PROOF THEORY

Proceedings of the Workshop held at
Unilog'2018 in Vichy, 25 June 2018

Edited by

Thomas Piecha
Peter Schroeder-Heister

University of Tübingen
2018

Contents

Preface

The workshop on proof theory took place in Vichy at the *Pôle Universitaire de Vichy* on 25 June 2018. It was part of Unilog'2018, the 6th World Congress and School on Universal Logic. The proceedings collect abstracts, slides and papers of the presentations given.

The workshop and its proceedings were supported by the French-German ANR-DFG project "Beyond Logic: Hypothetical Reasoning in Philosophy of Science, Informatics, and Law", DFG grant Schr 275/17-1. We would like to thank the organizers of Unilog'2018 for their support.

Thomas Piecha
Peter Schroeder-Heister

Programme

8.30–8.45	Peter Schroeder-Heister *Presentation of the Workshop*
8.45–9.30	Francesca Poggiolesi *Grounding as Meta-linguistic Relation*
9.30–10.00	Ulf Hlobil *Extensions of Non-Monotonic and Non-Transitive Atomic Bases*
10.00–10.30	Dorota Leszczyńska-Jasion and Szymon Chlebowski *Distributive Deductive Systems: the Case of the First-Order Logic*
11.00–11.30	Gerard R. Renardel de Lavalette *The Mathematics of Derivability*
11.30–12.00	René Gazzari *The Existence of Pure Proofs*
12.00–12.30	Lutz Straßburger *From Syntactic Proofs to Combinatorial Proofs*
14.15–15.00	Alexander Leitsch *CERES: Automated Deduction in Proof Theory*
15.00–15.30	Enrico Moriconi *Remarks on the Sequent Calculus*
15.30–16.00	Michael Arndt *Tomographs for Substructural Display Logic*

Abstracts

Grounding as Meta-linguistic Relation

Francesca Poggiolesi
Université Paris 1 Panthéon-Sorbonne, CNRS, ENS
UMR 8590 IHPST – Institut d'Histoire et de Philosophie des Sciences et des Techniques

The concept of *grounding* has a long and venerable history that starts with Aristotle and continue through philosophers such as Ockham or Bolzano. Quite recently we assist to an impressively flourishing and increasing interest for the notion of grounding, which is studied and analyzed from many different angles. Amongst them, scholars have been trying to capture the structural and formal properties of the concept in question by proposing several logics of grounding (see e.g. [1–3] and [6]). In these logics grounding is formalized either as an operator or as a predicate. The main aim of this talk is to present a different approach to the logic of grounding, which stems from some deep Bolzaian intuitions and where grounding is formalized as a meta-linguistic relation (see [4, 5]), just like the notion of derivability or that of logical consequence. Let me call such an approach **LG**. The central characteristics of **LG** can be resumed in the following list:

– **LG** allows a rigorous account of ground-theoretic equivalence.
– In **LG** grounding rules are unique; in particular it is possible to formulate an unique grounding rule for negation.
– In **LG** it is also possible to formulate grounding rules for implication which are quite different from everything that has been proposed so far and that seem to better reflect our intuitions on the issue.
– Finally **LG** allows to prove important results such as the soundness and completeness theorems, but also the deduction theorem.

References

[1] Correia, F.: Logical Grounds, *Review of Symbolic Logic* 7, 2014, pp. 31–59.
[2] Fine, K.: The pure logic of ground, *Review of Symbolic Logic* 25, 2012, pp. 1–25.
[3] Korbmacher, J.: Axiomatic Theories of Partial Grounds, *Journal of Philosophical Logic* 35, 2017, pp. 1–31.
[4] Poggiolesi, F.: On defining the notion of complete and immediate formal grounding, *Synthese* 193, 2016, pp. 3147–3167.
[5] Poggiolesi, F.: On constructing a logic for the notion of complete and immediate formal grounding, *Synthese* 195, 2018, pp. 1231–1254.
[6] Schnieder, B.: A logic for 'Because', *Review of Symbolic Logic* 4, 2011, pp. 445–465.

Extensions of Non-Monotonic and Non-Transitive Atomic Bases

Ulf Hlobil
Department of Philosophy, Concordia University, Montreal, Canada

The paper presents a proof-theoretic approach to nonmonotonic logics that are in line with inferentialism and logical expressivism [1, 2]. According to logical expressivism, it is the characteristic job of logical vocabulary to make explicit inferential relations among atomic sentences. According to semantic inferentialism, the meanings of atomic sentences are determined by such atomic consequence relations. The paper presents nonmonotonic, nontransitive logics that embody both of these philosophical ideas. These logics extend atomic bases, i.e. consequences relations over atomic sentences, by introducing logical vocabulary.

I provide sequent calculi for such non-monotonic, non-transitive base extensions. Tweaking G3cp and using reflexive bases, e.g., gives us supra-classical base extensions. Since the rules are invertible, the connectives have a straightforward expressivist interpretation. I show how a similar supra-intuitionistic sequent calculus can be turned into a natural deduction system.

Along the way, I compare and contrast my expressivist base extensions with extant work on atomic systems. In contrast to the systems presented here, most work on extensions of atomic bases focuses on monotonic and transitive bases [4]. Piecha and Schroeder-Heister [3] have noted that a certain kind of monotonicity isn't plausible if we take atomic bases to be meaning determining. I go further and argue we should also reject transitivity (multiplicative Cut) and so-called "definitional reflection."

References

[1] Brandom, R. B. (2008). *Between Saying and Doing: Towards an Analytic Pragmatism*. Oxford University Press, Oxford.

[2] Peregrin, J. (2014). *Inferentialism: Why Rules Matter*. Palgrave Macmillan, New York.

[3] Piecha, T. and Schroeder-Heister, P. (2016). *Atomic Systems in Proof-Theoretic Semantics: Two Approaches*. In Redmond, J., Pombo Martins, O., and Nepomuceno Fernández, Á., editors, *Epistemology, Knowledge and the Impact of Interaction*, pp. 47–62. Springer, Cham.

[4] Sandqvist, T. (2015). Base-Extension Semantics for Intuitionistic Sentential Logic. *Logic Journal of the IGPL* 23(5), 719–731.

Distributive Deductive Systems: the Case of the First-Order Logic

Dorota Leszczyńska-Jasion and Szymon Chlebowski
Department of Logic and Cognitive Science, Adam Mickiewicz University, Poznań, Poland

The objective of our research is a modern reflection on the notion of proof and on the effectiveness of its construction. In the project we take advantage of the fact that various proof systems can generate the same or closely related solutions for the same problem (a formula) with various complexity (understood both in terms of the time complexity and in terms of the size of the derivation tree). Therefore it seems like a lot can be achieved in the field of complexity of proof-search by dividing the initial problem into "subproblems" and assigning to each subproblem a "proof module" which is computationally optimal for the given subproblem.

A distributive deductive system (DDS, for short) for a given logic L consists of two layers: the *module-layer* of proof systems and the *meta-layer*. Proof systems of the first layer are understood as sets of rules enriched with procedures and/or heuristics of their use. The rules act on finite sequences of sequents. Each such proof system – called a *module* – simulates a proof method (or proof methods) to the effect of computational characteristics of the method. For example,

– Module A stores rules acting on finite sequences of left-sided sequents. The rules simulate the method of analytic tableaux in the original ac- count and system KE with the rule of cut.

– Module D enhances the method of resolution. The rules act upon finite sequences of right-sided reversed sequents.

– Module E stores rules acting on finite sequences of right-sided sequents, the rules are of synthesizing character.

Hence the different modules of a DDS represent (and characterize) a rich collection of various proof methods. The task of the meta-layer is to *distribute parts of a derivation among different modules*. Consequently, the meta-layer will *distribute the computational costs of conducting a derivation among the modules*. More specifically, the meta-layer analyses the input data (such as a single formula) using simple functions, such as the length of a formula, the number of distinct variables in a formula, but also the pattern of connectives nested in the scope of other connectives; then, taking into account the procedures and/or heuristics available, the meta-layer chooses the form of a sequent used for the input and hence also the module (modules) from the module-layer that will be used at the start. In case of big inputs, the obtained sequents may be analysed by the tools of the meta-layer many times. For example, the initial input is analysed to a collection of sequents and the meta-layer indicates that while part of the sequents can be efficiently treated with the rules of analytic tableaux (module A), for the other part it is more convenient to decide its inconsistency with resolution system module D.

The aim of our talk is to present the idea of distributive deductive systems and the results obtained so far for the case of the First-Order Logic.

The Mathematics of Derivability

Gerard R. Renardel de Lavalette
Faculty of Science and Engineering, University of Groningen, The Netherlands

Traditionally, the notion of derivability (or provability) in proof theory is defined in terms of derivations: sequences or tree-like structures consisting of formulae or sequents, satisfying certain conditions involving proof rules. The 'driving force' of derivations usually consists of conditional statements: implications in the object language ($\varphi \to \psi$), entailments in the metalanguage ($\varphi, \psi \vdash \varphi \wedge \psi$), or proof rules involving sequents (if $\Gamma \vdash \varphi$ and $\Gamma, \psi \vdash \chi$ then $\Gamma, \varphi \to \psi \vdash \chi$).

I propose an alternative definition of derivability, capitalizing on the dynamic character of conditional statements. It is based on set-valued functions $\mathcal{F}\colon \wp(\mathsf{EXP}) \to \wp(\mathsf{EXP})$, where EXP denotes a collection of expressions, with the intended meaning: *for all* $E \subseteq \mathsf{EXP}$, E *entails the expressions in* $\mathcal{F}(E)$. So when EXP is a collection of atomic formulae, then $\mathcal{F}$ represents the Horn sentence

$$\bigwedge_{\Gamma \subseteq \mathsf{EXP}} \bigwedge_{\varphi \in \mathcal{F}(\Gamma)} \left(\bigwedge \Gamma \to \varphi\right).$$

When EXP is a collection of formulae of some logical language, then $\mathcal{F}$ represents the collection of sequents $\Gamma \vdash \varphi$ for all $\Gamma \subseteq \mathsf{EXP}$ and all $\varphi \in \mathcal{F}(\Gamma)$. And when EXP is a collection of sequents, then $\mathcal{F}$ represents the proof rule *from* $\mathcal{S}$ *infer* $\Gamma \vdash \varphi$, for all collections of sequents

$$\mathcal{S} = \{\Gamma_i \vdash \varphi_i \mid i \in I\} \subseteq \mathsf{EXP}$$

and all sequents $\Gamma \vdash \varphi$ in $\mathcal{F}(\mathcal{S})$.

In [1], I experimented with this idea in the context of propositional Horn logic. This led to several results on uniform and polynomial interpolation. Along the way, a characterization of validity was established: $\mathcal{F} \models \mathcal{G}$ iff $\mathcal{G} \sqsubseteq \mathcal{F}^*$, i.e. $\mathcal{G}(P) \subseteq \mathcal{F}^*(P)$ for all sets P of atoms. In other words: (the Horn formula represented by) $\mathcal{F}$ entails (the Horn formula represented by) $\mathcal{G}$ if and only if $\mathcal{G}$ is contained in the reflexive transitive closure $\mathcal{F}^*$ of $\mathcal{F}$. Moreover, it appeared that the set-valued functions form a *weak lazy Kleene algebra* (a notion inspired by [2]), governed by axioms like:

- $(\mathcal{F} \sqcup \mathcal{G}) \circ \mathcal{H} = (\mathcal{F} \circ \mathcal{H}) \sqcup (\mathcal{G} \circ \mathcal{H})$,
- $\mathcal{I} \sqcup \mathcal{F} \circ \mathcal{F}^* \sqsubseteq \mathcal{F}^*$,
- if $\mathcal{F} \circ \mathcal{G} \sqsubseteq \mathcal{G}$, then $\mathcal{F}^* \circ \mathcal{G} \sqsubseteq \mathcal{G}$.

Here $\mathcal{I}$ is the identity function, and $\sqcup$ is defined by

$$(\mathcal{F} \sqcup \mathcal{G})(P) = \mathcal{F}(P) \cup \mathcal{G}(P).$$

The left distributive version

$$\mathcal{F} \circ (\mathcal{G} \sqcup \mathcal{H}) = (\mathcal{F} \circ \mathcal{G}) \sqcup (\mathcal{F} \circ \mathcal{H})$$

of the first axiom does not hold, and neither do the variants

$$\mathcal{I} \sqcup \mathcal{F}^* \circ \mathcal{F} \sqsubseteq \mathcal{F}^*$$

and

$$\textit{if } \mathcal{F} \circ \mathcal{G} \sqsubseteq \mathcal{F}, \textit{ then } \mathcal{F} \circ \mathcal{G}^* \sqsubseteq \mathcal{F}$$

of the second and third axiom.

In the paper abstracted here, the notions and results sketched above are extended to full Horn logic, where the atomic formulae contain terms and variables and where all formulae have implicit universal quantification at the outermost level for all occurring variables. For this purpose, the theory of set-valued functions is extended with substitutions $\sigma\colon \mathsf{EXP} \to \mathsf{EXP}$. The characterization of validity now reads

$$\mathcal{F} \vDash \mathcal{G} \text{ iff } \mathcal{G} \sqsubseteq \Big(\bigsqcup_{\sigma \in \mathsf{SUB}} (\sigma \cdot \mathcal{F}) \Big)^*,$$

where SUB denotes the set of all substitutions and where $\sigma \cdot \mathcal{F}$ is defined by

$$(\sigma \cdot \mathcal{F})(X) = \{\sigma(\varphi) \mid \exists Y \subseteq \mathsf{EXP}(X = \{\sigma(\psi) \mid \psi \in Y\} \,\&\, \varphi \in \mathcal{F}(Y))\}.$$

With the proper establishment of notions and results for the combination of set-valued functions with substitutions, we can scale up to the investigation of proof systems for algebraic theories and propositional logics, involving sequents. The next step to first-order logic requires another extension to deal with variable binders (like quantifiers). All in all, it is my goal to substantiate the claim that *set-valued functions are a core ingredient for the proper mathematical analysis of derivability*.

References

[1] Gerard R. Renardel de Lavalette: Interpolation in propositional Horn logic, *Journal of Logic and Computation*, doi:10.1093/logcom/exx042.

[2] Bernhard Möller: Kleene getting lazy, *Science of Computer Programming* 65(2), 2007, pp. 195–214.

The Existence of Pure Proofs

René Gazzari
Department of Computer Science, University of Tübingen, Germany

Topic of our talk is the notion of pure proofs from a proof theoretical point of view. In a first step, we explain how to deal with this informal philosophical notion in a formal way. We identify formal counterparts to the relevant philosophical concepts and notions and provide a formal definition of pure proofs, this means a definition of pure derivations (in the calculus of Natural Deduction).

The main goal of our talk is to show that every derivation can be transformed into a pure derivation, namely into a derivation satisfying the following condition: every non-logical symbol (the counterparts of mathematical notions) occurring in the derivation already occur in an essential assumption or in the conclusion of this derivation.

Partial results are easily obtained via well-known results: it is a technical lemma that we may replace unnecessary constant symbols by variables. Pureness with respect to relation symbols is a consequence of the existence of the Prawitz normal form and of the subformula property. The crucial aspect is the treatment of function symbols: to prove the existence of a pure derivation, we have to replace some (only the unnecessary) occurrences of terms in a derivation by variables, and to show that the resulting derivation satisfies our demands.

In the course of our argumentation, we overcome some technical difficulties: we introduce a formal notion of occurrences of terms in a derivation. We identify congruent occurrences of terms in a derivation, namely those occurrences which have to be of the same shape due to the inference rules according to which the derivation under discussion is generated. Finally, we show under which conditions such congruent occurrences can be replaced by variables (or other suitable terms). Applying this substitution theorem to derivations in Prawitz normal form, we obtain pure derivations. Our result also sheds light on the problem of the identity of proofs, another philosophically relevant problem of proof theory. When transforming a derivation into its pure version, we do not change its normal form, but an essential property of this derivation. This seems to be a good reason to reconsider, whether we should identify derivations having the same normal form.

Syntactic Proofs versus Combinatorial Proofs

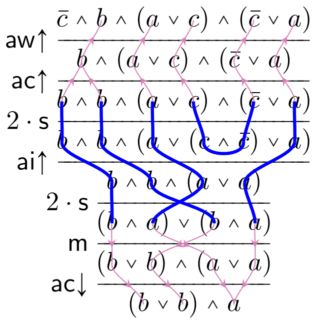
Lutz Straßburger
Inria Saclay – Île-de-France, Campus de l'École Polytechnique, Palaiseau, France

Proof theory is a central area of theoretical computer science, as it can provide the foundations not only for logic programming and functional programming, but also for the formal verification of software. Yet, despite the crucial role played by formal proofs, we have no proper notion of proof identity telling us when two proofs are "the same". This is very different from other areas of mathematics, like group theory, where two groups are "the same" if they are isomorphic, or topology, where two spaces are "the same" if they are homeomorphic.

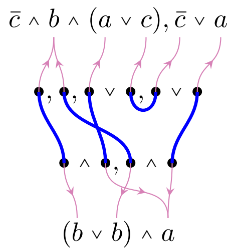
The problem is that proofs are usually presented by syntactic means, and depending on the chosen syntactic formalism, "the same" proof can look very different. In fact, one can say that at the current state of art, *proof theory is not a theory of proofs but a theory of proof systems*. This means that the first step must be to find ways to describe proofs independent from the proof systems. In other words, we need a "syntax-free" presentation of proofs.

Combinatorial proofs form such a canonical proof presentation that (1) comes with a polynomial correctness criterion, (2) is independent of the syntax of proof formalisms (like sequent calculi, tableaux systems, resolution, Frege systems, or deep inference systems), and (3) can handle cut and substitution, and their elimination. Below is an example showing how a combinatorial proof can be extracted from a deep inference derivation:

$$\begin{array}{rc}
 & \bar{c} \wedge b \wedge (a \vee c) \wedge (\bar{c} \vee a) \\ \hline
\mathsf{aw}\uparrow & b \wedge (a \vee c) \wedge (\bar{c} \vee a) \\ \hline
\mathsf{ac}\uparrow & b \wedge b \wedge (a \vee c) \wedge (\bar{c} \vee a) \\ \hline
2 \cdot \mathsf{s} & b \wedge b \wedge (a \vee (c \wedge \bar{c}) \vee a) \\ \hline
\mathsf{ai}\uparrow & b \wedge b \wedge (a \vee a) \\ \hline
2 \cdot \mathsf{s} & (b \wedge a) \vee (b \wedge a) \\ \hline
\mathsf{m} & (b \vee b) \wedge (a \vee a) \\ \hline
\mathsf{ac}\downarrow & (b \vee b) \wedge a
\end{array}
\quad \rightarrow$$



$\rightarrow$



In a nutshell, a combinatorial proof consists of a purely *linear* part (depicted above in blue/bold) and a part that corresponds to *contraction* and *weakening* (depicted above in purple/regular). Combinatorial proofs can be composed horizontally and vertically, and can be substituted into each other.

In this presentation, I will discuss the basic definition of combinatorial proofs, how they can be normalized, and how we can transform syntactic proofs into combinatorial proofs and back.

REFERENCES

[1] Dominic Hughes. Proofs Without Syntax. Annals of Mathematics, 164(3): 1065–1076, 2006.

[2] Dominic Hughes. Towards Hilbert's 24th problem: Combinatorial proof invariants: (preliminary version). Electr. Notes Theor. Comput. Sci., 165:37–63, 2006.

[3] Lutz Straßburger. Combinatorial Flows and Proof Compression. Research Report RR-9048, Inria Saclay, 2017. URL: https://hal.inria.fr/hal-01498468

[4] Lutz Straßburger. Combinatorial Flows and Their Normalisation. In FSCD 2017 (LIPIcs), Dale Miller (Ed.), Vol. 84. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 31:1–31:17.

CERES: Automated Deduction in Proof Theory

Alexander Leitsch
Institute of Logic and Computation, Vienna University of Technology, Austria

CERES (cut-elimination by resolution; see [1]) is a method of cut-elimination which strongly differs from cut-elimination a la Gentzen. Instead of reducing a proof φ stepwise (and thereby simplifying the cuts) CERES computes a formula $\mathrm{CL}(\varphi)$ represented as so-called *characteristic clause set*. $\mathrm{CL}(\varphi)$ encodes the structure of the derivations of cuts in φ and is always unsatisfiable. In classical logic any resolution refutation ρ of $\mathrm{CL}(\varphi)$ can be taken as a skeleton of a *CERES normal form* φ^* of φ (in φ^* all cuts are atomic). CERES was mainly designed as a computational tool for proof analysis and for performing cut-elimination in long and complex proofs; an implementation of the method was successfully applied to Fürstenberg's proof of the infinitude of primes [2].

There is, however, also an interesting theoretical aspect of the CERES method: reductive cut-elimination based on the rules of Gentzen can be shown to be "redundant" with respect to CERES in the following sense: if φ reduces to φ' then $\mathrm{CL}(\varphi)$ *subsumes* $\mathrm{CL}(\varphi')$ (subsumption is a principle of redundancy-elimination in automated deduction). This redundancy property can be used to prove that reductive methods (of a specific type) can never outperform CERES. Moreover, subsumption also plays a major role in proving the completeness of intuitionistic CERES (CERES-i) [3]. CERES-i is based on the concept of proof resolution, a generalization of clausal resolution to resolution of cut-free proofs. The completeness of CERES-i can then be proven via a subsumption property for cut-free proofs and a subsumption property for proof projections under reductive cut-elimination. The results demonstrate that principles invented in the area of automated deduction can be fruitfully applied to proof theory.

References

[1] Baaz, M. and Leitsch, A.: Cut-elimination and redundancy-elimination by resolution, *Journal of Symbolic Computation* 29, 2000, pp. 149–176.

[2] Baaz, M., Hetzl, S., Leitsch, A., Richter, C., and Spohr, H.: CERES: an analysis of Fürstenberg's proof of the infinity of primes, *Theoretical Computer Science* 403, 2008, pp. 160–175.

[3] Cerna, D., Leitsch, A., Reis, G., and Wolfsteiner, S.: Ceres in intuitionistic logic, *Annals of Pure and Applied Logic* 168, 2017, pp. 1783–1836.

Remarks on the Sequent Calculus

Enrico Moriconi

Dipartimento di Civiltà e Forme del Sapere, Università di Pisa

In the last section V of his thesis, after the proof of the Hauptsatz, Gentzen proved the equivalence between the main three types of formalization of the logical inference: the Hilbert-Ackermann system (H.A.), the Natural Deduction Calculus (N.D.), and the Sequent Calculus (S.C.). In this proof we can see, so to say, the birth of the same formalism of S.C., which is maybe the most important formalization of logical deduction ever provided. Also the handwritten version of the thesis, let's say *Ms. ULS*, contains a similar proof of equivalence, as we have learnt from the important researches made by Jan von Plato on the newly found Gentzen's texts. Admittedly, the last section of the thesis is normally rated "less important" than the other sections, but nonetheless it casts some important light on the emergence of the S.C., and more generally on some structural features of Gentzen's work. In the Thesis the equivalence proof proceeds through the following sequence of steps: i) a proof that every derivation within the H.A.-axiomatization can be transformed in an equivalent derivation of N.D.-calculus; ii) a proof that every N.D.-derivation can be transformed into an equivalent S.C.-derivation; iii) a proof that every S.C.-derivation can be transformed into an equivalent H.A.-derivation. The proof is conducted first for Intuitionistic logic and afterward for Classical logic. In this way, of course, the goal to prove the equivalence of all three calculi is accomplished. However, the main single component showing the origin of S.C. is *the translation of derivations built within N.D.-formalism into derivations built within the axiomatic logical calculus of Hilbert and Ackermann's book*. And it is interesting to note that in the pertinent part of *Ms. ULS* Gentzen provided a proof of the equivalence between N.D.-calculi and the H.A.-formalism by showing the possibility to translate every (classical) N.D.-derivation into an equivalent H.A.-derivation; in this way it is explicitly supplied a missing link which is only implicitly present, as a by-product of previous steps i)-iii), in the published version of the thesis. Gentzen proceeded as follows: given an N.D.-proof of, say A, one first lists all those assumptions which are not already discharged before the accomplishing of the inference leading to A. Let us indicate them by Γ. Then one substitutes A by $\Gamma \to A$. If A is an assumption, $A \to A$ takes its place. The steps of inference of N.D. are accordingly translated:

$$\frac{A \quad B}{A\&B}\,\mathrm{I\&} \qquad \rightsquigarrow \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A\&B}$$

Paired with the occurrence of the figure of *sequent*, here we see, probably for the first time, the disentangling of two meanings often conflated in the notion of implication: the propositional (object-language) connective, say $\supset$, and the (meta-level) notation for the formal derivability relation, say $\to$. Of course, in this step Gentzen was greatly helped by his work on Hertz-systems from the summer of 1931, which output his first published paper of 1932.

Beside trying to retrace the intricate threads leading to the proof of the equivalence, I mean to focus on the emergence of two paradigms in the conception of Cut. The paradigm of *structural reasoning*, which was preserved in the intermediate calculus ***LDK*** of *Ms. ULS*, where the Cut rule continues to play a fundamental role, and the *analytic* paradigm. In the latter paradigm analytic proofs were the new goal, and Gentzen was able to attain it thanks to the *Hauptsatz* proved for that "evolution" of ***LDK***-calculi which is constituted by the ***LK***-calculi. In the latter calculi, structural reasoning was sharply separated from logical meaning, and the general setting was purely inferential.

Tomographs for Substructural Display Logic

Michael Arndt
Department of Computer Science, University of Tübingen, Germany

The central feature of Belnap's Display Logic [1] is the possibility of displaying every formula occurring in any given sequent as the only formula in either the antecedent or succedent. This is accomplished by means of structural connectives that retain the positional information of the contextual formulae as they are moved aside. Goré accommodates substructural, intuitionistic and dual intuitionistic logic families by building upon a basic display calculus for Bi-Lambek logic. His version uses two nullary, two unary, and three binary structural connectives. Since the structural connectives are not independent of one another, display equivalences are required to mediate between the binary structural connectives.

I propose an alternative approach in which two graph-like ternary structural connectives express one set of three structural connectives each. Each of these new connectives represents all three sequents making up one of the two display equivalences. The notion of sequent disappears and is replaced by that of a structure tomograph consisting of systems of ternary connectives in which nodes mark the linking of the connectives and of formulae to those connectives. The turnstile of a sequent is represented by the highlighting of a single node linking connectives.

References

[1] Goré, R.: Substructural logics on display, *Logic Journal of the IGPL* 6(3), 1998, pp. 451–504.

Presentations

Extensions of Non-Monotonic and Non-Transitive Atomic Bases

Université Concordia University

Ulf Hlobil

ulf.hlobil@concordia.ca

Here is the plan:

Inferentialist Logical Expressivism

Brandom ([1], p. 30)

Logic is not properly understood as the study of a distinctive kind of formal inference. It is rather the study of inferential roles of vocabulary playing a distinctive expressive role: codifying in explicit form the inferences that are implicit in the use of ordinary, non-logical vocabulary. [...] The task of logic is in the first instance to help us say something about conceptual contents expressed by the use of nonlogical content.

- Overall goal: Developing this program by providing a way of doing formal logic that is in the spirit of Logical Expressivism. Here: bits and pieces of that.
- The consequence relation over atoms (base) determines the meanings of the atoms. We then introduce logical connectives by giving rules that specify their role in inferences.
- Previous attempts have failed to capture the nonmonotonicity of bases that Brandom claims is essential (see [2, 5]).

Contrast with Prawitz

Prawitz ([7], pp. 73–73)

If we accept this notion of a valid argument [in terms of reducibility] relative to some choice $\mathscr{C}$ of valid closed arguments for atomic sentences [i.e. consequence relation over atoms or bases], it seems reasonable to define a sentence B as a logical consequence of sentences $A_1, A_2, \ldots, A_n$ by [...]. This means again according to our definition that B is a logical consequence of $A_1, A_2, \ldots, A_n$ if and only if there exists an argument for B from $A_1, A_2, \ldots, A_n$ that is valid relative to any such choice $\mathscr{C}$.

- We are not interested in logical consequence but in consequence relative to particular bases (choices $\mathscr{C}$). Hence, we don't quantify over bases.
- We want to study how logical vocabulary makes explicit what follows from what according to a particular base within an enrichment or extension of that very base to cover logical vocabulary.

Goals

1. Rehearse (and slightly update) the sequent calculus method for extending non-monotonic, non-transitive bases to logically complex languages, which we developed elsewhere (see [3]).
2. Compare and contrast our ideas with some ideas that are prominent in work on atomic systems and proof-theoretic semantics.
4. Translate nonmonotonic, nontransitive sequent calculus into natural deduction system and includes all intuitionistically valid arguments.

Atomic Bases

We begin with an atomic system:

$\mathfrak{L}_0$: atomic language, $\{p_1, ..., p_n\}$.

$\mid\!\sim_0$: base consequence relation, $\mid\!\sim_0 \subset \mathcal{P}(\mathfrak{L}_0) \times \mathfrak{L}_0$ or $\mid\!\sim_0 \subset \mathcal{P}(\mathfrak{L}_0) \times \mathcal{P}(\mathfrak{L}_0)$.

- Permutation and contraction hold (sets). Usually also CO: $\Gamma \mid\!\sim_0 \Delta$ if $\emptyset \neq \Gamma \cap \Delta \subseteq \mathfrak{L}_0$. Otherwise: liberty.
- Similar to Schroeder-Heister, Piecha [6], Sandqvist [8] but:
  - No higher-order rules,
  - Don't look at supersets of bases,
  - No definitional reflection,
  - No transitivity,
  - No monotonicity.

Contrast with Other Atomic Systems

In a nutshell:

- Definitional reflection, transitivity, and monotonicity impose constraints on what one can mean by atomic sentences. Such constraints strike us as artificial and unjustified.
- Higher-order rules and appeal to larger atomic bases bring in complications whose mastery is not necessary to mean anything by an atomic sentence. Interesting things can be done with these notions in place. But we are interested in meaning and consequence in more basic cases.

Contrast with Other Atomic Systems

Why no Weakening?

- Meaning determining reasoning is defeasible (see Brandom).

Why no Cut?

- Cut and CO ($\Gamma, A \mid\!\sim A$) jointly imply MO. For, suppose $\Gamma \mid\!\sim A$. By CO, $\Gamma, A, B \mid\!\sim A$. By Cut, $\Gamma, B \mid\!\sim A$. The argument works even against share-context Cut if we have a conditional for which $\Gamma \mid\!\sim A \to B$ iff $\Gamma, A \mid\!\sim B$. Alternatively, we could reject CO. This would yield a system closer to relevance logic, which we presented elsewhere [4].

Why no higher-order rules?

- People must be able to reason with a sentence in order to mean anything by it. But they need not be able to assume and discharge rules to meaning anything by something.

Contrast with Other Atomic Systems

Why no Definitional Reflection?

- DR: If $\alpha \Leftarrow B_1$, ..., $\alpha \Leftarrow B_n$ (and that's all for a), then

$$\frac{\Gamma \mid\!\sim B_i}{\Gamma \mid\!\sim \alpha}\,\mathrm{Ra} \qquad \frac{\Gamma, B_1 \mid\!\sim C \quad \ldots \quad \Gamma, B_n \mid\!\sim C}{\Gamma, \alpha \mid\!\sim C}\,\mathrm{La}$$

- Everyday reasoning counter-example:
  - Definition of "looking like a zebra": it is a zebra $\Rightarrow$ it looks like a zebra (under standard observation conditions).
  - Also: it is a zebra $\mid\!\sim$ it isn't a painted donkey.
  - But: it looks like a zebra (under standard observation conditions) $\not\mid\!\sim$ it isn't a painted donkey.
- Advocates of Definitional Reflection must say that the definition is defective or partial (it doesn't take into account that sometime people want to trick us). We find that implausible and overly restrictive.

Contrast with Other Atomic Systems

Why no Definitional Reflection? (cont.)

- We suspect that what makes Definitional Reflection look plausible are intuitions regarding transitivity: "Suppose that $B_1, \ldots, B_n$ and only those (for atomic rules) give me α. Now suppose I have α. Surely, that means that I must have (in effect, in some covert way) some B_i. Suppose that, whichever B_i I have, it gives me C. So I get C if I assume that I got α (by transitivity)."
- These intuitions trade on the fact that the inversion of La together with the definitional clauses is an instance of Cut.

$$\frac{B_1 \mid\!\sim \alpha \qquad \alpha \mid\!\sim C}{B_1 \mid\!\sim C}\,\mathrm{La}$$

Contrast with Other Atomic Systems

Why no Definitional Reflection? (cont.)

- To rid ourselves of this bad intuition, we notice that (intuitively) La is not invertible: Definition: You are Bavarian $\Rightarrow$ you are German. You are from Berlin $\Rightarrow$ you are German. Etc. Also: You are German $\mid\!\sim$ you like Merkel's refugee policy. But: You are Bavarian $\not\mid\!\sim$ you like Merkel's refugee policy.
- We want to get more out of the conclusion of an atomic rule than what is already contained in its premises. Adding new conceptual resources can make genuinely new conclusion available.
- Schroeder-Heister [9] has shown that Definitional Reflection doesn't imply Cut. But we fail to see what motivates Definitional Reflection after we rejected Cut.

G3cp Plus Negation-Rules (Ketonen): NM-MS

Initial sequents: all those in $\mid\!\sim_0$.

$$\frac{\Gamma \mid\!\sim \Delta, A \qquad B, \Gamma \mid\!\sim \Delta}{\Gamma, A \to B \mid\!\sim \Delta}\text{LC} \qquad \frac{A, \Gamma \mid\!\sim \Delta, B}{\Gamma \mid\!\sim \Delta, A \to B}\text{RC}$$

$$\frac{\Gamma, A, B \mid\!\sim \Delta}{\Gamma, A\&B \mid\!\sim \Delta}\text{L\&} \qquad \frac{\Gamma \mid\!\sim \Delta, A \qquad \Gamma \mid\!\sim \Delta, B}{\Gamma \mid\!\sim \Delta, A\&B}\text{R\&}$$

$$\frac{A, \Gamma \mid\!\sim \Delta \qquad B, \Gamma \mid\!\sim \Delta}{A \vee B, \Gamma \mid\!\sim \Delta}\text{Lv} \qquad \frac{\Gamma \mid\!\sim \Delta, A, B}{\Gamma \mid\!\sim \Delta, A \vee B}\text{Rv}$$

$$\frac{\Gamma \mid\!\sim A, \Delta}{\Gamma, \neg A \mid\!\sim \Delta}\text{LN} \qquad \frac{\Gamma, A \mid\!\sim \Delta}{\Gamma \mid\!\sim \neg A, \Delta}\text{RN}$$

Properties of NM-MS

- **Supra-Classical**: If $\mid\!\sim_0$ obeys CO, we get all classical sequents.
- **Nice connectives** (invertible rules): e.g. $\Gamma, A\&B \mid\!\sim \Delta \;\Leftrightarrow\; \Gamma, A, B \mid\!\sim \Delta$ and $A \vee B, \Gamma \mid\!\sim \Delta \;\Leftrightarrow\; A, \Gamma \mid\!\sim \Delta$ and $B, \Gamma \mid\!\sim \Delta$.

Expressive?

- Conditionals (as conclusions) express that the consequent follows from the antecedent (in context): $\Gamma \mid\!\sim A \to B$ iff $\Gamma, A \mid\!\sim B$.
- Negations (as conclusions) express that the negatum is incompatible with the premises: $\Gamma \mid\!\sim \neg A$ iff $\Gamma, A \mid\!\sim$.
- **In general**: we can apply the rules backwards to reach a unique set of base sequents that all hold iff the original sequent holds. Thus, logical vocabulary expresses something about material inferences.

Single-Succedent: MN-SS

$$\frac{\Gamma, A \mid\!\sim B}{\Gamma \mid\!\sim A \to B}\,\text{CP} \qquad \frac{\Gamma, A \mid\!\sim}{\Gamma \mid\!\sim \neg A}\,\text{RN} \qquad \frac{\Gamma \mid\!\sim A}{\Gamma, \neg A \mid\!\sim}\,\text{LN}$$

$$\frac{\Gamma \mid\!\sim {}^{\uparrow}A \qquad \Gamma, B \mid\!\sim C}{\Gamma, A \to B \mid\!\sim C}\,\text{LC} \qquad \frac{\Gamma, A, B \mid\!\sim C}{\Gamma, A\&B \mid\!\sim C}\,\text{L\&} \qquad \frac{\Gamma \mid\!\sim A \qquad \Gamma \mid\!\sim B}{\Gamma \mid\!\sim A\&B}\,\text{R\&}$$

$$\frac{\Gamma, A \mid\!\sim C \qquad \Gamma, B \mid\!\sim C}{\Gamma, A \vee B \mid\!\sim C}\,\text{Lv} \qquad \frac{\Gamma \mid\!\sim A}{\Gamma \mid\!\sim A \vee B}\,\text{Rv} \qquad \frac{\Gamma \mid\!\sim B}{\Gamma \mid\!\sim A \vee B}\,\text{Rv}$$

$$\frac{\Gamma \mid\!\sim {}^{\uparrow}A}{\Gamma, B \to C \mid\!\sim {}^{[\uparrow]}A}\,\text{CK} \qquad \frac{\Gamma \mid\!\sim {}^{\uparrow}A}{\Gamma, \neg B \mid\!\sim {}^{[\uparrow]}A}\,\text{NK}$$

$$\frac{\Gamma \mid\!\sim {}^{\uparrow}}{\Gamma \mid\!\sim {}^{[\uparrow]}A}\,\text{ExFF} \qquad \frac{\Gamma \mid\!\sim {}^{\uparrow X}A \qquad \Gamma \mid\!\sim {}^{\uparrow Y}A}{\Gamma \mid\!\sim {}^{\uparrow X \cup Y}A}\,\text{UN} \qquad \frac{\Gamma, p_1 \ldots p_n \mid\!\sim A}{\Gamma \mid\!\sim {}^{\uparrow\{\{p_1 \ldots p_n\}\}}A}\,\text{PushUp}$$

Properties of NM-SS

The adjustments in red are needed for various reasons and the upward-arrow allows us to control weakening in a nice way (see [3]).

Some remarks:

- Supra-Intuitionist: If $\mid\!\sim_0$ obeys CO, we get all intuitionistic sequents.
- We still have: $\Gamma \mid\!\sim A \to B$ iff $\Gamma, A \mid\!\sim B$.
- Upward arrow: $X \subseteq \mathcal{P}(\mathfrak{L}_0)$, and $\Gamma \mid\!\sim^{\uparrow X} A$ iff $\forall \chi \in X\, (\chi, \Gamma \mid\!\sim A)$. Convention: $\Gamma \mid\!\sim^{\uparrow} A$ iff $\Gamma \mid\!\sim^{\uparrow \mathcal{P}(\mathfrak{L}_0)} A$.
- I show elsewhere how to introduce a $\Box$ such that: $\Gamma \mid\!\sim \Box A$ iff $\forall \chi \subseteq \mathfrak{L}\, (\chi, \Gamma \mid\!\sim A)$.

Formulate NM-SS as Natural Deduction

We now want to formulated NM-SS as a natural deduction system. This will give us a better idea of what it is like to reason according to a consequence relation like that of NM-SS. And many inferentialists like natural deduction systems. Some things to watch out for below:

- We restrict monotonicity in a way similar to that in which natural deduction systems for relevance logics do that: we keep track of assumptions in subscripts.
- We restrict transitivity by making sure that, in effect, major premises "stand proud" (as Tennant) puts it. In this way, we build in "normalizability."
- We use something close to generalized elimination rules.

Formulate NM-SS as Natural Deduction

- By marking a sentence in a proof-tree with subscript X we indicate that the conclusion depends on the assumptions in X. (We are a bit more explicit than relevance logic natural deduction here.)
- We can mark the subscripts with upward-arrows that are labeled by sets of sets of atoms, as in NM-SS.

We understand every base sequent as a rule:

$$\frac{p_1 \quad \cdots \quad p_n}{q_{\ \{p_1,\ldots,p_n\}}}\ \text{atom } p_1, \ldots, p_n \mid\!\sim_0 q$$

Iteration rule:

$$\frac{A_{\ X} \qquad B_{1\ Y} \quad \cdots \quad B_{n\ Z}}{A_{\ \{A, B_1, B_n\}^{\uparrow}}}\ \text{Iter}$$

Natural Deduction Rules

Introduction and Elimination Rules:

$$\frac{B_{\ X\cup\{A\}}}{A \to B_{\ X}}\ \to\text{-Int} \qquad \frac{A \to B_{\ Y} \qquad A_{\ X^{\uparrow}} \qquad C_{\ X\cup\{B\}}}{C_{\ X\cup\{A\to B\}}}\ \to\text{-Elim}$$

$$\frac{\bot_{\ X\cup\{A\}}}{\neg A_{\ X}}\ \neg\text{-Int} \qquad \frac{\neg A_{\ Y} \qquad A_{\ X}}{\bot_{\ X\cup\{\neg A\}}}\ \neg\text{-Elim}$$

$$\frac{A_{\ X} \qquad B_{\ X}}{A\&B_{\ X}}\ \&\text{-Int} \qquad \frac{A\&B_{\ Y} \qquad C_{\ X\cup\{A,B\}}}{C_{\ X\cup\{A\&B\}}}\ \&\text{-Elim}$$

$$\frac{A_{i\ X}}{A_1 \vee A_2{}_{\ X}}\ \vee\text{-Int} \qquad \frac{A \vee B_{\ Y} \qquad C_{\ X\cup\{A\}} \qquad C_{\ X\cup\{B\}}}{C_{\ X\cup\{A\vee B\}}}\ \vee\text{-Elim}$$

Natural Deduction Rules

Structural Rules:

$$\frac{A_{\ X\uparrow} \qquad B \to C_{\ Y}}{A_{\ (X\cup\{B\to C\})^{[\uparrow]}}}\ \text{pCW} \qquad \frac{A_{\ X\uparrow} \qquad \neg B_{\ Y}}{A_{\ (X\cup\{\neg B\})^{[\uparrow]}}}\ \text{pNW}$$

$$\frac{\bot_{\ X\uparrow}}{A_{\ X^{[\uparrow]}}}\ \text{ExFF}$$

$$\frac{A_{\ X\uparrow Y} \qquad A_{\ X\uparrow Z}}{A_{\ X\uparrow Y\cup Z}}\ \text{UN} \qquad \frac{A_{\ X\cup Y_0}}{A_{\ X\uparrow\{Y_0\}}}\ \text{PushUp}$$

Some Remarks on NJ-NM

- It is easy to prove that NJ-NM defines the same consequence relation, relative to a give base, as NM-SS.
- Hence, NJ-NM is supra-intuitionistic and obeys the deduction-detachment theorem.
- The sequent rules are translated into NJ-NM in a straightforward way, which is possible because of the subscripts.
- The subscripts of the major premises in elimination rules drops out of the proof. Hence, we can simply assume the major premises; they "stand proud." That is how "normalizability" and, hence, "Cut-freeness" is insured.

Example 1

Proof of $(A\&B \to C) \to (A \to (B \to C))$, just relying on Iter (aka CO):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\&B \to C}{A\&B \to C_{\ \{A\&B\to C\}}}\,\text{Iter} \qquad \dfrac{\dfrac{A \qquad B}{A_{\ \{A,B\}\uparrow}}\,\text{Iter} \quad \dfrac{A \qquad B}{B_{\ \{A,B\}\uparrow}}\,\text{Iter}}{A\&B_{\ \{A,B\}\uparrow}}\,\&\text{-Int} \qquad \dfrac{A \qquad B \qquad C}{C_{\ \{A,B,C\}}}\,\text{Iter}}{C_{\ \{A,B,A\&B\to C\}}}\,\to\text{-Elim}}{B \to C_{\ \{A,A\&B\to C\}}}\,\to\text{-Int}}{A \to (B \to C)_{\ \{A\&B\to C\}}}\,\to\text{-Int}}{(A\&B \to C) \to (A \to (B \to C))_{\ \emptyset}}\,\to\text{-Int}$$

Example 2

Proof of r from p and $s \vee q$, given a base in which $p, q \mid\!\sim_0 r$, and $p, s \mid\!\sim_0 r$, and $t \mid\!\sim_0 s$:

$$\dfrac{\dfrac{\dfrac{t}{s_{\ \{t\}}}\,\text{atom } t \mid\!\sim_0 s}{s \vee q_{\ \{t\}}}\,\vee\text{-Int} \qquad \dfrac{p \qquad q}{r_{\ \{p,q\}}}\,\text{atom } p,q \mid\!\sim_0 r \qquad \dfrac{p \qquad s}{r_{\ \{p,s\}}}\,\text{atom } p,s \mid\!\sim_0 r}{r_{\ \{p,\, s\vee q\}}}\,\vee\text{-Elim}$$

Notice: The conclusion doesn't depend on t. The stipulation that $t \mid\!\sim_0 s$ is superfluous, we could have used the iteration rule (applied to $s \vee q$) instead. All major premises of Elim-rules can be assumptions ("normalizability" is build in).

Let's Take Stock

- We could introduce our students to intuitionistic logic by teaching them NJ-NM over flat bases (just CO). Adding nonmonotonic, nontransitive material inferential relations among atoms would then seem entirely natural.
- No extra machinery like partial orders over models, default rules, graph theoretic machinery or the like (which is common in contemporary nonmonotonic logic) is needed.
- That montonicity and transitivity of *logical* consequence relations stem from ignoring what is specific to particular bases and assuming that they all obey CO becomes blindingly obvious if we look at intuitionistic and classical logic in this way.
- Logic is what happens when we ignore the fact that atomic sentences have genuine meanings. In order to see what logical vocabulary makes explicit, we shouldn't (always) do that.

Bibliography I

Robert B. Brandom.
Articulating reasons: an introduction to inferentialism.
Harvard University Press, Cambridge, Mass., 2000.

Robert B. Brandom.
Between saying and doing: towards an analytic pragmatism.
Oxford University Press, Oxford, 2008.

Ulf Hlobil.
A nonmonotonic sequent calculus for inferentialist expressivists.
In Pavel Arazim and Michal Danák, editors, *The Logica Yearbook 2015*, pages 87–105. College Publications, 2016.

Ulf Hlobil.
Choosing your nonmonotonic logic: A shopper' guide.
In Pavel Arazim and Tomáš Lávika, editors, *The Logica Yearbook 2017*. London: College Publications, forthcoming.

Bibliography II

Mark Norris Lance and Philip Kremer.
The logical structure of linguistic commitment 1: Four systems of non-relevant commitment entailment.
Journal of Philosophical Logic, 23(4):369–400, 1994.

Thomas Piecha and Peter Schroeder-Heister.
Atomic systems in proof-theoretic semantics: Two approaches.
In Juan Redmond, Olga Pombo Martins, and Ángel Nepomuceno Fernández, editors, *Epistemology, Knowledge and the Impact of Interaction*, pages 47–62. Springer, Cham, 2016.

Dag Prawitz.
On the idea of a general proof theory.
Synthese, 27(1-2):63–77, 1974.

Tor Sandqvist.
Base-extension semantics for intuitionistic sentential logic.
Logic Journal of the IGPL, 23(5):719–731, 2015.

Bibliography III

Peter Schroeder-Heister.
Definitional reflection and basic logic.
Annals of Pure and Applied Logic, 164(4):491 – 501, 2013.
Articles in honor of Giovanni Sambin's 60th birthday.

Distributive Deductive Systems: the Case of the First-Order Logic

Dorota Leszczyńska-Jasion, Szymon Chlebowski
Department of Logic and Cognitive Science
Adam Mickiewicz University, Poznań, Poland

June 25, 2018

6^{th} World Congress and School on Universal Logic
Vichy, France, June 16-26, 2018

Our work was financially supported by National Science Centre, Poland, grant no 2017/26/E/HS1/00127.

Overview

Description of the project

- **Distributive Deductive Systems for Classical and Non-classical Logics. Proof theory supported with computational methods**
- Objectives (1): proof-theoretical description and implementation (int.al., functional programming language) of complex deductive systems containing various proof systems as modules.
- Objectives (2): massive generation of problems (formulas, proofs, derivations), statistical analysis of the results, data analysis and data mining, computational experiments.
- The results of analyses / experiments will be used – if possible – to improve the structure and computational complexity of the first version of DDSs.

Description of the project

The systems are called ***distributive deductive systems***, as they contain functions which allow to distribute costs of a derivation among various modules.

- there are well-known classes of formulas that cause exponential, factorial (or even worse) growth of computational complexity on the grounds of a particular deductive system

$$S_n = \bigwedge(\pm p_1 \vee \pm p_2 \vee \ldots \vee \pm p_n)$$

where $\pm p_i \in \{p_i, \neg p_i\}$

the Pigeonhole Principle:

$$PHP_n := \bigwedge_{i=0}^{n} \bigvee_{j=1}^{n} p_{ij} \to \bigvee_{j=1}^{n} \bigvee_{i \neq k} p_{ij} \wedge p_{kj}$$

Description of the project

G. Boolos, “Don’t eliminate cut”, 1984, consider scheme H_n:

$$(x)(y)(z) + x + yz = ++xyz$$

$$(x)d(x) = +xx$$

$$L(1)$$

$$(x)(L(x) \to L(x+1))$$

with the conclusion:

$$L\left(\underbrace{d\dots d}_{2^n}(1)\right)$$

Boolos: the shortest tree-method proof has more than 2^{2^n} characters, whereas the shortest ND-proof of the conclusion of H_n from its premises contains less than $16(2^n + 8n + 21)$ characters (which is $\mathcal{O}(2^n)$).

Description of the project

- at the same time, usually the problematic “hard” formulas have “nice” proofs / refutations in some other systems
- if the task is to decide validity / entailment, then we should be able to match the given problem with a proof system

Modules of DDS for FOL

A distributive deductive system for a given logic L:

- the **module-layer** of proof systems
- the **meta-layer**

Modules of DDS for FOL

A distributive deductive system for a given logic L:

- the **module-layer** of proof systems
  Proof system is a set of rules + the notion of proof + possibly procedures / heuristics.

  Each such proof system – called a *module* – simulates a proof method (or proof methods) to the effect of computational characteristics of the method (p-simulation).
- the **meta-layer**
  The meta-layer contains meta-heuristics (small functions) which help to choose the module.

Table: the modules of the module-layer of a DDS

symbol	the proof methods characterized	types of sequents used in the module
A	**analytic tableaux** system KE	left-sided canonical sequents
B	**sequent calculus** semantic diagrams (Beth) natural deduction axiomatic system	both-sided (multi-conclusion) sequents
C	**R-S system** CNF	right-sided canonical sequents
D	**resolution**, **dual resolution** Davis-Putnam method	one-sided reversed sequents
E	**synthetic tableaux** truth-tables	right-sided canonical sequents with synthesizing rules

Module B

$$\begin{array}{c} S'\alpha'T \Rightarrow U \\ | \\ S'\alpha_1'\alpha_2'T \Rightarrow U \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{S \Rightarrow T'\alpha'U} \\ \swarrow & \searrow \\ S \Rightarrow T'\alpha_1'U & S \Rightarrow T'\alpha_2'U \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{S'\beta'T \Rightarrow U} \\ \swarrow & \searrow \\ S'\beta_1'T \Rightarrow U & S'\beta_2'T \Rightarrow U \end{array}$$

$$\begin{array}{c} S \Rightarrow T'\beta'U \\ | \\ S \Rightarrow T'\beta_1'\beta_2'U \end{array} \qquad \begin{array}{c} S'\kappa'T \Rightarrow U \\ | \\ S'\kappa *' T \Rightarrow U \end{array} \qquad \begin{array}{c} S \Rightarrow T'\kappa'U \\ | \\ S \Rightarrow T'\kappa *' U \end{array}$$

$$\begin{array}{c} S'\gamma'T \Rightarrow U \\ | \\ S'\gamma'\gamma(t_j)'T \Rightarrow U \end{array} \qquad \begin{array}{c} S \Rightarrow T'\delta'U \\ | \\ S \Rightarrow T'\delta'\delta(t_j)'U \end{array} \qquad \begin{array}{c} S'\delta'T \Rightarrow U \\ | \\ S'\delta(\tau_j)'T \Rightarrow U \end{array}$$

$$\begin{array}{c} S \Rightarrow T'\gamma'U \\ | \\ S \Rightarrow T'\gamma(\tau_j)'U \end{array}$$

Module D

$$\begin{array}{c} \Phi;\Leftarrow S'\beta' T;\Psi \\ | \\ \Phi;\Leftarrow S'\beta_1' T;\Leftarrow S'\beta_2' T;\Psi \end{array} \qquad \begin{array}{c} \Phi;\Leftarrow S'\alpha' T;\Psi \\ | \\ \Phi;\Leftarrow S'\alpha_1'\alpha_2' T;\Psi \end{array}$$

$$\begin{array}{c} \Phi;\Leftarrow S'\kappa' T;\Psi \\ | \\ \Phi;\Leftarrow S'\kappa*' T;\Psi \end{array} \qquad \begin{array}{c} \Phi;\Leftarrow S'\gamma' T;\Psi \\ | \\ \Phi;\Leftarrow S'\gamma(\tau_j)' T;\Psi \end{array}$$

$$\begin{array}{c} \Phi\,;\Leftarrow S\,'\,\delta\,'\,T\,;\Psi \\ | \\ \Phi\,;\Leftarrow S\,'\,\delta\,'\,T\,;\Leftarrow S\,'\,\delta(t_j)\,'\,T\,;\Psi \end{array}$$

$$\begin{array}{c} \Phi\,;\Leftarrow S\,'\,A\,'\,T\,;\Psi\,;\Leftarrow U\,'\,\overline{A}\,'\,V\,;\Omega \\ | \\ \Leftarrow\ \mathrm{rep}(S^{A}\,'\,T^{A}\,'\,U^{\overline{A}}\,'\,V^{\overline{A}})\,;\Phi\,;\Psi\,;\Omega\,;\Leftarrow S\,'\,A\,'\,T\,;\Leftarrow U\,'\,\overline{A}\,'\,V \end{array}$$

Module E

$$\begin{array}{ccc} & \diagup\ \diagdown & \\ p_i & & \neg p_i \end{array}$$

where p_i occurs in the formula to be derived

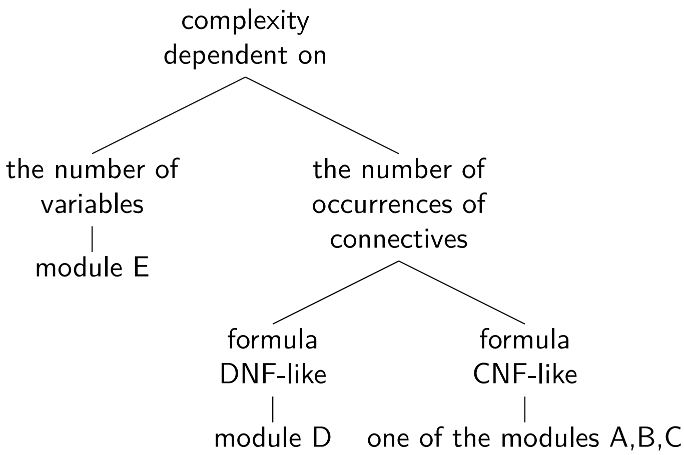
$$\frac{\neg B}{B \to C}\ \mathbf{r}^{\mathbf{1}}_{\to} \qquad \frac{C}{B \to C}\ \mathbf{r}^{\mathbf{2}}_{\to} \qquad \frac{\begin{array}{c} B \\ \neg C \end{array}}{\neg(B \to C)}\ \mathbf{r}^{\mathbf{3}}_{\to}$$

$$\frac{B}{B \vee C}\ \mathbf{r}^{\mathbf{1}}_{\vee} \qquad \frac{C}{B \vee C}\ \mathbf{r}^{\mathbf{2}}_{\vee} \qquad \frac{\begin{array}{c} \neg B \\ \neg C \end{array}}{\neg(B \vee C)}\ \mathbf{r}^{\mathbf{3}}_{\vee}$$

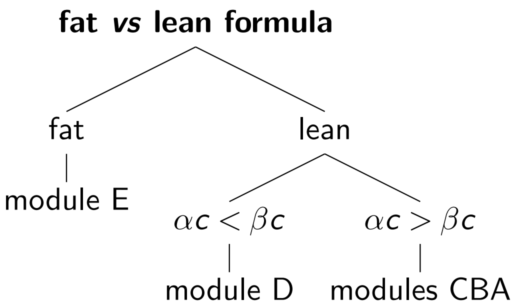
$$\frac{\neg B}{\neg(B \wedge C)}\ \mathbf{r}^{\mathbf{1}}_{\wedge} \qquad \frac{\neg C}{\neg(B \wedge C)}\ \mathbf{r}^{\mathbf{2}}_{\wedge} \qquad \frac{\begin{array}{c} B \\ C \end{array}}{B \wedge C}\ \mathbf{r}^{\mathbf{3}}_{\wedge} \qquad \frac{B}{\neg\neg B}\ \mathbf{r}_{\neg}$$

where the premises of $\mathbf{r}^{\mathbf{3}}_{\to}$, $\mathbf{r}^{\mathbf{3}}_{\vee}$, $\mathbf{r}^{\mathbf{3}}_{\wedge}$ may occur in any order. A synthesizing rule may be applied in the construction of a synthetic tableau for formula A provided that each premise *and conclusion* of the rule belongs to the set $Sub(A) \cup \neg Sub(A)$

How to use it: decision tree

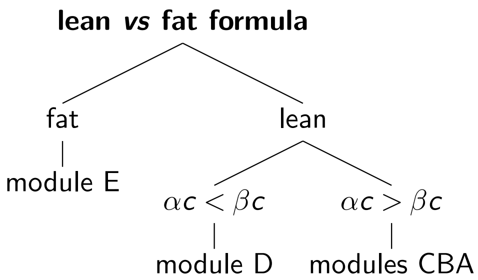


How to use it: small functions



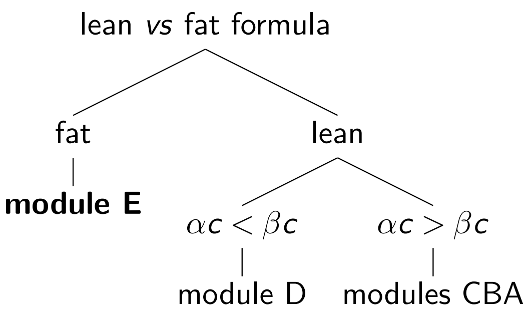
small = good complexity ($\mathcal{O}(n^2)$)

How to use it: small functions



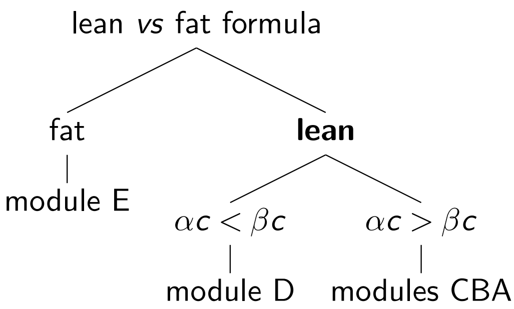
$$FMI = \frac{\texttt{no. of d. var.}}{\texttt{no. of occ. of var.}}$$

How to use it: small functions



- if $dp(\neg\neg A) = dp(A)$
- if $dp(A \wedge B) = r$, then $dp(A) = dp(B) = \frac{r}{2}$
- if $dp(\neg(A \vee B)) = r$, then $dp(\neg A) = dp(\neg B) = \frac{r}{2}$
- if $dp(\neg(A \rightarrow B)) = r$, then $dp(A) = dp(\neg B) = \frac{r}{2}$
- if $dp(A \vee B) = r$, then $dp(A) = dp(B) = r$
- if $dp(\neg(A \wedge B)) = r$, then $dp(\neg A) = dp(\neg B) = r$
- if $dp(A \rightarrow B) = r$, then $dp(\neg A) = dp(B) = r$

How to use it: small functions



- $\alpha c(p_i) = \alpha c(\neg p_i) = 1$ / $\beta c(p_i) = \beta c(\neg p_i) = 1$
- $\alpha c(\neg\neg A) = \alpha c(A)$ / $\beta c(\neg\neg A) = \beta c(A)$
- $\alpha c(\alpha) = \alpha c(\alpha_1) + \alpha c(\alpha_2)$ / $\beta c(\alpha) = \beta c(\alpha_1) \cdot \beta c(\alpha_2)$
- $\alpha c(\beta) = \alpha c(\beta_1) \cdot \alpha c(\beta_2)$ / $\beta c(\beta) = \beta c(\beta_1) + \beta c(\beta_2)$

Case study: formula S_n

$$S_n = \bigwedge(\pm p_1 \vee \pm p_2 \vee \ldots \vee \pm p_n)$$

$$FMI(S_n) = 2^{-n}$$

S_n is possibly a representative example of a fat formula?
The maximal (canonical) synthetic tableau for S_n has 2^n branches.

$$\alpha c(S_n) = \sum_{i=1}^{n} \alpha c(\pm p_1 \vee \ldots \vee \pm p_n) = n$$

$$\beta c(S_n) = \prod_{i=1}^{n} \beta c(\pm p_1 \vee \ldots \vee \pm p_n) = n^n$$

$$\alpha c(S_n) = n < n! < n^n = \beta c(S_n)$$

Case study: formula PHP_n

$$PHP_n := \bigwedge_{i=0}^{n} \bigvee_{j=1}^{n} p_{ij} \rightarrow \bigvee_{j=1}^{n} \bigvee_{i \neq k} p_{ij} \wedge p_{kj}$$

$$FMI(PHP_n) = \frac{1}{2n+1}$$

$$\alpha c(PHP_n) = \alpha c(\bigwedge_{i=0}^{n} \bigvee_{j=1}^{n} p_{ij}) \cdot \alpha c(\bigvee_{j=1}^{n} \bigvee_{i \neq k} p_{ij} \wedge p_{kj}) =$$

$$= \left(\sum_{i=0}^{n} \prod_{j=1}^{n} \alpha c(p_{ij}) \right) \cdot \left(\prod_{j=1}^{n} \prod_{i \neq k,=0}^{n} \alpha c(p_{ij} \wedge p_{kj}) \right) = (n+1) \cdot 2^{n^2(n+1)}$$

$$\beta c(PHP_n) = n^{n+1} + n^2(n+1)$$

$$\beta c(PHP_n) < \alpha c(PHP_n)$$

Case studies

1. PHP_n: the goal is a further analysis of the example,
2. scheme H_n (Boolos' example): the goal is to define the decision functions for quantifier formulas.

Implementation of DDS
Aims

- The general construction of the implementation is the same as that in the theoretical description.
- In writing the main interface (the meta-layer of a DDS) Python programming language will be used.
- The implementation of the module-layer of a DDS will be written in a functional programming language Haskell.

Implementation of DDS
Aims

- A large number of proofs for a great amount of formulas will be generated and analysed with the use of the implementation of proof-system modules and the interface controlling these modules.
- These proofs and their descriptions will constitute a data set which can be further processed using traditional statistical techniques and more sophisticated machine learning algorithms.
- The techniques will be used to obtain new knowledge about proof systems that can be hard to acquire in the traditional, analytic way. This does not mean that we wish to resign from the traditional approach—we rather tend to extend the repertoire of classical methods by modern computational tools.

Implementation of DDS

Aims

- On the basis of the results gained from computational experiments new decision-functions may be added to the meta-layer (the functions that are used to decide which module to choose).
- Also criteria of evaluation of the derivation trees will be developed at this step. These may be criteria such as: the length and size of a tree, the number of applications of rules together with their costs, the number of different modules used to solve the initial problem, the number of times the various rules of cut (or other strategic components) are used.
- The primary role of the computational methods is supplementary with respect to the theoretical part. To sum up, the computational methods will be used to generate large number of proofs (test cases) effortlessly, to describe automatically the generated structures, and to apply quantitative analyses and knowledge exploration techniques.

Implementation of the module layer of DDS

Aims

- Complete control over implementation—execution order of expressions and deductive machinery used.
- One structure to represent different proof formats (rose trees labelled by some kind of hypersequent structure).
- Construct a domain specific language.

Implementation of the module layer of DDS

Haskell

- The module layer will be implemented in Haskell Programming language.
- Mathematical basis of Haskell comprises of typed lambda calculus and some notions from category theory.
- *Powerful type system.* A lot of errors can be detected prior to the execution of the program. Supports polymorphism.
- *Concise Programs.* Programs in Haskell are of high-level nature. It is easy to adapt them for different tasks.
- *List comprehension.* Generally Haskell programs resemble ordinary math notation.

Implementation of the module layer of DDS

Haskell

- *Recursive functions.* In pure functional languages there are no traditional loops (*while,for*). The only way of looping is to use recursion.
- *Pure functions.* In Haskell, the term function is understood in the standard mathematical sense. Support for higher order functions.
- *Lazy evaluation.* No computation should be performed until the result is needed. It is possible to work out with possibly infinite structures.
- *Inductive data types.* One can use recursion, when defining new types.

Implementation of the module layer of DDS
Haskell

- *Algebraic data types.* Type constructors can be combined using the operations of sum and product.
- *Equational reasoning.* Since programs are functions, one can prove their properties by simple equational reasoning.
- *Domain-specific embedded languages.* New types can be used to construct extension of the language.

We think that due to the mentioned features, Haskell will serve well as a language for the implementation of variety of systems in a safe and controlled way.

What has been done?
Theory

- Theoretical description of all modules/proof systems for First-Order Logic.
- Small functions are defined on the propositional level.

What has been done?

Implementation

- Implementation of sequent calculus for CPL.
- Implementation of a hypersequent system for the logic K.
- Implementation of synthetic tableaux on the propositional level.

Thank You for Your attention!

Bibliography

G. Boolos, "Don't Eliminate Cut", *Journal of Philosophical Logic*, **13**:4, Springer 1984, 373–378.

S. Chlebowski, *Canonical and Dual Erotetic Calculi for First-Order Logic*, PhD thesis, Adam Mickiewicz University 2018.

S. Cook, R. Reckhow, "On the lengths of proofs in the propositional calculus", *Proceedings of the 6th Annual Symposium on the Theory of Computing*, 1974, p. 135–148.

M. D'Agostino, *Investigations into the complexity of some propositional calculi. Technical Monograph*, Oxford University Computing Laboratory 1990.

B. Dunham, H. Wang, "Towards feasible solutions of the tautology problem", *Annals of Mathematical Logic*, **10**:2, 1976, 117–154.

M. Komosinski, A. Kups, D. Leszczyńska-Jasion, M. Urbański, "Identifying efficient abductive hypotheses using multi-criteria dominance relation", *ACM Transactions on Computational Logic*, **15**:4, 2014.

Dorota Leszczyńska-Jasion, *From Questions to Proofs. Between the Logic of Questions and Proof Theory*, to appear in 2018.

M. Urbański, 'Remarks on Synthetic Tableaux for Classical Propositional Calculus", *Bulletin of the Section of Logic*, **30**:4, 2001, p. 194–204.

A. Wiśniewski, *Questions, Inferences, and Scenarios*, College Publications, 2013.

The Mathematics of Derivability: An Application in Horn Logic

Gerard R. Renardel de Lavalette*

Abstract

This is a draft paper, based on my presentation *The Mathematics of Derivability* given on June 25, 2018 in the Proof Theory workshop of the conference UNILOG'2018 in Vichy (France).

1 Introduction

Traditionally, the notion of derivability (or: provability) in proof theory is defined in terms of derivations: sequences or tree-like structures consisting of formulae or sequents, satisfying certain conditions involving proof rules. The 'driving force' of derivations usually consists of *conditional statements*: implications in the object language ($\varphi \to \psi$), entailments in the metalanguage ($\varphi, \psi \vdash \varphi \wedge \psi$), or proof rules involving sequents (if $\Gamma \vdash \varphi$ and $\Gamma, \psi \vdash \chi$ then $\Gamma, \varphi \to \psi \vdash \chi$).

I propose an alternative definition of derivability, capitalizing on the dynamic character of conditional statements. It is based on set-valued functions $\mathcal{F} : \wp(\mathsf{EXP}) \to \wp(\mathsf{EXP})$, where EXP denotes a collection of expressions, with the intended meaning: *for all $E \subseteq \mathsf{EXP}$, E entails the expressions in $\mathcal{F}(E)$*. I list some instantiations.

– EXP is a collection of atomic formulae: $\mathcal{F}$ represents the Horn sentence

$$\bigwedge_{\Gamma \subseteq \mathsf{EXP}} \bigwedge_{\varphi \in \mathcal{F}(\Gamma)} \Big(\bigwedge \Gamma \to \varphi\Big).$$

– EXP is a collection of formulae of some logical language: $\mathcal{F}$ represents the collection of sequents $\Gamma \vdash \varphi$ for all $\Gamma \subseteq \mathsf{EXP}$ and all $\varphi \in \mathcal{F}(\Gamma)$.

– EXP is a collection of sequents: $\mathcal{F}$ represents the proof rule *from $\mathcal{S}$ infer* $\Gamma \vdash \varphi$, for all collections of sequents $\mathcal{S} = \{\Gamma_i \vdash \varphi_i \mid i \in I\} \subseteq \mathsf{EXP}$ and all sequents $\Gamma \vdash \varphi$ in $\mathcal{F}(\mathcal{S})$.

In [1], I experimented with this idea in the context of propositional Horn logic. This led to several results on uniform and polynomial interpolation. Along the way, a characterization of validity was established: $\mathcal{F} \vDash \mathcal{G}$ iff $\mathcal{G} \sqsubseteq \mathcal{F}^*$, i.e. $\mathcal{G}(P) \subseteq \mathcal{F}^*(P)$ for all sets P of atoms. In other words: (the Horn formula represented by) $\mathcal{F}$ entails (the Horn formula represented by) $\mathcal{G}$ if and only if $\mathcal{G}$ is contained in the reflexive transitive closure $\mathcal{F}^*$ of $\mathcal{F}$. Moreover, it appeared that the set-valued functions form a *weak lazy Kleene algebra*, a notion inspired by [2].

In the present paper, I present a proof for an Interpolation Theorem in Horn logic, extending the results for propositional Horn logic given in [1]. For this purpose, the notions and results given in [1] are extended to full Horn logic, where the atomic formulae contain terms and variables and where all formulae have implicit universal quantification at the outermost level for all occurring variables. Moreover, the theory of set-valued functions is extended with substitutions.

*Bernoulli Institute of Mathematics, Computer Science and Artificial Intelligence, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands, g.r.renardel.de.lavalette@rug.nl

2 Logical preliminaries

2.1 Signatures, terms, atoms

Let $\mathsf{SIG} = \mathsf{SIG_f} \cup \mathsf{SIG_p}$, where $\mathsf{SIG_f}$ is a collection of function symbols and $\mathsf{SIG_p}$ is a collection of predicate symbols. The elements in SIG are called *signature elements*. Each signature element has its arity, indicating the number of arguments it expects.

Let VAR be an infinite collection of variables with $\#\mathsf{VAR} = \#\mathsf{SIG}$. The collections TERM of terms and ATOM of atomic formulae are defined as usual from SIG and VAR. Observe that $\#\mathsf{ATOM} = \#\mathsf{TERM} = \#\mathsf{VAR}$. We let s, t, u range over terms, φ, ψ, χ over atomic formulae, and A, B, X, Y, Z over sets of atomic formulae.

The function $\mathsf{sig} : \mathsf{TERM} \cup \mathsf{ATOM} \to \wp(\mathsf{SIG})$ gives the collection of signature elements in a term or formula, and is defined as usual. For $X \subseteq \mathsf{TERM} \cup \mathsf{ATOM}$, we write $\mathsf{sig}(X)$ for $\bigcup\{\mathsf{sig}(\varphi) \mid \varphi \in X\}$. We will also use $\mathsf{sig_f}, \mathsf{sig_p}$ to restrict the output of sig: so e.g. $\mathsf{sig_f}(\varphi) = \mathsf{sig}(\varphi) \cap \mathsf{SIG_f}$.

We define $\mathsf{TERM} : \wp(\mathsf{SIG}) \to \mathsf{TERM}$, $\mathsf{Atom} : \wp(\mathsf{TERM}) \times \wp(\mathsf{SIG}) \to \mathsf{ATOM}$ and $\mathsf{AT} : \wp(\mathsf{SIG}) \to \mathsf{ATOM}$ by

$$\begin{aligned}
\mathsf{Term}(S) &= \{t \in \mathsf{TERM} \mid \mathsf{sig}(t) \subseteq S\}\\
\mathsf{Atom}(T, S) &= \{p(t_1, \dots t_n) \mid p \in S \cap \mathsf{SIG_p}, t_1, \dots t_n \in T, n \text{ the arity of } p\}\\
\mathsf{AT}(S) &= \mathsf{Atom}(\mathsf{Term}(S), S)
\end{aligned}$$

So we have

$$\begin{aligned}
\mathsf{Term}(S) &= \{t \in \mathsf{TERM} \mid \mathsf{sig_f}(t) \subseteq S\}\\
\mathsf{AT}(S) &= \{\varphi \in \mathsf{ATOM} \mid \mathsf{sig_p}(\varphi) \subseteq S\}
\end{aligned}$$

We also define the collection $\mathsf{Hterm}(S, T)$ of terms in $T \subseteq \mathsf{TERM}$ with head in $S \subseteq \mathsf{SIG_f}$:

$$\mathsf{Hterm}(S, T) = \{t \in T \mid t = f(t_1, \dots, t_n) \text{ with } f \in S\}$$

2.2 Substitutions

As usual, substitutions $\sigma, \tau, \dots \in \mathsf{SUB}$ are functions $\sigma : \mathsf{VAR} \to \mathsf{TERM}$. We extend substitutions σ to all terms and atomic formulae by defining $\dot\sigma : \mathsf{TERM} \to \mathsf{TERM}$, $\dot\sigma : \mathsf{ATOM} \to \mathsf{ATOM}$ inductively by

$$\begin{aligned}
\dot\sigma(x) &= \sigma(x)\\
\dot\sigma(f(t_1, \dots, t_n)) &= f(\dot\sigma(t_1), \dots, \dot\sigma(t_1))\\
\dot\sigma(p(t_1, \dots, t_n)) &= p(\dot\sigma(t_1), \dots, \dot\sigma(t_1))
\end{aligned}$$

The domain, range and signature of a substitution are defined by

$$\begin{aligned}
\mathsf{dom}(\sigma) &= \{x \in \mathsf{VAR} \mid \sigma(x) \neq x\}\\
\mathsf{rg}(\sigma) &= \{\sigma(x) \mid \sigma(x) \neq x\}\\
\mathsf{sig}(\sigma) &= \mathsf{sig}(\mathsf{rg}(\sigma))
\end{aligned}$$

For $S \subseteq \mathsf{SIG_f}$, we define $\mathsf{SUB}(S) \subseteq \mathsf{SUB}$ as the collection of substitutions which map variables to terms with signature in S:

$$\mathsf{SUB}(S) = (\mathsf{VAR} \to \mathsf{Term}(S))$$

2.3 Horn logic

A *Horn clause* is a formula of the form $\bigwedge X \to \varphi$ with $X \subseteq \mathsf{ATOM}$ and $\varphi \in \mathsf{ATOM}$. A *Horn formula* is a conjunction of Horn clauses. So a Horn formula is of the form

$$\bigwedge_{k \in K} \Big(\bigwedge X_k \to \varphi_k \Big)$$

where K is some index set with $X_k \in \wp(\mathsf{ATOM})$ and $\varphi_k \in \mathsf{ATOM}$ for every $k \in K$. We impose no restrictions on the size of K and the X_k. HORN is the collection of Horn formulae.

2.4 Parameters in Horn formulae

The parameters in a Horn formula are the occurrences of signature elements in it. For the function parameters, we define

$$\mathsf{sig_f}\Big(\bigwedge_{k \in K} \Big(\bigwedge X_k \to \varphi_k \Big) \Big) = \mathsf{sig_f}\Big(\bigcup_{k \in K} X_k \cup \{\varphi_k \mid \kappa \in K\} \Big)$$

For the predicate parameters, we distinguish between positive occurrence (in the φ_k) and negative occurrence (in the X_k). So we define $\mathsf{sig_p}^+, \mathsf{sig_p}^- : \mathsf{HORN} \to \wp(\mathsf{SIG_p})$ by

$$\mathsf{sig_p}^+\Big(\bigwedge_{k \in K} \Big(\bigwedge X_k \to \varphi_k \Big) \Big) = \mathsf{sig_p}(\{\varphi_k \mid k \in K\})$$
$$\mathsf{sig_p}^-\Big(\bigwedge_{k \in K} \Big(\bigwedge X_k \to \varphi_k \Big) \Big) = \mathsf{sig_p}\Big(\bigcup_{k \in K} X_k \Big)$$

We also define $\mathsf{sig_p}^\pm(\Phi) = (\mathsf{sig_p}^-(\Phi), \mathsf{sig_p}^+(\Phi))$, $\mathsf{sig_p}^\mp(\Phi) = (\mathsf{sig_p}^+(\Phi), \mathsf{sig_p}^-(\Phi))$. So $\mathsf{sig_p}^\mp(\Phi)$ in the inverse of $\mathsf{sig_p}^\pm(\Phi)$.

In the sequel, we will extend functions, operations and relations on sets of predicate symbols to pairs of sets. For this purpose, we use the subscript $\cdot_2$, so we write e.g.

$$(A, B) \subseteq_2 \mathsf{AT}_2(S) \cup_2 \mathsf{AT}(\mathsf{sig_p}^\pm(\Phi))$$

where $S = (S^-, S^+)$, as an abbreviation of

$$A \subseteq \mathsf{AT}(S^-) \cup \mathsf{AT}(\mathsf{sig_p}^-(\Phi)) \text{ and } B \subseteq \mathsf{AT}(S^+) \cup \mathsf{AT}(\mathsf{sig_p}^+(\Phi))$$

2.5 Validity and entailment

A *model* is a pair $M = \langle U, i, j \rangle$ with

$U \neq \emptyset$,
$i(f) : U^n \to U$ for every n-ary function symbol $f \in \mathsf{SIG_f}$,
$j(p) \subseteq U^n$ for every n-ary predicate symbol $p \in \mathsf{SIG_p}$.

An M-*valuation* is a function $\alpha \in \mathsf{VAL}(M)$, where $\mathsf{VAL}(M) = \mathsf{VAR} \to U_M$. We shall in general abbreviate $\forall \alpha \in \mathsf{VAL}(M)$ to $\forall \alpha$ whenever M is evident from the context. The interpretation $[\![t]\!]^M_\alpha$ (also denoted $[\![t]\!]^i_\alpha$) of term t in M under valuation α is defined as usual, as is the interpretation $M, \alpha \vDash \varphi$ of the validity of propositional formulae $\varphi = p(\vec{t})$. For $X \subseteq \mathsf{ATOM}$ we define: $M, \alpha \vDash X$ iff $\forall \varphi \in X \; M, \alpha \vDash \varphi$. The interpretation of the universal closure $\overline{\forall}\varphi$ of a propositional formula φ is defined by

$$M \vDash \overline{\forall}\varphi \quad \text{iff} \quad \forall \alpha (M, \alpha \vDash \varphi)$$

So for the universal interpretation of Horn formula $\eta = \bigwedge_{k\in K}(\bigwedge X_k \to \varphi_k)$ we have

$$M \models \overline{\forall}\eta \quad \text{iff} \quad \forall\alpha\forall k \in K(M,\alpha \models X_k \implies M,\alpha \models \varphi_k)$$

Let $\mathcal{M} = \langle U, i, j\rangle$ be a model with $\alpha \in \mathsf{VAL}(M)$. We define the *diagram* $\mathsf{Diag}(M,\alpha)$ of M and α by

$$\mathsf{Diag}(M,\alpha) = \{\varphi \in \mathsf{ATOM} \mid M,\alpha \models \varphi\}(= \{p(\vec{t}) \in \mathsf{ATOM} \mid [\![\vec{t}]\!]^i_\alpha \in j(p)\})$$

Given $\sigma \in \mathsf{SUB}$, we define $\alpha^M_\sigma : \mathsf{VAR} \to U$ by

$$\alpha^M_\sigma(x) = [\![\sigma(x)]\!]^M_\alpha$$

and we claim

$$[\![\sigma(t)]\!]^M_\alpha = [\![t]\!]^M_{\alpha^M_\sigma}$$
$$M,\alpha \models \sigma(\varphi) \iff M,\alpha^M_\sigma \models \varphi$$

3 Set-valued functions

Set-valued functions are functions $\mathcal{F} \in \mathbb{F}(V) = (\wp(V) \to \wp(V))$ for some set V. X, Y, Z range over subsets of V. Inclusion $\mathcal{F} \sqsubseteq \mathcal{G}$ is defined by $\forall X \subseteq \ \mathcal{F}(X) \subseteq \mathcal{G}(X)$. The union $\mathcal{F} \sqcup \mathcal{G}$ is defined by $(\mathcal{F} \sqcup \mathcal{G})(X) = \mathcal{F}(X) \cup \mathcal{G}(X)$. The projection function $\mathcal{I}_X \in \mathbb{F}(V)$ is defined by $\mathcal{I}_X = \lambda Y.X \cap Y$. So $\mathcal{I} = \mathcal{I}_V$ is the identity function.

$\mathcal{F}$ is monotonic whenever $X \subseteq Y$ implies $\mathcal{F}(X) \subseteq \mathcal{F}(Y)$. The *monotonic closure* $\mathcal{F}^m$ of $\mathcal{F}$ is the least monotonic function that subsumes $\mathcal{F}$. It can be defined by

$$\mathcal{F}^m(X) = \bigcup\{F(Y) \mid Y \subseteq X\}$$

Monotonic composition $\mathcal{F} \overset{\mathsf{m}}{\circ} \mathcal{G}$ is defined by $\mathcal{F}^m \circ \mathcal{G}^m$.

The collection $\mathsf{PFP}(\mathcal{F}, X)$ of *prefixpoints of $\mathcal{F}$ extending X* is defined by

$$\mathsf{PFP}(\mathcal{F}, X) = \{Y \subseteq W \mid X \cup \mathcal{F}(Y) \subseteq Y\}$$

So $\mathsf{PFP}(\mathcal{F}) = \mathsf{PFP}(\mathcal{F}, \emptyset)$ is the set of *all* prefixpoints of $\mathcal{F}$, i.e. all Y with $\mathcal{F}(Y) \subseteq Y$. Moreover, we have $\mathsf{PFP}(\mathcal{F}, X) = \mathsf{PFP}(\mathcal{F}) \cap \{Y \mid Y \supseteq X\}$. When $\mathcal{F}$ is *extensive*, i.e. $X \subseteq \mathcal{F}(X)$ for all X, it is evident that $\mathsf{PFP}(\mathcal{F})$ equals $\mathsf{FP}(\mathcal{F})$, the collection of all fixpoints of $\mathcal{F}$, i.e. all Y with $\mathcal{F}(Y) = Y$.

PFP is antimonotonic in both arguments, and we have

$$\mathsf{PFP}(\bigcup_{i\in I}\mathcal{F}_i, X) = \bigcap_{i\in I}\mathsf{PFP}(\mathcal{F}_i, X) \tag{1}$$

With PFP, we can define the iterative closure $\mathcal{F}^*$ of $\mathcal{F}$ by:

$$\mathcal{F}^* = \lambda X.\bigcap\mathsf{PFP}(\mathcal{F}^m, X)$$

We claim

$$\mathcal{F}^* = \mathcal{I} \sqcup \mathcal{F}^m \circ \mathcal{F}^* \qquad \text{(defining property of } \mathcal{F}^*\text{)}$$
$$\mathcal{I} \sqcup \mathcal{F}^m \circ \mathcal{G} \sqsubseteq \mathcal{G} \implies \mathcal{F}^* \sqsubseteq \mathcal{G} \qquad \text{(inductive property of } \mathcal{F}^*\text{)}$$

3.1 Horn logic formulated with set-valued functions

Formula set valued functions $\mathcal{F} \in \mathbb{F}(\mathsf{ATOM})$ represent Horn formulae. A Horn formula $H = \bigwedge_{k \in K}(\bigwedge X_k \to \varphi_k)$ is uniquely represented by $\mathcal{F}$ defined by

$$\mathcal{F} = \lambda Y. \{\varphi_k \mid k \in K \;\&\; X_k = Y\}$$

Conversely, we define $\mathsf{horn}(\mathcal{F})$, the Horn formula represented by $\mathcal{F}$:

$$\mathsf{horn}(\mathcal{F}) = \bigwedge\left\{\bigwedge X \to \varphi \mid X \subseteq \mathsf{ATOM}, \varphi \in \mathcal{F}(X)\right\}$$

We define $\mathsf{dom}, \mathsf{rg} : \mathbb{F}(\mathsf{ATOM}) \to \wp(\mathsf{ATOM})$ and $\mathsf{base} : \mathbb{F}(\mathsf{ATOM}) \to \wp(\mathsf{ATOM})^2$ by

$$\begin{aligned} \mathsf{dom}(\mathcal{F}) &= \bigcup\{X \mid \mathcal{F}(X) \neq \emptyset\} \\ \mathsf{rg}(\mathcal{F}) &= \bigcup\{\mathcal{F}(X) \mid X \subseteq \mathsf{ATOM}\} \\ \mathsf{base}(\mathcal{F}) &= (\mathsf{dom}(\mathcal{F}), \mathsf{rg}(\mathcal{F})) \end{aligned}$$

The functions $\mathsf{sig_f}$, ${\mathsf{sig_p}}^+$ and ${\mathsf{sig_p}}^-$ are defined for $\mathbb{F}(\mathsf{ATOM})$ by

$$\begin{aligned} \mathsf{sig_f}(\mathcal{F}) &= \mathsf{sig_f}(\mathsf{base}(\mathcal{F})) \\ {\mathsf{sig_p}}^+(\mathcal{F}) &= \mathsf{sig_p}(\mathsf{rg}(\mathcal{F})) \\ {\mathsf{sig_p}}^-(\mathcal{F}) &= \mathsf{sig_p}(\mathsf{dom}(\mathcal{F})) \end{aligned}$$

As a consequence, we have $\mathsf{sig_f}(\mathcal{F}) = \mathsf{sig_f}(\mathsf{horn}(\mathcal{F}))$, and similarly for ${\mathsf{sig_p}}^+$ and ${\mathsf{sig_p}}^-$.

3.2 Restriction

Range restriction of a set-valued function $\mathcal{F}$ is obtained by left-composition with a projection function, for $\mathsf{rg}(\mathcal{I}_X \circ \mathcal{F}) \subseteq \mathsf{rg}(\mathcal{F}) \cap X$. However, domain restriction cannot obtained via right-composition, for in general $\mathsf{dom}(\mathcal{F} \circ \mathcal{I}_X) \not\subseteq \mathsf{dom}(\mathcal{F}) \cap X$. So we define

$$\mathcal{F} \restriction X = \lambda Y. \text{ if } Y \subseteq X \text{ then } \mathcal{F}(Y) \text{ else } \emptyset$$

Now $\mathsf{dom}(\mathcal{F} \restriction X) \subseteq \mathsf{dom}(\mathcal{F}) \cap X$. We define *full restriction* by

$$\mathcal{F}|(X, Y) = \mathcal{I}_X \circ \mathcal{F} \restriction Y$$

for which we have

$$\begin{aligned} \mathcal{F}|(X, Y) &\sqsubseteq \mathcal{F} \\ \mathsf{base}(\mathcal{F}|(X, Y)) &\subseteq_2 \mathsf{base}(\mathcal{F}) \cap_2 (X, Y) \\ (\mathcal{F}|(X, Y))^m &= \mathcal{I}_X \circ \mathcal{F}^m \circ \mathcal{I}_Y \end{aligned}$$

3.3 Axiomatization of set-valued functions

In [1], we presented *weak lazy Kleene algebras* $\langle A, +, \cdot, ^*, 0, 1\rangle$ as an axiomatization of set-valued functions. We list the axioms.

(WLKA1) $+$ is commutative: $a + b = b + a$

(WLKA2) $+$ is associative: $a + (b + c) = (a + b) + c$

(WLKA3) $+$ is idempotent: $a + a = a$

(WLKA4) $+$ has unit element 0: $a + 0 = a$

(WLKA5) $\cdot$ is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(WLKA6) $a \leq a \cdot 1 = 1 \cdot a$, $1 \cdot 1 = 1$, $a \cdot b \cdot 1 = a \cdot b$ and $(a \cdot 1)^* = a^* \cdot 1 = a^*$
(WLKA7) $\cdot$ is right-distributive over $+$: $(a+b) \cdot c = a \cdot c + b \cdot c$
(WLKA8) if $b \leq c$ then $a \cdot b \leq a \cdot c$
(WLKA9) $\cdot$ is left strict w.r.t. 0: $0 \cdot a = 0$
(WLKA10) $1 + a \cdot a^* \leq a^*$
(WLKA11) if $a \cdot b \leq b$ then $a^* \cdot b \leq b \cdot 1$

These axioms translate to properties of set-valued functions $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, where $+$ is interpreted by $\sqcup$, $\cdot$ by $\overset{m}{\circ}$, * by itself, 0 by $\lambda X.\emptyset$ and 1 by $\mathcal{I}$. Moreover, we derived in [1] some additional properties that we shall use here:

$$(\mathcal{F} \sqcup \mathcal{G})^* = \mathcal{F}^* \circ (\mathcal{G}^m \circ \mathcal{F}^*)^* \tag{2}$$

$$(\mathcal{F}^m \circ \mathcal{G}^m)^* \circ \mathcal{F}^m = \mathcal{F}^m \circ (\mathcal{G}^m \circ \mathcal{F}^m)^*, \text{ if } \mathcal{F}^m \text{ is union-distributive} \tag{3}$$

Here union-distributive means: for all X, Y $\mathcal{F}(X \cup Y) = \mathcal{F}(X) \cup \mathcal{F}(Y)$.
We shall also use the $\cdot^+$ operator, defined by $\mathcal{F}^+ = \mathcal{F}^m \circ \mathcal{F}^*$.

3.4 Set-valued functions and substitution

Given $\sigma \in \mathsf{SUB}$, we define the *lifting* $\underline{\sigma} \in \mathbb{F}(\mathsf{ATOM})$ and the *extended inverse* $\breve{\sigma} \in \mathbb{F}(\mathsf{ATOM})$ by

$$\underline{\sigma} = \lambda A.\{\sigma(\varphi) \mid \varphi \in A\}$$
$$\breve{\sigma} = \lambda A.\{\varphi \in \mathsf{ATOM} \mid \sigma(\varphi) \in A\}$$

We define $\sigma \cdot \mathcal{F}$ by

$$(\sigma \cdot \mathcal{F})(X) = \bigcup\{\underline{\sigma}(\mathcal{F}(Y)) \mid \underline{\sigma}(Y) = X\}$$

and claim that $\mathsf{horn}(\sigma \cdot \mathcal{F}) = \dot{\sigma}(\mathsf{horn}(\mathcal{F}))$. We will often use the monotonic closure $\sigma \overset{m}{\cdot} \mathcal{F} = (\sigma \cdot \mathcal{F})^m$ of $\sigma \cdot \mathcal{F}$. We have

$$\sigma \overset{m}{\cdot} \mathcal{F} = \underline{\sigma} \circ \mathcal{F}^m \circ \breve{\sigma} \tag{4}$$

$$\sigma \overset{m}{\cdot} (\tau \overset{m}{\cdot} \mathcal{F}) = (\sigma \circ \tau) \overset{m}{\cdot} \mathcal{F} \tag{5}$$

3.5 Validity and entailment for set-valued functions

We extend validity to set-valued functions:

$$\begin{aligned} M, \alpha \models \mathcal{F} \quad &\text{iff} \quad \forall X \subseteq \mathsf{ATOM}(M, \alpha \models X \implies M, \alpha \models \mathcal{F}(X)) \\ M \models \mathcal{F} \quad &\text{iff} \quad \forall \alpha\ M, \alpha \models \mathcal{F} \\ \mathcal{F} \models \mathcal{G} \quad &\text{iff} \quad \forall M(M \models \mathcal{F} \implies M \models \mathcal{G}) \end{aligned}$$

We claim

$$M, \alpha \models \mathcal{F} \quad \text{iff} \quad \mathsf{Diag}(M, \alpha) \in \mathsf{PFP}(\mathcal{F}^m) \tag{6}$$

which is demonstrated as follows:

$$\begin{aligned} M, \alpha \models \mathcal{F} &\iff \forall A \subseteq \mathsf{ATOM}\ (M, \alpha \models A \implies M, \alpha \models \mathcal{F}(A)) && \text{(by definition)} \\ &\iff \forall A \subseteq \mathsf{Diag}(M, \alpha)\ \mathcal{F}(A) \subseteq \mathsf{Diag}(M, \alpha) && \text{(definition of Diag)} \\ &\iff \mathcal{F}^m(\mathsf{Diag}(M, \alpha)) \subseteq \mathsf{Diag}(M, \alpha) && \text{(definition of } \mathcal{F}^m) \\ &\iff \mathsf{Diag}(M, \alpha)) \in \mathsf{PFP}(\mathcal{F}^m) && \text{(definition of PFP)} \end{aligned}$$

4 Characterizing entailment

For propositional Horn logic, we characterized in [1] entailment with the Kleene closure operator: $\mathcal{F} \models \mathcal{G}$ iff $\mathcal{G} \sqsubseteq \mathcal{F}^*$. For full Horn logic, we also need substitutions for the characterization. We define the *substitution closure* operator $\mathsf{SC} : \wp(\mathsf{SUB}) \times \mathbb{F}(\mathsf{ATOM}) \to \mathbb{F}(\mathsf{ATOM})$ by

$$\mathsf{SC}(\Sigma, \mathcal{F}) = \bigsqcup_{\sigma \in \Sigma} \sigma \overset{\mathsf{m}}{\cdot} \mathcal{F}$$

and we abbreviate $\mathsf{SC}(\mathsf{SUB}, \mathcal{F})$ to $\mathsf{SC}(\mathcal{F})$. Observe that we have

$$\mathsf{SC}(\mathcal{F}) = \mathsf{SC}(\mathsf{SUB}_{\mathsf{var}(\mathcal{F})}, \mathcal{F}) \tag{7}$$

where $\mathsf{SUB}_V = (V \to \mathsf{TERM})$. This follows from the fact that $\sigma \overset{\mathsf{m}}{\cdot} \mathcal{F} = (\sigma \restriction \mathsf{var}(\mathcal{F})) \overset{\mathsf{m}}{\cdot} \mathcal{F}$. We also define

$$\mathsf{KSC}(\mathcal{F}) = \mathsf{SC}(\mathcal{F})^*$$

Now we can show

Lemma 1. *$\mathcal{F} \models \mathcal{G}$ iff $\mathcal{G} \sqsubseteq \mathsf{KSC}(\mathcal{F})$.*

Proof. The 'if' part $\mathcal{G} \sqsubseteq \mathsf{KSC}(\mathcal{F}) \implies \mathcal{F} \models \mathcal{G}$ follows from

$$\mathcal{F} \models \mathcal{F}^* \tag{8}$$

$$\mathcal{F} \models \mathsf{SC}(\mathcal{F}) \tag{9}$$

$$\models \text{ is transitive} \tag{10}$$

$$\mathcal{G} \sqsubseteq \mathcal{F} \implies \mathcal{F} \models \mathcal{G} \tag{11}$$

(10) and (11) are easy to prove. For the other two, we reason as follows.

(8) is a consequence of $\mathsf{PFP}(\mathcal{F}^m) \subseteq \mathsf{PFP}(\mathcal{F}^*)$, which is demonstrated as follows: if $X \in \mathsf{PFP}(\mathcal{F}^m)$ then $\mathcal{F}^m(X) \subseteq X$, so $X = \bigcap \mathsf{PFP}(\mathcal{F}^m, X) = \mathcal{F}^*(X)$, hence $X \in \mathsf{PFP}(\mathcal{F}^*)$.

(9): first we recall that $M, \alpha \models \sigma \overset{\mathsf{m}}{\cdot} \mathcal{F}$ iff $M, \alpha^M_\sigma \models \mathcal{F}$, so $\mathcal{F} \models \sigma \overset{\mathsf{m}}{\cdot} \mathcal{F}$ for all $\sigma \in \mathsf{SUB}$. Now we use the fact the collection of set-valued functions entailed by $\mathcal{F}$ is closed under union, and we obtain $\mathcal{F} \models \mathsf{SC}(\mathcal{F})$.

For the proof of the 'only if' part, we use term models. Given $A \subseteq \mathsf{ATOM}$, the term model $\mathsf{TM}(A) = \langle \mathsf{TERM}, i, j_A \rangle$ is the model with ATOM as universe, where $i(f)$ maps terms $\vec{t}$ to the term $f(\vec{t})$ for all function symbols $f \in \mathsf{SIG_f}$, and where $j_A(p) = \{\vec{t} \mid p(\vec{t}) \in A\}$ for all predicate symbols $p \in \mathsf{SIG_p}$. The valuations in term models are the substitutions $\sigma \in \mathsf{SUB}$, and we have

$$[\![t]\!]^{\mathsf{TM}(A)}_\sigma = \dot{\sigma}(t)$$

$$\mathsf{TM}(A), \sigma \models \varphi \iff \dot{\sigma}(\varphi) \in A$$

As a consequence, we have (recall that $\breve{\sigma}(A) = \{\varphi \mid \dot{\sigma}(\varphi) \in A\}$)

$$\mathsf{Diag}(\mathsf{TM}(A), \sigma) = \breve{\sigma}(A) \tag{12}$$

We claim the following auxiliary results:

$$\breve{\sigma}(A) \in \mathsf{PFP}(\mathcal{F}^m) \iff A \in \mathsf{PFP}(\sigma \overset{\mathsf{m}}{\cdot} \mathcal{F}) \tag{13}$$

$$\mathsf{PFP}(\mathsf{SC}(\mathcal{F})) = \bigcap_{\sigma \in \mathsf{SUB}} \mathsf{PFP}(\sigma \overset{\mathsf{m}}{\cdot} \mathcal{F}) \tag{14}$$

$$\mathsf{TM}(A) \models \mathcal{F} \iff A \in \mathsf{PFP}(\mathsf{SC}(\mathcal{F})) \tag{15}$$

They are demonstrated as follows.

$$
\begin{array}{llll}
(13): & \breve{\sigma}(A) \in \mathsf{PFP}(\mathcal{F}^m) & \iff (\mathcal{F}^m \circ \breve{\sigma})(A) \subseteq \breve{\sigma}(A) & \text{(definition of PFP)} \\
& & \iff (\underline{\sigma} \circ \mathcal{F}^m \circ \breve{\sigma})(A) \subseteq A & (X \subseteq \breve{\sigma}(Y) \text{ iff } \underline{\sigma}(X) \subseteq Y) \\
& & \iff A \in \mathsf{PFP}(\sigma \overset{m}{\cdot} \mathcal{F}) & \text{(definition of } \overset{m}{\cdot} \text{ and PFP)}
\end{array}
$$

$$
\begin{array}{lll}
(14): & \mathsf{PFP}(\mathsf{SC}(\mathcal{F})) = \mathsf{PFP}\Big(\bigsqcup_{\sigma \in \mathsf{SUB}} \sigma \overset{m}{\cdot} \mathcal{F}\Big) & \text{(definition of SC)} \\
& \qquad = \bigcap_{\sigma \in \mathsf{SUB}} \mathsf{PFP}(\sigma \overset{m}{\cdot} \mathcal{F}) & (1)
\end{array}
$$

$$
\begin{array}{llll}
(15): & \mathsf{TM}(A) \vDash \mathcal{F} & \iff \forall \sigma \in \mathsf{SUB}\ \mathsf{TM}(A), \sigma \vDash \mathcal{F} & \text{(elementary)} \\
& & \iff \forall \sigma \in \mathsf{SUB}\ \mathsf{Diag}(\mathsf{TM}(A), \sigma) \in \mathsf{PFP}(\mathcal{F}^m) & (6) \\
& & \iff \forall \sigma \in \mathsf{SUB}\ \breve{\sigma}(A) \in \mathsf{PFP}(\mathcal{F}^m) & (12) \\
& & \iff \forall \sigma \in \mathsf{SUB}\ A \in \mathsf{PFP}(\sigma \overset{m}{\cdot} \mathcal{F}) & (13) \\
& & \iff A \in \mathsf{PFP}(\mathsf{SC}(\mathcal{F})) & (14)
\end{array}
$$

Now we can prove $\mathcal{F} \vDash \mathcal{G} \implies \mathcal{G} \sqsubseteq \mathsf{KSC}(\mathcal{F})$:

$$
\begin{array}{ll}
\mathcal{F} \vDash \mathcal{G} & \\
\implies \forall A \subseteq \mathsf{ATOM}\ (\mathsf{TM}(A) \vDash \mathcal{F} \implies \mathsf{TM}(A) \vDash \mathcal{G}) & \text{(elementary)} \\
\iff \forall A \subseteq \mathsf{ATOM}\ (A \in \mathsf{PFP}(\mathsf{SC}(\mathcal{F})) \implies A \in \mathsf{PFP}(\mathsf{SC}(\mathcal{G}))) & (15) \\
\iff \mathsf{PFP}(\mathsf{SC}(\mathcal{F})) \subseteq \mathsf{PFP}(\mathsf{SC}(\mathcal{G})) & \text{(elementary)} \\
\implies \mathsf{SC}(\mathcal{G})^* \sqsubseteq \mathsf{SC}(\mathcal{F})^* & \text{(by definition of } \cdot^*) \\
\implies \mathcal{G} \sqsubseteq \mathsf{KSC}(\mathcal{F}) \qquad (\mathcal{G} \sqsubseteq \mathsf{SC}(\mathcal{G})^*; \text{ definition of } \mathsf{KSC}(\mathcal{F})) & \square
\end{array}
$$

Observe that Lemma 1 can be read as a completeness result $\mathcal{F} \vDash \mathcal{G} \iff \mathcal{F} \vdash \mathcal{G}$, provided we define $\mathcal{F} \vdash \mathcal{G}$ by $\mathcal{G} \sqsubseteq \mathsf{KSC}(\mathcal{F})$.

5 Propositional interpolation

In [1], we proved interpolation for propositional Horn logic. This logic is obtained in the present setting when $\mathsf{SIG_f} = \emptyset$ and all $p \in \mathsf{SIG} = \mathsf{SIG_p}$ have arity 0, so variables and substitutions play no role, $\mathsf{ATOM} = \mathsf{SIG}$ and $\mathsf{sig_p} = \mathsf{sig}$. Interpolation follows from Lemma 2 that will also be used in the proof of interpolation for full Horn logic.

Lemma 2. *Let* $A = (A^-, A^+) \subseteq_2 \mathsf{ATOM}^2$, $\mathcal{F}, \mathcal{G} \in \mathbb{F}(\mathsf{ATOM})$. *Then*

$$(\mathcal{F} \sqcup \mathcal{G})^*|A \sqsubseteq (\mathcal{F}^+|(A \cup_2 A_{\mathcal{G}}^{-1}) \sqcup \mathcal{G})^*$$

where $A_{\mathcal{G}}$ *denotes* $\mathsf{Atom}_2(\mathsf{TERM}, \mathsf{sig_p}^{\pm}(\mathcal{G}))$.

Proof. We will use some auxiliary results:

$$\mathcal{F}^*|A \sqsubseteq (\mathcal{F}^+|A)^* \tag{16}$$

$$(\mathcal{F} \circ (\mathcal{G}|A^{-1} \circ \mathcal{H})^*)|A = \mathcal{F}|A \circ (\mathcal{G} \circ \mathcal{H}|A)^* \tag{17}$$

They are proved as follows.

(16): we have $\mathcal{F}^*|A \sqsubseteq (\mathcal{F}^*|A)^* = (\mathcal{F}^+|A \sqcup \mathcal{I}|A)^* \sqsubseteq ((\mathcal{F}^+|A)^*)^* = (\mathcal{F}^+|A)^*$.

(17) is demonstrated by

$$
\begin{array}{ll}
(\mathcal{F} \circ (\mathcal{G}|A^{-1} \circ \mathcal{H})^*)|A & \\
= \mathcal{I}_{A^+} \circ \mathcal{F} \circ (\mathcal{I}_{A^-} \circ \mathcal{G} \circ \mathcal{I}_{A^+} \circ \mathcal{H})^* \circ \mathcal{I}_{A^-} & \text{(definition of } |) \\
= \mathcal{I}_{A^+} \circ \mathcal{F} \circ \mathcal{I}_{A^-} \circ (\mathcal{G} \circ \mathcal{I}_{A^+} \circ \mathcal{H} \circ \mathcal{I}_{A^-})^* & ((3); \mathcal{I}_{A^-} \text{ is union-distributive}) \\
= \mathcal{F}|A \circ (\mathcal{G} \circ \mathcal{H}|A)^* & \text{(definition of } |)
\end{array}
$$

Now we can prove the statement of the Lemma. We use the abbreviation B for $A \cup_2 A_{\mathcal{G}}^{-1}$.

$$\begin{aligned}
(\mathcal{F} \sqcup \mathcal{G})^*|A &= \mathcal{F}^* \circ (\mathcal{G}^m \circ \mathcal{F}^*)^*|A && (2)\\
&\sqsubseteq \mathcal{F}^* \circ (\mathcal{G}^m|B^{-1} \circ \mathcal{F}^*)^*|B && (B = A \cup_2 A_{\mathcal{G}}^{-1})\\
&= \mathcal{F}^*|B \circ (\mathcal{G}^m \circ \mathcal{F}^*|B)^* && (17)\\
&\sqsubseteq (\mathcal{F}^+|B)^* \circ (\mathcal{G}^m \circ (\mathcal{F}^+|B)^*)^* && (16)\\
&= (\mathcal{F}^+|(A \cup_2 A_{\mathcal{G}}^{-1}) \sqcup \mathcal{G})^* && (2);\ Q = A \cup_2 A_{\mathcal{G}}^{-1} \qquad \square
\end{aligned}$$

Theorem 3 (propositional interpolation). *For any propositional Horn formula φ and for any $P \subseteq_2 \mathsf{sig_p}^{\pm}(\varphi)$, there is a uniform interpolant θ with*

1. $\mathsf{sig}^{\pm}(\theta) \subseteq P$*;*
2. *for all χ, ψ with $\mathsf{sig}^{\pm}(\varphi) \cap (\mathsf{sig}^{\mp}(\chi) \cup \mathsf{sig}^{\pm}(\psi)) \subseteq P$ we have*

$$\varphi \wedge \chi \vdash \psi \quad \textit{iff} \quad \theta \wedge \chi \vdash \psi.$$

Proof. Let $\mathcal{F}$ correspond with φ, i.e. $\mathsf{horn}(\mathcal{F}) = \varphi$. Define

$$\mathcal{J} = \mathcal{F}^+|P$$

We shall show that $\theta = \mathsf{horn}(\mathcal{J})$ satisfies the theorem. One straightforwardly checks that $\mathsf{sig}(\theta) = \mathsf{sig}(\mathcal{J}) \subseteq_2 P$, i.e. part (1) of the theorem. For part (2) we reason as follows. Let χ, ψ with $\mathsf{sig}^{\pm}(\varphi) \cap (\mathsf{sig}^{\mp}(\chi) \cup \mathsf{sig}^{\pm}(\psi)) \subseteq_2 P$ and let $\mathcal{G}$, $\mathcal{H}$ satisfy $\mathsf{horn}(\mathcal{G}) = \chi$, $\mathsf{horn}(\mathcal{H}) = \psi$. We set out to prove

$$\mathcal{F} \sqcup \mathcal{G} \vDash \mathcal{H} \quad \text{iff} \quad \mathcal{J} \sqcup \mathcal{G} \vDash \mathcal{H}$$

For the 'if' part, we observe that $\mathcal{J} \sqsubseteq \mathcal{F}^*$, so $\mathcal{F} \vDash \mathcal{J}$ by Lemma 1 and the fact that here $\mathsf{KSC}(\mathcal{F}) = \mathcal{F}^*$.

To prove the 'only if' part, we define $P_{\mathcal{F}} = \mathsf{sig}^{\pm}(\mathcal{F})$, and similarly for $P_{\mathcal{G}}$, $P_{\mathcal{H}}$. As a consequence, we have

$$\mathcal{F} = \mathcal{F}|P_{\mathcal{F}} \tag{18}$$

$$P_{\mathcal{F}} \cap_2 (P_{\mathcal{G}}^{-1} \cup_2 P_{\mathcal{H}}) \subseteq_2 P \tag{19}$$

Now it suffices to show that $(\mathcal{F} \sqcup \mathcal{G})^*|P_{\mathcal{H}} \sqsubseteq (\mathcal{J} \sqcup \mathcal{G})^*$.

$$\begin{aligned}
(\mathcal{F} \sqcup \mathcal{G})^*|P_{\mathcal{H}} &\sqsubseteq (\mathcal{F})^+|(P_{\mathcal{G}}^{-1} \cup_2 P_{\mathcal{H}}) \sqcup \mathcal{G})^* && \text{(propositional interpolation: Lemma 2)}\\
&\sqsubseteq (\mathcal{F}^+|P \sqcup \mathcal{G})^* && ((18), (19))\\
&= (\mathcal{J} \sqcup \mathcal{G})^* && (\text{definition of } \mathcal{J}) \qquad \square
\end{aligned}$$

6 Properties of substitutions

In this section, we prove three lemmata about substitutions. The first is about splitting substitutions in two parts, one of them restricted to terms built from some subset of $\mathsf{SIG_f}$. The second lemma tells when substitution and iteration commute, and the third lemma does the same for substitution and restriction.

Let $S \subseteq \mathsf{SIG_f}$. We want to split substitutions σ in two parts: a substitution σ^S with $\mathsf{sig}(\sigma^S) \subseteq S$, and a substitution τ (depending only on S, not on σ) such that $\sigma = \dot{\tau} \circ \sigma^S$. For this purpose, we introduce *term-indexed* variables x_t and a *de-indexing* substitution τ with $\tau(x_t) = t$. The term-indexed variables can be used to replace subterms $t = g(t_1, \ldots, t_n)$ which have as head a function symbol $g \in \mathsf{SIG_f} - S$. For this idea to work, we require that $\#(\mathsf{VAR} - \mathsf{dom}(\sigma)) = \#\mathsf{VAR}$, so there are enough 'fresh' variables available in $\mathsf{VAR} - \mathsf{dom}(\sigma)$. Without loss of generality, we assume that $\mathsf{VAR} - \mathsf{dom}(\sigma)$ contains variables x_t for all $t \in \mathsf{TERM}$ (here we use that $\#\mathsf{VAR} = \#\mathsf{TERM}$). Now we can prove

Lemma 4. $\mathsf{SC}(\mathcal{F}) = \tau \overset{\mathsf{m}}{\because} \mathsf{SC}(\mathsf{SUB}(\mathsf{sig_f}(\mathcal{F})), \mathcal{F})$*, where $\mathcal{T}$ is the de-indexing substitution with $\tau(x_t) = t$ for all $t \in \mathsf{Hterm}(\mathsf{SIG_f} - \mathsf{sig_f}(\mathcal{F}), \mathsf{TERM})$.*

Proof. Without loss of generality, we may assume that $\#(\mathsf{VAR} - \mathsf{var}(\mathcal{F})) = \#\mathsf{VAR}$. When needed, this can be realized via a renaming of the variables occurring in $\mathcal{F}$.
We have to prove

$$\bigsqcup_{\sigma \in \mathsf{SUB}} \sigma \overset{\mathsf{m}}{\because} \mathcal{F} = \tau \overset{\mathsf{m}}{\because} \Big(\bigsqcup_{\sigma^- \in \mathsf{SUB}(\mathsf{sig_f}(\mathcal{F}))} \sigma^- \overset{\mathsf{m}}{\because} \mathcal{F} \Big)$$

Since $\tau \overset{\mathsf{m}}{\because}$ and $\bigsqcup$ commute, and SUB contains *all* substitutions, the $\sqsupseteq$ inclusion is obvious. For the $\sqsubseteq$ inclusion, it suffices to show

$$\forall \sigma \in \mathsf{SUB}\ \exists \sigma^- \in \mathsf{SUB}(\mathsf{sig_f}(\mathcal{F}))\ \sigma = \tau \circ \sigma^-$$

To prove this, let $\sigma \in \mathsf{SUB}$. Since $\sigma \overset{\mathsf{m}}{\because} \mathcal{F}$ is determined by $\sigma \restriction \mathsf{var}(\mathcal{F})$, we may assume that $\mathsf{dom}(\sigma) = \mathsf{var}(\mathcal{F})$. We shall define σ^- such that $\sigma = \dot{\tau} \circ \sigma^-$ holds. For this purpose, we use variables $x_t \in \mathsf{VAR} - \mathsf{var}(\mathcal{F})$ for all $t \in HT = \mathsf{Hterm}(\mathsf{SIG_f} - \mathsf{sig_f}(\mathcal{F}), \mathsf{TERM})$. σ^- is obtained by eliminating all (sub)terms in the range of σ that are not in $\mathsf{Term}(\mathsf{sig_f}(\mathcal{F}))$. Define σ^- by $\sigma^-(x) = (\sigma(x))^-$ for all $x \in \mathsf{VAR}$, where $\cdot^- : \mathsf{TERM} \to \mathsf{Term}(\mathsf{sig_f}(\mathcal{F}))$ is defined by

$$\begin{aligned} x^- &= x && \text{if } x \in \mathsf{VAR} \\ t^- &= x_t && \text{if } t \in HT \\ f(t_t, \ldots, t_n)^- &= f(t_1^-, \ldots, t_n^-) && \text{for all } f \in \mathsf{SIG_f} \end{aligned}$$

Now one easily shows that $\dot{\tau}(t^-) = t$ for all $t \in \mathsf{TERM}$, so indeed $\sigma = \dot{\tau} \circ \sigma^-$. □

Lemma 5. *If $\dot{\sigma}$ is injective on $A \subseteq \mathsf{ATOM}^2$ and $\mathcal{F}|A = \mathcal{F}$, then substitution and iteration commute:*

$$(\sigma \overset{\mathsf{m}}{\because} \mathcal{F})^+ = \sigma \overset{\mathsf{m}}{\because} \mathcal{F}^+$$

Proof. First we define

$$\begin{aligned} \mathsf{Eq}_\sigma &= \lambda X \subseteq \mathsf{ATOM}.\{\varphi \mid \exists \psi \in X\ \dot{\sigma}(\psi) = \dot{\sigma}(\varphi)\} \\ \mathcal{F}^\sigma &= \mathsf{Eq}_\sigma \circ \mathcal{F}^m \circ \mathsf{Eq}_\sigma \end{aligned}$$

They satisfy the following properties:

$$\breve{\sigma} \circ \underline{\sigma} \circ \breve{\sigma} = \breve{\sigma} \tag{20}$$

$$\underline{\sigma} \circ \breve{\sigma} \circ \underline{\sigma} = \underline{\sigma} \tag{21}$$

$$\mathsf{Eq}_\sigma = \breve{\sigma} \circ \underline{\sigma} \tag{22}$$

$$\mathcal{I}_A \circ \mathsf{Eq}_\sigma \circ \mathcal{I}_A = \mathcal{I}_A, \text{if } \dot{\sigma} \text{ injective on } A \tag{23}$$

Now the Lemma follows directly from the next statements:

$$(\sigma \overset{\mathsf{m}}{\because} \mathcal{F})^+ = \sigma \overset{\mathsf{m}}{\because} (\mathcal{F}^\sigma)^+ \tag{24}$$

$$\dot{\sigma} \text{ is injective on } A \text{ and } \mathcal{F}|A = \mathcal{F} \implies (\mathcal{F}^\sigma)^+ = (\mathcal{F}^+)^\sigma \tag{25}$$

$$\sigma \overset{\mathsf{m}}{\because} \mathcal{F}^\sigma = \sigma \overset{\mathsf{m}}{\because} \mathcal{F} \tag{26}$$

They are proved as follows.

(26): $\sigma \overset{\mathrm{m}}{\because} \mathcal{F}^\sigma = \underline{\sigma} \circ \mathsf{Eq}_\sigma \circ \mathcal{F}^m \circ \mathsf{Eq}_\sigma \circ \breve{\sigma} = \underline{\sigma} \circ \mathcal{F}^m \circ \breve{\sigma} = \sigma \overset{\mathrm{m}}{\because} \mathcal{F}$.

$$
\begin{aligned}
(24):\quad (\sigma \overset{\mathrm{m}}{\because} \mathcal{F})^+ &= (\sigma \overset{\mathrm{m}}{\because} \mathcal{F}) \circ (\sigma \overset{\mathrm{m}}{\because} \mathcal{F})^* && \text{(definition of } \cdot^+\text{)}\\
&= \underline{\sigma} \circ \mathcal{F}^\sigma \circ \breve{\sigma} \circ (\underline{\sigma} \circ \mathcal{F}^m \circ \breve{\sigma})^* && \text{((26); (4))}\\
&= \underline{\sigma} \circ \mathcal{F}^\sigma \circ \breve{\sigma} \circ (\underline{\sigma} \circ \mathcal{F}^m \circ \mathsf{Eq}_\sigma \circ \breve{\sigma})^* && (\breve{\sigma} = \mathsf{Eq}_\sigma \circ \breve{\sigma} \text{ by (22), (20))}\\
&= \underline{\sigma} \circ \mathcal{F}^\sigma \circ (\breve{\sigma} \circ \underline{\sigma} \circ \mathcal{F}^m \circ \mathsf{Eq}_\sigma)^* \circ \breve{\sigma} && \text{((3); } \breve{\sigma} \text{ is left distributive)}\\
&= \underline{\sigma} \circ \mathcal{F}^\sigma \circ (\mathcal{F}^\sigma)^* \circ \breve{\sigma} && \text{((22); definition of } \mathcal{F}^\sigma\text{)}\\
&= \sigma \overset{\mathrm{m}}{\because} (\mathcal{F}^\sigma)^+ && \text{(definition of } \cdot^+\text{; (4))}
\end{aligned}
$$

$$
\begin{aligned}
(25):\quad (\mathcal{F}^\sigma)^+ &= \mathsf{Eq}_\sigma \circ \mathcal{F}^m \circ \mathsf{Eq}_\sigma \circ (\mathsf{Eq}_\sigma \circ \mathcal{F}^m \circ \mathsf{Eq}_\sigma)^* && \text{(definition of } \mathcal{F}^\sigma \text{ and } \cdot^+\text{)}\\
&= \mathsf{Eq}_\sigma \circ \mathcal{F}^m \circ \mathcal{I}_A \circ \mathsf{Eq}_\sigma \circ (\mathsf{Eq}_\sigma \circ \mathcal{I}_A \circ \mathcal{F}^m \circ \mathcal{I}_A \circ \mathsf{Eq}_\sigma)^*\\
& && (\mathcal{F} \text{ injective on } A\text{, so } \mathcal{F} = \mathcal{F} \circ \mathcal{I}_A = \mathcal{I}_A \circ \mathcal{F}))\\
&= \mathsf{Eq}_\sigma \circ \mathcal{F}^m \circ (\mathcal{I}_A \circ \mathsf{Eq}_\sigma \circ \mathsf{Eq}_\sigma \circ \mathcal{I}_A \circ \mathcal{F}^m)^* \circ \mathcal{I}_A \circ \mathsf{Eq}_\sigma\\
& && \text{((3); } \mathcal{I}_A \circ \mathsf{Eq}_\sigma \text{ is union-distributive)}\\
&= \mathsf{Eq}_\sigma \circ \mathcal{F}^m \circ (\mathcal{I}_A \circ \mathcal{F}^m)^* \circ \mathcal{I}_A \circ \mathsf{Eq}_\sigma\\
& && (\mathsf{Eq}_\sigma \circ \mathsf{Eq}_\sigma = \mathsf{Eq}_\sigma \text{ by (20) and (22); } \mathcal{I}_A \circ \mathsf{Eq}_\sigma \circ \mathcal{I}_A = \mathcal{I}_A \text{ by (23))}\\
&= \mathsf{Eq}_\sigma \circ \mathcal{F}^m \circ (\mathcal{F}^m)^* \circ \mathsf{Eq}_\sigma && \text{((2); } \mathcal{F} \circ \mathcal{I}_A = \mathcal{F}\text{)}\\
&= (\mathcal{F}^+)^\sigma && \text{(definition of } \cdot^+ \text{ and } \mathcal{F}^\sigma\text{)} \qquad \square
\end{aligned}
$$

Lemma 6. *Let $P = (P^-, P^+) \subseteq_2 \mathsf{SIG_p}^2$ and define $A = \mathsf{Atom}_2(\mathsf{TERM}, P)$. Then substitution and restriction to A commute: for any $\sigma \in \mathsf{SUB}$ we have*

$$(\sigma \overset{\mathrm{m}}{\because} \mathcal{F})|A = \sigma \overset{\mathrm{m}}{\because} (\mathcal{F}|A)$$

Proof. We observe that, by the definition of A, any $\dot{\sigma}$ is indifferent for A, i.e.

$$\forall \varphi \in \mathsf{ATOM}\ (\varphi \in A \iff \dot{\sigma}(\varphi) \in A)$$

As a consequence, we have $\mathcal{I}_A \circ \underline{\sigma} = \underline{\sigma} \circ \mathcal{I}_A$ and $\mathcal{I}_A \circ \breve{\sigma} = \breve{\sigma} \circ \mathcal{I}_A$, so

$$(\sigma \overset{\mathrm{m}}{\because} \mathcal{F})|A = \mathcal{I}_A \circ \underline{\sigma} \circ \mathcal{F} \circ \breve{\sigma} \circ \mathcal{I}_A = \underline{\sigma} \circ \mathcal{I}_A \circ \mathcal{F} \circ \mathcal{I}_A \circ \breve{\sigma} = \sigma \overset{\mathrm{m}}{\because} (\mathcal{F}|A) \qquad \square$$

7 Uniform interpolation

Now we can prove uniform interpolation for Horn logic.

Theorem 7. *For any Horn formula φ and for any $P = (P^-, P^+) \subseteq_2 \mathsf{sig_p}^\pm(\varphi)$, there is a uniform interpolant θ with*

1. $\mathsf{sig_p}^\pm(\theta) \subseteq_2 P$ *and* $\mathsf{sig_f}(\theta) \subseteq \mathsf{sig_f}(\varphi)$*;*
2. *for all χ, ψ with $\mathsf{sig_p}^\pm(\varphi) \cap_2 (\mathsf{sig_p}^\mp(\chi) \cup_2 \mathsf{sig_p}^\pm(\psi)) \subseteq_2 P$ we have*

$$\varphi \wedge \chi \vdash \psi \quad \textit{iff} \quad \theta \wedge \chi \vdash \psi.$$

Proof. Let $\mathcal{F}$ correspond with φ, i.e. $\mathsf{horn}(\mathcal{F}) = \varphi$. Define

$$\mathcal{J} = \mathsf{SC}(\mathsf{SUB}(\mathsf{sig_f}(\mathcal{F})), \mathcal{F})^+|A_P$$

where $A_P = \mathsf{Atom}_2(\mathsf{TERM}, P)$. We shall show that $\theta = \mathsf{horn}(\mathcal{J})$ satisfies the theorem. One straightforwardly checks that $\mathsf{sig_p}(\theta) = \mathsf{sig_p}(\mathcal{J}) \subseteq_2 S$ and $\mathsf{sig_f}(\theta) = \mathsf{sig_f}(\mathcal{J}) \subseteq \mathsf{sig_f}(\varphi)$, i.e. part

(1) of the theorem. For part (2) we reason as follows. Let χ, ψ with $\mathsf{sig_p}^{\pm}(\varphi) \cap (\mathsf{sig_p}^{\mp}(\chi) \cup \mathsf{sig_p}^{\pm}(\psi)) \subseteq_2 P$ and let $\mathcal{G}$, $\mathcal{H}$ satisfy $\mathsf{horn}(\mathcal{G}) = \chi$, $\mathsf{horn}(\mathcal{H}) = \psi$. We set out to prove

$$\mathcal{F} \sqcup \mathcal{G} \vDash \mathcal{H} \quad \text{iff} \quad \mathcal{J} \sqcup \mathcal{G} \vDash \mathcal{H}$$

For the 'if' part, we observe that $\mathcal{J} \sqsubseteq \mathsf{KSC}(\mathcal{F})$, so $\mathcal{F} \vDash \mathcal{J}$ by Lemma 1.

The proof for the 'only if' part is more involved. Define $A_{\mathcal{F}} = \mathsf{AT}_2(\mathsf{SIG_f} \cup_2 \mathsf{sig_p}^{\pm}(\mathcal{F}))$, and similarly for $A_{\mathcal{G}}$, $A_{\mathcal{H}}$. As a consequence, we have

$$\mathsf{SC}(F) = \mathsf{SC}(\mathcal{F})|A_{\mathcal{F}} \tag{27}$$

$$A_{\mathcal{F}} \cap_2 (A_{\mathcal{G}}^{-1} \cup_2 A_{\mathcal{H}}) \subseteq_2 A_P \tag{28}$$

For (27) we also use that $\mathsf{sig_p}^{\pm}(\mathsf{SC}(\mathcal{F})) = \mathsf{sig_p}^{\pm}(\mathcal{F})$. Now it suffices to show that $\mathsf{KSC}(\mathcal{F} \sqcup \mathcal{G})|A_{\mathcal{H}} \sqsubseteq \mathsf{KSC}(\mathcal{J} \sqcup \mathcal{G})$.

$\mathsf{KSC}(\mathcal{F} \sqcup \mathcal{G})\|A_{\mathcal{H}}$	
$= (\mathsf{SC}(\mathcal{F}) \sqcup \mathsf{SC}(\mathcal{G}))^*\|A_{\mathcal{H}}$	(def. of KSC; SC distributes over $\sqcup$)
$\sqsubseteq (\mathsf{SC}(\mathcal{F})^+\|(A_{\mathcal{G}}^{-1} \cup_2 A_{\mathcal{H}}) \sqcup \mathsf{SC}(\mathcal{G}))^*$	(prop. interpolation: Lemma 2)
$\sqsubseteq (\mathsf{SC}(\mathcal{F})^+\|A_P \sqcup \mathsf{SC}(\mathcal{G}))^*$	(27, 28)
$= ((\tau \overset{m}{\because} \mathsf{SC}(\mathsf{SUB}(\mathsf{sig_f}(\mathcal{F})), \mathcal{F}))^+\|A_P \sqcup \mathsf{SC}(\mathcal{G}))^*$	(substitution splitting: Lemma 4)
$= (\tau \overset{m}{\because} \mathsf{SC}(\mathsf{SUB}(\mathsf{sig_f}(\mathcal{F})), \mathcal{F})^+\|A_P \sqcup \mathsf{SC}(\mathcal{G}))^*$	(distributing τ over $\cdot^+$: Lemma 5)
$= (\tau \overset{m}{\because} (\mathsf{SC}(\mathsf{SUB}(\mathsf{sig_f}(\mathcal{F})), \mathcal{F})^+\|A_P) \sqcup \mathsf{SC}(\mathcal{G}))^*$	(distributing τ over $\cdot\|A_P$: Lemma 6)
$= (\tau \overset{m}{\because} \mathcal{J} \sqcup \mathsf{SC}(\mathcal{G}))^*$	(definition of $\mathcal{J}$)
$\sqsubseteq (\mathsf{SC}(\mathcal{J}) \sqcup \mathsf{SC}(\mathcal{G}))^*$	($\tau \overset{m}{\because} \mathcal{J} \sqsubseteq \mathsf{SC}(\mathcal{J})$)
$= \mathsf{KSC}(\mathcal{J} \sqcup \mathcal{G})$	(SC distributes over $\sqcup$; def. of KSC) $\square$

7.1 Counterexample for a stronger condition on occurrence of function symbols

The condition $\mathsf{sig_f}(\theta) \subseteq \mathsf{sig_f}(\varphi)$ about the occurrence of function symbols in the interpolant is rather weak compared with the condition $\mathsf{sig_p}^{\pm}(\theta) \subseteq^{\pm}_{\mathsf{sig_p}} (\varphi) \cap_2 (\mathsf{sig_p}^{\mp}(\chi) \cup_2 \mathsf{sig_p}^{\pm}(\psi))$ on predicate symbols. However, it is not possible to strengthen the function symbol condition to

$$\mathsf{sig_f}(\theta) \subseteq \mathsf{sig_f}(\varphi) \cap (\mathsf{sig_f}(\chi) \cup \mathsf{sig_f}(\psi))$$

We demonstrate this with a counterexample. Let $\varphi = p(a)$, $\chi = p(x) \to q$, $\psi = q$. Then indeed $\varphi \wedge \chi \vDash \psi$, but there is no interpolant θ with $\mathsf{sig_f}(\theta) \subseteq \mathsf{sig_f}(\varphi) \cap (\mathsf{sig_f}(\chi) \cup \mathsf{sig_f}(\psi)) = \{a\} \cap (\emptyset \cup \emptyset) = \emptyset$. The only interpolant (modulo equivalence) is $p(a)$, and $p(x)$ does not work. In first-order logic, the interpolant would be $\exists x\, p(x)$, but existential quantification is not available in Horn logic.

8 Further work

I list some ideas for further investigation.

- Try to find conditions that allow to strengthen the function parameter condition $\mathsf{sig_f}(\theta) \subseteq \mathsf{sig_f}(\varphi)$ to

  $$\mathsf{sig_f}(\theta) \subseteq \mathsf{sig_f}(\varphi) \cap (\mathsf{sig_f}(\chi) \cup \mathsf{sig_f}(\psi))$$
- Apply set-valued functions to propositional logics.
- Apply set-valued functions to (conditional) equational logic.
- Consider other types of conditional statements: sequents, proof rules.
- Apply set-valued functions to the theory of derived rules.

References

[1] Gerard R. Renardel de Lavalette. Interpolation in propositional Horn logic. *Journal of Logic and Computation*. doi:10.1093/logcom/exx042.

[2] Bernhard Möller. Kleene getting lazy. *Science of Computer Programming*, 65:195–214, 2007.

university of groningen

The mathematics of derivability

Gerard R. Renardel de Lavalette

Bernoulli Institute for Mathematics, CS and AI
University of Groningen
the Netherlands

UNILOG2018 Vichy – workshop Proof Theory

university of groningen

theorem

infinitary Horn logic (IHL): $t \in$ TERM, $\alpha \in$ ATOM as in predicate logic
formulae $\varphi = \overline{\forall} \bigwedge_{i \in I}(\bigwedge A_i \to \alpha_i) \quad (A_i \subseteq$ ATOM$)$

$\mathsf{pr}^{\pm}(\varphi) = (\mathsf{pr}^{+}(\varphi), \mathsf{pr}^{-}(\varphi))$: positively/negatively occurring predicate symbols.
$\mathsf{fun}(\varphi)$: function symbols.

uniform interpolation: for any φ in IHL and for any $P = (P^+, P^-) \subseteq_2 \mathsf{pr}(\varphi)$, there is a *uniform interpolant* θ with

1. $\mathsf{pr}(\theta) \subseteq_2 P$ and $\mathsf{fun}(\theta) \subseteq \mathsf{fun}(\varphi)$;
2. for all ψ, χ with $\mathsf{pr}^{\pm}(\varphi) \cap (\mathsf{pr}^{\mp}(\psi) \cup \mathsf{pr}^{\pm}(\chi)) \subseteq_2 P$ we have

$$\varphi \wedge \psi \vdash \chi \quad \text{iff} \quad \theta \wedge \psi \vdash \chi.$$

propositional non-uniform case (i.e. no terms; θ depends on φ, ψ, χ):
$\|\theta\|$ is *polynomial* in $\|\varphi\| + \|\chi\|$

university of groningen

theorem $\sim$ artefact

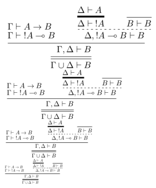
university of groningen

workbench

university of groningen

proof theory workbench

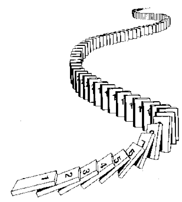
Instruments used for proof theory:



proof trees

1	$\forall x \exists y (Px \wedge Qy)$	ass.
2	$\exists y (Pa \wedge Qy)$	1, $(\forall E)$
3	$Pa \wedge Qb$	ass.
4	Pa	3, $(\wedge E)$
5	Pa	3–4, $(\exists E)$
6	$\forall x Px$	5, $(\forall I)$
7	$Pa \wedge Qc$	ass.
8	Qc	7, $(\wedge E)$
9	$\exists x Qx$	8, $(\exists I)$
10	$\exists x Qx$	7–9, $(\exists E)$
11	$\forall x Px \wedge \exists x Qx$	6, 10, $(\wedge I)$
12	$\forall x \exists y (Px \wedge Qy) \rightarrow \forall x Px \wedge \exists x Qx$	1–11, $(\rightarrow I)$

sequential derivations



induction

university of groningen

conditional statements

conditional statements are the *driving force* in derivations

They occur on different levels:

implications in the object language: $\varphi \rightarrow \psi$

sequents in the metalanguage: $\Gamma, \varphi, \psi \vdash \varphi \wedge \psi$

proof rules involving sequents: if $\Gamma \vdash \varphi$ and $\Gamma, \psi \vdash \chi$ then $\Gamma, \varphi \rightarrow \psi \vdash \chi$

university of groningen

set-valued functions

mathematical formulation of conditional statements:

set-valued functions $\mathcal{F} : \wp(\text{EXP}) \to \wp(\text{EXP})$

$$\begin{array}{lll} \text{EXP} = \text{ATOM}: & \mathcal{F} \sim & \text{Horn formula} \\ \text{EXP} = \text{FORM}: & \mathcal{F} \sim & \text{sequent} \\ \text{EXP} = \text{SEQ}: & \mathcal{F} \sim & \text{proof rule} \end{array}$$

From now on: EXP = ATOM

from $\mathcal{F}$ to Horn formula: $\bigwedge_{A \subseteq \text{ATOM}} \bigwedge_{\alpha \in \mathcal{F}(A)} (\bigwedge A \to \alpha)$

from Horn formula to $\mathcal{F}$: $\mathcal{F}(A) = \{\alpha_i \mid i \in I, A_i = A\}$

university of groningen

propositional case: example

propositional logic: no functions, no terms, so ATOM = SIG_p.

Let ATOM = $\{a, b, c, d\}$.

Now $a \wedge (a \to b) \wedge (a \to c) \wedge ((a \wedge c) \to d)$ corresponds to

$$\begin{array}{llll} \mathcal{F}: & \emptyset & \mapsto & \{a\} \\ & \{a\} & \mapsto & \{b, c\} \\ & \{a, c\} & \mapsto & \{d\} \\ & P & \mapsto & \emptyset \qquad \text{for } P \neq \emptyset, \{a\}, \{a, c\} \end{array}$$

university of groningen

propositional case: derivability

Derivability for propositional set-valued functions:

$\mathcal{F} \vdash \mathcal{G}$ is defined by $\mathcal{G} \sqsubseteq \mathcal{F}^*$

$\sqsubseteq$ is function inclusion:	$\mathcal{F} \sqsubseteq \mathcal{G}$ iff $\mathcal{F}(A) \subseteq \mathcal{G}(A)$ for all $A \subseteq$ ATOM
$\mathcal{F}^*$ is the *Kleene closure* of $\mathcal{F}$:	$\mathcal{F}^* =$ (the least $\mathcal{G}$ with $\mathcal{I} \sqcup \mathcal{F}^m \circ \mathcal{G} \sqsubseteq \mathcal{G}$) "*do* $\mathcal{F}$ *zero or more times*" [later also $\mathcal{F}^+$: "*one or more times*"]
$\mathcal{F}^m$ is the *monotonic closure* of $\mathcal{F}$:	$\mathcal{F}^m(A) = \bigcup\{\mathcal{F}(B) \mid B \subseteq A\}$
$\mathcal{I}$ is the identity function:	$\mathcal{I}(A) = A$

We have completeness: $\mathcal{F} \vdash \mathcal{G}$ iff $\mathcal{F} \models \mathcal{G}$

university of groningen

properties of set-valued functions

We consider the structure

$$\mathbf{SF}(\mathsf{EXP}) = \langle \mathbb{F}(\mathsf{EXP}), \sqcup, \overset{m}{\circ}, {}^*, \underline{\emptyset}, \mathcal{I} \rangle$$

of set-valued functions over EXP.

Here $\overset{m}{\circ}$ is *monotonic composition*: $\mathcal{F} \overset{m}{\circ} \mathcal{G} = \mathcal{F}^m \circ \mathcal{G}^m$.

Is $\mathbf{SF}(\mathsf{EXP})$ a Kleene algebra?

Recall that Kleene algebra axiomatizes composition, choice and iteration $(\cdot, +, ^*)$.

Well, almost: it turns out that $\mathbf{SF}(\mathsf{EXP})$ is a *weak lazy Kleene algebra*.

university of groningen

Kleene algebra

$$
\begin{aligned}
& a + b = b + a \\
& a + (b + c) = (a + b) + c \\
& a + a = a \\
\\
& a \cdot (b \cdot c) = (a \cdot b) \cdot c \\
\\
& (a + b) \cdot c = a \cdot c + b \cdot c \\
& a \cdot (b + c) = a \cdot b + a \cdot c \\
\\
& a + 0 = a \\
& a \cdot 0 = 0 \cdot a = 0 \\
& a = a \cdot 1 = 1 \cdot a \\
\\
& 1 + a \cdot a^* \leq a^* \\
& 1 + a^* \cdot a \leq a^* \\
& a \cdot b \leq b \Rightarrow a^* \cdot b \leq b \\
& a \cdot b \leq a \Rightarrow a \cdot b^* \leq a
\end{aligned}
$$

university of groningen

lazy Kleene algebra

proposed by Bernhard Möller in 2007 to axiomatize predicate transformers

$$
\begin{aligned}
& a + b = b + a \\
& a + (b + c) = (a + b) + c \\
& a + a = a \\
\\
& a \cdot (b \cdot c) = (a \cdot b) \cdot c \\
\\
& (a + b) \cdot c = a \cdot c + b \cdot c \\
& b \leq c \Rightarrow a \cdot b \leq a \cdot c \\
\\
& a + 0 = a \\
& 0 \cdot a = 0 \\
& a = a \cdot 1 = 1 \cdot a \\
\\
& 1 + a \cdot a^* \leq a^* \\
& a \cdot b \leq b \Rightarrow a^* \cdot b \leq b
\end{aligned}
$$

university of groningen

weak lazy Kleene algebra

$$a + b = b + a$$
$$a + (b + c) = (a + b) + c$$
$$a + a = a$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

$$(a + b) \cdot c = a \cdot c + b \cdot c$$
$$b \leq c \Rightarrow a \cdot b \leq a \cdot c$$

$$a + 0 = a$$
$$0 \cdot a = 0$$
$$a \leq a \cdot 1 = 1 \cdot a$$

$$1 + a \cdot a^* \leq a^*$$
$$a \cdot b \leq b \Rightarrow a^* \cdot b \leq b \cdot 1$$

university of groningen

useful laws in weak lazy Kleene algebra

$$1 \leq a^*$$
$$a \leq a^*$$
$$a \leq b \Rightarrow a^* \leq b^*$$
$$a^* \cdot a^* = (a^*)^* = a^*$$

$$c + a \cdot b \leq b \Rightarrow a^* \cdot c \leq b \cdot 1$$

$$(a + b)^* = a^* \cdot (b \cdot a^*)^*$$

$$(a \cdot b)^* \cdot a \leq a \cdot (b \cdot a)^*$$
$$(a \cdot b)^* \cdot a = a \cdot (b \cdot a)^*$$ if a is left distributive, i.e. $a \cdot (c + d) = a \cdot c + a \cdot d$ for all c, d

university of groningen

projection and restriction

projection: I_A with $\quad \mathcal{I}_A(B) \quad = \quad B \cap A$

restriction: $\mathcal{F}\restriction A$ with $\quad (\mathcal{F}\restriction A)(B) \quad = \quad \mathcal{F}(B)$ if $B \subseteq A$
$\quad = \quad \emptyset$ if $B \not\subseteq A$

We have

$$\mathcal{F}\restriction A \sqsubseteq \mathcal{F}, \quad \mathcal{I}_A \circ \mathcal{F} \sqsubseteq \mathcal{F}$$

$$\mathsf{pr}(\mathcal{I}_A \circ \mathcal{F}\restriction B) \subseteq_2 \mathsf{pr}(\mathcal{F}) \cap_2 (A, B)$$

Here $\mathsf{pr}(\mathcal{F}) = (\mathsf{pr}^+(\mathcal{F}), \mathsf{pr}^-(\mathcal{F}))$ with

$$\mathsf{pr}^-(\mathcal{F}) = \mathsf{dom}(\mathcal{F}) = \bigcup\{\mathcal{F}(A) \mid A \subseteq \mathsf{ATOM}\}$$
$$\mathsf{pr}^+(\mathcal{F}) = \mathsf{rg}(\mathcal{F}) = \bigcup\{A \mid A \subseteq \mathsf{ATOM}, \mathcal{F}(A) \neq \emptyset\}$$

university of groningen

propositional interpolation

if $P \subseteq_2 \mathsf{pr}(\mathcal{F})$ then $\mathcal{J} = \mathcal{F}^+|P$ is a uniform interpolant for $\mathcal{F}$ and P:

$\mathsf{pr}(\mathcal{J}) \subseteq_2 P$, and for all $\mathcal{G}$, $\mathcal{H}$ with $\mathsf{pr}(\mathcal{F}) \cap (\mathsf{pr}^{-1}(\mathcal{G}) \cup \mathsf{pr}(\mathcal{H})) \subseteq_2 P$ we have

$$\mathcal{F} \sqcup \mathcal{G} \vdash \mathcal{H} \text{ iff } \mathcal{J} \sqcup \mathcal{G} \vdash \mathcal{H}$$

proof: we have $\mathcal{F} \vdash \mathcal{J}$, so $\Leftarrow$ is easy. For $\Rightarrow$:

$$\begin{array}{l} \quad \mathcal{F} \sqcup \mathcal{G} \vdash \mathcal{H} \\ \Leftrightarrow \\ \quad \mathcal{H} \sqsubseteq (\mathcal{F} \sqcup \mathcal{G})^*|\mathsf{pr}(\mathcal{H}) \\ \Rightarrow \\ \quad \mathcal{H} \sqsubseteq ((\mathcal{F}^+|(\mathsf{pr}^{-1}(\mathcal{G}) \cup \mathsf{pr}(\mathcal{H}))) \sqcup \mathcal{G})^*|\mathsf{pr}(\mathcal{H}) \\ \Rightarrow \\ \quad \mathcal{H} \sqsubseteq ((\mathcal{F}^+|P) \sqcup \mathcal{G})^*|\mathsf{pr}(\mathcal{H}) \\ \Leftrightarrow \\ \quad \mathcal{J} \sqcup \mathcal{G} \vdash \mathcal{H} \end{array}$$

university of groningen

making it polynomial

in general, $\|\mathcal{J}\|$ is exponential in $\|\mathcal{F}\|$

however, for non-uniform interpolation a polynomial-size interpolant $\mathcal{J}_\mathsf{p}$ can be found

here we use *thin* functions $\mathcal{F}$ where $\mathcal{F}(A) \cap \mathcal{F}(B) = \emptyset$ whenever $A \neq B$

and a thinning operator Θ with $\Theta(\mathcal{F}) \sqsubseteq \mathcal{F}$, $(\Theta(\mathcal{F}))^*(\emptyset) = \mathcal{F}^*(\emptyset)$

then we get $\|\mathcal{J}_\mathsf{p}\| \leq \#\mathsf{pos}(\mathcal{F}) \cdot (\#\mathsf{neg}(\mathcal{F}) + 1) \cdot \#\mathsf{neg}(\mathcal{H})$

university of groningen

first-order Horn logic

terms in TERM are built from variables in VAR and functions in $\mathsf{SIG_f}$

atomic formulae in ATOM are built from terms in TERM and predicates in $\mathsf{SIG_p}$

set-valued functions $\mathcal{F} : \wp(\mathsf{ATOM}) \to \wp(\mathsf{ATOM})$

substitutions $\sigma : \mathsf{VAR} \to \mathsf{TERM}$

applied to $\mathcal{F}$: $(\sigma \cdot \mathcal{F})(X) = \{\sigma(\varphi) \mid \varphi \in \mathcal{F}^m(\{\psi \mid \sigma(\psi) \in X\})\}$

we have

$$(\sigma \cdot \mathcal{F})^+ = \sigma \cdot (\mathsf{Eq}_\sigma \circ \mathcal{F} \circ \mathsf{Eq}_\sigma)^+$$

where $\mathsf{Eq}_\sigma(X) = \{\varphi \mid \exists \psi \in X \ \sigma(\varphi) = \sigma(\psi)\}$

university of groningen

characterization of $\vdash$

universal closure: $\mathsf{UC}(\mathcal{F}) = \bigsqcup_{\sigma \in \mathsf{SUB}} \sigma \cdot \mathcal{F}$

Kleene universal closure: $\mathsf{UCK}(\mathcal{F}) = (\mathsf{UC}(\mathcal{F}))^*$

we define: $\mathcal{F} \vdash \mathcal{G}$ by $\mathcal{G} \sqsubseteq \mathsf{UCK}(\mathcal{F})$, i.e. $\mathcal{G} \sqsubseteq (\bigsqcup_{\sigma \in \mathsf{SUB}} \sigma \cdot \mathcal{F})^*$

we have completeness: $\mathcal{F} \vdash \mathcal{G}$ iff $\mathcal{F} \models \mathcal{G}$

university of groningen

substitution splitting

given $S \subseteq \mathsf{SIG_f}$, how to split $\sigma \in \mathsf{SUB}$ in an S-part $\sigma^S \in \mathsf{SUB}_S$ and τ outside S?

define $\mathsf{TERM_{elim}} = \{t \mid t = f(t_1, \ldots t_n)$ with $f \notin S\}$

define $\mathsf{VAR_{new}} = \{x_t \mid t \in \mathsf{TERM_{elim}}\}$

define σ^S by: $\sigma^S(x) = (\sigma(x)$ with subterms $s \in \mathsf{TERM_{elim}}$ replaced by $x_s)$

define τ by: $\tau(x_t) = t$ for all $x_t \in \mathsf{VAR_{new}}$

claim: now $\sigma = \tau \circ \sigma^S$

consequence: $\mathsf{UC}(\mathcal{F}) = \tau \cdot \mathsf{UC}_S(\mathcal{F})$

where $\mathsf{UC}_S(\mathcal{F}) = \bigsqcup\{\sigma \cdot \mathcal{F} \mid \sigma \in \mathsf{SUB}, \mathsf{sig}\sigma \subseteq S\}$

university of groningen

first-order interpolation

uniform interpolation: for any $\mathcal{F}$ and for any $P = (P^+, P^-) \subseteq_2 \mathsf{pr}(\mathcal{F})$, there is a *uniform interpolant* $\mathcal{J}$ with

1. $\mathsf{pr}(\mathcal{J}) \subseteq_2 P$ and $\mathsf{fun}(\mathcal{J}) \subseteq \mathsf{fun}(\mathcal{F})$;
2. for all $\mathcal{G}$, $\mathcal{H}$ with $\mathsf{pr}^{\pm}(\mathcal{F}) \cap (\mathsf{pr}^{\mp}(\mathcal{G}) \cup \mathsf{pr}^{\pm}(\mathcal{H})) \subseteq_2 P$ we have

$$\mathcal{F} \sqcup \mathcal{G} \vdash \mathcal{H} \quad \text{iff} \quad \mathcal{J} \wedge \mathcal{G} \vdash \mathcal{H}.$$

interpolant: $\mathcal{J} = ((\bigsqcup_{\sigma \in \mathsf{SUB}_{\mathcal{F}}} \sigma \overset{\mathsf{m}}{:} \mathcal{F})^+ | A_P)$

notation used:

$$A_P = \{\alpha \in \mathsf{ATOM} \mid \mathsf{pr}(\alpha) \subseteq_2 P\}$$
$$A_{\mathcal{F}} = \{\alpha \in \mathsf{ATOM} \mid \mathsf{pr}(\alpha) \subseteq_2 \mathsf{pr}(\mathcal{F})\}$$

university of groningen

first-order interpolation

$\mathcal{F} \sqcup \mathcal{G} \vdash \mathcal{H}$

$\Leftrightarrow$

$\mathcal{H} \sqsubseteq \mathsf{UCK}(\mathcal{F} \sqcup \mathcal{G}) | A_{\mathcal{H}}$

$\Leftrightarrow$

$\mathcal{H} \sqsubseteq (\mathsf{UC}(\mathcal{F}) \sqcup \mathsf{UC}(\mathcal{G}))^* | A_{\mathcal{H}}$

$\Leftrightarrow$

$\mathcal{H} \sqsubseteq (\mathsf{UC}(\mathcal{F})^+ | (A_{\mathcal{G}}^{-1} \cup A_{\mathcal{H}}) \sqcup \mathsf{UC}(\mathcal{G}))^* | A_{\mathcal{H}}$

$\Leftrightarrow$

$\mathcal{H} \sqsubseteq (\tau \overset{\mathsf{m}}{:} ((\bigsqcup_{\sigma \in \mathsf{SUB}_{\mathcal{F}}} \sigma \overset{\mathsf{m}}{:} \mathcal{F})^+ | A_P) \sqcup \mathsf{UC}(\mathcal{G}))^* | A_{\mathcal{H}}$

$\Leftrightarrow$

$\mathcal{H} \sqsubseteq (\tau \overset{\mathsf{m}}{:} \mathcal{J} \sqcup \mathsf{UC}(\mathcal{G}))^* | A_{\mathcal{H}}$

$\Rightarrow$

$\mathcal{H} \sqsubseteq (\mathsf{UC}(\mathcal{J}) \sqcup \mathsf{UC}(\mathcal{G}))^* | A_{\mathcal{H}}$

$\Leftrightarrow$

$\mathcal{H} \sqsubseteq \mathsf{UCK}(\mathcal{J} \sqcup \mathcal{G}) | A_{\mathcal{H}}$

$\Leftrightarrow$

$\mathcal{J} \sqcup \mathcal{G} \vdash \mathcal{H}$

university of groningen

further work

try to strenghten the function parameter condition $\mathsf{fun}(\theta) \subseteq \mathsf{fun}(\varphi)$ to

$$\mathsf{fun}(\theta) \subseteq \mathsf{fun}(\varphi)) \cap (\mathsf{fun}(\psi) \cup \mathsf{fun}(\chi))$$

apply set-valued functions to propositional logics

apply set-valued functions to (conditional) equational logic

consider other types of conditional statements: sequents, proof rules

apply set-valued functions to the theory of derived rules

...

The Existence of Pure Proofs

René Gazzari
University of Tübingen, Germany

UniLog 6, 2018, Vichy

June 25, 2018

Relevance of Pure Proofs

[M]athematicians and philosophers of mathematics think that it is somehow valuable for a proof to be 'pure', that is, not to use notions extraneous to what is being proved. (Arana)

- The problem of pure proofs is a philosophical problem about informal mathematics. We provide a proof-theoretical, and therefore formal approach to this problem.
- First, we provide a plausible definition of pure derivations, and then we prove that every derivation can be transformed into a pure derivation.

Representation of Informal Mathematics

We represent informal mathematics as follows:

1. *proofs:* by "a formal system which comes as close as possible to actual reasoning" (Gentzen): the calculus of Natural Deduction.
2. *notions:* by non-logical symbols (constant symbols, function symbols, and relation symbols) of a formal first order language.
3. *mathematical theories:* by a set of axioms faithfully formulated in the rich language with sufficiently many non-logical symbols.
4. *pureness:* by a formal definition of pure derivations (depending on occurrences of non-logical symbols in derivations).

Natural Deduction

1. as given by Gentzen
2. with inference rules for
   $\perp$ (reductio ad absurdum), implication (introduction // elimination), universal quantifier (introduction // elimination)
3. without inference rules for identity

Prawitz Normal Form

1. Theorem (Prawitz): every derivation **D** can be transformed into a cut-free derivation **F** stronger than **D**.
2. **F** is called the (Prawitz) Normal Form of **D**
3. **F** can be strictly stronger, as we may loose assumptions while normalisation.
4. every formula occurring in a normal derivation is a subformula of an open assumption or of the conclusion (subformula principle)
5. classical exception: negated assumptions ($\neg A$) discharged in an reductio ad absurdum step.

Pureness

1. An open assumption of **D** is called essential, if it is an open assumption of the normal form **F** of **D**.
2. An non-logical symbol ξ is called relevant in **D**, if ξ occurs in an essential assumption or in the conclusion of **D**.
3. A derivation **D** is (absolutely) pure, if every non-logical symbol occurring somewhere in **D** is relevant.

Well-Known Results

1. Technical Lemma (Completeness) Every constant symbol occurring in a derivation can be replaced by a fresh parameter.
   (Proof: induction over the structure of derivations.)
2. Consequence: every derivation can be transformed into a pure derivation with respect to constant symbols by replacing all non-relevant constant symbols.
3. Furthermore: due to the subformula principle, every derivation in Prawitz normal form is pure with respect to relation symbols (the classical exception makes no difference)

Introduction
Theory of Occurrences
Formal Results
Philosophical Issues
Natural Deduction
Notions of Pureness

Function Symbols

1. Can we eliminate non-relevant function symbols?
2. Yes!
3. Simple example (in Prawitz Normal Form):

$$\dfrac{\dfrac{\forall x.P(x)}{P(f(v))} \qquad \dfrac{\forall x.P(x) \to P(v)}{P(f(v)) \to P(v)}}{P(v)} \quad \rightsquigarrow \quad \dfrac{\dfrac{\forall x.P(x)}{P(w)} \qquad \dfrac{\forall x.P(x) \to P(v)}{P(w) \to P(v)}}{P(v)}$$

   The function symbol f is not relevant in $\mathbf{D}$.
4. But: we need an elaborate theory of occurrences (of terms in derivations) for the proof.

Concept (Occurrence)

1. An occurrence is determined by the following three aspects:
   - *context:* An occurrence is an occurrence in a syntactic entity.
   - *shape:* An occurrence is an occurrence of a syntactic entity.
   - *position:* An occurrence is an occurrence at a position in the context.
2. Context and shape are standard syntactic entities.
3. The position is given by nominal forms in which the intended positions of the occurrences are marked by nominal symbols $*_k$.

Example (Occurrence)

The (multiple) occurrence of the problematic term $f(v)$ is given as follows:

1. The position of the problematic term occurrence is given by the following nominal derivation:

$$\mathbf{D} \doteq \dfrac{\dfrac{\forall x.P(x)}{P(f(v))} \quad \dfrac{\forall x.P(x) \to P(v)}{P(f(v)) \to P(v)}}{P(v)} \rightsquigarrow \dfrac{\dfrac{\forall x.P(x)}{P(*_0)} \quad \dfrac{\forall x.P(x) \to P(v)}{P(*_0) \to P(v)}}{P(v)}$$

2. The context is the derivation $\mathbf{D}$, its shape is the term $f(v)$.

Nominal Forms

We need three types of nominal forms:

1. *nominal terms:* with nominal symbols $*_k$ as new atoms:

$$*_k \mid v \mid x \mid c \mid f(\mathtt{t}_0, \ldots \mathtt{t}_n)$$

2. *nominal formulae:* via nominal terms instead of standard terms:

$$\bot \mid (\mathtt{t} = \mathtt{s}) \mid P(\mathtt{t}_0, \ldots \mathtt{t}_{n-1}) \mid (\mathtt{A} \to \mathtt{B}) \mid (\forall x.\mathtt{A})$$

3. *nominal trees:* without inference rules, but with arbitrary discharge of assumptions

$$\mathtt{A} \quad \Big| \quad \frac{\mathtt{D}_0 \qquad \ldots \qquad \mathtt{D}_n}{\mathtt{A}}$$

Remarks (Nominal Forms)

1. Three types of nominal forms:
   - $\rightsquigarrow$ most definitions have three (analogous) versions.
   - $\rightsquigarrow$ we only provide one version in the talk
2. Classification of nominal forms:
   1. standard entity = no nominal symbol occurring
   2. simple = no nominal symbol occurs more than once
   3. unary = $*_0$ is the only nominal symbol
   4. single = simple and unary

Central Tool: General Substitution Function

1. Binary function on nominal trees and arbitrary sequences of nominal terms, resulting in nominal trees.
2. $\mathtt{D}[\mathtt{s}_0, \ldots \mathtt{s}_n]$ is the result of the simultaneous replacement of all occurrences of nominal symbols $*_0, \ldots *_n$ in $\mathtt{D}$ by the nominal terms $\mathtt{s}_0, \ldots \mathtt{s}_n$, respectively. (recursive definition)
3. *Example:*

$$\frac{P(*_0) \qquad P(*_2)}{P(*_0)} [c + *_0, *_1] \doteqdot \frac{P(c + *_0) \qquad P(*_2)}{P(c + *_0)}$$

Introduction
Theory of Occurrences
Formal Results

Theory of Occurrences
Excursus: Basic Proof Theory
Congruence Relation

Remarks (General Substitution Function)

1. Subsequently, only unary nominal forms needed; we write $*$ for $*_0$.

René Gazzari, elbron@gmx.net

Pure Proofs

Less-Structured Relation

1. A nominal tree $\mathtt{D}$ is less-structured than a nominal tree $\mathtt{D}'$ ($\mathtt{D} \leq \mathtt{D}'$), if there is a nominal term $\mathtt{s}$ such that $\mathtt{D}' \doteq \mathtt{D}[\mathtt{s}]$.
2. Example (with atomic trees):

$$\text{(minimal)} \quad P(*) \;\leq\; P(* + c) \;\leq\; P(c + c) \quad \text{(maximal)}$$

As: $P(*)[* + c] \doteq P(* + c)$, and $P(* + c)[c] \doteq P(c + c)$.

Introduction
Theory of Occurrences
Formal Results
Theory of Occurrences
Excursus: Basic Proof Theory
Congruence Relation

Elimination Forms

1. $\mathtt{D}$ is called an elimination form of a standard tree $\mathbf{D}$, if $\mathtt{D} \leq \mathbf{D}$.
2. $\mathtt{D}$ is called a nominal derivation, if $\mathtt{D}$ is an elimination form of a standard derivation $\mathbf{D}$.

René Gazzari, elbron@gmx.net
Pure Proofs

Occurrences

1. A triple $\mathfrak{o} = \langle \mathbf{D}, t, \mathtt{D} \rangle$ is called an occurrence of the term t in the derivation $\mathbf{D}$, if $\mathtt{D}$ is an elimination form of $\mathbf{D}$ in which the standard term t is eliminated ($\mathbf{D} \simeq \mathtt{D}[t]$).
   - $\mathbf{D}$ is the context of $\mathfrak{o}$
   - t is the shape of $\mathfrak{o}$
   - $\mathtt{D}$ is the position of $\mathfrak{o}$
2. A single occurrence $\mathfrak{o}$ is called extraneous, if its shape contains a non-logical symbol which is not relevant in the underlying derivation.

Lies-Within Relation

1. An occurrence $\mathfrak{o}$ lies-within an occurrence $\mathfrak{o}'$ ($\mathfrak{o} \leq \mathfrak{o}'$), if the position $\mathtt{D}'$ of $\mathfrak{o}'$ is less-structured than the position $\mathtt{D}$ of $\mathfrak{o}$. We also say that $\mathfrak{o}'$ contains $\mathfrak{o}$.
2. Example:

$$\frac{P(c+c)}{Q \to P(c+c)} \quad \leq \quad \frac{P(c+c)}{Q \to P(c+c)}$$

As:

$$\frac{P(*)}{Q \to P(c+c)} \quad \leq \quad \frac{P(*+c)}{Q \to P(c+c)}$$

Via: $\mathtt{s} \simeq * + c$

Basic Proof Theory

1. Needed: detailed specification of the position of occurrences.
2. Based on a theory of occurrences of subtrees in nominal trees (using a second kind of nominal symbols).
   - inference step = occurrence of the subtree generated in that step.
   - assumption = atomic inference step.
3. A term occurrence $\mathfrak{o}$ has a location, if there is an inference step such that all nominal symbols of $\mathfrak{o}$ occur in its conclusion.
   - in this case: the inference step is the location of $\mathfrak{o}$.
   - in this case: the nominal formula $\mathtt{A}$ in the conclusion of that inference step is the local position of $\mathfrak{o}$.
4. More specifications are possible!

Introduction
Theory of Occurrences
Formal Results
Theory of Occurrences
Excursus: Basic Proof Theory
Congruence Relation

Special Occurrences

The following special occurrences are crucial in our discussion:

1. Occurrences of a (co-)eigenvariable are single occurrences located as follows:

$$\begin{array}{c} \vdots \\ \dfrac{A(v)}{\forall x.A(x)} \\ \vdots \end{array}$$

2. Occurrences of a (co-)eigenterm are single occurrences located as follows:

$$\begin{array}{c} \vdots \\ \dfrac{\forall x.A(x)}{A(t)} \\ \vdots \end{array}$$

René Gazzari, elbron@gmx.net
Pure Proofs

Restricted Occurrences

1. A single occurrence $\mathfrak{o}$ is called restricted, if one of the following conditions is satisfied:
   - $\mathfrak{o}$ strictly contains an occurrence of an eigenvariable or of an eigenterm.
   - $\mathfrak{o}$ contains an occurrence of a co-eigenvariable or of a co-eigenterm.
2. We also attribute this property to the respective positions.

Congruence Relation

We introduce a congruence relation $\cong$:

1. *principle idea:* Relate single occurrences (their positions) which have necessarily the same shape (due to the inference rules).
2. *example:*

$$\frac{P(t) \qquad P(t)}{P(t) \wedge P(t)}$$

3. *relevance:* If we want to replace a term t by another term s in a derivation, then we have to replace complete congruence classes of t

Principle Definition

1. Define direct congruence $\cong_1$ between elimination forms of the underlying derivation.
   - ⇝ $\cong_1$ between single elimination forms having (almost) the same local positions at the right locations.
   - ⇝ Recursive definition depending on the structure of the underlying derivation; each inference rule has several subcases.
   - ⇝ We provide the two example clauses of the definition.

Direct Congruence

1. Example clause: introduction of the implication

$$\dfrac{\begin{matrix}[\mathtt{A}]_! \quad [A] \\ \mathtt{F} \\ \mathtt{B}\end{matrix}}{A \to \mathtt{B}} \quad \cong_1 \quad \dfrac{\begin{matrix}[A] \\ \mathtt{F} \\ B\end{matrix}}{\mathtt{A} \to \mathtt{B}}$$

- single discharged assumption $\doteqdot$ antecedent of the conclusion
- conclusion of the premise $\doteqdot$ succedent of the conclusion
- already congruent in the direct subtrees (recursive cases)

Direct Congruence

1. Example clause: introduction of the universal quantifier

$$\frac{\begin{array}{c}F\\ \mathtt{A}(v)\end{array}}{\forall x.A(x)} \cong_1 \frac{\begin{array}{c}F\\ \mathtt{A}(v)\end{array}}{\forall x.\mathtt{A}(x)}$$

   - (different) positions of the eigenvariable.
   - conclusion of the premise $\hat{=}$ kernel of the conclusion modulo substitution of the eigenvariable; additionally, the elimination forms are not restricted.
   - already congruent in the direct subtrees (recursive cases)

Congruence Relation

1. Congruence of single elimination forms is the reflexive, symmetric, and transitive closure of the direct congruence
2. Congruence of single occurrences defined via the congruence of their positions.

Representation of Congruence Classes

1. *needed:* one common elimination form representing all single elimination forms in a congruence class.
2. *definable:* A merge function μ resulting in common elimination form having nominal symbols at the same positions as any of the arguments. (recursive definition)
3. *common elimination form:* If $\mathcal{S}$ is a congruence class, then $\mu(\mathcal{S})$ is the common elimination form of all members of $\mathcal{S}$.

Proposition (Restricted Occurrences)

1. Let **D** in Prawitz Normal Form.
2. If $\mathfrak{o} = \langle \mathbf{D}, f(t_0, \ldots t_n), \mathtt{D} \rangle$ is a single and restricted occurrence of a complex term with main function symbol f then the following both statements hold:
3. There is a complex term $f(s_0, \ldots s_n)$ occurring in an open assumption or in the conclusion of **D**.
4. If **D** ends with an elimination step, then there is such an occurrence occurring in an open assumption.

$\rightsquigarrow$ Proved by induction over the number of elimination of the implication steps in **D**.

Side Conditions

If a term s is intended to replace some occurrences of a term t in a derivation, two side conditions have to be satisfied:

1. *regularity:* No quantifiable variable may occur free in a derivation.
   $\rightsquigarrow$ Demand that $V_q(s) \subseteq V_q(t)$.
2. *eigenvariable:* Eigenvariables may not occur in open assumptions.
   $\rightsquigarrow$ Demand that $V_p(s) \cap V_e(\mathbf{D}) \subseteq V_p(t)$.

Substitution Theorem

1. Let $\mathfrak{o} = \langle \mathbf{D}, t, \mathtt{D} \rangle$ be a single occurrence of a term t in a derivation $\mathbf{D}$ such that no restricted occurrence is congruent to $\mathfrak{o}$.
2. Let s be a term satisfying the regularity condition with respect to t and the eigenvariable condition.
3. If $\mathcal{S}$ is the set of positions of occurrences congruent to $\mathfrak{o}$, then $\mu(\mathcal{S})[s]$ is a derivation.

$\rightsquigarrow$ Proved by induction over the structure of $\mathbf{D}$.

Remarks (Substitution Theorem)

1. The substitution theorem is general as there are only few restriction on the involved terms.
2. The substitution theorem is strong, as we do not have to replace all occurrences of a given term, but minimal classes.
3. With the help of the substitution theorem, we are able to replace all extraneous occurrences of terms in a derivation.

Existence of Pure Derivations

1. Let **D** be an arbitrary derivation.
2. There is a pure derivation **F** stronger than **D**.

Proof Sketch (Existence of Pure Derivations)

1. Without loss of generality, **D** is in Prawitz Normal Form, and therefore pure with respect to relation symbols.
2. Assume: there is an extraneous occurrence $\mathfrak{o} = \langle \mathbf{D}, t, \mathtt{D} \rangle$ of a term t.
   - The congruence class of $\mathfrak{o}$ does not contain restricted occurrences. ($\mathfrak{o}$ is extraneous.)
   - A fresh parameter w (not occurring in **D**) satisfies both the regularity and the eigenvariable condition with respect to t.
   - Let $\mathcal{S}$ be the set of positions of the occurrences $\mathfrak{o}'$ congruent to $\mathfrak{o}$. $\mathbf{D}' \doteq \mu(\mathcal{S})[w]$ is a derivation.
3. $\mathbf{D}'$ has the same open assumptions and the same conclusion as **D**, as $\mathfrak{o}$ is extraneous. The number of extraneous occurrences in $\mathbf{D}'$ is strictly less than in **D**.
4. Repeating finitely many times such a substitution, we obtain the pure derivation **F** as demanded.

Introduction
Theory of Occurrences
Formal Results
Restricted Occurrences
Substitution Theorem
Existence of Pure Proofs

Concluding Remarks

1. *example:* Let **D** be an arithmetical proof of a statement about addition such that multiplication is not mentioned in any essential assumption. If multiplication somewhere occurs in **D**, then we can eliminate such occurrences.
2. *identity of proofs:* Eliminating non-relevant function symbols does not change the normal form (modulo some terms), but a relevant property of the derivation (pureness). Therefore, Pravitz normal form does not seem to be fine enough for defining the identity of proofs.

René Gazzari, elbron@gmx.net
Pure Proofs

Last Slide

Thank You!

Introduction
Theory of Occurrences
Formal Results
Restricted Occurrences
Substitution Theorem
Existence of Pure Proofs

Literature

- Andrew Arana, *On Formally Measuring Extraneous Notions*. Philosophia Mathematica (III) **17** (2009).
- Gerhard Gentzen, *Investgations into Logical Deduction*. in: M. E. Szabo (Editor), *The Collected Papers of Gerhard Gentzen*. North Holland Publishing Company, 1969.

René Gazzari, elbron@gmx.net
Pure Proofs

CERES: Automated Deduction in Proof Theory

Alexander Leitsch
Vienna University of Technology

proof theory and automated deduction

proof theory:

- analysis of proofs and provability
- sequent calculus, natural deduction
- cut-elimination, normalization

proof theory and automated deduction

proof theory:

- analysis of proofs and provability
- sequent calculus, natural deduction
- cut-elimination, normalization

automated deduction:

- proof search
- resolution calculus, paramodulation, superposition
- refinements, redundancy and deletion

proof theory and automated deduction

does proof theory benefit from automated deduction?

proof theory and automated deduction

does proof theory benefit from automated deduction?

yes.

proof theory and automated deduction

does proof theory benefit from automated deduction?

yes.

- Automation support in formal proof verification.
- Cut-elimination by resolution
- **theoretical** results from automated deduction yield new insights in cut-elimination.

Cut-elimination

- reductive methods: based on Gentzen's proof. Stepwise (and local) reduction of cuts. Local proof rewriting system.

Cut-elimination

- reductive methods: based on Gentzen's proof. Stepwise (and local) reduction of cuts. Local proof rewriting system.
- semantic methods: prove cut-free completeness.

Cut-elimination

- reductive methods: based on Gentzen's proof. Stepwise (and local) reduction of cuts. Local proof rewriting system.
- semantic methods: prove cut-free completeness.
- CERES: cut-elimination by resolution. Semi-semantic global method, based on resolution refutations of unsatisfiable clause sets. Works for **LK**, **LJ**, higher-order **LK**, finitely-valued **LK**, hypersequent calculus for Gödel logic, **LK** proof schemata.

  In this talk: **LK** and **LJ**.

The Method CERES: cut-elimination by resolution

Example: $\varphi =$

$$\frac{\varphi_1 \qquad \varphi_2}{(\forall x)(P(x) \rightarrow Q(x)) \vdash (\exists y)(P(a) \rightarrow Q(y))}\ cut$$

$\varphi_1 =$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u), P(u) \rightarrow Q(u) \vdash Q(u)}\ \rightarrow: l}{P(u) \rightarrow Q(u) \vdash P(u) \rightarrow Q(u)}\ \rightarrow: r}{P(u) \rightarrow Q(u) \vdash (\exists y)(P(u) \rightarrow Q(y))}\ \exists: r}{(\forall x)(P(x) \rightarrow Q(x)) \vdash (\exists y)(P(u) \rightarrow Q(y))}\ \forall: l}{(\forall x)(P(x) \rightarrow Q(x)) \vdash (\forall x)(\exists y)(P(x) \rightarrow Q(y))}\ \forall: r$$

$$S = \{P(u) \vdash\} \times \{\vdash Q(u)\}.$$

Example

$\varphi =$

$$\dfrac{\varphi_1 \qquad \varphi_2}{(\forall x)(P(x) \to Q(x)) \vdash (\exists y)(P(a) \to Q(y))}\ cut$$

$\varphi_2 =$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \vdash P(a) \qquad Q(v) \vdash Q(v)}{P(a), P(a) \to Q(v) \vdash Q(v)}\ \to: l}{P(a) \to Q(v) \vdash P(a) \to Q(v)}\ \to: r}{P(a) \to Q(v) \vdash (\exists y)(P(a) \to Q(y))}\ \exists: r}{(\exists y)(P(a) \to Q(y)) \vdash (\exists y)(P(a) \to Q(y))}\ \exists: l}{(\forall x)(\exists y)(P(x) \to Q(y)) \vdash (\exists y)(P(a) \to Q(y))}\ \forall: l$$

$$S' = \{\vdash P(a)\} \cup \{Q(v) \vdash\}.$$

cut-ancestors in axioms:

$$S_1 = \{P(u) \vdash\},\ S_2 = \{\vdash Q(u)\},\ S_3 = \{\vdash P(a)\},\ S_4 = \{Q(v) \vdash\}.$$

$$S = S_1 \times S_2 = \{P(u) \vdash Q(u)\}.$$

$$S' = S_3 \cup S_4 = \{\vdash P(a);\ Q(v) \vdash\}.$$

characteristic clause set:

$$\mathrm{CL}(\varphi) = S \cup S' = \{P(u) \vdash Q(u);\ \vdash P(a);\ Q(v) \vdash\}.$$

Projection of φ to $\mathrm{CL}(\varphi)$

- *Skip inferences leading to cuts.*
- Obtain cut-free proof of end-sequent + a clause in $\mathrm{CL}(\varphi)$.

$$\begin{array}{c}\text{proof } \varphi \text{ of } S\\ \Downarrow\\ \text{cut-free proof } \varphi(C) \text{ of } S \circ C.\end{array}$$

Let φ be the proof of the sequent
$S\colon (\forall x)(P(x) \to Q(x)) \vdash (\exists y)(P(a) \to Q(y))$ shown above.

$$\mathrm{CL}(\varphi) = \{P(u) \vdash Q(u);\ \ \vdash P(a);\ \ Q(v) \vdash\}.$$

Skip inferences in φ_1 leading to cuts:

$$\dfrac{\dfrac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u), P(u) \to Q(u) \vdash Q(u)}\ \to: l}{P(u), (\forall x)(P(x) \to Q(x)) \vdash Q(u)}\ \forall: l$$

$\varphi(C_1) =$

$$\dfrac{\dfrac{\dfrac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u), P(u) \to Q(u) \vdash Q(u)}\ \to: l}{P(u), (\forall x)(P(x) \to Q(x)) \vdash Q(u)}\ \forall: l}{P(u), (\forall x)(P(x) \to Q(x)) \vdash (\exists y)(P(a) \to Q(y)), Q(u)}\ w: r$$

φ proof of
$S\colon (\forall x)(P(x) \to Q(x)) \vdash (\exists y)(P(a) \to Q(y))$

$$\mathrm{CL}(\varphi) = \{P(u) \vdash Q(u);\ \ \vdash P(a);\ \ Q(v) \vdash\}.$$

For $C_2 = \vdash P(a)$ we obtain the projection $\varphi(C_2)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a) \vdash P(a)}{P(a) \vdash P(a), Q(v)}\ w:r}{\vdash P(a) \to Q(v), P(a)}\ \to: r}{\vdash (\exists y)(P(a) \to Q(y)), P(a)}\ \exists: l}{(\forall x)(P(x) \to Q(x)) \vdash (\exists y)(P(a) \to Q(y)), P(a)}\ w:l$$

The Method CERES

given proof φ of S (S skolemized),

- extract characteristic clause set $\mathrm{CL}(\varphi)$,
- compute the projections of φ to clauses in $\mathrm{CL}(\varphi)$,
- construct a resolution refutation γ of $\mathrm{CL}(\varphi)$,
- insert the projections of φ into $\gamma \Rightarrow$ CERES normal form of φ.

Example

φ proof of
$S\colon (\forall x)(P(x) \to Q(x)) \vdash (\exists y)(P(a) \to Q(y))$

$$\mathrm{CL}(\varphi) = \{C_1 : P(u) \vdash Q(u),\ C_2 : \ \vdash P(a),\ C_3 : Q(u) \vdash\}.$$

a resolution refutation δ of $\mathrm{CL}(\varphi)$:

$$\dfrac{\dfrac{\vdash P(a) \quad P(u) \vdash Q(u)}{\vdash Q(a)}\,R \qquad Q(v) \vdash}{\vdash}\,R$$

ground projection γ of δ - this is an **LK**-derivation!:

$$\dfrac{\dfrac{\vdash P(a) \quad P(a) \vdash Q(a)}{\vdash Q(a)}\,R \qquad Q(a) \vdash}{\vdash}\,R$$

via $\sigma = \{u \leftarrow a, v \leftarrow a\}$.

Example

end sequent S of φ, $S = B \vdash C$.
$\gamma =$

$$\dfrac{\dfrac{\vdash P(a) \quad P(a) \vdash Q(a)}{\vdash Q(a)}\,R \qquad Q(a) \vdash}{\vdash}\,R$$

CERES-normal form $\varphi(\gamma) =$

$$\dfrac{\dfrac{\dfrac{\overset{(\chi_2)}{B \vdash C, P(a)} \quad \overset{(\chi_1)}{P(a), B \vdash C, Q(a)}}{B, B \vdash C, C, Q(a)}\,cut \qquad \overset{(\chi_3)}{Q(a), B \vdash C}}{B, B, B \vdash C, C, C}\,cut}{S}\,\text{contractions}$$

atomic cut normal form (ACNF)

Characteristic Clause Set:

Let φ be an **LK**-derivation of S and let Ω be the set of all occurrences of cut formulas in φ. We define the set of clauses $\mathrm{CL}(\varphi)$ inductively:

Let ν be the occurrence of an initial sequent in φ and sq_ν the corresponding sequent. Then

$$S/\nu = \{\mathrm{sq}(\nu, \Omega)\}$$

where $\mathrm{sq}(\nu, \Omega)$ is the subsequent of sq_ν containing the ancestors of Ω.

Assume:
S/ν already constructed for $\mathrm{depth}(\nu) \leq k$.
$\mathrm{depth}(\nu) = k + 1$:

- unary rule: ν is the consequent of μ: $S/\nu = S/\mu$.
- binary rule: ν is the consequent of μ_1 and μ_2:
  - The auxiliary formulas of ν are ancestors of Ω, i.e. the formulas occur in $\mathrm{sq}(\mu_1, \Omega), \mathrm{sq}(\mu_2, \Omega)$:

    $$S/\nu = S/\mu_1 \cup S/\mu_2.$$

  - The auxiliary formulas of ν are not ancestors of Ω:

    $$S/\nu = S/\mu_1 \times S/\mu_2.$$

$\mathrm{CL}(\varphi) = S/\nu_0$ where ν_0 is the occurrence of the end-sequent.

unsatisfiability of $\mathrm{CL}(\varphi)$

If φ is a cut-free proof then there are no occurrences of cut formulas in φ and $\mathrm{CL}(\varphi) = \{\vdash\}$.

Proposition:
Let φ be an **LK**-derivation. Then $\mathrm{CL}(\varphi)$ is unsatisfiable.

structure of proof:

- construct a proof φ^* of $\vdash$ from $\mathrm{CL}(\varphi)$ (as a set of axioms) using the cut-formulas of φ.
- By soundness of **LK** $\mathrm{CL}(\varphi)$ is unsatisfiable.

projection lemma

Lemma:
Let φ be a proof of a sequent $S : \Gamma \vdash \Delta$.
Let $C : \bar{P} \vdash \bar{Q}$ be a clause in $\mathrm{CL}(\varphi)$. Then there exists a deduction

$$\varphi(C) \text{ of } \bar{P}, \Gamma \vdash \Delta, \bar{Q}$$

s.t.

$$\varphi(C) \text{ is cut-free and } \; l(\varphi(C)) \leq l(\varphi).$$

Projection of φ to C: construct $\varphi(C)$.

the remaining steps:

- Construct an R-refutation γ of $\mathrm{CL}(\varphi)$. γ exists by the completeness of resolution.
- insert the projections of φ into γ.
- add some contractions and obtain a proof with (only) atomic cuts (CERES normal form).

Generality of CERES

CERES does *not only* work for **LK**.

- any sound sequent calculus for classical logic (with some form of cut rule) does the job.
- unary rules do not "count".
- *necessary:* auxiliary formulas, principal formulas, ancestor relation

Why CERES?

- Efficient method for (semi-) automated proof analysis due to theorem provers (they refute the characteristic clause set). Analysis of Fürstenberg's (topology-based) proof of the infinitude of primes by CERES (BHLRS 2008).

Why CERES?

- Efficient method for (semi-) automated proof analysis due to theorem provers (they refute the characteristic clause set). Analysis of Fürstenberg's (topology-based) proof of the infinitude of primes by CERES (BHLRS 2008).

- **Analysis of cut-elimination methods**. Behavior of reductive cut-elimination "redundant" w.r.t. characteristic clause sets (subsumption property). CERES is in some sense more general and more efficient than reductive methods (asymptotic complexity analysis) (BL 2006, CLRW 2017).

Why CERES?

- Efficient method for (semi-) automated proof analysis due to theorem provers (they refute the characteristic clause set). Analysis of Fürstenberg's (topology-based) proof of the infinitude of primes by CERES (BHLRS 2008).
- **Analysis of cut-elimination methods**. Behavior of reductive cut-elimination "redundant" w.r.t. characteristic clause sets (subsumption property). CERES is in some sense more general and more efficient than reductive methods (asymptotic complexity analysis) (BL 2006, CLRW 2017).
- Identification of fast cut-elimination classes (i.e. classes where cut-elimination is of elementary complexity) (BL 2010).

subsumption: a detour to automated deduction

C subsumes C' if C' contains an instance of C.

- Let $C\colon \Gamma \vdash \Delta$ and $C'\colon \Gamma' \vdash \Delta'$ be clauses (atomic sequents).
- We define $C \subseteq C'$ if $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$ ($\subseteq$ denotes the multiset inclusion).
- C subsumes C' ($C \leq_{ss} C'$) via ϑ if there exists a substitution ϑ such that $C\vartheta \subseteq C'$.

Examples:

- $P(x) \vdash Q(x) \leq_{ss} P(a) \vdash Q(a), R(a)$ via $\vartheta = \{x \leftarrow a\}$.
- $P(x) \vdash P(f(x)) \not\leq_{ss} P(a) \vdash P(f(f(a)))$.
- $\vdash P(x), P(y) \not\leq_{ss} \vdash P(a)$ (holds for sets, we have multisets).

resolution and subsumption

The subsumption principle:

- resolvents of subsumed clauses are redundant.
- elimination of subsumed clauses preserves completeness of resolution.

The subsumption theorem: Let C_1, C_2, D_1, D_2 be clauses s.t.

- $C_1 \leq_{ss} D_1$,
- $C_2 \leq_{ss} D_2$.

Let D be a resolvent of D_1 and D_2. Then either

- $C_1 \leq_{ss} D$ or
- $C_2 \leq_{ss} D$ or
- there exists a resolvent C of C_1, C_2 such that $C \leq_{ss} D$.

resolution and subsumption

extension of subsumption to sets of clauses:

Let $\mathcal{C}, \mathcal{D}$ be sets of clauses. $\mathcal{C} \leq_{ss} \mathcal{D}$ if for every clause $D \in \mathcal{D}$ there exists a clause $C \in \mathcal{C}$ s.t. $C \leq_{ss} D$.

The extended subsumption theorem:

- if $\mathcal{C} \leq_{ss} \mathcal{D}$ and D is derivable by resolution from $\mathcal{D}$ then there exists a resolution derivation of a clause C from $\mathcal{C}$ s.t. $C \leq_{ss} D$.

This result yields the completeness of resolution + subsumption: $\vdash$ can only be subsumed by $\vdash$.

resolution and subsumption

Example:

$$\begin{array}{rcl} C_1: \ P(x) \vdash Q(x) & \leq_{ss} & D_1: \ P(f(z)) \vdash Q(f(z)), R(z), \\ C_2: \ Q(f(y)) \vdash R(y) & \leq_{ss} & D_2: \ Q(f(a)) \vdash R(a). \end{array}$$

$$\begin{array}{rcl} R(D_1, D_2) & = & P(f(a)) \vdash R(a), R(a), \\ R(C_1, C_2) & = & P(f(y)) \vdash R(y). \end{array}$$

$R(C_1, C_2) \leq_{ss} R(D_1, D_2)$ via $\{y \leftarrow a\}$.

the subsumption principle for proofs

Let γ and δ be resolution deductions. We define $\gamma \leq_{ss} \delta$ by induction on the number of nodes in δ:

If δ consists of a single node labelled with a clause D then $\gamma \leq_{ss} \delta$ if γ consists of a single node labelled with C and $C \leq_{ss} D$.

Let δ be

$$\frac{\begin{array}{cc}(\delta_1) & (\delta_2) \\ D_1 & D_2\end{array}}{D}\ R$$

and γ_1 be a deduction of C_1 with $\gamma_1 \leq_{ss} \delta_1$, γ_2 be a deduction of C_2 with $\gamma_2 \leq_{ss} \delta_2$. Then we distinguish the following cases:

$C_1 \leq_{ss} D$: then $\gamma_1 \leq_{ss} \delta$.
$C_2 \leq_{ss} D$: then $\gamma_2 \leq_{ss} \delta$.

the subsumption principle for proofs

Let δ be

$$\dfrac{\begin{matrix}(\delta_1) & (\delta_2)\\ D_1 & D_2\end{matrix}}{D}\ R$$

and γ_1 be a deduction of C_1 with $\gamma_1 \leq_{ss} \delta_1$, γ_2 be a deduction of C_2 with $\gamma_2 \leq_{ss} \delta_2$. $C_1 \not\leq_{ss} D, C_2 \not\leq_{ss} D$:

Let C be resolvent of C_1 and C_2 s.t. $C \leq_{ss} D$ and $\gamma =$

$$\dfrac{\begin{matrix}(\gamma_1) & (\gamma_2)\\ C_1 & C_2\end{matrix}}{C}\ R$$

Then $\gamma \leq_{ss} \delta$.

the subsumption principle for proofs

- $\mathcal{C}, \mathcal{D}$: sets of clauses and $\mathcal{C} \leq_{ss} \mathcal{D}$.
- Let δ be a resolution deduction of a clause D from $\mathcal{D}$.
- Then there exists a clause C and a resolution deduction γ of C from $\mathcal{C}$ s.t. $\gamma \leq_{ss} \delta$.

the subsumption principle for proofs

Example:

$$\begin{array}{rcl} \mathcal{C} & = & \{\vdash P(x);\ P(y) \vdash Q(y)\}, \\ \mathcal{D} & = & \{\vdash P(f(z)), R(z);\ P(y), R(y) \vdash Q(y)\}. \end{array}$$

$\gamma \leq_{ss} \delta$ for $\delta =$

$$\dfrac{\vdash P(f(x)), R(x) \qquad \dfrac{\vdash P(f(z)), R(z) \quad P(y), R(y) \vdash Q(y)}{P(z) \vdash P(f(z)), Q(z)}\ y \leftarrow z}{\vdash R(x), P(f(f(x))), Q(f(x))}\ z \leftarrow f(x)$$

and $\gamma =$

$$\dfrac{\vdash P(x) \quad P(y) \vdash Q(y)}{\vdash Q(y)}\ x \leftarrow y$$

subsumption and reductive cut-elimination

Let $\mathcal{R}$ be the proof rewrite system of the Gentzen rules - without the axiom rule and applied only to non-atomic cuts.

- $\varphi >_{\mathcal{R}} \varphi'$ if φ rewrites to φ' via a rule in $\mathcal{R}$.
- normal forms under $\mathcal{R}$: atomic cut normal forms (ACNFs).
- if $\varphi >^*_{\mathcal{R}} \psi$ and ψ is irreducible under $\mathcal{R}$ then ψ is an ACNF of φ.

Theorem: Let φ be an **LK**-proof of a skolemized end-sequent S and $\varphi >_{\mathcal{R}} \varphi'$. Then $\mathrm{CL}(\varphi) \leq_{ss} \mathrm{CL}(\varphi')$.

Corollary 1: If ψ is a normal form of φ under $\mathcal{R}$ then $\mathrm{CL}(\varphi) \leq_{ss} \mathrm{CL}(\psi)$.

subsumption and reductive cut-elimination

Theorem: Let φ be an **LK**-proof of a skolemized end-sequent S and $\varphi >_{\mathcal{R}} \varphi'$. Then $\mathrm{CL}(\varphi) \leq_{ss} \mathrm{CL}(\varphi')$.

Corollary 2:

- If ψ is a normal form of φ under $\mathcal{R}$ and δ is a resolution refutation of $\mathrm{CL}(\psi)$
  then there exists a resolution refutation γ of $\mathrm{CL}(\varphi)$ s.t. $\gamma \leq_{ss} \delta$.

Note: If $\gamma \leq_{ss} \delta$ then $\|\gamma\| \leq \|\delta\|$.

CERES versus $\mathcal{R}$

Complexity Results:

- corollary 2 can be used to show that a non-elementary speed-up of CERES via $\mathcal{R}$ is impossible!
- there are sequences of proofs φ_n where cut-elimination via CERES is elementary but the computation of Gentzen normal forms based on $\mathcal{R}$ has no elementary bound in $\|\varphi_n\|$.
- there are sequences of proofs φ_n where cut-elimination via CERES is elementary but the computation of Tait normal forms based on $\mathcal{R}$ has no elementary bound in $\|\varphi_n\|$.

subsumption and reductive cut-elimination

work in progress:

- $\mathcal{R}_0$: full set of Gentzen rules for cut-elimination.
- $\mathrm{CL}_s(\varphi)$: structural characteristic clause set of φ.

- If $\varphi >_{\mathcal{R}_0} \varphi'$ then there exists a resolution derivation of a set of clauses $\mathcal{D}$ from $\mathrm{CL}_s(\varphi)$ s.t. $\mathcal{D} \leq_{ss} \mathrm{CL}_s(\varphi')$.
- Let Φ be a cut-elimination sequence on φ based on $\mathcal{R}_0$; then there exists a resolution refutation $\gamma(\Phi)$ of $\mathrm{CL}_s(\varphi)$ such that $\|\gamma(\Phi)\|$ is polynomial in $\|\Phi\|$.
- Yields a polynomial simulation of cut-elimination via $\mathcal{R}_0$ by CERES.

CERES for **LJ**?

- straightforward applications fail:
- for **LJ**-proofs φ the CERES-normal forms are in **LK** but typically not in **LJ**

possible remedy: eliminate the atomic cuts? Also this may fail!

- works only for left-sided end-sequents and negative resolution.

An Example:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P \vdash P$}
\RightLabel{$\vee_r$}
\UnaryInfC{$P \vdash P \vee \neg P$}
\AxiomC{}
\UnaryInfC{$P \vdash P$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P, \neg P \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg P \vdash \neg P$}
\RightLabel{$\vee_r$}
\UnaryInfC{$\neg P \vdash P \vee \neg P$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P \vee \neg P \vdash P \vee \neg P$}
\AxiomC{}
\UnaryInfC{$P \vdash P$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg P, P \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg P \vdash \neg P$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg P, \neg\neg P \vdash$}
\RightLabel{w_r}
\UnaryInfC{$\neg P, \neg\neg P \vdash P$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\neg P \vdash \neg\neg P \rightarrow P$}
\AxiomC{}
\UnaryInfC{$P \vdash P$}
\RightLabel{w_l}
\UnaryInfC{$P, \neg\neg P \vdash P$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$P \vdash \neg\neg P \rightarrow P$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P \vee \neg P \vdash \neg\neg P \rightarrow P$}
\RightLabel{cut}
\BinaryInfC{$P \vee \neg P \vdash \neg\neg P \rightarrow P$}
\end{prooftree}
$$

An Example:

The characteristic clause set of φ is:
$\mathrm{CL}(\varphi) = \{\vdash P \;\; ; \;\; P \vdash \;\; ; \;\; P \vdash P\}$. Which admits the (only non-redundant) resolution refutation:

$$
\frac{\vdash P \qquad P \vdash}{\vdash}\; R
$$

An Example:

The projections are the following:

$$\varphi[\vdash P]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}}{\vdash P, \neg P}\,\neg_r}{\neg\neg P \vdash P}\,\neg_l}{\neg\neg P \vdash P, P}\,w_r}{\vdash P, \neg\neg P \to P}\,\to_r$$

and

$$\varphi[P \vdash]$$

$$\dfrac{\dfrac{\dfrac{}{P \vdash P}}{P, \neg\neg P \vdash P}\,w_l}{P \vdash \neg\neg P \to P}\,\to_r$$

Note that $\varphi[\vdash P]$ is classical.

An Example:

The final ACNF is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P}}{\vdash P, \neg P}\,\neg_r}{\neg\neg P \vdash P}\,\neg_l}{\neg\neg P \vdash P, P}\,w_r}{\vdash P, \neg\neg P \to P}\,\to_r \qquad \dfrac{\dfrac{\dfrac{}{P \vdash P}}{P, \neg\neg P \vdash P}\,w_l}{P \vdash \neg\neg P \to P}\,\to_r}{\vdash \neg\neg P \to P, \neg\neg P \to P}\,cut}{\vdash \neg\neg P \to P}\,c_r}{P \vee \neg P \vdash \neg\neg P \to P}\,w_l$$

not every resolution refutation is useful! Need resolution with tautology.

Solution of the problem

- don't separate resolution refutations from projections!
- resolve the projections directly.

- need a **resolution principle for cut-free LK-proofs**.

Proof resolution

- Let φ_1 be a cut-free proof of $\Gamma \vdash \Delta, A_1, \ldots, A_n$,
- φ_2 be a cut-free proof of $B_1, \ldots, B_m, \Pi \vdash \Lambda$,
- and σ be a most general unifier of $\mathcal{A}$: $\{A_1, \ldots, A_n, B_1, \ldots, B_m\}$ where $\mathcal{A}\sigma = \{A\}$.
- Then a ground resolvent of $\varphi_1\sigma$ and $\varphi_2\sigma$ is a resolvent of φ_1 and φ_2.

Note that

- $\varphi_1\sigma$ is a proof of $\Gamma\sigma \vdash \Delta\sigma, A, \ldots, A$,
- $\varphi_2\sigma$ is a proof of $A, \ldots, A, \Pi\sigma \vdash \Lambda\sigma$.
- ground resolution corresponds to propositional resolution in clause logic!

Apply this resolution principle to the proof projections in CERES!

Proof resolution

Let

- φ be a cut-free proof of $\Gamma \vdash \Delta, A, \ldots, A$,
- ψ be a cut-free proof of $A, \ldots, A, \Pi \vdash \Lambda$.
- A ground resolvent of φ and ψ is any cut-free proof of

$$\Gamma, \Pi \vdash \Delta, \Lambda$$

obtained by (kind of) reductive cut-elimination on A.

CERES-i

intuitionistic CERES CERES-i:

- given an **LJ**-proof φ of a skolemized end-sequent,
- compute the set $\mathcal{P}(\varphi)$ of all proof projections of φ,
- apply proof resolution to $\mathcal{P}(\varphi)$.

CERES-i is complete, i.e. there is always a resolution derivation of a cut-free intuitionistic proof from $\mathcal{P}(\varphi)$.

- completeness proof: uses a subsumption principle for cut-free proofs and a subsumption theorem for projections under reductive cut-elimination.
- proof is analogous to the completeness proof of resolution!

proof subsumption

complex definition! Here the $\vee:$ r-case:

- φ a proof of $\Gamma \vdash \Delta$ and $\varphi \leq_{ss} \psi'$, i.e. $(\varphi, \psi', \vartheta)$ is a proof subsumption.
- ψ' is a proof of $\Pi_1, \Pi_2 \vdash \Lambda_1, \Lambda_2$ and $(\Gamma \vdash \Delta)\vartheta = \Pi_1 \vdash \Lambda_1$.

case 1: $\psi =$

$$\frac{\begin{matrix}(\psi') \\ \Pi_1, \Pi_2 \vdash \Lambda_1, \Lambda_2', A\end{matrix}}{\Pi_1, \Pi_2 \vdash \Lambda_1, \Lambda_2', A \vee B} \vee: r_1$$

Then $(\varphi, \psi, \vartheta)$ is a proof subsumption.

proof subsumption

- φ a proof of $\Gamma \vdash \Delta$ and $\varphi \leq_{ss} \psi'$, i.e. $(\varphi, \psi', \vartheta)$ is a proof subsumption.
- ψ' is a proof of $\Pi_1, \Pi_2 \vdash \Lambda_1, \Lambda_2$ and $(\Gamma \vdash \Delta)\vartheta = \Pi_1 \vdash \Lambda_1$.

case 2: $\psi =$

$$\frac{\begin{matrix}(\psi') \\ \Pi_1, \Pi_2 \vdash \Lambda_1', A, \Lambda_2\end{matrix}}{\Pi_1, \Pi_2 \vdash \Lambda_1', A \vee B, \Lambda_2} \vee: r_1$$

Let $\varphi^* =$ (for $A_0\vartheta = A, B_0\vartheta = B$)

$$\frac{\begin{matrix}(\varphi) \\ \Gamma \vdash \Delta', A_0\end{matrix}}{\Gamma \vdash \Delta', A_0 \vee B_0} \vee: r_1$$

Then $(\varphi^*, \psi, \vartheta)$ is a proof subsumption.

proof subsumption: an example

$\varphi =$

$$\dfrac{\dfrac{P(x) \vdash P(x) \quad Q(x) \vdash Q(x)}{P(x), Q(x) \vdash P(x) \wedge Q(x)}\ \wedge : r}{P(x), Q(x) \vdash \exists y (P(y) \wedge Q(y))}\ \exists : r$$

$\psi =$

$$\dfrac{\dfrac{\dfrac{R(c) \vdash R(c) \quad \dfrac{P(a) \vdash P(a)}{R(b), P(a) \vdash P(a)}\ w : l}{R(c), R(c) \to R(b), P(a) \vdash P(a)}\ \to : l \quad Q(a) \vdash Q(a)}{R(c), R(c) \to R(b), P(a), Q(a) \vdash P(a) \wedge Q(a))}\ \wedge : r}{R(c), R(c) \to R(b), P(a), Q(a) \vdash \exists y (P(y) \wedge Q(y))}\ \exists : r$$

$(\varphi, \psi, \vartheta)$ is a proof subsumption for $\vartheta = \{x \leftarrow a\}$.

CERES-i: the completeness proof

the crucial results are:

- Let $(\varphi, \psi, \vartheta)$ be a proof subsumption and ψ be intuitionistic then φ is intuitionistic as well.

CERES-i: the completeness proof

the crucial results are:

- Let $(\varphi, \psi, \vartheta)$ be a proof subsumption and ψ be intuitionistic then φ is intuitionistic as well.
- The main subsumption lemma: Let $\varphi >_{\mathcal{R}} \varphi'$ and ψ' be a projection for φ'. Then there exists a projection ψ for φ and a substitution ϑ s.t. (ψ, ψ', ϑ) is a proof subsumption.

CERES-i: the completeness proof

the crucial results are:

- Let $(\varphi, \psi, \vartheta)$ be a proof subsumption and ψ be intuitionistic then φ is intuitionistic as well.
- The main subsumption lemma: Let $\varphi >_{\mathcal{R}} \varphi'$ and ψ' be a projection for φ'. Then there exists a projection ψ for φ and a substitution ϑ s.t. (ψ, ψ', ϑ) is a proof subsumption.

- Lifting Theorem for Proofs: Let $(\varphi, \varphi', \vartheta_1)$ and $(\psi, \psi', \vartheta_2)$ be proof subsumptions for cut-free proofs which are variable-disjoint. Let χ' be a ground proof resolution of φ' and ψ'. Then either
  1. $(\varphi, \chi', \vartheta_1)$ is a proof subsumption or
  2. $(\psi, \chi', \vartheta_2)$ is a proof subsumption or
  3. there exists a resolvent χ of φ and ψ and a substitution σ s.t. (χ, χ', σ) is a proof subsumption.

CERES-i: the completeness proof

- completeness for $\mathrm{ACNF}^{\mathrm{top}}$-forms: Let ψ be an $\mathrm{ACNF}^{\mathrm{top}}$ of an **LJ**-proof φ of $\Gamma \vdash \Delta$. Then resolving the projections of φ yields a cut-free **LJ**-proof of $\Gamma \vdash \Delta$.

CERES-i: the completeness proof

- completeness for $\mathrm{ACNF}^{\mathrm{top}}$-forms: Let ψ be an $\mathrm{ACNF}^{\mathrm{top}}$ of an **LJ**-proof φ of $\Gamma \vdash \Delta$. Then resolving the projections of φ yields a cut-free **LJ**-proof of $\Gamma \vdash \Delta$.
- completeness of CERES-i: Let φ be an **LJ**-proof of a skolemized end-sequent S. Then the application of CERES-i to φ yields a cut-free **LJ**-proof χ of S. proof by using above completeness result + main subsumption lemma + lifting theorem for proofs.

CERES-i: an example

$$
\dfrac{
\dfrac{\dfrac{}{P \vdash P^1}}{P \vdash P^1 \lor \neg P^2}\,\lor_r
\quad
\dfrac{\dfrac{\dfrac{\dfrac{}{P^2 \vdash P}}{P^2, \neg P \vdash}\,\neg_l}{\neg P \vdash \neg P^2}\,\neg_r}{\neg P \vdash P^1 \lor \neg P^2}\,\lor_r
}{P \lor \neg P \vdash P^1 \lor \neg P^2}\,\lor_l
\quad
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P^2}}{\neg P^2, P \vdash}\,\neg_l}{\neg P^2 \vdash \neg P}\,\neg_r}{\neg P^2, \neg\neg P \vdash}\,\neg_l}{\neg P^2, \neg\neg P \vdash P}\,w_r}{\neg P^2 \vdash \neg\neg P \to P}\,\to_r
\quad
\dfrac{\dfrac{\dfrac{}{P^1 \vdash P}}{P^1, \neg\neg P \vdash P}\,w_l}{P^1 \vdash \neg\neg P \to P}\,\to_r
}{P^1 \lor \neg P^2 \vdash \neg\neg P \to P}\,\lor_l
$$

$$
\dfrac{P \lor \neg P \vdash P^1 \lor \neg P^2 \qquad P^1 \lor \neg P^2 \vdash \neg\neg P \to P}{P \lor \neg P \vdash \neg\neg P \to P}\,cut
$$

CERES-i: an example

The projections are:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P^2}}{\vdash P^2, \neg P}\,\neg_r}{\neg\neg P \vdash P^2}\,\neg_l}{\neg\neg P \vdash P^2, P}\,w_r}{\vdash P^2, \neg\neg P \to P}\,\to_r
\quad \text{and} \quad
\dfrac{\dfrac{\dfrac{}{P^1 \vdash P}}{P^1, \neg\neg P \vdash P}\,w_l}{P^1 \vdash \neg\neg P \to P}\,\to_r
$$

and

$$
\dfrac{P \vdash P^1 \qquad \dfrac{P^2 \vdash P}{\neg P, P^2 \vdash}\,\neg_l}{P^2, P \lor \neg P \vdash P^1}\,\lor_l
$$

CERES-i: an example

resolve

$$\dfrac{\mathbf{P} \vdash \mathbf{P^1} \qquad \dfrac{P^2 \vdash P}{\neg P, P^2 \vdash}\,\neg_l}{P^2, P \lor \neg P \vdash P^1}\,\lor_l \qquad \text{and} \qquad \dfrac{\dfrac{\overline{\mathbf{P^1} \vdash \mathbf{P}}}{P^1, \neg\neg P \vdash P}\,w_l}{P^1 \vdash \neg\neg P \to P}\,\to_r$$

and get

$$\dfrac{\dfrac{\dfrac{P \vdash P \qquad \dfrac{P^2 \vdash P}{\neg P, P^2 \vdash}\,\neg_l}{P^2, P \lor \neg P \vdash P}\,\lor_l}{\neg\neg P, P^2, P \lor \neg P \vdash P}\,w_l}{P^2, P \lor \neg P \vdash \neg\neg P \to P}\,\to_r$$

CERES-i: an example

next resolve

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{P \vdash P^2}}{\vdash P^2, \neg P}\,\neg_r}{\neg\neg P \vdash P^2}\,\neg_l}{\neg\neg P \vdash P^2, P}\,w_r}{\vdash P^2, \neg\neg P \to P}\,\to_r \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{P \vdash P \qquad \dfrac{P^2 \vdash P}{\neg P, P^2 \vdash}\,\neg_l}{P^2, P \lor \neg P \vdash P}\,\lor_l}{\neg\neg P, P^2, P \lor \neg P \vdash P}\,w_l}{P^2, P \lor \neg P \vdash \neg\neg P \to P}\,\to_r$$

and get

$$\dfrac{\dfrac{P \vdash P \qquad \dfrac{\dfrac{\dfrac{\dfrac{P \vdash P}{\neg P, P \vdash}\,\neg_l}{\neg P \vdash \neg P}\,\neg_r}{\neg\neg P, \neg P \vdash}\,\neg_l}{\neg\neg P, \neg P \vdash P}\,w_r}{P \lor \neg P, \neg\neg P \vdash P}\,\lor_l}{P \lor \neg P \vdash \neg\neg P \to P}\,\to_r$$

Complexity: CERES-i versus reductive cut-elimination

- CERESIL outperforms Gentzen's cut-elimination method: there exists an infinite sequence of **LJ**-proofs φ_n s.t. Gentzen's method yields a nonelementary increase in proof size; CERESIL is polynomial on φ_n.
- Consider the following cut-elimination method R: (1) use the Gentzen method without the axiom rule and reduce to an atomic cut normal form and (2) eliminate the atomic cuts (using also the axiom rule). Result: a nonelementary speed-up of CERESIL via R is impossible (i.e. R cannot be much faster than CERESIL).

CERES-i: open problems

- is there a resolution refinement ρ s.t., given an **LJ**-proof φ any refutation via ρ can be used? i.e. any $\mathtt{CERES}$ normal form of φ based on ρ can be transformed into a cut-free **LJ**-proofs. conjecture: refinement based on indexing via atom occurrences in cuts.
- proof resolution is search intensive - find refinements to make the method useful in practice.

subsumption principle and proof theory

- complexity analysis of cut-elimination methods.
- prove completeness of cut-elimination methods.
- subsumption plays a major role in schematic CERES.

CERES: future research

- investigate other normalization methods using CERES.
- semi-automated analysis of (inductive proofs) via schematic CERES.
- proof mining: analysis of real mathematical proofs by CERES.

References

- M. Baaz, A.L.: Cut-elimination and redundancy-elimination by resolution. *Journal of Symbolic Computation* 29, 149-176 (2000)
- M. Baaz, A.L.: Towards a clausal analysis of cut-elimination. *Journal of Symbolic Computation* 41, 381-410 (2006)
- M. Baaz, S. Hetzl, A.L., C. Richter, H. Spohr: CERES: an analysis of Fürstenberg's proof of the infinity of primes. *Theoretical Computer Science* 403 (2-3), 160-175 (2008)
- M. Baaz, A.L.: Cut-elimination: syntax and semantics. *Studia Logica* 102(6): 1217-1244 (2014)
- D. Cerna, A.L., G. Reis, S. Wolfsteiner: CERES in intuitionistic logic. *Annals of Pure and Applied Logic* 168, pp. 1783–1836 (2017)

Thank You!

Remarks on the Sequent Calculus

Enrico Moriconi

Department of Civilisations and Forms of Knowledge
University of Pisa
enrico.moriconi@unipi.it

Universal Logic 2018
Vichy, 16-26/6/2018

OUTLINE

INTRODUCTION

In his thesis, after the proof of the Hauptsatz, Gentzen provided in the last long section V the proof of the equivalence between three types of formalization of the logical inference in both the intuitionistic and classical version: the Hilbert-Ackermann system **LHJ**/**K**, the Natural Deduction Calculus **NJ**/**K**, and the Sequent Calculus **LJ**/**K**.
Admittedly, this section of the thesis is normally rated "less important" than the other sections, but it is nonetheless interesting since it allows us to retrace some steps of the reasoning which brought to the invention of the Sequent Calculus, and more generally to highlight some structural features of Gentzen's work.

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

The equivalence proof Gentzen gave in the published version of the thesis proceeds through the following sequence of steps:

$$\mathcal{LHJ}/\mathcal{K} \rightsquigarrow \mathcal{NJ}/\mathcal{K} \rightsquigarrow \mathcal{LJ}/\mathcal{K} \rightsquigarrow \mathcal{LHJ}/\mathcal{K}$$

The step $\mathcal{LJ} \rightsquigarrow \mathcal{LHJ}$ has three main ideas:

1. Introduction of new Groundsequents (Gsq)
2. Introduction of the formula $A\&\neg A$ in the succedent of any sequent with empty succedent
3. Introduction of two new inference figures.

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

- The <u>first</u> step produces a “simplification” of $\mathcal{LJ}$ by introducing as new Gsqs some axioms of $\mathcal{LHJ}$, whose equivalence with the replaced Inference figures is easily proved by means of the Cut rule.
- The replaced inferences schemas are
$$\& - L, \vee - R, \forall - L, \exists - R, \neg - L, \supset -L.$$
- The new Gsqs are:
Gsq1-2 $A_1\&A_2 \to A_i$ Gsq3-4 $A_i \to A_1 \vee A_2$
Gsq5 $\forall x F(x) \to F(a)$ Gsq6 $F(a) \to \exists x F(x)$
Gsq7 $\neg A, A \to$ Gsq8 $A \supset B, A \to B$.

To show how the replacement works we give just an example, concerning $\supset -L$:

$$\dfrac{\Gamma \to A \qquad B, \Delta \to \Lambda}{A \supset B, \Gamma, \delta \to \Lambda}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\Gamma \to A \qquad \dfrac{\begin{array}{c}\text{Gs8}\\ A \supset B, A \to B\end{array}}{A, A \supset B \to B}\,\text{Exch}}{\Gamma, A \supset B \to B}\,\text{Cut} \qquad B, \Delta \to \Lambda}{\dfrac{\Gamma, A \supset B, \Delta \to \Lambda}{A \supset B, \Gamma, \Delta \to \Lambda}\,\text{Exch}}\,\text{Cut}$$

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

The <u>second</u> step is determined by the necessity that in any sequent the succedent is not empty. Otherwise, how could we translate a sequent in an implication of $\mathcal{LHJ}$? Thus, we write the formula $A \& \neg A$ at the right of $\to$ in any sequent with an empty succedent.

The third step is the more intricate one, even though the idea is simple. If we consider the axioms of $\mathcal{LHJ}$, it is immediate to acknowledge the close link with the inference figures of $\mathcal{LJ}$. For instance, the rule (Ctr-L)

$$\frac{A, A, \Gamma \to B}{A, \Gamma \to B}$$

corresponds obviously to the axiom schema 2.13 of $\mathcal{LHJ}$

$$(A \supset (A \supset B)) \supset (A \supset B)$$

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

However, the (possibly) occurring formulas in Γ are not involved in the accomplishment of the rule, constituting, so to speak, a *disturbing* element for the searched translation. To overcome this obstacle, Gentzen introduced two other inference rules:

$$\frac{\Gamma, A \to B}{\Gamma \to A \supset B}\ \text{Inf 10} \qquad \text{and} \qquad \frac{\Gamma \to A \supset B}{\Gamma, A \to B}\ \text{Inf 11}$$

It is immediate to see that these rules allow to move in the succedent all the formulas not active in the rule which is to be applied; after that, we can apply the rule, and at last all formulas previously moved to the right can be carried back in the antecedent.

Now, any sequent $A_1, \dots, A_n \rightarrow B$ is substituted by the formula $(A_1 \& \dots \& A_n) \supset B$: it is immediate to see that in this way all the Gsqs become axioms of $\mathcal{LHJ}$. And the proof is completed.

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

Now let us consider the translation $\mathcal{LK} \rightsquigarrow \mathcal{LHK}$, which is the only problematic case one encounters in extending the previous equivalence result to the classical case. Gentzen introduces an auxiliary calculus $\mathcal{LK}^\star$ such that:

1. Ctr-R and Exch-R are not allowed.
2. In all other schemas no substitution may be performed for Θ and Λ in the succedent of the inference schema; these places thus remain empty.
3. To the inference figures are then added the two following ones concerning negation:

$$\frac{\Gamma \rightarrow A, \Theta}{\Gamma, \neg A \rightarrow \Theta}\ \text{Inf 1} \qquad \frac{\Gamma, \neg A \rightarrow \Theta}{\Gamma \rightarrow A, \Theta}\ \text{Inf 2}$$

It is to be noted that in these cases Θ need not be empty.

All the inference figures undergo the following transformation:

1. By applying Inf 1 to the superior sequent(s) all the formulas occurring in Θ or Λ are negated and moved in the left of the antecedent, so that we isolate the active formula on the right-hand side.
2. Afterward, we apply the operational rule.
3. Lastly, thanks to Inf 2, the formulas occurring in Θ or Λ are carried back in the succedent.

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

To show the way in which any $\mathcal{LK}$-derivation is transformed in an $\mathcal{LK}^{\star}$-derivation we consider two cases: $\vee - R$ and $\supset -L$. In the following sometimes we denote $\Theta^{\neg}$ the set $\{\neg D_1, \ldots, \neg D_n\}$, where Θ is $\{D_1, \ldots, D_n\}$:

$$\frac{\Gamma \to \Theta, A}{\Gamma \to \Theta, A \vee B}\ \vee - R$$

$$\Downarrow$$

$$\frac{\Gamma \to \Theta, A}{\Gamma, \neg D_n \to \neg D_1, \ldots, \neg D_{n-1}, A}\ \text{Inf } 1$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \neg D_1, \ldots, \neg D_n \to A}\ \text{Inf } 1}{\Gamma, \neg D_1, \ldots, \neg D_n \to A \vee B}\ \vee - R}{\Gamma, \neg D_1, \ldots, \neg D_{n-1} \to D_n, A \vee B}\ \text{Inf } 2$$

$$\frac{\vdots}{\Gamma \to \Theta, A \vee B}\ \text{Inf } 2$$

Now $\supset -L$:

$$\frac{\Gamma \to \Theta, A \qquad B, \Delta \to \Lambda}{A \supset B, \Gamma, \Delta \to \Theta, \Lambda} \supset -L$$

$$\Downarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\Gamma \to \Theta, A\\ \vdots\end{array}}{\Theta^{\neg}, \Gamma \to A}\,\text{Inf 1} \qquad \dfrac{\begin{array}{c}B, \Delta \to \Lambda\\ \vdots\end{array}}{B, \Lambda^{\neg}, \Delta \to}\,\text{Inf 1}}{\dfrac{\begin{array}{c}A \supset B, \Theta^{\neg}, \Gamma, \Lambda^{\neg}, \Delta \to\\ \vdots\end{array}}{A \supset B, \Gamma, \Delta \to \Theta, \Lambda}\,\text{Inf 2}} \supset -L$$

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

The next step consists in transforming any $\mathcal{LK}^{\star}$-derivation in an $\mathcal{LJ} + (\to A \vee \neg A)$-derivation by translating any sequent

$$A_1, \ldots, A_m \to B_1, \ldots, B_n$$

into

$$A_1, \ldots, A_m \to B_1 \vee \ldots \vee B_n$$

If the succedent was empty, it remains empty.
What we have got is that, except for application of Inf 1 and Inf 2, all initial sequents and inference figures belong to $\mathcal{LJ}$. Let now consider Inf 1 and Inf 2.

Inf 1: if Θ is empty, we substitute Inf 1 with $\neg - L$. Otherwise, and denoting $\Theta^{\vee}$ the disjunction of the formulas in Θ, the translation produces:

$$\frac{\Gamma \to \Theta^{\vee} \vee A}{\Gamma, \neg A \to \Theta^{\vee}}$$

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

The last schema is then transformed in the following one, still belonging to $\mathcal{LJ}$

$$\begin{prooftree}
\AxiomC{$\Gamma \to \Theta^{\vee} \vee A$}
\AxiomC{$\Theta^{\vee} \to \Theta^{\vee}$}
\UnaryInfC{$\neg A, \Theta^{\vee} \to \Theta^{\vee}$}
\UnaryInfC{$\Theta^{\vee}, \neg A \to \Theta^{\vee}$}
\AxiomC{$A \to A$}
\UnaryInfC{$\neg A, A \to$}
\UnaryInfC{$A, \neg A \to$}
\UnaryInfC{$A, \neg A \to \Theta^{\vee}$}
\RightLabel{$\vee - L$}
\BinaryInfC{$\Theta^{\vee} \vee A, \neg A \to \Theta^{\vee}$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma, \neg A \to \Theta^{\vee}$}
\end{prooftree}$$

Inf 2: in consequence of previous modification, the schema assumes the following form:

$$\frac{\Gamma, \neg A \to \Theta^{\vee}}{\Gamma \to \Theta^{\vee} \vee A}$$

In turn, this schema is transformed as follows:

$$\dfrac{\to A \vee \neg A \qquad \dfrac{\dfrac{\dfrac{A \to A}{A, \Gamma \to A}}{A, \Gamma \to \Theta^{\vee} \vee A} \qquad \dfrac{\dfrac{\Gamma, \neg A \to \Theta^{\vee}}{\neg A, \Gamma \to \Theta^{\vee}}}{\neg A, \Gamma \to \Theta^{\vee} \vee A}}{A \vee \neg A, \Gamma \to \Theta^{\vee} \vee A}}{\Gamma \to \Theta^{\vee} \vee A}$$

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

The previous derivation, which occurs in $\mathcal{LJ}$+$(A \vee \neg A)$, can be transformed in an $\mathcal{LHJ}$+$(A \vee \neg A)$-derivation, that is to say in an $\mathcal{LK}$-derivation.

It is interesting to note that Inf 2 provides the classical flavour. In fact, it allows to prove "intuitionistically"–i.e., with just one formula occurring in the succedent– the Principle of Excluded Middle:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to A \vee \neg A}\,\vee - R}{\neg(A \vee \neg A), A \to}\,\neg - L;\text{Inf } 1}{\neg(A \vee \neg A) \to \neg A}\,\neg - R}{\neg(A \vee \neg A) \to A \vee \neg A}\,\vee - R}{\neg(A \vee \neg A), \neg(A \vee \neg A) \to}\,\neg - L;\text{Inf } 1}{\neg(A \vee \neg A) \to}\,\text{Ctr}}{\to (A \vee \neg A)}\,\text{Inf } 2
$$

By applying usual $\neg - R$ in the last step would result in producing $\neg\neg(A \vee \neg A)$.

Reflecting on the equivalence proofs Gentzen provided in the V section of the thesis, some points deserve attention:

1. The first feature one could note is the peculiar definition of "equivalence" that Gentzen adopted, which is different from the usual one (consisting in declaring two formulas equivalent *wenn die eine aus der anderen herleitbar ist*, V 1.2).
2. With this definition Gentzen aims at a *translation going from derivations to derivations*. Two derivations are said equivalent if the endformula (endsequent) of the first is equivalent with that of the second; two formulas are equivalent if either they are identical or one is got by substituting the falsum symbol ⅄ by $A \& \neg A$ in the other; a sequent is equivalent to the implication from the conjunction of antecedents to the disjunction of succedents.

3. Foreshadowing the “Reduction Strategy” he was going to employ shortly after in his proof of consistency, he devises a sequence of transformation steps which literaly transform derivations into equivalent derivations.
4. Gentzen is not interested in giving a proof that different formalizations of the logical deduction prove the same class of theorems, *extensionally* considered.
5. This is part and parcel of his belief that there is no realm of first-order logical truths to be captured by (his) logical calculi.
6. Keeping in mind that his base reference was to the building of the Natural Deduction Calculi, it is worth reminding that what he wanted to capture by means of them was not constituted by the unsettled class of logical truths, but by the concrete, though open-ended, cluster of schemas of reasoning actually exploited in mathematics.

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

7. This is what he clearly claims both in the thesis (page 183):

> *Wir wollen einen Formalismus aufstellen, der möglichst genau das wirkliche logische Schliessen bei mathematischen Beweisen wiedergibt*
> *[We wish to set up a formalism that reflects as accurately as possible the actual logical reasoning involved in mathematical proofs.].*

and in the first published paper on the consistency of elementary arithmetic (see p. 506).

- The Natural Deduction Calculi were the first (we are around 1932) formalization provided by Gentzen in his search for a way of representing, better than through axiomatic procedures, the reasoning actually employed in mathematical arguments.
- From the first pages of the *published* thesis, we know that reflecting on **NJ**/**K**-calculi Gentzen devised the idea of a *direct* proof, but –he soon realized– without having available the tools to get it. Only the intuitionistic version of the calculi, in fact, works well for the normal form theorem.

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

- From the important researches made by Jan von Plato on the newly found Gentzen's texts, we know that in the *handwritten* version of the thesis, called *Ms.ULS*, Gentzen's design was
  1. firstly, to present the calculus of natural deduction,
  2. secondly, to show its equivalence to axiomatic logic,
  3. thirdly, to establish normalization and the subformula property.
  4. Finally, to extend all of this to arithmetic (which however failed as there is no subformula property for derivations in a formal system of arithmetic).

- Thus, one of Gentzen's first concerns was to proof the equivalence of his new calculus with the formalism of **LHJ**/**K**: thanks to this proof, in fact, the property of semantic completeness proved by Gödel for **LHK** would also have been shared by his new calculi.
- First, he proves that derivations in the axiomatic calculus of **LHJ**/**K** can be reproduced in his system of **NJ**/**K** which in this way results to be at least as strong as the standard axiomatic calculus of **LHJ**/**K**.
- Then he provides a translation of every **NJ**/**K**-derivation into an equivalent **LHJ**/**K**-derivation. Notably, in this way it is *explicitly* supplied a missing link characterizing the published thesis: there, in fact, this step is only *implicitly* present, as a by-product of the steps

$$\mathcal{LHJ}/\mathcal{K} \rightsquigarrow \mathcal{NJ}/\mathcal{K} \rightsquigarrow \mathcal{LJ}/\mathcal{K} \rightsquigarrow \mathcal{LHJ}/\mathcal{K}$$

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

Derivations in **NJ**/**K** are reproduced in axiomatic logic as follows: for any formula A occurring in a given **NJ**/**K**-proof, we consider all the assumptions which still stand under A; *i.e.* such that A depends on them. Let they be $A_1, \ldots, A_n$, then A is substituted by the expression $(A_1 \& \ldots \& A_n) \supset A$. If A is an assumption, then $A \supset A$ takes its place. The **NJ**/**K**-step of inferences assume the following shape (we consider just the case of conjunction):

$$\frac{D \supset A \qquad E \supset B}{D \& E \supset A \& B} \qquad \frac{D \supset A_1 \& A_2}{D \supset A_i}$$

In the case of $\&I$, first one proves in **LHJ**/**K**

$$(D \supset A) \supset ((E \supset B) \supset (D\&E \supset A\&B))$$

then the previous rule $\&I$, which is the translation of usual **NJ**/**K**-rule of $\&I$, is replaced by two applications of MP, as follows:

$$\cfrac{\cfrac{(D \supset A) \supset ((E \supset B) \supset (D\&E \supset A\&B)) \qquad (D \supset A)}{(E \supset B) \supset (D\&E \supset A\&B)} \qquad (E \supset B)}{(D\&E \supset A\&B)}$$

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

In the case of $\&E$, first one proves in **LHJ**/**K**

$$((A_1\&A_2) \supset A_i) \supset ((D \supset A_1\&A_2) \supset (D \supset A_i))$$

then the previous rule $\&E$, which is the translation of usual **NJ**/**K**-rule of $\&E$, is replaced by two applications of MP, by exploiting also the **LHJ**/**K**-axiom $((A_1\&A_2) \supset A_i)$, as follows:

$$\cfrac{((A_1\&A_2) \supset A_i) \supset ((D \supset A_1\&A_2) \supset (D \supset A_i)) \qquad ((A_1\&A_2) \supset A_i)}{(D \supset A_1\&A_2) \supset (D \supset A_i)}$$

$$\cfrac{(D \supset A_1\&A_2) \supset (D \supset A_i) \qquad D \supset A_1\&A_2}{D \supset A_i}$$

The overall idea is very similar to the procedure Bernays provided in the first volume of *Grundlagen*, 1934, in order to prove the deduction theorem: to prove that $A \vdash B$ entails $\vdash A \supset B$ first put "$A \supset$" in front of any formula occurring in the given proof that $A \vdash B$ and then rearrange the resulting "proof-tree" in order to accommodate previous occurrences of axioms, and application of MP and UG.

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

- In this way, however, two, so to speak, "new" symbols, & and $\supset$, are introduced which are different from the same symbols belonging to the object language, henceforth requiring new additional inference figures which disturb the systematic character of introductions and eliminations. To overcome this difficulty, Gentzen exploits the notion of a "single-succedent sequent", where commas and the arrow substitute the two symbols & and $\supset$, and we get $A_1, \ldots, A_n \rightarrow A$.
- This notion was to him familiar as a result of his work on Hertz' systems. The structural framework provided by Hertz turn out useful to accommodate his new kind of expression, called by Gentzen "sentence" (*Satz* by Hertz) and then "sequent".

What has been thus provided, however, is not the previous expression in disguise, but a new metalinguistic entity –the commas and the arrows do not contribute to the building of formulas– which will be ruled by new inference rules anyway outside the systematic character of introductions and eliminations. Since these rules don't pertain to the logical symbols but to the structure of the sequents, they are called *structural* rules, whereas the other *operational* or *logical* rules. One could say that the distinction between these two kinds of rules is the characteristic feature of the sequent calculus, and also the witness that Gentzen succeeded in disentangling the linguistic component of implication from its metalinguistic, inferential component. That is to say, matters of *meaning* from matters of *reasoning*.

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

Here are the the most substantial sources of Sequent Calculus: first, the translation from a classical Natural Calculus to the axiomatic system of logic of Hilbert and Ackermann, and, secondly, Gentzen's experience with the "sequents" of Hertz systems.
However, to fully understand the rationale for the change of framework, moving from Natural Deduction to *Logistic*, or Sequent Calculus is not easy.

- Gentzen was perfectly aware that the kind of analysis which provided the backbone of the formulation of **NJ**/**K**-calculi was able to ensure their (experimental) completeness, but not the *analytic* nature of the inferences allowed.
- As it is suggested by difficulties in accommodating negation within the symmetries of Introduction and Elimination, derivations may fail to satisfy the so-called *subformula property*, the most important consequence of the Hauptsatz.
- From this same point of view, serious problems also come from the ($\supset E$)-rule, the so-called *modus ponens*. To a deeper analysis, in fact, using the $\supset$ operator appears to involve a peculiar form of cut.

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

- The way the rules of the **N**-calculi *analyse* the correct inferences associated with each individual logical operator was not completely *analytical*.
- In fact, Gentzen's inferential approach mixed the formalization of the meanings of the logical operators with an account of the consequence relation. Moving from **NJ**/**K**-calculi to **LJ**/**K**-calculi, Gentzen aimed to separate the task of determining the meanings of logical constants from the question of accounting for the *inferential* features of the deductive system.
- This question is clearly seen with respect to the $\supset$ symbol, since the meaning of "*A* implies *B*" overlaps with that of "*B* is derivable from *A*".

- Implication, which is the only symbol whose **I**-rule has an (open) derivation as its premise, in its **E**-rule, the *modus ponens*, involves a particular form of cut.
- In order to show this special nature of the implication within the natural deduction framework it makes sense to compare the normalization procedures concerning implication and disjunction. When an application of the $\supset$ **I**-rule is immediately followed by an application of the $\supset$ **E**-rule, as in:

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

$$\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \supset B} \qquad \begin{array}{c}\vdots \\ A\end{array}}{\begin{array}{c}B \\ \vdots\end{array}}$$

Here, of course, we face a sort of “detour” (*Umwege*), since the application of the **E**-rule does not contribute anything new, thus it can be eliminated, by converting (or reducing, or equating) the previous derivation to the following one:

$$\begin{array}{c} \vdots \\ [A] \\ \vdots \\ B \\ \vdots \end{array}$$

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

In the case of disjunction we can have the following case of detour

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\vdots \\ A\end{array}}{A \vee B} \qquad \begin{array}{c}[A] \\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B] \\ \vdots \\ C\end{array}}{C} \\ \vdots \end{array}$$

This normalizes to

$$\begin{array}{c} \vdots \\ [A] \\ \vdots \\ C \\ \vdots \end{array}$$

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

The configurations may seem quite analogous, but there are some important distinctions:

1. whereas in the case of implication we compose the minor premise with the direct grounds for obtaining the major premise,
2. in the case of disjunction we proceed the other way round by composing the direct grounds for obtaining the major premise with the (left) minor premise.
3. With implication, the normalization procedure gives back a derivation of B from A, which is the direct grounds for inferring the major premise of the elimination, $A \supset B$.
4. With disjunction, on the other hand, the direct grounds for inferring the major premise of the elimination is just A (or B), and what the normalization procedure gives back is not those direct grounds but the so-called general conclusion C which, according to the minor premises, can be inferred both from A and from B.

5. Whereas the $\supset E$-rule has a specific conclusion, given by the direct grounds for inferring its major premise, the conclusion of the $\vee E$-rule is a generic formula, *i.e.* any formula C which can be obtained both from A and B.
6. Since it is impossible to formulate the $\vee E$-rule with a specific conclusion (neither A nor B, in fact, can play this role), to compare the two rules properly, so that we can appreciate the specific nature of implication within the natural deduction framework, the only way is to transform the $\supset E$-rule.
7. We thus consider the sequence $\supset \mathbf{I}$– $\supset \mathbf{E}$ when the so-called general format has also been adopted for the $\supset \mathbf{E}$-rule, let's say $G \supset \mathbf{E}$:

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

$$\dfrac{\supset \mathbf{I}\,\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B} \qquad \begin{matrix}\vdots\\ A\end{matrix} \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\; G \supset \mathbf{E}$$

In the latter case, the normalization procedures gives back

$$\begin{array}{c} \vdots \\ [A] \\ \vdots \\ [B] \\ \vdots \\ C \end{array}$$

Here *two* kinds of composition occur: whereas the latter depends on the general format of the elimination rule, and is shared by both implication and disjunction; the former, on the other hand, is owned just by implication, and is a cut on A which is tied to the peculiar nature of the treatment of implication within natural deduction.

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

Gentzen proceeds as follows: given an N.D.-proof of, say A, one lists all those assumptions which are not already discharged before the accomplishing of the inference leading to A. Let us indicate them by Γ. Then one substitutes A by $\Gamma \to A$. If A is an assumption, $A \to A$ takes its place. The steps of inference of N.D. are accordingly translated:

$$\frac{A \qquad B}{A\&B}\ \mathrm{I}\& \qquad \rightsquigarrow \qquad \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A\&B}$$

Paired with the occurrence of the figure of *sequent*, here we see, probably for the first time, the disentangling of two meanings often conflated in the notion of implication: the propositional (object-language) connective, say $\supset$, and the (meta-level) notation for the formal derivability relation, say $\rightarrow$. Of course, in this step Gentzen was greatly helped by his work on Hertz-systems from the summer of 1931, which output his first published paper of 1932.

A sequent represents a "possibility of derivation": given such and such hypotheses, such sentence can hold. Gentzen's sequent calculus shows how to pass from a given possibility of proof to another possibility of proof, and not directly how to pass from a sentence to another sentence. Note that this is a character already present in his first paper of 1932. What in his calculus is equivalent to the proof of a given sentence is the proof of a sequent with a void antecedent and just that sentence in the consequent; which means: it is always possible to prove the sentence constituting the consequent.

It is interesting to note that, in the equivalence proofs provided in the last section of the thesis, Gentzen resurrected the LDK calculus where Cut plays a necessary role, even though this was a section of a work devoted to the proof of the eliminability of the Cut.

INTRODUCTION | REMARKS ON THE EQUIVALENCE PROOFS | NATURAL DEDUCTION AND ITS PROBLEMS | SEQUENT CALCULUS | TWO INTERPRETATIONS OF CUT

It is interesting to focus on the emergence of two paradigms in the conception of Cut. The paradigm of *structural reasoning*, which was preserved in the intermediate calculus **LDK** of *Ms.ULS* , where the Cut rule continues to play a fundamental role, and the *analytic* paradigm. Here analytic proofs was the new goal, and Gentzen was able to attain it thanks to the *Hauptsatz* proved for that "evolution" of **LDK**-calculi which is constituted by the **LK**-calculi. In the latter calculi, structural reasoning was sharply separated from logical meaning, and the general setting was purely inferential.

THANKS

FOR YOUR ATTENTION

Tomographs for Substructural Display Logic

Michael Arndt
University Tübingen

Universal Logic 2018, Vichy
Workshop on Proof Theory
June 25, 2018

Overview

- Introductory Remarks
  - Begriffsschrift
  - Residuation
- Display Calculus for Substructural Logics
  - Structural Connectives and their Properties
  - Display Property
  - Logical Connectives and Structural Overloading
  - Generalized Cut-Elimination
- Tomographs for Substructural Logics
  - Translating Sequents into Tomographs
  - Residuation in Tomographs
  - Example Derivations
- Explicit Negation
  - Galois Connections
  - Negation in the Display Caclulus
  - Negation in Tomographs
- Summary

Introductory Remarks

We introduce a two-dimensional notation for Goré's display calculus for substructural logics.

It requires only two ternary connectives to render all ten of the structural connectives of Goré's display calculus (six binary and four unary) as well as two structural constants.

Structural conjunction, implication, replication (due to the logic's non-commutative setting) and empty structure as well as two structural negations are rendered by one ternary connective, and their duals are rendered by another.

Recall the Begriffsschrift

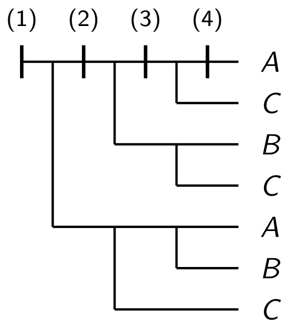
There are certain similarities to Frege's two-dimensional notation, the Begriffsschrift, regarding the role of the turnstile/judgement stroke.

Schroeder-Heister's observation of structural features in *Gentzen-style features in Frege* (1999) and *Frege's sequent calculus* (2014):

Moving the judgement stroke to the right along the top horizontal line corresponds to moving from the conclusion to the premise in the sequent calculus' (→R) rule.

Moreover, recall Frege's remarks on whether the Begriffsschrift should be based on a conjunctive or a conditional connective.

Begriffsschrift Sentences as Sequents



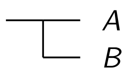
(4) $C \to (B \to A)\,,\; C \to B\,,\; C \vdash A$
(3) $C \to (B \to A)\,,\; C \to B \vdash C \to A$
(2) $C \to (B \to A) \vdash (C \to B) \to (C \to A)$
(1) $\vdash (C \to (B \to A)) \to ((C \to B) \to (C \to A))$

Frege's Design Choice

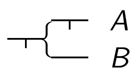
"Instead of expressing 'and' by means of the symbols for conditionality and negation, as is done here, conditionality could also be represented, conversely, by means of a symbol for 'and' and the symbol for negation. One might introduce, say,

$$\left\{ \begin{matrix} \Gamma \\ \Delta \end{matrix} \right.$$

as the symbol for the conjoined content of Γ *and* Δ*, and render*



by



I chose the other way, since inference seemed to me to be expressed more simply that way."

(G. Frege. Begriffsschrift, pp. 12–13)

Residuation

A *residuated groupoid* is a structure $\langle L, \leq, \odot, \obslash, \oslash \rangle$ where

(i) $\langle L, \leq \rangle$ is a lattice;
(ii) $\langle L, \odot \rangle$ is a groupoid;
(iii) for all $x, y, z \in L$: $\quad y \leq x \obslash z \quad$ iff $\quad x \odot y \leq z \quad$ iff $\quad x \leq z \oslash y$.

$x \obslash z$ *(left residual of* z *by* x*)* is the greatest y such that $x \odot y \leq z$,
$z \oslash y$ *(right residual of* z *by* y*)* is the greatest x such that $x \odot y \leq z$.

If the groupoid is commutative, then left and right residuals collapse.

Co-Residuation

A *co-residuated groupoid* is a structure $\langle L, \leq, \boxdot, \boxbslash, \boxslash \rangle$ where

(i) $\langle L, \leq \rangle$ is a lattice;
(ii) $\langle L, \boxdot \rangle$ is a groupoid;
(iii) for all $x, y, z \in L$: $\quad x \boxbslash z \leq y \quad$ iff $\quad z \leq x \boxdot y \quad$ iff $\quad z \boxslash y \leq x$.

$x \boxbslash z$ *(left co-residual of* z *by* x*)* is the least y such that $z \leq x \boxdot y$,
$z \boxslash y$ *(right co-residual of* z *by* y*)* is the least x such that $z \leq x \boxdot y$.

If the groupoid is commutative, then left and right co-residuals collapse.

Residuation and Co-residuation in Logic

The algebraic property of residuation has analogues in logic.

Import/Export:	$\Gamma \vDash A \to C$	iff	$A, \Gamma \vDash C$		
Deduction:	$\Gamma \vdash A \to C$	iff	$A, \Gamma \vdash C$		
Relevance logic:	$B \vdash A \to C$	iff	$A * B \vdash C$		
Linear logic:	$B \vdash A \multimap C$	iff	$A \otimes B \vdash C$	iff	$A \vdash C \multimapinv B$
Lambek logic:	$B \vdash A \setminus C$	iff	$A \otimes B \vdash C$	iff	$A \vdash C \,/\, B$
Display logic:	$\mathcal{Y} \vdash \mathcal{X} > \mathcal{Z}$	iff	$\mathcal{X} \,;\, \mathcal{Y} \vdash \mathcal{Z}$	iff	$\mathcal{X} \vdash \mathcal{Z} < \mathcal{Y}$
	$\mathcal{X} > \mathcal{Z} \vdash \mathcal{Y}$	iff	$\mathcal{Z} \vdash \mathcal{X} \,;\, \mathcal{Y}$	iff	$\mathcal{Z} < \mathcal{Y} \vdash \mathcal{X}$

Display Calculus for Substructural Logics

We will now present Goré's display calculus introduced in *Substructural logics on display* (1998).

Firstly, we discuss the matter of structures, the binary structural connectives and the residuation rules governing them.

We then present the structural rules of the calculus that can be added to the core calculus to obtain various Bi-Lambek logics.

Finally, we present its logical rules, which are extremely interesting, as each logical connective reflects a structural property expressed by a structural connective.

Structural Connectives in Display Logic

In Goré's display logic, all *sequents* have the form $\mathbf{S} \vdash \mathbf{S}$, where *structures* $\mathbf{S}$ are formed on the basis of *formulae* $\mathbf{F}$ as follows:

$$\mathbf{S} \quad ::= \quad \Phi \mid \mathbf{F} \mid (\mathbf{S} \,;\, \mathbf{S}) \mid (\mathbf{S} > \mathbf{S}) \mid (\mathbf{S} < \mathbf{S})$$

(By convention the outmost parentheses of structures may be omitted.)

The function of the *structural connectives* is as follows:

Φ	–	(co-)empty structure
;	–	binary structural (co-)composition
>	–	structural left (co-)residual
<	–	structural right (co-)residual

$A, B, C, \ldots$ range over formulae. $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$ range over structures.

Structural Overloading

Unfortunately, the definition obfuscates the fact that the structural connectives are overloaded. Each one of the symbols is actually used for two different connectives and, respectively, two different constants.

An improved definition would give sequents as $\mathbf{L} \vdash \mathbf{R}$, where *antecedent structures* $\mathbf{L}$ and *succedent structures* $\mathbf{R}$ are formed on the basis of *formulae* $\mathbf{F}$ as follows:

$$\begin{array}{lcl} \mathbf{L} & ::= & \grave{\Phi} \mid \mathbf{F} \mid (\mathbf{L} \grave{;} \mathbf{L}) \mid (\mathbf{R} \grave{>} \mathbf{L}) \mid (\mathbf{L} \grave{<} \mathbf{R}) \\ \mathbf{R} & ::= & \acute{\Phi} \mid \mathbf{F} \mid (\mathbf{R} \acute{;} \mathbf{R}) \mid (\mathbf{L} \acute{>} \mathbf{R}) \mid (\mathbf{R} \acute{<} \mathbf{L}) \end{array}$$

Indeed, under such a scheme, even formulae could and should be assigned a left/right-polarity.

It is important to keep in mind that Goré uses overloading. However, we will informally talk about **L**- and **R**-structures to clarify certain points.

Residuation Rules

The residuation rules codify the property of left and right residuation for structural **L**-composition within the framework of the sequent calculus.

Corresponding rules govern the property of left and right co-residuation for structural **R**-composition for the dual connectives.

$$
(\mathrm{rp}_1)\ \begin{array}{c} \mathcal{X} \,;\, \mathcal{Y} \vdash \mathcal{Z} \\ \hline\hline \mathcal{Y} \vdash \mathcal{X} > \mathcal{Z} \end{array}
\qquad\qquad
\begin{array}{c} \mathcal{Z} \vdash \mathcal{X} \,;\, \mathcal{Y} \\ \hline\hline \mathcal{X} > \mathcal{Z} \vdash \mathcal{Y} \end{array}\ (\mathrm{drp}_1)
$$

$$
(\mathrm{rp}_2)\ \begin{array}{c} \mathcal{X} \,;\, \mathcal{Y} \vdash \mathcal{Z} \\ \hline\hline \mathcal{X} \vdash \mathcal{Z} < \mathcal{Y} \end{array}
\qquad\qquad
\begin{array}{c} \mathcal{Z} \vdash \mathcal{X} \,;\, \mathcal{Y} \\ \hline\hline \mathcal{Z} < \mathcal{Y} \vdash \mathcal{X} \end{array}\ (\mathrm{drp}_2)
$$

Residuation Rules

The residuation rules and their duals are double line rules. That is, each of them abbreviates two rules, one to be read as it stands, the other one being obtained by reading the given rule upside down.

Since the premises of (rp_1) and (rp_2) are identical, as are the premises of (drp_1) and (drp_2), we can combine each pair into a "triple double line rule" governing the relationship between the respective three sequents:

$$
(\mathrm{rt})\ \begin{array}{c} \mathcal{X} \vdash \mathcal{Z} < \mathcal{Y} \\ \hline\hline \mathcal{X} \,;\, \mathcal{Y} \vdash \mathcal{Z} \\ \hline\hline \mathcal{Y} \vdash \mathcal{X} > \mathcal{Z} \end{array}
\qquad\qquad
\begin{array}{c} \mathcal{Z} < \mathcal{Y} \vdash \mathcal{X} \\ \hline\hline \mathcal{Z} \vdash \mathcal{X} \,;\, \mathcal{Y} \\ \hline\hline \mathcal{X} > \mathcal{Z} \vdash \mathcal{Y} \end{array}\ (\mathrm{drt})
$$

Displaying Formulae

The following example is an extended variant of an example for the residuation properties given in Goré's article.

It shows that any one of the formulae A, B, C and D occurring in sequent $C < D \vdash A > B$ can be *displayed*.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash B < (C < D)}{A \, ; (C < D) \vdash B}\,(\mathrm{rt})}{C < D \vdash A > B}\,(\mathrm{rt})}{C \vdash (A > B) \, ; D}\,(\mathrm{drt})}{(A > B) > C \vdash D}\,(\mathrm{drt})$$

Display Property

Theorem (Belnap)

For every sequent $\mathfrak{S}$ and every antecedent/succedent part $\mathcal{X}$ of $\mathfrak{S}$, there is a structurally equivalent sequent $\mathfrak{S}'$ that has $\mathcal{X}$ (alone) as its antecedent/succedent. $\mathcal{X}$ is said to be displayed *in $\mathfrak{S}'$.*

Proof.

Repeated applications of (rt) and (drt) allow arbitrarily nested structures to be displayed. □

Display Calculus – Core Structural Rules

$$(\mathsf{rt})\ \begin{array}{c} \mathcal{X} \vdash \mathcal{Z} < \mathcal{Y} \\ \hline\hline \mathcal{X} ; \mathcal{Y} \vdash \mathcal{Z} \\ \hline\hline \mathcal{Y} \vdash \mathcal{X} > \mathcal{Z} \end{array} \qquad\qquad \begin{array}{c} \mathcal{Z} < \mathcal{Y} \vdash \mathcal{X} \\ \hline\hline \mathcal{Z} \vdash \mathcal{X} ; \mathcal{Y} \\ \hline\hline \mathcal{X} > \mathcal{Z} \vdash \mathcal{Y} \end{array}\ (\mathsf{drt})$$

$$(\Phi^{-}_{+} \vdash)\ \begin{array}{c} \Phi ; \mathcal{X} \vdash \mathcal{Y} \\ \hline\hline \mathcal{X} \vdash \mathcal{Y} \\ \hline\hline \mathcal{X} ; \Phi \vdash \mathcal{Y} \end{array} \qquad\qquad \begin{array}{c} \mathcal{X} \vdash \Phi ; \mathcal{Y} \\ \hline\hline \mathcal{X} \vdash \mathcal{Y} \\ \hline\hline \mathcal{X} \vdash \mathcal{Y} ; \Phi \end{array}\ (\vdash \Phi^{-}_{+})$$

Display Calculus – Optional Structural Rules I

$$(\mathsf{ass} \vdash)\ \begin{array}{c} \mathcal{X} ; (\mathcal{Y} ; \mathcal{Z}) \vdash \mathcal{W} \\ \hline\hline (\mathcal{X} ; \mathcal{Y}) ; \mathcal{Z} \vdash \mathcal{W} \end{array} \qquad\qquad \begin{array}{c} \mathcal{W} \vdash (\mathcal{X} ; \mathcal{Y}) ; \mathcal{Z} \\ \hline\hline \mathcal{W} \vdash \mathcal{X} ; (\mathcal{Y} ; \mathcal{Z}) \end{array}\ (\vdash \mathsf{ass})$$

$$(\mathsf{com} \vdash)\ \begin{array}{c} \mathcal{Y} ; \mathcal{X} \vdash \mathcal{Z} \\ \hline \mathcal{X} ; \mathcal{Y} \vdash \mathcal{Z} \end{array} \qquad\qquad \begin{array}{c} \mathcal{Z} \vdash \mathcal{Y} ; \mathcal{X} \\ \hline \mathcal{Z} \vdash \mathcal{X} ; \mathcal{Y} \end{array}\ (\vdash \mathsf{com})$$

Display Calculus – Optional Structural Rules II

$$(\mathsf{w}^- \vdash)\ \frac{\mathcal{Y} \vdash \mathcal{Z}}{\mathcal{X} \,;\, \mathcal{Y} \vdash \mathcal{Z}} \qquad \frac{\mathcal{Z} \vdash \mathcal{Y}}{\mathcal{Z} \vdash \mathcal{X} \,;\, \mathcal{Y}}\ (\vdash \mathsf{w}^-)$$

$$(\mathsf{w}_+ \vdash)\ \frac{\mathcal{X} \vdash \mathcal{Z}}{\mathcal{X} \,;\, \mathcal{Y} \vdash \mathcal{Z}} \qquad \frac{\mathcal{Z} \vdash \mathcal{X}}{\mathcal{Z} \vdash \mathcal{X} \,;\, \mathcal{Y}}\ (\vdash \mathsf{w}_+)$$

$$(\mathsf{c} \vdash)\ \frac{\mathcal{X} \,;\, \mathcal{X} \vdash \mathcal{Z}}{\mathcal{X} \vdash \mathcal{Z}} \qquad \frac{\mathcal{Z} \vdash \mathcal{X} \,;\, \mathcal{X}}{\mathcal{Z} \vdash \mathcal{X}}\ (\vdash \mathsf{c})$$

Display Calculus – Optional Structural Rules III

$$(\mathsf{yet} \vdash)\ \frac{\mathcal{Y} \,;\, \mathcal{X} \vdash \Phi}{\mathcal{X} \,;\, \mathcal{Y} \vdash \Phi} \qquad \frac{\Phi \vdash \mathcal{Y} \,;\, \mathcal{X}}{\Phi \vdash \mathcal{X} \,;\, \mathcal{Y}}\ (\vdash \mathsf{yet})$$

$$(\mathsf{grn}_1 \vdash)\ \frac{\mathcal{X} > (\mathcal{Y} \,;\, \mathcal{Z}) \vdash \mathcal{W}}{\overline{(\mathcal{X} > \mathcal{Y}) \,;\, \mathcal{Z} \vdash \mathcal{W}}} \qquad \frac{\mathcal{W} \vdash (\mathcal{X} \,;\, \mathcal{Y}) < \mathcal{Z}}{\overline{\mathcal{W} \vdash \mathcal{X} \,;\, (\mathcal{Y} < \mathcal{Z})}}\ (\vdash \mathsf{grn}_1)$$

$$(\mathsf{grn}_2 \vdash)\ \frac{(\mathcal{X} \,;\, \mathcal{Y}) < \mathcal{Z} \vdash \mathcal{W}}{\overline{\mathcal{X} \,;\, (\mathcal{Y} < \mathcal{Z}) \vdash \mathcal{W}}} \qquad \frac{\mathcal{W} \vdash \mathcal{X} > (\mathcal{Y} \,;\, \mathcal{Z})}{\overline{\mathcal{W} \vdash (\mathcal{X} > \mathcal{Y}) \,;\, \mathcal{Z}}}\ (\vdash \mathsf{grn}_2)$$

Grishin's Rules

$$(\mathrm{grn}_2 \vdash)\ \begin{array}{c} (\mathcal{X} \mathbin{;} \mathcal{Y}) < \mathcal{Z} \vdash \mathcal{W} \\ \hline\hline \mathcal{X} \mathbin{;} (\mathcal{Y} < \mathcal{Z}) \vdash \mathcal{W} \end{array}$$

$$\begin{array}{c} A \mathbin{;} (C < D) \vdash B \\ \hline C < D \vdash A > B \\ \hline C \vdash (A > B) \mathbin{;} D \end{array}\ \begin{array}{l} (\mathrm{rt}) \\ (\mathrm{drt}) \end{array} \qquad\qquad \begin{array}{c} (A \mathbin{;} C) < D \vdash B \\ \hline A \mathbin{;} C \vdash B \mathbin{;} D \\ \hline C \vdash A > (B \mathbin{;} D) \end{array}\ \begin{array}{l} (\mathrm{drt}) \\ (\mathrm{rt}) \end{array}$$

$$\begin{array}{c} \mathcal{W} \vdash \mathcal{X} > (\mathcal{Y} \mathbin{;} \mathcal{Z}) \\ \hline\hline \mathcal{W} \vdash (\mathcal{X} > \mathcal{Y}) \mathbin{;} \mathcal{Z} \end{array}\ (\vdash \mathrm{grn}_2)$$

Grishin's Rules

$$(\mathrm{grn}_1 \vdash)\ \begin{array}{c} \mathcal{X} > (\mathcal{Y} \mathbin{;} \mathcal{Z}) \vdash \mathcal{W} \\ \hline\hline (\mathcal{X} > \mathcal{Y}) \mathbin{;} \mathcal{Z} \vdash \mathcal{W} \end{array}$$

$$\begin{array}{c} (B > A) \mathbin{;} C \vdash D \\ \hline B > A \vdash D < C \\ \hline A \vdash B \mathbin{;} (D < C) \end{array}\ \begin{array}{l} (\mathrm{rt}) \\ (\mathrm{drt}) \end{array} \qquad\qquad \begin{array}{c} B > (A \mathbin{;} C) \vdash D \\ \hline A \mathbin{;} C \vdash B \mathbin{;} D \\ \hline A \vdash (B \mathbin{;} D) < C \end{array}\ \begin{array}{l} (\mathrm{drt}) \\ (\mathrm{rt}) \end{array}$$

$$\begin{array}{c} \mathcal{W} \vdash (\mathcal{X} \mathbin{;} \mathcal{Y}) < \mathcal{Z} \\ \hline\hline \mathcal{W} \vdash \mathcal{X} \mathbin{;} (\mathcal{Y} < \mathcal{Z}) \end{array}\ (\vdash \mathrm{grn}_1)$$

Grishin's Rules

$$
\begin{array}{c}
C < D \vdash A > B \\
{\scriptstyle (\mathrm{rt})+(\mathrm{grn}_2 \vdash)+(\mathrm{drt})} \quad \vdots \quad {\scriptstyle (\mathrm{drt})+(\vdash \mathrm{grn}_2)+(\mathrm{rt})} \\
A \,;\, C \vdash B \,;\, D \\
{\scriptstyle (\mathrm{drt})+(\mathrm{grn}_1 \vdash)+(\mathrm{rt})} \quad \vdots \quad {\scriptstyle (\mathrm{rt})+(\vdash \mathrm{grn}_1)+(\mathrm{drt})} \\
B > A \vdash D < C
\end{array}
$$

Displayed Substructural Logics

	Intuitionistic					Classical
	Ass.	Com.	Ctr.	Wk.	Yet.	Grn.
Non-ass. Bi-Lambek						(✓)
Non-comm. Bi-Linear BL1	✓					
Non-comm. Bi-Linear BL2	✓					✓
Non-comm. Bi-Linear BL3	✓				✓	✓
Cyclic Bi-Linear	✓				✓	
Bi-Linear	✓	✓			[✓]	(✓)
Non-comm. Bi-relevant	✓		✓			(✓)
Bi-Affine	✓			✓		(✓)

Display Calculus – Logical Connectives

connective	name	type
$\otimes$	conjunction	intensional
$\multimap$	implication	intensional
$\multimapinv$	replication	intensional
$\wedge$	conjunction	extensional
$\oplus$	disjunction	intensional
$\rhd\!-$	dual replication	intensional
$-\!\lhd$	dual implication	intensional
$\vee$	disjunction	extensional

constant	name	type
$\mathbf{1}$	triviality	intensional
$\mathbf{0}$	absurdity	intensional
$\top$	triviality	extensional
$\bot$	absurdity	extensional

Display Calculus – Intensional Logical Rules I

$$(\otimes\vdash)\ \frac{A\,;B \vdash \mathcal{Z}}{A \otimes B \vdash \mathcal{Z}} \qquad \frac{\mathcal{X} \vdash A \qquad \mathcal{Y} \vdash B}{\mathcal{X}\,;\mathcal{Y} \vdash A \otimes B}\ (\vdash\otimes)$$

$$(\multimap\vdash)\ \frac{\mathcal{X} \vdash A \qquad B \vdash \mathcal{Y}}{A \multimap B \vdash \mathcal{X} > \mathcal{Y}} \qquad \frac{\mathcal{Z} \vdash A > B}{\mathcal{Z} \vdash A \multimap B}\ (\vdash\multimap)$$

$$(\multimapinv\vdash)\ \frac{B \vdash \mathcal{Y} \qquad \mathcal{X} \vdash A}{B \multimapinv A \vdash \mathcal{Y} < \mathcal{X}} \qquad \frac{\mathcal{Z} \vdash B < A}{\mathcal{Z} \vdash B \multimapinv A}\ (\vdash\multimapinv)$$

Display Calculus – Intensional Logical Rules II

$$(\oplus\vdash)\ \frac{A \vdash \mathcal{X} \qquad B \vdash \mathcal{Y}}{A \oplus B \vdash \mathcal{X} \,;\, \mathcal{Y}} \qquad\qquad \frac{\mathcal{Z} \vdash A \,;\, B}{\mathcal{Z} \vdash A \oplus B}\ (\vdash\oplus)$$

$$(\rhd\!\!-\vdash)\ \frac{A > B \vdash \mathcal{Z}}{A \rhd\!\!- B \vdash \mathcal{Z}} \qquad\qquad \frac{A \vdash \mathcal{X} \qquad \mathcal{Y} \vdash B}{\mathcal{X} > \mathcal{Y} \vdash A \rhd\!\!- B}\ (\vdash\rhd\!\!-)$$

$$(-\!\!\lhd\vdash)\ \frac{B < A \vdash \mathcal{Z}}{B -\!\!\lhd A \vdash \mathcal{Z}} \qquad\qquad \frac{\mathcal{Y} \vdash B \qquad A \vdash \mathcal{X}}{\mathcal{Y} < \mathcal{X} \vdash B -\!\!\lhd A}\ (\vdash-\!\!\lhd)$$

Display Calculus – Intensional Logical Rules III

$$(\mathbf{1}\vdash)\ \frac{\Phi \vdash \mathcal{Z}}{\mathbf{1} \vdash \mathcal{Z}} \qquad\qquad \frac{}{\Phi \vdash \mathbf{1}}\ (\vdash\mathbf{1})$$

$$(\mathbf{0}\vdash)\ \frac{}{\mathbf{0} \vdash \Phi} \qquad\qquad \frac{\mathcal{Z} \vdash \Phi}{\mathcal{Z} \vdash \mathbf{0}}\ (\vdash\mathbf{0})$$

Observation about the Extensional Logical Rules

Every intensional logical connective as well as every intensional constant corresponds to exactly one structural connective or constant.

logical	name	structural	name
$\otimes$	conjunction	;	composition
$\mathrel{-\!\!\triangleright}$	implication	$>$	left residual
$\mathrel{\triangleleft\!\!-}$	replication	$<$	right residual
$\mathbf{1}$	triviality	Φ	empty structure
$\oplus$	disjunction	;	co-composition
$\mathrel{\triangleright\!\!-}$	dual replication	$>$	left co-residual
$\mathrel{-\!\!\triangleleft}$	dual implication	$<$	right co-residual
$\mathbf{0}$	absurdity	Φ	co-empty structure

Recall that the structural symbols are overloaded!

Display Calculus – Extensional Logical Rules I

$$(\wedge\vdash_1)\ \frac{A \vdash \mathcal{Z}}{A \wedge B \vdash \mathcal{Z}} \qquad\qquad \frac{\mathcal{Z} \vdash A \qquad \mathcal{Z} \vdash B}{\mathcal{Z} \vdash A \wedge B}\ (\vdash\wedge)$$

$$(\wedge\vdash_2)\ \frac{B \vdash \mathcal{Z}}{A \wedge B \vdash \mathcal{Z}}$$

$$\frac{\mathcal{Z} \vdash A}{\mathcal{Z} \vdash A \vee B}\ (\vdash\vee_1)$$

$$(\vee\vdash)\ \frac{A \vdash \mathcal{Z} \qquad B \vdash \mathcal{Z}}{A \vee B \vdash \mathcal{Z}}$$

$$\frac{\mathcal{Z} \vdash B}{\mathcal{Z} \vdash A \vee B}\ (\vdash\vee_2)$$

Display Calculus – Extensional Logical Rules II

$$\dfrac{}{\mathcal{Z} \vdash \top}\ (\vdash \top)$$

$$(\bot \vdash)\ \dfrac{}{\bot \vdash \mathcal{Z}}$$

Display Calculus – Reasoning Rules

$$\dfrac{}{A \vdash A}\ (\mathrm{id})$$

$$\dfrac{\mathcal{X} \vdash A \qquad A \vdash \mathcal{Y}}{\mathcal{X} \vdash \mathcal{Y}}\ (\mathrm{cut})$$

Belnap's Conditions

(C1) Each formula occurring in a premise of a rule instance is a subformula of some formula in the conclusion.

(C2) Congruent parameters are occurrences of the same structure.

(C3) Each parameter is congruent to at most one structure variable in the conclusion. That is, no two structure variables in the conclusion are congruent to each other.

(C4) Congruent parameters are all either antecedent or succedent structures.

(C5) A schematic formula variable in the conclusion of an inference rule ρ is either the entire antecedent or the entire succedent. This formula is called a principal formula of ρ.

Belnap's Conditions

(C6) Each inference rule is closed under simultaneous substitution of arbitrary structures for congruent succedent parameters.

(C7) Each inference rule is closed under simultaneous substitution of arbitrary structures for congruent antecedent parameters.

(C8) For inference rules ρ and σ with respective conclusions $\mathcal{X} \vdash A$ and $A \vdash \mathcal{Y}$ with formula A principal in both inferences in the sense of C5 the following holds: if cut is applied to yield $\mathcal{X} \vdash \mathcal{Y}$, then

- (i) $\mathcal{X} \vdash \mathcal{Y}$ is identical to either $\mathcal{X} \vdash A$ or $A \vdash \mathcal{Y}$, or
- (ii) there is a derivation of $\mathcal{X} \vdash \mathcal{Y}$ from the premises of ρ and σ such that the cut formula of every cut occurring in that derivation is a proper subformula of A.

Generalized Cut-Elimination

Theorem (Belnap)

If a display calculus satisfies C1, then it has the subformula property, that is every formula occurring in a cut-free derivation appears as a subformula of some formula in the conclusion.
A display calculus satisfying C2–C8 enjoys cut-elimination.

Proof.

The proof follows Gentzen's original idea, successively eliminating topmost instances of the cut-rule by tracing the cut-formula upwards until the cut is a principal cut. Unlike in Gentzens proof, no multicut-rule is required to resolve the difficulties arising from a contraction rule applied to the cut-formula. Belnaps proof avoids this by computing the set of ancestor occurrences of the cut-formula and essentially applying the cut-rule to each member in that set. □

Tomographs for Substructural Logics

We will now present our two dimensional notation.

The name "tomograph" stems from a graph theoretical rendering of Goré's logic that employs bipartite graphs, using one type of vertices for logical and the other type for structural matters.

The required graphs are extremely simple in that they have the cut-property, which means that the graph is so weakly connected that removing any internal vertex results in a disconnected graph.

For the purpose of brevity, we simply provide an EBNF for a two-dimensional notation that generates such graphs.

Tomographs

Tomographs **T** are defined on the basis of *structures* **S** and *formulae* **F** as follows:

$$
\begin{array}{lcll}
\mathbf{T} & ::= & \mathbf{S}\ \mathbf{I}\ \mathbf{S} & \text{turnstile marker} \\
\mathbf{S} & ::= & \cdot \mid \cdot & \text{empty structure} \\
 & & \mid\ \mathbf{F} \rightarrow \mid\ \leftarrow \mathbf{F} & \text{formula content} \\
 & & \mid\ \substack{\mathbf{S}\\ \mathbf{S}} \succ \mid\ \mathbf{S} \succ \mathbf{S} \mid\ \mathbf{S} \succ \mathbf{S} & \text{fusion fork} \\
 & & \mid\ \prec \substack{\mathbf{S}\\ \mathbf{S}} \mid\ \mathbf{S} \prec \mathbf{S} \mid\ \mathbf{S} \prec \mathbf{S} & \text{fission fork}
\end{array}
$$

- Beginning at the turnstile marker, structures are grown both to the left (antecedent position) and to the right (succedent position).
- Structural elements must not overlap. Forks may be stretched vertically, strokes (horizontal lines) may be extended, but not tilted.

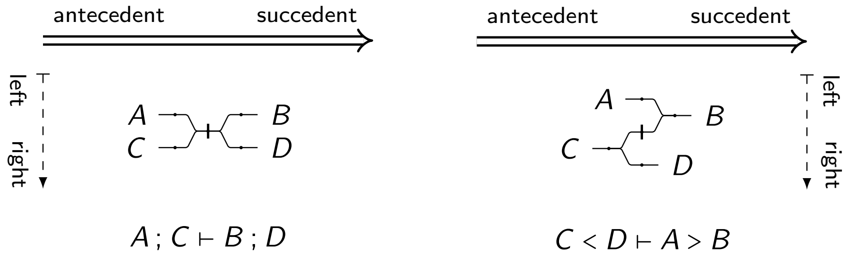
Translating Sequents into Tomographs

The functions $\langle\!\langle \cdot \rangle\!\rangle^{\vdash}$ and $\langle\!\langle \cdot \rangle\!\rangle$, by way of auxiliary functions $\langle\!\langle \cdot \|$ and $\| \cdot \rangle\!\rangle$, translate sequents of substructural display logic into tomographs.

$$
\begin{array}{lcl}
\langle\!\langle \mathcal{U} \vdash \mathcal{V} \rangle\!\rangle^{\vdash} & =_{\mathrm{def}} & \langle\!\langle \mathcal{U} \|\ \mathbf{I}\ \| \mathcal{V} \rangle\!\rangle \\
\langle\!\langle \mathcal{U} \vdash \mathcal{V} \rangle\!\rangle & =_{\mathrm{def}} & \langle\!\langle \mathcal{U} \|\ \| \mathcal{V} \rangle\!\rangle
\end{array}
$$

$$
\begin{array}{lcl@{\qquad}lcl}
\langle\!\langle \Phi \| & =_{\mathrm{def}} & \cdot & \| \Phi \rangle\!\rangle & =_{\mathrm{def}} & \cdot \\
\langle\!\langle A \| & =_{\mathrm{def}} & A \rightarrow & \| A \rangle\!\rangle & =_{\mathrm{def}} & \leftarrow A \\
\langle\!\langle \mathcal{X} ; \mathcal{Y} \| & =_{\mathrm{def}} & \substack{\langle\!\langle \mathcal{X} \| \\ \langle\!\langle \mathcal{Y} \|} \succ & \| \mathcal{X} ; \mathcal{Y} \rangle\!\rangle & =_{\mathrm{def}} & \prec \substack{\| \mathcal{X} \rangle\!\rangle \\ \| \mathcal{Y} \rangle\!\rangle} \\
\langle\!\langle \mathcal{X} > \mathcal{Z} \| & =_{\mathrm{def}} & \langle\!\langle \mathcal{Z} \| \prec \| \mathcal{X} \rangle\!\rangle & \| \mathcal{X} > \mathcal{Z} \rangle\!\rangle & =_{\mathrm{def}} & \langle\!\langle \mathcal{X} \| \succ \| \mathcal{Z} \rangle\!\rangle \\
\langle\!\langle \mathcal{Z} < \mathcal{Y} \| & =_{\mathrm{def}} & \langle\!\langle \mathcal{Z} \| \prec \| \mathcal{Y} \rangle\!\rangle & \| \mathcal{Z} < \mathcal{Y} \rangle\!\rangle & =_{\mathrm{def}} & \langle\!\langle \mathcal{Y} \| \succ \| \mathcal{Z} \rangle\!\rangle
\end{array}
$$

Reading Tomographs



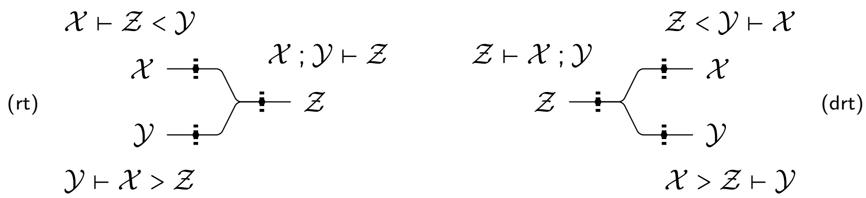
- Formulae on the *left* of a stroke are *antecedent* formulae: **F** –
- Formulae on the *right* of a stroke are *succedent* formulae: – **F**
- The occurrence of antecedent (succedent) formulae from top to bottom corresponds to the linear order of antecedent (succedent) formulae within a sequent from left to right.

Residuation in Tomographs

Residuation rule (rt) shifts the turnstile through the arms of fusions, and its dual (drt) shifts the turnstile through the arms of fissions.

Fusion Fork

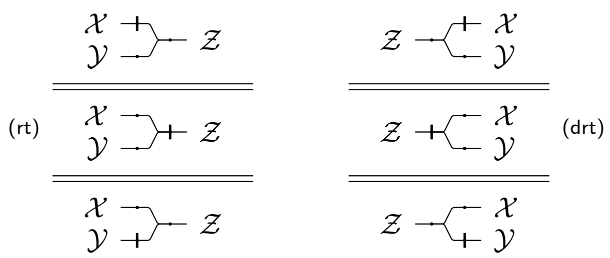
Fission Fork



Identity up to residuation in substructural display logic becomes identity of tomographs – up to turnstile postition if that information is retained.

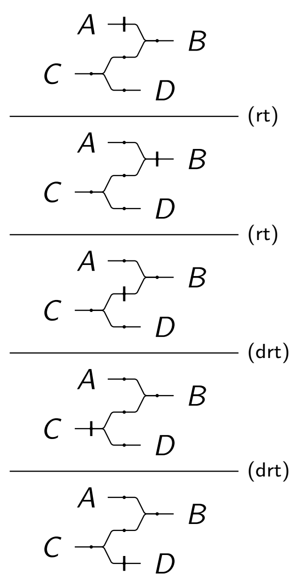
Residuation Rules for Tomographs

We can thus formulate rules governing residuation in a calculus for tomographs.



Returning to Goré's Example

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash B < (C < D)}{A \,;\, (C < D) \vdash B}\,(\mathsf{rt})}{C < D \vdash A > B}\,(\mathsf{rt})}{C \vdash (A > B)\,;\, D}\,(\mathsf{drt})}{(A > B) > C \vdash D}\,(\mathsf{drt})$$



Characteristc Property of the Translation

Proposition

Let $\mathfrak{S}$ and $\mathfrak{T}$ be two different sequents that are interderivable by any number of applications of the residuation rules (rt) *and* (drt). *Then*

(i) $\langle\!\langle \mathfrak{S} \rangle\!\rangle \equiv \langle\!\langle \mathfrak{T} \rangle\!\rangle$

(ii) $\langle\!\langle \mathfrak{S} \rangle\!\rangle^{\vdash} \not\equiv \langle\!\langle \mathfrak{T} \rangle\!\rangle^{\vdash}$

Proof.

By induction on the length of the derivation and inspecting the premises and conclusions of each possible instance of (rt) and (drt) and their translations. □

Characteristc Property of the Translation

The characteristic property states that tomographs obtained by a "forgetful" translation (forgetting the turnstile) of sequents that are interderivable by applications of the residuation rules alone are identical.

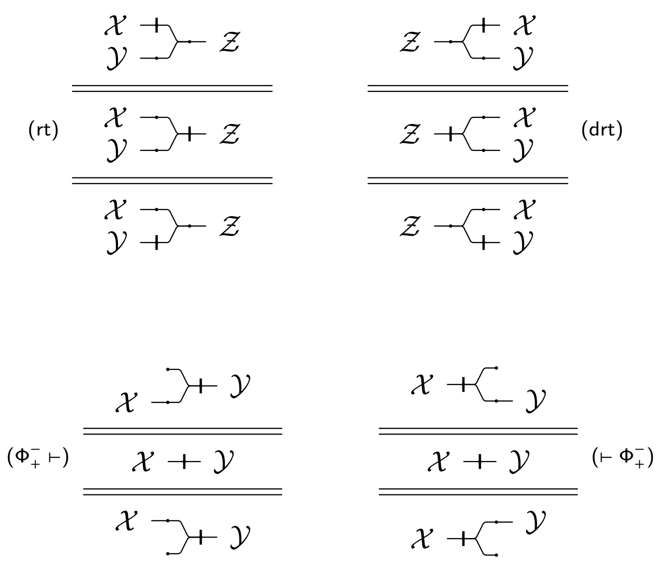
We will now present the rules for a tomograph calculus.

For each rule we provide both a version with turnstile markers and a version without them.

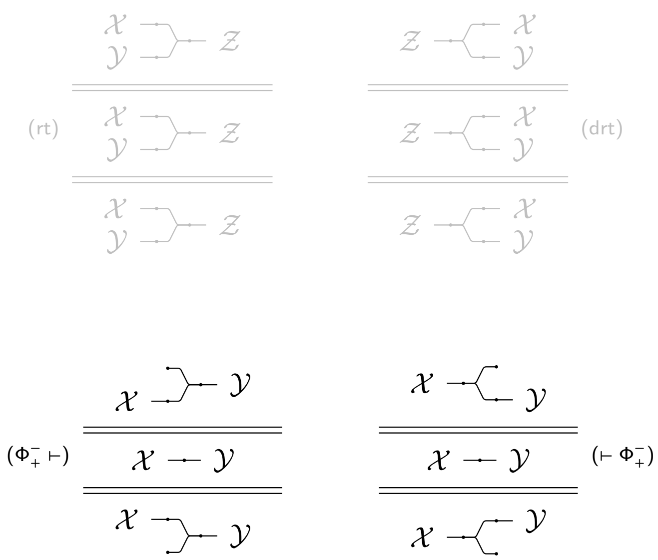
When using the former, the rules (rt) and (drt) are required to be able to move the turnstile marker to desired positions.

When using the latter, every rule can be applied at any position of a tomograph that matches the premise.

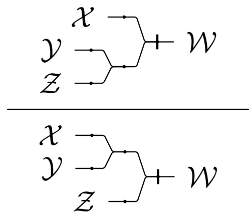
Display Tomographs – Core Structural Rules

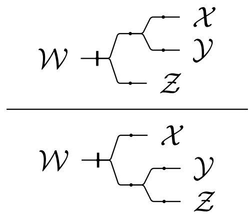


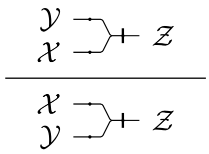
Display Tomographs – Core Structural Rules

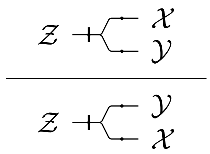


Display Tomographs – Optional Structural Rules I

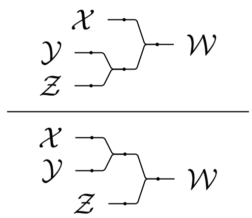
(ass ⊢) 

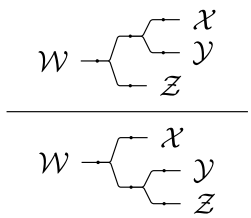
 (⊢ ass)

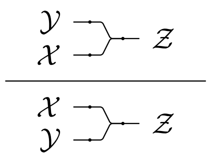
(com ⊢) 

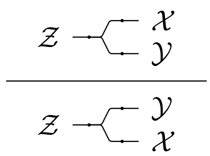
 (⊢ com)

Display Tomographs – Optional Structural Rules I

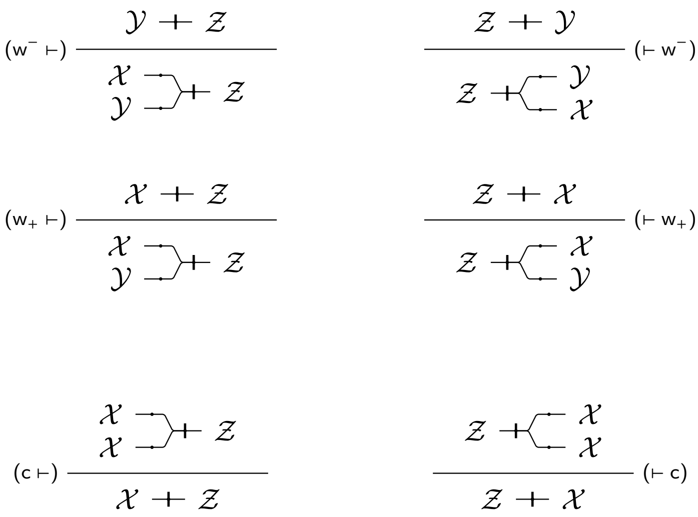
(ass ⊢) 

 (⊢ ass)

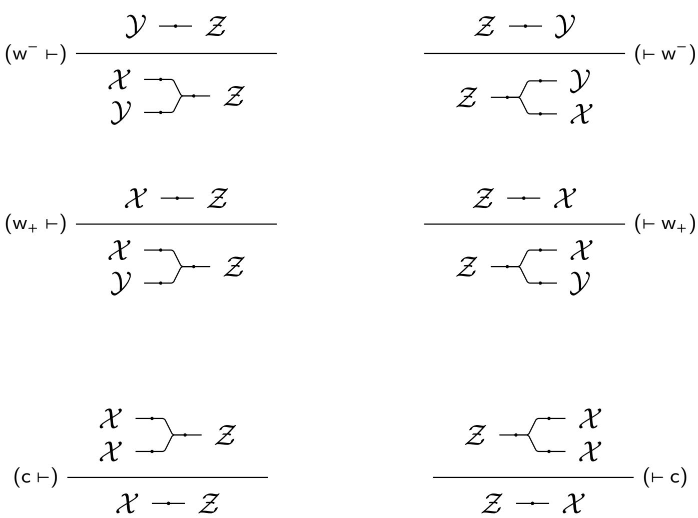
(com ⊢) 

 (⊢ com)

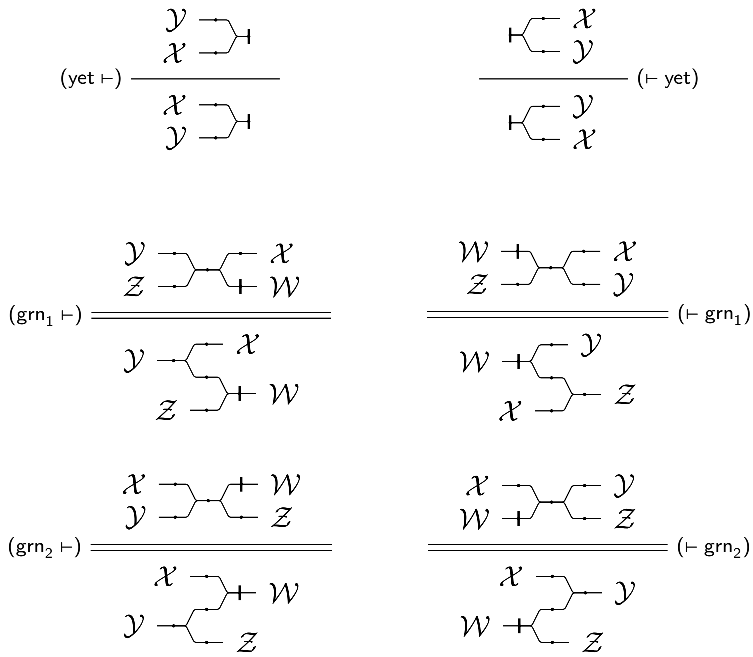
Display Tomographs – Optional Structural Rules II



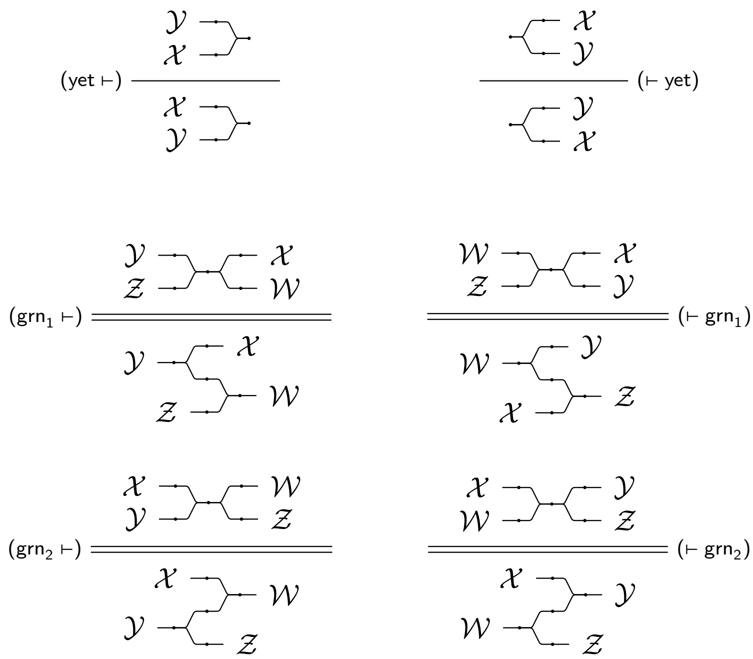
Display Tomographs – Optional Structural Rules II



Display Tomographs – Optional Structural Rules III



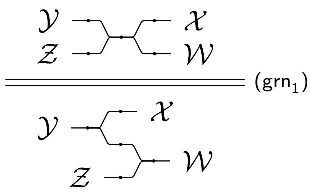
Display Tomographs – Optional Structural Rules III

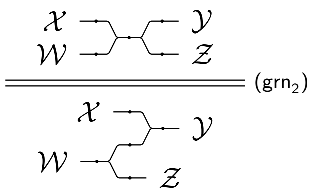


Display Tomographs – Optional Structural Rules III

Without turnstile, rules ($\mathsf{grn}_1 \vdash$) and ($\vdash \mathsf{grn}_1$) are indistinguishable, as are rules ($\mathsf{grn}_2 \vdash$) and ($\vdash \mathsf{grn}_2$).

Consequently, they can be combined into a single rule each.



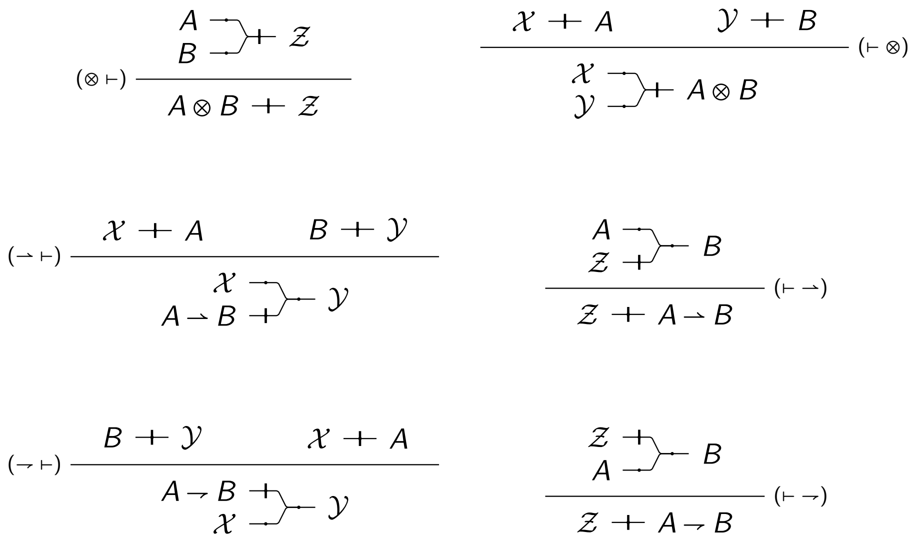


Display Tomographs – Logical Connectives

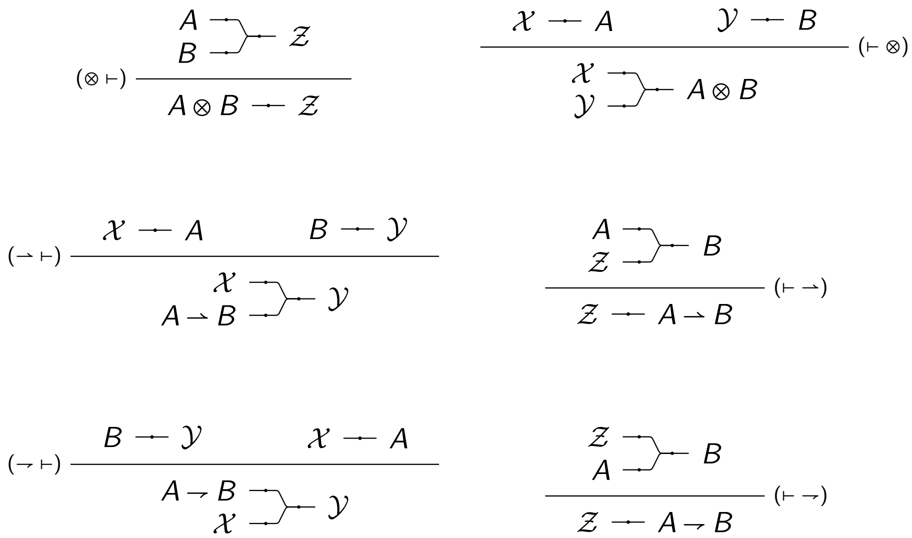
connective	name	type
$\otimes$	conjunction	intensional
$\rightharpoonup$	implication	intensional
$\rightharpoondown$	replication	intensional
$\wedge$	conjunction	extensional
$\oplus$	disjunction	intensional
$\leftharpoonup$	dual replication	intensional
$\leftharpoondown$	dual implication	intensional
$\vee$	disjunction	extensional

constant	name	type
$\mathbf{1}$	triviality	intensional
$\mathbf{0}$	absurdity	intensional
$\top$	triviality	extensional
$\bot$	absurdity	extensional

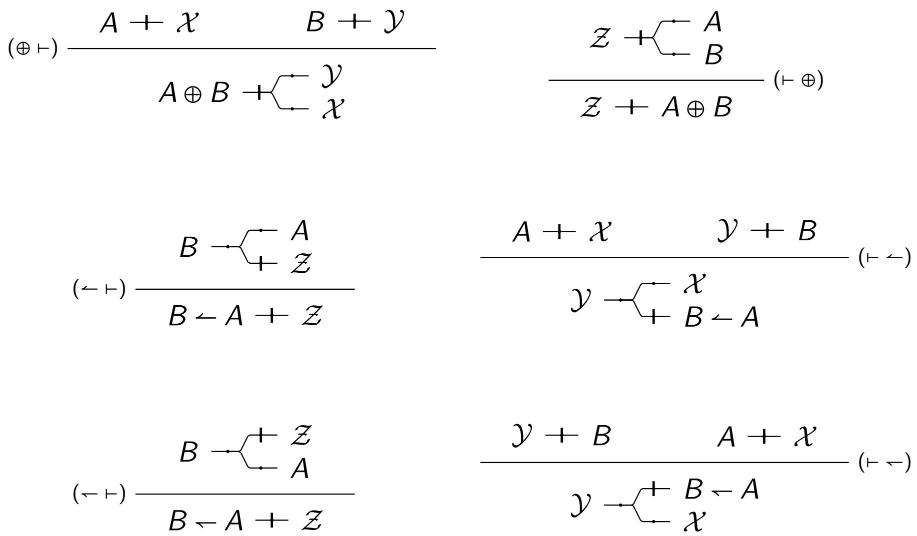
Display Tomographs – Intensional Logical Rules I



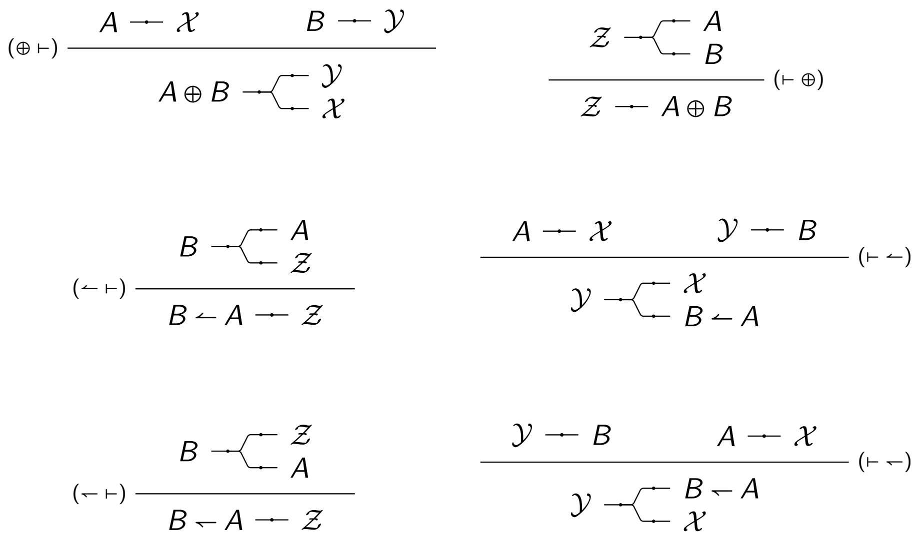
Display Tomographs – Intensional Logical Rules I



Display Tomographs – Intensional Logical Rules II



Display Tomographs – Intensional Logical Rules II



Display Tomographs – Intensional Logical Rules III

$$(\mathbf{1} \vdash)\ \frac{\vdash \mathcal{Z}}{\mathbf{1} \mathrel{\dashv\!\!\!\!\vdash} \mathcal{Z}} \qquad\qquad \frac{}{\vdash \mathbf{1}}\ (\vdash \mathbf{1})$$

$$(\mathbf{0} \vdash)\ \frac{}{\mathbf{0} \dashv} \qquad\qquad \frac{\mathcal{Z} \dashv}{\mathcal{Z} \mathrel{\dashv\!\!\!\!\vdash} \mathbf{0}}\ (\vdash \mathbf{0})$$

Display Tomographs – Intensional Logical Rules III

$$(\mathbf{1} \vdash)\ \frac{\multimapdotinv \mathcal{Z}}{\mathbf{1} \mathrel{-\!\!\!\bullet\!\!\!-} \mathcal{Z}} \qquad\qquad \frac{}{\multimapdotinv \mathbf{1}}\ (\vdash \mathbf{1})$$

$$(\mathbf{0} \vdash)\ \frac{}{\mathbf{0} \multimapdot} \qquad\qquad \frac{\mathcal{Z} \multimapdot}{\mathcal{Z} \mathrel{-\!\!\!\bullet\!\!\!-} \mathbf{0}}\ (\vdash \mathbf{0})$$

Display Tomographs – Extensional Logical Rules I

$$(\wedge\vdash_1)\ \frac{A \mathrel{-\!\!\!+\!\!\!-} \mathcal{Z}}{A \wedge B \mathrel{-\!\!\!+\!\!\!-} \mathcal{Z}}$$

$$(\wedge\vdash_2)\ \frac{B \mathrel{-\!\!\!+\!\!\!-} \mathcal{Z}}{A \wedge B \mathrel{-\!\!\!+\!\!\!-} \mathcal{Z}}$$

$$\frac{\mathcal{Z} \mathrel{-\!\!\!+\!\!\!-} A \qquad \mathcal{Z} \mathrel{-\!\!\!+\!\!\!-} B}{\mathcal{Z} \mathrel{-\!\!\!+\!\!\!-} A \wedge B}\ (\vdash\wedge)$$

$$(\vee\vdash)\ \frac{A \mathrel{-\!\!\!+\!\!\!-} \mathcal{Z} \qquad B \mathrel{-\!\!\!+\!\!\!-} \mathcal{Z}}{A \vee B \mathrel{-\!\!\!+\!\!\!-} \mathcal{Z}}$$

$$\frac{\mathcal{Z} \mathrel{-\!\!\!+\!\!\!-} A}{\mathcal{Z} \mathrel{-\!\!\!+\!\!\!-} A \vee B}\ (\vdash\vee_1)$$

$$\frac{\mathcal{Z} \mathrel{-\!\!\!+\!\!\!-} B}{\mathcal{Z} \mathrel{-\!\!\!+\!\!\!-} A \vee B}\ (\vdash\vee_2)$$

Display Tomographs – Extensional Logical Rules I

$$(\wedge\vdash_1)\ \frac{A \mathrel{-\!\!\bullet\!\!-} \mathcal{Z}}{A \wedge B \mathrel{-\!\!\bullet\!\!-} \mathcal{Z}}$$

$$(\wedge\vdash_2)\ \frac{B \mathrel{-\!\!\bullet\!\!-} \mathcal{Z}}{A \wedge B \mathrel{-\!\!\bullet\!\!-} \mathcal{Z}}$$

$$\frac{\mathcal{Z} \mathrel{-\!\!\bullet\!\!-} A \qquad \mathcal{Z} \mathrel{-\!\!\bullet\!\!-} B}{\mathcal{Z} \mathrel{-\!\!\bullet\!\!-} A \wedge B}\ (\vdash\wedge)$$

$$(\vee\vdash)\ \frac{A \mathrel{-\!\!\bullet\!\!-} \mathcal{Z} \qquad B \mathrel{-\!\!\bullet\!\!-} \mathcal{Z}}{A \vee B \mathrel{-\!\!\bullet\!\!-} \mathcal{Z}}$$

$$\frac{\mathcal{Z} \mathrel{-\!\!\bullet\!\!-} A}{\mathcal{Z} \mathrel{-\!\!\bullet\!\!-} A \vee B}\ (\vdash\vee_1)$$

$$\frac{\mathcal{Z} \mathrel{-\!\!\bullet\!\!-} B}{\mathcal{Z} \mathrel{-\!\!\bullet\!\!-} A \vee B}\ (\vdash\vee_2)$$

Display Tomographs – Extensional Logical Rules II

$$\frac{}{\mathcal{Z} \mathrel{-\!\!+\!\!-} \top}\ (\vdash \top)$$

$$(\bot \vdash)\ \frac{}{\bot \mathrel{-\!\!+\!\!-} \mathcal{Z}}$$

Display Tomographs – Extensional Logical Rules II

$$\frac{}{\mathcal{Z} \mathrel{-\!\!\cdot\!\!-} \top}\ (\vdash \top)$$

$$(\bot \vdash)\ \frac{}{\bot \mathrel{-\!\!\cdot\!\!-} \mathcal{Z}}$$

Display Tomographs – Reasoning Rules

$$\frac{}{A \mathrel{+} A}\ (\mathsf{id})$$

$$\frac{\mathcal{X} \mathrel{+} A \qquad A \mathrel{+} \mathcal{Y}}{\mathcal{X} \mathrel{+} \mathcal{Y}}\ (\mathsf{cut})$$

Display Tomographs – Reasoning Rules

$$\frac{}{A \mathrel{-\!\!\cdot\!\!-} A}\ (\mathsf{id})$$

$$\frac{\mathcal{X} \mathrel{-\!\!\cdot\!\!-} A \qquad A \mathrel{-\!\!\cdot\!\!-} \mathcal{Y}}{\mathcal{X} \mathrel{-\!\!\cdot\!\!-} \mathcal{Y}}\ (\mathsf{cut})$$

Display Tomographs – Discussion of the Rules

The rules that use turnstile markers are identical to the one of Goré's sequent calculus up to notation.

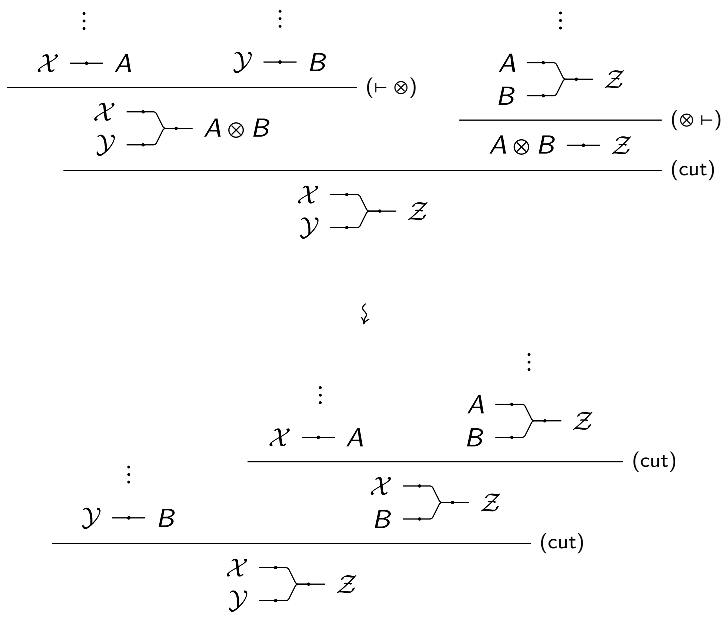
A structural rule without turnstile marker can be applied anywhere within a tomograph that matches the structural requirements of the premise(s).

A logical rule without turnstile marker can be applied anywhere along the "formula rim" of a tomograph:

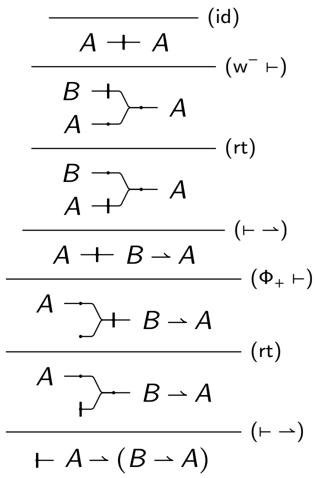
- In the case of a single premise rule it can be applied at any fork that has two formula occurrences. The fork is eliminated, the connective is introduced to connect the formulae.
- In the case of a two premise rule it can be applied at any two formula occurrences of two tomographs. The fork is introduced to compose the contexts, the connective is introduced to connect the formulae.

Cut elimination proceeds as it does in Goré's sequent calculus.

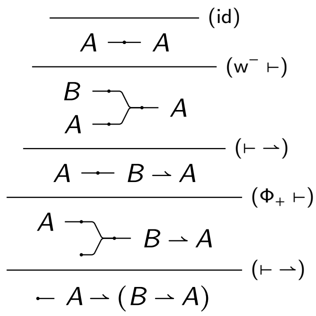
Display Tomographs – Eliminating Cuts



Example Derivation I

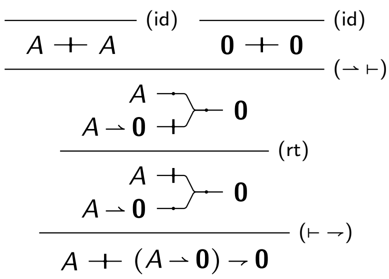


with turnstile focussing

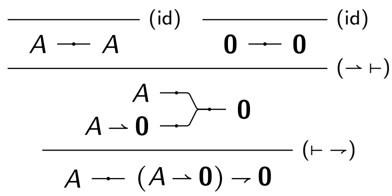


without turnstile focussing

Example Derivation II

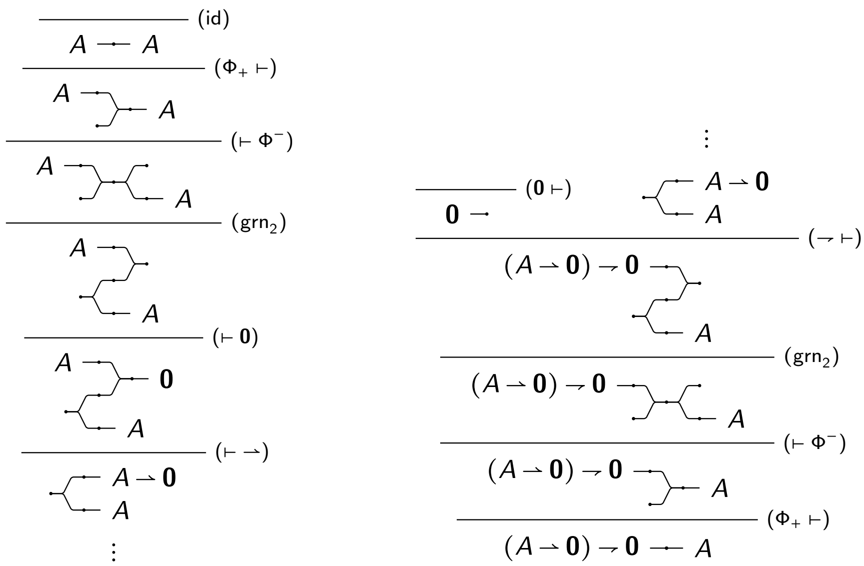


with turnstile focussing



without turnstile focussing

Example Derivation III



Explicit Negation

We now turn to the matter of including explicit structural negation into both the sequent calculus and the calculus for tomographs.

The former was demonstrated by Goré, and we give a brief account of galois connection and its dual, the algebraic basis for structural negation.

The two symbols for structural negations, $\sharp$ and $\flat$, used by Goré for both sets of connected negations are also overloaded.

Structural negations for tomographs are obtained according to the correlation between residuation with regard to the empty structure and the galois connection of two negations. The same holds for the dual case.

Galois Connections

A *Galois connection* on a structure $\langle L, \leq \rangle$ is a pair of functions $\zeta : L \to L$ and $\eta : L \to L$ such that

$(\star)$ for all $x, y \in L$: $\quad y \leq \zeta(x) \quad$ iff $\quad x \leq \eta(y)$.

A *dual Galois connection* on a structure $\langle L, \leq \rangle$ is a pair of functions $\kappa : L \to L$ and $\lambda : L \to L$ such that

$(\dagger)$ for all $x, y \in L$: $\quad \kappa(x) \leq y \quad$ iff $\quad \lambda(y) \leq x$.

The function pairs (ζ, η) and (κ, λ) are *pairs of polarities*, as each of them uniquely determines the other via the characteristic property.

Galois Connection as Structural Negation

Goré's display logic uses two unary structural connectives: $\sharp$ and $\flat$.

The display property governing these structural connectives is obtained by means of Galois connection and its dual.

$$\text{(gc)}\ \frac{\mathcal{X} \vdash \sharp\mathcal{Y}}{\mathcal{Y} \vdash \flat\mathcal{X}} \qquad\qquad \frac{\sharp\mathcal{X} \vdash \mathcal{Y}}{\flat\mathcal{Y} \vdash \mathcal{X}}\ \text{(dgc)}$$

The following rules relate the structural negations to the (dual) residuals and empty structure.

$$(\vdash \sharp)\ \frac{\mathcal{Y} \vdash \mathcal{X} > \Phi}{\mathcal{Y} \vdash \sharp\mathcal{X}} \qquad\qquad \frac{\mathcal{X} > \Phi \vdash \mathcal{Y}}{\sharp\mathcal{X} \vdash \mathcal{Y}}\ (\sharp \vdash)$$

$$(\vdash \flat)\ \frac{\mathcal{Y} \vdash \Phi < \mathcal{X}}{\mathcal{Y} \vdash \flat\mathcal{X}} \qquad\qquad \frac{\Phi < \mathcal{X} \vdash \mathcal{Y}}{\flat\mathcal{X} \vdash \mathcal{Y}}\ (\flat \vdash)$$

Explicit Negation in the Display Calculus

There are four natural negations in Intuitionistic Bi-Lambek logic.

connective	name	type
$\cdot^{\mathbf{0}}$	right negation	intensional
$^{\mathbf{0}}\cdot$	left negation	intensional
$\cdot^{\mathbf{1}}$	dual right negation	intensional
$^{\mathbf{1}}\cdot$	dual left negation	intensional

The easiest way to gain an intuition for these negations is via the following definitions.

$$A^{\mathbf{0}} =_{\text{def}} A \mathrel{-\!\!\triangleright} \mathbf{0} \qquad {}^{\mathbf{0}}A =_{\text{def}} \mathbf{0} \mathrel{\triangleleft\!\!-} A$$

$$A^{\mathbf{1}} =_{\text{def}} A \mathrel{\triangleright\!\!-} \mathbf{1} \qquad {}^{\mathbf{1}}A =_{\text{def}} \mathbf{1} \mathrel{-\!\!\triangleleft} A$$

Display Calculus – Intensional Logical Rules IV

$$(\cdot^{\mathbf{0}} \vdash)\ \frac{\mathcal{Z} \vdash A}{A^{\mathbf{0}} \vdash \sharp\mathcal{Z}} \qquad \frac{\mathcal{Z} \vdash \sharp A}{\mathcal{Z} \vdash A^{\mathbf{0}}}\ (\vdash \cdot^{\mathbf{0}})$$

$$({}^{\mathbf{0}}\cdot \vdash)\ \frac{\mathcal{Z} \vdash A}{{}^{\mathbf{0}}A \vdash \flat\mathcal{Z}} \qquad \frac{\mathcal{Z} \vdash \flat A}{\mathcal{Z} \vdash {}^{\mathbf{0}}A}\ (\vdash {}^{\mathbf{0}}\cdot)$$

$$(\cdot^{\mathbf{1}} \vdash)\ \frac{\sharp A \vdash \mathcal{Z}}{A^{\mathbf{1}} \vdash \mathcal{Z}} \qquad \frac{A \vdash \mathcal{Z}}{\sharp\mathcal{Z} \vdash A^{\mathbf{1}}}\ (\vdash \cdot^{\mathbf{1}})$$

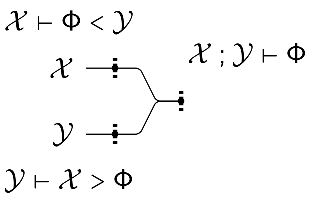
$$({}^{\mathbf{1}}\cdot \vdash)\ \frac{\flat A \vdash \mathcal{Z}}{{}^{\mathbf{1}}A \vdash \mathcal{Z}} \qquad \frac{A \vdash \mathcal{Z}}{\flat\mathcal{Z} \vdash {}^{\mathbf{1}}A}\ (\vdash {}^{\mathbf{1}}\cdot)$$

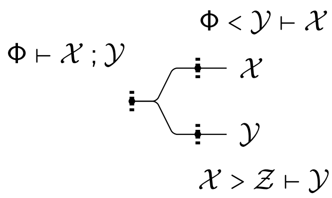
Galois Connections in Tomographs

Fusion Fork

Fission Fork

(rt)





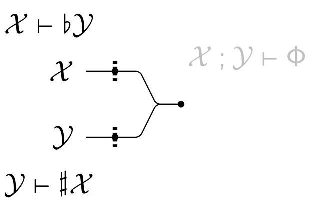
(drt)

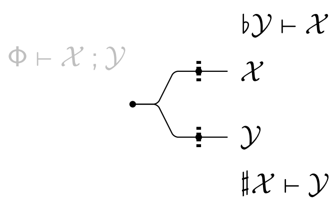
Galois Connections in Tomographs

Fusion Fork

Fission Fork

(gc)

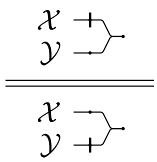
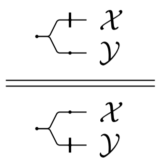




(dgc)

Galois Connection Rules for Tomographs

These are the rules governing Galois connection in a calculus for tomographs.

(gc)   (dgc)

These rules are only useful in the calculus that tracks the turnstile.

Explicit Negation in Display Tomographs

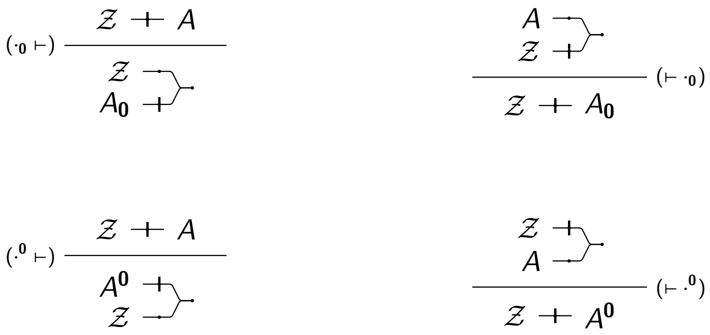
The are four negations in Intuitionistic Bi-Lambek logic are:

connective	name	type
${}^{\cdot}\mathbf{0}$	down negation	intensional
${}_{\cdot}\mathbf{0}$	up negation	intensional
$\mathbf{1}^{\cdot}$	dual down negation	intensional
$\mathbf{1}_{\cdot}$	dual up negation	intensional

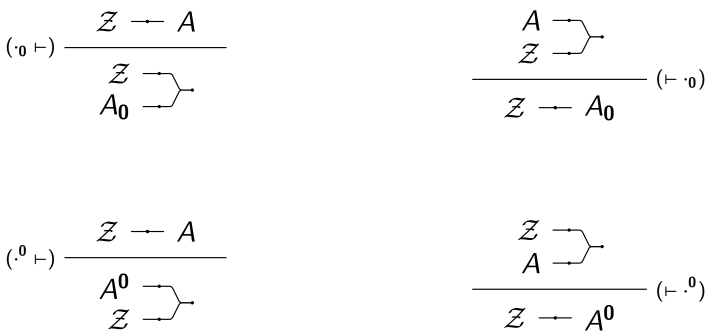
The same intuition for these negations is gained via the following definitions.

$$A_{\mathbf{0}} =_{\mathrm{def}} A \rightharpoonup \mathbf{0} \qquad A^{\mathbf{0}} =_{\mathrm{def}} A \rightharpoondown \mathbf{0}$$
$$_{\mathbf{1}}A =_{\mathrm{def}} A \leftharpoondown \mathbf{1} \qquad ^{\mathbf{1}}A =_{\mathrm{def}} A \leftharpoonup \mathbf{1}$$

Display Tomographs – Intensional Logical Rules IV



Display Tomographs – Intensional Logical Rules IV



Display Tomographs – Intensional Logical Rules V

$$(_1\cdot\vdash)\ \frac{\prec\begin{array}{l}- \ A\\ + \ \mathcal{Z}\end{array}}{{}_1A + \mathcal{Z}} \qquad \frac{A + \mathcal{Z}}{\prec\begin{array}{l}- \ \mathcal{Z}\\ + \ {}_1A\end{array}}\ (\vdash {}_1\cdot)$$

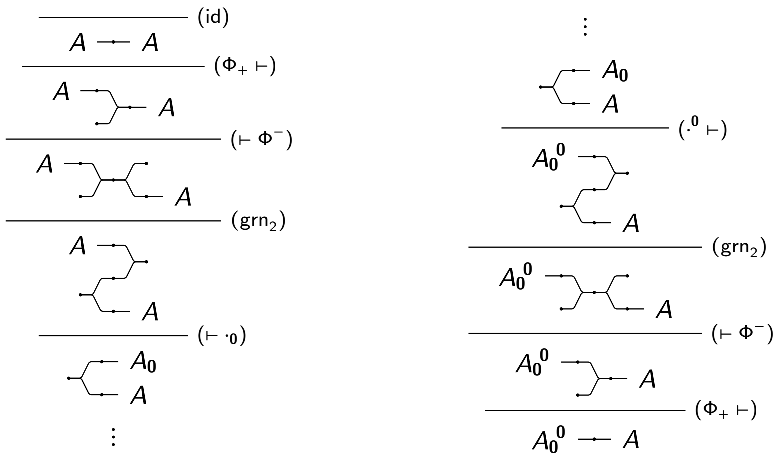
$$(^1\cdot\vdash)\ \frac{\prec\begin{array}{l}+ \ \mathcal{Z}\\ - \ A\end{array}}{{}^1A + \mathcal{Z}} \qquad \frac{A + \mathcal{Z}}{\prec\begin{array}{l}+ \ {}^1A\\ - \ \mathcal{Z}\end{array}}\ (\vdash {}^1\cdot)$$

Display Tomographs – Intensional Logical Rules V

$$(_1\cdot\vdash)\ \frac{\prec\begin{array}{l}- \ A\\ - \ \mathcal{Z}\end{array}}{{}_1A - \mathcal{Z}} \qquad \frac{A - \mathcal{Z}}{\prec\begin{array}{l}- \ \mathcal{Z}\\ - \ {}_1A\end{array}}\ (\vdash {}_1\cdot)$$

$$(^1\cdot\vdash)\ \frac{\prec\begin{array}{l}- \ \mathcal{Z}\\ - \ A\end{array}}{{}^1A - \mathcal{Z}} \qquad \frac{A - \mathcal{Z}}{\prec\begin{array}{l}- \ {}^1A\\ - \ \mathcal{Z}\end{array}}\ (\vdash {}^1\cdot)$$

Example Derivation IV



Summary

We have presented the notion of tomographs, a two-dimensional notation based on ternary structural elements – fission and fusion forks.

In this framework the property of (dual) residuated triples, which relates binary structural connectives in view of a turnstile, amounts to looking at the structural elements with one of three possible foci.

The binary relation of the turnstile itself becomes expendable, since the rules of the calculus can be formulated on the basis of relationships between structural elements and formulae alone.

At the same time, the structural representation of sequents is still the basis of a calculus in which structural and logical rules are applied to obtain new tomographs.

We have also addressed explicit structural negation, which can be treated by the same ternary structural elements – fission and fusion forks.

$$
\dfrac{\dfrac{\vdots\,(\mathfrak{n})}{\Gamma_1 \to \Theta_1, \mathfrak{F}(\mathfrak{n})}}{\overline{\Gamma_1 \to \mathfrak{F}(\mathfrak{n}), \Theta_1, \forall \mathfrak{x}\, \mathfrak{F}(\mathfrak{x})}}
\qquad
\dfrac{\dfrac{\vdots}{\mathfrak{F}(\mathfrak{n}), \Gamma_2 \to \Theta_2}}{\forall \mathfrak{x}\, \mathfrak{F}(\mathfrak{x}), \Gamma_2 \to \Theta_2}
\qquad
\dfrac{\dfrac{\vdots\,(\mathfrak{a})}{\Gamma_1 \to \Theta_1, \mathfrak{F}(\mathfrak{a})}}{\Gamma_1 \to \Theta_1, \forall \mathfrak{x}\, \mathfrak{F}(\mathfrak{x})}
\qquad
\dfrac{\dfrac{\vdots}{\mathfrak{F}(\mathfrak{n}), \Gamma_2 \to \Theta_2}}{\overline{\forall \mathfrak{x}\, \mathfrak{F}(\mathfrak{x}), \Gamma_2, \mathfrak{F}(\mathfrak{n}) \to \Theta_2}}
$$

$$
\vdots
$$

$$
\dfrac{\Gamma \to \mathfrak{F}(\mathfrak{n}), \Theta, \forall \mathfrak{x}\, \mathfrak{F}(\mathfrak{x}) \qquad \forall \mathfrak{x}\, \mathfrak{F}(\mathfrak{x}), \Delta \to \Lambda}{\Gamma, \Delta \to \mathfrak{F}(\mathfrak{n}), \Theta, \Lambda}
\qquad
\dfrac{\Gamma \to \Theta, \forall \mathfrak{x}\, \mathfrak{F}(\mathfrak{x}) \qquad \forall \mathfrak{x}\, \mathfrak{F}(\mathfrak{x}), \Delta, \mathfrak{F}(\mathfrak{n}) \to \Lambda}{\Gamma, \Delta, \mathfrak{F}(\mathfrak{n}) \to \Theta, \Lambda}
$$

$$
\vdots
$$

$$
\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_3 \to \mathfrak{F}(\mathfrak{n}), \Theta_3}}{\overline{\Gamma_3 \to \Theta_3, \mathfrak{F}(\mathfrak{n})}} \qquad \dfrac{\dfrac{\vdots}{\Gamma_3, \mathfrak{F}(\mathfrak{n}) \to \Theta_3}}{\overline{\mathfrak{F}(\mathfrak{n}), \Gamma_3 \to \Theta_3}}}{\dfrac{\Gamma_3, \Gamma_3 \to \Theta_3, \Theta_3}{\overline{\Gamma_3 \to \Theta_3}}}
$$

$$
\vdots
$$

$$
\dfrac{}{\to} \qquad \text{Die Endsequenz}
$$