exactly the same purposes within all simulated configurations. It is always not enough simply to start a simulation with the same seed value. Kleijnen, Ridder, and Rubinstein (2010) mention that synchronization issues for the common random numbers technique is of paramount importance, and controlling mechanisms should ensure that the same random variables are generated through the same random numbers by a random number generator.

In our research, random variables synchronization is achieved by generating and storing them in matrices as data sets, and by calling them later in respective places of the market mechanism in all simulation configurations.

Following the common random numbers technique, the artificial stock market is simulated 100×100 times (which provides overall 10000 market realizations): 100 market sessions use different fundamental value evolutions and 100 runs within each session use different sequences of the agents' appearance, their sides of the market, and the types of equilibrium. As the present simulation shows, the results are highly correlated among runs within the same session, and are barely correlated among sessions within the same run. This suggests that for the constructed ABM, keeping the same set of fundamental values is an essential element of reducing the variability of estimators, while sticking to the same sequences of agents, their side of the market, and equilibria creates on average the same variability of results as using different random streams for these purposes in all runs.

6.3 Market quality metrics and measures of individual performance

To assess the results of the simulated ABM, two groups of market quality metrics are used. The first group is based on the market measures discussed in Hoffmann (2014); but in our research, these measures are calculated for the multiperiod ABM using market statistics. However, purely

statistical methods are possible not for all of these measures, and then a certain combination of statistical and analytical methods needs to be applied to come up with an estimator. We call this group semianalytical estimators, which reflects their initial analytical nature. The second group of metrics is based on pure statistics from the market session results; computing these metrics for an analytical version of the model is not possible due to its infinite time horizon.
The first group of metrics is intended to be used for a comparison of the analytical model and the simple ABM: the estimators of the ABM should converge to the analytical results. The second group of metrics extends our analysis of the simulated ABM.

6.3.1 Semianalytical metrics

First, an analysis of the artificial stock market using the same market characteristics as in the analytical model of Hoffmann (2014) is performed. There are two ways of finding these metrics: the first one is semianalytical, while the second one is based completely on the simulated results. The goal is to get estimators of these values using a purely statistical computation of events in the artificial stock market (if possible) rather than using analytical computation formulae.
If a market session has N time steps t, it is possible to calculate semianalytical market metrics for each type of equilibrium k using statistical counting of events. In the following formulae, the superscript "sim" refers to the estimators of metrics through simulated results. i and j render to the type of an agent, and O_t is either an ask or bid order.
The conditional execution probabilities $p^{\text{sim}}_{i,k|j}$ are computed in the following way[66]:

$$p^{\text{sim}}_{i,k|j} = \frac{\sum\limits_{t=1}^{N} 1_{|O_t \text{ is executed, } \text{Eq}_t = k}}{\sum\limits_{t=1}^{N} 1_{|O_t \text{ is sent, } j_{t+1} \neq i, \text{ Eq}_t = k}}.$$

Full execution probabilities are a fraction of the executed and sent limit orders:

$$p^{\text{sim}}_{i,k} = \frac{\sum\limits_{t=1}^{N} 1_{|O_t \text{ is executed, } \text{Eq}_t = k}}{\sum\limits_{t=1}^{N} 1_{|O_t \text{ is sent, } \text{Eq}_t = k}}.$$

To find the probabilities of equilibrium events, two methods are available: the semianalytical method uses the analytical relationships from Proposition 2.5 and the statistically found conditional execution probabilities presented above, while the pure statistical method uses the

[66] In the following formula, $O_t = B_t$, if the buying side is analyzed, and $O_t = A_t$, if the selling side is considered. The same is applied to all of the following formulae, if not otherwise indicated.

following formulae[67]:

$$\varphi_i^{MO,\text{sim}} = \frac{\sum_{t=1}^{N} 1_{|O_t \text{ is executed}}}{N}, \qquad \varphi_i^{LO,\text{sim}} = \frac{\sum_{t=1}^{N} 1_{|O_t \text{ is sent}}}{N}.$$

The second (purely statistical) method could not be used for a multi-equilibrium configuration, but might be successfully applied to a single-equilibrium configuration[68].

The expected profit from sending a limit order as well as the "imaginary" expected profit from sending a limit order are the sum of all profits, if an order executed, divided by the number of posted limit orders during the session:

$$V_{i,k(\text{real})}^{LO,\text{sim}} = \frac{\sum_{t=1}^{N} V_{i,t(\text{real})|O_t \text{ is satisfied, } \text{Eq}_t=k}^{LO}}{\sum_{t=1}^{N} 1_{|O_t \text{ is sent, } \text{Eq}_t=k}}.$$

The risk of being picked-off is determined as the fraction of orders executed in a certain state and the whole number of orders executed, if they are posted under a certain equilibrium conditions[69]:

$$\pi_{i,k}^{\text{sim}} = \frac{\sum_{t=1}^{N} 1_{|O_t \text{ is executed}, \varepsilon_t=\mp\sigma, \text{ Eq}_t=k}}{\sum_{t=1}^{N} 1_{|O_t \text{ is executed, } \text{Eq}_t=k}}.$$

The costs of immediacy are the average of the costs incurred, if an order is executed[70]:

$$E(\tau_i)_k^{\text{sim}} = \frac{\sum_{t=1}^{N} \tau_{i|O_t \text{ is executed, } \text{Eq}_t=k}}{\sum_{t=1}^{N} 1_{|O_t \text{ is executed, } \text{Eq}_t=k}}.$$

[67] In the following formulae, for probabilities of market orders, $O_t = A_t$ for the buying side, and $O_t = B_t$, if the measure is related to the selling side, while for the probabilities of limit orders the notation is the same as described before.

[68] In the market simulation, the equilibrium type is endogenous in the period of a transaction. Even though an equilibrium type is pre-simulated for each t as part of the common random numbers technique, these are used for decision-making of an acting MEP only when limit orders are sent. If a limit order on the opposite side exists, an agent compares her private asset value with the existing order; if a transaction is profitable, an acting agent does not need to choose the equilibrium type to form a new limit order. Only if a transaction is not profitable, she uses a pre-simulated equilibrium type to set up a new limit order. Therefore, the realized equilibrium type in a period of a transaction is undefined. For simplicity, we set up an equilibrium type to the same value as for a limit order being executed, but this makes the distribution among equilibria biased, since not all of them are equally likely to lead to transactions. As a result, the probabilities of equilibrium events calculated statistically turn out to be incorrect. This problem disappears, if only one type of equilibrium within the whole simulation is possible. In such a case, we use straightforward statistical formulae. In these formulae, the denominator represents an unbiased number of periods when a certain equilibrium took place, which is equal to the whole duration of the market game.

[69] In the following formula, $\varepsilon = -\sigma$ $(+\sigma)$ for the buying (selling) side.

[70] In the following formula, $O_t = A_t$ $(O_t = B_t)$, if the buying (selling) side is considered.

Since the maker-taker ratio depends on the probabilities of equilibrium events, the semianalytical approach should be followed, to find these values, if a multi-equilibrium configuration is used:

$$MT_{i,k}^{\text{sim}} = \frac{\varphi_{i,k}^{LO,\text{ sim}} p_{i,k}^{\text{sim}}}{\varphi_{j,k}^{LO,\text{ sim}} p_{j,k}^{\text{sim}}}.$$

A pricing error is the average of the pricing errors, when the limit orders are sent:

$$PE_{i,b,k}^{sim} = \frac{\sum_{t=1}^{N} |v_{t+1} - B_t^m| |_{B_t \text{ is sent, Eq}_t=k}}{\sum_{t=1}^{N} 1_{|B_t \text{ is sent, Eq}_t=k}}, \qquad PE_{i,s,k}^{sim} = \frac{\sum_{t=1}^{N} |A_t^m - v_{t+1}| |_{A_t \text{ is sent, Eq}_t=k}}{\sum_{t=1}^{N} 1_{|A_t \text{ is sent, Eq}_t=k}}.$$

The final welfare of the market participants is calculated by the semianalytical approach, because it depends crucially on the probabilities of equilibrium events:

$$W_{i,k}^{\text{sim}} = \frac{\varphi_{i,k}^{LO,\text{sim}}}{\varphi_{i,k}^{LO,\text{sim}} + \varphi_{i,k}^{MO,\text{sim}}} V_{i,k}^{LO,\text{sim}} + \frac{\varphi_{i,k}^{MO,\text{sim}}}{\varphi_{i,k}^{LO,\text{sim}} + \varphi_{i,k}^{MO,\text{sim}}} (L - E(\tau_i)_k^{\text{sim}}).$$

6.3.2 Statistical metrics

Besides the semianalytical market characteristics described in the previous section, our analysis can be enriched by providing other market quality metrics.

Brogaard (2010) describes market quality to be three-dimensional: it includes liquidity measures, price discovery, as well as the volatility. However, for each individual agent the quality of the market is determined by their individual probability to succeed and may be measured by profits or relative returns. Therefore, to describe market quality from the perspective of different market participants, the additional parameters of individual activity and return measures are integrated into the research.

In the following, t refers to a period, s refers to a time step, (t, s) refers to a step s within a period t. ts denotes an uninterrupted series of time steps along the whole market session where a certain (specified) activity happens (either a limit order is posted or transaction happens).

6.3.2.1 Aggregate market metrics

(1) Size liquidity measures (trading activity measures)[71]

  (a) Number of transactions, N_{tr}.

  (b) Trading volume, i.e. the sum of transaction prices, given that each transaction is only for one unit of an asset.

[71] Jones (2013) describes liquidity as a complex multidimensional measure and distinguishes among three liquidity dimensions: price, size, and time. Our research takes into account the first two dimensions only, time measures lie outside of the scope of this dissertation.

(c) Acceptance ratio (Huber and Kirchler (2008)):

$$AR = \frac{N_{tr}}{N_{LO}},$$

where the number of transactions coincides with the number of market orders, and N_{LO} is the number of posted limit orders.

(d) Order-to-trade ratio is a measure of quoting activity, calculated as the opposite to acceptance ratio:

$$OT = \frac{N_{LO}}{N_{tr}}.$$

(e) Order imbalance[72] shows how severe the distribution of trade initiators among the market sides can be. If the measure is close to zero, the distribution is equal among the two sides. If it is positive (negative), the trade is initiated more frequently by the buying (selling) side of the market. For an individual asset, order imbalance could happen either due to some random reasons or it may be associated with public or private information. In the latter, liquidity is expected to reduce permanently. On the market level, asymmetric information is quite unlikely to be a cause of order imbalances, but the market-maker inventories still strain for some period. Irrespective of the cause for this imbalance, market-makers are most likely to worsen the offered trade terms.

There is a strong dependence between stock returns and order imbalances. Market prices reverse after order imbalances decline and continue after order imbalances move up. Reverse effects are especially important after long down-market and days with large negative imbalance. Moreover, price pressures due to imbalances extend to the aggregate market, they do not stay on the level of a separate stock. Order imbalance is strongly associated with contemporaneous absolute returns after taking control of the volume and liquidity. Therefore, it is important to analyze not only simple the trading volume, but also order imbalance measures.

The higher these imbalances are, the more overwhelming the market sentiment for a certain market side is. A high-quality market should have order imbalances close to zero, and if there are some imbalances, they should disappear almost immediately.

There are two versions of order imbalance measure:

(e1) Order imbalance in asset units is the number of the buyer-initiated trades minus the number of the seller-initiated trades during the whole market session:

$$OIB^{Num} = N_{tr}^{\text{buyer}} - N_{tr}^{\text{seller}},$$

where N_{tr}^{buyer} and N_{tr}^{seller} are a number of transactions initiated by a buyer or a

[72]See Chordia, Roll, and Subrahmanyam (2002).

seller, respectively.

(e2) Order imbalance in currency units is the trading volume of the buyer-initiated transactions minus the trading volume of the seller-initiated transactions:

$$OIB^{Curr} = \sum_{ts} P_{ts}^{\text{buy}} - \sum_{ts} P_{ts}^{\text{sell}},$$

where ts denotes a time series of buyer- or seller-initiated transactions, respectively, and P refers to the transaction price[73].

(2) Other price liquidity measures: trading costs, calculated as the price of a transaction minus the true (fundamental) value of an asset[74]:

$$TC_{t,s} = P_{t,s} - v_t,$$

where v_t is the fundamental value of a stock.

(3) Price efficiency measures (measures of fundamental information content in market prices): Relative absolute deviation (RAD) (Kirchler, Huber, and Stöckl (2012)) shows the absolute value of percentage difference between the average price during a period and the true value of an asset during this period:

$$RAD_t = \left| \frac{P_t^{\text{av}} - v_t}{v_t} \right|.$$

(4) Excess return (RET)[75] is the average price return over the market period:

$$RET_{ts} = log \frac{P_{ts+1}}{P_{ts}},$$

where ts is the uninterrupted sequence of transaction points during the whole market session. If excess returns are positive, the market participants are better-off on average.

6.3.2.2 Individual activity and performance metrics

In this section, i refers to an agent identifier.

(1) Individual activity measures

[73]Multiplication by the number of units is not necessary for this set-up, as transactions are possible for one unit of an asset only.

[74]This measure is similar to the costs of immediacy described in Section 2.5.6. However, trading costs are a non-signed measure, no distinction between selling and buying transactions is made. Since the two measures are calculated differently, their averages also diverge.

[75]A similar measure is presented in Chen and Swan (2008), but from the price return, the respective risk-free rate is subtracted.

(a) The number of transactions, $N_{tr,i}$, and the relative number of transactions:

$$\frac{N_{tr,i}}{\sum_i N_{tr,i}}.$$

(b) Sending of limit orders, $N_{LO,i}$, and the relative sending of limit orders:

$$\frac{N_{LO,i}}{\sum_i N_{LO,i}}.$$

(c) Trading via limit orders (passive transactions), $N_{tr,i}^{\text{via LOs}}$, and the relative trading via limit orders:

$$\frac{N_{tr,i}^{\text{via LOs}}}{\sum_i N_{tr,i}^{\text{via LOs}}}.$$

This is a liquidity supply measure and shows how often the agents' limit orders are hit by the other agents' market orders; it indicates the fraction of trades, where an agent acts as a liquidity supplier (market-maker).

(d) Trading via market orders (active transactions), $N_{tr,i}^{\text{via MOs}}$, and the relative trading via market orders:

$$\frac{N_{tr,i}^{\text{via MOs}}}{\sum_i N_{tr,i}^{\text{via MOs}}}.$$

This is a measure of liquidity demand and it shows how often the agents' market orders hit the other agents' limit orders, it indicates the fraction of trades, where an agent acts as a liquidity taker (market-taker).

(e) The maker-taker rate is the amount of the sent limit orders divided by the number of the market orders sent by a certain agent[76]:

$$MT_i = \frac{N_{tr,i}^{\text{via LOs}}}{N_{tr,i}^{\text{via MOs}}}.$$

This rate shows whether an agent is a market-maker (if the rate is higher than 1, an agent provides more liquidity than takes) or a market-taker (if the rate is smaller than 1).

(f) The order-to-trade ratio:

$$OT_i = \frac{N_{LO,i}}{N_{tr,i}}.$$

[76]This measure is different from the semianalytical maker-taker ratio. To distinguish the two measures, the current parameter is called the "rate": it depends on the characteristics of a certain individual agent only and shows her own relative making function, while the maker-taker ratio shows the making function relative to the other market participants.

(2) Price measures[77] or measures of transaction terms

(a) A relative average buying price:

$$P^{\text{buy}}_{\text{rel},i} = \frac{\sum_{ts} P_i | q = +1}{N_{tr,i} | q = +1} \cdot \frac{1}{P^{\text{av}}},$$

where $\frac{\sum_{ts} P_i | q=+1}{N_{tr,i} | q=+1}$ is the average buying price of a certain agent i over the whole market session and P^{av} is the average price during the market session for the whole market; it includes both the buying and selling transactions of all the agents. q indicates a dummy for a buying ($q = 1$) transaction.

(b) A relative average selling price:

$$P^{\text{sell}}_{\text{rel},i} = \frac{\sum_{ts} P_i | q = -1}{N_{tr,i} | q = -1} \cdot \frac{1}{P^{\text{av}}},$$

where $\frac{\sum_{ts} P_i | q=-1}{N_{tr,i} | q=-1}$ is the average selling price of a certain agent i over the whole market session. q indicates a dummy for a selling ($q = -1$) transaction.

(3) Individual performance measures

(a) Wealth change[78]:

$$\Delta W = W_{ts_{\text{last}},i} - W_{ts_1,i}.$$

(b) A relative return:

$$RR_i = R_i - R^m,$$

where R^m is an average return among the market participants, and R_i is the return of an individual trader:

$$R_i = \frac{W_{ts_{\text{last}},i}}{W_{ts_1,i}} - 1,$$

where $W_{ts_{\text{last}},i}$ and $W_{ts_1,i}$ are the final and the initial endowment of an agent i in currency units (cash position plus the number of assets multiplied by the final or initial asset price), respectively.

[77] Jones (2013) discusses similar measures of an individual adverse selection.

[78] Note the difference of our working definition of "wealth" from the definition of "welfare" in Hoffmann (2014), mentioned in Section 3.4.9 and Section 4.3.9.

6.4 Simulation configurations

For the simple ABM, we check four market configurations:

(1) Two identical MEPs act on the market (MEP+MEP).

(2) An MEP and an RT act on the market (MEP+RT).

(3) An MEP and an analytic HFT act on the market (MEP+ analytic HFT).

(4) An MEP and an IST act on the market (MEP+IST).

The fixed parameters used by the MEPs in the decision-making are ($\gamma = \frac{1}{2}, \sigma = \frac{4}{5}, L = 1$). For the last two configurations, the MEPs as well as the informed traders also need a share of the informed traders as the input parameter. We set up $\alpha = \frac{1}{2}$. Under these parameters, an MEP chooses Eq.3 orders, in the third configuration the agents use the order-setting rules of Eq.2, while the input data for the last configuration corresponds to Eq.4.

6.5 Analysis of simulation results

In this section, an analysis and the main findings of the simple ABM are presented.

6.5.1 MEP and MEP (ABM version of the simplified analytical model)

In the first simulation, the market is populated with two identical MEPs, which corresponds to the analytical model from Chapter 2. The results of this configuration simulation are provided in Table 39; the later observations are based on conclusions derived from this table.
The statistical metrics of all configurations are represented graphically using boxplots. The boxplots provided in this dissertation rely on the default Matlab configurations. The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the 95% confidence intervals for the means.
While analyzing the boxplots for different configurations and set-ups, we are interested in the mean and median values of each metrics for two different agents compared to each other, we abstain so far from comparing the metrics among the configurations.
The statistical significance of differences is checked using tests for the equality between the medians or means of two individual agents. On the boxplots, if the notches do not overlap, one can conclude with 95% of confidence that the medians differ; see Matlab documentation. If the 95% confidence intervals do not overlap, the difference between the means is statistically significant with the p-value of less than 0.05, see Payton, Greenstone, and Schenker (2003).
Economic insignificance may be concluded even if the measures are statistically different. Due

to the various sources of randomness in the simulated market, two agents may show not ideally identical behavior. For example, the distribution in sending limit orders to the market of 49% vs 51% does not seem to be too divergent from the 50% vs 50% distribution.

In the current configuration, when identical agents participate in trade, it is logical to expect that the difference between any measures should be either statistically or economically insignificant. Figure 43, where the number of buying transactions for both agents is plotted, shows that the notches of the boxplots are different, while the confidence intervals of the means are closer to each other. However, if the relative difference between the two medians or means is analyzed, one can conclude there exists economical insignificance in this difference, even though there is a certain statistical difference present. In Figure 44, on the opposite, the notches of the two medians are closer than the confidence intervals: the statistical difference is present, but according to the economic sense, the two agents are not significantly different in their making-taking activities. Figure 45, showing the absolute wealth change after the market session for both participating agents, suggests that economically there is no difference between the two agents.

In the following, when it is stated that the results of the two agents do not differ from each other, it means that the difference is insignificant from the economic point of view, but the statistical difference might be present.

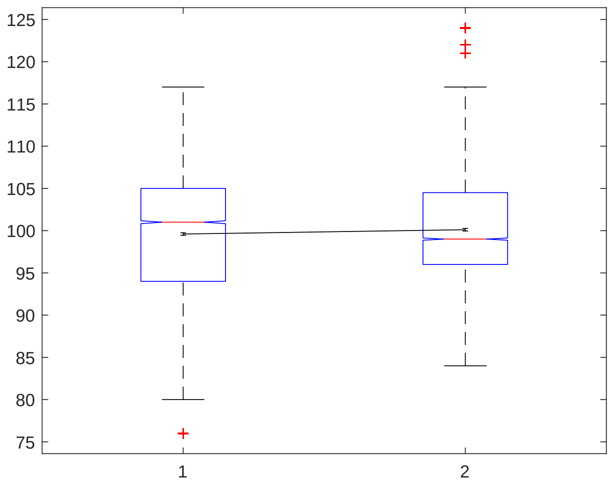


Figure 43: Number of buying transactions for the configuration (1) MEP and MEP

The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the 95% confidence intervals for the means.

Figure 46 provides all the analyzed statistical market metrics for the current configuration (1) MEP and MEP. The previously discussed three figures are also included in Figure 46 (Figure 43 corresponds to subplot (c), Figure 44 to subplot (m), and Figure 45 to subplot (n)).

Table 39: ABM with two MEPs: analytical vs simulated results

Market metrics	Eq.1	Eq.2	Eq.3	Eq.4	Average
p_b	0,125	0,125	0,25	0,25	0,1875
$\hat{p}_b$	0,1253	0,1264	0,2471	0,2517	0,1876
s	0,0477	0,0462	0,0611	0,0615	0,0825
p-value	0,3021	3,104e-05	1,9294e-11	5,7635e-05	0,6319
|error|	0,24%	1,12%	1,16%	0,68%	0,053%
p_s	0,25	0,125	0,25	0,125	0,1875
$\hat{p}_s$	0,2488	0,1260	0,2488	0,1241	0,1872
s	0,0574	0,045	0,0628	0,0441	0,0814
p-value	9,6614e-06	4,9204e-06	0,6354	0,0109	0,2418
|error|	0,48%	0,8%	0,48%	0,72%	0,16%
$p_{b\|s}$	0,25	0,25	0,5	0,5	0,375
$\hat{p}_{b\|s}$	0,2465	0,2549	0,4976	0,4993	0,3746
s	0,0868	0,0853	0,1014	0,1003	0,1554
p-value	9,7616e-09	3,4649e-16	7,3518e-04	0,3456	0,442
|error|	1,4%	1,96%	0,48%	0,14%	0,107%
$p_{s\|b}$	0,5	0,25	0,5	0,25	0,375
$\hat{p}_{s\|b}$	0,5041	0,2525	0,5036	0,2481	0,3771
s	0,0965	0,0840	0,1063	0,0844	0,1574
p-value	2,0754e-09	2,5941e-05	1,5347e-06	0,0011	2,0651e-04
|error|	0,82%	1%	0,72%	0,76%	0,56%
φ_b^{LO}	0,3871	0,4444	0,4	0,4516	0,4208
$\hat{\varphi}_b^{LO}$	0,3862	0,4440	0,3994	0,4519	0,4201
s	0,0153	0,0132	0,0154	0,0122	0,0351
p-value	1,1596e-16	7,1947e-06	5,9210e-09	3,6567e-04	1,0961e-07
|error|	0,23%	0,09%	0,15%	0,06%	0,166%
φ_s^{LO}	0,4516	0,4444	0,4	0,3871	0,4208
$\hat{\varphi}_s^{LO}$	0,4522	0,4433	0,4006	0,3871	0,4208
s	0,0177	0,0194	0,0219	0,0237	0,0345
p-value	2,3602e-06	1,1972e-16	2,8026e-04	0,8029	0,9557
|error|	0,133%	0,248%	0,15%	0,0%	0,0%
φ_b^{MO}	0,1129	0,0556	0,1	0,0484	0,0792
$\hat{\varphi}_b^{MO}$	0,1141	0,0561	0,1011	0,0482	0,0799
s	0,02333	0,0191	0,0230	0,0171	0,0351
p-value	4,2348e-14	2,5636e-11	1,2209e-04	0,0777	1,0961e-07
|error|	1,063%	0,899%	0,6%	0,413%	0,884%
φ_s^{MO}	0,0484	0,0556	0,1	0,1129	0,0792
$\hat{\varphi}_s^{MO}$	0,0478	0,0567	0,0994	0,1129	0,0792
s	0,0177	0,0194	0,0219	0,0237	0,0345
p-value	2,3602e-06	1,1972e-16	2,8026e-04	0,8029	0,9557
|error|	1,24%	1,978%	0,6%	0,0%	0,0%
TR	0,1613	0,1111	0,2	0,1613	0,1584
$\widehat{TR}$	0,1615	0,1128	0,2005	0,1611	0,1591
s	0,0233	0,0235	0,0243	0,0242	0,0392
p-value	7,3262e-05	2,3636e-23	0,0024	0,3149	1,5663e-06
|error|	0,12%	1,53%	0,25%	0,124%	0,442%
π_b	0	0	0,5	0,5	0,25
$\hat{\pi}_b$	0	0	0,4991	0,4982	0,2495
s	-	-	0,1443	0,1439	0,2693
p-value	-	-	0,3591	0,0765	0,5948
|error|	0%	0%	0,18%	0,36%	0,2%
π_s	0,5	0	0,5	0	0,25
$\hat{\pi}_s$	0,4989	0	0,4983	0	0,2494
s	0,1446	-	0,1457	-	0,2696
p-value	0,2860	-	0,0991	-	0,5568
|error|	0,22%	0%	0,34%	0%	0,24%

MT_b	0,4286	1	1	2,3333	1,1905
$\widehat{MT}_b$	0,4508	1,2244	1,0629	2,8302	1,3916
s	0,2283	0,4218	0,4373	1,8482	1,3780
p-value	5,4427e-43	5,7246e-91	3,526e-252	5,08e-305	0
|error|	5,18%	22,44%	6,29%	21,296%	16,892%
V_b^{LO}	0,2	0,2	0,2	0,2	0,2
$\hat{V}_b^{LO}$	0,2006	0,2022	0,1977	0,2018	0,2006
s	0,0764	0,0740	0,0742	0,0750	0,0749
p-value	0,3021	3,1047e-05	1,0658e-05	6,4953e-04	0,03493
|error|	0,3%	1,1%	1,15%	0,9%	0,3%
V_s^{LO}	0,2	0,2	0,2	0,2	0,2
$\hat{V}_s^{LO}$	0,1986	0,2023	0,2	0,1987	0,1999
s	0,0723	0,0720	0,0757	0,0706	0,0726
p-value	0,0049	4,9204e-06	0,9406	0,0109	0,6800
|error|	0,7%	1,15%	0%	0,65%	0,05%
$V_{b,multi}^{LO}$	0,0774	0,0889	0,0800	0,0903	0,0842
$\hat{V}_{b,multi}^{LO}$	0,0778	0,0899	0,0791	0,0913	0,0845
s	0,0309	0,0336	0,0305	0,0344	0,0330
p-value	0,0857	0	1,6732e-05	5,5432e-05	0,0017
|error|	0,517%	1,12%	1,125%	1,1%	0,35%
$V_{s,multi}^{LO}$	0,0903	0,0889	0,0800	0,0774	0,0842
$\hat{V}_{s,multi}^{LO}$	0,0899	0,0899	0,0802	0,0771	0,0843
s	0,0332	0,0326	0,0311	0,0282	0,0318
p-value	0,0580	0	0,2873	0,1220	0,03234
|error|	0,443%	1,125%	0,25%	0,38%	0,119%
$V_b^{LO,\text{ real}}$	0,0750	0,0750	-0,0500	-0,0500	0,0125
$\hat{V}_b^{LO,\text{ real}}$	0,0752	0,0758	-0,0494	-0,0499	0,0129
s	0,0286	0,0277	0,0562	0,0572	0,0770
p-value	0,3021	3,1047e-05	0,1377	0,8846	0,1238
|error|	0,26%	1,06%	1,2%	0,2%	3,2%
$V_s^{LO,\text{ real}}$	-0,0500	0,0750	-0,0500	0,0750	0,0125
$\hat{V}_s^{LO,\text{ real}}$	-0,0496	0,0759	-0,0498	0,0745	0,0127
s	0,0568	0,0270	0,0577	0,0265	0,0768
p-value	0,3735	4,9204e-06	0,6758	0,0109	0,3951
|error|	0,8%	1,2%	0,4%	0,66%	1,6%
$V_{b,multi}^{LO,\text{ real}}$	0,0290	0,0333	-0,0200	-0,0226	0,0049
$\hat{V}_{b,multi}^{LO,\text{ real}}$	0,0292	0,0337	-0,0197	-0,0226	0,0051
s	0,0116	0,0126	0,0225	0,0259	0,0326
p-value	0,0857	1,3421e-05	0,1122	0,9581	0,0942
|error|	0,69%	1,2%	1,5%	0%	4,08%
$V_{s,multi}^{LO,\text{ real}}$	-0,0226	0,0333	-0,0200	0,0290	0,0049
$\hat{V}_{s,multi}^{LO,\text{ real}}$	-0,0225	0,0337	-0,0200	0,0289	0,0050
s	0,0258	0,0122	0,0232	0,0106	0,0325
p-value	0,5614	2,9254e-05	0,8551	0,1220	0,4070
|error|	0,44%	1,2%	0%	0,34%	2,04%
$E(\tau_b)$	-0,2	0,6	-0,2	0,6	0,2
$\widehat{E(\tau_b)}$	-0,2003	0,6000	-0,1997	0,6000	0,1998
s	0,2313	2,1679e-13	0,2331	2,1668e-13	0,4324
p-value	0,8762	0	0,8468	0	0,8938
|error|	0,15%	0%	0,15%	0%	0,1%
$E(\tau_s)$	0,6	0,6	-0,2	-0,2	0,2
$\widehat{E(\tau_s)}$	0,6000	0,6000	-0,2008	-0,1991	0,1997
s	2,1667e-13	2,1679e-13	0,2308	0,2302	0,4320
p-value	0	0	0,6141	0,5872	0,8597
|error|	0%	0%	0,4%	0,45%	0,15%
PE_b	1,4	1,4	0,8	0,8	1,1
$\widehat{PE}_b$	1,4009	1,3997	0,8000	0,7999	1,1001
s	0,1100	0,1098	0,0276	0,0276	0,3107
p-value	0,2309	0,7090	0,8161	0,4798	0,9168
|error|	0,0064%	0,021%	0%	0,012%	0,009%

PE_s	0,8	1,4	0,8	1,4	1,1
$\widehat{PE}_s$	0,8001	1,3996	0,7999	1,4006	1,1000
s	0,0276	0,1099	0,0277	0,1109	0,3106
p-value	0,6219	0,6045	0,6523	0,4622	0,9669
|error|	0,0125%	0,028%	0,0125%	0,0428%	0%
PE	1,0769	1,4	0,8	1,0769	1,0885
$\widehat{PE}$	1,0761	1,3986	0,7999	1,0763	1,0887
s	0,0513	0,0782	0,0198	0,0516	0,2188
p-value	0,0187	0,0098	0,5653	0,1061	0,3403
|error|	0,074%	0,1%	0,0125%	0,056%	0,018%
W_b	0,4258	0,2222	0,4000	0,2194	0,3168
$\hat{W}_b$	0,4296	0,2247	0,4008	0,2212	0,3191
s	0,0861	0,0653	0,0856	0,0675	0,1234
p-value	5,4298e-10	9,0114e-08	0,2119	1,3583e-04	2,2128e-07
|error|	0,892%	1,125%	0,2%	0,82%	0,726%
W_s	0,2194	0,2222	0,4000	0,4258	0,3168
$\hat{W}_s$	0,2180	0,2251	0,3992	0,4251	0,3169
s	0,0654	0,0635	0,0855	0,0862	0,1222
p-value	0,0037	2,5132e-10	0,1827	0,257	0,8731
|error|	0,638%	1,305%	0,2%	0,164%	0,032%
W	0,3226	0,2222	0,4000	0,3226	0,3168
$\hat{W}$	0,3238	0,2249	0,4	0,3231	0,3180
s	0,0614	0,0504	0,0686	0,0632	0,0873
p-value	0,0047	9,9845e-14	0,9589	0,2080	2,2984e-04
|error|	0,372%	1,215%	0,0%	0,155%	0,379%

s denotes the standard deviation of a simulated sample. The p-value relates to the t-test with the null hypothesis that the simulated mean equals the analytical value. If the p-value is small, this hypothesis can be rejected with a high statistical significance against the alternative hypothesis that the two values differ significantly. For the π_i measure, there is no variability in results: no t-test results are available for some equilibria. |error| is the absolute value of the simulated results relative error from the analytical value. This error is treated as the measure of the economic significance of an error.

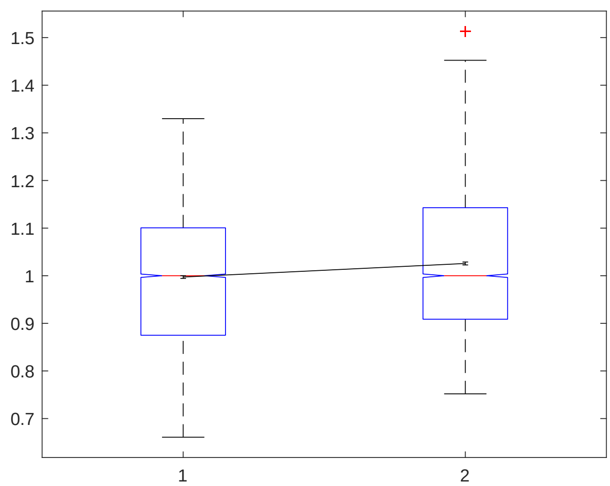


Figure 44: Maker-taker ratio for the configuration (1) MEP and MEP

The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the 95% confidence intervals for the means.

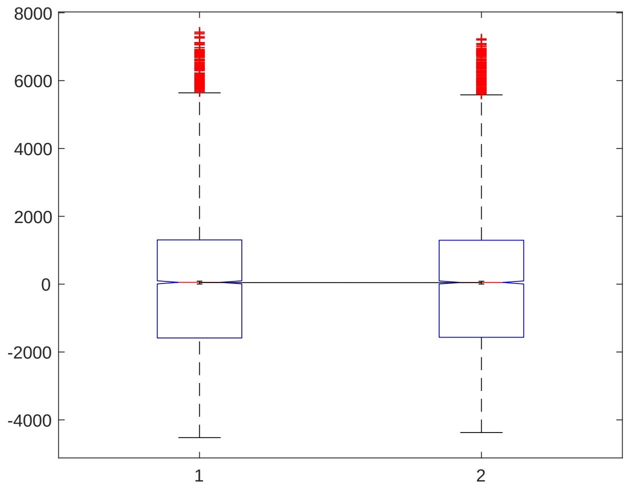
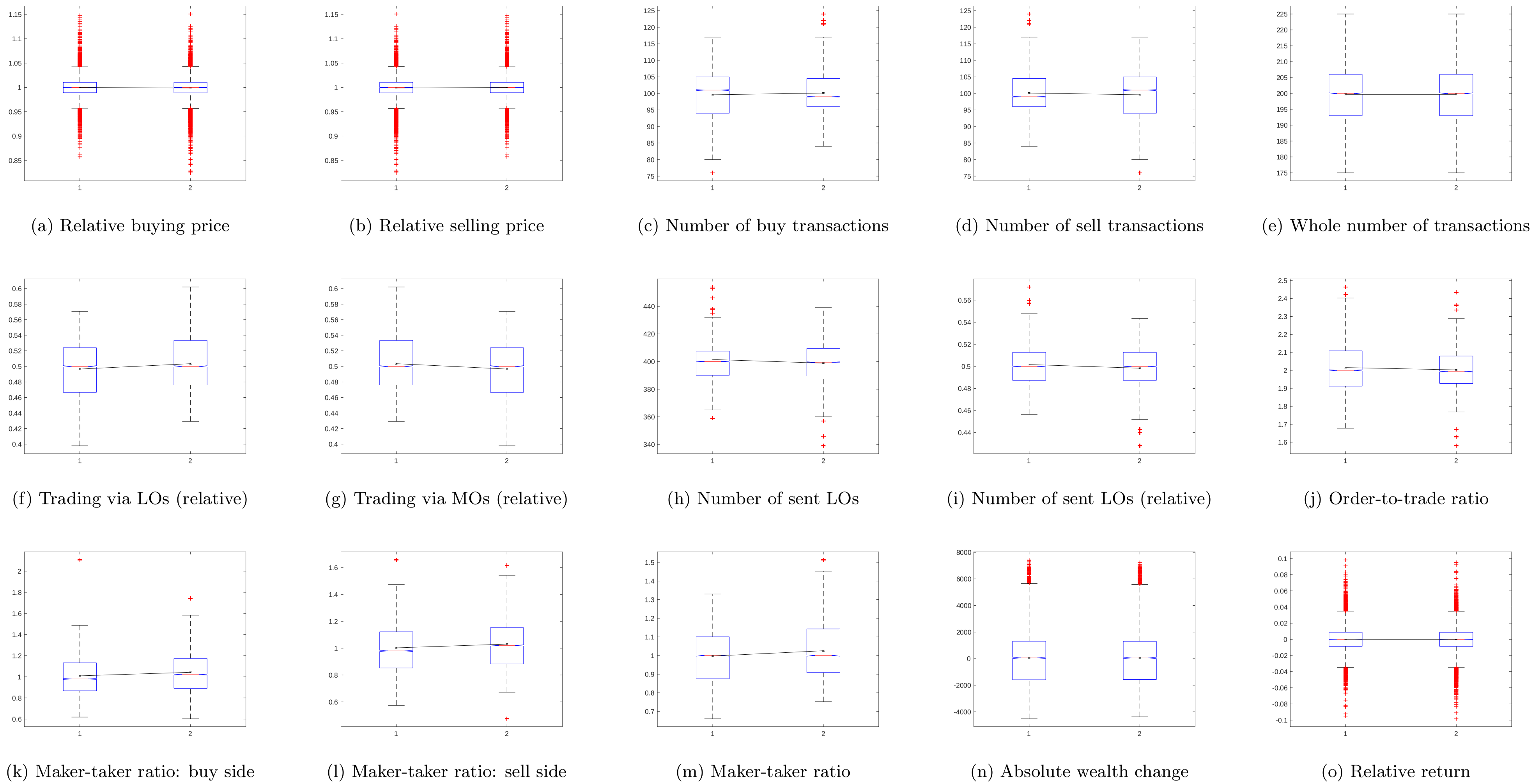


Figure 45: Absolute wealth change for the configuration (1) MEP and MEP

The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the means 95% confidence intervals.

Observation 6.1. *The simple ABM closely mimics the results of the corresponding analytical model, if their assumptions are harmonized.*
On average, the two MEPs participating in the market show identical behavior and results. If some difference appears, this is explained by the random nature of simulations.

Evidence. Table 39 reflects the fact that the results of the simulated ABM closely resemble the results of the analytical model, if the assumptions of the two are harmonized. Even though from the statistical point of view the simulated mean of a measure in many cases does differ from its analytical value, from the economic point of view this difference is not considerable. For the majority of measures, the absolute value of the relative error is not higher than 2%. The error is the most significant for the maker-taker ratio and is the result of an accumulated error.
Figure 46 shows the boxplots and error bars for some metrics per an individual agent. From there it follows that the two identical agents participating in the market display a similar activity and have similar profits.
The transaction terms of the two agents do not differ from each other on average, even though in individual market runs the agents pay more or receive less from trade (graphs (a), (b)).
The number of buying and selling transactions, the whole number of transactions are similar for the agents (graphs (c), (d), (e)). The difference in the relative number of transactions conducted via limit orders and market orders (graphs (f), (g)) is not economically significant and occurs due to the randomness of the artificial stock market. The agents send approximately the same number of limit orders to the market (graph (h)); they have the same order-to-trade ratio (graph (j)): the number of the sent orders is approximately two times higher than the



(a) Relative buying price
(b) Relative selling price
(c) Number of buy transactions
(d) Number of sell transactions
(e) Whole number of transactions
(f) Trading via LOs (relative)
(g) Trading via MOs (relative)
(h) Number of sent LOs
(i) Number of sent LOs (relative)
(j) Order-to-trade ratio
(k) Maker-taker ratio: buy side
(l) Maker-taker ratio: sell side
(m) Maker-taker ratio
(n) Absolute wealth change
(o) Relative return

Figure 46: Market metrics per agent for the configuration (1) MEP and MEP

The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the 95% confidence intervals for the means.

number of transactions. The maker-taker rate on average is very close to 1 for the buy side, the sell side, as well as for the both sides of the market (graphs (k), (l), (m)).
Both the absolute wealth change and relative return are zero on average (graphs (n), (o)). The market is a zero-sum game in relative terms: what one individual trader gains is lost by the second one. In this configuration, none of the agents has special abilities or additional information endowments; profits or losses in separate market games stem from randomness. □

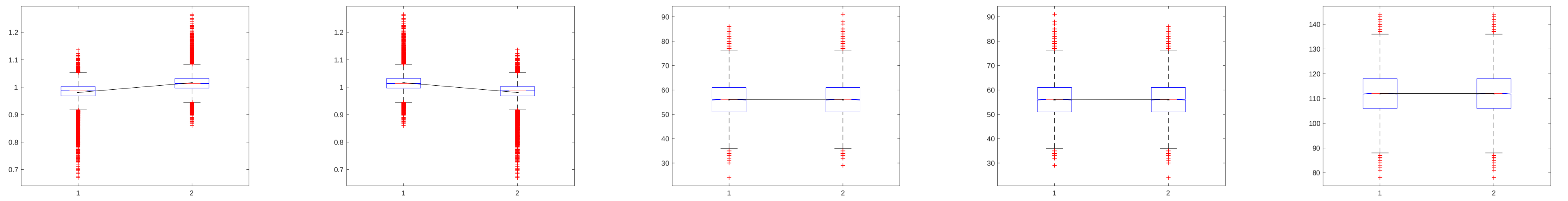
Since the simulated results closely represent the analytical results, we make the conclusion that the ABM can be used in more complex configurations to analyze how the market composition influences the market quality. The ABM allows for an analysis of further metrics using statistical tools; not all of these metrics are computable for the analytical model. In this way, the ABM is a powerful tool that can be used in addition to the analytical models for an analysis of different configurations from various points of view.

6.5.2 MEP and RT

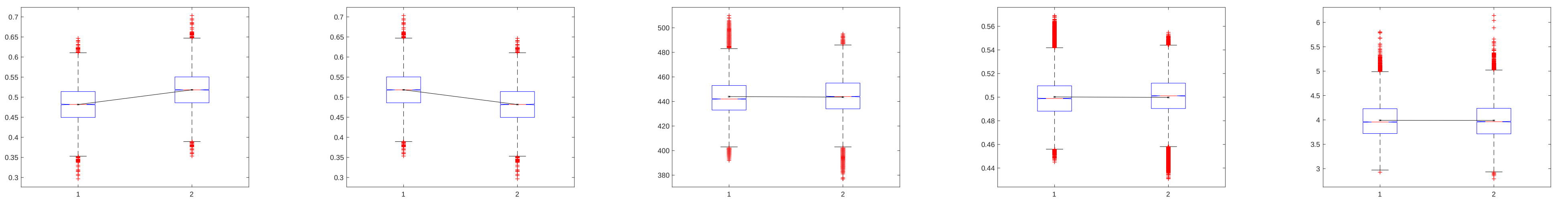
In this configuration, the market is populated with two agents: an MEP and an RT. An MEP is assumed to be either uninformed about the RT's presence on the market or to make her trading decisions as if there were no RTs. This assumption is necessary, as the equilibrium map used by MEPs is valid only if there are exclusively MEPs participating in the market.
Figure 47 provides results of statistical metrics for this configuration. Moreover, we present here the semianalytical metrics, statistical individual and aggregate metrics in Table 40, Table 41, and Table 42, as well as in graphical form in Figure 48 and Figure 49. For comparison reasons, the aforementioned tables and graphs include results of all configurations simulated within this chapter; we therefore refer to these tables and graphs in later sections when discussing the results of further configurations.

Observation 6.2. *An MEP has an informational advantage over an RT, which allows her to exploit the RT's ignorance and win the market game. Both types of agents exhibit a similar activity, while an RT is a slightly more relative market-maker with a higher risk of being picked-off. The RT's entry worsens the market quality in terms of liquidity, the informational efficiency of prices, and average market wealth, while the costs of immediacy improve.*

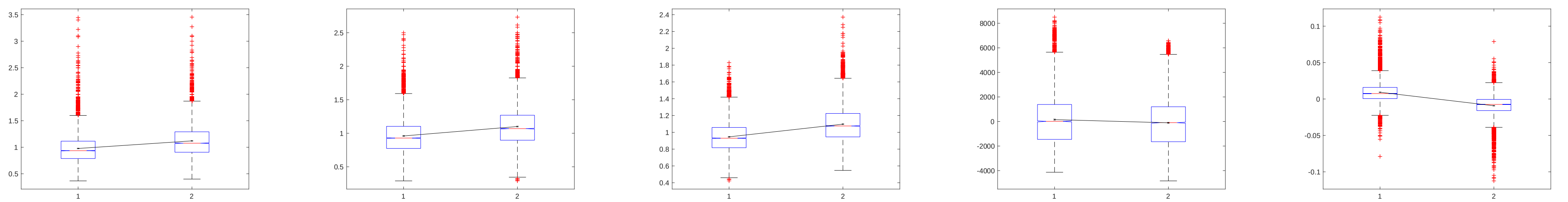
Evidence. Figure 47 shows boxplots and error bars for market metrics with a subdivision per each individual agent. This figure illustrates that an MEP is better-off than a simple RT according to many metrics. An MEP is able to exploit the RT's ignorance for her own advantage. The trade conditions of an MEP are better than those of an RT (graphs (a), (b)): this influences the final wealth of the two agents in opposite directions. The number of transactions both on the buy and sell sides is approximately equal (graphs (c), (d), (e)), but the agents are not similar in terms of providing liquidity to the market. The number of transactions that an MEP makes through market orders (limit orders) is slightly higher (lower) than 50%. For an RT the opposite holds true (graphs (f), (g)).



(a) Relative buying price (b) Relative selling price (c) Number of buy transactions (d) Number of sell transactions (e) Whole number of transactions



(f) Trading via LOs (relative) (g) Trading via MOs (relative) (h) Number of sent LOs (i) Number of sent LOs (relative) (j) Order-to-trade ratio



(k) Maker-taker ratio: buy side (l) Maker-taker ratio: sell side (m) Maker-taker ratio (n) Absolute wealth change (o) Relative return

Figure 47: Market metrics per agent for the configuration (2) MEP and RT

The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the 95% confidence intervals for the means.

Table 40: Semianalytical market metrics for various market configurations and agents

Market metrics	MEP and MEP	MEP and RT			MEP and analytic HFT			MEP and IST		
	Average	MEP	RT	Average	MEP	analytic HFT	Average	MEP	IST	Average
$\hat{p}_b$	0,2507 0,0313	0,1227 0,0236	0,1305 0,0261	0,1266 0,0252	0,1242 0,0245	0,2522 0,0278	0,1882 0,0691	0,1242 0,0245	0,2522 0,0278	0,1882 0,0691
$\hat{p}_s$	0,2498 0,0302	0,1214 0,0221	0,1318 0,0266	0,1266 0,0250	0,1235 0,0232	0,2532 0,0266	0,1883 0,0695	0,1235 0,0232	0,2532 0,0266	0,1883 0,0695
$\hat{\varphi}_b^{LO}$	0,3985 0,0242	0,4383 0,0249	0,4460 0,0239	0,4421 0,0247	0,3832 0,0225	0,4524 0,0239	0,4178 0,0416	0,3832 0,0225	0,4524 0,0239	0,4178 0,0416
$\hat{\varphi}_s^{LO}$	0,4015 0,0250	0,4460 0,0254	0,4454 0,0236	0,4457 0,0245	0,3905 0,0240	0,4511 0,0236	0,4208 0,0385	0,3905 0,0240	0,4511 0,0236	0,4208 0,0385
$\hat{\varphi}_b^{MO}$	0,1002 0,0133	0,0581 0,0117	0,0545 0,0100	0,0563 0,0111	0,1132 0,0127	0,0486 0,0096	0,0809 0,0342	0,1132 0,0127	0,0486 0,0096	0,0809 0,0342
$\hat{\varphi}_s^{MO}$	0,0998 0,0133	0,0576 0,0114	0,0541 0,0101	0,0559 0,0109	0,1130 0,0131	0,0479 0,0097	0,0805 0,0345	0,1130 0,0131	0,0479 0,0097	0,0805 0,0345
$\widehat{TR}$	0,2000 0,0191	0,1157 0,0140	0,1086 0,0148	0,1122 0,0148	0,2262 0,0183	0,0965 0,0137	0,1614 0,0668	0,2262 0,0183	0,0965 0,0137	0,1614 0,0668
$\hat{\pi}_b$	0,4995 0,0704	0,5009 0,0973	0,5848 0,0920	0,5429 0,1036	0,0000 0,0000	0,0000 0,0000	0,0000 0,0000	0,0000 0,0000	0,4995 0,0659	0,2498 0,2541
$\hat{\pi}_s$	0,4996 0,0709	0,5014 0,0973	0,5832 0,0926	0,5423 0,1034	0,0000 0,0000	0,0000 0,0000	0,0000 0,0000	0,0045 0,0654	0,4987 0,0665	0,2516 0,2558
$\widehat{MT}_b$	0,9892 0,2069	0,9776 0,2725	0,9932 0,3777	0,9854 0,3300	0,9708 0,3134	1,0012 0,1826	0,9860 0,2565	0,9708 0,3134	1,0012 0,1826	0,9860 0,2565
$\hat{V}_b^{LO}$	0,2008 0,0377	0,0983 0,0277	-0,2452 0,1293	-0,0734 0,1956	0,1683 0,0401	0,3999 0,0679	0,2841 0,1285	0,1586 0,0386	0,3207 0,0626	0,2397 0,0963
$\hat{V}_s^{LO}$	0,2001 0,0376	0,0972 0,0268	-0,2577 0,1389	-0,0803 0,2037	0,1823 0,0423	0,4325 0,0684	0,3074 0,1374	0,1726 0,0409	0,3531 0,0632	0,2628 0,1048
$\hat{V}_{b,multi}^{LO}$	0,0800 0,0155	0,0430 0,0119	-0,1092 0,0578	-0,0331 0,0868	0,0644 0,0155	0,1808 0,0319	0,1226 0,0634	0,0607 0,0149	0,1450 0,0291	0,1029 0,0481
$\hat{V}_{s,multi}^{LO}$	0,0803 0,0155	0,0433 0,0118	-0,1147 0,0620	-0,0357 0,0907	0,0711 0,0168	0,1949 0,0314	0,1330 0,0668	0,0674 0,0163	0,1591 0,0288	0,1133 0,0515
$\hat{V}_b^{LO,\text{ real}}$	-0,0499 0,0286	-0,0245 0,0191	-0,3758 0,1456	-0,2001 0,2041	0,0441 0,0254	0,1476 0,0540	0,0959 0,0668	0,0343 0,0249	0,0685 0,0520	0,0514 0,0442
$\hat{V}_s^{LO,\text{ real}}$	-0,0497 0,0287	-0,0243 0,0188	-0,3896 0,1553	-0,2069 0,2135	0,0588 0,0284	0,1793 0,0545	0,1190 0,0743	0,0491 0,0278	0,0999 0,0522	0,0745 0,0489
$\hat{V}_{b,multi}^{LO,\text{ real}}$	-0,0199 0,0114	-0,0107 0,0083	-0,1674 0,0652	-0,0891 0,0911	0,0169 0,0097	0,0668 0,0246	0,0418 0,0312	0,0131 0,0095	0,0310 0,0236	0,0221 0,0201
$\hat{V}_{s,multi}^{LO,\text{ real}}$	-0,0199 0,0115	-0,0108 0,0084	-0,1733 0,0693	-0,0921 0,0951	0,0229 0,0111	0,0808 0,0246	0,0519 0,0346	0,0192 0,0108	0,0450 0,0234	0,0321 0,0224
$\widehat{E(\tau_b)}$	-0,1993 0,1135	-2,9160 0,8774	-0,2023 0,1557	-1,5591 1,4960	0,7081 0,2007	0,4784 0,2205	0,5933 0,2401	0,3944 0,2007	0,3999 0,2205	0,3972 0,2108
$\widehat{E(\tau_s)}$	-0,1992 0,1126	-2,8410 0,8197	-0,2015 0,1558	-1,5213 1,4457	0,5853 0,2034	0,3568 0,1858	0,4710 0,2258	0,2716 0,2034	0,2784 0,1858	0,2750 0,1948
$\widehat{PE}_b$	0,7999 0,0137	0,7998 0,0130	3,4367 0,4956	2,1182 1,3643	1,1954 0,0843	0,9263 0,0427	1,0608 0,1502	1,1198 0,0774	0,3693 0,1172	0,7446 0,3882
$\widehat{PE}_s$	0,7999 0,0138	0,8002 0,0130	3,4412 0,4970	2,1207 1,3665	1,2726 0,2160	0,9868 0,2097	1,1297 0,2564	1,1942 0,2160	0,3949 0,2010	0,7946 0,4508
$\widehat{PE}$	0,7999 0,0100	0,8000 0,0094	3,4392 0,4748	2,1196 1,3617	1,2343 0,0841	0,9564 0,1002	1,0953 0,1669	1,1573 0,0882	0,3820 0,1586	0,7697 0,4083
$\hat{W}_b$	0,4015 0,0462	0,5502 0,1588	-0,0875 0,1204	0,2314 0,3486	0,1966 0,0674	0,4118 0,0780	0,3042 0,1300	0,2606 0,0679	0,3480 0,0747	0,3043 0,0837
$\hat{W}_s$	0,3992 0,0466	0,5306 0,1477	-0,0997 0,1283	0,2155 0,3442	0,2345 0,0700	0,4528 0,0765	0,3437 0,1315	0,2975 0,0709	0,3886 0,0732	0,3430 0,0853
$\hat{W}_i$	0,4004 0,0326	0,5404 0,0901	-0,0936 0,0861	0,2234 0,3290	0,2155 0,0203	0,4323 0,0320	0,3239 0,1117	0,2790 0,0211	0,3683 0,0267	0,3237 0,0507
$\hat{W}$	0,400351 0,032567	- -	- -	0,223409 0,020272	- -	- -	0,323926 0,01902	- -	- -	0,323667 0,018554

This table is also discussed in Sections 6.5.3, 6.5.4, and 6.5.5.
The first value in each cell is the mean of a distribution, while the second value is the standard deviation.

Table 41: Statistical market metrics for various market configurations and agents

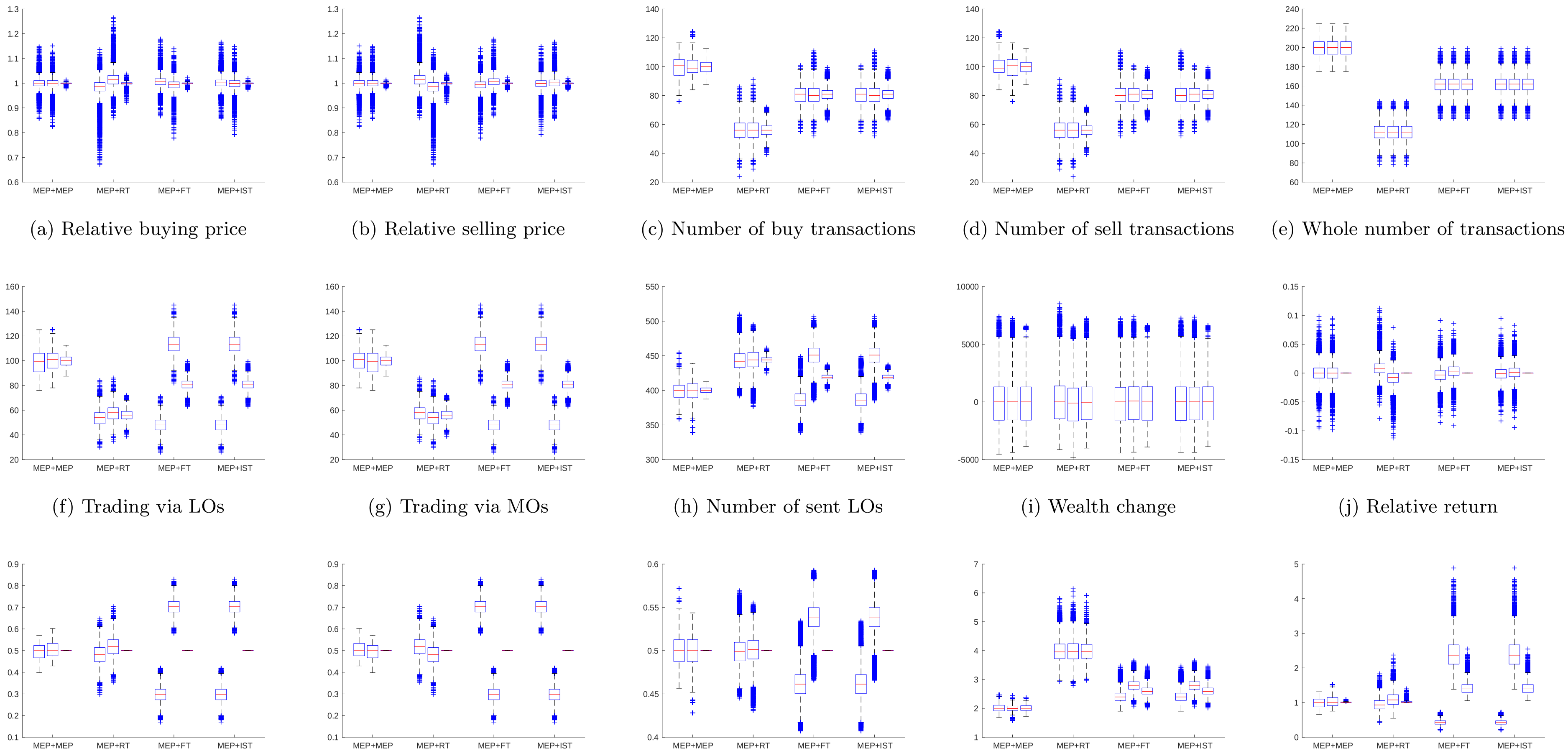
Market metrics	MEP and MEP	MEP and RT			MEP and analytic HFT			MEP and IST		
	Average	MEP	RT	Average	MEP	analytic HFT	Average	MEP	IST	Average
Sent LOs	400,1450 19,3143	443,9611 18,2149	443,5684 18,1229	443,7648 18,1686	388,5543 35,8480	450,0265 35,9051	419,2904 35,8756	388,5543 35,8480	450,0265 35,9051	419,2904 35,8756
Trade via MOs	99,8550 8,7942	58,0289 7,1963	53,9796 7,2682	56,0043 7,2327	113,4357 33,5655	47,9835 33,6186	80,7096 33,5913	113,4357 33,5655	47,9835 33,6186	80,7096 33,5913
Trade via LOs	99,8550 8,7942	53,9796 7,1963	58,0289 7,2682	56,0043 7,2327	47,9835 33,5655	113,4357 33,6186	80,7096 33,5913	47,9835 33,5655	113,4357 33,6186	80,7096 33,5913
Number of transactions	199,7100 9,1099	112,0085 8,8272	112,0085 8,8873	112,0085 8,8587	161,4192 9,1299	161,4192 9,0903	161,4192 9,1100	161,4192 9,1299	161,4192 9,0903	161,4192 9,1100
Selling transactions	99,8550 7,9948	55,9903 7,8239	56,0182 7,6825	56,0043 7,7538	80,8517 7,2890	80,5675 7,2726	80,7096 7,2807	80,8517 7,2890	80,5675 7,2726	80,7096 7,2807
Buying transactions	99,8550 7,9948	56,0182 7,8239	55,9903 7,6825	56,0043 7,7538	80,5675 7,2890	80,8517 7,2726	80,7096 7,2807	80,5675 7,2890	80,8517 7,2726	80,7096 7,2807
Trade via MOs (relative)	0,5000 0,0376	0,5183 0,0510	0,4817 0,0514	0,5000 0,0512	0,7028 0,2058	0,2972 0,2062	0,5000 0,2060	0,7028 0,2058	0,2972 0,2062	0,5000 0,2060
Trade via LOs (relative)	0,5000 0,0376	0,4817 0,0510	0,5183 0,0514	0,5000 0,0512	0,2972 0,2058	0,7028 0,2062	0,5000 0,2056	0,2972 0,2058	0,7028 0,2062	0,5000 0,2056
Sent LOs (relative)	0,5000 0,0234	0,5002 0,0199	0,4998 0,0198	0,5000 0,0198	0,4633 0,0424	0,5367 0,0425	0,5000 0,0424	0,4633 0,0424	0,5367 0,0425	0,5000 0,0424
Order-to-trade ratio	2,0089 0,1494	3,9915 0,3934	3,9888 0,3947	3,9902 0,3941	2,4170 0,2850	2,7982 0,2849	2,6076 0,2850	2,4170 0,2850	2,7982 0,2849	2,6076 0,2850
Maker-taker ratio	1,0115 0,1534	0,9461 0,2135	1,0973 0,2146	1,0217 0,2141	0,4267 1,0410	2,4164 1,0437	1,4215 1,0423	0,4267 1,0410	2,4164 1,0437	1,4215 1,0423
Buying price (relative)	0,9993 0,0224	0,9808 0,0433	1,0162 0,0414	0,9985 0,0424	1,0068 0,0264	0,9915 0,0265	0,9992 0,0264	1,0012 0,0250	0,9971 0,0249	0,9992 0,0250
Selling price (relative)	0,9993 0,0224	1,0162 0,0433	0,9808 0,0414	0,9985 0,0424	0,9915 0,0264	1,0068 0,0265	0,9992 0,0264	0,9971 0,0250	1,0012 0,0249	0,9992 0,0250
Wealth change	45,2320 2136,4115	153,6852 2233,3851	-106,2895 2002,5274	23,6979 2132,6240	-10,4820 2221,2273	85,2904 2013,2742	37,4042 2137,3258	24,7204 2220,0350	57,2049 2012,0560	40,9627 2136,0714
Relative return	0,0000 0,0158	0,0093 0,0182	-0,0093 0,0167	0,0000 0,0174	-0,0034 0,0142	0,0034 0,0138	0,0000 0,0140	-0,0012 0,0138	0,0012 0,0134	0,0000 0,0136

This table is also discussed in Sections 6.5.3, 6.5.4, and 6.5.5.
The first value in each cell is the mean of a distribution, while the second value is the standard deviation.

Table 42: Aggregate statistical market metrics for different market configurations

Market metrics	MEP and MEP	MEP and RT	MEP and analytic HFT	MEP and IST
Trading volume	7970,8940 2705,9400	4469,2681 1540,0196	6444,0921 2209,5137	6444,0505 2209,3703
Number of transactions	199,7100 9,1101	112,0085 8,8589	161,4192 9,1102	161,4192 9,1102
OIB_{money}	-22,9266 572,5059	-158,4006 436,6060	150,9448 499,4377	111,5938 499,5198
OIB_{number}	0,4100 13,1620	0,4499 10,1039	0,3824 11,4447	0,3824 11,4447
Order-to-trade ratio	2,0089 0,1154	3,9902 0,3602	2,6076 0,1793	2,6076 0,1793
Acceptance ratio	0,2497 0,0142	0,1263 0,0112	0,1926 0,0129	0,1926 0,0129
RAD	0,0308 0,0502	0,0708 0,1536	0,0277 0,0457	0,0277 0,0376
RET	-8,7925e-04 0,0851	-0,0021 0,1457	-0,0011 0,1071	-0,0011 0,0984
Trading costs	-0,0014 1,1500	-0,0267 3,1944	5,3687e-05 0,9920	-2,7409e-04 0,8664

This table is also discussed in Sections 6.5.3, 6.5.4, and 6.5.5.
The first value in each cell is the mean of a distribution, while the second value is the standard deviation.



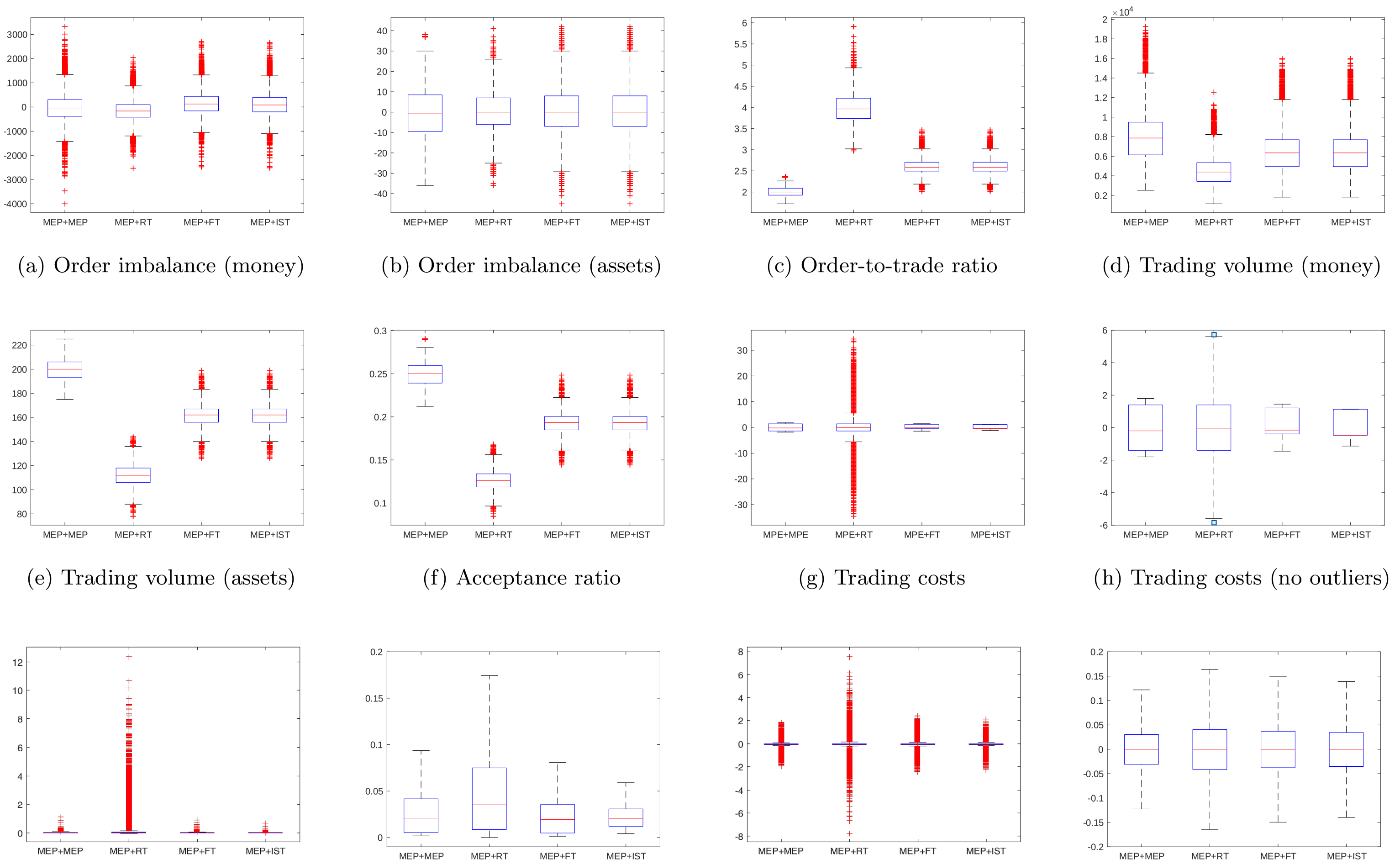
(a) Relative buying price (b) Relative selling price (c) Number of buy transactions (d) Number of sell transactions (e) Whole number of transactions

(f) Trading via LOs (g) Trading via MOs (h) Number of sent LOs (i) Wealth change (j) Relative return

(k) Trading via LOs (relative) (l) Trading via MOs (relative) (m) Sent LOs (relative) (n) Order-to-trade ratio (o) Maker-taker rate

Figure 48: Statistical metrics for various configurations per agent

This figure is also discussed in Sections 6.5.3, 6.5.4, and 6.5.5.
On the x-axis, different market configurations are grouped: each cluster of three boxplots corresponds to a different configuration. The first boxplot in a cluster corresponds to an MEP, the second one to her trading partner (MEP, RT, analytic HFT, or IST, depending on the configuration), and the third boxplot shows the average results for the two market participants. The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The blue points outside the whiskers are outliers.



(a) Order imbalance (money) (b) Order imbalance (assets) (c) Order-to-trade ratio (d) Trading volume (money)

(e) Trading volume (assets) (f) Acceptance ratio (g) Trading costs (h) Trading costs (no outliers)

(i) RAD (j) RAD (no outliers) (k) RET (l) RET (no outliers)

Figure 49: Aggregate market metrics for various configurations

This figure is also discussed in Sections 6.5.3, 6.5.4, and 6.5.5.
On the x-axis, different market configurations are shown: each boxplot corresponds to a different market configuration. The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers.

On average, the order-to-trade ratio and the relative number of the sent limit orders for both agents are approximately equal (graphs (i), (j)). An RT is only slightly a market-maker, while an MEP is a relative market-taker. Both on the buy and sell side, as well as the overall maker-taker rate of an RT (MEP) is higher (smaller) than 1 (graphs (k), (l), (m)). Even though the absolute wealth change on average for both agents is quite close to zero, it is positive for an MEP and negative for an RT (graph (n)). The statistical difference in relative returns is significant: the average relative return of an RT (MEP) is negative (positive) (graph (o)).

If the current market configuration (MEP+RT) is compared to the basic configuration (MEP+MEP), it is possible to conclude that the RTs' entry changes the market game and its results, as it is evident from Tables 40, 41, 42 and from Figures 47, 48, 49. The main effects of the RTs' entry are the following:

The execution probability of a limit order (p_i) worsens, while an RT has a higher execution probability than her MEP trading partner. The probability to send a limit order (φ_i^{LO}) increases, while an RT has a higher probability to send a limit order than an MEP; both of them are greater than in the basic configuration. The probability to make a market order (φ_i^{MO}) decreases, while an RT has a lower probability to send a market order than an MEP; both of them are lower than in the basic set-up. The entry of an RT changes the balance of equilibrium events probabilities: there is a lower (higher) probability to post a market order (limit order), while the RT's probability to send a market order (limit order) is lower (higher) than that of an MEP. The number of sent limit orders by both types of agents is very close: an RT trades more through limit orders, while an MEP trades more through market orders. The maker-taker rate for an RT is greater, meaning that she makes fewer market orders in comparison to an MEP, since the amount of the sent limit orders is almost the same for both of them. The trading rate worsens; for an RT it is smaller than for an MEP. The whole number of transactions decreases, while the number of the sent limit orders goes up. The order-to-trade ratio increases, and the acceptance ratio worsens: fewer limit orders are executed. The maker-taker ratio does not considerably change between the configurations and agents. The maker-taker rate suggests that an RT is more of a market-maker than an MEP.

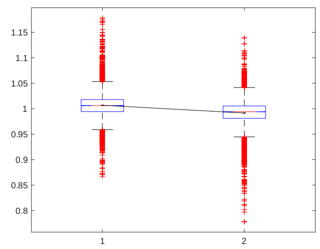
The change in order imbalance is statistically insignificant, given the high standard deviation of means. The average risk of being picked-off grows, the risk is higher for an RT than for an MEP. On average, the bargaining power becomes negative. While the MEP's expected profit from sending a limit order stays in the positive range, for an RT it becomes negative (an RT does not take it into account when making her trading decisions). With the RTs' entry, an MEP receives a smaller imaginary expected profit but a higher real expected profit from sending a limit order. The average costs of immediacy for the market improve: the trading profits increase. The RTs' trading profit is smaller than that of an MEP. The average pricing error and relative absolute deviation increase. The MEPs' pricing error stays almost unchanged, while an RT experiences a higher pricing error. The average welfare on the market decreases with the RTs' entry. □

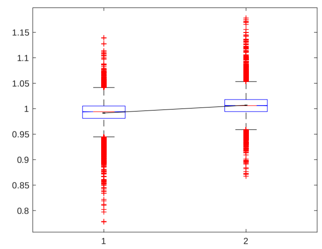
6.5.3 MEP and analytic HFT (ABM version of Hoffmann (2014))

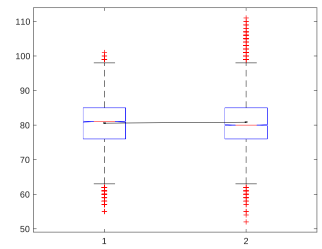
In the third configuration, an analytic HFT enters the market where slow traders (MEPs) trade. To judge how the market results and its quality change due to the analytic HFT entry, the current configuration is compared to the basic one (MEP+MEP). Besides of already provided results, some further representations are available in Figure 50.

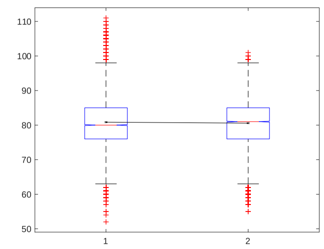
Observation 6.3. *An analytic HFT has the speed advantage (which leads to the informational advantage) over an MEP, which allows her to exploit the MEP's slowness and win the market game at the cost of an MEP. An analytic HFT is a relative market-maker, as she trades more through limit orders and posts more limit orders.*
With the analytic HFTs' entry, the market quality worsens based on liquidity, the costs of immediacy, and pricing error. However, the price efficiency improves based on the relative absolute deviation criteria.
An analytic HFT influences the market more gently than an RT: the market quality with an analytic HFT is better than with an RT.

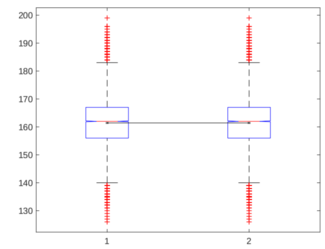
Evidence. Figure 50 shows boxplots and error bars for some market metrics under this configuration with a subdivision per each individual agent. The boxplots illustrate that an analytic HFT is better-off in the market according to many metrics; an analytic HFT is able to exploit the MEP's presence to her own advantage. The transaction terms are better for an analytic HFT (graphs (a), (b)); this influences the final wealth of the two agents in opposite directions. The number of transactions on the buy and sell sides is approximately equal (graphs (c), (d), (e)), but the agents are not similar in terms of providing liquidity to the market: an MEP makes relatively more transactions through market orders; for an analytic HFT the opposite holds (graphs (f), (g)). An analytic HFT sends more limit orders to the market: the order-to-trade ratio as well as the relative number of the sent limit orders are much higher than for an MEP (graphs (i), (j)): an analytic HFT is a relative market-maker, while an MEP is a relative market-taker. The maker-taker rate of an analytic HFT is higher than 2, while that of an MEP is smaller than 1 (graphs (k), (l), (m)). Even though on average the absolute wealth change for both agents is quite close to zero, it is positive for an analytic HFT and negative for an MEP (graph (n)). The relative return of an MEP is negative, while that of an analytic HFT is positive (graph (o)).
The analytic HFTs' entry changes the market game and its results, as it is evident from Tables 40, 41, 42 and from Figures 48, 49, 50:
The execution probability of a limit order (p_i) worsens, while an analytic HFT has a higher execution probability than her MEP trading partner. The average probability to send a limit order (φ_i^{LO}) increases, while for an MEP it slightly decreases. An analytic HFT has a higher probability to send a limit order than an MEP. The average probability to make a market order (φ_i^{MO}) decreases, while for an MEP it is higher than in the basic configuration. An analytic HFT has a lower probability to post a market order than an MEP. The number of the sent limit orders is much higher for an analytic HFT than for an MEP: an analytic HFT

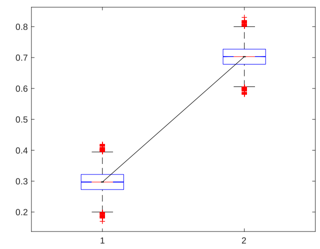

(a) Relative buying price

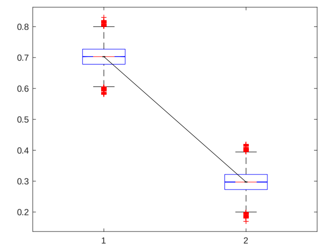

(b) Relative selling price

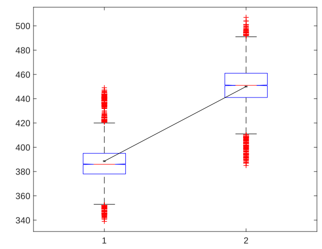

(c) Number of buy transactions

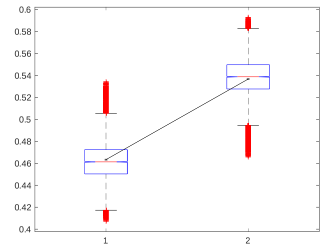

(d) Number of sell transactions

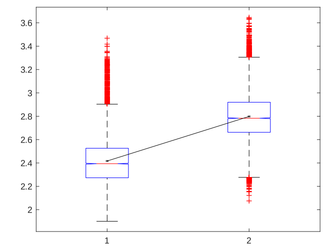

(e) Whole number of transactions

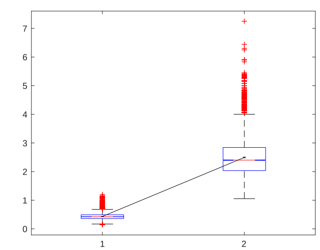

(f) Trading via LOs (relative)

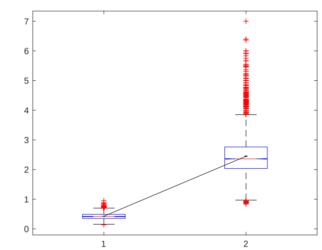

(g) Trading via MOs (relative)

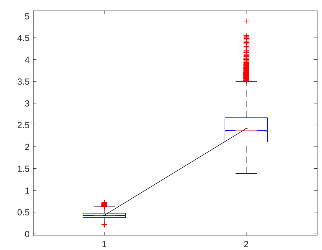

(h) Number of sent LOs

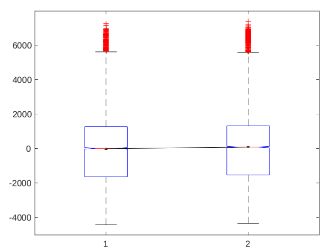

(i) Number of sent LOs (relative)


(j) Order-to-trade ratio


(k) Maker-taker ratio: buy side


(l) Maker-taker ratio: sell side


(m) Maker-taker ratio


(n) Absolute wealth change

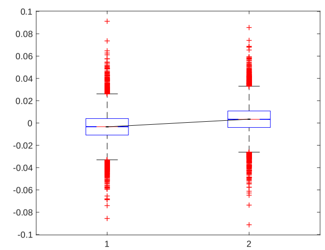

(o) Relative return

Figure 50: Market metrics per agent for the configuration (3) MEP and analytic HFT

The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the 95% confidence intervals for the means.

trades more through limit orders, an MEP trades more through market orders. The average trading rate worsens. The trading rate of an analytic HFT is smaller than that of an MEP: the MEP's trading rate improves with the analytic HFTs' entry. The whole number of transactions decreases when an analytic HFT enters the market, while the average number of the sent limit orders goes up. The order-to-trade ratio increases and the acceptance ratio worsens: fewer limit orders are satisfied. The maker-taker ratio does not lead to any considerable change between the configurations and agents. The maker-taker rate shows that an analytic HFT is more market-maker than an MEP.

The order imbalance becomes positive in currency units and decreases in asset units; it is hard to draw any reliable conclusions because of high standard deviations. The average risk of being picked-off nullifies, as an MEP uses a low fill-rate equilibrium in this market state under the configuration with analytic HFTs, while an analytic HFT can adapt her order, if the next coming agent is an MEP. The bargaining power (as well as the real expected profit from sending a limit order) is positive and on average greater with an analytic HFT being present. With the analytic HFT's entry, the bargaining power of an MEP deteriorates, while the power of an analytic HFT is higher than the initial level of an MEP. The average costs of immediacy worsen; for an analytic HFT they are smaller than those of an MEP. The average pricing error increases: the pricing error of an MEP grows, while an analytic HFT experiences a less pronounced pricing error. However, the relative absolute deviation decreases. The average market welfare decreases with the analytic HFTs' entry. □

6.5.4 MEP and IST (ABM version of the modified analytical model)

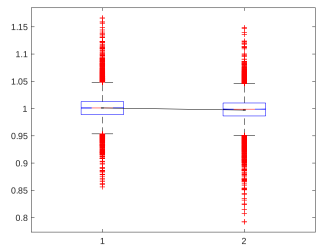
In this section, the configuration (4) MEP and IST is analyzed and compared with the basic configuration (1) MEP and MEP. Besides already provided results, some further representation is available in Figure 51.

Observation 6.4. *An IST has the informational advantage over an MEP, which allows her to exploit the MEP's uninformativeness and win the market game at the cost of an MEP. An IST is a relative market-maker, since she trades more through limit orders and posts more limit orders.*

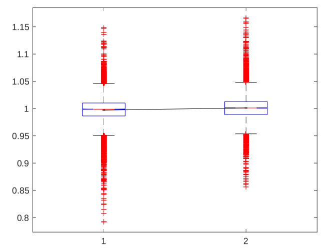
When an IST enters the market, the market quality worsens based on liquidity and the costs of immediacy. The informational efficiency of prices improves based on the relative absolute deviation measure.

An IST influences the market more gently than an RT: the market quality with an IST is better than that with an RT.

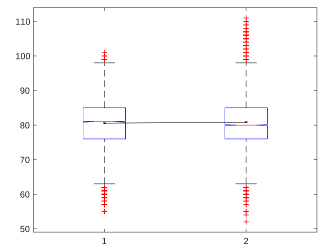
Evidence. Figure 51 shows boxplots and error bars for some market metrics under the current configuration with a subdivision per each individual agent. The boxplots illustrate that an IST is better-off than an MEP according to many characteristics; an IST is able to exploit the MEP's presence to her own advantage.



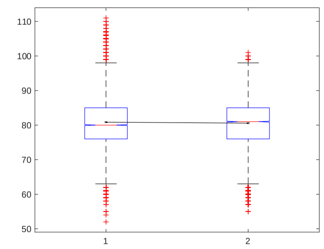
(a) Relative buying price



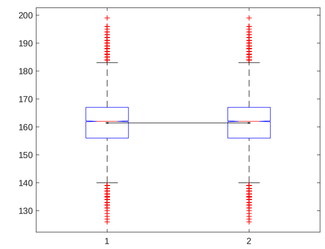
(b) Relative selling price



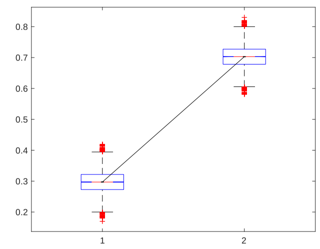
(c) Number of buy transactions



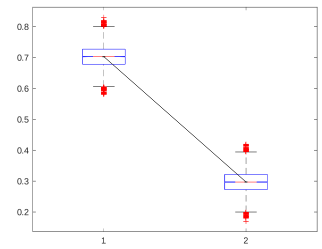
(d) Number of sell transactions



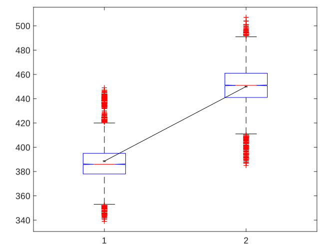
(e) Whole number of transactions



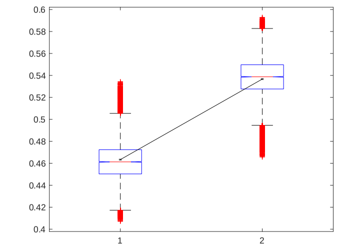
(f) Trading via LOs (relative)



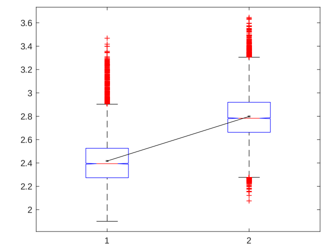
(g) Trading via MOs (relative)



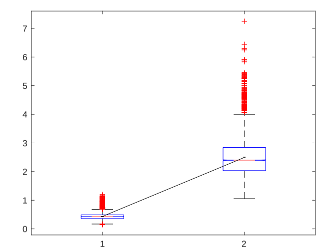
(h) Number of sent LOs



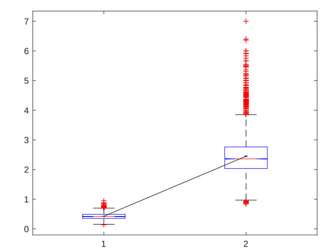
(i) Number of sent LOs (relative)



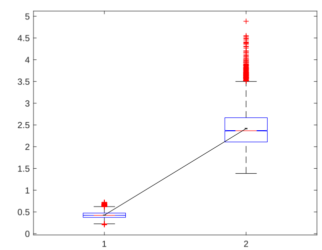
(j) Order-to-trade ratio



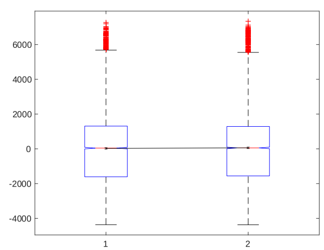
(k) Maker-taker ratio: buy side



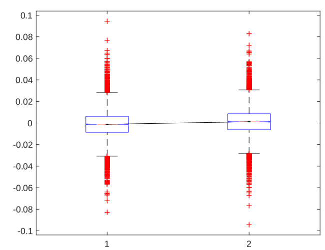
(l) Maker-taker ratio: sell side



(m) Maker-taker ratio



(n) Absolute wealth change



(o) Relative return

Figure 51: Market metrics per agent for the configuration (4) MEP and IST

The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the 95% confidence intervals for the means.

The trading terms of an MEP are worse (graphs (a), (b)); this influences the final wealth of the two agents in opposite directions. The number of transactions is approximately equal (graphs (c), (d), (e)), but the agents are not similar in terms of providing liquidity to the market: the number of transactions that an MEP makes through market orders is higher; for an IST the opposite holds true (graphs (f), (g)). An IST sends more limit orders to the market: the order-to-trade ratio as well as the relative number of the sent limit orders is much higher than those for an MEP (graphs (i), (j)): an IST is a relative market-maker, while an MEP is a relative market-taker. The maker-taker ratio of an IST is higher than 2, while that of an MEP is smaller than 1 (graphs (k), (l), (m)). Even though on average the absolute wealth change for both agents is quite close to zero, it is positive for an IST and negative for an MEP (graph (n)). The relative return of an MEP is negative, while that of an IST is positive (graph (o)).
The ISTs' entry changes the market game and its results, as it is evident from Tables 40, 41, 42 and from Figures 48, 49, 51. The ISTs' influence on the market is very similar to the analytic HFT's influence, with a couple of distinctions:
The execution probability of a limit order (p_i) worsens, while an IST has a higher execution probability than her MEP trading partner. The average probability to send a limit order (φ_i^{LO}) increases, while for an MEP it slightly decreases. An IST has a higher probability to send a limit order than an MEP. The average probability to make a market order (φ_i^{MO}) drops, while for an MEP it is higher than in the basic configuration. An IST has a lower probability to send a market order than an MEP. The number of the sent limit orders is much higher for an IST than for an MEP: an IST trades more through limit orders, an MEP trades more through market orders. The average trading rate worsens. The trading rate of an IST is smaller than that of an MEP: MEP's trading rate improves with the ISTs' entry. The whole number of transactions decreases when an IST enters the market, while the average number of the sent limit orders goes up. The order-to-trade ratio increases: fewer limit orders are satisfied. The acceptance ratio worsens. The maker-taker rate shows that an IST is more a market-maker than an MEP.
The order imbalance becomes positive in currency units and decreases in asset units. However, definite conclusions are impossible due to high standard deviations. The average risk of being picked-off decreases, since an ST uses a low fill-rate equilibrium in this market state under the configurations with ISTs. The risk of being picked-off is therefore the risk of an IST to be picked-off weighted by her share on the market. The bargaining power (as well as the real expected profit from sending a limit order) is positive and on average greater with the IST's presence. With the IST's entry, the bargaining power of an MEP deteriorates, while the power of an IST is greater than the initial level of an MEP. The average costs of immediacy worsen: they increase. The costs of immediacy of an IST and an MEP are similar. The trading costs turn to trading profits (negative trading costs). The average pricing error decreases. The pricing error of an MEP increases compared to the basis level, while an IST experiences a smaller pricing error. The relative absolute deviation decreases. The average welfare on the market decreases with the IST's entry. □

6.5.5 Analytic HFT vs IST: influence on MEPs and on the market as a whole

Based on the previous analysis and the additional comparison of the configurations (3) MEP and analytic HFT and (4) MEP and IST, as well as on the comparison of the agent types using Tables 40, 41, 42 and Figures 48 and 49, it is possible to make the following main conclusions:

Observation 6.5. *Given the state of the market for which the configurations are analyzed, there is no difference between an analytic HFT and an IST in terms of activity and their relative market-making function. An analytic HFT has a higher bargaining power, better transaction terms, higher costs of immediacy, and pricing errors, but she turns out to have a greater welfare than an IST (these effects are also carried over to the economy level).*
An MEP would choose an IST as a trading partner, since an MEP's costs of immediacy are lower and the market welfare is higher with an IST.
The markets with informed traders are better than the market with RTs based on the pricing error and liquidity measures. The RTs' maker-taker ratio is only slightly higher than that of informed traders. The amount of the sent limit orders is the highest on the market with RTs, but a high number of limit orders does not lead to a higher number of transactions, therefore such liquidity can be considered as liquidity "dust"[79]*.*
The minimum aggregate welfare and liquidity measures are achieved in the configuration with an RT, followed by the configurations with informed traders, while the maximum welfare is reached with two identical MEPs. The order imbalance measures, trading costs, and relative absolute deviation improve with the informed traders' entry, while they have the worst values with an RT on the market.

6.6 Summary

We have created a powerful ABM of financial markets that follows the assumptions of the analytical model and mimics its results. Given the achieved similarity of results, we treat the ABM as a reliable tool for further research of more sophisticated market configurations. Additionally, the influence of the RT's, analytic HFT's, and IST's entry on the market quality and their trading partner (an MEP) was investigated. The main conclusions from our research are the following:
The entry to the market of any agent different from an MEP, where another MEP is trading, worsens liquidity and welfare. All entering agents perform a market-making function. RTs additionally worsen price efficiency, while improving the costs of immediacy. Analytic HFTs negatively influence the costs of immediacy and pricing error, while improving price efficiency.

[79] By liquidity "dust" a mutual increase of the posted limit orders and decrease of transactions is meant. Even though liquidity seem to be enhanced, this enhance is inefficient, since the amount of real trades is either uninfluenced or reduced. The RTs' entry overflows the system with limit orders and this may create a pressure on MEPs. The fact that the number of transactions decreases points at a worsened market quality. However, having an additional potential liquidity improves chances for a potential transaction and therefore has (also) a positive influence on the market quality. The amount of transactions, however, also shows the market quality. Liquidity dust can be considered also as a phantom liquidity.

ISTs influence the market similarly to analytic HFTs, but they additionally improve the pricing error.

An MEP, being relatively more informed, exploits an RT's ignorance to her own advantage by picking-off the RT's orders. The informed traders' informational advantage adversely affects the MEP's results. An analytic HFT, compared to an IST, has a higher bargaining power and better transaction terms. In spite of her higher costs of immediacy and pricing errors, she gets a higher profit than an IST. An MEP would prefer an IST as a trading partner.

The market quality and welfare with informed traders is better than those with an RT.

The choice of the HFT-related policy depends on the target of such a policy: it might be designed to prevent a damage on the usual long-term investors' side, on the market as a whole, or to improve some specific market quality measure. Informed traders in general, including both slow and fast traders, damage the pricing errors, the costs of immediacy, and the slow uninformed traders' welfare, while improving the informational content of market prices. From this point of view, FT is similar in effects to IST, and the implementation of some policy would not be logical. However, slow uninformed traders are worse-off when there are FTs rather than ISTs in the market. From this perspective, the potential policy should encourage investment in informed trading rather than in technologies providing a time advantage in decision-making, execution, and the cancellation of orders.

7 A More Realistic Agent-Based Model of Financial Markets

As was discussed in Chapter 1, analytical models of financial markets are usually too restrictive and do not reflect all market dynamics, even though they lead to closed-form solutions. Having investigated the analytical model with HFTs in Chapter 3 and having created the simple ABM with HFTs in Chapter 6, it is possible to construct a less restrictive ABM allowing for more HFT properties and for the analysis of limit order book dynamics. Developing such a model is the main goal of the current chapter.

We extend the simple ABM from Chapter 6 to a more complex structure, which is non-tractable analytically but can potentially give useful insights about the HFT influence on the real financial market. After discussing how this more realistic model differs from the simple ABM, details about the trading mechanism, market session, and participants are provided. We list the HFT characteristics implemented within the new ABM and systemize all market configurations checked within our research. The new ABM allows to calculate additional market metrics, which are described in this chapter. We compare and contrast our findings with the existing literature. In the final step, we address possible HFT-related policies and regulations: it is a burning issue if some of them have a potential to minimize a damage from HFT activities without any side effects for the market quality and its participants.

7.1 Construction of the more realistic ABM

In this section, we outline the key differences between the more realistic and the simple ABMs and describe the design of the new ABM. Since many of the market building blocks were discussed in detail in Chapter 6, we concentrate here on the modified elements only. If not otherwise specified, the more realistic ABM follows the assumptions or construction principles of the simple ABM.

7.1.1 Simple ABM vs more realistic ABM

The main differences between the simple ABM from Chapter 6 and the modified, more realistic ABM can be summarized as follows:

- The new market is driven by changes in the expected dividends instead of the direct fundamental price of an asset.

- The initial auction is incorporated. Many exchanges start a trading day with an initial auction and then proceed with continuous trading[80].

[80] As described in Muscarella and Piwowar (2001), Paris Bourse is an example of an exchange that starts with the pre-opening call phase, during which quotes for potential transactions are accumulated. The market opens with the theoretical equilibrium price that is determined at the end of the pre-opening phase. After that, a continuous market lasts up to the end of a business day.

- A limit order book allows for more than one order per time step.
- The number of activity steps per period is increased. The simple ABM unrealistically assumes that there is only one step per period, i.e. there is only one agent acting between two points when information becomes available. In reality, many agents could act and many transactions could happen between two points of information dissemination.
- At least one fundamentalist[81] is active on the market, and there is no information lag associated with her knowledge[82]. The fundamentalist's presence is essential to force the market price of an asset to be dragged by its fundamental value.
- The option of canceling previously posted orders is incorporated. A crucial assumption of many models is that traders cannot revise or cancel their limit orders once they are sent to a limit order book. Due to this, the risk of being picked-off in a market is higher compared to the case where cancellation opportunities are given. If a new potential bid (ask) is smaller (higher) than the previous bid (ask) of the same agent, it is irrational to buy (sell) at a higher (lower) price. Moreover, if a new bid (ask) is higher (lower) than the existing ask (bid), it is irrational to sell (buy) at a lower (higher) price and buy (sell) at a higher (lower) price. These contradictions between new and the existing orders should be eliminated by an acting agent, if she has an opportunity for that. New orders are based on the most recent information, while the existing orders are outdated. As a result, the previous orders are to be deleted and new orders stay in a limit order book.
- Random traders (RTs) are included into all market configurations to ensure a reasonable liquidity, since RTs replenish a limit order book even when the other agents abstain from trade. Moreover, it is interesting to check how RTs are influenced by HFT activities compared to informed slow traders, i.e. slow fundamentalists in our study.
- More than one agent per type is present on the market to ensure that the results can be generalized and are not an outcome of the random nature of simulations. With this in mind, the results of identical agents are expected to be economically similar.

7.1.2 Trading mechanism

There are two main changes in the trading mechanism:
First, the opening auction precedes continuous trading. The opening auction has the aim of setting the price through the Walrasian mechanism by determining the equilibrium price of supply and demand. The agents send initial bid and ask orders. An RT sends an initial order randomly relying on the previous stock price, while a fundamentalist bases her initial order on

[81] A fundamental trader bases her orders on information about future fundamental value development and can be either a slow or fast trader, see Section 7.1.5.1 for a detailed discussion.

[82] Under the construction of this ABM, a fundamentalist has exact information about the next nine dividend realizations. Further information can be found in Section 7.1.3 and Section 7.1.5.1.

the expected present value[83]. The agents participate on both sides of a limit order book during the opening auction. Both the demand and supply functions have a step-form. The equilibrium price is therefore either a single intersection price (if the demand and supply functions intersect) or the average price of maximum quantities that are possible to satisfy. Market clearing happens at this equilibrium price.
Second, this study removes the analytical assumption of only one limit order in a limit order book. Orders once posted stay in a limit order book, unless they are accepted by another trader or canceled by a posting agent[84]. In this way, a limit order book represents a queue of limit orders on both sides of the market, while limit orders follow the time and price priority rules. Chiarella and Iori (2002) describe the price priority rule as the higher (lower) bid (ask) having priority in transactions over all the other bids (asks). However, if two bids (asks) have the same value, they are satisfied according to the sequence of their market entry, which is the time priority rule. Since there are more than one bids and asks in a limit order book, it is important to discriminate the best ask and the best bid from all the other limit orders: these two prices are potential transaction prices, in case an acting agent decides to accept one of the existing orders.

7.1.3 Economic environment

The fundamental value is modeled in the form of the expected present value, i.e. the present value of an expected dividend stream. For this purpose, a dividend process is simulated as a random walk:

$$D_t = |D_{t-1} + 0.1N(0,1)| \;\; \forall t = 0...T,$$

where T is the number of simulated periods, $N(0,1)$ denotes a standard normally distributed random variable, and the starting value of a dividend is $D_0 = 0.2$; the dividend process is assumed to be non-negative, meaning that whenever $D_t < 0$, the absolute value is taken[85]. To find the present value of future dividends, the risk-adjusted interest rate $r_e = 0.005$ is used. Based on the definitions of the risk-free and risk-adjusted interest rates, the risk-adjusted interest rate is assumed to be higher than the risk-free interest rate, $r_f = 0.001$[86].
For the sake of simplicity, it is postulated that information about dividends that agents receive for their decision-making is exact; dividend values do not suffer from any noise or inaccuracy[87]. The expected present value is computed from a stream of dividends based on Gordon's formula. This formula considers the last entry in a time series of dividends to be infinite and discounts

[83] See Section 7.1.3.
[84] The cancellation option is added to some of the simulation configurations.
[85] The dividend process is modeled similarly to Tóth and Scalas (2008).
[86] Some authors use the opposite relationship between two interest rates, e.g. Tóth and Scalas (2008).
[87] See Huber (2007), p.2539.

both the known dividends and the assumed perpetual[88]:

$$EPV_{j,k} = \frac{D_{k+j-1}}{r_e(1+r_e)^{j-2}} + \sum_{i=k}^{k+j-2} \frac{D_i}{(1+r_e)^{i-k}}.$$

Here, $EPV_{j,k}$ is the conditional expected present value of a stock for a trader with information level j in a period k. For simplicity, it is assumed that there is only one fundamental value on the market, which corresponds to the predicted value by a fundamentalist with information level 9. The information level shows how many dividend realizations are known to a trader for sure. Starting from the next period (in our case, starting from the period 10), an agent assumes an infinite stream of the last known dividend (9th dividend in this case).

The endowment of the market participants similar to the simple ABM: each agent possesses 1000 units of cash and 100 units of stock.

Each period consists of more than one step. Depending on the market configuration, it can be 100 steps or 500 steps per period: more than one agent can act between two points of fundamental information dissemination. Each market session lasts 70 periods.

In contrast to Chapter 6, where conclusions were based on 100×100 simulations, here the limit of the computational speed comes into play that is amplified by the length (and a relatively higher amount of activities) of a more realistic trading session. We set up the amount of sessions to 70, while each session is repeated 50 runs: the conclusions about a more realistic ABM are based on 70×50 simulations (overall 3500 market runs).

7.1.4 Market session sequence

The market session sequence in the more realistic ABM is closer to real financial markets. In the following, the scheme of a market routine is reconceptualized to avoid any ambiguity:

(1) The whole time series of dividends D_t is simulated.

(2) Expected present value is computed.

(3) The initial endowment is distributed among the agents.

(4) The opening auction takes place.

(5) A continuous trading session happens:

   (a) A trader is chosen randomly to act in the market.

   (b) An agent decides on the value of her order.

   (c) A transaction happens and a price is formed, if a new order of an acting agent matches some existing order from a limit order book and if transacting agents have enough endowment necessary for a transaction. Otherwise, a new limit order is

[88] The notations for the following formula are taken from Tóth and Scalas (2008).

posted, but only if a posting agent has enough endowment required for a potential future transaction.

(d) The wealth bookkeeping mechanism changes the participating agents' endowments according to their side (for a buyer, cash is decreased and the stock endowment is increased, the opposite is true for a seller). If a new limit order is posted, the endowment stays unchanged.

(e) Dividends and interests are paid out.

(6) At the end of a market session, the agents' final wealth and relative return are calculated.

Figure 52 shows the main steps of a market session.

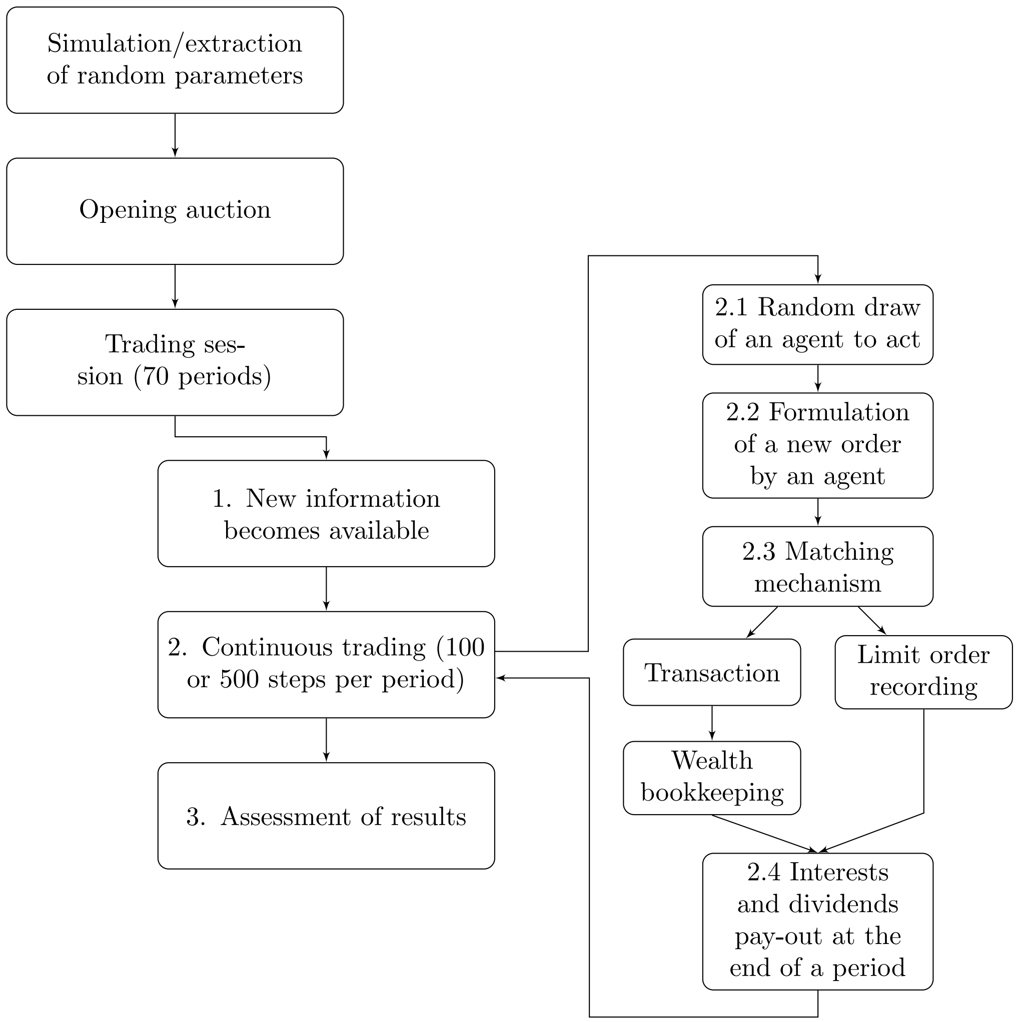


Figure 52: Temporal flow for the more realistic ABM

The first step - simulation/extraction of random parameters - is closely related to the common random numbers technique discussed in Section 6.2. To ensure that the same random variables are used for exactly the same purposes, random streams are pre-created and saved in matrices. During a market session, the respective random variable is loaded from the saved matrices.

7.1.5 Market participants and their trading strategies

In all configurations, there are 10 agents on the market: 5 of them are RTs[89] and 5 of them are informed agents; the informed agents can be either slow or fast traders (only two traders can be fast). In contrast to the simple ABM, the informed agents participating in the more realistic ABM do not use the analytical equilibrium map as their decision-making rule. The informed slow traders are modeled as slow fundamentalists and the informed fast traders are modeled as realistic HFTs. The algorithms provided in Tóth and Scalas (2008) are used as the starting point in creating the traders' algorithms for our research.

For the sake of simplicity, it is assumed that the agents do not change their type. However, the model can be extended by assuming that the agents can:

- Change their type randomly (so-called mutation process)[90];
- Be convinced by the other market participants about the superiority of another trading strategy (conviction or herding processes);
- Choose another strategy depending on their own past performance.

The possibility to change the agent type results in a positive feedback loop: the largest group is most likely to recruit other agents, generating herding behavior and a single-type situation as an extreme case[91].

In the following, the fundamentalist's and realistic HFT's strategies are described in greater detail.

7.1.5.1 Fundamentalist

A fundamentalist uses information about the expected present value and the orders available in a limit order book for her trading decisions.

As is mentioned in Schredelseker (1980), the most important assumption about fundamentalists is their belief that information about the future expected present value development is not yet reflected in the prices and that the price will converge with its fundamental value. In this way, fundamentalists are similar to the rational informed investors from Levy, Levy, and Solomon (2000). Harris (2003) defines fundamentalists as traders making transactions only if prices differ from the fundamental value; they get a profit when the price adjusts back to its fundamental value.

If the expected present value calculated by a fundamentalist is smaller (larger) than the best bid (ask), the expected present value is treated as a new marketable ask (bid). A new limit order is posted by a fundamentalist, if the expected present value is within the current spread (best ask and best bid).

[89] Their behavior pattern is identical to that described in Section 6.1.5.4.
[90] Example of this and the next extension is available in Gilli and Winker (2003), pp.301-302.
[91] See Gilli, Maringer, and Schumann (2011), p.229.

In order to place a new limit order to a certain market side, the distances from the expected present value to the best ask and best bid are calculated and compared. An agent can reach a highest turnover by posting an order in the direction of the highest difference. If the expected present value is smaller (higher) than the best ask (bid), an agent perceives a stock to be cheaper (more expensive) than the price at which the other traders would sell (buy) it. Creating an order which is close to the best order provides a better probability of execution; choosing the side with the highest difference from the expected present value provides a higher turnover. New limit orders are set according to the following formulae:

$$A = EPV + 0,25 \times (EPV - BestBid) \times |N(0,1)|,$$
$$B = EPV - 0,25 \times (BestAsk - EPV) \times |N(0,1)|,$$

where $N(0,1)$ is the standard normally distributed random variable. Such a mechanism guarantees that in the next step a new limit order is chosen as the best order, because $A < BestAsk$ and $B > BestBid$[92]:

$$A < EPV + (EPV - BestBid) < EPV + (BestAsk - EPV) < BestAsk,$$
$$B > EPV - (BestAsk - EPV) > EPV - (EPV - BestBid) > BestBid.$$

A new limit order is written down to a limit order book only if it is positive and if the posting agents' endowment is enough for a potential transaction.

7.1.5.2 Realistic high-frequency trader

The existing agent-based models with HFTs are mainly based on the assumption that HFTs are zero-intelligence agents (e.g., Vuorenmaa and Wang (2014), Cvitanic and Kirilenko (2010)) with some exogenously set trading frequency. Leal, Napoletano, Roventini, and Fagiolo (2016) depart from this assumption by allowing HFTs to place orders based on market volumes. Zhang (2010) claims that HFTs are usually agnostic to fundamental information, since they are solely interested in the statistical properties of short-term returns, while holding their positions only very shortly.

However, most of the HFT firms are professional proprietary market players. It is natural to presume that such firms have analysts able to generate knowledge about the fundamental asset prices. In Jovanovic and Menkveld (2016), HFTs are believed to be better informed. Foucault, Hombert, and Rosu (2016) model a fast trader as an informed speculator, and Carrion (2013) concludes that including only the short-term perspective into HFTs' decision-making rules might lead to an incomplete analysis[93]. Jones (2013) mentions that "conversations with market

[92] In theoretical terms, it is possible that a new ask (bid) is higher (smaller) than the best ask (best bid), if an extremely large random variable is drawn from the normal distribution. The probability of such an event is small, and it is prevented by the use of control filters.

[93] However, most of the policy concerns are focused on proprietary traders who trade rapidly and who are not long-term shareholders.

participants indicate that many HFTs do carry substantial inventory positions overnight..."[94], which suggests that at least a part of HFTs do actually care about the fundamental price development. According to Chaboud, Chiquoine, Hjalmarsson, and Vega (2014), at least on the exchange market, both human and machine traders are equally informed.

The goal of this analysis is to investigate how professional fundamental fast traders influence the market if there are also comparable professional slow traders in the market (those who experience trade latency). In this respect, the conceptualized HFT may be defined as fast fundamental trading. A realistic HFT follows the same rules for her order-setting and decision-making processes as a fundamentalist described in Section 7.1.5.1. A realistic HFT differs from a slow fundamentalist only in terms of trade speed[95], yet some other HFT features might be available to realistic HFTs[96].

As summarized by Huh (2014), the models that investigate the HFT's influence on the market understand HFTs as being purely market-makers or market-takers, but not both. The market-making behavior is assumed when HFTs have an opportunity to adjust or cancel their quotes when new information becomes available (e.g., Jovanovic and Menkveld (2016)). The market-taking behavior is constructed by allowing HFTs to send market orders (e.g., Foucault, Hombert, and Rosu (2016), Martinez and Rosu (2011)). These two patterns of the HFT behavior produce opposite effects: market-making increases liquidity and reduces spreads, while market-taking imposes adverse selection problems on liquidity-providing agents. The recent papers that advocate the opposite conclusions about the HFTs' influence, are most likely to model HFT oppositely. It is more realistic that HFTs are both market-makers and market-takers. In the more realistic ABM, realistic HFTs are allowed to be both liquidity makers and liquidity takers depending on the market conditions: a realistic HFT decides endogenously whether she wants to take the liquidity from the market or provide it.

7.2 HFT characteristics and implemented HFT dimensions

Two main characteristics of HFT were chosen to be implemented in a more realistic ABM. These characteristics could be modeled with several dimensions and included to the ABM configuration either separately or in some mix:

(1) The informational advantage resulting from the speed advantage. HFTs have access to the newest snapshot of a limit order book seconds before slow traders have access to the same snapshot. As a result, HFTs could act based on the information before slow traders get hold of the same information. Slow traders use a "stale" limit order book for their decision-making: the orders forming the basis for slow traders' orders might already

[94] See Jones (2013), p.5.

[95] A similar assumption is made in the ABM by Vuorenmaa and Wang (2014): both agents are assumed to be uninformed, with no knowledge of the fundamental value. In this way, the only advantage of an HFT over a low-frequency trader is the speed advantage, but not the advantage in terms of fundamental information or decision-making.

[96] See details in Section 7.2.

be unavailable in the current version of a limit order book. Transactions, however, are executed with the existing orders, and limit orders are sent to the current limit order book. An HFT has a potential to avoid staleness of a limit order book and see its true current version[97].

(2) Active order cancellation. HFTs cancel limit orders more frequently than slow traders, therefore it is important to add such a feature to the market. Three versions of limit orders' cancellation are integrated into our analysis: (i) all agents can cancel limit orders, (ii) only HFTs can cancel orders, (iii) all agents can cancel limit orders but slow traders experience latency in cancellation[98].

There are two other core HFT characteristics which are not incorporated in this research, but it is worth briefly mentioning them :

(1) The reaction advantage, in the sense that HFTs potentially have an opportunity to jump into the trading process before a slow trader starts acting on the market. One can allow a realistic HFT to enter the market before each slow trader.

(2) Various HFT strategies. This research relies on only one trading strategy used by an HFT: a fundamental strategy. However, HFTs do not always base their trading decisions solely on fundamental information. For example, market-making HFTs are quite unlikely to have any knowledge about the fundamental value (or at least they are unlikely to use this information profitably). Alternatively, HFTs can follow the price reversal strategy driven by market imbalances. Moreover, HFT strategies might be event-driven or time-driven. Biais, Foucault, and Moinas (2011) discuss two more HFT strategies: smoking and spoofing. Smoking is described as an HFT filling a certain market side with alluring quotes to attract a market order from the opposite side. However, before the expected market order hits the market, an HFT deletes less profitable of these alluring orders. The opposite side is attracted by orders which do not exist anymore, and it receives less profitable transaction conditions, while for an HFT a transaction is quite profitable. While spoofing, an HFT posts large orders at safe prices on the opposite market side. These orders are easy to cancel but they create an additional pressure to buy or sell on the opposite side. It causes naive investors to make transactions under less profitable conditions offered by HFTs.

Based on the target HFT characteristics described above, seven HFT dimensions were designed and varied along the market configurations to analyze how these dimensions influence the market quality and market participants' results:

(1) The market population or types of agents acting in the market. As discussed in Section 7.1.5, both fundamentalists and RTs are necessary for healthy market functioning.

[97] This HFT property potentially addresses characteristics (3), (4), and (5) from Section 1.1.

[98] This HFT property potentially addresses characteristics (5) and (6) from Section 1.1.

Fundamentalists are responsible for the price convergence to its fundamental value; too few fundamentalists might be not enough for convergence. RTs are accountable for limit order book's replenishment. Common for all the market configurations is that five agents (identifiers $1-5$) are RTs and five agents (identifiers $6-10$) are fundamentalists. The RTs are slow traders. The fundamentalists can either all be slow traders or the last two fundamentalists (identifiers $9-10$) may be realistic HFTs. Two versions of the market composition are distinguished for ABM simulations:

(a) Agents with identifiers $1-5$ are slow RTs, agents with identifiers $6-10$ are slow fundamentalists,

(b) Agents with identifiers $1-5$ are slow RTs, with identifiers $6-8$ are slow fundamentalists, and with identifiers $9-10$ are fast fundamentalists (realistic HFTs).

(2) The latency parameter refers to the amount of time steps by which a limit order book shown to the slow traders is stale in comparison to the current version of a limit order book used for transactions and limit orders' recording. Our study investigates three latency values:

(a) A non-latent market: there is no delay in information inside a limit order book (latency$=0$),

(b) A modestly latent market: a limit order book available to the slow traders is 8 steps behind its current version (latency$=8$),

(c) A very latent market: a limit order book available to the slow traders is 20 steps behind its current version (latency$=20$).

(3) The possibility to cancel the previous limit orders. At the end of each step, an acting agent checks which orders she wants to delete from a limit order book given the version of a limit order book at her disposal. Here, three options are available:

(a) The previous orders cannot be canceled,

(b) The orders can be canceled by realistic HFTs only (i.e. an exclusive cancellation right),

(c) The orders can be canceled by all agents (i.e. a non-exclusive cancellation right).

(4) The slow traders' waiting time in canceling orders shows how many steps a slow trader has to wait until an order is canceled. This parameter is applied only in case of $3c$. The realistic HFTs are unaffected by this parameter; their orders are canceled right away. Two waiting time values are taken into consideration:

(a) Zero waiting time: a slow trader's order is canceled at the end of the current step when this slow trader acts and decides to cancel an order,

(b) The waiting time of 8 steps: a slow trader sends an inquiry to cancel an order at the current step, but she waits for eight steps until this inquiry is executed (it is executed if a limit order still exists, i.e. if no one took this limit order in the meantime).

(5) The degree of limit order book visibility to the slow traders. This dimension implements latency in a different form. The slow traders are disadvantaged in a way that they do not observe some activities in a limit order book timely. However, this is a relative disadvantage to the HFTs only, the other slow traders have the same problem. To implement this (relative) disadvantage, the present configuration allows for two options:

(a) All limit order book activities within the last latency period are not observable by the slow traders. This applies to the posted and canceled quotes by both slow traders and realistic HFTs,

(b) A slow trader is able to partially see the current version of a limit order book: the other slow traders' activities are visible timely; the realistic HFTs' quotes and cancellations are observable by the slow traders with a delay equal to a latency period. This dimension potentially addresses the fact that there is a lot activity going on in a limit order book caused by HFT.

(6) The accumulation option before the realistic HFTs' quotes are canceled. The realistic HFTs have a speed advantage over the slow traders, but not over similar realistic HFTs. To account for this, two possible parameters for the realistic HFTs' order cancellation are introduced:

(a) The realistic HFTs can cancel their orders in their active step, even if the next coming agent is another realistic HFT,

(b) The realistic HFTs do not have a cancellation speed advantage over the other realistic HFTs. All realistic HFTs who want to cancel their previous orders, have to wait until a slow trader comes to the market. All of the realistic HFTs' cancellation inquiries are accumulated until the next slow trader acts and executed if they were not used by the other realistic HFTs in the meantime.

(7) The market speed or the number of potential trading steps within one market period. To secure that the sensitivity analysis is comprehensive, it should come under scrutiny whether the HFT influences an accelerated market in a similar manner. An accelerated market can be described as a market with a longer duration or a higher speed of actions between two points of information dissemination. In the accelerated market, there is more activity per period, as more agents have a chance to act. Two duration spans of the market period are tested. As specified in the market configurations in Section 7.4, the accelerated market is studied only once:

(a) The standard market duration with the standard speed of acting agents (100 steps per period),

(b) An accelerated market (500 steps per period).

7.3 Additional market quality metrics and measures of individual performance

It was impossible to calculate some of the important market quality measures for the simple ABM, because it is restricted by the assumption of a single limit order for both market sides at each point of time. Since the more realistic ABM is not subject to this restriction, further market metrics are available in addition to those mentioned in Section 6.3.2.
In the following equations, t refers to a period, s refers to a time step, (t, s) refers to a step s within a period t. ts denotes an uninterrupted series of time steps when some certain (specified) activity happens (either limit orders are posted or transactions occur).

7.3.1 Aggregate market metrics

(1) Spreads as price liquidity measures[99]

(a) The quoted spread serves as the measure of the overall liquidity supply on the market:

$$QS_{t,s} = BestAsk_{t,s} - BestBid_{t,s},$$

which is determinable only if both a bid and ask exist in a limit order book.

(b) The quoted spread (Chordia, Roll, and Subrahmanyam (2000)):

$$QS^{CH}_{t,s} = \big(BestAsk_{t,s} - BestBid_{t,s}\big)\big|\big(tr_{t,s} = 1\big),$$

where $tr_{t,s}$ is a dummy variable showing whether there is a transaction at a step s of a period t.

(c) The quoted spread (log-version):

$$QS^{Log}_{t,s} = log(BestAsk_{t,s} - BestBid_{t,s}).$$

(d) The proportional quoted spread[100]:

$$PQS_{t,s} = \frac{BestAsk_{t,s} - BestBid_{t,s}}{m_{t,s}},$$

where midquote $m_{t,s}$ is calculated as:

$$m_{t,s} = \frac{BestBid_{t,s} + BestAsk_{t,s}}{2}.$$

[99] Bid-ask spreads are components of trading costs, see Jones (2013), p.11.
[100] Riordan and Storkenmaier (2012) divide this measure additionally by two and call it a quoted spread.

(e) The effective half spread:

$$EHS_{t,s} = P_{t,s} - m_{t,s}.$$

(f) The effective spread (Chordia, Roll, and Subrahmanyam (2000)) is used to measure actual trading costs:

$$ES^{CH}_{t,s} = 2 \times abs(P_{t,s} - m_{t,s}).$$

The multiplier of two is necessary, since the distance in the parentheses captures the costs of one side of a trade only, while bid-ask spreads should measure the round-trip costs of trading by convention[101].

(g) The effective half spread (Huh (2014))[102]:

$$EHS^{HU}_{t,s} = \frac{q(P_{t,s} - m_{t,s})}{m_{t,s}},$$

where $q = 1$ ($q = -1$) for a buy (sell) transaction.

(h) The realized half spread[103]:

$$RHS_{t,s} = EHS^{HU}_{t,s} - PI^{HU}_{t,s},$$

where PI is a price impact explained later in the list.
Hendershott and Moulton (2011) claim that the realized spread has only a temporary effect: it shows the profit earned by a liquidity provider.

(2) Other price liquidity measures: the costs of adverse selection (price pressure or price impact)[104]:
Price pressure is a charge for a risk of price changes to which liquidity providers are exposed and which has to be compensated for by liquidity takers (see Jones (2013), p.11). The costs of adverse selection can be described as the time period it takes for a price to reflect the information content of a transaction[105]. Low costs of adverse selection are

[101] See Jones (2013), p.13.

[102] Similar measures are also described in Riordan and Storkenmaier (2012). Hendershott, Jones, and Menkveld (2011) provide a similar measure but the full (not half) spread: they additionally multiply the outcome of the above provided formula by two.

[103] Similar relationship for the realized and effective spreads is presented and analyzed in Hendershott, Jones, and Menkveld (2011) and Hendershott and Moulton (2011). Riordan and Storkenmaier (2012) find the realized spread similarly to $EHS^{HU}_{t,s}$, but instead of the midquote in step (t, s) they suggest calculating it in a couple of steps $(t, s + k)$.

[104] According to Jones (2013), the implementation shortfall measure is a preferred measure to price impacts; it is calculated as an average price for a large order execution compared to the stock price prior to execution, see p.13. However, the present ABM is simplified by allowing only one-stock trades, therefore this measure is not informative for the modeled artificial stock market.

[105] See Jovanovic and Menkveld (2016), p.25.

a sign of a high quality and liquid financial market. The costs of adverse selection are measured by a price impact and estimated through a standard spread decomposition.

(a) The price impact (log-version):

$$PI_{t,s} = q_{t,s}(log(m_{t,s+30}) - log(m_{t,s})).$$

(b) The price impact (Huh (2014))[106]:

$$PI^{HU}_{t,s} = \frac{q_{t,s}(m_{t,s+30} - m_{t,s})}{m_{t,s}}.$$

(3) Price efficiency measures (the measures of fundamental information content in market prices) - a correlation between the (average) transaction prices and fundamental value:

$$RHO = corr(P^{\mathrm{av}}_t, v_t),$$

where P^{av}_t is an average transaction price per period.

7.3.2 Individual performance metrics

In this section, i refers to the agent identifier.

(1) Individual activity measures

(a) The number of canceled limit orders per market session, $N_{canc,i}$.

(b) Participation:

$$Part_i = \frac{N_{tr,i}}{N^{av}_{tr}} - 1,$$

where N^{av}_{tr} is the average number of transactions for the whole market session for all the market participants.

(2) Individual performance measures: trading profit (sum of the three components)
To analyze the main source of HFT profitability, it is possible to break down trading profit (TP) into three parts: two spread components (which add up to the net spread) and the positioning profit component[107]:

$$TP_i = RPP_i - PSAO_i + ESPO_i,$$

where:

[106] A similar measure of losses to liquidity demanders due to adverse selection is proposed by Riordan and Storkenmaier (2012), Hendershott and Moulton (2011), and Hendershott, Jones, and Menkveld (2011): they call this measure "adverse selection" and estimate it through the five-minute price impact.

[107] See Menkveld (2013), p.724.

(a) RPP is a realized positioning profit, it reflects value changes associated with the net position:

$$RPP_i = \sum_{ts=2}^{t\cdot s} \left(S_{ts,i} - S_{1,i}\right) \cdot \left(m_{ts} - m_{ts-1}\right),$$

where ts represents a time series of steps during the whole trading session, $S_{ts,i}$ is an i-agent's asset endowment at any step s of a period t and S_1 is her initial asset endowment, $m_{ts} - m_{ts-1}$ is the midquote change.

(b) $PSAO$ is the paid spread at aggressive orders:

$$PSAO_i = \sum_{ts=2}^{t\cdot s} m_{ts|MO_i} \cdot EHS_{ts}^{HU},$$

where $m_{ts|MO_i}$ is a midquote at any step s of a period t, where an agent i makes a transaction through a market order.

(c) $ESPO$ is the earned spread at passive orders:

$$ESPO_i = \sum_{ts=2}^{t\cdot s} m_{ts|LO_i} \cdot EHS_{ts}^{HU},$$

where $m_{ts|LO_i}$ is a midquote at any step s of a period t, where an agent i makes a passive transaction (her limit orders from the past steps are satisfied).

7.4 Simulation configurations

Eleven configurations of the more realistic ABM are simulated. Table 43 systemizes these configurations based on the dimensions discussed in Section 7.2. The first two configurations are considered to be benchmark models: basic non-latent and latent markets. In the first non-latent market, no agents are given any HFT-related properties. The second configuration is also conceptualized as an HFT-free market, while it has latency, meaning that limit order book information is delayed but no agents have any kind of speed advantage.

To make the analysis more self-explanatory, we subdivide the non-benchmark configurations into two groups: the first group, more general configurations (3)-(5), introduce latency and HFT to the market, as well as check whether acceleration changes the results considerably. The second group, configurations (6)-(11), introduces various cancellation options. The first group will be studied in greater detail, describing effects on the market and differences between the market participants for each configuration individually (see Section 7.5.1). Moreover, the same configurations are analyzed pairwise (see Section 7.5.2). For the second group, an individual analysis is not performed; relevant comments are provided along the pairwise comparison in Section 7.5.3, if necessary.

Table 43: Simulation configurations for the more realistic ABM

Configurations		Dimensions and choices of their alternatives															
N	Abbreviation	1		2			3			4		5		6		7	
		a	b	a	b	c	a	b	c	a	b	a	b	a	b	a	b
(1)	NLat	X		X			X			n/a		n/a		n/a		X	
(2)	Lat	X			X		X			n/a		n/a		n/a		X	
(3)	Lat+HFT		X		X		X			n/a		X		n/a		X	
(4)	VLat+HFT		X			X	X			n/a		X		n/a		X	
(5)	Acc+VLat+HFT		X			X	X			n/a		X		n/a			X
(6)	Canc		X		X				X	X		X		X		X	
(7)	Canc8		X		X				X		X	X		X		X	
(8)	HFTCanc		X		X			X		n/a		X		X		X	
(9)	AccumCanc		X		X			X		n/a		X			X	X	
(10)	CancNVis		X		X				X		X		X	X		X	
(11)	HFTCancNVis		X		X			X		n/a			X	X		X	

"X"-sign shows which alternative of an HFT dimension is applied in a certain configuration, while "n/a" means that this dimension is not applicable (e.g., cancellation latency is not applicable if orders cannot be canceled or if they can be canceled only by realistic HFTs, as realistic HFTs do not experience any latency when the next coming agent is a slow trader).

7.5 Analysis of simulation results

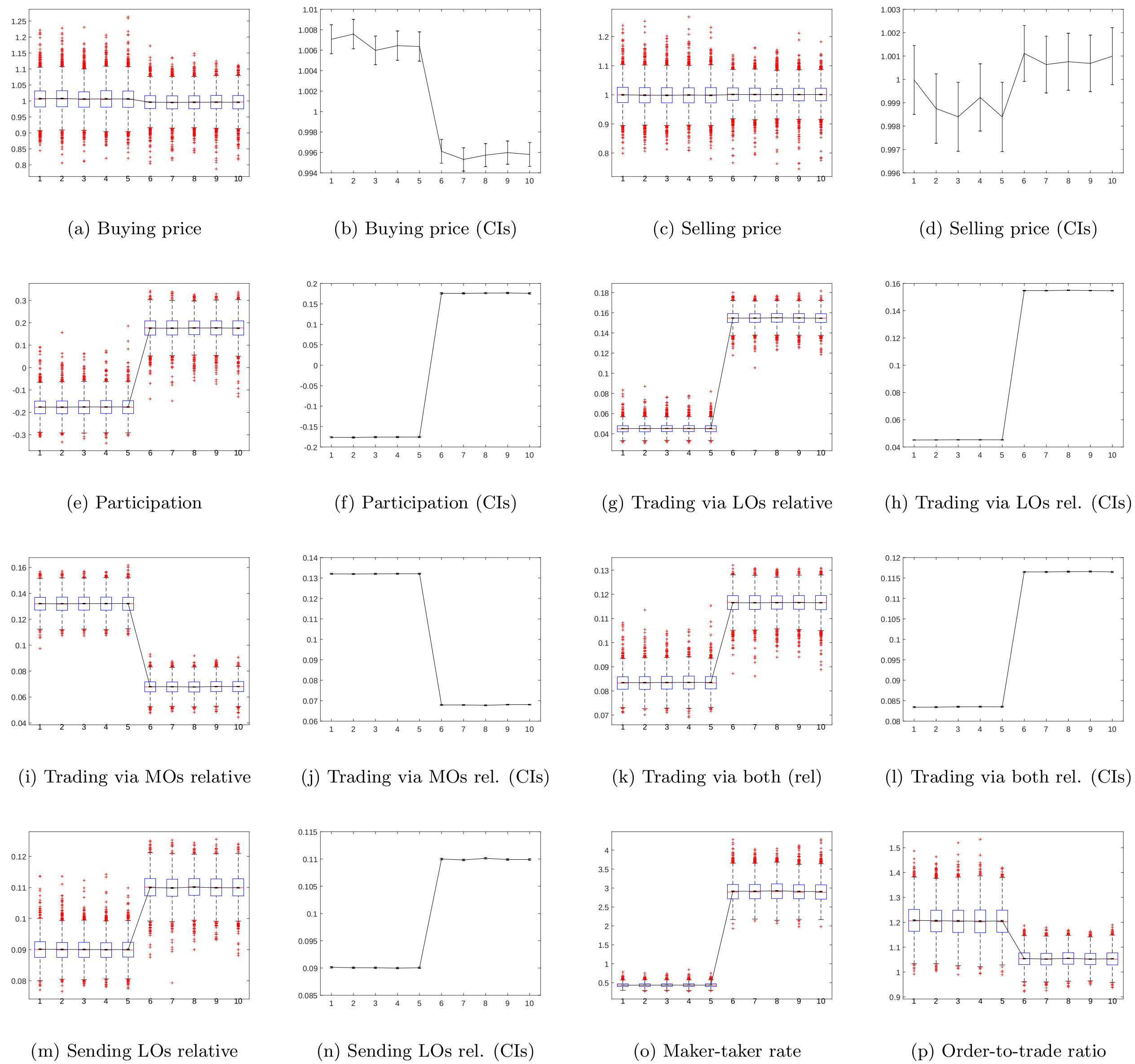
When analyzing and presenting conclusions from simulations in this section, we first concentrate on basic configurations and examine them individually, after which we compare pairwise configurations from the first group and the second group one after another in the sequence of their complication. A comparison of our results with the findings of the existing research presented in Section 1.4 is based on Table 1.

7.5.1 Individual analysis of configurations from the first group

In this section, we present a detailed analysis of the market configurations from the first group concentrating mainly on differences in various market participants' behavior and results.

7.5.1.1 Non-latent market: fundamentalists use their informational advantage profitably

In this section, the results of the market configuration (1) NLat are analyzed. Results of this configurations are presented graphically in Figure 53 and Figure 54.



(a) Buying price (b) Buying price (CIs) (c) Selling price (d) Selling price (CIs)

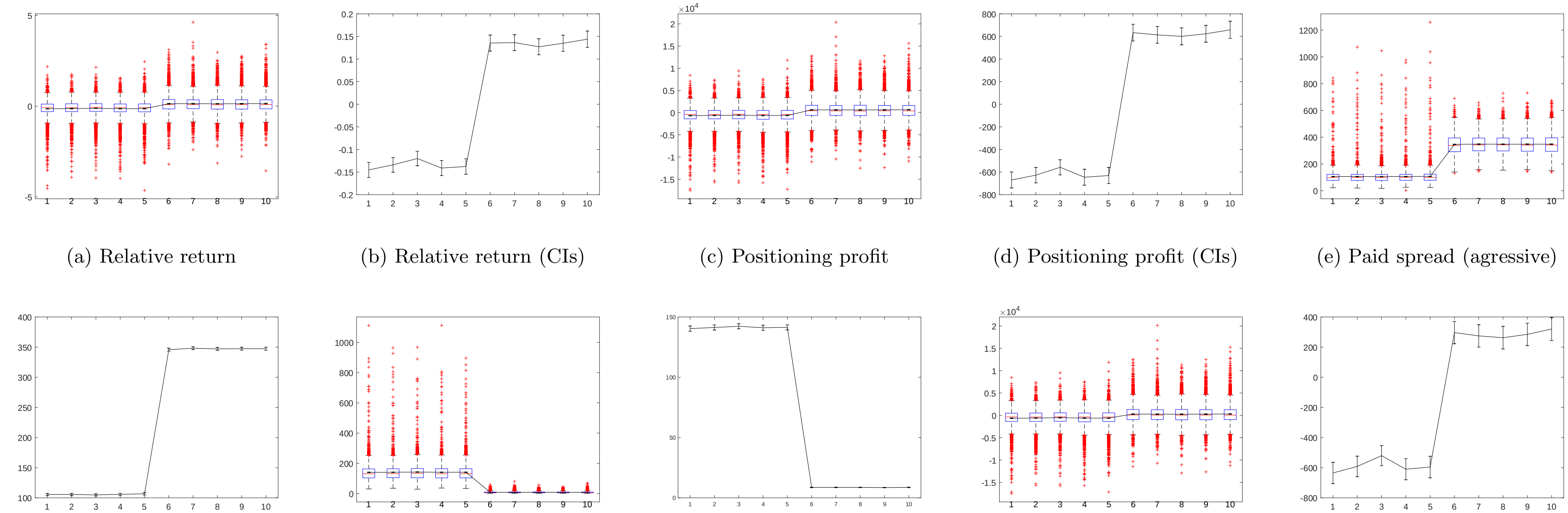
(e) Participation (f) Participation (CIs) (g) Trading via LOs relative (h) Trading via LOs rel. (CIs)

(i) Trading via MOs relative (j) Trading via MOs rel. (CIs) (k) Trading via both (rel) (l) Trading via both rel. (CIs)

(m) Sending LOs relative (n) Sending LOs rel. (CIs) (o) Maker-taker rate (p) Order-to-trade ratio

Figure 53: Results of the configuration (1) NLat (non-latent market): liquidity and activity metrics

In this figure, some of the graphs are boxplots and others plot only confidence intervals: this is done for a more detailed investigation of the selected metrics. On the x-axis, the agents' identifiers are shown. The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the means 95% confidence intervals.



(a) Relative return (b) Relative return (CIs) (c) Positioning profit (d) Positioning profit (CIs) (e) Paid spread (agressive)

(f) Paid spread (agressive) (CIs) (g) Earned spread (passive) (h) Earned spread (passive) (CIs) (i) Trading profit (j) Trading profit (CIs)

Figure 54: Results of the configuration (1) NLat (non-latent market): individual returns and profits

In this figure, some of the graphs are boxplots and others plot only confidence intervals: this is done for a more detailed investigation of the selected metrics. On the x-axis, the agents' identifiers are shown. The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the means 95% confidence intervals.

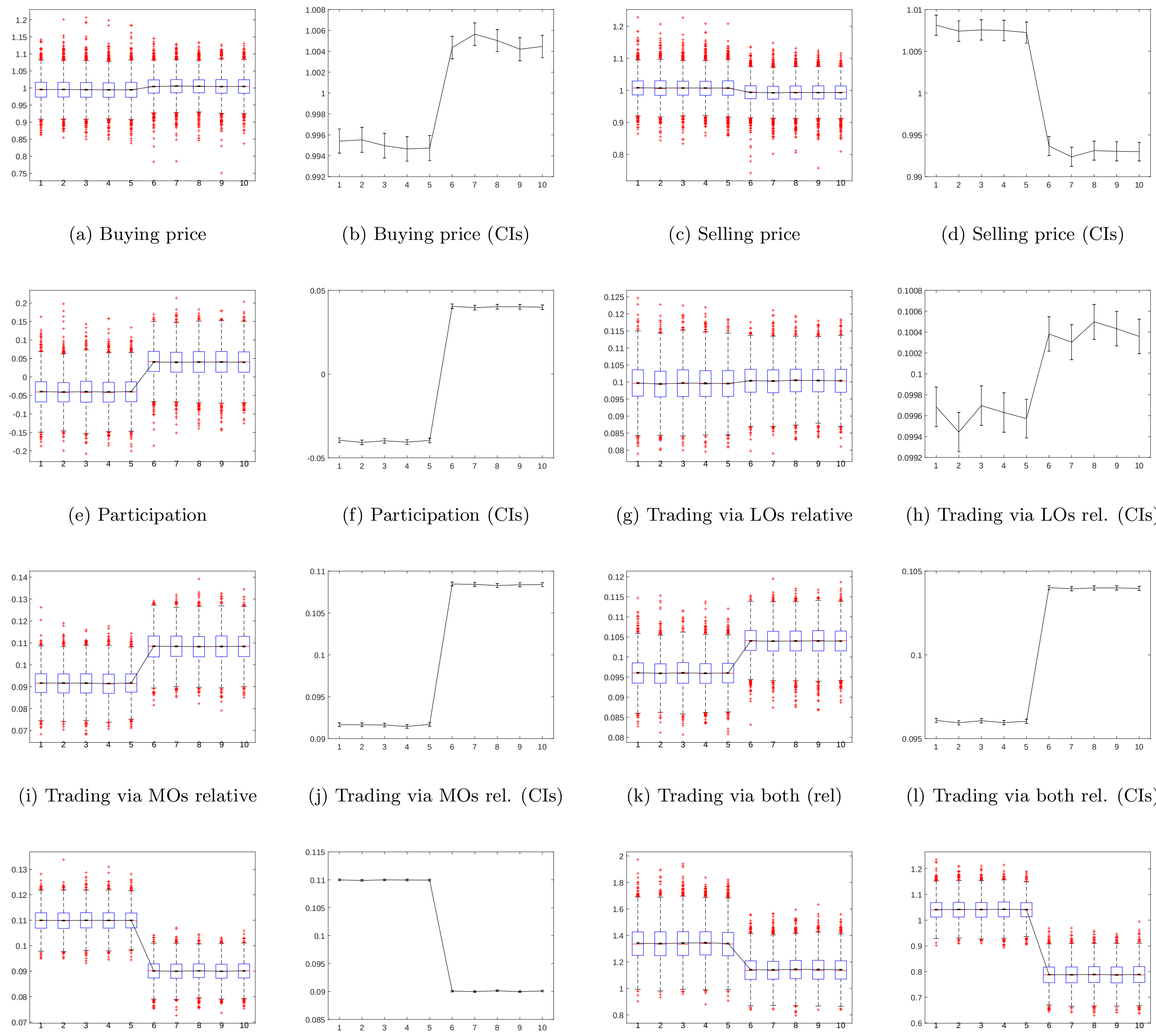
Observation 7.1. *In the non-latent market, the informational advantage allows fundamentalists to win the game: fundamentalists anticipate asset value changes correctly and adjust their positions respectively. Fundamentalists are relative market-makers with a higher number of sent limit orders and conducted transactions, with a higher paid spreads from aggressive orders and lower earned spreads from passive orders but with a higher positioning profit outweighing the spread components of a trading profit.*

Evidence. Observations are based on Figure 53 and Figure 54. The selling (buying) price on the part of fundamentalists is higher (considerably smaller) than that of RTs (graphs (a)-(d) in Figure 53); this influences the agents' final wealth respectively. The fundamentalists participate more in trade (graphs (e), (f), (k), (l) in Figure 53). They trade relatively more through limit orders (graphs (g), (h) in Figure 53), while the RTs through market orders (graphs (i), (j) in Figure 53). The fundamentalists send more limit orders to the market (graphs (m), (n) in Figure 53). The fundamentalists (RTs) are relative market-makers (-takers) (graph (o) in Figure 53). More intensive participation in trade makes the fundamentalists' order-to-trade ratio even lower than that of the RTs (graph (p) in Figure 53): there are more transactions per one limit order sent by a fundamentalist than by an RT. The fundamentalists provide a lot of liquidity to the market but also take a lot of it. The majority of transactions, however, are conducted through limit orders.
The fundamentalists' trading profit is higher than that of the RTs (graphs (i), (j) in Figure 54). The RTs win a higher spread from passive orders (graphs (g), (h) in Figure 54) and pay a lower spread in aggressive orders (graphs (e), (f) in Figure 54), while the positioning profit outweighs spread components: the fundamentalists' (RTs') positioning profit is positive (negative) (graphs (c), (d) in Figure 54). The fundamentalists have a higher relative return than the RTs (graphs (a), (b) in Figure 54). □

The fundamentalists' relative market-making activity can be explained by the fact that the fundamentalists have the fundamental value as an attractor: their new order is more likely to be outside the current spread, which automatically leads to a limit order rather than to a direct transaction. The RTs, on the contrary, base their orders on the last transaction price, and chances of a new order to be within the current spread are higher, therefore the RTs trade relatively more through market orders.
Since information asymmetry is incorporated into this market configuration, it leads to adverse selection: the fundamentalists win at the cost of the RTs.

7.5.1.2 Latent market: fundamentalists fail to "catch" the right moment for a transaction

In this section, the results of the market configuration (2) Lat are analyzed. Simulated results of this configuration are available in Figure 55 and Figure 56.



(a) Buying price (b) Buying price (CIs) (c) Selling price (d) Selling price (CIs)

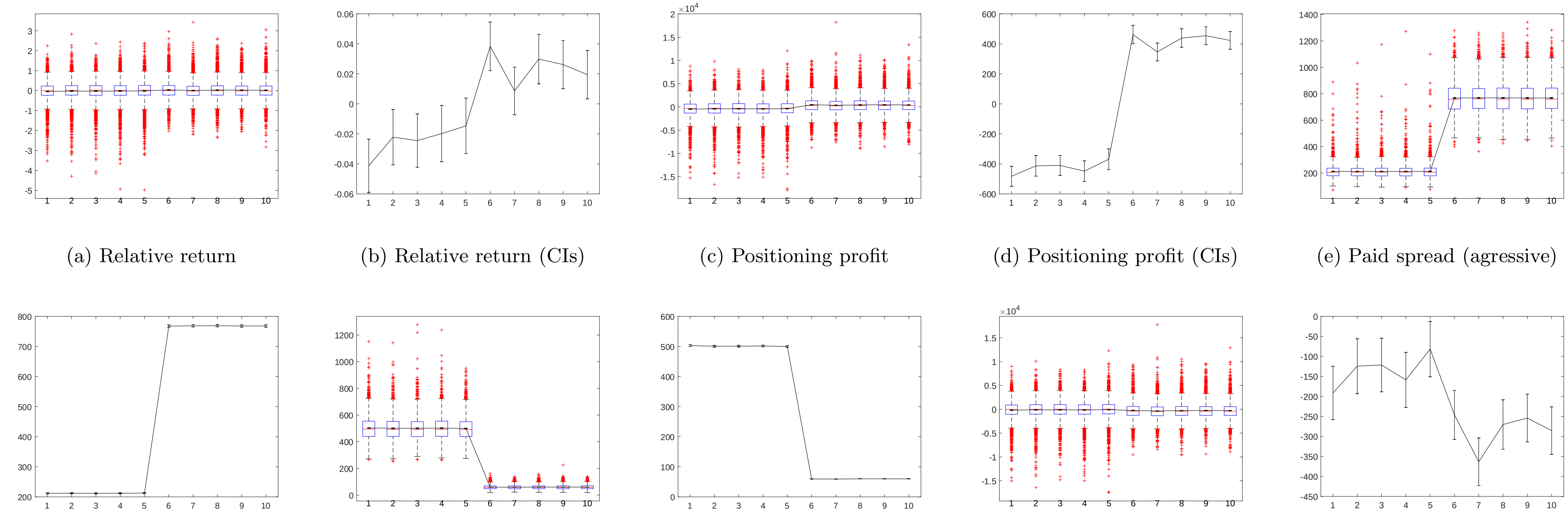
(e) Participation (f) Participation (CIs) (g) Trading via LOs relative (h) Trading via LOs rel. (CIs)

(i) Trading via MOs relative (j) Trading via MOs rel. (CIs) (k) Trading via both (rel) (l) Trading via both rel. (CIs)

(m) Sending LOs relative (n) Sending LOs rel. (CIs) (o) Maker-taker rate (p) Order-to-trade ratio

Figure 55: Results of the configuration (2) Lat (latent market): liquidity and activity metrics

In this figure, some of the graphs are boxplots and others plot only confidence intervals: this is done for a more detailed investigation of the selected metrics. On the x-axis, the agents' identifiers are shown. The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the means 95% confidence intervals.



(a) Relative return (b) Relative return (CIs) (c) Positioning profit (d) Positioning profit (CIs) (e) Paid spread (agressive)

(f) Paid spread (agressive) (CIs) (g) Earned spread (passive) (h) Earned spread (passive) (CIs) (i) Trading profit (j) Trading profit (CIs)

Figure 56: Results of the configuration (2) Lat (latent market): individual returns and profits

In this figure, some of the graphs are boxplots and others plot only confidence intervals: this is done for a more detailed investigation of the selected metrics. On the x-axis, the agents' identifiers are shown. The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the means 95% confidence intervals.

Observation 7.2. *Latency changes the market game, agents' behavior and their results. Since the agents' decision-making strategies are based on a stale limit order book, trading is not as successful as it is in a non-latent market. Trading based on fundamental information still brings a higher positioning profit, but also results in higher spread losses. The former does not outweigh the latter, though, which results in the fundamentalists' trading loss.*
The most important changes when latency is increased are the following:

(a) The fundamentalists become relative market-takers, while still participating in trade more actively.

(b) The fundamentalists' trading profit becomes also negative. The reason for this are net spread losses: the fundamentalists pay higher spreads in aggressive orders and earn lower spreads in passive orders. The fundamentalists cannot catch the right transaction price anymore: they sell when the price is low and buy when it is high. Their positioning profit is still positive, but it does not outweigh spread losses.

(c) Even though the average trading profit is slightly more negative for the fundamentalists than for the RTs, the fundamentalists have a slightly higher relative return. The fundamentalists' advantage should increase with latency dropping, while random trading should become comparatively more profitable with latency growing.

Evidence. Observations are based on Figure 55 and Figure 56. Due to a delay in the limit order book information, the fundamentalists cannot get the right price for the conducted transactions. The fundamentalists' selling (buying) price becomes lower (higher) than that of the RTs (graphs (a)-(d) in Figure 55), which influences the agents' final endowment and wealth respectively.
The fundamentalists still participate in trade more actively (graphs (e), (f), (k), (l) in Figure 55); they continue to trade relatively more through limit orders (graphs (g), (h) in Figure 55) and start to trade relatively more through market orders (graphs (i), (j) in Figure 55). The RTs send more limit orders (graphs (m), (n) in Figure 55). As a combined effect, the fundamentalists (RTs) become relative market-takers (-makers) (graph (o) in Figure 55). The fundamentalists' order-to-trade ratio is still lower than that for the RTs (graph (p) in Figure 55): there are more transactions per limit order sent by a fundamentalist than by an RT.
The trading profit of the both types of agents is negative, for the fundamentalists it is even more negative than for the RTs (graphs (i), (j) in Figure 56). The fundamentalists' informational advantage still provides a positive positioning profit (graphs (c)-(d) in Figure 56), but the fundamentalists earn smaller spreads in passive orders (graphs (g)-(h) in Figure 56) and pay higher spreads in aggressive orders (graphs (e)-(f) in Figure 56). The fundamentalists have a slightly higher relative return than the RTs (graphs (a), (b) in Figure 56). □

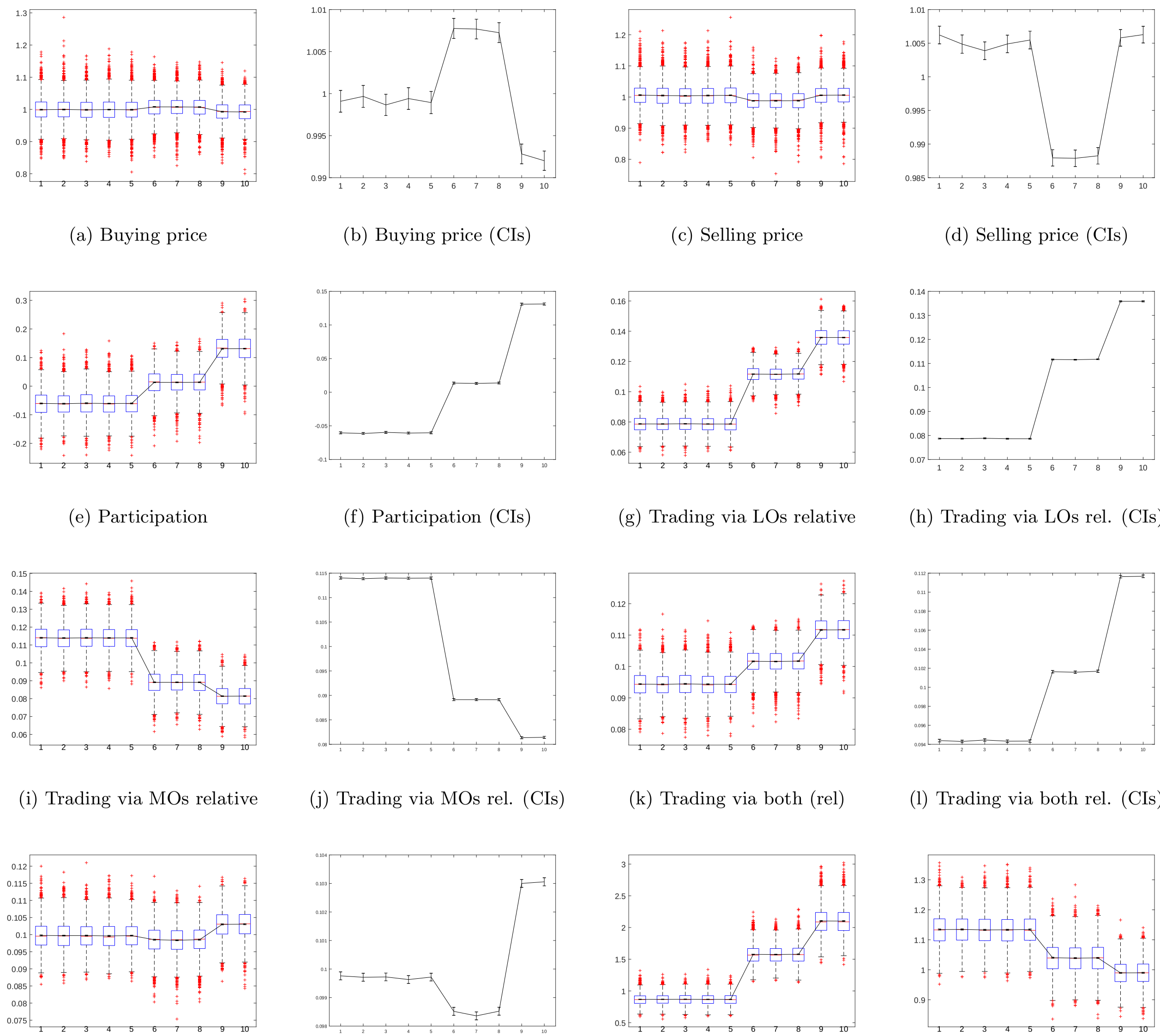
A delay in the limit order book information is not a relative disadvantage for the fundamentalists per se, as both agents see a limit order book with the same latency. However, latency causes a bigger negative impact on the fundamental decision-making rules than on random rules. With reference to the net spread measure, making transactions randomly in the latent market is a better choice than basing them on the fundamental information. At the same time, latency reduces the adverse selection problem, since the informational advantage cannot be used as profitably as in the previous, non-latent configuration.

7.5.1.3 Latent market with realistic HFTs: realistic HFTs win at the cost of the other agents

In this section, the results of the market configuration (3) Lat+HFT are analyzed. For this configuration of the artificial stock market, it is important to choose the most meaningful benchmark of the two configurations discussed earlier in this section. Comparing the current configuration with a non-latent market makes little sense, since a non-latent market is hardly achievable and rather an idealistic configuration. If realistic HFTs are present on the market, it means they can exploit market latency; HFT would not be possible in a non-latent market. The realistic HFTs' entry changes the market game. The realistic HFTs differ from the slow fundamentalists not only in their degree of activity but also in their final return, which they are able to exploit due to the speed advantage. This section provides a comparison of the realistic HFTs with the slow fundamentalists. Results of this configuration are presented in Figure 57 and Figure 58.

Observation 7.3. *The realistic HFTs are able to exploit their speed advantage at the cost of the other agents: the slow fundamentalists are able to keep their return on the zero level, while the RTs' returns are significantly negative. All the three trading profit components are better for the realistic HFTs than for the slow fundamentalists: the realistic HFTs get profit both from net spread due to having access to the current version of a limit order book and from correct positioning due to the fundamental information. The realistic HFTs buy low and sell high, while the slow fundamentalists are lost in latency and receive unattractive transaction conditions.*
Nevertheless, the realistic HFTs provide more liquidity than the slow market participants: even though the realistic HFTs participate in trade more than the others, they trade more through limit orders.

Evidence. Observations are based on Figure 57 and Figure 58. The realistic HFTs' buying (selling) price is smaller (higher) than that of the slow fundamentalists (graphs (a)-(d) in Figure 57), which influences the agents' final endowment and wealth respectively.
The realistic HFTs participate in trade even more than the slow fundamentalists (graphs (e), (f), (k), (l) in Figure 57). The realistic HFTs trade more than the slow fundamentalists through limit orders (graphs (g), (h) in Figure 57) and less through market orders (graphs (i), (j) in Figure 57). The realistic HFTs send more limit orders (graphs (m), (n) in Figure 57). All fundamentalists are relative market-makers, while the realistic HFTs have a higher maker-taker rate than the slow fundamentalists (graph (o) in Figure 57).
A more intensive participation in trade makes the realistic HFTs' order-to-trade ratio even lower than for the slow fundamentalists (graph (p) in Figure 57): a realistic HFT makes more



(a) Buying price (b) Buying price (CIs) (c) Selling price (d) Selling price (CIs)

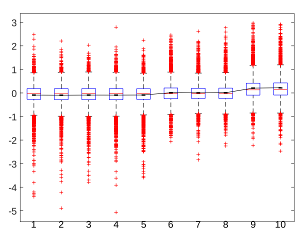
(e) Participation (f) Participation (CIs) (g) Trading via LOs relative (h) Trading via LOs rel. (CIs)

(i) Trading via MOs relative (j) Trading via MOs rel. (CIs) (k) Trading via both (rel) (l) Trading via both rel. (CIs)

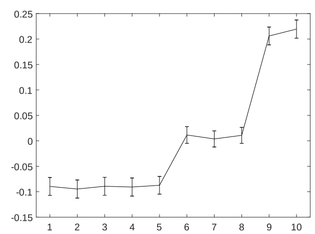
(m) Sending LOs relative (n) Sending LOs rel. (CIs) (o) Maker-taker rate (p) Order-to-trade ratio

Figure 57: Results of the configuration (3) Lat+HFT (latent market with HFTs): liquidity and activity metrics

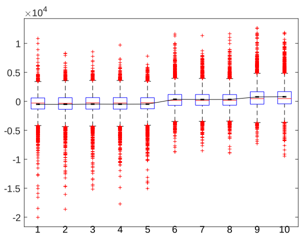
In this figure, some of the graphs are boxplots and others plot only confidence intervals: this is done for a more detailed investigation of the selected metrics. On the x-axis, the agents' identifiers are shown. The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the means 95% confidence intervals.



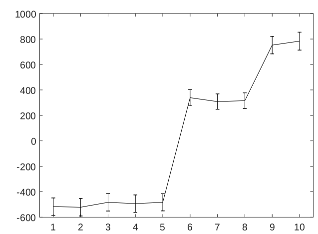
(a) Relative return



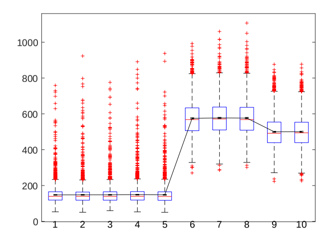
(b) Relative return (CIs)



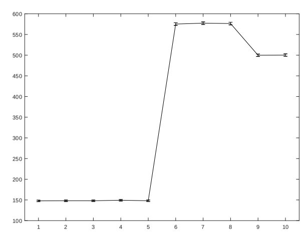
(c) Positioning profit



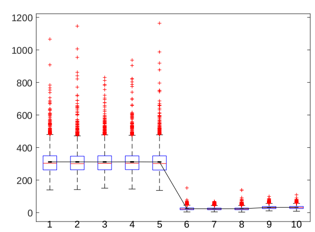
(d) Positioning profit (CIs)



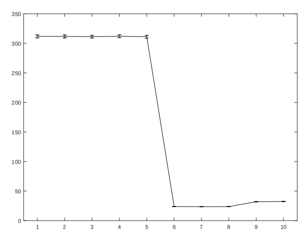
(e) Paid spread (agressive)



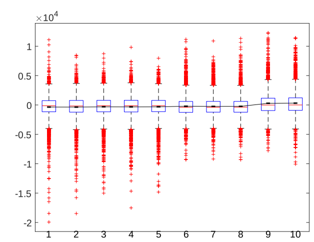
(f) Paid spread (agressive) (CIs)



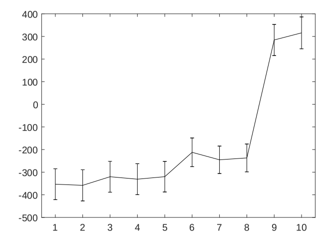
(g) Earned spread (passive)



(h) Earned spread (passive) (CIs)



(i) Trading profit



(j) Trading profit (CIs)

Figure 58: Results of the configuration (3) Lat+HFT (latent market with HFTs): individual returns and profits

In this figure, some of the graphs are boxplots and others plot only confidence intervals: this is done for a more detailed investigation of the selected metrics. On the x-axis, the agents' identifiers are shown. The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the means 95% confidence intervals.

transactions per a limit order than a slow fundamentalist.

The realistic HFTs' trading profit is positive, while the other market participants experience a negative trading profit (graphs (i), (j) in Figure 58). All the realistic HFTs' trading profit components are better than for the slow fundamentalists. The realistic HFTs pay a smaller spread from aggressive orders (graphs (e), (f) in Figure 58), both positioning profit and earned spread from passive orders are higher (graphs (c), (d), (g), (h) in Figure 58). The realistic HFTs win the market game: their relative returns are much higher than those of the slow fundamentalists (which are around zero), while the RTs' returns are negative (graphs (a), (b) in Figure 58). □

Our results are in line with the observations described in Jones (2013): better informed traders (in this case, realistic HFTs and slow fundamentalists compared to RTs) buy low and sell high and end up with profits, while lower informed traders buy high and sell low and experience losses. Thus, the adverse selection problem causes less informed traders to participate in unattractive trades. The realistic HFTs' entry increases the adverse selection problem compared to the benchmark market.

With respect to the realistic HFTs being responsible for a higher share of trading volume and benefiting market liquidity due to their relative market-making function and a high amount of limit orders sent, the conclusions of our study coincide with the analytical results reported in Boco, Germain, and Rousseau (2017) and Foucault, Hombert, and Rosu (2016), the empirical results described in Biais, Declerck, and Moinas (2016), Hendershott, Jones, and Menkveld (2011), and Hasbrouck and Saar (2013), and with the ABM results by Arifovic, Chiarella, He, and Wei (2016). In contrast to Bernales (2014), claiming that algorithmic traders serve as liquidity-takers when the majority of agents are less skilled, in our model with eight slow traders and two realistic HFTs, it is the realistic HFTs who serve as market-makers.

With respect to realistic HFTs winning at the cost of slow traders, the results of our research are close to the conclusions of the analytical models by Boco, Germain, and Rousseau (2017) and Hoffmann (2014), the empirical results by Gerig (2012) and Baron, Brogaard, and Kirilenko (2012), as well as the ABM results reported in Bernales (2014). In comparison to Brogaard (2010), the HFT activity in the modeled ABM does clearly damage non-HFT traders. In contrast to Menkveld (2013), who claims that HFTs are mainly market-makers gaining spreads but having losses on their net positions, the realistic HFTs in our model gain both spreads and positioning profit, since they are not only market-makers, but also market-takers, even though predominantly they perform a relative market-making function. The realistic HFTs pay lower spreads in aggressive orders and earn more in passive orders compared to the other market participants, which suggests that both activities are more profitable for the realistic HFTs: even though rebates are not considered in our research, this result is different from Carrion (2013), who reports profitability of HFTs' market-making activities and losses from HFTs' aggressive trading. Alternatively to Rojcek and Ziegler (2016), the welfare of investors (fundamentalists) are negatively affected by the HFT activities. Arifovic, Chiarella, He, and Wei (2016) conclude that the main source of HFTs' advantage is informational advantage and learning[108], while this model shows that the speed advantage also brings a considerable additional profit and relative return to the realistic HFTs as compared to the otherwise identical fundamentalists.

7.5.1.4 Very latent market with realistic HFTs: slow fundamentalists as main victims of HFTs

In this section, the results of the market configuration (4) VLat+HFT are analyzed. The main aim of this configuration is to check whether the results described in Section 7.5.1.3 hold true for a market with higher latency. As the results show, a further latency impairment affects mainly the slow fundamentalists. Graphical results of the current configuration are available in Figure 59 and Figure 60.

Observation 7.4. *The effect of the slow fundamentalists' informational advantage disappears when the market is very latent: the slow fundamentalists turn out to be the most important victims of the HFT activities, while in a modestly latent market the RTs are harmed the most. The slow fundamentalists do not participate in trade very actively, they reduce the amount of the sent limit orders. Their results both in terms of trading profit and relative returns are the worst; the realistic HFTs' and the RTs' results are almost unchanged.*

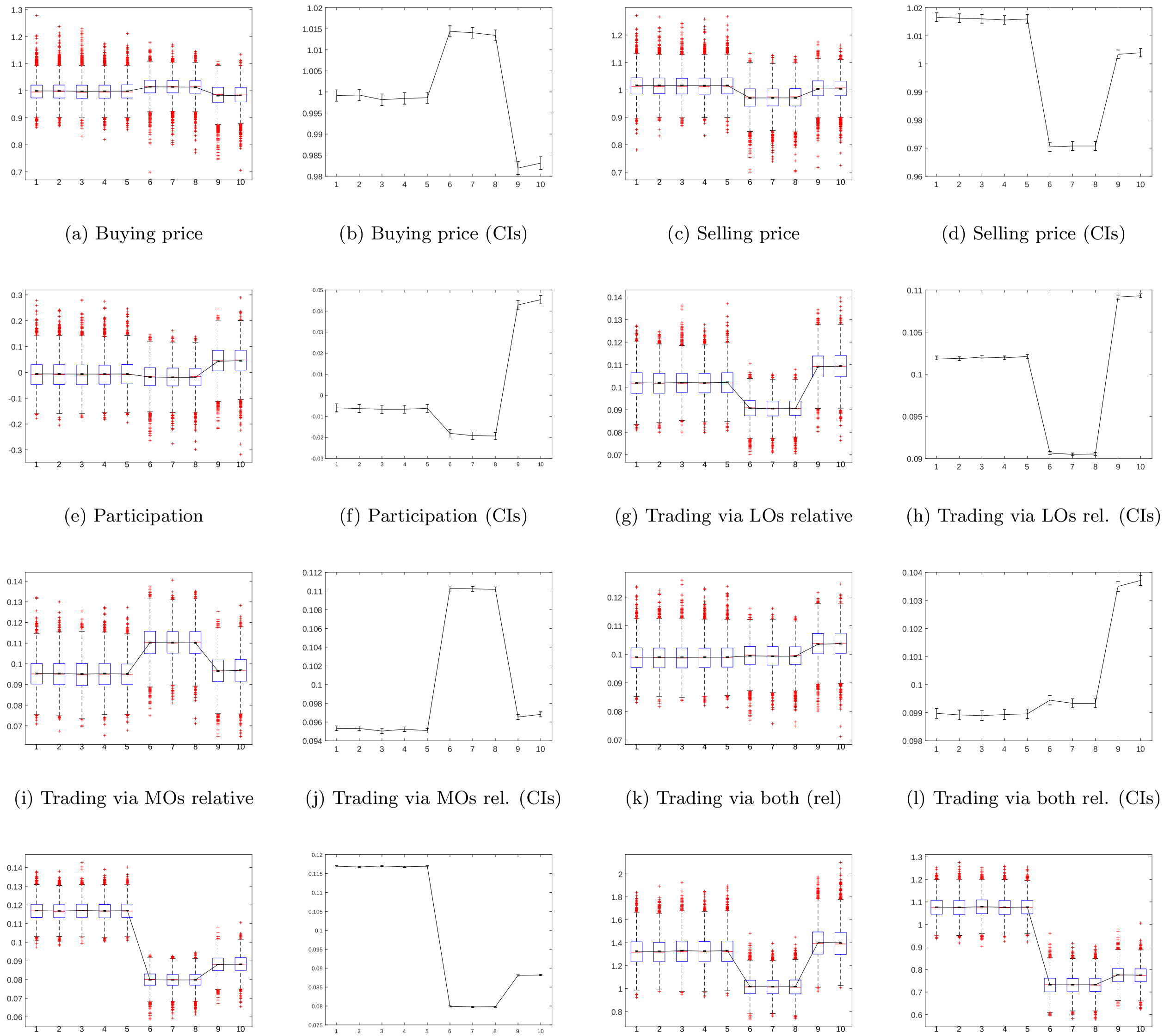
Evidence. Observations are based on Figure 59 and Figure 60. The participation in trade and trading via limit orders for the slow fundamentalists becomes the lowest among all the market participants (graphs (e)-(h) in Figure 59); they prefer to trade relatively more through market orders when latency increases (graphs (i), (j) in Figure 59). The slow fundamentalists' maker-taker ratio decreases, while the RTs become relatively more market-makers compared to the modestly latent market (graph (o) in Figure 59). The order-to-trade ratio for the slow fundamentalists diminishes as well (graph (p) in Figure 59): this change is mainly influenced by the smaller amount of sent limit orders. The trading profit for the slow fundamentalists is negative and drops below the RTs' level (graph (j) in Figure 60). The slow fundamentalists become relative market losers (graphs (a), (b) in Figure 60). □

When the latency is high, investors prefer to wait and reduce their trading activity. In spite of this, they turn out to be the most damaged group of traders due to HFT activities.

7.5.1.5 Accelerated very latent market with realistic HFTs: fundamentalists are able to recover their results

In this section, the results of the market configuration (5) Acc+VLat+HFT are analyzed. There are more activity steps within one period of the market session between two points of information dissemination in an accelerated market. Results of this configuration are presented graphically in Figure 61 and Figure 62.

[108] Learning is not considered in our ABM.



(a) Buying price (b) Buying price (CIs) (c) Selling price (d) Selling price (CIs)

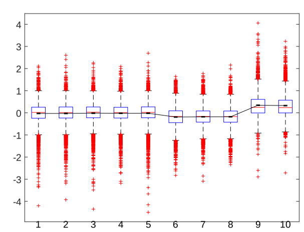
(e) Participation (f) Participation (CIs) (g) Trading via LOs relative (h) Trading via LOs rel. (CIs)

(i) Trading via MOs relative (j) Trading via MOs rel. (CIs) (k) Trading via both (rel) (l) Trading via both rel. (CIs)

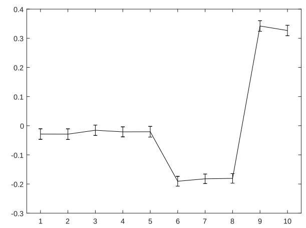
(m) Sending LOs relative (n) Sending LOs rel. (CIs) (o) Maker-taker rate (p) Order-to-trade ratio

Figure 59: Results of the configuration (4) VLat+HFT (very latent market with HFTs): liquidity and activity metrics

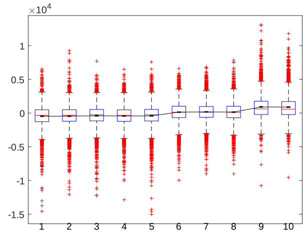
In this figure, some of the graphs are boxplots and others plot only confidence intervals: this is done for a more detailed investigation of the selected metrics. On the x-axis, the agents' identifiers are shown. The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the means 95% confidence intervals.



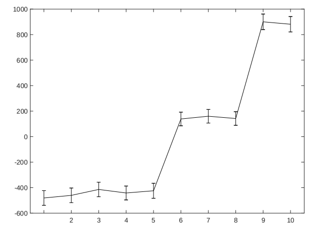
(a) Relative return



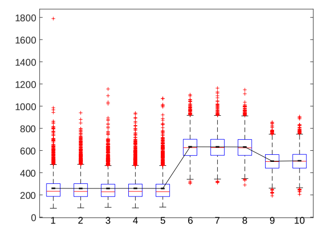
(b) Relative return (CIs)



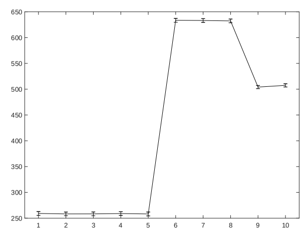
(c) Positioning profit



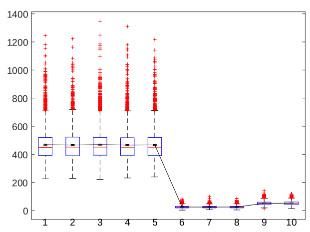
(d) Positioning profit (CIs)



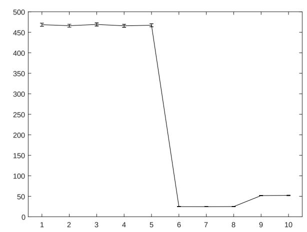
(e) Paid spread (agressive)



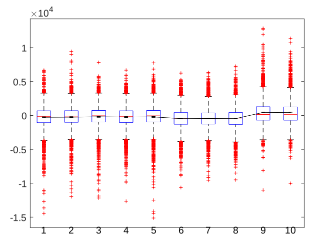
(f) Paid spread (agressive) (CIs)



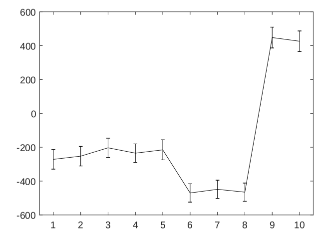
(g) Earned spread (passive)



(h) Earned spread (passive) (CIs)



(i) Trading profit



(j) Trading profit (CIs)

Figure 60: Results of the configuration (4) VLat+HFT (very latent market with HFTs): individual returns and profits

In this figure, some of the graphs are boxplots and others plot only confidence intervals: this is done for a more detailed investigation of the selected metrics. On the x-axis, the agents' identifiers are shown. The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the means 95% confidence intervals.

Observation 7.5. *Market acceleration improves the slow fundamentalists' activity measures and results, while leaving the relative results of the other market participants unchanged. The realistic HFTs still beat the market at the cost of RTs and slow fundamentalists, but the slow fundamentalists' losses reach the RTs' level.*

Evidence. Observations are based on Figure 61 and Figure 62. The slow fundamentalists' buying price decreases (graphs (a)-(b) in Figure 61). The relative participation parameters recover from the depressed level under the very latent market configuration back to the modestly latent market level (graphs (e)-(l), (o), (p) in Figure 61). The slow fundamentalists post more limit orders than RTs (graphs (m), (n) in Figure 61). The slow fundamentalists' trading profit stays on the negative level but slightly increases compared to the very latent market configuration: new values are within the range of the RTs' results (graph (j) in Figure 62). Due to a higher market activity, better net spreads (as well as a lower average buying price), slightly higher positioning profits, and relative returns are received, though these measures stay in the negative range and are very close to the RTs' results (graphs (a), (b) in Figure 62). □

Accelerating the market allows slow fundamentalists to find their pace and to control the situation in a similar manner to the modestly latent market.

7.5.2 Pairwise analysis of configurations from the first group

In this section, a pairwise comparison of the market configurations in the sequence of their complication is offered. The main aim is to investigate the realistic HFT's influence on the market and her trading partners. The conclusions in this section are based on Figure 63, when the agent types are compared, and on Figure 64, when the aggregate market measures are analyzed. In focus are the results which were not presented and discussed in Section 7.5.1.

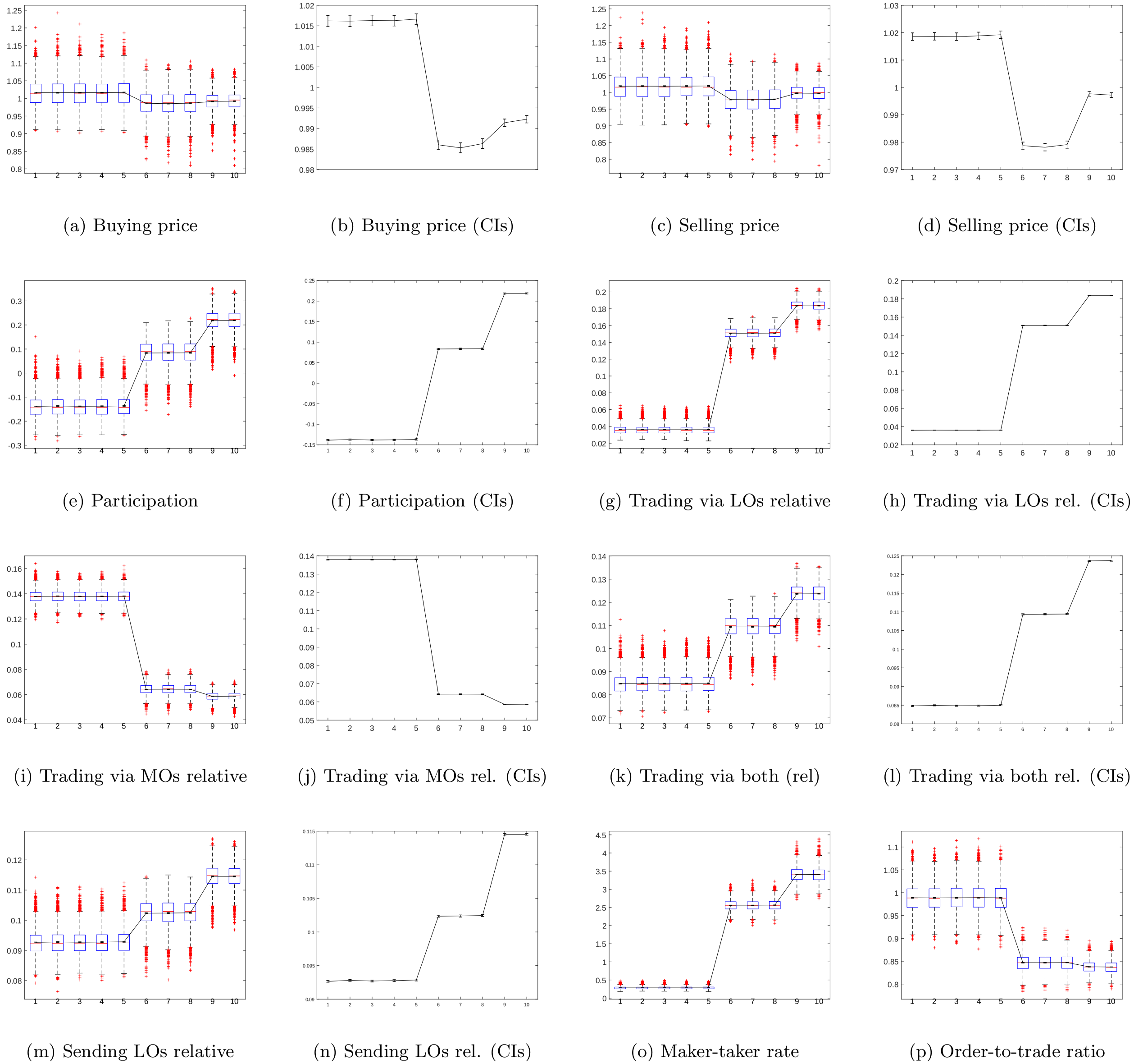
7.5.2.1 Latent market vs non-latent market: damaged market quality but smoothed inequality

In this section, the results of the market configurations (2) Lat and (1) NLat are compared.

Observation 7.6. *The fundamentalists' informational advantage over the RTs diminishes when latency is introduced into the market: the difference between the two agent types smooths*[109]*. The quality of the market worsens based on the spread measures, price impact, informational content of prices, and trading costs, but improves based on the size liquidity measures and order imbalance.*

Evidence. Introducing latency to the market has the following effects on the different agent types (references to the subplots in Figure 63):
The fundamentalist's advantage in terms of better buying/selling prices disappears and turns to

[109] The first (non-latent) benchmark shows a severe returns distribution. As the agents have asymmetric information, the RTs experience adverse selection from the fundamentalists' side. The introduced latency improves the adverse selection problem, since information cannot be used as profitably as it was in a non-latent market.



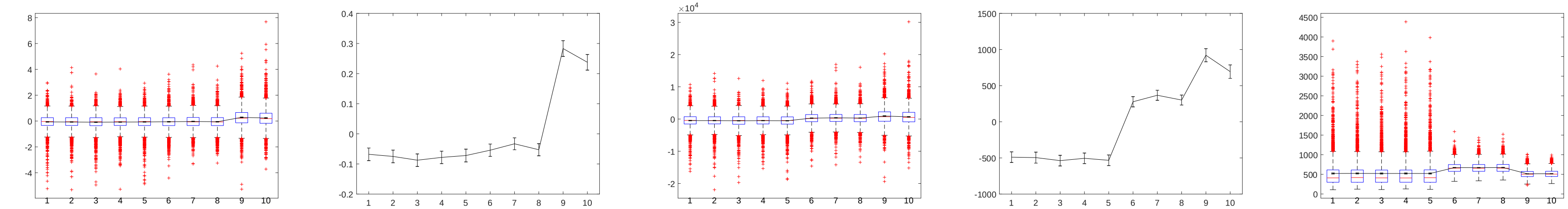
(a) Buying price (b) Buying price (CIs) (c) Selling price (d) Selling price (CIs)

(e) Participation (f) Participation (CIs) (g) Trading via LOs relative (h) Trading via LOs rel. (CIs)

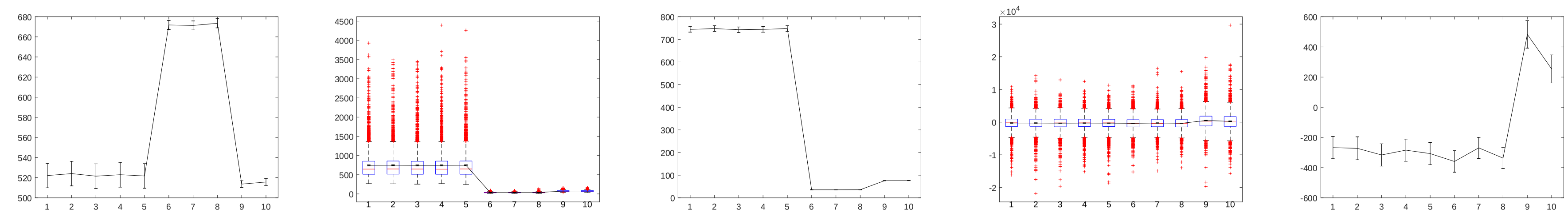
(i) Trading via MOs relative (j) Trading via MOs rel. (CIs) (k) Trading via both (rel) (l) Trading via both rel. (CIs)

(m) Sending LOs relative (n) Sending LOs rel. (CIs) (o) Maker-taker rate (p) Order-to-trade ratio

Figure 61: Results of the configuration (5) VLat+HFT+Acc (accelerated very latent market with HFTs): liquidity and activity metrics
In this figure, some of the graphs are boxplots and others plot only confidence intervals: this is done for a more detailed investigation of the selected metrics. On the x-axis, the agents' identifiers are shown. The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the means 95% confidence intervals.



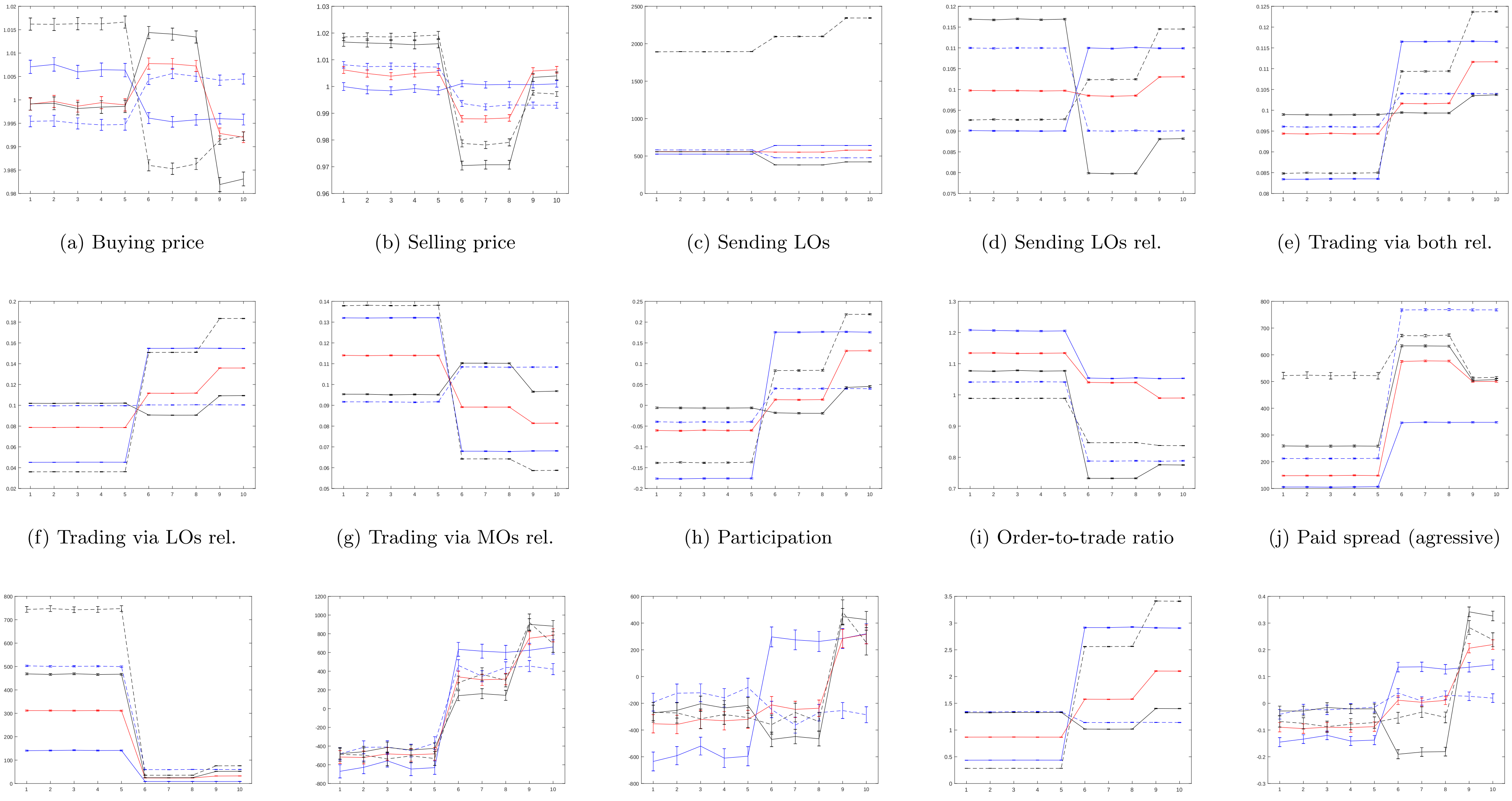
(a) Relative return (b) Relative return (CIs) (c) Positioning profit (d) Positioning profit (CIs) (e) Paid spread (agressive)



(f) Paid spread (agressive) (CIs) (g) Earned spread (passive) (h) Earned spread (passive) (CIs) (i) Trading profit (j) Trading profit (CIs)

Figure 62: Results of the configuration (5) VLat+HFT+Acc (accelerated very latent market with HFTs): individual returns and profits

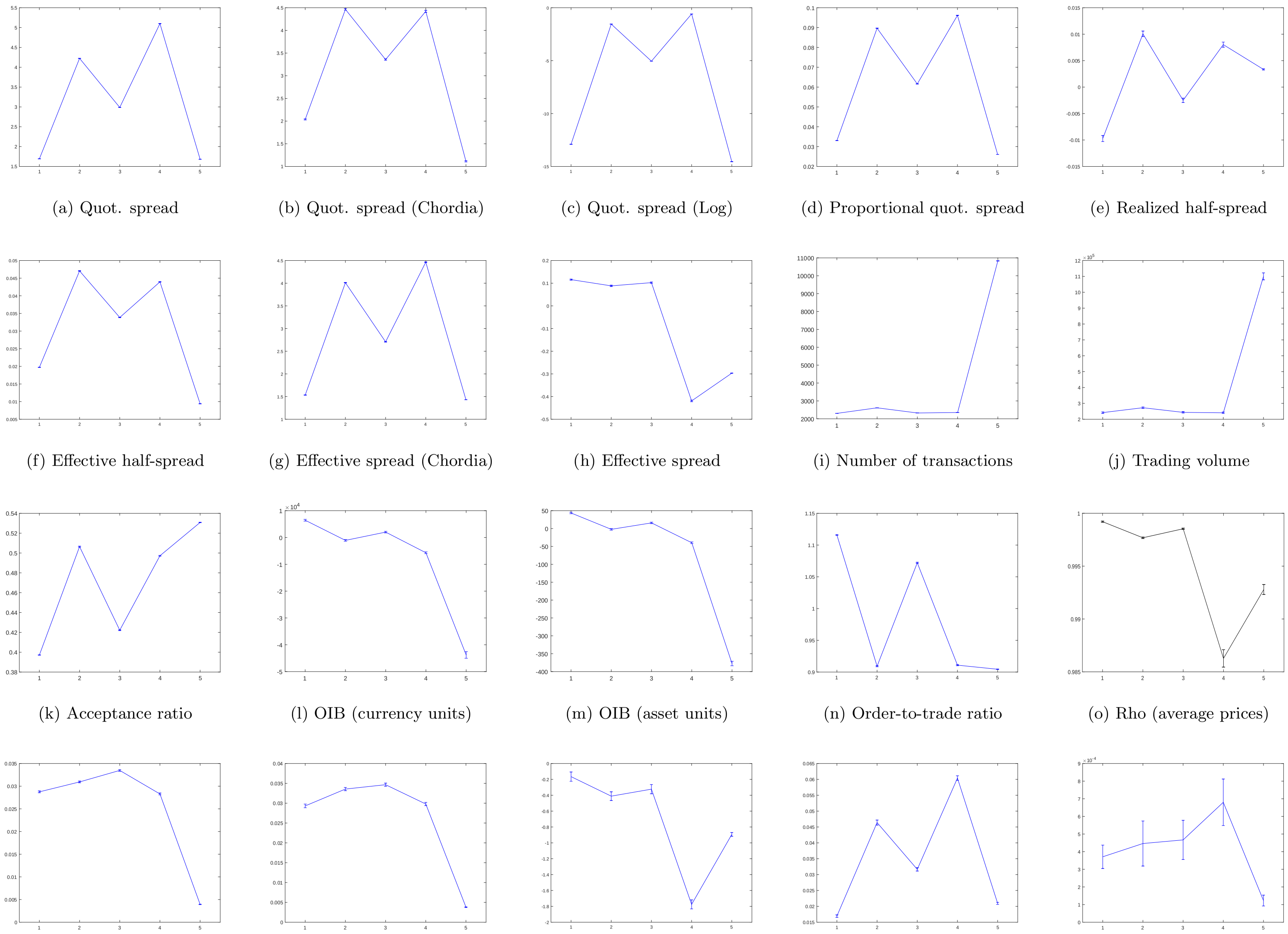
In this figure, some of the graphs are boxplots and others plot only confidence intervals: this is done for a more detailed investigation of the selected metrics. On the x-axis, the agents' identifiers are shown. The lower and upper boundaries of the blue box represent the 25th and 75th percentiles, respectively. The red line inside the blue box is the median. The notches of the box represent the 95% confidence interval of the median. The upper and lower horizontal black boundaries are whiskers which include 99.3% of the data. The red points outside the whiskers are outliers. The black line in the middle of a graph is the line plot with the error bar: it connects the means for each individual agent and shows the means 95% confidence intervals.



(a) Buying price
(b) Selling price
(c) Sending LOs
(d) Sending LOs rel.
(e) Trading via both rel.
(f) Trading via LOs rel.
(g) Trading via MOs rel.
(h) Participation
(i) Order-to-trade ratio
(j) Paid spread (agressive)
(k) Earned spread (passive)
(l) Positioning profit
(m) Trading profit
(n) Maker-taker rate
(o) Relative return

Figure 63: Combined results of the first group of configurations: illustration per agent

This figure is also discussed in Section 7.6.
Error bars are illustrated: the means as well as the 95% confidence intervals for each agent are provided. On the x-axis, the agents' identifiers are plotted. Different lines correspond to different configurations: (1) solid blue line – NLat, (2) dashed blue line – Lat, (3) red line – Lat+HFT, (4) solid black line – Vlat+HFT, (5) dashed black line – Acc+Vlat+HFT.



(a) Quot. spread (b) Quot. spread (Chordia) (c) Quot. spread (Log) (d) Proportional quot. spread (e) Realized half-spread

(f) Effective half-spread (g) Effective spread (Chordia) (h) Effective spread (i) Number of transactions (j) Trading volume

(k) Acceptance ratio (l) OIB (currency units) (m) OIB (asset units) (n) Order-to-trade ratio (o) Rho (average prices)

(p) PI (q) PI (Huh) (r) Trading costs (s) RAD (t) RET

Figure 64: Combined results of the first group of configurations: the aggregate market metrics

This figure is also discussed in Section 7.6.
Error bars are illustrated: the means as well as the 95% confidence intervals for each configuration are provided. On the x-axis, the sequential number of a configuration is presented: (1) NLat, (2) Lat, (3) Lat+HFT, (4) Vlat+HFT, (5) Acc+Vlat+HFT.

disadvantage (graphs (a), (b)). The making-taking roles reverse: the fundamentalists start to send fewer limit orders than the RTs (graphs (c), (d)) and start to trade relatively more through market orders (graph (g)). The maker-taker rate (graph (n)) shows that the fundamentalists become relative market-takers, even though the difference becomes less pronounced. Overall, the activity measures are closer for the fundamentalists and the RTs (graphs (e), (f), (h)). The order-to-trade ratio for the both types of agents decreases. There are more transactions per each limit order sent, which can be considered as an improvement in market quality, since a greater amount of limit orders leads to transactions (graph (i)).

The paid spread in aggressive orders as well as the earned spread in passive orders increase for both agents (graphs (j), (k)). This is a sign of a deteriorating market quality, since liquid markets provide narrow spreads. The positioning profit for the two types of agents becomes less diverse: for the fundamentalists, it declines and for the RTs it rises (graph (l)). If the results are considered from the trading profit perspective, a fundamentalist fails to exploit her informational advantage (graph (m)). The relative returns smooth (graph (o)), while the fundamentalists slightly win and the RTs slightly lose.

The aggregate market liquidity and quality measures are influenced by a latency increase in the following way (references to the subplots in Figure 64):

All the spread measures are increasing, with one exception (graphs (a)-(h)). Liquidity goes up: both the number of transactions and trading volume increase (graphs (i), (j)). The acceptance ratio improves (graph (k)), while the order-to-trade ratio drops (graph (n)). The order imbalance decreases, moving closer to zero in absolute terms (graphs (l), (m)): in the latent market, the concentration of agents on one side of the market is less probable.

The correlation between the market prices and fundamental value decreases (graph (o)), while the relative absolute deviation grows (graph (s)): the quality of information in the transaction prices diminishes. The price impact of the past transactions on the coming transactions enhances (graphs (p), (q)). The trading costs become more negative and diverge from the zero level (graph (r)). The variance of excess returns increases, but the statistical difference of the two means is insignificant (graph (t)). □

Riordan and Storkenmaier (2012) demonstrate that decreased latency improves price efficiency. In this regard, our results show a similar relation: latency increase damages price efficiency.

The two compared configurations are quite different in terms of the agents' activities, their participation in trade and providing liquidity to the market, as well as in terms of the final results. This justifies the importance of choosing the correct benchmark for the HFT-influenced configurations. As was previously discussed, HFT per se is possible on a latent market only. Therefore, the latent configuration provides the benchmark for the next comparison.

7.5.2.2 Latent market with realistic HFTs vs latent market: improved spreads and price discovery but enhanced adverse selection and harmed RTs

In this section, the results of the market configurations (3) Lat+HFT and (2) Lat are compared. The realistic HFTs observe a current limit order book and the latest transaction price and make the right decisions based on this up-to-date information. The slow fundamentalists observe a stale limit order book and make their decisions based on the outdated information.

Observation 7.7. *The general effect of the realistic HFTs' entry on the market quality is twofold:*
On the one hand, quality improves based on the spread measures and informational content of prices: the realistic HFTs, having fundamental information and using the latest version of a limit order book, accelerate price discovery.
On the other hand, market quality worsens based on the size liquidity measures, order imbalance, and price impact.
The realistic HFTs serve the market-making function. The realistic HFTs' entry ignites additional market liquidity from both the realistic HFTs and the slow fundamentalists. However, this potential liquidity fails to translate to real transactions due to the increased adverse selection; the liquidity induced by the realistic HFTs is liquidity dust[110].
The adverse selection influences mainly the slow fundamentalists (as their transaction terms are the worst), but their positioning profits due to the informational endowment allow them to get a relative return higher than that of the RTs. The realistic HFTs exaggerate profit and return inequality among the agent types: the main losers from the HFT activities are the RTs, as the slow fundamentalists' results are affected inconsiderably.

Evidence. The realistic HFTs' entry has the following effects on the market participants (references to the subplots in Figure 63):
Realistic HFTs' relative buying (selling) price is the lowest (the same as for the RTs). The prices for both the slow fundamentalists and RTs worsen compared to the realistic HFTs-free configuration (graphs (a), (b)). The slow fundamentalists' prices are the most disadvantageous among the market participants: RTs get better transaction conditions than the slow fundamentalists.
The difference in sending limit orders smooths out among the market participants (graphs (c), (d)). The relative number of transactions in which the RTs and the slow fundamentalists participate decreases, while the realistic HFTs participate in trade more than the others (graphs (e), (h)). The trading activity moves from the active to the passive type for the slow fundamentalists and from the passive to the active type for the RTs (graphs (g), (f)). The realistic HFTs take relatively less liquidity than the slow traders (graph (g)). The realistic HFTs' entry ignites some additional volume of orders from the fundamentalists (graph (c)). The difference in terms of providing liquidity to the market becomes visible again (graph (f)). The fundamentalists are

[110]The realistic HFTs' entry overflows the system with phantom limit orders and this may create a pressure on the slow traders.

relative market-makers (graph (n)); they participate in trade more actively than the RTs (graph (h)). The order-to-trade ratio for all the agents increases, there are fewer transactions per each limit order sent: fewer limit orders lead to a transaction, meaning there is more liquidity dust (graph (i)). Realistic HFTs' market-making function enhances by more than that of the slow fundamentalists (graph (n)).

The paid spread in aggressive orders as well as the earned spread in passive orders decrease for all the agents (graphs (j), (k)). This is a sign of an enhanced market liquidity. The positioning profit (graph (l)) does not show considerable changes when the realistic HFTs enter the market. The RTs' overall trading profit and relative return drop, for the slow fundamentalists these measures stay on approximately the same level, while for the realistic HFTs these values are quite high due to the speed advantage (graphs (m), (o)). The realistic HFTs win the game, while the RTs lose more than the slow fundamentalists.

The aggregate market liquidity and quality measures change with realistic HFTs' entry in the following directions (references to the subplots in Figure 64):

All the spread measures decrease, with one exception (graphs (a)-(h)). The liquidity of the market goes down, in terms of both the trading volume and number of transactions (graphs (i), (j)). The acceptance ratio worsens, while the order-to-trade ratio increases (graphs (k), (n)): a higher proportion of limit orders is left unexecuted. The order imbalance changes direction but diverges further from zero in absolute terms (graphs (l), (m)). The correlation between the market prices and fundamental value grows, while the relative absolute deviation decreases (graphs (o), (s)): the quality of information included in the prices improves. The price impact of the past transactions on the future transactions increases even further (graphs (p), (q)). The trading costs improve insignificantly (graph (r)). The variance of excess returns slightly decreases, but the statistical difference of the two means is insignificant (graph (t)). □

The improved spreads on the market are in line with the results obtained by Huh (2014) but opposite to those by Hendershott and Moulton (2011), Leal, Napoletano, Roventini, and Fagiolo (2016), and Arifovic, Chiarella, He, and Wei (2016). Our ABM shows that even with a non-predominant share, the realistic HFTs' entry reduces spreads, in contrary to Bernales (2014).

With respect to the price efficiency improvement, the previous research agrees on one point: the HFT activity adds a lot to the price discovery process: quotes and prices reflect the fundamental value better (see section "Price efficiency" in Table 1). The results of our simulated ABM are similar.

The impact on the market behavior and activities is close to the results of Bernales (2014): with the HFTs' entry, the slow traders start to submit more limit orders, the fast traders execute more limit orders and have more cancellations. The authors claim that the change in behavior is affected not only by the HFTs' advantages (informational, or speed advantage, or both), but also by the share of the HFTs and slow traders on the market.

Even though the realistic HFTs benefit market liquidity by creating additional potential liquidity in the form of limit orders, which is in line with the previous research (see Section 7.5.1.3), a decrease in trading volume with the realistic HFTs' entry is rather a contradictory result. Jovanovic and Menkveld (2016) report an increase in trading volume, while the model by Hoffmann (2014) predicts some combination of the initial market parameters leading to a trading volume decrease. Biais, Foucault, and Moinas (2011) explain this inconsistency: the HFT's nature brings an increase in trading volume, but the problems related to adverse selection lead to a respective decrease. The final effect depends on the combination of the two forces and to a greater extent on the severeness of adverse selection. Huh (2014) claims that the HFTs overflow the trading system with orders which do not benefit anyone: it is in line with our research showing that the realistic HFTs create liquidity dust.

Enhanced order imbalances potentially prove the realistic HFTs' destabilizing force: the similar observation is reported in Leal, Napoletano, Roventini, and Fagiolo (2016). However, Subrahmanyam and Zheng (2016) describe the HFTs as performing a stabilizing function.

A high price impact is a sign of an increased adverse selection in our ABM. In this respect, our results confirm the conclusions from the analytical models by Hoffmann (2014) and Biais, Foucault, and Moinas (2011), as well as the empirical research by Hendershott and Moulton (2011), and Huh (2014). However, the majority of analyzed studies claim that the HFT activity should diminish the adverse selection problem (e.g., Jovanovic and Menkveld (2016), Menkveld (2013), Gerig (2012), and Riordan and Storkenmaier (2012)).

7.5.2.3 Very latent market with realistic HFTs vs latent market with realistic HFTs: slow fundamentalists as main preys and moderate market-making function of realistic HFTs

In this section, the results of the market configurations (4) VLat+HFT and (3) Lat+HFT are compared.

Observation 7.8. *A latency increase, even with the realistic HFTs being present, worsens market quality. It deteriorates based on the spread measures, price efficiency, trading costs, and order imbalance. At the same time, there is less adverse selection shown by the decreased price impact. The liquidity measure is partially improved, since there is a higher share of executed limit orders.*

The realistic HFTs' market-making function declines: a realistic HFT provides as much liquidity as an RT.

The higher the latency is, the more the realistic HFTs' results diverge from the slow fundamentalists' and the RTs' returns. A realistic HFT is able to exploit the other traders' ignorance or slowness to her own advantage. When latency worsens, the main victims of the HFT activity change: though the RTs still lose, the slow fundamentalists' results diminish drastically.

Evidence. If more severe latency is assumed, the market participants experience the following changes (references to the subplots in Figure 63):

The relative price discrepancies between the slow fundamentalists and the realistic HFTs enhance (graphs (a), (b)). The RTs' relative selling price improves. The difference in sending limit orders becomes more pronounced again (graphs (c), (d)). The relative number of transactions in which the RTs (fundamentalists) participate boosts (decreases) (graphs (e), (h)). The realistic HFTs send fewer limit orders than the RTs (graph (c)). The fundamentalists' market-making function drops. Even if the realistic HFTs are relative market-makers, they provide approximately as much liquidity as the RTs (graphs (f), (n)). The order-to-trade ratio for all the agents decreases, there are fewer transactions per each limit order sent (graph (i)).
The paid spread in aggressive orders as well as the earned spread in passive orders rise for all the agents (graphs (j), (k)). This is a sign of a deteriorating market quality. The positioning profit (graph (l)) shows a significant decrease only for the slow fundamentalists. The RTs' overall trading profit and the relative return increase (though stay in the negative range), for the slow fundamentalists it drops considerably, while for the realistic HFTs it improves even further (graphs (m), (o)). Increased latency improves (worsens) the realistic HFTs' (the slow fundamentalists') spread earnings and positioning profits. The inequality of profits and returns grows, with the realistic HFTs winning at the cost of the RTs and even more at the cost of the slow fundamentalists.
With an increased latency of the market, the aggregate market liquidity and quality measures change in the following directions (references to the subplots in Figure 64):
All the spread measures are growing, with one exception (graphs (a)-(h)). The trading volume and the number of transactions only slightly change, while the acceptance ratio improves and the order-to-trade ratio deteriorates, meaning that more sent limit orders lead to transactions (graphs (i)-(k), (n)). The order imbalance diverges from zero in absolute terms even more (graphs (l), (m)). The correlation between the market prices and fundamental value decreases, and the relative absolute deviation increases (graphs (o), (s)): the quality of information included in the transaction prices goes down. The price impact of the past transactions on the future transactions drops (graphs (p), (q)). The trading costs go down considerably and move away from the zero level (graph (r)). Any conclusion about the significance of excess returns change is hardly possible (graph (t)). □

Independently of the severeness of latency, the realistic HFTs experience profits at the cost of the slow traders. The decision about the main victim of the HFT activity depends a lot on latency: with modest latency, the slow fundamentalists might be able to stay in the positive range due to their informational advantage, but higher latency brings about high losses in their net spreads which outweigh positioning profits. The higher the latency is, the less significant the realistic HFTs' market-making role becomes.

7.5.2.4 Accelerated very latent market with realistic HFTs vs very latent market with realistic HFTs: improved liquidity and the slow fundamentalists' recovered results

In this section, the results of the market configurations (5) Acc+VLat+HFT and (4) VLat+HFT are compared.

Observation 7.9. *When the very latent market with the realistic HFTs is accelerated, most of the market characteristics serve quality improvement (spread measures, liquidity, acceptance measures, price efficiency, price impact, and trading costs), and only order imbalances continue worsening.*
Having more chances to trade on the market and use their informational advantage, the slow fundamentalists are able to improve their results (mainly through the improvement of the positioning profit): even though the returns stay on the negative level (the realistic HFTs adversely select orders by the slow fundamentalists), the slow fundamentalists suffer less than in the non-accelerated market.

Evidence. If the market is accelerated, or if more agents can act between two points of information dissemination, the following changes happen in respect to the agents' results (references to the subplots in Figure 63):
The slow fundamentalists are able to improve their transaction conditions (graphs (a), (b)), while the conditions for both the realistic HFTs and RTs (for the buying price only) are negatively affected. The fundamentalists become relative market-makers again (graphs (c), (d), (f), (g), (n)). The relative number of transactions in which the RTs (fundamentalists) participate decreases (increases) (graphs (e), (h)). The realistic HFTs get the status of the main market-maker (graph (n)). The order-to-trade ratio smooths (graph (i)).
The paid spreads in aggressive orders as well as the earned spreads in passive orders boost for all the agents (graphs (j), (k)). This is a sign of a deteriorating market quality. The positioning profit (graph (l)) improves for the slow fundamentalists. The overall trading profit (graph (m)) slightly increases for the slow fundamentalists. The RTs' relative return decreases; for the slow fundamentalists it increases considerably, while for the realistic HFTs it decreases only slightly (graph (o)). Any inequality between the agents smooths.
In the accelerated market, the aggregate market liquidity and quality measures change in the following directions (references to the subplots in Figure 64):
All the spread measures decrease, with one exception (graphs (a)-(h)). The trading volume and number of transactions increase considerably (as a trading period lasts much longer); the acceptance ratio rises while the order-to-trade ratio goes down, meaning that more sent limit orders lead to transactions (graphs (i)-(k), (n)). The order imbalance decreases even more, diverging further away from the zero level (graphs (l), (m)). The correlation between the market prices and fundamental value boosts, while the relative absolute deviation drops (graphs (o), (s)): the quality of information included in the transaction prices goes up. The price impact of the past transactions on the future transactions diminishes (graphs (p), (q)). The trading

costs move closer to the zero level (graph (r)). The excess returns decrease considerably (graph (t)). □

Results of this analysis show that a simple prolongation of a trading session or a market acceleration can change the game outcome considerably: the market quality improves, and the informed slow traders can get more positioning profit. A change in the market session duration leads to the redistribution of the HFT-related disadvantage among slow market participants.

7.5.3 Pairwise analysis of configurations from the second group

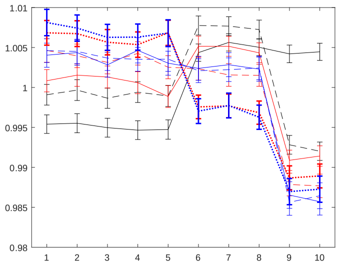
In this section, pairwise comparisons of the configurations from the second group is provided in the sequence of their complication. The aim is to investigate how different forms of cancellation (especially by realistic HFTs) influence the market and its participants. All conclusions in this section are based on Figure 65, when different agent types are compared, and on Figure 66, when the aggregate market measures are analyzed.

7.5.3.1 Non-exclusive cancellation right: less adverse selection for fundamentalists but damaged liquidity

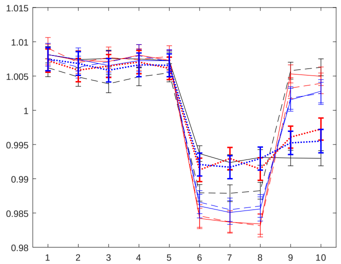
In this section, the configurations (6) Canc and (3) Lat+HFT are compared to investigate how the cancellation feature, allowed for all the market participants, changes the game, market quality, and distribution of the final results among the market participants.

Observation 7.10. *Providing the agents with the opportunity to cancel orders negatively influences market liquidity and quality in terms of the price efficiency, spreads, size liquidity measures, price impact, relative absolute deviation, and trading costs.*
The terms of trade for the slow fundamentalists are improved. Surprisingly, the realistic HFTs cancel fewer limit orders than the RTs.
The making-taking roles are influenced inconsiderably, as well as the final absolute positioning and trading profits and relative return.

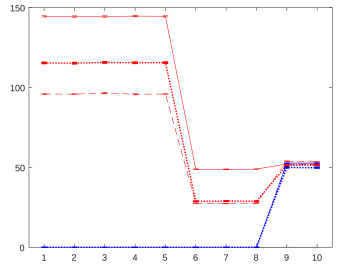
Evidence. The cancellation feature influences the agents' behavior and results in the following way (references to the subplots in Figure 65):
The slow fundamentalists' buying and selling prices improve: the more stale orders the slow fundamentalists are able to cancel, the better trade terms they can get, since they are less adversely selected by the realistic HFTs. The other traders are influenced by this feature insignificantly (graphs (a), (b)).
The RTs cancel much more limit orders than the fundamentalists, while the realistic HFTs cancel only slightly more limit orders than the slow fundamentalists (graph (c)). The difference in the relative number of the sent limit orders between the agents is exaggerated: the share of limit orders sent by the RTs rises, while for the slow fundamentalists it falls by more than for the realistic HFTs (graph (d)). The fundamentalists' relative participation in trade slightly decreases, while for the RTs it grows (graphs (e), (h)). As the amount of transactions becomes



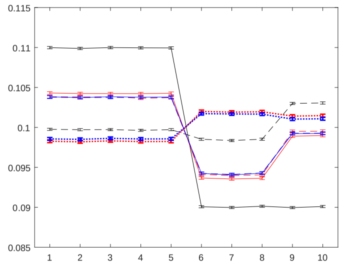
(a) Buying price



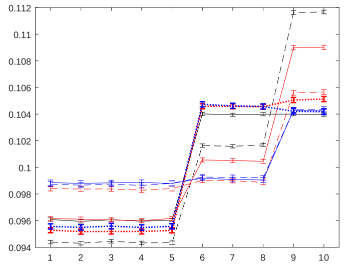
(b) Selling price



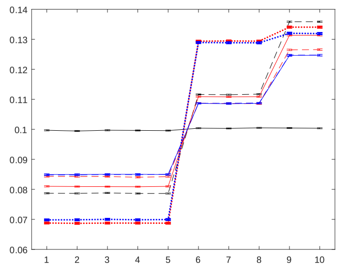
(c) Cancelled LOs (per run)



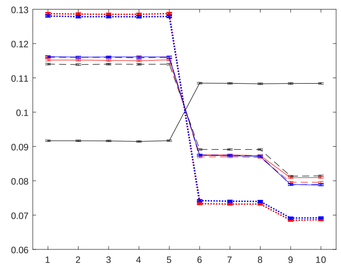
(d) Sending LOs rel.



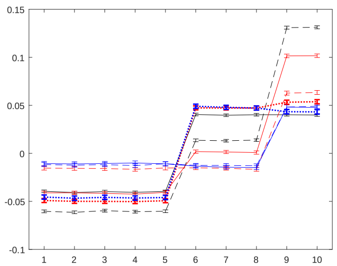
(e) Trading via both rel.



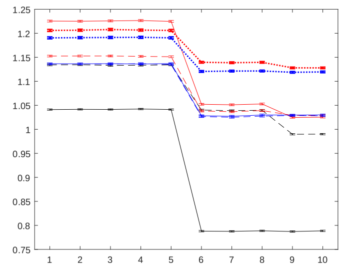
(f) Trading via LOs rel.



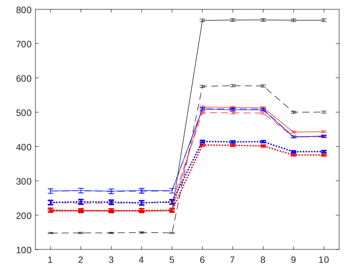
(g) Trading via MOs rel.



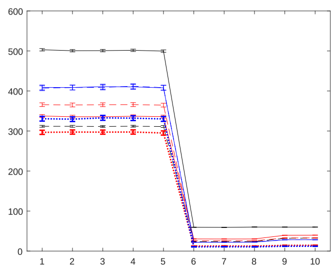
(h) Participation



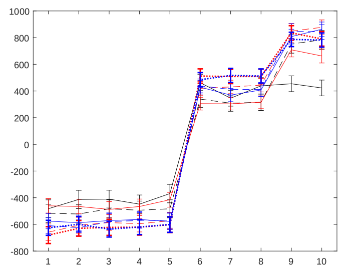
(i) Order-to-trade ratio



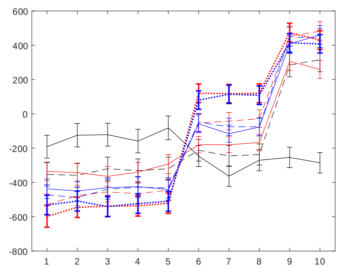
(j) Paid spread (agressive)



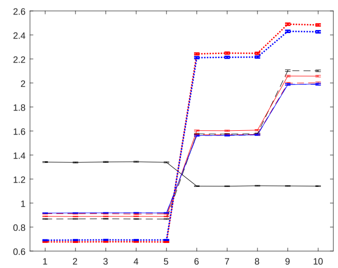
(k) Earned spread (passive)



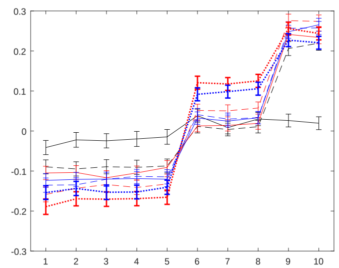
(l) Positioning profit



(m) Trading profit



(n) Maker-taker rate



(o) Relative return

Figure 65: Combined results of the second group of configurations: illustration per agent

Error bars are illustrated: the means as well as the 95% confidence intervals for each agent are provided. On the x-axis, the agents' identifiers are presented. Different lines corresponds to different configurations: (1) solid black line - Lat, (2) dashed black line - Lat+HFT, (3) solid red line - Canc, (4) dashed red line - Canc8, (5) solid blue line - HFTCanc, (6) dashed blue line - AccumCanc, (7) dotted red line - CancNVis, (8) dotted blue line - HFTCancNVis.

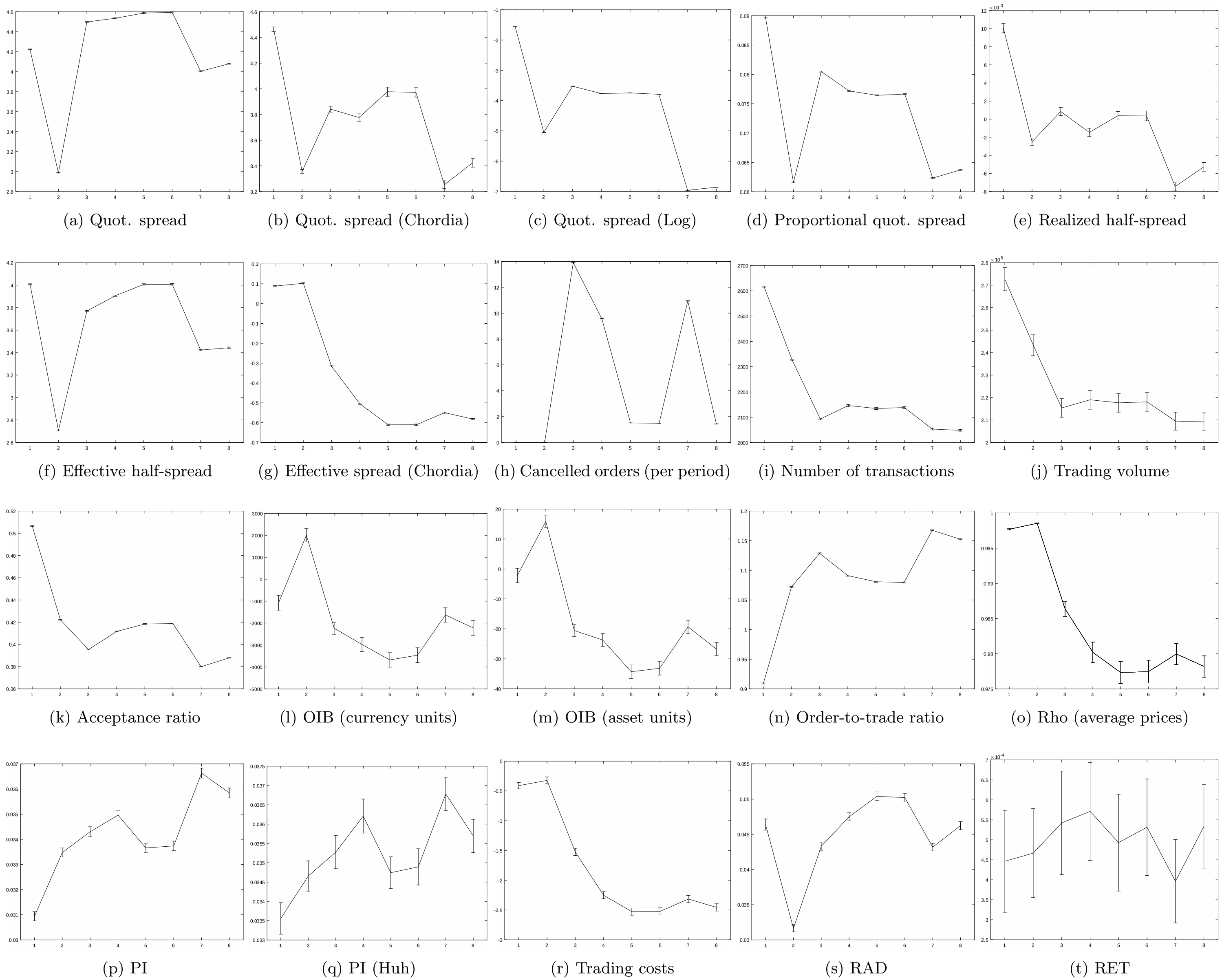


Figure 66: Combined results of the second group of configurations: the aggregate market metrics

Error bars are illustrated: the means as well as the 95% confidence intervals for each configuration are provided. On the x-axis, the sequential number of a configuration is presented: (1) Lat, (2) Lat+HFT, (3) Canc, (4) Canc8, (5) HFTCanc, (6) AccumCanc, (7) CancNVis, (8) HFTCancNVis.

smaller, the order-to-trade ratio for all the participants boosts (graph (i)). The agents' making-taking roles remain unchanged (graph (n)).
Due to the cancellation property, the RTs start to pay higher spreads in aggressive orders, while the paid spread by the fundamentalists decreases (graph (j)). The earned spread goes up for the RTs and only slightly for the fundamentalists (graph (k)). The positioning profit, trading profit, and relative return are not affected considerably (graphs (l), (m), (o)).
Regarding the general market characteristics, the following changes can be observed (references to the subplots in Figure 66):
All of the spread measures increase except one (graphs (a)-(g)). The liquidity of the market goes down, as can be seen by the number of transactions, trading volume, acceptance ratio (graphs (i)-(k)), proved also by the increased order-to-trade ratio (graph (n)). The order imbalance measures do not change in absolute terms (graphs (l), (m)). The informational content of prices worsens and the relative absolute deviation increases (graphs (o), (s)). The price impact increases (graphs (p), (q)), while the trading costs become more negative (graph (r)). □

The cancellation right should be beneficial to all the market participants, as they receive a chance to cancel stale orders. However, as the results of the simulated market suggest, the cancellation option does not affect the results of the individual agents considerably both in absolute and relative terms. Our findings are close to those by Subrahmanyam and Zheng (2016), who do not find any significant difference between the cancellation frequencies of HFTs and non-HFTs. As in Leal, Napoletano, Roventini, and Fagiolo (2016), the effects of order cancellation are predominantly negative.

7.5.3.2 Cancellation latency: RTs' adversely selected orders before cancellation is executed

In this section, we compare the configurations (7) Canc8 and (6) Canc to analyze whether cancellation latency changes the results described in Section 7.5.3.1.

Observation 7.11. *Cancellation latency improves the size liquidity measures, spreads and trading costs, while negatively affecting market quality based on the other metrics (order imbalance, informational content of prices, and price impact).*
Cancellation latency influences positively the fundamentalists' trade terms and negatively those of the RTs: the fundamentalists can catch the RTs' stale orders before they are canceled. The trading profits and relative returns are impacted only slightly, but the direction of change suggests that the RTs pay for the fundamentalists' relative success.

Evidence. Cancellation latency influences the agents' behavior and return in the following way (references to the subplots in Figure 65):
The fundamentalists' buying prices improve, while those of the RTs worsen (graph (a)). The selling prices are affected insignificantly for all the market participants (graph (b)).
The RTs still cancel much more limit orders than the fundamentalists, but cancellation latency

decreases the amount of cancellations by the RTs a lot, while for the other traders it stays on the same level (graph (c)). The improvement in trade conditions for the fundamentalists means that cancellation latency allows the fundamentalists to adversely select the RTs' stale orders before they are canceled.

Cancellation latency influences the relative number of the sent limit orders only slightly: negatively for the RTs and positively for the fundamentalists (graph (d)). The fundamentalists' relative participation in trade drops, while for the RTs it increases (graphs (e), (h)). As the amount of transactions becomes higher, the order-to-trade ratio for all the participants declines (graph (i)). The fundamentalists start to trade relatively less through limit orders, while the RTs slightly more (graphs (f), (g)). The maker-taker ratio changes only slightly: it increases for the RTs and goes down for the fundamentalists: the realistic HFTs become even more market-takers (graph (n)).

The paid spread from aggressive orders by the fundamentalists drops slightly and remains unchanged for the RTs (graph (j)). The earned spread changes inconsiderably: for the RTs it goes up, for the fundamentalists it decreases (graph (k)). The positioning and trading profits as well as the relative return display a falling (increasing) tendency for the RTs (fundamentalists) (graphs (l), (m), (o)).

In regard to the general market characteristics, the following changes can be observed (references to the subplots in Figure 66):

The spread measures decrease except two (graphs (a)-(g)). The liquidity of the market goes up, as seen by the number of transactions, trading volume, and the acceptance ratio (graphs (i)-(k)), proved also by the decreasing order-to-trade ratio (graph (n)). The amount of canceled orders falls (graph (h)). The order imbalance measures diverge further from the zero level (graphs (l), (m)).

The informational content of prices worsens and the relative absolute deviation increases (graphs (o), (s)). The price impact increases (graphs (p), (q)), while the trading costs become more negative (graph (r)). □

7.5.3.3 Exclusive cancellation by realistic HFTs: damaged informational efficiency with slightly harmed RTs but advantaged fundamentalists

In this section, a comparison of the configuration (8) HFTCanc (the realistic HFTs having an exclusive right to cancel the previously sent limit orders) with the basic configuration (3) Lat+HFT (no one has the right to cancel orders) and the close configuration (6) Canc (non-exclusive cancellation right, i.e. all the market participants can cancel their orders) is conducted to analyze how the realistic HFTs' cancellation activities change the game and distribution of the final results among the market participants. The HFTs are blamed for canceling a higher share of limit orders on the market in comparison to human traders. Therefore, it is essential to assess the configuration based on the exclusive cancellation right possessed by the realistic HFTs to fill in the gap in the research of this topic.

Observation 7.12. *As was concluded for the non-exclusive cancellation right, cancellation in general influences market liquidity and quality negatively (in terms of the spreads, size liquidity measures, price impact, and relative absolute deviation), see Section 7.5.3.1.*
The realistic HFTs' exclusive cancellation right compared to non-exclusive cancellation damages the informational efficiency of the market prices even further, while the trading costs become more negative. The RTs' trading terms deteriorate, while those of the fundamentalists (including both fast and slow fundamentalists) improve. The trading profits and relative returns are influenced only slightly. However, the results suggest that the RTs have to pay for the fundamentalists' relative increased advantage.
In terms of the spread measures, however, an exclusive cancellation right might be a better choice than non-exclusive cancellation.
Irrespective of whether the cancellation right is exclusive or non-exclusive, the realistic HFTs cancel approximately the same amount of limit orders.

Evidence. The realistic HFTs' exclusive cancellation right influences the trading agents' behavior and results in the following way (references to the subplots in Figure 65):
Compared to the exclusive cancellation case, the RTs' buying prices increase even further, while the fundamentalists' buying prices improve (graph (a)). The selling prices are between the two extremes: no cancellation and the non-exclusive cancellation right (graph (b)).
The realistic HFTs cancel approximately the same amount of orders under the two cancellation rights (graph (c)).
The relative sending of limit orders (graph (d)), trading via limit orders (graph (f)) and market orders (graph (g)), the order-to-trade ratio (graph (i)), the maker-taker rate (graph (n)) are close to the non-exclusive right configuration. The fundamentalists' relative participation in trade decreases, while that of the RTs increases (graphs (e), (h)), and they trade approximately on the same level. The order-to-trade ratio for the slow participants drops, while for the realistic HFTs it stays almost unchanged (graph (i)).
The paid and earned spreads are close to the non-exclusive right configuration (graphs (j), (k)).
The positioning and trading profits decrease for the RTs and increase for the fundamentalists (graphs (l), (m)). The relative return moves together with its distribution to the direction of the RT's disadvantage and the fundamentalist's advantage (graph (o)).
Regarding the general market characteristics, the following changes can be observed (references to the subplots in Figure 66):
Some spread measures show that the spreads are maximized when only the realistic HFTs can cancel orders; yet the others indicate that an exclusive cancellation right is a better choice than a non-exclusive right (graphs (a)-(g)). It is clear that cancellation in both forms worsens the spreads. However, based on some spread measures, providing an exclusive right to cancel orders to the realistic HFTs might be a better option than empowering all agents to cancel their limit orders.
Liquidity is damaged by the cancellation opportunities in general. A smaller amount of can-

celed orders provides better size liquidity measures (graphs (i)-(k), (n)). The order imbalance measures go further away from zero (graphs (l)-(m)), the informational content of prices is negatively affected even more (graph (o)) and the relative absolute deviation increases (graph (s)) compared to the non-exclusive right configuration.
The price impact is insignificantly higher than in the non-cancellation configuration but lower than with the non-exclusive right (graphs (p), (q)). The trading costs become more negative (graph (r)). □

Cancellation in general, including the realistic HFTs' exclusive cancellation, damages the market quality metrics, which is in line with Huh (2014). However, this rather stems from a respective activity than from the nature of an individual HFT. The fact that an HFT is generally more active in the market causes the problem.

7.5.3.4 Realistic HFTs' speed advantage in order cancellation over other realistic HFTs: no significant difference

In this section, a comparison of two configurations takes place, namely the configuration (9) AccumCanc (which includes the accumulation of the realistic HFTs' orders until a slow trader acts on the market) with the configuration (8) HFTCanc (which allows the direct cancellation of the realistic HFTs' limit orders in the sequence these inquiries are sent to the market, meaning that the realistic HFTs have the speed advantage over the identical realistic HFTs in the cancellation of orders).

Observation 7.13. *There is no significant difference in the agents' behavior and market results, if the realistic HFTs are assumed to have (or not to have) the speed advantage over the identical realistic HFTs in the cancellation of the previously sent limit orders.*

Evidence. As is shown by the individual per-agent measures (Figure 65) and the general market measures (Figure 66), there is no significant difference between the two configurations. □

Under the construction of the current ABM, it is important that the agents, all of them or just the realistic HFTs, have a cancellation right. Whether the realistic HFTs have an additional cancellation speed advantage over the other realistic HFTs seems to play an insignificant role: the cancellation speed competition between the realistic HFTs does not have any negative or positive influence on the market and its participants.

7.5.3.5 Stale information about the realistic HFTs' activities and non-exclusive cancellation right: better spreads, improved fundamentalists' results but worse size liquidity measures

In this section, the configuration (10) CancNVis is compared with the configuration (7) Canc8 to answer the question whether the fact that the latency period applies not only to the realistic HFTs' activities but also to the slow trader's activities influences the market game and the final distribution of returns. In both of these configurations, cancellation is possible by all the market participants.

Observation 7.14. *If the slow traders fail to observe only the realistic HFTs' activities in a limit order book, while the other slow traders' activities are visible, the market liquidity and price impact worsen, while the market quality in terms of the spread and order imbalance measures is improved. The slow fundamentalists' transaction terms improve and those of the RTs deteriorate. The RTs cancel more orders in the more informative configuration of a limit order book. The paid and earned spreads decrease. Overall, the positioning and trading profits as well as the relative return move to the RT's disadvantage and the slow fundamentalist's advantage: the RTs pay for the improved results of the slow fundamentalists.*

Evidence. If the slow traders have a delay of information from a limit order book about the realistic HFTs' activities, in opposite to the case when such delay is general (includes the activities of both the slow traders and the realistic HFTs), the agents' behavior and results are influenced in the following way (references to the subplots in Figure 65):
The RTs' (fundamentalists') buying prices worsen (improve) (graph (a)). The slow fundamentalists' (realistic HFTs') selling prices rise (fall) (graph (b)). The fact that the slow traders' activities are visible timely to the other slow traders improves the slow fundamentalists' trading conditions, while relatively negatively influencing both RTs and realistic HFTs.
The amount of the canceled limit orders grows for the RTs (graph (c)). The relative amount of the sent limit orders by the RTs drops, while by the fundamentalists increases. The improvement of the realistic HFTs' market-making activity is less considerable (graph (d)). The relative participation in trade on the part of the slow fundamentalists increases, for the realistic HFTs it declines slightly and for the RTs considerably (graphs (e), (h)). The order-to-trade ratio for all the participants increases (graph (i)). The fundamentalists trade relatively more through limit orders, while the RTs trade more through market orders (graphs (f), (g)). The maker-taker relation stays the same but becomes more pronounced: the fundamentalists become relative more market-makers, while the RTs increase their taking function. The realistic HFTs are the most important market-makers (graph (n)).
The paid and earned spreads from aggressive orders for all the agents drop (graphs (j), (k)). The slow fundamentalists' positioning profit goes up slightly (graph (l)). The RTs' trading profit goes down, while the slow fundamentalists experience a higher trading profit (graph (m)). The relative return follow the path of its distribution to the direction of the RT's disadvantage, while the slow fundamentalists get a higher relative return and the realistic HFTs are inconsiderably disadvantaged (graph (o)).
Regarding the general market characteristics, the following changes are observable (references to the subplots in Figure 66):
All the spread measures decline (graphs (a)-(g)). The liquidity of the market goes down, as seen by the number of transactions, the trading volume, and the acceptance ratio (graphs (i)-(k)), proved also by an increase in the order-to-trade ratio (graph (n)). The amount of canceled

orders boosts (graph (h)). The order imbalance measures go slightly up converging closer to the zero level (graphs (l)-(m)). The informational content of the market prices and the relative absolute deviation remain unchanged (graphs (o), (s)). The price impact increases (graphs (p), (q)). The trading costs decrease insignificantly (graph (r)). □

The results presented in this section are intuitive, as the slow fundamentalists have improved their informational endowment by having access to a more complete version of a limit order book, where only the realistic HFTs' activities are delayed (compared to the case when information about all activities, both the realistic HFTs' and slow traders', enters a limit order book with delay). Having more information, the slow fundamentalists are able to use this information to react correctly to the changing market conditions, and the adverse selection bias shifts more to the RTs who do not use any fundamental information in their decision-making.

7.5.3.6 Stale information about realistic HFTs' activities and exclusive cancellation right: similar results to configuration with non-exclusive cancellation right

In this section, the configuration (11) HFTCancNVis is compared with the configuration (10) CancNVis to check whether providing the exclusive cancellation right to the realistic HFTs, as opposite to the non-exclusive cancellation right, modifies the previous results discussed in Section 7.5.3.5.

Observation 7.15. *The limit order book's staleness with respect to the realistic HFT activities has the same consequences for both cancellation rights: when the realistic HFTs cancel orders exclusively, the results are similar to those presented in Section 7.5.3.5 (the configuration with the non-exclusive cancellation right). The difference between the two configurations is very slight: the exclusive cancellation right causes depressed market quality in terms of the spreads, order imbalance, and the informational content of prices. The trading costs and price impact improve. Moreover, the fundamentalists provide relatively less liquidity to the market, and the relative returns of the participants show a narrowing gap between the winning realistic HFTs and the losing RTs.*

Evidence. Regarding the individual agents' behavior and results, the following considerable changes can be observed (references to the subplots in Figure 65):
The relative participation in trade as well as trading through limit orders on the part of the realistic HFTs decreases (graphs (e), (f), (h)). The order-to-trade ratio for all the agents drops (graph (i)). The market-making functions of all the fundamentalists decline, especially the realistic HFTs' (graph (n)).
The paid spread for all the agents increases, the earned spread increases for the RTs only (graphs (j), (k)). The distribution of the relative return moves to the direction of the RT's advantage and the fundamentalist's disadvantage (graph (o)).
Regarding the general market characteristics, the following changes are the most vivid (references to the subplots in Figure 66):

All the spread measures increase except one (graphs (a)-(g)). The amount of the sent limit orders to the market decreases, whereas the amount of transactions stays approximately on the same level (graphs (i)-(j)). The acceptance ratio goes up and the order-to-trade ratio goes down (graphs (k), (n)). The order imbalance measures move further away from the zero level (graphs (l), (m)). The informational efficiency of the market prices decreases inconsiderably and the relative absolute deviation increases (graphs (o), (s)). The price impact drops (graphs (p), (q)). The trading costs fall (graph (r)). □

7.6 Broad picture for selected market metrics

In this section, the effects of the configuration change on some of the discussed market metrics are summed up.

The fundamentalists' participation in trade is generally higher than that of the RTs (Figure 63 (h)). Only in a very latent market the slow fundamentalists' participation in trade is lower than the RTs'. The realistic HFTs participate in trade more than all the other market participants. In the non-latent market, the fundamentalists are market-makers, while the RTs are market-takers (Figure 63 (n)). The latency increase causes the fundamentalists (RTs) to reduce (enhance) their relative market-making function. When the realistic HFTs enter the latent market, the fundamentalists increase their market-making function again, with the realistic HFTs being more active makers than the slow fundamentalists. If the latency increases further, the fundamentalists start taking more liquidity, but the realistic HFTs take relatively less than the slow fundamentalists.

The best informational efficiency of the market prices is reached in the non-latent market (Figure 64 (o)). A higher degree of latency makes prices less informative, but the realistic HFTs' entry recovers informational efficiency considerably. A similar effect can be achieved by increasing the duration of the market period or by accelerating the market.

In the non-latent market, the fundamentalists exploit their informational advantage at the cost of the RTs. In the latent market, the reaction time friction brings relative returns of all the agents to approximately the same level, depending on the severeness of latency (see Figure 63 (o)). With moderate latency, the realistic HFTs' entry harms only the RTs, but if latency is considerable, a bigger disadvantage is experienced by the slow fundamentalists. The realistic HFTs win the market at the cost of their trading partners.

In the non-latent market, the fundamentalists get their trading profit due to the informational advantage, the RTs experience losses (Figure 63 (m)). In the latent market, all the agents' trading profit is on average negative, the fundamentalists' loss is greater than that of the RTs. The realistic HFTs are able to rock the market and win the game exacerbating slow fundamentalists' losses.

The spread in passive orders (Figure 63 (k)) is earned mainly by the RTs. The realistic HFTs' entry decreases the RTs' passive spread earnings, while additional latency increases these earnings. Realistic HFTs' spread earnings are only slightly higher than those of the slow fundamen-

talists.
The spread in aggressive orders (Figure 63 (j)) is paid mainly by the fundamentalists; higher latency increases these payments. The realistic HFTs' entry lowers the spread paid by the slow fundamentalists, while the realistic HFTs pay a smaller spread than the slow fundamentalists.
The fundamentalists' positioning profit is always non-negative in comparison to that of the RTs (Figure 63 (l)). Latency decreases the fundamentalists' positioning profits and the RTs' losses.
The realistic HFTs get a higher positioning profit than the slow fundamentalists; a further latency increase boosts the realistic HFTs' positioning profits.

7.7 HFT-related policies

HFT has an overwhelming capacity to influence the market stability and its integrity. However, HFT firms have few obligations and restrictions, even when it comes to influencing market volatility. Recently there were some attempts to limit HFT activities through restrictive mechanisms such as a levy tax on shortly-lasting transactions.
The modeled ABM can serve as a testbed for a number of policy interventions related to HFT.
Some important concerns about the HFT-related policies should be taken into account.
According to SEC (2010a), the Commission is focused on safeguarding the fairness of the market structure to the interests of long-term investors. Sometimes their interests may align with short-term traders' interests (including HFTs), but often they diverge. SEC (2010b) proposed regulations that oblige HFT firms to stay active on volatile markets. The following control measures were suggested:

(1) Limits on the number of orders for a certain period of time;

(2) A kill switch or a circuit breaker[111] which could stop trading in extreme cases;

(3) Limits on the intra-day position;

(4) Profit-and-loss limits.

Other possible policies discussed in the literature that potentially limit the adverse impact of HFT on the market quality and stability include:

- Consolidated order-level audit trails[112];
- Order cancellation or excess message fees, securities transaction taxes;
- Price limits[113];

[111]Discussed in Brewer, Cvitanic, and Plott (2013).
[112]This and the next policy is discussed in more detail in Jones (2013).
[113]Discussed in Yeh and Yang (2010), Deb, Kalev, and Marisetty (2010), Gode and Sunder (2004), and Maskawa (2007).

- Restriction on short sales[114] (including the up-tick rule, i.e. the requirement for a short sale to be entered at a price higher than the previous transaction price; this rule prevents short sellers from adding a down momentum to a price that already experiences a downward move);

- Minimum resting times or minimum quote durations[115];

- Switching to call auction market mechanism[116].

CFA Institute (2014), instead of restricting certain trading activities by introducing taxes or charges, suggests promoting internal risk management and controls over HFT algorithms and strategies that address manipulation and other threats to the integrity of financial markets.
Bernales (2014) claims that both latency restriction and cancellation fees for fast traders are unnecessary market frictions damaging the market quality. However, a cancellation fee for fast traders reduces adverse selection and generates positive changes in their behavior, since fast traders start to act more as liquidity providers; therefore cancellation fees is a better policy choice compared to latency restrictions. Rojcek and Ziegler (2016) study many possible regulatory policies to prevent HFT's disadvantages and conclude that such policies, though increasing market quality measures, do not influence the welfare of long-term investors, increasing only the welfare transfer from HFTs to speculators. Therefore, such policies are inadequate. They believe that a transaction tax is the least harmful policy which reduces welfare exactly by the tax amount and would not considerably influence the market quality. Pasquale (2015) claims that a transaction tax could at least partially prevent unethical behavior on the part of HFTs and support other sectors of economy, but regulators had better eliminate many of the HFT strategies than try to accommodate them.
As mentioned in Ciallella (2015), the European algorithmic trading and HFT regulation, Directive 2014/65/EU (or so-called MiFIDII), requires firms who engage in algorithmic trading to implement internal controls (like trading thresholds and limits as well as testing mechanisms that control for potentially erroneous orders) to ensure resiliency of the trading system. A description of the trading strategies and activities, as well as the data about all the placed, executed, and canceled orders have to be made available to authorities upon request. Firms engaging in algorithmic trading and pursuing market-making functions to the trading platform are required to provide such functions continuously.
According to the German Stock Exchange Act (Börsengesetz vom 16. Juli 2007 (2007)), algotraders have to test their algorithms in a test environment provided by an exchange; if orders influence a system a lot (limits of the capital market will be reached) an exchange has to prevent execution of such an order.
Regulation (EU) No 596/2014 on Market Abuse (2014) defines HFT- and algotrading-activities

[114] Analyzed in Grundy, Lim, and Verwijmeren (2012), Anufriev and Tuinstra (2013), Easton, Pinder, and Uylangco (2013), and Brogaard, Hendershott, and Riordan (2017).
[115] Investigated in Huh (2014), Brewer, Cvitanic, and Plott (2013), and Jones (2013).
[116] Discussed in Brewer, Cvitanic, and Plott (2013).

as market manipulating if they lead to a disruption or delay of the trading system, or if they overload or destabilize the system, or if they create a false or misleading signal.

7.8 Summary

In this chapter, we extended the agent-based model from Chapter 6 to a more realistic model. The trading mechanism was changed and a limit order book was granted the option to be filled with many orders on both market sides. The heterogeneity of the market participants was improved, which should represent the composition of the real financial market in a better way. Informed and uninformed slow traders were modeled. Uninformed slow traders are random traders similar to the previous version of the agent-based model, but informed traders do not base their decisions on any type of analytical equilibrium map. Rather, they are conceptualized as fundamentalists, making a strategic order placement based on their information about the fundamental value development. Fast traders, i.e. realistic HFTs, are also fundamentalists with additional information about the other agents' recent activities (they see a current version of a limit order book) and with some cancellation privilege, depending on the configuration.

The results of the more realistic agent-based model simulation allow for the following main conclusions:

HFT influences the market through a complex system of effects. Some market characteristics might improve and other might deteriorate; different HFT activity characteristics might have an opposite influence on the market quality. The results crucially depend on the initial state, i.e. on the parameters assumed for the market. The final conclusions about the HFT's positive or negative influence depend on whether the effects are assessed from a certain agent's perspective or from the aggregate market perspective.

In a non-latent market, due to the informational advantage, fundamentalists win at the cost of random traders, but they most exclusively make the market, i.e. they make profit from the strategic placement of limit orders.

With latency increase, random trading becomes less damaging, while fundamental trading becomes less profitable: even though positioning profits from strategic order placement are still present, slow fundamentalists cannot catch the right moment to make a transaction due to staleness of a limit order book that slow fundamentalists observe. Realistic HFTs, having both informational advantages in terms of fundamental value and limit order book, rock the market. A realistic HFT gets higher spread earnings and positioning profit; at the same time, she provides liquidity to the market, i.e. the majority of transactions are made through limit orders. The bigger the latency problem on the market is, the smaller is the realistic HFTs' market-making function and the higher profits she gets at the cost of the other market participants. The main victim of the HFT activity depends on the latency severeness compared to the trading session duration or speed of the market. Fundamentalists' profits depend on the staleness of a limit order book that they use. If the latency problem is modest, the positioning profit that they get due to their fundamental information covers the potential spread losses due to the non-correct time of transaction. In this case, random traders are the main victims on the market (both due to the HFT activity and the fundamentalists' activities in general). When latency goes up, the slow fundamentalists' positioning profit cannot cover spread losses, which makes them the main victims: being a random trader turns to be also unprofitable but is in fact the lesser of the two evils.

Latency weakens the slow fundamentalists' adverse selection or informational advantage compared to random traders; it also worsens the market quality in terms of spreads and the informational efficiency of prices, while improving the size liquidity measures and order imbalance. The HFT entry recovers spreads and informational efficiency damaged by latency. However, HFT increases adverse selection and negatively influences the trading volume (in spite of the increased volume of limit orders sent by fundamentalists) and boosts inequality between the market participants.

Cancellation in general (and not only cancellation done by HFTs) negatively influences the market quality in terms of informational efficiency and trading volume, while improving the trading costs (as trade terms can be improved). However, a higher cancellation rate by HFTs (in our study, the exclusive cancellation form) damages the informational efficiency of prices even further; at the same time, it reduces spreads. With a higher cancellation rate by HFTs, slow fundamentalists can improve their trade terms.

After investigating the influence of different HFT characteristics on the market quality, a brief discussion of HFT-related policies was offered, since recently many regulators have voiced the necessity of HFT activities regulation.

According to our agent-based model results, potential HFT-related policies should aim at a latency decrease, because it negatively influences the market quality and the agents' results. HFT increases informational content of transactions, but the same effect might be achieved through an acceleration of the market or increase of the market session duration. HFT provides liquidity (dust) and improves spreads, therefore it would be only natural to allow HFTs to play on the market unrestricted, if these quality measures are critical. However, HFT negatively influences the slow investors' results. The slow informed traders' returns, again, might be recovered through the market acceleration or prolongation of the market session. An order cancellation deteriorates the market quality; the minimum resting time or minimum quote duration might help to make up for this damage. However, cancellation plays a positive role for the slow investors' returns. If orders rest before being canceled, it deteriorates the order imbalance, price impact, and informational content of prices. The fact that HFTs are responsible for a high number of cancellations undermines informational efficiency and trading costs further, but positively influences spread measures. Depending on the policy goal with respect to the mentioned market quality metrics, the order cancellation from the HFT side might be restricted by a levy.

8 Conclusion

The motivation behind our research was to investigate the HFT's influence on the market and to bridge the gap between the analytical dynamic limit order book models and agent-based models for financial markets. For this purpose, as the first step, we constructed the simplified version of the analytical dynamic limit order book model with slow traders with the aim to study the symmetry issue between the market sides implicitly assumed in the literature. Our study suggests that the buying side of the market never sells and the selling side never buys under the optimality conditions. Depending on the share of each market side and the volatility states, different fill-rate equilibria might exist: low fill-rate orders are used by the numerous market side or in the high volatility states to be protected from the excessive risk of adverse selection. The symmetric fill-rate equilibria turned out to be symmetric for all the other market metrics as well as under the neutral market sentiment; the mixed equilibria, while providing the same imaginary profit from sending a limit order, behaved as asymmetric based on the other market metrics. Depending on the market perspective and the critical market metrics, the agents might choose different optimal equilibria from the possible set at the intersection point. Additional market metrics for the simplified model under the modified assumption of only two market participants were introduced. This assumption harmonized the analytical model with its agent-based alternative. The target metrics were later used to show the convergence of the agent-based model outcomes with the corresponding analytical results.

After constructing the simplified model with slow traders, the set-up of Hoffmann (2014) was followed and his equilibrium model was reconstructed, accounting for the order-setting rules of both market sides. It turned out that the cases presented in the original paper are area-equilibria only. Other equilibrium order-setting combinations were excluded from further consideration, since line- and point-equilibria correspond to infinite decimal parameters, which are hardly distinguished by market agents, either human or artificial. A graphical comparison of the market metrics for slow traders, fast traders, and in slow traders-only market was provided. The latter case serves as a benchmark to assess the fast traders' entry influence. The agents' choice of order-setting strategies depends on the share of fast traders and volatility: in the small share states, slow traders use a specialized strategy, while high volatility causes both agent types to follow low fill-rate strategies (as they strive to protect themselves from the increased risk of adverse selection). The fast traders' entry negatively affects slow traders: their bargaining power, welfare, and risk of being picked-off deteriorate in almost all states. However, if the volatility is high, fast traders improve the major qualities of the whole economy (the trading rate, the pricing error, and the aggregate market welfare). The aggregate market welfare may drop, however, in the low share - low volatility states. Even though a fast trader wins the market at the cost of slow traders, the model of Hoffmann (2014) suggests that a fast trader serves the market-making function, while the degree of her market-making increases in volatile markets, which is perceived to be a vital positive trait of HFT.

Having replicated the results of Hoffmann (2014), we investigated whether the fast traders' influence on the market is different from the influence of simple informed traders without any speed advantage and the possibility to adjust orders. For this purpose, we designed a modified analytical model with informed slow traders. Informed slow traders are unconditionally informed about the next fundamental price change but cannot revise their quotes. As with fast traders, the choice of orders depends on the share (in the low share states, both agent types prefer to use specialized strategies) and on the volatility (in the low volatility states, a slow trader follows high fill-rate strategies). An informed slow trader has a greater bargaining power and she wins the game at the cost of slow traders, performing the role of a market-maker. The informed slow traders' entry adversely affects the slow traders' bargaining power and their costs of immediacy. If the informed slow traders' share stays on a low level, the aggregate market welfare goes down with the informed slow traders' entry. If, additionally, the volatility is low, the informed slow traders' entry undermines the trading rate and pricing error, while increasing the slow traders' risk of being picked-off. Only in the high volatility states, the pricing error might improve, and if the share of informed slow traders is high enough, the slow trader's costs of immediacy and welfare may revive.

Having compared the market metrics for the set-ups with fast traders and informed slow traders, it became obvious that a fast trader may influence the market quality and efficiency as well as her trading partners more positively or negatively than an informed slow trader depending on the market state, the considered market criteria, and the perspective assumed for the analysis. A decision map that makes it possible to determine the most beneficial market set-up was designed based on three dimensions discussed above. If the share of informed traders is high, a fast trader allows slow traders to exercise a higher bargaining power than an informed slow trader. If the share of informed traders is low, the slow traders' costs of immediacy and the trading rate are better with fast traders. If the volatility is high, a fast trader triggers off a higher trading rate. If the volatility is low, a fast trader causes a better informational efficiency and a smaller pricing error. Regarding the aggregate market welfare, a fast trader is a better alternative than an informed slow trader in the low share - low volatility states and in the high share - high volatility states, while a fast trader can never be an optimal partner based on the slow trader's welfare.

As the next step, we created the simple agent-based model as an intermediate stage between the analytical models and agent-based models. It was possible to bridge the gap between the two types of models by showing that the results of the two converge, if the assumptions are synchronized. The constructed agent-based model was applied to enrich the analysis by increasing the heterogeneity of the market participants and the number of market metrics analyzed. The random trader's or informed trader's entry to the market negatively affects liquidity and welfare, even though all the entering agents serve the market-making function. HFT negatively influence the costs of immediacy and the pricing error, while improving the price efficiency. Informed slow traders improve the pricing error. Both informed traders adversely affect a slow uninformed trader, but she would choose an informed slow trader as a trading partner. The profit realized by a fast trader is higher than that of an informed slow trader. However, both informed traders improve the market quality and welfare compared to the configuration with a random trader.

Similar to the analytical decision map, the more realistic agent-based model illustrated that the effects of the HFT entry are multidimensional and depend on the initial market state as well as on the specific HFT feature analyzed. The simulation results suggest that HFTs outperform the market due to their informational and speed advantage. Moreover, HFTs are relative market-makers, even though they participate much more actively in the market both as makers and takers. However, the bigger the latency problem is, the less significant is the HFTs' market-making function and the higher are the profits they get at the cost of the other market participants. If the latency is modest, the informed slow traders' positioning profits cover their net spread losses. In this situation, random traders are the main losers on the market. In the high latency environments, the informed slow traders' net spread losses are very high and they are not covered by the positioning profits. In such a situation, trading randomly is a better option than trading based on fundamental information. If the HFTs' activities are assessed from the aggregate market perspective, the results are two-fold: on the one hand, HFT helps to recover the spreads and price efficiency damaged by market latency; on the other hand, a fast trader increases adverse selection and inequality among the market participants, while the trading volume goes down. The results of our simulations suggest that a high cancellation rate of fast traders orders damages the informational efficiency of prices but improves the spreads; with a high cancellation activity on the part of HFTs, the informed slow traders' transaction terms can revive.

Our agent-based model results suggest that a potential market policy should aim at the latency reduction, as it harms the market quality in general and the participants' results in particular. HFT boosts informational efficiency, provides additional liquidity, and improves spreads. However, the HFT activities negatively influence the slow traders' profits. Restrictions on HFT activities bring the market quality a step back. The cancellation option negatively affects the market quality, therefore the minimum resting time might be beneficial. However, the cancellation option allows slow traders to eliminate their stale orders and improve the final results; the cancellation latency impairs the order imbalance, price impacts, and informational efficiency. HFT has similar effects on the market as slow informed trading. The slow uninformed traders, however, are better-off with informed slow traders rather than with fast traders. Therefore, a possible policy might have the goal of increasing investments into research supporting informed trading rather than in technologies supporting common HFT strategies.

Our study adds value to both analytical limit order book models and agent-based models of financial markets. As in the analytical models by Boco, Germain, and Rousseau (2017) and Foucault, Hombert, and Rosu (2016), the empirical studies by Biais, Declerck, and Moinas (2016), Hendershott, Jones, and Menkveld (2011), and Hasbrouck and Saar (2013), and in the agent-based model by Arifovic, Chiarella, He, and Wei (2016), a realistic fast trader serves the market-making function. The fact that a realistic HFT wins at the cost of a slow trader is similar to the one presented in the analytical models by Boco, Germain, and Rousseau (2017) and Hoffmann (2014), the empirical results by Gerig (2012) and Baron, Brogaard, and Kirilenko (2012), as well as the agent-based model results reported in Bernales (2014). However, such a conclusion clashes with the outcomes by Rojcek and Ziegler (2016). As in Riordan and Storkenmaier (2012), our research demonstrates price efficiency improvement due to latency decrease. HFT as a spreads-improving force is in line with the results by Huh (2014), but opposite to Hendershott and Moulton (2011), Leal, Napoletano, Roventini, and Fagiolo (2016), and Arifovic, Chiarella, He, and Wei (2016). Our study confirms the generally agreed on effect of HFT improving the informational efficiency. The HFTs' destabilizing force manifests itself in our research as the enhanced order imbalances, which is close to the results by Leal, Napoletano, Roventini, and Fagiolo (2016). The result of HFT increasing the adverse selection bias corresponds to the outcomes reported by Hoffmann (2014), Biais, Foucault, and Moinas (2011), Hendershott and Moulton (2011), and Huh (2014). The bulk of the literature analyzed claims though that HFT improves the adverse selection problem (e.g., Jovanovic and Menkveld (2016), Menkveld (2013), Gerig (2012), and Riordan and Storkenmaier (2012)). As in Leal, Napoletano, Roventini, and Fagiolo (2016), the order cancellation option influences the market quality mostly negatively.

This dissertation sheds light on HFT and its multidimensional nature and shows that it can have both positive and negative effects on various market quality metrics and on different market participants, while the initial market state also plays a significant role in the final effect produced. To decide which HFT-related policies should be implemented, it is important to set a clear goal by prioritizing a certain market participant or a specific market quality metric. Possible side effects on the other market players or market quality dimensions should be investigated beforehand.

Some essential and promising topics for future research are summarized in Menkveld (2016). If human traders can hardly compete with machines, it would be both challenging and beneficial to compare the human market with the machine market regarding their effectiveness. In the latter market, the behavioral bias is absent: would it boost the market quality or efficiency? Moreover, HFT is usually believed to be responsible for market manipulation. Can market manipulation exist under the equilibrium conditions at all? What regulations are the most efficient to prevent such manipulation and the use of damaging HFT strategies?

References

Aldridge, I. ETFs, high-frequency trading and flash crashes. *Journal of Portfolio Management*, 43(1):17–28, 2016.

Anufriev, M. and Tuinstra, J. The impact of short-selling constraints on financial market stability in a heterogeneous agents model. *Journal of Economic Dynamics & Control*, 37: 1523–1543, 2013.

Arifovic, J., Chiarella, C., He, X., and Wei, L. High frequency trading and learning. *Working Paper*, 2016.

Baron, M., Brogaard, J., and Kirilenko, A. The trading profits of high frequency traders. *Working paper*, 2012.

Bauwens, L. and Giot, P. *Econometric Modelling of Stock Market Intraday Activity*, volume 38. Kluwer Academic Publishers, Boston, 2001.

Bellia, M., Pelizzon, L., Subrahmanyam, M., Uno, J., and Yuferova, D. Coming early to the party. *SAFE Working Paper*, No.182, 2017.

Bernales, A. Algorithmic and high frequency trading in dynamic limit order markets. *Working Paper*, 2014.

Biais, B., Foucault, T., and Moinas, S. Equilibrium high frequency trading. *International Conference of the French Finance Association (AFFI)*, 2011.

Biais, B., Declerck, F., and Moinas, S. Who supplies liquidity, how and when? *BIS Working Paper*, No.563, 2016.

Black, F. Noise. *The Journal of Finance*, 41(3):529–543, 1986.

Boco, H., Germain, L., and Rousseau, F. High frequency trading: Strategic competition between slow and fast traders. *Working Paper*, 2017.

Brandimarte, P. *Handbook in Monte Carlo Simulation. Applications in Financial Engineering, Risk Management, and Economics.* John Wiley & Sons, Hoboken, New Jersey, 2014.

Brewer, P., Cvitanic, J., and Plott, C. Market microstructure design and flash crashes: A simulation approach. *Journal of Applied Economics*, 16(2):223–250, 2013.

Brogaard, J. High frequency trading and its impact on market quality. *Working Paper*, 2010.

Brogaard, J., Hendershott, T., and Riordan, R. High frequency trading and price discovery. *European Central Bank, Working Paper Series*, No.1602, 2013.

Brogaard, J., Hendershott, T., and Riordan, R. High frequency trading and the 2008 short-sale ban. *Journal of Financial Economics*, 124(1):22–42, 2017.

Börsengesetz vom 16. Juli 2007. *(BGBl. I S. 1330, 1351).* (Zuletzt geändert durch Artikel 8 des Gesetzes vom 23. Juni 2017 (BGBl. I S. 1693)), 2007.

Carrion, A. Very fast money: High-frequency trading on the NASDAQ. *Journal of Financial Markets*, 16:680–711, 2013.

Cervone, R., Galavotti, S., and LiCalzi, M. Symmetric equilibria in double auctions with markdown buyers and markup sellers. In Hernández Iglesias, C., Posada, M., and López-Paredes, A., editors, *Artificial Economics*, volume 631, pages 81–92. Springer, Dordrecht and New York, 2009.

CFA Institute. High frequency trading. Investor issues and perspectives. *Policy Brief*, 2014.

Chaboud, A., Chiquoine, B., Hjalmarsson, E., and Vega, C. Rise of the machines: Algorithmic trading in the foreign exchange market. *The Journal of Finance*, 69(5):2045–2084, 2014.

Chen, Z. and Swan, P. Liquidity asset pricing model in a segmented equity market. In Lhabitant, F. and Gregoriou, G., editors, *Stock Market Liquidity: Implications for Market Microstructure and Asset Pricing*, pages 417–444. John Wiley & Sons, Inc., Hoboken, New Jersey, 2008.

Chiarella, C. and Iori, G. A simulation analysis of the microstructure of double auction markets. *Quantitative Finance*, 2:346–353, 2002.

Chordia, T., Roll, R., and Subrahmanyam, A. Commonality in liquidity. *Journal of Financial Economics*, 56(1):3–28, 2000.

Chordia, T., Roll, R., and Subrahmanyam, A. Order imbalance, liquidity, and market returns. *Journal of Financial Economics*, 64(1):111–130, 2002.

Ciallella, G. Chapter 6 - Describing and regulating high-frequency trading: A European perspective. In Gregoriou, G., editor, *The Handbook of High Frequency Trading*, pages 95–110. Academic Press, 2015.

Clapham, B., Haferkorn, M., and Zimmermann, K. Does speed matter? The role of high-frequency trading for order book resiliency. *Paris December 2015 Finance Meeting EUROFIDAI - AFFI*, 2017.

Clark, C. How to keep markets safe in the era of high-speed trading. *Chicago Fed Letter*, No.303, 2012.

Colliard, J. and Foucault, T. Trading fees and efficiency in limit order markets. *The Review of Financial Studies*, 25(11):3389–3421, 2012.

Cvitanic, J. and Kirilenko, A. High frequency traders and asset prices. *Working Paper*, 2010.

Deb, S., Kalev, P., and Marisetty, V. Are price limits really bad for equity markets? *Journal of Banking & Finance*, 34:2462–2471, 2010.

Easley, D., Lopez de Prado, M., and O'Hara, M. The microstructure of the 'flash crash': Flow toxicity, liquidity crashes and the probability of informed trading. *The Journal of Portfolio Management*, 37(2):118–128, 2011.

Easton, S., Pinder, S., and Uylangco, K. A case study of short-sale constraints and limits to arbitrage. *Journal of Banking & Finance*, 37:3924–3929, 2013.

Foucault, T. Order flow composition and trading costs in a dynamic limit order market. *Journal of Financial Markets*, 2:99–134, 1999.

Foucault, T., Hombert, J., and Rosu, I. News trading and speed. *Journal of Finance*, 71(1): 335–382, 2016.

Gerig, A. High-frequency trading synchronizes prices in financial markets. *DERA Working Paper Series*, 2012.

German High Frequency Trading Act. Gesetz zur Vermeidung von Gefahren und Missbräuchen im Hochfrequenzhandel (Hochfrequenzhandelsgesetz), 2013.

Gilli, M. and Winker, P. A global optimization heuristic for estimating agent based models. *Computational Statistics & Data Analysis*, 42:299–312, 2003.

Gilli, M., Maringer, D., and Schumann, E. *Numerical Methods and Optimization in Finance*. Elsevier, Amsterdam, 2011.

Gode, D. and Sunder, S. Double auction dynamics: Structural effects of non-binding price controls. *Journal of Economic Dynamics & Control*, 28:1707–1731, 2004.

Gomber, P., Arndt, B., Lutat, M., and Uhle, T. High-frequency trading. *Policy Platform, House of Finance, Goethe-Universität Frankfurt*, 2011.

Grinstead, C. and Snell, J. *Introduction to Probability, Second Edition.* American Mathematical Society, Providence, RI, 2003.

Groth, S. Does algorithmic trading increase volatility? Empirical evidence from the fully-electronic trading platform Xetra. *Wirtschaftsinformatik Proceedings*, 112, 2011.

Grundy, B., Lim, B., and Verwijmeren, P. Do option markets undo restrictions on short sales? Evidence from the 2008 short-sale ban. *Journal of Financial Economics*, 106:331–348, 2012.

Harris, L. *Trading and Exchanges: Market Microstructure for Practitioners*. Oxford University Press, Oxford and New York, 2003.

Hasbrouck, J. and Saar, G. Low-latency trading. *Journal of Financial Markets*, 16:646–679, 2013.

Hendershott, T. and Moulton, P. Automation, speed, and stock market quality: The NYSE's hybrid. *Journal of Financial Markets*, 14(4):568–604, 2011.

Hendershott, T. and Riordan, R. Algorithmic trading and information. *Working Papers from NET Institute*, No.09-08, 2009.

Hendershott, T. and Riordan, R. Algorithmic trading and the market for liquidity. *Journal of Financial and Quantitative Analysis*, 48(4):1001–1024, 2013.

Hendershott, T., Jones, C., and Menkveld, A. Does algorithmic trading improve liquidity? *Journal of Finance*, 66(1):1–33, 2011.

Hoffmann, A., Jager, W., and Von Eije, J. Social simulation of stock markets: Taking it to the next level. *Journal of Artificial Societies and Social Simulation*, 10(2), 2007.

Hoffmann, P. A dynamic limit order market with fast and slow traders. *Journal of Financial Economics*, 113(1):156–169, 2014.

Huber, J. 'J'-shaped returns to timing advantage in access to information - experimental evidence and a tentative explanation. *Journal of Economic Dynamics & Control*, 31:2536–2572, 2007.

Huber, J. and Kirchler, M. When more information can be harmful: Evidence from experimental asset markets. In *Information, Interaction and (In)Efficiency in Financial Markets*, pages 62–94. Linde, Wien, 2008.

Huh, Y. Machines vs. machines: High frequency trading and hard information. *FEDS Working Paper*, No.2014-33, 2014.

Izumi, K., Toriumi, F., and Matsui, H. Evaluation of automated-trading strategies using an artificial market. *Neurocomputing*, 72:3469–3476, 2009.

Jones, C. What do we know about high-frequency trading? *Columbia Business School Research Paper*, No.13-11, 2013.

Jovanovic, B. and Menkveld, A. Middlemen in limit order markets. *Working Paper*, 2016.

Kalimullina, L. From analytical markets to artificial stock markets: A new type of agent-based models. *Working paper*, 2017.

Kalimullina, L. and Schöbel, R. The J-curve and transaction taxes: Insights from an artificial stock market. In Amblard, F., Miguel, F., Blanchet, A., and Gaudou, B., editors, *Advances in Artificial Economics. Lecture Notes in Economics and Mathematical Systems, vol. 676*, pages 91–103. Springer International Publishing, 2015.

Kirchler, M., Huber, J., and Stöckl, T. Thar she bursts: Reducing confusion reduces bubbles. *The American Economic Review*, 102(2):865–883, 2012.

Kirilenko, A., Kyle, A., Samadi, M., and Tuzun, T. The flash crash: High-frequency trading in an electronic market. *The Journal of Finance*, 72(3):967–998, 2017.

Kleijnen, J., Ridder, A., and Rubinstein, R. Variance reduction techniques in Monte Carlo methods. *CentEr Discussion Paper*, No.2010-117, 2010.

Law, A. *Simulation Modeling and Analysis, Fifth Edition.* McGraw-Hill Education, New York, 2015.

Lawrenz, J. Understanding non-monotonic payoffs for heterogeneously informed agents. In Hanke, M. and Huber, J., editors, *Information, Interaction and (In)Efficiency in Financial Markets*, pages 115–134. Linde, Wien, 2008.

Leal, S. and Napoletano, M. Market stability vs. market resilience: Regulatory policies experiments in an agent-based model with low- and high-frequency trading. *Journal of Economic Behavior & Organization*, Corrected Proof, 2017.

Leal, S., Napoletano, M., Roventini, A., and Fagiolo, G. Rock around the clock: An agent-based model of low- and high-frequency trading. *Journal of Evolutionary Economics*, 26(1): 49–76, 2016.

LeBaron, B., Arthur, B., and Palmer, R. Time series properties of an artificial stock market. *Journal of Economic Dynamics & Control*, 23:1487–1516, 1999.

Levy, M., Levy, H., and Solomon, S. *The Microscopic Simulation of Financial Markets: From Investor Behavior to Market Phenomena.* Academic Press, San Diego, 2000.

Ma, H. and Leung, H. *Bidding Strategies in Agent-Based Continuous Double Auctions.* Birkhäuser, Basel and Boston, 2008.

Maskawa, J. Correlation of coming limit price with order book in stock markets. *Physica A*, 383:90–95, 2007.

Maskin, E. and Tirole, J. Markov perfect equilibrium: I. Observable actions. *Journal of Economic Theory*, 100:191–219, 2001.

Menkveld, A. High frequency trading and the new market makers. *Journal of Financial Markets*, 16:712–740, 2013.

Menkveld, A. The economics of high-frequency trading: Taking stock. *Annual Review of Financial Economics*, 8:1–24, 2016.

Moosa, I. and Ramiah, V. Chapter 2 - The profitability of high-frequency trading: Is it for real? In Gregoriou, G., editor, *The Handbook of High Frequency Trading*, pages 2–45. Academic Press, 2015.

Muscarella, C. and Piwowar, M. Market microstructure and securities values: Evidence from the Paris Bourse. *Journal of Financial Markets*, 4(3):209–229, 2001.

Pasquale, F. Law's acceleration of finance: Redefining the problem of high-frequency trading. *36 Cardozo Law Review 2085, University of Maryland Legal Studies Research Paper*, No.2015-29, 2015.

Payton, M., Greenstone, M., and Schenker, N. Overlapping confidence intervals or standard error intervals: What do they mean in terms of statistical significance? *Journal of Insect Science*, 3(1):34, 2003.

Posada, M. and Hernández, C. The effect of transaction costs on artificial continuous double auction markets. In Li Calzi, M., Milone, L., and Pellizzari, P., editors, *Progress in Artificial Economics*, volume 645, pages 65–74. Springer-Verlag, Berlin and Heidelberg, 2010.

Raberto, M., Cincotti, S., Dose, C., Focardi, S., and Marchesi, M. Price formation in an artificial market: Limit order book versus matching of supply and demand. In Lux, T., Reitz, S., and Samanidou, E., editors, *Nonlinear Dynamics and Heterogeneous Interacting Agents*, pages 305–315. Springer Science & Business Media, Berlin, 2005.

Regulation (EU) No 596/2014 on Market Abuse. *The European Parliament and the Council.* 2014.

Riordan, R. and Storkenmaier, A. Latency, liquidity and price discovery. *Journal of Financial Markets*, 15:416–437, 2012.

Rojcek, J. and Ziegler, A. High-frequency trading in limit order markets: Equilibrium impact and regulation. *Swiss Finance Institute Research Paper*, No.15-23, 2016.

Schredelseker, K. Unequally distributed information and stock market theory. Wuppertal, 1980.

Schredelseker, K. Anlagestrategie und Informationsnutzen am Aktienmarkt. *Schmalenbachs Zeitschrift für betriebswirtschaftliche Forschung*, 36:44–59, 1984.

SEC. *Part III Securities and Exchange Commission. 17CFR Part 242 Concept Release on Equity Market Structure, Proposed Rule.* Federal Register 75(13), 2010a.

SEC. *Speech by SEC Chairman: Remarks Before the Security Traders Association, Sept. 22, 2010.* U.S. Securities and Exchange Commission, 2010b.

SEC. *Equity Market Structure Literature Review. Part II: High Frequency Trading.* U.S. Securities and Exchange Commission, 2014.

Sloan, B. and Unwin, A. Common random numbers in multivariate simulations. *European Journal of Operational Research*, 48:252–259, 1990.

Subrahmanyam, K. and Zheng, H. Limit order placement by high-frequency traders. *Borsa Istanbul Review*, 16(4):185–209, 2016.

Tóth, B. and Scalas, E. The value of information in financial markets: An agent-based simulation. In Hanke, M. and Huber, J., editors, *Information, Interaction and (In)Efficiency in Financial Markets*, pages 95–114. Linde, Wien, 2008.

Vuorenmaa, T. and Wang, L. An agent-based model of the flash crash of May 6, 2010, with policy implications. *Valo Research and Trading Working Paper*, 2014.

Yeh, C. and Yang, C. Examining the effectiveness of price limits in an artificial stock market. *Journal of Economic Dynamics & Control*, 34:2089–2108, 2010.

Yim, K., Oh, G., and Kim, S. The effect of fast traders in continuous double auction market. *Working paper*, 2015.

Zhang, F. High-frequency trading, stock volatility, and price discovery. *Working paper*, 2010.