Gorenstein Ideals of Codimension 4 and Combinatorics of Stanley-Reisner Rings

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Dedicated to my beloved parents and my lovely Husband

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INTRODUCTION

Gorenstein rings form an important class of commutative rings. The foundations of the theory of Gorenstein rings go back to the classic work of Bass [Bas63]. For futher developments in the theory of Gorenstein rings we refer to [Hun99].

If R is a finitely generated graded Gorenstein algebra over an algebraically closed field $\mathbb{K}$, then one can represent R as a quotient ring S/I of a polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$ where $I \subset S$ is a homogeneous ideal, which in this case is called the Gorenstein ideal, see Proposition 1.1.5.16. The difference $k := \dim(S) - \dim(R)$ is called the codimension of I. The Gorenstein algebra R has "as S-module" the minimal graded free resolution as follows

$$0 \longrightarrow S^{b_k^S} \longrightarrow S^{b_{k-1}^S} \longrightarrow \ldots \longrightarrow S^{b_1^S} \longrightarrow S^{b_0^S} \longrightarrow S/I \longrightarrow 0,$$

with $b_k^S = b_0^S = 1$ and for b_i^S the equation $b_i^S = b_{k-i}^S$ holds for all $1 \leq i \leq k$, see Theorem 1.1.5.8 and Theorem 1.1.5.9. The structure of this resolution is known for $k \leq 3$. In the cases $k = 1$, or $k = 2$ the Gorenstein ideal is generated by k elements, and thus in particular I is a complete intersection ideal, see Chapter 2, Subsection 2.1.1. In the case $k = 3$, the structure theorem of Buchsbaum and Eisenbud [BE77] yields that the minimal number of generators of I is an odd number $2m + 1 \geq 3$ and that this minimal system of generators of I are given by the $2m + 1$ Pfaffians of order $2m$ of a skew-symmetric $(2m + 1) \times (2m + 1)$-matrix A, see Chapter 2, Subsection 2.1.2. Therefore the minimal graded free resolution is given as

$$0 \longrightarrow S \longrightarrow S^{2m+1} \xrightarrow{A} S^{2m+1} \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

The structure of the minimal graded free resolutions of Gorenstein ideals of codimension 4 is not yet fully understood. Some progress in this direction is due to Gulliksen and Negård [GN72]. In this article they study the Gorenstein ring S/I, where S is the polynomial ring in rs variables x_{ij}, $1 \leq i \leq r$, $1 \leq j \leq s$, over a field $\mathbb{K}$ and I is a Gorenstein ideal, which is generated by the t-minors of the matrix (x_{ij}) for $1 \leq i \leq r$ and $1 \leq j \leq s$. For $n = r = s$ and $t = n - 1$ the authors give an explicit minimal graded free resolution of the quotient module S/I, where I has codimension 4. Further important results for

Gorenstein ideals of codimension 4 are due to Kustin and Miller [KM82] and [KM83]. The results of Kustin and Miller have an interesting application in a new construction of Calabi-Yau manifolds [Kap11] and a classification of singular Fano varieties [PR04]. In [Rei15] Reid developed further the results of Kustin and Miller and he partially generalizes the Buchsbaum-Eisenbud theorem [BE77], see Chapter 4, Section 4.1.

Reid considers the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ over an algebraically closed field $\mathbb{K}$ and $I \subset S$ a Gorenstein ideal of codimension 4 generated by $l+1$ elements. He proposes that the minimal graded free resolution of the quotient module S/I is given as

$$\mathbf{F}: \quad 0 \longrightarrow F_4 \xrightarrow{d_4} F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} S/I \longrightarrow 0,$$

where $F_0 = S$, $F_4 = S$, $F_1 = S^{k+1}$, $F_3 = \mathrm{Hom}(F_1, F_4) \cong F_1^*$ and $F_2 = S^{2k}$. Moreover, $F_2 \longrightarrow F_1$ is dual to $F_3 \longrightarrow F_2$. By choice of appropriate bases of F_2 and F_3, we obtain the matrix A of d_2, which has the form

$$A = \begin{bmatrix} B & C \end{bmatrix},$$

where B and C are $(k+1) \times k$-matrices satisfying the following condition

$$\begin{bmatrix} B & C \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} B & C \end{bmatrix}^t = 0.$$

This is equivalent to $BC^t + CB^t = 0$ or to BC^t being a skew-symmetric matrix.

The construction of Stanley-Reisner rings is a basic tool within algebraic combinatorics and combinatorial algebra. Its properties were investigated by Richard Stanley, Melvin Hochster, and Gerald Reisner in the early 1970s, see [Hoc77], [Sta78] and [Sta80].

Given a simplicial d-polytope P with n vertices $\{v_1, \dots, v_n\}$, let $\Delta(P)$ be the boundary complex of P. For a field $\mathbb{K}$, we define the corresponding *Stanley–Reisner ring* of $\Delta(P)$, or *face ring*, denoted by $\mathbb{K}[\Delta(P)]$, as the quotient ring of the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ and the ideal $I_{\Delta(P)}$ generated by square-free monomials corresponding to the nonfaces of $\Delta(P)$:

$$\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_n]/I_{\Delta(P)},$$

where

$$I_{\Delta(P)} = (x_{i_1} \dots x_{i_r} : \ i_1 < i_2 < \dots < i_r, \ \{v_{i_1}, \dots, v_{i_r}\} \notin \Delta(P)).$$

The ideal $I_{\Delta(P)}$ is called the *Stanley–Reisner ideal* or the *face ideal* of $\Delta(P)$.

In this thesis we consider Stanley-Reisner rings that are at the same time Gorenstein rings. Therefore their Stanley-Reisner ideals are Gorenstein ideals, see Proposition 1.1.5.16. We dedicate special attention to Stanley-Reisner rings associated to simplicial d-polytopes with $d+4$-vertices, which represent an important illustration of the structure theory of Kustin-Miller [KM82] and Reid [Rei15] in codimension 4.

The aim of this thesis is to achieve some progress on the structure of minimal graded free resolutions of Gorenstein ideals of codimension 4 by using Stanley-Reisner rings. We apply the homological methods of commutative algebra on simplicial d-polytopes with $d+4$ vertices. We want to connect the structure theory of Gorenstein rings with combinatorial problems. The starting point of our investigation is the relation between the classification of simplicial d-polytopes with $d+3$ vertices and the structure theorem of Buchsbaum and Eisenbud of Gorenstein ideals of codimension 3 [BE77], see Chapter 3, Subsection 3.2.1.

In *Chapter 1*, we give basic definitions of minimal graded free resolutions of graded finitely generated modules over a graded polynomial ring with a homogeneous maximal ideal and Hilbert series. Then we recall complete intersections, Cohen-Macaulay rings and Gorenstein rings, and we shall show the relationship between them. After that we introduce the Stanley-Reisner rings.

In *Chapter 2*, we explain the structure of the minimal graded free resolution of the quotient module S/I, where S is a polynomial ring and I is a Gorenstein ideal of codimension 3. Buchsbaum and Eisenbud study this case in [BE77]. Then we discuss corresponding combinatorial concepts. We introduce the Gale diagram of a simplicial d-polytope P with $d+3$ vertices, following mainly [Grü03, Section 5.4 and Chapter 6] and [Zie95, Section 6.5]. After that we elucidate how we can determine a minimal monomial set of generators of the associated Stanley-Reisner Gorenstein ideal to P using a Gale diagram. We conclude the chapter by describing the minimal graded free resolution of the Stanley-Reisner ring.

In *Chapter 3*, we discuss a suggestion of Reid [Rei15] on how to generalize the Buchsbaum-Eisenbud theorem [BE77] to Gorenstein ideals of codimension 4. We illustrate this by some examples. Then for $d = 3, 4$, we compute explicitly the minimal graded free resolution of the Stanley-Reisner rings associated to simplicial d-polytopes with $d+4$ vertices using Gale diagrams.

In *Chapter 4*, we consider a simplicial d-polytope P with vertex set $V = \{v_1, \dots, v_n\}$, so that $0 \in \operatorname{int}(P)$ and the boundary complex $\Delta(P)$. We apply a radial projection of P from the origin onto the unit sphere S^{d-1}. The image of V under this projection is denoted by $V' := \{v'_1, \dots, v'_n\}$, where v'_i is the image of v_i, for $i = 1, \dots, n$. Then we use the stereographic projection at each point v'_i, for $i = 1, \dots, n$. For every v'_i, we obtain a simplicial $(d-1)$-polytope, which has at most $n-1$ vertices, see Proposition 3.2.1.4. The resulting polytope is denoted by P_i, for the stereographic projection at the projection point v'_i and the corresponding vertex set we denote by $V'' := \{v''_{i_1}, \dots, v''_{i_k}\}$, where v''_{i_l} is the image of v'_{i_l} under this projection. For every such polytope there is an associated Stanley-Reisner ring $\mathbb{K}[\Delta(P_i)] = \mathbb{K}[x_{i_1}, \dots, x_{i_k}]/I_{\Delta(P_i)}$, where $I_{\Delta(P_i)}$ is the associated

Gorenstein Stanley-Reisner ideal to P_i and $\Delta(P_i)$ is the boundary complex of P_i.

Now we summarise the main results of this chapter. The first aim is to determine the minimal set of monomial generators of the associated Gorenstein Stanley-Reisner ideals $I_{\Delta(P_i)}$. We give an algorithm, that allows us to determine this set if the minimal set of monomial generators of the Gorenstein Stanley-Reisner ideal $I_{\Delta(P)}$ is known.

Next we turn to Gorenstein ideals of codimension 4. In order to answer a question of Reid (see [Rei13, Open problems 4.9.4], [Rei15, Section 2.6]), about Stanley-Reisner ideals of codimension 4 we introduce the following notion, see Theorem 4.2.0.2.

Definition. *4.2.0.1 Let $S = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring, $f \in S$ a polynom and let I and I' be ideals in S. We say I is a* complete intersection *of I' and f, if $I = I' + (f)$ and f modulo I' is non-zero divisor in the residue class ring S/I'.*

So our second aim in this chapter is to prove the following theorem

Theorem. *4.2.0.2 Let P be a simplicial d-polytope with $d+4$ vertices, $\Delta(P)$ be the boundary complex of P and $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_{d+4}]/I_{\Delta(P)}$ be the Stanley-Reisner ring of $\Delta(P)$. If the Gorenstein Stanley-Reisner ideal $I_{\Delta(P)}$ is minimally generated by the polynomials $f_1, \dots, f_6$, then there exists $i \in \{1, \dots, 6\}$ such that $T(f_i) \cap T(f_j) = \emptyset$ for all $i \neq j$ and $I' = (f_j : j \in \{1, \dots 6\} \setminus \{i\})$ is a Gorenstein ideal of codimension 3.*

Finally, the third aim is to give a counterexample to a conjecture of Reid (see [Rei13, Open problems 4.9.4]), that every Gorenstein ideal of codimension 4 with even number of generators is a complete intersection of a Gorenstein ideal of codimension 3 and an extra polynom.

In *Chapter 5*, we introduce affine Gale diagrams of simplicial d-polytopes with $d+4$ vertices, see Definition 5.1.0.1. Gale diagrams of these polytopes are in $\mathbb{R}^3$, but affine Gale diagrams are of one dimension lower that the well-known Gale diagrams. For $d = 3$, 4, Grünbaum and Sreedharan construct in [GS67] all simplicial d-polytopes with $d+4$ vertices. There are exactly 5 combinatorial types of simplicial 3-polytopes with 7 vertices and 37 combinatorial types of simplicial 4-polytopes with 8 vertices. For all these polytopes we sketch affine Gale diagrams. That should help us to achieve the following main aim.

Let P be a d-polytope with vertex set $V = \{v_1, \dots, v_{d+4}\}$ and $\hat{\mathfrak{B}} = \{\hat{v}_1, \dots, \hat{v}_{d+4}\}$ the Gale diagram of P. Let $\mathfrak{B}^* = \{v_1^*, \dots, v_{d+4}^*\}$, where each of the v_i^* is declared to be either black or white, be an affine Gale diagram of P. There is a canonical bijection between $\hat{\mathfrak{B}}$ and $\mathfrak{B}^*$ with the point v_i corresponding to v_i^*. Therefore there is also a canonical bijection between the points v_i of V and the points v_i^* of $\mathfrak{B}^*$, see Remark 5.1.0.2.

We characterize the minimal set of monomial generators of Gorenstein Stanley-Reisner ideals $I_{\Delta(P)}$ using affine Gale diagrams, for an arbitrary d.

Theorem. *5.3.0.2 Let P be a simplicial d-polytope with vertex set $V = \{v_1, \dots, v_{d+4}\}$, $\Delta(P)$ its boundary complex and the configuration $\mathfrak{B}^* = \{v_1^*, \dots, v_{d+4}^*\}$ an affine Gale diagram of P. Let $\mathbb{K}$ be an algebraically closed field and $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_{d+4}]/I_{\Delta(P)}$ the Stanley-Reisner ring of $\Delta(P)$. A monomial $x_{i_1} \dots x_{i_k}$ is an element of the minimal set of monomial generators of the Gorenstein Stanley-Reisner ideal $I_{\Delta(P)}$ if and only if the set $\mathfrak{B}^* \setminus \{v_{i_1}^*, \dots, v_{i_k}^*\}$ satisfies the following condition: The black and white points can be split by an affine hyperplane. Morevore, there is no superset of $\mathfrak{B}^* \setminus \{v_{i_1}^*, \dots, v_{i_k}^*\}$, which satisfies the previous condition.*

Finally *Chapter 6*, we explain the complex of Gulliksen and Negård [GN72] and then focus an the corresponding combinatorial concepts of neigbourly and cyclic polytopes.

Definition. *6.2.0.1 A* neighbourly *d-polytope is a convex d-polytope, such that any set of vertices of cardinality $\lfloor d/2 \rfloor$ spans a face. A polytope is called k-*neighbourly *if any set of k vertices spans a face.*

Definition. *6.2.0.2 Let $t_1 < t_2 < \dots < t_n$ be real numbers. The* cyclic *d-polytope with n vertices $C = C_d(t_1, \dots, t_n)$ is the convex hull of the subset $\{f(t_1), f(t_2), \dots, f(t_n)\} \subset \mathbb{R}^d$, where $f : \mathbb{R} \to \mathbb{R}^d$ is defined by $f(t) = (t, t^2, \dots, t^d)$ for $t \in \mathbb{R}$.*

We start by considering cyclic d-polytopes with $d+3$ vertices. For Gorenstein Stanley-Reisner ideals of codimension 3 associated to these polytopes, the structure theory of Buchsbaum and Eisenbud states, the minimal number of generators of each such ideal is an odd number $2m+1 \geq 3$ and that this minimal system of generators is given by the $2m+1$ Pfaffians of order $2m$ of a skew-symmetric $(2m+1) \times (2m+1)$-matrix A. We describe in this chapter this matrix explicitly concerning cyclic $2d$-polytopes with $2d+3$ vertices.

Theorem. *6.2.1.1 Let P be a cyclic $(2d-2)$-polytope with the vertex set $V = \{v_1, \dots, v_{2d+1}\}$ and the boundary complex $\Delta(P)$. Let $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_{2d+1}]/I_{\Delta(P)}$ be the Stanley-Reisner ring of $\Delta(P)$ and $I_{\Delta(P)}$ the Gorenstein Stanley-Reisner ideal associated to P. Then the monomial generators of $I_{\Delta(P)}$ are $2d$-th order Pfaffians of the following skew-symmetric $(2d+1) \times (2d+1)$-matrix A of degree d.*

$$A = \begin{bmatrix}
 & & & & & -x_{2d+1} & x_{d+1} & & \cdots & & 0 \\
 & & & & & 0 & -x_1 & x_{d+2} & & & \\
 & & \mathbf{0} & & & \vdots & & & & & \vdots \\
 & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & 0 \\
 & & & & & & & & & -x_{d-1} & x_{2d} \\
x_{2d+1} & & \cdots & \cdots & & 0 & & \cdots & \cdots & & -x_d \\
-x_{d+1} & x_1 & & & & & & & & & \\
0 & -x_{d+2} & x_2 & & & & & & & & \\
\vdots & & & & & \vdots & & & \mathbf{0} & & \\
 & & & \ddots & \ddots & & & & & & \\
\vdots & & & & & 0 & & & & & \\
0 & \cdots & & & -x_{2d} & x_d & & & & &
\end{bmatrix}$$

We can determine minimal sets of monomial generators of Gorenstein Stanley-Reisner ideals associated to all simplicial d-polytopes with $d+3$ through minimal sets of monomial generators of Gorenstein Stanley-Reisner ideals associated to cyclic d-polytopes with $d+3$ vertices. We achieve that as follows: If we take a product of a monomial (or more) from the minimal set of monomial generators of the Gorenstein Stanley-Reisner ideal associated to a cyclic polytope with a new variable (or more), then we obtain a minimal set of monomial generators of the Gorenstein Stanley-Reisner ideal of codimension 3 associated to a polytope of dimension $d+k$ with $d+k+3$ vertices, where k is the number of new variables, see Example 6.2.1.2.

As we want to use the same idea for d-polytopes with $d+4$ vertices, we are interested in Gorenstein Stanley-Reisner ideals associated to cyclic d-polytopes with $d+4$ vertices. Since every cyclic polytope is a neighbourly polytope, see Corollary 6.2.0.6, we start with considering neighbourly polytopes in chapter 6. Grünbaum and Sreedharan construct all simplicial neighbourly 4-polytopes with 8 vertices in [GS67]. There are exactly three combinatorial types of such polytopes, two of them P^8_{36}, P^8_{37} are not cyclic, and the other one P^8_{35} is cyclic, see Chapter 3, Subsection 3.2.2. In 1981 Barnette [Bar81] construct a family of neighbourly polytopes that are not cyclic in any dimension. After that in 1982 Shemer [She82] shows that the number of combinatorially different neighbourly $2d$-polytopes with $2d+4$ vertices grows superexponentially as $d \to \infty$. In 1987 all neighbourly 6-polytopes with 10 vertices are classified by Bokowski and Shemer [BS87]. There are 37 combinatorial types. In 2011 Devyatov [Dev11] classified neighbourly $2d$-polytopes with $2d+4$ vertices, which have a planer Gale diagram of a special type with exactly $d+3$ black points in convex position. Finbow in [FS04], [Fin10] and [Fin15] published a list of the simplicial neighbourly 5-polytopes with 9 vertices. There are exactly 126 combinatorially distinct types.

In 1996 Teria and Hibi [TH96] compute the Betti numbers of the minimal graded free resolution of the Stanley-Reisner ring of the boundary complex of a cyclic polytope. Then in 2010 Böhm and Papadakis [BP12] study the structure of Stanley-Reisner rings associated to cyclic polytopes and show how to express the Stanley-Reisner ring of cyclic d-polytope with $n+1$ vertices in terms of the Stanley-Reisner rings of a cyclic d-polytope with n vertices and a cyclic $(d-2)$-polytope with $n-1$ vertices.

Let C be a cyclic $2d$-polytope with $2d+4$ vertices and $\Delta(C)$ the boundary complex of C. Let $\mathbb{K}[\Delta(C)]$ be the associated Stanley-Reisner ring to C. The minimal graded free resolution of $\mathbb{K}[\Delta(C)]$ over $S := \mathbb{K}[x_1, \dots, x_{2d+4}]$, as explained in [TH96], is of the form

$$0 \longrightarrow S(-(2d+4)) \longrightarrow S(-(d+3))^{b_3^S} \longrightarrow S(-(d+2))^{b_2^S} \longrightarrow$$
$$S(-(d+1))^{b_1^S} \longrightarrow S \longrightarrow \mathbb{K}[\Delta(C)] \longrightarrow 0,$$

where $b_1^S = (d+2)^2$, $b_2^S = 2(d+3)(d+1)$ and $b_3^S = (d+2)^2$.

That means that Gorenstein Stanley-Reisner ideals associated to cyclic $2d$-polytopes with $2d+4$ vertices are generated by $(d+2)^2$ monomials of degree $d+1$.

Hence, we verify in this chapter that the cyclic polytopes have also a crucial role for the associated Gorenstein ideals of codimension 4.

Conjecture. *6.2.0.7 Let P be a simplicial neighbourly $2d$-polytope with $2d+4$ vertices. The polytope P is cyclic if and only if there exists a $(d+2)\times(d+2)$-matrix A, so that all its $(d+1)$-minors minimally generate the Gorenstein Stanley-Reisner ideal associated to P.*

The "only if" part of this conjecture is that the minimal graded free resolutions of the Stanley-Reisner rings associated to cyclic $2d$-polytopes with $2d+4$ vertices can be considered as a special version of the Gulliksen-Negård complex to a $(d+2)\times(d+2)$-matrix. We give a complete proof of this direction, but for the "if" part, we only give a partial argument. In [Dev11] Devyatov classified special neighbourly $2d$-polytopes with $2d+4$ vertices which are not cyclic. We prove for each polytope of Devyatov's polytopes that the associated Gorenstein Stanley-Reisner ideal is generated by exactly $(d+2)^2$ monomials and all have degrees $d+1$, but there is no square $(d+2)\times(d+2)$-matrix, so that its $(d+1)$-minors generate it. That means that the minimal graded free resolutions of the associated Stanley-Reisner rings to Devyatov's polytopes can not be regarded as a version of the Gulliksen-Negård complex.

Theorem. *6.2.2.1 Let P be a cyclic $2d$-polytope with the vertex set V=$\{v_1,\dots,v_{d+2},w_1,\dots,$ $w_{d+2}\}$ and the boundary complex $\Delta(P)$. Let $\mathbb{K}[\Delta(P)]=\mathbb{K}[x_1,\dots,x_{d+2},y_1,\dots,y_{d+2}]/I_{\Delta(P)}$ be the Stanley-Reisner ring of $\Delta(P)$, where $I_{\Delta(P)}$ is the Gorenstein Stanley-Reisner ideal associated to P. Consider a $(d+2)\times(d+2)$-matrix (or its transpose) of the form*

$$A=\begin{bmatrix} x_1 & 0 & 0 & 0 & \cdots & y_{d+2} \\ y_1 & x_2 & 0 & 0 & \cdots & 0 \\ 0 & y_2 & x_3 & 0 & \cdots & 0 \\ 0 & 0 & y_3 & x_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x_{d+2} \end{bmatrix}.$$

Then the $(d+1)$-minors construct a minimal set of monomial generators of $I_{\Delta(P)}$.

Hence, we characterize the minimal sets of monomial generators of the Gorenstein ideals associated to special neighbourly $2d$-polytopes with $2d+4$ vertices, which are different from cyclic polytopes of Devyatov's [Dev11], see Theorem 6.2.3.4. We refer to the affine Gale diagrams of these polytopes as T-diagrams. These diagrams have a special type, namely, with exactly $d+3$ black points lie in convex position and the remaining $d+1$ white points lie inside the $(d+3)$-gon formed by the black points, see Definition 6.2.3.1.

Proposition. *6.2.3.5 Let P be a special neighbourly simplicial $2d$-polytope with $2d+4$ vertices, which were classified by Devyatov [Dev11], with the vertex set V=$\{v_1,\dots,v_{d+3},w_1,\dots,$*

$w_{d+1}\}$ *and the boundary complex* $\Delta(P)$. *Let* $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \ldots, x_{d+3}, y_1, \ldots, y_{d+1}]/I_{\Delta(P)}$ *be the Stanley-Reisner ring of* $\Delta(P)$ *and* $I_{\Delta(P)}$ *the Gorenstein Stanley-Reisner ideal associated to* P. *Then* $I_\Delta(P)$ *is minimally generated by exactly* $(d+2)^2$ *monomial generators and all have degree* $d+1$.

At the end of this chapter we show the "if" part of Conjecture 6.2.0.7 for the class of Devyatov's polytopes.

Theorem. *6.2.4.1 Let* P *be a special neighbourly simplicial* $2d$*-polytope with* $2d+4$ *vertices, which were classified by Devyatov [Dev11], with the vertex set* V=$\{v_1, \ldots, v_{d+3}, w_1, \ldots, w_{d+1}\}$ *and the boundary complex* $\Delta(P)$. *Let* $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \ldots, x_{d+3}, y_1, \ldots, y_{d+1}]/I_{\Delta(P)}$ *be the Stanley-Reisner ring of* $\Delta(P)$ *and* $I_{\Delta(P)}$ *the Gorenstein Stanley-Reisner ideal associated to* P. *Then there is no* $(d+2) \times (d+2)$*-matrix, so that its* $(d+1)$*-minors are monomial generators of* $I_{\Delta(P)}$.

So we give an important step to toward proving the "if" part of the conjecture.

CHAPTER

ONE

PRELIMINARIES

In this chapter, we recall basic notions from commutative algebra and from convex geometry. All of this chapter's content is well known. Our primary references are [Pee11], [HH11], [BH93], [BIV89] and [Sta96]. In the first section, we give some basic definitions of minimal graded free resolutions of graded finitely generated modules over a graded polynomial ring, Hilbert series and Hilbert functions. We close the section with recalling complete intersections, Cohen-Macaulay rings and Gorenstein rings, and we shall show the relationship between them. In the second section, we introduce the "Stanley-Reisner rings" because the construction of the Stanley-Reisner ring is a basic tool within algebraic combinatorics and combinatorial commutative algebra.

1.1 Basic notions on commutative algebra

List of general notation

$\mathbb{K}$	an algebraically closed field
$S := \mathbb{K}[x_1, \dots, x_n]$	the polynomial ring with standard $\mathbb{Z}$-grading by $\deg(x_i) = 1$ for $1 \leq i \leq n$
$\mathfrak{m} = (x_1, \dots, x_n)$	the homogeneous maximal ideal of S
I	a graded ideal of S
$R := S/I$	the quotient ring of S modulo I

Throughout this thesis, we consider polynomial rings with finitely many variables over fields and "dimension of a ring" is understood as the Krull dimension.

1.1.1 Graded polynomial rings, modules and homomorphisms

In this subsection, we define monomial ideals of a polynomial ring and show that there exists a unique minimal monomial system of generators for each monomial ideal. We deal with monomial ideals in next chapters, then we explain and introduce a grading on the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ and discuss graded R-modules and homomorphisms, where $R = S/I$ and I is a graded ideal of S.

Definition 1.1.1.1. Any product $x_1^{a_1} \dots x_n^{a_n}$ from S with $a_i \geq 0$ is called a *monomial*.

Definition 1.1.1.2. An ideal I of S is called *monomial ideal* if it can be generated by monomials. If a monomial ideal is generated by monomials not divisible by the square of any of the variables, then it is called *squarefree*.

By Hilbert's basis theorem we know that the polynomial ring S is Noetherian, hence any monomial ideal is finitely generated.

Proposition 1.1.1.3. [HH11, Proposition 1.1.5] *Let I be a monomial ideal and $\{u_1, \dots, u_m\}$ be a monomial system of generators of I. Then a monomial $v \in S$ belongs to I if and only if there exists a monomial $w \in S$ such that $v = wu_i$ for some i.*

Proposition 1.1.1.4. [HH11, Proposition 1.1.6] *Each monomial ideal has a unique minimal set of monomial generators. More precisely, let V denote the set of monomials in I which are minimal with respect to divisibility. Then V is the unique minimal set of monomial generators.*

Definition 1.1.1.5. The *degree* of a monomial $x_1^{a_1} \dots x_n^{a_n}$ is defined as $\deg(x_1^{a_1} \dots x_n^{a_n}) := \sum_{i=1}^n a_i$. We denote by S_i the $\mathbb{K}$-vector space generated by all monomials of degree i, for $i \geq 0$. In particular, $S_0 = \mathbb{K}$. The elements of S_i are called *homogeneous* of degree i. Note that 0 is a homogeneous element with arbitrary degree. We have a direct sum decomposition $S = \bigoplus_{i \geq 0} S_i$ of S as a $\mathbb{K}$-vector space such that $S_i S_j \subseteq S_{i+j}$ for all $i,\ j \geq 0$. Thus S is *standard graded*. Every polynomial $f \in S$ can be written uniquely as a finite sum $f = \sum f_i$ of non-zero elements $f_i \in S_i$, and in this case f_i is called the *homogeneous component* of f of degree i.

Definition 1.1.1.6. A proper ideal I of S is called *graded* or *homogeneous* if it satisfies the following equivalent conditions.

1. The ideal I has a system of homogeneous generators.
2. If $\tilde{I}$ is the ideal generated by all homogeneous elements in I, then $I = \tilde{I}$.
3. The ideal $I = \bigoplus_{i \geq 0} I_i = \bigoplus_{i \geq 0}(S_i \cap I)$. In this case, the $\mathbb{K}$-spaces I_i are called the *homogeneous components* of I.
4. If $f \in I$, then every homogeneous component of f is in I.

Let I be a graded ideal in S. Note that $S_i I_j \subseteq I_{i+j}$ for all $i,\ j \in \mathbb{N}$. The quotient ring $R = S/I$ inherits the grading from S by $R_i := S_i/I_i$ for every $i \in \mathbb{N}$.

Definition 1.1.1.7. An S-module M is called *graded*, if it has a direct sum decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ as a $\mathbb{K}$-vector space and $S_i M_j \subseteq M_{i+j}$ for all $i,\ j \in \mathbb{Z}$. The $\mathbb{K}$-spaces M_i are called the *homogeneous components* of M. Elements of M_i are called *homogeneous* of degree i. Every element $m \in M$ can be written uniquely as a finite sum $m = \sum m_i$, where $m_i \in M_i$, and in this case m_i is called the *homogeneous component* of m of degree i. For $p \geq 0$ denote by $M(-p)$ the graded S-module such that $M(-p)_i = M_{i-p}$ for all i. We say that $M(-p)$ is the module M *shifted p degrees*, and call p the *shift.*

Proposition 1.1.1.8. [Pee11, Proposition 2.1] *Let M be a graded S-module.*
1. *There exists a system of homogeneous generators of M.*
2. *The degrees of the elements in a system of homogeneous generators determine the grading of M.*

Proposition 1.1.1.9. [Pee11, Proposition 2.3] *The module $S(-p)$ is the free S-module generated by one element in degree p, for $p \geq 0$.*

Proof. $S(-p)_p = S_0$. □

Remark 1.1.1.10. The element $1 \in S(-p)$ has degree p and is called the *1-generator* of $S(-p)$.

Definition 1.1.1.11. Let N be a submodule of a graded S-module M. We say that N is *graded* or *homogeneous* if it satisfies the following equivalent conditions:
1. The submodule N has a system of homogeneous generators.
2. If $\tilde{N}$ is the submodule generated by all homogeneous elements in N, then $N = \tilde{N}$.
3. The submodule $N = \bigoplus_{i\in\mathbb{Z}}(M_i \cap N)$.
4. If $f \in N$, then every homogeneous component of f is in N.

Let N be a graded submodule of a graded S-module M, then M/N inherits the grading from M via
$$M/N = \bigoplus_{i\in\mathbb{Z}}(M/N)_i \quad \text{with } (M/N)_i := M_i/N_i.$$

Definition 1.1.1.12. Let M and T be graded S-modules. We say that a homomorphism $\varphi : M \to T$ has *degree* $i \in \mathbb{Z}$ if $\deg(\varphi(m)) = i + \deg(m)$, for each homogeneous element $m \in M$. Since 0 has arbitrary degree, then $\deg(\varphi(m)) = i + \deg(m)$ is only a condition on the homogeneous elements outside $\mathrm{Ker}(\varphi)$. The $\mathbb{K}$-space of all homomorphisms of degree i from M to T is denoted by $\mathrm{Hom}_i(M, T)$. A homomorphism $\phi \in \mathrm{Hom}(M, T)$ is called *graded* (or *homogeneous*) if there exists $i \in \mathbb{Z}$ such that $\phi \in \mathrm{Hom}_i(M, T)$; we also say that ϕ is a *homomorphism of graded modules.*

Proposition 1.1.1.13. [Pee11, Proposition 2.9] *If $\phi : M \to T$ is a homomorphism of graded S-modules, then* $\mathrm{Ker}(\phi)$, $\mathrm{Im}(\phi)$, *and* $\mathrm{Coker}(\phi)$ *are graded.*

Theorem 1.1.1.14. [Pee11, Theorem 2.10] *Let T be an S-module. Then T is a finitely generated graded S-module if and only if $T \cong M/N$, where M is a finite direct sum of shifted free S-modules, N is a graded submodule of M (called the* module of relations*), and the isomorphism has degree* 0.

Definition 1.1.1.15. Let M be a finitely generated S-module. An exact sequence of the form
$$F_1 \xrightarrow{A} F_0 \longrightarrow M \longrightarrow 0,$$
is called a *finite presentation of M*, where F_0 and F_1 are finitely generated free S-modules. Since the sequence is exact, M is isomorphic to $F_0/\mathrm{Im}(A)$. The matrix A is called a *presentation matrix* of M. The presentation is called *graded* if M, F_0, F_1 are graded and the two homomorphisms have degree 0.

1.1.2 Graded complexes and minimal free resolutions

Definition 1.1.2.1. A *complex* $\mathbf{F}$ *over* S is a sequence of homomorphisms of S-modules

$$\mathbf{F}: \ \dots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} F_{-1} \longrightarrow \dots,$$

such that $d_{i-1} \circ d_i = 0$ for all $i \in \mathbb{Z}$. The set of maps $d = \{d_i\}_{i\in\mathbb{Z}}$ is called the *differential of* $\mathbf{F}$. If $F_i = 0$ for all $i < 0$, it is called a *left complex*, that is,

$$\mathbf{F}: \ \dots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \dots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0,$$

with $i \geq 0$. Furthermore, $\mathbf{F}$ is called a *left complex over* M (or a *complex over* M) if it is a left complex and we have a homomorphism $d_0 : F_0 \to M$, called an *augmentation map*.

Buchsbaum and Eisenbud gave in 1973 a criterion for the exactness of a finite complex of finitely generated free modules over a commutative noetherian ring. The criterion consists of a condition on the rank of the homomorphisms d_i and another condition which involves only one of the maps d_i at a time. For more details see [BE73].

Definition 1.1.2.2. Let $\mathbf{F}$ be a complex of S-modules F_i. If the S-modules F_i are graded and each d_i is a homomorphism of degree 0, we say that the complex $\mathbf{F}$ is *graded*. In this case the S-modules F_i are actually bigraded since

$$F_i = \bigoplus_{j\in\mathbb{Z}} F_{i,j} \qquad \text{for } i \in \mathbb{Z}.$$

An element $f \in F_{i,j}$ is said to have *homological degree* i and *internal degree* j. We denote the homological degree of f by $\operatorname{hdeg}(f)$, and the internal degree of f by $\deg(f)$.

Lemma 1.1.2.3. [Pee11, Lemma 9.2] *Let T be a graded S-module. If T is a direct summand of a finitely generated graded free S-module, then T is free.*

Construction 1.1.2.4. Let $\mathbf{F}$ be a complex of S-modules F_i. If each module F_i is a finitely generated graded free S-module, then we can write it as

$$F_i = \bigoplus_{p\in\mathbb{Z}} S(-p)^{c_{i,p}}.$$

Then the graded complex $\mathbf{F}$ of finitely generated free modules takes the form

$$\mathbf{F}: \ \dots \longrightarrow \oplus_{p\in\mathbb{Z}} S(-p)^{c_{i,p}} \xrightarrow{d_i} \oplus_{p\in\mathbb{Z}} S(-p)^{c_{i-1,p}} \longrightarrow \dots.$$

Definition 1.1.2.5. The *i-th homology* $\mathrm{H}_i(\mathbf{F})$ of a complex $\mathbf{F}$ is defined as

$$\mathrm{H}_i(\mathbf{F}) = \operatorname{Ker}(d_i)/\operatorname{Im}(d_{i+1}).$$

The elements in $\operatorname{Ker}(d_i)$ are called *cycles* and the elements in $\operatorname{Im}(d_i)$ are called *boundaries*. The complex is *exact* at F_i (or at *step* i) if $\mathrm{H}_i(\mathbf{F}) = 0$. The complex is *exact* if $\mathrm{H}_i(\mathbf{F}) = 0$ for all $i \in \mathbb{Z}$. A left complex is *acyclic* if $\mathrm{H}_i(\mathbf{F}) = 0$ for all $i > 0$; it is *acyclic over* M if it is acyclic and $\mathrm{H}_0(\mathbf{F}) = M$.

In the graded case, since the differential is graded, it follows that the homology is bigraded by

$$\mathrm{H}_i(\mathbf{F}) = \bigoplus_{j\in\mathbb{Z}} \mathrm{H}_i(\mathbf{F})_j \quad \text{for } i \in \mathbb{Z}.$$

Definition 1.1.2.6. A *free resolution* of a finitely generated S-module M is a sequence of homomorphisms of S-modules

$$\mathbf{F}: \ \ldots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \ldots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0,$$

such that

1. $\mathbf{F}$ is a complex of finitely generated free S-modules F_i,
2. $\mathbf{F}$ is exact,
3. $M \cong F_0/\mathrm{Im}(d_1)$.

Throughout this thesis, we use the following notation for free resolutions

$$\mathbf{F}: \ \ldots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0.$$

Remark 1.1.2.7. Every resolution is an acyclic left complex over M.

Now we introduce the Ext modules of finitely generated S-modules M and N. These Ext modules are very important because many invariants (such as grade, depth and projective dimension) can be defined in terms of vanishing of suitable Ext's, see [GS76] and [BH93]. We shall use them to define Gorenstein ideals.

Definition 1.1.2.8. Let M and N be finitely generated S-modules and

$$\mathbf{F}: \ \ldots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0,$$

a free resolution of M. Now we consider the complex

$$\mathrm{Hom}_S(\mathbf{F}, N): \ 0 \longrightarrow \mathrm{Hom}_S(F_0, N) \xrightarrow{\partial_0} \mathrm{Hom}_S(F_1, N) \xrightarrow{\partial_1} \ldots,$$

where the homomorphisms ∂_i are defined by $\partial_i(f) = f \circ d_{i+1}$ for $f \in \mathrm{Hom}_S(F_i, N)$. Then $\mathrm{Ext}^i_S(M, N)$ is defined as the module $\mathrm{H}_i(\mathrm{Hom}_S(\mathbf{F}, N))$ over S. If $\mathbf{F}$ is finite that is

$$\mathbf{F}: \ 0 \longrightarrow F_n \xrightarrow{d_n} \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0,$$

then we can consider in addition to the above complex the following complex

$$\mathrm{Hom}_S(N, \mathbf{F}): \ 0 \longrightarrow \mathrm{Hom}_S(N, F_n) \xrightarrow{\partial^*_n} \mathrm{Hom}_S(N, F_{n-1}) \xrightarrow{\partial^*_{n-1}} \ldots,$$

where the homomorphisms ∂^*_i are defined by $\partial^*_i(f) = f \circ d_{i+1}$ for $f \in \mathrm{Hom}_S(N, F_i)$. Then $\mathrm{Ext}^i_S(N, M)$ is defined as the module $\mathrm{H}_i(\mathrm{Hom}_S(N, \mathbf{F}))$ over S.

Definition 1.1.2.9. Let $\mathbf{F}$ be a free resolution of M. If M is graded, $\mathbf{F}$ is a graded complex, and the isomorphism $M \cong F_0/\mathrm{Im}(d_1)$ has degree 0, then we say that the resolution is *graded*. Fix a homogeneous basis of each free module F_i. Then the differential d_i is given by a matrix A_i, whose entries are homogeneous elements in S. These matrices are called *differential matrices*.

Construction 1.1.2.10. [Pee11, Construction 4.2] We explain the construction of a graded free resolution of a finitely generated graded S-module M by induction on the homological degree.

Step 0**:** Set $M_0 := M$. Choose homogeneous generators $m_1, \dots, m_r$ of M_0. Let $a_1, \dots, a_r$ be their degrees, respectively. Set $F_0 := S(-a_1) \oplus \cdots \oplus S(-a_r)$. For $1 \le j \le r$ denote by f_j the 1-generator of $S(-a_j)$.
Thus, $\deg(f_j) = a_j$. Define

$$\begin{aligned} d_0 : \quad & F_0 \to M \\ & d_0(f_j) = m_j \quad \text{for} \quad 1 \le j \le r. \end{aligned}$$

This is a homomorphism of degree 0.

Step $i+1$**:** Set $M_{i+1} := \mathrm{Ker}(d_i)$. Choose homogeneous generators $u_1, \dots, u_s$ of M_{i+1}. Let $c_1, \dots, c_s$ be their degrees, respectively. Set $F_{i+1} := S(-c_1) \oplus \cdots \oplus S(-c_s)$. For $1 \le j \le s$ denote by g_j the 1-generator of $S(-c_j)$. Thus, $\deg(g_j) = c_j$. Define

$$\begin{aligned} d_{i+1} : \quad & F_{i+1} \to M_{i+1} \subset F_i \\ & d_{i+1}(g_j) = u_j \qquad \text{for} \quad 1 \le j \le s. \end{aligned}$$

This is a surjective homomorphism of degree 0.

The constructed complex is exact since $\mathrm{Ker}(d_i)$=$\mathrm{Im}(d_{i+1})$ by construction.

Example 1.1.2.11. Let $S = \mathbb{K}[x,y]$ and $I = (x,y)$. We will construct a graded free resolution of S/I over S.

Step 0**:** Set $F_0 := S$ and let $d_0 : S \to S/I$.

Step 1**:** The elements x, y are homogeneous generators of $\mathrm{Ker}(d_0)$. Their degree is 1. Set $F_1 := S(-1) \oplus S(-1)$. Denote by (f_1, f_2) the 1-generators of $S(-1) \oplus S(-1)$. Hence $\deg(f_1) = \deg(f_2) = 1$. Let $d_1 : F_1 \to S$ be the homomorphism defined by $d_1(f_1) = x$ and $d_1(f_2) = y$. We obtain the beginning of the resolution:

$$S(-1) \oplus S(-1) \xrightarrow{[x \ y]} S \longrightarrow S/I \longrightarrow 0.$$

Step 2**:** We need to find homogeneous generators of $\mathrm{Ker}(d_1)$. Let $\alpha f_1 + \beta f_2 \in \mathrm{Ker}(d_1)$, with α, $\beta \in S$. We want to solve the equation $\alpha x + \beta y = 0$, where α, $\beta \in S$ are the unkowns. The element $-yf_1 + xf_2$ is a homogeneous generator of $\mathrm{Ker}(d_1)$. Its degree is 2, $\deg(-y) + \deg(f_1) = \deg(x) + \deg(f_2) = 2$. Set $F_2 := S(-2)$. Denote by g_1 the 1-generator of $S(-2)$. Hence $\deg(g_1) = 2$. Let $d_2 : F_2 \to F_1$ be the homomorphism S-modules that is uniquely defined by $d_2(g_1) = -yf_1 + xf_2$. We obtain the next step in the resolution:

$$S(-2) \xrightarrow{[-y\ x]^t} S(-1) \oplus S(-1) \xrightarrow{[x\ y]} S \longrightarrow S/I \longrightarrow 0.$$

Step 3: Now we need to find homogeneous generators of $\mathrm{Ker}(d_2)$. Let $\mu g_1 \in \mathrm{Ker}(d_2)$ with $\mu \in S$. Hence $\mu(-yf_1 + xf_2) = -\mu y f_1 + \mu x f_2 = 0$, it then follows that $\mu y = 0$ and $\mu x = 0$. We conclude that $\mu = 0$. Thus $F_3 = 0$. We obtain the graded free resolution

$$0 \longrightarrow S(-2) \xrightarrow{[-y\ x]^t} S(-1) \oplus S(-1) \xrightarrow{[x\ y]} S \longrightarrow S/I \longrightarrow 0.$$

Now we define when a graded free resolution is minimal and describe the properties of minimal graded free resolutions. Theorem 1.1.2.19 shows that the minimal graded free resolution is the smallest graded free resolution in the sense that the ranks of its free modules are less than or equal to the ranks of the corresponding free modules in an arbitrary graded free resolution of the resolved module.

Definition 1.1.2.12. Let S be a graded polynomial ring with $\mathfrak{m}$ its homogeneous maximal ideal. A graded free resolution of a finitely generated graded S-module M is *minimal* if

$$d_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i \quad \text{for all } i \geq 0.$$

This means, that no invertible elements (non-zero constants) appear in the differential matrices.

Remark 1.1.2.13. Recall that $\mathfrak{m} = (x_1, \dots, x_n)$ is the unique homogeneous maximal ideal of S.

Example 1.1.2.14. The resolution in Example 1.1.2.11 is minimal.

Theorem 1.1.2.15. [Pee11, Theorem 7.3] *The graded free resolution constructed in Construction 1.1.2.10 is minimal if and only if at each step we choose a minimal homogeneous system of generators of the kernel of the differential.*

Definition 1.1.2.16. A complex of the form $0 \to S(-p) \xrightarrow{\cdot 1} S(-p) \to 0$ is called a *short trivial complex.*

Definition 1.1.2.17. Let $(\mathbf{F}, d)$ be a complex of S-module F_i and $(\mathbf{G}, \delta)$ be a complex of S-module G_i. Then a *homomorphism of complexes* $\varphi : \mathbf{F} \to \mathbf{G}$ is a set homomorphisms $\varphi_i : F_i \to G_i$ for all $i \in \mathbb{Z}$, such that $\varphi \circ d = \delta \circ \varphi$. That is $\varphi_{i-1} \circ d_i = \delta_i \circ \varphi_i$ for all $i \in \mathbb{Z}$. If the complexes $\mathbf{F}$ and $\mathbf{G}$ are graded, then we call φ a *homomorphism of graded complexes* if $\varphi_i : F_i \to G_i$ is a homomorphism of a fixed degree q for all $i \in \mathbb{Z}$.

Definition 1.1.2.18. Let $(\mathbf{F}, d)$ and $(\mathbf{G}, \delta)$ be complexes, then their *direct sum* is the complex $(\mathbf{F} \bigoplus \mathbf{G}, d \bigoplus \delta)$ with modules $(\mathbf{F} \bigoplus \mathbf{G})_i := F_i \oplus G_i$ and differential $d \bigoplus \delta$ with homomorphisms $(d \bigoplus \delta)_i := d_i \bigoplus \delta_i$ for all $i \in \mathbb{Z}$. A direct sum of short trivial complexes in different homological degrees is called a *trivial complex*.

Theorem 1.1.2.19. [Pee11, Theorem 7.5] *Let M be a graded finitely generated S-module. There exists a minimal graded free resolution of M and up to an isomorphism, there exists a unique minimal graded free resolution of M.*

Remark 1.1.2.20. According this theorem we may say "the minimal graded free resolution of M".

Example 1.1.2.21. Let $S = \mathbb{K}[x, y]$ and $I = (x, xy)$. We construct a graded free resolution of S/I over S.

Step 0**:** Set $F_0 := S$ and let $d_0 : S \to S/I$.

Step 1**:** The elements x and xy are homogeneous generators of $\mathrm{Ker}(d_0)$. Their degrees are 1 and 2, respectively. Set $F_1 := S(-1) \oplus S(-2)$. Denote by f_1 and f_2 the 1-generators of $S(-1)$, $S(-2)$, respectively. Hence $\deg(f_1) = 1$, $\deg(f_2) = 2$. Let $d_1 : F_1 \to S$ be the homomorphism defined by $d_1(f_1) = x$ and $d_1(f_2) = xy$. We obtain the beginning of the resolution:

$$S(-1) \oplus S(-1) \xrightarrow{[x\ xy]} S \longrightarrow S/I \longrightarrow 0.$$

Step 2**:** We need to find homogeneous generators of $\mathrm{Ker}(d_1)$. Let $\alpha f_1 + \beta f_2 \in \mathrm{Ker}(d_1)$, with α, $\beta \in S$. We want to solve the eqaution $\alpha x + \beta y = 0$, where α, $\beta \in S$ are the unkowns. The element $-yf_1 + f_2$ is a homogeneous generator of $\mathrm{Ker}(d_1)$. Its degree is $\deg(-y) + \deg(f_1) = \deg(1) + \deg(f_2) = 2$. Set $F_2 := S(-2)$. Denote by g_1 the 1-generator of $S(-2)$. Hence $\deg(g_1) = 2$. Let $d_2 : F_2 \to F_1$ be the homomorphism S-modules that is uniquely defined by $d_2(g_1) = -yf_1 + f_2$. We obtain the next step in the resolution:

$$S(-2) \xrightarrow{[-y\ 1]^t} S(-1) \oplus S(-2) \xrightarrow{[x\ xy]} S \longrightarrow S/I \longrightarrow 0.$$

Step 3**:** Now we need to find homogeneous generators of $\mathrm{Ker}(d_2)$. Let $\mu g_1 \in \mathrm{Ker}(d_2)$ with $\mu \in S$. Hence $\mu(-yf_1 + f_2) = -\mu y f_1 + \mu f_2 = 0$, then it follows $\mu y = 0$ and $\mu = 0$. Thus $F_3 = 0$. We obtain the graded free resolution

$$0 \longrightarrow S(-2) \xrightarrow{[-y\ 1]^t} S(-1) \oplus S(-2) \xrightarrow{[x\ xy]} S \longrightarrow S/I \longrightarrow 0.$$

It is not minimal, because the presentation matrix of d_2 contains the entry 1.
We change the basis in $S(-1) \oplus S(-2)$ by setting

$$h_1 = f_1, \quad h_2 = -yf_1 + f_2.$$

With respect to the new basis, the resolution is

$$0 \longrightarrow S(-2) \xrightarrow{[0\ 1]^t} S(-1) \oplus S(-2) \xrightarrow{[x\ 0]} S \longrightarrow S/I \longrightarrow 0.$$

Thus, the resolution is the direct sum of the short trivial complex

$$0 \longrightarrow S(-2) \longrightarrow S(-2) \longrightarrow 0.$$

The minimal graded free resolution is

$$0 \longrightarrow S(-1) \xrightarrow{[x]} S \longrightarrow S/I \longrightarrow 0.$$

In this case we say that *the two copies of* $S(-2)$ *cancel.*

Remark 1.1.2.22. The minimal graded free resolution $\mathbf{F}$ of a graded finitely generated S-module M is very important, because it describes the structure of M since it has the form

$$\cdots \longrightarrow F_2 \xrightarrow{\begin{bmatrix}\text{a minimal system}\\ \text{of homogeneous}\\ \text{relations on the}\\ \text{relations in } d_1\end{bmatrix}} F_1 \xrightarrow{\begin{bmatrix}\text{a minimal system}\\ \text{of homogeneous}\\ \text{relations on the}\\ \text{relations in } M\end{bmatrix}} F_0 \xrightarrow{\begin{bmatrix}\text{a minimal system}\\ \text{of homogeneous}\\ \text{generators of } M\end{bmatrix}} M \longrightarrow 0.$$

The minimality of the relations encoded in d_i follows from Theorem 1.1.2.15. Not surprisingly, many properties of M can be read off the structure of $\mathbf{F}$.

Now we define the Betti numbers, the projective dimension of a graded finitely generated module M over a graded polynomial ring, because they help us to understand the structure of the minimal graded free resolution.

Definition 1.1.2.23. Let $\mathbf{F}$ be a minimal graded free resolution of a graded finitely generated S-module M

$$\mathbf{F}: \ \ldots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0,$$

The i'th *Betti number of* M over S is

$$b_i^S(M) := \operatorname{rank}(F_i).$$

By Theorem 1.1.2.19 the Betti numbers do not depend on the choice of the minimal graded free resolution of M.

Definition 1.1.2.24. Let $\mathbf{F}$ be a minimal graded free resolution of a graded finitely generated S-module M, that means, each free module F_i is a direct sum of modules of the form $S(-p)$. We define the *graded Betti numbers of* M by

$$b_{i,p}^S(M) = \text{number of summands in } F_i \text{ of the form } S(-p).$$

Proposition 1.1.2.25. [Pee11, Proposition 12.3] *Let c be the minimal degree of an element in a minimal system of homogeneous generators of M. We have that $b_{i,p}^S(M) = 0$ for $p < i + c$.*

Example 1.1.2.26. In the Example 1.1.2.11, we have $b_{0,p}^S(S/I) = 0$ for $p < 0$, $b_{1,p}^S(S/I) = 0$ for $p < 1$ and $b_{2,p}^S(S/I) = 0$ for $p < 2$.

Definition 1.1.2.27. We define the *length* of a graded free resolution $\mathbf{F}$ by $\max\{i \in \mathbb{N} : F_i \neq 0\}$. We say that $\mathbf{F}$ is a *finite resolution* if its length is finite, otherwise we say that $\mathbf{F}$ is an *infinite resolution*. The *projective dimension* of M is

$$\operatorname{proj.dim}_S(M) = \max\{i : b_i^S(M) \neq 0\}.$$

Thus, $\operatorname{proj.dim}_S(M)$ is the length of the minimal free resolution of M.

Example 1.1.2.28. In Example 1.1.2.11, the projective dimension of S/I is 2.

Remark 1.1.2.29. By Theorem 1.1.2.19 we observe that $\mathrm{proj.dim}_S(M)$ is the length of the shortest graded free resolution of M.

Corollary 1.1.2.30. [BH93, Corollary 2.2.14] *Let $S = \mathbb{K}[x_1, \dots, x_n]$, where $\mathbb{K}$ is a field. Then*

1. *(Hilbert's syzygy theorem) Every finitely generated S-module M has a finite graded free resolution of length $\leq n$.*
2. $\mathrm{proj.dim}_S(M) \leq n$ *for every finite S-module M.*
3. *Every finitely generated S-module has a finite free resolution of length $\leq n$.*

Definition 1.1.2.31. Let S be a graded polynomial ring with $\mathfrak{m}$ its homogeneous maximal ideal and I be an ideal of S. The *socle* of S/I is $\mathrm{soc}(S/I) := \{f \in S/I : \ \mathfrak{m}f = 0\}$ (see Remark 1.1.2.13), we denote it by $\mathrm{soc}(S/I)$.

Lemma 1.1.2.32. [Pee11, Corollary 14.12] *Let S be a graded polynomial ring with $\mathfrak{m}$ its homogeneous maximal ideal and I an ideal of S. Let $\mathbf{F}$ be a minimal graded free resolution of a graded finitely generated S-module S/I with* $\mathrm{proj.dim}_S(S/I) = n$. *Then* $b_n^S(S/I) = \dim_{\mathbb{K}}(\mathrm{soc}(S/I))$, *with $\mathbb{K} \cong S/\mathfrak{m}$.*

Theorem 1.1.2.33. *(Serre's Theorem) Every finitely generated graded R-module has finite projective dimension if and only if R is a polynomial ring, that is, $R = S/I$ for some ideal I generated by linear forms.*

Proof. See [Mat86, Theorem 19.2] for a proof of Serre's theorem. □

Now we come back to a graded free resolution $\mathbf{F}$ of a finitely generated graded S-module M, in which each differential matrix has entries of the same degree, for defining a pure free resolution of M (see [Pee11] and [Put16]). Especially interesting are the linear free resolutions in which all differentials have linear entries, and we use the following notation. Let $c_{i,p(M)}$ be the number of copies of $S(-p)$ in F_i. Note that if the resolution is minimal then the numbers $c_{i,p}(M)$ coincide with the graded Betti numbers $b_{i,p}^S(M)$.

Definition 1.1.2.34. The set of i'th *shifts* in $\mathbf{F}$ is

$$\{p \in \mathbb{N}_0 : \ c_{i,p} \neq 0\}.$$

Denote by t_i the *minimal* i'th shift, and by T_i the *maximal* i'th shift, that is

$$t_i = \min\{p \in \mathbb{N}_0 : \ c_{i,p} \neq 0\} \quad \text{and} \quad T_i = \max\{p \in \mathbb{N}_0 : \ c_{i,p} \neq 0\}.$$

Definition 1.1.2.35. We say that $\mathbf{F}$ is *pure* if it has the form

$$\dots \longrightarrow S(-p_i)^{c_{i,p_i}} \xrightarrow{d_i} S(-p_{i-1})^{c_{i-1,p_{i-1}}} \longrightarrow \dots,$$

that is, for each i the set of i'th shifts consists of one number denoted p_i, that is $t_i = T_i = p_i$.

Proposition 1.1.2.36. [Pee11, Proposition 17.5] *Let* $\mathbf{F}$ *be a graded free resolution of a finitely generated graded* S*-module* M*. Then* $\mathbf{F}$ *is pure if and only if for each* i *there exists a number* p_i *such that* $c_{i,r} = 0$ *for* $r \neq p_i$*.*

Corollary 1.1.2.37. [Pee11, Corollary 17.6] *If there exists a pure graded free resolution of* M*, then the minimal graded free resolution of* M *is pure.*

Example 1.1.2.38. Let $S = \mathbb{K}[x,y]$ and $I = (x^3, x^2y, xy^2)$. The minimal graded free resolution of a finitely generated graded S-module S/I is

$$0 \longrightarrow S(-4)^2 \xrightarrow{\begin{bmatrix} y & 0 \\ -x & y \\ 0 & -1 \end{bmatrix}} S(-3)^3 \xrightarrow{[x^3\ x^2y\ xy^2]} S \longrightarrow S/I \longrightarrow 0.$$

This resolution is pure with $p_0 = 0$, $p_1 = 3$, $p_2 = 4$.

1.1.3 Hilbert series and Hilbert polynomial

The Hilbert function and the Hilbert series of a finitely generated graded algebra over a field are very important, because they measure the growth of the dimension of the homogeneous components of the algebra. We will be using these notions in the following situation: the quotient by a homogeneous ideal of a graded polynomial ring (graded by the total degree). Hilbert series are important in computational algebraic geometry, as it is the easiest known way for computing the dimension and the degree of an algebraic variety defined by explicit polynomial equations. Since the Hilbert series of an algebra or a module is a special case of the Hilbert-Poincaré series of a graded vector space, we will explain as a first step what the Hilbert-Poincaré series is, see [Sta78].

Definition 1.1.3.1. The *Poincaré series* of a graded finitely generated S-module M is

$$\mathrm{P}_M^S(t) = \sum_{i \geq 0} b_i^S(M) t^i.$$

The properties of the Poincaré series are usually of interest for infinite free resolutions.

Example 1.1.3.2. In Example 1.1.2.11 the Poincaré series of S/I over S is

$$\mathrm{P}_{S/I}^S(t) = 1 + 2t + t^2.$$

Since $R = S/I$ and $S = \mathbb{K}[x_1, \ldots, x_n]$, we have a grading and hence we can measure the dimension of the quotient ring R by measuring the dimension of its graded components. We observe that R_i is a $\mathbb{K}$-vector space because $R_0 R_i \subseteq R_i$ and $R_0 = \mathbb{K}$. Its basis is called a *basis in degree* i.

Definition 1.1.3.3. The *Hilbert function* of R is the function $\mathbb{N}_0 \to \mathbb{N}_0$ defined by $i \longmapsto \dim_{\mathbb{K}}(R_i)$ for $\dim_{\mathbb{K}}(R_i) < \infty$ for all $i \geq 0$. The *Hilbert series* is defined by

$$\mathrm{Hilb}_R(t) = \sum_{i \geq 0} \dim_{\mathbb{K}}(R_i) t^i.$$

Example 1.1.3.4. Let $S = \mathbb{K}[x,y]$ and $I = (x^3, y^2)$. Then S/I is graded with basis $\{1\}$ in degree 0, $\{x,y\}$ in degree 1, $\{x^2, xy\}$ in degree 2 and $\{x^2y\}$ in degree 3. Then the Hilbert series is

$$\mathrm{Hilb}_{S/I}(t) = 1 + 2t + 2t^2 + t^3.$$

Example 1.1.3.5. In Example 1.1.2.11 S/I is graded with basis $\{1\}$ in degree 0. Hence the Hilbert series of S/I is

$$\mathrm{Hilb}_{S/I}(t) = 1.$$

Proposition 1.1.3.6. [Pee11, Proposition 1.8] *Let I be an ideal ofS generated by monomials. Then the $\mathbb{K}$-vector space $(S/I)_i = S_i/I_i$ has the basis*

$$\{\text{monomial } m \in S : \ m \notin I, \ \deg(m) = i\},$$

for all $i \geq 0$. Hence, $dim_{\mathbb{K}}((S/I)_i)$ equals the number of monomials of degree i not in I.

Let M be a graded S-module. Then M_i is a $\mathbb{K}$-vector space because $S_0 M_i \subset M_i$ and $S_0 = \mathbb{K}$. A basis of M_i is called a *basis in degree i*. If N is a finitely generated graded S-module, then $\dim_{\mathbb{K}}(N_i) < \infty$ for all $i \in \mathbb{Z}$ and $N_i = 0$ for $i \ll 0$.

Definition 1.1.3.7. Let $N = \bigoplus_{i \in \mathbb{Z}} N_i$ be a graded S-module. The generating function $i \longmapsto \dim_{\mathbb{K}}(N_i)$ is called the *Hilbert function* of N and the series

$$\mathrm{Hilb}_N(t) = \sum_{i \in \mathbb{Z}} \dim_{\mathbb{K}}(N_i) t^i$$

is called the *Hilbert series* of N.

In the case that the module N is shifted p degrees, its Hilbert function is

$$\mathrm{Hilb}_{N(-p)}(t) = t^p \,\mathrm{Hilb}_N(t).$$

Hilbert series can be computed using graded free resolutions, as it is illustrated by the following results.

Proposition 1.1.3.8. [Pee11, Proposition 16.1] *Let*

$$0 \longrightarrow K \longrightarrow N \longrightarrow W \longrightarrow 0,$$

be a short exact sequence of graded finitely generated S-modules and homomorphisms of degree 0, then

$$\mathrm{Hilb}_N(t) = \mathrm{Hilb}_K(t) + \mathrm{Hilb}_W(t).$$

Theorem 1.1.3.9. [Pee11, Theorem 16.2]*(Hilbert) Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring and $\mathbf{F}$ be a graded free resolution of a finitely generated graded S-module M. Write*

$$F_i = \bigoplus_{p \in \mathbb{Z}} S(-p)^{c_{i,p}}.$$

For each p suppose that $c_{i,p} = 0$ for $i \gg 0$. Then

$$\mathrm{Hilb}_M(t) = \frac{\sum_{i \geq 0} \sum_{p \in \mathbb{Z}} (-1)^i \, c_{i,p} \, t^p}{(1-t)^n}.$$

Theorem 1.1.3.9 can be applied in the following cases:

1. The resolution $\mathbf{F}$ is finite.
2. The resolution $\mathbf{F}$ is minimal. In this case $c_{i,p} = b^S_{i,p}(M)$ and we apply Proposition 1.1.2.25.

Example 1.1.3.10. Let $S = \mathbb{K}[x,y]$ and $R = S/(x^3, xy, y^5)$. The graded free resolution can be computed using by Construction 1.1.2.10 and we obtain

$$0 \longrightarrow S(-4) \oplus S(-6) \longrightarrow S(-3) \oplus S(-2) \oplus S(-5) \longrightarrow S \longrightarrow R \longrightarrow 0.$$

By Theorem 1.1.3.9 the Hilbert series is

$$\mathrm{Hilb}_R(t) = \frac{1 - t^3 - t^2 - t^5 + t^4 + t^6}{(1-t)^2} = 1 + 2t + 2t^2 + t^3 + t^4.$$

Theorem 1.1.3.11. *Let S be a graded polynomial ring. If S is generated by n homogeneous elements of positive degrees $a_1, \dots, a_n$, then the Hilbert series is a rational function*

$$\mathrm{Hilb}_S(t) = \frac{P(t)}{\prod_{i=1}^n (1 - t^{a_i})},$$

where $P(t)$ is a polynomial with integer coefficients.

Proof. See [AM69, Theorem 11.1], or [Smo72, Theorem 4.2]. □

Corollary 1.1.3.12. *Let $S = \mathbb{K}[x_1, \dots, x_n]$ be the graded polynomial ring. The Hilbert series may be rewritten as*

$$\mathrm{Hilb}_S(t) = \frac{P(t)}{(1-t)^n},$$

where n is the dimension of S.

Remark 1.1.3.13. Let $S = \mathbb{K}[x_1, \dots, x_n]$ and $R = S/I$, the graded $\mathbb{K}$-algebra R is finitely generated by elements of positive degree. Thus R satisfies the above Theorem 1.1.3.11, see [BH93, Proposition 4.4.1].

1.1.4 Cohen-Macaulay rings and complete intersection

The polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ over a field $\mathbb{K}$ is a graded ring, in particular it has the form $S = \bigoplus_{i \geq 0} S_i$. Moreover S is a local ring with the unique homogeneous maximal ideal $\mathfrak{m} = \bigoplus_{i>0} S_i = (x_1, \dots, x_n)$. Since S is also a regular ring by Proposition 1.1.4.7, then regular local rings can be considered as analogues of polynomial rings with finitely many variables over fields.

In what follows, "dimension of a module M over a ring S" is understood as the Krull dimension of $S/\mathrm{Ann}(M)$.

Definition 1.1.4.1. Let M be a finitely generated module over a graded polynomial ring S. A sequence of elements $a_1, \dots, a_r$ in an ideal I of S is called an *M-regular sequence* or an *M-sequence* in I, if a_{i+1} is not a zero divisor of $M/(a_1, \dots, a_i)M$ for $1 \leq i < r$ and $(a_1, \dots, a_r)M \neq M$. The sequence is said to be *maximal* if there does not exist any element $a_{r+1} \in I$ such that $a_1, \dots, a_r, a_{r+1}$ is an M-sequence in I.

Example 1.1.4.2. In the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ the sequence $x_1, \dots, x_n$ of indeterminates is a S-regular sequence.

Definition 1.1.4.3. Let S be a graded polynomial ring with $\mathfrak{m}$ its homogeneous maximal ideal. An S-sequence $a_1, \dots, a_r$ in $\mathfrak{m}$ is called a *system of parameters* if $a_1, \dots, a_r$ are generators of $\mathfrak{m}$.

Definition 1.1.4.4. Let S be a graded polynomial ring with $\mathfrak{m}$ its homogeneous maximal ideal and M be a finitely generated S-module with of dimension d. A *system of parameters* of M is a set $\{a_1, \dots, a_d\}$ of elements of $\mathfrak{m}$ such that $M/(a_1, \dots, a_d)M$ has finite length.

Definition 1.1.4.5. A Noetherian local ring is called *regular* if it has a system of parameters generating its unique maximal ideal; such a system of parameters is called a *regular system of parameters*.

Remark 1.1.4.6. [BH93, Definition 2.2.1] Equivalent definition to Definition 1.1.4.5 is, a Noetherian local ring is regular if and only if its dimension is equal to the minimal number of generators of its unique maximal ideal.

Proposition 1.1.4.7. *A polynomial ring with finitely many variables over a field $\mathbb{K}$ is a regular ring.*

Proof. That follows immediately from [BH93, Theorem 2.2.13] or [BIV89, Theorem 14.31], since $\mathbb{K}$ is regular. □

Theorem 1.1.4.8. [BH93, Theorem 1.5.17] *Let S be a graded polynomial ring with $\mathfrak{m}$ its homogeneous maximal ideal and $\dim(S) = n$. Then there exist homogeneous elements $x_1, \dots, x_n$ of $\mathfrak{m}$, such that they form a system of parameters of degree 1. Such elements are algebraically independent over $\mathbb{K}$, where $\mathbb{K} \cong S/\mathfrak{m}$.*

Proposition 1.1.4.9. [BH93, Proposition 2.2.4] *Let S be a regular graded polynomial ring with $\mathfrak{m}$ its homogeneous maximal ideal and $I \subset S$ be an ideal. Then S/I is regular if and only if I is generated by a subset of a regular system of parameters.*

Note that S/I is a graded regular ring if and only if I is generated by linear forms.

Theorem 1.1.4.10. *(Rees) Let M be a finitely generated module over a graded polynomial polynomial ring S and I be an ideal such that $IM \neq M$. Then all maximal M-sequences in I have the same length s and it is given by*

$$s = \min\{i : \operatorname{Ext}^i_S(S/I, M) \neq 0\}.$$

Proof. See [BH93, Theorem 1.2.5]. □

Definition 1.1.4.11. Let M be a finitely generated module over a graded polynomial ring S and I be an ideal such that $IM \neq M$. Then the *grade* of I on M is the common length of the maximal M-sequences in I, denoted by $\operatorname{grade}(I, M)$.

To complement this definition we put $\operatorname{grade}(I, M) = \infty$ if $IM = M$. By Theorem 1.1.4.10 we have

$$\operatorname{grade}(I, M) = \infty \quad \text{if and only if} \quad \operatorname{Ext}^i_S(S/I, M) = 0 \ \text{ for all } i.$$

Definition 1.1.4.12. Let S be a graded polynomial ring with $\mathfrak{m}$ its homogeneous maximal ideal and M be a finitely generated S-module. Then the *depth* of M is the grade of $\mathfrak{m}$ on M, denoted by $\operatorname{depth}(M)$.

Corollary 1.1.4.13. *Let M be a finitely generated non-zero module over a graded polynomial ring S. Let $\mathbb{K}$ be the residue field. Then*

$$\operatorname{depth}(M) = \min\{i : \operatorname{Ext}^i_S(\mathbb{K}, M) \neq 0\}.$$

Proof. It follows immediately from Theorem 1.1.4.10. □

Proposition 1.1.4.14. [BIV89, Proposition 14.18] *Let S be a graded polynomial ring and M be a finitely generated S-module. If $M \neq 0$, then* $\operatorname{depth}(M) \leq \dim(M)$.

Definition 1.1.4.15. Let S be a graded polynomial ring and M be a finitely generated non-zero S-module. Then the *grade* of M is given by

$$\operatorname{grade}(M) = \min\{i : \operatorname{Ext}^i_S(M, S) \neq 0\}.$$

It follows directly from [BH93, Proposition 1.2.10 (e)] that $\operatorname{grade}(M) = \operatorname{grade}(\operatorname{Ann}(M), S)$. For an ideal I of S it is customary to set

$$\operatorname{grade}(I) = \operatorname{grade}(S/I) = \operatorname{grade}(I, S).$$

Now grade I has two different meanings, but we never use it to denote the grade of the module I.

Proposition 1.1.4.16. [BH93, Proposition 1.2.14] *Let S be a graded polynomial ring and $I \subset S$ be an ideal. Then*

$$\operatorname{grade}(I) \leq \operatorname{height}(I).$$

Theorem 1.1.4.17. *(Auslander-Buchsbaum) Let S be a graded polynomial ring and M be a finitely generated non-zero S-module. If* $\operatorname{proj.dim}_S(M) < \infty$*, then*

$$\operatorname{proj.dim}_S(M) + \operatorname{depth}(M) = \operatorname{depth}(S).$$

In particular $\operatorname{depth}(M) \leq \operatorname{depth}(S)$.

Proof. See [BH93, Theorem 1.3.3] for a proof of Auslander-Buchsbaum's theorem. □

Theorem 1.1.4.18. *(Auslander-Buchsbaum-Serre) Let S be a graded polynomial ring and M be a finitely generated S-module. Then S is regular if and only if* $\text{proj.dim}_S(M) < \infty$ *for every finitely generated S-module M.*

Proof. See [Ser56] or [AB57, Theorem 1.10 and Corollary 4.8]. □

Definition 1.1.4.19. Let S be a graded polynomial ring. A finitely generated non-zero S-module M is a *Cohen-Macauly module* if $\text{depth}(M) = \dim(M)$. If S itself is a Cohen-Macaulay module, then it is called a *Cohen-Macauly ring*. (For $M = 0$ we have $\dim(M) = -\infty$ and $\text{depth}(M) = \infty$.)

Theorem 1.1.4.20. [BH93, Corollary 2.2.6] *A regular graded polynomial ring is a Cohen-Macaulay ring.*

Theorem 1.1.4.21. [BH93, Corollary 2.1.4] *Let S be a Cohen-Macaulay ring and $I \neq S$ be an ideal. Then* $\text{grade}(I) = \text{height}(I)$*, and if S/I is a graded ring, then*

$$\text{height}(I) + \dim(S/I) = \dim(S).$$

Definition 1.1.4.22. Let S be a graded polynomial ring. A finitely generated non-zero S-module M is *perfect* if $\text{proj.dim}_S(M) = \text{grade}(M)$. An ideal I of S is called *perfect* if S/I is a perfect module, that is $\text{proj.dim}_S(S/I) = \text{grade}(I)$.

Theorem 1.1.4.23. [BH93, Corollary 2.2.10] *Let S be a graded polynomial ring and I be an ideal of S. If I is generated by a S-sequence, then I is perfect.*

Theorem 1.1.4.24. [BH93, Corollary 2.2.15] *Let $\mathbb{K}$ be a field, $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring, $\mathfrak{m} = (x_1, \dots, x_n)$ be the homogeneous maximal ideal and M be a finitely generated graded S-module. Then the following are equivalent:*

1. *M is Cohen-Macaulay,*
2. *M is perfect,*
3. *$M_\mathfrak{m}$ is Cohen-Macaulay,*
4. *$M_\mathfrak{m}$ is perfect.*

Definition 1.1.4.25. Let R be a graded polynomial ring, such that R has the form S/I, where S is a regular graded polynomial ring over a field $\mathbb{K}$ with $\mathfrak{m}$ its homogeneous maximal ideal. Then R is called a *complete intersection* if I is generated by an S-sequence.

Definition 1.1.4.26. Let S be a graded polynomial ring. An ideal I of S with $\text{grade}(I) = r$ is called a *complete intersection ideal*, if I is generated by r elements. An equivalent definition is: A graded ideal I of S is called a *complete intersection ideal* if I is generated by a S-regular sequence $x_1, \dots, x_r$ of polynomials.

Theorem 1.1.4.27. [BH93, Section 2.3] *Let S be a graded polynomial ring I be an ideal of S. For a complete intersection ideal I the ring S/I is a Cohen-Macaulay ring.*

1.1.5 Gorenstein rings and Gorenstein ideals

We introduce Gorenstein rings and show that Gorenstein rings are Cohen-Macaulay rings. There are many equivalent definitions for Gorenstein rings, we give now some of them. In this thesis we are intersted by a graded polynomial rings, thus we use such rings in all definitions as assumption instead of a Noetherian local rings. In [Bas63] there are many equivalent conditions for Gorenstein rings, it is defined and presented using injective resolutions of rings. In this subsection we define a Gorenstein ring using Cohen-Macaulay rings and systems of parameters. For a more extensive treatment see [Bas63], [HK71] or [BH93].

Definition 1.1.5.1. Let S be a graded ring. We say that S is a *Gorenstein ring* if and only if the following conditions are satisfied

1. S is a Cohen-Macaulay ring.
2. There exists a homogeneous system of parameters $x_1, \dots, x_r$ in S, such that the ideal $(x_1, \dots, x_r)$ is irreducible, i.e.

$$\text{if} \quad (x_1, \dots, x_r) = I \cap J, \quad \text{then} \quad I = (x_1, \dots, x_r) \quad \text{or} \quad J = (x_1, \dots, x_r).$$

Proposition 1.1.5.2. *Any regular graded polynomial ring is a Gorenstein ring.*

Proof. Suppose that S is a regular graded polynomial ring in the variables $x_1, \dots, x_n$ of dimension n with $\mathfrak{m}$ its homogeneous maximal ideal. Since S is regular, it is a Cohen-Macaulay ring by Theorem 1.1.4.20, and

$$S/(x_1, \dots, x_n) \cong S/\mathfrak{m} \cong \mathbb{K},$$

where $\mathbb{K}$ is the residue field. We know that (0) is irreducible in $\mathbb{K}$. □

Example 1.1.5.3. Let $S = \mathbb{K}[x]$ and $I = (x^2)$. The ring $R = S/I$ is a Gorenstein ring and has dimension 0 and three ideals $(0) \subset (x) \subset R$. It is clear that (0) cannot be obtained as an intersection of two non-zero ideals in R. On the other hand R is not regular. That means not every Gorenstein ring is regular.

Proposition 1.1.5.4. *Let R be a zero-dimensional graded ring with $\mathfrak{m}$ its homogeneous maximal ideal. Then the following are equivalent:*

1. *R Gorenstein ring.*
2. $\operatorname{soc}(R)$ *is a* 1*-dimensional vector space over* $\mathbb{K}$*, where* $\mathbb{K}$ *is the residue field.*
3. *R is injective as an R-module.*
4. *The ideal* (0) *in R is irreducible.*

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ follow directly from [Eis95, Proposition 21.5]. To prove $(2) \Leftrightarrow (4)$, assume that (0) is irreducible and $\dim_{\mathbb{K}}(\operatorname{soc}(R)) \geq 2$. Choose linear independet vectors v, u in $\operatorname{soc}(R)$. Then we have $(v) \cap (u) = (0)$, which is a contradiction since we assumed that (0) was irreducible. Conversely assume that $\dim_{\mathbb{K}}(\operatorname{soc}(R)) = 1$. Let v be a basis of $\operatorname{soc}(R)$ and suppose $(0) = I \cap J$. Choose $r \geq 0$ maximal such that $\mathfrak{m}^r I \neq 0$ and $s \geq 0$ maximal such that $\mathfrak{m}^s J \neq 0$. Since R has dimension 0, $\mathfrak{m}^n = 0$ for all $n \gg 0$. Also

$\mathfrak{m}^r I \subseteq (0 : \mathfrak{m}) \cap I$ and $\mathfrak{m}^s J \subseteq (0 : \mathfrak{m}) \cap J$. Therefore $\mathrm{soc}(R) \cap I \neq (0)$ and $\mathrm{soc}(R) \cap J \neq (0)$. But $\mathrm{soc}(R)$ is a 1-dimensional vector space, so $v \in I$ and $v \in J$, and this is a contradiction to $v \neq 0$. $\square$

Corollary 1.1.5.5. *Let S be a regular graded polynomial ring of dimension n and I be an ideal of S such that $\sqrt{I} = \mathfrak{m}$, so that $R = S/I$ has dimension 0. Then R is a Gorenstein ring if and only if $b_n^S(R) = 1$.*

Proof. See Proposition 1.1.5.4 and Lemma 1.1.2.32. $\square$

Theorem 1.1.5.6. [BH93, Theorem 3.2.10] *Let S be a regular graded polynomial ring of dimension n and I be an ideal of S. Then $R = S/I$ is Gorenstein ring if and only if it is Cohen-Macaulay ring and its* canonical module $\mathrm{Ext}_S^{n-q}(R,S)$ *is free of rank* 1*, where* $\dim(R) = q$*. The number* $\dim_{\mathbb{K}}(\mathrm{Ext}_S^{n-q}(R,S))$ *is called the* type *of R.*

Remark 1.1.5.7. The type of a ring $R = S/I$ of depth 0 is the dimension of its socle.

Theorem 1.1.5.8. [Pee11, Theorem 25.7] *Let S be a regular graded polynomial ring of dimension n, I be an ideal of S and let $q := \dim(S/I)$. Then the quotient S/I is a Gorenstein ring if and only if*

$$\mathrm{proj.dim}_S(S/I) = n - q \quad \text{and} \quad b_{n-q}^S(S/I) = 1.$$

Theorem 1.1.5.9. [Pee11, Theorem 25.6] *Let S be a regular graded polynomial ring of dimension n and I an ideal of S. If S/I is Gorenstein ring of dimension q, then*

$$b_i^S(S/I) = b_{n-q-i}^S(S/I) \quad \text{for} \quad 0 \leq i \leq n-q.$$

Proposition 1.1.5.10. [Eis95, Corollary 21.19] *Let S be a regular graded polynomial ring and $x_1, \ldots, x_r$ be an S-regular sequence. Then $S/(x_1, \ldots, x_r)$ is a Gorenstein ring, i.e. complete intersections are Gorenstein rings.*

The converse is false: We see an example of a Gorenstein ring S/I where I has codimension 3 and is generated by five quadrics, we will see that in the second chapter, see Example 2.1.2.6. However, there is no such example in codimension 2.

Corollary 1.1.5.11. *Let S be a regular graded polynomial ring and $x_1, \ldots, x_r$ be a system of parameters. Then $S/(x_1, \ldots, x_r)$ is a Gorenstein ring.*

Proposition 1.1.5.12. [Hun99, Corollary 3.5] *Let S be a regular graded polynomial ring and let I be an ideal of height k generated by k elements. Then S/I is Gorenstein. In general the reverse is not true, an Example 2.1.2.6 shows.*

Proposition 1.1.5.13. *Let S be a graded polynomial ring S. Then we have the following*

$$S \text{ is regular} \Rightarrow S \text{ is complete intersection} \Rightarrow S \text{ is Gorenstein} \Rightarrow S \text{ is Cohen-Macaulay}.$$

Proof. See [BH93, Proposition 3.1.20], Theorem 1.1.4.20, Theorem 1.1.4.27 and Proposition 1.1.5.10. $\square$

Example 1.1.5.14. Let $S = \mathbb{K}[x, y]$ and $I = (x^2, xy, y^2)$. Then $R = S/I$ is a Cohen-Macaulay ring, but it is not a Gorenstein ring. The ring R is Noetherian, Artinian and local ring, since R has finite length and (x, y) is the unique maximal ideal in R. Thus $\dim(R) = 0$. Since every element in (x, y) is a zero divisor, there is no regular sequence in R, i.e. $\operatorname{depth}(R) = 0$, that means that R is a Cohen-Macaulay ring. But it is not a Gorenstein ring, since $R = \mathbb{K}[x, y]/(x^2, xy, y^2) = \mathbb{K} \oplus \mathbb{K}x + \mathbb{K}y$ and $\operatorname{soc}(R) = (x, y)$, hence $\dim_{\mathbb{K}}(\operatorname{soc}(R)) = 2 \neq 1$, which implies by Proposition 1.1.5.4 that R is not a Gorenstein ring.

Definition 1.1.5.15. Let S be a regular graded polynomial ring. An ideal I of S is called *Gorenstein ideal* (*of grade* g) if I is perfect and $\operatorname{Ext}^g_S(S/I, S) \cong S/I$.

Proposition 1.1.5.16. [BH93, Theorem 3.3.7 (b)] *If S is a Gorenstein ring and I is perfect, then I is a Gorenstein ideal if and only if S/I is a Gorenstein ring.*

Proposition 1.1.5.17. *Let S be a graded polynomial ring over $\mathbb{K}$ of dimension n and I be an ideal of S. Let S/I be a quotient ring of dimension q and generated by elements of degree 1. If S/I is a Gorenstein ring, then the Hilbert series may be rewritten as:*

$$\operatorname{Hilb}_{S/I}(t) = \frac{P(t)}{(1-t)^{n-q}},$$

with $P(t) = \sum_{i \geq 0} p_i t^i$; $p_i \in \mathbb{Z}$. If $\deg(P(t)) = m$, then $t^m P(t^{-1}) = P(t)$, with $p_0 = p_m = 1$ and $p_i = p_{m-i}$ for all i.

Proof. See Corollary 1.1.3.12 and Theorem 1.1.5.8. □

Corollary 1.1.5.18. *If S/I is artian, then $\operatorname{Hilb}_{S/I}(t) = P(t)$.*

1.2 Basic combinatorial concepts

We introduce basic concepts of combinatorial commutative algebra, a new branch of commutative algebra created by Hochster and Stanley (see [Hoc77], [Sta78], [Sta80] and [Sta96]). The combinatorial objects considered are simplicial complexes to which one assigns algebraic objects, the Stanley-Reisner rings. It turns out that most a lot of important algebraic notions such as "Cohen-Macaulay ring", "Gorenstein ring" and "Hilbert series" are proper concepts in solving purely combinatorial problems.

1.2.1 Simplicial complexes

Definition 1.2.1.1. Let $V = \{v_1, \ldots, v_n\}$ be a set of vertices. A *(finite) simplicial complex* Δ on V is a collection of subsets of V such that if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$, and $\{v_i\} \in \Delta$ for all $i = 1, \ldots, n$. Each element of Δ is called a *face*. The *dimension of a face* F is the number $|F| - 1$ and denoted by $\dim(F)$. The *dimension of the simplicial complex* Δ is $\dim(\Delta) = \max\{\dim(F) : F \in \Delta\}$. A *vertex* of Δ is a face of dimension 0. An *edge* of Δ is a face of dimension 1.

Definition 1.2.1.2. A maximal face is called a *facets* of Δ (with respect to inclusion). The set of facets of Δ is denoted by $\mathcal{F}(\Delta)$. A *nonface* of Δ is a subset F of V with $F \notin \Delta$. Let $\mathcal{N}(\Delta)$ denote the set of minimal nonfaces of Δ.

The empty set $\emptyset$ is a face of dimension -1 of any nonempty simplicial complex. It is clear that $\mathcal{F}(\Delta)$ determines Δ. When $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$, we write $\Delta = (F_1, \ldots, F_m)$. This simplicial complex is said to be generated by $F_1, \ldots, F_m$.

Definition 1.2.1.3. A simplicial complex is called *simplex* if it generated by on face.

Definition 1.2.1.4. Let $d = \max\{|F| : F \in \Delta\}$, then $\dim(\Delta) = d - 1$. Let Δ be a simplicial complex of dimension $d-1 \geq 0$ on a vertex set V. Then we denote the number of faces of Δ of dimension i by $f_i = f_i(\Delta)$. We have $f_0 = |V|$, and $f_{-1} = 1$, since $\emptyset \in \Delta$. The sequence $F(\Delta) = (f_0, \ldots, f_{d-1})$ is called the *f-vector* of Δ.

Example 1.2.1.5. Figure 1.1 represents the simplicial complex Δ of dimension 2 on the vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ with

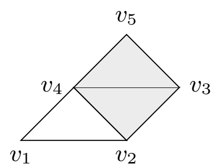


Figure 1.1: The geometric realization of Δ.

$$\mathcal{F}(\Delta) = \{\{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3, v_4\}, \{v_3, v_4, v_5\}\},$$

$$\mathcal{N}(\Delta) = \{\{v_1, v_3\}, \{v_1, v_5\}, \{v_2, v_5\}, \{v_1, v_2, v_4\}\},$$

$$F(\Delta) = (5, 7, 2).$$

Example 1.2.1.6. A matroid is a simplicial complex on its ground set: faces correspond to independent sets, and facets to bases.

1.2.2 Polytopes

Definition 1.2.2.1. A *polytope* P is the convex hull, $\operatorname{conv}(V)$, of a finite set of points V in $\mathbb{R}^d$. A *hyperplane* of an d-dimensional space W is a subspace of dimension $d-1$, or equivalently, of codimension 1 in W. The set of points lying on one side of a hyperplane (including the hyperplane) is a *closed half space*. A *polyhedral* set or *polyhedron* is the intersection of a fnite number of closed half spaces.

Theorem 1.2.2.2. [BH93, Theorem 5.2.3] *A subset of $\mathbb{R}^d$ is a polytope if and only if it is a bounded polyhedron.*

Definition 1.2.2.3. A hyperplane H is called a *supporting hyperplane* of a polyhedron P if $H \cap P \neq \emptyset$ and P is contained in one of the closed half spaces determined by H. A *face* of P is $H \cap P$, if H is a supporting hyperplane of P. The empty set and P as faces are the *improper faces*. All the other faces of P are called *proper faces*. The faces of a polyhedron (polytope) are again polyhedra (polytopes). The dimension of its affine hull is called *dimension* of P and is denoted by $\dim(P)$. A *d-polyhedron* is a polyhedron of dimension d. A j-face is a face whose dimension as a polyhedron is j. We set $\dim(\emptyset) = -1$. If $\dim(P) = d$, then faces of dimension 0, 1, $d-1$ are called *vertices*, *edges*, *facets*, respectively.

Theorem 1.2.2.4. [BH93, Theorem 5.2.4] *Let P be a polyhedron. Then the following holds*

1. *P has only a finite number of faces.*
2. *Let F be a face of P and F' a face of F. Then F' is a face of P.*
3. *Any proper face of P is a face of some facet of P.*

Definition 1.2.2.5. The *boundary* of a polyhedron P is the union of all faces of P except P itself and it is denoted by ∂P.

Definition 1.2.2.6. A *d-simplex* is a d-polytope P with exactly $d+1$ vertices. A d-polytope P is said to be *simplicial* if its facets are $(d-1)$-simplices.

Remark 1.2.2.7. The facets of a d-simplex Δ are $(d-1)$-simplices.

Example 1.2.2.8. In the case $d = 2$ and 3:

(a) 2-simplex **(b)** 3-simplex

Figure 1.2: Illustration of low-dimensional simplices.

Example 1.2.2.9. In the case $d = 3$:

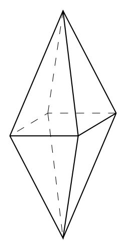


Figure 1.3: Illustration of low-dimensional simplicial polytope.

Proposition 1.2.2.10. [BH93, Proposition 5.2.6] *Every j-face of a d-simplex P is a j-simplex, and every $j+1$ vertices of P are the vertices of a j-face of P.*

Definition 1.2.2.11. [BH93, Corollary 5.2.7] The boundary ∂P of a simplicial d-polytope P is the geometric realization of a $(d-1)$-dimensional simplicial complex of the set of vertices of P $\mathrm{vert}(P)$, called the *boundary complex* of P. It is denoted by $\Delta = \Delta(P)$.

1.2.3 Stanley-Reisner rings and ideals

Definition 1.2.3.1. Let P be a simplicial d-polytope with the vertex set $\{v_1, \ldots, v_n\}$ and the boundary complex $\Delta(P)$. For a field $\mathbb{K}$, the corresponding *Stanley–Reisner ring* of $\Delta(P)$, or *face ring*, is defined as

$$\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \ldots, x_n]/I_{\Delta(P)},$$

where the ideal

$$I_{\Delta(P)} = (x_{i_1} \ldots x_{i_r} : \; i_1 < i_2 < \ldots < i_r, \; \{v_{i_1}, \ldots, v_{i_r}\} \notin \Delta(P))$$

is generated by square-free monomials corresponding to the nonfaces of $\Delta(P)$ and it is called the *Stanley–Reisner ideal* or the *face ideal* of $\Delta(P)$.

Remark 1.2.3.2. Observe that the minimal set of monomial generators of $I_{\Delta(P)}$ are in bijection with the set of minimal nonfaces of $\Delta(P)$.

Theorem 1.2.3.3. [Sta96, Theorem 1.3] *Let P be a simplicial d-polytope with the boundary complex $\Delta(P)$. Let $\mathbb{K}$ be a field. Then*

$$\dim(\mathbb{K}[\Delta(P)]) = \dim(\Delta(P)) + 1.$$

Proof. The Krull dimension of $\mathbb{K}[\Delta(P)]$ is maximal cardinality of an algebraically independet set of vertices $v_{i_1}, \ldots, v_{i_j}$ by Theorem 1.1.4.8. On the other hand that is maximal cardinality of any face of $\Delta(P)$. $\square$

Definition 1.2.3.4. Let P be a simplicial d-polytope with the boundary complex $\Delta(P)$. Then $\Delta(P)$ is *pure* if all its facets are of the same dimension, namely $\dim(\Delta(P))$. The complex $\Delta(P)$ is called a *Cohen-Macaulay complex* over $\mathbb{K}$ if $\mathbb{K}[\Delta(P)]$ is a Cohen-Macaulay ring.

Corollary 1.2.3.5. [HH11, Lemma 8.1.5] *Any Cohen-Macaulay complex is pure.*

Let P be a simplicial d-polytope with the boundary complex $\Delta(P)$. We are intersted in the Hilbert series of $\mathbb{K}[\Delta(P)]$ as a homogeneous graded algebra. For all $i \in \mathbb{Z}$ we have

$$\mathbb{K}[\Delta(P)]_i = \bigoplus_{a \in \mathbb{Z}^n, \; |a|=i} \mathbb{K}[\Delta(P)]_a,$$

where $|a| := a_1 + \ldots + a_n$ for $a = (a_1, \ldots, a_n)$.

Theorem 1.2.3.6. [Sta96, Theorem 1.4] *Let P be a simplicial d-polytope and $\Delta(P)$ be the boundary complex of P with f-vactor $F(\Delta(P)) = (f_0, \ldots, f_{d-1})$ (see Definition 1.2.1.4). Define* $\deg(x_i) = 1$*, then the Hilbert series of* $\mathbb{K}[\Delta(P)]$ *is*

$$\mathrm{Hilb}_{\mathbb{K}[\Delta(P)]}(t) = \sum_{i=-1}^{d-1} \frac{f_i t^{i+1}}{(1-t)^{i+1}}.$$

And its Hilbert function is given by

$$m \longmapsto \begin{cases} 1, & \text{for } m = 0, \\ \sum_{i=0}^{d-1} f_i \binom{m-1}{i}, & \text{for } m > 0. \end{cases}$$

Definition 1.2.3.7. Let P be a simplicial d-polytope with the boundary complex $\Delta(P)$. Now we define *Euler characteristic* of $\Delta(P)$, it is

$$\mathcal{X}(\Delta(P)) = \sum_{i=0}^{d-1} (-1)^i f_i.$$

And the *reduced Euler characteristic* of $\Delta(P)$ is

$$\tilde{\mathcal{X}}(\Delta(P)) = \mathcal{X}(\Delta(P)) - 1.$$

Corollary 1.2.3.8. [BH93, Corollary 5.2.17] *Let P be a simplicial d-polytope and $\Delta(P)$ be the boundary complex of P with f-vector $(f_0, \ldots, f_{d-1})$. Then*

$$\mathcal{X}(\Delta(P)) = \sum_{i=0}^{d-1} (-1)^i f_i = 1 - (-1)^d \quad \text{and} \quad \tilde{\mathcal{X}}(\Delta(P)) = (-1)^{d-1}.$$

For a more extensive treatment see [Sta96, Chapter II].

Now an important question is: For which simplicial complexes is the corresponding Stanley-Reisner ring Cohen-Macaulay or Gorenstein?

Definition 1.2.3.9. Let $\mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$. A *multicomplex* on $V = \{v_1, \ldots, v_n\}$ is a set Γ of monomials $x_1^{a_1} \ldots x_n^{a_n} \in \mathbb{K}[x_1, \ldots, x_n]$ of such that $y \in \Gamma$, $z | y$ implies $z \in \Gamma$, for $z \in \mathbb{K}[x_1, \ldots, x_n]$.

Definition 1.2.3.10. For a multicomplex Γ, let $h_i = \#\{y \in \Gamma : \deg(y) = i > 0\}$, and define the *$h$-vector* of Γ as $h(\Gamma) = (h_0, h_1, \ldots)$. An h-vector may be infinite, and if $\Gamma \neq \varnothing$, then $h_0 = 1$. If $h_i = 0$ for $i > d$ we write also $h(\Gamma) = (h_0, h_1, \ldots, h_d)$, where $d \in \mathbb{N}$.

Definition 1.2.3.11. Let P be a simplicial polytope. We define the *h-vector of the boundary complex* $\Delta(P)$ as follows

$$h(\Delta(P)) \coloneqq h(\mathbb{K}[\Delta(P)]).$$

In this case $h(\Delta(P))$ is a finite vector, $h_i = 0$ for $i > d = \dim(\Delta(P)) + 1$.

This definition is equivalent to the following explicit expression for $h(\Delta(P)) = (h_0, \dots, h_d)$ in terms of the f-vector $(f_0, f_1, \dots, f_{d-1})$ of $\Delta(P)$

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}, \quad 0 \le k \le d,$$

where $f_{-1} = 1$.

Example 1.2.3.12. If $\Delta(P)$ is the boundary complex of an octahedron P, then $f(\Delta(P)) = (6, 12, 8)$ and $h(\Delta(P)) = (1, 3, 3, 1)$.

Theorem 1.2.3.13. [BH93, Theorem 5.2.16]*(Sommerville) Let P be a simplicial polytope and $\Delta(P)$ be the boundary complex of P with $(h_0, \dots, h_d)$ be the h-vector of $\Delta(P)$. Then $h_i = h_{d-i}$ for $0 \le i \le d$.*

Corollary 1.2.3.14. [Sta96, Corollary 3.2] *Let P be a simplicial d-polytope and $\Delta(P)$ the boundary complex of P. If $\Delta(P)$ is a Cohen-Macaulay complex, then $h(\Delta(P))$ is the h-vector of some nonempty multicomplex Γ.*

Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$ and the boundary complex $\Delta(P)$. Let $\mathbb{K}$ be a field and $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_n]/I_{\Delta(P)}$ be the Stanley-Reisner ring of $\Delta(P)$ with $\mathfrak{m}$ its homogeneous maximal ideal, generated by the residue classes of x_i. We observe that $(\mathbb{K}[\Delta(P)], \mathfrak{m})$ is a local ring and hence by Theorem 1.1.4.24 $\mathbb{K}[\Delta(P)]$ is a Cohen-Macaulay ring if and only if $\mathbb{K}[\Delta(P)]_{\mathfrak{m}}$ is a Cohen-Macaulay ring. Thus $\Delta(P)$ is a Cohen-Macaulay complex.

Theorem 1.2.3.15. [MT11, Theorem 3.5] *Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$ and the boundary complex $\Delta(P)$. Let $S = \mathbb{K}[x_1, \dots, x_n]$ be a graded polynomial ring, then $\Delta(P)$ is a matroid if and only if $S/I^k_{\Delta(P)}$ is Cohen-Macaulay ring for $k \ge 1$.*

Example 1.2.3.16. If $\Delta(P)$ is a matroid, then $\mathbb{K}[\Delta(P)]$ is a Cohen-Macaulay ring. That means that matroids are a very special case of Cohen-Macaulay simplicial complexes.

Definition 1.2.3.17. Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$ and the boundary complex $\Delta(P)$. For $F \in \Delta(P)$ define the *link*

$$\mathrm{lk}(F) = \{F' \in \Delta(P) : \ F' \cup F \in \Delta(P), \ F' \cap F = \varnothing\}.$$

Notice that $\Delta(P) = \mathrm{lk}(\varnothing)$.

Definition 1.2.3.18. Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$ and the boundary complex $\Delta(P)$. Then $\Delta(P)$ is an *Euler complex* if $\Delta(P)$ is pure and $\tilde{\mathcal{X}}(\mathrm{lk}(F)) = (-1)^{\dim(\mathrm{lk}(F))}$ for all $F \in \Delta(P)$.

Definition 1.2.3.19. Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$ and the boundary complex $\Delta(P)$. Then the complex $\Delta(P)$ is called a *Gorenstein complex* over a field $\mathbb{K}$ if $\mathbb{K}[\Delta(P)]$ is a Gorenstein ring.

Definition 1.2.3.20. Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$ and the boundary complex $\Delta(P)$. For $F \in \Delta(P)$, the *star* of F is the set

$$\mathrm{st}(F) = \{F' \in \Delta(P) : \ F' \cup F \in \Delta(P)\}.$$

And we define the $\mathrm{core}(\Delta(P))$ to be $\Delta(P)_{\mathrm{core}(V)}$, where $\mathrm{core}(V) = \{v \in V : \ \mathrm{st}(v) \neq \Delta(P)\}$.

Notice $\mathbb{K}[\Delta(P)] \cong \mathbb{K}[\mathrm{core}(\Delta(P))][x_i : \ v_i \in V \smallsetminus \mathrm{core}(V)]$. It follows that $\Delta(P)$ is a Gorenstein complex if and only if $\mathrm{core}(\Delta(P))$ is a Gorenstein complex.

Theorem 1.2.3.21. [BH93, Theorem 5.6.2] *Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$ and the boundary complex $\Delta(P)$ with $\Delta(P) = \mathrm{core}(\Delta(P))$. Then $\Delta(P)$ is a Gorenstein complex over a field $\mathbb{K}$ if and only if $\Delta(P)$ is an Euler complex which is a Cohen-Macaulay complex over $\mathbb{K}$.*

Proof. We observe that $\mathbb{K}[\Delta(P)] \cong \mathrm{Ext}_S^{n-d}(\mathbb{K}[\Delta(P)], S)$ from [BH93, Lemma 5.6.3]. This implies that $\mathbb{K}[\Delta(P)]$ is a Gorenstein ring by Theorem 1.1.5.6. Then it follows from Proposition 1.1.5.16 that $I_{\Delta(P)}$ is a Gorenstein ideal. □

In this thesis, we consider Stanley-Reisner rings that are at the same time Gorenstein rings. Therefore their Stanley-Reisner ideals are Gorenstein ideals, by Proposition 1.1.5.16.

1.2.4 Graded Betti numbers of Stanley-Reisner rings

Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$ and the boundary complex $\Delta(P)$. Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \dots, x_n]$ and $\mathbb{K}[\Delta(P)] = S/I_{\Delta(P)}$ be a Stanley-Reisner ring. Since $\mathbb{K}[\Delta(P)]$ is a $\mathbb{Z}^n$-graded S-module, it has a minimal free graded resolution, according Theorem 1.1.2.19.

$$\mathbf{F}: \ 0 \longrightarrow F_m \xrightarrow{d_m} F_{m-1} \longrightarrow \dots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0,$$

where $F_i = \oplus_{j=1}^{b_i^S(\mathbb{K}[\Delta(P)])} S(-a_{ij})$ for $i = 0, \dots, \mathrm{proj.dim}_S(\mathbb{K}[\Delta(P)]) = m$ with certain $a_{ij} \in \mathbb{N}^n$, and where the maps d_i are homogeneous of degree 0. Minimality of the resolution means that $d_i(F_i) \subseteq (x_1, \dots, x_n)F_{i-1}$ for all i. The numbers $b_{i,p}^S(\mathbb{K}[\Delta(P)]) = |\{j : \ a_{ij} = p\}|$ with $p \in \mathbb{Z}^n$ are called the *graded Betti numbers* of $\mathbb{K}[\Delta(P)]$. By Theorem 1.1.2.19 it can be easily seen that the minimal graded free resolution is uniquely determined up to isomorphism.

Lemma 1.2.4.1. [BH93, Corollary 5.5.2] *The shifts a_{ij} in the minimal graded free resolution of $\mathbb{K}[\Delta(P)]$ are square-free.*

CHAPTER
TWO

GORENSTEIN IDEALS OF CODIMENSION 3

In this chapter, we clarify minimal graded free resolutions of Gorenstein ideals of codimension 1, 2 and 3, respectively. In [BE77] Buchsbaum and Eisenbud study the structure of these resolutions of Gorenstein ideals of codimension 3. Then we discuss corresponding combinatorial concepts. We explain Gale diagrams of simplicial d-polytopes with $d+k$ vertices, for $k = 1, 2, 3$, here we follow mainly [Grü03, Section 5.4 and Chapter 6] and [Zie95, Section 6.5]. After that we consider Stanley-Reisner rings associated to simplicial d-polytopes with $d+3$ vertices and corresponding Stanley-Reisner Gorenstein ideals of codimension 3. We explain how we can determine minimal sets of monomial generators of these ideals using Gale diagrams. Then we conclude the chapter by descriping minimal graded free resolutions of Stanley-Reisner ideals of codimension 3.

2.1 Gorenstein ideals of codimension 3 in commutative Algebra

Let S be a regular graded polynomial ring with $\mathfrak{m}$ its homogeneous maximal ideal. An ideal I of S is called *Gorenstein ideal* of grade g (codimension g) if I is perfect and $\operatorname{Ext}^g_S(S/I, S) \cong S/I$. Serre proved that an ideal of codimension 1 is Gorenstein if and only if it is principal and showed that if an ideal has codimension 2, then it is a complete intersection, see [Eis95, Corollary 21.20]. According to [Bur68] and [BH93, Theorem 1.4.17], Hilbert-Burch proved a structure theorem for codimension 2. If R is a regular local ring, and I is an ideal of codimension 2 in R requiring n generators, then I is perfect if and only if I is generated by the $n \times n$-minors of an $n \times (n+1)$ matrix. For Gorenstein ideals of codimension 3 there exists a structure theorem due to Buchsbaum and Eisenbud [BE77], which we describe in next Subsections 2.1.1 and 2.1.2.

2.1.1 Gorenstein ideals of codimensin 1 and 2

As a starting point we determine the minimal number of monomial generators of Gorenstein ideals in codimension 1 and 2. Then we describe the structure of the minimal graded free resolution of S/I over S, where $S = \mathbb{K}[x_1, \dots, x_n]$ and I is a Gorenstein ideal of codimension 1 respectively 2.

Corollary 2.1.1.1. [Eis95, Corollary 21.20]*(Serre) Let S be a regular graded polynomial ring with $\mathfrak{m}$ its homogeneous maximal ideal, and I be an ideal of S.*

1. *If* $\operatorname{codim}(I) = 1$*, then S/I is a Cohen-Macaulay ring if and only if S/I is a Gorenstein ring if and only if I is principal.*
2. *If* $\operatorname{codim}(I) = 2$*, then S/I is a Gorenstein ring if and only if I is generated by a S-regular sequence of length* 2.

For the case where I is Gorenstein ideal and has codimension 1, we have $I = (a)$, with $a \in S = \mathbb{K}[x_1, \dots, x_n]$ and $\deg(a) = k$. We use Construction 1.1.2.10 to consturct the minimal graded free resolution of S/I over S.

Step 0**:** Set $F_0 := S$ and let $d_0 : S \to S/I$.

Step 1**:** The element a is a homogeneous generator of $\operatorname{Ker}(d_0)$ of degree k. Set $F_1 := S(-k)$. Denote by f_1 the 1-generator of $S(-k)$. Hence $\deg(f_1) = k$. $d_1 : F_1 \to S$ be the homomorphism defined by $d_1(f_1) = a$ and $d_1(f_2) = y$. We obtain the beginning of the resolution:

$$S(-k) \xrightarrow{[a]} S \longrightarrow S/I \longrightarrow 0.$$

Step 2**:** We need to find homogeneous generators of $\operatorname{Ker}(d_1)$. Let $\alpha f_1 \in \operatorname{Ker}(d_1)$, with $\alpha \in S$. Hence $\alpha a = 0$, and it follows $\alpha = 0$. Thus $F_2 = 0$. We obtain the minimal graded free resolution of S/I over S.

$$0 \longrightarrow S(-k) \xrightarrow{[a]} S \longrightarrow S/I \longrightarrow 0.$$

By Theorem 1.1.3.9 we can write the Hilbert series of S/I

$$\operatorname{Hilb}_{S/I}(t) = \frac{1}{(1-t)^n} - \frac{t^k}{(1-t)^n} = \frac{1-t^k}{(1-t)^n}.$$

We observe that $b_1^S(S/I) = 1$. Then Theorem 1.1.5.8 implies that S/I is a Gorenstein ring, since I is perfect. Hence we can use Proposition 1.1.5.17 and the Hilbert series is

$$\operatorname{Hilb}_{S/I}(t) = \frac{t^{k-1} + t^{k-2} + \dots + 1}{(1-t)^{n-1}}.$$

If I is a Gorenstein ideal of codimension 2, then we can write $I = (a, b)$ with $a, b \in S$ and $\deg(a) = k_1$, $\deg(b) = k_2$. As above we construct the minimal graded free resolution of S/I over S.

Step 0**:** Set $F_0 := S$ and let $d_0 : S \to S/I$.

Step 1**:** The elements a and b are homogeneous generators of $\operatorname{Ker}(d_0)$. Their degrees are k_1 and k_2. Set $F_1 := S(-k_1) \oplus S(-k_2)$. Denote by f_1 and f_2 the 1-generators of $S(-k_1)$ and $S(-k_2)$, respectively. Hence $\deg(f_1) = k_1$ and $\deg(f_2) = k_2$. Let $d_1 : F_1 \to S$ be the homomorphism S-modules that is uniquely defined by $d_1(f_1) = a$ and $d_1(f_2) = b$. We obtain the resolution:

$$S(-k_1) \oplus S(-k_2) \xrightarrow{[a\ b]} S \longrightarrow S/I \longrightarrow 0.$$

Step 2: We need to find homogeneous generators of $\mathrm{Ker}(d_1)$. Let $\alpha f_1 + \beta f_2 \in \mathrm{Ker}(d_1)$, with $\alpha, \beta \in S$. Hence $\alpha a + \beta b = 0$, and it follows that the kernel of d_1 is generated by $-bf_1 + af_2$. Its degree is $k_1 + k_2$. Set $F_2 := S(-(k_1 + k_2))$. Denote by g_1 the 1-generator of $S(-(k_1 + k_2))$. Hence $\deg(g_1) = k_1 + k_2$. Let $d_2 : F_2 \to F_1$ be the homomorphism S-modules that is uniquely defined by $d_2(g_1) = -bf_1 + af_2$. We obtain the resolution:

$$S(-(k_1 + k_2)) \xrightarrow{[-b\ a]^t} S(-k_1) \oplus S(-k_2) \xrightarrow{[a\ b]} S \longrightarrow S/I \longrightarrow 0.$$

Step 3: We need to find homogeneous generators of $\mathrm{Ker}(d_2)$. Let $\lambda g_1 \in \mathrm{Ker}(d_2)$, with $\lambda \in S$. Hence $-\lambda bf_1 + \lambda af_2 = 0$, then it follows that $\lambda b = 0$ and $\lambda a = 0$. Thus $\lambda = 0$, and hence $F_3 = 0$. We obtain the minimal graded free resolution of S/I over S.

$$0 \longrightarrow S(-(k_1 + k_2)) \xrightarrow{[-b\ a]^t} S(-k_1) \oplus S(-k_2) \xrightarrow{[a\ b]} S \longrightarrow S/I \longrightarrow 0.$$

By Theorem 1.1.3.9 the Hilbert series of S/I takes the form

$$\mathrm{Hilb}_{S/I}(t) = \frac{1 - t^{k_1} - t^{k_2} + t^{k_1+k_2}}{(1-t)^n}.$$

Since $b_2^S(S/I) = 1$, S/I is a Gorenstein ring by Theorem 1.1.5.8. Hence we can use Proposition 1.1.5.17 and the Hilbert series becomes

$$\mathrm{Hilb}_{S/I}(t) = \frac{t^{(k_1+k_2-2)} + \ldots + 1}{(1-t)^{n-2}}.$$

2.1.2 Structure Theorem for Gorenstein ideals codimension 3

In [BE77] Buchsbaum and Eisenbud show that an ideal I of codimension 3 of S is Gorenstein if and only if I is an ideal of $(m-1)$-th order Pfaffians of some $m \times m$ skew-symmetric matrix A of rank $m-1$. For discussing that we recall some definitions from linear algebra.

Definition 2.1.2.1. Let S be a commutative ring and F be a finitely generated S-module. An S-module homomorphism $\varphi : F \to F^*$ is called *alternating* if the matrix A of φ with respect to some basis of F and its corresponding dual basis of F^* is skew-symmetric and all its diagonal elements are 0.

Definition 2.1.2.2. The determinant of a skew-symmertic matrix A can always be written as the square of a polynomial in the matrix entries. This polynomial is called the *Pfaffian* of the matrix and it denoted by $\mathrm{Pf}(A)$. The *$(m-1)$-th order Pfaffians* of an $m \times m$-skew-symmetric matrix is the determinant of the $(m-1) \times (m-1)$-submatrices obtained by deleting a row and the corresponding column of the matrix.

More about Pfaffians can be found in [Bou80].

Remark 2.1.2.3. The Pfaffian is zero for $(2m+1)\times(2m+1)$ skew-symmetric matrices (respectively $\operatorname{rank}(F)$ is odd) and for $2m \times 2m$ skew-symmetric matrices (respectively $\operatorname{rank}(F)$ is even) a homogeneous polynomial of degree m.

Theorem 2.1.2.4 (Buchsbaum-Eisenbud)**.** *Let S be a graded polynomial ring with $\mathfrak{m}$ its homogeneous maximal ideal and I an ideal of S. Then we obtain the following:*

1. *Suppose $2m+1 \geq 3$ is an integer and $\varphi : S^{2m+1} \longrightarrow S^{2m+1}$ an alternating homomorphism, such that the matrix A of φ has rank $2m$ and $\varphi(S^{2m+1}) \subseteq \mathfrak{m}S^{2m+1}$. If I is generated by the $(m-1)$-th order Pfaffians of the skew-symmetric matrix A, then* $\operatorname{grade}(I) \leq 3$. *If* $\operatorname{grade}(I) = 3$, *then the complex*

$$0 \longrightarrow S \longrightarrow S^{2m+1} \xrightarrow{A} S^{2m+1} \longrightarrow S \longrightarrow S/I \longrightarrow 0$$

   is acyclic and I is a Gorenstein ideal.

2. *Conversely, let I be a Gorenstein ideal of* $\operatorname{grade}(I) = 3$. *Then there exist a free module F of odd rank and an alternating homomorphism $\varphi : F \to F^*$, so that the matrix A of φ, such that*

$$0 \longrightarrow S \longrightarrow F \xrightarrow{A} F^* \longrightarrow S \longrightarrow S/I \longrightarrow 0$$

   is a minimal free S resolution of S/I over S. In particular, any Gorenstein ideal of grade 3 *is minimally generated by an odd number of $(m-1)$-th order Pfaffians of the skew-symmetric matrix A.*

Remark 2.1.2.5. For ideals I of Cohen-Macaulay rings $\operatorname{grade}(I)$ coincides with the codimension (=height=rank) of I, see Theorem 1.1.4.21.

Example 2.1.2.6. Let $S = \mathbb{K}[x,y,z]$ be the graded polynomial ring and $I = (xy,\ xz,\ yz,\ x^2 - y^2,\ x^2 - z^2)$. Since $R = S/I$ has finite length and the ideal (x,y,z) is the unique maximal ideal of R, the ring R is Artinian and local, and $\dim(R) = 0$. It follows from Theorem 1.1.4.21 that $\operatorname{height}(I) = 3$, since S is a Cohen-Macaulay ring by Proposition 1.1.4.7 and Theorem 1.1.4.20. Every element in (x,y,z) is zero divisor, hence there is no regular sequence in R, that means $\operatorname{depth}(R) = 0$, for that R is a Cohen-Macaulay ring. By Theorem 1.1.4.17 we have $\operatorname{proj.dim}_S(R) = 3$. So the minimal free resolution of S/I over S is

$$0 \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow S/I \longrightarrow 0.$$

Now we construct this minimal free resolution with the help of Construction 1.1.2.10.
Step 0**:** Set $F_0 := S$ and let $d_0 : S \to S/I$.
Step 1**:** The elements xy, xz, yz, $x^2 - y^2$, $x^2 - z^2$ are homogeneous generators of $\operatorname{Ker}(d_0)$. Their degrees are 2. Set $F_1 := S^5(-2)$. Denote by f_1, f_2, f_3, f_4, f_5 the 1-generators of $S(-2)$. Hence $\deg(f_i) = 2$ for $i = 1, \ldots, 5$. Let $d_1 : F_1 \to S$ be the homomorphism S-modules that is uniquely defined by $d_1(f_1) = xy$, $d_1(f_2) = xz$, $d_1(f_3) = yz$, $d_1(f_4) = x^2 - y^2$ and $d_1(f_5) = x^2 - z^2$. We obtain the beginning of the resolution:

$$S(-2)^5 \xrightarrow{[xy\ xz\ yz\ x^2-y^2\ x^2-z^2]} S \longrightarrow S/I \longrightarrow 0.$$

Step 2: We need to find homogeneous generators of $\mathrm{Ker}(d_1)$. Let $a_1f_1 + a_2f_2 + a_3f_3 + a_4f_4 + a_5f_5 \in \mathrm{Ker}(d_1)$, with $a_i \in S$ for $i = 1, \ldots, 5$. We want to solve the equation

$$a_1xy + a_2xz + a_3yz + a_4(x^2 - y^2) + a_5(x^2 - z^2) = 0,$$

where $a_i \in S$ for $i = 1, \ldots, 5$ are the unkowns. All solutions $(a_1, a_2, a_3, a_4, a_5)$ are generated by

$$v_1 = (0, -x, y, z, 0), \quad v_2 = (-z, 0, x, 0, 0), \quad v_3 = (-z, y, 0, 0, 0),$$

$$v_4 = (-x, 0, z, 0, y), \quad v_5 = (-y, z, 0, -x, x).$$

All v_i, for $i = 1, \ldots, 5$ linear independent, hence all solutions $(a_1, a_2, a_3, a_4, a_5)$ of the above equation are minimally generated by v_i, for $i = 1, \ldots, 5$.
Thus $g_1 = -xf_2 + yf_3 + zf_4$, $g_2 = -zf_1 + xf_3$, $g_3 = -zf_1 + yf_2$, $g_4 = -xf_1 + zf_3 + yf_5$ and $g_5 = -yf_1 + zf_2 - xf_4 + xf_5$ are homogeneous generators of $\mathrm{Ker}(d_1)$. Their degrees are $\deg(g_i) = 3$ for $i = 1, \ldots, 5$. Set $F_2 := S^5(-3)$. Denote by h_i for $i = 1, \ldots, 5$ the 1-generators of $S^5(-3)$. Hence $\deg(h_i) = 3$, for $i = 1, \ldots, 5$. Let $d_2 : F_2 \to F_1$ be the homomorphism S-modules that is uniquely defined by $d_2(h_i) = g_i$ for $i = 1, \ldots, 5$. We obtain the resolution:

$$S^5(-3) \xrightarrow{\begin{bmatrix} 0 & -x & y & z & 0 \\ x & 0 & -z & 0 & -y \\ -y & z & 0 & -x & x \\ -z & 0 & x & 0 & 0 \\ 0 & y & -x & 0 & 0 \end{bmatrix}} S^5(-2) \xrightarrow{[xy\ xz\ yz\ x^2-y^2\ x^2-z^2]} S \longrightarrow S/I \longrightarrow 0.$$

Step 3: We need to find homogeneous generators of $\mathrm{Ker}(d_2)$. Let $b_1h_1 + b_2h_2 + b_3h_3 + a_4h_4 + b_5h_5 \in \mathrm{Ker}(d_2)$, with $b_i \in S$ for $i = 1, \ldots, 5$. Hence $(-zb_2 - zb_3 - xb_4 - yb_5)f_1 + (-xb_1 + yb_3 + zb_5)f_2 + (yb_1 + xb_2 + zb_4)f_3 + (zb_1 - xb_5)f_4 + (yb_4 + xb_5)f_5 = 0$, and therefore b_i for $i = 1, \ldots, 5$ statisfy the equations

$$-zb_2 - zb_3 - xb_4 - yb_5 = 0, \quad -xb_1 + yb_3 + zb_5 = 0, \quad yb_1 + xb_2 + zb_4 = 0,$$

$$zb_1 - xb_5 = 0, \quad yb_4 + xb_5 = 0.$$

We solve these five equations and hence all solutions $(b_1, b_2, b_3, b_4, b_5)$ are minimally generated by only one generator

$$W_1 = (xy, -y^2 + z^2, x^2 - z^2, -xz, yz).$$

Thus $xyh_1 + (-y^2 + z^2)h_2 + (x^2 - z^2)h_3 - xzh_4 + yzh_5$ is a homogeneous generator of $\mathrm{Ker}(d_2)$. Its degree is 5. Set $F_3 := S(-5)$. Denote by q_1 the 1-generator of $S(-5)$. Hence $\deg(q_1) = 5$. Let $d_3 : F_3 \to F_2$ be the homomorphism S-modules that is uniquely defined by $d_3(q_1) = xyh_1 + (-y^2 + z^2)h_2 + (x^2 - z^2)h_3 - xzh_4 + yzh_5$. We obtain the resolution:

$$S(-5) \xrightarrow{[xy\ -y^2+z^2\ x^2-z^2\ -xz\ yz]^t} S^5(-3) \xrightarrow{A} S^5(-2) \xrightarrow{B} S \longrightarrow S/I \longrightarrow 0.$$

Step 4: Now we need to find homogeneous generators of $\mathrm{Ker}(d_3)$. Let $c_1 q_1 \in \mathrm{Ker}(d_3)$ with $c_1 \in S$. Hence

$$c_1(xyh_1 + (-y^2+z^2)h_2 + (x^2-z^2)h_3 - xzh_4 + yzh_5) = 0,$$

and therefore c_1 statisfies the equations

$$c_1 xy = 0,\quad c_1(-y^2+z^2) = 0,\quad c_3(x^2-z^2) = 0,\quad -c_1 xz = 0,\quad c_1 yz = 0.$$

We conclude that $c_1 = 0$ and thus $F_3 = 0$. We obtain the minimal graded free resolution

$$0 \longrightarrow S(-5) \xrightarrow{C^t} S^5(-3) \xrightarrow{A} S^5(-2) \xrightarrow{B} S \longrightarrow S/I \longrightarrow 0,$$

where

$$A = \begin{bmatrix} 0 & -x & y & z & 0 \\ x & 0 & -z & 0 & -y \\ -y & z & 0 & -x & x \\ -z & 0 & x & 0 & 0 \\ 0 & y & -x & 0 & 0 \end{bmatrix},$$

$$B = [xy\quad xz\quad yz\quad x^2-y^2\quad x^2-z^2] \text{ and } C = [xy\quad -y^2+z^2\quad x^2-z^2\quad -xz\quad yz].$$

Since $b_2^S(S/I) = 1$ and I is perfect, it follows from Theorem 1.1.5.8 that S/I is Gorenstein ring. Hence I is Gorenstein ideal by Proposition 1.1.5.16. Moreover we have A is 5×5-skew-symmetric matrix and all its diagonal elements are 0.

4×4-submatrices of A	4-th order Pfaffians of A
$A_1 = \begin{bmatrix} 0 & -z & 0 & -y \\ z & 0 & -x & x \\ 0 & x & 0 & 0 \\ y & -x & 0 & 0 \end{bmatrix}$	$\mathrm{Pf}(A_1) = xy$
$A_2 = \begin{bmatrix} 0 & y & z & 0 \\ -y & 0 & -x & x \\ -z & x & 0 & 0 \\ 0 & -x & 0 & 0 \end{bmatrix}$	$\mathrm{Pf}(A_2) = xz$
$A_3 = \begin{bmatrix} 0 & -x & z & 0 \\ x & 0 & 0 & -y \\ -z & 0 & 0 & 0 \\ 0 & y & 0 & 0 \end{bmatrix}$	$\mathrm{Pf}(A_3) = yz$
$A_4 = \begin{bmatrix} 0 & -x & y & 0 \\ x & 0 & -z & -y \\ -y & z & 0 & x \\ 0 & y & -x & 0 \end{bmatrix}$	$\mathrm{Pf}(A_4) = -x^2 + y^2$

$$A_5 = \begin{bmatrix} 0 & -x & y & z \\ x & 0 & -z & 0 \\ -y & z & 0 & -x \\ -z & 0 & x & 0 \end{bmatrix} \qquad \mathrm{Pf}(A_5) = x^2 - z^2$$

where A_i is the 4×4-submatrix of A obtained by deleting the i-th row and i-th column of A. That means that I is generated by an odd number of polynomials, namely the five 4-th order Pfaffians of the skew-symmetric matrix A.

In contrary to Gorenstein ideals of codimension 2, not all Gorenstein ideals of codimension 3 are complete intersection, as Example 2.1.2.6 shows.

Now we construct the minimal graded free resolution of S/I over S, where S is the graded polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ and I is a Gorenstein ideal of codimension 3. In this case I is generated by an odd number $2m+1$ of $2m$-th order Pfaffians $\mathrm{Pf}(A_i)$ of the $(2m+1) \times (2m+1)$-skew-symmetric matrix A. By Remark 2.1.2.3 we obtain that all $2m$-th order Pfaffians are homogeneous polynomials of degree m.

Step 0**:** Set $F_0 := S$ and let $d_0 : S \to S/I$.

Step 1**:** The elements $\mathrm{Pf}(A_i)$ are homogeneous generators of $\mathrm{Ker}(d_0)$, for $i = 1, \ldots, 2m+1$. Their degrees are $n_i := m$ for all $i = 1, \ldots, 2m+1$. Set $F_1 := \bigoplus_{i=1}^{2m+1} S(-n_i)$. Denote by f_i the 1-generators of $S(-n_i)$. Hence $\deg(f_i) = n_i$. Let $d_1 : F_1 \to S$ be the homomorphism S-modules that is uniquely defined by $d_1(f_i) = \mathrm{Pf}(A_i)$. We obtain the beginning of the resolution:

$$\bigoplus_{i=1}^{2m+1} S(-n_i) \xrightarrow{[\mathrm{Pf}(A_1)\ \ldots\ \mathrm{Pf}(A_{2m+1})]} S \longrightarrow S/I \longrightarrow 0.$$

Step 2**:** We need to find homogeneous generators of $\mathrm{Ker}(d_1)$. Let $\alpha_1 f_1 + \ldots + \alpha_{2m+1} f_{2m+1} \in \mathrm{Ker}(d_1)$, with $\alpha_i \in S$. Hence $\sum_{i=1}^{2m+1} \alpha_i \mathrm{Pf}(A_i) = 0$ and it follows that all solutions $(\alpha_1, \ldots, \alpha_{2m+1})$ are generated by the columns of the $(2m+1) \times (2m+1)$-skew-symmetric matrix A (see Buchsbaum-Eisenbud Theorem 2.1.2.4). Thus we obtain $2m+1$ homogeneous generators of $\mathrm{Ker}(d_1)$. Let their degrees be k_i. Set $F_2 := \bigoplus_{i=1}^{2m+1} S(-k_i)$. Denote by g_i the 1-generators of $S(-k_i)$. Hence $\deg(g_i) = k_i$. Let $d_2 : F_2 \to F_1$ be the homomorphism S-modules that is uniquely defined by $(g_i) \mapsto (i\text{-th row of } A)(f_1, \ldots, f_{2m+1})^t$. We obtain

$$\bigoplus_{i=1}^{2m+1} S(-k_i) \xrightarrow{A} \bigoplus_{i=1}^{2m+1} S(-n_i) \xrightarrow{[\mathrm{Pf}(A_1)\ \ldots\ \mathrm{Pf}(A_{2m+1})]} S \longrightarrow S/I \longrightarrow 0.$$

Step 3**:** We need to find homogeneous generators of $\mathrm{Ker}(d_2)$. Let $\sum_{i=1}^{2m+1} \lambda_i g_i \in \mathrm{Ker}(d_2)$, with $\lambda_i \in S$ and $i = 1, \ldots, 2m+1$. Hence $\sum_{i=1}^{2m+1} \lambda_i((i\text{-th row of } A)(f_1, \ldots, f_{2m+1})^t) = 0$. If we solve this equation, all solutions $(\lambda_1, \ldots, \lambda_{2m+1})$ are minimally generated by only one generator

$$W_1 = (\mathrm{Pf}(A_1), \ldots, \mathrm{Pf}(A_{2m+1})),$$

so there is one homogeneous generator $z := \sum_{i=1}^{2m+1} \mathrm{Pf}(A_i) g_i$ of $\mathrm{Ker}(d_2)$ of degree $r := \max\{(k_i + n_i) : \text{ for } i = 1, \ldots, 2m+1\}$. Set $F_3 := S(-r))$. Denote by h_1 the

1-generator of $S(-r)$. Hence $\deg(h_1) = r$. Let $d_3 : F_3 \to F_2$ be the homomorphism S-modules that is uniquely defined by $d_3(h_1) = z$. We obtain

$$S(-r) \xrightarrow{[\mathrm{Pf}(A_1)\ \dots\ \mathrm{Pf}(A_{2m+1})]^t} \bigoplus_{i=1}^{2m+1} S(-k_i) \xrightarrow{A}$$

$$\bigoplus_{i=1}^{2m+1} S(-n_i) \xrightarrow{[\mathrm{Pf}(A_1)\ \dots\ \mathrm{Pf}(A_{2m+1})]} S \longrightarrow S/I \longrightarrow 0.$$

Step 4: We need to find homogeneous generators of $\mathrm{Ker}(d_3)$. Let $\mu h_1 \in \mathrm{Ker}(d_3)$, with $\mu \in S$. Hence $\mu z = 0$. That means $\mu = 0$. So $F_4 = 0$. The minimal graded free resolution is

$$0 \longrightarrow S(-r) \xrightarrow{[\mathrm{Pf}(A_1)\ \dots\ \mathrm{Pf}(A_{2m+1})]^t} \bigoplus_{i=1}^{2m+1} S(-k_i) \xrightarrow{A}$$

$$\bigoplus_{i=1}^{2m+1} S(-n_i) \xrightarrow{[\mathrm{Pf}(A_1)\ \dots\ \mathrm{Pf}(A_{2m+1})]} S \longrightarrow S/I \longrightarrow 0.$$

By Theorem 1.1.3.9 we can write the Hilbert series of S/I in the form

$$\mathrm{Hilb}_{S/I}(t) = \frac{1 - \sum_{i=1}^{2m+1} t^{n_i} + \sum_{i=1}^{2m+1} t^{k_i} - t^r}{(1-t)^n}.$$

Example 2.1.2.7. Let S/I be the quotient module in Example 2.1.2.6. Then the Hilbert series of S/I is

$$\mathrm{Hilb}_{S/I}(t) = \frac{1 - 5t^2 + 5t^3 - t^5}{(1-t)^3} = 1 + 3t + t^2.$$

2.2 Combinatorics of Gorenstein ideals of codimension 3

We introduce the Gale transform and Gale diagram of simplicial d-polytopes with $d+3$ vertices, here we follow mainly [Grü03, Section 5.4 and Chapter 6], [Zie95, Section 6.5], [Stu88], [McM79] and [HRGZ97]. After that we consider Stanley-Reisner rings associated to simplicial d-polytopes with $d+3$ vertices and corresponding Gorenstein Stanley-Reisner ideals of codimension 3. We read off the minimal graded free resolutions of these ideals from Gale diagrams, that means that the Gale diagrams uniquely determine the minimal graded free resolutions.

2.2.1 Gale transforms and Gale diagrams

The Gale diagram construction assigns a finite set of vectors to a given polytope. We construct the Gale diagram of a polytope through a concept of a positive set of vectors and a new matrix. Many properties of polytopes can be read off from resulting correspondence between the sets of normalized vectors and polytopes. Using Gale diagrams allow us to visualize higher-dimensional polytopes as long as they do not have too many vertices.

We turn first to the description of the Gale transform and its properties, see [Gal] and [Grü03, Section 5.4]. Let $V = \{v_1, \dots, v_n\}$ with $\mathrm{aff}(V) = \mathbb{R}^d$ the affine hull of V. Let G be the real $(d+1) \times n$-matrix

$$G = \begin{bmatrix} 1 & 1 & \dots & 1 \\ v_1 & v_2 & \dots & v_n \end{bmatrix}.$$

Since $\dim(\mathrm{aff}(V)) = d$, the matrix G contains $d+1$ affinely independent rows (vectors), hence it follows that the rank of G is $d+1$. Let $\mathrm{Ker}(G)$ be the kernel of G. Then $\dim_{\mathbb{R}}(\mathrm{Ker}(G)) = n-d-1$. Let $B_1, \dots, B_{n-d-1} \in \mathbb{R}^n$ be a basis for the vector space $\mathrm{Ker}(G)$ and B be the $n \times (n-d-1)$-matrix with B_i as the i-th column for $i = 1, \dots, n-d-1$.

$$B = \begin{bmatrix} B_1 & B_2 & \dots & B_{n-d-1} \end{bmatrix}.$$

Definition 2.2.1.1. Let $\bar{v}_i \in \mathbb{R}^{n-d-1}$ be the i-th row of the matrix B. Then $\mathfrak{B} = \{\bar{v}_1, \dots, \bar{v}_n\}$ is called *Gale transform* of $V = \{v_1, \dots, v_n\}$.

Definition 2.2.1.2. The *Gale diagram* of $V = \{v_1, \dots, v_n\}$ is defined as $\hat{\mathfrak{B}} = \{\hat{v}_1, \dots, \hat{v}_n\}$, where $\mathfrak{B} = \{\bar{v}_1, \dots, \bar{v}_n\}$ is the Gale transform of V and where

$$\begin{cases} \hat{v}_i = 0 & \text{if } \bar{v}_i = 0, \\ \hat{v}_i = \frac{\bar{v}_i}{\|\bar{v}_i\|} & \text{if } \bar{v}_i \neq 0, \end{cases}$$

and $\| \bar{v}_i \|$ is the (Euclidean) length of the vector $\bar{v}_i$.

Let P be a polytope with the vertex set V. Throughout this thesis, "Gale diagram of the polytope P" is understood as the Gale diagram of the vertex set V of P .

Example 2.2.1.3. Let P be the 3-polytope with the vertex set $V = \{v_1 = (1,0,-2),\ v_2 = (-1,-1,-2),\ v_3 = (-1,1,-2),\ v_4 = (-1,1,2),\ v_5 = (-1,-1,2),\ v_6 = (1,0,2)\}$, see Figure 2.1.

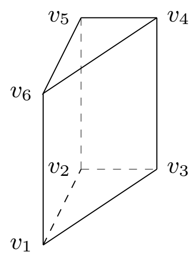


Figure 2.1: Illustration of the polytope P.

We consider the following 4×6-matrix

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ -2 & -2 & -2 & 2 & 2 & 2 \end{bmatrix}.$$

So $\dim_{\mathbb{R}}(\mathrm{Ker}(G)) = 2$ and $\{(0, -1, 1, -1, 1, 0), (-1, 0, 1, -1, 0, 1)\}$ is a basis for $\mathrm{Ker}(G)$. Let B be the 6×2-matrix with these two base vectors as columns.

$$B^t = \begin{bmatrix} 0 & -1 & 1 & -1 & 1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}.$$

Then the Gale transform $\mathfrak{B}$ of V is the set of vectors $\bar{v}_1 = (0,-1)$, $\bar{v}_2 = (-1,0)$, $\bar{v}_3 = (1,1)$, $\bar{v}_4 = (-1,-1)$, $\bar{v}_5 = (1,0)$, $\bar{v}_6 = (0,1)$, hence the Gale diagram of the polytope P is $\hat{\mathfrak{B}} = \{\hat{v}_1 = (0,-1),\ \hat{v}_2 = (-1,0),\ \hat{v}_3 = (\frac{1}{2},\frac{1}{2}),\ \hat{v}_4 = (-\frac{1}{2},-\frac{1}{2}),\ \hat{v}_5 = (1,0),\ \hat{v}_6 = (0,1)\}$, see Figure 2.2.

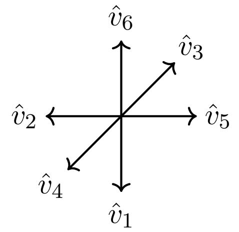


Figure 2.2: Gale diagram of the polytope P.

Let P be a d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$. Now we explain how to read off all faces of P from the Gale diagram of the polytope P.

Lemma 2.2.1.4. [Gal, Lemma 5.4] *Let P be a d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$. Then for any subset J of $\{1, \dots, n\}$, $F := \mathrm{conv}(\{v_j : j \in J\})$ is a face of P if and only if*

$$\mathrm{conv}(\{v_j :\ j \in \{1, \dots, n\} \smallsetminus J\}) \cap \mathrm{aff}(\{v_j :\ j \in J\}) = \varnothing.$$

Definition 2.2.1.5. Let P be a d-polytope. The *interior* of P, denoted by $\mathrm{int}(P)$, is the set of all points v in the polytope such that exists an $\epsilon > 0$ so that $B_\epsilon(v) \subseteq S$, where

$$B_\epsilon(v) = \{y \in \mathbb{R}^d : \| y - v \| < 0\}.$$

Remark 2.2.1.6. A polytope has a nonempty interior if and only if it is full-dimensional.

Definition 2.2.1.7. Let P be a d-polytope. The *relative interior* of P, denoted by $\mathrm{relint}(P)$, is the set of all points v in the polytope such that exists an $\epsilon > 0$ with $B_\epsilon(v) \cap \mathrm{aff}(P) \subseteq S$.

Remark 2.2.1.8. The relative interior of convex polytopes is a nonempty set and if the polytope P is full-dimensional, then $\mathrm{int}(P) = \mathrm{relint}(P)$.

Theorem 2.2.1.9. [BG69, Theorem 19] *Let P be a d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$ and $\hat{\mathfrak{B}} = \{\hat{v}_1, \dots, \hat{v}_n\}$ be the Gale diagram of P. Then for any subset J of $\{1, \dots, n\}$, $F := \mathrm{conv}(\{v_j : j \in J\})$ is a face of P if and only if either $J = \{1, \dots, n\}$ or $0 \in \mathrm{relint}(\mathrm{conv}(\{\hat{v}_k :\ k \notin J\}))$.*

Example 2.2.1.10. We read off all faces of the triangular prism P (see Figure 2.1) from the Gale diagram of P (see in Figure 2.2). We note that the $0 \in \mathrm{relint}(\mathrm{conv}(\{\hat{v}_k : k \neq i, \text{ for } i = 1, \ldots, 6\}))$. This implies that v_1, v_2, v_3, v_4, v_5 and v_6 are faces of P. Now let us find the edges of P. In this case, we take all pairs $v_i v_j$ such that $0 \in \mathrm{relint}(\mathrm{conv}(\{\hat{v}_k : k \neq i, j\}))$. For example, $v_5 v_6$ is an edge of P since $0 \in \mathrm{relint}(\mathrm{conv}(\{\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4\}))$. However, $v_1 v_5$ is not an edge of P since $0 \notin \mathrm{relint}(\mathrm{conv}(\{\hat{v}_2, \hat{v}_3, \hat{v}_4, \hat{v}_6\}))$. So we can determine all the other edges. Now let us find the facets of P. $v_1 v_2 v_3$ is a face (facet) because $0 \in \mathrm{relint}(\mathrm{conv}(\{\hat{v}_4, \hat{v}_5, \hat{v}_6\}))$, but $v_1 v_2 v_4$ is not a face because $0 \notin \mathrm{relint}(\mathrm{conv}(\{\hat{v}_3, \hat{v}_5, \hat{v}_6\}))$. In this way we can determine all faces (facets).

Corollary 2.2.1.11. *Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \ldots, v_n\}$ and the boundary complex $\Delta(P)$. Let $\hat{\mathfrak{B}} = \{\hat{v}_1, \ldots, \hat{v}_n\}$ be the Gale diagram of P. Then $F := \mathrm{conv}(\{v_j : j \in J \subseteq \{1, \ldots, n\}\})$ is a nonface of $\Delta(P)$ if and only if $0 \notin \mathrm{relint}(\mathrm{conv}(\{\hat{v}_k : k \notin J\}))$.*

Proof. See Theorem 2.2.1.9. □

Remark 2.2.1.12. Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \ldots, v_n\}$ and the boundary complex $\Delta(P)$. Let $\hat{\mathfrak{B}} = \{\hat{v}_1, \ldots, \hat{v}_n\}$ be the Gale diagram of P. Then $F := \mathrm{conv}(\{v_j : j \in J \subseteq \{1, \ldots, n\}\})$ is a minimal nonface of $\Delta(P)$ if and only if $0 \notin \mathrm{relint}(\mathrm{conv}(\{\hat{v}_k : k \notin J\}))$ for a maximal set $\{\hat{v}_k : k \notin J\}$.

2.2.2 Gale diagrams of polytopes with few vertices

Let P and P' be two d-polytopes. We define P and P' to be *combinatorially equivalent*, if there is a bijection between their faces that preserves the inclusion relation. If P and P' are combinatorially equivalent, we write $P \simeq P'$. To explain this definition we use Theorem 2.2.1.9.

Definition 2.2.2.1. Let $V = \{v_1, \ldots, v_n\}$ and $V' = \{v'_1, \ldots, v'_n\}$ be the vertex sets of P and P', respectively. Then we say that P and P' are *combinatorially equivalent* under a bijection φ of $\mathcal{F}(P)$ and $\mathcal{F}(P')$ such that $v'_{\vartheta(i)} = \varphi(v_i)$ for $i = 1, \ldots, n$ and a permutation ϑ of $1, \ldots, n$, if and only if for every $J \subset \{1, \ldots, n\}$, the condition $0 \in \mathrm{relint}(\mathrm{cov}(\{\bar{v}_j : j \in J\}))$ is equivalent to $0 \in \mathrm{relint}(\mathrm{cov}(\{\bar{v'}_{\vartheta(j)} : j \in J\}))$. If this condition holds, we say that the Gale transforms $\mathfrak{B}$ and $\mathfrak{B}'$ of P and P', respectively, are *isomorphic* and write $\mathfrak{B} \simeq \mathfrak{B}'$.

Remark 2.2.2.2. Two polytopes P and P' are combinatorially equivalent if and only if the Gale transforms of the sets of vertices are isomorphic.

Definition 2.2.2.3. The Gale diagrams are *isomorphic* if and only if the associated Gale transforms are isomorphic. This is denoted by $\hat{\mathfrak{B}} \simeq \hat{\mathfrak{B}}'$.

We study Gale diagrams of polytopes with few vertices, d-polytopes with only d-plus-a-few vertices. Every d-polytope with $d+1$ vertices is a d-simplex. Therefore it is known that there are exactly $\lfloor d^2/4 \rfloor$ combinatorial types of d-polytopes with $d+2$ vertices, among these, $\lfloor d/2 \rfloor$ different combinatorial types are simplicial d-polytopes with $d+2$ vertices, see [BG69], [Grü03], [Som58], [Sch05] and [Ewa96].

The Gale diagrams of any d-polytope with $d+1$ vertices is in the 0-dimensional space $\mathbb{R}^0$, so all vectors are 0-vector (only one point), therefore the Gale diagram for any d-polytope with $d+2$ vertices is in 1-dimensional space. Since the vectors of the Gale diagram are normalized vectors of the Gale transform of the vertex set and are elements of the 1-dimensional space $\mathbb{R}$, they are contained in the set $\{0, 1, -1\}$, those three points having multiplicities m_0, m_1, m_{-1} assigned in such a way that $m_0 \geq 0$, $m_1 \geq 2$, $m_{-1} \geq 2$, and $m_0 + m_1 + m_{-1} = d + 2$, see [Grü03, Theorems 5.4.2 and 5.4.3] or [Stu88, Theorem 2.4].

Definition 2.2.2.4. Let P and P' be d-polytopes with $d+2$ vertices, let (m_0, m_1, m_{-1}) and (m'_0, m'_1, m'_{-1}) be the associated multiplicities. We say that P and P' are *combinatorially equivalent* if either $(m_0, m_1, m_{-1}) = (m'_0, m'_1, m'_{-1})$ or $(m_0, m_1, m_{-1}) = (m'_0, m'_{-1}, m'_1)$.

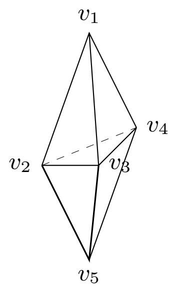
Remark 2.2.2.5. Let P be a d-polytope with $d+2$ vertices, then P is simplicial if and only if $m_0 = 0$.

Proof. This follows directly from the definition of a d-simplicial polytope and Corollary 2.2.1.9. $\square$

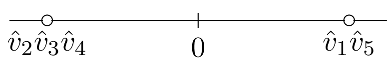
Example 2.2.2.6. Let P be a 3-polytope with the vertex set $V = \{v_1 = (0,0,1),\ v_2 = (-1,0,0),\ v_3 = (1,-1,0),\ v_4 = (0,1,0),\ v_5 = (0,0,-1)\}$, see Figure 2.3a. We consider the following 4×5-matrix

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

The kernel of G is generated by $(3, -2, -2, -2, 3)$ and hence $\dim_{\mathbb{R}}(\mathrm{Ker}(G)) = 1$. Then the Gale transform $\mathfrak{B}$ of V is the set of the vectors $\bar{v}_1 = 3$, $\bar{v}_2 = -2$, $\bar{v}_3 = -2$, $\bar{v}_4 = -2$, $\bar{v}_5 = 3$, hence the Gale diagram of P is $\hat{\mathfrak{B}} = \{\hat{v}_1 = 1, \hat{v}_2 = -1, \hat{v}_3 = -1, \hat{v}_4 = -1, \hat{v}_5 = 1\}$, see Figure 2.3b. The associated multiplicities are $m_{-1} = 2$, $m_0 = 0$ and $m_1 = 3$.



(a) Illustration of the 3-polytope P with 5 vertices.



(b) Gale diagram of P.

Figure 2.3: Simplicial 3-polytope P with 5 vertices and the Gale diagram of P.

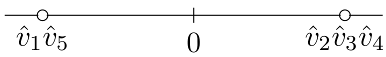


Figure 2.4: Combinatorially equivalent Gale diagram to Figure 2.3b.

We turn now to the much more interesting discussion of d-polytopes with $d+3$ vertices. Explicit formulas for the number of d-polytopes with $d+3$ vertices were obtained by Lloy, see [Llo70]. In this thesis, we are interested in simplicial polytopes, hence we take now simplicial d-polytopes with $d+3$ vertices. This case has been studied in [Grü03, Chapter 6]. The Gale diagram for any such polytope is a set of vectors of a 2-dimensional space $\mathbb{R}^2$. Since the vectors of the Gale diagram are normalized vectors of the Gale transform of the set of vertices, then the Gale diagrams of these polytopes are contained in the set $C^+ = \{0\} \cup C$, where C denotes the unit circle centered at the origin 0 of $\mathbb{R}^2$. When drawing a Gale diagram $\hat{\mathfrak{B}}$ of a polytope, we show that all vectors of $\hat{\mathfrak{B}}$ are diameters of the unit circl C and all these diameters have at least one endpoint in $\hat{\mathfrak{B}}$. If a point of $\hat{\mathfrak{B}}$ has k multiplicity for $k > 1$, then it will be marked by k. For a more extensive treatment see [Grü03, Chapter 6].

Definition 2.2.2.7. Each combinatorial type of d-polytopes with $d+3$ vertices has representatives for which the consecutive diameters of its Gale diagram are equidistant. We shall call such Gale diagrams *standard diagrams*. That is 0 is not a vector in the Gale diagram.

Example 2.2.2.8. The standard diagram of the triangular prism in Example 2.2.1.3 is

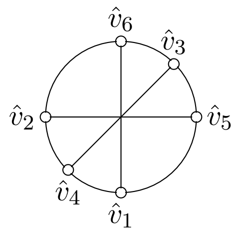


Figure 2.5: Standard diagram of the triangular prism, see Figure 2.1.

Theorem 2.2.2.9. *Let P and P' be d-polytopes with $d+3$ vertices and let $\hat{\mathfrak{B}}$ and $\hat{\mathfrak{B}}'$ be Gale diagrams (without loss of generality Standard diagrams) of P, P', respectively. If the only difference between $\hat{\mathfrak{B}}$ and $\hat{\mathfrak{B}}'$ is in the position of one of the diameters, so that its position in $\hat{\mathfrak{B}}'$ being obtained by rotating the corresponding diameter in $\hat{\mathfrak{B}}$ through an angle sufficiently small not to meet any other diameter, then $\hat{\mathfrak{B}}$ and $\hat{\mathfrak{B}}'$ are isomorphic.*

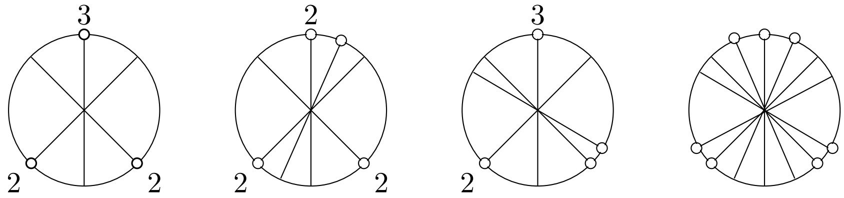
Proof. Let $V = \{v_1, \ldots, v_{d+3}\}$ and $V' = \{v'_1, \ldots, v'_{d+3}\}$ be the vertex sets of P and P', respectively. Since the only difference between $\hat{\mathfrak{B}}$ and $\hat{\mathfrak{B}}'$ is in the position of one of the diameters, we can put without loss of generality $\hat{v'}_j = \hat{v}_j$, for $i = 2, \ldots, d+3$ and $\hat{v'}_1$ is obtained by rotating the corresponding diameter $\hat{v}_1$ in $\hat{\mathfrak{B}}$ through an angle sufficiently small not to meet any other diameter. We prove now that $\hat{\mathfrak{B}} \simeq \hat{\mathfrak{B}}'$ by induction on the cardinality of the set $J \subseteq \{1, \ldots, d+3\}$. If $1 \notin J$, then there is nothing to pove it. Therefore we consider that $1 \in J$. For $|J| = 2$ we have $J = \{1, j\}$, for $j \in \{2, \ldots, d+3\}$. If $0 \in \mathrm{relint}(\mathrm{cov}(\{\hat{v}_1,\, \hat{v}_j\}))$, then $\hat{v}_1$ and $\hat{v}_j$ are two endpoints of a diameter. By rotating this diameter through a very small angle, 0 lies in the convex hull of $\hat{v'}_1$ and $\hat{v'}_j$. For $|J| = 3$ we have $J = \{1, j_1, j_2\}$, for $j_1, j_2 \in \{2, \ldots, d+3\}$ with $j_1 \neq j_2$. If $0 \in \mathrm{relint}(\mathrm{cov}(\{\hat{v}_1,\, \hat{v}_{j_1},\, \hat{v}_{j_2}\}))$. By rotating the corresponding diameter of $\hat{v}_1$ through a very small angle not to meet any other diameter, $0 \in \mathrm{relint}(\mathrm{cov}(\{\hat{v'}_1,\, \hat{v'}_{j_1},\, \hat{v'}_{j_2}\}))$, see Figure 2.6.

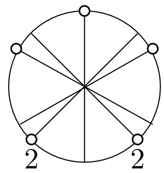
(a) Standard diagram of P. **(b)** Standard diagram of P'.

Figure 2.6: Standard diagrams.

The induction hypothesis is that the statement holds for $3 < |J| \leq d+1$. Now we prove the statement for $|J| = d+2$. If $0 \in \mathrm{relint}(\mathrm{cov}(\{\hat{v}_1,\, \hat{v}_j :\ \text{for all } j \in J\}))$, then there are three points of them by Carathéodory's theorem such that 0 lies in their convex hull. Assume that $0 \in \mathrm{relint}(\mathrm{cov}(\{\hat{v}_1,\, \hat{v}_{j_1}, \hat{v}_{j_2}\}))$, then the statement is true by the base step, see Figure 2.6. If $0 \in \mathrm{relint}(\mathrm{cov}(\{\hat{v}_{j_1},\, \hat{v}_{j_2},\, \hat{v}_{j_3} :\ j_i \neq 1, \text{ for } i = 1,2,3\}))$, then by rotating the corresponding diameter of $\hat{v}_1$ through a very small angle not to meet any other diameter, 0 stays in $\mathrm{relint}(\mathrm{cov}(\{\hat{v'}_{j_1},\, \hat{v'}_{j_2},\, \hat{v'}_{j_3} :\ j_i \neq 1, \text{ for } i = 1,2,3\}))$, hence $0 \in \mathrm{relint}(\mathrm{cov}(\{\hat{v'}_1,\, \hat{v'}_j :\ \text{for all } j \in J\}))$. For $|J| = d+3$ there is nothing to pove it. □

Example 2.2.2.10.





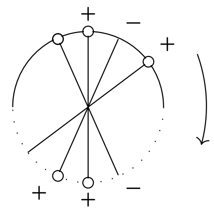
By Theorem 2.2.2.9 the first four Gale diagrams (standard diagrams) are isomorphic, but the fifth is not isomorphic to any of them.

Definition 2.2.2.11. Each combinatorial type of d-polytopes with $d+3$ vertices may be represented by a Gale diagram. If a Gale diagram has the least possible number of diameters among all isomorphic diagrams, then it is called *contracted*. If a Gale diagram has the largest possible number of diameters among all isomorphic diagrams, then it is called *distended*.

Example 2.2.2.12. The first Gale diagram in Example 2.2.2.10 is contracted and the fourth is distended. We note that the points of the contracted diagram are situated on alternate endpoints of the diameters.

Theorem 2.2.2.13. *Let P be a simplicial d-polytope with $d+3$ vertices and $\hat{\mathfrak{B}}$ be its Gale diagram. Then $0 \notin \hat{\mathfrak{B}}$, and no diameter of $\hat{\mathfrak{B}}$ has both endpoints in $\hat{\mathfrak{B}}$. Therefore the contracted Gale diagram of P has an odd number (bigger or equal 3) of diameters, the points of $\hat{\mathfrak{B}}$ being situated on alternate endpoints of the diameters.*

Proof. Since P is a simplicial d-polytop, each facet has exactly d vertices. If $0 \in \hat{\mathfrak{B}}$, then $0 \in \operatorname{relint}(\operatorname{conv}(\{0\}))$. In this case there is a facet which has $d+2$ vetices. If there is a diameter of $\hat{\mathfrak{B}}$ which has both endpoints, such as $\hat{v}_1$ and $\hat{v}_2$, then $0 \in \operatorname{relint}(\operatorname{conv}(\{\hat{v}_1, \hat{v}_2\}))$. In this case there is a facet which has $d+1$ vetices. Since the points of a contracted diagram $\hat{\mathfrak{B}}$ being situated on alternate endpoints of the diameters, see Theorem 2.2.2.9, Example 2.2.2.10 and Example 2.2.2.12], we have an odd number (bigger or equal 3) of diameters. We assume the number of diameters is even and thus equal to $2k$ for some $k \in \mathbb{N}$. Then we obtain the following diagram.



This is a contradiction, since there is no diameter of $\hat{\mathfrak{B}}$ that has both endpoints in $\hat{\mathfrak{B}}$. □

2.2.3 Stanley-Reisner rings associated to simplicial d-polytopes with $d+3$ vertices

Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_{d+3}\}$ and the boundary complex $\Delta(P)$. Let $\mathbb{K}$ be a field and $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_{d+3}]/I_{\Delta(P)}$ be the Stanley-Reisner ring of $\Delta(P)$, where $I_{\Delta(P)}$ is the Stanley-Reisner ideal. We will determine the minimal monomial set of generators of $I_{\Delta(P)}$, so that $I_{\Delta(P)}$ is a Gorenstein ideal of codimension 3. We would like to read off these generators from the Gale diagram of P. We know from Definition 1.2.3.1 and Remark 1.2.3.2 that the minimal set of monomial generators of $I_{\Delta(P)}$ are minimal nonfaces of $\Delta(P)$.

Example 2.2.3.1. Let P be a simplicial 3-polytope with the vertex set $V = \{v_1, \dots, v_5\}$, which is in Example 2.2.2.6 and the boundary complex $\Delta(P)$. Then the Stanley-Reisner ideal of the Stanley-Reisner ring $\mathbb{K}[\Delta(P)]$ is $I_{\Delta(P)} = (x_1x_5,\, x_2x_3x_4)$.

At the beginning we read off the minimal sets of monomial generators of Gorenstein ideals of codimension 1 and 2. Therefore we consider a simplicial d-polytope P with the vertex set $V = \{v_1, \dots, v_{d+1}\}$ and the boundary complex $\Delta(P)$. Let $\mathbb{K}[\Delta(P)]$ be the Stanley-Reisner ring of $\Delta(P)$ and $I_{\Delta(P)}$ the Gorenstein Stanley-Reisner ideal. By Proposition 1.1.5.16 $\mathbb{K}[\Delta(P)]$ is a Gorenstein ring. It follows from Proposition 1.1.5.13, Theorem 1.1.4.21 and Theorem 1.2.3.3 that $\operatorname{codim}(I_{\Delta(P)}) = 1$. The Gale diagram of a simplicial d-polytope with $d+1$ vertices is in the 0-dimensional space $\mathbb{R}^0$, so all vectors are the 0-vector trivially, see the Figure 2.7. This zero point has multiplicities $d+1$. By Theorem 1.1.5.8 we have $\operatorname{proj.dim}_S(\mathbb{K}[\Delta(P)]) = 1$ and $b_1^S(\mathbb{K}[\Delta(P)]) = 1$, hence the minimal graded free resolution of $\mathbb{K}[\Delta(P)]$ over S is

$$\mathbf{F}:\ 0 \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} \mathbb{K}[\Delta(P)] \longrightarrow 0.$$

By Remark 2.2.1.12 we have $I_{\Delta(P)} = (x_1 \dots x_{d+1})$. We use Construction 1.1.2.10, to obtain the minimal graded free resolution of $\mathbb{K}[\Delta(P)]$ over S is

$$\mathbf{F}:\ 0 \longrightarrow S(-(d+1)) \xrightarrow{d_1} S \xrightarrow{d_0} \mathbb{K}[\Delta(P)] \longrightarrow 0.$$

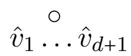


Figure 2.7: Gale diagram of a simplicial d-polytope with $d+1$ vertices.

Now we consider another case. Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_{d+2}\}$ and the boundary complex $\Delta(P)$. Let $\mathbb{K}[\Delta(P)]$ be the Stanley-Reisner ring of $\Delta(P)$ and $I_{\Delta(P)}$ the Gorenstein Stanley-Reisner ideal. By Proposition 1.1.5.16 $\mathbb{K}[\Delta(P)]$ is a Gorenstein ring. Proposition 1.1.5.13, Theorem 1.1.4.21 and Theorem 1.2.3.3 imply that $\operatorname{codim}(I_{\Delta(P)}) = 2$. The Gale diagram of a simplicial d-polytope with $d+2$ vertices is in the 1-dimensional space $\mathbb{R}$, see Figure 2.8. By Theorem 1.1.5.8 we have $\operatorname{proj.dim}_S(\mathbb{K}[\Delta(P)]) = 2$ and $b_2^S(\mathbb{K}[\Delta(P)]) = 1$, hence the minimal graded free resolution of $\mathbb{K}[\Delta(P)]$ over S is

$$\mathbf{F}:\ 0 \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} \mathbb{K}[\Delta(P)] \longrightarrow 0.$$

By Remark 2.2.1.12 we have $I_{\Delta(P)} = (x_{i_1}\dots x_{i_k}, x_{i_{k+1}}\dots x_{i_{d+2}})$. We construct now the minimal graded free resolution of $\mathbb{K}[\Delta(P)]$ over S

$$\mathbf{F}:\ 0 \longrightarrow S(-(d+2)) \xrightarrow{d_2} S(-((d+2)-k)) \oplus S(-k) \xrightarrow{d_1} S \xrightarrow{d_0} \mathbb{K}[\Delta(P)] \longrightarrow 0.$$

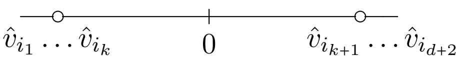


Figure 2.8: Gale diagram of a simplicial d-polytope with $d+2$ vertices.

For a more extensive treatement see [BP12].

We consider now an important case, let P be a simplicial d-polytope with the vertex set $V = \{v_1,\dots,v_{d+3}\}$ and the boundary complex $\Delta(P)$. Let $\mathbb{K}[\Delta(P)]$ be the Stanley-Reisner ring of $\Delta(P)$ and $I_{\Delta(P)}$ the Gorenstein Stanley-Reisner ideal. By Proposition 1.1.5.16 $\mathbb{K}[\Delta(P)]$ is a Gorenstein ring. It follows from Proposition 1.1.5.13, Theorem 1.1.4.21 and Theorem 1.2.3.3 that $\operatorname{codim}(I_{\Delta(P)}) = 3$. The Gale diagram of a simplicial d-polytope with $d+3$ vertices is in the 2-dimensional space $\mathbb{R}^2$, see Figure 2.9. We obtain by Theorem 1.1.5.8 that $\operatorname{proj.dim}_S(\mathbb{K}[\Delta(P)]) = 3$ and $b_3^S(\mathbb{K}[\Delta(P)]) = 1$, so that the minimal graded free resolution of $\mathbb{K}[\Delta(P)]$ over S is

$$\mathbf{F}:\ 0 \longrightarrow F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} \mathbb{K}[\Delta(P)] \longrightarrow 0.$$

By Remark 2.2.1.12 and Theorem 2.2.2.13 we observe that the Gorenstein ideal $I_{\Delta(P)}$ is generated by an odd number bigger or equal to 3 of monomials. We explain now how we can obtain them from Gale diagram of P. We consider at first the contracted Gale diagram of P, then an arbitrary straight line which passes through origin 0. This line split the point in two sets on its sides. Since the contracted Gale diagram has an odd number of diameters, and no diameter of the diagram has both endpoints, there is an odd number of different points in the contracted Gale diagram, i.e. $2m+1$ points for some natural number m. We observe that $2m+1 \le d+3$, because it may be that there is a point which has multiplicities greater than one. In all cases for every position i of the straight line, we have two sets of points $\hat{B'}_i$ and $\hat{B}''_i$ on its sides, for $i = 1,\dots,2m+1$. Let $|\hat{B'}_i| = a_i$ and $|\hat{B}''_i| = b_i$ with $a_i + b_i = d+3$. We have either $a_i > b_i$ or $b_i > a_i$. Without loss of generality we consider $b_i > a_i$. By Remark 2.2.1.12 we observe that $0 \notin \operatorname{relint}(\operatorname{conv}(\{\hat{v}_k : \text{ for all } \hat{v}_k \in \hat{B}''_i\}))$, for each $i = 1,\dots,2m+1$, see Figure 2.9. Then the Gorenstein ideal $I_{\Delta(P)}$ is generated by $2m+1$ sequare-free monomials f_i. Their degrees are a_i for $i = 1,\dots,2m+1$.

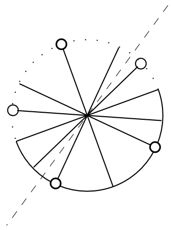


Figure 2.9: Gale diagram of P.

The minimal graded free resolution of $\mathbb{K}[\Delta(P)]$ over S is

$$\mathbf{F}: \quad 0 \longrightarrow S(-(d+3)) \xrightarrow{d_3} \bigoplus_{i=1}^{2m+1} S(-b_i) \xrightarrow{d_2} \bigoplus_{i=1}^{2m+1} S(-a_i) \xrightarrow{d_1}$$

$$S \xrightarrow{d_0} \mathbb{K}[\Delta(P)] \longrightarrow 0.$$

Remark 2.2.3.2. Let P be a simplicial d-polytope with $d+3$ vertices. Let $\Delta(P)$ be the boundary complex of P. We observe that all monomials of the minimal set of monomial generators of the Gorenstein Stanley-Reisner ideal $I_{\Delta(P)}$ have the same degree, see Remark 2.1.2.3 and Buchsbaum-Eisenbud Theorem 2.1.2.4. Since the number of diameters of the contracted Gale diagram of P is the minimal number of the monomial generators of $I_{\Delta(P)}$ and all have the same degree and no diameter of the diagram has both endpoints, we distinguish between two cases. The first one for $d = 2k$ even with $k \in \mathbb{N}$, then the contracted Gale diagram of P has $2k+3$ deffirent diameters. The second one for $d = 2k+1$ with $k \in \mathbb{N}$, then the contracted Gale diagram of P has $k+2$ different diameters.

Example 2.2.3.3. Let P be a simplicial 2-polytope with 5 vertices, such as $V = \{v_1 = (0,2), v_2 = (1,1), v_3 = (1,-1), v_4 = (-1,-1), v_5 = (-1,1)\}$, see Figure 2.10a, and the boundary complex $\Delta(P)$. We consider the 3×5-matrix

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 & -1 \\ 2 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

The kernel of G is generated by two vectors $(-2,3,-2,1,0)$ and $(-2,2,-1,0,1)$ and hence $\dim_{\mathbb{R}}(\mathrm{Ker}(G)) = 2$. Then the Gale transform $\mathfrak{B}$ of V is the set of the vectors $\bar{v}_1 = (-2,-2)$, $\bar{v}_2 = (3,2)$, $\bar{v}_3 = (-2,-1)$, $\bar{v}_4 = (1,0)$, $\bar{v}_5 = (0,1)$, hence the Gale diagram of P is $\hat{\mathfrak{B}} = \{\hat{v}_1 = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), \hat{v}_2 = (\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}), \hat{v}_3 = (\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{-5}}), \hat{v}_4 = (1,0), \hat{v}_5 = (0,1)\}$, see Figure 3.5f. We consider now the corresponding Stanley-Reisner ring $\mathbb{K}[\Delta(P)]$ of $\Delta(P)$, such that $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_5]/I_{\Delta(P)}$, where $I_{\Delta(P)}$ is the corresponding Gorenstein Stanley-Reisner ideal of codimension 3. We observe that the minimal set of monomial generators of $I_{\Delta(P)}$ is $\{x_1x_3,\ x_3x_5,\ x_2x_5,\ x_2x_4,\ x_1x_4\}$. These monomial generators are of degree 2.

(a) Illustration of the polytope P.

(b) Gale diagram of P.

Figure 2.10: Simplicial 2-polytope P with 5 vertices and its Gale diagram.

The minimal graded free resolution of $\mathbb{K}[\Delta(P)]$ over S is

$$\mathbf{F}: \quad 0 \longrightarrow S(-5) \xrightarrow{d_3} \bigoplus_{i=1}^{5} S(-3) \xrightarrow{d_2} \bigoplus_{i=1}^{5} S(-2) \xrightarrow{d_1} S \xrightarrow{d_0} \mathbb{K}[\Delta(P)] \longrightarrow 0.$$

We observe that the Buchsbaum-Eisenbud Theorem 2.1.2.4 is valid. The 5×5-skew symmetic matrix A of the map d_2 has the form

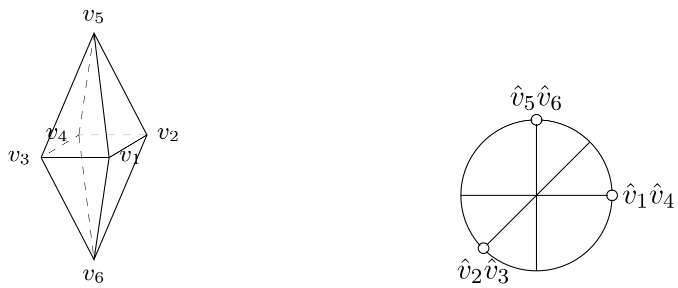
$$A = \begin{bmatrix} 0 & 0 & -x_3 & x_4 & 0 \\ 0 & 0 & 0 & -x_5 & x_1 \\ x_3 & 0 & 0 & 0 & -x_2 \\ -x_4 & x_5 & 0 & 0 & 0 \\ 0 & -x_1 & x_2 & 0 & 0 \end{bmatrix}$$

and $I_{\Delta(P)}$ is minimallly generated by the 4-th order Pfaffians of A.

Example 2.2.3.4. Let P be a simplicial 3-polytope with 6 vertices, such as $V = \{v_1 = (1,1,0), v_2 = (-1,1,0), v_3 = (1,-1,0), v_4 = (-1,-1,0), v_5 = (0,0,1), v_6 = (0,0,-1)\}$, see Figure 2.11a, and the boundary complex $\Delta(P)$. We consider the 4×6-matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

So $\dim_{\mathbb{R}}(\mathrm{Ker}(G)) = 2$ and $\{(1,-1,-1,1,0,0), (0,-1,-1,0,1,1)\}$ is a basis for $\mathrm{Ker}(G)$. Then the Gale transform $\mathfrak{B}$ of V is the set of the vectors $\bar{v}_1 = (1,0)$, $\bar{v}_2 = (-1,-1)$, $\bar{v}_3 = (-1,-1)$, $\bar{v}_4 = (1,0)$, $\bar{v}_5 = (0,1)$, $\bar{v}_6 = (0,1)$, hence the Gale diagram of P is $\hat{\mathfrak{B}} = \{\hat{v}_1 = (1,0),\ \hat{v}_2 = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}),\ \hat{v}_3 = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}),\ \hat{v}_4 = (1,0),\ \hat{v}_5 = (0,1),\ \hat{v}_6 = (0,1)\}$, see Figure 2.11b. We consider now the corresponding Stanley-Reisner ring $\mathbb{K}[\Delta(P)]$ of $\Delta(P)$, such that $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \ldots, x_6]/I_{\Delta(P)}$, where $I_{\Delta(P)}$ is the corresponding Gorenstein Stanley-Reisner ideal of codimension 3. We observe that the minimal set of monomial generators of $I_{\Delta(P)}$ is $\{x_1x_4,\ x_2x_3,\ x_5x_6\}$. These monomial generators are of degree 2. By Construction 1.1.2.10 we obtain that the minimal graded free resolution of $\mathbb{K}[\Delta(P)]$ over S is



(a) Illustration of the polytope P. **(b)** Gale diagram of P.

Figure 2.11: Simplicial 3-polytope P with 6 vertices and its Gale diagram.

$$\mathbf{F}: \quad 0 \longrightarrow S(-6) \xrightarrow{d_3} \bigoplus_{i=1}^{3} S(-4) \xrightarrow{d_2} \bigoplus_{i=1}^{3} S(-2) \xrightarrow{d_1} S \xrightarrow{d_0} \mathbb{K}[\Delta(P)] \longrightarrow 0.$$

We observe that the Buchsbaum-Eisenbud Theorem 2.1.2.4 is valid. The 3×3-skew symmetic matrix A of the map d_2 is the following

$$A = \begin{bmatrix} 0 & -x_1x_4 & -x_2x_3 \\ x_1x_4 & 0 & -x_5x_6 \\ x_2x_3 & x_5x_6 & 0 \end{bmatrix}$$

and $I_{\Delta(P)}$ is minimally generated by the 2-th order Pfaffians of A.

CHAPTER

THREE

GORENSTEIN IDEALS OF CODIMENSION 4

Let S be the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ over an algebraically closed field $\mathbb{K}$, graded in positive degrees. The structure of the minimal graded free resolution of the quotient module S/I, where I is a Gorenstein ideal of codimension 4, is still not fully understood. Some progress in this direction is due to Kustin and Miller, see [KM82] and [KM83]. The results of Kustin and Miller give an interesting application in the construction of new Calabi-Yau manifolds [Kap11] and the classification of singular Fano varieties [PR04]. In recent work of Reid [Rei15] the results of Kustin and Miller were developed further and he partially generalizes the Buchsbaum-Eisenbud Theorem [BE77]. In this chapter, we discuss the structure theorem of Reid. Then for $d = 3, 4$, we compute explicitly the minimal graded free resolution of the Stanley-Reisner rings associated to simplicial d-polytopes with $d + 4$ vertices using Gale diagrams.

In what follows, the minimal graded free resolution of a Stanley-Reisner ring is understood as the minimal graded free resolution of a quotient module.

3.1 Gorenstein ideals of codimension 4 in commutative Algebra

Generalizing of the Buchsbaum-Eisenbud theorem [BE77] in codimension 3 to codimension 4 has been a notoriously elusive problem since the 1970s. In 2015 this has been partially generalized by Reid [Rei15].

3.1.1 Structure theory for Gorenstein ideals of codimension 4

Let S be the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ over an algebraically closed field $\mathbb{K}$. The structure of the minimal graded free resolution of the quotient module S/I, where I is a Gorenstein ideal of codimension 4 and generated by $k + 1$ elements, is

$$\mathbf{F}: \quad 0 \longrightarrow F_4 \xrightarrow{d_4} F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} S/I \longrightarrow 0,$$

where $F_0 = S$, $F_4 = S$, $F_1 = S^{k+1}$, $F_3 = \mathrm{Hom}(F_1, F_4) \cong F_1^*$ and $F_2 = S^{2k}$. Moreover, $F_2 \longrightarrow F_1$ is dual to $F_3 \longrightarrow F_2$. By choice of appropriate bases of F_2 and F_3, we obtain the matrix A of d_2, which has the form

$$A = \begin{bmatrix} B & C \end{bmatrix},$$

where B and C are $(k+1) \times k$-matrices satisfying the following condition

$$\begin{bmatrix} B & C \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} B & C \end{bmatrix}^t = 0.$$

This is equivalent to $BC^t + CB^t = 0$ or to BC^t being a skew-symmetric matrix.

Example 3.1.1.1. Let $S = \mathbb{K}[x_1, \dots, x_8]$ be the polynomial ring and $I = (x_1, x_2, x_3, x_4)$ an ideal of S. We observe that $\dim(S/I) = 4$. Since S is a Cohen-Macaulay ring, it follows from Theorem1.1.4.21 that $\mathrm{codim}(I) = 4$ and from Proposition 1.1.5.12 that S/I is a Gorenstein ring. So it is Cohen-Macaulay. Moreover, by Theorem 1.1.4.17 we have $\mathrm{proj.dim}_S(S/I) = 4$. So I is perfect, and therefore I is a Gorenstein ideal by Proposition 1.1.5.16. Morover we observe that I is a complete intersection ideal. The minimal free resolution is

$$0 \longrightarrow S(-4) \xrightarrow{Q^t} S^4(-3) \xrightarrow{A^t} S^6(-2) \xrightarrow{A} S^4(-1) \xrightarrow{Q} S \longrightarrow S/I \longrightarrow 0,$$

where

$$A = \begin{bmatrix} -x_4 & 0 & 0 & 0 & x_3 & -x_2 \\ 0 & -x_4 & 0 & -x_3 & 0 & x_1 \\ 0 & 0 & -x_4 & x_2 & -x_1 & 0 \\ x_1 & x_2 & x_3 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}.$$

We observe that $A = \begin{bmatrix} B & C \end{bmatrix}$, where

$$B = \begin{bmatrix} -x_4 & 0 & 0 \\ 0 & -x_4 & 0 \\ 0 & 0 & -x_4 \\ x_1 & x_2 & x_3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore $BC^t + CB^t = 0$ and BC^t is the skew-symmetric matrix

$$BC^t = \begin{bmatrix} 0 & x_3x_4 & -x_2x_4 & 0 \\ -x_3x_4 & 0 & x_1x_4 & 0 \\ x_2x_4 & -x_1x_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.2 Combinatorics of Gorenstein ideals of codimension 4

3.2.1 Radial projection and stereographic projection

Projections are fundamental techniques in this subsection and in the next chapter. To introduce these projections we need the definition of a strongly convex spherical polytope on S^{d-1}.

Definition 3.2.1.1. Let S^{d-1} be the unit sphere in the Euclidean space centered at the origin 0. A *strongly convex spherical polytope* on S^{d-1} is a non-empty intersection of finitely many hemispheres on S^{d-1} that does not contain any pair of antipodal points.

Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$, the boundary complex $\Delta(P)$ and $0 \in \mathrm{int}(P)$. We want to apply a radial projection of the polytope P from the origin 0 onto the unit sphere S^{d-1}. Consider the *radial projection*

$$\phi : \Delta(P) \longrightarrow S^{d-1}, \quad v_i \mapsto r(v_i) \cap S^{d-1},$$

where $r(v_i)$ is the ray with endpoint 0 containing v_i. Put $v_i' := \phi(v_i)$ for all $v_i \in V$ and $V' := \{v_1', \dots, v_n'\}$. Then the image under ϕ of each proper face of P is a strongly convex spherical polytope on S^{d-1}. For a more extensive treatment see [She71]. After that we apply the stereographic projection at each point v_i' for $i = 1, \dots, n$, respectively. This projection is defined on the entire sphere, except at one point (the projection point). It preserves angles at which curves meet, but it preserves neither distances nor the areas of figures.

Definition 3.2.1.2. Let $d \geq 1$ and $v := (0, \dots, 0, 1) \in \mathbb{R}^d$. The *stereographic projection* is defined by

$$\psi : S^{d-1} \smallsetminus \{v\} \longrightarrow \mathbb{R}^{d-1} \times \{0\}, \quad x \mapsto (\frac{x_1}{1 - x_d}, \dots, \frac{x_{d-1}}{1 - x_d}, 0).$$

In this case we draw a line from the North Pole of the sphere (we could rotate the sphere such that v_i is at the pole); the line pass through both a point on the sphere and a point on the plane. A point on the sphere is mapped to the corresponding point on the plane.

Definition 3.2.1.3. Let $d > 1$ and let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$, so that $V' = \{v_1', \dots, v_n'\}$ is the image set of V on S^{d-1}. For every v_i' we rotate the sphere such that v_i' is at the pole, then we define the *vertex projection* Ψ to be the map induced by ψ on $V' \smallsetminus \{v_i'\}$.

Proposition 3.2.1.4. *Let P be a simplicial d-polytope with the vertex set $V \uplus \{v\}$, so that $V' \uplus \{v'\}$ is the image set of $V \uplus \{v\}$ on the S^{d-1}. Let T be the image of a vertex projection Ψ of P from v'. Then the boundary vertices of T are the vertices of a simplicial polytope P'', and for $U \subset V$, $Q := \mathrm{conv}(U \cup \{v\})$ is a k-face of P if and only if $F := \mathrm{conv}(\Psi(U'))$ is a $(k-1)$-face of a simplicial polytope P'' and $F \cap \Psi(V') = \psi(U')$, where U' is the image set of U on S^{d-1}.*

Proof. See [Gon12, Lemma 0.3.6] and [Gon12, Corollary 0.3.7]. □

Remark 3.2.1.5. Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \ldots, v_n\}$. We observe that the simplicial $(d-1)$-polytope P'', which we obtain in Proposition 3.2.1.4, has at most $n-1$ vertices. If the stereographic projection is at the projection point v_i', then this polytope is denoted by P_i, where its vertex set $V'' := \{v''_{i_1}, \ldots, v''_{i_k}\}$ with $v''_{i_l} := \Psi(\Phi(v_{i_l}))$.

3.2.2 Stanley-Reisner ideals of codimension 4 for $d = 3, 4$

In [GS67], it is shown that there are 43 combinatorial types of simplicial d-polytopes with $d+4$ vertices for $d \leq 4$. Let P be a simplicial d-polytope with the vertex set V and the boundary complex $\Delta(P)$. In this subsection, we determine the minimal set of monomial generators of the Stanley-Reisner ideal that is associated to P using Gale diagrams. After that we can compute the minimal free resolution of the corresponding Stanley Reisner ring using the computer algebra system Singular. That help us in the next Chapter 4 to prove that every Gorenstein ideal of codimension 4 which is generated by an even number of monomials is not a complete intersection[1]. of a Gorenstein ideal of codimension 3 and an extra monomial. Such ideals provide a counterexample to a conjecture of Reid in [Rei13] and [Rei15].

We consider now simplicial 3-polytopes with 7 vertices and determine the minimal set of monomial generators of the corresponding Stanley-Reisner ideals using Gale diagrams. These polytopes are classified and there are only 5 different combinatorial types, see [GS67].

Definition 3.2.2.1. Let P be a d-polytope, H a hyperplane such that $H \cap \operatorname{int}(P) = \varnothing$, and let v be a point in $\mathbb{R}^d$. We say that v is *beyond* H, if v belongs to the open halfspace determined by H which does not meet P. If F is a facet of P, we shall say that v is *beyond* F if v is beyond $\operatorname{aff}(F)$.

Definition 3.2.2.2. Let P be a d-polytope. A *Schlegel diagram* of P is a projection of a polytope from R^d into R^{d-1} through a point beyond one of its facets or faces.

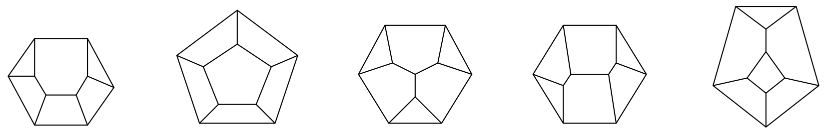
Remark 3.2.2.3. A Schlegel diagram of P based at the facet F is a polytopal subdivision of F in R^{d-1} that is combinatorially equivalent to the original polytope.

Example 3.2.2.4. The Schlegel diagrams of the two types of pentahedra (pyramids and roofs) are

Figure 3.1: Schlegel diagrams of 3-polytopes with 5 facets.

[1] See Definition 4.2.0.1.

Remark 3.2.2.5. The Schlegel diagrams of the combinatorial types of simplicial 3-polytopes with 7 vertices are



d_1 d_2 d_3 d_4 d_5

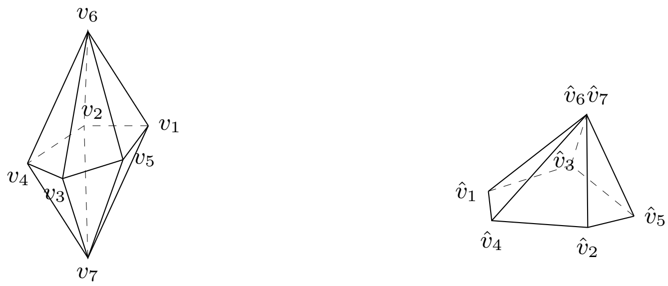
Let D_2 be one of the simplicial 3-polytopes with 7 vertices $V = \{v_1, \ldots, v_7\}$, which its Schlegel diagram d_2 is considered in Remark 3.2.2.5, such that $0 \in \operatorname{int}(D_2)$ and let $\Delta(D_2)$ be its boundary complex. Now we would like to determine the minimal set of monomial generators of the Gorenstein Stanley-Reisner ideal $I_{\Delta(D_2)}$ using the Gale diagram. To achieve this we use the following strategy. We apply a radial projection of D_2 from the origin 0 onto the unit sphere S^2. Let $V' = \{v'_1, \ldots, v'_7\}$ be the image of V under the radial projection. Then we use the stereographic projection at each point v'_i, for $i = 1, \ldots, 7$. By doing that, we get simplicial 2-polytopes D_{2_i} (see Remark 3.2.1.5), which have at most 6 vertices. After that we draw the Gale diagram of every $\operatorname{vert}(D_{2_i})$, then we determine the minimal set of monomial generators of every $I_{\Delta(D_{2_i})}$ using Gale diagrams, see Chapter 2. If two monomial generators of two different ideals $I_{\Delta(D_{2_i})}$ exist, which have at least two common divisors, such as $x_ix_jx_k$ and $x_ix_jx_l$, then $x_ix_jx_kx_l$ is a monomial generator of $I_{\Delta(D_2)}$. Otherwise all other monomial generators of all $I_{\Delta(D_{2_i})}$ are monomial generators of $I_{\Delta(D_2)}$, see Chapter 4. Then we can compute explicitly the minimal graded free resolution of the Stanley-Reisner ring associated to D_2 using the computer algebra system Singular.

By Remark 3.2.2.5 we have the Schlegel diagram of D_2, hence we can draw it. Let $V := \{v_1 = (-1,1,0), v_2 = (-1,-1,0), v_3 = (1,1,0), v_4 = (2,1,0), v_5 = (1,2,0), v_6 = (0,0,3), v_7 = (0,0,-3)\}$ be the vertex set of D_2, see Figure 3.3a.

On the one hand, we can determine immediately the set of minimal nonfaces of $\Delta(D_2)$, which is at the same time the minimal set of monomial generators of $I_{\Delta(D_2)}$, see Remark 1.2.3.2. On the other hand, we can determine them using Gale diagrams, because sometimes we have only the Gale diagram of the vertices set without any information about the polytope. Consider for example the 4×7-matrix

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & -3 \end{pmatrix}.$$

The set $\{(1,0,-3,2,0,0,0), (-1,1,-2,0,2,0,0), (0,-1,-1,0,0,1,1)\}$ is a basis for the kernel of G. Then the Gale transform $\mathfrak{B}$ of V is the set of the vectors $\bar{v}_1 = (1,-1,0)$, $\bar{v}_2 = (0,1,-1)$, $\bar{v}_3 = (-3,-2,-1)$, $\bar{v}_4 = (2,0,0)$, $\bar{v}_5 = (0,2,0)$, $\bar{v}_6 = (0,0,1)$, $\bar{v}_7 = (0,0,1)$, hence the Gale diagram of D_2 is the set $\hat{\mathfrak{B}} = \{\hat{v}_1 = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0), \hat{v}_2 = (0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), \hat{v}_3 = (\frac{-3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}), \hat{v}_4 = (1,0,0), \hat{v}_5 = (0,1,0), \hat{v}_6 = (0,0,1), \hat{v}_7 = (0,0,1)\}$, see Figure 3.3b.



(a) Illustration of D_2. **(b)** Gale diagram of D_2.

Figure 3.3: Simplicial 3-polytope D_2 with 7 vertices and its Gale diagram.

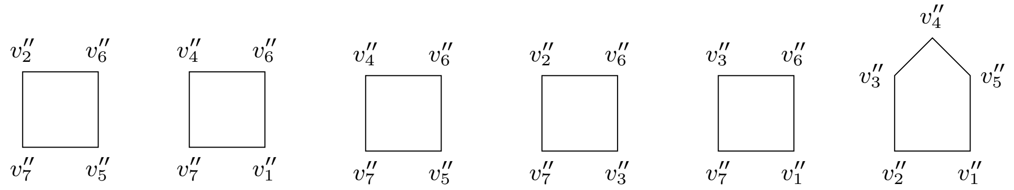
Now we assume that only the Gale diagram of D_2 is given and the polytope itself is not known. We can determine the faces of the polytope by Theorem 2.2.1.9 and using the computer algebra system Maple through the following commands

```
>  with(convex):

>  C := poshull([1/sqrt(2),-1/sqrt(2),0], [0,1/sqrt(2),-1/sqrt(2)],
                          [0,1,0], [0,0,1]);

>  containsrelint(C, [0,0,0]);

                                    true
```

We obtain that the faces of D_2 are:

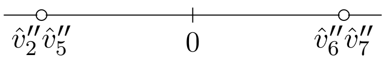
Polytope	**Facets**	
D_2	$v_1v_2v_6$	$v_1v_2v_7$
	$v_2v_4v_6$	$v_2v_4v_7$
	$v_3v_4v_6$	$v_3v_4v_7$
	$v_4v_5v_6$	$v_4v_5v_7$
	$v_1v_5v_6$	$v_1v_5v_7$

We apply a radial projection of D_2 from the origin onto the unit sphere S^3. Then we use the stereographic projection at each point v_i', for $i = 1, \dots, 7$. For each $i = 1, \dots, 7$ we obtain simplicial 2-polytopes D_{2i}, see Figure 3.4.

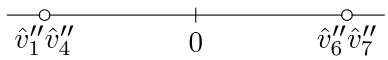


(a) D_{2_1} **(b)** D_{2_2} **(c)** D_{2_3} **(d)** D_{2_4} **(e)** D_{2_5} **(f)** $D_{2_6} = D_{2_7}$

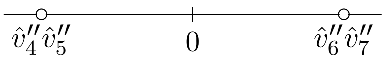
Figure 3.4: Resulting polytopes from stereographic projection.



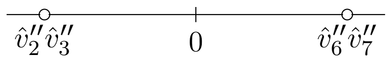
(a) Gale diagram of D_{2_1}.



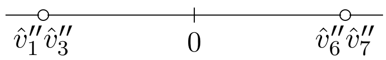
(b) Gale diagram of D_{2_2}.



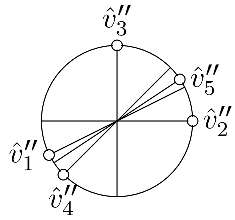
(c) Gale diagram of D_{2_3}.



(d) Gale diagram of D_{2_4}.



(e) Gale diagram of D_{2_5}.



(f) Gale diagram of $D_{2_{6,7}}$.

Figure 3.5: Gale diagrams of polytopes in Figure 3.4.

Now let $S = \mathbb{K}[x_1, \ldots, x_7]$ and $\mathbb{K}[\Delta(D_2)] = S/I_{\Delta(D_2)}$ be the Stanley-Reisner ring of $\Delta(D_2)$. The minimal sets of monomial generators of $I_{\Delta(D_{2_i})}$ for $i = 1, \ldots, 7$ are given in the following equations

$$
\begin{aligned}
I_{\Delta(D_{2_1})} &= (x_2x_5,\ x_6x_7),\\
I_{\Delta(D_{2_2})} &= (x_1x_4\,, x_6x_7),\\
I_{\Delta(D_{2_3})} &= (x_4x_5,\ x_6x_7),\\
I_{\Delta(D_{2_4})} &= (x_2x_3,\ x_6x_7),\\
I_{\Delta(D_{2_5})} &= (x_1x_3,\ x_6x_7),\\
I_{\Delta(D_{2_{6,7}})} &= (x_1x_4,\ x_4x_2,\ x_2x_5,\ x_5x_3,\ x_3x_1),
\end{aligned}
$$

see Chapter 2, Subsection 2.2.3. We can write now the minimal set of monomial generators of $I_{\Delta(D_2)}$

$$I_{\Delta(D_2)} = (x_1x_4,\, x_4x_2,\, x_2x_5,\, x_5x_3,\, x_3x_1,\, x_6x_7).$$

We use computer algebra system Singular to compute the minimal graded free resolution of $\mathbb{K}[\Delta(D_2)]$.

```
> ring R = 0,(x1, x2, x3, x4, x5, x6, x7), Dp;
> ideal I = (x1*x4, x2*x4, x2*x5, x3*x5, x1*x3, x6*x7);
> def s = res(I, 0);
> s;

 1      6      10      6      1
r <--  r <--  r <--   r <--  r

0      1      2       3      4

> list(s);
[1]:
   _[1] = x6*x7
   _[2] = x3*x5
   _[3] = x2*x5
   _[4] = x2*x4
   _[5] = x1*x4
   _[6] = x1*x3
[2]:
   _[1] = - x2*gen(2) + x3*gen(3)
   _[2] = - x4*gen(3) + x5*gen(4)
   _[3] = - x1*gen(4) + x2*gen(5)
   _[4] = - x1*gen(2) + x5*gen(6)
   _[5] = - x3*gen(5) + x4*gen(6)
   _[6] = - x3*x5*gen(1) + x6*x7*gen(2)
   _[7] = - x2*x5*gen(1) + x6*x7*gen(3)
   _[8] = - x2*x4*gen(1) + x6*x7*gen(4)
   _[9] = - x1*x4*gen(1) + x6*x7*gen(5)
   _[10] = - x1*x3*gen(1) + x6*x7*gen(6)
[3]:
   _[1] = - x1*x3*gen(2) - x1*x4*gen(1) + x2*x4*gen(4) - x2*x5*gen(5) -
            x3*x5*gen(3)
   _[2] = - x6*x7*gen(1) - x2*gen(6) + x3*gen(7)
   _[3] = - x6*x7*gen(2) - x4*gen(7) + x5*gen(8)
   _[4] = - x6*x7*gen(3) - x1*gen(8) + x2*gen(9)
   _[5] = - x6*x7*gen(4) - x1*gen(6) + x5*gen(10)
   _[6] = - x6*x7*gen(5) - x3*gen(9) + x4*gen(10)
```

```
[4]:
   _[1] = - x1*x3*gen(3) - x1*x4*gen(2) + x2*x4*gen(5) - x2*x5*gen(6) -
            x3*x5*gen(4) + x6*x7*gen(1)
[5]:
   _[1] = 0
[6]:
   _[1] = gen(1)
[7]:
   _[1] = 0

> print(betti(s), "betti");

           0     1     2     3     4
------------------------------------
    0:     1     -     -     -     -
    1:     -     6     5     -     -
    2:     -     -     5     6     -
    3:     -     -     -     -     1
------------------------------------
total:     1     6    10     6     1
```

So we can explicitly write the minimal graded free resolution of $\mathbb{K}[\Delta(D_2)]$ as

$$0 \longrightarrow S(-7) \longrightarrow S^6(-5) \longrightarrow S^5(-4) \oplus S^5(-3) \longrightarrow$$
$$S^6(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(D_2)] \longrightarrow 0.$$

We repeat this method for all combinatorial types of simplicial 3-polytopes with 7 vertices. We obtain that the minimal sets of monomial generators of the corresponding Gorenstein Stanley-Reisner ideals $I_{\Delta(D_1)}$, $I_{\Delta(D_3)}$, $I_{\Delta(D_4)}$ and $I_{\Delta(D_5)}$ are given as

$$I_{D_1} = (x_3x_7,\ x_4x_7,\ x_4x_6,\ x_5x_7,\ x_3x_5,\ x_3x_6,\ x_1x_2x_4,\ x_1x_2x_5,\ x_1x_2x_6).$$
$$I_{D_3} = (x_1x_7,\ x_2x_3,\ x_2x_4,\ x_2x_7,\ x_3x_5,\ x_3x_7,\ x_1x_5x_6,\ x_1x_4x_6,\ x_4x_5x_6).$$
$$I_{D_4} = (x_1x_7,\ x_2x_7,\ x_2x_5,\ x_1x_5,\ x_1x_3,\ x_6x_7,\ x_3x_4x_5,\ x_3x_4x_6,\ x_2x_4x_6).$$
$$I_{D_5} = (x_5x_6,\ x_2x_7,\ x_4x_7,\ x_3x_4,\ x_1x_2,\ x_6x_7,\ x_1x_3x_5).$$

We can use computer algebra system Singular to compute the minimal graded free resolutions of the Stanley-Reisner rings associated to D_1, D_3, D_4 and D_5. For $i = 1, 3, 4$, the minimal free resolution of $\mathbb{K}[\Delta(D_i)]$ is

$$0 \longrightarrow S(-7) \longrightarrow S^6(-5) \oplus S^3(-4) \longrightarrow S^8(-4) \oplus S^8(-3) \longrightarrow$$
$$S^3(-3) \oplus S^6(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(D_i)] \longrightarrow 0.$$

For $i = 5$, the minimal graded free resolution of $\mathbb{K}[\Delta(D_5)]$ is

$$0 \longrightarrow S(-7) \longrightarrow S^6(-5) \oplus S(-4) \longrightarrow S^6(-4) \oplus S^6(-3) \longrightarrow$$

$$S(-3) \oplus S^6(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(D_5)] \longrightarrow 0.$$

Furthermore we can write a summary for the minimal graded free resolutions of the Stanley-Reisner rings associated to all combinatorial types of simplicial 3-polytopes with 7 vertices.

$$0 \longrightarrow S(-(d+4)) \longrightarrow S^{b^S_{3,d+2}}(-(d+2)) \oplus S^{b^S_{3,d+1}}(-(d+1)) \longrightarrow S^{b^S_{2,d+1}}(-(d+1)) \oplus$$

$$S^{b^S_{2,d}}(-d) \longrightarrow S^{b^S_{1,d}}(-d) \oplus S^{b^S_{1,d-1}}(-(d-1)) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(D_5)] \longrightarrow 0,$$

where $d = 3$, $b^S_{1,d-1} = b^S_{3,d+2} = d+3$, $b^S_{1,d} = b^S_{3,d+1} = (k+1)-(d+3)$ and $b^S_{2,d} = b^S_{2,d+1} = k$. For every case we notice that the corresponding Gorenstein Stanley-Reisner ideal is generated by $k+1$ elements, for $k \in \mathbb{N}$.

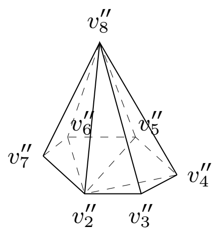
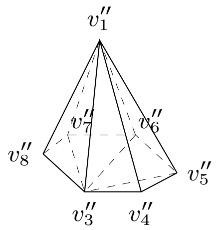
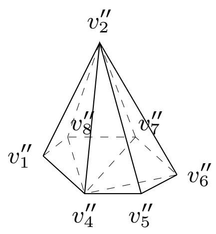
In [GS67], not only the simplicial 3-polytopes with 7 vertices are classified, but also all simplicial 4-polytopes with 8 vertices. There are 37 combinatorial types of these polytopes. We determine also the minimal graded free resolutions of the Stanley-Reisner rings associated to them using Gale diagrams and our previous discussion about the simplicial 3-polytopes with 7 vertices. We explain this explicitly for the polytope P^8_{35}.

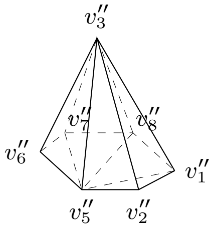
Let P^8_{35} be a simplicial 4-polytopes with the vertex set $V = \{v_1, \dots, v_8\}$, the boundary complex $\Delta(P^8_{35})$ and $0 \in \operatorname{int}(P^8_{35})$. In [GS67], the facets of P^8_{35} are given as

Polytope	**Facets**		$P^8_{35_i}$ **Type**
P^8_{35}	$v_1v_2v_3v_4$	$v_1v_2v_3v_8$	
	$v_1v_2v_6v_7$	$v_1v_2v_7v_8$	$P^8_{35_1}$: d_1
	$v_1v_2v_5v_6$	$v_2v_3v_7v_8$	$P^8_{35_2}$: d_1
	$v_1v_2v_4v_5$	$v_1v_3v_4v_8$	$P^8_{35_3}$: d_1
	$v_2v_3v_4v_5$	$v_3v_4v_7v_8$	$P^8_{35_4}$: d_1
	$v_2v_3v_5v_6$	$v_1v_4v_5v_8$	$P^8_{35_5}$: d_1
	$v_2v_3v_6v_7$	$v_4v_5v_7v_8$	$P^8_{35_6}$: d_1
	$v_3v_4v_6v_7$	$v_1v_5v_6v_8$	$P^8_{35_7}$: d_1
	$v_3v_4v_5v_6$	$v_1v_6v_7v_8$	$P^8_{35_8}$: d_1
	$v_4v_5v_6v_7$	$v_5v_6v_7v_8$	

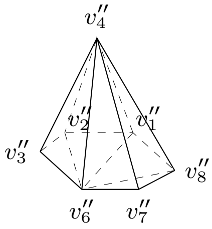
Since the polytope P^8_{35} has dimension 4, we cannot sketch it. Let $V' = \{v'_1, \dots, v'_8\}$ be the image of V under the radial projection of P^8_{35} from the origin 0 onto S^3. We use the same method as above, i.e. stereographic projections at point v_i' of $V' \smallsetminus \{v_i'\}$ for each $i = 1, \ldots, 8$, see Remark 3.2.1.5. These new polytopes are denoted by $P^8_{35_i}$ for $i = 1, \ldots, 8$, respectively. From the facets of P^8_{35} we know which point is connected with which point and which points are in the same face. Then we can sketch easily the polytopes $P^8_{35_i}$, for $i = 1, \ldots, 8$. If a polytope $P^8_{35_i}$ has 7 vertices, then we already know the minimal set of monomial generators of the corresponding Stanley-Reisner ideal. If a polytope $P^8_{35_i}$ has less than 7 vertices, then we can determine the minimal set of monomial generators of the corresponding Stanley-Reisner ideal, see Chapter 2, Subections 2.2.2 and 2.2.3.

Our aim is to determine the minimal set of the monomial generators of the Gorenstein Stanley-Reisner ideal $I_{\Delta(P^8_{35})}$. Once we know the minimal set of monomial generators of each $I_{\Delta(P^8_{35_i})}$, we can determine the minimal set of monomial generators of $I_{\Delta(P^8_{35})}$. If $x_j x_k$ is a monomial generator of an ideal $I_{\Delta(P^8_{35_i})}$, then either $x_j x_k$ or $x_i x_j x_k$ is a monomial generator of $I_{\Delta(P^8_{35})}$. If $x_j x_k$ is a face, then $x_i x_j x_k$ is a monomial generator of $I_{\Delta(P^8_{35})}$, otherwise $x_j x_k$.

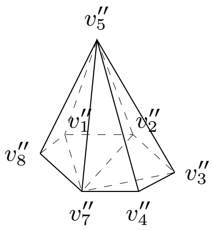
Polytope $P^8_{35_i}$	$P^8_{35_i}$ **Type**	$I_{\Delta(P^8_{35_i})}$
	$P^8_{35_1}$: d_1	$I_{\Delta(P^8_{35_1})} = (x_1x_3x_7, x_1x_3x_6, x_1x_3x_5, x_1x_4x_7, x_1x_4x_6, x_1x_5x_7, x_2x_4x_8, x_2x_5x_8, x_2x_6x_8)$
	$P^8_{35_2}$: d_1	$I_{\Delta(P^8_{35_2})} = (x_2x_4x_8, x_2x_4x_7, x_2x_4x_6, x_2x_5x_8, x_2x_5x_7, x_2x_4x_6, x_1x_3x_5, x_1x_3x_6, x_1x_3x_7)$
	$P^8_{35_3}$: d_1	$I_{\Delta(P^8_{35_3})} = (x_1x_3x_5, x_3x_5x_8, x_3x_5x_7, x_1x_3x_6, x_3x_6x_8, x_1x_3x_7, x_2x_4x_6, x_2x_4x_7, x_2x_4x_8)$



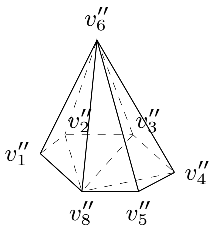
$P^8_{35_4}$: d_1 $\quad I_{\Delta(P^8_{35_4})} = \big(x_2x_4x_6,\, x_2x_4x_7,\, x_2x_4x_8,\, x_1x_4x_6,\, x_1x_4x_7,\, x_4x_6x_8,\, x_1x_3x_5,\, x_3x_5x_8,\, x_3x_5x_7\big)$



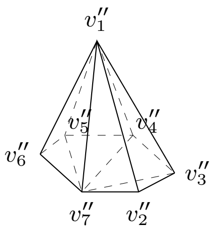
$P^8_{35_5}$: d_1 $\quad I_{\Delta(P^8_{35_5})} = \big(x_3x_5x_7,\, x_2x_5x_7,\, x_1x_5x_7,\, x_3x_5x_8,\, x_2x_5x_8,\, x_1x_3x_5,\, x_4x_6x_8,\, x_1x_4x_6,\, x_2x_4x_6\big)$



$P^8_{35_6}$: d_1 $\quad I_{\Delta(P^8_{35_6})} = \big(x_4x_6x_8,\, x_1x_4x_6,\, x_2x_4x_6,\, x_3x_6x_8,\, x_1x_3x_6,\, x_2x_6x_8,\, x_3x_5x_7,\, x_2x_5x_7,\, x_1x_5x_7\big)$



$P^8_{35_7}$: d_1 $\quad I_{\Delta(P^8_{35_7})} = \big(x_1x_5x_7,\, x_2x_5x_7,\, x_3x_5x_7,\, x_1x_4x_7,\, x_2x_4x_7,\, x_1x_3x_7,\, x_4x_6x_8,\, x_3x_6x_8,\, x_2x_6x_8\big)$



$P^8_{35_8}$: d_1 $\quad I_{\Delta(P^8_{35_8})} = \big(x_2x_6x_8,\, x_2x_5x_8,\, x_2x_4x_8,\, x_3x_6x_8,\, x_3x_5x_8,\, x_4x_6x_8,\, x_1x_3x_7,\, x_1x_4x_7,\, x_1x_5x_7\big)$

Then the Gorenstein Stanley-Reisner ideal $I_{\Delta(P_{35}^8)}$ is

$$I_{\Delta(P_{35}^8)} = \big(x_2x_4x_7,\ x_2x_4x_6,\ x_2x_5x_7,\ x_3x_5x_8,\ x_3x_5x_7,\ x_3x_6x_8,\ x_1x_3x_7,\ x_1x_3x_6,\ x_1x_3x_5,\ x_1x_4x_7,\ x_1x_4x_6,\ x_1x_5x_7,\ x_2x_4x_8,\ x_2x_5x_8,\ x_2x_6x_8,\ x_4x_6x_8\big).$$

Finally we can compute explicitly the minimal graded free resolution of the Stanley-Reisner ring associated to P_{35}^8 using computer algebra system Singular and we obtain the following

$$0 \longrightarrow S(-8) \longrightarrow S^{16}(-5) \longrightarrow S^{30}(-4) \xrightarrow{A} S^{16}(-3) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_{35}^8)] \longrightarrow 0.$$

This resolution has the same structure of the minimal graded free resolution as in the structure theorem of Reid (see Chapter 3, Subsection 3.1.1 and [Rei15]). We observe that the matrix A has the form $A = [B\ C]$ with B and C are 16×15-matrices.

$$B = \begin{bmatrix}
0 & & & & & & & & & & & & & & 0 \\
 & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & \\
0 & & & & & & & & & & & & & & \\
x_3 & x_2 & & & & & x_2 & & & & & & & & \vdots \\
0 & & & & & & & & & & & & & & \\
-x_4 & & x_2 & x_5 & & & & & & & & & & & \\
0 & x_4 & x_3 & & -x_5 & & & & & & & & & -0.5x_4 & \\
 & & & & & & & & & & & & & & 0 \\
 & & & & & & & & x_8 & & -x_2 & & & & \\
 & & & -x_6 & & x_2 & & & x_7 & & & & & & \vdots \\
\vdots & & & & & & & & & x_8 & x_3 & & -x_4 & & \\
 & & & & x_6 & x_3 & & -x_4 & & -x_7 & & & & & 0 \\
 & & & & & & & & & & & & & -x_8 & -x_7 \\
 & & & & & & & & & & & x_8 & x_5 & & -x_6 \\
0 & & & & & & x_6 & x_5 & & & & -x_7 & & -0.5x_6 & 0
\end{bmatrix}$$

and

$$C = \begin{bmatrix}
x_2 & -x_3 & x_4 & & & & & & & & & & & & 0 \\
0 & & -x_5 & x_2 & -x_3 & x_6 & & & & & & & & & \\
 & x_5 & & & x_4 & & -x_5 & x_6 & & & & & & & \\
 & & & & & -x_7 & & & x_2 & -x_3 & x_8 & & & & \vdots \\
\vdots & & & & & & & & & & x_1 & & & & \\
 & & & & & & & -x_7 & & x_4 & & -x_5 & x_8 & & \\
 & & & & & & & & & & & & x_1 & & \\
 & & & & & & & & & & & & & & 0 \\
 & -0.5x_7 & & & & & 0.5x_7 & & & & & x_6 & & -x_7 & x_8 \\
 & 0.5x_1 & & & & & 0.5x_1 & & & & & & & x_1 & 0 \\
0 & & & & & & & & & & & & & & x_1 \\
-x_1 & & & & & & & & & & & & & & 0 \\
 & & & & & & & & & & & & & & \\
\vdots & & & & & & & & x_1 & & & & & & \\
 & & & & -x_1 & & & & & & & & & & \\
0 & & & & & & & & & & & & & & 0
\end{bmatrix},$$

where BC^t is a 16×16-skew-symmetirc matrix.

$$BC^t = \begin{bmatrix}
 & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & \\
 & & & & & & & & & x_1x_2 & & -x_1x_3 & & & & \\
 & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & x_1x_4 & & & -x_1x_5 & \\
 & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & \\
 & & & & -x_1x_2 & & & & & & & & & x_1x_8 & & \\
 & & & & & & & & & & & & & x_1x_7 & x_1x_6 & \\
 & & & & x_1x_3 & & -x_1x_4 & & & & & & & & & \\
 & & & & & & & & & & & & & & & \\
 & & & & & & & & & -x_1x_8 & -x_1x_7 & & & & & \\
 & & & & & & x_1x_5 & & & & -x_1x_6 & & & & & \\
 & & & & & & & & & & & & & & &
\end{bmatrix}.$$

So we have computed the minimal graded free resolutions of the Stanley-Reisner rings associated to all simplicial 4-polytopes with 8 vertices, which are classifed in [GS67], see the following table.

P^8	**minimal free resolution of associated Stanley-Reisner ring**
P_1^8	$0 \longrightarrow S(-8) \longrightarrow S^6(-6) \oplus S^3(-4) \longrightarrow S^8(-5) \oplus S^8(-3) \longrightarrow S^3(-4) \oplus S^6(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_1^8)] \longrightarrow 0$
P_2^8	$0 \longrightarrow S(-8) \longrightarrow {}^6(-6) \oplus S^3(-4) \longrightarrow S^8(-5) \oplus S^8(-3) \longrightarrow S^3(-4) \oplus S^6(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_2^8)] \longrightarrow 0$
P_3^8	$0 \longrightarrow S(-8) \longrightarrow S^6(-6) \oplus S^3(-4) \longrightarrow S^8(-5) \oplus S^8(-3) \longrightarrow S^3(-4) \oplus S^6(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_3^8)] \longrightarrow 0$
P_4^8	$0 \longrightarrow S(-8) \longrightarrow S^2(-4) \oplus S^2(-5) \oplus S^5(-6) \longrightarrow S^6(-5) \oplus S^4(-4) \oplus S^6(-3) \longrightarrow$ $S^2(-4) \oplus S^2(-3) \oplus S^5(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_4^8)] \longrightarrow 0$
P_5^8	$0 \longrightarrow S(-8) \longrightarrow S^2(-4) \oplus S^2(-5) \oplus S^5(-6) \longrightarrow S^6(-5) \oplus S^4(-4) \oplus S^6(-3) \longrightarrow$ $S^2(-4) \oplus S^2(-3) \oplus S^5(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_5^8)] \longrightarrow 0$
P_6^8	$0 \longrightarrow S(-8) \longrightarrow S^5(-6) \oplus S^2(-5) \oplus S^2(-4) \longrightarrow S^6(-5) \oplus S^4(-4) \oplus S^6(-3) \longrightarrow$ $S^2(-4) \oplus S^2(-3) \oplus S^5(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_6^8)] \longrightarrow 0$
P_7^8	$0 \longrightarrow S(-8) \longrightarrow S^5(-6) \oplus S(-5) \oplus S(-4) \longrightarrow S^5(-5) \oplus S^2(-4) \oplus S^5(-3) \longrightarrow$ $S(-4) \oplus S(-3) \oplus S^5(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_7^8)] \longrightarrow 0$
P_8^8	$0 \longrightarrow S(-8) \longrightarrow S^4(-6) \oplus S^4(-5) \oplus S(-4) \longrightarrow S^4(-5) \oplus S^8(-4) \oplus S^4(-3) \longrightarrow$ $S(-4) \oplus S^4(-3) \oplus S^4(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_8^8)] \longrightarrow 0$
P_9^8	$0 \longrightarrow S(-8) \longrightarrow S^4(-6) \oplus S^4(-5) \oplus S(-4) \longrightarrow S^4(-5) \oplus S^8(-4) \oplus S^4(-3) \longrightarrow$ $S(-4) \oplus S^4(-3) \oplus S^4(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_9^8)] \longrightarrow 0$
P_{10}^8	$0 \longrightarrow S(-8) \longrightarrow S^4(-6) \oplus S^4(-5) \oplus S(-4) \longrightarrow S^4(-5) \oplus S^8(-4) \oplus S^4(-3) \longrightarrow$ $S(-4) \oplus S^4(-3) \oplus S^4(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_{10}^8)] \longrightarrow 0$
P_{11}^8	$0 \longrightarrow S(-8) \longrightarrow S^3(-6) \oplus S^7(-5) \oplus S(-4) \longrightarrow S^3(-5) \oplus S^{14}(-4) \oplus S^3(-3) \longrightarrow$ $S(-4) \oplus S^7(-3) \oplus S^3(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_{11}^8)] \longrightarrow 0$
P_{12}^8	$0 \longrightarrow S(-8) \longrightarrow S^3(-6) \oplus S^7(-5) \oplus S(-4) \longrightarrow S^3(-5) \oplus S^{14}(-4) \oplus S^3(-3) \longrightarrow$ $S(-4) \oplus S^7(-3) \oplus S^3(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_{12}^8)] \longrightarrow 0$
P_{13}^8	$0 \longrightarrow S(-8) \longrightarrow S^5(-6) \oplus S(-5) \longrightarrow S^5(-5) \oplus S^5(-3) \longrightarrow S(-3) \oplus S^5(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_{13}^8)] \longrightarrow 0$
P_{14}^8	$0 \longrightarrow S(-8) \longrightarrow S^4(-6) \oplus S^4(-5) \oplus S(-4) \longrightarrow S^4(-5) \oplus S^8(-4) \oplus S^4(-3) \longrightarrow$ $S(-4) \oplus S^4(-3) \oplus S^4(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_{14}^8)] \longrightarrow 0$
P_{15}^8	$0 \longrightarrow S(-8) \longrightarrow S^4(-6) \oplus S^4(-5) \oplus S(-4) \longrightarrow S^4(-5) \oplus S^8(-4) \oplus S^4(-3) \longrightarrow$ $S(-4) \oplus S^4(-3) \oplus S^4(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_{15}^8)] \longrightarrow 0$
P_{16}^8	$0 \longrightarrow S(-8) \longrightarrow S^4(-6) \oplus S^3(-5) \longrightarrow S^3(-5) \oplus S^6(-4) \oplus S^3(-3) \longrightarrow$ $S^3(-3) \oplus S^4(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_{16}^8)] \longrightarrow 0.$
P_{17}^8	$0 \longrightarrow S(-8) \longrightarrow S^4(-6) \oplus S^2(-5) \longrightarrow S^2(-5) \oplus S^6(-4) \oplus S^2(-3) \longrightarrow$ $S^2(-3) \oplus S^4(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_{17}^8)] \longrightarrow 0$
P_{18}^8	$0 \longrightarrow S(-8) \longrightarrow S^3(-6) \oplus S^6(-5) \longrightarrow S^2(-5) \oplus S^{12}(-4) \oplus S^2(-3) \longrightarrow$ $S^6(-3) \oplus S^3(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_{18}^8)] \longrightarrow 0$
P_{19}^8	$0 \longrightarrow S(-8) \longrightarrow S^3(-6) \oplus S^6(-5) \longrightarrow S^2(-5) \oplus S^{12}(-4) \oplus S^2(-3) \longrightarrow$ $S^6(-3) \oplus S^3(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_{19}^8)] \longrightarrow 0$
P_{20}^8	$0 \longrightarrow S(-8) \longrightarrow S^3(-6) \oplus S^6(-5) \longrightarrow S^2(-5) \oplus S^{12}(-4) \oplus S^2(-3) \longrightarrow$ $S^6(-3) \oplus S^3(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P_{20}^8)] \longrightarrow 0$

P^8_{21}	$0 \longrightarrow S(-8) \longrightarrow S^3(-6)\oplus S^6(-5) \longrightarrow S^2(-5)\oplus S^{12}(-4)\oplus S^2(-3) \longrightarrow$ $S^6(-3)\oplus S^3(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{21})] \longrightarrow 0$
P^8_{22}	$0 \longrightarrow S(-8) \longrightarrow S^3(-6)\oplus S^7(-5)\oplus S(-4) \longrightarrow S^3(-5)\oplus S^{14}(-4)\oplus S^3(-3) \longrightarrow$ $S(-4)\oplus S^7(-3)\oplus S^3(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{22})] \longrightarrow 0$
P^8_{23}	$0 \longrightarrow S(-8) \longrightarrow S^2(-6)\oplus S^9(-5) \longrightarrow S(-5)\oplus S^{18}(-4)\oplus S(-3) \longrightarrow$ $S^9(-3)\oplus S^2(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{23})] \longrightarrow 0$
P^8_{24}	$0 \longrightarrow S(-8) \longrightarrow S^2(-6)\oplus S^9(-5) \longrightarrow S(-5)\oplus S^{18}(-4)\oplus S(-3) \longrightarrow$ $S^9(-3)\oplus S^2(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{24})] \longrightarrow 0$
P^8_{25}	$0 \longrightarrow S(-8) \longrightarrow S^2(-6)\oplus S^9(-5) \longrightarrow S(-5)\oplus S^{18}(-4)\oplus S(-3) \longrightarrow$ $S^9(-3)\oplus S^2(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{25})] \longrightarrow 0$
P^8_{26}	$0 \longrightarrow S(-8) \longrightarrow S^3(-6)\oplus S^4(-5) \longrightarrow S^{12}(-4) \longrightarrow S^4(-3)\oplus S^3(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{26})] \longrightarrow 0$
P^8_{27}	$0 \longrightarrow S(-8) \longrightarrow S^2(-6)\oplus S^8(-5) \longrightarrow S^{18}(-4) \longrightarrow S^8(-3)\oplus S^2(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{27})] \longrightarrow 0$
P^8_{28}	$0 \longrightarrow S(-8) \longrightarrow S^2(-6)\oplus S^8(-5) \longrightarrow S^{18}(-4) \longrightarrow S^8(-3)\oplus S^2(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{28})] \longrightarrow 0$
P^8_{29}	$0 \longrightarrow S(-8) \longrightarrow S^2(-6)\oplus S^8(-5) \longrightarrow S^{18}(-4) \longrightarrow S^8(-3)\oplus S^2(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{29})] \longrightarrow 0$
P^8_{30}	$0 \longrightarrow S(-8) \longrightarrow S(-6)\oplus S^{12}(-5) \longrightarrow S^{24}(-4) \longrightarrow S^{12}(-3)\oplus S(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{30})] \longrightarrow 0$
P^8_{31}	$0 \longrightarrow S(-8) \longrightarrow S(-6)\oplus S^{12}(-5) \longrightarrow S^{24}(-4) \longrightarrow S^{12}(-3)\oplus S(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{31})] \longrightarrow 0$
P^8_{32}	$0 \longrightarrow S(-8) \longrightarrow S(-6)\oplus S^{12}(-5) \longrightarrow S^{24}(-4) \longrightarrow S^{12}(-3)\oplus S(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{32})] \longrightarrow 0$
P^8_{33}	$0 \longrightarrow S(-8) \longrightarrow S(-6)\oplus S^{12}(-5) \longrightarrow S^{24}(-4) \longrightarrow S^{12}(-3)\oplus S(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{33})] \longrightarrow 0$
P^8_{34}	$0 \longrightarrow S(-8) \longrightarrow S^4(-6) \longrightarrow S^6(-4) \longrightarrow S^4(-2) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{34})] \longrightarrow 0$
P^8_{35}	$0 \longrightarrow S(-8) \longrightarrow S^{16}(-5) \longrightarrow S^{30}(-4) \longrightarrow S^{16}(-3) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{35})] \longrightarrow 0$
P^8_{36}	$0 \longrightarrow S(-8) \longrightarrow S^{16}(-5) \longrightarrow S^{30}(-4) \longrightarrow S^{16}(-3) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{36})] \longrightarrow 0$
P^8_{37}	$0 \longrightarrow S(-8) \longrightarrow S^{16}(-5) \longrightarrow S^{30}(-4) \longrightarrow S^{16}(-3) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{37})] \longrightarrow 0$

We observe that if the minimal graded free resolution of the Stanley-Reisner ring associated to a simplicial 4-polytopes P^8 with 8 vertices has the form

$$0 \longrightarrow S \longrightarrow S^6 \longrightarrow S^{10} \longrightarrow S^6 \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8)] \longrightarrow 0,$$

then the Gale diagram of the polytope P^8 is the same as the Gale diagram of the polytope D_2, but there is a vertex of the Gale diagram of P^8, which has two multiplicities.

If the minimal graded free resolution has the form

$$0 \longrightarrow S \longrightarrow S^7 \longrightarrow S^{12} \longrightarrow S^7 \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8)] \longrightarrow 0,$$

then the Gale diagram of the polytope P^8 is the same as the Gale diagram of the polytope D_5, but there is a vertex of the Gale diagram of P^8, which has two multiplicities.

If the minimal graded free resolution has the form

$$0 \longrightarrow S \longrightarrow S^9 \longrightarrow S^{16} \longrightarrow S^9 \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8)] \longrightarrow 0,$$

then the Gale diagram of the polytope P^8 is the same as the Gale diagram of the polytope D_1 or D_3 or D_4, but there is a vertex of the Gale diagram of P^8, which has two multiplicities.

CHAPTER

FOUR

GORENSTEIN IDEALS OF CODIMENSION 4 WITH AN EVEN NUMBER OF GENERATORS

Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$, the boundary complex $\Delta(P)$ and $0 \in \operatorname{int}(P)$. We apply a radial projection of P from the origin 0 onto the unit sphere S^{d-1}. The image of V under this projection is denoted by $V' := \{v'_1, \dots, v'_n\}$, where v'_i is the image of v_i, for $i = 1, \dots, n$. We then use a stereographic projection at each point v'_i, for $i = 1, \dots, n$. For every v'_i, we obtain a simplicial $(d-1)$-polytope P_i, which has at most $n-1$ vertices. Let $V'' := \{v''_{i_1}, \dots, v''_{i_k}\}$ be the vertex set of P_i, where v''_{i_l} is the image of v'_{i_l} under the stereographic projection, see Proposition 3.2.1.4 and Remark 3.2.1.5. For every such polytope P_i we define a corresponding Stanley-Reisner ring $\mathbb{K}[\Delta(P_i)] = \mathbb{K}[x_{i_1}, \dots, x_{i_k}]/I_{\Delta(P_i)}$, where $I_{\Delta(P_i)}$ is the Gorenstein Stanley-Reisner ideal associated to P_i and $\Delta(P_i)$ is the boundary complex of P_i. The first aim in this chapter is to determine the minimal sets of monomial generators of the corresponding Gorenstein Stanley-Reisner ideals $I_{\Delta(P_i)}$, for $i = 1, \dots, n$. We give an algorithm, that allows us to determine these sets if the minimal set of monomial generators of the Gorenstein Stanley-Reisner ideal $I_{\Delta(P)}$ is known. The second aim is to answer a question of Reid (see [Rei13, Open problems 4.9.4], [Rei15, Section 2.6]), about Stanley-Reisner ideals of codimension 4, whether every Gorenstein ideal of codimension 4 with 6 generators is a complete intersection[1]of a Gorenstein ideal of codimension 3 with 5 generators, and an extra polynom. The third aim is to give a counterexample to a conjecture of Reid (see [Rei13, Open problems 4.9.4]), that every Gorenstein ideal of codimension 4 with an even number of generators is a complete intersection of a Gorenstein ideal of codimension 3 and an extra polynom.

4.1 Construction of monomial generators of Gorenstein ideals associates to projections of polytopes

Let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring over an algebraically closed field $\mathbb{K}$ and $\mathbb{K}[\Delta(P)]$ the corresponding Stanley-Reisner ring of $\Delta(P)$. Assume that the minimal set

[1] See Definition 4.2.0.1.

of monomial generators of the Gorenstein Stanley-Reisner ideal $I_{\Delta(P)}$ is known. In this section, we prove some propositions, which help us to give an algorithm to determine the minimal sets of monomial generators of $I_{\Delta(P_i)}$, through the minimal set of monomial generators of $I_{\Delta(P)}$, for $i = 1, \ldots, n$.

Definition 4.1.0.1. Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \ldots, v_n\}$. A subset $\{v_{i_1}, \ldots v_{i_k}\} \subseteq V$ is called *primitive in* P, if

1. For every facet F of P, $\operatorname{conv}(\{v_{i_1}, \ldots, v_{i_k}\}) \nsubseteq F$.
2. There is a facet F of P, such that $\operatorname{conv}(\{v_{i_1}, \ldots, v_{i_k}\} \smallsetminus \{v_j\}) \subseteq F$, for every $j \in \{i_1, \ldots, i_k\}$

Theorem 4.1.0.2. *Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \ldots, v_n\}$ and the boundary complex $\Delta(P)$. Then the Stanley-Reisner ideal is given as*

$$I_{\Delta(P)} = (x_{i_1} \ldots x_{i_r} : \ i_1 < i_2 < \ldots < i_r, \ \{v_{i_1}, \ldots, v_{i_r}\} \text{ primitive in } P), \text{ where}$$

$$\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \ldots, x_n]/I_{\Delta(P)}.$$

Proof. See Remark 1.2.3.2. □

Now let P be a simplicial d-polytope with the vertex set $V = \{v_1, \ldots, v_n\}$ and $0 \in \operatorname{int}(P)$. We apply a radial projection from the origin onto the unit sphere S^{d-1}. The point v_i' is the image of v_i under this projection, for $i = 1, \ldots, n$. We then use a stereographic projection at each point v_i', for $i = 1, \ldots, n$. For every v_i', we obtain a simplicial $(d-1)$-polytope P_i, which has at most $n-1$ vertices. Let $V'' := \{v''_{i_1}, \ldots, v''_{i_m}\}$ be the vertex set of P_i, where v''_{i_l} is the image of v'_{i_l} under the stereographic projection, see Chapter 2, Subsection 3.2.1.

Proposition 4.1.0.3. *Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \ldots, v_n\}$. Let P_i be the new polytope obtained by the stereographic projection at the projection point v_i', where $V'' = \{v''_{i_1}, \ldots, v''_{i_m}\}$ is its vertex set with $d \le m \le (n-1)$. If $\{v''_{i_1}, \ldots, v''_{i_k}\}$ primitive in P_i, then either $\{v_i, v_{i_1}, \ldots, v_{i_k}\}$ or $\{v_{i_1}, \ldots v_{i_k}\}$ is primitive in P.*

Proof. Assume that $\{v_i, v_{i_1}, \ldots, v_{i_k}\}$ is not primitive in P. That means that either the first condition of Definition 4.1.0.1 is not fulfilled or the second one. We consider now the first case, then there is a facet F of P such that $\operatorname{conv}(\{v_i, v_{i_1}, \ldots, v_{i_k}\}) \subseteq F$. By Proposition 3.2.1.4 there is a facet F_i of P_i such that $\operatorname{conv}(\{v''_{i_1}, \ldots, v''_{i_k}\}) \subseteq F_i$, a contradiction to $\{v''_{i_1}, \ldots v''_{i_k}\}$ being primitive in P_i. Now suppose that the second condition of Definition 4.1.0.1 is not true, that is, there is at least a $j \in \{i, i_1, \ldots, i_k\}$ such that for all facets F of P, we have $\operatorname{conv}(\{v_i, v_{i_1}, \ldots, v_{i_k}\} \smallsetminus \{v_j\}) \nsubseteq F$. There are two cases. In the case $j \neq i$ we have by Proposition 3.2.1.4 that $\operatorname{conv}(\{v''_{i_1}, \ldots, v''_{i_k}\} \smallsetminus \{v''_j\}) \nsubseteq F_i$, for all facets F_i of P_i. This leads to a contradiction to the primitivity of $\{v''_{i_1}, \ldots v''_{i_k}\}$ in P_i. Now assume that $j = i$ and that is $\operatorname{conv}(\{v_{i_1}, \ldots, v_{i_k}\}) \nsubseteq F$, for every facet F of P. Since $\{v''_{i_1}, \ldots v''_{i_k}\}$ is primitive in P_i, for every $j \in \{i_1, \ldots, i_k\}$ there exists a facet F_i of P_i such that $\operatorname{conv}(\{v''_{i_1}, \ldots, v''_{i_k}\} \smallsetminus \{v''_j\}) \subseteq F_i$. By Proposition 3.2.1.4 there exists a facet F of P such that $\operatorname{conv}(\{\{v_{i_1}, \ldots, v_{i_k}\} \smallsetminus \{v_j\}\} \cup \{v_i\}) \subseteq F$ and this is equivalent to $\operatorname{conv}(\{v_{i_1}, \ldots, v_{i_k}\} \smallsetminus \{v_j\}) \subseteq F$. □

Proposition 4.1.0.4. *Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$. Let P_i be the new polytope obtained by the stereographic projection at the projection point v_i', where $V'' = \{v''_{i_1}, \dots, v''_{i_m}\}$ is its vertex set with $d \le m \le (n-1)$. If $\{v_{i_1}, \dots v_{i_k}\}$ is primitive in P with $v_i \notin \{v_{i_1}, \dots v_{i_k}\}$ and $\{v_i, v_{j_1}, \dots v_{j_s}\}$ is not primitive in P for all subsets $\{v_{j_1}, \dots v_{j_s}\} \subset \{v_{i_1}, \dots v_{i_k}\}$, then $\{v''_{i_1}, \dots v''_{i_k}\}$ is primitive in P_i.*

Proof. Assume that $\{v''_{i_1}, \dots v''_{i_k}\}$ is not primitive in P_i. Either the first condition of Definition 4.1.0.1 is not fulfilled or the second one. Assume that the first condition of Definition 4.1.0.1 is not true. This means that there is a facet F_i of P_i such that $\operatorname{conv}(\{v''_{i_1}, \dots, v''_{i_k}\}) \subseteq F_i$. It follows from Proposition 3.2.1.4 that $\operatorname{conv}(\{v_i, v_{i_1}, \dots, v_{i_k}\}) \subseteq F$, where F a facet of P. This is equivalent to $\operatorname{conv}(\{v_{i_1}, \dots, v_{i_k}\}) \subseteq F$, which leads to a contradiction to the primitivity of $\{v_{i_1}, \dots v_{i_k}\}$ in P. If the second condition of Definition 4.1.0.1 is not true, then there is at least one $j \in \{i_1, \dots, i_k\}$ such that $\operatorname{conv}(\{v''_{i_1}, \dots, v''_{i_k}\} \setminus \{v''_j\}) \nsubseteq F_i$, for every facet F_i of P_i. By Proposition 3.2.1.4 this is equivalent to $\operatorname{conv}(\{\{v_{i_1}, \dots, v_{i_k}\} \setminus \{v_j\}\} \cup \{v_i\}) \nsubseteq F$, for every facet F of P. But we have that $\{v_i, v_{j_1}, \dots v_{j_s}\}$ is not primitive in P for all subsets $\{v_{j_1}, \dots v_{j_s}\} \subset \{v_{i_1}, \dots v_{i_k}\}$, which means that $\{\{v_{i_1}, \dots, v_{i_k}\} \setminus \{v_j\}\} \cup \{v_i\}$ is not primitive in P. Hence there is at least one $h \in \{i, i_1, \dots, i_k\} \setminus \{j\}$ such that $\operatorname{conv}(\{(\{v_{i_1}, \dots, v_{i_k}\} \setminus \{v_j\}) \cup \{v_i\}\} \setminus \{v_h\}) \nsubseteq F$, for every facet F of P. We have two cases. In the case $h = i$ we obtain a contradiction to $\{v_{i_1}, \dots v_{i_k}\}$ being primitive in P. In the case $h \neq i$ we have $\operatorname{conv}(\{v_i\} \cup \{v_{i_1}, \dots, v_{i_k}\} \setminus \{v_j, v_h\}) \nsubseteq F$, for every facet F of P. Recursively we obtain $\operatorname{conv}(\{v_i, v_{i_s}\}) \nsubseteq F$, for every facet F of P. By assumption $\{v_i, v_{i_s}\}$ is not primitive in P. Then there is an $l \in \{i, i_s\}$ such that either $\operatorname{conv}(\{v_i\}) \nsubseteq F$ or $\operatorname{conv}(\{v_{i_s}\}) \nsubseteq F$, a contradiction, since the vertices are faces. □

Proposition 4.1.0.5. *Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$. Let P_i be the new polytope obtained by the stereographic projection at the projection point v_i', where $V'' = \{v''_{i_1}, \dots, v''_{i_m}\}$ is its vertex set with $d \le m \le (n-1)$. If $\{v_{i_1}, \dots v_{i_k}\}$ is primitive in P with $v_i \notin \{v_{i_1}, \dots v_{i_k}\}$ and there exists at least one subset $\{v_i, v_{j_1}, \dots v_{j_s}\}$ of V, which is primitive in P for a subset $\{v_{j_1}, \dots v_{j_s}\} \subset \{v_{i_1}, \dots v_{i_k}\}$, then $\{v''_{i_1}, \dots v''_{i_k}\}$ is not primitive in P_i.*

Proof. Since there exists at least one subset $\{v_i, v_{j_1}, \dots v_{j_s}\}$ of V, which is primitive in P for a subset $\{v_{j_1}, \dots v_{j_s}\} \subset \{v_{i_1}, \dots v_{i_k}\}$, then we have $\operatorname{conv}(\{v_i, v_{j_1}, \dots, v_{j_k}\}) \nsubseteq F$, for every facet F in P. It follows by Proposition 3.2.1.4, that $\operatorname{conv}(\{v''_{j_1}, \dots, v''_{j_k}\}) \nsubseteq F_i$, for every facet F_i in P_i. Hence $\{v''_{i_1}, \dots v''_{i_k}\}$ is not primitive in P_i. □

Proposition 4.1.0.6. *Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$. Let P_i be the new polytope obtained by the stereographic projection at the projection point v_i', where $V'' = \{v''_{i_1}, \dots, v''_{i_m}\}$ is its vertex set with $d \le m \le (n-1)$. If $\{v_{i_1}, \dots v_{i_k}\}$ is primitive in P with $v_i \in \{v_{i_1}, \dots v_{i_k}\}$ and $k \ge 3$, then $\{v''_{i_1}, \dots v''_{i_k}\} \setminus \{v''_i\}$ is primitive in P_i.*

Proof. Since $\{v_{i_1}, \dots v_{i_k}\}$ is primitive in P, $\operatorname{conv}(\{v_{i_1}, \dots, v_{i_k}\}) \nsubseteq F$, for every facet F in P. By Proposition 3.2.1.4 we have $\operatorname{conv}(\{v''_{i_1}, \dots, v''_{i_k}\} \setminus \{v''_i\}) \nsubseteq F_i$, for every F_i in P_i, because $v_i \in \{v_{i_1}, \dots v_{i_k}\}$. Moreover there is a facet F of P for every $j \in \{i_1, \dots, i_k\}$ such that $\mathrm{conv}(\{v_{i_1}, \dots, v_{i_k}\} \setminus \{v_j\}) \subseteq F$. It follows from Proposition 3.2.1.4 that for every $j \in \{i_1, \dots, i_k\} \setminus \{i\}$, there is a facet F_i of P_i such that $\mathrm{conv}(\{v''_{i_1}, \dots, v''_{i_k}\} \setminus \{v''_i, v''_j\}) \subseteq F_i$. So $\{v''_{i_1}, \dots v''_{i_k}\} \setminus \{v''_i\}$ is primitive in P_i. □

Definition 4.1.0.7. Let $S = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring and $f \in S$ a monomial. We define $T(f) \coloneqq \{i \in \{1, \dots, n\} : \ x_i | f\}$.

Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$ and the boundary complex $\Delta(P)$. Let $\{f_1, \dots, f_k\}$ be the minimal set of monomial generators of $I_{\Delta(P)}$. Set $I_{\Delta(P)} \coloneqq (f_1, \dots, f_k)$. Let P_i be the new polytope, which is obtained by stereographic projection at the projection point v'_i with $\Delta(P_i)$ its boundary complex. For every such polytope P_i we define a corresponding Stanley-Reisner ring $\mathbb{K}[\Delta(P_i)] = \mathbb{K}[x_{i_1}, \dots, x_{i_k}]/I_{\Delta(P_i)}$, where $I_{\Delta(P_i)}$ is the Gorenstein Stanley-Reisner ideal associated to P_i. Now we can introduce an algorithm to compute the minimal sets of monomial generators of the Gorenstein Stanley-Reisner ideals $I_{\Delta(P_i)}$ for $i = 1, \dots, n$, if the minimal set of monomial generators of $I_{\Delta(P)}$ is known.

Algorithm 1: Compute of minimal set of monomial generators of $I_{\Delta(P_i)}$.

```
Input  : T(f_1),...,T(f_k)} and i      (The stereographic projection is at v'_i)
Output: Minimal set of monomial generators of I_Δ(P_i)

j ← 1;
N' ← ∅;
M ← {∅};
for j ← 1 to k do
    if i ∈ T(f_j) then
        overline{T(f)} ← T(f_j) ∖ {i};
        N' ← {T(f_l) ∈ N : overline{T(f)} ⊈ T(f_l)};
        M ← M ∪ {overline{T(f)}};
        if |overline{T(f)}| ≥ 2 then
            N ← N' ∪ overline{T(f_j)};
        else
            N ← N';
    else
        M' ← ρ(T(f_j));      (where ρ(T(f_j)) is the power set of T(f_j))
        if M' ∩ M = ∅ then
            N ← N' ∪ T(f_j);
        else
            N ← N';
```

Proof of the correctness of the algorithm. That follows immediately from Propositions 4.1.0.3, 4.1.0.4, 4.1.0.5 and 4.1.0.6. □

Corollary 4.1.0.8. *Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$ and the boundary complex $\Delta(P)$. Let P_i be the new polytope, which is obtained by stereographic projection at the projection point v_i' with $\Delta(P_i)$ its boundary complex. Then the minimal number of monomial generators of $I_{\Delta(P_i)}$ is smaller than or equal to the minimal number of monomial generators of $I_{\Delta(P)}$.*

Proof. This follows from Algorithm 1. □

Example 4.1.0.9. Let P be a simplicial 4-polytope with 8 vertices $V = \{v_1, \dots, v_8\}$ and the boundary complex $\Delta(P)$. Let $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_8]/I_{\Delta(P)}$ with $I_{\Delta(P)} := (x_7x_8,\ x_6x_8,\ x_3x_8,\ x_5x_7,\ x_4x_7,\ x_4x_6,\ x_1x_2x_3x_6,\ x_1x_2x_4x_5,\ x_1x_2x_3x_5)$. For $i = 8$ we have by algorithm 1 that $I_{\Delta(P_8)} = (x_1x_2x_4x_5)$. For $i = 1$ we have $I_{\Delta(P_1)} = (x_7x_8,\ x_6x_8,\ x_3x_8,\ x_5x_7,\ x_4x_7,\ x_4x_6,\ x_2x_3x_6,\ x_2x_4x_5,\ x_2x_3x_5)$.

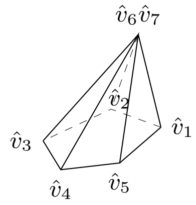
4.2 Gorenstein ideals of codimension 4 with 6 generators

Every Gorenstein ideal of codimension 4 with 4 generators is a complete intersection ideal. That means it is a complete intersection of a Gorenstein ideal of codimension 3 with 3 generators and an extra polynom. We show that this is not true for every Gorenstein ideal of codimension 4 with an even number of monomial generators. We answer in this section a question of Reid (see [Rei13, Open problems 4.9.4], [Rei15, Section 2.6]), for Gorenstein Stanley-Reisner ideals of codimension 4: Every Gorenstein ideal of codimension 4 with 6 monomial generators is a complete intersection of a Gorenstein ideal of codimension 3 with 5 generators and an extra polynom.

Definition 4.2.0.1. Let $S = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring, $f \in S$ a polynom and let I and I' be ideals of S. We say that I is a *complete intersection* of I' and f, if $I = I' + (f)$ and f modulo I' is a non-zero divisor in the residue class ring S/I'.

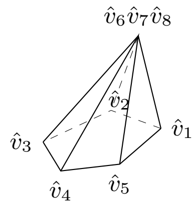
Theorem 4.2.0.2. Let P be a simplicial d-polytope with $d+4$ vertices, $\Delta(P)$ be the boundary complex of P and $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_{d+4}]/I_{\Delta(P)}$ the associated Stanley-Reisner ring to P. If the Gorenstein Stanley-Reisner ideal $I_{\Delta(P)}$ is minimally generated by 6 monomials, such like $f_1, \dots, f_6$, then there exists $i \in \{1, \dots, 6\}$ such that $T(f_i) \cap T(f_j) = \varnothing$ for all $i \neq j$ and $I' = (f_j : j \in \{1, \dots 6\} \setminus \{i\})$ is a Gorenstein ideal of codimension 3.

Proof. The proof is by induction on d. The base case is for $d = 3$, because there is no simplicial 2-polytope with 6 vertices, such that the Gorenstein Stanley-Reisner ideal of its boundary complex is generated by 6 monomials. In Chapter 3, Subsection 3.2.2, we have seen that there is a simplicial 3-polytope with 7-vertices P, such that the Gorenstein Stanley-Reisner ideal of its boundary complex is generated by 6 monomials. The Gale diagram of P has the following form



and $I_{\Delta(P)} = (x_1x_2,\ x_2x_3,\ x_3x_4,\ x_4x_5,\ x_5x_1,\ x_6x_7)$.

We observe that $I' = (x_1x_2,\ x_2x_3,\ x_3x_4,\ x_4x_5,\ x_5x_1)$ is a Gorenstein ideal of codimension 3, see Example 2.2.3.3. For $f_j \in \{x_1x_2,\ x_2x_3,\ x_3x_4,\ x_4x_5,\ x_5x_1\}$ we have $T(x_6x_7) \cap T(f_j) = \varnothing$. Now let $d = 4$. There is a simplicial 4-polytope with 8-vertices P^8_{13}, see [GS67], such that $I_{\Delta(P^8_{13})}$ is generated by 6 monomials, see Chapter 3, Subsection 3.2.2. The Gale diagram of P^8_{13} is



and $I_{\Delta(P^8_{13})} = (x_1x_2,\ x_2x_3,\ x_3x_4,\ x_4x_5,\ x_5x_1,\ x_6x_7x_8)$.

We observe that $I' = (x_1x_2,\ x_2x_3,\ x_3x_4,\ x_4x_5,\ x_5x_1)$ is a Gorenstein ideal of codimension 3, see Example 2.2.3.3, $T(x_6x_7x_8) \cap T(f_j) = \varnothing$ for $f_j \in \{x_1x_2,\ x_2x_3,\ x_3x_4,\ x_4x_5,\ x_5x_1\}$.

Induction hypothesis Let P' be a simplicial k-polytope with $k+4$ vertices and its boundary complex $\Delta(P')$. We assume that the claim is true for such $4 < k < d$. If the corresponding Gorenstein Stanley-Reisner ideal $I_{\Delta(P')} = (f'_1, \dots, f'_6)$, then it exists $i \in \{1, \dots, 6\}$ such that $T(f'_i) \cap T(f'_j) = \varnothing$ for all $i \neq j$ and $I' = (f'_j : \ j \in \{1, \dots 6\} \setminus \{i\})$ is a Gorenstein ideal of codimension 3.

Induction step: Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_{d+4}\}$ and the boundary complex $\Delta(P)$ with $0 \in \operatorname{int}(P)$. Let $I_{\Delta(P)}$ be the Gorenstein Stanley-Reisner ideal of $\Delta(P)$ with $I_{\Delta(P)} = (f_1, \dots, f_6)$. Now we apply a radial projection of P from the origin onto the unit sphere S^{d-1}. The image of V under this projection is denoted by $V' := \{v'_1, \dots, v'_{d+4}\}$, where v'_i is the image of v_i, for $i = 1, \dots, d+4$. Then we use the stereographic projection at each point v'_i, for $i = 1, \dots, d+4$. The resulting polytopes are denoted by P_i according to the projection point v'_i. These $(d-1)$-polytopes P_i are simplicial, see Proposition 3.2.1.4 and Remark 3.2.1.5, and each of them has at most $d+3$ vertices. We distinguish between four cases depending on the number of the vertices of P_i.

In the first case, if $|V''| = d+3 = (d-1)+4$, where V'' is the vertex set of P_i. Since the stereographic projection at each point v'_i, thats mean the image of v_i under the function composition of the radial projection and stereographic projection is not vertex of P_i. By

Theorem 1.2.3.3 and Theorem 1.1.4.21 we have $\mathrm{codim}(I_{\Delta(P_i)}) = 4$. Hence the polytope P_i is a simplicial k-polytope with $k+4$ vertices. Since $I_{\Delta(P)}$ is minimal generated by 6 elements, the minimal number of generators of $I_{\Delta(P_i)}$ is smaller than or equal to 6, by Corollary 4.1.0.8. We distinguish the following cases:

1. The minimal number of monomial generators of $I_{\Delta(P_i)}$ is smaller than 4. This case is excluded, because $\mathrm{codim}(I_{\Delta(P_i)}) = 4$.
2. The minimal number of monomial generators of $I_{\Delta(P_i)}$ is equal to 4. Then $I_{\Delta(P_i)}$ is a complete intersection ideal. Let $\{g_1, \dots, g_4\}$ be the minimal set of monomial generators of $I_{\Delta_{P_i}}$. Since a complete intersection ideal is generated by a regular sequence, there is a $j \in \{1, \dots, 4\}$ and $j \neq i_l$ such that $T(g_j) \cap T(g_{i_l}) = \varnothing$ for $i_l \in \{1, \dots, 4\}$ and $l \in \{1, \dots, 3\}$. Let p_{i_k} and p'_{i_k} be two monomials which divide g_{i_k} for $i_k = 1, 2, 3, 4$. By Propositions 4.1.0.3, 4.1.0.4, 4.1.0.5 and 4.1.0.6 we have only the following possibilities
$$I_{\Delta(P)} = (x_i g_j,\, g_{i_1},\, g_{i_2},\, g_{i_3}, g_j p_{i_2},\, p_{i_3} g_j), \tag{4.1}$$
$$I_{\Delta(P)} = (x_i g_j,\, g_{i_1},\, x_i g_{i_2},\, g_{i_3}, g_j p_{i_2},\, p_{i_3} g_j), \tag{4.2}$$
$$I_{\Delta(P)} = (x_i g_j,\, g_{i_1},\, x_i g_{i_2},\, x_i g_{i_3}, g_j p_{i_2},\, p_{i_3} g_j), \tag{4.3}$$
$$I_{\Delta(P)} = (x_i g_j,\, x_i g_{i_1},\, x_i g_{i_2},\, x_i g_{i_3}, g_j p_{i_2},\, p_{i_3} g_j), \tag{4.4}$$
$$I_{\Delta(P)} = (g_j,\, g_{i_1},\, x_i g_{i_2},\, x_i g_{i_3}, g_j p_{i_1},\, p'_{i_1} g_{i_3}), \tag{4.5}$$
$$I_{\Delta(P)} = (x_i g_j,\, x_i g_{i_1},\, x_i g_{i_2},\, x_i g_{i_3}, g_j p_{i_1},\, p'_{i_1} g_{i_3}), \tag{4.6}$$
$$I_{\Delta(P)} = (x_i g_j,\, g_{i_1},\, x_i g_{i_2},\, x_i g_{i_3}, g_j p_{i_1},\, p'_{i_1} g_{i_3}). \tag{4.7}$$
If we apply a radial projection of P with the vertex set V from the origin onto the unit sphere S^{d-1}, and use the stereographic projection at an appropriate projection point of V', where V' is the image of V under the radial projection, see Remark 3.2.1.5, then we obtain an ideal of codimension 4 with 6 generators, and we observe that it does not satisfy the induction hypothesis.
3. The minimal number of monomial generators of $I_{\Delta(P_i)}$ is equal to 5. In this case the ideal is not Gorenstein, see [Kun74].
4. The minimal number of monomial generators of $I_{\Delta(P_i)}$ is equal to 6. Let $\{g_1, \dots, g_6\}$ be the minimal set of monomial generators of $I_{\Delta_{P_i}}$. By induction hypothesis there exists $j \in \{1, \dots, 6\}$ with $j \neq i_l$ such that $T(g_j) \cap T(g_{i_l}) = \varnothing$ for $i_l \in \{1, \dots, 6\}$ and $l \in \{1, \dots, 5\}$, and $I' := (g_{i_1}, g_{i_2}, g_{i_3}, g_{i_4}, g_{i_5})$ is a Gorenstein ideal of codimension 3. That means that all g_{i_l} for $i_l \neq j$ have the same degree, see Buchsbaum-Eisenbud Theorem 2.1.2.4. Now we can determine the minimal set of monomial generators of $I_{\Delta(P)}$ by Propositions 4.1.0.3, 4.1.0.4, 4.1.0.5 and 4.1.0.6. There are the following possibilities
$$I_{\Delta(P)} = (x_i g_j,\, g_{i_1},\, g_{i_2},\, g_{i_3},\, g_{i_4},\, g_{i_5}), \tag{4.8}$$
$$I_{\Delta(P)} = (x_i g_j,\, x_i g_{i_1},\, g_{i_2},\, g_{i_3},\, g_{i_4},\, g_{i_5}), \tag{4.9}$$
$$I_{\Delta(P)} = (x_i g_j,\, x_i g_{i_1},\, x_i g_{i_2},\, g_{i_3},\, g_{i_4},\, g_{i_5}), \tag{4.10}$$
$$I_{\Delta(P)} = (x_i g_j,\, x_i g_{i_1},\, x_i g_{i_2},\, x_i g_{i_3},\, g_{i_4},\, g_{i_5}), \tag{4.11}$$

$$I_{\Delta(P)} = (x_i g_j,\, x_i g_{i_1},\, x_i g_{i_2},\, x_i g_{i_3},\, x_i g_{i_4},\, g_{i_5}), \tag{4.12}$$
$$I_{\Delta(P)} = (x_i g_j,\, x_i g_{i_1},\, x_i g_{i_2},\, x_i g_{i_3},\, x_i g_{i_4},\, x_i g_{i_5}), \tag{4.13}$$
$$I_{\Delta(P)} = (g_j,\, x_i g_{i_1},\, g_{i_2},\, g_{i_3},\, g_{i_4},\, g_{i_5}), \tag{4.14}$$
$$I_{\Delta(P)} = (g_j,\, x_i g_{i_1},\, x_i g_{i_2},\, g_{i_3},\, g_{i_4},\, g_{i_5}), \tag{4.15}$$
$$I_{\Delta(P)} = (g_j,\, x_i g_{i_1},\, x_i g_{i_2},\, x_i g_{i_3},\, g_{i_4},\, g_{i_5}), \tag{4.16}$$
$$I_{\Delta(P)} = (g_j,\, x_i g_{i_1},\, x_i g_{i_2},\, x_i g_{i_3},\, x_i g_{i_4},\, g_{i_5}), \tag{4.17}$$
$$I_{\Delta(P)} = (g_j,\, x_i g_{i_1},\, x_i g_{i_2},\, x_i g_{i_3},\, x_i g_{i_4},\, x_i g_{i_5}). \tag{4.18}$$

All are excluded except the first one (4.8). Let us for example check the equation (4.9). Assume $|T(g_i)| = 2$. Since $I' := (g_{i_1}, g_{i_2}, g_{i_3}, g_{i_4}, g_{i_5})$ is a Gorenstein ideal of codimension 3, we observe that the polytope P is a simplicial 4-polytope with 8 vertices, see Example 2.2.3.3. If we compute the monomial generators of the Gorenstein Stanley-Reisner ideal, then they are different from the generators we computed in Chapter 3, Subsection 3.2.2, a contradiction. In the case $|T(g_i)| \geq 3$ we consider that $g_j := x_i x_{h_1} \dots x_{h_k}$, then we apply at first a radial projection on P with the vertex set V, after then a stereographic projection at v'_{h_r}, such that $x_{h_r}|g_j$, where v'_{h_r} is the image of v_{h_r} under the radial projection. Since $T(g_j) \cap T(g_{i_l}) = \varnothing$ for $j \neq i_l$, we obtain the following Gorenstein Stanley-Reisner ideal $I_{\Delta(P_{h_r})} = (x_i x_{h_1} \dots x_{h_{r-1}} \dots x_{h_{r+1}} \dots x_{h_k}, x_i g_{i_1}, g_{i_2}, g_{i_3}, g_{i_4}, g_{i_5})$. We have that P_{h_r} is a simplicial $(d-1)$-polytope with $d+3$ vertices, for which the induction hypothesis does not apply.

In the second case, , if $|V''| = d + 2 = (d - 1) + 3$, where V'' is the vertex set of P_i. Since the stereographic projection at each point v'_i, thats mean the image of v_i under the function composition of the radial projection and stereographic projection is not vertex of P_i. Consider that the image of v_j under this function composition is also not vertex of P_i. It follows from Theorem 1.2.3.3 and Theorem 1.1.4.21, that $\operatorname{codim}(I_{\Delta(P_i)}) = 3$. Hence the polytope P_i is a simplicial k-polytope with $k + 3$ vertices. Since $I_{\Delta(P)}$ is generated by 6 elements, the minimal number of generators of $I_{\Delta(P_i)}$ is smaller than or equal to 6, by Corollary 4.1.0.8. Moreover by the Buchsbaum Eisenbud Theorem 2.1.2.4, the Gorenstein Stanley-Reisner ideal of the Stanley-Reisner ring of $\Delta(P_i)$ is minimally generated by an odd number of monomials and all have the same degree. Hence the number of generators is either 3 or 5.

1. The minimal number of monomial generators of $I_{\Delta(P_i)}$ is equal to 3. Let $\{g_1, g_2, g_3\}$ be the minimal set of monomial generators of $I_{\Delta_{P_i}}$. Then I is a complete intersection ideal and $T(g_1) \cap T(g_2) \cap T(g_3) = \varnothing$. We observe that $\{v_i, v_j\}$ is primitive in P. Let p_i and p'_i be two monomials which divide g_i for $i = 1, 2, 3$. By Propositions 4.1.0.3, 4.1.0.4, 4.1.0.5 and 4.1.0.6 we have the following possibilities

$$I_{\Delta(P)} = (x_i x_j,\, x_i g_1,\, x_i g_2,\, g_3,\, x_j p_1,\, x_j p_3), \tag{4.19}$$
$$I_{\Delta(P)} = (x_i x_j,\, g_1,\, g_2,\, g_3,\, x_j p_1,\, x_j p_3), \tag{4.20}$$
$$I_{\Delta(P)} = (x_i x_j,\, g_1,\, x_i g_2,\, g_3,\, x_j p_1,\, x_j p_2), \tag{4.21}$$

$$I_{\Delta(P)} = (x_i x_j,\, g_1,\, x_i g_2,\, g_3,\, x_j p_1,\, x_j p_1'), \tag{4.22}$$
$$I_{\Delta(P)} = (x_i x_j,\, g_1,\, g_2,\, g_3,\, x_j p_1,\, x_j p_1'), \tag{4.23}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i g_1,\, x_i g_2,\, x_1 g_3,\, x_j p_1,\, x_j p_3). \tag{4.24}$$

If $|T(g_i)| = 2$ for $i = 1, 2, 3$, then P in this case is a simplicial 4-polytope with 8 vertices. If we compute the monomial generators of the Gorenstein Stanley-Reisner ideal, then they are different from the generators we computed in chapter3, Subection 3.2.2, a contradiction. Otherwise we apply a radial projection of P with the vertex set V from the origin onto the unit sphere S^{d-1}, and use a stereographic projection at an appropriate projection point of V', where the image of V under this projection is denoted by V', see Remark 3.2.1.5. We obtain an ideal of codimension 4 with 6 generators, then it do not satisfy the induction hypothesis.

2. The minimal number of monomial generators of $I_{\Delta(P_i)}$ is equal to 5. Let $\{g_1, \dots, g_5\}$ be the minimal set of monomial generators of $I_{\Delta_{P_i}}$. By Example 2.2.3.3, we observe that $T(g_k) \cap T(g_{k+1}) \neq \varnothing$ (if we all the generators arrange in a certain way), for $k \in \{1, \dots, 5\}$. Moreover we observe that $\{v_i,\, v_j\}$ is primitive in P. Now we can determine the minimal set of monomial generators of $I_{\Delta(P)}$ by Propositions 4.1.0.3, 4.1.0.4, 4.1.0.5 and 4.1.0.6. There are the following possibilities

$$I_{\Delta(P)} = (x_i x_j,\, g_1,\, g_2,\, g_3,\, g_4,\, g_5), \tag{4.25}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i g_1,\, g_2,\, g_3,\, g_4,\, g_5), \tag{4.26}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i g_1,\, x_i g_2,\, g_3,\, g_4,\, g_5), \tag{4.27}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i g_1,\, x_i g_2,\, x_i g_3,\, g_4,\, g_5), \tag{4.28}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i g_1,\, x_i g_2,\, x_i g_3,\, x_i g_4,\, g_5), \tag{4.29}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i g_1,\, x_i g_2,\, x_i g_3,\, x_i g_4,\, x_i g_5). \tag{4.30}$$

All are excluded except the first one (4.25). Assume that $|T(g_k)| \geq 3$. Since $T(g_k) \cap T(g_{k+1}) \neq \varnothing$ for $k \in \{1, \dots, 5\}$, we have $s \in T(g_4) \cap T(g_5)$. Now we apply a radial projection of P with the vertex set V from the origin onto the unit sphere S^{d-1}. Then we use a stereographic projection at v_s', where v_s' is the image of v_s. We obtain a contradiction to the induction hypothesis. If $|T(g_k)| = 2$, then P is a simplicial 3-polytope with 7 vertices. We have only one type of simplicial 3-polytopes with 7 vertices such that the Gorenstein Stanley-Reisner ideal of its boundary complex is generated by 6 vertices. It is a complete intersection of a Gorenstein ideal of codimension 3 and an extra polynom.

In the third case, if $|V''| = d + 1 = (d - 1) + 2$, where V'' is the vertex set of P_i. Since the stereographic projection at each point v_i', thats mean the image of v_i under the function composition of the radial projection and stereographic projection is not vertex of P_i. Consider that the image of v_j and the image of v_l under this function composition are also not vertex of P_i. It follows $\operatorname{codim}(I_{\Delta(P_i)}) = 2$, by Theorem 1.2.3.3 and Theorem 1.1.4.21. Hence the polytope P_i is a simplicial k-polytope with $k+2$ vertices. Since $I_{\Delta(P)}$ is minimally generated by 6 elements, then the minimal number of generators of $I_{\Delta(P_i)}$ is smaller than or equal to 6, by Corollary 4.1.0.8. In this case the Gale diagram of P_i has dimension 1 and the Gorenstein Stanley-Reisner ideal of Stanley-Reisner ring of $\Delta(P_i)$ is $I_{\Delta_{P_i}} = (g_1, g_2)$ with $T(g_1) \cap T(g_2) = \varnothing$, see Chapter 2, Subsections 2.2.2 and 2.2.3. We observe that $\{v_1, v_2\}$ and $\{v_1, v_3\}$ are primitive in P. But $\{v_1, v_2, v_3\}$ is not primitive in P. Let p_i and p_i' be two monomials which divide g_i for $i = 1, 2$. Now we can determine the minimal set of monomial generators of $I_{\Delta(P)}$ using Propositions 4.1.0.3, 4.1.0.4, 4.1.0.5 and 4.1.0.6. There are the following possibilities

$$I_{\Delta(P)} = (x_i x_j,\, x_i x_l,\, g_1,\, g_2,\, x_j p_1',\, x_l p_1), \quad \text{for } \{v_2, v_3\} \text{ not primitive in } P, \tag{4.31}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i x_l,\, x_j x_l,\, g_1, g_2,\, p_1 p_2), \quad \text{for } \{v_2, v_3\} \text{ primitive in } P, \tag{4.32}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i x_l,\, x_j x_l,\, x_i g_1,\, g_2,\, p_1 p_2), \quad \text{for } \{v_2, v_3\} \text{ primitive in } P, \tag{4.33}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i x_l,\, x_j x_l,\, x_i g_1,\, x_i g_2,\, p_1 p_2), \quad \text{for } \{v_2, v_3\} \text{ primitive in } P, \tag{4.34}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i x_l,\, g_1,\, g_2,\, x_j p_1,\, x_j p_2), \quad \text{for } \{v_2, v_3\} \text{ not primitive in } P, \tag{4.35}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i x_l,\, x_i g_1,\, g_2,\, x_j p_2,\, x_j p_2), \quad \text{for } \{v_2, v_3\} \text{ not primitive in } P \tag{4.36}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i x_l,\, g_1,\, g_2,\, x_j p_1,\, x_j p_2), \quad \text{for } \{v_2, v_3\} \text{ not primitive in } P, \tag{4.37}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i x_l,\, x_i g_1,\, g_2,\, x_j p_1,\, x_j p_2), \quad \text{for } \{v_2, v_3\} \text{ not primitive in } P. \tag{4.38}$$

All are excluded except the first one (4.31). The reason is same as above.

In the fourth case, if $|V''| = d = (d-1) + 1$, where V'' is the vertex set of P_i. Since the stereographic projection at each point v_i', thats mean the image of v_i under the function composition of the radial projection and stereographic projection is not vertex of P_i. Consider that the images of v_j, v_k and v_l under this function composition are also not vertex of P_i. It follows from Theorem 1.2.3.3 and Theorem 1.1.4.21 that $\operatorname{codim}(I_{\Delta(P_i)}) = 1$. Hence this polytope P_i is a simplicial k-polytope with $k+1$ vertices. Since $I_{\Delta(P)}$ is generated by 6 elements, the minimal number of generators of $I_{\Delta(P_i)}$ is smaller than or equal to 6, by Corollary 4.1.0.8. In this case the Gale diagram of P_i has dimension 0 and the Gorenstein Stanley-Reisner ideal of the Stanley-Reisner ring of $\Delta(P_i)$ is $I_{\Delta_{P_i}} = (g_1)$, as explained in Chapter 2, Subsections 2.2.2 and 2.2.3. Let $g_1 := x_{i_5} \dots x_{i_{d+4}}$ and let p_1 and p_1' be two monomials which divide g_1. We observe that $\{v_i, v_j\}$, $\{v_i, v_k\}$, $\{v_i, v_l\}$ are primitive in P. There are the following possibilities for the minimal set of monomial generators of $I_{\Delta(P)}$ By Propositions 4.1.0.3, 4.1.0.4, 4.1.0.5 and 4.1.0.6.

$$I_{\Delta(P)} = (x_i x_j,\, x_i x_k,\, x_i x_l,\, x_j x_k,\, x_j x_l,\, g_1), \text{ for } \{v_j, v_k\},\ \{v_j, v_l\} \text{ primitive in } P, \tag{4.39}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i x_k,\, x_i x_l,\, x_j x_k,\, x_j x_l,\, x_i g_1), \text{ for } \{v_j, v_k\},\ \{v_j, v_l\} \text{ primitive in } P, \tag{4.40}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i x_k,\, x_i x_l,\, x_j x_k,\, g_1,\, x_j p_1), \text{ for } \{v_j, v_k\} \text{ primitive in } P, \tag{4.41}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i x_k,\, x_i x_l,\, x_j x_k,\, x_i g_1,\, x_j p_1), \text{ for } \{v_j, v_k\} \text{ primitive in } P, \tag{4.42}$$
$$I_{\Delta(P)} = (x_i x_j,\, x_i x_k,\, x_i x_l,\, x_i g_1,\, x_j p_1, x_k p_1'). \tag{4.43}$$

In the equation (4.43), $\{v_j, v_k\}$, $\{v_j, v_l\}$, $\{v_k, v_l\}$ are not primitive in P.

All are excluded, because if we apply a radial projection of P with the vertex set V from the origin onto the unit sphere S^{d-1} and use a stereographic projection at an appropriate projection point, we obtain a contradiction to the induction hypothesis. □

4.3 Counterexample

In this section, we show that not all Gorenstein ideals of codimension 4 with even number of monomial generators are a complete intersection of a Gorenstein ideal of codimension 3 and a polynom. We give a counterexample for that.
Simplicial 4-polytopes with 8 vertices have been classified by Grünbaum and Sreedharan in [GS67]. There is for example P^8_{35}. The Gorenstein Stanley-Reisner ideal associated to P^8_{35} is generated by an even number of monomials, but they are not complete intersection of a Gorenstein ideal of codimension 3 and a polynom. In Chapter 3, Subection 3.2.2 we have determined the minimal set of monomial generators of this ideal.

CHAPTER

FIVE

CHARACTERIZATION OF MONOMIAL GENERATORS OF GORENSTEIN IDEALS OF CODIMENION 4

Let P be a simplicial d-polytope with $d+4$ vertices and $\Delta(P)$ be the boundary complex of P. Let $\mathbb{K}$ be an algebraically closed field and $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \ldots, x_{d+4}]/I_{\Delta(P)}$ the Stanley-Reisner ring of $\Delta(P)$. In Chapter 3, Subection 3.2.2, we explained for cases $d = 3, 4$ how to determine the minimal set of monomial generators of the associated Gorenstein Stanley-Reisner ideals $I_{\Delta(P)}$ using Gale diagrams. Gale diagrams of simplicial d-polytopes with $d+4$ vertices are subsets of $\mathbb{R}^3$, but affine Gale diagrams are of one dimension lower that the well-known Gale diagrams. Therefore simplicial d-polytopes with $d+4$ vertices can be represented by planar point configurations. In this chapter, we sketch the associated affine Gale diagrams of all combinatorial types of simplicial d-polytopes with $d+4$ vertices, for $d = 3, 4$. Then we characterize the minimal set of monomial generators of the Gorenstein Stanley-Reisner ideal $I_{\Delta(P)}$ using affine Gale diagrams, for an arbitrary d. Our primary references in this chapter are [Zie95, Section 6.4], [Stu88], [Dev11, Section 3] and [Gal].

5.1 Affine Gale diagrams

Ziegler gave in [Zie95, Section 6.4] a formal definition of an affine Gale diagram of a d-polytope with n vertices. Let P be a d-polytope with the vertex set $V = \{v_1, \ldots, v_n\}$. We showed in Chapter 2, Subsection 2.2.1, that the Gale diagram $\hat{\mathfrak{B}} = (\hat{v}_1, \ldots, \hat{v}_n)$ of P is contained in the unit $(n-d-2)$-sphere. Now we choose an arbitrary hyperplane e, so that no vector of $\hat{\mathfrak{B}}$ pokes out through the hyperplane e in $\mathbb{R}^{n-d-2}$. Either e or another hyperplane, which is parallel to e, divides the sphere into two hemispheres. If we look at this sphere from outside, we only see one hemisphere, which we call the northern hemisphere. Then we project the configuration $\hat{\mathfrak{B}}$ from the origin to the hyperplane e. The image of $\hat{v}_i$ under this project is denoted by v_i^*. If v_i^* is belong to the southern hemisphere, then we mark it with a black point and denote it by i. If v_i^* is belong to the northern hemisphere, we mark it with a white point and denote it by $\bar{i}$. So we obtain a configuration $\mathfrak{B}^* = \{v_1^*, \dots, v_n^*\}$ of black and white points in $\mathbb{R}^{n-d-2}$, see Figure 5.1.

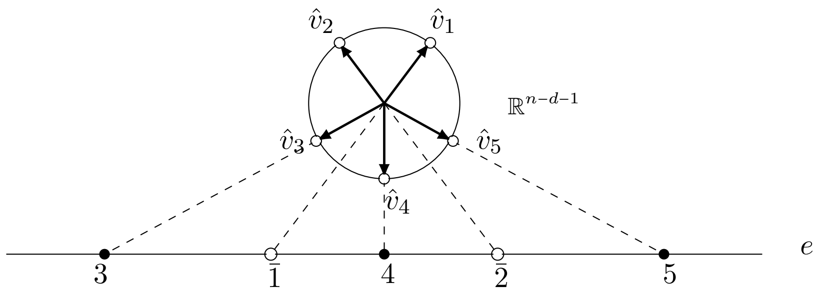


Figure 5.1: Affine Gale diagram.

Definition 5.1.0.1. Let P be a d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$. A configuration $\mathfrak{B}^* = \{v_1^*, \dots, v_n^*\}$ of black and white points in affine space $\mathbb{R}^{n-d-2}$ is called an *affine Gale diagram* of P.

Remark 5.1.0.2. Let P be a d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$, the configuration $\hat{\mathfrak{B}} = \{\hat{v}_1, \dots, \hat{v}_n\}$ be the Gale diagram of P and $\mathfrak{B}^* = \{v_1^*, \dots, v_n^*\}$ be an affine Gale diagram of P. There is a canonical bijection between $\hat{\mathfrak{B}}$ and $\mathfrak{B}^*$, where the point v_i corresponds to v_i^*. Therefore there is also a canonical bijection between the points v_i of V and the points v_i^* of $\mathfrak{B}^*$.

Remark 5.1.0.3. An affine Gale diagram $\mathfrak{B}^*$ of the polytope P is not determined by the combinatorics of the vertex set of P, since the choice of a hyperplane e is involved. On the other hand, the combinatorics of an affine Gale diagram determines the combinatorics of a Gale diagram, and therefore it determines the combinatorics of the original set of points, see [Dev11].

Definition 5.1.0.4. We say that two affine Gale diagrams are *combinatorially equivalent* if there is a bijection between the two sets of points preserving the colours and the orientations of Gale transforms vectors. That means, interchanging black and white points does not change the combinatorial type of the polytope.

In [Zie95], Ziegler has characterized affine Gale diagrams of polytopes by the following corollary.

Corollary 5.1.0.5. [Zie95, Corollary 6.20] *A point configuration $A = \{a_1, \dots, a_n\} \subset \mathbb{R}^{n-d-2}$, each of them declared to be either black or white, that affinely spans $\mathbb{R}^{n-d-2}$, is the affine Gale diagram of a d-polytope with n vertices if and only if the following condition is satisfied: for every oriented hyperplane H in $\mathbb{R}^{n-d-2}$ spanned by some points of A, the number of black points on the positive side of H plus the number of white points on the negative side of H is at least two.*

In this chapter, we discuss only simplicial d-polytopes. From Chapter 2, Subsection 2.2.2, we know that Gale diagrams of simplicial d-polytopes with $d+1$ vertices are in the 0-dimensional space, so all vectors are equal to the 0-vector trivially. Gale diagrams of simplicial d-polytopes with $d+2$ vertices are subsets of the 1-dimensional space, so affine Gale diagrams are subsets of the 0-dimensional space and may be represented by a "cloud" of black and white points, see Figure 5.2.

Example 5.1.0.6. Let P be a simplicial 3-polytope with 5 vertices, which has been already considered in Example 2.2.2.6.

Figure 5.2: Affine Gale diagram of the polytope P.

For a simplicial 3-polytope with $d+3$ vertices, we know that Gale diagrams are in a 2-dimensional space, so affine Gale diagrams are in 1-dimensional space, i.e., a line, see Figure 5.3. The special case of simplicial polytopes has been treated in Grünbaum [Grü03, Section 6.2].

Example 5.1.0.7. Let P be a simplicial 3-polytope with 6 vertices, which has already considered in Example 2.2.3.4, where $\hat{\mathfrak{B}} = \{\hat{v}_1 = (1,0), \hat{v}_2 = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), \hat{v}_3 = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), \hat{v}_4 = (1,0), \hat{v}_5 = (0,1), \hat{v}_6 = (0,1)\}$. Now we choose an arbitrary line e, see Figure 5.3, then we project the configuration $\hat{\mathfrak{B}}$ from the origin to the hyperplane e. Let us declare the left hemisphere to be the southern hemisphere. The Gale vector $\hat{v}_2\hat{v}_3$ intersect this hemisphere. We mark the images of point $\hat{v}_2\hat{v}_3$, under this projection with black dot and label them 23. We mark the images of points $\hat{v}_1\hat{v}_4$ and $\hat{v}_5\hat{v}_6$ under this projection with black dots and label them $\bar{1}\bar{4}$ and $\bar{5}\bar{6}$, respectively. The affine Gale diagram of P is the configuration $\mathfrak{B}^* = \{v_1^* = \bar{1}\bar{4}, v_2^* = 23, v_3^* = \bar{5}\bar{6}\}$.

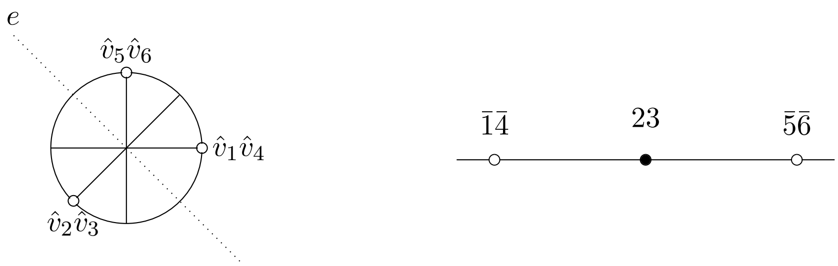
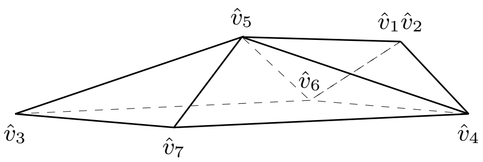


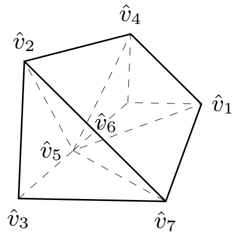
Figure 5.3: Affine Gale diagram of the polytope P.

5.2 Affine Gale diagrams of simplicial d-polytopes with $d+4$ vertices, for $d = 3, 4$

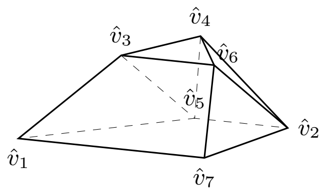
Now we are interested in simplicial d-polytopes with $d+4$ vertices. In this case affine Gale diagrams are subsets of $\mathbb{R}^2$. As a starting point, we begin with simplicial 3 polytopes with 7 vertices. In Chapter 3, Subsection 3.2.2, we gave a method to compute the Gale diagram of these polytopes. The Schlegel diagrams of these polytopes are presented in Remark 3.2.2.5. Since these polytopes are in $\mathbb{R}^3$, we know the coordinates of their vertices, hence we can easily compute their Gale diagrams.



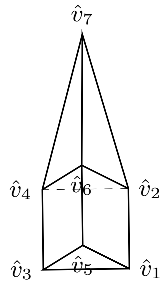
Gale diagram of D_1.



Gale diagram of D_3.

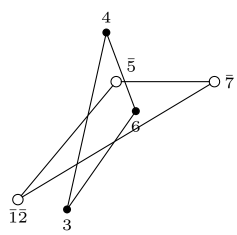


Gale diagram of D_4.

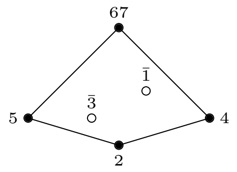


Gale diagram of D_5.

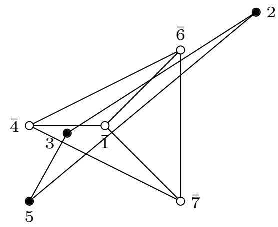
Now we choose an arbitrary plane e and project the configuration of Gale vectors of the vertex sets of D_1, D_2, D_3, D_4 and D_5, respectively, from the origin to the hyperplane e. Then we can compute the white and black points. We obtain the following affine Gale diagrams.

Polytope	**Affine Gale diagram**
D_1	

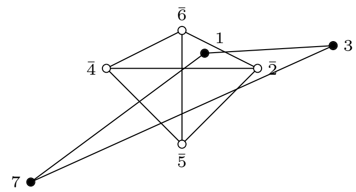
D_2



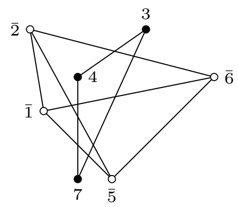
D_3



D_4



D_5



Now we focus on simplicial 4-polytopes with 8 vertices, which have been classified by Grünbaum and Sreedharan in [GS67]. In this paper, they only give the facets of these polytopes. Therefore it is very difficult to compute Gale diagrams of some of them, as we don't have the coordinates of the vertices of these polytopes. To determine affine Gale diagrams of these polytopes we need the following criterion.

Notation 5.2.0.1. Let P be a d-polytope with the vertex set $V = \{v_1, \dots, v_n\}$. Given a subset $M \subset V$, we denote the subset of its corresponding points of $\mathfrak{B}^*$ by M^*.

Ziegler gives in [Zie95] a corollary that can be used to read off the properties of a polytope from its affine Gale diagram.

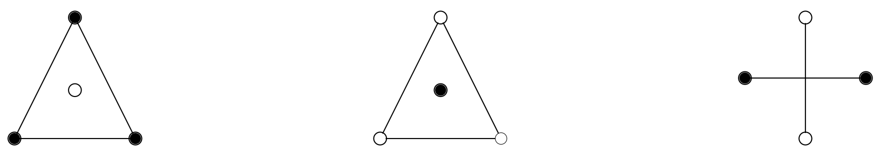
Corollary 5.2.0.2. *[HRGZ97, Page 260] Let V be the vertex set of a simplicial d-polytope P and $\mathfrak{B}^*$ an affine Gale diagram of P. A set $M \subset V$ is the vertex set of a face of P if and only if the set $\mathfrak{B}^* \setminus M^*$ satisfies the following condition: If we remove the points of M^* from the diagram, then the relative interiors of the sets*

$$\mathrm{conv}(\{A \in \mathfrak{B}^* \setminus M^* : A \text{ is a black point}\}) \text{ and } \mathrm{conv}(\{A \in \mathfrak{B}^* \setminus M^* : A \text{ is a white point}\})$$

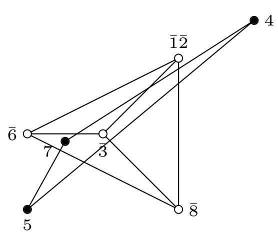
have nonempty intersection.

Proof. Let $V = \{v_1, \dots, v_n\}$ and $M = \{v_j : j \in J \subseteq \{1, \dots, n\}\}$. Assume that $F :=$ conv$(\{v_j : j \in J \subseteq \{1, \dots, n\}\})$ is a face of P. Then it follows from Theorem 2.2.1.9 that $0 \in$ relint(conv$(\{\hat{v}_k : k \notin J\})$), because $M \subset V$. That means that there is a strictly positive linear dependence $\sum_{k \notin J} b_k \hat{v}_k = 0$ with $b_i > 0$, see [Mar84, Theorem 3] and [Mar84, Lemma 1]. Now if we turn to the affine Gale diagram and project all the Gale vectors from the origin to an arbitrary hyperplane e, we obtain a configuration of black and white points. The bicolored points $\{v_k^* : k \notin J\}$ are affine dependent, with positive coefficients on the black points, and with negative coefficients on the white points, see definition [Zie95, Definition 6.17]. Equivalently, the convex hull of all the black points not in our set and the convex hull of all the white points not in the set, intersect in their relative interiors, see [Zie95, Section 6.1 (a)]. ☐

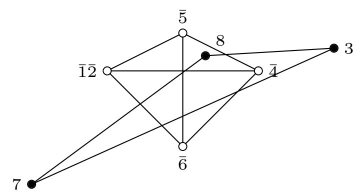
Corollary 5.2.0.3. *Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_{d+4}\}$. We can formulate Criterium 2.2.1.9 differently. A set $M \subset V$ is the vertex set of a face of P if and only if there are four points of the set $\mathfrak{V}^* \setminus M^*$, which have one of the following forms*



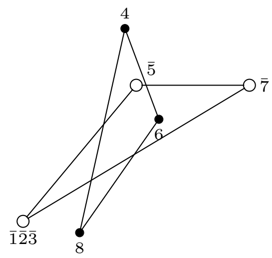
With this criterion, we can determine affine Gale diagrams of all combinatorial types of simplicial 4-polytopes with 8 vertices. We take each polytope alone, then we choose an arbitrary plane e and we take the following steps: we suggest, that images of some points of the vertex set of a polytope are white points and the other black points, then we use Corollary 5.2.0.3 and check the criterion. We can make that, since the affine Gale diagram of the polytope does not need a specific plane. After that we can sketch the affine Gale diagram and we obtain the following table.

Polytope	**Affine Gale diagram**
P_1^8	

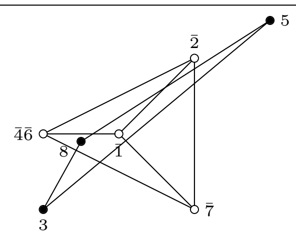
P_2^8



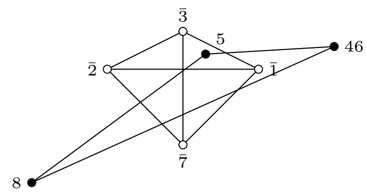
P_3^8



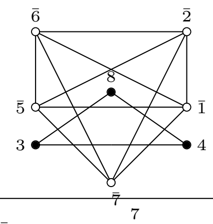
P_4^8



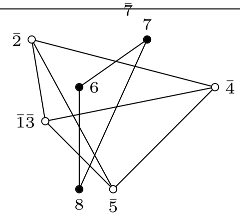
P_5^8



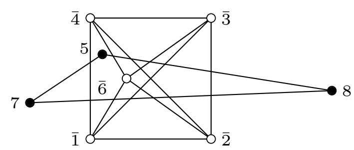
P_6^8

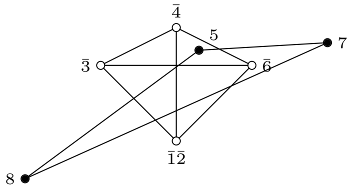
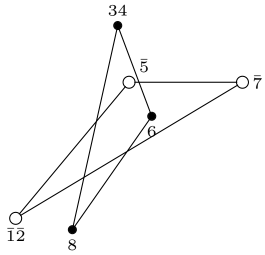
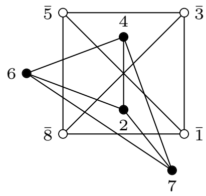
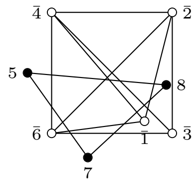
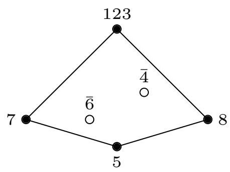
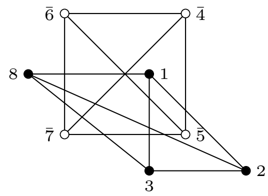
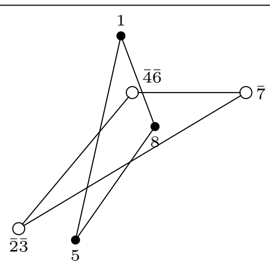


P_7^8

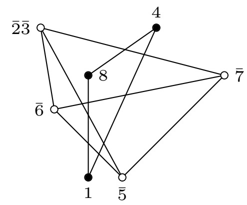


P_8^8

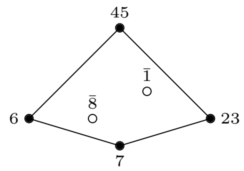


P^8_9	
P^8_{10}	
P^8_{11}	
P^8_{12}	
P^8_{13}	
P^8_{14}	
P^8_{15}	

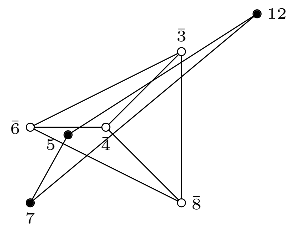
P^8_{16}



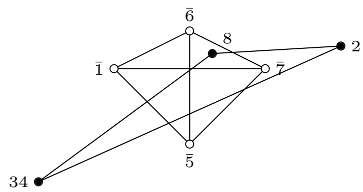
P^8_{17}



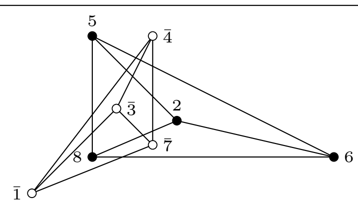
P^8_{18}



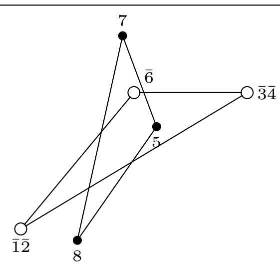
P^8_{19}



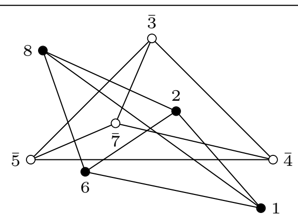
P^8_{20}



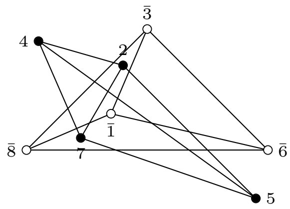
P^8_{21}



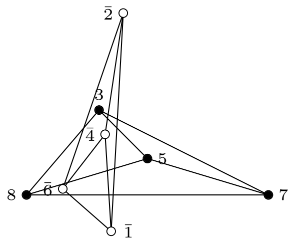
P^8_{22}



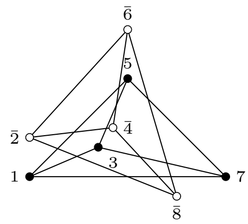
P^8_{23}



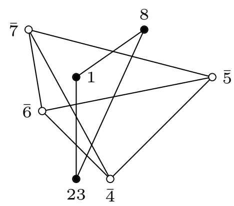
P^8_{24}



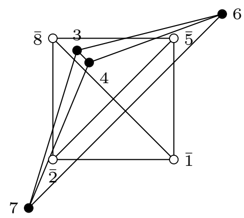
P^8_{25}



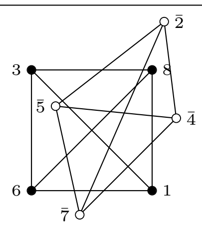
P^8_{26}



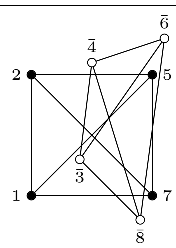
P^8_{27}



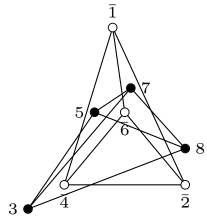
P^8_{28}



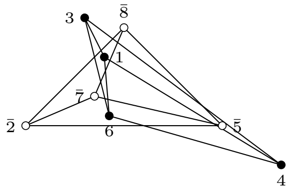
P^8_{29}



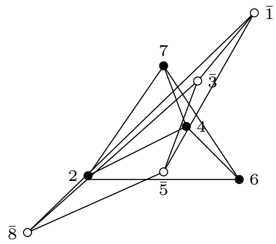
P^8_{30}



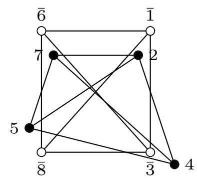
P^8_{31}



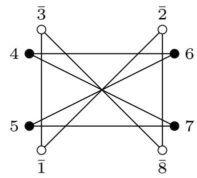
P^8_{32}



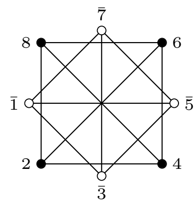
P^8_{33}



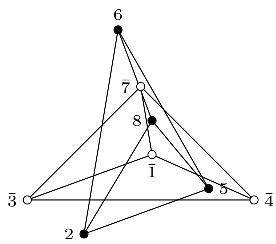
P^8_{34}

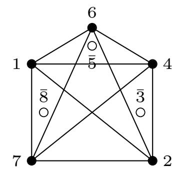


P^8_{35}



P^8_{36}



P^8_{37}	

Remark 5.2.0.4. We would like to give an example of Remark 5.1.0.3. By working out these affine Gale diagrams, we observed that P^8_2 and P^8_6 both have the same Gale diagram. But the affine Gale diagrams are not necessarily the same, since they depend on the choice of a plane e. The polytope P^8_6 can have the same affine Gale diagram of P^8_2 when we choose an appropriate plane e.

5.3 Generators of Gorenstein Stanley-Reisner ideals of codimension 4

In this section, we characterize the Gorenstein Stanley-Reisner ideal corresponding to a given simplicial d-polytope with $d+4$ vertices using affine Gale diagrams. This characterization applies not only to simplicial d-polytopes with $d+4$ vertices, but also for simplicial d-polytopes with n vertices. In this section, we deal with this criterion to simplicial d-polytopes with $d+4$ vertices, because in the next chapter 6 we deal with special d-polytopes with $d+4$ vertices.

Remark 5.3.0.1. The notiation in the next theorem is as in Remark 5.1.0.2.

Theorem 5.3.0.2. *Let P be a simplicial d-polytope with the vertex set $V = \{v_1, \dots, v_{d+4}\}$, the boundary complex $\Delta(P)$ and an ffine Gale diagram $\mathfrak{B}^* = \{v^*_1, \dots, v^*_{d+4}\}$. Let $\mathbb{K}$ be an algebraically closed field and $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_{d+4}]/I_{\Delta(P)}$ the Stanley-Reisner ring of $\Delta(P)$. A monomial $x_{i_1} \dots x_{i_k}$ is an element of the minimal set of monomial generators of the Gorenstein Stanley-Reisner ideal $I_{\Delta(P)}$ if and only if the set $\mathfrak{B}^* \smallsetminus \{v^*_{i_1}, \dots, v^*_{i_k}\}$ satisfies the following condition: The black and white points can be split by an affine hyperplane. Morevore, there is no superset of $\mathfrak{B}^* \smallsetminus \{v^*_{i_1}, \dots, v^*_{i_k}\}$, which satisfies the previous condition.*

Proof. Since the minimal set of monomial generators of the Gorenstein Stanley-Reisner ideal $I_{\Delta(P)}$ is the set of minimal nonfaces, see Remark 1.2.3.2, the theorem holds by Corollary 5.2.0.2. □

Example 5.3.0.3. Let P^{37}_8 be a simplicial 4-polytope with 8 vertices $V = \{v_1, \dots, v_8\}$ and $\mathbb{K}[\Delta(P^{37}_8)] = \mathbb{K}[x_1, \dots, x_{d+4}]/I_{\Delta(P^{37}_8)}$ the associated Stanley-Reisner ring. Let $\mathfrak{B}^*$ be the affine Gale diagram considered in the previous table. We shall use the criterion in Theorem 5.3.0.2 to determine the minimal set of monomial generators of the associated Gorenstein Stanley-Reisner ideal $I_{\Delta(P^{37}_8)}$. For example, if we take the set $\{v^*_{i_1} = 1,\ v^*_{i_2} =$

$\bar{3}$, $v^*_{i_3} = 7\}$, we observe that its complement $\{2, 4, \bar{5}, 6, \bar{8}\}$ satisfies the above condition. That means, if we remove the points $\{v^*_{i_1} = 1$, $v^*_{i_2} = \bar{3}$, $v^*_{i_3} = 7\}$ from the affine Gale diagram, then the remaining black and white points $\{2, 4, \bar{5}, 6, \bar{8}\}$ can be splitted by a straight line. Therefore, there is no set in $\mathfrak{B}^*$, which is contained in the set $\{v^*_{i_1} = 1$, $v^*_{i_2} = \bar{3}$, $v^*_{i_3} = 7\}$ and its complement satisfies the previous condition.

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CHAPTER

SIX

ON THE STRUCTURE OF GORENSTEIN IDEALS OF CODIMENSION 4 ASSOCIATED TO CYCLIC POLYTOPES

In this chapter, we begin by explaining the complex of Gulliksen and Negård [GN72], which describes the structure of a minimal graded free resolution of a quotient module obtained from a polynomial ring modulo a Gorenstein ideal of codimension 4. In this case, we can use this complex to prove that the Gorenstein ideal is generated by minors of a squar matrix. Our first aim in this chapter is to prove that minimal graded free resolutions of Stanley-Reisner rings associated to cyclic $2d$-polytopes with $2d+4$ vertices can be considered as a special versions of the Gulliksen-Negård complex. The second aim is to characterize the minimal sets of monomial generators of Gorenstein ideals associated to special neighbourly $2d$-polytopes with $2d+4$ vertices, which are different from cyclic polytopes and were classified by Devyatov [Dev11]. Moreover, we raise a conjecture that only for cyclic $2d$-polytopes with $2d+4$ vertices minimal graded free resolutions of associated Stanley-Reisner rings can be considered as a special version of Gulliksen-Negård complex. Finally, our third aim is to make an important step, that may help to prove our conjecture, namely we prove that the minimal graded free resolutions of Stanley-Reisner rings associated to Devyatov's polytopes, which are not cyclic, can not be regarded as a version of the Gulliksen-Negård complex.

6.1 The complex of Gulliksen and Negård

In [GN72], Gulliksen and Negård considered the polynomial ring $S = \mathbb{K}[x_{11}, \dots, x_{rs}]$ with rs variables and a Gorenstein ideal I, which is generated by the t-minors of the matrix (x_{ij}) for $1 \le i \le r$ and $1 \le j \le s$. The authors gave for $n = r = s$ and $t = n-1$ an explicit minimal graded free resolution of the quotient module S/I, where I has codimension 4.

Let S be a polynomial ring in n^2 variables and $\mathcal{M}_n(S)$ the ring of $n \times n$-matrices with entries in S. Then $\mathcal{M}_n(S)$ is a free S-module of rank n^2, i.e. $\mathcal{M}_n(S) \cong S^{n^2}$. Let $A \in \mathcal{M}_n(S)$, then there is a complex of S-modules as the following

$$\mathbf{F}: \ 0 \longrightarrow S \xrightarrow{d_4} S^{n^2} \xrightarrow{d_3} F_2 \xrightarrow{d_2} S^{n^2} \xrightarrow{d_1} S \longrightarrow 0.$$

To determine F_2, we consider the zero-sequence ($\pi \circ \imath = 0$)

$$S \xrightarrow{\imath} S^{n^2} \bigoplus S^{n^2} \xrightarrow{\pi} S,$$

where $\imath(s) = (sE, sE)$, E being the unit matrix of $\mathcal{M}_n(S) \cong S^{n^2}$, and $\pi(V_1, V_2) = \mathrm{trace}(V_1 - V_2)$, for V_1 and V_2 in S^{n^2}. Let E_{ij}, $1 \le i, j \le n$, be the canonical basis of S^{n^2}. Then $\mathrm{Ker}(\pi)$ is generated by the elementes $(E_{ij}, 0)$, $i \ne j$, $(0, E_{uv})$, $u \ne v$, (E_{ij}, E_{11}), $1 \le i \le n$, and $(0, E_{uu} - E_{11})$, $2 \le u \le n$. Since $\mathrm{Im}(\imath)$ is generated by $\sum_{i=1}^n (E_{ii}, E_{ii}) = \sum_{i=1}^n (E_{ii}, E_{11}) + \sum_{u=2}^n (0, E_{uu} - E_{11})$, $F_2 := \mathrm{Ker}(\pi)/\mathrm{Im}(\imath)$ is a free S-module of rank $2n^2 - 2$. Then we get the following complex of S-modules:

$$\mathbf{F}: \ 0 \longrightarrow S \xrightarrow{d_4} S^{n^2} \xrightarrow{d_3} S^{2n^2-2} \xrightarrow{d_2} S^{n^2} \xrightarrow{d_1} S \longrightarrow 0.$$

Now we want to determine the maps d_i, for $i = 1, \ldots, 4$. Let $A^{\#^T}$ be the matrix of cofactors of A, i.e.

$$A^{\#^t} = \begin{bmatrix} A_{11} & \cdots & (-1)^{1+n} A_{1n} \\ \vdots & \ddots & \vdots \\ (-1)^{1+n} A_{n1} & \cdots & A_{nn} \end{bmatrix}^t$$

where the A_{ij} are the $(n-1)$-minors of A (determinants of $(n-1) \times (n-1)$-submatrices). We put $d_1(V_1) := \mathrm{trace}(A^{\#^T} V_1)$, $d_4(s) := sA^{\#^t}$. To define d_2, d_3, we consider the zero sequence ($\varphi \circ \psi = 0$)

$$S^{n^2} \xrightarrow{\psi} S^{n^2} \oplus S^{n^2} \xrightarrow{\varphi} S^{n^2},$$

where $\psi(V_1) = (AV_1, V_1 A)$ and $\varphi(V_1, V_2) = V_1 A - AV_2$. Clearly $\mathrm{Im}(\imath) \subset \mathrm{Ker}(\varphi)$ and $\mathrm{Im}(\psi) \subset \mathrm{Ker}(\pi)$ so that we may define d_2, d_3 as the maps induced by φ and ψ, resp. Since $\mathrm{Im}(\imath) \subset \mathrm{Ker}(\varphi)$ and $\mathrm{Ker}(\varphi) \subseteq S^{n^2} \oplus S^{n^2}$, it follows that $\varphi|_{\mathrm{Im}(\imath)} = 0$ and $\varphi|_{\mathrm{Ker}(\pi)}$ is well defined. So we define $d_2 : \mathrm{Ker}(\pi)/\mathrm{Im}(\imath) \to S^{n^2}$, $d_2((V_1, V_2) + \mathrm{Im}(\imath)) = \varphi(V_1, V_2) = V_1 A - AV_2$. We consider the sequence

$$\begin{tikzcd} S^{n^2} \arrow[r, "\psi"] \arrow[rr, bend right, "d_3"'] & \mathrm{Ker}(\pi) \arrow[r, "\phi"] & \mathrm{Ker}(\pi)/\mathrm{Im}(\imath), \end{tikzcd}$$

since $\mathrm{Im}(\psi) \subset \mathrm{Im}(\pi)$, we have $d_3(V_1) = \phi \circ \psi(V_1) = \phi(AV_1, V_1 A) = (AV_1, V_1 A) + \mathrm{Im}(\imath)$. A trivial calculation shows that $d_i \circ d_{i+1} = 0$, for $i = 1, 2, 3$, whence $\mathbf{F}$ is in fact a complex over S. Since $d_1(V_1) = \mathrm{trace}(A^{\#^T} V_1)$, it is $\mathrm{Im}(d_1) = I_{n-1}(A)$, where we denote by $I_{n-1}(A)$ the ideal generated by the $(n-1)$-minors of A.

Theorem 6.1.0.1. [GN72, Theorem 2.26] *Let S be a polynomial ring over a field and A an $n \times n$-matrix with enteries in S. If* $\mathrm{height}(I_{n-1}(A)) \ge 4$, *then* $\mathbf{F}$ *is acyclic.*

The minimal graded free resolution of $S/I_{n-1}(A)$, where $I_{n-1}(A)$ has codimension 4 is the following

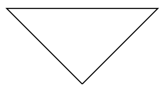
$$\mathbf{F}: \quad 0 \longrightarrow S \xrightarrow{d_4} S^{n^2} \xrightarrow{d_3} S^{2n^2-2} \xrightarrow{A} S^{n^2} \xrightarrow{d_1} S \longrightarrow S/I_{n-1}(A) \longrightarrow 0.$$

6.2 Gorenstein ideals of codimension 4 associated to neighbourly polytopes

Definition 6.2.0.1. A *neighbourly* d-polytope is a convex d-polytope, such that any set of vertices of cardinality $\lfloor d/2 \rfloor$ spans a face. A polytope is called k*-neighbourly* if any set of k vertices spans a face.

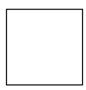
Definition 6.2.0.2. Let $1 \leq i \leq n$, $t_i \in \mathbb{R}$ with $t_1 < t_2 < \cdots < t_n$. The *cyclic* d-polytope with n vertices $C = C_d(t_1, \ldots, t_n)$ is the convex hull of the subset $\{f(t_1), f(t_2), \ldots, f(t_n)\} \subset \mathbb{R}^d$, where $f : \mathbb{R} \to \mathbb{R}^d$ with $f(t) = (t, t^2, \ldots, t^d)$ for $t \in \mathbb{R}$.

Example 6.2.0.3. Let P be a 2-polytope with the vertex set $V = \{(-1,1), (0,0), (1,1)\}$. Then P is a cyclic polytope, because $d = 2$, $t_1 = -1$, $t_2 = 0$ and $t_3 = 1$.



Remark 6.2.0.4. A cyclic d-polytope with n vertices is a simplicial d-polytope, which up to combinatorial equivalence does not depend on the choice the point t_i in Definition 6.2.0.2. But not every simplicial d-polytope is cyclic, see Example 6.2.0.5.

Example 6.2.0.5. Let P be a simplicial 2-polytope with the vertex set $V = \{(1,1), (-1,1), (-1,-1), (1,-1)\}$. This polytope is not cyclic, because the equation $t^2 = -1$ has no solution.



Corollary 6.2.0.6. [Zie95, Corollary 0.8] *A cyclic polytope is an example of a neigbourly polytope.*

We are interested in Gorenstein Stanley-Reisner ideals associated to cyclic d-polytopes with $d+4$ vertices. Since every cyclic polytope is a neighbourly polytope, see Corollary 6.2.0.6, we consider only neighbourly polytopes in this section. Grünbaum and Sreedharan construct all simplicial neighbourly 4-polytopes with 8 vertices in [GS67]. There are exactly three combinatorial types of such polytopes, two of them P^8_{36}, P^8_{37} are not cyclic, and the other one P^8_{35} is cyclic, see Chapter 3, Subsection 3.2.2. In 1981 Barnette [Bar81] construct a family of neighbourly polytopes that are not cyclic in any dimension. After that in 1982 Shemer [She82] shows that the number of combinatorially different neighbourly $2d$-polytopes with $2d+4$ vertices grows superexponentially as $d \to \infty$. In 1987 all neighbourly 6-polytopes with 10 vertices are classified by Bokowski and Shemer [BS87]. There are 37 combinatorial types of them. In 2011 Devyatov [Dev11] classified neighbourly $2d$-polytopes with $2d+4$ vertices, which have a planer Gale diagram of a special type with exactly $d+3$ black points in convex position. Four years ago Finbow in [FS04], [Fin10] and [Fin15] published a list of the simplicial neighbourly 5-polytopes with 9 vertices. There are exactly 126 combinatorially distinct types of such polytopes.

In 1996 Teria and Hibi [TH96] compute the Betti numbers of the minimal graded free resolution of the Stanley-Reisner ring of the boundary complex of a cyclic polytope. Then in 2010 Böhm and Papadakis [BP12] study the structure of Stanley-Reisner rings associated to cyclic polytopes and show how to express the Stanley-Reisner ring of cyclic d-polytope with $n+1$ vertices in terms of the Stanley-Reisner rings of a cyclic d-polytope with n vertices and a cyclic $(d-2)$-polytope with $n-1$ vertices.

Let C be a cyclic $2d$-polytope with $2d+4$ vertices and $\Delta(C)$ the boundary complex of C. Let $\mathbb{K}[\Delta(C)]$ be the associated Stanley-Reisner ring to C. Then the minimal graded free resolution of $\mathbb{K}[\Delta(C)]$ over $S := \mathbb{K}[x_1,\dots,x_{2d+4}]$, as it is clarified in [TH96], is of the form

$$0 \longrightarrow S(-(2d+4)) \longrightarrow S(-(d+3))^{b_3^S} \longrightarrow S(-(d+2))^{b_2^S} \longrightarrow$$
$$S(-(d+1))^{b_1^S} \longrightarrow S \longrightarrow \mathbb{K}[\Delta(C)] \longrightarrow 0,$$

where $b_1^S = (d+2)^2$, $b_2^S = 2(d+3)(d+1)$ and $b_3^S = (d+2)^2$.

That means that Gorenstein Stanley-Reisner ideals associated to cyclic $2d$-polytope with $2d+4$ are generated by $(d+2)^2$ monomials of degree $d+1$.

Because of that, we verify whether in this chapter that cyclic polytopes have also an important role for associated Gorenstein ideals of codimension 4.

Conjecture 6.2.0.7. Let P be a simplicial neighbourly $2d$-polytope with $2d+4$ vertices. The polytope P is cyclic if and only if there exists a $(d+2)\times(d+2)$-matrix A, so that all its $(d+1)$-minors generate minimally the Gorenstein Stanley-Reisner ideal associated to P.

The direct assertion of this conjecture means that the minimal graded free resolutions of Stanley-Reisner rings associated to cyclic $2d$-polytopes with $2d+4$ vertices can be considered as a special version of the Gulliksen-Negård complex to a $(d+2)\times(d+2)$-matrix. We prove this direction completely. For the reversal assertion, we show Conjecture 6.2.0.7 partially. In [Dev11], Devyatov classified special neighbourly $2d$-polytopes with $2d+4$ vertices which are not cyclic. We prove for each polytope of Devyatov's polytopes that the associated Gorenstein Stanley-Reisner ideal is generated by exactly $(d+2)^2$ monomials and all have degrees $d+1$, but there is no square $(d+2)\times(d+2)$-matrix, so that its $(d+1)$-minors generate it. That means that the minimal graded free resolutions of associated Stanley-Reisner rings to Devyatov's polytopes can not be regarded as a version of the Gulliksen-Negård complex.

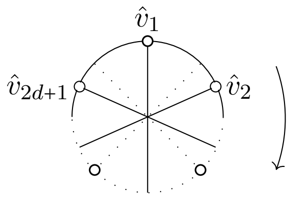
6.2.1 Gorenstein ideals of codimension 3 associated to cyclic polytopes

Our starting point is cyclic d-polytopes with $d+3$ vertices. For Gorenstein Stanley-Reisner ideals of codimension 3 associated to these polytopes, the structure theory of Buchsbaum and Eisenbud states, the minimal number of generators of each such ideal is an odd $2m+1 \geq 3$ and that this minimal system of generators is given by the $2m+1$ Pfaffians of order $2m$ of a skew-symmetric $(2m+1)\times(2m+1)$-matrix A. We describe now this matrix explicitly.

Theorem 6.2.1.1. *Let P be a cyclic $(2d-2)$-polytope with the vertex set $V=\{v_1,\dots,v_{2d+1}\}$ and the boundary complex $\Delta(P)$. Let $\mathbb{K}[\Delta(P)]=\mathbb{K}[x_1,\dots,x_{2d+1}]/I_{\Delta(P)}$ be the Stanley-Reisner ring of $\Delta(P)$ and $I_{\Delta(P)}$ the Gorenstein Stanley-Reisner ideal associated to P. Then the monomial generators of $I_{\Delta(P)}$ are $2d$-th order Pfaffians of the following skew-symmetric $(2d+1)\times(2d+1)$-matrix A of degree d.*

$$A=\left[\begin{array}{ccccccccccc}
 & & & & & -x_{2d+1} & x_{d+1} & & \cdots & & 0\\
 & & & & & 0 & -x_1 & x_{d+2} & & & \\
 & & 0 & & & \vdots & & & & & \vdots\\
 & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & 0\\
 & & & & & & & & & -x_{d-1} & x_{2d}\\
x_{2d+1} & & \cdots & \cdots & & 0 & & \cdots & \cdots & & -x_d\\
-x_{d+1} & x_1 & & & & & & & & & \\
0 & -x_{d+2} & x_2 & & & & & & & & \\
\vdots & & & & & \vdots & & & 0 & & \\
 & & & \ddots & \ddots & & & & & & \\
\vdots & & & & & 0 & & & & & \\
0 & \cdots & & & -x_{2d} & x_d & & & & &
\end{array}\right]$$

Proof. The Gorenstein Stanley-Reisner ideal associated to P is generated by $(2d+1)$ monomials of degree d, see [TH96, Proposition 3.1], therefore the Gale diagram of a cyclic polytope is contracted and distended in the same time, see Definition 2.2.2.11. That means the Gale diagram of P has the following



Using Gale diagram we can determine all monomial generators of $I_{\Delta(P)}$, see Chapter 2, Subsection 2.2.3. These generators are the following

$$\begin{aligned}\mathrm{Pf}(A_1) &:= x_1 \dots x_d,\\ \mathrm{Pf}(A_2) &:= x_2 \dots x_{d+1},\\ &\vdots\\ \mathrm{Pf}(A_{d+1}) &:= x_{d+1} \dots x_{2d},\\ &\vdots\\ \mathrm{Pf}(A_{2d+1}) &:= x_{2d+1} \dots x_{d-1}.\end{aligned}$$

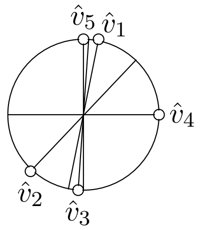
By Eisenbud-Buchsbaum Theorem 2.1.2.4 the generators are $2d$-th order Pfaffians of a skew-symmetric. We obtain this matrix using construction of the monimal graded free resolution of $\mathbb{K}[\Delta(P)]$. We consider the Step 2 of Construction 1.1.2.10. Let $\alpha_1 f_1 + \dots + \alpha_{2d+1} f_{2d+1} \in \mathrm{Ker}(d_1)$, where $d_1 : \bigoplus_{i=1}^{2d+1} S \to \bigoplus_{i=1}^{2d+1} S$ and $\alpha_i \in S$. Hence $\sum_{i=1}^{2d+1} \alpha_i \mathrm{pf}(A_i) = 0$ and it follows all solutions $(\alpha_1, \dots, \alpha_{2d+1})$ are generated by the columns of a $(2d+1) \times (2d+1)$-skew-symmetric matrix A, see Buchsbaum-Eisenbud Theorem 2.1.2.4. Thus we obtain $2d+1$ homogeneous generators of $\mathrm{Ker}(d_1)$. Their degrees are $d-1$, see [TH96, Proposition 3.1]. That means each α_i has degree 1. It follows all solutions $(\alpha_1, \dots, \alpha_{2d+1})$ are generated by

$$\begin{gathered}(0, \dots, 0, -x_{2d+1}, x_{d+1}, 0, \dots, 0), \quad (0, \dots, 0, -x_1, x_{d+2}, 0, \dots, 0),\\ \vdots\\ (-x_{2d+1}, 0, \dots, 0), \quad (-x_{d+1}, x_1, 0, \dots, 0), \quad (0, -x_{d+2}, x_2, 0, \dots, 0),\\ \vdots\\ (0, \dots, -x_{2d}, x_d, 0, \dots, 0).\end{gathered}$$

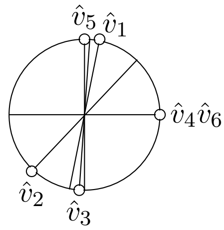
□

We can determine minimal sets of monomial generators of Gorenstein Stanley-Reisner ideals associated to all simplicial d-polytopes with $d+3$ through minimal sets of monomial generators of Gorenstein Stanley-Reisner ideals associated to cyclic d-polytopes with $d+3$ vertices. We achieve that as follows: If we take a product of a monomial (or more) from the minimal set of monomial generators of the Gorenstein Stanley-Reisner ideal associated to a cyclic polytope with a new variable (or more), then we obtain a minimal set of monomial generators of the Gorenstein Stanley-Reisner ideal of codimension 3 associated to a polytope of dimension $d+k$ with $d+k+3$ vertices, where k is the number of new variables, see Example 6.2.1.2.

Example 6.2.1.2. Let C be a cyclic 2-polytope with the vertex set $V = \{v_1, \dots, v_5\}$, the boundary complex $\Delta(C)$ and the Gale diagram $\hat{\mathfrak{B}} = \{\hat{v}_1, \dots, \hat{v}_5\}$. Let $\mathbb{K}[\Delta(C)] = \mathbb{K}[x_1, \dots, x_5]/I_{\Delta(C)}$ be the Stanley-Reisner ring associated to C, where $I_{\Delta(C)}$ is the corresponding Gorenstein Stanley-Reisner ideal. From Chapter 2, Subsection 2.1.2, we get $I_{\Delta(C)} = (x_1x_4,\ x_3x_4,\ x_2x_3,\ x_2x_5,\ x_1x_5)$. If we take a product of a monomial (or more) from the minimal set of monomial generators of $I_{\Delta(C)}$ with a new variable, then we obtain the minimal set of monomial generators of the associated Gorenstein Stanley-Reisner ideal to a simplicial 3-polytope P with 6 vertices $\{v_1, \dots, v_6\}$. So the corresponding Gorenstein Stanley-Reisner ideal is $I_{\Delta(P)} = (x_1x_4x_6,\ x_3x_4x_6,\ x_2x_3, x_2x_5,\ x_1x_5)$.



(a) Gale diagram of the polytope C.



(b) Gale diagram of the polytope P.

6.2.2 Sufficiency statement of Conjecture 6.2.0.7

Theorem 6.2.2.1. *Let P be a cyclic $2d$-polytope with the vertex set V=$\{v_1,\ldots,v_{d+2},w_1,\ldots,$ $w_{d+2}\}$ and the boundary complex $\Delta(P)$. Let $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1,\ldots,x_{d+2},y_1,\ldots,y_{d+2}]/I_{\Delta(P)}$ be the Stanley-Reisner ring of $\Delta(P)$, where $I_{\Delta(P)}$ is the Gorenstein Stanley-Reisner ideal associated to P. Consider a $(d+2)\times(d+2)$-matrix (or its transpose) of the form*

$$A = \begin{bmatrix} x_1 & 0 & 0 & 0 & \cdots & y_{d+2} \\ y_1 & x_2 & 0 & 0 & \cdots & 0 \\ 0 & y_2 & x_3 & 0 & \cdots & 0 \\ 0 & 0 & y_3 & x_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x_{d+2} \end{bmatrix}.$$

Then the $(d+1)$-minors construct a minimal set of monomial generators of $I_{\Delta(P)}$.

Proof. We claim that the following monomials

$$A_{pq} = x_1\ldots x_{d+2}\prod_{p\le i\le q}\left(\frac{y_i}{x_i}\right)\cdot y_q^{-1}, \quad \text{for } p\le q,$$

$$A_{pq} = (-1)^d y_1\ldots y_{d+2}\prod_{q+1\le j\le p-1}\left(\frac{x_j}{y_j}\right)\cdot y_q^{-1}, \quad \text{for } p>q$$

are all $(d+1)$-minors of A. To check that, we take a matrix A', so that its entries are $a'_{qp} = (-1)^{p+q}A_{pq}$ for $p,\ q = 1,\ldots,\ d+2$.
We should check, whether the equality $A'\cdot A = \det(A)E_{(d+2)}$ is true. If we multiply the two matrices A and A', we get the following equalities:

$$x_{(p-1)}A_{(p-1)q} = y_{(p-1)}A_{pq}, \quad \text{for} \quad p\ne q \quad \text{and} \quad p,\ q = 1,\ldots,\ d+2.$$

In other words, $x_{(p-1)}A_{(p-1)q} - y_{(p-1)}A_{pq} = 0$, for $p\ne q$ and $x_pA_{pp} - y_pA_{(p+1)p} = x_1\ldots x_{d+2} - (-1)^d y_1\ldots y_{d+2} = \det(A)$, for $p = q$. That means $A'\cdot A = \det(A)E_{(d+2)}$. Therefore it follows $A' = A^{\#}$, where $A^{\#}$ is the adjugate matrix of A, and hence A_{pq} are $(d+1)$-minors of A.

Now we prove that all these minors are generators of the Gorenstein Stanley-Reisner ideal $I_{\Delta(P)}$.

The cobminatorial type of a Gale diagram of a cyclic $2d$-polytope with $2d+4$ vertices is the following figure, see [Dev11].

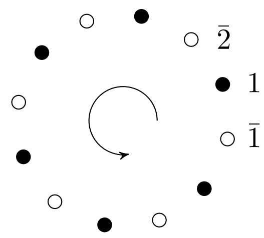


Figure 6.2: Planar Gale diagram of cyclic $2d$-polytope with $2d+4$ vertices.

There are $(d+2)$ black points and $(d+2)$ white points. We enumerate these points in the diagram, so that each consecutive black and white points have the same number. The minimal set of monomial generators of the Gorenstein Stanley-Reisner ideal associated to P contains $(d+2)^2$ elements, according to Theorem 5.3.0.2.

There are $(d+2)$ different monomials, which correspond to $(d+1)$ black points, $(d+2)$ different monomials, which correspond to d black points and a white point, etc. That means there are $(d+2)^2$ monomials.

We can now define a bijection between the set of the above minors A_{pq} and the minimal set of monomial generators of $I_{\Delta(P)}$. So that the monomials, which are corresponding to $(d+1)$ black points, are corresponding to A_{pp}, for $p = 1, \dots, d+2$, respectively. The monomials, which correspond to d black points and a white point, correspond to $A_{p(p+1)}$ and $A_{(d+2)1}$, for $p = 1, \dots, d+1$, respectively, etc. $\square$

6.2.3 Characterization of monomial generators of Gorenstein ideals associated to T-polytopes

All neighbourly $2d$-polytopes with $2d+4$ vertices, which have been classified by Devyatov [Dev11], have a Gale diagram with exactly $d+3$ black points in convex position. In this subsection, we prove for each polytope of Devyatov's polytopes that the corresponding Gorenstein Stanley-Reisner ideal is generated by exactly $(d+2)^2$ monomials and all have degree $d+1$, but there is no square $(d+2)\times(d+2)$-matrix, so that its $(d+1)$-minors generate it. That is not a complete proof of the reversal direction of Conjecture 6.2.0.7, but it is an important step to achieve that, since all these polytopes are not cyclic.

At first we need some definitions from [Dev11].

Definition 6.2.3.1. Let P be a neigbourly $2d$-polytope with the vertex set $V = \{v_1, \dots, v_{2d+4}\}$ and an affine Gale diagram $\mathfrak{B}^* = \{v_1^*, \dots, v_{2d+4}^*\}$, each of them declared to be either black or white. Then $\mathfrak{B}^*$ is called *T-diagram* if the following conditions are satisfed:

1. $2d > 4$;

2. the points of $\mathfrak{B}^*$ are in general position;
3. the diagram $\mathfrak{B}^*$ contains exactly $d+3$ black points, and they are in convex position and vertices of a $(d+3)$-gon;
4. all white points of $\mathfrak{B}^*$ lie inside the $(d+3)$-gon formed by the black points;
5. Each triangle formed by three black vertices of $\mathfrak{B}^*$ contains exactly one white point.

Definition 6.2.3.2. The convex hull of a set of consequent vertices of the $(d+3)$-gon is called *lune* of a T-diagram.

Notation 6.2.3.3. Let P be a neighbourly simplicial $2d$-polytope with $2d+4$ vertices, which are classified by Devyatov [Dev11], with the vertex set $V = \{v_1, \dots, v_{d+3}, w_1, \dots, w_{d+1}\}$ and the boundary complex $\Delta(P)$. Let $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_{d+3}, y_1, \dots, y_{d+1}]/I_{\Delta(P)}$ be the Stanley-Reisner ring of $\Delta(P)$ and $I_{\Delta(P)}$ the Gorenstein Stanley-Reisner ideal associated to P. Let $\mathfrak{B}^* = \{v_1^*, \dots, v_{d+3}^*, w_1^*, \dots, w_{d+1}^*\}$ be the T-diagram of P, where the v_i^* are declared as black points and the w_j^* are declared as white points. We denote the black points v_i^* by i and the white points by $\bar{j}$. By Remark 5.1.0.2 there are canonical bijections between the variables x_i and the black points i of $\mathfrak{B}^*$ and between the variables y_j and the white points $\bar{j}$ of $\mathfrak{B}^*$. Analogously a subset $M \subset V$ corresponds to $M^* \subset \mathfrak{B}^*$.

Now we characterize minimal sets of monomial generators of Gorenstein ideals associated to T-polytopes. In what follows (in this and the next subsection), a "monomial generator" is understood as the momomial generator in the minimal set of generators of $I_{\Delta(P)}$.

Theorem 6.2.3.4. *Let P be a special neighbourly simplicial $2d$-polytope with $2d+4$ vertices, which were classified by Devyatov [Dev11]. We consider the assumptions in Notation 6.2.3.3. Let $\{i_1, \dots, i_k\}$ be a set of black points, and $\{\bar{j}_1, \dots, \bar{j}_s\}$ a set of white points of $\mathfrak{B}^*$.*

1. *For $1 \leq \#\{i_1, \dots, i_k\} \leq d$, a monomial $x_{i_1} \dots x_{i_k} y_{j_1} \dots y_{j_s}$ is a generator of $I_{\Delta(P)}$ of degree $d+1$ if and only if the set $M := \{i_1, \dots, i_k, \bar{j}_1, \dots, \bar{j}_s\}$ of $\mathfrak{B}^*$ contains $d+1$ points and the black points $\{i_1, \dots, i_k\}$ are consecutive points of the $(d+3)$-gon and the white points $\{\bar{j}_1, \dots, \bar{j}_s\}$ are all white points, which are located in the convex hull (lune) of all black points of $\mathfrak{B}^* \setminus \{i_1, \dots, i_k\}$.*
2. *For $\#\{i_1, \dots, i_k\} = d+1$, a monomial $x_{i_1} \dots x_{i_k}$ is a generator of $I_{\Delta(P)}$ of degree $d+1$ if and only if the set $M := \{i_1, \dots, i_k\}$ is a subset of $\mathfrak{B}^*$ and these black points are consequent points of the $(d+3)$-gon.*
3. *For $\#\{i_1, \dots, i_k\} = 0$, a monomial $y_{j_1} \dots y_{j_s}$ is a generator of $I_{\Delta(P)}$ of degree $d+1$ if and only if the set $M := \{\bar{j}_1, \dots, \bar{j}_s\}$ is a subset of $\mathfrak{B}^*$ and contains all white points.*

Proof. Since all white points of $\mathfrak{B}^*$ lie inside the $(d+3)$-gon formed by the black point and each triangle formed by three black vertices of $\mathfrak{B}^*$ contains exactly one white point, the assertion holds by Theorem 5.3.0.2. □

Proposition 6.2.3.5. *Let P be a special neighbourly simplicial $2d$-polytope with $2d+4$ vertices, which were classified by Devyatov [Dev11]. We consider the assumptions in Notation 6.2.3.3. The associated Gorenstein Stanley-Reisner ideal $I_\Delta(P)$ is minimally generated by exactly $(d+2)^2$ monomial generators and all have degree $d+1$.*

Proof. We would like to determine the minimal set of monomial generators of $I_{\Delta}(P)$ using a T-diagram. Let $\{i_1,\dots,i_k\}$ be a set of black points and $\{\bar{j}_1,\dots,\bar{j}_s\}$ a set of white points of the T-diagram $\mathfrak{B}^*$.

1. For $1 \le \#\{i_1,\dots,i_k\} \le d$, by Theorem 6.2.3.4, since there are exactly $d+3$ black points in the T-diagram, there are $d(d+3)$ monomial generators of the form $x_{i_1}\dots x_{i_k}y_{j_1}\dots y_{j_s}$ of degree $d+1$.
2. For $\#\{i_1,\dots,i_k\} = d+1$, by Theorem 6.2.3.4, there are $d+1$ monomial generators of the form $x_{i_1}\dots x_{i_k}$ of degree $d+1$
3. For $\#\{i_1,\dots,i_k\} = 0$, by Theorem 6.2.3.4, there is a one monomial generator of the form $y_{j_1}\dots y_{j_s}$ of degree $d+1$.

So we obtain $d(d+3)+(d+1)+1 = (d+2)^2$ monomial generators of $I_{\Delta}(P)$ of degree $d+1$.

By Definition 6.2.3.1 (4) and (5), we observe that the complement set of the black and white points, which correspond to the variables of a monomial generator of degree smaller than $d+1$, does not satisfy the condition in Theorem 5.3.0.2.There is always at least a lune of 3 black points, which contains a white point. Hence there is by Theorem 5.3.0.2 no monomial generator of $I_{\Delta}(P)$ of degree smaller than $d+1$.

If there is a set of the T-diagram, that contains more than $d+1$ black points so that its complement satisfies the condition in Theorem 5.3.0.2, that means all these black points have to be consecutive points of the $(d+3)$-gon, then the set is not minimal, see Theorem 6.2.3.4.

If the set contains more than $d+1$ black and white points, so that its complement satisfies the condition in Theorem 5.3.0.2, then the set is also not minimal. By Theorem 6.2.3.4, the corresponding white points to variables of a monomial generator of $I_{\Delta}(P)$ are located in the lune of the complement set of the black points, which correspond to other variables of the same generator. Hence there is no monomial generator of $I_{\Delta}(P)$ of degree more than $d+1$. □

Definition 6.2.3.6. Let P be a special neighbourly simplicial $2d$-polytope with $2d+4$ vertices, which were classified by Devyatov [Dev11]. We consider the assumptions in Notation 6.2.3.3. Let i be a black point and $\bar{j}$ a white point of the T-diagram $\mathfrak{B}^*$. We call i and $\bar{j}$ *neighbors*, if $\bar{j}$ lies inside a lune of a set of three consecutive vertices of $(d+3)$-gon, in which i lies in the middle of this sequence. If i and $\bar{j}$ are neighbors, then we write $i \in S_{\bar{j}}$, where $S_{\bar{j}}$ is a subset of $\mathfrak{B}^*$.

Example 6.2.3.7. In Example 6.2.5.2, the points 4 and $\bar{2}$ are neighbors, we write $4 \in S_{\bar{2}}$.

According to this definition, we can distribute the black points in different sets as follows.

Remark 6.2.3.8. Let P be a special neighbourly simplicial $2d$-polytope with $2d+4$ vertices, which were classified by Devyatov [Dev11]. We consider the assumptions in Notation 6.2.3.3. Let $\bar{j}$ be a white point of the T-diagram $\mathfrak{B}^*$. Two black points i and k of $\mathfrak{B}^*$ lie in the same set $S_{\bar{j}} \subset \mathfrak{B}^*$, if they are neighbors to the same white point $\bar{j}$.

Example 6.2.3.9. In Example 6.2.5.2, the black points 3 and 4 are in $S_{\bar{2}}$.

Remark 6.2.3.10. According to Definition 6.2.3.1 it follows that each black point has exactly one white point as a neighbor. Since we have $(d+1)$ white points and there is exactly one white point inside each triangle with black vertices, see Definition 6.2.3.1, each white point has either one or two neighbors.

Definition 6.2.3.11. If $|S_{\bar{j}}| = 1$, then we say that the white point $\bar{j}$ has *type 1*, otherwise *type 2*.

Corollary 6.2.3.12. *Let P be a special neighbourly simplicial $2d$-polytope with $2d+4$ vertices, which were classified by Devyatov [Dev11]. We consider the assumptions in Notation 6.2.3.3. Let i be a black point and $\bar{j}$ a white point of the T-diagram $\mathfrak{B}^*$ with $i \in S_{\bar{j}}$. Then the corresponding variables x_i and y_j can not appear in one of the monomial generators of $I_\Delta(P)$.*

Proof. See Theorem 6.2.3.4. □

Proposition 6.2.3.13. *Let P be a special neighbourly simplicial $2d$-polytope with $2d+4$ vertices, which were classified by Devyatov [Dev11]. We consider the assumptions in Notation 6.2.3.3. Let i be a black point and $\bar{j}$ a white point of the T-diagram $\mathfrak{B}^*$, where $\bar{j}$ has type 2. Then the corresponding variables x_i and y_j appear exactly $((d+1)(d+2))/2$ times in the monomial generators of degree $d+1$ of $I_\Delta(P)$.*

Proof. Let i be a black point. Coming back to the proof of Proposition 6.2.3.5, we observe that

1. For $1 \leq \#\{i_1,\ldots,i_k\} \leq d$, the corresponding variable x_i turns up exactly $1+2+\ldots+d$ times in the monomial generators of degree $d+1$ of $I_\Delta(P)$.
2. For $\#\{i_1,\ldots,i_k\} = d+1$, the corresponding variable x_i turns up exactly $d+1$ times in the monomial generators of degree $d+1$ of $I_\Delta(P)$.
3. For $\#\{i_1,\ldots,i_k\} = 0$, the corresponding variable x_i turns up exactly 0 times in the monomial generators of degree $d+1$ of $I_\Delta(P)$.

So the corresponding variable x_i appears exactly $1+2+\ldots+(d+1) = ((d+1)(d+2))/2$ times in the monomial generators of degree $d+1$ of $I_\Delta(P)$.

Let $\bar{j}$ be a white point has type 2. Coming back to the proof of Proposition 6.2.3.5, we observe that

1. For $1 \leq \#\{i_1,\ldots,i_k\} \leq d$, by Corollary 6.2.3.12, the corresponding variable y_j turns up exactly $(d+1)+\ldots+2$ times in the monomial generators of degree $d+1$ of $I_\Delta(P)$, since by Theorem 6.2.3.4, the corresponding white points to variables of a monomial generator of $I_\Delta(P)$ are all points, which are located in the lune of the complement set of the black points, which are corresponding to another variables of the same generator.
2. For $\#\{i_1,\ldots,i_k\} = d+1$, the corresponding variable y_j turns up exactly 0 times in the monomial generators of degree $d+1$ of $I_\Delta(P)$.
3. For $\#\{i_1,\ldots,i_k\} = 0$, the corresponding variable y_j turns up exactly 1 time in the monomial generators of degree $d+1$ of $I_\Delta(P)$.

So the corresponding variable y_j appears also exactly $((d+1)(d+2))/2$ times in the monomial generators of degree $d+1$ of $I_\Delta(P)$. □

6.2.4 Necessity statement of Conjecture 6.2.0.7

We show the reversal direction partially. In [Dev11], Devyatov classified special neighbourly $2d$-polytopes with $2d+4$ vertices which are not cyclic. For every one of Devyatov's polytopes, we prove there is no square $(d+2)\times(d+2)$-matrix, so that its $(d+1)$-minors generate the Gorenstein Stanley-Reisner ideal associated to a Devyatov's polytope. That means that the minimal graded free resolutions of Stanley-Reisner rings associated to Devyatov's polytopes can not be regarded as a version of the Gulliksen-Negård complex.

Theorem 6.2.4.1. *Let P be a neighbourly simplicial $2d$-polytope with $2d+4$ vertices, which were classified by Devyatov [Dev11], where V=$\{v_1,\dots,v_{d+3},w_1,\dots,w_{d+1}\}$ is the vertex set and $\Delta(P)$ is the boundary complex Let $\mathbb{K}[\Delta(P)]$=$\mathbb{K}[x_1,\dots,x_{d+3},y_1,\dots,y_{d+1}]/I_{\Delta(P)}$ be the Stanley-Reisner ring of $\Delta(P)$, where $I_{\Delta(P)}$ is the Gorenstein Stanley-Reisner ideal associated to P. Then there is no $(d+2)\times(d+2)$-matrix, so that its $(d+1)$-minors are monomial generators of $I_{\Delta(P)}$.*

Proof. Proof by contradiction. Assume there is a $(d+2)\times(d+2)$-matrix A such that its $(d+1)$-minors are monomial generators of $I_{\Delta(P)}$ and $A^{\#}$ its adjugate matrix. Let $\mathfrak{B}^*$ be a T-diagram of P, $i\in\mathfrak{B}^*$ a black point and $\bar{j}\in\mathfrak{B}^*$ a white point, which has type 2 with $i\in S_{\bar{j}}$ (in each T-diagram there always exists at least one white point, which has type 2 by Remark 6.2.3.10). Now we put as zero the corresponding variables x_i and y_j in $\mathbb{K}[\Delta(P)]$. For the case $x_i := y_j := 0$, we denote A respectively $A^{\#}$ by A_{x_i,y_j} respectively $A^{\#}_{x_i,y_j}$. According to the Propositions 6.2.3.5, 6.2.3.12 and 6.2.3.13 there remain only $(d+2)$ monomial generators of $I_{\Delta(P)}$ of degree $d+1$, which don't have x_i and y_j as variabels. Since the black points are vertices of $(d+3)$-gon, see Definition 6.2.3.1, because of that we can number the vertices of $(d+3)$-gon respectively sequentially, as the following Figure 6.3.

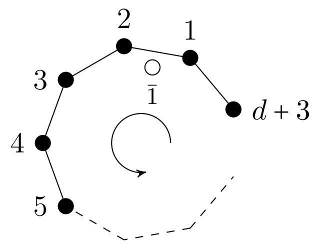


Figure 6.3: $(d+3)$-gon with the white point $\bar{1}$[1].

Without loss of generality we choose $i := 1$, $\bar{j} := 1$ and $S_{\bar{j}} := \{1,2\}$. Then the $(d+2)$

[1]In Figure 6.3 there should be $d+1$ white points inside the $(d+3)$-gon.

remaining monomial generators of $I_{\Delta(P)}$ of degree $d+1$ are

$$\begin{gathered} x_3x_4\dots x_{d+3}, \\ x_2x_3\dots x_{d+2}, \\ x_2x_3\dots x_{d+1}y_{d+1}, \\ x_2x_3\dots x_d y_{d+1}y_d, \\ \vdots \\ x_2x_3y_{d+1}\dots y_3, \\ x_2y_{d+1}\dots y_3y_2. \end{gathered}$$

We denote this set by $(*)$.

Assume that $\det(A_{x_i,y_j}) \neq 0$, it means $\operatorname{rank}(A_{x_i,y_j}) = \operatorname{rank}(A^{\#}_{x_i,y_j}) = d+2$, hence the remaining elements lie on the main diagonal of $A^{\#}_{x_i,y_j}$. Therefore

$$\det(A^{\#}_{x_i,y_j}) = x_2^{(d+1)}x_3^{(d+2)}x_4^d\dots x_{d+3}y_{d+1}^d\dots y_2.$$

Since $A^{\#}_{x_i,y_j}\dots A_{x_i,y_j} = \det(A_{x_i,y_j})E_{d+2}$ we have $\det(A^{\#}_{x_i,y_j}) = (\det(A_{x_i,y_j}))^{d+1}$, i.e. the powers of variables in $\det(A^{\#}_{x_i,y_j})$ should be divisible by $d+1$, a contradiction. So $\det(A_{x_i,y_j}) = 0$, because the entries of the matrix $A^{\#}_{x_i,y_j}$ are monomial generators of $I_{\Delta(P)}$ of $\mathbb{K}[\Delta(P)]$, we have $A^{\#}_{x_i,y_j} \neq 0$. Since $\det(A_{x_i,y_j}) = 0$, it follows $\operatorname{rank}(A^{\#}_{x_i,y_j}) = 1$. That means, the remaining monomial generators belong to either the same column or the same row of $A^{\#}_{x_i,y_j}$ or not.

Assume that the remaining monomial generators of $I_{\Delta(P)}$ of $\mathbb{K}[\Delta(P)]$ do not belong to the same column or row. Then there is at least a 2×2-minor H, such that the all entries are not equal to zero, but $H = 0$. Without loss of generality we substitute $y_j := 1$ into the remaining monomial generators for all $j \in \{1, 2, \cdots, d+1\}$.

$$H = \begin{vmatrix} x_2x_3\dots x_{k_1} & x_2x_3\dots x_{k_2} \\ x_2x_3\dots x_{k_3} & x_2x_3\dots x_{k_4} \end{vmatrix}$$

such that $2 \geq k_1 > k_2 > k_3 > k_4 \geq (d+3)$. Then it follows

$$(x_2^2x_3^2\dots x_{k_4}^2x_{k_4+1}\dots x_{k_3}\dots x_{k_2})(x_{k_2+1}\dots x_{k_1} - 1) = 0.$$

So we have $x_{k_2+1}\dots x_{k_1} = 1$. If we substitute this value into the first generator in (*), this leads to a contradiction with Theorem 6.2.3.4. Therefore the remaining monomial generators belong to the same column or row of $A^{\#}_{x_i,y_j}$.

Since the entries of the matrix $A^{\#}_{x_i,y_j}$ are the monomial generators of degree $d+1$, $\det(A^{\#}_{x_i,y_j})$ is a polynomial of degree $(d+1)(d+2)$. On the other hand $\det(A^{\#}_{x_i,y_j}) = (\det(A_{x_i,y_j}))^{d+1}$, thus $\det(A_{x_i,y_j})$ is a polynomial of degree $(d+2)$. Let $A^{-1}_{x_i,y_j}$ be the inverse matrix of A_{x_i,y_j}, then the entries of the matrix $A^{-1}_{x_i,y_j}$ have degree -1, therefore the entries of the matrix $A_{x_i,y_j} = (A^{-1}_{x_i,y_j})^{-1}$ have degree 1.

The entries of A_{x_i,y_j} are linear polynomials in $2d+2$ variables. Suppose that each polynomial has this form:

$$a^{(n)}_{m2}x_2 + a^{(n)}_{m3}x_3 + \cdots + a^{(n)}_{m(d+3)}x_{d+3} + b^{(n)}_{m2}y_2 + \cdots + b^{(n)}_{m(d+1)}y_{d+1} \tag{6.1}$$

with $a_{mk_1},\ b_{mk_2} \in \mathbb{K}$, $k_1 = 1,\dots,d+3$, $k_2 = 1,\dots,d+1$ and $n,\ m = 1,\dots,d+2$, so that n refers to the corresponding row and m refers to the m-th entry. Suppose without loss of generality that the remaining generators lie on the first column of $A^{\#}_{x_i,y_j}$[2],. Since $A_{x_i,y_j} \cdot A^{\#}_{x_i,y_j} = \det(A_{x_i,y_j})E_{d+2} = 0$, we get a homogeneous system of equations, consisting of $d+2$ equations. For each $n = 1,\dots,d+2$ we obtain an equation of the form

$$\begin{aligned}
&\left(a^{(n)}_{12}x_2 + a^{(n)}_{13}x_3 + \ldots + a^{(n)}_{1(d+3)}x_{d+3} + b^{(n)}_{12}y_2 + \ldots + b^{(n)}_{1(d+1)}y_{d+1}\right)\, x_3x_4\ldots x_{d+3} + \\
&\left(a^{(n)}_{22}x_2 + a^{(n)}_{23}x_3 + \ldots + a^{(n)}_{2(d+3)}x_{d+3} + b^{(n)}_{22}y_2 + \ldots + b^{(n)}_{2(d+1)}y_{d+1}\right)\, x_2x_3\ldots x_{d+2} + \\
&\qquad\qquad\vdots \\
&\left(a^{(n)}_{(d+2)2}x_2 + \ldots + a^{(n)}_{(d+2)(d+3)}x_{d+3} + b^{(n)}_{(d+2)2}y_2 + \ldots + b^{(n)}_{(d+2)(d+1)}y_{d+1}\right)\, x_2y_{d+1}\ldots y_3y_2 = 0
\end{aligned}$$

If we solve these equations, we get the following solution

$$a^{(n)}_{12} = -a^{(n)}_{2(d+3)} \text{ and } b^{(n)}_{r(d+3-r)} = -a^{(n)}_{(r+1)(d+4-r)}, \quad \text{for} \quad r = 2,\dots,d+1.$$

That means, the solution space of each equation has dimension $d+1$. Since the solution space of a system of equations is the intersection of the solution spaces of the individual equations, and all equations have the same solution space. It follows the solution space of the system of equations has dimension $d+1$ with a basis B, which is generated by $(1,0,\dots,0,-1,0,\dots,0) \in \mathbb{K}^{(d+2)(2d+2)}$ (where -1 on the $(3d+4)$-th position) and all vectors $(0,\dots,0,-1,0,\dots,0,1,0,\dots,0) \in \mathbb{K}^{(d+2)(2d+2)}$ (where -1 on the $[(r-1)(2d+2) + d+4-r]$-th position and 1 on the $[(r-1)(2d+2)+2d+4-r]$-th position). Since there is at least a $(d+1)$-minor of A_{x_i,y_j}, which is not zero, such as $x_3x_4\ldots x_{d+3}$, we have $\operatorname{rank}(A_{x_i,y_j}) = d+1$. That means there are $d+1$ rows respectively columns in the matrix A_{x_i,y_j}, which are linear independent. Now we take all vectors of B and substitute them into the polynomials in (6.1), so we obtain almost all enteries of the matrix A_{x_i,y_j}.

$$\begin{bmatrix}
* & * & * & * & \cdots & * & * \\
x_2 & -x_{d+3} & 0 & 0 & \ldots & 0 & 0 \\
0 & y_{d+1} & -x_{d+2} & 0 & \ldots & 0 & 0 \\
0 & 0 & y_d & -x_{d+1} & \ldots & 0 & 0 \\
0 & 0 & 0 & y_{d-1} & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & y_2 & x_3
\end{bmatrix}
\begin{bmatrix}
x_3x_4\ldots x_{d+3} & 0 & \cdots & 0 \\
x_2x_3\ldots x_{d+2} & \vdots & \ddots & \vdots \\
x_2x_3\ldots x_{d+1}y_{d+1} & 0 & \cdots & 0 \\
x_2x_3\ldots y_{d+1}y_d & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
x_2x_3y_{d+1}\ldots y_3 & 0 & \cdots & 0 \\
x_2y_{d+1}\cdots y_3y_2 & 0 & \cdots & 0
\end{bmatrix} = 0.$$

Now we take another black point k, which lies in $S_{\bar{j}}$, in our case it is $k = 2$, because $S_{\bar{j}} = \{1,2\}$. Exactly as above we put the corresponding variables x_k and y_j in $\mathbb{K}[\Delta(P)]$

[2]By neglecting the order of the monomial generators. Hence we can take the order as in $(*)$.

as zero. In this case we denote A and $A^{\#}$, where x_k and y_j are replaced by 0, by A_{x_k,y_j} and $A^{\#}_{x_k,y_j}$, respectively. Therefore, as explained above, there remain only $(d+2)$ monomial generators of $I_{\Delta(P)}$ of degree $d+1$, which don't have x_k and y_j as variabels. They are

$$\begin{gathered} x_{d+3}x_{d+2}\dots x_4x_3, \\ x_1x_{d+3}\dots x_4, \\ x_1x_{d+3}\dots x_5y_2, \\ x_1x_{d+3}\dots x_6y_2y_3, \\ \vdots \\ x_1y_2\dots y_d, \\ x_1y_2\dots y_dy_{d+1}. \end{gathered}$$

Then the remaining monomial generators belong to the same column or row of $A^{\#}_{x_k,y_j}$. We may notice that the monomial $x_{d+3}x_{d+2}\dots x_4x_3$ is a common element between the first set of the remaining generators (in the case $x_i = 0$ and $y_j = 0$) and the second one (in the case $x_k = 0$ and $y_j = 0$). Thus, if the remaining monomial generators in the first set are in a column of $A^{\#}_{x_k,y_j}$, then the second one are in a row and the other way around. Since we have assumed in the first case that the remaining generators are in a column, it follows in the second case that remaining generators are in the row, where the common element is. If we arrange the elements appropriately [3] and multiply the matrix $A^{\#}_{x_k,y_j}$ by A_{x_k,y_j}, we obtain the following

$$\begin{bmatrix} x_{d+3}x_{d+2}\dots x_4x_3 & 0 & \cdots & 0 \\ x_1y_2\dots y_dy_{d+1} & \vdots & \ddots & \vdots \\ x_1x_{d+3}y_2\dots y_d & 0 & \cdots & 0 \\ x_1x_{d+3}x_{(d+2)}\dots y_{d-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_1x_{d+3}\dots x_5y_2 & 0 & \cdots & 0 \\ x_1x_{d+3}\dots x_4 & 0 & \cdots & 0 \end{bmatrix}^t \begin{bmatrix} \boxed{\begin{matrix} * \\ * \\ * \\ * \\ * \\ \vdots \\ * \end{matrix}} & \begin{matrix} 0 & 0 & 0 & \cdots & 0 & x_1 \\ -x_{d+3} & 0 & 0 & \cdots & 0 & 0 \\ y_{d+1} & -x_{d+2} & 0 & \cdots & 0 & 0 \\ 0 & y_d & -x_{d+1} & \cdots & 0 & 0 \\ 0 & 0 & y_{d-1} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y_2 & x_3 \end{matrix} \end{bmatrix} = 0.$$

So according to above the required matrix is

$$A = \begin{bmatrix} \boxed{*} & 0 & 0 & 0 & \cdots & 0 & x_1 \\ x_2 & -x_{d+3} & 0 & 0 & \dots & 0 & 0 \\ 0 & y_{d+1} & -x_{d+2} & 0 & \dots & 0 & 0 \\ 0 & 0 & y_d & -x_{d+1} & \dots & 0 & 0 \\ 0 & 0 & 0 & y_{d-1} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & y_2 & x_3 \end{bmatrix}.$$

This leads to a contradiction with Theorem 6.2.3.4, because we obtain by deleting the $(d+2)$-th row and the second column the $(d+1)$-minor: $x_1x_2x_4\dots x_{d+2}$. This minor is not a monomial generator of degree $d+1$ of $I_\Delta(P)$ by Theorem 6.2.3.4. □

[3] If we do not arrange the elements in a nice sequence, we get a matrix, with elementary row and column transformations we get the desired matrix.

6.2.5 Examples

Example 6.2.5.1. Let P be a cyclic 4-polytope with the vertex set $V = \{v_1, \dots, v_4, w_1, \dots, w_4\}$ and the boundary complex $\Delta(P)$. Let $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_4, y_1, \dots, y_4]/I_{\Delta(P)}$ be the Stanley-Reisner ring of $\Delta(P)$, where $I_{\Delta(P)}$ is the Gorenstein Stanley-Reisner ideal associated to P. In this case P is P^8_{35}. Consider the following 4×4-matrix (or the transpose)

$$A = \begin{bmatrix} x_1 & 0 & 0 & y_4 \\ y_1 & x_2 & 0 & 0 \\ 0 & y_2 & x_3 & 0 \\ 0 & 0 & y_3 & x_4 \end{bmatrix}.$$

The 3-minors of the matrix A are given in the following table

3×3-**minors**	of	A	
$A_{11} = x_2x_3x_4$	$A_{12} = x_3x_4y_1$	$A_{13} = x_4y_1y_2$	$A_{14} = y_1y_2y_3$
$A_{21} = y_2y_3y_4$	$A_{22} = x_1x_3x_4$	$A_{23} = x_1x_4y_2$	$A_{24} = x_1y_2y_3$
$A_{31} = x_2y_3y_4$	$A_{32} = y_1y_3y_4$	$A_{33} = x_1x_3x_4$	$A_{34} = x_1x_2y_3$
$A_{41} = x_2x_3y_4$	$A_{42} = x_3y_1y_4$	$A_{43} = y_1y_2y_4$	$A_{44} = x_1x_2x_3$

We sketched in Chapter 5, Section 5.2 the affine Gale diagram of this polytope, as the following

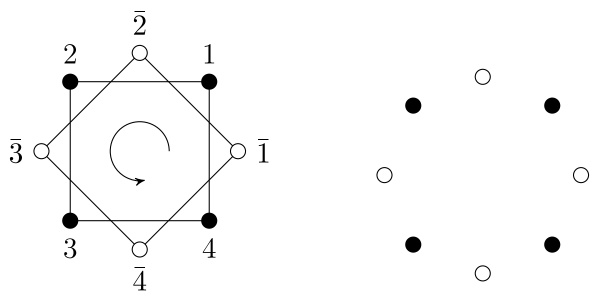


Figure 6.4: Gale diagram of cyclic 4-polytope P^8_{35} with 8 vertices.

The Gorenstein Stanley-Reisner ideal $I_{\Delta(P^8_{35})}$ has exactley 4^2 monomial generators, which have degree 3, see [TH96]. By Theorem 5.3.0.2 we can determine the minimal set of monomial generators of $I_{\Delta(P^8_{35})}$ of $\mathbb{K}[\Delta(P^8_{35})]$ using the affine Gale diagram and we obtain the following

$$
\begin{aligned}
I_{\Delta(P^8_{35})} = <\, & x_1x_2x_3, x_2x_3x_4, x_3x_4x_1, x_4x_1x_2, && \text{3 black points}\\
& x_1x_2y_4, x_2x_3y_1, x_3x_4y_2, x_4x_1y_3, && \text{2 black points and 1 white point}\\
& x_1y_3y_4, x_2y_1y_4, x_3y_1y_2, x_4y_2y_3, && \text{1 black point and 2 white points}\\
& y_1y_2y_3, y_2y_3y_4, y_3y_4y_1, y_4y_1y_2 > . && \text{3 white points}
\end{aligned}
$$

Using the computer algebra system Singular we can obtain the minimal graded free resolution of the Stanley-Reisner ring of $\mathbb{K}[\Delta(P^8_{35})]$ over $S = \mathbb{K}[x_1, \dots, x_4, y_1, \dots, y_4]$

$$0 \longrightarrow S(-8) \longrightarrow S^{16}(-5) \longrightarrow S^{30}(-4) \longrightarrow S^{16}(-3) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{35})] \longrightarrow 0.$$

There is a canonical bijection between the minimal set of monomial generators of $I_{\Delta(P^8_{35})}$ and the set all 3-minors of A. So the minimal graded free resolution of $\mathbb{K}[\Delta(P^8_{35})]$ can be considered as a special version of Gulliksen-Negård complex.

Example 6.2.5.2. Let P be one of the Devyatov's polytopes and neighbourly simplicial 4-polytope with the vertex set $V = \{v_1, \dots, v_5, w_1, \dots, w_3\}$ and the boundary complex $\Delta(P)$. Let $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_5, y_1, \dots, y_3]/I_{\Delta(P)}$ be the Stanley-Reisner ring of $\Delta(P)$, where $I_{\Delta(P)}$ is the Gorenstein Stanley-Reisner ideal associated to P. This polytope is P^8_{37}.

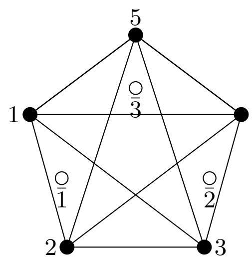


Figure 6.5: T-diagram of P^8_{37}.

According to Proposition 6.2.3.5, $I_{\Delta(P^8_{37})}$ has exaclty 4^2 monomial generators of degree 3. We can determine them by Theorem 6.2.3.4 as follows

$$
\begin{aligned}
I_{\Delta(P^8_{37})} = <\, & x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2, && \text{3 black points}\\
& x_1x_2y_2, x_2x_3y_3, x_3x_4y_1, x_4x_5y_1, x_1x_5y_2, && \text{2 black points and 1 white point}\\
& x_1y_2y_3, x_2y_2y_3, x_3y_1y_3, x_4y_1y_3, x_5y_1y_2, && \text{1 black point and 2 white points}\\
& y_1y_2y_3 > . && \text{3 white points}
\end{aligned}
$$

Using the computer algebra system Singular we can obtain the minimal graded free resolution of the Stanley-Reisner ring of $\mathbb{K}[\Delta(P^8_{37})]$ over $S = \mathbb{K}[x_1, \dots, x_5, y_1, \dots, y_3]$

$$0 \longrightarrow S(-8) \longrightarrow S^{16}(-5) \longrightarrow S^{30}(-4) \longrightarrow S^{16}(-3) \longrightarrow S \longrightarrow \mathbb{K}[\Delta(P^8_{37})] \longrightarrow 0.$$

Exactly as in proof of Theorem 6.2.4.1, we assume that there is a 4×4-matrix A, so that its 3-minors are monomial generators of $I_\Delta(P^8_{37})$. Let $i := 1$, $k := 2$ and $\bar{j} := 1$ [4], we obtain that $S_{\bar{j}} = \{1, 2\}$.
As first step we put $x_1 := 0$ and $y_1 := 0$, then there remain only 4 monomial generators of degree 3 of $I_{\Delta(P^8_{37})}$, they are

$$x_2x_3x_4,\ x_3x_4x_5,\ x_2x_3y_3,\ x_2y_2y_3.$$

These monomials should belong to the same column or row of $A^{\#}_{x_1,y_1}$, see proof of Theorem 6.2.4.1. Without loss of generality we put them on the first column, then we obtain the following

$$\begin{bmatrix} \boxed{* \quad * \quad * \quad *} \\ x_2 & -x_5 & 0 & 0 \\ 0 & y_3 & -x_4 & 0 \\ 0 & 0 & y_2 & x_3 \end{bmatrix} \begin{bmatrix} x_3x_4x_5 & 0 & 0 & 0 \\ x_2x_3x_4 & 0 & 0 & 0 \\ x_2x_3y_3 & 0 & 0 & 0 \\ x_2y_2y_3 & 0 & 0 & 0 \end{bmatrix} = 0.$$

As second step, $x_2 := 0$ and $y_1 := 0$, then there remain only 4 monomial generators of degree 3 of $I_{\Delta(P^8_{37})}$, they are

$$x_3x_4x_5,\ x_4x_5x_1,\ x_1x_5y_2,\ x_1y_2y_3.$$

We notice that $x_3x_4x_5$ is a common element, therefore, they belong to the same row of $A^{\#}_{x_2,y_1}$, in particular on the first row. Then we obtain the following

$$\begin{bmatrix} x_3x_4x_5 & 0 & 0 & 0 \\ x_1y_2y_3 & 0 & 0 & 0 \\ x_1x_5y_2 & 0 & 0 & 0 \\ x_4x_5x_1 & 0 & 0 & 0 \end{bmatrix}^t \begin{bmatrix} * & 0 & 0 & x_1 \\ * & -x_5 & 0 & 0 \\ * & y_3 & -x_4 & 0 \\ * & 0 & y_2 & -x_3 \end{bmatrix} = 0.$$

So the required matrix is

$$A = \begin{bmatrix} \boxed{*} & 0 & 0 & x_1 \\ x_2 & -x_5 & 0 & 0 \\ 0 & y_3 & -x_4 & 0 \\ 0 & 0 & y_2 & -x_3 \end{bmatrix}.$$

We obtain by deleting the 4-th row and the 2-th column the 3-minor $x_1x_2x_4$. This minor is not a monomial generator of degree 3 of $I_{\Delta(P^8_{37})}$ by Theorem 6.2.3.4, a contradiction to all 3-minors of A being monomial generators of $I_{\Delta(P^8_{37})}$. So the minimal graded free resolution of $\mathbb{K}[\Delta(P^8_{37})]$ can not be regarded as version of Gulliksen-Negård complex.

[4] We use the same notation as in proof of Theorem 6.2.4.1.

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INDEX

ZUSAMMENFASSUNG IN DEUTSCHER SPRACHE

Gorenstein Ringe bilden eine wichtige Klasse von Ringen in der kommutativen Algebra. Die Grundlagen der Theorie der Gorenstein Ringen gehen auf die klassische Arbeit von Bass [Bas63] zurück. Weitere Entwicklungen dieser Theorie sind im Artikel von Huneke [Hun99] zusammengefasst.

Ist R eine endlich erzeugte graduierte Gorenstein Algebra über einem algebraisch abgeschlossenen Körper $\mathbb{K}$, so kann man R als Restklassenring eines Polynomrings S=$\mathbb{K}[x_1, \ldots, x_n]$ nach einem homogenen Ideal $I \subset S$ darstellen, wobei I in diesem Fall Gorenstein Ideal genannt wird. Die Differenz $k := \dim(S) - \dim(R)$ heißt die Kodimension von I. Als S-Modul besitzt die Gorenstein Algebra R die minimale freie Auflösung

$$0 \longrightarrow S^{b_k^S} \longrightarrow S^{b_{k-1}^S} \longrightarrow \ldots \longrightarrow S^{b_1^S} \longrightarrow S^{b_0^S} \longrightarrow S/I \longrightarrow 0,$$

wobei $b_k^S = b_0^S = 1$ gilt und b_i^S die Gleichung $b_i^S = b_{k-i}^S$ für alle $1 \leq i \leq k$ erfüllt, siehe Satz 1.1.5.8 und Satz 1.1.5.9. Die Struktur dieser Auflösung ist für $k \leq 3$ bekannt. Im Fall $k = 1$ und $k = 2$ wird das Gorenstein Ideal genau von k Elementen erzeugt (d.h. I ist ein vollständiger Durchschnitt), see Kapitel 2, Abschnitt 2.1.1. Im Fall $k = 3$ besagt der Satz von Buchsbaum und Eisenbud [BE77], dass man dieses minimale Erzeugendensystem von I ist durch die $2m + 1$ Pfaffschen Determinanten der Ordnung $2m$ einer schiefsymmetrischen $(2m + 1) \times (2m + 1)$-Matrix A, siehe Kapitel 3, Abschnitt 2.1.2. Dabei erhält man die minimale freie Auflösung der Gestalt

$$0 \longrightarrow S \longrightarrow S^{2m+1} \xrightarrow{A} S^{2m+1} \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

Die Struktur der freien Auflösungen von Gorenstein Idealen der Kodimension 4 ist bis heute nicht vollständig geklärt. Einige Fortschritte in dieser Richtung wurden von Gulliksen und Negård [GN72] gemacht. In diesem Artikel untersuchen sie den Gorenstein Ring S/I, in dem S der Polynomring in rs Variablen x_{ij}, $1 \leq i \leq r$, $1 \leq j \leq s$, über einen Körper $\mathbb{K}$ ist und I ein Gorenstein Ideal ist, das von den t-Minoren einer Matrix (x_{ij}) erzeugt wird, für $1 \leq i \leq r$ und $1 \leq j \leq s$. Sie konstruieren eine explizite minimale freie Auflösung von S/I der Länge 4 für den Fall $n = r = s$, $t = n - 1$, wobei I ein Gorenstein Ideal der Kodimension 4 ist. Zehn Jahre später haben Kustin und Miller [KM82] und

[KM83] einige wichtige Ergebnisse über Gorenstein Ideale der Kodimension 4 erzielt. Die Resultate von Kustin und Miller finden eine interessante Anwendung in der Konstruktion neuer Calabi-Yau-Mannigfaltigkeiten [Kap11] und der Klassifikation singulärer Fano-Varietäten [PR04]. In [Rei13] entwickelte Reid die Ergebnisse von Kustin und Miller weiter und verallgemeinert teilweise das Buchsbaum-Eisenbud Theorem [BE77], siehe Kapitel 4, Abschnitt 4.1.

Reid betrachtet den Polynomring $S = \mathbb{K}[x_1, \dots, x_n]$ über einem algebraisch abgeschlossenen Körper $\mathbb{K}$ und ein Gorenstein Ideal $I \subset S$, das von $l+1$ Elementen erzeugt wird. Er schlägt vor, dass die Struktur der minimalen freien Auflösung des Quotientenmoduls S/I wie folgt ist

$$\mathbf{F}: \quad 0 \longrightarrow F_4 \xrightarrow{d_4} F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} S/I \longrightarrow 0,$$

wobei $F_0 = S$, $F_4 = S$, $F_1 = S^{l+1}$, $F_3 = \mathrm{Hom}(F_1, F_4) \cong F_1^*$ und $F_2 = S^{2l}$. Außerdem ist $F_2 \longrightarrow F_1$ dual zu $F_3 \longrightarrow F_2$. Durch Wahl geeigneter Basen von F_2 und F_3 erhalten wir die Matrix A von d_2, die die Form hat

$$A = \begin{bmatrix} B & C \end{bmatrix},$$

wobei B und C je $(l+1) \times l$-Matrizen sind, die die folgende Bedingung erfüllen

$$\begin{bmatrix} B & C \end{bmatrix} \begin{bmatrix} 0 & E \\ E & 0 \end{bmatrix} \begin{bmatrix} B & C \end{bmatrix}^t = 0.$$

Das ist äquivalent zu $BC^t + CB^t = 0$ oder zu der Aussage, dass BC^t eine schiefsymmetrische Matrix ist.

Die Struktur von Stanley-Reisner Ringen ist ein grundlegendes Werkzeug der algebraischen Kombinatorik und kombinatorischen Algebra. Seine Eigenschaften wurden von Richard Stanley, Melvin Hochster und Gerald Reisner in den 1970er Jahren untersucht, siehe [Hoc77], [Sta78] and [Sta80].

Seien P ein simpliziales d-Polytop mit n Ecken $\{v_1, \dots, v_n\}$ und $\Delta(P)$ der Randkomplex des Polytops P. Für einen Körper $\mathbb{K}$ definieren wir den assoziierten *Stanley-Reisner Ring* von $\Delta(P)$, als den Quotienten des Polynomringes $S = \mathbb{K}[x_1, \dots, x_n]$ nach dem Ideal $I_{\Delta(P)}$, das von quadratfreien Monomen erzeugt. Wir bezeichnen den Stanley-Reisner Ring mit $\mathbb{K}[\Delta(P)]$.

$$\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_n]/I_{\Delta(P)},$$

wobei

$$I_{\Delta(P)} = (x_{i_1} \dots x_{i_r} : \ i_1 < i_2 < \dots < i_r, \ \{v_{i_1}, \dots, v_{i_r}\} \notin \Delta(P)).$$

Das ideal $I_{\Delta(P)}$ wird *Stanley-Reisner Ideal* von $\Delta(P)$ genannt.

In dieser Dissertation wir betrachten Stanley-Reisner Ringe, die gleichzeitig Gorenstein Ringe sind. Deshalb sind ihre Stanley-Reisner Ideale nach Proposition 1.1.5.16 Gorenstein. Besondere Aufmerksamkeit widmen wir den zu simplizialen d-Polytopen mit $d+4$ Ecken assoziierten Stanley-Reisner Ringen, die eine wichtige Illustration für die Strukturtheorie von Kustin-Miller [KM82] und Reid [Rei15] in der Kodimension 4 darstellt.

Das Ziel der Dissertation ist, einige Fortschritte für die Struktur der minimalen freien Auflösungen von Gorenstein Idealen der Kodimension 4 durch die Untersuchung der Stanley-Reisner Ringe zu erzielen. Andererseits beabsichtigen wir, die homologischen Methoden der kommutativen Algebra mit der Untersuchung von simplizialen d-Polytopen mit $d+4$ Ecken zu verknünpfen. Die Strukturtheorie von Gorenstein Ringen soll mit kombinatorischen Fragestellungen verknüpft werden. Der Ausgangspunkt für die Untersuchungen ist die Verbindung zwischen der Klassifikation simplizialer d-Polytope mit $d+3$ Ecken und dem Struktursatz von Buchsbaum und Eisenbud für Gorenstein-Ideale der Kodimension 3, [BE77], siehe Kapitel 3, Abschnitt 3.2.1.

Im Folgenden verstehen wir unter einem „Erzeugendensystem“ ein Erzeugendensystem, das aus Monomen besteht.

Im *ersten Kapitel* erinnern wir an die Definitionen von minimalen freien Auflösungen der endlich erzeugten Modulen über einen graduierten Polynomring mit einem homogenem maximalem Ideal und der Hilbert-Reihe. Dann erinnern wir an vollständige Durchschnitte, Cohen-Macaulay Ringe und Gorenstein-Ringe, und wir zeigen die Beziehung zwischen ihnen. Danach führen wir „Stanley-Reisner Ringe “ ein.

Im *zweiten Kapitel* erklären wir die Struktur der minimalen freien Auflösung des Quotientenmoduls S/I, wobei S ein Polynomring ist and I Gorenstein Ideal der Kodimension 3 ist. Dieser Fall wird auch von Buchsbaum und Eisenbud in [BE77] behandelt. Dann führen wir entsprechende kombinatorische Konzepte ein. Wir führen das Gale-Diagramm der Eckenmenge V eines simplizialen d-Polytopes P mit $d+3$ Ecken ein, siehe [Grü03, Section 5.4 und Chapter 6] und [Zie95, Section 6.5]. Danach erläutern wir, wie das minimale Erzeugendensystem von zu P assoziierten Gorenstein Stanley-Reisner Idealen mittels Gale-Diagrammen bestimmt werden kann. Wir schließen das Kapitel ab, indem wir die minimale freie Auflösung des Stanley-Reisner Rings beschreiben.

Im *dritten Kapitel* diskutieren wir einen Vorschlag von Reid [Rei15] zur Verallgemeinerung des Buchsbaum-Eisenbud Theorems [BE77] auf Gorensteins Ideale der Kodimension 4. Wir illustrieren dies anhand einiger Beispiele. Dann berechnen wir für $d = 3, 4$ explizit die minimale freie Auflösung von zu simplizialen d-Polytopen mit $d+4$ Ecken assoziierten Stanley-Reisner Ringen mittels Gale-Diagrammen

Im *vierten Kapitel* betrachten wir ein simpliziales d-Polytop P mit Eckenmenge $V = \{v_1, \ldots, v_n\}$, so dass $0 \in \operatorname{int}(P)$ und den Randkomplex $\Delta(P)$. Wir wenden eine radiale Projektion von P vom Ursprung der Einheitssphäre S^{d-1} an. Das Bild von V unter dieser Projektion wird mit $V' := \{v'_1, \ldots, v'_n\}$ bezeichnet, wobei v'_i das Bild von v_i für $i = 1, \ldots, n$ ist. Dann verwenden wir die stereographische Projektion an jedem Punkt v'_i, für $i = 1, \ldots, n$. Für jedes v'_i erhalten wir ein simpliziales $(d-1)$-Polytop, das höchstens $n-1$ Ecken hat, siehe Proposition 3.2.1.4. Dieses resultierende Polytop wird mit P_i bezeichnet, wenn die stereographische Projektion am Projektionspunkt v'_i, und seine Eckenmenge ist $V'' := \{v''_{i_1}, \ldots, v''_{i_k}\}$, wobei v''_{i_l} das Bild von v'_{i_l} unter dieser Projektion ist. Für jedes solches Polytop gibt es einen assoziierten Stanley-Reisner Ring $\mathbb{K}[\Delta(P_i)] = \mathbb{K}[x_{i_1}, \ldots, x_{i_k}]/I_{\Delta(P_i)}$, wobei $I_{\Delta(P_i)}$ das assoziierte Gorenstein Stanley-Reisner Ideal zu P_i und $\Delta(P_i)$ der Randkomplex von P_i ist.

Jetzt fassen wir die Hauptergebnisse dieses Kapitels zusammen. Das erste Ziel ist, das minimale Erzeugendensystem der assoziierten Gorenstein Stanley-Reisner Ideale $I_{\Delta(P_i)}$ zu bestimmen. Wir geben einen Algorithmus an, der es uns erlaubt, dieses minimal Erzeugendensystem zu bestimmen, wenn das minimale Erzeugendensystem des Gorenstein Stanley-Reisner Ideals $I_{\Delta(P)}$ bekannt ist.

Jetzt wenden wir uns Gorenstein Idealen der Kodimension 4 zu. Wir führen die folgende Definition 4.2.0.1 ein, um eine Frage über Stanley-Reisner Ideale der Kodimension 4 von Reid zu beantworten (siehe [Rei13, Open problems 4.9.4], [Rei15, Section 2.6]), siehe Satz 4.2.0.2.

Definition. *4.2.0.1 Seien $S = \mathbb{K}[x_1, \ldots, x_n]$ ein Polynomring, $f \in S$ ein Polynom und seien I und I' Ideale in S. Wir sagen, I ist ein* vollständiger Durchschnitt *von I' und f, wenn $I = I' + (f)$ und f modulo I' nicht-Nullteiler in dem Restklassenring S/I' ist.*

Also ist unser zweites Ziel in diesem Kapitel, den folgenden Satz zu beweisen:

Satz. *4.2.0.2 Seien P ein simpliziales d-Polytop mit $d+4$ Ecken, $\Delta(P)$ der Randkomplex von P und $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \ldots, x_{d+4}]/I_{\Delta(P)}$ der Stanley-Reisner Ring von $\Delta(P)$. Wenn das Gorenstein Stanley-Reisner Ideal $I_{\Delta(P)}$ minimal von den Polynomen $f_1, \ldots, f_6$ erzeugt wird, dann existiert ein $i \in \{1, \ldots, 6\}$, so dass $T(f_i) \cap T(f_j) = \varnothing$ für alle $i \neq j$ und $I' = (f_j : j \in \{1, \ldots, 6\} \setminus \{i\})$ ein Gorenstein Ideal der Kodimension* 3 *ist.*

Schließlich ist das dritte Ziel dieses Kapitels, ein Gegenbeispiel einer Vermutung von Reid zu geben (siehe [Rei13, Open problems 4.9.4]), nämlich dass jedes Gorenstein Ideal der Kodimension 4 mit gerader Anzahl von Erzeugenden als vollständiger Durchschnitt von einem Gorenstein Ideal der Kodimension 3 und einem zusätzlichem Polynom darstellbar ist.

Im *fünften Kapitel* führen wir affine Gale-Diagramme simplizialer d-Polytopen mit $d+4$ Ecken ein, siehe Definition 5.1.0.1. Gale-Diagramme solcher Polytope sind in $\mathbb{R}^3$, aber affine Gale-Diagramme sind Teilmengen von Vektorräumen, deren Dimension um eins kleiner ist als die Dimension des Vektorraumes, der eine natürliche Obermenge des zugehörigen Gale-Diagramm bildet. Für $d = 3, 4$ konstruieren Grünbaum und Sreedharan alle simpliziale d-Polytope mit $d+4$ Ecken in [GS67]. Es gibt genau 5 kombinatorische Typen simplizialer 3-Polytopen mit 7 Ecken und 37 kombinatorische Typen von simplizialen 4-Polytopen mit 8 Ecken. Für all diese Polytope skizzieren wir affine Gale-Diagramme. Das wird uns helfen, das folgende Hauptziel zu erreichen.

Seien P ein d-Polytop mit der Eckenmenge $V = \{v_1, \dots, v_{d+4}\}$ und $\hat{\mathfrak{B}} = \{\hat{v}_1, \dots, \hat{v}_n\}$ das Gale-Diagramm von V. Sei $\mathfrak{B}^* = \{v_1^*, \dots, v_n^*\}$ ein affines Gale-Diagramm von P, wobei jeder Punkt v_i^* entweder schwarz oder weiß ist. Es gibt eine kanonische Bijektion zwischen $\hat{\mathfrak{B}}$ und $\mathfrak{B}^*$, unter der v_i auf v_i^* abgebildet wird. Daher gibt es auch eine kanonische Bijektion zwischen den Punkten v_i von V und den Punkten v_i^* von$\mathfrak{B}^*$, siehe Bemerkung 5.1.0.2.

Wir charakterisieren das minimale Erzeugendensystem der Gorenstein Stanley-Reisner Ideale $I_{\Delta(P)}$ mittels affiner Gale-Diagramme für ein beliebiges d.

Satz. *5.3.0.2 Seien P ein simpliziales d-Polytop mit der Eckenmenge $V = \{v_1, \dots, v_{d+4}\}$, $\Delta(P)$ sein Randkomplex und die Konfiguration $\mathfrak{B}^* = \{v_1^*, \dots, v_{d+4}^*\}$ ein affines Gale-Diagramm von P. Seien $\mathbb{K}$ ein algebraisch abgeschlossener Körper und $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \dots, x_{d+4}]/I_{\Delta(P)}$ der Stanley-Reisner Ring von $\Delta(P)$. Ein Monom $x_{i_1} \dots x_{i_k}$ ist genau dann ein Element des minimalen Erzeugendensystems des Gorenstein Stanley-Reisner Ideals $I_{\Delta(P)}$, wenn die Menge $\mathfrak{B}^* \setminus \{v_{i_1}^*, \dots, v_{i_k}^*\}$ die folgende Bedingung erfüllt: Die schwarzen und weißen Punkte sind durch eine affine Hyperebene geteilt. Außerdem gibt es keine echte Obermenge, die die vorherige Bedingung erfüllt.*

Im *sechsten Kapitel* beginnen wir mit der Erklärung des Komplexes von Gulliksen und Negård [GN72]. Dann wenden wir uns den entsprechenden kombinatorischen Konzepten zu und führen die Begriffe der benachbarten und zyklischen Polytope ein.

Definition. *6.2.0.1 Ein benachbartes d-Polytop ist ein Konvexes d-Polytop, so dass jede Menge seiner $\lfloor d/2 \rfloor$ Ecken eine Fläche überspannt. Manchmal wird ein Polytop genau dann k-benachbart genannt, wenn jede Menge seiner k Ecken eine Fläche überspannt.*

Definition. *6.2.0.2 Seien $t_1 < t_2 < \dots < t_n$ reele Zahlen. Das zyklische d-Polytop mit n Ecken $C = C_d(t_1, \dots, t_n)$ ist die konvexe Hülle von der Teilmenge $\{f(t_1), f(t_2), \dots, f(t_n)\} \subset \mathbb{R}^d$, wobei $f : \mathbb{R} \to \mathbb{R}^d$ durch $f(t) = (t, t^2, \dots, t^d)$ für $t \in \mathbb{R}$ definiert ist.*

Unser Startpunkt ist ein zyklisches d-Polytop mit $d+3$ Ecken. Für assoziierte Gorenstein Stanley-Reisner Ideale der Kodimension 3 zu diesen Polytopen bedeutet die Strukturtheorie von Buchsbaum und Eisenbud, dass die minimale Anzahl von Erzeugenden eines solchen Ideals eine ungerade Anzahl $2m + 1 \geq 3$ ist und dass dieses minimales Erzeugendensystem als die $2m + 1$ Pfaffschen Determinanten der Ordnung $2m$ einer schiefsymmetrischen $(2m + 1) \times (2m + 1)$-Matrix A gefunden werden kann. Wir beschreiben in diesem Kapitel diese Matrix explizit bezüglich für zyklische $2d$-Polytope mit $2d + 3$ Ecken.

Satz. *6.2.1.1 Sei P ein zyklisches $(2d-2)$-Polytop mit Eckenmenge $V=\{v_1,\dots,v_{2d+1}\}$ und sein Randkomplex $\Delta(P)$. Sei $\mathbb{K}[\Delta(P)]=\mathbb{K}[x_1,\dots,x_{2d+1}]/I_{\Delta(P)}$ der Stanley-Reisner Ring von $\Delta(P)$ und $I_{\Delta(P)}$ das assoziierte Gorenstein Stanley-Reisner Ideal zu P. Dann sind die Erzeugende von $I_{\Delta(P)}$ Pfaffische Determinante der Ordnung $2d$ der schiefsymmetrischen $(2d+1)\times(2d+1)$-Matrix von Grad d*

$$A=\begin{bmatrix}
 & & & & & -x_{2d+1} & x_{d+1} & & \cdots & & 0\\
 & & & & & 0 & -x_1 & x_{d+2} & & & \\
 & & \text{\Large 0} & & & & & & & & \vdots\\
 & & & & & \vdots & & & & & \\
 & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & 0\\
 & & & & & & & & & -x_{d-1} & x_{2d}\\
x_{2d+1} & & \cdots & \cdots & & 0 & & \cdots & \cdots & & -x_d\\
-x_{d+1} & x_1 & & & & & & & & & \\
0 & -x_{d+2} & x_2 & & & & & & & & \\
\vdots & & & & & \vdots & & & \text{\Large 0} & & \\
 & & & \ddots & \ddots & & & & & & \\
\vdots & & & & & 0 & & & & & \\
0 & \cdots & & & -x_{2d} & x_d & & & & &
\end{bmatrix}$$

Wir können minimale Erzeugendensysteme von assoziierten Gorenstein Stanley-Reisner Idealen zu allen simplizialen d-Polytopen mit $d+3$ Ecken durch minimale Erzeugendensysteme von assoziierten Gorenstein Stanley-Reisner Idealen zu zyklischen d-Polytopen mit $d+3$ Ecken bestimmen. Wir erreichen das wie folgt: Wenn wir ein Produkt eines Monoms (oder mehr) aus dem minimalen Erzeugendensystem des assoziierten Gorenstein Stanley-Reisner Ideals zu einem zyklischen Polytop mit einer neuen Variablen (oder mehr) nehmen, dann erhalten wir ein minimales Erzeugendensystem von dem assoziierten Gorenstein Stanley-Reisner Ideal der Kodimension 3 zu einem Polytop der Dimension $d+k$ mit $d+k+3$ Ecken, wobei k die Anzahl der neuen Variablen ist, siehe Beispiel 6.2.1.2.

Daher sind wir interessiert an assoziierten Gorenstein Stanley-Reisner Idealen zu zyklischen d-Polytopen mit $d+4$ Ecken. Da jedes zyklische Polytop ein benachbartes Polytop ist, siehe Korollar 6.2.0.6, betrachten wir in diesem Kapitel nur benachbarte Polytope. Grünbaum und Sreedharan konstruieren alle simplizialen 2-benachbarten 4-Polytope mit 8 Ecken in [GS67]. Es gibt genau drei kombinatorische Typen solcher Polytopen, von denen zwei P^8_{36}, P^8_{37} nicht zyklisch sind und das andere P^8_{35} zyklisch ist, siehe Kapitel 2, Abschnitt 3.2.2. In 1981 konstruiert Barnette [Bar81] eine Familie benachbarten Polytopen für jede Dimension, die nicht zyklisch sind. Danach zeigt Shemer [She82] in 1982, dass die Anzahl der kombinatorischen verschiedenen d-benachbarte $2d$-Polytope mit $2d+4$ Ecken superexponentiell mit $d\to\infty$ wächst. In 1987 werden alle 3-benachbarte 6-Polytope mit 10 Ecken von Bokowski und Shemer [BS87] klassifiziert. Es gibt 37 kombinatorische Typen von ihnen. In 2011 klassifizierte Devyatov [Dev11] d-benachbarte $2d$-Polytope mit $2d+4$ Ecken, die ein affines Gale Diagramm eines speziellen Typs mit genau $d+3$ schwarzen Punkten in konvexer Position haben. Vor vier Jahren veröffentlichten Finbow in [FS04], [Fin10] und [Fin15] eine Liste der simplizialen 2-benachbarten 5-Polytope mit 9 Ecken. Es gibt genau 126 kombinatorisch verschiedene Typen solcher Polytope.

In 1996 berechnen Teria und Hibi [TH96] die Betti Zahlen der minimalen freien Auflösung des Stanley-Reisner Rings des Randkomplexes eines zyklischen Polytops. Dann untersuchen Böhm und Papadakis [BP12] in 2010 die Struktur von assoziierten Stanley-Reisner Ringen zu zyklischen Polytopen und zeigen, wie man den assoziierten Stanley-Reisner Ring zu eines zyklischen d-Polytops mit $n+1$ Ecken in Termen der assoziierten Stanley-Reisner Ringen zu einem zyklischen d-Polytop mit n Ecken und einem zyklischen $(d-2)$-Polytop mit $n-1$ Ecken ausdrückt.

Sei C ein zyklisches $2d$-Polytop mit $2d+4$ Ecken und Randkomplex $\Delta(C)$. Sei $\mathbb{K}[\Delta(C)]$ der assoziierte Stanley-Reisner Ring zu C. Dann hat die minimale freie Auflösung von $\mathbb{K}[\Delta(C)]$ über $S \coloneqq \mathbb{K}[x_1, \dots, x_{2d+4}]$, wie in [TH96] erklärt ist, die Form:

$$0 \longrightarrow S(-(2d+4)) \longrightarrow S(-(d+3))^{b_3^S} \longrightarrow S(-(d+2))^{b_2^S} \longrightarrow$$

$$S(-(d+1))^{b_1^S} \longrightarrow S \longrightarrow \mathbb{K}[\Delta(C)] \longrightarrow 0,$$

wobei $b_1^S = (d+2)^2$, $b_2^S = 2(d+3)(d+1)$ und $b_3^S = (d+2)^2$.

Das heißt, assoziierte Gorenstein Stanley-Reisner Ideale zu zyklischen $2d$-Polytopen mit $2d+4$ sind von $(d+2)^2$ Monomen vom Grad $d+1$ erzeugt.

Deshalb verifizieren wir, ob in diesem Kapitel zyklische Polytope auch eine wichtige Rolle für assoziierte Gorenstein-Ideale der Kodimension 4 spielen.

Vermutung. *6.2.0.7 Sei P ein simpliziales neighbourly $2d$-Polytop mit $2d+4$ Ecken. Das Polytop P ist genau dann zyklisch, wenn es eine $(d+2) \times (d+2)$-Matrix A gibt, so dass alle ihre $(d+1)$-Minoren das assoziierte Gorenstein Stanley-Reisner Ideal zu P minimal erzeugen.*

Die direkte Richtung dieser Vermutung bedeutet, dass die minimalen freien Auflösungen von assoziierten Stanley-Reisner Ringen zu zyklischen $2d$-Polytopen mit $2d+4$ Ecken als eine spezielle Version des Gulliksen-Negård Komplexes zu einer $(d+2) \times (d+2)$-Matrix aufgefasst werden können. Wir beweisen diese Richtung vollständig. Für die Umkehrrichtung zeigen wir Vermutung 6.2.0.7 teilweise. In [Dev11] klassifizierte Devyatov spezielle neighbourly $2d$-Polytope mit $2d+4$ Ecken, die nicht zyklisch sind. Wir beweisen für jedes Polytop von Devyatovs Polytopen, dass das zugehörige Gorenstein Stanley-Reisner Ideal von genau $(d+2)^2$ Monomen erzeugt wird und alle Grad $d+1$ haben, aber es gibt keine quadratische $(d+2) \times (d+2)$-Matrix, so dass ihre $(d+1)$-Minoren es erzeugen. Das bedeutet, dass die minimale freie Auflösungen der assoziierten Stanley Reisner-Ringen zu Devyatovs Polytopen nicht als eine Version des Gulliksen-Negård Komplexes aufgefasst werden kann.

Satz. *6.2.2.1 Sei P ein zyklisches $2d$ Polytop mit Eckenmenge $V = \{v_1, \ldots, v_{d+2}, w_1, \ldots, w_{d+2}\}$ und der Randkomplex $\Delta(P)$. Sei $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \ldots, x_{d+2}, y_1, \ldots, y_{d+2}]/I_{\Delta(P)}$ der Stanley-Reisner Ring $\Delta(P)$ und $I_{\Delta(P)}$ das assoziierte Gorenstein Stanley-Reisner Ideal zu P. Betrachte eine $(d+2) \times (d+2)$-Matrix (oder ihre transponierte) der Form*

$$A = \begin{bmatrix} x_1 & 0 & 0 & 0 & \cdots & y_{d+2} \\ y_1 & x_2 & 0 & 0 & \cdots & 0 \\ 0 & y_2 & x_3 & 0 & \cdots & 0 \\ 0 & 0 & y_3 & x_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x_{d+2} \end{bmatrix}.$$

Dann bilden die $(d+1)$-Minoren ein minimales Erzeugendensystem von $I_{\Delta(P)}$.

Daher charakterisieren wir das minimale Erzeugendensystem der assoziierten Gorenstein Ideale zu speziellen neighbourly $2d$-Polytopen mit $2d+4$ Ecken, die nicht zyklische Polytope sind und bei Devyatov [Dev11] klassifiziert wurden, siehe Satz 6.2.3.4. Wir beziehen uns auf die affine Gale Diagramme dieser Polytope als T-Diagramme. Diese Diagramme haben einen speziellen Typ, nämlich mit genau $d+3$ schwarzen Punkten in konvexer Position und $d+1$ weißen Punkten, die innerhalb des regelmäßigen $(d+3)$-Polygons liegen, das durch den schwarzen Punkt gebildet wird, siehe Definition 6.2.3.1.

Proposition. *6.2.3.5 Sei P ein simpliziales d-benachbartes $2d$-Polytop mit $2d+4$ Ecken, die bei Devyatov [Dev11] klassifiziert wurden, mit Eckenmenge $V=\{v_1, \ldots, v_{d+3}, w_1, \ldots, w_{d+1}\}$ und der Randkomplex $\Delta(P)$. Sei $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \ldots, x_{d+3}, y_1, \ldots, y_{d+1}]/I_{\Delta(P)}$ der Stanley-Reisner Ring von $\Delta(P)$ und $I_{\Delta(P)}$ das assoziierte Gorenstein Stanley-Reisner Ideal zu P. Dann wird $I_\Delta(P)$ von genau $(d+2)^2$ Monomen erzeugt und alle haben Grad $d+1$.*

Wir zeigen die Umkehrrichtung der Vermutung 6.2.0.7 teilweise. Deshalb beweisen wir den folgende Satz.

Satz. *6.2.4.1 Sei P ein simpliziales d-benachbartes $2d$-Polytop mit $2d+4$ Ecken, die bei Devyatov [Dev11] klassifiziert wurden, mit Eckenmenge $V = \{v_1, \ldots, v_{d+3}, w_1, \ldots, w_{d+1}\}$ und der Randkomplex $\Delta(P)$. Sei $\mathbb{K}[\Delta(P)] = \mathbb{K}[x_1, \ldots, x_{d+3}, y_1, \ldots, y_{d+1}]/I_{\Delta(P)}$ der Stanley-Reisner Ring von $\Delta(P)$ und $I_{\Delta(P)}$ das assoziierte Gorenstein Stanley-Reisner Ideal zu P. Dann gibt es keine $(d+2) \times (d+2)$-Matrix, so dass ihre $(d+1)$-Minoren minimale Erzeugende von $I_{\Delta(P)}$ sind.*

Unsere Vorarbeiten sind ein wichtiger Schritt hin zu einem Beweis unserer Vermutung 6.2.0.7, den wir dem geneigten Leser überlassen.