On Terminal Fano Varieties with a Torus Action of Complexity One

Dissertation
der Mathematisch-Naturwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt von
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aus Bolzano, Italien

Tübingen
2017

Gedruckt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Eberhard Karls Universität Tübingen.

Tag der mündlichen Qualifikation:	23.06.2017
Dekan:	Prof. Dr. Wolfgang Rosenstiel
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CONTENTS

ACKNOWLEDGEMENTS

First and foremost, I want to express my immense gratitude to my advisor, Prof. Dr. Jürgen Hausen, for his tireless encouragement and trust. I am thankful for his inspiring guidance and uncountable valuable advices, regarding both mathematical and personal matters. This thesis would simply not have existed without his enthusiasm and kindness.

I am also greatly indebted to Prof. Dr. Alexander Kasprzyk for taking an interest in my work and inviting me to the University of Nottingham. I really enjoyed the hospitality and the many fruitful discussions.

Furthermore, I would like to thank my colleagues Lukas Braun, Anne Fahrner, Christoff Hische, Timo Hummel, Simon Keicher, Faten Komaira, Karin Schaller, Milena Wrobel. Not only they made my everyday life at work rich and entertaining, with hilarious coffee breaks and spontaneous chats, they also created a stimulating mathematical environment, provided me with uncountable insightful comments and helped me proof-read this thesis.

I express my gratitude to all my friends for their personal support and for making this period of my life a memorable adventure. In particular, a special mention goes out to my flatmates Jan, Stefan, Max and Jakub, my cousins Thomas, Carlo and Emil, friends I met in Germany like Babette, Esther, Silvia and Maximilian, friends I left in Italy like Ilaria, Matteo, Alice and Antonio.

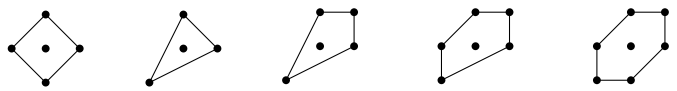
Most importantly, this wonderful experience could not have happened without my family. I am forever grateful to my parents Mauro and Patrizia, my brother Stefano and my grandmother Rosa. Their unbounded faith, support and love made me who I am today.

INTRODUCTION

In this dissertation we provide various classification results on terminal Fano varieties with an effective torus action of complexity one.

A Fano variety is an irreducible normal projective variety, such that its anticanonical divisor class is ample. Research on these varieties was initiated by their namesake G. Fano [20, 21] in the 1930s. The present interest in Fano varieties is motivated by their role in the Minimal Model Program, a structured approach towards the classification of projective varieties up to birational equivalence, initiated by S. Mori [42, 43]. The simplest examples of Fano varieties are the projective spaces $\mathbb{P}_n$. The only smooth Fano variety of dimension one is the projective line. Fano varieties of dimension two are called del Pezzo surfaces, named after P. del Pezzo who first studied them in [18]. There are 10 families of smooth del Pezzo surfaces, namely $\mathbb{P}_1 \times \mathbb{P}_1$ and the blow-ups $\mathrm{Bl}_r(\mathbb{P}_2)$ of $\mathbb{P}_2$ in r points in general position, for $0 \leq r \leq 8$. In dimension three there are 105 families, a result obtained by Iskovskikh [29, 30] and Mori/Mukai [44, 45]. For higher dimensions, the classification problem is still widely open.

A rich source of examples in algebraic geometry is given by toric varieties. A variety is toric if it contains an algebraic torus as an open dense subset, such that the action of the torus on itself extends to the whole variety. Toric Fano varieties are in correspondence with certain lattice polytopes, called Fano polytopes, and their classification reduces to a combinatorial problem. The smooth toric Fano varieties of dimension two are well-known; there are five of them and their Fano polytopes look as follows:



$\mathbb{P}_1 \times \mathbb{P}_1$ $\qquad$ $\mathbb{P}_2$ $\qquad$ $\mathrm{Bl}_1(\mathbb{P}_2)$ $\qquad$ $\mathrm{Bl}_2(\mathbb{P}_2)$ $\qquad$ $\mathrm{Bl}_3(\mathbb{P}_2)$

In dimension three and four there are 18 and 124 smooth toric Fano varieties, respectively; these results are mainly due to V. Batyrev in [4, 5]. Dimension five provides 866 smooth toric Fano varieties and their classification was done by M. Kreuzer and B. Nill in [39]. Finally, in [46] M. Øbro provided an algorithm for the classification of smooth toric Fano varieties of any dimension; explicit lists have been computed by himself up to dimension eight and by A. Paffenholz [48] in dimension nine.

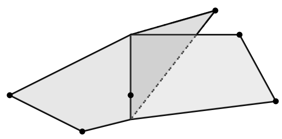
The next step is to weaken the smoothness condition and allow terminal singularities. These arise naturally in the context of the Minimal Model Program. Restricting to the toric case, there is a purely combinatorial criterion for terminality: a toric Fano variety is terminal if and only if the only lattice points in its Fano polytope are the vertices and the origin. A. Kasprzyk provided in [34] the complete list of terminal toric Fano threefolds, which counts 634 varieties up to isomorphism. For higher dimensions, only partial results are known.

We go one step beyond toric varieties and consider rational varieties X with an effective action of a torus T of complexity one, i.e. $\dim(T) = \dim(X) - 1$ holds. Our approach is via the Cox ring

$$\mathcal{R}(X) \ := \bigoplus_{[D] \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)),$$

which can be associated with any normal complete variety X with finitely generated divisor class group $\mathrm{Cl}(X)$, see [1] for the details of this definition. The work [24, 26, 27] by J. Hausen, E. Huggenberger and H. Süß provides an explicit description of the Cox ring of complete rational varieties with a torus action of complexity one.

For such varieties we introduced in [8], together with B. Bechtold, J. Hausen and E. Huggenberger, the anticanonical complex. Similar to the toric Fano polytope, this polyhedral complex characterizes terminality in terms of its lattice points. For the precise formulation see Section 2.1.



The anticanonical complex is a fundamental tool for our first two classification results. We only need to provide lists of Cox rings, since a Fano variety is completely determined by its Cox ring.

Theorem. *The following table lists the Cox rings $\mathcal{R}(X)$ of the non-toric rational terminal $\mathbb{Q}$-factorial Fano threefolds X having Picard number one and an effective two-torus action; the* $\mathrm{Cl}(X)$*-degrees of the generators $T_1, \ldots, T_r$ are denoted as columns $w_i \in \mathrm{Cl}(X)$ of a matrix $[w_1, \ldots, w_r]$.*

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$[w_1, \ldots, w_r]$
1.01	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle$	$\mathbb{Z}$	$[1\ 1\ 1\ 1\ 1]$
1.02	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle$	$\mathbb{Z}$	$[1\ 5\ 2\ 4\ 3]$
1.03	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle$	$\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$	$\left[\begin{smallmatrix}1&1&1&1&1\\ \bar{2}&\bar{3}&\bar{1}&\bar{4}&\bar{0}\end{smallmatrix}\right]$
1.04	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^3\rangle$	$\mathbb{Z}$	$[1\ 5\ 3\ 3\ 2]$
1.05	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^4\rangle$	$\mathbb{Z}$	$[1\ 3\ 2\ 2\ 1]$
1.06	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^4\rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left[\begin{smallmatrix}1&3&2&2&1\\ \bar{1}&\bar{1}&\bar{1}&\bar{1}&\bar{0}\end{smallmatrix}\right]$
1.07	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^6\rangle$	$\mathbb{Z}$	$[2\ 4\ 3\ 3\ 1]$

1.08	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^2T_4+T_5^2\rangle$	$\mathbb{Z}$	$[1\ 3\ 1\ 2\ 2]$
1.09	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^2T_4+T_5^2\rangle$	$\mathbb{Z}$	$[1\ 5\ 2\ 2\ 3]$
1.10	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^2T_4+T_5^2\rangle$	$\mathbb{Z}$	$[3\ 7\ 4\ 2\ 5]$
1.11	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^2T_4+T_5^3\rangle$	$\mathbb{Z}$	$[2\ 1\ 1\ 1\ 1]$
1.12	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^2T_4+T_5^3\rangle$	$\mathbb{Z}$	$[3\ 3\ 1\ 4\ 2]$
1.13	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^2T_4+T_5^3\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$	$\left[\frac{2}{1}\,\frac{1}{2}\,\frac{1}{1}\,\frac{1}{1}\,\frac{1}{0}\right]$
1.14	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^2T_4+T_5^6\rangle$	$\mathbb{Z}$	$[3\ 3\ 2\ 2\ 1]$
1.15	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^2T_4^2+T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left[\frac{1}{1}\,\frac{3}{1}\,\frac{1}{0}\,\frac{1}{0}\,\frac{2}{1}\right]$
1.16	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^2T_4^2+T_5^3\rangle$	$\mathbb{Z}$	$[3\ 3\ 2\ 1\ 2]$
1.17	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3T_4+T_5^2\rangle$	$\mathbb{Z}$	$[1\ 3\ 1\ 1\ 2]$
1.18	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3T_4+T_5^2\rangle$	$\mathbb{Z}$	$[2\ 4\ 1\ 3\ 3]$
1.19	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3T_4+T_5^2\rangle$	$\mathbb{Z}$	$[3\ 7\ 2\ 4\ 5]$
1.20	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3T_4+T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left[\frac{1}{1}\,\frac{3}{1}\,\frac{1}{0}\,\frac{1}{0}\,\frac{2}{1}\right]$
1.21	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3T_4+T_5^4\rangle$	$\mathbb{Z}$	$[2\ 2\ 1\ 1\ 1]$
1.22	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3T_4+T_5^4\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left[\frac{2}{1}\,\frac{2}{1}\,\frac{1}{1}\,\frac{1}{1}\,\frac{1}{0}\right]$
1.23	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3T_4^2+T_5^2\rangle$	$\mathbb{Z}$	$[3\ 5\ 2\ 1\ 4]$
1.24	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3T_4^3+T_5^2\rangle$	$\mathbb{Z}$	$[2\ 4\ 1\ 1\ 3]$
1.25	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^4T_4+T_5^3\rangle$	$\mathbb{Z}$	$[3\ 3\ 1\ 2\ 2]$
1.26	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^4T_4^2+T_5^3\rangle$	$\mathbb{Z}$	$[3\ 3\ 1\ 1\ 2]$
1.27	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^5T_4+T_5^2\rangle$	$\mathbb{Z}$	$[2\ 4\ 1\ 1\ 3]$
1.28	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^5T_4+T_5^3\rangle$	$\mathbb{Z}$	$[3\ 3\ 1\ 1\ 2]$
1.29	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^6T_4+T_5^2\rangle$	$\mathbb{Z}$	$[3\ 5\ 1\ 2\ 4]$
1.30	$\frac{\mathbb{K}[T_1,\dots,T_6]}{\langle T_1T_2+T_3T_4+T_5^2,\,aT_3T_4+T_5^2+T_6^2\rangle}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left[\frac{1}{1}\,\frac{1}{1}\,\frac{1}{0}\,\frac{1}{0}\,\frac{1}{1}\,\frac{1}{0}\right]$
1.31	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3+T_4^3+T_5^2\rangle$	$\mathbb{Z}$	$[1\ 1\ 4\ 2\ 3]$
1.32	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3+T_4^3+T_5^2\rangle$	$\mathbb{Z}$	$[2\ 3\ 1\ 2\ 3]$
1.33	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^2+T_4^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left[\frac{2}{1}\,\frac{4}{1}\,\frac{3}{1}\,\frac{3}{0}\,\frac{1}{0}\right]$
1.34	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3+T_4^2\rangle$	$\mathbb{Z}$	$[1\ 5\ 2\ 3\ 1]$
1.35	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3+T_4^2\rangle$	$\mathbb{Z}$	$[1\ 5\ 2\ 3\ 2]$
1.36	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3+T_4^2\rangle$	$\mathbb{Z}$	$[1\ 5\ 2\ 3\ 3]$
1.37	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3+T_4^2\rangle$	$\mathbb{Z}$	$[1\ 5\ 2\ 3\ 4]$
1.38	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3+T_4^2\rangle$	$\mathbb{Z}$	$[5\ 7\ 4\ 6\ 1]$
1.39	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3+T_4^2\rangle$	$\mathbb{Z}$	$[5\ 7\ 4\ 6\ 3]$
1.40	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^3+T_4^3\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$	$\left[\frac{1}{1}\,\frac{2}{2}\,\frac{1}{2}\,\frac{1}{0}\,\frac{1}{0}\right]$
1.41	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^4+T_4^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left[\frac{1}{1}\,\frac{3}{1}\,\frac{1}{0}\,\frac{2}{1}\,\frac{1}{0}\right]$
1.42	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^4+T_4^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left[\frac{1}{1}\,\frac{3}{1}\,\frac{1}{0}\,\frac{2}{1}\,\frac{2}{1}\right]$
1.43	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^4+T_4^3\rangle$	$\mathbb{Z}$	$[5\ 7\ 3\ 4\ 1]$
1.44	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^4+T_4^3\rangle$	$\mathbb{Z}$	$[5\ 7\ 3\ 4\ 2]$
1.45	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^5+T_4^2\rangle$	$\mathbb{Z}$	$[3\ 7\ 2\ 5\ 1]$

1.46	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^5+T_4^2\rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix}3&7&2&5&4\end{smallmatrix}\right]$
1.47	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3^6+T_4^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left[\begin{smallmatrix}\bar{2}&\bar{4}&\bar{1}&\bar{3}&\bar{1}\\1&1&1&0&0\end{smallmatrix}\right]$

Our second classification result concerns the larger class of combinatorially minimal varieties, i.e. those that do not have any contractible divisor. It turns out that, in our setting, the Picard number is bounded by two. In addition to the varieties of Picard number one, which are all combinatorially minimal, we obtain the following varieties.

Theorem. *The following table lists the Cox rings $\mathcal{R}(X)$ of the non-toric rational combinatorially minimal terminal $\mathbb{Q}$-factorial Fano threefolds X with an effective two-torus action and with Picard number bigger than one. The* $\mathrm{Cl}(X)$*-degrees of the generators $T_1,\dots,T_r$ are denoted as columns $w_i\in\mathrm{Cl}(X)$ of a matrix $[w_1,\dots,w_r]$.*

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$[w_1,\dots,w_r]$
2.01	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2+T_3T_4+T_5T_6\rangle$	$\mathbb{Z}^2$	$\left[\begin{smallmatrix}1&0&1&0&1&0\\0&1&0&1&0&1\end{smallmatrix}\right]$
2.02	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle$	$\mathbb{Z}^2$	$\left[\begin{smallmatrix}1&0&1&0&1&0\\0&1&0&1&0&1\end{smallmatrix}\right]$
2.03	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2+T_3T_4+T_5T_6\rangle$	$\mathbb{Z}^2\oplus\mathbb{Z}/3\mathbb{Z}$	$\left[\begin{smallmatrix}1&0&1&0&1&0\\0&1&0&1&0&1\\\bar{2}&\bar{1}&\bar{1}&\bar{2}&\bar{0}&\bar{0}\end{smallmatrix}\right]$
2.04	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle$	$\mathbb{Z}^2\oplus\mathbb{Z}/3\mathbb{Z}$	$\left[\begin{smallmatrix}1&0&1&0&1&0\\0&1&0&1&0&1\\\bar{1}&\bar{1}&\bar{2}&\bar{2}&\bar{0}&\bar{0}\end{smallmatrix}\right]$
2.05	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2+T_3^2T_4+T_5T_6\rangle$	$\mathbb{Z}^2$	$\left[\begin{smallmatrix}2&0&1&0&1&1\\0&1&0&1&0&1\end{smallmatrix}\right]$
2.06	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2+T_3T_4^2+T_5^2T_6^2\rangle$	$\mathbb{Z}^2$	$\left[\begin{smallmatrix}1&1&2&0&1&0\\1&1&0&1&0&1\end{smallmatrix}\right]$
2.07	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3+T_4T_5+T_6^2\rangle$	$\mathbb{Z}^2$	$\left[\begin{smallmatrix}1&1&0&2&0&1\\0&0&2&1&1&1\end{smallmatrix}\right]$
2.08	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3+T_4T_5^2+T_6^2\rangle$	$\mathbb{Z}^2$	$\left[\begin{smallmatrix}1&1&0&2&0&1\\1&0&1&0&1&1\end{smallmatrix}\right]$
2.09	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^2+T_4T_5+T_6^2\rangle$	$\mathbb{Z}^2$	$\left[\begin{smallmatrix}1&1&0&2&0&1\\0&0&1&1&1&1\end{smallmatrix}\right]$
2.10	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^2+T_4T_5^2+T_6^2\rangle$	$\mathbb{Z}^2$	$\left[\begin{smallmatrix}1&1&0&2&0&1\\0&0&1&0&1&1\end{smallmatrix}\right]$
2.11	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2+T_3^2T_4+T_5^2\rangle$	$\mathbb{Z}^2$	$\left[\begin{smallmatrix}2&0&1&0&1&1\\1&1&0&2&1&0\end{smallmatrix}\right]$
2.12	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3+T_4^2+T_5^2\rangle$	$\mathbb{Z}^2\oplus\mathbb{Z}/2\mathbb{Z}$	$\left[\begin{smallmatrix}1&1&0&1&1&0\\0&0&2&1&1&1\\\bar{0}&\bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{0}\end{smallmatrix}\right]$

Our third classification result concerns smooth projective varieties. In [19], together with A. Fahrner and J. Hausen, we look at varieties having small Picard number and arbitrary dimension. For toric varieties, the projective spaces are the only smooth examples with Picard number one, and we have Kleinschmidt's description [36] of all smooth toric projective varieties with Picard number two. We follow that line and study smooth projective rational varieties with a torus action of complexity one. The case of Picard number one is done by A. Liendo and H. Süß [40, Thm. 6.5]: the only non-toric examples are the smooth projective quadrics in dimension three and four. Our work settles the case of Picard number two.

Theorem. *The following table lists the Cox rings $\mathcal{R}(X)$ of the non-toric rational smooth projective varieties X of complexity one with Picard number equal to two. The grading by $\mathrm{Cl}(X)=\mathbb{Z}^2$ is given by the matrix $[w_1,\dots,w_r]$ of generator degrees* $\deg(T_i),\deg(S_j)$ *and the isomorphy type is specified by an ample class $u\in\mathrm{Cl}(X)$.*

No.	$\mathcal{R}(X)$	$[w_1,\dots,w_r]$	u	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\dots,T_7]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix}0&0&1&1&1&1&1\\1&1&0&a&2-a&b&2-b\end{bmatrix}$ $1\le a\le b$	$\begin{bmatrix}1\\1+b\end{bmatrix}$	4
2	$\frac{\mathbb{K}[T_1,\dots,T_7]}{\langle T_1T_2T_3+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix}0&0&1&1&0&1&0\\1&1&0&1&1&1&1\end{bmatrix}$	$\begin{bmatrix}1\\2\end{bmatrix}$	4
3	$\frac{\mathbb{K}[T_1,\dots,T_6]}{\langle T_1T_2T_3^2+T_4T_5+T_6^2\rangle}$	$\begin{bmatrix}0&0&1&1&1&1\\1&1&0&2-a&a&1\end{bmatrix}$ $a\ge 1$	$\begin{bmatrix}1\\1+a\end{bmatrix}$	3
4	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^{l_2}+T_3T_4^{l_4}+T_5T_6^{l_6}\rangle}$ $m\ge 0$	$\begin{bmatrix}0&1&a&1&b&1&\vert&c_1&\dots&c_m\\1&0&1&0&1&0&\vert&1&\dots&1\end{bmatrix}$ $0\le a\le b,\ c_1\le\dots\le c_m,$ $l_2=a+l_4=b+l_6$	$\begin{bmatrix}d+1\\1\end{bmatrix}$ $d:=\max(b,c_m)$	$m+3$
5	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$ $m\ge 0$	$\begin{bmatrix}0&2a+1&a&1&a&1&\vert&1&\dots&1\\1&1&1&0&1&0&\vert&0&\dots&0\end{bmatrix}$ $a\ge 0$	$\begin{bmatrix}2a+2\\1\end{bmatrix}$	$m+3$
6	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$ $m\ge 1$	$\begin{bmatrix}0&2c+1&a&b&c&1&\vert&1&\dots&1\\1&1&1&1&1&0&\vert&0&\dots&0\end{bmatrix}$ $a,b,c\ge 0,\quad a<b,$ $a+b=2c+1$	$\begin{bmatrix}2c+2\\1\end{bmatrix}$	$m+3$
7	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\ge 1$	$\begin{bmatrix}0&0&0&0&-1&1&\vert&1&\dots&1\\1&1&1&1&1&1&\vert&0&\dots&0\end{bmatrix}$	$\begin{bmatrix}1\\2\end{bmatrix}$	$m+3$
8	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\ge 2$	$\begin{bmatrix}0&0&0&0&0&0&\vert&1&1&\dots&1\\1&1&1&1&1&1&\vert&0&a_2&\dots&a_m\end{bmatrix}$ $0\le a_2\le\dots\le a_m,\ a_m>0$	$\begin{bmatrix}1\\a_m+1\end{bmatrix}$	$m+3$
9	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\ge 2$	$\begin{bmatrix}0&a_2&\dots&a_6&\vert&1&\dots&1\\1&1&\dots&1&\vert&0&\dots&0\end{bmatrix}$ $0\le a_3\le a_5\le a_6\le a_4\le a_2,$ $a_2=a_3+a_4=a_5+a_6$	$\begin{bmatrix}a_2+1\\1\end{bmatrix}$	$m+3$
10	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\ge 1$	$\begin{bmatrix}1&1&1&1&1&\vert&0&\dots&0\\-1&1&0&0&0&\vert&1&\dots&1\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$	$m+2$
11	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\ge 2$	$\begin{bmatrix}1&1&1&1&1&\vert&0&a_2&\dots&a_m\\0&0&0&0&0&\vert&1&1&\dots&1\end{bmatrix}$ $0\le a_2\le\dots\le a_m,\ a_m>0$	$\begin{bmatrix}a_m+1\\1\end{bmatrix}$	$m+2$
12	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\ge 2$	$\begin{bmatrix}1&1&1&1&1&\vert&0&0&\dots&0\\0&2c&a&b&c&\vert&1&1&\dots&1\end{bmatrix}$ $0\le a\le c\le b,\ a+b=2c$	$\begin{bmatrix}1\\2c+1\end{bmatrix}$	$m+2$
13	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\left\langle \begin{array}{c}T_1T_2+T_3T_4+T_5T_6,\\ aT_3T_4+T_5T_6+T_7T_8\end{array}\right\rangle}$ $a\in\mathbb{K}^*\setminus\{1\}$	$\begin{bmatrix}1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	4

The anticanonical divisor of the varieties can be computed explicitly. This enables us to determine, for every dimension, the finitely many families of non-toric smooth rational Fano varieties of Picard number two that admit a torus action of complexity one.

Theorem. *The following table lists the Cox rings $\mathcal{R}(X)$ of the non-toric rational smooth Fano varieties X of complexity one with Picard number equal to two. The grading by* $\mathrm{Cl}(X)=\mathbb{Z}^2$ *is given by the matrix $[w_1,\dots,w_r]$ of generator degrees* $\deg(T_i),\deg(S_j)$ *and we list the anticanonical class $-\mathcal{K}_X$.*

No.	$\mathcal{R}(X)$	$[w_1,\dots,w_r]$	$-\mathcal{K}_X$	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\dots,T_7]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix}0&0&1&1&1&1&1\\1&1&0&1&1&1&1\end{bmatrix}$	$\begin{bmatrix}3\\4\end{bmatrix}$	4

2	$\frac{\mathbb{K}[T_1,\dots,T_7]}{\langle T_1T_2T_3+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0&0&1&1&0&1&0 \\ 1&1&0&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$	4
3	$\frac{\mathbb{K}[T_1,\dots,T_6]}{\langle T_1T_2T_3^2+T_4T_5+T_6^2\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1 \\ 1&1&0&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	3
4.A	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|cccc} 0&1&0&1&0&1&c&0&\dots&0 \\ 1&0&1&0&1&0&1&1&\dots&1 \end{array}\right]$ $c \in \{-1,0\}$, $c := 0$ *if* $m = 0$	$\begin{bmatrix} 2+c \\ 2+m \end{bmatrix}$	$m+3$
4.B	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0&1&1&1&1&1&1&\dots&1 \\ 1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 3+m \\ 2+m \end{bmatrix}$	$m+3$
4.C	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0&1&0&1&0&1&0&\dots&0 \\ 1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2+m \end{bmatrix}$	$m+3$
5	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|ccc} 0&2a+1&a&1&a&1&1&\dots&1 \\ 1&1&1&0&1&0&0&\dots&0 \end{array}\right]$ $0 \leq 2a < m$	$\begin{bmatrix} 2a+m+2 \\ 2 \end{bmatrix}$	$m+3$
6	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|ccc} 0&2c+1&a&b&c&1&1&\dots&1 \\ 1&1&1&1&1&0&0&\dots&0 \end{array}\right]$ $a,b,c \geq 0,\quad a < b,$ $a+b = 2c+1,$ $m > 3c+1$	$\begin{bmatrix} 3c+2+m \\ 3 \end{bmatrix}$	$m+3$
7	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $1\leq m\leq 3$	$\left[\begin{array}{cccccc\|ccc} 0&0&0&0&-1&1&1&\dots&1 \\ 1&1&1&1&1&1&0&\dots&0 \end{array}\right]$	$\begin{bmatrix} m \\ 4 \end{bmatrix}$	$m+3$
8	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccc\|cccc} 0&0&0&0&0&0&1&1&\dots&1 \\ 1&1&1&1&1&1&0&a_2&\dots&a_m \end{array}\right]$ $0 \leq a_2 \leq \dots \leq a_m,$ $a_m \in \{1,2,3\},$ $4+\sum_{k=2}^m a_k > ma_m$	$\begin{bmatrix} m \\ 4+\sum_{k=2}^m a_k \end{bmatrix}$	$m+3$
9	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccc\|ccc} 0&a_2&\dots&a_6&1&\dots&1 \\ 1&1&\dots&1&0&\dots&0 \end{array}\right]$ $0 \leq a_3 \leq a_5 \leq a_6 \leq a_4 \leq a_2,$ $a_2 = a_3+a_4 = a_5+a_6,$ $2a_2 < m$	$\begin{bmatrix} 2a_2+m \\ 4 \end{bmatrix}$	$m+3$
10	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $1\leq m\leq 2$	$\left[\begin{array}{ccccc\|ccc} 1&1&1&1&1&0&\dots&0 \\ -1&1&0&0&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 3 \\ m \end{bmatrix}$	$m+2$
11	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1&1&1&1&1&0&a_2&\dots&a_m \\ 0&0&0&0&0&1&1&\dots&1 \end{array}\right]$ $0 \leq a_2 \leq \dots \leq a_m,$ $a_m \in \{1,2\},$ $3+\sum_{k=2}^m a_k > ma_m$	$\begin{bmatrix} 3+\sum_{k=2}^m a_k \\ m \end{bmatrix}$	$m+2$
12	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1&1&1&1&1&0&0&\dots&0 \\ 0&2c&a&b&c&1&1&\dots&1 \end{array}\right]$ $0 \leq a \leq c \leq b,\ a+b = 2c,$ $3c < m$	$\begin{bmatrix} 3 \\ 3c+m \end{bmatrix}$	$m+2$
13	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\left\langle \begin{smallmatrix} T_1T_2+T_3T_4+T_5T_6, \\ aT_3T_4+T_5T_6+T_7T_8 \end{smallmatrix} \right\rangle}$ $a\in\mathbb{K}^*\setminus\{1\}$	$\begin{bmatrix} 1&0&1&0&1&0&1&0 \\ 0&1&0&1&0&1&0&1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	4

A closer look at the geometry of these Fano varieties reveals that they are obtained from a series of lower dimensional varieties via iterating the following procedure: we take a certain $\mathbb{P}_1$-bundle over the given variety, apply a natural sequence of flips and then contract a prime divisor. In terms of Cox rings, this construction simply means duplicating a free weight.

Corollary. *Every smooth rational non-toric Fano variety with a torus action of complexity one and Picard number two arises via iterated duplication of a free weight from a smooth rational projective (not necessarily Fano) variety with a torus action of complexity one, Picard number two and dimension at most seven.*

Similar to the Fano varieties, we can figure out the almost Fano ones, i.e. those with a big and nef anticanonical divisor. Without the assumption of a torus action, the classification of smooth almost Fano varieties of Picard number two is widely open; for the threefold case, we refer to the work of P. Jahnke, T. Peternell and I. Radloff [32, 33]. In the setting of a torus action of complexity one, the following result together with the previous theorem settles the problem in any dimension; by a truly almost Fano variety we mean an almost Fano variety which is not Fano.

Theorem. *The following table lists the Cox rings $\mathcal{R}(X)$ of the non-toric rational smooth truly almost Fano varieties X of complexity one with Picard number equal to two. The grading by $\mathrm{Cl}(X) = \mathbb{Z}^2$ is given by the matrix $[w_1, \ldots, w_r]$ of generator degrees $\deg(T_i)$ and $\deg(S_j)$, and the isomorphy type is specified by an ample class $u \in \mathrm{Cl}(X)$.*

<table>
<tr><th>No.</th><th>$\mathcal{R}(X)$</th><th>$[w_1, \ldots, w_r]$</th><th>u</th><th>$\dim(X)$</th></tr>
<tr><td>4.A</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 1 & 0 & 1 & 0 & 1 & c_1 & \ldots & c_m \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 1 \end{array}\right]$
$c_1 \leq \ldots \leq c_m$
$d := \max(0, c_m)$
$(2+m)d = 2 + c_1 + \cdots + c_m$</td><td>$\begin{bmatrix} 1 \\ 1+d \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>4.B</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccc|cccc} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & \ldots & 1 \end{array}\right]$</td><td>$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>4.C</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccc|cccc} 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & \ldots & 1 \end{array}\right]$</td><td>$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>4.D</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6\rangle}$
$m\geq 0$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 1 & 0 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 1 \end{array}\right]$</td><td>$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>4.E</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^3+T_3T_4+T_5T_6\rangle}$
$m\geq 0$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 1 & 2 & 1 & 2 & 1 & 2 & \ldots & 2 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 1 \end{array}\right]$</td><td>$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>4.F</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6^2\rangle}$
$m\geq 0$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 1 \end{array}\right]$</td><td>$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>5</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$
$m\geq 0$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 2a+1 & a & 1 & a & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$
$m = 2a$</td><td>$\begin{bmatrix} m+2 \\ 1 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>6</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 2c+1 & a & b & c & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$
$a, b, c \geq 0, \quad a < b,$
$a + b = 2c + 1,$
$m = 3c + 1$</td><td>$\begin{bmatrix} 2c+2 \\ 1 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>7</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$
$m=4$</td><td>$\left[\begin{array}{cccccc|cccc} 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$</td><td>$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$</td><td>7</td></tr>
<tr><td>8</td><td>$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$
$m\geq 2$</td><td>$\left[\begin{array}{cccccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \ldots & a_m \end{array}\right]$
$0 \leq a_2 \leq \ldots \leq a_m, \ a_m > 0,$
$4 + a_2 + \ldots + a_m = m a_m$</td><td>$\begin{bmatrix} 1 \\ a_m+1 \end{bmatrix}$</td><td>$m+3$</td></tr>
</table>

9	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccc\|ccc} 0 & a_2 & \ldots & a_6 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \end{array}\right]$ $0\leq a_3\leq a_5\leq a_6\leq a_4\leq a_2,$ $a_2=a_3+a_4=a_5+a_6,$ $m=2a_2$	$\begin{bmatrix} a_2+1 \\ 1 \end{bmatrix}$	$m+3$
10	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m=3$	$\left[\begin{array}{ccccc\|ccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	5
11	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \ldots & a_m \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \end{array}\right]$ $0\leq a_2\leq\ldots\leq a_m,\ a_m>0,$ $3+a_2+\ldots+a_m=ma_m$	$\begin{bmatrix} 1 \\ a_m+1 \end{bmatrix}$	$m+2$
12	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 3$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 2c & a & b & c & 1 & 1 & \ldots & 1 \end{array}\right]$ $0\leq a\leq c\leq b,\ a+b=2c,$ $m=3c$	$\begin{bmatrix} 1 \\ 2c+1 \end{bmatrix}$	$m+2$

The present dissertation is organized as follows. *The first Chapter* recalls basic notions and statements for the convenience of the reader. We review the language of Cox rings and bunched rings. This enables the description of projective varieties with a torus action of complexity one via certain defining matrices and bunches of cones. We follow closely [1]. *The second Chapter* provides the classification of $\mathbb{Q}$-factorial terminal Fano threefolds with a torus action of complexity one having Picard number one. First we introduce the anticanonical complex. We investigate this object in the special situation of a rational variety of complexity one. This description provides the tools for the explicit classification in dimension three for Picard number one. Parts of this Chapter already appeared in [8]. *The third Chapter* deals with terminal Fano threefolds with a torus action of complexity one that are combinatorially minimal, i.e. that do not admit any divisorial contraction. A first step is to bound the Picard number. Then we go through all possible cases and obtain the explicit classification. *The fourth Chapter* focuses on smooth varieties with a torus action of complexity one, having Picard number two and arbitrary dimension. In this context, we are able to classify all rational projective varieties and find, among them, the Fano ones. Furthermore we discuss a geometric procedure that allows to find all these varieties starting from lower dimensional prototypes. The results of this Chapter are already published in [19]. *The Appendix* provides a compendium of the classification results from Chapter 2 and 3. For each variety we give the defining data and accompanying relevant information.

CHAPTER
ONE

BACKGROUND

Throughout the whole thesis $\mathbb{K}$ is an algebraically closed field of characteristic zero.

This Chapter provides the fundamental notions and concepts required in the rest of the thesis and does not contain original results by the author. The main reference is the book [1] by I. Arzhantsev, U. Derenthal, J. Hausen and A. Laface.

In Section 1.1 we review some basic facts about G-varieties and emphasize the toric case. Section 1.2 briefly recalls the theory of Cox rings. In Section 1.3 we present the language of bunched rings and their correspondence with Mori dream spaces. Geometric properties of these varieties are investigated in Section 1.4. Lastly, in Section 1.5, we recall the construction and description of T-varieties of complexity one.

1.1 G-varieties and toric geometry

An *(affine) algebraic group* is an (affine) variety G over $\mathbb{K}$ together with a group structure, such that the group operations $G \times G \to G$, $(g,h) \mapsto gh$, and $G \to G$, $g \mapsto g^{-1}$, are morphisms. A *character* of G is a morphism $\chi\colon G \to \mathbb{K}^*$ of algebraic groups. The characters of G form a group, denoted by $\mathbb{X}(G)$.

Definition 1.1.1. Let G be an affine algebraic group. A variety X, together with a morphism $G \times X \to X$, is called a *G-variety*.

Definition 1.1.2. Let G be a reductive algebraic group and X a G-variety. The *ring of invariants* is the $\mathbb{K}$-algebra

$$\mathcal{O}(X)^G \ := \ \{f \in \mathcal{O}(X);\ f(g \cdot x) = f(x) \text{ for all } g \in G \text{ and } x \in X\}.$$

A *good quotient* for the action of G on X is a morphism $\pi\colon X \to Y$ such that

- π is *affine*, i.e. preimages of open affine subsets of Y are again affine;
- π is *G-invariant*, i.e. $\pi(x) = \pi(g \cdot x)$ holds for all $g \in G$ and $x \in X$;
- the pullback $\pi^*\colon \mathcal{O}_Y \to (\pi_*\mathcal{O}_X)^G$ is an isomorphism.

If a good quotient $\pi\colon X \to Y$ for the G-action exists, then the quotient space Y is unique up to isomorphism and we denote it by $X /\!\!/ G$.

Definition 1.1.3. A *quasitorus* is an affine algebraic group isomorphic to some $(\mathbb{K}^*)^n \times C$, where C is some finite abelian group. A connected quasitorus is a *torus*. We denote by $\mathbb{T}^n := (\mathbb{K}^*)^n$ the *standard n-torus.*

Remark 1.1.4. The category of quasitori and that of finitely generated abelian groups are equivalent via the functors

$$G \mapsto \mathbb{X}(G), \qquad K \mapsto \operatorname{Spec}(\mathbb{K}[K]).$$

Now we turn to the correspondence between quasitori actions on affine varieties and affine algebras graded by finitely generated abelian groups. The action of a quasitorus H on an affine variety X defines a grading on the algebra of regular functions of X by

$$\mathcal{O}(X) = \bigoplus_{\chi \in \mathbb{X}(H)} \mathcal{O}(X)_\chi \qquad \text{with} \quad \mathcal{O}(X)_\chi = \{f \in \mathcal{O}(X);\ f(h \cdot x) = \chi(h) f(x)\}.$$

Vice versa, let K be a finitely generated abelian group and R a K-graded affine $\mathbb{K}$-algebra with K-homogeneous generators $f_1, \ldots, f_r \in R$. Then we get a closed embedding

$$X := \operatorname{Spec}(R) \to \mathbb{K}^r, \qquad x \mapsto (f_1(x), \ldots, f_r(x)),$$

and $X \subseteq \mathbb{K}^r$ is invariant under the *diagonal action* of the quasitorus $H := \operatorname{Spec}(\mathbb{K}[K])$ given by the characters χ^{w_i}, where $w_i := \deg(f_i)$, i.e.

$$h \cdot x := (\chi^{w_1}(h) x_1, \ldots, \chi^{w_r}(h) x_r).$$

Now we turn to a special case of G-varieties, the toric varieties. For a detailed background, refer e.g. to [15, 16, 22, 47].

Definition 1.1.5. A *toric variety* is a normal irreducible variety Z together with a *base-point* $z_0 \in Z$ and an effective action of a torus T_Z, such that the map $T_Z \to Z$ defined via $t \mapsto t \cdot z_0$ is an open embedding.

We recall a few basic notions from convex geometry and their relation to toric geometry.

Let N and M be two lattices, dual to each other. Define the rational vector spaces $N_\mathbb{Q} := N \otimes \mathbb{Q}$ and $M_\mathbb{Q} := M \otimes \mathbb{Q}$. By a *cone* $\sigma \subseteq N_\mathbb{Q}$ we always mean a convex polyhedral cone. The pair (N, σ) is called a *lattice cone*. The *dual cone* of σ is the cone $\sigma^\vee \subseteq M_\mathbb{Q}$ of linear forms $u \in M_\mathbb{Q}$ for which $u|_\sigma \geq 0$ holds. A cone σ is called *pointed* if $\sigma \cap -\sigma = \{0\}$ holds, i.e. it does not contain any line. The *dimension* of a cone is defined as the dimension of its linear hull. A *face* of the cone σ is a subset $\tau \subseteq \sigma$ such that there exists a linear form $u \in \sigma^\vee$ with $\tau = \sigma \cap \ker(u)$. Faces are again cones. A face of dimension one is called a *ray*, whereas a face of codimension one is called a *facet*.
A *quasifan* Σ in $N_\mathbb{Q}$ is a finite collection of cones such that any two cones intersect in a common face and any face of a cone in Σ is again an element of Σ. If all cones are pointed, we call Σ a *fan*. The pair (N, Σ) is called a *lattice fan*. The *support* $|\Sigma|$ of a (quasi)fan Σ is the union of its cones. Σ is called *complete* if $|\Sigma| = N_\mathbb{Q}$ holds. We denote by $\Sigma^{(k)}$ the set of k-dimensional cones of the (quasi)fan Σ.

Remark 1.1.6. The categories of lattice cones and that of affine toric varieties are covariantly equivalent via a functor mapping a lattice cone (N, σ) to the affine variety $Z_\sigma := \mathrm{Spec}\mathbb{K}[\sigma^\vee \cap M]$, which is toric with dense torus $T_{Z_\sigma} := \mathrm{Spec}\mathbb{K}[M]$.
This process can be extended to the categories of toric varieties and lattice fans. Given a lattice fan (N, Σ), we construct a toric variety Z_Σ by glueing together the affine pieces Z_σ and Z_τ along the common open subset $Z_{\sigma\cap\tau}$ for all $\sigma, \tau \in \Sigma$. Note that complete toric varieties correspond to complete lattice fans.

1.2 Cox rings

Let X be a normal irreducible variety over $\mathbb{K}$. A *prime divisor* on X is an irreducible subvariety $D \subset X$ of codimension one. The free abelian group generated by the prime divisors on X is denoted by $\mathrm{WDiv}(X)$, its elements are the *Weil divisors* on X. A Weil divisor $D \in \mathrm{WDiv}(X)$ is called *principal* if there exists a rational function $f \in \mathbb{K}(X)^*$ such that

$$D \;=\; \mathrm{div}(f) \;:=\; \sum_{E \text{ prime}} \mathrm{ord}_E(f) \cdot E \;\in\; \mathrm{WDiv}(X)$$

holds, where $\mathrm{ord}_E(f)$ denotes the vanishing order of f along the prime divisor E. The subgroup of principal divisors $\mathrm{PDiv}(X) \le \mathrm{WDiv}(X)$ is the image of the homomorphism $\mathbb{K}(X)^* \to \mathrm{WDiv}(X)$, sending f to $\mathrm{div}(f)$.
For any open subset $U \subseteq X$ we define the *restriction* of $D \in \mathrm{WDiv}(X)$ to U as the Weil divisor $D|_U \in \mathrm{WDiv}(U)$, where we set $D|_U := D \cap U$ if this intersection is non-trivial and $D|_U := 0$ otherwise. A Weil divisor $D \in \mathrm{WDiv}(X)$ is called a *Cartier divisor* if it is locally principal, i.e. there exists an open cover $\{U_i\}_{i\in I}$ of X such that $D|_{U_i}$ is principal for every $i \in I$. We write $\mathrm{CDiv}(X) \le \mathrm{WDiv}(X)$ for the group of Cartier divisors.
The *divisor class group* and the *Picard group* of X are respectively

$$\mathrm{Cl}(X) := \mathrm{WDiv}(X)/\,\mathrm{PDiv}(X), \qquad \mathrm{Pic}(X) := \mathrm{WDiv}(X)/\,\mathrm{CDiv}(X).$$

It is clear that $\mathrm{Pic}(X) \subseteq \mathrm{Cl}(X)$ holds. We define the *Picard number* $\rho(X)$ as the rank of the Picard group of X. A Weil divisor $D = a_1D_1 + \ldots + a_nD_n$ is *effective* if the D_i are prime divisors and $a_i \ge 0$ holds for all $i = 1, \ldots, n$. In this case we write $D \ge 0$.

Definition 1.2.1. To any Weil divisor D on X we associate its *divisorial sheaf* $\mathcal{O}_X(D)$ of $\mathcal{O}_X$-modules; for any open subset $U \subseteq X$ we define

$$\Gamma(U, \mathcal{O}_X(D)) \;:=\; \{f \in \mathbb{K}(X)^*; \mathrm{div}(f|_U) + D|_U \ge 0\} \cup \{0\}.$$

To any subgroup $K \le \mathrm{WDiv}(X)$ we associate its *sheaf of divisorial $\mathcal{O}_X$-algebras*

$$\mathcal{S} \;:=\; \bigoplus_{D\in K} \mathcal{S}_D \qquad \text{with} \quad \mathcal{S}_D \;:=\; \mathcal{O}_X(D).$$

Multiplication in $\mathcal{S}$ is given by the usual multiplication in $\mathbb{K}(X)$. Note that, for $f_1 \in \Gamma(X, \mathcal{O}_X(D_1))$ and $f_2 \in \Gamma(X, \mathcal{O}_X(D_2))$, we have $f_1f_2 \in \Gamma(X, \mathcal{O}_X(D_1 + D_2))$.

The following constructions are given for any normal irreducible variety X with only constant invertible global functions and finitely generated divisor class group $\mathrm{Cl}(X)$. Note that, for complete varieties, $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ is always satisfied.

Construction 1.2.2 (Cox ring)**.** Fix a subgroup $K \leq \mathrm{WDiv}(X)$ such that the canonical projection $c\colon K \to \mathrm{Cl}(X)$, $D \mapsto [D]$, is surjective and set $K^0 := \ker(c)$. Choose a character $\chi\colon K^0 \to \mathbb{K}(X)^*$ such that $\mathrm{div}(\chi(E)) = E$ holds for any $E \in K^0$. Let $\mathcal{S}$ be the sheaf of divisorial algebras associated to K as in Definition 1.2.1. Consider the sheaf of ideals $\mathcal{I}$ locally generated by the sections $1 - \chi(E)$, where E runs through all elements of K^0.
The *Cox sheaf* of X associated to K and χ is the quotient sheaf $\mathcal{R} := \mathcal{S}/\mathcal{I}$ together with the $\mathrm{Cl}(X)$-grading

$$\mathcal{R} \ := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{R}_{[D]}, \qquad \mathcal{R}_{[D]} := \pi\left(\bigoplus_{D'\in c^{-1}([D])} \mathcal{S}_{D'}\right),$$

where $\pi\colon \mathcal{S} \to \mathcal{R}$ is the canonical projection. The algebra of global sections $\mathcal{R}(X)$ of the Cox sheaf $\mathcal{R}$ is the *Cox ring* of X:

$$\mathcal{R}(X) \ := \ \Gamma(X, \mathcal{R}) \ = \bigoplus_{[D]\in \mathrm{Cl}(X)} \Gamma(X, \mathcal{R}_{[D]}).$$

Note that, if $\mathrm{Cl}(X)$ is torsion-free, then the Cox sheaf can be defined by taking a group K that is isomorphic to $\mathrm{Cl}(X)$ and setting $\mathcal{R}_{[D]} := \mathcal{S}_D = \mathcal{O}_X(D)$.

One can show that Construction 1.2.2 does not depend on the choices of K and χ.

Definition 1.2.3. Let K be an abelian group and consider a K-graded integral $\mathbb{K}$-algebra $R = \bigoplus_{w\in K} R_w$.

a) An element $0 \neq f \in R \setminus R^*$ is called *K-prime* if it is homogeneous and whenever $f|gh$ for some homogeneous $g, h \in R$, then $f|g$ or $f|h$.
b) R is called *factorially K-graded* if every homogeneous non-zero $f \in R \setminus R^*$ is a product of K-primes.

Theorem 1.2.4. *In the above setting, the Cox ring $\mathcal{R}(X)$ is factorially $\mathrm{Cl}(X)$-graded. Moreover if $\mathrm{Cl}(X)$ is torsion-free, then $\mathcal{R}(X)$ is factorial.*

In general, the Cox ring $\mathcal{R}(X)$ of a variety X does not need to be finitely generated. This motivates the next definition.

Definition 1.2.5. Let X be a normal irreducible variety X with only constant invertible global functions and finitely generated divisor class group $\mathrm{Cl}(X)$. If its Cox ring $\mathcal{R}(X)$ is finitely generated, then X is called a *Mori dream space*.

Now we turn our attention to the geometric aspects of Construction 1.2.2.

Remark 1.2.6. Assume that X is a Mori dream space. Then $\mathcal{R}$ is locally of finite type and $\mathcal{R}(X)$ is a factorially $\mathrm{Cl}(X)$-graded finitely generated affine $\mathbb{K}$-algebra. The quasitorus $H := \mathrm{Spec}(\mathbb{K}[\mathrm{Cl}(X)])$ defines an action on the affine variety $\overline{X} := \mathrm{Spec}(\mathcal{R}(X))$. We call $\overline{X}$ the *total coordinate space* of X. Moreover, consider $\widehat{X} := \mathrm{Spec}_X(\mathcal{R})$, the relative spectrum of the Cox sheaf $\mathcal{R}$. Note that $\widehat{X}$ is an irreducible normal variety. The canonical morphism $\widehat{X} \to \overline{X}$ is an H-invariant open embedding and the complement $\overline{X} \backslash \widehat{X}$ is of codimension at least two. The morphism $p\colon \widehat{X} \to X$ defined by the H-action is a good quotient and is called the *characteristic space* of X.

1.3 Bunched rings and Mori dream spaces

This Section recalls the theory of bunched rings, originally developed in [10, 23]. These objects answer the problem of constructing a variety with prescribed Cox ring. It turns out that non-isomorphic varieties may share the same Cox ring, provided they are isomorphic in codimension one. Bunched rings encode a Cox ring and a choice of isomorphy type.

Construction 1.3.1. Let K be a finitely generated abelian group and R a factorially K-graded affine $\mathbb{K}$-algebra. Consider a system $\mathfrak{F} = (f_1, \ldots, f_r)$ of pairwise non-associated K-prime generators of R. Define a homomorphism $Q\colon \mathbb{Z}^r \to K$ of abelian groups, which maps the canonical basis vector $e_i \in E := \mathbb{Z}^r$ to the degree $w_i := \deg(f_i) \in K$. This grading defines a quasitorus action of $H := \mathrm{Spec}(\mathbb{K}[K])$ on $\overline{X} := \mathrm{Spec}(R)$. Furthermore, there is an H-invariant closed embedding

$$\overline{X} \to \mathbb{K}^r, \qquad x \mapsto (f_1(x), \ldots, f_r(x)).$$

Definition 1.3.2. In the situation of Construction 1.3.1 we define the following:

(i) An $\mathfrak{F}$*-face* is a face $\gamma_0 \preceq \gamma$ of the positive orthant $\gamma := \mathbb{Q}^r_{\geq 0}$, such that there is some point $x \in \overline{X}$ for which $x_i \neq 0$ holds if and only if $e_i \in \gamma_0$.

(ii) A *projected* $\mathfrak{F}$*-face* is an element of $\Omega_{\mathfrak{F}} := \{Q(\gamma_0);\ \gamma_0 \text{ is an } \mathfrak{F}\text{-face}\}$.

(iii) An $\mathfrak{F}$*-bunch* is a non-empty collection $\Phi \subseteq \Omega_{\mathfrak{F}}$ that satisfies the following conditions:
- for any two $\tau_1, \tau_2 \in \Phi$ we have $\tau_1^\circ \cap \tau_2^\circ \neq \emptyset$.
- if $\tau_1 \in \Phi$ and $\tau_2 \in \Omega_{\mathfrak{F}}$ satisfy $\tau_1^\circ \subseteq \tau_2^\circ$, then $\tau_2 \in \Phi$ holds.

(iv) An $\mathfrak{F}$-bunch Φ is called *true* if the image of every facet of γ belongs to Φ.

(v) The K-grading of R is called *almost free* if any $r-1$ of the weights $w_1, \ldots, w_r$ generate K as an abelian group.

(vi) A *bunched ring* is a triple $(R, \mathfrak{F}, \Phi)$, where
- R is an almost freely, factorially K-graded affine $\mathbb{K}$-algebra, with $\mathbb{K}^*$ as its group of homogeneous units;
- $\mathfrak{F}$ is a system of pairwise non-associated K-prime generators of R;
- Φ is a true $\mathfrak{F}$-bunch.

Starting with a bunched ring $(R, \mathfrak{F}, \Phi)$, we construct a variety having R as its Cox ring. The $\mathfrak{F}$-bunch fixes the isomorphy type of the variety and encodes further algebraic aspects through following sets.

Definition 1.3.3. In the situation of Construction 1.3.1, let $(R, \mathfrak{F}, \Phi)$ be a bunched ring. We define the set of *relevant faces* and the *covering collection* as

$$\begin{aligned} \mathrm{rlv}(\Phi) &:= \{\gamma_0 \preceq \gamma;\ \gamma_0 \text{ is an } \mathfrak{F}\text{-face with } Q(\gamma_0) \in \Phi\}, \\ \mathrm{cov}(\Phi) &:= \{\gamma_0 \in \mathrm{rlv}(\Phi);\ \gamma_0 \text{ minimal}\}. \end{aligned}$$

Construction 1.3.4. In the situation of Construction 1.3.1, let $(R, \mathfrak{F}, \Phi)$ be a bunched ring. To any $\mathfrak{F}$-face γ_0 we associate

$$\overline{X}_{\gamma_0} := \overline{X}_{f_1^{u_1} \cdots f_r^{u_r}} \qquad \text{for some } u = (u_1, \ldots, u_r) \in \gamma_0^\circ.$$

One can easily see that $\overline{X}_{\gamma_0}$ is independent from the choice of u. Then we define the varieties

$$\begin{aligned} \widehat{X} &:= \widehat{X}(R, \mathfrak{F}, \Phi) := \bigcup_{\gamma_0 \in \mathrm{rlv}(\Phi)} \overline{X}_{\gamma_0}, \\ X &:= X(R, \mathfrak{F}, \Phi) := \widehat{X} /\!\!/ H. \end{aligned}$$

Note that $\widehat{X}$ is an H-invariant subset of $\overline{X}$. The H-action on $\widehat{X}$ admits a good quotient by [1, Prop. 3.1.3.8], thus X is well-defined. We denote the quotient map by $p\colon \widehat{X} \to X$. The pieces

$$X_{\gamma_0} := p(\overline{X}_{\gamma_0}) \subseteq X$$

form an affine cover of X. Moreover, every element f_i of $\mathfrak{F}$ defines a prime divisor $D_X^i := p(V_{\widehat{X}}(f_i))$ on X.

Recall that an A_2*-variety* is a variety X with the property that any two points of X admit a common affine open neighborhood.

Theorem 1.3.5. *Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring and $\overline{X}$, $\widehat{X}$, X be as above. Then X is a normal, irreducible A_2-variety with*

$$\dim(X) = \dim(\overline{X}) - \dim(K_{\mathbb{Q}}), \qquad \mathrm{Cl}(X) \cong K,$$

$$\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*, \qquad \mathcal{R}(X) \cong R.$$

In particular, X is a Mori dream space. Moreover, the map $p\colon \widehat{X} \to X$ is a characteristic space.

Remark 1.3.6. In Construction 1.3.1, any vector $u \in \mathrm{cone}(w_1, \ldots, w_r)^\circ \subseteq K_{\mathbb{Q}}$ defines an $\mathfrak{F}$-bunch through

$$\Phi(u) := \{\tau \in \Omega_{\mathfrak{F}};\ u \in \tau^\circ\}.$$

Motivated by the following Proposition, we call a bunched ring *projective* if its bunch is of the form $\Phi(u)$.

Proposition 1.3.7. *Let X be a Mori dream space. If X is projective, then X is isomorphic to a variety $X(R, \mathfrak{F}, \Phi(u))$ for some projective bunched ring $(R, \mathfrak{F}, \Phi(u))$.*

Every variety X defined by a bunched ring comes with a closed embedding into a toric variety Z, uniquely determined by the following Construction.

Construction 1.3.8. In the situation of Construction 1.3.1, let $(R, \mathfrak{F}, \Phi)$ be a bunched ring. By setting $F := E^*$ and $M := \ker(Q)$, we obtain the following exact sequences

$$0 \longrightarrow L \longrightarrow F \xrightarrow{\ P\ } N$$

$$0 \longleftarrow K \xleftarrow{\ Q\ } E \xleftarrow{\ P^*\ } M \longleftarrow 0$$

Set $\delta := \gamma^\vee \subset F_\mathbb{Q}$. For each $\gamma_0 \preceq \gamma$, let $\gamma_0^* := \gamma_0^\perp \cap \delta \preceq \delta$ be the corresponding face. We define the *enveloping collection* and the following fans living in $F_\mathbb{Q}$ and $N_\mathbb{Q}$ respectively:

$$\begin{aligned}
\operatorname{Env}(\Phi) &:= \{\gamma_0 \preceq \gamma;\ \exists\, \gamma_1 \in \operatorname{rlv}(\Phi) \text{ with } \gamma_1 \preceq \gamma_0 \text{ and } Q(\gamma_1)^\circ \subseteq Q(\gamma_0)^\circ\},\\
\widehat{\Sigma} &:= \{\delta_0 \preceq \delta;\ \exists\, \gamma_0 \in \operatorname{Env}(\Phi) \text{ with } \delta_0 \preceq \gamma_0^*\},\\
\Sigma &:= \{P(\gamma_0^*);\ \gamma_0 \in \operatorname{Env}(\Phi)\}.
\end{aligned}$$

Let $\overline{\Sigma}$ be the fan consisting of δ and all its faces and denote by $\overline{Z} := \mathbb{K}^r$ the toric variety associated to $\overline{\Sigma}$. Since $\widehat{\Sigma}$ is a subfan of $\overline{\Sigma}$, there is an open embedding of the corresponding varieties $\widehat{Z} \subseteq \overline{Z}$. Moreover, there is a map of fans $\widehat{\Sigma} \to \Sigma$ arising from $P\colon F \to N$. Denoting by Z the toric variety associated to Σ, we obtain a toric morphism $p\colon \widehat{Z} \to Z$. The varieties $\overline{X}$, $\widehat{X}$ and X from Construction 1.3.4 fit nicely in the following commutative diagram:

$$\begin{array}{ccc}
\overline{X} & \xrightarrow{\ \bar{\imath}\ } & \overline{Z}\\
\big\uparrow & & \big\uparrow\\
\widehat{X} & \xrightarrow{\ \hat{\imath}\ } & \widehat{Z}\\
{\scriptstyle /\!\!/ H}\big\downarrow & & \big\downarrow{\scriptstyle /\!\!/ H}\\
X & \xrightarrow{\ \imath\ } & Z
\end{array}$$

where $\bar{\imath}$, $\hat{\imath}$, $\imath$ are closed embeddings.

Definition 1.3.9. In the setting of Construction 1.3.8, we call Z the *minimal toric ambient variety* of X.

In the last part of this Section, we introduce the class of Fano varieties and that of complete intersections, which have nice properties in the setting of bunched rings. For the definition of an ample divisor, see Section 1.4.

Definition 1.3.10. Let X be an irreducible normal projective variety. We call X a *Fano variety* if its anticanonical divisor class $-\mathcal{K}_X$ is ample.

Remark 1.3.11. In the setting of Construction 1.3.4 assume that $X = X(R, \mathfrak{F}, \Phi)$ is a Fano variety. Then $-\mathcal{K}_X \in \mathrm{Mov}(X)^\circ$ holds and the $\mathfrak{F}$-bunch is uniquely determined by $\Phi = \Phi(-\mathcal{K}_X)$.

Definition 1.3.12. Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring with $\mathfrak{F} = (f_1, \ldots, f_r)$. We say that it is a *complete intersection* if the kernel of the epimorphism $\mathbb{K}[T_1, \ldots, T_r] \to R$, mapping T_i to f_i, is generated by K-homogeneous polynomials $g_1, \ldots, g_s$ where $s = r - \dim(R)$. In this case we define the *degree vectors* of $(R, \mathfrak{F}, \Phi)$ as $(w_1, \ldots, w_r)$ and $(u_1, \ldots, u_s)$, where $w_i := \deg(f_i) \in K$ and $u_j := \deg(g_j) \in K$.

Proposition 1.3.13. *Let the bunched ring $(R, \mathfrak{F}, \Phi)$ be a complete intersection with degree vectors $(w_1, \ldots, w_r)$ and $(u_1, \ldots, u_s)$. Then the anticanonical divisor class of $X = X(R, \mathfrak{F}, \Phi)$ is given in $\mathrm{Cl}(X) = K$ as*

$$-\mathcal{K}_X \;=\; \sum_{i=1}^{r} w_i - \sum_{j=1}^{s} u_j \;.$$

1.4 Geometry of Mori dream spaces

Many geometrical properties of Mori dream spaces are naturally encoded in the combinatorics of their defining bunched rings. Here we present some of these descriptions.

Construction 1.4.1. Consider the situation of Construction 1.3.4. To any $\mathfrak{F}$-face $\gamma_0 \preceq \gamma$ we associate the locally closed subset

$$\overline{X}(\gamma_0) \;:=\; \{z \in \overline{X};\; f_i(z) \neq 0 \iff e_i \in \gamma_0 \text{ for each } 1 \leq i \leq r\} \;\subseteq\; \overline{X}.$$

These sets give a disjoint covering of $\overline{X}$. By taking only the subsets defined by relevant faces and considering their images under $p\colon \widehat{X} \to X$ we obtain a disjoint covering of X in locally closed subsets

$$X \;=\; \bigcup_{\gamma_0 \in \mathrm{rlv}(\Phi)} X(\gamma_0), \qquad X(\gamma_0) \;:=\; p\left(\overline{X}(\gamma_0)\right).$$

We call any set $\overline{X}(\gamma_0)$ a *stratum* of $\overline{X}$ and any set $X(\gamma_0)$ a *stratum* of X.

Let X be a normal irreducible variety. A point $x \in X$ is called $\mathbb{Q}$-*factorial* if, near x, every Weil divisor has a non-zero multiple that is principal. A point $x \in X$ is called *factorial* if, near x, every Weil divisor is principal. The variety X is called *($\mathbb{Q}$-)factorial* if all of its points are ($\mathbb{Q}$-)factorial.

Proposition 1.4.2. *Let $X = X(R, \mathfrak{F}, \Phi)$ arise from Construction 1.3.4. Consider $\gamma_0 \in \mathrm{rlv}(X)$ and $x \in X(\gamma_0)$. Then the following statements hold:*

(i) x is $\mathbb{Q}$-factorial if and only if $Q(\gamma_0)$ is full-dimensional;

(ii) x is factorial if and only if Q maps $\mathrm{lin}(\gamma_0) \cap \mathbb{Z}^{n+m}$ onto $\mathrm{Cl}(X)$;

(iii) x is smooth if and only if x is factorial and all $z \in p^{-1}(x)$ are smooth in $\widehat{X}$.

Let X be a normal irreducible complete variety. Classical geometry associates to any such variety some cones in $\mathrm{Cl}(X)_{\mathbb{Q}}$. These are particularly easy to describe in the setting of bunched rings.

Recall that the *effective cone* is the cone $\mathrm{Eff}(X)$ generated by the classes of effective divisors on X. The *stable base locus* of a divisor $D \in \mathrm{WDiv}(X)$ is given by

$$\mathrm{sB}(D) \;:=\; \bigcap_{n\in\mathbb{Z}_{\geq 1}} \; \bigcap_{f\in\Gamma(X,\mathcal{O}(nD))} \mathrm{supp}(\mathrm{div}(f)+nD),$$

where the *support* of a Weil divisor $E=\sum_i a_iE_i$ is the set-theoretical union of the prime divisors E_i with non-zero coefficient a_i. A Weil divisor $D \in \mathrm{WDiv}(X)$ is called *movable* if its stable base locus $\mathrm{sB}(D)$ has codimension at least 2 in X and *semiample* if $\mathrm{sB}(D)$ is empty. The *moving cone* is the cone $\mathrm{Mov}(X)$ generated by the classes of movable divisors on X, whereas the *semiample cone* $\mathrm{SAmple}(X)$ by the classes of semiample divisors. Lastly we call a divisor $D \in \mathrm{WDiv}(X)$ *ample*, if X is covered by affine sets of the form

$$X_{nD,f} \;:=\; X \setminus \mathrm{supp}(\mathrm{div}(f)+nD)$$

for some $n \in \mathbb{Z}_{\geq 1}$. The classes of ample divisors generate the *ample cone* $\mathrm{Ample}(X)$.

Proposition 1.4.3. *Let $X = X(R,\mathfrak{F},\Phi)$ arise from Construction 1.3.4. Then the cones of effective, movable, semiample and ample divisor classes are given in $\mathrm{Cl}(X)_{\mathbb{Q}} = K_{\mathbb{Q}}$ as*

$$\mathrm{Eff}(X) \;=\; Q(\gamma), \qquad \mathrm{Mov}(X) \;=\; \bigcap_{\gamma_0 \text{ facet of } \gamma} Q(\gamma_0),$$

$$\mathrm{SAmple}(X) \;=\; \bigcap_{\tau\in\Phi} \tau, \qquad \mathrm{Ample}(X) \;=\; \bigcap_{\tau\in\Phi} \tau^{\circ}.$$

In particular we have $\mathrm{Ample}(X) \subseteq \mathrm{SAmple}(X) \subseteq \mathrm{Mov}(X) \subseteq \mathrm{Eff}(X)$.

Now we want to investigate the dimension of the strata. A similar idea appears in the diploma thesis of B. Bechtold [6]. First we introduce a notion of height among the $\mathfrak{F}$-faces and give an alternative description of the strata. This will lead to a combinatorial computation of their dimension.

Definition 1.4.4. Let Σ be a quasi-fan and $\widetilde{\Sigma} \subseteq \Sigma$. The $\widetilde{\Sigma}$-*height* of a cone $\sigma \in \widetilde{\Sigma}$ is

$$\mathrm{ht}_{\widetilde{\Sigma}}(\sigma) := \max\{k \in \mathbb{Z}_{\geq 0} : \exists \text{ chain } \sigma_0 \prec \ldots \prec \sigma_k = \sigma \text{ in } \widetilde{\Sigma}\}.$$

From now on we denote by $\mathfrak{F}(R)$ the set of $\mathfrak{F}$-faces of the variety $X = X(R,\mathfrak{F},\Phi)$.

Definition 1.4.5. In the setting of Construction 1.4.1, for any $\gamma_0 \in \mathrm{rlv}(\Phi)$ define

$$V_{\overline{X}}(\gamma_0) := \bigcap_{e_i \notin \gamma_0} V_{\overline{X}}(f_i) \qquad \text{and} \qquad V_X(\gamma_0) := \bigcap_{e_i\notin\gamma_0} D_X^i.$$

Remark 1.4.6. With the definitions above one has, for any fixed $\mathfrak{F}$-face γ_0,

$$\overline{X}_{\gamma_0} = \overline{X} \setminus \bigcup_{e_i \in \gamma_0} V_{\overline{X}}(f_i) \qquad \text{and} \qquad \overline{X}(\gamma_0) = V_{\overline{X}}(\gamma_0) \setminus \bigcup_{e_i \in \gamma_0} V_{\overline{X}}(f_i)\,.$$

$$X_{\gamma_0} = X \setminus \bigcup_{e_i \in \gamma_0} D_X^i \qquad \text{and} \qquad X(\gamma_0) = V_X(\gamma_0) \setminus \bigcup_{e_i \in \gamma_0} D_X^i\,.$$

In particular $\overline{X}(\gamma_0)$ and $V_{\overline{X}}(\gamma_0)$ share the same dimension.

Lemma 1.4.7. *For any $\gamma_0 \in \mathrm{rlv}(\Phi)$ we have*

$$\dim X(\gamma_0) = \dim \overline{X}(\gamma_0) - \dim Q(\gamma_0).$$

Proof. First note that the restriction $p\colon \overline{X}(\gamma_0) \to X(\gamma_0)$ is again a good quotient, in particular surjective. Let $A \subseteq \overline{X}(\gamma_0)$ be an irreducible component and define $Y := \overline{p(A)}$. Then $p\colon A \to Y$ is a dominant morphism of irreducible varieties. Every fiber is a single closed orbit, whose dimension equals $\dim Q(\gamma_0)$. Therefore we have $\dim(A) - \dim(Y) = \dim Q(\gamma_0)$. If we choose A of maximal dimension, then so is Y, and we obtain

$$\dim X(\gamma_0) = \dim(Y) = \dim(A) - \dim Q(\gamma_0) = \dim \overline{X}(\gamma_0) - \dim Q(\gamma_0).$$

□

Assumption 1.4.8. *For the variety $X = X(R, \mathfrak{F}, \Phi)$ the two following statements hold:*
- *$\{0\}$ is an $\mathfrak{F}$-face;*
- *for all $\gamma_0, \gamma_1 \in \mathfrak{F}(R)$, such that $\gamma_1 \prec \gamma_0$ and this inclusion is maximal among $\mathfrak{F}$-faces, it holds that $\dim V_{\overline{X}}(\gamma_1) = \dim V_{\overline{X}}(\gamma_0) - 1$.*

Proposition 1.4.9. *If Assumption 1.4.8 holds, then for any $\gamma_0 \in \mathrm{rlv}(\Phi)$ we have*

$$\dim \overline{X}(\gamma_0) = \mathrm{ht}_{\mathfrak{F}(R)}(\gamma_0).$$

Proof. By the description of strata we have

$$V_{\overline{X}}(\gamma_0) \cap \bigcup_{e_i \in \gamma_0} V_{\overline{X}}(f_i) \;=\; \bigcup_{\substack{\gamma_0 \succ \gamma_1 \in \mathfrak{F}(R) \\ \text{maximal}}} V_{\overline{X}}(\gamma_1).$$

Hence for any $\{0\} \neq \gamma_0 \in \mathfrak{F}(R)$ we have

$$\dim\Big(V_{\overline{X}}(\gamma_0) \cap V_{\overline{X}}\Big(\prod_{e_i \in \gamma_0} f_i \Big)\Big) \;=\; \dim V_{\overline{X}}(\gamma_0) - 1. \tag{1.1}$$

Now we want to show that $\dim V_{\overline{X}}(\gamma_0) = \mathrm{ht}_{\mathfrak{F}(R)}(\gamma_0)$ holds for any $\mathfrak{F}$-face $\gamma_0 \preceq \gamma$. We proceed by induction over $k := \mathrm{ht}_{\mathfrak{F}(R)}(\gamma_0)$.
For the base case consider $k = 0$. Then $\gamma_0 = \{0\}$ holds. The set $V_{\overline{X}}(0)$ is just a point, hence 0-dimensional.

For the inductive step, consider a chain $\{0\} = \gamma_{(0)} \prec \ldots \prec \gamma_{(k)} = \gamma_0$ of maximal length. In particular we have $\mathrm{ht}_{\mathfrak{F}(R)}(\gamma_{(k-1)}) = k-1$. By assumption and using equation (1.1), we obtain

$$\begin{aligned} k-1 &= \max_{\substack{\gamma_0 \succ \gamma_1 \in \mathfrak{F}(R) \\ \text{maximal}}} \mathrm{ht}_{\mathfrak{F}(R)}(\gamma_1) = \max_{\substack{\gamma_0 \succ \gamma_1 \in \mathfrak{F}(R) \\ \text{maximal}}} \dim V_{\overline{X}}(\gamma_1) = \\ &= \dim\Big(V_{\overline{X}}(\gamma_0) \cap \bigcup_{e_i \in \gamma_0} V_{\overline{X}}(f_i)\Big) = \dim V_{\overline{X}}(\gamma_0) - 1 \end{aligned}$$

and we arrive at $\dim V_{\overline{X}}(\gamma_0) = k = \mathrm{ht}_{\mathfrak{F}(R)}(\gamma_0)$. Remark 1.4.6 yields $\dim \overline{X}(\gamma_0) = k$. □

Corollary 1.4.10. *If Assumption 1.4.8 holds, then for any* $\gamma_0 \in \mathrm{rlv}(\Phi)$ *we have*

$$\dim X(\gamma_0) = \mathrm{ht}_{\mathfrak{F}(R)}(\gamma_0) - \dim Q(\gamma_0).$$

Proof. This is a direct consequence of Lemma 1.4.7 and Proposition 1.4.9. □

Corollary 1.4.11. *Let* $X = X(R, \mathfrak{F}, \Phi)$ *be* $\mathbb{Q}$*-factorial such that Assumption 1.4.8 holds. Then for any* $\gamma_0 \in \mathrm{rlv}(\Phi)$ *we have*

$$\dim X(\gamma_0) = \mathrm{ht}_{\mathrm{rlv}(\Phi)}(\gamma_0).$$

In particular the strata of dimension 0 *are precisely those defined by elements of the covering collection.*

Proof. By $\mathbb{Q}$-factoriality one has

$$\mathrm{ht}_{\mathfrak{F}(R)}(\gamma_0) = \mathrm{rk}\,(\mathrm{Cl}(X)) = \dim Q(\gamma_0) \qquad \forall\, \gamma_0 \in \mathrm{cov}(\Phi).$$

Therefore the height can be counted on the relevant faces from the covering collection upwards, since any $\mathfrak{F}$-face containing a covering one is relevant, i.e.

$$\mathrm{ht}_{\mathfrak{F}(R)}(\gamma_0) = \mathrm{ht}_{\mathrm{rlv}(\Phi)}(\gamma_0) + \mathrm{rk}\,(\mathrm{Cl}(X)).$$

By Proposition 1.4.2 any relevant face projects via Q to a full-dimensional cone and with Corollary 1.4.10 we obtain

$$\dim X(\gamma_0) = \mathrm{ht}_{\mathrm{rlv}(\Phi)}(\gamma_0).$$

□

1.5 T-varieties of complexity one

We take a closer look at rational projective varieties with an effective torus action of complexity one. This means that the general torus orbit has codimension one. First we recall the approach provided by [26, 24]. The Cox rings of these varieties are precisely the rings obtained in the following way.

Construction 1.5.1. Fix $r \in \mathbb{Z}_{\geq 1}$, a sequence $n_0, \ldots, n_r \in \mathbb{Z}_{\geq 1}$, set $n := n_0 + \ldots + n_r$, and fix integers $m \in \mathbb{Z}_{\geq 0}$ and $0 < s < n + m - r$. The *defining matrices* (A, P) are

- a $2 \times (r+1)$-matrix $A := [a_0, \ldots, a_r]$ with pairwise linearly independent column vectors $a_0, \ldots, a_r \in \mathbb{K}^2$,
- an integral matrix P of size $(r+s) \times (n+m)$, the columns of which are pairwise different primitive vectors generating $\mathbb{Q}^{r+s}$ as a cone. The matrix P is divided into blocks

$$P \quad = \quad \begin{bmatrix} L & 0 \\ d & d' \end{bmatrix},$$

where d is an $(s \times n)$-matrix, d' an $(s \times m)$-matrix and L an $(r \times n)$-matrix built from tuples $l_i := (l_{i1}, \ldots, l_{in_i}) \in \mathbb{Z}^{n_i}_{\geq 1}$ as follows

$$L \quad = \quad \begin{bmatrix} -l_0 & l_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \ldots & l_r \end{bmatrix}.$$

Consider the polynomial ring $\mathbb{K}[T_{ij}, S_k]$ in the variables T_{ij} and S_k, where $0 \leq i \leq r$, $1 \leq j \leq n_i$, $1 \leq k \leq m$. For every $0 \leq i \leq r$, define a monomial

$$T_i^{l_i} \quad := \quad T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}.$$

Denote by $\mathfrak{I}$ the set of all triples $I = (i_1, i_2, i_3)$ with $0 \leq i_1 < i_2 < i_3 \leq r$ and define for any $I \in \mathfrak{I}$ a trinomial

$$g_I \quad := \quad g_{i_1, i_2, i_3} \quad := \quad \alpha_{i_2 i_3} T_{i_1}^{l_{i_1}} + \alpha_{i_3 i_1} T_{i_2}^{l_{i_2}} + \alpha_{i_1 i_2} T_{i_3}^{l_{i_3}},$$

where we set $\alpha_{ij} := \det(a_i, a_j)$. Let P^* be the transpose of P. Consider the factor group $K := \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$, and the projection $Q\colon \mathbb{Z}^{n+m} \to K$. We define a K-grading on $\mathbb{K}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) \ := \ Q(e_{ij}), \qquad \deg(S_k) \ := \ Q(e_k).$$

Then the trinomials g_I are K-homogeneous, all of the same degree $\mu \in K$. In particular, we obtain a K-graded factor ring

$$R(A, P) \quad := \quad \mathbb{K}[T_{ij}, S_k;\ 0 \leq i \leq r,\ 1 \leq j \leq n_i, 1 \leq k \leq m] \ / \ \langle g_I;\ I \in \mathfrak{I} \rangle.$$

The rings $R(A, P)$ are (precisely) those which occur as Cox rings of normal rational projective (or, more generally, complete A_2-) varieties with a torus action of complexity one; see [24, Theorem 1.5]. We recall basic properties and retrieve these varieties by means of bunched rings.

Remark 1.5.2. The K-graded ring $R(A, P)$ of Construction 1.5.1 is a complete intersection: with $g_i := g_{i,i+1,i+2}$ we have

$$\langle g_I;\ I \in \mathfrak{I} \rangle \ = \ \langle g_0, \ldots, g_{r-2} \rangle, \qquad \dim(R(A, P)) \ = \ n + m - (r - 1).$$

We can always assume that P is *irredundant* in the sense that $l_{i1} + \ldots + l_{in_i} \geq 2$ holds for $i = 0, \ldots, r$; note that a redundant P allows the elimination of variables in $R(A, P)$.

Remark 1.5.3. The *anticanonical class* of the K-graded ring $R(A,P)$ from Construction 1.5.1 is

$$\kappa(A,P) \ := \ \sum_{i,j} Q(e_{ij}) \ + \ \sum_{k} Q(e_k) \ - \ (r-1)\mu \ \in \ K$$

and the *moving cone* of $R(A,P)$ in $K_{\mathbb{Q}}$ is

$$\mathrm{Mov}(A,P) \ := \ \bigcap_{i,j} \mathrm{cone}\big(Q(e_{uv}, e_t;\ (u,v) \neq (i,j)\big) \ \cap \ \bigcap_{k} \mathrm{cone}\big(Q(e_{uv}, e_t;\ t \neq k)\big).$$

The K-graded ring $R(A,P)$ is the Cox ring of a Fano variety if and only if $\kappa(A,P)$ belongs to the relative interior of $\mathrm{Mov}(A,P)$.

Construction 1.5.4. Let (A,P) be defining matrices as in Construction 1.5.1 and consider the K-graded ring $R := R(A,P)$. The variables T_{ij} and S_k define a system $\mathfrak{F}$ of pairwise non-associated K-prime generators for R. For every true $\mathfrak{F}$-bunch Φ we obtain a bunched ring $(R,\mathfrak{F},\Phi)$. With Construction 1.3.4 we have the varieties

$$\widehat{X} := \widehat{X}(A,P,\Phi) := \widehat{X}(R,\mathfrak{F},\Phi), \qquad X := X(A,P,\Phi) := X(R,\mathfrak{F},\Phi).$$

Remark 1.5.5. The following elementary operations on the columns and rows of the defining matrix P do not change the isomorphy type of the associated variety $X(A,P,\Phi)$. We call them *admissible operations*:

(i) swap two columns inside a block $v_{ij_1},\ldots,v_{ij_{n_i}}$;
(ii) swap two whole column blocks $v_{ij_1},\ldots,v_{ij_{n_i}}$ and $v_{i'j_1},\ldots,v_{i'j_{n_{i'}}}$;
(iii) add multiples of the upper r rows to one of the last s rows;
(iv) any elementary row operation among the last s rows;
(v) swap two columns inside the d' block.

The operations of type (iii) and (iv) do not change the associated ring $R(A,P)$, whereas the types (i), (ii), (v) correspond to certain renumberings of the variables of $R(A,P)$ and do not affect the graded isomorphy type.

Definition 1.5.6. A T-variety X is called of *complexity one* if T is a torus and $\dim(T) = \dim(X) - 1$ holds.

Theorem 1.5.7. *Let $X = X(A,P,\Phi)$ be as in Construction 1.5.4. Then X is an irreducible normal A_2-variety of complexity one with*

$$\dim(X) = s+1, \qquad \mathrm{Cl}(X) \cong K, \qquad \Gamma(X,\mathcal{O}) = \mathbb{K}$$
$$-\mathcal{K}_X = \kappa(A,P), \qquad \mathrm{Mov}(X) = \mathrm{Mov}(A,P), \qquad \mathcal{R}(X) = R(A,P).$$

Moreover, each irreducible normal A_2-variety X with $\Gamma(X,\mathcal{O}) = \mathbb{K}$ (e.g. projective) and a torus action of complexity one arises from Construction 1.5.4.

Remark 1.5.8. In the setting of Construction 1.5.4, assume that $X = X(A, P, \Phi)$ is projective. Proposition 1.3.7 tells us that $\Phi = \Phi(u)$ holds for some $u \in \mathrm{Eff}(X)$. In this case we will often write $X = X(A, P, u)$.
Moreover, if X is a Fano variety, then Remark 1.3.11 states that the $\mathfrak{F}$-bunch is uniquely determined and can be written as $\Phi = \Phi(-\mathcal{K}_X)$. In this case the notation $X = X(A, P)$ suffices.

Example 1.5.9 (E_6-singular cubic surface)**.** Consider $X := V(z_1z_2^2 + z_2z_0^2 + z_3^3) \subseteq \mathbb{P}_3$, which is invariant under the $\mathbb{K}^*$-action

$$t \cdot [z_0, \ldots, z_3] \;=\; [z_0, t^{-3}z_1, t^3z_2, tz_3]$$

on $\mathbb{P}_3$. The divisor class group and the Cox ring of X are given by

$$\mathrm{Cl}(X) \;=\; \mathbb{Z}, \qquad\qquad \mathcal{R}(X) \;=\; \mathbb{K}[T_1, T_2, T_3, T_4]/\langle T_1T_2^3 + T_3^3 + T_4^2\rangle,$$

where the $\mathrm{Cl}(X)$-degrees of T_1, T_2, T_3, T_4 are 3, 1, 2, 3. For the explicit computation see [1, Example 4.4.1.8]. By defining the matrices

$$A \;=\; \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \qquad P \;=\; \begin{bmatrix} -1 & -3 & 3 & 0 \\ -1 & -3 & 0 & 2 \\ -1 & -2 & 1 & 1 \end{bmatrix},$$

we retrieve the variety X as $X \cong X(A, P)$.

Now we discuss a possible resolution of singularities for varieties of complexity one. The references for complete proofs are [1, Sec. 3.4.4] and [27]. Recall that a *resolution (of singularities)* of a normal projective variety X is a proper morphism $\varphi\colon X' \to X$ with X' smooth and projective such that the restriction $\varphi^{-1}(U) \to U$ with $U := X \setminus X^{\mathrm{sing}}$ is an isomorphism.

Construction 1.5.10. Consider a variety $X = X(A, P, \Phi)$ as in Construction 1.5.4 and let Z be its minimal ambient toric variety. By defining the matrices $P_0 := [L, 0]$ and $P_1 := [E_r, 0]$, where E_r is the $(r \times r)$ unit matrix, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^{n+m} & \xrightarrow{\;P\;} & \mathbb{Z}^{r+s} \\ & {\scriptstyle P_0}\searrow \quad \swarrow{\scriptstyle P_1} & \\ & \mathbb{Z}^r & \end{array}$$

Now, let $e_1, \ldots, e_r \in \mathbb{Z}^r$ be the canonical basis vectors, set $e_0 := -e_1 \ldots - e_r$ and

$$\varrho_i \;:=\; \mathrm{cone}(e_i), \qquad 0 \le i \le r,$$

and consider the fan $\Delta(r) := \{0, \varrho_0, \ldots, \varrho_r\}$ in $\mathbb{Q}^r$. Note that P_1 sends the ij-th column v_{ij} of P into the ray ϱ_i and all columns v_k to zero. Define $\lambda_i := P_1^{-1}(\varrho_i) \subset \mathbb{Q}_{r+s}$. With

$\lambda := \{0\} \times \mathbb{Q}^s \subset \mathbb{Q}_{r+s}$ we have $\lambda_i = \lambda + \mathrm{cone}(e_i)$. The *tropical variety* of $X \cap \mathbb{T}^n \subseteq Z$ is then given as

$$\mathrm{trop}(X) \;=\; \lambda_0 \cup \cdots \cup \lambda_r \;\subseteq\; \mathbb{Q}^{r+s}.$$

Define a new fan Σ' in $\mathbb{Q}^{r+s}$ as

$$\Sigma' \;:=\; \Sigma \sqcap \mathrm{trop}(X) \;=\; \{\sigma \cap P_1^{-1}(\varrho_i);\ \sigma \in \Sigma, 0 \leq i \leq r\}.$$

Then we have a map of fans $\Sigma' \to \Sigma$ and the associated birational toric morphism $Z' \to Z$ fits into a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

where $X' \subseteq Z'$ is the proper transform, i.e. the closure of $X \cap \mathbb{T}^{r+s}$ in Z'. The restriction $X' \to X$ is called *weak tropical resolution*. We call a variety X *weakly tropical* if $\Sigma' = \Sigma$. Any regular subdivision $\Sigma'' \to \Sigma'$ provides a toric resolution $Z'' \to Z'$ and induces a resolution $X'' \to X'$.

Theorem 1.5.11. *The varieties X' and X'' arising from Construction 1.5.10 are again normal rational varieties of complexity one. In particular their Cox rings are of the form $R(A, P')$ and $R(A, P'')$.*

In the last part of this Section. the goal is to prove that Assumption 1.4.8 holds for all varieties of complexity one. We follow an idea of B. Bechtold from [6].

Proposition 1.5.12. *Let X be a normal complete rational A_2-variety of complexity one. Then Assumption 1.4.8 holds.*

Remark 1.5.13. Consider a ring $R(A, P)$ as in Construction 1.5.1. Then we have

$$\overline{X} \;=\; V(g_I; I \in \mathfrak{I}) \;=\; V(g_0, \ldots, g_{r-2}) \;\subseteq\; \mathbb{K}^{n+m}.$$

A face $\gamma_0 \preceq \gamma$ is an $\mathfrak{F}$-face if and only if it fulfills one of the two following mutually exclusive properties:

(i) for every $i = 0, \ldots, r$ there is a j_i such that $e_{ij_i} \notin \gamma_0$ holds;

(ii) there is at most one $i = 0, \ldots, r$ such that $e_{ij} \notin \gamma_0$ holds for some j.

Lemma 1.5.14. *In the notation of Construction 1.5.1 and Remark 1.5.2, define $\overline{X} := V(g_0, \ldots, g_{r-2}) \subseteq \mathbb{K}^{n+m}$. Then $\overline{X} \cap V(T_{01}) = V(\widetilde{g}_0, \ldots, \widetilde{g}_{r-2}, T_{01})$ holds for binomials*

$$\widetilde{g}_i \;:=\; g_{0,i+1,i+2}|_{T_{01}=0} \;=\; \alpha_{i+2,0} T_{i+1}^{l_{i+1}} + \alpha_{0,i+1} T_{i+2}^{l_{i+2}}.$$

Proof. For $i = 0$ one has $\widetilde{g}_0 = g_0|_{T_{01}=0}$. Define recursively $\widetilde{g}_i := \alpha_{i+1,i+2}\widetilde{g}_{i-1} + \alpha_{0,i+1} g_i$. The equalities from [27, Lemma 2.3] ensure that we obtain the desired binomials and that the equality holds. □

Lemma 1.5.15. *In the notation of Construction 1.5.1 and Remark 1.5.2, let* $x \in \overline{X}$ *and define* $\overline{X} := V(g_0, \ldots, g_{r-2}) \subseteq \mathbb{K}^{n+m}$*. If* $T_i^{l_i}(x) = T_j^{l_j}(x) = 0$ *for some* $i \neq j$*, then* $T_k^{l_k}(x) = 0$ *holds for all* $0 \leq k \leq r$*.*

Proof. See [27, Lemma 2.5]. □

Proof of Proposition 1.5.12. By Theorem 1.5.7 there exist defining matrices (A, P) and a true bunch Φ such that $X = X(A, P, \Phi)$ holds. In particular the Cox ring $\mathcal{R}(X)$ is isomorphic to some $\mathbb{K}[T_1, \ldots, T_n]/\langle g_0, \ldots, g_{r-2}\rangle$ and $\overline{X} = V(g_0, \ldots, g_{r-2}) \subseteq \mathbb{K}^{n+m}$. For every $\mathfrak{F}$-face $\gamma_0 \preceq \gamma$,

$$\begin{aligned} V_{\overline{X}}(\gamma_0) &= V(g_0, \ldots, g_{r-2}, T_{ij}, S_k; e_{ij} e_{,k} \notin \gamma_0) \\ &= V(g_0^{\gamma_0}, \ldots, g_{r-2}^{\gamma_0}, T_{ij}, S_k; e_{ij} e_{,k} \notin \gamma_0) \end{aligned}$$

holds, where $g_t^{\gamma_0}$ is obtained from g_t by imposing $T_{ij} = 0$ whenever $e_{ij} \notin \gamma_0$ holds. Consider an $\mathfrak{F}$-face $\gamma_1 \prec \gamma_0$, such that the inclusion is maximal. Pick some $e_\beta \in \gamma_0 \setminus \gamma_1$. Note that β is an element of the index-set $\{ij, k\}$.
Case 1: $\beta = 1, \ldots, m$ or $\beta = ij$ for some i, j such that T_{ij} does not appear in any $g_t^{\gamma_0}$. Let γ_0^β be the facet of γ_0 that does not contain e_β. Clearly γ_0^β is again an $\mathfrak{F}$-face, since it fulfills the same property from Remark 1.5.13 as γ_0. By maximality we arrive at $\gamma_1 = \gamma_0^\beta$. Therefore the assertion follows from $V_{\overline{X}}(\gamma_0) \cong V_{\overline{X}}(\gamma_1) \times \mathbb{K}$.
Case 2: $\beta = ij$ for some i, j such that T_{ij} appears in some polynomial. In particular γ_0 is an $\mathfrak{F}$-face of type (ii) from Remark 1.5.13. Define $\gamma_2 := \gamma_0^\beta$.
Subcase 2.a: the only basis vectors missing from γ_0 are of type e_k. Then γ_2 is again an $\mathfrak{F}$-face of type (ii) and by maximality one has $\gamma_2 = \gamma_1$. With Lemma 1.5.14 we obtain $V_{\overline{X}}(\gamma_1)$ as complete intersection of $r - 1$ binomials. Therefore we arrive at the assertion through

$$\dim V_{\overline{X}}(\gamma_1) = \dim \gamma_1 - (r-1) = \dim \gamma_0 - 1 - (r-1) = \dim V_{\overline{X}}(\gamma_0) - 1.$$

Subcase 2.b: there is precisely one $i_1 = 0, \ldots, r$ such that $e_{i_1 j_1} \notin \gamma_0$ for some j_1. In particular $\beta = i_2 j_2$ holds for some $i_2 \neq i_1$. By Lemma 1.5.15, for any $t = 0, \ldots, r$ with $t \neq i_1, i_2$ there exists some $e_{tj_t} \in \gamma_1$. Let γ_3 be the maximal face of γ_2 that does not contain any of the e_{tj_t}. Note that γ_3 is an $\mathfrak{F}$-face of type (i). By maximality we achieve $\gamma_1 = \gamma_3$ and therefore

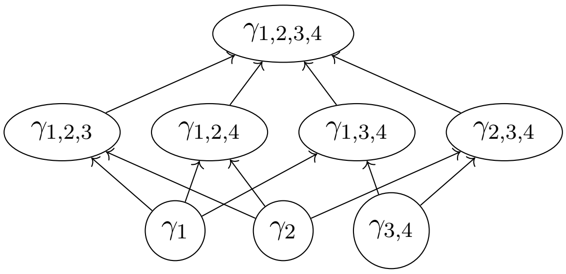
$$\begin{aligned} \dim V_{\overline{X}}(\gamma_1) &= \dim \gamma_1 = \dim \gamma_3 \\ &= \dim \gamma_2 - (r-1) = \dim \gamma_0 - 1 - (r-1) \\ &= \dim V_{\overline{X}}(\gamma_0) - 1. \end{aligned}$$

□

Corollary 1.5.16. *Let* X *be a normal complete rational* A_2*-variety of complexity one. If* X *is* $\mathbb{Q}$*-factorial, then for any* $\gamma_0 \in \mathrm{rlv}(\Phi)$ *we have*

$$\dim X(\gamma_0) = \mathrm{ht}_{\mathrm{rlv}(\Phi)}(\gamma_0).$$

Example 1.5.17. We present the directed graph of relevant faces for the recurring example, the E_6-singular cubic surface from Example 1.5.9.



The nodes are given by the relevant faces of X. There is an arrow from an edge γ_0 to an edge γ_1 if and only if $\gamma_0 \prec \gamma_1$ holds and this inclusion is maximal among the relevant faces. Using Corollary 1.5.16 we obtain

$$\begin{aligned}
\dim X(\gamma_{1,2,3,4}) &= 2,\\
\dim X(\gamma_{1,2,3}) = \dim X(\gamma_{1,2,4}) &= 1,\\
\dim X(\gamma_{1,3,4}) = \dim X(\gamma_{2,3,4}) &= 1,\\
\dim X(\gamma_1) = \dim X(\gamma_2) = \dim X(\gamma_{3,4}) &= 0.
\end{aligned}$$

CHAPTER

TWO

TERMINAL FANO THREEFOLDS OF COMPLEXITY ONE WITH $\rho(X) = 1$

In this chapter we provide the complete classification of $\mathbb{Q}$-factorial terminal Fano threefolds of complexity one having Picard number one. The results of this chapter have been published in [8].

With that goal in mind, Section 2.1 introduces the anticanonical complex, a certain combinatorial object that controls the discrepancies of a normal Fano variety X with a complete intersection Cox ring $\mathcal{R}(X)$. Section 2.2 investigates this object in the special situation of a rational Fano variety of complexity one and Section 2.3 provides some preliminary results for the terminal case. Section 2.4 is dedicated to the classification steps that deliver the $\mathbb{Q}$-factorial terminal Fano threefolds of complexity one having Picard number one. Their full list is given in Section 2.5.

2.1 The anticanonical complex

This Section deals with the development of the anticanonical complex. We follow the spirit of the Fano polytopes from toric geometry and extend the combinatorial approach to normal Fano varieties X with a complete intersection Cox ring $\mathcal{R}(X)$.

The aim is to characterize the behaviour of singularities of X in terms of lattice points of the anticanonical complex A_X^c. For a precise formulation, we introduce the concept of discrepancies following [41, Chapter 4].

Definition 2.1.1. Let X be a normal variety, whose canonical divisor $\mathcal{K}_X$ is $\mathbb{Q}$-Cartier, i.e. some positive multiple of $\mathcal{K}_X$ is a Cartier divisor. Let $\varphi\colon Y \to X$ be a resolution of singularities. Then we consider the *ramification formula*

$$\mathcal{K}_Y \;=\; \varphi^*\mathcal{K}_X \;+\; \sum a_i E_i,$$

where the E_i are the prime components of the exceptional divisor and the a_i are called the *discrepancies* of the resolution. The variety X is called

- *log terminal*, if $a_i > -1$ for all i;
- ε*-log terminal*, if $a_i > -1 + \varepsilon$ for all i, for a given $0 < \varepsilon < 1$;
- *canonical*, if $a_i \geq 0$ for all i;
- *terminal*, if $a_i > 0$ for all i.

The definition does not depend on the choice of resolution, see e.g. [28, Prop. 6.2.6].

We already introduced toric varieties in Section 1.1. If a toric variety is Fano, then there is a correspondence to a certain convex lattice polytope, called *Fano polytope*, that describes the singularity type of the variety, see e.g. [12].

Definition 2.1.2. Let $B \subset N_{\mathbb{Q}}$ be a full-dimensional convex polytope. Then B is called a *Fano polytope* if the origin lies in its interior B° and all vertices of B are primitive elements of N.

Construction 2.1.3. Consider a projective toric variety Z and let Σ be its fan in $N_{\mathbb{Q}}$. The *Fano polytope* of Z is

$$P_Z \ := \ \mathrm{conv}(v_\varrho;\ \varrho \in \Sigma^{(1)}) \ \subset \ N_{\mathbb{Q}},$$

i.e. the convex hull over the primitive generators of the rays of Σ. One sees that P_Z is indeed a Fano polytope in the sense of Definition 2.1.2.

Theorem 2.1.4. *Let Z be a projective toric variety, Σ its fan and P_Z its Fano polytope. Then we have*

$$Z \text{ is Fano} \qquad \Longleftrightarrow \qquad \Sigma = \mathrm{Fan}(P_Z).$$

If Z is indeed a Fano variety, then it is log terminal and we also have the following equivalences:

(i) Z has at most ε-log terminal singularities if and only if 0 *is the only lattice point of εP_Z.*

(ii) Z has at most canonical singularities if and only if 0 *is the only lattice point in the interior of P_Z.*

(iii) Z has at most terminal singularities if and only if 0 *and the primitive generators v_ϱ for $\varrho \in \Sigma^{(1)}$ are the only lattice points of P_Z.*

Inspired by this approach to the discrepancies, we define a combinatorial object that encodes the type of singularities for a larger family of varieties.

Construction 2.1.5. Let X be a normal Fano variety and assume that its Cox ring $\mathcal{R}(X)$ is a complete intersection, i.e.

$$\mathcal{R}(X) \ = \ \mathbb{K}[T_\varrho;\ \varrho \in \mathrm{R}]/\langle g_1, \ldots, g_s \rangle$$

holds with pairwise non-associated $\mathrm{Cl}(X)$-homogeneous prime generators T_ϱ for some finite index set R and relations g_i, such that the dimension of $\mathcal{R}(X)$ equals $|\mathrm{R}| - s$. By Construction 1.3.8 we have $X \subset Z \subset Z_c$, where Z_c is a complete toric variety and Z is the minimal open toric subvariety of Z_c containing X. The divisor class group and Cox ring of Z are given by

$$\mathrm{Cl}(Z) \cong \mathrm{Cl}(X), \qquad \mathcal{R}(Z) = \mathbb{K}[T_\varrho;\ \varrho \in \mathrm{R}].$$

Consider the degree homomorphism $Q\colon \mathbb{Z}^{\mathrm{R}} \to \mathrm{Cl}(X)$ sending the ϱ-th canonical basis vector $e_\varrho \in \mathbb{Z}^{\mathrm{R}}$ to $\deg(T_\varrho) \in \mathrm{Cl}(X)$ and let $P^*\colon \mathbb{Z}^n \to \mathbb{Z}^{\mathrm{R}}$ be a linear embedding with image $\ker(Q)$. Then we have

$$\mathrm{Cl}(Z) \cong \mathbb{Z}^{\mathrm{R}}/P^*(\mathbb{Z}^n) \cong \mathrm{Cl}(X).$$

Denote by $P\colon \mathbb{Z}^{\mathrm{R}} \to \mathbb{Z}^n$ the dual map of P^*. Set $e_Z := \sum e_\varrho$. Then the canonical classes of Z and X are given as

$$\mathcal{K}_Z = -Q(e_Z), \qquad \mathcal{K}_X = \sum_{i=1}^{s} \deg(g_i) + \mathcal{K}_Z.$$

Let $\gamma_{\mathrm{R}} \subseteq \mathbb{Q}^{\mathrm{R}}$ be the positive orthant, spanned by the e_ϱ. Define the polytope

$$B(-\mathcal{K}_X) := Q^{-1}(-\mathcal{K}_X) \cap \gamma_{\mathrm{R}} \subseteq \mathbb{Q}^{\mathrm{R}}.$$

Denote with $B(g_i)$ the Newton polytopes of the relations g_i, i.e. $B(g_i) \subseteq \mathbb{Q}^{\mathrm{R}}$ is the convex hull over the exponent vectors of g_i. Let $B := B(g_1) + \ldots + B(g_s)$ be their Minkowski sum and $\mathrm{trop}(X) \subseteq \mathbb{Q}^n$ be the tropical variety of $X \cap \mathbb{T}$, endowed with a fan structure that refines the normal fan $\mathcal{N}(B)$ of B.

Definition 2.1.6. The *anticanonical polyhedron* of X is the dual polyhedron $A_X \subseteq \mathbb{Q}^n$ of the polytope

$$B_X := (P^*)^{-1}(B(-\mathcal{K}_X) + B - e_Z) \subseteq \mathbb{Q}^n.$$

The *anticanonical complex* of X is the coarsest common refinement of polyhedral complexes

$$A_X^c := \mathrm{faces}(A_X) \sqcap \Sigma \sqcap \mathrm{trop}(X).$$

The *relative interior* of A_X^c is the interior of its support with respect to the tropical variety $\mathrm{trop}(X)$.

Example 2.1.7. We continue Example 1.5.9. X is the E_6-singular cubic surface with divisor class group and Cox ring given by

$$\mathrm{Cl}(X) = \mathbb{Z}, \qquad \mathcal{R}(X) = \mathbb{K}[T_1, T_2, T_3, T_4]/\langle T_1T_2^3 + T_3^3 + T_4^2\rangle,$$

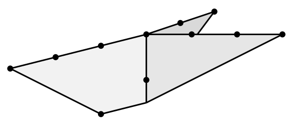
where the $\mathrm{Cl}(X)$-degrees of T_1, T_2, T_3, T_4 are 3, 1, 2, 3. The minimal ambient toric variety Z is an open subset of $Z_c = \mathbb{P}_{3,1,2,3}$ and the tropical variety in $\mathbb{Q}^3$ is

$$\mathrm{trop}(X) = \mathrm{cone}(e_1, \pm e_3) \cup \mathrm{cone}(e_2, \pm e_3) \cup \mathrm{cone}(-e_1 - e_2, \pm e_3),$$

where $e_i \in \mathbb{Q}^3$ is the i-th canonical basis vector. The anticanonical polyhedron $A_X \subseteq \mathbb{Q}^3$ has the vertices

$$(-1,-1,-1),\ (-3,-3,-2),\ (3,0,1),\ (0,2,1),\ (0,0,1),\ (0,0,-1/5).$$

The anticanonical complex $A_X^c = A_X \sqcap \operatorname{trop}(X)$ lives on the three cones of $\operatorname{trop}(X)$ and thus is of dimension two.



Definition 2.1.8. Let $Z_{\Sigma'} \to Z_\Sigma$ be a toric modification given by a subdivision $\Sigma' \to \Sigma$ of fans. Define the sets of rays $\mathrm{R} := \Sigma^{(1)}$ and $\mathrm{R}' := \Sigma'^{(1)}$. The toric Cox constructions $P\colon \mathbb{Z}^{\mathrm{R}} \to \mathbb{Z}^n$ and $P'\colon \mathbb{Z}^{\mathrm{R}'} \to \mathbb{Z}^n$ define homomorphisms of tori

$$\mathbb{T}^{\mathrm{R}'} \xrightarrow{p'} \mathbb{T}^n \xleftarrow{p} \mathbb{T}^{\mathrm{R}}.$$

Let $g \in \mathbb{K}[T_\varrho;\ \varrho \in \mathrm{R}]$ be without monomial factors. The *push-down* of g is the unique $p_*(g) \in \mathbb{K}[T_1,\ldots,T_n]$ without monomial factors such that $T^\mu p^*(p_*(g)) = g$ holds for some Laurent monomial $T^\mu \in \mathbb{K}[T_\varrho^{\pm 1};\ \varrho \in \mathrm{R}]$. The *shift* of g is the unique $g' \in \mathbb{K}[T_{\varrho'};\ \varrho' \in \mathrm{R}']$ without monomial factors satisfying $p'_*(g') = p_*(g)$.

Definition 2.1.9. Let X be as in Construction 2.1.5.

(i) We call the modification $X' \to X$ arising from a subdivision $\Sigma' \to \Sigma$ of fans a *tropical resolution of singularities* if Σ' subdivides $\Sigma \sqcap \operatorname{trop}(X)$ and X' is smooth with complete intersection Cox ring defined by the shifts g'_i of g_i:

$$\mathcal{R}(X') \ = \ \mathbb{K}[T_{\varrho'};\ \varrho' \in \mathrm{R}'] / \langle g'_1, \ldots, g'_s \rangle.$$

(ii) We say that X is *strongly tropically resolvable* if every subdivision of $\Sigma \sqcap \operatorname{trop}(X)$ admits a regular refinement providing a tropical resolution of singularities.

Theorem 2.1.10. *Let X be a (strongly tropically resolvable) normal Fano variety with a complete intersection Cox ring.*

(i) A_X^c contains the origin in its relative interior and all primitive generators of the fan Σ are vertices of A_X^c.

(ii) X has at most log terminal singularities if (and only if) the anticanonical complex A_X^c is bounded.

(iii) X has at most ε-log terminal singularities if (and only if) 0 is the only lattice point in εA_X^c.

(iv) X has at most canonical singularities if (and only if) 0 is the only lattice point in the relative interior of A_X^c.

(v) X has at most terminal singularities if (and only if) 0 and the primitive generators v_ϱ for $\varrho \in \Sigma^{(1)}$ are the only lattice points of A_X^c.

A detailed proof of this Theorem can be found in [8, Section 2]. These statements generalize the characterizations of toric singularities in terms of lattice polytopes from Theorem 2.1.4. In the toric case, i.e. in the absence of relations g_i, our anticanonical polytope A_X is just the Fano polytope of X and the anticanonical complex is the subdivison of A_X by the fan Σ.

We conclude the Section with some observations that may be drawn for the intersection of A_X^c with the lineality space of $\mathrm{trop}(X)$.

Definition 2.1.11. Let $\mathrm{trop}_0(X) \subseteq \mathrm{trop}(X)$ denote the lineality space of the tropical variety. The *lineality part* of the anticanonical complex is the polyhedral complex $A_{X,0}^c := A_X^c \sqcap \mathrm{trop}_0(X)$.

Proposition 2.1.12. *Let X be a log terminal Fano variety and let $|A_{X,0}^c|$ denote the support of the lineality part of the anticanonical complex A_X^c.*

- *(i) $|A_{X,0}^c|$ is a full dimensional polytope in $\mathrm{trop}_0(X)$ having the origin as an interior point.*
- *(ii) If X is ε-log terminal then the origin is the only lattice point of $\varepsilon|A_{X,0}^c|$.*
- *(iii) If X is canonical then the origin is the only interior lattice point of $|A_{X,0}^c|$.*
- *(iv) If X is terminal then the origin is the only lattice point of $|A_{X,0}^c|$.*

2.2 Specializing to complexity one

In this Section we consider a $\mathbb{Q}$-factorial rational Fano variety $X = X(A, P)$ with torus action of complexity one and investigate the structure of its anticanonical complex A_X^c. It turns out that the vertices of A_X^c can be computed explicitly from the defining matrix P. Hence, as soon as we assume some restrictions on the singularities of X, Theorem 2.1.10 delivers bounding conditions on some entries of P. The main reference is [8, Section 4].

Recall that we have $X \subseteq Z \subseteq Z_c$, where Z_c is a complete toric variety (not necessarily Fano) and Z is the minimal open toric subvariety of Z_c containing X as a closed subvariety. The fans Σ_c of Z_c and Σ of Z share the same set of rays ϱ and the primitive generators $v_\varrho \in \varrho$ are precisely the columns of the matrix

$$P = \begin{pmatrix} -l_0 & l_1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ -l_0 & 0 & & l_r & 0 \\ d_0 & d_1 & & d_r & d' \end{pmatrix}$$

from Construction 1.5.1. The tropical variety $\mathrm{trop}(X)$ with its quasifan structure also lives in $\mathbb{Q}^{r+s}$. Given $\lambda := \{0\} \times \mathbb{Q}^s \subseteq \mathbb{Q}^{r+s}$, the canonical basis vectors $e_1, \ldots, e_r$ and $e_0 := -e_1 - \ldots - e_r$, we have

$$\mathrm{trop}(X) = \lambda_0 \cup \ldots \cup \lambda_r \subseteq \mathbb{Q}^{r+s}, \qquad \text{where} \quad \lambda_i := \mathrm{cone}(e_i) + \lambda.$$

Note that this defines the coarsest possible quasifan structure on $\mathrm{trop}(X)$, and λ is the lineality space of this quasifan.

Definition 2.2.1. A cone $\sigma \in \Sigma$ is called

- *big*, if $\sigma \cap \operatorname{relint}(\lambda_i) \neq \emptyset$ holds for each $i = 0, \dots, r$;
- *elementary big* if it is big, has no rays inside λ and precisely one inside each λ_i;
- a *leaf cone* if $\sigma \subseteq \lambda_i$ holds for some i.

Remark 2.2.2. The big cones and the leaf cones are precisely those $\sigma \in \Sigma$ such that $\operatorname{relint}(\sigma)$ intersects $\operatorname{trop}(X)$. The latter property, by Tevelev's criterion [50, Lemma 2.2], means that the big cones and the leaf cones correspond precisely to the toric orbits of Z intersecting X. Observe that all maximal cones of Σ are big cones or leaf cones. Furthermore, recall the characterization of $\mathfrak{F}$-faces given in Remark 1.5.13. Big cones are Gale dual to $\mathfrak{F}$-faces of type (i), and leaf cones to $\mathfrak{F}$-faces of type (ii).

Definition 2.2.3. Let $\sigma \in \Sigma$ be an elementary big cone. We assign the following positive integers to the rays $\varrho = \operatorname{cone}(v_{ij}) \in \sigma^{(1)}$ of σ and to σ itself:

$$l_\varrho \;:=\; l_{ij}, \qquad \ell_{\sigma,\varrho} \;:=\; l_\varrho^{-1} \prod_{\varrho' \in \sigma^{(1)}} l_{\varrho'}, \qquad \ell_\sigma \;:=\; \sum_{\varrho \in \sigma^{(1)}} \ell_{\sigma,\varrho} - (r-1) \prod_{\varrho \in \sigma^{(1)}} l_\varrho.$$

Moreover, in $\mathbb{Q}^{r+s}$, we define vectors and a ray:

$$v_\sigma \;:=\; \sum_{\varrho \in \sigma^{(1)}} \ell_{\sigma,\varrho} v_\varrho, \qquad v'_\sigma \;:=\; \ell_\sigma^{-1} v_\sigma, \qquad \varrho_\sigma \;:=\; \operatorname{cone}(v_\sigma).$$

Finally, we denote by c_σ the greatest common divisor of the entries of $v_\sigma \in \mathbb{Z}^{r+s}$.

The first structural statement describes the rays of the coarsest common refinement $\Sigma \sqcap \operatorname{trop}(X)$ of the fan Σ and the tropical variety $\operatorname{trop}(X)$ regarded as a quasifan.

Proposition 2.2.4. *Let $X = X(A, P)$ be a $\mathbb{Q}$-factorial Fano variety.*

(i) For every elementary big cone $\sigma \in \Sigma$, we have $\sigma \cap \lambda = \varrho_\sigma$; in particular, ϱ_σ lies in the lineality space λ.

(ii) The set of rays of $\Sigma \sqcap \operatorname{trop}(X)$ consists of the rays $\varrho \in \Sigma$ and the rays ϱ_σ, where $\sigma \in \Sigma$ runs through the elementary big cones.

Proof. For (i), one directly computes the intersection $\sigma \cap \lambda$. We prove (ii). Since all rays of Σ lie on $\operatorname{trop}(X)$, the rays of Σ are also rays of $\Sigma \sqcap \operatorname{trop}(X)$. By (i), the ϱ_σ, where $\sigma \in \Sigma$ is elementary big, are rays of $\Sigma \sqcap \operatorname{trop}(X)$. Let $\varrho' \in \Sigma \sqcap \operatorname{trop}(X)$ be any ray not belonging to Σ. Then there exist cones $\sigma \in \Sigma$ and $\tau \in \operatorname{trop}(X)$ which satisfy $\sigma \cap \tau = \varrho'$ and which are minimal with this property. The latter means $\operatorname{relint}(\varrho') = \operatorname{relint}(\sigma) \cap \operatorname{relint}(\tau)$.
To obtain $\tau = \lambda$ we have to exclude the case $\tau = \lambda_i$ for some $i = 0, \dots, r$. Indeed if $\tau = \lambda_i$ holds, then no ray $\varrho \preceq \sigma$ lies in λ_i, because otherwise we have $\varrho \subseteq \sigma \cap \lambda_i = \varrho'$, contradicting $\varrho' \notin \Sigma$. Thus, σ has no rays inside λ_i. Since all rays of σ lie on $\operatorname{trop}(X)$, we conclude $\operatorname{relint}(\sigma) \cap \operatorname{relint}(\lambda_i) = \emptyset$, a contradiction.
We show that σ is an elementary big cone. First, σ must be big because otherwise we have $\operatorname{relint}(\sigma) \cap \lambda = \emptyset$. Since X is $\mathbb{Q}$-factorial, σ is simplicial. Thus there exists an elementary big face η of σ. But then $\varrho_\eta = \eta \cap \lambda \preceq \sigma \cap \lambda = \varrho'$ which implies $\varrho' = \varrho_\eta$. By minimality of σ, we conclude $\sigma = \eta$. □

In the next two Propositions we take a closer look at the discrepancies of a tropical resolution of singularities along the divisors corresponding to the rays ϱ_σ.

Proposition 2.2.5. *Let $\varphi\colon X' \to X$ be a tropical resolution of singularities given by subdivision $\Sigma' \to \Sigma$ of fans. Then the discrepancy α_ϱ along a divisor D_ϱ corresponding to a ray $\varrho \in \Sigma'$ satisfies*

$$\alpha_\varrho = \frac{\|v_\varrho\|}{\|v'_\varrho\|} - 1 \text{ if } \varrho \not\subseteq A_X^c, \qquad \alpha_\varrho \le -1 \text{ if } \varrho \subseteq A_X^c.$$

Proof. See [8, Prop. 2.3]. □

Proposition 2.2.6. *Let $X = X(A,P)$ be a $\mathbb{Q}$-factorial Fano variety and $\sigma \in \Sigma$ an elementary big cone.*

(i) If ϱ_σ leaves A_X, e.g. if σ defines a log terminal singularity, then its leaving point is $v'_{\varrho_\sigma} = \ell_\sigma^{-1} v_\sigma = v'_\sigma$.

(ii) For any tropical resolution $\varphi\colon X' \to X$ of singularities, the discrepancy along the divisor corresponding to ϱ_σ is $a_{\varrho_\sigma} = -1 + c_\sigma^{-1}\ell_\sigma$.

Proof. Recall that the intersection point v'_{ϱ_σ} of the ray ϱ_σ with the boundary ∂A_X^c is defined by

$$\langle u, v'_{\varrho_\sigma}\rangle \;=\; -1, \qquad \text{where} \quad u \;:=\; (P^*)^{-1}(e_{-\mathcal{K}_X} + e - e_\Sigma)$$

with any vertex $e_{-\mathcal{K}_X} + e - e_\Sigma$ of $B(-\mathcal{K}_X) + B - e_\Sigma$ minimizing $v'_\sigma := \sum_{\varrho\in\sigma^{(1)}} \ell_{\sigma,\varrho} e_\varrho$. For $v_\sigma = P(v'_\sigma)$, we obtain

$$\langle u, v_\sigma\rangle \;=\; \langle e_{-\mathcal{K}_X}, v'_\sigma\rangle + \langle e, v'_\sigma\rangle - \langle e_\Sigma, v'_\sigma\rangle \;=\; \langle e, v'_\sigma\rangle - \langle e_\Sigma, v'_\sigma\rangle.$$

To compute further, set $u'_i := \sum_{\varrho\in\mathrm{R}_i} l_\varrho e_\varrho$ for $i = 0,\dots,r$, where R_i denotes the set of rays of Σ contained in λ_i. Denoting by ϱ_i the unique ray of σ in λ_i, we have

$$\langle u'_i, v'_\sigma\rangle \;=\; l_{\varrho_i}\ell_{\sigma,\varrho_i} \;=\; \prod_{\varrho\in\sigma^{(1)}} l_\varrho.$$

Consequently, for any point $e \in B = B(g_0) + \ldots + B(g_{r-2})$, we obtain

$$\langle e, v'_\sigma\rangle \;=\; (r-1) \prod_{\varrho\in\sigma^{(1)}} l_\varrho.$$

Thus, we obtain $\langle u, v_\sigma\rangle = -\ell_\sigma$ and the leaving point is $v'_{\varrho_\sigma} = \ell_\sigma^{-1} v_\sigma = v'_\sigma$ as claimed in (i). Assertion (ii) is then a direct consequence of Proposition 2.2.5. □

As an application, we obtain first bounding conditions on the entries l_ϱ of the defining matrix P in terms of the singularities of X.

Corollary 2.2.7. *Let $X = X(A,P)$ be a $\mathbb{Q}$-factorial Fano variety and $\sigma \in \Sigma$ an elementary big cone. If the singularity defined by σ is* *(i)* *log terminal, then* $\sum_{\varrho\in\sigma^{(1)}} l_\varrho^{-1} > r-1$,

(ii) ε*-log terminal, then* $\sum_{\varrho\in\sigma^{(1)}} l_\varrho^{-1} > r-1+\varepsilon c_\sigma \prod_{\varrho\in\sigma^{(1)}} l_\varrho^{-1}$,

(iii) *canonical, then* $\sum_{\varrho\in\sigma^{(1)}} l_\varrho^{-1} \geq r-1+c_\sigma \prod_{\varrho\in\sigma^{(1)}} l_\varrho^{-1}$,

(iv) *terminal, then* $\sum_{\varrho\in\sigma^{(1)}} l_\varrho^{-1} > r-1+c_\sigma \prod_{\varrho\in\sigma^{(1)}} l_\varrho^{-1}$.

Corollary 2.2.8. *Let $X = X(A,P)$ be a $\mathbb{Q}$-factorial Fano variety and consider an elementary big cone $\sigma = \varrho_0 + \ldots + \varrho_r \in \Sigma$ defining a log terminal singularity. Assume $l_{\varrho_0} \geq \ldots \geq l_{\varrho_r}$. Then $l_{\varrho_3} = \ldots = l_{\varrho_r} = 1$ holds and $(l_{\varrho_0}, l_{\varrho_1}, l_{\varrho_2})$ is a platonic triple, i.e. one of*

$$(l_{\varrho_0}, l_{\varrho_1}, 1), \qquad (l_{\varrho_0}, 2, 2), \qquad (3,3,2), \qquad (4,3,2), \qquad (5,3,2).$$

According to these possibilities, the number ℓ_σ is given as

$$\begin{aligned}\ell_\sigma &= l_{\varrho_0}l_{\varrho_1} + l_{\varrho_0}l_{\varrho_2} + l_{\varrho_1}l_{\varrho_2} - l_{\varrho_0}l_{\varrho_1}l_{\varrho_2} \\ &= \begin{cases} l_{\varrho_0} + l_{\varrho_1}, & \text{if } (l_{\varrho_0}, l_{\varrho_1}, l_{\varrho_2}) = (l_{\varrho_0}, l_{\varrho_1}, 1), \\ 4, & \text{if } (l_{\varrho_0}, l_{\varrho_1}, l_{\varrho_2}) = (l_{\varrho_0}, 2, 2), \\ 3, & \text{if } (l_{\varrho_0}, l_{\varrho_1}, l_{\varrho_2}) = (3,3,2), \\ 2, & \text{if } (l_{\varrho_0}, l_{\varrho_1}, l_{\varrho_2}) = (4,3,2), \\ 1, & \text{if } (l_{\varrho_0}, l_{\varrho_1}, l_{\varrho_2}) = (5,3,2). \end{cases}\end{aligned}$$

Corollary 2.2.9. *Let $X = X(A,P)$ be a log terminal $\mathbb{Q}$-factorial Fano variety. Assume that P is irredundant and Σ contains a big cone. Then the number $r-1$ of relations is bounded by*

$$r - 1 \ \leq\ \dim(X) + \rho(X).$$

Proof. Since X is $\mathbb{Q}$-factorial, $\mathrm{Pic}(X)$ is of rank $n+m-r-s$. Let $I \subseteq \{0,\ldots,r\}$ be the set of indices with $n_i > 1$ and set $n_I := \sum_{i\in I} n_i$. Then the rank of $\mathrm{Pic}(X)$ equals $n_I + m - |I| - s$. Since there exists a big cone, there is also an elementary big cone $\sigma = \varrho_0 + \ldots + \varrho_r \in \Sigma$. Since P is irredundant, $l_{\varrho_i} > 1$ holds for all $i \notin I$. Corollary 2.2.8 yields $|I| \geq r-2$. We conclude

$$\rho(X) \ =\ m + n_I - |I| - s \ \geq\ 2|I| - |I| - s \ \geq\ r - 2 - s \ =\ r - 1 - \dim(X).$$

□

Definition 2.2.10. Let A_X^c be the anticanonical complex of $X = X(A,P)$. Recall that the lineality part of A_X^c is the polyhedral complex $A_{X,0}^c = A_X^c \sqcap \lambda$. The *i-th leaf* of A_X^c is the polyhedral complex $A_X^c \sqcap \lambda_i$.

Corollary 2.2.11. *Let $X = X(A,P)$ be a log terminal $\mathbb{Q}$-factorial Fano variety. Then the vertices of the anticanonical complex A_X^c are precisely the points v_ϱ and v'_σ, where ϱ runs through the rays and σ through the elementary big cones of Σ. In particular, for the supports of the lineality part and the leaves of A_X^c, we obtain*

$$\begin{aligned}|A_X^c \sqcap \lambda| &= \mathrm{conv}(v_\varrho, v'_\sigma;\ \varrho \in \Sigma \text{ with } \varrho \subseteq \lambda,\ \sigma \in \Sigma \text{ elementary big}), \\ |A_X^c \sqcap \lambda_i| &= \mathrm{conv}(v_\varrho, v'_\sigma;\ \varrho \in \Sigma \text{ with } \varrho \subseteq \lambda_i,\ \sigma \in \Sigma \text{ elementary big}).\end{aligned}$$

Remark 2.2.12. Let $X = X(A,P)$ be a $\mathbb{Q}$-factorial Fano variety and X' the variety arising from the tropical refinement $\Sigma \sqcap \operatorname{trop}(X)$. Then $A^c_{X'}$ and A^c_X both generate $\Sigma \sqcap \operatorname{trop}(X)$ but do not in general coincide, because the rays ϱ_σ of big elementary cones $\sigma \in \Sigma$ intersect the boundary of $A^c_{X'}$ in integral points, whereas the intersection points v'_σ with A^c_X do not need to be integral.

Example 2.2.13. Consider again the variety X from Example 1.5.9, whose anticanonical complex A^c_X was obtained, using the general definition, in Example 2.1.7. Here we use Corollary 2.2.11 to directly compute the vertices of A^c_X. The points v_ϱ correspond to the columns of the defining matrix P, hence

$$(-1,-1,-1),\ (-3,-3,-2),\ (3,0,1),\ (0,2,1).$$

There are two elementary big cones, namely

$$\sigma_1 := \operatorname{cone}(v_{01}, v_{11}, v_{21}), \qquad \sigma_2 := \operatorname{cone}(v_{02}, v_{11}, v_{21}).$$

Using Definition 2.2.3 we compute

$$v'_{\sigma_1} = (0,0,1), \qquad v'_{\sigma_2} = (0,0,-1/5).$$

With Theorem 2.1.10 we conclude that X is canonical and non-terminal.

2.3 Constraints by terminality

This Section contains a simple result that is not available in the original reference and specializes Corollary 2.2.8 to the terminal case. This allows to shorten some parts of the proofs of next Section.

Proposition 2.3.1. *Let $X = X(A,P)$ be a $\mathbb{Q}$-factorial Fano variety and consider an elementary big cone $\sigma = \varrho_0 + \ldots + \varrho_r \in \Sigma$ defining a terminal singularity. Then at most two l_{ϱ_i} differ from one.*

Proof. Assume $l_{\varrho_0} \geq \ldots \geq l_{\varrho_r}$. By Corollary 2.2.8 the triple $(l_{\varrho_0}, l_{\varrho_1}, l_{\varrho_2})$ is platonic and $l_{\varrho_3} = \ldots = l_{\varrho_r} = 1$ holds. The same Corollary lists the value of ℓ_σ, which by Definition 2.2.3 is the denominator of the entries of v'_σ. In particular for the sporadic platonic triples

$$(3,3,2), \qquad (4,3,2), \qquad (5,3,2),$$

we have $\ell_\sigma = 3,2,1$ respectively. Moreover, the numerator is a sum in which every addend is a multiple of all l_{ϱ_i} except one. Hence the vertex v'_σ of $A^c_{X,0}$ turns out to be a lattice point, contradicting terminality by Theorem 2.1.10. The same happens for $(l_{\varrho_0}, 2, 2)$ if l_{ϱ_0} is even, so we may assume $l_{\varrho_0} \geq 3$ to be odd. Now we show that this case provides an integral point on A^c_X and therefore does not define a terminal singularity. The primitive lattice point v_{ϱ_i} has coordinates

$$\begin{aligned} v_{\varrho_0} &= (-l_{\varrho_0}, \ldots, -l_{\varrho_0}, d_{01}, \ldots, d_{0s})^T, \\ v_{\varrho_i} &= (0, \ldots, 0, l_{\varrho_i}, 0, \ldots, 0, d_{i1}, \ldots, d_{is})^T \end{aligned}$$

for all $i = 1, \ldots, r$, where l_{ϱ_i} occupies the i-th coordinate. Let $u := v'_\sigma$ be the vertex of $A_{X,0}$ defined by σ. It has coordinates $u_j = 0$ for $j = 1, \ldots, r$ and

$$u_{r+t} = d_{0t} + \frac{l_{\varrho 0}}{2} d_{1t} + \frac{l_{\varrho 0}}{2} d_{2t} + l_{\varrho 0} \sum_{k=3}^{r} d_{kt} \qquad \forall\, t = 1, \ldots, s\,.$$

Using the fact that $l_{\varrho 0}$ is odd, we see that on the edge connecting $v_{\varrho 0}$ with u lies at least one lattice point, namely

$$\frac{l_{\varrho 0} - 1}{l_{\varrho 0}} u + \frac{1}{l_{\varrho 0}} v_{\varrho 0} = (-1, \ldots, -1, q_1, \ldots, q_s)^T,$$

where for all $t = 1, \ldots, s$ holds

$$q_t = d_{0t} + \frac{l_{\varrho 0} - 1}{2} d_{1t} + \frac{l_{\varrho 0} - 1}{2} d_{2t} + (l_{\varrho 0} - 1) \sum_{k=3}^{r} d_{kt}.$$

Hence the platonic triple $(l_{\varrho 0}, l_{\varrho 1}, l_{\varrho 2})$ is of the first type, i.e. $l_{\varrho 2} = 1$. □

Corollary 2.3.2. *Let $X = X(A, P)$ be a $\mathbb{Q}$-factorial Fano variety, with P irredundant. If there is an elementary big cone $\sigma \in \Sigma$ defining a terminal singularity, then $n \geq 2r$.*

Proof. The cone σ has exactly one ray in each of the $r + 1$ leaves. By Proposition 2.3.1, there are at most two monomials with just one variable. Thus $n \geq 2(r-1)+2 = 2r$. □

2.4 Picard number one

Here we show how to obtain the classification of terminal $\mathbb{Q}$-factorial Fano threefolds X of Picard number one coming with an effective action of a two-dimensional torus.
Let X be rational. This allows us to work in terms of the defining data (A, P) of X from Construction 1.5.1, where we always choose P to be irredundant. The main step is to derive suitable effective bounds on the entries of P. According to Theorem 2.1.10, terminality of X is equivalent to the fact that the anticanonical complex A_X^c contains no lattice points except the origin and the vertices given by the columns of the defining matrix P. A first observation towards bounds for the shape of P is that terminality leads to the following situations.

Lemma 2.4.1. *Let $X = X(A, P)$ be a non-toric terminal $\mathbb{Q}$-factorial Fano threefold with $\rho(X) = 1$, where P is irredundant. Then, after suitable admissible operations, P fits into one of the following cases:*
(i) $m = 0$, $r = 2$ and $n = 5$, where $\overline{n} = (2, 2, 1)$.
(ii) $m = 0$, $r = 3$ and $n = 6$, where $\overline{n} = (2, 2, 1, 1)$.
(iii) $m = 0$, $r = 2$ and $n = 5$, where $\overline{n} = (3, 1, 1)$.
(iv) $m = 1$, $r = 2$ and $n = 4$, where $\overline{n} = (2, 1, 1)$.

Proof. Since X is non-toric, there is at least one relation in the Cox ring. This means $r \geq 2$. Moreover we have $n + m = r + s + \rho(X) = r + 3$, so Corollary 2.3.2 delivers $r + m \leq 3$. In particular $m = 0, 1$.
If $m = 1$ holds, then we have $r = 2$ and $n = 4$. This leads to case (*iv*) of the assertion. If $m = 0$ holds, we distinguish $r = 2$ and $r = 3$. In the former case $n = 5$ holds, so we end up with either (*i*) or (*iii*). If the latter holds, then we obtain $n = 6$ and with Proposition 2.3.1 we arrive at (*ii*). □

In the rest of this Chapter we treat extensively all cases of Lemma 2.4.1. In each of them we take similar steps towards the complete classification.
First we fix as many entries of the defining matrix P as possible, through admissible operations from Remark 1.5.5. Then we take a closer look at $A^{c}_{X,0}$. Its shape typically shows some regularity, e.g. some of the edges are parallel to each other or to one of the main axes. This delivers estimates for most values of P. Sometimes one or two entries are still unbounded. For these cases an additional tool is provided by Theorems 2.4.4 and 2.4.5. We find suitable lattice polytopes, whose volumes are expressed as polynomials in the entries of P, having exactly one or two interior lattice points. Bounding these quantities leads to the last estimates needed. Sometimes it is necessary to go through several specific configurations of the entries of the matrix P, each giving rise to a different polytope. All cases are given in detail in the proofs for the sake of completeness.

Remark 2.4.2. The following table summarizes the relevant structural elements of each case. For each case we list:

- the Setting that comes from using admissible operations in order to minimize the number of non-fixed entries of P;
- the Lemma that computes the vertices of $A^{c}_{X,0}$;
- the Remark on the shape and regularity of $A^{c}_{X,0}$;
- the Proposition(s) containing the final estimates and bounds.

	Setting	Lemma	Remark	Proposition(s)
Case (*i*)	2.4.7	2.4.9	2.4.10	2.4.19/2.4.20/2.4.21
Case (*ii*)	2.4.23	2.4.24	2.4.25	2.4.28
Case (*iii*)	2.4.30	2.4.31	2.4.32	2.4.35
Case (*iv*)	2.4.37	2.4.38	2.4.39	2.4.44/2.4.45/2.4.46

Before engaging in the details of the classification, we recall a result by Averkov, Krümpelmann and Nill [2], concerning the volume of certain lattice simplices. For this purpose, let us define the *Sylvester sequence* $(s_d)_{d\in\mathbb{Z}_{\geq 1}}$ through

$$s_1 \ := \ 2, \qquad s_d \ := \ 1 + \prod_{j=1}^{d-1} s_j \quad \text{for } d \geq 2.$$

Theorem 2.4.3 ([2, Thm. 2.2]). *Let $S \subset \mathbb{Q}^d$ be a lattice simplex containing exactly one lattice point in its interior. Then we have*

$$\mathrm{vol}(S) \ \le \ \frac{2(s_d - 1)^2}{d!} \ .$$

In particular, if $d = 3$ then $\mathrm{vol}(S) \le 12$ holds.

While this Theorem has the advantage of working in any dimension, it only applies to simplices with one interior lattice point. Since we are considering polytopes in dimension three, we will exploit the full classification of lattice polytopes with one interior lattice point provided by A. Kasprzyk in [35]. Moreover, the recent result of the same author with G. Balletti, published in [3], provides a bound on the volume of three-dimensional lattice polytopes with two interior lattice points.

Theorem 2.4.4 ([35]). *Let $B \subset \mathbb{Q}^3$ be a three-dimensional lattice polytope containing exactly one lattice point in its interior. Then $\mathrm{vol}(B) \le 12$ holds.*

Theorem 2.4.5 ([3, Thm. 1.1]). *Let $B \subset \mathbb{Q}^3$ be a three-dimensional lattice polytope containing exactly two lattice points in its interior. Then $\mathrm{vol}(B) \le 18$ holds.*

Some of the bounds on the l-values in this Section are improvements of their counterparts in the original source [8] partly because here we make use of the two results above.

Case (i) of Lemma 2.4.1

Lemma 2.4.6. *Let $X = X(A, P)$ be a non-toric terminal $\mathbb{Q}$-factorial Fano threefold with $\rho(X) = 1$ such that P is irredundant and we have $r = 2$, $m = 0$ and $\overline{n} = (2, 2, 1)$. Then $l_{01} = l_{02} = 1$ or $l_{11} = l_{12} = 1$ hold.*

Proof. Since P is irredundant, we have $l_{21} \ge 2$. Since $\rho(X) = 1$ holds, every triple of rays (v_{0i}, v_{1j}, v_{21}), for $i, j \in \{1, 2\}$, defines an elementary big cone. Therefore Proposition 2.3.1 applies to all of them and yields $l_{01} = l_{02} = 1$ or $l_{11} = l_{12} = 1$. □

As a consequence of Lemma 2.4.6, we can focus our search for terminal varieties $X(A, P)$ on defining matrices P of the following type.

Setting 2.4.7. Let $X = X(A, P)$ be a non-toric terminal $\mathbb{Q}$-factorial Fano threefold with $\rho(X) = 1$ such that P is irredundant and we have $r = 2$, $m = 0$ and $\overline{n} = (2, 2, 1)$. We can assume that $l_{01} = l_{02} = 1$ holds. Then, after suitable admissible operations, P is of the form

$$P \ = \ \begin{bmatrix} -1 & -1 & l_{11} & l_{12} & 0 \\ -1 & -1 & 0 & 0 & l_{21} \\ 0 & 1 & d_{111} & d_{112} & d_{121} \\ 0 & 0 & d_{211} & d_{212} & d_{221} \end{bmatrix},$$

where $l_{11} \ge l_{12}$ and $l_{21} \ge 2$ hold. Moreover, denoting by P_{ij} the matrix obtained by removing the column v_{ij} from P, we have positive *weights*

$$w_{01} \ := \ \det(P_{01}), \qquad w_{02} \ := \ -\det(P_{02}),$$

$$w_{11} \ := \ \det(P_{11}), \quad w_{12} \ := \ -\det(P_{12}), \quad w_{21} \ := \ \det(P_{21}).$$

Observe that the weight vector $(w_{01}, w_{02}, w_{11}, w_{12}, w_{21})$ lies in the kernel of P. The last three weights are explicitly given by

$$w_{11} \ = \ -l_{21}d_{212} - l_{12}d_{221}, \quad w_{12} \ = \ l_{21}d_{211} + l_{11}d_{221}, \quad w_{21} \ = \ -l_{11}d_{212} + l_{12}d_{211}$$

and the first two weights can be expressed in a compact form in terms of the others as follows:

$$w_{02} \ = \ -d_{111}w_{11} - d_{112}w_{12} - d_{121}w_{21}, \qquad w_{01} \ = \ l_{21}w_{21} - w_{02}.$$

Remark 2.4.8. In Setting 2.4.7 we can achieve, by further admissible operations, the following for the entries of the third and fourth row of P:

$$0 \ \le \ d_{121}, d_{221} \ < \ l_{21}, \qquad d_{121} \ < \ d_{221} \text{ if } d_{221} \neq 0, \qquad 0 \ \le \ d_{112} \ < \ w_{11},$$

$$-\frac{(l_{21} + d_{121})w_{21} + d_{112}w_{12}}{w_{11}} \ < \ d_{111} \ < \ -\frac{d_{121}w_{21} + d_{112}w_{12}}{w_{11}}.$$

For the third estimate we add a suitable multiple of $d_{221}(p_1 - p_2) + l_{21}p_4$ to p_3, where p_i denotes the i-th row of P (this preserves the first two estimates). The inequalities for d_{111} follow directly from $w_{02} > 0$ and $w_{01} > 0$.

A first series of bounding conditions on the entries of the defining matrix P is derived from the fact that, by terminality, the lineality part $A^c_{X,0}$ of the anticanonical complex A^c_X has the origin as its only lattice point; we also write $A^c_{X,0}$ for the support of the lineality part, which in our situation is a rational two-dimensional polytope. Here is how it precisely looks.

Lemma 2.4.9. *Let $X = X(A, P)$ be as in Setting 2.4.7. The vertices of $A^c_{X,0}$, regarded as a subset of the lineality space $\mathbb{Q}^2$ of the tropical variety, are*

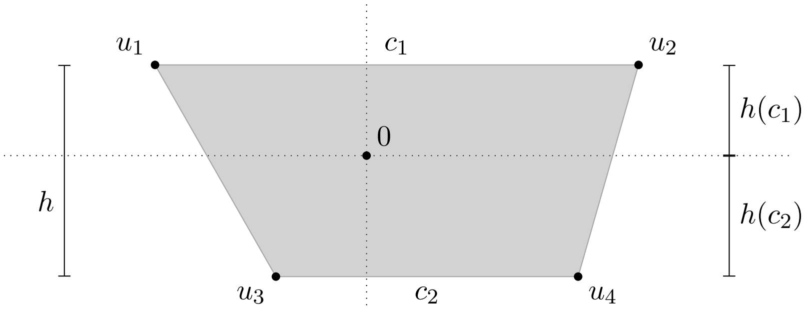
$$\begin{aligned}
u_1 \ &:= \ \left[\frac{l_{21}d_{111} + l_{11}d_{121}}{l_{21} + l_{11}}, \ \frac{l_{21}d_{211} + l_{11}d_{221}}{l_{21} + l_{11}}\right], \\
u_2 \ &:= \ \left[\frac{l_{11}l_{21} + l_{21}d_{111} + l_{11}d_{121}}{l_{21} + l_{11}}, \ \frac{l_{21}d_{211} + l_{11}d_{221}}{l_{21} + l_{11}}\right], \\
u_3 \ &:= \ \left[\frac{l_{21}d_{112} + l_{12}d_{121}}{l_{21} + l_{12}}, \ \frac{l_{21}d_{212} + l_{12}d_{221}}{l_{21} + l_{12}}\right], \\
u_4 \ &:= \ \left[\frac{l_{12}l_{21} + l_{21}d_{112} + l_{12}d_{121}}{l_{21} + l_{12}}, \ \frac{l_{21}d_{212} + l_{12}d_{221}}{l_{21} + l_{12}}\right].
\end{aligned}$$

Proof. We just compute the lineality part $A^c_{X,0}$ according to Corollary 2.2.11. □

Remark 2.4.10. Observe that $A^{\mathrm{c}}_{X,0}$ as described in Lemma 2.4.9 is a trapezoid. The edges $c_1 := \overline{u_1 u_2}$ and $c_2 := \overline{u_3 u_4}$ are parallel to the x-axis and the remaining two edges are $\overline{u_1 u_3}$ and $\overline{u_2 u_4}$. Length and y-value $h(c_i)$ of the line segments c_i are

$$|c_1| = \frac{l_{11} l_{21}}{l_{11} + l_{21}}, \qquad h(c_1) = \frac{w_{12}}{l_{11} + l_{21}},$$
$$|c_2| = \frac{l_{12} l_{21}}{l_{12} + l_{21}}, \qquad h(c_2) = -\frac{w_{11}}{l_{12} + l_{21}}.$$

Since we assumed $l_{11} \geq l_{12}$ in Setting 2.4.7, the lower segment c_2 is shorter than the upper segment c_1. Note that the values $|c_i|$ and $h(c_i)$ are invariant under admissible row operations of type 1.5.5 (iii).



Lemma 2.4.11. *Let $X = X(A, P)$ be as in Setting 2.4.7. Let $h := h(c_1) - h(c_2)$ denote the total height of the trapezoid $A^{\mathrm{c}}_{X,0}$. Then we have*

$$\frac{l_{12} l_{21}}{l_{12} + l_{21}} < 2, \qquad \frac{l_{11} l_{21}}{l_{11} + l_{21}} < \frac{2(l_{12} + l_{21}) - l_{12} l_{21}}{w_{11}} \cdot h + \frac{l_{12} l_{21}}{l_{12} + l_{21}}.$$

Moreover, following estimates hold:

$$w_{01} < w_{11} + w_{12} + w_{21}, \qquad w_{02} < w_{11} + w_{12} + w_{21}.$$

Proof. For the first inequality, note that the lower bounding segment c_2 of $A^{\mathrm{c}}_{X,0}$ is of length at most 2, because otherwise the segment $A^{\mathrm{c}}_{X,0} \cap \{y = 0\}$ is of length at least 2 as well, which would imply existence of lattice points different from the origin in A^{c}_X. Similarly, since $A^{\mathrm{c}}_{X,0} \cap \{y = 0\}$ has length strictly smaller than 2, we arrive at the second inequality:

$$|c_1| < \frac{2 - |c_2|}{|h(c_2)|} \cdot h + |c_2|.$$

Explicitly computing $A^{\mathrm{c}}_{X,0} \cap \{y = 0\}$ gives the bounding x-values $-w_{01}/(w_{11} + w_{12} + w_{21})$ and $w_{02}/(w_{11} + w_{12} + w_{21})$. Since the origin is the only lattice point in $A^{\mathrm{c}}_{X,0} \cap \{y = 0\}$, we arrive at estimates number three and four. ☐

Lemma 2.4.12. *Let $X = X(A, P)$ be as in Setting 2.4.7. If $l_{21} \geq 3$ holds, then we obtain the estimate*

$$l_{12} < \frac{l_{21} + 2}{l_{21} - 2} \leq 5.$$

Proof. Estimates three and four from Lemma 2.4.11 imply

$$l_{21}w_{21} \;=\; l_{11}w_{11} + l_{12}w_{12} \;=\; w_{01} + w_{02} \;<\; 2w_{11} + 2w_{12} + 2w_{21}.$$

We deduce

$$(l_{11} - 2)w_{11} + (l_{12} - 2)w_{12} \;<\; 2w_{21}, \qquad (l_{21} - 2)w_{21} \;<\; 2w_{11} + 2w_{12}.$$

Using $l_{21} \geq 3$ we obtain

$$(l_{11} - 2)w_{11} + (l_{12} - 2)w_{12} \;<\; \frac{4}{l_{21} - 2}w_{11} + \frac{4}{l_{21} - 2}w_{12},$$

which implies

$$l_{11}w_{11} + l_{12}w_{12} \;<\; \frac{l_{21} + 2}{l_{21} - 2}w_{11} + \frac{l_{21} + 2}{l_{21} - 2}w_{12}$$

and in particular

$$l_{12} \;<\; \frac{l_{21} + 2}{l_{21} - 2}.$$

□

Remark 2.4.13. Let $X = X(A, P)$ be as in Setting 2.4.7. For $t > 0$ the assumption $h(c_2) > -t$ leads to

$$-t - \frac{l_{12}}{l_{21}}(t + d_{221}) \;<\; d_{212} \;<\; 0.$$

Remark 2.4.14. Let $X = X(A, P)$ be as in Setting 2.4.7. If $h(c_1) < 1$ holds, then we have

$$-\frac{l_{11}}{l_{21}}d_{221} \;<\; d_{211} \;<\; -\frac{l_{11}}{l_{21}}d_{221} + 1 + \frac{l_{11}}{l_{21}}.$$

Lemma 2.4.15. *Let $X = X(A, P)$ be as in Setting 2.4.7. Assume $l_{21} \geq 3$. If $h(c_1) < 1$ and $h(c_2) > -2$ hold, then we have*

$$l_{11} \;<\; 2\frac{l_{21}}{l_{21} - 2}.$$

This bounds l_{11} in terms of l_{21} in the case $h(c_1) < 1$ and $h(c_2) > -2$. In particular, we then have $l_{11} \leq 5$ and we have $l_{11} \leq 2$ as soon as $l_{21} \geq 6$.

Proof. Observe that $w_{01} + w_{02} = l_{11}w_{11} + l_{12}w_{12}$ holds. Thus, the third and the fourth inequalities of Lemma 2.4.11 give us the condition

$$l_{11}w_{11} + l_{12}w_{12} \;<\; 2w_{11} + 2w_{12} + 2w_{21}.$$

We arrive at the assertion by writing this out and estimating d_{212} as well as d_{211} according to Remarks 2.4.13 and 2.4.14. □

Lemma 2.4.16. *Let $X = X(A, P)$ be as in Setting 2.4.7. Suppose that $h(c_1) < 1$ and $h(c_2) \le -1$ hold. Then we have $l_{12} = 1$. Moreover, if $h(c_2) \le -t$ holds for some $t \in \mathbb{Z}_{\ge 2}$, then*

$$\frac{l_{11} l_{21}}{l_{11} + l_{21}} \ < \ \frac{t+1}{t-1} - \frac{2}{t-1} \cdot \frac{l_{21}}{1 + l_{21}}.$$

Proof. Since $h(c_2) \le -1$ holds, we must have $|c_2| < 1$ and thus obtain $l_{12} = 1$. The line segment $A^c_{X,0} \cap \{y = -1\}$ is of length strictly smaller than 1 and $A^c_{X,0} \cap \{y = -t\}$ is of length at least $|c_2|$. Since $h(c_1) < 1$ holds, we conclude

$$\frac{l_{11} l_{21}}{l_{11} + l_{21}} \ = \ |c_1| \ < \ \frac{1 - |c_2|}{t-1}(1 + h(c_1)) + 1 \ < \ \frac{t+1}{t-1} - \frac{2}{t-1} \cdot \frac{l_{21}}{1 + l_{21}}.$$

□

Remark 2.4.17. Let $X = X(A, P)$ be as in Setting 2.4.7. Assume $l_{12} = 1$ and $d_{112} = d_{212} = 0$. Then $w_{11} > 0$ and $w_{12} > 0$ imply

$$0 \ < \ d_{211}, \qquad\qquad -\frac{l_{21}}{l_{11}} d_{211} \ < \ d_{221} \ < \ 0.$$

Moreover, the conditions $h(c_1) < 1$ and $h(c_2) > -t$ are equivalent to the following conditions

$$d_{211} \ < \ -\frac{l_{11}}{l_{21}}(d_{221} - 1) + 1, \qquad\qquad d_{221} \ > \ -t(l_{21} + 1).$$

Lemma 2.4.18. *Let $X = X(A, P)$ be as in Setting 2.4.7. Suppose that $h(c_1) \ge 1$ holds. Then either $l_{11} = l_{12} = 1$ or $l_{11} = l_{21} = 2$ hold.*

Proof. First observe that in this case, the segment $A^c_{X,0} \cap \{y = 1\}$ can be of length at most 1, because otherwise we have lattice points different from the origin and the vertices in A^c_X. This means $l_{11} = 1$ or $l_{11} = l_{21} = 2$. □

A second series of estimates makes use of the whole anticanonical complex A^c_X. The strategy is to detect via A^c_X suitable three-dimensional lattice polytopes with precisely one or two interior lattice points and to use the volume bounds given in Theorem 2.4.4 and Theorem 2.4.5 to control the entries of the defining matrix P. We will distinguish several cases, using the notation of Remark 2.4.10.

Proposition 2.4.19. *Let $X = X(A, P)$ be as in Setting 2.4.7. Suppose $l_{11} = l_{12} = 1$. Then we achieve by admissible operations $d_{112} = d_{212} = 0$ and obtain the estimates*

$$3 \ \le \ (l_{21} + 1) d_{211} \ < \ 36, \qquad\qquad 0 \ \le \ d_{111} \ < \ d_{211},$$

$$-d_{211} l_{21} \ < \ d_{221} \ < \ 0, \qquad\qquad \frac{d_{111} d_{221}}{d_{211}} - l_{21} \ < \ d_{121} \ < \ 0.$$

Proof. Consider the convex hull B of $A^c_{X,0}$ and v_{21}. We may regard B as a polytope in $\mathbb{Q}^3$ by omitting the first coordinate. Then B is contained in the polytope C, given by the vertices

$$(l_{21}, d_{121}, d_{221}), \quad (-1, d_{111}, d_{211}), \quad (-1, 1+d_{111}, d_{211}), \quad (-1,0,0), \quad (-1,1,0).$$

Now, C is a lattice polytope having $(0,0,0)$ as only interior lattice point. According to Theorem 2.4.4, its volume is at most 12. This gives the condition

$$2(l_{21}+1)d_{211} \ = \ 6\mathrm{vol}(C) \ < \ 72.$$

The remaining estimates follow from positivity of the weights w_{ij}. □

Proposition 2.4.20. *Let $X = X(A,P)$ be as in Setting 2.4.7. Suppose $l_{21} = 2$.*

(i) If $h(c_1) < 1$ and $h(c_2) > -1$ hold, turn P by means of admissible operations into the shape of Remark 2.4.8. Then we are in one of the following situations:
(a) $d_{121} = 0$, $d_{221} = 1$, with $l_{11} + l_{12} \leq 68$.
(b) $d_{121} = 1$, $d_{221} = 0$, with $l_{11} + l_{12} \leq 69$.
In both situations d_{111}, d_{112} are bounded according to Remark 2.4.8 and we have

$$\frac{l_{11}}{2}(d_{121}-1) \ < \ d_{211} \ \leq \ \frac{l_{11}}{2}d_{121}, \qquad -\frac{l_{12}}{2}(d_{221}+1) \ \leq \ d_{212} \ < \ -\frac{l_{12}}{2}d_{221}.$$

(ii) If $h(c_1) < 1$ and $h(c_2) \leq -1$ hold, then we have $l_{12} = 1$. Moreover, after adjusting $d_{112} = d_{212} = 0$ by admissible operations, we achieve one of the following three situations:
(a) $l_{11} = 1$ holds and Proposition 2.4.19 applies.
(b) $l_{11} = 2$ holds and we have estimates

$$-6 \ \leq d_{221} \leq \ -3, \qquad\qquad d_{211} \ = \ 1 - d_{221}.$$

(c) $3 \leq l_{11} \leq 24$ holds and we have estimates

$$\frac{-5l_{11}+2}{l_{11}-2} \ < \ d_{221} \ \leq \ -3, \qquad -\frac{l_{11}}{2}d_{221} \ < \ d_{211} \ < \ -\frac{l_{11}}{2}d_{221} + \frac{l_{11}}{2}.$$

In both cases (b) and (c), the remaining entries of P are bounded by

$$0 \ \leq \ d_{121} \ < \ -d_{221}, \qquad \frac{d_{121}d_{211}}{d_{221}} + 2\frac{d_{211}}{d_{221}} \ < \ d_{111} \ < \ \frac{d_{121}d_{211}}{d_{221}}.$$

(iii) If $h(c_1) \geq 1$ holds, then we have $l_{11} = l_{12} = 1$ and Proposition 2.4.19 applies.

Proof. First we prove (i). Observe that Remark 2.4.8 yields $d_{221} \in \{0,1\}$ because of $l_{21} = 2$. If $d_{221} = 1$ holds, then Remark 2.4.8 implies $d_{121} = 0$. If $d_{221} = 0$ holds, then we have $d_{121} = 1$ since v_{21} is a primitive lattice point. Note that the same Remark also bounds the entries d_{111} and d_{112}. Moreover, we recover bounds for d_{211} and d_{212}

by writing down explicitly the positivity of the weights w_{11} and w_{12}, as well as the assumptions on the heights of c_1 and c_2:

$$\frac{l_{11}}{2}(d_{121} - 1) \ < \ d_{211} \ \leq \ \frac{l_{11}}{2} d_{121}, \qquad -\frac{l_{12}}{2}(d_{221} + 1) \ \leq \ d_{212} \ < \ -\frac{l_{12}}{2} d_{221}.$$

Case $d_{121} = 0$, $d_{221} = 1$: consider the lattice simplex C in $\mathbb{Q}^3$ given as the convex hull of the following points:

$$(l_{11}, d_{111}, d_{211}), \quad (l_{12}, d_{112}, d_{212}), \quad (-2, 0, 1), \quad (-2, 2, 1).$$

We have $\mathrm{vol}(C) = (w_{11} + w_{12} + w_{21})/3$. Now, put the leaf $A_X^c \cap \lambda_1$ of the anticanonical complex into $\mathbb{Q}^3$ by removing the second coordinate (which always equals zero) from its points. For $a = 0, -1, -2$, consider

$$H_a^+ \ := \{(x, y, z);\ x \geq a\} \ \subseteq \ \mathbb{Q}^3, \qquad H_a^0 \ := \{(x, y, z);\ x = a\} \ \subseteq \ \mathbb{Q}^3.$$

Then $C \cap H_0^+$ equals $A_X^c \cap \lambda_1$ and H_0^0 cuts out the lineality part $A_{X,0}^c$. In particular, by terminality of X and Theorem 2.1.10, the intersection $C \cap H_0^+$ has no interior lattice point and inside $C \cap H_0^0$ the origin is the only lattice point. The intersection $C \cap H_{-1}^0$ is a quadrilateral with vertices

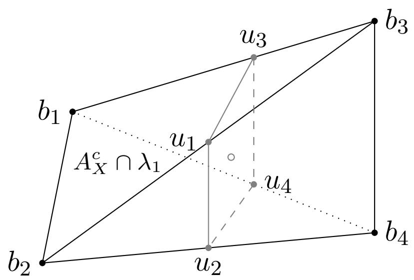
$$\Big(-1, \frac{d_{111}}{l_{11}+2}, \frac{l_{11} + d_{211} + 1}{l_{11} + 2}\Big), \qquad \Big(-1, \frac{2l_{11} + d_{111} + 2}{l_{11} + 2}, \frac{l_{11} + d_{211} + 1}{l_{11} + 2}\Big),$$
$$\Big(-1, \frac{d_{112}}{l_{12}+2}, \frac{l_{12} + d_{212} + 1}{l_{12} + 2}\Big), \qquad \Big(-1, \frac{2l_{12} + d_{112} + 2}{l_{12} + 2}, \frac{l_{12} + d_{212} + 1}{l_{12} + 2}\Big).$$

The bounds on d_{211} and d_{212} ensure that the points of $C \cap H_{-1}^0$ never have an integral z-value. Thus we can apply Theorem 2.4.4 to $\mathrm{vol}(C)$ and arrive at $w_{11} + w_{12} + w_{21} \leq 36$. This implies $w_{21} \leq 34$, which in turn yields $l_{11}w_{11} + l_{12}w_{12} \leq 68$. Since all weights are positive, we conclude $l_{11} + l_{12} \leq 68$.

Case $d_{121} = 1$ *and* $d_{221} = 0$: here we consider the lattice simplex C given in $\mathbb{Q}^3$ as the convex hull of the following points:

$$b_1 := (l_{11}, d_{111}, d_{211}), \quad b_2 := (l_{12}, d_{112}, d_{212}),$$
$$b_3 := (-2, 1, 0), \quad b_4 := (-2, 3, 0).$$

Similarly as in the previous case, $C \cap H_0^+$ equals $A_X^c \cap \lambda_1$ and H_0^0 cuts out the lineality part $A_{X,0}^c$. The intersection $C \cap H_0^+$ has no interior lattice point and inside $C \cap H_0^0$ the origin is the only lattice point.



Here the intersection $\overline{C} := C \cap H_{-1}^0$ is a quadrilateral with vertices

$$\Big(-1, \frac{l_{11}+d_{111}+1}{l_{11}+2}, \frac{d_{211}}{l_{11}+2}\Big), \qquad \Big(-1, \frac{3l_{11}+d_{111}+3}{l_{11}+2}, \frac{d_{211}}{l_{11}+2}\Big),$$

$$\Big(-1, \frac{l_{12}+d_{112}+1}{l_{12}+2}, \frac{d_{212}}{l_{12}+2}\Big), \qquad \Big(-1, \frac{3l_{12}+d_{112}+3}{l_{12}+2}, \frac{d_{212}}{l_{12}+2}\Big),$$

The first two points have a positive z-value, the other two a negative one, all in absolute value strictly smaller than one. Noting that the length of $\overline{C} \cap \{z = 0\}$ is bigger than one and smaller than two, we conclude that $\overline{C}$ contains precisely two lattice points, say q_1 and q_2. It turns out that the origin lies in $\mathrm{conv}(b_1, b_2, q_1, q_2)$ as only interior lattice point. Theorem 2.4.4 yields $d_{211}(l_{12}+1) - d_{212}(l_{11}+1) < 72$ and the assertion follows.

We turn to *(ii)*. By Lemma 2.4.16 we have $l_{12} = 1$. With admissible operations we achieve $d_{112} = d_{212} = 0$. If $l_{11} = 1$ holds, we can apply Proposition 2.4.19.

Case $l_{11} = 2$: Here u_1 and u_2 are half-integral points. Therefore we have $h(c_1) = 1/2$, which implies $d_{211} + d_{221} = 1$. The constraint $h(c_2) \leq -1$ is equivalent to $d_{221} \leq -3$. Estimates on d_{111} and d_{121} are found by positivity of the weights and admissible operations. For the lower bound on d_{221} we note that u_3 lies under the bisection of the quadrant $\mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\leq 0}$. Requiring that no lattice point lies in $A_{X,0}^c$ except for the origin only leaves a confined area to place c_2, namely $h(c_2) \geq -2$ must hold. This provides the bound $d_{221} \geq -6$.

Case $l_{11} \geq 3$: positivity of the weights and the constraints on the heights, together with suitable admissible operations, lead to all bounds for the d_{ijk} stated in (c) except for the lower bound on d_{221}. For that, observe that the segment line $A_{X,0}^c \cap \{y = -1\}$ has to be of length strictly smaller than 1 and conclude

$$\frac{-5l_{11}+2}{l_{11}-2} \ < \ d_{221}.$$

The next step is to bound l_{11}. For this, we consider the simplex $D \subseteq \mathbb{Q}^3$ given as the convex hull of following points:

$$(l_{11}, d_{111}, d_{211}), \quad (1,0,0), \quad (-2, d_{121}, d_{221}), \quad (-2, d_{121}+2, d_{221}).$$

Now, put the leaf $A_X^c \cap \lambda_1$ of the anticanonical complex into $\mathbb{Q}^3$ by removing the second coordinate (which always equals zero). With the same notation as in part (i) of the proof, we see that $D \cap H_0^+$ equals $A_X^c \cap \lambda_1$ and H_0^0 cuts out the lineality part $A_{X,0}^c$. For large $l_{11} \geq 10$, the only possible values for d_{221} are $-3, -4, -5$. Moreover we already have $0 \leq d_{121} < -d_{221}$. Thus the allowed pairs (d_{121}, d_{221}) are

$$(0,-3), \quad (1,-3), \quad (2,-3), \quad (1,-4), \quad (3,-4),$$

$$(0,-5), \quad (1,-5), \quad (2,-5), \quad (3,-5), \quad (4,-5).$$

Subcase $(0,-3)$: u_3 is a lattice point, hence X is not terminal.

Subcase $(1, -3)$: u_4 is a lattice point, hence X is not terminal.

Subcase $(2, -3)$: $A^{c}_{X,0}$ contains the lattice point $(1, -1) = (u_3 + u_4)/2$.

Subcase $(1, -4)$: here D contains exactly two lattice points in its interior, namely the origin and $(-1, 1, -2)$. By Theorem 2.4.5 we have $\mathrm{vol}(D) \leq 18$, which yields $l_{11} < 25$.

Subcase $(3, -4)$: the lattice point $(1, -1)$ lies in $\mathrm{conv}(0, u_3, u_4) \subset A^{c}_{X,0}$.

Subcase $(0, -5)$: the lattice point $(0, -1)$ lies in $\mathrm{conv}(0, u_3) \subset A^{c}_{X,0}$.

Subcase $(1, -5)$: here D contains exactly two lattice points in its interior, namely the origin and $(-1, 1, -3)$. By Theorem 2.4.5 we have $\mathrm{vol}(D) \leq 18$, which yields $l_{11} < 20$.

Subcase $(2, -5)$: here D contains exactly two lattice points in its interior, namely the origin and $(-1, 2, -3)$. By Theorem 2.4.5 we have $\mathrm{vol}(D) \leq 18$, which yields $l_{11} < 20$.

Subcase $(3, -5)$: the lattice point $(1, -1)$ lies in $\mathrm{conv}(0, u_3, u_4) \subset A^{c}_{X,0}$.

Subcase $(4, -5)$: the lattice point $(1, -1)$ lies in $\mathrm{conv}(0, u_3, u_4) \subset A^{c}_{X,0}$.

Lastly we verify *(iii)*. By Lemma 2.4.18 we have $(l_{11}, l_{12}) \in \{(1, 1), (2, 1), (2, 2)\}$. If both exponents are equal to 1, then Proposition 2.4.19 applies straightforward. If both exponents are equal to 2, then $|c_1| = |c_2| = 1$. Therefore the segment line $A^{c}_{X,0} \cap \{y = 1\}$ is of length one and hence contains at least one lattice point. Lastly we show that the case $(l_{11}, l_{12}) = (2, 1)$ is also not possible. Here $|c_1| = 1$ holds and two of the vertices are

$$u_1 \;=\; \Big(\frac{1}{2}d_{111} + \frac{1}{2}d_{121}\,,\, \frac{1}{2}d_{211} + \frac{1}{2}d_{221}\Big), \qquad u_2 \;=\; u_1 + (1, 0).$$

We assume $h(c_1)$ to be non-integral, otherwise we would have a lattice point on c_1 itself. Nonetheless an integral point p is always in the lineality part, precisely at the height $h(c_1) - 1/2$ and it can be given explicitly as $p := \alpha u_1 + \beta u_2$ where

$$\alpha \;:=\; -k - \frac{d_{111} + d_{121} + 2}{2(d_{211} + d_{221})}, \qquad \beta \;:=\; 1 + k + \frac{d_{111} + d_{121}}{2(d_{211} + d_{221})}$$

for an appropriate $k \in \mathbb{Z}_{\geq 0}$ that makes $0 \leq \alpha, \beta < 1$. Then we have

$$p \;=\; \Big(\frac{1}{2}d_{111} + \frac{1}{2}d_{121} + k + 1\,,\, h(c_1) - \frac{1}{2}\Big),$$

which is an integral point since we can always assume, via an admissible operation, that d_{111} and d_{121} have the same parity. □

Proposition 2.4.21. *Let $X = X(A, P)$ be as in Setting 2.4.7. Suppose $l_{21} \geq 3$.*

(i) If $h(c_1) < 1$ and $h(c_2) > -2$ hold, then we are in one of the following three situations:

(a) We have $3 \leq l_{21} \leq 5$ and the other l_{ij} are bounded according to the table

l_{21}	3	4	5
l_{12}	≤ 4	≤ 2	≤ 2
l_{11}	≤ 5	≤ 3	≤ 2

In this case we turn P by means of admissible operations into the shape of Remark 2.4.8. Then we have $0 \le d_{121}, d_{221} < l_{21}$ and the estimates

$$-2 - \frac{l_{12}}{l_{21}}(d_{221}+2) \ < \ d_{212} \ < \ 0, \qquad -\frac{l_{11}}{l_{21}}d_{221} \ < \ d_{211} \ < \ -\frac{l_{11}}{l_{21}}d_{221} + 1 + \frac{l_{11}}{l_{21}}$$

and the remaining two entries d_{111}, d_{112} are bounded according to Remark 2.4.8.

(b) *We have $l_{21} \ge 6$ and $l_{11} = l_{12} = 1$. Then all entries d_{ijk} can be bounded according to Proposition 2.4.19.*

(c) *We have $l_{21} \ge 6$, $l_{11} = 2$ and $l_{12} = 1$. Then we achieve $d_{112} = d_{212} = 0$ by suitable admissible operations and values and bounds for the remaining entries are given by the table*

d_{111}	0	1	1	1	1	2	2
d_{211}	1	2	3	4	5	3	5
l_{21}	≤ 45	≤ 51	≤ 33	≤ 50	≤ 33	≤ 35	≤ 101

and by the estimates

$$-2(l_{21}+1) \ < \ d_{221} \ < \ 0, \qquad \frac{d_{111}d_{221}}{d_{211}} - l_{21} \ < \ d_{121} \ < \ \frac{d_{111}d_{221}}{d_{211}}.$$

(ii) *If $h(c_1) < 1$ and $h(c_2) \le -2$ hold, then we are in one of the following two situations:*

(a) *We have $l_{11} = l_{12} = 1$. Then l_{21} and the entries d_{ijk} can be bounded according to Proposition 2.4.19.*

(b) *We have $l_{11} = 2$, $l_{12} = 1$ and $l_{21} = 3, 4$. Then we achieve $d_{112} = d_{212} = 0$ by admissible operations and obtain the following estimates*

$$-4(l_{21}+1) \ < \ d_{221} \ < \ 0, \qquad -\frac{2}{l_{21}}d_{221} \ < \ d_{211} \ < \ -\frac{2}{l_{21}}(d_{221}-1)+1,$$

$$0 \ \le \ d_{121} \ < \ -d_{221}, \qquad \frac{d_{211}(d_{121}+l_{21})}{d_{221}} \ < \ d_{111} \ < \ \frac{d_{211}d_{121}}{d_{221}}.$$

(iii) *If $h(c_1) \ge 1$ holds, then we have $l_{11} = l_{12} = 1$ and Proposition 2.4.19 applies.*

Proof. Let us verify (i) first. Lemmas 2.4.12 and 2.4.15 provide bounds on l_{11} and l_{12} in terms of l_{21}, namely those from the table of case (a) if $l_{21} < 6$, otherwise $l_{11} = 1, 2$ and $l_{12} = 1$. The other estimates of case (a) follow directly from Remarks 2.4.8, 2.4.13 and 2.4.14. From now on we have $l_{21} \ge 6$ and $l_{12} = 1$, thus we arrive at $d_{112} = d_{212} = 0$ by admissible operations. If $l_{11} = 1$ then we are in case (b) and Proposition 2.4.19 applies. If $l_{11} = 2$ then we have to prove the estimates of case (c). Writing down explicitly the inequalities $h(c_1) < 1$ and $h(c_2) > -2$, as well as the positivity of the weights, already gives bounds for d_{121} and d_{221} and following estimates

$$-\frac{2d_{221}}{l_{21}} \ < \ d_{211} \ < \ -\frac{2d_{221}}{l_{21}} + \frac{l_{21}+2}{l_{21}}, \qquad 0 \ \le \ d_{111} \ < \ d_{211}.$$

We still need to find an upper bound for l_{21}. Note that by substituting the lower estimate for d_{221} in the upper estimate for d_{211} one obtains

$$0 \ < \ d_{211} \ < \ 5 + \frac{6}{l_{21}} \ \le \ 6.$$

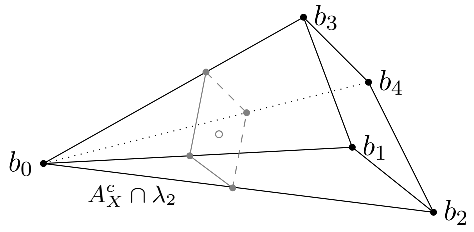
Thus we have a finite range (independent from l_{21}) for d_{211} and therefore for d_{111} too, namely

$$d_{111} \in \{0, 1, 2\}, \qquad d_{111} < d_{211} \leq 5.$$

The cases $d_{111} = 3, 4$ are discharged, because there the origin lies outside of the lineality part $A^c_{X,0}$. Moreover, if $d_{111} = 0$ holds, then $d_{211} = 1$ holds by terminality. We look at the lattice polytope C in $\mathbb{Q}^3$ given as the convex hull of the following points:

$$b_0 := (l_{21}, d_{121}, d_{221}), \quad b_3 := (-1, 0, 0), \quad b_4 := (-1, 1, 0).$$
$$b_1 := (-2, d_{111}, d_{211}), \quad b_2 := (-2, d_{111} + 2, d_{211}).$$

Now, put the leaf $A^c_X \cap \lambda_2$ of the anticanonical complex into $\mathbb{Q}^3$ by removing the first coordinate (which always equals zero) from its points.



For $a = 0, -1, -2$, consider

$$H^+_a := \{(x, y, z);\ x \geq a\} \subseteq \mathbb{Q}^3, \qquad H^0_a := \{(x, y, z);\ x = a\} \subseteq \mathbb{Q}^3.$$

Then $C \cap H^+_0$ equals $A^c_X \cap \lambda_2$ and H^0_0 cuts out the lineality part $A^c_{X,0}$. In particular, by terminality of X and Theorem 2.1.10, the intersection $C \cap H^+_0$ has no interior lattice point and inside $C \cap H^0_0$ the origin is the only lattice point. Other interior lattice points of C may only appear in $C \cap H^0_{-1}$. More than once we will use the following two simplices:

$$C_1 := \operatorname{conv}(b_0, b_1, b_2, b_3), \qquad C_2 := \operatorname{conv}(b_0, b_2, b_3, b_4).$$

They fulfill $C = C_1 \cup C_2$ and $C_1^\circ \cap C_2^\circ = \emptyset$. We go through all possible pairs (d_{111}, d_{112}).

Case $d_{111} = 0$, $d_{112} = 1$: C° contains the origin as only lattice point. We bound $\operatorname{vol}(C)$ according to Theorem 2.4.4 and arrive at $l_{21} < 46$.

Case $d_{111} = 1$, $d_{112} = 2$: define $q_1 := (-1, 1, 1)$ and $q_2 := (-1, 2, 1)$. If $d_{221} \geq d_{121} - 1$ holds, then C has precisely two interior lattice points, namely the origin and q_1. We apply Theorem 2.4.5 and obtain $l_{21} < 34$. If $d_{221} < d_{121} - 1$ holds, then C contains also q_2. We note that the origin and q_1 lie in C_1, whereas q_2 lies in C_2. Therefore Theorem 2.4.5 applies to C_1 and we conclude $l_{21} < 52$.

Case $d_{111} = 1$, $d_{112} = 3$: consider $q_1 := (-1, 1, 1)$ and $q_2 := (-1, 2, 1)$. If $d_{221} \geq d_{121} - l_{21} - 2$ holds, then C contains precisely two interior lattice points, namely the origin and q_1. We apply Theorem 2.4.5 and obtain $l_{21} < 22$. If $d_{221} < d_{121} - l_{21} - 2$ holds, then C contains in addition q_2. We note that the origin and q_1 lie in C_1, whereas q_2 lies in C_2. We apply Theorem 2.4.5 to C_1 and arrive at $l_{21} < 34$.

Case $d_{111} = 1$, $d_{112} = 4$: if $d_{221} \geq 2d_{121}+1$ holds, then C has precisely three lattice points in its interior, namely the origin, $q_1 := (-1, 1, 1)$ and $q_2 := (-1, 1, 2)$. In this situation q_1 is not contained in the simplex $\operatorname{conv}(b_0, b_1, b_2, b_3)$, whereas the two other points are. By bounding its volume according to Theorem 2.4.5 we reach $l_{21} < 25$. Conversely, if $d_{221} < 2d_{121} + 1$ holds, then C° contains the points above and also $q_3 := (-1, 2, 2)$. We note that the origin and q_2 lie in the interior of $\operatorname{conv}(b_0, b_1, b_3, q_3)$, whereas q_1 lies on its boundary. Therefore Theorem 2.4.5 applies and we conclude $l_{21} < 51$.

Case $d_{111} = 1$, $d_{112} = 5$: this situation can be settled in the exact same way as the case directly above. The only differences are as follows: the inequality that allows q_3 to be an interior point of C is $d_{221} < 2d_{121} - l_{21} - 3$. The new bounds are $l_{21} < 20$ and $l_{21} < 34$.

Case $d_{111} = 2$, $d_{112} = 3$: consider the points $q_1 := (-1, 1, 1)$ and $q_2 := (-1, 2, 1)$. If $d_{221} \geq d_{121} - 1$ holds, then C° contains precisely two lattice points, namely the origin and q_1. We apply Theorem 2.4.5 and obtain $l_{21} < 22$. If $d_{221} < d_{121} - 1$ holds, then C° contains also q_2. We see that only q_2 lies in the interior of $\operatorname{conv}(b_0, b_2, b_4, q_1)$, whereas the origin does not. Therefore Theorem 2.4.4 yields $l_{21} < 36$.

Case $d_{111} = 2$, $d_{112} = 5$: here the polytope C contains three lattice points other than the origin, namely

$$q_1 := (-1, 1, 1), \qquad q_2 := (-1, 1, 2), \qquad q_3 := (-1, 2, 2).$$

The simplex $\operatorname{conv}(b_0, b_1, b_3, q_3)$ contains the origin and q_2 in its interior, whereas q_1 lies on its boundary. We apply Theorem 2.4.5 and arrive at $l_{21} < 102$.

Now we prove (*ii*). By Lemma 2.4.16 we have l_{12}=1, therefore we can always achieve $d_{112} = d_{212} = 0$ by admissible operations. The same Lemma gives us $l_{11} = 1$ if $l_{21} \geq 5$ or if $h(c_2) \leq -4$. This case is covered by Proposition 2.4.19. Let us therefore assume $l_{21} \in \{3, 4\}$ and $h(c_2) > -4$, together with $l_{11} > 1$. Then Lemma 2.4.16 implies $l_{11} = 2$. Moreover Remark 2.4.17 provides estimates on d_{211} and d_{221} in terms of l_{21}. The last bounds on d_{111} and d_{121} are obtained as in Remark 2.4.8.

Lastly we turn to (*iii*). We have $l_{21} \geq 3$ and $h(c_1) \geq 1$. Then by Lemma 2.4.18 we must have $l_{11} = l_{12} = 1$. Hence Proposition 2.4.19 applies. □

Case (ii) of Lemma 2.4.1

Lemma 2.4.22. *Let $X = X(A, P)$ be a non-toric terminal $\mathbb{Q}$-factorial Fano threefold with $\rho(X) = 1$ such that P is irredundant and we have $r = 3$, $m = 0$ and $\overline{n} = (2, 2, 1, 1)$. Then $l_{01} = l_{02} = 1$ and $l_{11} = l_{12} = 1$ hold.*

Proof. Since P is irredundant, we have $l_{21}, l_{31} \geq 2$. Since $\rho(X) = 1$, every set of rays of the type $(v_{0k}, v_{1j}, v_{21}, v_{31})$, for $k, j \in \{1, 2\}$, defines an elementary big cone. Therefore Proposition 2.3.1 applies to all of them. In particular $l_{ij} = 1$ for all $i = 0, 1$ and $j = 1, 2$. □

Setting 2.4.23. Let $X = X(A, P)$ be a non-toric terminal $\mathbb{Q}$-factorial Fano threefold with $\rho(X) = 1$ such that P is irredundant and we have $r = 3$, $m = 0$ and $\overline{n} = (2, 2, 1, 1)$.

As a consequence of Lemma 2.4.22 we have $l_{01} = l_{02} = l_{11} = l_{12} = 1$ and, after suitable admissible operations, the matrix P is of the form

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{21} & 0 \\ -1 & -1 & 0 & 0 & 0 & l_{31} \\ 0 & 1 & d_{111} & 0 & d_{121} & d_{131} \\ 0 & 0 & d_{211} & 0 & d_{221} & d_{231} \end{bmatrix},$$

where $l_{21} \geq l_{31} \geq 2$ holds. Moreover, denoting by P_{ij} the matrix obtained by removing the column v_{ij} from P, we have positive *weights*

$$\begin{aligned} w_{01} &:= \det(P_{01}), & w_{11} &:= \det(P_{11}), & w_{21} &:= \det(P_{21}), \\ w_{02} &:= -\det(P_{02}), & w_{12} &:= -\det(P_{12}), & w_{31} &:= -\det(P_{31}). \end{aligned}$$

In particular some weights have a compact form, such as $w_{11} = -l_{31}d_{221} - l_{21}d_{231}$, as well as $w_{21} = l_{31}d_{211}$ and $w_{31} = l_{21}d_{211}$. Then we have forms

$$w_{02} = -d_{111}w_{11} - d_{121}w_{21} - d_{131}w_{31},$$
$$w_{01} = -w_{02} + l_{21}l_{31}d_{211}, \qquad w_{12} = -w_{11} + l_{21}l_{31}d_{211}.$$

Lemma 2.4.24. *Let $X = X(A, P)$ be as in Setting 2.4.23. The vertices of $A^{c}_{X,0}$, regarded as a subset of the lineality space $\mathbb{Q}^2$ of the tropical variety, are*

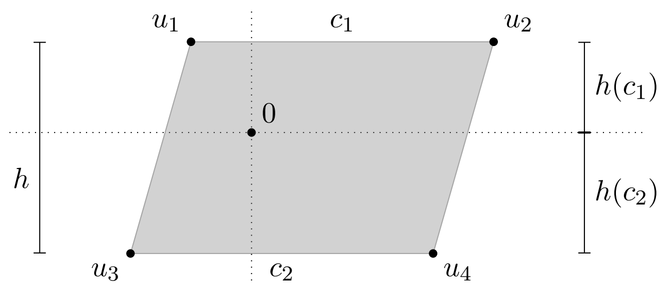
$$\begin{aligned} u_1 &:= \left[\frac{l_{21}d_{131} + l_{31}d_{121}}{l_{21} + l_{31}}, \frac{l_{21}d_{231} + l_{31}d_{221}}{l_{21} + l_{31}}\right], \\ u_2 &:= \left[\frac{l_{21}d_{111} + l_{11}d_{121} + l_{21}l_{31}}{l_{21} + l_{31}}, \frac{l_{21}d_{231} + l_{31}d_{221}}{l_{21} + l_{31}}\right], \\ u_3 &:= \left[\frac{l_{21}d_{131} + l_{31}d_{121} + l_{21}l_{31}d_{111}}{l_{21} + l_{31}}, \frac{l_{21}d_{231} + l_{31}d_{221} + l_{21}l_{31}d_{211}}{l_{21} + l_{31}}\right], \\ u_4 &:= \left[\frac{l_{21}d_{131} + l_{31}d_{121} + l_{21}l_{31}d_{111} + l_{21}l_{31}}{l_{21} + l_{31}}, \frac{l_{21}d_{231} + l_{31}d_{221} + l_{21}l_{31}d_{211}}{l_{21} + l_{31}}\right]. \end{aligned}$$

Proof. We just compute the lineality part $A^{c}_{X,0}$ according to Corollary 2.2.11. □

Remark 2.4.25. Observe that $A^{c}_{X,0}$ as described in Lemma 2.4.24 is a parallelogram. The edges $c_1 := \overline{u_1u_2}$ and $c_2 := \overline{u_3u_4}$ are parallel to the x-axis and the remaining two edges are $\overline{u_1u_3}$ and $\overline{u_2u_4}$. Length and y-value $h(c_i)$ of the line segments c_i are

$$|c_1| = |c_2| = \frac{l_{21}l_{31}}{l_{21} + l_{31}}, \qquad h(c_1) = -\frac{w_{11}}{l_{21} + l_{31}}, \qquad h(c_2) = \frac{w_{12}}{l_{21} + l_{31}}.$$

Note that the length of the segments c_1 and c_2 is at least 1, since we assumed $l_{21} \geq l_{31} \geq 2$.



In the following we denote by p_i the i-th row of P.

Lemma 2.4.26. *Let $X = X(A, P)$ be as in Setting 2.4.23. All d-values of the matrix P can be bounded in dependence on the l-values simply by taking into account the positivity of the weights, some admissible operations and the previous Remark. The explicit estimates are*

$$\begin{aligned} 0 \leq d_{131} &< l_{31} \\ 0 \leq d_{231} &< l_{31} \\ -\frac{l_{21}}{l_{31}}(d_{231}+1) - 1 < d_{221} &< -\frac{l_{21}}{l_{31}} d_{231} \\ -\frac{d_{221}}{l_{21}} - \frac{d_{231}}{l_{31}} < d_{211} &< -\frac{d_{221}}{l_{21}} - \frac{d_{231}}{l_{31}} + \frac{l_{21}+l_{31}}{l_{21}l_{31}}, \\ 0 \ \leq \ d_{111} &< \ d_{211}l_{31}, \\ \kappa - l_{21} \ < \ d_{121} &< \ \kappa, \end{aligned}$$

where

$$\kappa := \frac{d_{111}(l_{21}d_{231} + l_{31}d_{221}) - l_{21}d_{211}d_{131}}{l_{31}d_{211}}.$$

Proof. First we can bound d_{131} and d_{231} by adding a suitable multiple of $p_2 - p_3$ to p_4 and p_5. The combination $d_{231}(p_2 - p_3) + l_{31}p_5$, added suitably to p_4, bounds d_{111}. Note that this operation does not alter d_{131} nor d_{231}. The weights w_{01} and w_{02} give estimates for d_{121}. The other weights provide a lower bound for d_{211} and an upper bound for d_{221}. The missing bounds for these two last d-values are obtained through $h(c_1) > -1$ and $h(c_2) < 1$. These inequalities hold since the width of the parallelogram $A^c_{X,0}$ is at least 1. Therefore $A^c_{X,0}$ must lie in the stripe $-1 < y < 1$. □

Lemma 2.4.27. *Let $X = X(A, P)$ be as in Setting 2.4.23. Suppose $l_{31} > 2$. Then $l_{31} = 3$ and $l_{21} = 3, 4, 5$.*

Proof. This is a direct consequence of the shape of $A^c_{X,0}$. Its width, calculated in Remark 2.4.25, must remain smaller than 2 in order to avoid lattice points other than the origin on its intersection with the x-axis. □

Proposition 2.4.28. *Let $X = X(A, P)$ be as in Setting 2.4.23. Then $l_{31} = 2, 3$ holds and we are in one of the following situations:*

(i) We have $l_{31} = 2$. Then $d_{211} = 1$ holds and we can achieve $d_{111} = 0$ by a suitable admissible operation. The other entries of P are bounded according to the table

d_{131}	0	1	1
d_{231}	1	0	1
l_{21}	≤ 33	≤ 24	≤ 11

and the estimates

$$-\frac{l_{21}}{2}(d_{231}+1)-1 \ < \ d_{221} \ < \ -\frac{l_{21}}{2}d_{231},$$
$$-\frac{l_{21}}{2}d_{131}-l_{21} \ < \ d_{121} \ < \ -\frac{l_{21}}{2}d_{131}.$$

(ii) We have $l_{31} = 3$. Then $3 \leq l_{21} \leq 5$ holds, we obtain $0 \leq d_{131}, d_{231} < 3$ and we have the estimates

$$-\frac{l_{21}}{3}(d_{231}+1)-1 \ < \ d_{221} \ < \ -\frac{l_{21}}{3}d_{231},$$
$$-\frac{d_{221}}{l_{21}}-\frac{d_{231}}{3} \ < \ d_{211} \ < \ -\frac{d_{221}}{l_{21}}-\frac{d_{231}}{3}+\frac{3+l_{21}}{3l_{21}},$$
$$0 \ \leq \ d_{111} \ < \ d_{211}l_{31},$$
$$\kappa - l_{21} \ < \ d_{121} \ < \ \kappa,$$

where

$$\kappa := \frac{d_{111}(l_{21}d_{231}+3d_{221})-l_{21}d_{211}d_{131}}{3d_{211}}.$$

Proof. By Lemma 2.4.27 we have $l_{31} = 2$ or $l_{31} = 3$. In the latter case the same Lemma bounds l_{21} as well, and Lemma 2.4.26 provides constraints for all d-values.
From now on let $l_{31} = 2$, so $l_{21} \geq 2$ holds. Note that all bounds from Lemma 2.4.26 already hold, so we only need an upper bound for l_{21}. The total height h of $A_{X,0}$ is

$$h := h(c_2) - h(c_1) = \frac{2l_{21}d_{211}}{l_{21}+2}.$$

Terminality prevents non-zero lattice points from lying in $A^c_{X,0}$. In particular $h < 2$ holds. Thus $d_{211} = 1$. According to Lemma 2.4.26, $d_{111} \in \{0, 1\}$. Since we can subtract p_5 from p_4 without altering the other bounds, we may assume $d_{111} = 0$. Define the lattice polytope B in $\mathbb{Q}^3$ as the convex hull of the following points:

$$b_0 := (l_{21}, d_{121}, d_{221}),$$
$$b_1 := (-2, d_{131}, d_{231}), \qquad b_2 := (-2, d_{131}, d_{231}+2),$$
$$b_3 := (-2, d_{131}+2, d_{231}), \qquad b_4 := (-2, d_{131}+2, d_{231}+2).$$

Consider the leaf $A^c_X \cap \lambda_2$ of the anticanonical complex as a subset of $\mathbb{Q}^3$ by ignoring the first and third coordinates, which always equal zero. For $a = 0, -1, -2$, consider

$$H_a^+ := \{(x,y,z);\ x \geq a\} \ \subseteq \ \mathbb{Q}^3, \qquad H_a^0 := \{(x,y,z);\ x = a\} \ \subseteq \ \mathbb{Q}^3.$$

Then $B \cap H_0^+$ equals $A_X^c \cap \lambda_2$ and H_0^0 cuts out the lineality part $A_{X,0}^c$. In particular, by terminality of X and Theorem 2.1.10, the intersection $B \cap H_0^+$ has no interior lattice point and inside $B \cap H_0^0$ the origin is the only lattice point. According to Lemma 2.4.26 we have $d_{131}, d_{231} \in \{0, 1\}$, where at least one of the two values equals 1, since v_{31} is a primitive point by construction. Thus we have three distinct cases to analyze. The goal is to find a lattice polytope containing exactly one or two interior lattice points and bound its volume using Theorem 2.4.4 or Theorem 2.4.5 respectively.

Case $d_{131} = 0$, $d_{231} = 1$: the polytope B has exactly two interior lattice point in H_{-1}^0, namely $q_0 := (-1, 0, 1)$ and $q_1 := (-1, 1, 1)$. Consider the simplex $C := \mathrm{conv}(b_0, b_1, b_2, q_1)$. The point q_0 lies in the interior of C, whereas the origin does not. Hence we can bound $\mathrm{vol}(C) = (l_{21} - d_{121} + 2)/3$ with Theorem 2.4.4 and obtain $l_{21} < 34$.

Case $d_{131} = 1$, $d_{231} = 0$: B has two interior lattice points in H_{-1}^0, namely $q_0 := (-1, 0, 1)$ and $q_1 := (-1, 1, 1)$. We define two simplices C_1, C_2 as follows:

$$C_1 \;:=\; \mathrm{conv}(b_0, b_1, b_2, b_3), \qquad C_2 \;:=\; \mathrm{conv}(b_0, b_1, b_3, q_4).$$

These are not disjoint nor cover the whole polytope B. The point p_1 lies in the interior of the intersection $B_1 \cap B_2$, while p_2 does not. Hence in at least one of the C_i lie at most two lattice interior points and the bound is obtained by applying Theorem 2.4.5 on $\mathrm{vol}(C_i) = (2l_{21} + 4)/3$, which results in $l_{21} < 25$.

Case $d_{131} = 1$, $d_{231} = 1$: here the only lattice point in $B^\circ \cap H_{-1}^0$ is $q := (-1, 1, 1)$. In particular, B is a lattice polytope containing exactly two lattice points in its interior, namely the origin and q. We apply Theorem 2.4.5 to $\mathrm{vol}(B) = (4l_{21} + 8)/3$ and obtain the condition $l_{21} < 12$. □

Case (iii) of Lemma 2.4.1

Lemma 2.4.29. *Let $X = X(A, P)$ be a non-toric terminal $\mathbb{Q}$-factorial Fano threefold with $\rho(X) = 1$ such that P is irredundant and we have $r = 2$, $m = 0$ and $\overline{n} = (3, 1, 1)$. Then $l_{01} = l_{02} = l_{03} = 1$ holds.*

Proof. Since P is irredundant, we have $l_{11}, l_{21} \geq 2$. Since $\rho(X) = 1$ holds, the elementary big cones are precisely those of the form $\mathrm{cone}(v_{0j}, v_{11}, v_{21})$, for $j = 1, 2, 3$. Therefore Proposition 2.3.1 applies to all of them. In particular $l_{0j} = 1$ for all $j = 1, 2, 3$. □

Setting 2.4.30. Let $X = X(A, P)$ be a non-toric terminal $\mathbb{Q}$-factorial Fano threefold with $\rho(X) = 1$ such that P is irredundant and we have $r = 2$, $m = 0$ and $\overline{n} = (3, 1, 1)$. By Lemma 2.4.29 we have $l_{01} = l_{02} = l_{03} = 1$ and, after suitable admissible operations, the matrix P is of the form

$$P \;=\; \begin{bmatrix} -1 & -1 & -1 & l_{11} & 0 \\ -1 & -1 & -1 & 0 & l_{21} \\ 0 & 1 & 0 & d_{111} & d_{121} \\ 0 & 0 & 1 & d_{211} & d_{221} \end{bmatrix},$$

where $l_{11} \geq l_{21} \geq 2$ holds. Moreover, denoting by P_{ij} the matrix obtained by removing the column v_{ij} from P, we have positive *weights*

$$w_{01} := \det(P_{01}), \quad w_{02} := -\det(P_{02}), \quad w_{03} := \det(P_{03}),$$
$$w_{11} := -\det(P_{11}), \qquad w_{21} := \det(P_{21}).$$

We can write the weights explicitly as $w_{11} = l_{21}$, $w_{21} = l_{11}$ and $w_{01} = l_{11}l_{21} - w_{02} - w_{03}$, where

$$w_{02} = -l_{21}d_{111} - l_{11}d_{121}, \quad w_{03} = -l_{21}d_{211} - l_{11}d_{221}.$$

Lemma 2.4.31. *Let $X = X(A, P)$ be as in Setting 2.4.30. The vertices of $A^{c}_{X,0}$, regarded as a subset of the lineality space $\mathbb{Q}^2$ of the tropical variety, are*

$$\begin{aligned} u_1 &:= \left[\frac{l_{11}d_{121} + l_{21}d_{111}}{l_{11} + l_{21}}, \frac{l_{11}d_{221} + l_{21}d_{211}}{l_{11} + l_{21}}\right], \\ u_2 &:= \left[\frac{l_{11}d_{121} + l_{21}d_{111} + l_{11}l_{21}}{l_{11} + l_{21}}, \frac{l_{11}d_{221} + l_{21}d_{211}}{l_{11} + l_{21}}\right], \\ u_3 &:= \left[\frac{l_{11}d_{121} + l_{21}d_{111}}{l_{11} + l_{21}}, \frac{l_{11}d_{221} + l_{21}d_{211} + l_{11}l_{21}}{l_{11} + l_{21}}\right]. \end{aligned}$$

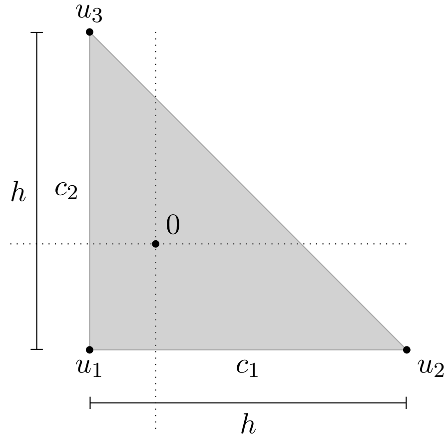
Proof. We just compute the lineality part $A^{c}_{X,0}$ according to Corollary 2.2.11. □

Remark 2.4.32. Observe that $A^{c}_{X,0}$ as described in Lemma 2.4.31 is an isosceles right triangle. The edge $c_1 := \overline{u_1u_2}$ is parallel to the x-axis whereas $c_2 := \overline{u_1u_3}$ is parallel to the y-axis. They have the same length h, i.e.,

$$h \;:=\; |c_1| \;=\; |c_2| \;=\; \frac{l_{11}l_{21}}{l_{11} + l_{21}},$$

Note that $h \geq 1$ holds, since we assumed $l_{11} \geq l_{21} \geq 2$. Moreover we can write u_1 in terms of the weights as

$$u_1 = \left(\frac{-w_{02}}{l_{11} + l_{21}}, \frac{-w_{03}}{l_{11} + l_{21}}\right).$$



Lemma 2.4.33. *Let $X = X(A, P)$ be as in Setting 2.4.30. All d-values of the matrix P can be bounded in dependence to the l-values simply by taking into account the positivity of the weights, some admissible operations and Remark 2.4.32. The explicit estimates are*

$$
\begin{aligned}
0 \leq d_{121} &< l_{21} \\
0 \leq d_{221} &< l_{21} \\
-\frac{l_{11}}{l_{21}} d_{121} - l_{11} < d_{111} &< -\frac{l_{11}}{l_{21}} d_{121}, \\
-\frac{l_{11}}{l_{21}} d_{221} - l_{11} < d_{211} &< -\frac{l_{11}}{l_{21}} d_{221}.
\end{aligned}
$$

Proof. The entries d_{121} and d_{221} are bounded by suitably adding $p_2 - p_1$ to p_3 and p_4 respectively. In this way the shape obtained in Setting 2.4.30 is not altered. Positivity of the weights w_{02} and w_{03} gives the upper bounds for the other two d-values. The corresponding lower bounds are obtained by noting that the first coordinate of u_2 and the second coordinate of u_3 must be positive to ensure that the origin lies in the interior of $A^c_{X,0}$. □

Lemma 2.4.34. *Let $X = X(A, P)$ be as in Setting 2.4.30. Assume that $l_{21} > 3$ holds. Then $l_{21} = 4, 5$ and $l_{11} < 3l_{21}/(l_{21} - 3)$.*

Proof. This is a direct consequence of $h < 3$, a necessary condition to avoid non-zero lattice points in $A^c_{X,0}$. □

Proposition 2.4.35. *Let $X = X(A, P)$ as in Setting 2.4.30. Then we have $2 \leq l_{21} \leq 5$ and we are left with the following situations:*

(i) *We have $l_{21} = 2$. Then we achieve $d_{121} = 1$ by suitable admissible operations and the remaining entries of P are bounded as follows:*

$$
l_{11} \ \leq \ 24, \qquad d_{221} \ \in \ \{0, 1\},
$$

$$
-\frac{3l_{11}}{2} < d_{111} < -\frac{l_{11}}{2}, \qquad -\frac{l_{11}}{2} d_{221} - l_{11} < d_{211} < -\frac{l_{11}}{2} d_{221}.
$$

(ii) *We have $l_{21} = 3$. Then we achieve $0 \leq d_{121} \leq d_{221} < 3$ by suitable admissible operations, the value l_{11} is bounded according to the table*

d_{121}	0	0	1	1	2
d_{221}	1	2	1	2	2
l_{11}	≤ 71	≤ 50	≤ 32	≤ 68	≤ 50

and for the remaining entries we obtain the conditions

$$
\begin{aligned}
-\frac{l_{11}}{3}(d_{121} + 1) - 1 \ &< \ d_{111} \ < \ -\frac{l_{11}}{3} d_{121}, \\
-\frac{l_{11}}{3}(d_{221} + 1) - 1 \ &< \ d_{211} \ < \ -\frac{l_{11}}{3} d_{221}.
\end{aligned}
$$

(iii) We have $l_{21} = 4$ *or* $l_{21} = 5$. *Then Lemma 2.4.34 bounds* l_{11} *and Lemma 2.4.33 provides estimates for all* d*-values.*

Proof. By Lemma 2.4.34 the inequality $l_{21} \leq 5$ holds. In particular, for $l_{21} > 3$, the value of l_{11} is bounded as well, and so are the d-values with Lemma 2.4.33.

Case $l_{21} = 2$. With Lemma 2.4.33 we have $d_{121}, d_{221} \in \{0, 1\}$. Since v_{21} is a primitive lattice point, at least one of these values must be equal to one. Without loss of generality we assume $d_{121} = 1$. Consider the leaf $A_X^c \cap \lambda_1$ of the anticanonical complex as a subset of $\mathbb{Q}^3$ by ignoring the second coordinate, which always equals zero. For $a = 0, -1, -2$, consider

$$H_a^+ := \{(x, y, z);\ x \geq a\} \subseteq \mathbb{Q}^3, \qquad H_a^0 := \{(x, y, z);\ x = a\} \subseteq \mathbb{Q}^3.$$

Moreover define B as the convex hull of the following points:

$$b_0 := (l_{11}, d_{111}, d_{211}), \qquad b_1 := (-2, 1, d_{221}),$$
$$b_2 := (-2, 3, d_{221}), \qquad b_3 := (-2, 3, d_{221} + 2).$$

Then $B \cap H_0^+$ equals $A_X^c \cap \lambda_1$ and H_0^0 cuts out the lineality part $A_{X,0}^c$. In particular, by terminality of X and Theorem 2.1.10, the intersection $B \cap H_0^+$ has no interior lattice point. The simplex B contains exactly two lattice points in its interior, namely the origin in H_0^0 and $(-1, 1, d_{221})$ in H_{-1}^0. Therefore we may apply Theorem 2.4.5 on the volume of B, which is given by $\mathrm{vol}(B) = (2l_{11} + 4)/3$, and arrive at $l_{11} < 25$. Lastly, with Lemma 2.4.33 we bound the entries d_{111} and d_{211}.

Case $l_{21} = 3$. According to Lemma 2.4.33 we already bound all d-values. Moreover we can assume $d_{121} \leq d_{221}$, by swapping p_3 with p_4 as well as v_{02} with v_{03}. Hence there are five combinations for the couple (d_{121}, d_{221}). Each one of them will deliver a specific upper bound for l_{11}. To see that, define the lattice polytope B in $\mathbb{Q}^3$ as the convex hull of the following points:

$$b_0 := (l_{11}, d_{111}, d_{211}), \qquad b_1 := (-3, d_{121}, d_{221}),$$
$$b_2 := (-3, d_{121} + 3, d_{221}), \qquad b_3 := (-3, d_{121}, d_{221} + 3).$$

Consider the leaf $A_X^c \cap \lambda_1$ of the anticanonical complex as a subset of $\mathbb{Q}^3$ by ignoring the second coordinate, which always equals zero. Consider

$$H_0^+ := \{(x, y, z);\ x \geq 0\} \subseteq \mathbb{Q}^3, \qquad H_0^0 := \{(x, y, z);\ x = 0\} \subseteq \mathbb{Q}^3.$$

Then $B \cap H_0^+$ equals $A_X^c \cap \lambda_1$ and H_0^0 cuts out the lineality part $A_{X,0}^c$. In particular, by terminality of X and Theorem 2.1.10, the intersection $B \cap H_0^+$ has no interior lattice point and inside $B \cap H_0^0$ the origin is the only lattice point.

Subcase $d_{121} = 0$ *and* $d_{221} = 1$. Here the number of interior points of B varies. If $l_{11} \geq -2d_{211}$ holds, then B contains four integral points other than the origin. The origin appears as only interior point in one of the two following simplices: $\mathrm{conv}(b_0, b_1, b_2, (-1, 0, 1))$

or $\operatorname{conv}(b_0, b_2, b_3, (-1, 0, 1))$. Bounding their volumes according to Theorem 2.4.4 we obtain $l_{11} < 33$ and $l_{11} < 27$ respectively. On the contrary, if $l_{11} < -2d_{211}$ holds, then B has six interior lattice points and the origin is the only lattice point in the interior of the simplex

$$\operatorname{conv}(b_0, (-1, 0, 0), (-1, 1, 0), (-1, 0, 1)).$$

For this situation we obtain $l_{11} < 72$.

Subcase $d_{121} = 0$ *and* $d_{221} = 2$. For large $l_{11} \geq 16$, the polytope B contains six lattice points other than the origin in its interior:

$$q_1 := (-1, 0, 1), \qquad q_2 := (-1, 1, 1), \qquad q_3 := (-1, 0, 2),$$
$$q_4 := (-2, 0, 2), \qquad q_5 := (-2, 1, 2), \qquad q_6 := (-2, 0, 3),$$

The origin is the only lattice point that lies in the interior of $C := \operatorname{conv}(b_0, b_1, b_2, q_1, q_2)$. We bound $\operatorname{vol}(C) = (2 - 4d_{211} - 2l_{11})/3$ according to Theorem 2.4.4 and obtain $l_{11} < 51$.

Subcase $d_{121} = 1$ *and* $d_{221} = 1$. First note that in this subcase we can assume $d_{111} \leq d_{211}$ by admissible operations. If $l_{11} \geq -2d_{111}$ holds, then B° contains the following integral points:

$$q_1 := (-2, 1, 1), \quad q_2 := (-2, 2, 1), \quad q_3 := (-2, 1, 2), \quad q_4 := (-1, 1, 1).$$

Only the origin lies in $\operatorname{conv}(b_0, b_1, b_2, q_4)$ and bounding the volume of this simplex by Theorem 2.4.4 we obtain $l_{11} < 33$. If $-2d_{111} > l_{11} \geq -2d_{211}$ holds, then two additional points are in B:

$$q_5 := (-1, 0, 1), \qquad q_6 := (-1, 0, 2).$$

The origin is the only interior lattice point of $\operatorname{conv}(b_0, b_1, b_2, q_4, q_5)$. Bounding its volume by Theorem 2.4.4 we arrive at $l_{11} < 24$. Lastly, if $l_{11} < -2d_{211}$, three additional point lie in the interior of B, namely:

$$q_7 := (-1, 0, 0), \qquad q_8 := (-1, 1, 0), \qquad q_9 := (-1, 2, 0).$$

Here the origin lies, as only interior lattice point, in the simplex $\operatorname{conv}(b_0, q_6, q_7, q_9)$ and we arrive at $l_{11} < 18$. Considering the worst case we conclude $l_{11} < 33$.

Subcase $d_{121} = 1$ *and* $d_{221} = 2$. If $l_{11} \leq -2d_{111} - 1$ holds, then B° contains only the following four non-zero lattice points:

$$q_1 := (-2, 1, 2), \qquad q_2 := (-2, 2, 2),$$
$$q_3 := (-2, 1, 3), \qquad q_4 := (-1, 1, 1).$$

The origin lies, as only interior lattice point, in one of the two following simplices:

$$\operatorname{conv}(b_0, q_4, b_1, b_2), \qquad \operatorname{conv}(b_0, q_4, b_1, b_3).$$

By bounding their volume accoring to Theorem 2.4.4 we arrive at $l_{11} < 69$ and $l_{11} < 33$ respectively. If $l_{11} > -2d_{111} - 1$ holds, then B° contains two new lattice points in addition to the ones given above, namely:

$$q_5 := (-1, 0, 1), \qquad q_6 := (-1, 0, 2).$$

The origin lies in the polytope $\mathrm{conv}(b_0, b_1, b_2, q_4, q_5)$ and is the only lattice point in its interior, hence we obtain $l_{11} < 51$ by Theorem 2.4.4. By comparing all inequalities we conclude $l_{11} < 69$.

Subcase $d_{121} = 2$ and $d_{221} = 2$. The polytope B has the origin and two other integral points in its interior, namely $q_1 := (-1, 1, 1)$ and $q_2 := (-2, 2, 2)$. We define simplices $C_i := \mathrm{conv}(q_2, b_j; j \neq i)$ for $i = 1, 2, 3$. Note that q_1, q_2 and the origin all lie on a common line. The point b_0 does not lie on the same line, because otherwise $l_{11} = 1$ holds but P is irredundant. Hence one of the C_i contains two interior lattice points and its volume can be bounded according to Theorem 2.4.5. We have

$$\mathrm{vol}(C_1) = \frac{1}{2}d_{111} + \frac{1}{2}d_{211} + \frac{3}{2}l_{11} + 1, \qquad \mathrm{vol}(C_2) = \mathrm{vol}(C_3) = -\frac{1}{2}d_{211} + 1.$$

The worst bound is obtained by C_2, C_3 and delivers $l_{11} < 51$. □

Case (iv) of Lemma 2.4.1

Lemma 2.4.36. *Let $X = X(A, P)$ be a non-toric terminal $\mathbb{Q}$-factorial Fano threefold with $\rho(X) = 1$ such that P is irredundant and we have $r = 2$, $m = 1$ and $\overline{n} = (2, 1, 1)$. Then $l_{01} = l_{02} = 1$ hold.*

Proof. Since P is irredundant, we have $l_{11}, l_{21} \geq 2$. Moreover $\rho(X) = 1$ holds, hence $\mathrm{cone}(v_{0j}, v_{11}, v_{21})$, for $j = 1, 2$, are precisely the elementary big cones. Therefore Proposition 2.3.1 applies and $l_{01} = l_{02} = 1$ holds. □

Setting 2.4.37. Let $X = X(A, P)$ be a non-toric terminal $\mathbb{Q}$-factorial Fano threefold with $\rho(X) = 1$ such that P is irredundant and we have $r = 2$, $m = 1$ and $\overline{n} = (2, 1, 1)$. By Lemma 2.4.36 we have $l_{01} = l_{02} = 1$ and after suitable admissible operations the matrix P is of the form

$$P = \begin{bmatrix} -1 & -1 & l_{11} & 0 & 0 \\ -1 & -1 & 0 & l_{21} & 0 \\ 0 & 1 & d_{111} & d_{121} & d'_{11} \\ 0 & 0 & d_{211} & d_{221} & d'_{21} \end{bmatrix},$$

where the following relations hold

$$2 \leq l_{21} \leq l_{11}, \qquad 0 \leq d_{121}, d_{221} < l_{21},$$

$$0 \leq d'_{11} < d'_{21}, \qquad \gcd(d'_{11}, d'_{21}) = 1.$$

Moreover, denoting by P_α the matrix obtained by removing the column v_α from P, we have positive *weights*

$$w_{01} := \det(P_{01}), \qquad w_{02} := -\det(P_{02}),$$

$$w_{11} := \det(P_{11}), \qquad w_{21} := -\det(P_{21}), \qquad w_1 := \det(P_1).$$

Lemma 2.4.38. *Let $X = X(A,P)$ be as in Setting 2.4.37. The vertices of $A^c_{X,0}$, regarded as a subset of the lineality space $\mathbb{Q}^2$ of the tropical variety, are*

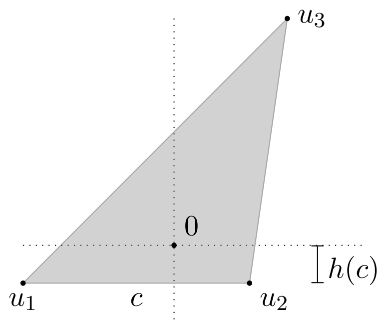
$$\begin{aligned} u_1 &:= \left[\frac{l_{11}d_{121}+l_{21}d_{111}}{l_{11}+l_{21}},\ \frac{l_{11}d_{221}+l_{21}d_{211}}{l_{11}+l_{21}}\right], \\ u_2 &:= \left[\frac{l_{11}d_{121}+l_{21}d_{111}+l_{11}l_{21}}{l_{11}+l_{21}},\ \frac{l_{11}d_{221}+l_{21}d_{211}}{l_{11}+l_{21}}\right], \\ u_3 &:= \left[d'_{11},\ d'_{21}\right]. \end{aligned}$$

Proof. We just compute the lineality part $A^c_{X,0}$ according to Corollary 2.2.11. □

Remark 2.4.39. Observe that $A^c_{X,0}$ as described in Lemma 2.4.38 is a triangle. The edge $c := \overline{u_1u_2}$ is parallel to the x-axis, lies underneath it and has length and height

$$|c| = \frac{l_{11}l_{21}}{l_{11}+l_{21}} \geq 1, \qquad h(c) = \frac{-w_1}{l_{11}+l_{21}}.$$

The third vertex u_3 is a lattice point that lies in the first orthant, above the bisection.



Remark 2.4.40. Let $X = X(A,P)$ be as in Setting 2.4.37. Positivity of w_{02} and w_1 deliver respectively

$$d_{111} < -\frac{l_{11}}{l_{21}}d_{121}, \qquad d_{211} < -\frac{l_{11}}{l_{21}}d_{221}.$$

Moreover, positivity of w_{01} yields

$$d_{111} > -\frac{l_{11}}{l_{21}}d_{121} - l_{11} + \frac{d'_{11}}{d'_{21}}\left(d_{211} + \frac{l_{11}}{l_{21}}d_{221}\right).$$

Remark 2.4.41. Let $X = X(A,P)$ be as in Setting 2.4.37. The condition $h(c) > -t$ for some given $t \in \mathbb{Q}_{>0}$ is equivalent to

$$d_{211} \;>\; -\frac{l_{11}}{l_{21}}(d_{221}+t) - t.$$

Lemma 2.4.42. *Let $X = X(A,P)$ be as in Setting 2.4.37. Then the following holds:*

$$\frac{l_{11}l_{21}}{l_{11}+l_{21}} \;<\; 4.$$

In particular we have $l_{21} \leq 7$ and, for $l_{21} \geq 5$, we bound l_{11} via

$$l_{11} \;<\; \frac{4l_{21}}{l_{21}-4}.$$

Proof. If $h(c) > -1$ holds, the assertion follows since the length of $A^{c}_{X,0} \cap \{y = 0\}$ is smaller than 2. If $h(c) \leq -1$ holds, then the assertion follows from the more restrictive fact that at height $\lceil h(c) \rceil$ the intersection of the lineality part is shorter than 1. □

Remark 2.4.43. Let $X = X(A,P)$ be as in Setting 2.4.37. If $h(c) > -1$ and $|c| > 1$, then terminality prevents integral points from lying in the intersection of the lineality part with $\{y = 1\}$. This implies

$$d'_{21} \;<\; \frac{l_{11}l_{21}+l_{11}+l_{21}}{l_{11}l_{21}-l_{11}-l_{21}} \;\leq\; 11.$$

Proposition 2.4.44. *Let $X = X(A,P)$ be as in Setting 2.4.37. Assume $d'_{21} > 1$ and $h(c) \leq -1$. Then $l_{11} = 3$, $l_{21} = 2$, $0 \leq d_{121}, d_{221} < 2$, $(d'_{11}, d'_{21}) \in \{(1,2),(1,3),(2,3)\}$ and we have following estimates*

$$-\frac{3}{2}d_{221} + \frac{5}{2}(d'_{21} - 6) \;<\; d_{211} \;<\; -\frac{3}{2}d_{221},$$
$$-\frac{3}{2}d_{121} - 3 + \frac{d'_{11}}{d'_{21}}\Big(d_{211} + \frac{3}{2}d_{221}\Big) \;<\; d_{111} \;<\; -\frac{3}{2}d_{121}.$$

Proof. First note that $h(c) < -1$ holds, since $|c| \geq 1$. The lineality part $A^{c}_{X,0}$ has a non-empty intersection with $\{y = 2\}$ since $d'_{21} \geq 2$. At height $\{y = -1\}$ it has a width smaller than one, by terminality. Since $h(c) < -1$ holds, we conclude

$$\frac{l_{11}l_{21}}{l_{11}+l_{21}} \;<\; \frac{4}{3}.$$

This implies already $l_{21} = 2$ and $l_{11} = 2, 3$.

Case $l_{11} = 2$: here $|c| = 1$ holds and two of the vertices are

$$u_1 \;=\; \Big(\frac{1}{2}d_{111} + \frac{1}{2}d_{121}\,,\; \frac{1}{2}d_{211} + \frac{1}{2}d_{221}\Big), \qquad u_2 \;=\; u_1 + (1,0).$$

We assume $h(c)$ to be non-integral, otherwise we have a lattice point on c itself. Nonetheless an integral point q always lies in the interior of $\mathrm{conv}(u_1, u_2, 0) \subset A^c_{X,0}$, precisely at height $h(c) + 1/2$. It can be given explicitly as $q := \alpha u_1 + \beta u_2$ where

$$\alpha \; := \; -\frac{k}{2} + \frac{d_{111} + d_{121} + 2}{2(d_{211} + d_{221})}, \qquad\qquad \beta \; := \; 1 + \frac{k}{2} - \frac{d_{111} + d_{121}}{2(d_{211} + d_{221})}$$

for an appropriate $k \in \mathbb{Z}_{\geq 0}$ that makes $0 \leq \alpha, \beta < 1$ and

$$q \; = \; \Big(\frac{1}{2}d_{111} + \frac{1}{2}d_{121} + \frac{k}{2} + 1 \, , \, h(c) + \frac{1}{2}\Big)$$

an integral point. Hence this case does not provide terminal varieties.

Case $l_{11} = 3$: here we have $|c| = 6/5$. To avoid lattice points in the lineality part at height $\lceil h(c) \rceil$ we obtain $d'_{21} < 4$. In particular, for $d'_{21} = 2$ we have $h(c) > -4$ and for $d'_{21} = 3$ we have $h(c) > -3$. Remarks 2.4.40 and 2.4.41 deliver the estimates for d_{111} and d_{211}. □

Proposition 2.4.45. *Let $X = X(A, P)$ be as in Setting 2.4.37 Assume $d'_{21} > 1$ and $h(c) > -1$. Then $d'_{11} > 0$, $l_{11} \geq l_{21}$ and $0 \leq d_{121}, d_{221} < l_{21}$. Moreover $l_{21} \leq 5$ holds and we are in one of the following situations:*

(i) $l_{21} = 2$, $d'_{21} = 2, \dots, 10$ and $d'_{11} \in \{1, d'_{21} - 1\}$.
For $d'_{21} = 2, 3$ we obtain bounds for l_{11} according to the table

u_3	$(1,2)$			$(1,3)$			$(2,3)$		
d_{121}	0	1	1	0	1	1	0	1	1
d_{221}	1	0	1	1	0	1	1	0	1
l_{11}	≤ 24	–	≤ 24	≤ 15	≤ 69	–	–	≤ 51	≤ 15

.

For the other values of d'_{21} we have

$$l_{11} \; < \; 2 \cdot \frac{d'_{21} + 1}{d'_{21} - 2}.$$

(ii) $l_{21} = 3$, $u_3 \in \{(1,2), (1,3), (2,3), (3,4)\}$ and the exponent l_{11} is bounded according to the table:

u_3	$(1,2)$	$(1,3)$	$(2,3)$	$(3,4)$
l_{11}	≤ 14	≤ 4	≤ 4	$= 3$

.

(iii) $l_{21} = 4, 5$ and we have the estimates:

$$l_{11} \; < \; \frac{3 l_{21}}{l_{21} - 3} \qquad\qquad u_3 \; = \; (1,2).$$

In all these situations d_{111} and d_{211} are bounded by Remarks 2.4.40 and 2.4.41.

Proof. Remarks 2.4.40 and 2.4.41 provide bounds for d_{111} and d_{211}. Now consider the lineality part $A_{X,0}^{c}$. It has a non-empty intersection with $\{y = 2\}$ since $d'_{21} \geq 2$. At height $\{y = 1\}$ it has a width smaller than one, by terminality. Since $h(c) > -1$ holds, we conclude

$$\frac{l_{11} l_{21}}{l_{11} + l_{21}} \ < \ 3.$$

This implies $l_{21} \leq 5$.

Case $l_{21} = 4, 5$: the same inequality bounds l_{11}, with 11 and 7 respectively. With Remark 2.4.43 we conclude $d'_{21} = 2$ and thus $d'_{11} = 1$.

Case $l_{21} = 3$: here Remark 2.4.43 delivers $d'_{21} \leq 4$. For $u_3 \in \{(1, 3), (1, 4)\}$, the fact that the lineality part does not contain $(0, 1)$ means that it cuts the x-axis right from the point

$$\Big(-\frac{1}{d'_{21} - 1}, 0\Big).$$

To avoid the point $(1, 0)$ we get

$$l_{11} \ < \ \frac{3}{2} \cdot \frac{d'_{21} + 1}{d'_{21} - 2} \ \leq \ 6.$$

The same reasoning and result holds for $u_3 \in \{(2, 3), (3, 4)\}$, by taking the integral point $(1, 1)$ instead of $(0, 1)$.
Hence assume $u_3 = (1, 2)$. Consider the leaf $C := A_X^{c} \cap \lambda_2$. If $d_{121} + d_{221} = 3$ holds, we show that C contains an integral point and hence X is not terminal. Otherwise, we give a rational point q_0 in the lineality space λ such that

$$\frac{1}{3} \cdot v_{21} + \frac{2}{3} \cdot q_0 \ = \ q_1$$

holds for some integral point q_1. By terminality we must impose $q_0 \notin A_{X,0}^{c}$. Herewith we obtain a sharper bound on $|c|$, which in turn delivers a bound on l_{11}. When this is the case, we simply provide the points q_0, q_1 and the estimates for $|c|$ and l_{11}.

Subcase $d_{121} = 0$, $d_{221} = 1$: with $q_0 := (0, 0, 0, -1/2)$ and $q_1 := (0, 1, 0, 0)$ we obtain $|c| < 5/2$, hence $l_{11} < 15$.

Subcase $d_{121} = 0$, $d_{221} = 2$: with $q_0 := (0, 0, 0, 1/2)$ and $q_1 := (0, 1, 0, 1)$ we obtain $|c| < 2$, hence $l_{11} < 6$.

Subcase $d_{121} = 1$, $d_{221} = 0$: with $q_0 := (0, 0, -1/2, 0)$ and $q_1 := (0, 1, 0, 0)$ we obtain $|c| < 9/4$, hence $l_{11} < 9$.

Subcase $d_{121} = 1$, $d_{221} = 1$: with $q_0 := (0, 0, -1/2, -1/2)$ and $q_1 := (0, 1, 0, 0)$ we obtain $|c| < 9/5$, hence $l_{11} < 5$.

Subcase $d_{121} = 1$, $d_{221} = 2$: on the segment $\mathrm{conv}(v_{21}, v_1) \subset A_{X,0}^{c}$ lie the lattice points $(-2, 1, 2)$ and $(-1, 1, 2)$, contradicting terminality.

Subcase $d_{121} = 2$, $d_{221} = 0$: with $q_0 := (0, 0, 1/2, 0)$ and $q_1 := (0, 1, 1, 0)$ we obtain $|c| < 9/4$, hence $l_{11} < 9$.

Subcase $d_{121} = 2$, $d_{221} = 1$: the lattice point $(1,1,1)$ lies in $\operatorname{conv}(0, v_{21}, v_1) \subset A^c_{X,0}$. Thus the corresponding varieties are not terminal.

Subcase $d_{121} = 2$, $d_{221} = 2$: with $q_0 := (0,0,1/2,1/2)$ and $q_1 := (0,1,1,1)$ we obtain $|c| < 2$, hence $l_{11} < 6$.

Case $l_{21} = 2$, $l_{11} \geq 3$: since $|c| > 1$, Remark 2.4.43 yields $l_{21} \leq 10$. The length of the segment $A^c_{X,0} \cap \{y = 0\}$ is at least

$$\frac{l_{11} l_{21}}{l_{11} + l_{21}} \cdot \frac{d'_{21}}{d'_{21} + 1} \;\geq\; \frac{6}{5} \cdot \frac{d'_{21}}{d'_{21}+1} \;=:\; t.$$

Consider the triangles:

$$\begin{aligned} A_\mathrm{l} &= \operatorname{conv}((0,0), u_3, (-t/2, 0)), \\ A_\mathrm{r} &= \operatorname{conv}((0,0), u_3, (t/2, 0)). \end{aligned}$$

At least one between A_l and A_r is subset of the lineality part $A^c_{X,0}$. For any $u_3 = (d'_{11}, d'_{21})$ with $d'_{11} \notin \{1, d'_{21} - 1\}$ both A_l and A_r contain integral points, a contradiction to terminality. If $d'_{11} = \{1, d'_{21} - 1\}$ holds and $d'_{21} \geq 4$, then we look at the intersection of $A^c_{X,0}$ with the x-axis. The fact that this segment does not contain integral points other than the origin delivers the bounding condition

$$l_{11} \;<\; 2 \cdot \frac{d'_{21} + 1}{d'_{21} - 2}\;.$$

We still need to handle following possibilities for u_3:

$$(1,2), \quad (1,3), \quad (2,3).$$

We go through all combinations of u_3, d_{121} and d_{221} and provide constraints on l_{11} based on volumes of appropriate lattice polytopes. For this purpose consider the convex hull B in $\mathbb{Q}^3$ of the following points:

$$\begin{aligned} b_0 &:= (l_{11}, d_{111}, d_{211}), \quad b_3 := (0, d'_{11}, d'_{21}), \\ b_1 &:= (-2, d_{121}, d_{221}), \quad b_2 := (-2, d_{121} + 2, d_{221}). \end{aligned}$$

Put the leaf $A^c_X \cap \lambda_1$ of the anticanonical complex into $\mathbb{Q}^3$ by removing the first coordinate (which always equals zero) from its points. For $a = 0, -1, -2$, consider

$$H^+_a \;:=\; \{(x,y,z);\; x \geq a\} \;\subseteq\; \mathbb{Q}^3, \qquad H^0_a \;:=\; \{(x,y,z);\; x = a\} \;\subseteq\; \mathbb{Q}^3.$$

Then $B \cap H^+_0$ equals $A^c_X \cap \lambda_1$ and H^0_0 cuts out the lineality part $A^c_{X,0}$. In particular, by terminality of X and Theorem 2.1.10, the intersection $B \cap H^+_0$ has no interior lattice point and inside $B \cap H^0_0$ the origin is the only lattice point. Other interior lattice points of B may only appear in $B \cap H^0_{-1}$.

Subcase $u_3 = (1,2)$, $d_{221} = 1$: the simplex B contains only the origin and $(-1,1,1)$ as lattice points. With Theorem 2.4.5 we arrive at $l_{11} < 25$.

Subcase $u_3 = (1,2)$, $d_{121} = 1$, $d_{221} = 0$: here the point $(0,1,1,1)$ lies on the segment joining v_{21} and v_1, hence contradicting terminality.

Subcase $u_3 = (1,3)$, $d_{121} = 0$, $d_{221} = 1$: the origin and $(-1,1,1)$ are the only lattice points in the interior of B. We apply Theorem 2.4.5 and conclude $l_{11} < 16$.

Subcase $u_3 = (1,3)$, $d_{121} = 1$, $d_{221} = 0$: here B° contains two lattice points other than the origin, namely $q_1 := (-1,1,0)$ and $q_2 := (-1,1,1)$. The origin and q_1 lie in the union of the simplices

$$\mathrm{conv}(b_0, b_1, b_2, q_2), \qquad \mathrm{conv}(b_0, b_1, b_3, q_2).$$

We bound l_{11} by considering the worse of the two estimates obtained by applying Theorem 2.4.5. This returns $l_{11} < 70$ for both polytopes.

Subcase $u_3 = (1,3)$, $d_{121} = 1$, $d_{221} = 1$: we find the lattice point $(0,1,1,2)$ between v_{21} and v_1. Thus terminality is not fulfilled.

Subcase $u_3 = (2,3)$, $d_{121} = 0$, $d_{221} = 1$: the point $(0,1,1,2)$ lies on the segment joining v_{21} and v_1, hence contradicting terminality.

Subcase $u_3 = (2,3)$, $d_{121} = 1$, $d_{221} = 0$: here B° contains two lattice points other than the origin, namely $q_1 := (-1,1,0)$ and $q_2 := (-1,2,1)$. The origin and q_1 lie in the union of the simplices

$$\mathrm{conv}(b_0, b_1, b_2, q_2), \qquad \mathrm{conv}(b_0, b_1, b_3, q_2).$$

We bound l_{11} by considering the worse of the two estimates obtained by applying Theorem 2.4.5. This gives $l_{11} < 42$ and $l_{11} < 52$, respectively.

Subcase $u_3 = (2,3)$, $d_{121} = 1$, $d_{221} = 1$: the origin and $(-1,1,1)$ are the only lattice points in the interior of B. We apply Theorem 2.4.5 and conclude $l_{11} < 16$.

Case $l_{21} = 2$, $l_{11} = 2$: the vertices u_1 and u_2 are half-integral points. The segment c has length equal to 1 and height $h(c) > -1$. Terminality yields $h(c) = -1/2$, hence $d_{211} = -d_{221} - 1$. The bounds already found on d_{111} allow only two values for it, namely $-d_{121} - 1$ and $-d_{121} - 2$. For $d_{111} = -d_{121} - 1$ we have $d'_{11} = 1$ (to avoid points of the type $(1,k)$ in the lineality part, for some $k \in \mathbb{Z}_{\geq 0}$) and consequently $d'_{21} = 2, 3$, to prevent $(0,1)$ from lying in $A^c_{X,0}$. For $d_{111} = -d_{121} - 2$, the only choices for u_3 that do not include $(1,2)$ and $(1,1)$ in the lineality part $A^c_{X,0}$ are the points $(1,2)$ and $(2,3)$. These situations are already included in the estimates of the assertion. □

Proposition 2.4.46. *Let $X = X(A,P)$ be as in Setting 2.4.37 and assume $d'_{21} = 1$. Then we have $d'_{11} = 0$, $l_{11} \geq l_{21}$, $0 \leq d_{121}, d_{221} < l_{21}$ and*

$$-\frac{l_{11}}{l_{21}} d_{121} - l_{21} \; < \; d_{111} \; < \; -\frac{l_{11}}{l_{21}} d_{121},$$
$$-\frac{l_{11}}{l_{21}} (d_{221} + 1) - 1 \; < \; d_{211} \; < \; -\frac{l_{11}}{l_{21}} d_{221}.$$

Moreover, $l_{21} \leq 7$ holds and l_{11} is bounded according to the table

l_{21}	2	3	4	5	6	7
l_{11}	≤ 51	≤ 105	≤ 11	≤ 19	≤ 11	≤ 9

.

Proof. Since the origin is contained in the relative interior of $A^c_{X,0}$ and $u_3 = (0, 1)$ holds, u_1 has a negative y-value and u_2 a positive one. Therefore c intersects the y-axis and $h(c) > -1$ follows, in order to avoid the point $(0, -1)$. Remarks 2.4.40 and 2.4.41 deliver the bounds for d_{111} and d_{211}. Now we still need to bound the exponents l_{11} and l_{21}. Lemma 2.4.42 implies $l_{21} \leq 7$ and gives an explicit bound for l_{11} in the cases $l_{21} = 5, 6, 7$. For $l_{21} = 2, 3, 4$ define the lattice simplex $B \subset \mathbb{Q}^3$ as the convex hull over following points:

$$b_1 := (-l_{21}, d_{121}, d_{221}), \qquad b_2 := (-l_{21}, d_{121} + l_{21}, d_{221}),$$
$$b_3 := (0, 0, 1), \qquad b_0 := (l_{11}, d_{111}, d_{211}).$$

Consider the leaf $A^c_{X,0} \cap \lambda_1$ as a subset of $\mathbb{Q}^3$ by ignoring the second coordinate, which always equals zero. For $a = -l_{21}, \ldots, 0$ define

$$H_a^+ := \{(x, y, z);\ x \geq a\} \subseteq \mathbb{Q}^3, \qquad H_a^0 := \{(x, y, z);\ x = a\} \subseteq \mathbb{Q}^3.$$

Then $B \cap H_0^+$ equals $A^c_X \cap \lambda_1$ and H_0^0 cuts out the lineality part $A^c_{X,0}$. By terminality, $B \cap H_0^+$ does not contain lattice points and the origin is the only lattice point in $A^c_{X,0}$. We go through all finitely many possible b_1 and look for polytopes with precisely one or two interior lattice points, in order to bound their volumes and obtain bounds on l_{11}.

Case $l_{21} = 2$, $d_{121} = 0$, $d_{221} = 1$: the the origin is the only interior lattice point of B, hence $\mathrm{vol}(B)$ is bounded according to Theorem 2.4.4 and we reach $l_{11} < 34$.

Case $l_{21} = 2$, $d_{121} = 1$, $d_{221} = 0$: B° contains exactly two lattice points, namely the origin and $(-1, 1, 0)$. Its volume is bounded by 18 according to Theorem 2.4.5 and we obtain $l_{11} < 52$.

Case $l_{21} = 2$, $d_{121} = 1$, $d_{221} = 1$: the simplex B has the origin as only interior lattice point. By Theorem 2.4.4 we arrive at $l_{11} < 34$.

Case $l_{21} = 3$, $d_{121} = 0$, $d_{221} = 1$: the point $(0, 1, 0, 1)$ lies in on the segment that connects v_{21} with v_1, contradicting terminality.

Case $l_{21} = 3$, $d_{121} = 0$, $d_{221} = 2$: the point $(0, 1, 0, 1)$ lies in $\mathrm{conv}(0, v_{21}, v_1) \subset A^c_{X,0}$, thus contradicting terminality.

Case $l_{21} = 3$, $d_{121} = 1$, $d_{221} = 0$: three non-zero lattice points lie always in B°, namely

$$q_1 := (-2, 1, 0), \qquad q_2 := (-2, 2, 0), \qquad q_3 := (-1, 1, 0).$$

If $d_{211} \leq 2d_{111} + l_{11} + 1$ holds, then they are the only ones and $\mathrm{conv}(b_0, b_1, b_3, q_3)$ has the origin as only interior lattice point. Bounding its volume with 12 we arrive at $l_{11} < 105$. If $d_{211} > 2d_{111} + l_{11} + 1$ holds, the point $q_4 := (-1, 0, 0)$ also lies in B°. We bound the volume of $\mathrm{conv}(b_0, b_2, b_3, q_4)$, which contains the origin and q_3 an only interior lattice points, and arrive at $l_{11} < 78$.

Case $l_{21} = 3$, $d_{121} = 1$, $d_{221} = 1$: if $2d_{211} \geq -l_{11} - 1$ holds, then B contains the origin as only interior lattice point and Theorem 2.4.4 yields $l_{11} < 21$. If $2d_{211} < -l_{11} - 1$ holds, then $q := (-1, 1, 0)$ lies also in B°, together with one between $(-1, 0, 0)$ or $(-1, 2, 0)$. The simplices $\mathrm{conv}(b_0, b_1, b_3, q)$ and $\mathrm{conv}(b_0, b_2, b_3, q)$ contain at most two interior lattice points and one of the contains the origin. With Theorem 2.4.5 we conclude $l_{11} < 106$.

Case $l_{21} = 3$, $d_{121} = 1$, $d_{221} = 2$: other than the origin, B° contains only the lattice point $(-1, 1, 1)$. Hence we can directly apply Theorem 2.4.5 and obtain $l_{11} < 33$.

Case $l_{21} = 3$, $d_{121} = 2$, $d_{221} = 0$: three non-zero lattice points lie always in B, namely

$$q_1 := (-2, 2, 0), \qquad q_2 := (-2, 3, 0), \qquad q_3 := (-1, 1, 0).$$

If $d_{211} + 2d_{111} + 3l_{11} \leq 1$ holds, then they are the only ones and $\operatorname{conv}(b_0, b_1, b_3, q_3)$ has the origin as only interior lattice point. Bounding its volume with 12 we arrive at $l_{11} < 31$. If $d_{211} + 2d_{111} + 3l_{11} \geq 2$ holds, the point $q_4 := (-1, 2, 0)$ also lies in B°. We bound the volume of $\operatorname{conv}(b_0, b_1, b_3, q_4)$, which contains the origin and q_3 an only interior lattice points, and arrive at $l_{11} < 78$.

Case $l_{21} = 3$, $d_{121} = 2$, $d_{221} = 1$: here the exact same process works as in case $l_{21} = 3$, $d_{121} = 1$, $d_{221} = 1$.

Case $l_{21} = 3$, $d_{121} = 2$, $d_{221} = 2$: other than the origin, B contains only the lattice point $(-1, 1, 1)$. Hence we can directly apply Theorem 2.4.5 and obtain $l_{11} < 33$.

Case $l_{21} = 4$: here we only work with the leaf $C := A_X^c \cap \lambda_2$. If d_{121} is even, then we can always show that C contains an integral point and hence X is not terminal. If d_{121} is odd, we give a rational point q_0 in the lineality space λ such that

$$\frac{1}{4} \cdot v_{21} + \frac{3}{4} \cdot q_0 \;=\; q_1$$

holds for some integral point q_1. By Theorem 2.1.10 we must impose $q_0 \notin A_{X,0}^c$. As a consequence we obtain a sharper bound on $|c|$, which in turn delivers a bounding condition on l_{11}. When this is the case, we simply provide the points q_0, q_1 and the estimates for $|c|$ and l_{11}.

Case $l_{21} = 4$, $d_{121} = 0$, $d_{221} = 1$: the point $(0, 2, 0, 1)$ lies between v_{21} and v_1.

Case $l_{21} = 4$, $d_{121} = 0$, $d_{221} = 3$: the point $(0, 2, 0, 2)$ lies between v_{21} and v_1.

Case $l_{21} = 4$, $d_{121} = 1$, $d_{221} = 0$: with $q_0 := (0, 0, -1/3, 0)$ and $q_1 := (0, 1, 0, 0)$ we obtain $|c| < 8/3$, hence $l_{11} < 8$.

Case $l_{21} = 4$, $d_{121} = 1$, $d_{221} = 1$: with $q_0 := (0, 0, -1/3, -1/3)$ and $q_1 := (0, 1, 0, 0)$ we obtain $|c| < 5/2$, hence $l_{11} < 7$.

Case $l_{21} = 4$, $d_{121} = 1$, $d_{221} = 2$: with $q_0 := (0, 0, -1/3, -2/3)$ and $q_1 := (0, 1, 0, 0)$ we obtain $|c| < 12/5$, hence $l_{11} < 6$.

Case $l_{21} = 4$, $d_{121} = 1$, $d_{221} = 3$: with $q_0 := (0, 0, -1/3, 1/3)$ and $q_1 := (0, 1, 0, 1)$ we obtain $|c| < 3$, hence $l_{11} < 12$.

Case $l_{21} = 4$, $d_{121} = 2$, $d_{221} = 1$: the point $(0, 2, 1, 1)$ lies between v_{21} and v_1.

Case $l_{21} = 4$, $d_{121} = 2$, $d_{221} = 3$: the point $(0, 2, 1, 2)$ lies between v_{21} and v_1.

Case $l_{21} = 4$, $d_{121} = 3$, $d_{221} = 0$: with $q_0 := (0, 0, 1/3, 0)$ and $q_1 := (0, 1, 1, 0)$ we obtain $|c| < 8/3$, hence $l_{11} < 8$.

Case $l_{21} = 4$, $d_{121} = 3$, $d_{221} = 1$: with $q_0 := (0, 0, 1/3, -1/3)$ and $q_1 := (0, 1, 1, 0)$ we obtain $|c| < 5/2$, hence $l_{11} < 7$.

Case $l_{21} = 4$, $d_{121} = 3$, $d_{221} = 2$: with $q_0 := (0, 0, 1/3, -2/3)$ and $q_1 := (0, 1, 1, 0)$ we obtain $|c| < 12/5$, hence $l_{11} < 6$.

Case $l_{21} = 4$, $d_{121} = 3$, $d_{221} = 3$: with $q_0 := (0, 0, 1/3, 1/3)$ and $q_1 := (0, 1, 1, 1)$ we obtain $|c| < 3$, hence $l_{11} < 12$. □

2.5 Classification

Theorem 2.5.1. *The following table lists the Cox rings $\mathcal{R}(X)$ of the non-toric rational terminal $\mathbb{Q}$-factorial Fano threefolds X with $\rho(X) = 1$ and an effective two-torus action; the $\mathrm{Cl}(X)$-degrees of the generators $T_1, \ldots, T_r$ are denoted as columns $w_i \in \mathrm{Cl}(X)$ of a matrix $[w_1, \ldots, w_r]$.*

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$[w_1, \ldots, w_r]$
1.01	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^2 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 1 & 1 & 1 & 1 & 1 \end{smallmatrix}\right]$
1.02	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^2 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 1 & 5 & 2 & 4 & 3 \end{smallmatrix}\right]$
1.03	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$	$\left[\begin{smallmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{3} & \bar{1} & \bar{4} & \bar{0} \end{smallmatrix}\right]$
1.04	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^3 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 1 & 5 & 3 & 3 & 2 \end{smallmatrix}\right]$
1.05	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^4 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 1 & 3 & 2 & 2 & 1 \end{smallmatrix}\right]$
1.06	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^4 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left[\begin{smallmatrix} 1 & 3 & 2 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{smallmatrix}\right]$
1.07	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^6 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 2 & 4 & 3 & 3 & 1 \end{smallmatrix}\right]$
1.08	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^2T_4 + T_5^2 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 1 & 3 & 1 & 2 & 2 \end{smallmatrix}\right]$
1.09	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^2T_4 + T_5^2 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 1 & 5 & 2 & 2 & 3 \end{smallmatrix}\right]$
1.10	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^2T_4 + T_5^2 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 3 & 7 & 4 & 2 & 5 \end{smallmatrix}\right]$
1.11	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^2T_4 + T_5^3 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 2 & 1 & 1 & 1 & 1 \end{smallmatrix}\right]$
1.12	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^2T_4 + T_5^3 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 3 & 3 & 1 & 4 & 2 \end{smallmatrix}\right]$
1.13	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^2T_4 + T_5^3 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\left[\begin{smallmatrix} 2 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{2} & \bar{1} & \bar{1} & \bar{0} \end{smallmatrix}\right]$
1.14	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^2T_4 + T_5^6 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 3 & 3 & 2 & 2 & 1 \end{smallmatrix}\right]$
1.15	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^2T_4^2 + T_5^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left[\begin{smallmatrix} 1 & 3 & 1 & 1 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{smallmatrix}\right]$
1.16	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^2T_4^2 + T_5^3 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 3 & 3 & 2 & 1 & 2 \end{smallmatrix}\right]$
1.17	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^3T_4 + T_5^2 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 1 & 3 & 1 & 1 & 2 \end{smallmatrix}\right]$
1.18	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^3T_4 + T_5^2 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 2 & 4 & 1 & 3 & 3 \end{smallmatrix}\right]$
1.19	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^3T_4 + T_5^2 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 3 & 7 & 2 & 4 & 5 \end{smallmatrix}\right]$
1.20	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^3T_4 + T_5^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left[\begin{smallmatrix} 1 & 3 & 1 & 1 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{smallmatrix}\right]$
1.21	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^3T_4 + T_5^4 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 2 & 2 & 1 & 1 & 1 \end{smallmatrix}\right]$
1.22	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^3T_4 + T_5^4 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left[\begin{smallmatrix} 2 & 2 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{smallmatrix}\right]$
1.23	$\mathbb{K}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^3T_4^2 + T_5^2 \rangle$	$\mathbb{Z}$	$\left[\begin{smallmatrix} 3 & 5 & 2 & 1 & 4 \end{smallmatrix}\right]$

1.24	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^3T_4^3 + T_5^2 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 2 & 4 & 1 & 1 & 3 \end{bmatrix}$
1.25	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^4T_4 + T_5^3 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 3 & 3 & 1 & 2 & 2 \end{bmatrix}$
1.26	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^4T_4^2 + T_5^3 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 3 & 3 & 1 & 1 & 2 \end{bmatrix}$
1.27	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^5T_4 + T_5^2 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 2 & 4 & 1 & 1 & 3 \end{bmatrix}$
1.28	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^5T_4 + T_5^3 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 3 & 3 & 1 & 1 & 2 \end{bmatrix}$
1.29	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^6T_4 + T_5^2 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 3 & 5 & 1 & 2 & 4 \end{bmatrix}$
1.30	$\frac{\mathbb{K}[T_1, \dots, T_6]}{\langle T_1T_2 + T_3T_4 + T_5^2, aT_3T_4 + T_5^2 + T_6^2 \rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$
1.31	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2T_3 + T_4^3 + T_5^2 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 4 & 2 & 3 \end{bmatrix}$
1.32	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2T_3 + T_4^3 + T_5^2 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 2 & 3 & 1 & 2 & 3 \end{bmatrix}$
1.33	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^2 + T_4^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 2 & 4 & 3 & 3 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$
1.34	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^3 + T_4^2 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 1 & 5 & 2 & 3 & 1 \end{bmatrix}$
1.35	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^3 + T_4^2 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 1 & 5 & 2 & 3 & 2 \end{bmatrix}$
1.36	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^3 + T_4^2 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 1 & 5 & 2 & 3 & 3 \end{bmatrix}$
1.37	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^3 + T_4^2 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 1 & 5 & 2 & 3 & 4 \end{bmatrix}$
1.38	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^3 + T_4^2 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 5 & 7 & 4 & 6 & 1 \end{bmatrix}$
1.39	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^3 + T_4^2 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 5 & 7 & 4 & 6 & 3 \end{bmatrix}$
1.40	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^3 + T_4^3 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 0 & 0 \end{bmatrix}$
1.41	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^4 + T_4^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$
1.42	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^4 + T_4^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$
1.43	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^4 + T_4^3 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 5 & 7 & 3 & 4 & 1 \end{bmatrix}$
1.44	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^4 + T_4^3 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 5 & 7 & 3 & 4 & 2 \end{bmatrix}$
1.45	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^5 + T_4^2 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 3 & 7 & 2 & 5 & 1 \end{bmatrix}$
1.46	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^5 + T_4^2 \rangle$	$\mathbb{Z}$	$\begin{bmatrix} 3 & 7 & 2 & 5 & 4 \end{bmatrix}$
1.47	$\mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3^6 + T_4^2 \rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 2 & 4 & 1 & 3 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$

where $a \in \mathbb{K}^* \setminus \{1\}$ *in No. 1.30. Any two of the Cox rings listed in the table correspond to non-isomorphic varieties. No. 1.01 is the only smooth one.*

Proof. By Lemma 2.4.1 there are only four possible cases for the matrix P defining a terminal $\mathbb{Q}$-factorial Fano threefold $X = X(A, P)$ with effective two-torus action and Picard number $\rho(X) = 1$. The Propositions listed in the last column of the table of Remark 2.4.2 give bounds on all entries of P in all cases. Among all of these candidates, one can figure out the terminal ones by checking the condition of Theorem 2.1.10 (v); we do it by computer using [25], where the anticanonical complex A_X^c is implemented. By comparing the data, one directly sees that any two varieties listed above are non-isomorphic. □

Appendix A contains detailed information about all varieties of Theorem 2.5.1; in particular one finds possible defining matrices P.

Remark 2.5.2. For $\mathbb{K} = \mathbb{C}$, any Fano variety X with at most log terminal singularities, has finitely generated divisor class group $\mathrm{Cl}(X)$; see [31, Sec. 2.1]. If X comes in addition with a torus action of complexity one, then X is rational and its Cox ring is finitely generated; see [1, Remark 4.4.1.5].

Remark 2.5.3. By Remark 2.5.2, the assumption of rationality can be omitted in Theorem 2.5.1, for $\mathbb{K} = \mathbb{C}$. Alternatively, rationality can be replaced by the property "$\mathrm{Cl}(X)$ is finitely generated".

CHAPTER

THREE

COMBINATORIALLY MINIMAL TERMINAL FANO THREEFOLDS OF COMPLEXITY ONE

In the work [34] of A. Kasprzyk, one of the steps in the classification of terminal Fano toric threefolds consists in finding all minimal terminal Fano polytopes, i.e. those that do not contain any smaller terminal Fano polytope. These correspond to varieties that do not admit the contraction of a prime divisor. From the minimal polytopes one can obtain all others systematically. Following the same line, we study here terminal Fano threefolds of complexity one that do not admit the contraction of a prime divisor.

In Section 3.1 we introduce the concept of combinatorial minimality, with focus on the (Fano) varieties of complexity one. Section 3.2 specializes to the three-dimensional case and provides a bound on the Picard number in the $\mathbb{Q}$-factorial terminal situation. Section 3.3 is dedicated to the actual classification work. Its results are given in Section 3.4.

3.1 Combinatorially minimal T-varieties of complexity one

A *small quasimodification* is a birational map $X \dashrightarrow X'$ of complete varieties that restricts to a regular isomorphism $U \to U'$ between open subsets $U \subseteq X$ and $U' \subseteq X'$ having complements of codimension at least two in X and X' respectively. We say that a Mori dream space X is *combinatorially minimal* if it has no contractible prime divisors in the sense that any birational map $X \dashrightarrow X'$ which is defined in codimension two is a small quasimodification.

Remark 3.1.1. A projective Mori dream space X is combinatorially minimal if and only if its cone of movable divisor classes coincides with its cone of effective divisor classes. In particular for a variety $X = X(A, P, u)$ of complexity one, this precisely means that every extremal ray of the effective cone $\mathrm{Eff}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ hosts the degrees of at least two of the generators T_{ij}, S_k of the Cox ring $\mathcal{R}(X) = R(A, P)$, see [23].

In the following we give a characterization for the defining matrix P of a combinatorially minimal variety $X = X(A, P, u)$. Moreover we derive bounds on the Picard number in the case where the variety is also $\mathbb{Q}$-factorial, log terminal and Fano.

First we recall the result [1, Lemma 2.2.3.2]. Consider two mutually dual exact sequences of finite dimensional rational vector spaces

$$0 \longrightarrow L_{\mathbb{Q}} \longrightarrow F_{\mathbb{Q}} \longrightarrow N_{\mathbb{Q}} \longrightarrow 0$$

$$0 \longleftarrow K_{\mathbb{Q}} \xleftarrow[Q]{} E_{\mathbb{Q}} \longleftarrow M_{\mathbb{Q}} \longleftarrow 0$$

Denote by $(f_1, \ldots, f_r)$ a basis for $F_{\mathbb{Q}}$ and by $(e_1, \ldots, e_r)$ its dual basis for $E_{\mathbb{Q}}$. Moreover define $\delta := \operatorname{cone}(f_1, \ldots, f_r)$ and $\gamma := \operatorname{cone}(e_1, \ldots, e_r)$. We call an element $e \in E_{\mathbb{Q}}$ an *$L_{\mathbb{Q}}$-invariant separating linear form* for $\delta_1, \delta_2 \preceq \delta$ if

$$e|_{L_{\mathbb{Q}}} = 0, \quad e|_{\delta_1} = 0, \quad e|_{\delta_2} = 0, \quad \delta_1 \cap e^{\perp} = \delta_2 \cap e^{\perp} = \delta_1 \cap \delta_2.$$

Lemma 3.1.2 (Invariant Separation Lemma [1, Lemma 2.2.3.2])**.** *Consider $\delta_1, \delta_2 \preceq \delta$ and their corresponding faces $\gamma_i := \delta_i^{\perp} \cap \gamma \preceq \gamma$. Then the following statements are equivalent:*

- *there exists an $L_{\mathbb{Q}}$-invariant separating linear form for δ_1, δ_2;*
- *$Q(\gamma_1)^\circ \cap Q(\gamma_2)^\circ \neq \emptyset$.*

Consider the weights $w_{ij} := \deg(T_{ij})$ and $w_k := \deg(S_k)$. We call a variable T_{ij}, S_k *extremal* if its weight $w_{ij}, w_k \in \mathrm{Cl}_{\mathbb{Q}}(X)$ sits on an extremal ray of the effective cone $\mathrm{Eff}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$. Moreover, we call a weight $w \in \{w_{ij}, w_k\}$ *exceptional*, if $\mathbb{Q}_{\geq 0} w$ is an extremal ray of $\mathrm{Eff}(X)$ and no other weight lies on $\mathbb{Q}_{\geq 0} w$. All variables with exceptional weights are extremal, but the converse does not hold. Note that, by Remark 3.1.1, X is combinatorially minimal if and only if no weight is exceptional.

Only in this Section, we rename the weights $w_1, \ldots, w_{n+m}$ and let $v_1, \ldots, v_{n+m}$ be the corresponding columns of P.

Lemma 3.1.3. *The weight $w \in \{w_{ij}, w_k\}$ is exceptional if and only if $\mathbb{Q}^{r+s}$ is already generated as a cone by all columns of P except the one that corresponds to w.*

Proof. An exceptional weight w_i is characterized by the fact that $\operatorname{cone}(w_j; j \neq i)$ and $\operatorname{cone}(w_i)$ allow a separating linear form. By the Invariant Separation Lemma 3.1.2, this is equivalent to the fact that the corresponding Gale dual cones intersect in their relative interiors, i.e.

$$\operatorname{cone}(v_j; j \neq i)^\circ \cap \operatorname{cone}(v_i)^\circ \neq \emptyset.$$

This is the case if and only if $v_i \in \operatorname{cone}(v_j; j \neq i)^\circ$ holds. Since the cone over all columns is $\mathbb{Q}^{r+s}$, the last condition is equivalent to $\operatorname{cone}(v_j; j \neq i) = \mathbb{Q}^{r+s}$. □

Proposition 3.1.4. *The variety $X = X(A, P, u)$ is combinatorially minimal if and only if for every column v_ϱ of P the following holds:*

$$\operatorname{cone}(v_{\varrho'};\ \varrho' \in \Sigma^{(1)} \setminus \{\varrho\}) \ \neq\ \mathbb{Q}^{r+s}.$$

Proof. The assertion follows directly from Lemma 3.1.3. □

Given a matrix P and an index-set $I \subset \mathbb{Z}_{\geq 1}$, we denote with P_I the matrix obtained from P by deleting the i-th column, for all $i \in I$.

Corollary 3.1.5. *Let $X = X(A, P, u)$ be a $\mathbb{Q}$-factorial projective variety of complexity one with Picard number $\rho(X) \geq 2$. Assume that* $\operatorname{cone}(w_i) = \operatorname{cone}(w_j)$ *holds for two weights w_i, w_j, with $i \neq j$. Then* $\det(P_I) = 0$ *holds for every $I \subset \{1, \ldots, n+m\}$ such that $i, j \in I$ and $|I| = \rho(X)$.*

Proof. Since X is $\mathbb{Q}$-factorial, we have $n + m = r + s + \rho(X)$. Define

$$\tau_i := \operatorname{cone}(v_1, \ldots, \widehat{v_i}, \ldots, v_{n+m}),$$
$$\tau_j := \operatorname{cone}(v_1, \ldots, \widehat{v_j}, \ldots, v_{n+m}).$$

According to Lemma 3.1.3 neither of them equals $\mathbb{Q}^{r+s}$. Moreover, by the Invariant Separation Lemma, there is a linear form separating them, i.e. $\tau_i \cap \tau_j$ is a proper face of both cones. This means that $\tau_i \cap \tau_j = \operatorname{cone}(v_1, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_{n+m})$ is not full-dimensional. In particular any collection of $r + s$ columns of P that does not contain v_i nor v_j is linearly dependent. □

We turn to bounding conditions on the Picard number. In the sequel, let α denote the difference between the number of extremal rays of the effective cone $\operatorname{Eff}(X)$ and the Picard number $\rho(X)$.

Lemma 3.1.6. *Let $X = X(A, P, u)$ be combinatorially minimal and $\mathbb{Q}$-factorial. Then one of the following holds:*

- $\dim(X) \geq \rho(X)$ *and* $m \geq 2\rho(X) - 2$;
- $\dim(X) \geq \alpha + 2 + m/2$ *and* $m < 2\rho(X) - 2$.

Proof. The effective cone $\operatorname{Eff}(X) \subseteq \operatorname{Cl}_{\mathbb{Q}}(X)$ is of full dimension and has $\rho(X)+\alpha$ vertices. Since X is combinatorially minimal, the number $n+m$ of variables T_{ij}, S_k is bounded from below by $n + m \geq 2\rho(X) + 2\alpha$. If the number m of variables S_k satisfies $m \geq 2\rho(X) - 2$, then the assertion follows from

$$\dim(X) \;=\; n + m + 2 - (r + 1) - \rho(X) \;\geq\; m + 2 - \rho(X) \;\geq\; \rho(X).$$

So, consider the case $m < 2\rho(X) - 2$. There are at least two extremal variables of type T_{ij} having their weights on different rays and we have in total at least $2\rho(X) + 2\alpha - m$ extremal variables of type T_{ij}. For each of these T_{ij}, we must have $n_i \geq 2$. This gives

$$n - (r + 1) \;=\; (n_0 - 1) + \ldots + (n_r - 1) \;\geq\; \frac{2\rho(X) + 2\alpha - m}{2}.$$

The assertion then follows from

$$\begin{aligned} \dim(X) &= n + m + 2 - (r + 1) - \rho(X) \\ &\geq \frac{2\rho(X) + 2\alpha - m}{2} + m + 2 - \rho(X) \\ &= \alpha + \frac{m}{2} + 2. \end{aligned}$$

□

For $X = X(A, P, u)$, we denote by η the number of extremal variables of type T_{ij} and by $\zeta = n - \eta$ the number of non-extremal variables of type T_{ij}.

Lemma 3.1.7. *Let $X = X(A, P, u)$ be $\mathbb{Q}$-factorial and combinatorially minimal. Then we have*

$$\rho(X) \ \leq \ \dim(X) + r - 1 - \zeta - 2\alpha.$$

Proof. Just observe that $\dim(X)+\rho(X)+r-1 = \eta+\zeta+m \geq 2\rho(X)+2\alpha+\zeta$ holds. □

The above estimate is useful for large ζ, for example $\zeta \geq r-2$. For small ζ, we need statements concerning the cases $\alpha = 0$ and $\alpha = 1$. Observe that the following three Lemmas, including the estimate for the case $\alpha = 0$, do not require combinatorial minimality.

Lemma 3.1.8. *Let $X = X(A, P)$ be $\mathbb{Q}$-factorial, log terminal and Fano. Let b be the number of elementary big cones. Then $m + b \geq \dim(X)$.*

Proof. The lineality part $A^c_{X,0}$ has $m + b$ vertices. According to Proposition 2.1.12, $A^c_{X,0}$ is full-dimensional in the lineality space λ of $\mathrm{trop}(X)$, i.e. it has dimension s. This is only possible with at least $s + 1$ vertices, hence the assertion follows. □

Lemma 3.1.9. *Let $X = X(A, P)$ be $\mathbb{Q}$-factorial, log terminal and Fano. If $m < \dim(X)$, then we have*

$$\frac{n + m}{2} \ \leq \ \dim(X) + \rho(X).$$

Proof. Lemma 3.1.8 ensures the existence of an elementary big cone. Therefore we can apply Corollary 2.2.9 and obtain that $r - 1$, the number of relations, equals at most $\dim(X) + \rho(X)$. The assertion follows from

$$n + m \ = \ \dim(X) + \rho(X) + r - 1 \ \leq \ 2(\dim(X) + \rho(X)).$$

□

Recall that μ indicates the unique degree of the relations in the Cox ring. We know that μ lies in the effective cone $\mathrm{Eff}(X)$, but not necessarily in its interior.

Lemma 3.1.10. *Let $X = X(A, P)$ be a $\mathbb{Q}$-factorial log terminal Fano variety, such that $\mu \in \mathrm{Eff}(X)^\circ$. If $\alpha = 0$, $\zeta \leq r - 2$ and $m < \dim(X)$ hold, then we have*

$$\rho(X) \ \leq \ \frac{2}{r - 1 - \zeta} \dim(X) - \frac{m + \zeta}{r - 1 - \zeta}.$$

Proof. In the relations of $R(A, P)$, consider the monomials $T_i^{l_i}$ consisting only of extremal variables T_{ij}. Since $\mu \in \mathrm{Eff}(X)^\circ$ and $\alpha = 0$ hold, each such monomial comprises at least $\rho(X)$ variables. There are at least $r + 1 - \zeta$ such monomials. We obtain

$$n \ = \ \eta + \zeta \ \geq \ (r - 1 - \zeta)\rho(X) + 2\rho(X) + \zeta.$$

Since X is log terminal with $m < \dim(X)$, we can apply Lemma 3.1.9 and see that $2\dim(X) + 2\rho(X)$ is bigger or equal to $n + m$. Combining the two estimates gives the assertion. □

For the case $\alpha = 1$, we use geometrical properties of d-dimensional polyhedral cones with $d+1$ extremal rays. Here we gather and prove the relevant facts.

Lemma 3.1.11. *Let $d \geq 3$ and $\sigma \subseteq \mathbb{Q}^d$ be a pointed convex polyhedral d-dimensional cone with $d+1$ extremal rays. Let $v_1, \ldots, v_{d+1} \in \mathbb{Q}^d$ be primitive generators of the extremal rays of σ and $w_1, \ldots, w_{d+1} \in \mathbb{Q}$ the Gale dual configuration. Set $D := \{1, \ldots, d+1\}$ and*

$$D_- := \{i \in D;\ w_i < 0\}, \quad D_0 := \{i \in D;\ w_i = 0\}, \quad D_+ := \{i \in D;\ w_i > 0\}.$$

Moreover, for any subset $I \subseteq D$, denote by $I^c \subseteq D$ its complement and define cones $\sigma_I := \mathrm{cone}(v_i;\ i \in I) \subseteq \mathbb{Q}^d$ and $\tau_I := \mathrm{cone}(w_i;\ i \in I^c) \subseteq \mathbb{Q}$. Then the following statements hold:

(i) σ_I is a proper face of σ if and only if $\tau_I = \{0\}$ or $\tau_I = \mathbb{Q}$ holds.
(ii) There are at least two i with $w_i > 0$ and at least two j with $w_j < 0$.
(iii) We have $\sigma_I^\circ \subseteq \sigma^\circ$ if and only if $D_- \cup D_0 \subseteq I$ or $D_+ \cup D_0 \subseteq I$ holds.
(iv) We have $\sigma_I^\circ \cap \sigma_J^\circ \neq \emptyset$ if and only if $D_- \subseteq I$, $D_+ \subseteq J$, $I \cap D_0 = J \cap D_0$ or $D_+ \subseteq I$, $D_- \subseteq J$, $I \cap D_0 = J \cap D_0$ holds.

In particular, $\sigma_- := \mathrm{cone}(v_i;\ i \in D_- \cup D_0)$ and $\sigma_+ := \mathrm{cone}(v_i;\ i \in D_+ \cup D_0)$ form the unique pair of minimal cones satisfying $\sigma_\pm^\circ \subseteq \sigma^\circ$, $\mathrm{cone}(\sigma_-, \sigma_+) = \sigma$ and $\sigma_-^\circ \cap \sigma_+^\circ \neq \emptyset$.

Proof. Let $P\colon \mathbb{Q}^{d+1} \to \mathbb{Q}^d$ be the linear map sending e_i to v_i and $Q\colon \mathbb{Q}^{d+1} \to \mathbb{Q}$ the one sending e_i to w_i. For I consider $\delta_I := \mathrm{cone}(e_i;\ i \in I) \subseteq \mathbb{Q}^{d+1}$ and $\gamma_I := \mathrm{cone}(e_i;\ i \in I^c) \subseteq \mathbb{Q}^{d+1}$. The Invariant Separation Lemma 3.1.2 yields for any two $I, J \subseteq \{1, \ldots, d+1\}$ the following statements:

- There is a $\ker(P)$-invariant separating linear form for δ_I and δ_J if and only if $\tau_I^\circ \cap \tau_J^\circ \neq \emptyset$ holds,
- We have $\sigma_I^\circ \cap \sigma_J^\circ \neq \emptyset$ if and only if there is a $\ker(Q)$-invariant separating linear form for γ_I and γ_J.

Observe that σ_I, σ_J intersect in a common face if and only if δ_I, δ_J admit a $\ker(P)$-invariant separating linear form.
Now, assertion (i) is an immediate consequence of the first of the above two items. Since $\{0\}$ is a face of σ, we see that there must be positive and negative w_i. Assertion (ii) reflects the fact that every ray $\sigma_{\{i\}}$ is a face of σ. Assertion (iii) is a special case of (iv) which in turn is obtained by adapting the second of the above items to the setting of the Lemma. □

We are ready to estimate the Picard number for the case $\alpha = 1$ and small ζ. Again, this statement does not assume combinatorial minimality.

Lemma 3.1.12. *Let $X = X(A, P)$ be a $\mathbb{Q}$-factorial log terminal Fano variety, such that $\mu \in \mathrm{Eff}(X)^\circ$. Assume $\alpha = 1$ and $\zeta \leq r - 2$. Then we have*

$$\rho(X) \ \leq\ \dim(X) + 3 + \zeta - r - m \ \leq\ \dim(X) + 1 - m.$$

Proof. Let $\omega_-, \omega_+ \subseteq \mathrm{Eff}(X)$ be the minimal pair of subcones as in Lemma 3.1.11 and denote by a_-, a_+ their respective numbers of extremal rays. Then we have

$$a_- \;\geq\; 2, \qquad a_+ \;\geq\; 2, \qquad a_- + a_+ \;\geq\; \rho(X) + 1.$$

Consider a monomial $T_i^{l_i}$ with only extremal variables T_{ij} and let ω_i be the cone generated by the $\deg(T_{ij})$. We say that $T_i^{l_i}$ is of type $(-)$ if $\omega_- \subseteq \omega_i$ holds and of type $(+)$ otherwise. Lemma 3.1.11, with $\mu \in \mathrm{Eff}(X)^\circ$, shows that any $T_i^{l_i}$ of type $(+)$ satisfies $\omega_+ \subseteq \omega_i$. Let b_- and b_+ denote the respective numbers of monomials of these types that occur in the $(r-1)$ relations. Then we have

$$b_- + b_+ \;\geq\; r + 1 - \zeta, \qquad \eta \;\geq\; b_- a_- + b_+ a_+.$$

For the first estimate, we use that there are at least $r + 1 - \zeta$ monomials involving only extremal variables. For the second one, note that every monomial of type $(\pm)$ has at least $a_\pm$ distinct variables. We conclude

$$\begin{aligned} \dim(X) + \rho(X) + r - 1 \;&=\; \eta + \zeta + m \\ &\geq\; b_- a_- + b_+ a_+ + \zeta + m \\ &\geq\; 2(\rho(X)+1) + (b_- - 2)a_- + (b_+ - 2)a_+ + \zeta + m \\ &\geq\; 2(\rho(X)+1) + 2(r + 1 - \zeta - 4) + \zeta + m. \end{aligned}$$

□

Proposition 3.1.13. *For any $X = X(A,P,u)$, the degree μ of the relations g_I lies in the effective cone $\mathrm{Eff}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$. Moreover μ lies in its interior if and only if $0 \notin \mathrm{cone}(v_{k_1}, \ldots, v_{k_t})^\circ$ holds for any collection $1 \leq k_1 < \ldots < k_t \leq m$.*

Proof. The first assertion is trivial. The degree μ lies on a facet τ of $\mathrm{Eff}(X)$ if and only if all weights w_{ij} lie on τ as well. By the Invariant Separation Lemma, this is precisely the case, when there exist $v_{k_1}, \ldots, v_{k_t}$ such that $0 \in \mathrm{cone}(v_{k_1}, \ldots, v_{k_t})^\circ$. □

3.2 The 3-dimensional case

In this Section we specialize to three-dimensional $\mathbb{Q}$-factorial, terminal and combinatorially minimal Fano varieties $X = X(A,P)$ and give a bound on their Picard number. Moreover, we establish the analogue of Lemma 2.4.1, i.e. we find a finite list of possible shapes for the defining matrix P. First we allow X to have log terminal singularities.

Proposition 3.2.1. *Let $X = X(A,P)$ be a 3-dimensional $\mathbb{Q}$-factorial, log terminal and combinatorially minimal Fano variety. Then $\rho(X) \leq 3$ holds.*

Proof. Lemma 3.1.6 tells us that, besides $\rho(X) \leq 3$, we have to consider the following two situations:

$$\alpha \;=\; 0 \text{ and } m \;\leq\; 2, \qquad\qquad \alpha \;=\; 1 \text{ and } m \;=\; 0.$$

Case $\mu \in \mathrm{Eff}(X)^\circ$: If $\alpha = 0$ holds, then Lemma 3.1.7 with $\zeta \ge r-2$ gives $\rho(X) \le 4$ and Lemma 3.1.10 with $\zeta \le r-3$ gives $\rho(X) \le 3$. If $\alpha = 1$ holds, Lemma 3.1.7 with $\zeta \ge r-2$ gives $\rho(X) \le 2$ and Lemma 3.1.12 with $\zeta \le r-3$ gives $\rho(X) \le 3$. So we have to exclude $\rho(X) = 4$, which only appears in the case $\alpha = 0$ with $\zeta \ge r-2$. Lemma 3.1.7 yields in this case $\zeta \le r-2$, thus we have $\zeta = r-2$. In particular we have a relation involving only extremal variables, say the one with monomials $T_0^{l_0}, T_1^{l_1}, T_2^{l_2}$. Together with $\mu \in \mathrm{Eff}(X)^\circ$ and $\alpha = 0$, this implies $n_0, n_1, n_2 \ge 4$. On the other hand, Lemma 3.1.10 gives us $m+\zeta \le 2$ and thus $r \le 4$. This shows $n+m = 7+r-1 \le 10$, a contradiction to $n \ge 12$.

Case $\mu \notin \mathrm{Eff}(X)^\circ$: According to Proposition 3.1.13 we have $m \ge 2$, therefore $\alpha = 0$ and $m = 2$. In particular the weights w_{ij} generate a $(\rho(X)-1)$-dimensional facet of $\mathrm{Eff}(X)$. For $r \le \zeta$, Lemma 3.1.7 already yields $\rho(X) \le 2$, so assume $r \ge \zeta+1$. Since there are at least $r+1-\zeta$ monomials having only extremal variables and each of these monomials must have at least $\rho(X)-1$ variables we conclude

$$\dim(X) + \rho(X) + r - 1 \;=\; \eta + \zeta + m \;\ge\; (r+1-\zeta)(\rho(X)-1) + \zeta + m$$

and thus $(r-\zeta)(\rho(X)-2) \le \dim(X) - m = 1$. It follows $\rho(X) \le 3$. □

Restricting to the terminal case, we can prove that the degree μ of the relations lies in the interior of the effective cone $\mathrm{Eff}(X)$.

Proposition 3.2.2. *Let $X = X(A,P)$ be a non-toric, 3-dimensional, combinatorially minimal, terminal, $\mathbb{Q}$-factorial Fano variety, where the defining matrix P is irredundant. Then $\mu \in \mathrm{Eff}(X)^\circ$ holds.*

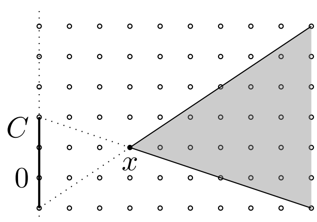
Definition 3.2.3. Consider a convex set $C \subset \{0\} \times \mathbb{Q}^d \subset \mathbb{Q}^{d+1}$ and a point $x \in \mathbb{Q}^{d+1}$ with first coordinate $x_1 > 0$. The *shadow* of C from x is

$$\mathrm{sw}(C,x) \;:=\; \{y \in \mathbb{Q}^{d+1}\ ;\ x \in \mathrm{conv}(y, C)\} \;\subseteq\; \mathbb{Q}^{d+1}.$$

Moreover, for any $t \in \mathbb{Q}$ with $t \ge x_1$, we define the *sliced shadow* at height t as

$$\mathrm{sw}_t(C,x) \;:=\; \{y \in \mathrm{sw}(C,x)\ ;\ y_1 = t\}.$$

Example 3.2.4. The following picture clarifies the origin of the name *shadow*. Consider $C := \mathrm{conv}((0,2),(0,-1)) \subset \mathbb{Q}^2$ and $x := (3,1) \in \mathbb{Q}^2$. The area in grey is $\mathrm{sw}(C,x)$:



The convex hull over the line segment C and any point from the grey area contains x.

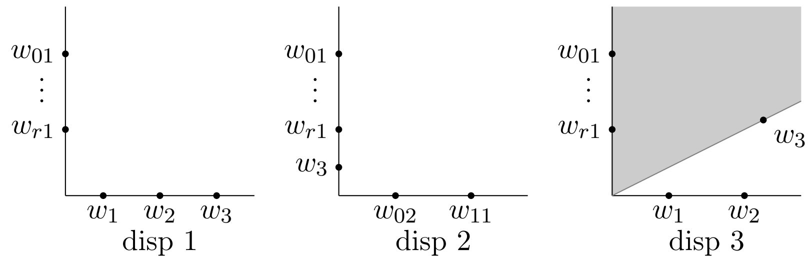
Lemma 3.2.5. *Consider the triangle* $C := \mathrm{conv}(e_2, e_3, -e_2 - e_3) \subset \mathbb{Q}^3$. *Let* $x \in \mathbb{Z}^3$ *be a lattice point with first coordinate* $x_1 \geq 2$. *If the only lattice points of* $\mathrm{conv}(C, x)$ *are its four vertices and the origin, then* $x_1 = 3$ *holds.*

Proof. Note that $\mathrm{conv}(C, x)$ always contains its four vertices and the origin. A further lattice point y lies in it if and only if $x \in \mathrm{sw}(C, y)$ holds. Therefore we look for a point x that does not lie in any $\mathrm{sw}(C, y)$ for $x \neq y \in \mathbb{Z}^3$.
At height $t = 2$ all integral points are of the form $(2, a, b)$ for some $a \in \{2u, 2u + 1\}$ and $b \in \{2v, 2v + 1\}$, and they lie in the respective $\mathrm{sw}_2(C, (1, u, v))$.
For every $t \geq 4$, the union of the shadows $\mathrm{sw}_t(C, z)$, for all $z \in \mathbb{Z}^3$ with $z_1 = 1$, contains all integral points at the height t, except for the multiples of $q_1 := (3, 3u + 1, 3v - 1)$ and $q_2 := (3, 3u - 1, 3v + 1)$. Since $0 \in C$ holds, these points lie in $\mathrm{sw}(C, q_1)$ and $\mathrm{sw}(C, q_2)$, respectively. Thus the assertion follows. □

Recall that a variety is called *weakly tropical* if the fan of its minimal ambient toric variety is supported on $\mathrm{trop}(X)$. This means that there are only leaf cones.

Proof of Proposition 3.2.2. By Proposition 3.2.1 we only have to consider $\rho(X) \leq 3$. The assertion is clear for $\rho(X) = 1$, since $\mathrm{Eff}(X) = \mathbb{Q}_{\geq 0}$ and $\mu > 0$ hold.

Turn to the case $\rho(X) = 2$. Then we have $n + m = r + 4$, which implies $m \leq 3$. Suppose that $\mu \notin \mathrm{Eff}(X)^\circ$ holds. Then Proposition 3.1.13 yields $m \geq 2$ and we have only two possible constellations:
(a) $m = 2$, $r = 2$ and $\overline{n} = (2, 1, 1)$,
(b) $m = 3$, $r \geq 2$ and $\overline{n} = (1, \ldots, 1)$.
By combinatorial minimality, constellation (a) can only happen if the weights of the two free variables lie on one of the two extremal rays of $\mathrm{Eff}(X)$ and all the weights w_{ij} lie on the other extremal ray. This means that the variety is a product of $\mathbb{P}_1$ with a del Pezzo surface. By [27, Prop. 5.10] there are no non-toric terminal del Pezzo $\mathbb{K}^*$-surfaces, hence this case is not compatible with the assumptions. Constellation (b) allows three weight dispositions:



Note that, by almost freeness of the grading, in all three cases one can assume $\mathrm{Eff}(X) = \mathbb{Q}^2_{\geq 0}$. The first two dispositions correspond to products of varieties, hence we rule them out just like before. Consider disposition 3. By Proposition 2.3.1 and irredundancy of P, there cannot be elementary big cones, hence X is weakly tropical. Therefore $-\mathcal{K}_X$ lies in $\mathrm{cone}(w_{01}, w_3)^\circ$, the cone colored in grey in the picture above. Since $m = 3$, we can assume, with admissible operations, that the last three columns of P are $(0,\dots,0,1,0)$, $(0,\dots,0,0,1)$ and $(0,\dots,0,-1,-1)$. These three points are also the vertices of the lineality part $A^c_{X,0}$. Consider now the i-th leaf $A^c_X \cap \lambda_i$ of the anticanonical complex, given as the convex hull of $A^c_{X,0}$ and v_{i1}, since $n_i = 1$. Terminality implies that $A^c_X \cap \lambda_i$ does not contain additional integral points. By Lemma 3.2.5 we follow $l_{i1} = 3$ for all $i = 0,\dots,r$. This yields $w_{i1} = (0,1)$ and hence $\mu = (0,3)$. The torsion-free part of the anticanonical class in $K_{\mathbb{Q}}$ is

$$-\mathcal{K}_X \;=\; (r+1)\begin{pmatrix}0\\1\end{pmatrix} + w_1 + w_2 + w_3 - (r-1)\begin{pmatrix}0\\3\end{pmatrix} \;=\; \begin{pmatrix}w_1^1 + w_2^1 + w_3^1\\ 4-2r+w_3^2\end{pmatrix}.$$

Since $r \geq 2$, the anticanonical class does not lie in $\mathrm{cone}(w_{01}, w_3)$, a contradiction.

Lastly consider $\rho(X) = 3$. Assume that $\mu \notin \mathrm{Eff}(X)^\circ$ holds. Proposition 3.1.13 yields $m \geq 2$. We also have the relation $n + m = r + 5$, hence $m \leq 4$. Note that $m = 3$ is excluded by Lemma 3.1.6. Therefore we have three constellations:

(a) $m = 2$, $r = 2$ and $\overline{n} = (2,2,1)$,
(b) $m = 2$, $r = 3$ and $\overline{n} = (2,2,1,1)$,
(c) $m = 4$, $r \geq 2$ and $\overline{n} = (1,\dots,1)$.

We treat both constellations (a) and (b) at once. By combinatorial minimality the effective cone $\mathrm{Eff}(X)$ is simplicial and the three extremal rays are

$$\mathrm{cone}(w_{01}) = \mathrm{cone}(w_{11}), \qquad \mathrm{cone}(w_{02}) = \mathrm{cone}(w_{12}), \qquad \mathrm{cone}(w_1) = \mathrm{cone}(w_2).$$

We apply Remark 3.3.1 to the relevant face $\gamma_{01,12,1,2}$ and achieve $\mathrm{Eff}(X) = \mathbb{Q}^3_{\geq 0}$. Thus we are looking at a product of $\mathbb{P}_1$ with a surface. This is a contradiction, as already seen before. In the constellation (c) every monomial consists of only one variable, hence the respective weights all lie on the same extremal ray of $\mathrm{Eff}(X)$, while the weights of the free variables lie on the other two extremal rays, two each. Since the grading is almost free, we can assume that $\mathrm{Eff}(X) = \mathbb{Q}^3_{\geq 0}$ holds, hence X is product of three curves, again a contradiction to complexity one. □

Remark 3.2.6. The result of Proposition 3.2.2 does not hold if we allow $X = X(A,P)$ to have non-terminal singularities. Consider any $\mathbb{Q}$-factorial log terminal Fano $\mathbb{K}^*$-surface S with $\rho(S) = 1$, e.g. the E_6-surface from Example 1.5.9. Then $S \times \mathbb{P}_1$ is a $\mathbb{Q}$-factorial log terminal (but not terminal) Fano variety of complexity one with Picard number two, such that the degree μ lies on an extremal ray of its effective cone.

Here comes the Lemma that states the existence of finitely many possible shapes for the defining matrix P and provides the list of those shapes.

Lemma 3.2.7. *Let $X = X(A,P)$ be a non-toric, three-dimensional, combinatorially minimal, terminal, $\mathbb{Q}$-factorial Fano variety, where the matrix P is irredundant and $\rho(X) > 1$ holds. Then, after suitable admissible operations, P fits into one of the following cases:*

(i) We have $\rho(X) = 3$ and one of the following constellations:
- *(a) $m = 0$, $r = 2$ and $n = 7$, where $\overline{n} = (3, 3, 1)$.*
- *(b) $m = 0$, $r = 3$ and $n = 8$, where $\overline{n} = (3, 3, 1, 1)$.*
- *(c) $m = 0$, $r = 3$ and $n = 8$, where $\overline{n} = (2, 2, 2, 2)$.*
- *(d) $m = 0$, $r = 4$ and $n = 9$, where $\overline{n} = (2, 2, 2, 2, 1)$.*
- *(e) $m = 0$, $r = 5$ and $n = 10$, where $\overline{n} = (2, 2, 2, 2, 1, 1)$.*

(ii) We have $\rho(X) = 2$ and one of the following constellations:
- *(a) $m = 0$, $r = 2$ and $n = 6$, where $\overline{n} = (2, 2, 2)$.*
- *(b) $m = 0$, $r = 3$ and $n = 7$, where $\overline{n} = (2, 2, 2, 1)$.*
- *(c) $m = 0$, $r = 4$ and $n = 8$, where $\overline{n} = (2, 2, 2, 1, 1)$.*
- *(d) $m = 0$, $r = 2$ and $n = 6$, where $\overline{n} = (3, 2, 1)$.*
- *(e) $m = 0$, $r = 3$ and $n = 7$, where $\overline{n} = (3, 2, 1, 1)$.*
- *(f) $m = 0$, $r = 2$ and $n = 6$, where $\overline{n} = (4, 1, 1)$.*
- *(g) $m = 1$, $r = 2$ and $n = 5$, where $\overline{n} = (2, 2, 1)$.*
- *(h) $m = 1$, $r = 3$ and $n = 6$, where $\overline{n} = (2, 2, 1, 1)$.*
- *(i) $m = 1$, $r = 2$ and $n = 5$, where $\overline{n} = (3, 1, 1)$.*
- *(j) $m = 2$, $r = 2$ and $n = 4$, where $\overline{n} = (2, 1, 1)$.*

Proof. First of all, note that we have $\rho(X) \leq 3$ and $\mu \in \mathrm{Eff}(X)^\circ$, by Proposition 3.2.1 and Proposition 3.2.2 respectively.

Case $\rho(X) = 3$: here we have the relation

$$n + m = r + 5 \tag{3.1}$$

coming from $n + m = r + s + \rho(X)$. Since $n \geq r + 1$ always holds, we obtain $m \leq 4$.

The case $m = 4$ does not fit combinatorial minimality in the following sense: according to (3.1) we would have $n = r + 1$, meaning that every monomial consists only of one variable; therefore all their weights must lie in the interior of the effective cone and consequently the extremal variables are maximal 4, contradicting the fact that there are at least $2\rho(X) = 6$ extremal variables.

The case $m = 3$ is excluded by Lemma 3.1.6. Therefore we have $m < 3 = \dim(X)$ and there is always a big elementary cone. This implies $r \leq 5 - m$ by Corollary 2.3.2. Combining Proposition 2.3.1 with the assumption that P is irredundant we see that at most two of the n_i equal one.

If $m = 2$ holds, then we have following constellations:

- $\overline{n} = (3, 1, 1)$: here only 5 variables can be extremal, a contradiction to combinatorial minimality.
- $\overline{n} = (2, 2, 1)$: there are exactly 6 extremal variables, so $\alpha = 0$ holds. Every possible disposition contradicts $\mu \in \mathrm{Eff}(X)^\circ$.
- $\overline{n} = (2, 2, 1, 1)$: there are exactly 6 extremal variables, so $\alpha = 0$ holds. Every possible disposition contradicts $\mu \in \mathrm{Eff}(X)^\circ$.

If $m = 1$ holds, then we have following constellations:

- $\overline{n} = (4, 1, 1)$: here only 5 variables can be extremal, a contradiction to combinatorial minimality.

- $\overline{n} = (3, 2, 1)$: there are exactly 6 extremal variables, so $\alpha = 0$ holds. Every possible disposition contradicts $\mu \in \mathrm{Eff}(X)^\circ$.
- $\overline{n} = (2, 2, 2)$: there are exactly seven variables that can be extremal, so $\alpha = 0$ holds. Every possible disposition contradicts $\mu \in \mathrm{Eff}(X)^\circ$ or combinatorial minimality.
- $r > 2$: any constellation is obtained from the previous ones by adding a monomial $T_r^{l_r}$ with $n_r = 1$. This means that w_{r1} lies on the half-line spanned by μ and the same contradictions as above arise.

If $m = 0$ holds, then we have following constellations:

- $\overline{n} = (5, 1, 1)$: here only 5 variables can be extremal, a contradiction to combinatorial minimality.
- $\overline{n} = (4, 2, 1)$: there are exactly 6 extremal variables, so $\alpha = 0$ holds end every w_{ij} with $n_i \neq 1$ lies on an extremal ray of $\mathrm{Eff}(X)$. In particular $n_1 = 2$ implies $\mu \notin \mathrm{Eff}(X)^\circ$, a contradiction.
- $\overline{n} = (3, 3, 1)$: this is case (a).
- $\overline{n} = (3, 2, 2)$: of the 7 variables at least 6 are extremal. In particular we can assume that both variables T_{11} and T_{12} are extremal. This implies $\mu \notin \mathrm{Eff}(X)^\circ$, a contradiction.
- $\overline{n} = (4, 2, 1, 1)$: same contradiction as in $\overline{n} = (4, 2, 1)$.
- $\overline{n} = (3, 3, 1, 1)$: this is case (b).
- $\overline{n} = (3, 2, 2, 1)$: same contradiction as in $\overline{n} = (3, 2, 2)$.
- $\overline{n} = (2, 2, 2, 2)$: this is case (c).
- $\overline{n} = (3, 2, 2, 1, 1)$: same contradiction as in $\overline{n} = (3, 2, 2)$.
- $\overline{n} = (2, 2, 2, 2, 1)$: this is case (d).
- $\overline{n} = (2, 2, 2, 2, 1, 1)$: this is case (e).

Case $\rho(X) = 2$: here we have $n + m = r + 4$, which implies $m \leq 3$. If $m = 3$ holds, then so does $n = r + 1$, i.e., every monomial consists of only one variable. In particular, since $\mu \in \mathrm{Eff}(X)^\circ$, their weights all lie in the interior of the effective cone. There are at most three exceptional weights, namely the ones corresponding to the free variables. This contradicts the fact that there are at least $2\rho(X)$ extremal variables. Therefore we have $m < 3 = \dim(X)$ and there is always a big elementary cone. By Corollary 2.3.2 we obtain $r \leq 4 - m$. Combining Proposition 2.3.1 with the assumption that P is irredundant we see that at most two of the n_i equal one. All possible constellations are listed in the assertion. □

In the next Proposition we show that we can exclude the constellations with Picard number equal to three.

Proposition 3.2.8. *Let $X = X(A, P)$ be non-toric, three-dimensional, combinatorially minimal, terminal, $\mathbb{Q}$-factorial and Fano. Then $\rho(X) \leq 2$ holds.*

Proof. If $\rho(X) > 1$, we can assume P irredundant and apply Lemma 3.2.7. Therefore we need to exclude the constellations of part (i), where $\rho(X) = 3$ holds.

Case (a): the weights w_{0i} and w_{1j}, for $i, j = 1, 2, 3$, all lie on extremal rays of $\mathrm{Mov}(X)$ by combinatorial minimality. In particular $\mathrm{Mov}(X)$ is simplicial. We can always assume that $\operatorname{cone}(w_{0i}) = \operatorname{cone}(w_{1i})$ holds. The elementary big cones are precisely those of type $\operatorname{cone}(v_{0i}, v_{1j}, v_{21})$, with $i \neq j$. By Proposition 2.3.1 the corresponding triples of exponents must all contain at least one element equal to 1. Up to rearranging of variables, there are only two possible exponents configurations:

A $l_0 = (1, 1, 1)$;

B $l_0 = (1, 1, l_{03})$ and $l_1 = (1, 1, l_{13})$.

In configuration A, admissible operations lead to

$$P = \begin{bmatrix} -1 & -1 & -1 & l_{11} & l_{12} & l_{13} & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & l_{21} \\ 0 & 1 & 0 & d_{111} & d_{112} & d_{113} & d_{121} \\ 0 & 0 & 1 & d_{211} & d_{212} & d_{213} & d_{221} \end{bmatrix}.$$

By applying Corollary 3.1.5 on the sets $\{2,5,6\}$, $\{1,4,6\}$, $\{2,4,5\}$, $\{3,5,6\}$, $\{3,4,6\}$, $\{1,4,5\}$, we obtain

$$d_{111} = -\frac{l_{11}}{l_{21}} d_{121}, \qquad d_{112} = -\frac{l_{12}}{l_{21}} d_{121} - l_{12}, \qquad d_{113} = -\frac{l_{13}}{l_{21}} d_{121},$$
$$d_{211} = -\frac{l_{11}}{l_{21}} d_{221}, \qquad d_{212} = -\frac{l_{12}}{l_{21}} d_{221}, \qquad d_{213} = -\frac{l_{13}}{l_{21}} d_{221} - l_{13}.$$

This matrix delivers very special vertices for the lineality part $A^{c}_{X,0}$; some of them are

$$\Big(\frac{l_{11}l_{21}}{l_{11}+l_{21}}, 0\Big), \qquad \Big(-\frac{l_{12}l_{21}}{l_{12}+l_{21}}, 0\Big), \qquad \Big(0, -\frac{l_{13}l_{21}}{l_{13}+l_{21}}\Big).$$

Terminality and $l_{21} \geq 2$ imply $l_{11} = l_{12} = l_{13} = 1$ and by admissible operations we can assume $d_{111} = d_{211} = 0$. The previous equations force $d_{121} = d_{221} = 0$, contradicting the primitivity of v_{21}.

Turning to configuration B, admissible operations lead to the shape

$$P = \begin{bmatrix} -1 & -1 & -l_{03} & 1 & 1 & l_{13} & 0 \\ -1 & -1 & -l_{03} & 0 & 0 & 0 & l_{21} \\ 0 & 1 & d_{103} & 0 & d_{112} & d_{113} & d_{121} \\ 0 & 0 & d_{203} & 0 & d_{212} & d_{213} & d_{221} \end{bmatrix}.$$

We apply Corollary 3.1.5 to $\{3,5,6\}$ and $\{3,6,7\}$ and obtain $d_{221} = d_{212} = 0$. Similarly we have $0 = \det(P_{\{2,3,5\}}) = -d_{121}d_{213}$ but neither of the two values may equal zero. If $d_{121} = 0$, then v_{21} is not a primitive vector, whereas if $d_{213} = 0$, then the columns of P do not generate $\mathbb{Q}^4$ as a cone.

Case (*b*): the situation is similar to the previous case. Combining terminality, Proposition 2.3.1 and admissible operations, we can achieve the form

$$P = \begin{bmatrix} -1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & l_{21} & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & l_{31} \\ 0 & 1 & 0 & 0 & d_{112} & d_{113} & d_{121} & d_{131} \\ 0 & 0 & 1 & 0 & d_{203} & d_{212} & d_{213} & d_{221} \end{bmatrix},$$

with $\mathrm{cone}(w_{0j}) = \mathrm{cone}(w_{1j})$ for $j = 1, 2, 3$. Applying Corollary 3.1.5 to $\{2, 5, 7\}$ and $\{3, 6, 7\}$, we get $d_{113} = d_{212} = 0$ and, with $\{1, 4, 8\}$, also $d_{112} = d_{213}$. Using once again the same Corollary, this time with $\{1, 2, 5\}$ and $\{1, 3, 6\}$, we obtain

$$d_{131} = -\frac{l_{31}}{l_{21}} d_{121}, \qquad d_{231} = -\frac{l_{31}}{l_{21}} d_{221}.$$

The vertices of the lineality part $A^c_{X,0} \subset \mathbb{Q}^2$ yield to a contradiction to terminality. For example, the vertex defined by the big elementary cone $P(\delta_{01,12,21,31})$ is

$$\Big(0, 0, 0, \frac{l_{21}l_{31}}{l_{21} + l_{31}} d_{112}, 0\Big).$$

Since $d_{112} \neq 0$ and $l_{21}l_{31} \geq 2$ hold, the absolute value of the non-zero entry is bigger or equal to one, i.e. there is a non-zero integral point in $A^c_{X,0}$.

Case (c): all weights w_{ij}, for $i = 0, 1, 2, 3$ and $j = 1, 2$, lie on extremal rays. In particular we can always assume $\mathrm{cone}(w_{0j}) = \mathrm{cone}(w_{1j})$ and $\mathrm{cone}(w_{2j}) = \mathrm{cone}(w_{3j})$ for $j = 1, 2$. As a consequence, following cones are among the elementary big ones:

$$\mathrm{cone}(v_{0,i}, v_{1,3-i}, v_{2,j}, v_{3,3-j}), \quad i, j = 1, 2.$$

Therefore there are only three configurations for the exponents, up to renumbering of variables and of monomials:

A $l_0 = l_1 = (1, 1)$;

B $l_0 = l_2 = (1, 1)$;

C $l_{01} = l_{11} = l_{21} = l_{31} = 1$.

In configuration A we can achieve for P the shape

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{21} & l_{22} & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & l_{31} & l_{32} \\ 0 & 1 & 0 & d_{112} & d_{121} & d_{122} & d_{131} & d_{132} \\ 0 & 0 & 0 & d_{212} & d_{221} & d_{222} & d_{231} & d_{232} \end{bmatrix}.$$

We apply Corollary 3.1.5 on the set $\{6, 7, 8\}$ and obtain $d_{212} = 0$. Terminality implies that $\mathrm{conv}(0, v_{11}, v_{12}) \subset A^c_X$ does not contain integral points other than its vertices, hence $d_{112} = \pm 1$ holds. The vertices of $A^c_{X,0}$ defined by the above-mentioned big elementary cones are:

$$\begin{aligned} u_1 &= \Big(\frac{l_{21}l_{32} + l_{21}d_{132} + l_{32}d_{121}}{l_{21} + l_{32}}, \frac{l_{21}d_{232} + l_{32}d_{221}}{l_{21} + l_{32}}\Big), \\ u_2 &= u_1 + \frac{l_{21}l_{32}}{l_{21} + l_{32}}\Big(1 - d_{112}, 0\Big), \\ u_3 &= \Big(\frac{l_{22}l_{31} + l_{22}d_{131} + l_{31}d_{122}}{l_{22} + l_{31}}, \frac{l_{22}d_{231} + l_{31}d_{222}}{l_{22} + l_{31}}\Big), \\ u_4 &= u_3 + \frac{l_{22}l_{31}}{l_{22} + l_{31}}\Big(1 - d_{112}, 0\Big). \end{aligned}$$

Therefore $d_{112} = -1$ holds. At least one among the still unbounded l-values must be equal to 1, otherwise the length of $A^c_{X,0} \cap \{y = 0\}$ is at least 2, in contradiction to terminality of X. Without loss of generality let $l_{21} = 1$. Then we can assume $d_{121} = d_{221} = 0$. Using once again Corollary 3.1.5, this time on $\{4, 6, 8\}$ and afterwards on $\{1, 3, 8\}$, we obtain $d_{231} = d_{131} = 0$. Since v_{31} is primitive, we also have $l_{31} = 1$. Now it is possible to show that $l_{22} = l_{32}$ must hold. First, we see that Corollary 3.1.5 applies to $\{4, 5, 7\}$ and $\{1, 3, 4\}$. This allows to write v_{22} and v_{32} in the following forms

$$\begin{aligned} v_{22} &= (0, l_{22}, 0, d_{122}, d_{222}) = \left(0, l_{22}, 0, -\frac{l_{22}}{l_{32}} d_{132}, -\frac{l_{22}}{l_{32}} d_{232}\right), \\ v_{32} &= (0, 0, l_{32}, d_{132}, d_{232}) = \left(0, 0, l_{32}, -\frac{l_{32}}{l_{22}} d_{122}, -\frac{l_{32}}{l_{22}} d_{222}\right). \end{aligned}$$

Primitivity of these vectors implies that both l_{22}/l_{32} and l_{32}/l_{22} are integers, thus $l_{22} = l_{32}$. Assume now $l_{22} > 1$. Then $-\mathcal{K}_X \in Q(\gamma_{2,3,5,7})^\circ = Q(\gamma_{1,4,5,7})^\circ$ holds, hence $P(\delta_{1,4,6,8})$ and $P(\delta_{2,3,6,8})$ are elementary big cones. These define vertices $(-l_{31}/2, 0)$ and $(l_{31}/2, 0)$ for the lineality part, so the variety X is not terminal. The remaining case $l_{22} = l_{32} = 1$ is invalid as well, because the resulting variety is not $\mathbb{Q}$-factorial: among others $\gamma_{1,4}$ is a relevant face whose projected cone is not full-dimensional.
Now we turn to configuration B. First we use admissible operations to achieve

$$P = \begin{bmatrix} -1 & -1 & l_{11} & l_{12} & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & l_{31} & l_{32} \\ 0 & 1 & d_{111} & d_{112} & 0 & d_{122} & d_{131} & d_{132} \\ 0 & 0 & d_{211} & d_{212} & 0 & d_{222} & d_{231} & d_{232} \end{bmatrix}.$$

By subsequently using Corollary 3.1.5 on the sets $\{4, 6, 8\}$, $\{3, 6, 8\}$, $\{4, 5, 7\}$, $\{2, 4, 8\}$, $\{1, 3, 6\}$ and $\{1, 3, 4\}$, we can express the d-values of the columns v_{11}, v_{12} and v_{32} in terms of some l-values and entries of v_{31}. Following the idea already used for the previous configuration, primitivity of these vectors ultimately yields $l_{11} = l_{12} = l_{31} = l_{32}$. The lineality part $A^c_{X,0}$ is now a trapezoid with two edges parallel to the x-axis, both of length l_{11}. Terminality implies $l_{11} = 1$, hence we are again in configuration A.
Lastly we want to discharge configuration C. We use the same principles used up to this point: first we use admissible operations to let appear as many zeros as possible in P, then we use Corollary 3.1.5 to express some d-values in terms of others and last we show that, by primitivity of the columns of P, $l_{02} = l_{12}$ and $l_{22} = l_{32}$ hold. If one of the two is equal to one, we reduce to configuration A. If both are strictly greater than one, then we have

$$-\mathcal{K}_X = \sum_{i,j} w_{ij} - 2\mu = w_{11} + w_{31} + (2 - l_{12})w_{12} + (2 - l_{32})w_{32},$$

and as such X is not Fano, since $-\mathcal{K}_X$ does not lie in the interior of the moving cone.
Case (d): the disposition of the weights is the same as in the previous case, with the new weight w_{41} lying in the interior of the effective cone. In this case we only have two configurations:

A $l_0 = l_1 = l_2 = (1,1)$;
B $l_0 = l_1 = (1,1)$ and $l_{21} = l_{31} = 1$.
Take configuration A; after admissible operations we get

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & l_{31} & l_{32} & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} \\ 0 & 1 & 0 & d_{112} & 0 & d_{122} & d_{131} & d_{132} & d_{141} \\ 0 & 0 & 0 & d_{212} & 0 & d_{222} & d_{231} & d_{232} & d_{241} \end{bmatrix}.$$

With Corollary 3.1.5, applied to $\{6,7,8\}$, $\{4,6,8\}$, $\{4,5,7\}$, $\{2,4,8\}$ and $\{1,3,8\}$, we are able to express some d-values in terms of others, namely

$$d_{212} = 0, \qquad d_{231} = -\frac{l_{31}}{l_{41}} d_{241}, \qquad d_{232} = -\frac{l_{32}}{l_{41}} d_{241} - l_{32} d_{222},$$

$$d_{131} = -\frac{l_{31}}{l_{41}} d_{141}, \qquad d_{112} = -1.$$

Consider the vertices of the lineality part $A^c_{X,0}$ defined by big elementary cones of type cone$(v_{0i}, v_{1,3-i}, v_{2j}, v_{3,3-j}, v_{41})$ for $i,j = 1,2$. These define a trapezoid, with two edges, L_1 and L_2, parallel to the x-axis, whose lengths $l(L_i)$ and heights $h(L_i)$ are

$$\begin{aligned} l(L_1) &= 2\frac{l_{31} l_{41}}{l_{31} + l_{41}}, & l(L_2) &= 2\frac{l_{32} l_{41}}{l_{32} + l_{41}}, \\ h(L_1) &= d_{222}\frac{l_{31} l_{41}}{l_{31} + l_{41}}, & h(L_2) &= d_{222}\frac{l_{32} l_{41}}{l_{32} + l_{41}}. \end{aligned}$$

Since $l_{41} \geq 2$, both lengths are at least 1, thus the heights are smaller than 1 in absolute value, in order to avoid non-zero integral points in $A^c_{X,0}$. This implies $l_{31} = l_{32} = 1$. Looking back at d_{131} and d_{231}, their representation implies that l_{41} divides both d_{141} and d_{241}. This contradicts the primitivity of v_{41}.
Turn to configuration B; after achieving the form

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & l_{22} & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & l_{32} & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} \\ 0 & 1 & 0 & d_{112} & 0 & d_{122} & 0 & d_{132} & d_{141} \\ 0 & 0 & 0 & d_{212} & 0 & d_{222} & 0 & d_{232} & d_{241} \end{bmatrix}$$

via admissible operations, we just apply Corollary 3.1.5 to the index-sets $\{6,7,8\}$, $\{4,6,8\}$ and $\{1,3,8\}$ to obtain $d_{212} = d_{241} = 0$ and $d_{141} = -l_{41}(d_{112}+1)$. It follows a contradiction to v_{41} primitive.

Case (*e*): the disposition of the weights is the same as in the previous case, with the new weight w_{51} lying on $\mathrm{cone}(w_{41})$. Due to the fact that there are two monomials with only one variable, Proposition 2.3.1 implies $l_{i1} = l_{i2} = 1$ for $i = 0, \ldots, 3$. Therefore admissible operations lead to

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & l_{41} & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{51} \\ 0 & 1 & 0 & d_{112} & 0 & d_{122} & 0 & d_{132} & d_{141} & d_{151} \\ 0 & 0 & 0 & d_{212} & 0 & d_{222} & 0 & d_{232} & d_{241} & d_{251} \end{bmatrix}.$$

Now apply Corollary 3.1.5 to the index-sets $\{6, 8, 9\}$, $\{4, 6, 8\}$, $\{3, 5, 7\}$, $\{1, 3, 10\}$, $\{2, 4, 8\}$ and $\{1, 3, 5\}$ and then look at the vertices defined by big elementary cones of the type $\mathrm{cone}(v_{0i}, v_{1,3-i}, v_{2j}, v_{3,3-j}, v_{41}, v_{51})$ for $i, j = 1, 2$. Their convex hull is a trapezoid, with two edges parallel to the x-axis, both of length at least 2. This implies that the intersection $A^c_{X,0} \cap \{y = 0\}$ contains at least one integral point other than the origin, a contradiction to terminality of X. □

3.3 The 3-dimensional case with $\rho(X) = 2$

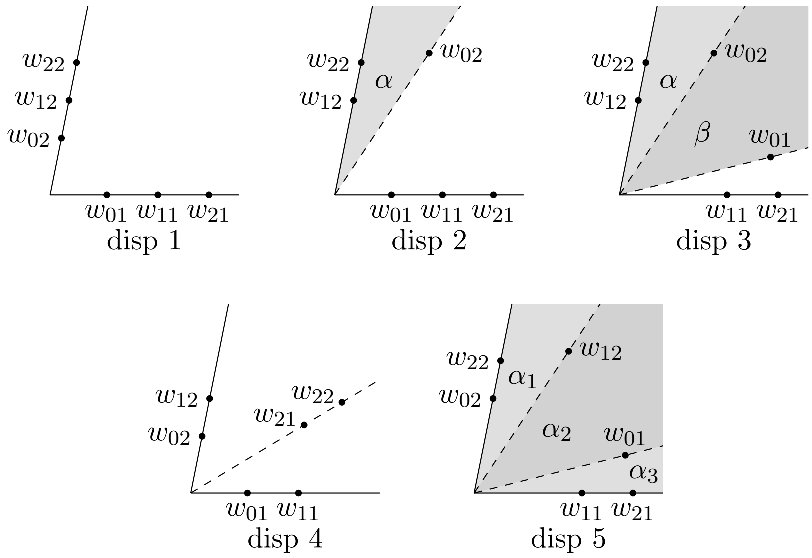
The goal of this Section is the classification of combinatorially minimal $\mathbb{Q}$-factorial terminal Fano threefolds. First we study the effect of terminality on the strata of such varieties, more precisely on the corresponding subsets of weights.

Remark 3.3.1. According to Proposition 1.4.2, the stratum $X(\gamma_0) \subset X$ consists of factorial points of $X = X(A, P, \Phi)$ if and only if $Q(\mathrm{lin}(\gamma_0) \cap \mathbb{Z}^{n+m}) = \mathrm{Cl}(X)$ holds. In dimension three, terminal singularities occur as isolated points, see e.g. [41, Cor. 4.6.6]. According to Corollary 1.5.16, every $\gamma_0 \in \mathrm{rlv}(\Phi) \setminus \mathrm{cov}(\Phi)$ defines a stratum $X(\gamma_0)$ of positive dimension. This has to be smooth, in particular factorial. Therefore its weights generate $\mathrm{Cl}(X)$ as an abelian group.

We go through the cases of Lemma 3.2.7 (*ii*) and keep the numbering introduced there. Throughout this Section, Q denotes only the torsion-free part of the actual degree matrix Q; the same holds for the degree $\mu = (\mu^1, \mu^2)$ and the weights $w = (w^1, w^2) \in \mathbb{Z}^2$.

Case (a) of Lemma 3.2.7 (ii)

We have $r = 2$, $m = 0$, $n = 6$ and $\overline{n} = (2, 2, 2)$. Combinatorial minimality prescribes at least two weights on each of the two extremal rays of $\mathrm{Eff}(X)$. All six weights may be placed on these rays, therefore we end up with five possible dispositions:



In disposition 2 and 3 define $\alpha := \mathrm{cone}(w_{02}, w_{12})^\circ$ and $\beta := \mathrm{cone}(w_{01}, w_{02})^\circ$. In disposition 5 define $\alpha_1 := \mathrm{cone}(w_{02}, w_{12})^\circ$, $\alpha_2 := \mathrm{cone}(w_{01}, w_{12})^\circ$ and $\alpha_3 := \mathrm{cone}(w_{01}, w_{11})^\circ$. With Proposition 2.3.1 we obtain a list of possible exponent configurations:

A $l_0 = (1, 1)$;
B $l_2 = (1, 1)$;
C $l_{01} = l_{11} = l_{21} = 1$;
D $l_{11} = l_{21} = 1$;
E $l_{02} = l_{12} = 1$;
F $l_{01} = l_{21} = 1$.

Due to terminality, every disposition allows only a few of these configurations, sometimes even just for restricted situations, depending on the position of the anticanonical class. The following table summarizes the totality of possible situations:

	config A	config B	config C	config D	config E	config F
disp 1	✓		✓			
disp 2	✓	✓	✓	$✓_{\alpha}$		
disp 3	✓	✓	$✓_{\alpha,\beta}$	$✓_{\alpha}$		
disp 4	✓	✓			✓	
disp 5	✓	✓	$✓_{\alpha_1,\alpha_2}$			$✓_{\alpha_1}$

The combinations of dispositions and configurations that need to be studied are marked with the sign ✓. A subscript indicates that the anticanonical class $-\mathcal{K}_X$ has to lie in the given cone(s).
This case provides the first six varieties of the table of Theorem 3.4.1, namely No. 1 and 3 from situation 1A, No. 2 and 4 from 1C, No. 5 from 2A and No. 6 from 4B.

Disposition 1: since all weights are located on the two extremal rays, we can assume $\mathrm{Eff}(X) = \mathbb{Q}^2_{\geq 0}$. For each w_{ij}, the two weights $w_{k\ell}$ such that $k \neq i$ and $\ell \neq j$ lie on the other extremal ray. The three together form a relevant face, to which Remark 3.3.1 applies. Hence the degree matrix assumes the form

$$Q = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right].$$

Situation 1A: admissible operations, together with equations from $P \cdot Q^T = 0$, yield

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & d_{112} & 0 & -d_{112}-1 \\ 0 & 0 & 0 & d_{212} & 0 & -d_{212} \end{bmatrix},$$

where we can also assume $0 \leq d_{112} < d_{212}$. Therefore we only need to bound d_{212}. For this, take a look at the lineality part $A^c_{X,0}$. Its vertices are

$$u_1 = \frac{1}{2}(d_{112}, d_{212}), \qquad u_2 = u_1 + \left(\frac{1}{2}, 0\right),$$
$$u_3 = \frac{1}{2}(-1, 0), \qquad u_4 = u_3 + (1, 0),$$
$$u_5 = \frac{1}{2}(-d_{112}-1, -d_{212}), \qquad u_6 = u_5 + \left(\frac{1}{2}, 0\right).$$

The value d_{212} is odd, otherwise one between u_1 and u_2 would be a lattice point, contradicting terminality. Since $A^c_{X,0}$ contains no integral point other than the origin, there are only two possibilities for (d_{112}, d_{212}), namely $(0, 1)$ and $(1, 3)$. Both define valid varieties, respectively No. 1 and No. 3.

Situation 1C: here the anticanonical class is $-\mathcal{K}_X = (2, 3-l_{02})$. Since X is a Fano variety and $\mathrm{Mov}(X) = \mathbb{Q}^2_{\geq 0}$ holds, we have $l_{02} < 3$. From now on we assume $l_{02} = 2$, because $l_{02} = 1$ has been already discussed in situation 1A. Admissible operations and $P \cdot Q^T = 0$ yield

$$P = \begin{bmatrix} -1 & -2 & 1 & 2 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & d_{112} & 0 & -d_{112}-1 \\ 0 & 0 & 0 & d_{212} & 0 & -d_{212} \end{bmatrix},$$

with $0 \leq d_{112} < d_{212}$. In order to bound d_{212}, take a look at the lineality part $A^c_{X,0}$. Its vertices are

$$u_1 = \frac{1}{3}(d_{112}, d_{212}), \qquad u_2 = \frac{1}{2}(d_{112}+1, d_{212}),$$
$$u_3 = \left(-\frac{1}{2}, 0\right), \qquad u_4 = \left(\frac{1}{3}, 0\right)$$
$$u_5 = \frac{1}{3}(-d_{112}-1, -d_{212}), \qquad u_6 = \frac{1}{2}(-d_{112}, -d_{212}).$$

Consider $C := \operatorname{conv}(u_3, u_4, u_5) \subset A^c_{X,0}$. The point u_5 lies under the bisection of the third orthant and, because of the terminality of X, C does not contain integral points. We conclude $d_{212} < 20$. Using the MDSpackage [25] we see that (d_{112}, d_{212}) can assume the values $(0, 1)$ and $(1, 3)$. These data correspond to varieties No. 2 and 4 respectively.

Disposition 2: we can apply Remark 3.3.1 to the relevant faces $\gamma_{01,11,22}$, $\gamma_{01,12,21}$ and $\gamma_{01,12,22}$ and obtain

$$Q = \left[\begin{array}{cc|cc|cc} 1 & w^1_{02} & w^1_{11} & 0 & w^1_{21} & 0 \\ 0 & w^2_{02} & 0 & 1 & 0 & 1 \end{array}\right].$$

Situation 2A: homogeneity of the relation delivers $1 + w^1_{02} = l_{11}w^1_{11} = l_{21}w^1_{21}$ and $w^2_{02} = l_{12} = l_{22}$. We show that the anticanonical class lies in $\operatorname{cone}(w_{11}, w_{02})^\circ$. If we suppose otherwise, then $\gamma_{02,12,22}$ is a relevant face and in particular $w^1_{02} = 1$. This yields $l_{11}, l_{21} \in \{1, 2\}$, but then the anticanonical class does not lie in the prescribed cone. So $-\mathcal{K}_X \in \operatorname{cone}(w_{11}, w_{02})^\circ$ holds, the face $\gamma_{02,11,21}$ is relevant and we conclude $l_{12} = 1$. Without loss of generality assume $l_{11} \le l_{21}$. The requirement $0 < \det(-\mathcal{K}_X, w_{02})$ yields

$$\frac{w^1_{02}}{1 + w^1_{02}} < \frac{l_{11} + l_{21}}{2l_{11}l_{21}}.$$

Since the left side is at least $1/2$, we get $l_{11} = 1$. Now Remark 3.3.1 with $\gamma_{11,12,21,22}$ implies $l_{21} = w^1_{02} + 1$. Substituting these equalities in the inequality above we arrive at $l_{21} < 3$, therefore we have $l_{21} = 2$ (for $l_{21} = 1$ refer to situation 2C). Taking P into account, we use admissible operations and equalities from $P \cdot Q^T = 0$ and achieve

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 1 \\ 0 & 1 & d_{111} & 0 & -2d_{111} - 1 & -1 \\ 0 & 0 & d_{211} & 0 & -2d_{211} & 0 \end{bmatrix},$$

where $0 \le d_{111} < d_{211}$ holds. In order to find an upper bound for d_{211} we turn to the lineality part $A^c_{X,0}$ of the anticanonical complex. Its vertices are

$$u_1 = \frac{1}{2}(d_{111} - 1, d_{211}), \qquad u_2 = u_1 + \left(\frac{1}{2}, 0\right),$$

$$u_3 = \frac{1}{2}(-1, 0), \qquad u_4 = u_3 + \left(\frac{5}{6}, 0\right),$$

$$u_5 = \frac{1}{3}(-2d_{111} - 1, -2d_{211}), \qquad u_6 = u_5 + \left(\frac{2}{3}, 0\right).$$

Consider $C := \operatorname{conv}(u_2, u_3, u_4) \subset A^c_{X,0}$. The point u_3 lies over the bisection of the first orthant and, because of the terminality of X, C does not contain integral points. We conclude $d_{211} < 20$. With the MDSpackage [25] we find out that (d_{111}, d_{211}) assumes the value $(0, 1)$ and delivers variety No. 5.

Situation 2B: homogeneity of the relation yields $l_{02} = l_{12} = w_{02}^2 = 1$ for the second component and $l_{01} + w_{02}^1 = l_{11}w_{11}^1 = w_{21}^1$ for the first component. With Remark 3.3.1 applied to $\gamma_{11,12,21,22}$ we conclude $w_{11}^1 = 1$, hence $w_{21}^1 = l_{21}$ and $w_{02}^1 = l_{11} - l_{01}$. We can discharge the possibility that $-\mathcal{K}_X \in \mathrm{cone}(w_{02}, w_{22})^\circ$ holds, since in that case $\gamma_{02,12,22}$ is a relevant face and $w_{02}^1 = 1$ follows, contradicting the fact that the anticanonical class lies in that prescribed cone. Thus $-\mathcal{K}_X \in \mathrm{cone}(w_{01}, w_{02})^\circ$ holds. In particular $\det(-\mathcal{K}_X, w_{02}) > 0$ holds, which implies $l_{11} = l_{01} + 1$. We use admissible operations and $P \cdot Q^T = 0$ and reach

$$P = \begin{bmatrix} -l_{01} & -1 & l_{01}+1 & 1 & 0 & 0 \\ -l_{01} & -1 & 0 & 0 & 1 & 1 \\ d_{101} & 0 & -d_{101} & -1 & 0 & 1 \\ d_{201} & 0 & -d_{201} & 0 & 0 & 0 \end{bmatrix},$$

with $0 \leq d_{101} < d_{201}$. We find bounds on d_{201} and l_{01} by considering the lineality part $A_{X,0}^c$, whose vertices are

$$u_1 = \frac{1}{l_{01}+1}(d_{101} - l_{01}, d_{201}), \qquad u_2 = u_1 + \left(\frac{l_{01}}{l_{01}+1}, 0\right),$$
$$u_3 = \frac{1}{2}(-1, 0), \qquad u_4 = \frac{1}{2l_{01}+1}(l_{01}^2 + d_{101} + l_{01}, d_{201})$$
$$u_5 = \frac{1}{l_{01}+2}(-d_{101}, -d_{201}), \qquad u_6 = u_5 + \left(\frac{l_{01}+1}{l_{01}+2}, 0\right).$$

In particular, consider the width l_C of $C := \mathrm{conv}(0, u_1, u_2, u_4)$ at the height $h(u_4)$ of u_4, i.e.

$$l_C = \frac{l_{01}(l_{01}+2)}{2l_{01}+1}.$$

Since $l_{01} > 1$ (otherwise we are in configuration A), the width l_C is greater than 1 and as a consequence $h(u_4) < 1$ holds by terminality. This gives $d_{201} \leq 2l_{01}$. The length of the line segment $A_{X,0}^c \cap \{y = 0\}$ increases when l_{01} increases. By terminality, it cannot be greater than 2, hence we conclude $l_{01} < 5$. The MDSpackage [25] finds a lattice point in A_X^c for each variety defined by such data, hence this situation does not provide terminal varieties.

Situations 2C *and* 2D: homogeneity delivers $w_{11}^1 = w_{21}^1 = l_{01} + l_{02}w_{02}^1$. Hence, by Remark 3.3.1, the relevant face $\gamma_{11,12,21,22}$ yields $w_{11}^1 = 1$. Since all terms on the right side of the equation are greater or equal to one, we reach a contradiction.

Disposition 3: we start with the following degree matrix Q:

$$Q = \left[\begin{array}{cc|cc|cc} w_{01}^1 & w_{02}^1 & w_{11}^1 & 0 & w_{21}^1 & 0 \\ w_{01}^2 & w_{02}^2 & 0 & w_{12}^2 & 0 & w_{22}^2 \end{array}\right],$$

where we could assume $\mathrm{Eff}(X) = \mathbb{Q}_{\geq 0}^2$ thanks to Remark 3.3.1 applied to $\gamma_{11,12,21,22}$. We can assume that $-\mathcal{K}_X \in \mathrm{cone}(w_{01}, w_{12})^\circ$ holds, thus $\gamma_{01,12,22}$ is a relevant face and

$w^1_{01} = 1$ follows. If $-\mathcal{K}_X \in \mathrm{cone}(w_{02}, w_{12})^\circ$ holds, then $\gamma_{02,11,21}$ is a relevant face and $w^2_{02} = 1$ would hold, contradicting $\det(w_{01}, w_{02}) > 0$. Hence $-\mathcal{K}_X \in \mathrm{cone}(w_{01}, w_{02})^\circ$, $\gamma_{02,12,22}$ is a relevant face and $w^1_{02} = 1$ holds.

Situation 3A: by homogeneity of the relation $2 = \mu^1 = l_{11}w^1_{11} = l_{21}w^1_{21}$ holds. At least one of the exponents is equal to two, since $\gamma_{11,12,21,22}$ is relevant and Remark 3.3.1 can be applied to it. Let $l_{11} = 2$, so $l_{21} \in \{1, 2\}$. Using admissible operations we can assume $d_{101} = d_{201} = d_{202} = 0$ and $d_{102} = 1$. Then Proposition 3.1.4 with $\{3, 5\}$ and $\{4, 6\}$, together with equations coming from $P \cdot Q^T = 0$, delivers

$$d_{121} = -\frac{1}{2}l_{21}(d_{111} + 1), \qquad d_{221} = -\frac{l_{21}d_{211}}{2}, \qquad d_{222} = -\frac{l_{22}d_{212}}{l_{12}}.$$

The vertices of the lineality part $A^c_{X,0} \subset \mathbb{Q}^2$ are

$$u_1 = \frac{l_{21}(2d_{112} - l_{12}d_{111} - l_{12}, 2d_{212} - l_{12}d_{211})}{2(l_{12} + l_{21})}, \qquad u_2 = u_1 + \left(\frac{l_{12}l_{21}}{l_{12} + l_{21}}, 0\right),$$
$$u_3 = \frac{l_{21}}{l_{21} + 2}(-1, 0), \qquad u_4 = u_3 + \left(\frac{2l_{21}}{l_{21} + 2}, 0\right),$$
$$u_5 = \frac{(l_{12}(2d_{122} + l_{22}d_{111}), l_{22}(l_{12}d_{211} - 2d_{212}))}{l_{12}(l_{22} + 2)}, \qquad u_6 = u_5 + \left(\frac{2l_{22}}{l_{22} + 2}, 0\right).$$

We go through both cases $l_{21} = 1, 2$.
First assume that $l_{21} = 1$ holds. Then we achieve $d_{121} = d_{221} = 0$ by admissible operations, and $P \cdot Q^T = 0$ also yields $d_{111} = -1$. In order for u_3 and u_4 to be both vertices, $l_{12} = 1$ must hold. For $l_{22} > 1$, the intersection of $\mathrm{conv}(0, u_5, u_6)$ with the line $\{y = -1\}$ has length one, thus contains an integral point and contradicts terminality, whereas $l_{22} = 1$ will be handled in situation 3B.
Now assume that $l_{21} = 2$ holds. By admissible operations we achieve $0 \le d_{111}, d_{211} < 2$. Since v_{11} and v_{21} are primitive, we arrive at $d_{111} = 0$ and $d_{211} = 1$. In order for u_3 and u_4 to be both vertices, at least one between l_{12} and l_{22} is equal to one. Since Remark 3.3.1 applies to $\gamma_{11,12,21,22}$, they cannot be both equal to one. Without loss of generality say $l_{12} = 1$ and $l_{22} \ge 2$. Therefore $A^c_{X,0} \cap \{y = 0\}$ has length one and the length of the edge $\overline{u_5u_6}$ is at least one. This means $|u^2_5| < 1$, i.e. $d_{212} = 0, 1$. Using homogeneity in the second component and once again $P \cdot Q^T = 0$ we arrive at

$$Q = \left[\begin{array}{cc|cc|cc} 1 & 1 & 1 & 0 & 1 & 0 \\ l_{22} + d_{122} & -d_{122} & 0 & l_{22} & 0 & 1 \end{array}\right].$$

In particular the anticanonical class is $-\mathcal{K}_X = (2, l_{22} + 1)$. The inequalities coming from $\det(w_{02}, -\mathcal{K}_X) > 0$ and $\det(w_{01}, w_{02}) > 0$ are incompatible with d_{122} being an integer:

$$-\frac{1}{2}l_{22} - \frac{1}{2} < d_{122} < -\frac{1}{2}l_{22}.$$

Therefore we reach a contradiction.

Situation 3B: homogeneity delivers $l_{01} + l_{02} = l_{11}w_{11}^1 = w_{21}^1$ and $l_{12}w_{12}^2 = w_{22}^2$. With $\gamma_{11,12,21,22}$ relevant face we conclude $w_{11}^1 = 1 = w_{12}^2$. Therefore the anticanonical class is $-\mathcal{K}_X = (3, w_{01}^1 + w_{02}^1 + 1)$. Now $\det(w_{01}, w_{02}) > 0$ and $\det(w_{02}, -\mathcal{K}_X) > 0$ yield $w_{02}^2 < 1$, a contradiction to the disposition.

Situations 3C *and* 3D: the same argument as situations 2C and 2D above works here too.

Disposition 4: without loss of generality we assume that $-\mathcal{K}_X \in \mathrm{cone}(w_{01}, w_{21})^\circ$ holds. In particular $\gamma_{01,11,21}$ and $\gamma_{01,11,22}$ are relevant faces, to which Remark 3.3.1 applies; we arrive at

$$Q = \left[\begin{array}{cc|cc|cc} w_{01}^1 & 0 & w_{11}^1 & 0 & w_{21}^1 & w_{21}^1 \\ 0 & w_{02}^2 & 0 & w_{12}^2 & 1 & 1 \end{array}\right].$$

Situation 4A: we use homogeneity of the relation and Remark 3.3.1 together with relevant faces $\gamma_{01,02,11,12}$ and $\gamma_{01,02,12,21,22}$ in the usual ways and achieve

$$Q = \left[\begin{array}{cc|cc|cc} l_{11} & 0 & 1 & 0 & 1 & 1 \\ 0 & l_{11} & 0 & 1 & 1 & 1 \end{array}\right].$$

Moreover we use admissible operations on P to achieve $d_{101} = d_{201} = d_{202} = 0$ and $d_{201} = 1$. The equations of $P \cdot Q^T = 0$ allow us to write

$$\begin{aligned} v_{12} &= (l_{12}, 0, d_{112}, d_{212}), \\ v_{11} &= v_{12} + (0, 0, l_{21} + l_{22}, 0). \end{aligned}$$

This means that there are integral points on the segment between v_{11} and v_{12}, contradicting terminality by Theorem 2.1.10.

Situation 4B: we have $l_{02}w_{02}^2 = l_{12}w_{12}^2 = 2$. We may assume that one of those exponents is greater than one, otherwise see situation 4E. Without loss of generality let $l_{12} = 2$ and $w_{12}^2 = 1$. Using all the equations coming from $P \cdot Q^T = 0$ we can write

$$P = \left[\begin{array}{cccccc} -l_{01} & -l_{02} & l_{11} & 2 & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 1 & 1 \\ -\frac{l_{01}d_{111}}{l_{11}} - \frac{1}{2}l_{01} & -\frac{1}{2}l_{02}(d_{112}+1) & d_{111} & d_{112} & 0 & 1 \\ -\frac{l_{01}d_{211}}{l_{11}} & -\frac{1}{2}l_{02}d_{212} & d_{211} & d_{212} & 0 & 0 \end{array}\right],$$

$$Q = \left[\begin{array}{cc|cc|cc} \frac{2}{l_{01}}w_{21}^1 & 0 & \frac{2}{l_{11}}w_{21}^1 & 0 & w_{21}^1 & w_{21}^1 \\ 0 & \frac{2}{l_{02}} & 0 & 1 & 1 & 1 \end{array}\right].$$

In particular we see that $l_{02} = 1, 2$ holds.

First we rule out the case $l_{02} = 1$; if it holds, we can reach $d_{112} = 1$ and $d_{212} = 0$ by means of admissible operations. The vertices of $A^c_{X,0}$ are

$$u_1 = \frac{1}{l_{11}+1}(d_{111} - l_{11}, d_{211}), \qquad u_2 = u_1 + \left(\frac{l_{11}}{l_{11}+1}, 0\right),$$
$$u_3 = \frac{1}{3}(-1, 0), \qquad u_4 = u_3 + \left(\frac{2}{3}, 0\right),$$
$$u_5 = \frac{2l_{01}}{(l_{01}+2)l_{11}}(-d_{111}, -d_{211}), \qquad u_6 = u_5 + \left(\frac{2l_{01}}{l_{01}+2}, 0\right).$$

The only way to ensure that u_3 and u_4 are vertices is to set $l_{11} = 1$. Hence the weights relative to the relevant face $\gamma_{02,11,21}$ are $(2, 0)$, $(0, 2)$ and $(1, 1)$. These points do not generate $\mathbb{Z}^2$ as a lattice. Therefore the stratum $X(\gamma_{02,11,21})$ consists of singular points, contradicting terminality by Remark 3.3.1.
Now assume $l_{02} = 2$. Again, we look at the vertices of $A^c_{X,0}$ and have to set $l_{11} = 1$, after which we achieve $d_{111} = d_{211} = 0$ by admissible operations. Moreover, since v_{01} is primitive, $l_{01} = 2$ holds. We have

$$P = \begin{bmatrix} -2 & -2 & 1 & 2 & 0 & 0 \\ -2 & -2 & 0 & 0 & 1 & 1 \\ -1 & -1-d_{112} & 0 & d_{112} & 0 & 1 \\ 0 & -d_{212} & 0 & d_{212} & 0 & 0 \end{bmatrix}$$

and the vertices of the lineality part are

$$u_1 = \frac{1}{2}(d_{112} - 1, d_{212}), \qquad u_2 = u_1 + (1, 0),$$
$$u_3 = \frac{1}{2}(-1, 0), \qquad u_4 = u_3 + (1, 0),$$
$$u_5 = \frac{1}{3}(-d_{112} - 1, -d_{212}), \qquad u_6 = u_5 + \left(\frac{2}{3}, 0\right).$$

Since $\mathrm{conv}(u_1, u_2, u_3, u_4)$ does not contain integral points other than the origin, we conclude $d_{212} = 1$ and, with an admissible operation, $d_{112} = 0$. These data define a valid variety, namely No. 6.

Situation 4E: homogeneity implies $\mu^2 = w^2_{02} = w^2_{12} = l_{21} + l_{22}$. Moreover, Remark 3.3.1 applied on $\gamma_{01,02,11,12}$ prescribes $w^2_{02} = 1$. This means $l_{21} + l_{22} = 1$, a contradiction.

Disposition 5: applying Remark 3.3.1 to $\gamma_{02,11,21}$ and $\gamma_{02,11,22}$ we obtain the degree matrix

$$Q = \left[\begin{array}{cc|cc|cc} w^1_{01} & 0 & 1 & w^1_{12} & w^1_{21} & 0 \\ w^2_{01} & 1 & 0 & w^2_{12} & 0 & w^2_{22} \end{array}\right].$$

Situation 5A: we divide this situation into three subcases. They differ from one another by the Mori chamber $\alpha_i \subset \mathrm{Eff}(X)$ in which the anticanonical class $-\mathcal{K}_X$ lies. In all three cases we use admissible operations and bring the defining matrix P into the following shape:

$$P = \begin{bmatrix} -1 & -1 & l_{11} & l_{12} & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{21} & l_{22} \\ 0 & 1 & d_{111} & d_{112} & d_{121} & d_{122} \\ 0 & 0 & d_{211} & d_{212} & d_{221} & d_{222} \end{bmatrix}.$$

Situation 5A *with* $-\mathcal{K}_X \in \alpha_1$: here $\gamma_{02,12,22}$ is a relevant face, hence Remark 3.3.1 yields $w_{12}^1 = 1$. Homogeneity of the relation implies

$$w_{01}^1 = l_{11} + l_{12}, \qquad w_{21}^1 = \frac{l_{11} + l_{12}}{l_{21}},$$
$$w_{12}^2 = \frac{w_{01}^2 + 1}{l_{12}}, \qquad w_{22}^2 = \frac{w_{01}^2 + 1}{l_{22}}.$$

Note that $\gamma_{01,02,21,22}$ is also a relevant face. Hence $w_{21}^1 = 1$ holds, i.e. $l_{21} = l_{11}+l_{12}$. Since the anticanonical class lies in α_1, we have $\det(w_{12}, -\mathcal{K}_X) > 0$. This implies $l_{12} > 2l_{22}$, in particular $l_{12} \geq 3$. Through equations from $P \cdot Q^T = 0$ we have

$$d_{111} = -d_{112} - d_{121}, \qquad d_{211} = -d_{212} - d_{221}, \qquad d_{212} = -\frac{l_{12}d_{222}}{l_{22}}.$$

Consider the vertices u_1 and u_2 of the lineality part $A_{X,0}^c$, defined by the elementary big cones $\operatorname{cone}(v_{01}, v_{1j}, v_{21})$, for $j = 1, 2$ respectively. The segment line $\overline{u_1u_2}$ intersects the x-axis in the point $(0, 0, (l_{11} + l_{12})/3, 0)$. Since $l_{12} \geq 3$ holds, the lattice point $(0, 0, 1, 0)$ lies in $\operatorname{conv}(0, u_1, u_2) \subset A_{X,0}^c$, a contradiction to terminality.

Situation 5A *with* $-\mathcal{K}_X \in \alpha_2$: we use homogeneity of the relation and Remark 3.3.1 on the relevant face $\gamma_{01,02,21,22}$ to arrive at

$$Q = \left[\begin{array}{cc|cc|cc} l_{21} & 0 & 1 & (l_{21} - l_{11})/l_{12} & 1 & 0 \\ w_{01}^2 & 1 & 0 & (w_{01}^2 + 1)/l_{12} & 0 & (w_{01}^2 + 1)/l_{22} \end{array}\right].$$

In particular $l_{21} \geq l_{11} + l_{12}$ holds. The matrix Q allows us to compute the anti-canonical class $-\mathcal{K}_X$ according to Proposition 1.3.13. Since $-\mathcal{K}_X \in \alpha_2$ holds, we have $\det(-\mathcal{K}_X, w_{12}) > 0$. Writing down this condition explicitly we obtain $l_{21} < l_{11} + 2l_{22}$. Now we turn to the matrix P. Using equations from $P \cdot Q^T = 0$ we determine

$$d_{111} = -\frac{d_{112}(l_{21} - l_{11})}{l_{12}} - d_{121},$$
$$d_{211} = -\frac{d_{222}(l_{21} - l_{11})}{l_{22}} - d_{221},$$
$$d_{212} = -\frac{l_{12}d_{222}}{l_{22}}.$$

Now consider the vertices u_1 and u_2 of the lineality part $A_{X,0}^c$, defined by the elementary big cones $\operatorname{cone}(v_{02}, v_{1j}, v_{21})$, for $j = 1, 2$ respectively. The segment line $\overline{u_1u_2}$ intersects the x-axis in the point

$$\Big(0, 0, \frac{l_{12}l_{21}}{2l_{12} + l_{21} - l_{11}}, 0\Big).$$

By terminality the lattice point $(0,0,1,0)$ does not lie $\mathrm{conv}(0,u_1,u_2) \subset A^c_{X,0}$. This implies $l_{11} = l_{12} = 1$. In this special situation, we achieve

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{21} & l_{22} \\ 0 & 1 & 0 & d_{112} & -d_{112}(l_{21}-1) & -l_{22}d_{112}-1 \\ 0 & 0 & 0 & d_{212} & -d_{212}(l_{21}-1) & -l_{22}d_{212} \end{bmatrix},$$

$$Q = \left[\begin{array}{cc|cc|cc} l_{21} & 0 & 1 & l_{21}-1 & 1 & 0 \\ l_{22}-1 & 1 & 0 & l_{22} & 0 & 1 \end{array}\right],$$

where we can assume $0 \leq d_{112} < d_{212}$ and $l_{21}, l_{22} \geq 2$. Consider the leaf $A^c_X \cap \lambda_2$ of the anticanonical complex, embedded in $\mathbb{Q}^3$ by removing the first coordinate (which always equals zero) from its points. Define $B \subset \mathbb{Q}^3$ as the convex hull of the following points

$$b_1 := (l_{21}, d_{121}, d_{221}), \qquad b_2 := (l_{22}, d_{122}, d_{222}),$$
$$a_1 := (-1, 0, 0), \qquad a_2 := (-1, 1, 0),$$
$$a_3 := (-1, d_{112}, d_{212}), \qquad a_4 := (-1, d_{112}+1, d_{212}).$$

Then the leaf $A^c_X \cap \lambda_2$ corresponds to the intersection $B \cap \{(x,y,z) \in \mathbb{Q}^3; x \geq 0\}$. By terminality, the only integral points of the leaf are b_1, b_2 and the origin. Hence B contains the origin as only interior point and, with Theorem 2.4.4, $\mathrm{vol}(B)$ is bounded by 12. This gives the condition

$$d_{212}(l_{21} + l_{22} + 2) \; < \; 36.$$

Therefore all entries of P are bounded. We use the MDSpackage [25] to check all possibilities. It turns out that none of the matrices defines a terminal variety.

Situation 5A *with* $-\mathcal{K}_X \in \alpha_3$: here $\gamma_{01,11,21}$ is a relevant face, so Remark 3.3.1 yields $w^2_{01} = 1$. Using homogeneity of the relation and $\gamma_{01,02,21,22}$ relevant face we arrive at

$$Q = \left[\begin{array}{cc|cc|cc} l_{21} & 0 & 1 & (l_{21}-l_{11})/l_{12} & 1 & 0 \\ 1 & 1 & 0 & 2/l_{12} & 0 & 2/l_{22} \end{array}\right].$$

Since the anticanonical class lies in α_3, we have $\det(-\mathcal{K}_X, w_{01}) > 0$. This condition is equivalent to the inequality

$$l_{11}l_{22} + 2l_{12}l_{21} - 2l_{12}l_{22} + l_{21}l_{22} < 0.$$

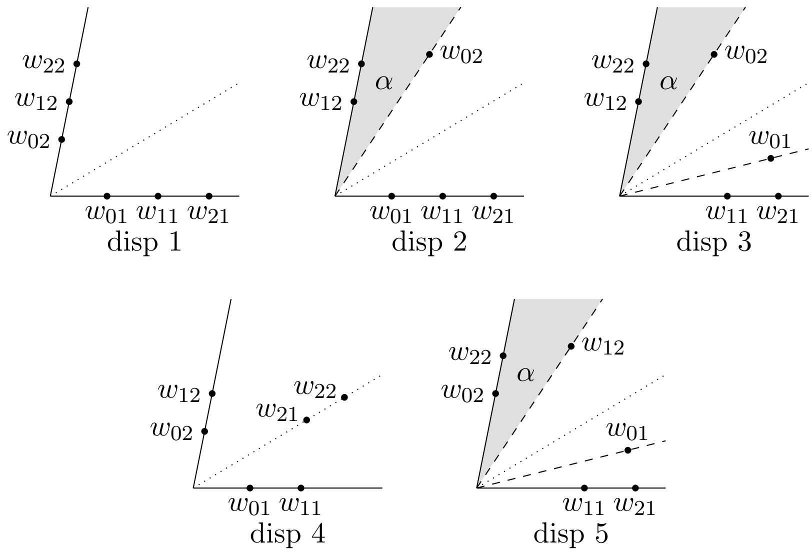
By looking at the matrix Q we see that $l_{12}, l_{22} \in \{1,2\}$ holds. None of the possible combinations satisfies the condition above, hence we reach a contradiction.

Situation 5B: homogeneity implies $w^1_{21} = l_{01}w^1_{01}$ and $w^2_{22} = l_{12}w^2_{12}$. Using Remark 3.3.1, respectively with $\gamma_{01,02,21,22}$ and $\gamma_{11,12,21,22}$, we obtain $w^1_{01} = 1$ and $w^2_{12} = 1$. Since $w^2_{01}, w^1_{12} > 0$ holds, we arrive at a contradiction with the disposition of the weights, because $\det(w_{01}, w_{12}) > 0$ holds.

Situations 5C *and* 5F: homogeneity yields $w^1_{01} = w^1_{21} = l_{11} + l_{12}w^1_{12}$. By Remark 3.3.1 with $\gamma_{01,02,21,22} \in \mathrm{rlv}(X)$, we have $w^1_{01} = 1$ but then $l_{11} + l_{12}w^1_{12} = 1$ holds. This is a contradiction, since all values appearing on the left side are at least one.

Case (b) of Lemma 3.2.7 (ii)

We have $r = 3$, $m = 0$, $n = 8$ and $\overline{n} = (2, 2, 2, 1)$. Combinatorial minimality requires at least two weights on each of the two extremal rays of $\mathrm{Eff}(X)$. There are six weights that can be placed on these rays, therefore we have five possible dispositions:



The weight w_{31} always lies on the dotted line, which is the cone spanned by the degree μ of the relations. In disposition 2, 3 and 5 we define $\alpha := \mathrm{cone}(w_{02}, w_{12})^\circ$.
With Proposition 2.3.1 we obtain a list of possible exponent configurations:

A $l_0 = l_1 = (1, 1)$;
B $l_1 = l_2 = (1, 1)$;
C $l_0 = (1, 1)$ and $l_{11} = l_{21} = 1$;
D $l_1 = (1, 1)$ and $l_{01} = l_{21} = 1$;
E $l_2 = (1, 1)$ and $l_{02} = l_{12} = 1$.

Due to terminality, every disposition allows only a few of these configurations, sometimes even just for restricted situations, depending on the position of the anticanonical class. The following table summarizes the totality of possible situations:

	config A	config B	config C	config D	config E
disp 1	✓				
disp 2	✓	✓	✓$_\alpha$		
disp 3	✓	✓	✓$_\alpha$		
disp 4	✓	✓			✓
disp 5	✓	✓		✓$_\alpha$	

The combinations of dispositions and configurations that need to be studied are marked with the sign ✓. A subscript indicates that the anticanonical class $-\mathcal{K}_X$ has to lie in the given cone.
In the following we show that none of these situations provides a valid variety.

Situation 1A: by almost freeness of the grading we can assume $\mathrm{Eff}(X) = \mathbb{Q}^2_{\geq 0}$. With Remark 3.3.1 applied to $\gamma_{01,12,22}$ and $\gamma_{02,11,21}$ we achieve $w_{01} = (1,0)$ and $w_{02} = (0,1)$. Hence $\mu = (1,1)$ holds, which implies $l_{31} = 1$, a contradiction to P irredundant.

Disposition 2: we can apply Remark 3.3.1 to $\gamma_{01,11,22}$, $\gamma_{01,12,21}$ and $\gamma_{01,12,22}$ in order to arrive at

$$Q = \left[\begin{array}{cc|cc|cc|c} 1 & w^1_{02} & w^1_{11} & 0 & w^1_{21} & 0 & w^1_{31} \\ 0 & w^2_{02} & 0 & 1 & 0 & 1 & w^2_{31} \end{array}\right].$$

Situation 2A *and* 2B: the degree of the relations is $\mu = (w^1_{11}, 1)$. Therefore $l_{31}w^2_{31} = 1$ holds, which contradicts irredundancy of P.

Situation 2C: here $-\mathcal{K}_X \in \mathrm{cone}(w_{02}, w_{12})^\circ$ is required. Therefore $\gamma_{02,12,22}$ is a relevant face and $w^1_{02} = 1$ holds by Remark 3.3.1. With admissible operations we achieve

$$P = \begin{bmatrix} -1 & -1 & 1 & l_{12} & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & l_{22} & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & l_{31} \\ 0 & 1 & 0 & d_{112} & 0 & d_{122} & d_{131} \\ 0 & 0 & 0 & d_{212} & 0 & d_{222} & d_{231} \end{bmatrix}.$$

Using the equations coming from $P \cdot Q^T = 0$ we can fix most of the values of the defining matrices. In particular in P we get

$$v_{22} = (0, l_{12}, 0, -d_{112} - \frac{l_{12}}{2}, -d_{212}) \qquad \text{and} \qquad v_{31} = (0,0,2,-1,0).$$

This allows us to express the vertices of the lineality part of the anticanonical complex in the following way:

$$\begin{aligned} u_1 &= \left(\frac{2d_{112} - l_{12}}{l_{12}+2}, \frac{2d_{212}}{l_{12}+2}\right), & u_2 &= u_1 + \left(\frac{2l_{12}}{l_{12}+2}, 0\right), \\ u_3 &= \left(-\frac{1}{3}, 0\right), & u_4 &= u_3 + \left(\frac{2}{3}, 0\right), \\ u_5 &= \left(\frac{-2d_{112} - 2l_{12}}{l_{12}+2}, \frac{-2d_{212}}{l_{12}+2}\right), & u_6 &= u_5 + \left(\frac{2l_{12}}{l_{12}+2}, 0\right). \end{aligned}$$

We notice that the length of the edges u_1u_2 and u_5u_6 is at least $2/3$, hence u_3 and u_4 are not both vertices, a contradiction.

Situation 3A: by Remark 3.3.1 applied to $\gamma_{01,12,22}$ we can achieve $w^1_{12} = w^1_{22} = 0$ and $w_{01} = (1,0)$. Since homogeneity requires $w^2_{12} = l_{22}w^2_{22}$, we also obtain $w^2_{22} = 1$.
Suppose for a moment that $-\mathcal{K}_X \in \mathrm{cone}(w_{02}, w_{12})^\circ$ holds. Then $\gamma_{02,12,22}$ is relevant, so $w^1_{02} = 1$. Homogeneity implies $l_{21} = l_{31} = 2$ (otherwise see situation 3C) and the anticanonical class is

$$-\mathcal{K}_X = \left(2, 1 - w^2_{02} - \frac{1}{2}l_{22}\right).$$

Since $l_{22}, w^2_{02} \geq 1$, the anticanonical class does not lie in the prescribed cone.

Hence from now on let $-\mathcal{K}_X \in \mathrm{cone}(w_{01}, w_{02})^\circ$. By homogeneity, the inequalities $w_{11}^2 < 0$ and $w_{31}^2 \geq 1$ turn into $l_{22} > w_{02}^2$ and $w_{02}^2 \geq l_{31}$. Since $l_{31} \geq 2$ holds by irredundancy of P, we conclude $l_{22} \geq 3$. With Proposition 3.1.4 applied to $\{4,6\}$ and $\{3,5\}$ we can write

$$d_{221} \;=\; -\frac{l_{21}}{l_{31}} d_{231} \qquad \text{and} \qquad d_{221} \;=\; -\frac{l_{22}}{l_{31}} d_{232} - l_{22} d_{212}.$$

The vertices of $A_{X,0}^c$ defined by the elementary big cones $P(\delta_{0i,11,22,31})$ for $i = 1, 2$ are

$$u_1 \;=\; \left(\frac{l_{22}d_{131} + l_{31}d_{122}}{l_{22} + l_{31}}, -\frac{l_{22}l_{31}d_{212}}{l_{22} + l_{31}} \right) \qquad u_2 \;=\; u_1 + \left(\frac{l_{22}l_{31}}{l_{22} + l_{31}}, 0 \right).$$

Since $l_{31} \geq 2$ and $l_{22} \geq 3$ hold, the intersection of $\mathrm{conv}(0, u_1, u_2) \subset A_{X,0}^c$ with $\{y = -d_{212}\}$ exists and has length equal to one. Therefore it contains a lattice point and this contradicts terminality.

Situation 3B: here we can assume that the weights w_{12} and w_{22} lie in $\mathrm{cone}(e_2)$, i.e. $w_{12}^1 = w_{22}^1 = 0$. By homogeneity of the relations we also get $w_{11}^1 = w_{21}^1 = l_{31}w_{31}^1$. Since w_{11} and w_{21} lie on the same half-line, we conclude $w_{11}^2 = w_{21}^2$ and consequently $w_{12}^2 = w_{22}^2$. These equalities, together with Remark 3.3.1 applied to $\gamma_{01,12,22}$, deliver $w_{01} = (1, 0)$ and $w_{12}^2 = 1$. At this point, homogeneity requires $l_{02}w_{02}^2 = \mu^2 = w_{11}^2 + 1$. This contradicts the configuration of the weights, because $w_{02}^2 \geq 1$ but $w_{11}^2 < 0$.

Situation 3C: here $-\mathcal{K}_X \in \mathrm{cone}(w_{02}, w_{12})^\circ$ holds. We proceed the same way as in situation 2C, namely we use the relevant faces $\gamma_{01,12,22}$ and $\gamma_{02,12,22}$, then we work with P and lastly find the six vertices of the lineality part of the anticanonical complex. The same kind of contradiction follows here too.

Situation 4A: by means of admissible operations, we achieve the following defining matrix

$$P \;=\; \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{21} & l_{22} & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & l_{31} \\ 0 & 1 & 0 & d_{112} & d_{121} & d_{122} & d_{131} \\ 0 & 0 & 0 & d_{212} & d_{221} & d_{222} & d_{231} \end{bmatrix}.$$

Applying Proposition 3.1.4 to $\{5,6\}$ we obtain $d_{212} = 0$. We can assume $-\mathcal{K}_X \in \mathrm{cone}(w_{01}, w_{21})^\circ$ and compute the vertices of the lineality part $A_{X,0}^c$. A contradiction is reached: it turns out that three vertices lie on the same line segment.

Situation 4B: here Remark 3.3.1 applies to $\gamma_{01,11,21}$ and $\gamma_{01,11,22}$ and yields $w_{01}^2 = w_{11}^2 = 0$, $w_{21}^2 = w_{22}^2 = 1$ and $w_{21}^1 = w_{22}^1$. Moreover, homogeneity of the relations delivers

$$2 = \mu^2 = l_{02}w_{02}^2 = w_{12}^2 = l_{31}w_{31}^2.$$

This means already $l_{02} = l_{31} = w_{12}^2 = 2$ and $w_{02}^2 = w_{31}^2 = 1$. In particular, multiplying Q from left with an appropriate unimodular matrix, we achieve $w_{02}^1 = w_{12}^1 = 0$. We use homogeneity in the first component to arrive at

$$Q \;=\; \left[\begin{array}{cc|cc|cc|c} \frac{2w_{21}^1}{l_{01}} & 0 & 2w_{21}^1 & 0 & w_{21}^1 & w_{21}^1 & w_{21}^1 \\ 0 & 1 & 0 & 2 & 1 & 1 & 1 \end{array}\right].$$

At the same time we can use admissible operations on P and achieve

$$P = \begin{bmatrix} -l_{01} & -2 & 1 & 1 & 0 & 0 & 0 \\ -l_{01} & -2 & 0 & 0 & 1 & 1 & 0 \\ -l_{01} & -2 & 0 & 0 & 0 & 0 & 2 \\ d_{101} & d_{102} & 0 & 1 & 0 & d_{122} & d_{131} \\ d_{201} & d_{202} & 0 & 0 & 0 & d_{222} & d_{231} \end{bmatrix}.$$

With $P \cdot Q^T = 0$ we obtain the following equalities:

$$d_{102} = -d_{122} - d_{131} - 2 \qquad \text{and} \qquad d_{202} = -d_{222} - d_{231}.$$

We look specifically at the vertices of $A^c_{X,0} \subset \mathbb{Q}^2$ defined by the two elementary big cones $\operatorname{cone}(v_{02}, v_{11}, v_{2j}, v_{31})$ for $j = 1, 2$, which are respectively

$$\left(-1 - \frac{1}{2}d_{122}\ ,\ -\frac{1}{2}d_{222}\right) \qquad \text{and} \qquad \left(-1 + \frac{1}{2}d_{122}\ ,\ \frac{1}{2}d_{222}\right).$$

The integral point $(-1, 0)$ lies on the segment joining the two vertices, hence $A^c_{X,0}$ contains a forbidden lattice point and X is not terminal.

Situation 4E: we assume $-\mathcal{K}_X \in \operatorname{cone}(w_{01}, w_{21})^\circ$ without loss of generality. With the usual game on homogeneity, relevant faces, admissible operations and equations from $P \cdot Q^T = 0$ we arrive at

$$P = \begin{bmatrix} -l_{01} & -1 & l_{11} & 1 & 0 & 0 & 0 \\ -l_{01} & -1 & 0 & 0 & 1 & 1 & 0 \\ -l_{01} & -1 & 0 & 0 & 0 & 0 & 2 \\ d_{101} & 0 & d_{111} & 0 & 1 & 0 & -1 \\ d_{201} & 0 & d_{211} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The vertices of the lineality part $A^c_{X,0} \subset \mathbb{Q}^2$ are

$$u_1 = \frac{1}{l_{11}+2}(2d_{111} - l_{11}, 2d_{211}), \qquad u_2 = u_1 + \left(\frac{2l_{11}}{l_{11}+2}, 0\right),$$
$$u_3 = \frac{1}{3}(-1, 0), \qquad u_4 = u_3 + \left(\frac{2}{3}, 0\right),$$
$$u_5 = \frac{1}{l_{01}+2}(2d_{101} - l_{01}, 2d_{201}), \qquad u_6 = u_5 + \left(\frac{2l_{01}}{l_{01}+2}, 0\right).$$

We see that, for any choice of l_{01} and l_{11}, at least one between u_3 and u_4 is not a vertex, a contradiction.

Disposition 5: here we can assume $-\mathcal{K}_X \in \operatorname{cone}(w_{01}, w_{02})^\circ$ without loss of generality.

Situation 5A: Applying Remark 3.3.1 to $\gamma_{02,11,21}$ and $\gamma_{02,11,22}$ we obtain $\mathrm{Eff}(X) = \mathbb{Q}^2_{\geq 0}$, with $w_{02} = (0,1)$ and $w_{11} = (1,0)$. Using homogeneity of all monomials we arrive at the following grading matrix:

$$Q = \left[\begin{array}{cc|cc|cc|c} w_{01}^1 & 0 & 1 & w_{01}^1 - 1 & \frac{w_{01}^1}{l_{21}} & 0 & \frac{w_{01}^1}{l_{31}} \\ w_{01}^2 & 1 & 0 & w_{01}^2 + 1 & 0 & \frac{w_{01}^2+1}{l_{22}} & \frac{w_{01}^2+1}{l_{31}} \end{array}\right].$$

Assume for a moment $\det(w_{12}, -\mathcal{K}_X) > 0$. Then $-\mathcal{K}_X \in \mathrm{cone}(w_{02}, w_{12})^\circ$ holds, $\gamma_{02,12,22}$ is a relevant face and Remark 3.3.1 yields w_{01}^2 and $l_{31} = 2$. This contradicts the determinantal assumption. Therefore $-\mathcal{K}_X$ lies in $\mathrm{cone}(w_{01}, w_{12})$. We turn to the matrix P. With admissible operations we achieve

$$\begin{aligned} v_{01} &= (-1,-1,-1,0,0), \\ v_{02} &= (-1,-1,-1,1,0), \\ v_{11} &= (1,0,0,0,0). \end{aligned}$$

Moreover, Corollary 3.1.5 applied to $P_{\{3,5\}}$ delivers

$$d_{231} = -l_{31}d_{212} - \frac{l_{31}d_{222}}{l_{22}}.$$

Consider $\mathrm{cone}(v_{0j}, v_{11}, v_{22}, v_{31})$ for $j = 1,2$. These are big elementary cones and define two vertices u_1 and u_2 of the lineality part $A^c_{X,0} \subset \mathbb{Q}^2$. The triangle $C := \mathrm{conv}(0, u_1, u_2) \subset \mathbb{Q}^2$ has an edge $c := \overline{u_1u_2}$ parallel to the x-axis. The height $h(c)$ and length $\ell(c)$ of c are

$$h(c) = -d_{212}\frac{l_{22}l_{31}}{l_{22}+l_{31}}, \qquad \ell(c) = \frac{l_{22}l_{31}}{l_{22}+l_{31}}.$$

By terminality, C does not contain integral points other than the origin, hence $\ell(c) < 1$ holds. This implies $l_{22} = 1$. By admissible operations we achieve $d_{122} = d_{222} = 0$. Using the fact that $P \cdot Q^T = 0$ holds, we write explicitly

$$w_{01}^1 = \frac{d_{212}l_{21}}{d_{221}}, \qquad w_{01}^2 = -\frac{d_{112}l_{31} + d_{131} + l_{31}}{d_{112}l_{31} + d_{131}}.$$

Now consider w_{21}^1 and w_{31}^2, which satisfy

$$w_{21}^1 = \frac{d_{212}}{d_{221}}, \qquad w_{31}^2 = -\frac{1}{d_{112}l_{31} + d_{131}}.$$

Since they are both integers, we conclude $d_{131} = -d_{112}l_{31} - 1$ and $d_{212} = bd_{221}$ for some integer $b \in \mathbb{Z}_{\geq 1}$. For the anticanonical class we have

$$-\mathcal{K}_X = \left(b + b\frac{l_{21}}{l_{31}}, l_{31} + 1\right).$$

The condition $\det(-\mathcal{K}_X, w_{21}) > 0$ implies $l_{21} = 1, 2$. We leave $l_{21} = 1$ for situation 5B, thus we have $l_{21} = 2$. The same determinantal condition yields $b = 1$ and, by looking at w_{31}^1, we conclude $l_{31} = 2$. Furthermore, $P \cdot Q^T = 0$ also delivers $d_{112} = d_{121} - 1$. To finally achieve a contradiction to terminality, take a look at the vertices u_3 and u_4 of the lineality part, defined by $\mathrm{cone}(v_{02}, v_{1j}, v_{21}, v_{31})$ for $j = 1, 2$ respectively:

$$u_3 = \frac{1}{2}(-d_{121} + 3, -d_{221}), \qquad u_4 = \frac{1}{2}(d_{121} + 1, d_{221}).$$

The midpoint of the segment line connecting these two vertices is $(1, 0)$, a contradiction to Theorem 2.1.10.

Situation 5B: Combining Remark 3.3.1 applied to many relevant $\mathfrak{F}$-faces, homogeneity of the monomials and admissible operations on P, we arrive at $\mathrm{Eff}(X) = \mathbb{Q}^2_{\geq 0}$ and

$$P = \begin{bmatrix} -l_{01} & -l_{02} & 1 & 1 & 0 & 0 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 1 & 1 & 0 \\ -l_{01} & -l_{02} & 0 & 0 & 0 & 0 & l_{31} \\ d_{101} & d_{102} & 0 & 1 & 0 & d_{122} & d_{131} \\ d_{201} & d_{202} & 0 & 0 & 0 & d_{222} & d_{231} \end{bmatrix}.$$

$$Q = \left[\begin{array}{cc|cc|cc|c} w_{01}^1 & 0 & 1 & l_{01}w_{01}^1 - 1 & l_{01}w_{01}^1 & 0 & \frac{l_{01}w_{01}^1}{l_{31}} \\ \frac{l_{31}-l_{02}}{l_{01}} & 1 & 0 & l_{31} & 0 & l_{31} & 1 \end{array}\right].$$

Equations given by $P \cdot Q^T = 0$ deliver

$$d_{201} = -\frac{l_{01}d_{231}}{l_{31}}, \qquad d_{231} = -\frac{l_{31}^2 d_{222}}{l_{02}} - \frac{l_{31}d_{202}}{l_{02}}.$$

These help us write down the explicit coordinates of the vertices of $A^c_{X,0} \subset \mathbb{Q}^2$. Five vertices $u_1, \ldots, u_5$ are always present and given by

$$\begin{aligned} u_1 &= \frac{1}{l_{01} + l_{31}}(l_{01}d_{131} + l_{31}d_{101} + l_{01}l_{31}d_{122}, l_{01}l_{31}d_{222}), \quad u_2 = u_1 + \left(\frac{l_{01}l_{31}}{l_{01} + l_{31}}, 0\right), \\ u_3 &= \frac{1}{l_{01} + l_{31}}(l_{01}d_{131} + l_{31}d_{101} + l_{01}l_{31}, 0), \\ u_4 &= \frac{1}{l_{02} + l_{31}}(l_{02}d_{131} + l_{31}d_{102}, -l_{31}^2 d_{222}), \qquad u_5 = u_4 + \left(\frac{l_{02}l_{31}}{l_{02} + l_{31}}, 0\right). \end{aligned}$$

The coordinates of the sixth vertex u_6 depend on the position of the anticanonical class, i.e. if it lies in $\alpha := \mathrm{cone}(w_{02}, w_{12})$ or in $\beta := \mathrm{cone}(w_{01}, w_{12})$, because these two cases allow different big elementary cones.

Assume for a moment $-\mathcal{K}_X \in \alpha$. Then $\gamma_{02,12,22}$ is a relevant face, implying $l_{01}w_{01}^1 = 2$ and $l_{31} = 2$ by Remark 3.3.1. Moreover the sixth vertex is defined by $P(\delta_{01,11,21,31})$ and yields $l_{02} < l_{01}$, hence $l_{01} = 2$ and $l_{02} = 1$, which contradicts $w_{01}^2 \in \mathbb{Z}$.

Therefore $-\mathcal{K}_X \in \beta$ holds and the vertex u_6 is defined by the elementary big cone $P(\delta_{02,11,22,31})$. Consider the polyhedron $C := \mathrm{conv}(u_1, u_2, u_4, u_5) \subset A^c_{X,0}$, which is a trapezoid with edges $c_1 := \overline{u_1u_2}$ and $c_2 := \overline{u_4u_5}$ parallel to the x-axis. The height of

c_2 is negative and smaller than -1. If both l_{01} and l_{02} are greater than 1, then both edges c_i are longer than 1 and C contains integral points at height $\{y = -1\}$. Hence we follow $l_{02} = 1$ and $l_{01} \geq 2$. (the case $l_{01} = 1$ is covered in situation 5D). With admissible operations we achieve $d_{102} = d_{202} = 0$. We find the bounding condition $l_{01} \leq 4$ by imposing that the segment $A^c_{X,0} \cap \{y = 0\}$ has length at most 2. For all cases of l_{01} we find out that the segment $A^c_{X,0} \cap \{y = 1\}$ is longer than 1, hence $A^c_{X,0}$ contains a non-zero lattice point, contradicting terminality.

Situation 5D: $-\mathcal{K}_X \in \mathrm{cone}(w_{12}, w_{02})^\circ$ is required. Here $\gamma_{02,11,21}$, $\gamma_{02,11,22}$ and $\gamma_{02,12,22}$ are relevant faces to which Remark 3.3.1 applies. Therefore, after setting $\mathrm{Eff}(X) = \mathbb{Q}^2_{\geq 0}$, we obtain $w^1_{11} = w^1_{12} = w^2_{02} = 1$. In particular $\mu^1 = 2$ holds, hence $l_{31} = 2$ and

$$Q \;=\; \left[\begin{array}{cc|cc|cc|c} 2 & 0 & 1 & 1 & 2 & 0 & 1 \\ w^2_{01} & 1 & 0 & w^2_{12} & 0 & w^2_{22} & w^2_{31} \end{array}\right].$$

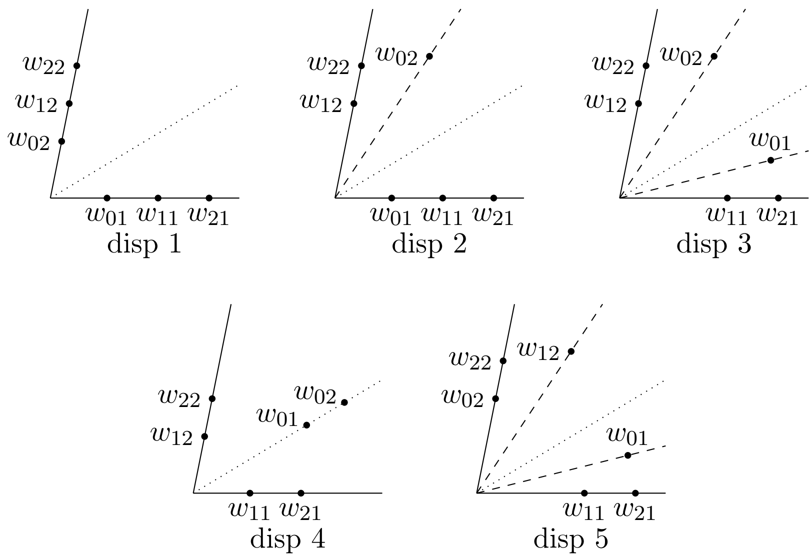
Using homogeneity of the relations we can write all weights of the second row of Q in terms of w^2_{01}. For the anticanonical class $-\mathcal{K}_X$ we obtain

$$-\mathcal{K}_X \;=\; \left(3, 1 + \frac{1}{2}w^2_{01} - \frac{1}{2}l_{02} + \frac{w^2_{01} + l_{02}}{l_{22}}\right).$$

It turns out that $\det(w_{12}, -\mathcal{K}_X) < 0$ holds, in contradiction to the hypothesis of this situation on the position of the anticanonical class.

Case (c) of Lemma 3.2.7 (ii)

We have $r = 4$, $m = 0$, $n = 8$ and $\overline{n} = (2, 2, 2, 1, 1)$. Combinatorial minimality prescribes at least two weights on each of the two extremal rays of $\mathrm{Eff}(X)$. There are six weights that can be placed on these rays, therefore we end up with five possible dispositions:



The weights w_{31}, w_{41} always lie on the thin dotted line, which is the cone spanned by the degree μ. Note that in every disposition the following cones are elementary big:

$$\mathrm{cone}(v_{01}, v_{11}, v_{22}, v_{31}, v_{41}), \qquad \mathrm{cone}(v_{02}, v_{12}, v_{21}, v_{31}, v_{41}).$$

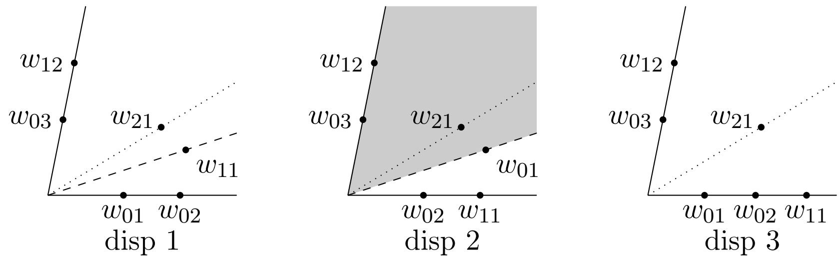
In particular, every ray v_{ij} appears at least once in an elementary big cone. With Proposition 2.3.1 we obtain $l_i = (1, 1)$ for $i = 0, 1, 2$. Admissible operations on P deliver

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & l_{31} & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & l_{41} \\ 0 & 1 & 0 & d_{112} & 0 & d_{122} & d_{131} & d_{141} \\ 0 & 0 & 0 & d_{212} & 0 & d_{222} & d_{231} & d_{241} \end{bmatrix},$$

where we can also assume $0 \le d_{141}, d_{241} < l_{41} \le l_{31}$.
While treating any of the above mentioned dispositions, it is possible to conclude non-terminality in two steps: first one needs to apply Corollary 3.1.5 whenever possible, then look at the lineality part $A^c_{X,0}$ of the anticanonical complex and see that it always contains a non-zero integral point.

Case (d) of Lemma 3.2.7 (ii)

We have $r = 2$, $m = 0$, $n = 6$ and $\overline{n} = (3, 2, 1)$. Combinatorial minimality requires at least two weights on each of the two extremal rays of $\mathrm{Eff}(X)$. There are five weights that can be placed on these rays, therefore we end up with three possible dispositions:



In every disposition we find the following elementary big cones:

$$\mathrm{cone}(v_{0i}, v_{1j}, v_{21}), \quad \forall (i, j) \in \{1, 2, 3\} \times \{1, 2\} \quad \text{with } (i, j) \neq (3, 2).$$

In particular we assumed, without loss of generality, that the anticanonical class in disposition 2 lies in $\mathrm{cone}(w_{01}, w_{03})^\circ$, the light-grey cone in the picture. With Proposition 2.3.1, we find three possible configurations of the exponents:

A $l_0 = (1, 1, 1)$;
B $l_1 = (1, 1)$;
C $l_{01} = l_{02} = l_{11} = 1$.

We will see that this case contributes with four varieties, namely No. 7 (from 1A), No. 8 (from 2A), No. 9 (from 1C) and No. 10 (from 3C).

Disposition 1: we apply Remark 3.3.1 to $\gamma_{01,02,12}$, $\gamma_{01,03,12}$ and $\gamma_{02,03,12}$ and achieve, for the grading matrix,

$$Q = \begin{bmatrix} 1 & 1 & 0 & w_{11}^1 & 0 & w_{21}^1 \\ 0 & 0 & w_{03}^2 & w_{11}^2 & 1 & w_{21}^2 \end{bmatrix}.$$

Situation 1A: homogeneity of the monomials implies $l_{21} = 2$, $w_{21}^1 = 1$, $w_{11}^1 = 2/l_{11}$ and $w_{03}^2 = 2w_{21}^2$. With $\gamma_{01,02,03,21} \in \mathrm{rlv}(\Phi)$ and Remark 3.3.1, we conclude $w_{21}^2 = 1$. In particular we have $w_{11} = (2/l_{11}, w_{11}^2)$ and $w_{21} = (1,1)$. Since the disposition requires $0 < \det(w_{11}, w_{21})$, we obtain $l_{11} = w_{11}^2 = 1$. Using admissible operations and equations resulting from $P \cdot Q^T = 0$ we can fix the remaining entries of both matrices. We get

$$P = \begin{bmatrix} -1 & -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -2 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{bmatrix}.$$

These data define variety No. 7.

Situation 1B: we have

$$\begin{aligned} w_{11}^1 &= l_{01} + l_{02}, & w_{11}^2 &= l_{03}w_{03}^2 - 1, \\ w_{21}^1 &= \frac{l_{01} + l_{02}}{l_{21}}, & w_{21}^2 &= \frac{l_{03}w_{03}^2 - 1}{l_{21}}, \end{aligned}$$
$$d_{203} = -\frac{l_{03}d_{221}}{l_{21}},$$

where the equations on the weights come from homogeneity of the relation and the last equality is due to Corollary 3.1.5 applied to $\{1,2\}$. We need to understand whether the anticanonical class $-\mathcal{K}_X$ lies in $\mathrm{cone}(w_{01}, w_{11})^\circ$ or in $\mathrm{cone}(w_{03}, w_{11})^\circ$. In order to do that, consider the lineality part $A_{X,0}^c \subset \mathbb{Q}^2$. In both cases, the following rational points are vertices of $A_{X,0}^c$:

$$\begin{aligned} u_1 &= \left(\frac{d_{121}l_{01} + d_{101}l_{21}}{l_{01} + l_{21}}, \frac{d_{221}l_{01} + d_{201}l_{21}}{l_{01} + l_{21}} \right), & u_2 &= u_1 + \left(\frac{l_{01}l_{21}}{l_{01} + l_{21}}, 0 \right), \\ u_3 &= \left(\frac{d_{121}l_{03} + d_{103}l_{21}}{l_{03} + l_{21}}, 0 \right), \\ u_4 &= \left(\frac{d_{121}l_{02} + d_{102}l_{21}}{l_{02} + l_{21}}, \frac{d_{221}l_{02} + d_{202}l_{21}}{l_{02} + l_{21}} \right), & u_5 &= u_4 + \left(\frac{l_{02}l_{21}}{l_{02} + l_{21}}, 0 \right). \end{aligned}$$

Assume for a moment that $-\mathcal{K}_X \in \mathrm{cone}(w_{01}, w_{11})^\circ$ holds. Then $\gamma_{01,02,11}$ is a relevant face and Remark 3.3.1 applies, yielding $w_{11}^2 = 1$, hence $l_{03}w_{03}^2 = 2$ and $l_{21} = 2$. If $l_{03} = 2$ and $w_{03}^2 = 1$ hold, then we arrive at a contradiction since the anticanonical class is $(2 + (l_{01} + l_{02})/2, 2)$, but this point does not lie in the prescribed cone. If $l_{03} = 1$ and $w_{03}^2 = 2$ hold, then there is a sixth vertex of $A_{X,0}^c$, namely $u_6 = u_3 + (2/3, 0)$. Since both $\overline{u_1u_2}$ and $\overline{u_4u_5}$ are edges of $A_{X,0}^c$, with a length of at least $2/3$, the points u_3 and u_6 cannot be both vertices: a contradiction.

Therefore we can assume $-\mathcal{K}_X \in \operatorname{cone}(w_{03}, w_{11})^\circ$. In particular $\gamma_{03,11,12,21}$ is a relevant face; with Remark 3.3.1 we obtain $l_{21} = l_{01} + l_{02}$. Equalities from $P \cdot Q^T = 0$ deliver

$$d_{121} = -d_{101} - d_{102}, \qquad d_{221} = -d_{201} - d_{202}.$$

These estimates put the vertices $u_1, \ldots, u_5$ of the lineality part in a special position. As a matter of fact, the intersection of the x-axis with the segment line $\overline{u_2 u_5}$ is the point

$$\frac{1}{3}(l_{01} + l_{02}, 0).$$

Since $l_{01} + l_{02} \geq 3$ holds (otherwise we are in situation 1C), we obtain $(1, 0) \in A^c_{X,0}$, a contradiction to the terminality of X.

Situation 1C: we apply some admissible operations to simplify the form of P and obtain the following shape:

$$P = \begin{bmatrix} -1 & -1 & -l_{03} & 1 & l_{12} & 0 \\ -1 & -1 & -l_{03} & 0 & 0 & l_{21} \\ 0 & 1 & d_{103} & 0 & d_{112} & d_{121} \\ 0 & 0 & d_{203} & 0 & d_{212} & d_{221} \end{bmatrix}.$$

Using $P \cdot Q^T = 0$ with the first row of Q we get $w^1_{11} = 2$, $w^1_{21} = 1$, $l_{21} = 2$, $d_{121} = -1$ and $d_{221} = 0$. Similarly, the second row of Q, together with the first and second rows of P, yield

$$w^2_{11} = l_{03} w^2_{03} - l_{12} \quad \text{and} \quad w^2_{21} = \frac{1}{2} l_{03} w^2_{03}.$$

Note that $\gamma_{01,02,03,21} \in \operatorname{rlv}(\Phi)$ is not in the covering collection, hence we apply Remark 3.3.1 and obtain $\gcd(w^2_{03}, w^2_{03} l_{03}/2) = 1$. This leaves us with two cases: either $w^2_{03} = 2$ and l_{03} is odd, or $w^2_{03} = 1$ and l_{03} is even.
The first case can be discharged as follows: $w^2_{11} > 0$ here means $2w^2_{03} > l_{12}$, and, since $\det(w_{11}, -\mathcal{K}_X) > 0$ holds, we also have $4l_{03} < 6 + l_{12}$. But these inequalities combined imply $l_{03} < 3$, hence $l_{03} = 1$, a situation already studied in situation 1A.
Therefore we assume that $w^2_{03} = 1$ holds and l_{03} is even. Here, too, we obtain two inequalities, namely $l_{12} < l_{03}$ and $2l_{03} < 4 + l_{12}$. We conclude $l_{03} = 2$ and hence $l_{12} = 1$. The last equalities from $P \cdot Q^T = 0$ deliver $d_{103} = 1 - d_{112}$ and $d_{203} = -d_{212}$. The matrices look as follows:

$$P = \begin{bmatrix} -1 & -1 & -2 & 1 & 1 & 0 \\ -1 & -1 & -2 & 0 & 0 & 1 \\ 0 & 1 & 1 - d_{112} & 0 & d_{112} & -1 \\ 0 & 0 & -d_{212} & 0 & d_{212} & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Among the vertices of the lineality part $A^c_{X,0}$ of the anticanonical complex we find

$$u_1 := \left(-\frac{1}{3}, 0\right), \quad u_2 := \left(\frac{1}{3}, 0\right), \quad u_3 := \left(-\frac{1}{2} d_{112}, -\frac{1}{2} d_{212}\right).$$

With admissible operations we can assume $0 \le d_{112} < d_{212}$ and, since terminality implies that $\mathrm{conv}(u_1, u_2, u_3)$ does not contain any integral point, we arrive at $d_{212} < 12$. We check the varieties given by these parameters with the MDSpackage [25]. It turns out that there is one terminal variety among them, namely with $d_{112} = 0$ and $d_{212} = 1$; this occurs as No. 10 in Theorem 3.4.1.

Disposition 2: the $\mathfrak{F}$-face $\gamma_{02,03,11}$ is relevant and is not in the covering collection, hence with Remark 3.3.1 we can assume that the effective cone is the positive orthant. By applying the same Remark, on the relevant faces $\gamma_{01,03,12}$, $\gamma_{02,03,12}$ and $\gamma_{02,03,11}$, we arrive at

$$Q = \begin{bmatrix} 1 & 1 & 0 & w_{11}^1 & 0 & w_{21}^1 \\ w_{01}^2 & 0 & 1 & 0 & w_{12}^2 & w_{21}^2 \end{bmatrix}.$$

Situation 2A: homogeneity of the relation implies $2 = \mu^1 = l_{21} w_{21}^1$, i.e. $l_{21} = 2$ and $w_{21}^1 = 1$. Therefore $2w_{21}^2 = \mu^2 = w_{01}^2 + 1$ holds. Together with $\det(w_{01}, w_{21}) \ge 0$ we conclude $w_{01}^2 = w_{21}^2 = 1$ and $\mu^2 = 2$. Since $\gamma_{01,11,12,21}$ is a relevant face, at least one between w_{11}^1 and w_{12}^2 must equal 1. Without loss of generality let $w_{12}^2 = 1$, hence $l_{12} = 2$. If $w_{11}^1 = 1$ holds, then the anticanonical class is $(2, 2)$: this contradicts $\mathbb{Q}$-factoriality by Proposition 1.4.2 since $(2, 2) \in \mathrm{cone}(w_{01})^\circ$. Therefore we have $w_{11}^1 = 2$ and $l_{11} = 1$. The last entries of P are determined by $P \cdot Q^T = 0$. This delivers variety No. 8.

Situation 2B: by homogeneity of the relation we achieve $w_{11}^1 = l_{21} w_{21}^1$ and $w_{12}^2 = l_{21} w_{21}^2$. The relevant faces $\gamma_{02,11,12,21}$ and $\gamma_{03,11,12,21}$, together with Remark 3.3.1, imply $w_{21} = (1, 1)$. This contradicts $\det(w_{01}, w_{21}) > 0$ and $w_{01}^2 > 0$.

Situation 2C: through suitable admissible operations the matrix P assumes the following form:

$$P = \begin{bmatrix} -1 & -1 & -l_{03} & 1 & l_{12} & 0 \\ -1 & -1 & -l_{03} & 0 & 0 & l_{21} \\ 0 & 1 & d_{103} & 0 & d_{112} & d_{121} \\ 0 & 0 & d_{203} & 0 & d_{212} & d_{221} \end{bmatrix}.$$

By imposing $P \cdot Q^T = 0$ we obtain several equalities, which altogether deliver the following degree matrix Q:

$$Q = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 1 \\ w_{01}^2 & 0 & 1 & 0 & \frac{w_{01}^2 + l_{03}}{l_{12}} & \frac{w_{01}^2 + l_{03}}{2} \end{bmatrix}.$$

We can assume $l_{03}, l_{12} \ge 2$, otherwise we go back to configurations A and B respectively. Moreover we have $l_{12} \ge 3$, since $l_{12} = 2$, together with $\gamma_{02,11,12,21} \in \mathrm{rlv}(\Phi)$, implies $w_{01}^2 = l_{03} = 1$. At this point a contradiction is reached, since this disposition requires $\det(w_{01}, -\mathcal{K}_X) > 0$.

Disposition 3: using Remark 3.3.1 applied to the relevant faces $\gamma_{01,02,12}$, $\gamma_{01,03,11}$, $\gamma_{01,03,12}$, $\gamma_{02,03,12}$, we can achieve the degree matrix

$$Q = \begin{bmatrix} 1 & 1 & 0 & w_{11}^1 & 0 & w_{21}^1 \\ 0 & 0 & 1 & 0 & 1 & w_{21}^2 \end{bmatrix}.$$

Situations 3A *and* 3B: here $\mu^2 = w_{03}^2 = 1$ holds. This implies $l_{21} = 1$, a contradiction to irredundancy of P.

Situation 3C: the matrix P, via admissible operations, takes the form

$$P = \begin{bmatrix} -1 & -1 & -l_{03} & 1 & l_{12} & 0 \\ -1 & -1 & -l_{03} & 0 & 0 & l_{21} \\ 0 & 1 & d_{103} & 0 & d_{112} & d_{121} \\ 0 & 0 & d_{203} & 0 & d_{212} & d_{221} \end{bmatrix}.$$

By imposing $P \cdot Q^T = 0$ we obtain several equalities, which altogether deliver the following matrices P and Q:

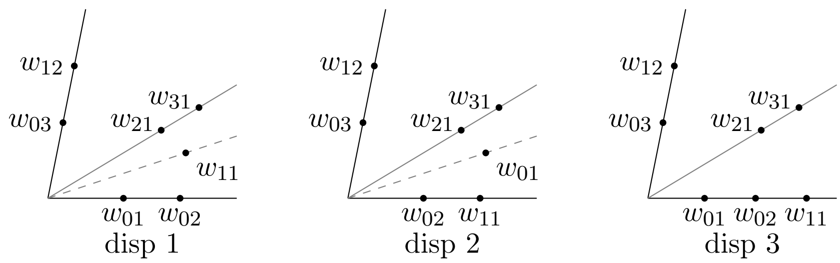
$$P = \begin{bmatrix} -1 & -1 & -l_{03} & 1 & l_{03} & 0 \\ -1 & -1 & -l_{03} & 0 & 0 & 2 \\ 0 & 1 & \frac{1}{2}l_{03} - d_{112} & 0 & d_{112} & -1 \\ 0 & 0 & -d_{212} & 0 & d_{212} & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & \frac{1}{2}l_{03} \end{bmatrix}.$$

Note that l_{03} must be even. Moreover, since the anticanonical class $-\mathcal{K}_X$ is equal to $(3, 2 - \frac{1}{2}l_{03})$ and lies in the positive orthant, the inequality $l_{03} < 4$ holds, thus $l_{03} = 2$. By terminality, the lineality part $A^c_{X,0}$ does not contain integral points other than the origin. This implies $d_{212} = 1$ and, with an admissible operation, $d_{112} = 0$. The resulting matrix P defines indeed a valid variety, namely No. 10.

Case (e) of Lemma 3.2.7 (ii)

We have $r = 3$, $m = 0$, $n = 7$ and $\overline{n} = (3, 2, 1, 1)$. Combinatorial minimality requires at least two weights on each of the two extremal rays of $\mathrm{Eff}(X)$. There are five weights that can be placed on these rays, therefore we arrive at with three possible dispositions:



By specifying $w_{11} \in \mathrm{cone}(w_{01}, w_{21})^\circ$ in disp 1 and $w_{01} \in \mathrm{cone}(w_{11}, w_{21}) \setminus \mathrm{cone}(w_{11})$ in disp 2, the three pictures represent mutually exclusive situations. We prove that none of these dispositions defines a valid variety.

Some preliminary considerations can be done in general. Note, for example, that in all dispositions, every ray of the type v_{ij} appears in at least one elementary big cone. Hence Proposition 2.3.1 delivers $l_0 = (1, 1, 1)$ and $l_1 = (1, 1)$. With admissible operations we achieve

$$P = \begin{bmatrix} -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & l_{21} & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & l_{31} \\ 0 & 1 & 0 & 0 & d_{112} & d_{121} & d_{131} \\ 0 & 0 & 1 & 0 & d_{212} & d_{221} & d_{231} \end{bmatrix},$$

with $0 \leq d_{131}, d_{231} < l_{31}$.

Disposition 1: we apply Corollary 3.1.5 to the set $\{3, 5\}$ and obtain

$$d_{221} = -\frac{l_{21} d_{231}}{l_{31}}.$$

Now we take a look at the vertices of $A^c_{X,0} \subset \mathbb{Q}^2$. The disposition shows that there are five elementary big cones and the respective vertices are:

$$u_1 = \left(\frac{d_{121} l_{31} + d_{131} l_{21}}{l_{21} + l_{31}}, \ \frac{l_{21} l_{31}}{l_{21} + l_{31}} \right).$$

$$u_2 = \left(\frac{d_{121} l_{31} + d_{131} l_{21}}{l_{21} + l_{31}}, \ 0 \right), \qquad u_3 = u_1 + \left(\frac{l_{21} l_{31}}{l_{21} + l_{31}}, 0 \right),$$

$$u_4 = u_1 + \frac{l_{21} l_{31}}{l_{21} + l_{31}} (d_{112}, \ d_{212}), \qquad u_5 = u_3 + \left(\frac{l_{21} l_{31}}{l_{21} + l_{31}}, 0 \right),$$

Since $l_{21}, l_{31} \geq 2$ holds, the parallelogram $\operatorname{conv}(u_2, u_3, u_4, u_5)$ contains at least one lattice point at height $\{y = -1\}$. This contradicts terminality.

Disposition 2: we apply Corollary 3.1.5 to the sets $\{3, 5\}$, $\{2, 4\}$ and obtain

$$d_{221} = -\frac{l_{21} d_{231}}{l_{31}}, \qquad d_{112} = -\frac{d_{121}}{l_{21}} - \frac{d_{131}}{l_{31}}.$$

Using these equalities, the vertex of $A^c_{X,0} \subset \mathbb{Q}^2$ defined by $\operatorname{cone}(v_{01}, v_{12}, v_{21}, v_{31})$ is

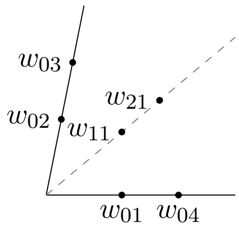
$$\left(0 \ , \ d_{212} \frac{l_{21} l_{31}}{l_{21} + l_{31}} \right).$$

This implies $d_{212} \neq 0$. Since $l_{21}, l_{31} \geq 2$ holds, the point $(0, d_{212})$ lies in $A^c_{X,0}$, a contradiction to terminality by Proposition 2.1.12.

Disposition 3: the same considerations of the previous disposition apply here.

Case (f) of Lemma 3.2.7 (ii)

We have $r = 2$, $m = 0$, $n = 6$ and $\overline{n} = (4, 1, 1)$. We show that no valid variety arises from this case. By combinatorial minimality there is only one disposition of the weights, as sketched in the picture:



In particular, all cones of type $\operatorname{cone}(v_{0j}, v_{11}, v_{21})$ are elementary big and Proposition 2.3.1 delivers $l_{0j} = 1$ for $j = 1, \dots, 4$. With admissible operations, and using the fact that $\operatorname{conv}(0, v_{0j}; j = 1, 2, 3, 4) \subset A_X^c$ does not contain integral points other than the vertices, we achieve

$$P = \begin{bmatrix} -1 & -1 & -1 & -1 & l_{11} & 0 \\ -1 & -1 & -1 & -1 & 0 & l_{21} \\ 0 & 1 & 0 & 1 & d_{111} & d_{121} \\ 0 & 0 & 1 & 1 & d_{211} & d_{221} \end{bmatrix}.$$

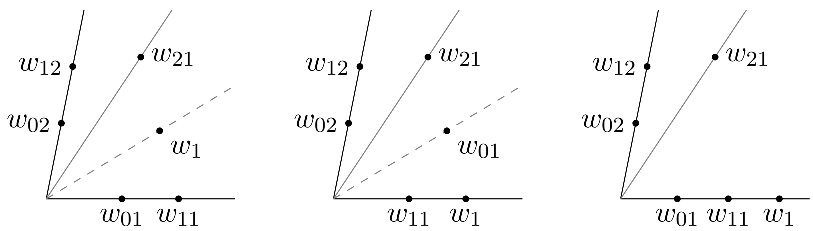
Regarding the weights, we can set $w_{01}^2 = w_{04}^2 = 0$. Applying Remark 3.3.1 to $\gamma_{01,02,04}$ and $\gamma_{01,03,04}$ we obtain $w_{02}^2 = w_{03}^2 = 1$, which allows us to further assume $w_{02}^1 = w_{03}^1 = 0$. Now, homogeneity of the relation yields $2 = \mu^2 = l_{11} w_{11}^2 = l_{21} w_{21}^2$, hence $l_{11} = l_{21} = 2$ and $w_{11}^2 = w_{21}^2 = 1$. Since the last two weights lie on the same half-line, we also have $w_{11}^1 = w_{21}^1$. Up to this point, the matrix Q can be written down as follows:

$$Q = \begin{bmatrix} w_{01}^1 & 0 & 0 & w_{04}^1 & a & a \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix},$$

for some $a \in \mathbb{Z}_{\geq 1}$. In particular the anticanonical class is $-\mathcal{K}_X = (2a, 2)$. Hence $-\mathcal{K}_X$ lies in $\operatorname{cone}(w_{11}) = Q(\gamma_{11,21})$, which is a non-fulldimensional projected relevant $\mathfrak{F}$-face. This contradicts $\mathbb{Q}$-factoriality of X by Proposition 1.4.2.

Case (g) of Lemma 3.2.7 (ii)

We have $r = 2$, $m = 1$, $n = 5$ and $\overline{n} = (2, 2, 1)$. Taking combinatorial minimality into account, there are three possible weight dispositions:



disp 1 disp 2 disp 3

where in disposition 1 we may assume without loss of generality that $-\mathcal{K}_X \in \operatorname{cone}(w_{02}, w_1)^\circ$ holds. Therefore all three dispositions present the following elementary big cones:

$$\operatorname{cone}(v_{01}, v_{11}, v_{21}), \qquad \operatorname{cone}(v_{01}, v_{12}, v_{21}), \qquad \operatorname{cone}(v_{02}, v_{11}, v_{21}).$$

Following Proposition 2.3.1, we find three configurations for the exponents:

A $l_0 = (1,1)$;
B $l_1 = (1,1)$;
C $l_{01} = l_{11} = 1$.

Note that, in both dispositions 1 and 3, configurations A and B coincide after admissible operations on P of type (i) and (ii). We show that this case delivers variety No. 11 from the table of Theorem 3.4.1, specifically in situation 2A.

Situation 1A: here Remark 3.3.1 applied to $\gamma_{01,02,11,12}$ ensures that $\mathrm{Eff}(X) = \mathbb{Q}^2_{\geq 0}$ holds. Together with homogeneity of the relation and the relevant faces $\gamma_{01,02,12,21}$ and $\gamma_{01,02,11,21}$ we arrive at $l_{11} = l_{12} = l_{21}$ and

$$Q \;=\; \begin{bmatrix} l_{21} & 0 & 1 & 0 & 1 & w_1^1 \\ 0 & l_{21} & 0 & 1 & 1 & w_1^2 \end{bmatrix}.$$

Now we turn to P. With admissible operations we achieve $d_{101} = d_{201} = 0$ as well as $d'_{11} = 0$ and $d'_{21} = 1$. Moreover we can assume $0 \leq d_{202} < d_{102}$. Using the equations coming from the third row of $P \cdot Q^T = 0$ we obtain

$$d_{111} = -d_{121} \qquad \text{and} \qquad d_{112} = -d_{121} - l_{21}d_{102}.$$

The three elementary big cones listed above and the ray in the lineality space of $\mathrm{trop}(X)$ define the four vertices of the lineality part $A^c_{X,0} \subset \mathbb{Q}^2$ of the anticanonical complex:

$$\begin{aligned} u_1 &= \frac{1}{2}\left(0\ ,\ d_{211} + d_{221}\right), \\ u_2 &= \frac{1}{2}\left(-l_{21}d_{102}\ ,\ d_{212} + d_{221}\right), \\ u_3 &= \frac{1}{2}\left(l_{21}d_{102}\ ,\ d_{211} + d_{221} + l_{21}d_{202}\right), \\ u_4 &= \left(0\ ,\ 1\right). \end{aligned}$$

Note that u_1 lies on the y-axis. By terminality $(0,-1) \notin A^c_{X,0}$ holds, hence we arrive at $d_{211} + d_{221} = -1$. This can be used to simplify the form of u_3. We have $u_3^1 \geq 1$ as well as $u_3^1 > u_3^2 \geq -1/2$. Since $(1,0)$ and $(1,1)$ do not lie on the lineality part $A^c_{X,0}$, the previous inequalities imply $d_{102} = 1$, $d_{202} = 0$ and $l_{21} = 2$. Now we look at $d_{212} + d_{221}$: if it is even, then u_2 is a lattice point, if it is odd, then a lattice point lies between v_{11} and v_{12}. In both cases we obtain a contradiction to terminality of X.

Situation 1C: by Remark 3.3.1 applied to $\gamma_{01,02,11,12}$, we can assume that $\mathrm{Eff}(X) = \mathbb{Q}^2_{\geq 0}$ holds. Together with homogeneity of the relation we are able to express the first row of the grading matrix Q as follows

$$\begin{bmatrix} w_{01}^1 & 0 & w_{01}^1 & 0 & \frac{w_{01}^1}{l_{21}} & w_1^1 \end{bmatrix}.$$

Using again Remark 3.3.1 with $\gamma_{01,02,11,12}$ we conclude $w_{01}^1 = 1$ and hence $l_{21} = 1$, a contradiction to P irredundant.

Disposition 2: we apply Remark 3.3.1 to the relevant faces $\gamma_{02,11,1}$ and $\gamma_{02,12,1}$ and achieve the following degree matrix:

$$Q = \begin{bmatrix} w_{01}^1 & 0 & w_{11}^1 & 0 & w_{21}^1 & 1 \\ w_{01}^2 & 1 & 0 & w_{12}^2 & w_{21}^2 & 0 \end{bmatrix}.$$

Situation 2A: homogeneity of the relation delivers

$$w_{11}^1 = \frac{w_{01}^1}{l_{11}}, \qquad w_{12}^1 = \frac{w_{01}^1}{l_{21}},$$

$$w_{21}^2 = \frac{w_{01}^2+1}{l_{12}}, \qquad w_{22}^1 = \frac{w_{01}^2+1}{l_{21}}.$$

We apply Remark 3.3.1 to $\gamma_{01,02,11,12}$ and obtain $w_{01}^1 = l_{11}$. Note that there are two possibilities for the anticanonical class $-\mathcal{K}_X$: it lies either in $\mathrm{cone}(w_1, w_{01})^\circ$ or in $\mathrm{cone}(w_{01}, w_{02})^\circ$. We rule out the first cone because then $\gamma_{01,11,1}$ would be a relevant face and Remark 3.3.1 would imply $w_{01}^2 = 1$, $l_{21} = 2$ and $l_{12} = 1, 2$. This in turn forces $-\mathcal{K}_X$ outside of $\mathrm{cone}(w_1, w_{01})^\circ$, a contradiction.
Hence the anticanonical class lies in $\mathrm{cone}(w_{01}, w_{02})^\circ$. This means in particular that $\gamma_{01,02,12,21}$ is relevant, which implies $l_{11} = l_{21}$. We turn to the matrix P. With admissible operations we set $d_{101} = d_{201} = 0$, as well as $d'_{11} = 0$ and $d'_{21} = 1$. Looking at $P \cdot Q^T = 0$ we also obtain $d_{111} = -d_{121}$ and $d_{211} = -d_{221} - 1$. With more admissible operations we can assume $0 \le d_{202} < d_{102}$ and $0 \le d_{121}, d_{221} < l_{21}$. The three elementary big cones listed above and the lattice point $v_1 \in \mathrm{trop}_0(X)$ define the four vertices of the lineality part $A_{X,0}^c \subset \mathbb{Q}^2$, namely:

$$\begin{aligned} u_1 &= \left(0\ ,\ -\frac{1}{2}\right), \\ u_2 &= \frac{1}{l_{12}+l_{21}}\left(l_{21}d_{112} + l_{12}d_{121}\ ,\ l_{21}d_{212} + l_{12}d_{221}\right), \\ u_3 &= \frac{1}{2}\left(l_{21}d_{102}\ ,\ l_{21}d_{202} - 1\right), \\ u_4 &= (0\ ,\ 1)\,. \end{aligned}$$

By the previous inequalities, we have $u_3^1 \ge 1$ as well as $u_3^1 > u_3^2 \ge -1/2$. Since $(1, 0)$ and $(1, 1)$ do not lie in the lineality part, we get $d_{102} = 1$, $d_{202} = 0$ and $l_{21} = 2$. By the same inequalities we have $d_{121}, d_{221} \in \{0, 1\}$. They are not both equal to zero, because v_{21} is primitive. Since v_{11} is also primitive, we get $d_{121} = 1$. Proposition 3.1.4 used on $\{3, 6\}$ yields $d_{212} = -l_{12}d_{221}/2$. This means that u_2 lies on the x-axis. Hence terminality requires $-1 < u_2^1 < 0$, i.e. $-l_{12} \le d_{112} < -l_{12}/2$. Now, the only value of P still unbounded is l_{12}. First assume that $l_{12} = 1$. Then $d_{112} = -1$ and $d_{221} = 0$ hold and these data define a valid variety, namely No. 11. So assume $l_{12} > 1$. Since the anticanonical class lies in $\mathrm{cone}(w_{01}, w_{02})$, we have $0 < \det(w_{01}, -\mathcal{K}_X)$ and obtain

$$1 \le w_{01}^2 < \frac{l_{12}+2}{2l_{12}-2}.$$

This implies immediately $l_{12} < 4$ and $w_{01}^2 = 1$. The case $l_{12} = 3$ is not valid, because w_{12}^2 would not be an integer. The case $l_{12} = 2$ delivers a variety that is not terminal, since we obtain $d_{112} = -2$, $d_{221} = 1$ and finally an integral point on the segment line joining v_{12} and v_1, namely $(1, 0, -1, 0)$.

Situation 2B: as a first step we exploit homogeneity of g_0:

$$w_{11}^1 = l_{01} w_{01}^1, \qquad w_{12}^2 = l_{01} w_{01}^2 + l_{02},$$
$$w_{21}^1 = \frac{l_{01} w_{01}^1}{l_{21}}, \qquad w_{21}^2 = \frac{l_{01} w_{01}^2 + l_{02}}{l_{21}}.$$

We apply Remark 3.3.1 to $\gamma_{01,02,11,12}$ and $\gamma_{02,11,1}$ to obtain $w_{01}^1 = 1$ and $l_{21} = l_{01}$. Applying it again, this time to $\gamma_{11,12,21,1}$, leads to a contradiction to the disposition, since we get $w_{21}^2 = 1$ and this means $w_{01}^2 = 1 - \frac{l_{02}}{l_{01}} < 1$.

Situation 2C: it is sufficient to look at the first row of Q. By homogeneity, $w_{01}^1 = w_{11}^1 = l_{21} w_{21}^1$ holds. With Remark 3.3.1 applied to $\gamma_{01,02,11,12}$ we get $\mu^1 = w_{01}^1 = 1$ and conclude $l_{21} = 1$, a contradiction to P irredundant.

Situation 3A: by almost freeness we can assume $\mathrm{Eff}(X) = \mathbb{Q}^2_{\geq 0}$ and, exploiting the homogeneity of the relation, we can write

$$Q = \begin{bmatrix} w_{01}^1 & 0 & \frac{w_{01}^1}{l_{11}} & 0 & \frac{w_{01}^1}{l_{21}} & w_1^1 \\ 0 & w_{02}^2 & 0 & \frac{w_{02}^2}{l_{12}} & \frac{w_{02}^2}{l_{21}} & 0 \end{bmatrix}.$$

Using Remark 3.3.1 first on $\gamma_{02,12,1}$ and $\gamma_{01,02,11,12}$ and then on $\gamma_{01,02,11,21}$ and $\gamma_{01,02,12,21}$ we arrive at $l_{11} = l_{12} = l_{21}$ and

$$Q = \begin{bmatrix} l_{21} & 0 & 1 & 0 & 1 & 1 \\ 0 & l_{21} & 0 & 1 & 1 & 0 \end{bmatrix}.$$

We use admissible operations to achieve $v_{01} = (-1, -1, 0, 0)$ and $v_1 = (0, 0, 0, 1)$. The equations resulting from $P \cdot Q^T = 0$ fix some relations between the d-values, so that we can write the matrix P as

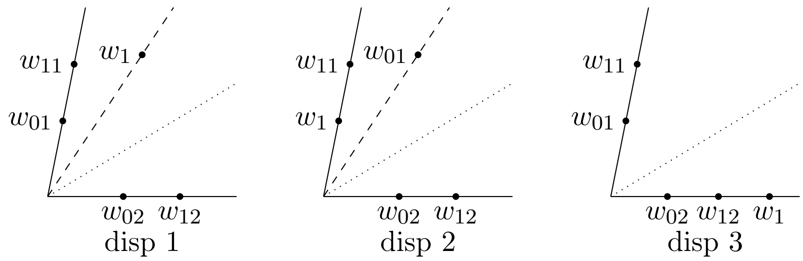
$$P = \begin{bmatrix} -1 & -1 & l_{21} & l_{21} & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{21} & 0 \\ 0 & d_{102} & -d_{121} & -l_{21} d_{102} - d_{121} & d_{121} & 0 \\ 0 & d_{202} & -d_{221} - 1 & -l_{21} d_{202} - d_{221} & d_{221} & 1 \end{bmatrix}.$$

The vertex u of the anticanonical complex arising from the elementary big cone $P(\delta_{01,12,21})$ is $-\frac{1}{2}(l_{21} d_{102}, l_{21} d_{202})$. Since $l_{21} \geq 2$ holds, the integral point $(-d_{102}, -d_{202})$ lies between the vertex u and the origin. This is a contradiction to terminality.

Situation 3C: the same proof as in situation 1C applies.

Case (h) of Lemma 3.2.7 (ii)

We have $r = 3$, $m = 1$, $n = 6$ and $\overline{n} = (2, 2, 1, 1)$. Combinatorial minimality prescribes at least two weights on each of the two extremal rays of $\mathrm{Eff}(X)$. Since there are five weights that can be placed on these rays, we find the following three dispositions:



In these pictures, the weights w_{21} and w_{31} always lie on the thin dotted half-line, which is spanned by the degree μ of the relations. The following cones are elementary big in all three dispositions:

$$\mathrm{cone}(v_{01}, v_{12}, v_{21}, v_{31}), \qquad \mathrm{cone}(v_{02}, v_{11}, v_{21}, v_{31}).$$

Therefore Proposition 2.3.1 implies $l_{01} = l_{02} = l_{11} = l_{12} = 1$. With suitable admissible operations we can achieve the form:

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{21} & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & l_{31} & 0 \\ 0 & d_{102} & 0 & d_{112} & d_{121} & d_{131} & 0 \\ 0 & d_{202} & 0 & d_{212} & d_{221} & d_{231} & 1 \end{bmatrix},$$

where we also have $0 \le d_{121}, d_{221} < l_{21}$ and, if $d_{102} \neq 0$, then $0 \le d_{202} < d_{102}$.

Disposition 1: we assume that $-\mathcal{K}_X \in \mathrm{cone}(w_{02}, w_1)$ holds, otherwise one only needs to swap v_{01} with v_{02} and v_{11} with v_{12}. This implies that $\mathrm{cone}(v_{01}, v_{11}, v_{21}, v_{31})$ is elementary big. Moreover we can assume $w_{02}^2 = w_{12}^2 = 0$. By using the fact that all monomials have the same degree, we achieve the following degree matrix:

$$Q = \begin{bmatrix} w_{01}^1 & w_{02}^1 & w_{01}^1 & w_{02}^1 & \frac{w_{01}^1 + w_{02}^1}{l_{21}} & \frac{w_{01}^1 + w_{02}^1}{l_{31}} & w_1^1 \\ w_{01}^2 & 0 & w_{01}^2 & 0 & \frac{w_{01}^2}{l_{21}} & \frac{w_{01}^2}{l_{31}} & w_1^2 \end{bmatrix}.$$

Now we can use Remark 3.3.1. In particular, with $\gamma_{01,02,11,12,21}$ and $\gamma_{01,02,11,12,31}$ we obtain $l_{21} = l_{31} = w_{01}^2$. By applying it to $\gamma_{02,12,1}$, one gets $\det(w_{02}, w_1) = 1$, and concludes $w_{02}^1 = w_1^2 = 1$. Since $w_{21}^2 = w_{31}^2 = 1$ holds, by suitably adding the second row to the first one we achieve the following degree matrix:

$$Q = \begin{bmatrix} w_{01}^1 & 1 & w_{01}^1 & 1 & 0 & 0 & w_1^1 \\ l_{21} & 0 & l_{21} & 0 & 1 & 1 & 1 \end{bmatrix}.$$

By homogeneity, $w_{01}^1 = -1$ holds. A contradiction to the disposition of the weights arises as follows: $\det(w_1, w_{01}) > 0$ is not possible, since it implies $w_1^1 l_{21} + 1 > 0$ but w_1^1 is negative.

Disposition 2: we can assume that w_{02} and w_{12} lie on cone(e_1), so $w_{02}^2 = w_{12}^2 = 0$ holds. By homogeneity of the relations, the second row of Q looks as follows:

$$\begin{bmatrix} w_{01}^2 & 0 & w_{01}^2 & 0 & \frac{w_{01}^2}{l_{21}} & \frac{w_{01}^2}{l_{31}} & w_1^2 \end{bmatrix}.$$

Applying Remark 3.3.1 to $\gamma_{01,02,11,12,21}$ and $\gamma_{01,02,11,12,31}$ one gets $l_{21} = w_{01}^2 = l_{31}$. Using $P \cdot Q^T = 0$, specifically for the second row of Q, we obtain the following equalities:

$$d_{121} = -d_{131}, \qquad d_{221} = -d_{231} - 1.$$

Altogether, these equalities simplify the entries of the vertices of $A_{X,0}^c \subset \mathbb{Q}^2$:

$$\begin{aligned} u_1 &= \left(\frac{l_{21}}{2} d_{102} \ , \ \frac{l_{21}}{2} d_{202} - \frac{1}{2}\right), \\ u_2 &= \left(\frac{l_{21}}{2} d_{112} \ , \ \frac{l_{21}}{2} d_{212} - \frac{1}{2}\right), \\ u_3 &= \left(0 \ , \ -\frac{1}{2}\right), \\ u_4 &= (0 \ , \ 1). \end{aligned}$$

The first three are given respectively by cone$(v_{01}, v_{12}, v_{21}, v_{31})$, cone$(v_{02}, v_{11}, v_{21}, v_{31})$ and cone$(v_{01}, v_{11}, v_{21}, v_{31})$, whereas the last one is v_1, the last column of P. Note that we can assume $d_{202} > 0$. Otherwise $d_{202} = 0$ implies $d_{102} = \pm 1$ by terminality and the determinant obtained by $P_{\{3,7\}}$ cannot be zero, contradicting Corollary 3.1.5 and combinatorial minimality. With $d_{102} > d_{202} > 0$ we conclude that $u_1^1 > u_1^2$ and $u_1^1 > 1$ hold. Hence, to avoid $(1,1)$ from lying in the lineality part we have $u_1^2 < 1$, which translates into $d_{202} = 1$ and $l_{21} = 2$. Furthermore, as soon as $d_{102} \geq 3$ holds, we have integral points in conv$(0, u_1, u_3, u_4)$. The only possibility left is $d_{102} = 2$, but even in this case $(1,0)$ lies in the lineality part. This leaves no valid variety.

Disposition 3: Corollary 3.1.5 can be applied to $\{1,3\}$, $\{2,4\}$, $\{2,7\}$ and delivers

$$d_{112} = -d_{102}, \qquad d_{131} = -\frac{d_{121} l_{31}}{l_{21}}, \qquad d_{231} = -\frac{d_{221} l_{31}}{l_{21}}.$$

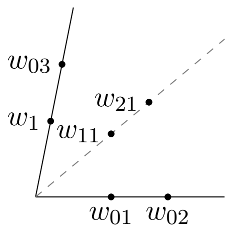
The elementary big cones cited above define the following two vertices of $A_{X,0}^c$:

$$\begin{aligned} u_1 &= \frac{l_{21} l_{31}}{l_{21} + l_{31}} \left(-d_{102}, d_{212}\right), \\ u_2 &= \frac{l_{21} l_{31}}{l_{21} + l_{31}} \left(d_{102}, d_{202}\right). \end{aligned}$$

Since $l_{21}, l_{31} \geq 2$ holds, the points $(-d_{102}, d_{212})$ and (d_{102}, d_{202}) lie in the lineality part of A_X^c, contradicting terminality.

Case (i) of Lemma 3.2.7 (ii)

We have $r = 2$, $m = 1$, $n = 5$ and $\overline{n} = (3, 1, 1)$. This case provides No. 12 in the list of Theorem 3.4.1. First note that combinatorially minimal only allows one disposition of the weights, up to renumbering of variables:



This leads to all cones of type $\mathrm{cone}(v_{0j}, v_{11}, v_{21})$, for $j = 1, 2, 3$, being elementary big. Hence Proposition 2.3.1 implies $l_0 = (1, 1, 1)$. With admissible operations we arrive at

$$P = \begin{bmatrix} -1 & -1 & -1 & l_{11} & 0 & 0 \\ -1 & -1 & -1 & 0 & l_{21} & 0 \\ 0 & d_{102} & d_{103} & d_{111} & d_{121} & 0 \\ 0 & d_{202} & d_{203} & d_{211} & d_{221} & 1 \end{bmatrix}$$

and $0 \le d_{121}, d_{221} < l_{21}$. More can be said about Q, first by assuming that w_{01} and w_{02} lie on $\mathrm{cone}(1, 0)$ and then by using Remark 3.3.1 with $\gamma_{01,02,1}$. We achieve

$$Q = \begin{bmatrix} w_{01}^1 & w_{02}^1 & 0 & w_{11}^1 & w_{21}^1 & 0 \\ 0 & 0 & w_{03}^2 & w_{11}^2 & w_{21}^2 & 1 \end{bmatrix}.$$

Remark 3.3.1, applied on $\gamma_{01,03,1}$ and $\gamma_{02,03,1}$, implies $w_{01}^1 = w_{02}^1 = 1$. Then, from the homogeneity of the monomials we obtain

$$w_{11}^1 = \frac{2}{l_{11}}, \qquad w_{11}^2 = \frac{w_{03}^2}{l_{11}}, \qquad w_{21}^1 = \frac{2}{l_{21}}, \qquad w_{21}^2 = \frac{w_{03}^2}{l_{21}}.$$

These imply in particular $l_{11} = l_{21} = 2$. Considering once again Remark 3.3.1, with $\gamma_{01,02,03,11}$ and $\gamma_{01,02,03,21}$, we see that $\gcd(w_{03}^2, w_{11}^2) = \gcd(w_{03}^2, w_{21}^2) = 1$ holds, hence $w_{03}^2 = 2$. The matrix Q is now

$$Q = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{bmatrix}.$$

Corollary 3.1.5 applied to the set $\{1, 2\}$ and $P \cdot Q^T = 0$ applied to the third and forth rows of P yield

$$d_{103} = \frac{1}{2} d_{102}, \qquad\qquad d_{111} = -d_{102} - d_{121},$$
$$d_{203} = \frac{1}{2}(d_{202} - 1), \qquad\qquad d_{211} = -d_{202} - d_{221}.$$

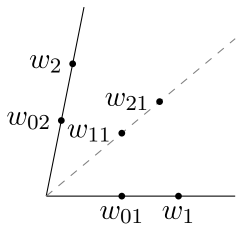
It is clear that $d_{102} \neq 0$ holds, otherwise v_{01}, v_{02}, v_{03} all lie on a common line. Therefore we can assume $d_{102} > 0$ and $0 \leq d_{202} < d_{102}$ by using more admissible operations, that do not destroy any of the equalities found up to now. In order to bound d_{102}, we look at the vertices of the lineality part $A^c_{X,0} \subset \mathbb{Q}^2$, which are

$$u_1 = \left(-\frac{1}{2}d_{102},\ -\frac{1}{2}d_{202}\right), \qquad u_3 = \left(0,\ -\frac{1}{2}\right),$$
$$u_2 = \left(\frac{1}{2}d_{102},\ \frac{1}{2}d_{202}\right), \qquad u_4 = (0,\ 1).$$

By terminality, $A^c_{X,0}$ does not cointain integral points other than the origin. In particular $(1,1) \notin A^c_{X,0}$ implies $d_{202} < 2$, hence $d_{202} = 1$. Finally, with $(1,0) \notin A^c_{X,0}$ we conclude $d_{102} < 4$, hence $d_{102} = 2$. Among the very few varieties left, the MDSpackage [25] shows that only the one listed as No. 12 in Theorem 3.4.1 fulfills all assumptions.

Case (j) of Lemma 3.2.7 (ii)

We have $r = 2$, $m = 2$, $n = 4$ and $\overline{n} = (2, 1, 1)$. We show that this case does not provide valid examples for the family of varieties studied. Due to combinatorial minimality, there is only one possible disposition for the weights, as illustrated below:



Since both cone(v_{01}, v_{11}, v_{21}) and cone(v_{02}, v_{11}, v_{21}) are elementary big, Proposition 2.3.1 yields $l_{01} = l_{02} = 1$. Using admissible operations we can achieve

$$P = \begin{bmatrix} -1 & -1 & l_{11} & 0 & 0 & 0 \\ -1 & -1 & 0 & l_{21} & 0 & 0 \\ 0 & d_{102} & d_{111} & d_{121} & 1 & 0 \\ 0 & d_{202} & d_{211} & d_{221} & 0 & 1 \end{bmatrix},$$

In order to bring v_1 and v_2 into the form above, we ruled out the case $v_1 = -v_2$ with Propositions 3.1.13 and 3.2.2 combined and then noted that, by terminality, conv($0, v_1, v_2$) cannot contain integral points other than its vertices.
Applying Corollary 3.1.5 to the sets $\{1, 5\}$ and $\{2, 6\}$ we obtain equalities for d_{102} and d_{211} respectively. These simplify the coordinates of the vertex u of the anticanonical complex, corresponding to the elementary big cone cone(v_{02}, v_{11}, v_{21}), as follows:

$$u = \left(0, d_{202}\frac{l_{11}l_{21}}{l_{11} + l_{21}}\right).$$

Since $l_{11}, l_{21} \geq 2$ and u does not coincide with the origin, the last coordinate of u is bigger than one in absolute value. Hence the point $(0, -1)$ lies in $A^c_{X,0}$, contradicting terminality.

3.4 Classification

Theorem 3.4.1. *The following table lists the Cox rings $\mathcal{R}(X)$ of the non-toric rational combinatorially minimal terminal $\mathbb{Q}$-factorial Fano threefolds X with an effective two-torus action and with Picard number at least two. The* $\mathrm{Cl}(X)$*-degrees of the generators $T_1, \ldots, T_r$ are denoted as columns $w_i \in \mathrm{Cl}(X)$ of a matrix $[w_1, \ldots, w_r]$.*

No.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	$[w_1, \ldots, w_r]$
2.01	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2 + T_3T_4 + T_5T_6 \rangle$	$\mathbb{Z}^2$	$\begin{bmatrix} 1&0&1&0&1&0 \\ 0&1&0&1&0&1 \end{bmatrix}$
2.02	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^2 + T_3T_4^2 + T_5T_6^2 \rangle$	$\mathbb{Z}^2$	$\begin{bmatrix} 1&0&1&0&1&0 \\ 0&1&0&1&0&1 \end{bmatrix}$
2.03	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2 + T_3T_4 + T_5T_6 \rangle$	$\mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix} 1&0&1&0&1&0 \\ 0&1&0&1&0&1 \\ \bar{2}&\bar{1}&\bar{1}&\bar{2}&\bar{0}&\bar{0} \end{bmatrix}$
2.04	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^2 + T_3T_4^2 + T_5T_6^2 \rangle$	$\mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix} 1&0&1&0&1&0 \\ 0&1&0&1&0&1 \\ \bar{1}&\bar{1}&\bar{2}&\bar{2}&\bar{0}&\bar{0} \end{bmatrix}$
2.05	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2 + T_3^2T_4 + T_5T_6 \rangle$	$\mathbb{Z}^2$	$\begin{bmatrix} 2&0&1&0&1&1 \\ 0&1&0&1&0&1 \end{bmatrix}$
2.06	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2 + T_3T_4^2 + T_5^2T_6^2 \rangle$	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&2&0&1&0 \\ 1&1&0&1&0&1 \end{bmatrix}$
2.07	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2T_3 + T_4T_5 + T_6^2 \rangle$	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&0&2&0&1 \\ 0&0&2&1&1&1 \end{bmatrix}$
2.08	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2T_3 + T_4T_5^2 + T_6^2 \rangle$	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&0&2&0&1 \\ 1&0&1&0&1&1 \end{bmatrix}$
2.09	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2T_3^2 + T_4T_5 + T_6^2 \rangle$	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&0&2&0&1 \\ 0&0&1&1&1&1 \end{bmatrix}$
2.10	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2T_3^2 + T_4T_5^2 + T_6^2 \rangle$	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&0&2&0&1 \\ 0&0&1&0&1&1 \end{bmatrix}$
2.11	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2 + T_3^2T_4 + T_5^2 \rangle$	$\mathbb{Z}^2$	$\begin{bmatrix} 2&0&1&0&1&1 \\ 1&1&0&2&1&0 \end{bmatrix}$
2.12	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2T_3 + T_4^2 + T_5^2 \rangle$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 1&1&0&1&1&0 \\ 0&0&2&1&1&1 \\ \bar{0}&\bar{1}&\bar{1}&\bar{1}&\bar{0}&\bar{0} \end{bmatrix}$

Any two of the Cox rings listed in the table correspond to non-isomorphic varieties. No. 2.01 and No. 2.02 are the only smooth ones.

Proof. Lemma 3.2.7 lists all possible cases for the matrix P defining a terminal $\mathbb{Q}$-factorial combinatorially minimal Fano threefold $X = X(A, P)$ with effective two-torus action and Picard number at least two. Proposition 3.2.8 ensures that $\rho(X) = 2$ holds. The cases treated in Section 3.3 deliver the list from the assertion. Furthermore, by comparing the data, one directly sees that any two varieties listed above are non-isomorphic. □

Appendix A contains detailed information about all varieties of Theorem 3.4.1; in particular one finds possible defining matrices P.

Remark 3.4.2. By Remark 2.5.2, the assumption of rationality can be omitted in Theorem 3.4.1, for $\mathbb{K} = \mathbb{C}$. Alternatively, rationality can be replaced by the property "$\mathrm{Cl}(X)$ is finitely generated".

CHAPTER

FOUR

SMOOTH T-VARIETIES OF COMPLEXITY ONE WITH $\rho(X) = 2$

This Chapter contributes to the classification of smooth (almost) Fano varieties with torus action. The toric case already produced classification results up to dimension nine [4, 51, 5, 39, 46, 48], based on the description via lattice polytopes. We are interested in varieties of complexity one. The results of this chapter have been published in [19], a joint work of the author of this thesis with A. Fahrner and J. Hausen.

Here we look at varieties of arbitrary dimension with small Picard number. Recall that in toric geometry the projectives spaces are the only smooth examples of Picard number one, and we have Kleinschmidt's description [36] of all smooth toric varieties with Picard number two, which in particular allows to figure out the (almost) Fano ones in this setting. We follow that approach and study first arbitrary smooth projective rational varieties with a torus action of complexity one. The case of Picard number one is basically settled by a result of Liendo and Süß [40, Thm. 6.5]: the only non-toric examples are the smooth projective quadrics in dimension three and four. For Picard number two we need to provide an analogue of Kleinschmidt's description for complexity one T-varieties.

In Section 4.1, we derive first bounding conditions on the defining data for smooth varieties of complexity one and Picard number two. Section 4.2 is devoted to proving the main classification results, which are listed in Section 4.3. In Section 4.4, we introduce and discuss duplication of free weights and show how to obtain the Fano varieties of Theorem 4.3.2 via this procedure from lower dimensional varieties. Finally, in Section 4.5, we describe the Fano varieties of Theorem 4.3.2 in more geometric terms.

4.1 First structural constraints

We derive conditions on the defining matrices of smooth rational projective varieties $X = X(A, P, u)$ with a torus action of complexity one and $\rho(X) = 2$. Throughout the whole Chapter, P is irredundant in the sense of Remark 1.5.2. Moreover, let Z be the minimal toric ambient variety of X, with fan Σ, as in Construction 1.3.8.

Recall from Proposition 1.4.2 that X is smooth if and only if it is factorial and quasismooth, where we call X *quasismooth* if $\widehat{X}$ is smooth.

First we study the impact of X being factorial on the defining matrix P. Recall that X is called *weakly tropical* if and only if Σ consists only of leaf cones. In the notation of Construction 1.5.10, this means that every cone $\sigma \in \Sigma$ is contained in a leaf $\lambda_i \subset \mathrm{trop}(X)$ for some $0 \le i \le r$.

Lemma 4.1.1. *Let $X = X(A, P, u)$ be non-toric and factorial. If X is weakly tropical, then $n_i \ge 2$ holds for all $i = 0, \dots, r$.*

Proof. Suppose that $n_i = 1$ holds for some i. Since X is weakly tropical, there exists a cone $\sigma \in \Sigma$ of dimension $s + 1$ contained in the leaf λ_i. Because of $n_i = 1$ we have $\sigma = \varrho_{i1} + \tau$ with a facet $\tau \preceq \sigma$ such that $\tau \subseteq \lambda$. Now, $\sigma = P(\gamma_0^*)$ holds for some $\gamma_0 \in \mathrm{rlv}(X)$. Since the points of $X(\gamma_0)$ are factorial, σ is a regular cone. This implies $l_{i1} = 1$, contradicting irredundancy of P. □

Lemma 4.1.2. *Let $X = X(A, P, u)$ be non-toric and factorial. If X is weakly tropical, then $\rho(X) \ge r + 3$ holds.*

Proof. Lemma 4.1.1 ensures $n_i \ge 2$ for all $i = 1, \dots, r$, hence $n \ge 2 \cdot (r + 1)$ holds. The s-dimensional lineality space $\lambda = \{0\} \times \mathbb{Q}^s \subseteq \mathrm{trop}(X)$ is a union of cones of Σ. Thus P must have at least $s + 1$ columns v_k which means $m \ge s + 1$. Together this yields

$$\rho(X) \;=\; n + m - (r - 1) - (s + 1) \;\ge\; r + 3.$$

□

Lemma 4.1.3. *Let $X = X(A, P, u)$ be non-toric and not weakly tropical. If X is $\mathbb{Q}$-factorial, then there is an elementary big cone in Σ.*

Proof. Since X is not weakly tropical, there exists a big cone $\sigma \in \Sigma$. We have $\sigma = P(\gamma_0^*)$ with $\gamma_0 \in \mathrm{rlv}(X)$. Since the points of $X(\gamma_0)$ are $\mathbb{Q}$-factorial, the cone σ is simplical. For every $i = 0 \dots, r$ choose a ray $\varrho_i \preceq \sigma$ with $\varrho_i \in \lambda_i$. Then $\sigma_0 := \varrho_0 + \dots + \varrho_r \preceq \sigma$ is as wanted. □

Corollary 4.1.4. *Let $X = X(A, P, u)$ be non-toric and factorial. If $\rho(X) \le 4$ holds, then there exists an elementary big cone $\sigma \in \Sigma$.*

Next we investigate the effect on the defining matrix P of quasismoothness of X. This will lead to constraints on P via the Jacobian of the defining relations of $\overline{X}$.

Remark 4.1.5. Let (A, P) be defining matrices. Then the Jacobian J_g of the defining relations $g_0, \dots, g_{r-2}$ has the shape $J_g = (J, 0)$ with a zero block of size $(r - 1) \times m$ corresponding to the variables $S_1, \dots, S_m$ and a block

$$J \;:=\; \begin{bmatrix} \delta_{10} & \delta_{11} & \delta_{12} & 0 & & & & & \\ 0 & \delta_{21} & \delta_{22} & \delta_{23} & 0 & & & & \\ & & & & \vdots & & & & \\ & & & & & \delta_{r-2,r-3} & \delta_{r-2,r-2} & \delta_{r-2,r-1} & 0 \\ & & & & & 0 & \delta_{r-1,r-2} & \delta_{r-1,r-1} & \delta_{r-1,r} \end{bmatrix}$$

of size $(r-1) \times n$, where each vector $\delta_{a,i}$ is a nonzero multiple of the gradient of the monomial $T_i^{l_i}$:

$$\delta_{a,i} \;=\; \alpha_{a,i}\left(l_{i1}\frac{T_i^{l_i}}{T_{i1}},\;\ldots,\;l_{in_i}\frac{T_i^{l_i}}{T_{in_i}}\right), \qquad \alpha_{a,i} \;\in\; \mathbb{K}^*.$$

For given $1 \le a,b \le r-1$, $0 \le i \le r$ and $z \in \overline{X}$, we have $\delta_{a,i}(z) = 0$ if and only if $\delta_{b,i}(z) = 0$. Moreover, the Jacobian $J_g(z)$ of a point $z \in \overline{X}$ is of full rank if and only if $\delta_{a,i}(z) = 0$ holds for at most two different $i = 0, \ldots, r$.

Lemma 4.1.6. *Assume that $X = X(A,P,u)$ is non-toric and that there is an elementary big cone $\sigma = \varrho_{0j_0} + \ldots + \varrho_{rj_r} \in \Sigma$. If X is quasismooth, then $l_{ij_i} \ge 2$ holds for at most two $i = 0, \ldots, r$.*

Proof. We have $\sigma = P(\gamma_0^*)$ with a relevant face $\gamma_0 \in \mathrm{rlv}(X)$. Since X is quasismooth, any $z \in \overline{X}(\gamma_0)$ is a smooth point of $\overline{X}$. Thus, $J_g(z)$ is of full rank $r-1$. Consequently, $\delta_{a,i}(z) = 0$ holds for at most two different i. This means $l_{ij_i} \ge 2$ for at most two different i. □

Corollary 4.1.7. *Let $X = X(A,P,u)$ be non-toric and quasismooth. If there is an elementary big cone in Σ, then $n_i = 1$ holds for at most two different $i = 0, \ldots, r$.*

Lemma 4.1.8. *Let (A,P) be defining matrices. Consider the rays $\gamma_k := \mathrm{cone}(e_k)$ and $\gamma_{ij} := \mathrm{cone}(e_{ij})$ of the orthant $\gamma \subseteq \mathbb{Q}^{r+s}$ and the two-dimensional faces*

$$\gamma_{k_1,k_2} := \gamma_{k_1} + \gamma_{k_2}, \quad \gamma_{ij,k} := \gamma_{ij} + \gamma_k, \quad \gamma_{i_1j_1,i_2j_2} := \gamma_{i_1j_1} + \gamma_{i_2j_2}.$$

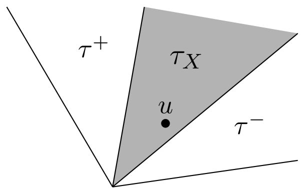
- *(i) All γ_k, resp. γ_{k_1,k_2}, are $\mathfrak{F}$-faces and each $\overline{X}(\gamma_k)$, resp. $\overline{X}(\gamma_{k_1,k_2})$, consists of singular points of $\overline{X}$.*
- *(ii) A given γ_{ij}, resp. $\gamma_{ij,k}$, is an $\mathfrak{F}$-face if and only if $n_i \ge 2$ holds. In that case, $\overline{X}(\gamma_{ij})$, resp. $\overline{X}(\gamma_{ij,k})$, consists of smooth points of $\overline{X}$ if and only if $r = 2$, $n_i = 2$ and $l_{i,3-j} = 1$ hold.*
- *(iii) A given γ_{ij_1,ij_2} with $j_1 \ne j_2$ is an $\mathfrak{F}$-face if and only if $n_i \ge 3$ holds. In that case, $\overline{X}(\gamma_{ij_1,ij_2})$ consists of smooth points of $\overline{X}$ if and only if $r = 2$, $n_i = 3$ and $l_{ij} = 1$ for the $j \ne j_1, j_2$ hold.*
- *(iv) A given $\gamma_{i_1j_1,i_2j_2}$ with $i_1 \ne i_2$ is an $\mathfrak{F}$-face if and only if we have $n_{i_1}, n_{i_2} \ge 2$ or $n_{i_1} = n_{i_2} = 1$ and $r = 2$. In the former case, $\overline{X}(\gamma_{i_1j_1,i_2j_2})$ consists of smooth points of $\overline{X}$ if and only if one of the following holds:*
  - *$r = 2$, $n_{i_t} = 2$ and $l_{i_t,3-j_t} = 1$ for a $t \in \{1,2\}$,*
  - *$r = 3$, $n_{i_1} = n_{i_2} = 2$, $l_{i_1,3-j_1} = l_{i_2,3-j_2} = 1$.*

Proof. The statements follow from the structure of the defining relations $g_0, \ldots, g_{r-2}$ of $R(A,P)$ and the shape of the Jacobian J_g. □

We now restrict to the case, in which the rational divisor class group $\mathrm{Cl}(X)_{\mathbb{Q}} = K_{\mathbb{Q}}$ of $X = X(A, P, u)$ is of dimension two. Set $\tau_X := \mathrm{Ample}(X)$. Then the effective cone $\mathrm{Eff}(X)$ is of dimension two and is uniquely decomposed into three convex sets

$$\mathrm{Eff}(X) \;=\; \tau^+ \cup \tau_X \cup \tau^-,$$

such that τ^+, τ^- do not intersect the ample cone τ_X and $\tau^+ \cap \tau^-$ consists of the origin. Recall that u lies in τ_X and that, due to $\tau_X \subseteq \mathrm{Mov}(X)$, each of τ^+ and τ^- contains at least two of the weights w_{ij}, w_k.



Remark 4.1.9. Consider $X = X(A, P, u)$ such that $\mathrm{Cl}(X)_{\mathbb{Q}}$ is of dimension two. Then, for every $\mathfrak{F}$-face $\{0\} \neq \gamma_0 \preceq \gamma$ precisely one of the following inclusions holds

$$Q(\gamma_0) \;\subseteq\; \tau^+, \qquad \tau_X \;\subseteq\; Q(\gamma_0)^\circ, \qquad Q(\gamma_0) \;\subseteq\; \tau^-.$$

The $\mathfrak{F}$-faces $\gamma_0 \preceq \gamma$ satisfying the second inclusion are exactly those with $\gamma_0 \in \mathrm{rlv}(X)$, i.e. the relevant ones.

Lemma 4.1.10. *Let $X = X(A, P, u)$ be non-toric with* $\mathrm{rk}\,(\mathrm{Cl}(X)) = 2$.

(i) Suppose that X is $\mathbb{Q}$-factorial. Then $w_k \notin \tau_X$ holds for all $1 \le k \le m$ and for all $0 \le i \le r$ with $n_i \ge 2$ we have $w_{ij} \notin \tau_X$, where $1 \le j \le n_i$.

(ii) Suppose that X is quasismooth, $m > 0$ holds and there is $0 \le i_1 \le r$ with $n_{i_1} \ge 3$. Then the w_{ij}, w_k with $n_i \ge 3$, $j = 1, \ldots, n_i$ and $k = 1, \ldots, m$ lie either all in τ^+ or all in τ^-.

(iii) Suppose that X is quasismooth and there is $0 \le i_1 \le r$ with $n_{i_1} \ge 4$. Then the w_{ij} with $n_i \ge 4$ and $j = 1, \ldots, n_i$ lie either all in τ^+ or all in τ^-.

(iv) Suppose that X is quasismooth and there exist $0 \le i_1 < i_2 \le r$ with $n_{i_1}, n_{i_2} \ge 3$. Then the w_{ij} with $n_i \ge 3$, $j = 1, \ldots, n_i$ lie either all in τ^+ or all in τ^-.

(v) Suppose that X is quasismooth. Then $w_1, \ldots, w_m$ lie either all in τ^+ or all in τ^-.

Proof. We prove (i). By Lemma 4.1.8 (i) and (ii), the rays $\gamma_k, \gamma_{ij} \preceq \gamma$ with $n_i \ge 2$ are $\mathfrak{F}$-faces. Since X is $\mathbb{Q}$-factorial, the ample cone $\tau_X \subseteq K_{\mathbb{Q}}$ of X is of dimension two and thus neither $\tau_X \subseteq Q(\gamma_{ij})^\circ$ nor $\tau_X \subseteq Q(\gamma_k)^\circ$ is possible. Remark 4.1.9 yields the assertion.
We turn to (ii). By Lemma 4.1.8 (i) and (ii), all $\gamma_k, \gamma_{ij}, \gamma_{ij,k} \preceq \gamma$ in question are $\mathfrak{F}$-faces and the corresponding strata in $\overline{X}$ consist of singular points. Because X is quasismooth, none of these $\mathfrak{F}$-faces is relevant. Thus, Remark 4.1.9 gives $w_{i_1 1} \in \tau^+$ or $w_{i_1 1} \in \tau^-$; say we have $w_{i_1 1} \in \tau^+$. Then, applying again Remark 4.1.9, we obtain $w_k, w_{ij} \in \tau^+$ for $k = 1, \ldots, m$, all i with $n_i \ge 3$ and $j = 1, \ldots, n_i$.

Assertion (*iii*) is proved analogously: treat first γ_{i_11,i_12} with Lemma 4.1.8 (*iii*), then $\gamma_{i_11,ij}$ with Lemma 4.1.8 (*iii*) and (*iv*). Similarly, we obtain (*iv*) by treating first γ_{i_11,i_21} and then all $\gamma_{i_11,ij}$ and $\gamma_{i_21,ij}$ with Lemma 4.1.8 (*iii*) and (*iv*). Finally, we obtain (*v*) using Lemma 4.1.8 (*i*). □

Proposition 4.1.11. *Let $X = X(A,P,u)$ be non-toric, quasismooth and $\mathbb{Q}$-factorial with $\rho(X) = 2$. Assume that there is an elementary big cone in Σ and that we have $n_0 \geq \ldots \geq n_r$. If $m > 0$ holds, then there is a $\gamma_{ij,k} \in \mathrm{rlv}(X)$, we have $r = 2$ and the constellation $\overline{n}$ of the n_i is $(n_0, 2, 2)$, $(2, 2, 1)$ or $(2, 1, 1)$.*

Proof. According to Lemma 4.1.10 (*v*), we may assume $w_1, \ldots, w_m \in \tau^+$. We claim that there is a $w_{i_1j_1} \in \tau^-$ with $n_{i_1} \geq 2$. Otherwise, use Corollary 4.1.7 to see that there exist weights w_{ij} with $n_i \geq 2$ and Lemma 4.1.10 (*i*) to see that they all lie in τ^+. Since all monomials $T_i^{l_i}$ have the same degree in K, we obtain in addition $w_{i1} \in \tau^+$ for all i with $n_i = 1$. But then no weights w_{ij}, w_k are left to lie in τ^-, a contradiction.
Having verified the claim, we may take a weight $w_{i_1j_1} \in \tau^-$ with $n_{i_1} \geq 2$. Then $\gamma_{i_1j_1,1} \in \mathrm{rlv}(X)$ is as desired. Moreover, Lemma 4.1.8 (*ii*) yields $r = 2$ and $n_{i_1} = 2$. If $n_0 \geq 3$ holds, then Lemma 4.1.10 (*ii*) gives $w_{ij} \in \tau^+$ for all i with $n_i \geq 3$. Moreover, as all monomials share the same degree, we have $w_{i1} \in \tau^+$ for all i with $n_i = 1$. For the same reason, one of the w_{i_11}, w_{i_12} must lie in τ^+. As τ^- contains at least two weights, there is a $w_{i_2j_2} \in \tau^-$ with $n_{i_2} = 2$ and $i_1 \neq i_2$. Thus, the constellation $\overline{n}$ is as claimed. □

Proposition 4.1.12. *Let $X = X(A,P,u)$ be non-toric, quasismooth and $\mathbb{Q}$-factorial with $\rho(X) = 2$. Assume that there is an elementary big cone in Σ and that we have $n_0 \geq \ldots \geq n_r$. If $m = 0$ holds, then there is a $\gamma_{i_1j_1,i_2j_2} \in \mathrm{rlv}(X)$, we have $r \leq 3$ and the constellation $\overline{n}$ of the n_i is one of the following*

$$\begin{array}{ll} r = 2\colon & (n_0, 2, 2),\ (3, 2, 1),\ (3, 1, 1),\ (2, 2, 2),\ (2, 2, 1); \\ r = 3\colon & (2, 2, 2, 2),\ (2, 2, 2, 1),\ (2, 2, 1, 1). \end{array}$$

Proof. We first show $n_1 \leq 2$. Otherwise we have $n_1 \geq 3$ and, according to point (*iv*) of Lemma 4.1.10, we may assume that all the w_{ij} with $n_i \geq 3$ lie in τ^+. In particular, w_{11} lies in τ^+. By homogeneity of all monomials, also $w_{i1} \in \tau^+$ holds for all i with $n_i = 1$. At least two weights $w_{i_1j_1}$ and $w_{i_2j_2}$ must belong to τ^-. For these, only $n_{i_1} = n_{i_2} = 2$ and $i_1 \neq i_2$ is possible. Applying Lemma 4.1.8 (*iv*) to $\gamma_{11,i_1j_1} \in \mathrm{rlv}(X)$ gives $r = 2$, contradicting $n_0 \geq n_1 \geq 3$ and $n_{i_1} = n_{i_2} = 2$.
We treat the case $n_0 \geq 4$. By Lemma 4.1.10 (*iii*), we can assume $w_{01}, \ldots, w_{0n_0} \in \tau^+$. As before, we obtain $w_{i1} \in \tau^+$ for all i with $n_i = 1$ and we find two weights $w_{i_1j_1}, w_{i_2j_2} \in \tau^-$ with $n_{i_1} = n_{i_2} = 2$ and $i_1 \neq i_2$. Then $\gamma_{01,i_1j_1} \in \mathrm{rlv}(X)$ is as wanted. Lemma 4.1.8 (*iv*) gives $r = 2$ and we end up with $(n_0, 2, 2)$.
Now let $n_0 = 3$. Lemma 4.1.10 (*i*) guarantees that no w_{0j} lies in τ_X. If weights w_{0j} occur in both cones τ^+ and τ^-, say $w_{01} \in \tau^+$ and $w_{02} \in \tau^-$, then $\gamma_{01,02}$ is as wanted. Lemma 4.1.8 (*iii*) yields $r = 2$ and we obtain the constellations $(n_0, 2, 2)$, $(3, 2, 1)$ and $(3, 1, 1)$. So, assume that all weights w_{0j} lie in one of τ^+ and τ^-, say in τ^+. Then

we proceed as in the case $n_0 \geq 4$ to obtain a $\gamma_{01,i_1j_1} \in \mathrm{rlv}(X)$ and $r = 2$ with the constellation $(3, 2, 2)$.
Finally, let $n_0 \leq 2$. Corollary 4.1.7 yields $n_0 = 2$. According to Lemma 4.1.10 (*i*) no w_{ij} with $n_i = 2$ lies in τ_X. So we may assume $w_{01} \in \tau^+$. Moreover, all w_{ij} with $n_i = 1$ lie together in one among τ^+, τ_X or τ^-. Since each of τ^+ and τ^- contains two weights, we obtain $n_1 = 2$ and some $\gamma_{0j_1,1j_2}$ is as wanted. Lemma 4.1.8 (*iv*) shows $r \leq 3$. ☐

As a byproduct we retrieve a special case of [17, Cor. 4.18].

Corollary 4.1.13. *Let $X = X(A, P, u)$ be smooth with $\rho(X) = 2$. Then the divisor class group $\mathrm{Cl}(X)$ is torsion-free.*

Proof. By Corollary 4.1.4, there is an elementary big cone in Σ. Thus, Propositions 4.1.11 and 4.1.12 deliver a two-dimensional $\gamma_0 \in \mathrm{rlv}(X)$. The corresponding weights generate K as a group. This gives $\mathrm{Cl}(X) \cong K \cong \mathbb{Z}^2$. ☐

The following result lists the possible shapes of the defining matrix P.

Proposition 4.1.14. *Let X be a non-toric smooth rational projective variety with a torus action of complexity one and Picard number $\rho(X) = 2$. Then $X \cong X(A, P, u)$, where P is irredundant and fits into one of the following cases:*

- *(I) We have $r = 2$ and one of the following constellations:*
  - *(a) $m \geq 0$ and $n = 4 + n_0$ with $\overline{n} = (n_0, 2, 2)$, where $n_0 \geq 3$.*
  - *(b) $m = 0$ and $n = 6$ with $\overline{n} = (3, 2, 1)$.*
  - *(c) $m = 0$ and $n = 5$ with $\overline{n} = (3, 1, 1)$.*
  - *(d) $m \geq 0$ and $n = 6$ with $\overline{n} = (2, 2, 2)$.*
  - *(e) $m \geq 0$ and $n = 5$ with $\overline{n} = (2, 2, 1)$.*
  - *(f) $m \geq 1$ and $n = 4$ with $\overline{n} = (2, 1, 1)$.*
- *(II) We have $r = 3$ and one of the following constellations:*
  - *(a) $m = 0$ and $n = 8$ with $\overline{n} = (2, 2, 2, 2)$.*
  - *(b) $m = 0$ and $n = 7$ with $\overline{n} = (2, 2, 2, 1)$.*
  - *(c) $m = 0$ and $n = 6$ with $\overline{n} = (2, 2, 1, 1)$.*

Proof. The variety X is isomorphic to some $X(A, P, u)$, where after suitable admissible operations we may assume $n_0 \geq \ldots \geq n_r$. Thus, Propositions 4.1.11 and 4.1.12 apply. ☐

4.2 Towards the classification

We obtain a complete classification by going through the cases established in Proposition 4.1.14: we deal with a smooth projective variety $X = X(A, P, u)$ of Picard number two coming with an effective torus action of complexity one.

From Corollary 4.1.13 we know that $\mathrm{Cl}(X) = K = \mathbb{Z}^2$ holds. With $w_{ij} = Q(e_{ij})$ and $w_k = Q(e_k)$, the columns of the $2 \times (n + m)$ degree matrix Q will be written as

$$w_{ij} \; = \; (w_{ij}^1, w_{ij}^2) \; \in \; \mathbb{Z}^2, \qquad\qquad w_k \; = \; (w_k^1, w_k^2) \; \in \; \mathbb{Z}^2.$$

Recall that all relations $g_0, \ldots, g_{r-2}$ of $R(A,P)$ have the same degree in $K = \mathbb{Z}^2$, namely

$$\mu \ = \ (\mu^1, \mu^2) \ := \ \deg(g_0) \ \in \ \mathbb{Z}^2.$$

We will frequently work with the following faces of the positive orthant $\gamma = \mathbb{Q}^{n+m}_{\geq 0}$ introduced in Lemma 4.1.8:

$$\gamma_{ij,k} \ = \ \mathrm{cone}(e_{ij}, e_k) \ \preceq \ \gamma, \qquad \gamma_{i_1j_1,i_2j_2} \ = \ \mathrm{cone}(e_{i_1j_1}, e_{i_2j_2}) \ \preceq \ \gamma.$$

Remark 4.2.1. Consider a face $\gamma_0 \preceq \gamma$ of type $\gamma_{ij,k}$ or $\gamma_{i_1j_1,i_2j_2}$. Write e', e'' for the two generators of γ_0 and $w' = Q(e')$, $w'' = Q(e'')$ for the corresponding columns of the degree matrix Q such that (w', w'') is positively oriented in $\mathbb{Z}^2$. Then Proposition 1.4.2 tells us

$$\gamma_0 \ \in \ \mathrm{rlv}(X) \quad \Rightarrow \quad \det(w', w'') \ = \ 1.$$

So, if $\gamma_0 \in \mathrm{rlv}(X)$, then we may multiply Q from the left with a unimodular (2×2)-matrix transforming w' and w'' into $(1,0)$ and $(0,1)$. This change of coordinates on $\mathrm{Cl}(X)$ does not affect the defining data (A,P). If $w' = (1,0)$ and $w'' = (0,1)$ hold and $e \in \gamma$ is a canonical basis vector with corresponding column $w = Q(e)$, then we have

$$\mathrm{cone}(e', e) \in \mathrm{rlv}(X) \quad \Rightarrow \quad w = (w^1, 1),$$
$$\mathrm{cone}(e'', e) \in \mathrm{rlv}(X) \quad \Rightarrow \quad w = (1, w^2).$$

We are ready to go through the cases of Proposition 4.1.14; we keep the numbering introduced there.

Case (I) (a) of Proposition 4.1.14

We have $r = 2$, $m \geq 0$ and $\overline{n} = (n_0, 2, 2)$, where $n_0 \geq 3$. This leads to No. 1 and No. 2 in Theorems 4.3.1 and 4.3.2.

In a first step we show that there occur weights w_{0j} in each of τ^+ and τ^-. Otherwise, we may assume that all w_{0j} lie in τ^+, see Lemma 4.1.10 (*i*). Then Lemma 4.1.10 (*ii*) says that also all w_k lie in τ^+. Moreover, we have $\deg(T_i^{l_i}) \in \tau^+$ for $i = 0,1,2$. Thus, we may assume $w_{11}, w_{21} \in \tau^+$ and obtain $w_{12}, w_{22} \in \tau^-$, as there must be at least two weights in τ^-. Finally, we may assume that $\mathrm{cone}(w_{01}, w_{12})$ contains $w_{02}, \ldots, w_{0n_0}$ and w_{22}. Applying Remark 4.2.1 first to $\gamma_{01,12}$, then to all $\gamma_{0j,12}$, $\gamma_{12,k}$ and $\gamma_{01,22}$, $\gamma_{12,21}$ yields

$$Q \ = \ \left[\begin{array}{cccc|cc|cc|ccc} 0 & w_{02}^1 & \ldots & w_{0n_0}^1 & w_{11}^1 & 1 & w_{21}^1 & 1 & w_1^1 & \ldots & w_m^1 \\ 1 & 1 & \ldots & 1 & w_{11}^2 & 0 & 1 & w_{22}^2 & 1 & \ldots & 1 \end{array}\right],$$

with $w_{0j}^1 \geq 0$ and $w_{22}^2 \geq 0$. Since $\gamma_{01,12}, \gamma_{01,22} \in \mathrm{rlv}(X)$ holds, Lemma 4.1.8 (*iv*) implies $l_{11} = l_{21} = 1$. Applying $P \cdot Q^T = 0$ to the first row of P and the second row of Q gives

$$0 \ < \ 3 \ \leq \ n_0 \ \leq \ l_{01} + \ldots + l_{0n_0} \ = \ w_{11}^2 \ = \ 1 + w_{22}^2 w_{11}^1,$$

where the last equality is due to $\gamma_{11,22} \in \mathrm{rlv}(X)$ and thus $\det(w_{22}, w_{11}) = 1$. We conclude $w_{22}^2 > 0$ and $w_{11}^1 > 0$. Because of $\gamma_{0j,22} \in \mathrm{rlv}(X)$, we obtain $\det(w_{22}, w_{0j}) = 1$. This implies $w_{0j}^1 = 0$ for all $j = 2, \ldots, n_0$. Applying $P \cdot Q^T = 0$ to the first row of P and the first row of Q gives $w_{11}^1 + l_{12} = 0$, a contradiction.

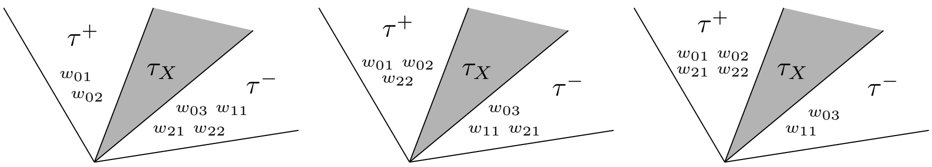
Knowing that each of τ^+ and τ^- contains weights w_{0j}, we can assume $w_{01}, w_{02} \in \tau^+$ and $w_{03} \in \tau^-$. Lemma 4.1.10 *(ii)* and *(iii)* show $n_0 = 3$ and $m = 0$. There is at least one other weight in τ^-, say $w_{11} \in \tau^-$. By applying Lemma 4.1.8 *(iii)* to $\gamma_{0j,03} \in \mathrm{rlv}(X)$ for $j = 1, 2$ and *(iv)* to suitable $\gamma_{0j_1,i_2j_2} \in \mathrm{rlv}(X)$, we obtain

$$l_{01} = l_{02} = 1, \qquad l_{11} = l_{12} = 1, \qquad l_{21} = l_{22} = 1.$$

Moreover, Remark 4.2.1 applied to $\gamma_{01,03}$, $\gamma_{02,03}$ and $\gamma_{01,11}$ brings Q into the shape

$$Q \;=\; \left[\begin{array}{ccc|cc|cc} 0 & w_{02}^1 & 1 & 1 & w_{12}^1 & w_{21}^1 & w_{22}^1 \\ 1 & 1 & 0 & w_{11}^2 & w_{12}^2 & w_{21}^2 & w_{22}^2 \end{array}\right].$$

Observe that the second component of the degree of the relation is $\mu^2 = 2$. There are three possible dispositions of the weights w_{2j}:



(i) *(ii)* *(iii)*

We will see that dispositions *(i)* and *(ii)* give No. 1 and No. 2 of Theorem 4.3.1 respectively and disposition *(iii)* does not provide any smooth variety.

In *(i)* we assume $w_{21}, w_{22} \in \tau^-$. Then $\gamma_{01,21}, \gamma_{01,22} \in \mathrm{rlv}(X)$ holds and Remark 4.2.1 shows $w_{21}^1 = w_{22}^1 = 1$. This implies $\mu^1 = 2$. Similarly, considering $\gamma_{02,21}, \gamma_{02,22} \in \mathrm{rlv}(X)$, we obtain $w_{02}^1 = 0$ or $w_{21}^2 = w_{22}^2 = 0$. The latter contradicts $\mu^2 = 2$ and thus $w_{02}^1 = 0$ holds. We conclude $l_{03} = \mu^1 = 2$. Furthermore $w_{12}^1 = \mu^1 - w_{11}^1 = 1$. Together, we have

$$g_0 \;=\; T_{01}T_{02}T_{03}^2 + T_{11}T_{12} + T_{21}T_{22}, \qquad Q \;=\; \left[\begin{array}{ccc|cc|cc} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & a & 2-a & b & 2-b \end{array}\right],$$

where $a, b \in \mathbb{Z}$. Observe that $w_{12} \in \tau^-$ must hold; otherwise, $\gamma_{03,12} \in \mathrm{rlv}(X)$ holds and Remark 4.2.1 yields $w_{12}^2 = 1$, contradicting $w_{12} = (1,1) = w_{11} \in \tau^-$. The semiample cone is $\mathrm{SAmple}(X) = \mathrm{cone}((0,1),(1,d))$, where $d = \max(a, 2-a, b, 2-b)$. The anticanonical class is $-\mathcal{K}_X = (3,4)$. Hence X is an almost Fano variety if and only if $d = 1$, which is equivalent to $a = b = 1$. In this situation X is already a Fano variety.

In *(ii)* we assume $w_{21} \in \tau^-$ and $w_{22} \in \tau^+$. Remark 4.2.1, applied to $\gamma_{01,21}, \gamma_{03,22} \in \mathrm{rlv}(X)$, shows $w_{21}^1 = w_{22}^2 = 1$. The latter implies $w_{21}^2 = \mu^2 - w_{22}^2 = 1$. We claim $w_{11}^2 \neq 0$. Otherwise, we have $w_{12}^2 = \mu^2 = 2$. This gives $\det(w_{03}, w_{12}) = 2$. We conclude $\gamma_{03,12} \notin \mathrm{rlv}(X)$ and $w_{12} \in \tau^-$. Then $\gamma_{01,12} \in \mathrm{rlv}(X)$ implies $w_{12}^1 = 1$. Thus, $w_{22} = (1,1)$ and $w_{12} = (1,2)$ hold, contradicting $w_{22} \in \tau^+$ and $w_{12} \in \tau^-$. Now, $\gamma_{11,22} \in \mathrm{rlv}(X)$ yields $w_{11}^2 w_{22}^1 = 0$ and thus $w_{22}^1 = 0$. We obtain $\mu^1 = 1$ and as a consequence $l_{03} = 1, w_{02}^1 = 0$ and $w_{12}^1 = 0$. Therefore $w_{12} \in \tau^+$ holds. Now $\gamma_{03,12} \in \mathrm{rlv}(X)$ implies $w_{12}^2 = 1$ and

$w_{11}^2 = \mu^2 - w_{12}^2 = 1$. We arrive at

$$g_0 \;=\; T_{01}T_{02}T_{03} + T_{11}T_{12} + T_{21}T_{22}, \qquad Q \;=\; \left[\begin{array}{ccc|cc|cc} 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right].$$

The anticanonical class is $-\mathcal{K}_X = (2,4)$ and the semiample cone is $\mathrm{SAmple}(X) = \mathrm{cone}((0,1),(1,1))$. In particular X is Fano.

We turn to *(iii)*, where both weights w_{21} and w_{22} lie in τ^+. Homogeneity of g_0 implies $w_{12} \in \tau^+$. Thus, $\gamma_{03,12}, \gamma_{03,21}, \gamma_{03,22} \in \mathrm{rlv}(X)$ holds and Remark 4.2.1 delivers $w_{12}^2 = w_{21}^2 = w_{22}^2 = 1$. We conclude $w_{11}^2 = \mu^2 - w_{12}^2 = 1$. Similarly, $\gamma_{02,11}, \gamma_{11,21}, \gamma_{11,22} \in \mathrm{rlv}(X)$ yields $w_{02}^1 = w_{21}^1 = w_{22}^1 = 0$. This gives $0 \neq l_{03} = \mu^1 = w_{21}^1 + w_{22}^1 = 0$, which is not possible.

Case (I) (b) of Proposition 4.1.14

We have $r = 2$, $m = 0$, $n = 6$ and $\overline{n} = (3,2,1)$. This leads to No. 3 in Theorems 4.3.1 and 4.3.2.

Since there are at least two weights in τ^+ and another two in τ^-, we can assume $w_{01}, w_{02} \in \tau^+$ and $w_{03}, w_{12} \in \tau^-$. By Lemma 4.1.8 *(iii)* and *(iv)* we obtain $l_{01} = l_{02} = l_{11} = l_{12} = 1$. We may assume that $\mathrm{cone}(w_{01}, w_{03})$ contains w_{02}. Applying Remark 4.2.1 first to $\gamma_{01,03}$, then to $\gamma_{02,03}$ and $\gamma_{01,12}$, we obtain

$$Q \;=\; \left[\begin{array}{ccc|cc|c} 0 & w_{02}^1 & 1 & w_{11}^1 & 1 & w_{21}^1 \\ 1 & 1 & 0 & w_{11}^2 & w_{12}^2 & w_{21}^2 \end{array}\right],$$

with $w_{02}^1 \geq 0$. For the degree μ of g_0, we have $\mu^2 = 2$. We conclude $w_{11}^2 = 2 - w_{12}^2$ and $l_{21}w_{21}^2 = 2$, which in turn implies $l_{21} = 2$ and $w_{21}^2 = 1$. For $\gamma_{02,12} \in \mathrm{rlv}(X)$, Remark 4.2.1 gives $\det(w_{12}, w_{02}) = 1$ and thus $w_{02}^1 = 0$ or $w_{12}^2 = 0$ must hold.
We treat the case $w_{02}^1 = 0$. Then $\mu = (l_{03}, 2)$ holds. We conclude $w_{11}^1 = l_{03} - 1$ and $w_{21}^1 = l_{03}/2$. With $c := l_{03}/2 \in \mathbb{Z}_{\geq 1}$ and $a := w_{12}^2 \in \mathbb{Z}$, we obtain the degree matrix

$$Q \;=\; \left[\begin{array}{ccc|cc|c} 0 & 0 & 1 & 2c-1 & 1 & c \\ 1 & 1 & 0 & 2-a & a & 1 \end{array}\right].$$

We show $w_{11} \in \tau^-$. Otherwise, $w_{11} \in \tau^+$ holds, we have $\gamma_{03,11} \in \mathrm{rlv}(X)$ and Remark 4.2.1 yields $a = 1$. But then $w_{01} = (0,1) \in \tau^+$ and $w_{11} = (2c-1,1) \in \tau^+$ imply $w_{12} = (1,1) \in \tau^+$, a contradiction. So we have $w_{11} \in \tau^-$. Then $\gamma_{01,11} \in \mathrm{rlv}(X)$ holds. Remark 4.2.1 gives $\det(w_{11}, w_{01}) = 1$, which means $c = 1$ and, as a consequence, $l_{03} = 2$. Together, we have

$$g_0 \;=\; T_{01}T_{02}T_{03}^2 + T_{11}T_{12} + T_{21}^2, \qquad Q \;=\; \left[\begin{array}{ccc|cc|c} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2-a & a & 1 \end{array}\right],$$

where we may assume $a \geq 2-a$ that means $a \in \mathbb{Z}_{\geq 1}$. The semiample cone is $\mathrm{SAmple}(X) = \mathrm{cone}((0,1),(1,a))$, and the anticanonical class is $-\mathcal{K}_X = (2,3)$. In particular, X is an almost Fano variety if and only $a = 1$ holds. In this situation X is already a Fano variety.

We turn to the case $w_{12}^2 = 0$. Here, $w_{11}^2 = \mu^2 = 2$ leads to $\det(w_{03}, w_{11}) = 2$ and thus the $\mathfrak{F}$-face $\gamma_{03,11}$ does not belong to $\mathrm{rlv}(X)$; see Remark 4.2.1. Hence $w_{11} \in \tau^-$ and thus $\gamma_{01,11} \in \mathrm{rlv}(X)$. This gives $w_{11}^1 = 1$ and thus $w_{11} = (1, 2)$. Because of $w_{02} = (w_{02}, 1) \in \tau^+$, we must have $w_{02}^1 = 0$ and the previous consideration applies.

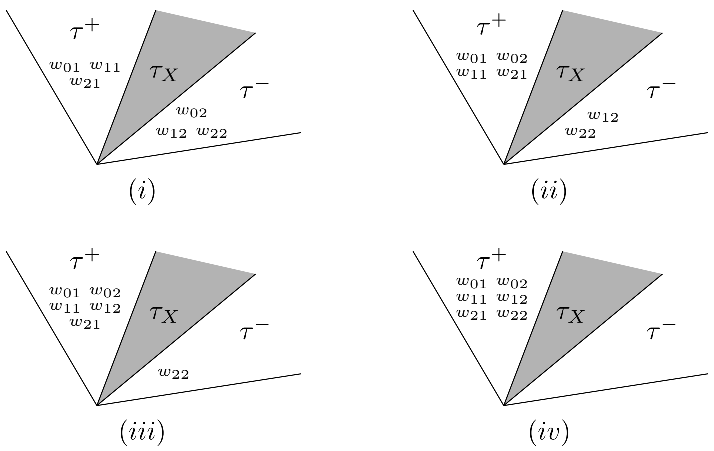
Case (I) (c) of Proposition 4.1.14

We have $r = 2$, $m = 0$, $n = 5$ and $\overline{n} = (3, 1, 1)$. This case does not provide smooth varieties.

Each of τ^+ and τ^- contains at least two weights. We may assume $w_{01}, w_{02} \in \tau^+$ and $w_{03}, w_{11}, w_{21} \in \tau^-$. Then $\gamma_{01,03}, \gamma_{02,03} \in \mathrm{rlv}(X)$ holds and Lemma 4.1.8 (*iii*) yields $l_{01} = l_{02} = 1$. By Remark 4.2.1 we can assume $w_{03} = (1, 0)$ and $w_{01}^2 = w_{02}^2 = 1$. This implies $\mu^2 = 2$ and, as a consequence, $l_{11} = l_{21} = 2$. By [24, Thm. 1.1], this leads to torsion in $\mathrm{Cl}(X)$, a contradiction to Corollary 4.1.13.

Case (I) (d) of Proposition 4.1.14

We have $r = 2$, $m \geq 0$, $n = 6$ and $\overline{n} = (2, 2, 2)$. Suitable admissible operations lead to one of the following dispositions for the weights w_{ij}:



Disposition (*i*) amounts to No. 4 in Theorems 4.3.1, 4.3.2 and 4.3.3, disposition (*ii*) to No. 5, disposition (*iii*) to Nos. 6 and 7, and disposition (*iv*) to Nos. 8 and 9.

Disposition (*i*). We have $w_{01}, w_{11}, w_{21} \in \tau^+$ and $w_{02}, w_{12}, w_{22} \in \tau^-$. We may assume $w_k \in \tau^+$ for all $k = 1, \ldots, m$. If $m > 0$, we have $\gamma_{i2,1} \in \mathrm{rlv}(X)$ and Lemma 4.1.8 (*ii*) gives $l_{i1} = 1$ for $i = 0, 1, 2$. If $m = 0$, we use $\gamma_{i_1 1, i_2 2} \in \mathrm{rlv}(X)$ and Lemma 4.1.8 (*iv*) to obtain $l_{i_1 2} = 1$ or $l_{i_2 1} = 1$ for all $i_1 \neq i_2$. Thus, for $m = 0$, we may assume $l_{01} = l_{11} = 1$ and are left with $l_{21} = 1$ or $l_{22} = 1$.
We treat the case $m \geq 0$ and $l_{01} = l_{11} = l_{21} = 1$. Here we may assume $w_{11}, w_{21}, w_{22} \in \mathrm{cone}(w_{01}, w_{12})$. Applying Remark 4.2.1 first to $\gamma_{01,12}$ and then to $\gamma_{01,22}$, $\gamma_{12,21}$ and all

$\gamma_{12,k}$ gives

$$Q = \left[\begin{array}{cc|cc|cc|ccc} 0 & w_{02}^1 & w_{11}^1 & 1 & w_{21}^1 & 1 & w_1^1 & \dots & w_m^1 \\ 1 & w_{02}^2 & w_{11}^2 & 0 & 1 & w_{22}^2 & 1 & \dots & 1 \end{array}\right].$$

Using $w_{11}, w_{21}, w_{22} \in \operatorname{cone}(w_{01}, w_{12})$ and the fact that the determinants of (w_{02}, w_{01}), (w_{12}, w_{11}) and (w_{22}, w_{21}) are positive, we obtain

$$w_{11}^1,\ w_{21}^1,\ w_{22}^2 \ \geq\ 0, \qquad w_{02}^1,\ w_{11}^2 \ >\ 0, \qquad 1 \ >\ w_{22}^2 w_{21}^1.$$

The degree μ of the relation satisfies

$$0 \ <\ \mu^1 \ =\ l_{02} w_{02}^1 \ =\ w_{11}^1 + l_{12} \ =\ w_{21}^1 + l_{22},$$

$$0 \ <\ \mu^2 \ =\ 1 + l_{02} w_{02}^2 \ =\ w_{11}^2 \ =\ 1 + l_{22} w_{22}^2.$$

In particular, $w_{02}^2 \geq 0$ holds and thus all components of the w_{ij} are non-negative. With $\gamma_{02,11}, \gamma_{02,21}, \in \operatorname{rlv}(X)$ and Remark 4.2.1, we obtain

$$w_{02}^1 w_{11}^2 \ =\ 1 + w_{02}^2 w_{11}^1, \qquad\qquad w_{02}^1 - 1 \ =\ w_{02}^2 w_{21}^1.$$

We show $w_{22}^2 = 0$. Otherwise, because of $1 > w_{22}^2 w_{21}^1$, we have $w_{21}^1 = 0$. This implies $w_{02}^1 = 1$ and thus

$$w_{11}^2 \ =\ 1 + w_{02}^2 w_{11}^1 \ =\ 1 + l_{02} w_{02}^2.$$

This gives $w_{02}^2 = 0$ or $w_{11}^1 = l_{02}$. The first is impossible because of $l_{02} w_{02}^2 = l_{22} w_{22}^2$ and the second because of $l_{02} = l_{02} w_{02}^1 = w_{11}^1 + l_{12}$.
Knowing $w_{22}^2 = 0$, we directly conclude $w_{11}^2 = 1$ and $w_{02}^2 = 0$ from $\mu^2 = 1$. This gives $w_{02}^1 = 1$. With $a := w_{11}^1 \in \mathbb{Z}_{\geq 0}$, $b := w_{21}^1 \in \mathbb{Z}_{\geq 0}$ and $c_k := w_k^1 \in \mathbb{Z}$ we are in the situation

$$g_0 \ =\ T_{01} T_{02}^{l_{02}} + T_{11} T_{12}^{l_{12}} + T_{21} T_{22}^{l_{22}}, \qquad Q \ =\ \left[\begin{array}{cc|cc|cc|ccc} 0 & 1 & a & 1 & b & 1 & c_1 & \dots & c_m \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right],$$

where we may assume $0 \leq a \leq b$ and $c_1 \leq \ldots \leq c_m$. Observe that $l_{02} = a + l_{12} = b + l_{22}$ holds. The anticanonical class and the semiample cone of X are given by

$$\begin{aligned} -\mathcal{K}_X &= (3 + b + c_1 + \ldots + c_m - l_{12},\ 2 + m), \\ \operatorname{SAmple}(X) &= \operatorname{cone}((1,0), (d,1)), \end{aligned}$$

where $d := \max(b, c_m)$. Consequently, X is a Fano variety if and only if the following inequality holds

$$3 + b + c_1 + \ldots + c_m - l_{12} \ >\ (2 + m)d.$$

A necessary condition for this is $0 \leq d \leq 1$ with $l_{12} = 1$ if $d = 1$ and $l_{12} \leq 2$ if $d = 0$. The tuples $(a, b, d, l_{02}, l_{12}, l_{22})$ fulfilling that condition are

$$(0,0,0,2,2,2), \qquad (0,0,0,1,1,1), \qquad (1,1,1,2,1,1).$$

Each of these three tuples allows indeed a Fano variety X; the respectively possible choices of the c_k lead to Nos. 4.A, 4.B and 4.C of Theorem 4.3.2 and are as follows:

$$c_1 = \ldots = c_m = 0, \qquad -1 \le c_1 \le 0 = c_2 = \ldots = c_m, \qquad c_1 = \ldots = c_m = 1.$$

Moreover X is a truly almost Fano variety if and only if the following equality holds

$$3 + b + c_1 + \ldots + c_m - l_{12} \;=\; (2+m)d.$$

This implies $0 \le d \le 2$ and the only possible parameters fulfilling that condition are listed as Nos. 4.A to 4.F in the table of Theorem 4.3.3.
We turn to the case $m = 0$, $l_{01} = l_{11} = 1$ and $l_{21} \ge 2$. Lemma 4.1.8 (*iv*) applied to $\gamma_{01,22}, \gamma_{11,22} \in \mathrm{rlv}(X)$ gives $l_{02} = l_{12} = 1$. If $l_{22} = 1$, then suitable admissible operations bring us to the previous case. So, let $l_{22} \ge 2$. We may assume $w_{11} \in \mathrm{cone}(w_{01}, w_{12})$. We apply Remark 4.2.1 first to $\gamma_{01,12}$, then to $\gamma_{01,22}$, $\gamma_{12,21}$ and arrive at

$$g_0 \;=\; T_{01}T_{02} + T_{11}T_{12} + T_{21}^{l_{21}}T_{22}^{l_{22}}, \qquad Q \;=\; \left[\begin{array}{cc|cc|cc} 0 & w_{02}^1 & w_{11}^1 & 1 & w_{21}^1 & 1 \\ 1 & w_{02}^2 & w_{11}^2 & 0 & 1 & w_{22}^2 \end{array}\right],$$

where $w_{11}^1 \ge 0$ and $w_{11}^2 = \det(w_{12}, w_{11}) > 0$ hold. We have $\mu = w_{02} + w_{01} = w_{11} + w_{12}$ and thus $w_{02} = w_{11} + w_{12} - w_{01}$. Because of $\gamma_{02,11} \in \mathrm{rlv}(X)$, we obtain

$$1 \;=\; \det(w_{02}, w_{11}) \;=\; \det(w_{12} - w_{01}, w_{11}) \;=\; w_{11}^1 + w_{11}^2.$$

We conclude $w_{11} = (0,1)$ and $\mu = (1,1)$. Using $\mu = l_{21}w_{21} + l_{22}w_{22}$ and $l_{21}, l_{22} \ge 2$ we see $w_{21}^1, w_{22}^2 < 0$. On the other hand, $0 < \det(w_{22}, w_{21}) = 1 - w_{21}^1 w_{22}^2$, a contradiction. Thus $l_{22} \ge 2$ does not occur.

Disposition (*ii*). We have $w_{01}, w_{02}, w_{11}, w_{21} \in \tau^+$ and $w_{12}, w_{22} \in \tau^-$. We may assume that $w_{02}, w_{12} \in \mathrm{cone}(w_{01}, w_{22})$ holds. Applying Remark 4.2.1 first to $\gamma_{01,22} \in \mathrm{rlv}(X)$ and then to $\gamma_{01,12}, \gamma_{02,22}, \gamma_{11,22} \in \mathrm{rlv}(X)$ we obtain

$$Q \;=\; \left[\begin{array}{cc|cc|cc|ccc} 0 & w_{02}^1 & w_{11}^1 & 1 & w_{21}^1 & 1 & w_1^1 & \ldots & w_m^1 \\ 1 & 1 & 1 & w_{12}^2 & w_{21}^2 & 0 & w_1^2 & \ldots & w_m^2 \end{array}\right],$$

where we have $w_{02}^1, w_{12}^2 \ge 0$ due to $w_{02}, w_{12} \in \mathrm{cone}(w_{01}, w_{22})$. Moreover, $w_{21}^2 > 0$ holds, as we infer from the conditions

$$0 \;\le\; \mu^1 \;=\; l_{02}w_{02}^1 \;=\; l_{11}w_{11}^1 + l_{12} \;=\; l_{21}w_{21}^1 + l_{22},$$

$$0 \;<\; \mu^2 \;=\; l_{01} + l_{02} \;=\; l_{11} + l_{12}w_{12}^2 \;=\; l_{21}w_{21}^2.$$

We show $l_{11} \ge 2$. Otherwise, the above conditions give $l_{12}w_{12}^2 > 0$ and thus $w_{12}^2 > 0$. For $\gamma_{02,12} \in \mathrm{rlv}(X)$, Remark 4.2.1 gives $\det(w_{12}, w_{02}) = 1$ which means $w_{12}^2 w_{02}^1 = 0$ and thus $w_{02}^1 = 0$. This implies $l_{21}w_{21}^1 + l_{22} = 0$ and thus $w_{21}^1 < 0$; a contradiction to $1 = \det(w_{12}, w_{21}) = w_{21}^2 - w_{12}^2 w_{21}^1$ which in turn holds due to $\gamma_{12,21} \in \mathrm{rlv}(X)$ and Remark 4.2.1.

Lemma 4.1.8 (*iv*) applied to $\gamma_{02,12}, \gamma_{01,12}, \gamma_{21,12} \in \mathrm{rlv}(X)$ shows $l_{01} = l_{02} = l_{22} = 1$. Putting together $\mu^2 = 2 = l_{11} + l_{12}w_{12}^2$ and $l_{11} \neq 1$, we conclude $l_{11} = 2$ and $w_{12}^2 = 0$. With $\gamma_{12,21} \in \mathrm{rlv}(X)$ and Remark 4.2.1 we obtain $w_{21}^2 = 1$ and hence $l_{21} = \mu^2 = 2$. From

$$0 \ \leq\ \mu^1 \ =\ w_{02}^1 \ =\ 2w_{11}^1 + 1 \ =\ 2w_{21}^1 + 1$$

we conclude $w_{11}^1 = w_{21}^1 \geq 0$ and thus $w_{02}^1 > 0$. Lemma 4.1.8 (*ii*) implies that possible weights of type w_k lie in τ^-. Thus Remark 4.2.1 and $\gamma_{01,k}$ imply $w_k^1 = 1$ for all k. Moreover, since $\gamma_{02,k} \in \mathrm{rlv}(X)$ holds, the latter implies $w_k^2 = 0$. All in all, we arrive at

$$g_0 \ =\ T_{01}T_{02} + T_{11}^2T_{12} + T_{21}^2T_{22}, \qquad Q \ =\ \left[\begin{array}{cc|cc|cc|ccc} 0 & 2a+1 & a & 1 & a & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots & 0 \end{array}\right],$$

with $a \in \mathbb{Z}_{\geq 0}$. The anticanonical class is $-\mathcal{K}_X = (2a+2+m, 2)$ and the semiample cone is $\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(2a+1,1))$. Therefore X is an almost Fano variety if and only if $m \geq 2a$ holds and X is a Fano variety if and only if $m > 2a$ holds.

Disposition (*iii*). We have $w_{01}, w_{02}, w_{11}, w_{12}, w_{21} \in \tau^+$ and $w_{22} \in \tau^-$. As there must be another weight in τ^-, we obtain $m > 0$. Lemma 4.1.10 (*v*) yields $w_1, \dots, w_m \in \tau^-$. We may assume $w_{02}, w_{11}, w_{12}, w_k \in \mathrm{cone}(w_{01}, w_1)$, where $k = 2, \dots, m$. Applying Remark 4.2.1 first to $\gamma_{01,1} \in \mathrm{rlv}(X)$ and then to the remaining faces $\gamma_{01,22}, \gamma_{01,k}, \gamma_{ij,1}$ from $\mathrm{rlv}(X)$ leads to the degree matrix

$$Q \ =\ \left[\begin{array}{cc|cc|cc|cccc} 0 & w_{02}^1 & w_{11}^1 & w_{12}^1 & w_{21}^1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & w_{22}^2 & 0 & w_2^2 & \dots & w_m^2 \end{array}\right]$$

with at most w_{21}^1, w_{22}^2 negative. We infer $l_{01} = l_{02} = l_{11} = l_{12} = l_{22} = 1$ from Lemma 4.1.8 (*ii*). For $\gamma_{02,22}, \gamma_{11,22}, \gamma_{12,22} \in \mathrm{rlv}(X)$ Remark 4.2.1 tells us

$$w_{22}^2 \ =\ 0 \qquad \text{or} \qquad w_{02}^1 \ =\ w_{11}^1 \ =\ w_{12}^1 \ =\ 0.$$

We treat the case $w_{22}^2 = 0$. Here $l_{21} = \mu^2 = 2$ holds. Thus $\mu^1 = w_{02}^1 = 2w_{21}^1 + 1$ holds. Because of $w_{02}^1 \geq 0$, we conclude $w_{02}^1 > 0$ and $w_{21}^1 \geq 0$. Remark 4.2.1 applied to $\gamma_{02,k} \in \mathrm{rlv}(X)$ gives $w_k^2 = 0$ for all $k = 2, \dots, m$. We arrive at

$$g_0 \ =\ T_{01}T_{02} + T_{11}T_{12} + T_{21}^2T_{22}, \qquad Q \ =\ \left[\begin{array}{cc|cc|cc|ccc} 0 & 2c+1 & a & b & c & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \end{array}\right],$$

where $a, b, c \in \mathbb{Z}_{\geq 0}$ and $a + b = 2c + 1$. Furthermore, the anticanonical class is $-\mathcal{K}_X = (3c+2+m, 3)$ and we have $\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(2c+1,1))$. In particular, X is an almost Fano variety if and only if $3c + 1 \leq m$ holds and a Fano variety if and only if the corresponding strict inequality holds.

Now we consider the case $w_{02}^1 = w_{11}^1 = w_{12}^1 = 0$. We have $\mu^1 = 0$, which implies $l_{21} = 1$, $w_{21}^1 = -1$. Consequently, $\mu^2 = 2$ gives $w_{22}^2 = 1$. Since $\gamma_{21,k} \in \mathrm{rlv}(X)$ for $2 \leq k \leq m$, we conclude $w_k^2 = 0$ for all k. Therefore we obtain

$$g_0 \ =\ T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}, \qquad Q \ =\ \left[\begin{array}{cc|cc|cc|ccc} 0 & 0 & 0 & 0 & -1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \end{array}\right].$$

Finally, we have $-\mathcal{K}_X = (m, 4)$ and $\mathrm{SAmple}(X) = \mathrm{cone}((1,1),(0,1))$. Thus, X is a Fano variety if and only if $m < 4$ holds. Moreover, X is an almost Fano variety if and only if $m \leq 4$ holds.

Disposition (*iv*). All w_{ij} lie in τ^+. Then we have $m \geq 2$ and one and hence all w_k lie in τ^-, see Lemma 4.1.10 (*v*). Applying Lemma 4.1.8 (*ii*) to $\gamma_{ij,1} \in \mathrm{rlv}(X)$, we conclude $l_{ij} = 1$ for all i, j. Thus we have the relation

$$g_0 \ = \ T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}.$$

We may assume that $\mathrm{cone}(w_{01}, w_1)$ contains all w_{ij}, w_k. Remark 4.2.1 applied to $\gamma_{01,1} \in \mathrm{rlv}(X)$ leads to $w_1 = (1, 0)$ and $w_{01} = (0, 1)$. All other weights lie in the positive orthant. For $\gamma_{ij,1}, \gamma_{01,k} \in \mathrm{rlv}(X)$ Remark 4.2.1 shows $w_{ij}^2 = w_k^1 = 1$ for all i, j, k. Consider the case where all w_k^2 vanish. Then the degree matrix is of the form

$$Q \ = \ \left[\begin{array}{cc|cc|cc|ccc} 0 & a_2 & a_3 & a_4 & a_5 & a_6 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right],$$

where $a_i \in \mathbb{Z}_{\geq 0}$ and $a_2 = a_3 + a_4 = a_5 + a_6$. We have $-\mathcal{K}_X = (2a_2 + m, 4)$ and $\mathrm{SAmple}(X) = \mathrm{cone}((1, 0), (a_2, 1))$. Hence X is a Fano variety if and only if $2a_2 < m$ holds and an almost Fano variety if and only if $2a_2 \leq m$ holds.
Finally, let $w_k^2 > 0$ for some k. Note that we may assume $0 \leq w_2^2 \leq \ldots \leq w_m^2$; in particular $w_m^2 > 0$. Since $\gamma_{ij,m} \in \mathrm{rlv}(X)$ for all i, j, Remark 4.2.1 yields $w_{ij}^1 = 0$ for all i, j. Thus we obtain the degree matrix

$$Q \ = \ \left[\begin{array}{cc|cc|cc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \ldots & a_m \end{array}\right],$$

where $0 \leq a_2 \leq \ldots \leq a_m$ and $a_m > 0$. The anticanonical class and the semiample cone are given as

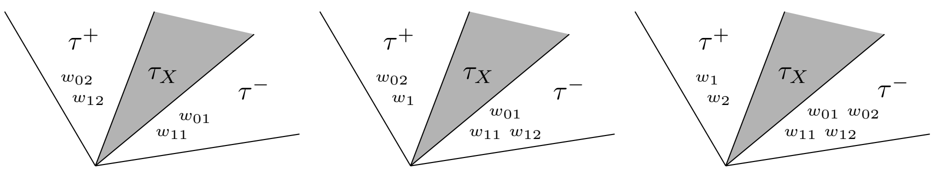
$$-\mathcal{K}_X \ = \ (m,\, 4 + a_2 + \ldots + a_m), \qquad \mathrm{SAmple}(X) \ = \ \mathrm{cone}((0, 1), (1, a_m)).$$

In particular, X is a Fano variety if and only if $4 + a_2 + \ldots + a_m > ma_m$ holds. Note that for the latter $a_m \leq 3$ is necessary. Moreover, X is a truly almost Fano variety if and only if the equality $4 + a_2 + \ldots + a_m = ma_m$ holds.

Case (I) (e) of Proposition 4.1.14

We have $r = 2$, $m \geq 0$, $n = 5$ and $\overline{n} = (2, 2, 1)$. This leads to Nos. 10, 11 and 12 in Theorems 4.3.1, 4.3.2 and 4.3.3.

We divide this case into the following three dispositions, according to the way some weights lie with respect to τ_X.



(*i*) (*ii*) (*iii*)

We show that disposition (*i*) does not provide any smooth variety, whereas (*ii*) delivers No. 10 and (*iii*) delivers Nos. 11 and 12.

In disposition (i) we have $w_{01}, w_{11} \in \tau^-$ and $w_{02}, w_{12} \in \tau^+$. We may assume that $w_{11} \in \mathrm{cone}(w_{01}, w_{12})$ holds. Remark 4.2.1 applied to $\gamma_{01,12} \in \mathrm{rlv}(X)$ leads to $w_{01} = (1,0)$ and $w_{12} = (0,1)$. Observe $w_{11}^1, w_{11}^2 \geq 0$. Due to $\det(w_{11}, w_{12}) > 0$, we even have $w_{11}^1 > 0$ and $\det(w_{01}, w_{02}) > 0$ gives $w_{02}^2 > 0$. Since $T_0^{l_0}$ and $T_1^{l_1}$ share the same degree, we have

$$l_{01}w_{01} + l_{02}w_{02} \;=\; l_{11}w_{11} + l_{12}w_{12}.$$

Lemma 4.1.8 (iv) says $l_{02} = 1$ or $l_{11} = 1$, which allows us to resolve for w_{02} or for w_{11} in the above equation. Using $\gamma_{02,11} \in \mathrm{rlv}(X)$, we obtain

$$\begin{aligned} l_{02} = 1 \quad &\Longrightarrow \quad 1 = \det(w_{11}, w_{02}) = \det(w_{11}, l_{12}w_{12} - l_{01}w_{01}) = l_{12}w_{11}^1 + l_{01}w_{11}^2, \\ l_{11} = 1 \quad &\Longrightarrow \quad 1 = \det(w_{11}, w_{02}) = \det(l_{01}w_{01} - l_{12}w_{12}, w_{02}) = l_{01}w_{02}^2 + l_{12}w_{02}^1. \end{aligned}$$

We show $l_{02} > 1$. Otherwise, $l_{02} = 1$ holds. The above consideration shows $w_{11}^2 = 0$ and $l_{12} = w_{11}^1 = 1$. Thus, $l_{21}w_{21}^2 = l_{12} = 1$ holds and we obtain $l_{21} = 1$; a contradiction to P being irredundant. Thus, $l_{02} > 1$ and $l_{11} = 1$ must hold. Because of $w_{02}^2 > 0$, we must have $w_{02}^1 \leq 0$. With

$$1 \;=\; \det(w_{11}, w_{02}) \;=\; w_{11}^1 w_{02}^2 - w_{11}^2 w_{02}^1$$

we see $w_{11}^2 w_{02}^1 = 0$ and $w_{11}^1 = w_{02}^2 = 1$. But then we arrive at $1 = l_{11}w_{11}^1 = l_{21}w_{21}^1$. Again this means $l_{21} = 1$; a contradiction to P being irredundant.

In disposition (ii) we have $w_{01}, w_{11}, w_{12} \in \tau^-$ and $w_{02}, w_1 \in \tau^+$. In particular $m \geq 1$. Lemma 4.1.10 (v) yields $w_2, \ldots, w_m \in \tau^+$. Applying Remark 4.2.1 first to $\gamma_{11,1} \in \mathrm{rlv}(X)$ an then to $\gamma_{01,1}, \gamma_{12,1}, \gamma_{02,11}, \gamma_{11,k} \in \mathrm{rlv}(X)$ leads to

$$Q \;=\; \left[\begin{array}{cc|cc|c|cccc} 1 & w_{02}^1 & 1 & 1 & w_{21}^1 & 0 & w_2^1 & \ldots & w_m^1 \\ w_{01}^2 & 1 & 0 & w_{12}^2 & w_{21}^2 & 1 & 1 & \ldots & 1 \end{array}\right].$$

Applying Lemma 4.1.8 (ii) to $\gamma_{01,1}, \gamma_{12,1}, \gamma_{11,1} \in \mathrm{rlv}(X)$ we obtain $l_{02} = l_{11} = l_{12} = 1$. For the degree μ of the relation g_0 we note

$$\mu^1 \;=\; l_{01} + w_{02}^1 \;=\; 2 \;=\; l_{21}w_{21}^1, \qquad\qquad \mu^2 \;=\; l_{01}w_{01}^2 + 1 \;=\; w_{12}^2 \;=\; l_{21}w_{21}^2.$$

From $\mu^1 = 2$ we infer $l_{21} = 2$ and $w_{21}^1 = 1$. Consequently, μ^2 is even and both l_{01}, w_{01}^2 are odd. Using again $\mu^1 = 2$ gives $w_{02}^1 \neq 0$. For $\gamma_{02,12} \in \mathrm{rlv}(X)$ Remark 4.2.1 yields $\det(w_{12}, w_{02}) = 1$ which means $w_{02}^1 w_{12}^2 = 0$. We conclude $w_{12}^2 = 0 = \mu^2$. This implies $w_{21}^2 = 0$, $w_{01}^2 = -1$, $l_{01} = 1$ and $w_{02}^1 = 1$. We obtain

$$g_0 \;=\; T_{01}T_{02} + T_{11}T_{12} + T_{21}^2, \qquad Q \;=\; \left[\begin{array}{cc|cc|c|ccc} 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & \ldots & 1 \end{array}\right],$$

where $w_2^1 = \ldots = w_m^1 = 0$ follows from Remark 4.2.1 applied to $\gamma_{01,k} \in \mathrm{rlv}(X)$. The semiample cone is given as $\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(1,1))$ and the anticanonical class as $-\mathcal{K}_X = (3, m)$. Therefore X is a Fano variety if and only if $m < 3$, i.e $m = 1, 2$. Moreover, X is an almost Fano variety if and only if $m \leq 3$.

In disposition (iii) we have $w_{01}, w_{02}, w_{11}, w_{12} \in \tau^-$ and $w_1, w_2 \in \tau^+$. In particular $m \geq 2$. Lemma 4.1.10 (v) ensures $w_3, \ldots, w_m \in \tau^+$. We can assume that all w_{ij}, w_k lie in $\mathrm{cone}(w_{01}, w_1)$. Applying Remark 4.2.1, first to $\gamma_{01,1}$ and then to all relevant faces of the types $\gamma_{ij,1}$ and $\gamma_{01,k}$, we achieve

$$w_{01} = (1,0), \quad w_1 = (0,1), \quad w_{02}^1 = w_{11}^1 = w_{12}^1 = 1, \quad w_2^2 = \ldots = w_m^2 = 1.$$

Lemma 4.1.8 (ii) applied to all $\gamma_{ij,1}$ shows $l_{ij} = 1$ for all i, j. We conclude $\mu^1 = 2$, which in turn implies $l_{21} = 2$ and $w_{21}^1 = 1$. In particular, we have the relation

$$g_0 \;=\; T_{01}T_{02} + T_{11}T_{12} + T_{21}^2.$$

We treat the case where $w_1^1 = \ldots = w_m^1 = 0$ holds. All columns of the degree matrix lie in $\mathrm{cone}(w_{01}, w_1)$ and thus Q is of the form

$$Q \;=\; \left[\begin{array}{cc|cc|c|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 2c & a & b & c & 1 & 1 & \ldots & 1 \end{array}\right],$$

where $a, b, c \in \mathbb{Z}_{\geq 0}$ and $a + b = 2c$. The anticanonical class is $-\mathcal{K} = (3, m + 3c)$ and we have $\mathrm{SAmple}(X) = \mathrm{cone}((0,1), (1, 2c))$. Therefore X is a Fano variety if and only if $m > 3c$. Moreover, X is an almost Fano variety if and only if $m \geq 3c$.
We treat the case where $w_k^1 > 0$ holds for some k. Then we obtain $w_{02}^2 = 0$ by applying Remark 4.2.1 to $\gamma_{02,k}$. This yields $\mu^2 = 0$ and thus $w_{ij}^2 = 0$ for all i, j. Consequently, the degree matrix is given as

$$Q \;=\; \left[\begin{array}{cc|cc|c|cccc} 1 & 1 & 1 & 1 & 1 & 0 & w_2^1 & \ldots & w_m^1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \end{array}\right],$$

where we can assume $0 \leq w_2^1 \leq \ldots \leq w_m^1$. The semiample cone and the anticanonical divisor are given as

$$\mathrm{SAmple}(X) = \mathrm{cone}((1,0), (w_m^1, 1)), \qquad -\mathcal{K} = (3 + w_2^1 + \ldots + w_m^1, m).$$

We see that X is an almost Fano variety if and only if $m w_m^1 \leq 3 + w_2^1 + \ldots + w_m^1$ and that X is a Fano variety if and only if the corresponding strict inequality holds.

Case (I) (f) of Proposition 4.1.14

We have $r = 2$, $m \geq 1$, $n = 4$ and $\overline{n} = (2, 1, 1)$. This case does not provide any smooth variety.

We can assume $w_{01} \in \tau^-$ and $w_1 \in \tau^+$. Lemma 4.1.10 (v) ensures $w_2, \ldots, w_m \in \tau^+$. Applying Remark 4.2.1 first to $\gamma_{01,1} \in \mathrm{rlv}(X)$ and then to the remaining $\gamma_{01,k} \in \mathrm{rlv}(X)$, we achieve

$$Q \;=\; \left[\begin{array}{cc|c|c|cccc} 1 & w_{02}^1 & w_{11}^1 & w_{21}^1 & 0 & w_2^1 & \ldots & w_m^1 \\ 0 & w_{02}^2 & w_{11}^2 & w_{21}^2 & 1 & 1 & \ldots & 1 \end{array}\right].$$

Moreover $\gamma_{01,1} \in \mathrm{rlv}(X)$ implies $l_{02} = 1$ by Lemma 4.1.8 (ii). Recall from Corollary 4.1.13 that $\mathrm{Cl}(X)$ is torsion-free. Thus [24, Thm. 1.1] implies that l_{11} and l_{21} are coprime.
Consider the case $w_{02} \in \tau^-$. Then $\gamma_{02,1} \in \mathrm{rlv}(X)$ holds, Lemma 4.1.8 (ii) yields $l_{01} = 1$ and Remark 4.2.1 shows $w_{02}^1 = 1$. We conclude $\mu^1 = 2$ and thus obtain $l_{11} = l_{21} = 2$, a contradiction.
Now let $w_{02} \in \tau^+$, which implies $\gamma_{01,02,11} \in \mathrm{rlv}(X)$. Since X is factorial, Proposition 1.4.2 (ii) shows that w_{02}^2 and w_{11}^2 are coprime. Now we look at

$$\mu^2 \;=\; w_{02}^2 \;=\; l_{11}w_{11}^2 \;=\; l_{21}w_{21}^2.$$

We infer that l_{21} divides w_{02}^2 and w_{11}^2. This contradicts coprimality of w_{02}^2 and w_{11}^2, because irredundancy of P requires $l_{21} \geq 2$.

Case (II) of Proposition 4.1.14

We have $r = 3$, $m = 0$ and $\overline{n} = (2, 2, n_2, n_3)$ with $2 \geq n_2 \geq n_3 \geq 1$. This leads to No. 13 in Theorems 4.3.1 and 4.3.2.

We treat constellations (a), (b) and (c) all at once. First observe that for every $w_{i_1j_1}$ with $n_{i_1} = 2$, there is at least one $w_{i_2j_2}$ with $n_{i_2} = 2$ and $i_1 \neq i_2$ such that $\tau_X \subseteq Q(\gamma_{i_1j_1,i_2j_2})^\circ$ and thus $\gamma_{i_1j_1,i_2j_2} \in \mathrm{rlv}(X)$. Since $r = 3$, we conclude $l_{ij} = 1$ for all i with $n_i = 2$; see Lemma 4.1.8 (iv).
We can assume $w_{01}, w_{11} \in \tau^-$ and $w_{02}, w_{12} \in \tau^+$ as well as $w_{11} \in \mathrm{cone}(w_{01}, w_{12})$. Applying Remark 4.2.1 to $\gamma_{01,12}, \in \mathrm{rlv}(X)$, we obtain $w_{01} = (1, 0)$ and $w_{12} = (0, 1)$. Moreover $w_{11}^1, w_{11}^2 \geq 0$ holds and, because of $w_{11} \notin \tau^+$, we even have $w_{11}^1 > 0$. For the degree μ of the relations we note

$$\mu^1 \;=\; w_{02}^1 + 1 \;=\; w_{11}^1, \qquad\qquad \mu^2 \;=\; w_{02}^2 \;=\; w_{11}^2 + 1.$$

Therefore we can express w_{02} in terms of w_{11}. Remark 4.2.1 applied to $\gamma_{02,11} \in \mathrm{rlv}(X)$ gives $1 = \det(w_{11}, w_{02}) = w_{11}^1 + w_{11}^2$. We conclude $w_{11} = (1, 0)$ and $w_{02} = (0, 1)$. In particular, the degree of g_0 and g_1 is $\mu = (1, 1)$.
In constellations (b) and (c), we have $n_3 = 1$ and $\mu = (1, 1)$. This implies $l_{31} = 1$, a contradiction to P being irredundant. Thus, constellations (b) and (c) do not occur.

We are left with constellation (a). This means that we have $n_0 = \ldots = n_3 = 2$. As seen before, $l_{ij} = 2$ for all i, j. Thus, the relations are

$$g_0 \;=\; T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}, \qquad g_1 \;=\; aT_{11}T_{12} + T_{21}T_{22} + T_{31}T_{32},$$

where $a \in \mathbb{K}^* \setminus \{1\}$. In this situation, we may assume $w_{21}, w_{31} \in \tau^-$. Applying Remark 4.2.1 to the relevant faces $\gamma_{02,21}, \gamma_{02,31}$, we conclude $w_{21}^1 = w_{31}^1 = 1$. Since $\mu^1 = 1$ and $l_{ij} = 1$, we obtain $w_{22}^1 = w_{32}^1 = 0$. Thus, w_{22} and w_{32} lie in τ^+. Again Remark 4.2.1, this time applied to $\gamma_{01,22}, \gamma_{01,32} \in \mathrm{rlv}(X)$, yields $w_{22}^2 = w_{32}^2 = 1$. Since $\mu^2 = 1$ and $l_{ij} = 1$, we conclude $w_{21}^2 = w_{31}^2 = 0$. Hence we obtain the degree matrix

$$Q \;=\; \left[\begin{array}{cc|cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right].$$

The semiample cone is $\mathrm{SAmple}(X) = (\mathbb{Q}_{\geq 0})^2$ and the anticanonical divisor is $-\mathcal{K}_X = (2, 2)$. In particular, X is a Fano variety.

4.3 Classification results

Theorem 4.3.1. *Every smooth rational projective non-toric variety of Picard number two that admits a torus action of complexity one is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and an ample class $u \in \mathrm{Cl}(X)$, where we always have $\mathrm{Cl}(X) = \mathbb{Z}^2$ and the grading is fixed by the matrix $[w_1, \ldots, w_r]$ of generator degrees $\deg(T_i), \deg(S_j) \in \mathrm{Cl}(X)$.*

No.	$\mathcal{R}(X)$	$[w_1, \ldots, w_r]$	u	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & a & 2-a & b & 2-b \end{bmatrix}$ $1 \leq a \leq b$	$\begin{bmatrix} 1 \\ 1+b \end{bmatrix}$	4
2	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	4
3	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2T_3^2+T_4T_5+T_6^2\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2-a & a & 1 \end{bmatrix}$ $a \geq 1$	$\begin{bmatrix} 1 \\ 1+a \end{bmatrix}$	3
4	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^{l_2}+T_3T_4^{l_4}+T_5T_6^{l_6}\rangle}$ $m \geq 0$	$\left[\begin{array}{cccccc\|ccc} 0 & 1 & a & 1 & b & 1 & c_1 & \ldots & c_m \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 1 \end{array}\right]$ $0 \leq a \leq b,\ c_1 \leq \ldots \leq c_m,$ $l_2 = a + l_4 = b + l_6$	$\begin{bmatrix} d+1 \\ 1 \end{bmatrix}$ $d := \max(b, c_m)$	$m+3$
5	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$ $m \geq 0$	$\left[\begin{array}{cccccc\|ccc} 0 & 2a+1 & a & 1 & a & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$ $a \geq 0$	$\begin{bmatrix} 2a+2 \\ 1 \end{bmatrix}$	$m+3$
6	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$ $m \geq 1$	$\left[\begin{array}{cccccc\|ccc} 0 & 2c+1 & a & b & c & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$ $a, b, c \geq 0,\quad a < b,$ $a + b = 2c + 1$	$\begin{bmatrix} 2c+2 \\ 1 \end{bmatrix}$	$m+3$
7	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m \geq 1$	$\left[\begin{array}{cccccc\|ccc} 0 & 0 & 0 & 0 & -1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+3$
8	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m \geq 2$	$\left[\begin{array}{cccccc\|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \ldots & a_m \end{array}\right]$ $0 \leq a_2 \leq \ldots \leq a_m,\ a_m > 0$	$\begin{bmatrix} 1 \\ a_m+1 \end{bmatrix}$	$m+3$
9	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m \geq 2$	$\left[\begin{array}{cccc\|ccc} 0 & a_2 & \ldots & a_6 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \end{array}\right]$ $0 \leq a_3 \leq a_5 \leq a_6 \leq a_4 \leq a_2,$ $a_2 = a_3 + a_4 = a_5 + a_6$	$\begin{bmatrix} a_2+1 \\ 1 \end{bmatrix}$	$m+3$
10	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m \geq 1$	$\left[\begin{array}{ccccc\|ccc} 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & \ldots & 1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$m+2$
11	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m \geq 2$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \ldots & a_m \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \end{array}\right]$ $0 \leq a_2 \leq \ldots \leq a_m,\ a_m > 0$	$\begin{bmatrix} a_m+1 \\ 1 \end{bmatrix}$	$m+2$
12	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m \geq 2$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 2c & a & b & c & 1 & 1 & \ldots & 1 \end{array}\right]$ $0 \leq a \leq c \leq b,\ a + b = 2c$	$\begin{bmatrix} 1 \\ 2c+1 \end{bmatrix}$	$m+2$

<table>
<tr><td>13</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_8]}{\left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5T_6, \\ aT_3T_4+T_5T_6+T_7T_8 \end{array} \right\rangle}$
$a\in\mathbb{K}^*\setminus\{1\}$</td><td>$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$</td><td>$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$</td><td>4</td></tr>
</table>

Moreover, each of the listed data defines a smooth rational non-toric projective variety of Picard number two coming with a torus action of complexity one.

The anticanonical divisor of the varieties listed in Theorem 4.3.1 can be computed through Proposition 1.3.13. This enables us to determine for every dimension the finitely many families of non-toric smooth rational Fano varieties of Picard number two that admit a torus action of complexity one.

Theorem 4.3.2. *Every smooth rational non-toric Fano variety of Picard number two that admits a torus action of complexity one is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$, where the grading by $\mathrm{Cl}(X) = \mathbb{Z}^2$ is given by the matrix $[w_1, \dots, w_r]$ of generator degrees $\deg(T_i), \deg(S_j) \in \mathrm{Cl}(X)$ and we list the (ample) anticanonical class $-\mathcal{K}_X$.*

<table>
<tr><th>No.</th><th>$\mathcal{R}(X)$</th><th>$[w_1, \dots, w_r]$</th><th>$-\mathcal{K}_X$</th><th>$\dim(X)$</th></tr>
<tr><td>1</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_7]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7\rangle}$</td><td>$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$</td><td>$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$</td><td>4</td></tr>
<tr><td>2</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_7]}{\langle T_1T_2T_3+T_4T_5+T_6T_7\rangle}$</td><td>$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$</td><td>$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$</td><td>4</td></tr>
<tr><td>3</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_6]}{\langle T_1T_2T_3^2+T_4T_5+T_6^2\rangle}$</td><td>$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$</td><td>$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$</td><td>3</td></tr>
<tr><td>4.A</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$
$m\geq 0$</td><td>$\left[\begin{array}{cccccc|cccc} 0 & 1 & 0 & 1 & 0 & 1 & c & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & \dots & 1 \end{array}\right]$
$c \in \{-1, 0\}$,
$c := 0$ *if* $m = 0$</td><td>$\begin{bmatrix} 2+c \\ 2+m \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>4.B</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$
$m\geq 0$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right]$</td><td>$\begin{bmatrix} 3+m \\ 2+m \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>4.C</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$
$m\geq 0$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right]$</td><td>$\begin{bmatrix} 1 \\ 2+m \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>5</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 2a+1 & a & 1 & a & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots & 0 \end{array}\right]$
$0 \leq 2a < m$</td><td>$\begin{bmatrix} 2a+m+2 \\ 2 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>6</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 2c+1 & a & b & c & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \end{array}\right]$
$a, b, c \geq 0, \quad a < b,$
$a+b = 2c+1,$
$m > 3c+1$</td><td>$\begin{bmatrix} 3c+2+m \\ 3 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>7</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$
$1\leq m\leq 3$</td><td>$\left[\begin{array}{cccccc|ccc} 0 & 0 & 0 & 0 & -1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \end{array}\right]$</td><td>$\begin{bmatrix} m \\ 4 \end{bmatrix}$</td><td>$m+3$</td></tr>
<tr><td>8</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$
$m\geq 2$</td><td>$\left[\begin{array}{cccccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_m \end{array}\right]$
$0 \leq a_2 \leq \dots \leq a_m,$
$a_m \in \{1, 2, 3\},$
$4 + \sum_{k=2}^m a_k > m a_m$</td><td>$\begin{bmatrix} m \\ 4 + \sum_{k=2}^m a_k \end{bmatrix}$</td><td>$m+3$</td></tr>
</table>

9	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccc\|ccc} 0 & a_2 & \dots & a_6 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & 0 & \dots & 0 \end{array}\right]$ $0 \leq a_3 \leq a_5 \leq a_6 \leq a_4 \leq a_2$, $a_2 = a_3 + a_4 = a_5 + a_6$, $2a_2 < m$	$\begin{bmatrix} 2a_2 + m \\ 4 \end{bmatrix}$	$m+3$
10	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $1\leq m\leq 2$	$\left[\begin{array}{ccccc\|ccc} 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & \dots & 1 \end{array}\right]$	$\begin{bmatrix} 3 \\ m \end{bmatrix}$	$m+2$
11	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_m \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \end{array}\right]$ $0 \leq a_2 \leq \dots \leq a_m$, $a_m \in \{1,2\}$, $3 + \sum_{k=2}^m a_k > ma_m$	$\begin{bmatrix} 3 + \sum_{k=2}^m a_k \\ m \end{bmatrix}$	$m+2$
12	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 2c & a & b & c & 1 & 1 & \dots & 1 \end{array}\right]$ $0 \leq a \leq c \leq b,\ a+b = 2c$, $3c < m$	$\begin{bmatrix} 3 \\ 3c+m \end{bmatrix}$	$m+2$
13	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5T_6, \\ aT_3T_4+T_5T_6+T_7T_8 \end{array} \right\rangle}$ $a\in\mathbb{K}^*\setminus\{1\}$	$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	4

Moreover, each of the listed data defines a smooth rational non-toric Fano variety of Picard number two coming with a torus action of complexity one.

For $\mathbb{K} = \mathbb{C}$, the assumption of rationality can be omitted in Theorem 4.3.2, due to [31, Sec. 2.1] and [1, Rem. 4.4.1.5].

Similar to the Fano varieties, we can figure out the *almost Fano* varieties from Theorem 4.3.1, i.e. those with a big and nef anticanonical divisor. Without the assumption of a torus action, the classification of smooth almost Fano varieties of Picard number two is widely open; for the threefold case, we refer to the work of Jahnke, Peternell and Radloff [32, 33]. In the setting of a torus action of complexity one, the following result together with Theorem 4.3.2 settles the problem in any dimension; by a *truly almost Fano variety* we mean an almost Fano variety which is not Fano.

Theorem 4.3.3. *Every smooth rational projective non-toric truly almost Fano variety of Picard number two that admits a torus action of complexity one is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and an ample class $u \in \mathrm{Cl}(X)$, where we always have $\mathrm{Cl}(X) = \mathbb{Z}^2$ and the grading is fixed by the matrix $[w_1,\dots,w_r]$ of generator degrees $\deg(T_i), \deg(S_j) \in \mathrm{Cl}(X)$.*

No.	$\mathcal{R}(X)$	$[w_1,\dots,w_r]$	u	$\dim(X)$
4.A	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|ccc} 0 & 1 & 0 & 1 & 0 & 1 & c_1 & \dots & c_m \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right]$ $c_1 \leq \dots \leq c_m$, $d := \max(0, c_m)$, $(2+m)d = 2 + c_1 + \dots + c_m$	$\begin{bmatrix} 1 \\ 1+d \end{bmatrix}$	$m+3$
4.B	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|cccc} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & \dots & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & \dots & 1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+3$
4.C	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|cccc} 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & \dots & 1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$m+3$

4.D	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0 & 1 & 0 & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+3$
4.E	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4+T_5T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0 & 1 & 2 & 1 & 2 & 1 & 2 & \dots & 2 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$	$m+3$
4.F	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6^2\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+3$
5	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0 & 2a+1 & a & 1 & a & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots & 0 \end{array}\right]$ $m=2a$	$\begin{bmatrix} m+2 \\ 1 \end{bmatrix}$	$m+3$
6	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|ccc} 0 & 2c+1 & a & b & c & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \end{array}\right]$ $a,b,c\geq 0,\quad a<b,$ $a+b=2c+1,$ $m=3c+1$	$\begin{bmatrix} 2c+2 \\ 1 \end{bmatrix}$	$m+3$
7	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m=4$	$\left[\begin{array}{cccccc\|cccc} 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	7
8	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccc\|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_m \end{array}\right]$ $0\leq a_2\leq\dots\leq a_m,\ a_m>0,$ $4+a_2+\dots+a_m=ma_m$	$\begin{bmatrix} 1 \\ a_m+1 \end{bmatrix}$	$m+3$
9	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccc\|ccc} 0 & a_2 & \dots & a_6 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & 0 & \dots & 0 \end{array}\right]$ $0\leq a_3\leq a_5\leq a_6\leq a_4\leq a_2,$ $a_2=a_3+a_4=a_5+a_6,$ $m=2a_2$	$\begin{bmatrix} a_2+1 \\ 1 \end{bmatrix}$	$m+3$
10	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m=3$	$\left[\begin{array}{ccccc\|ccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	5
11	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_m \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \end{array}\right]$ $0\leq a_2\leq\dots\leq a_m,\ a_m>0,$ $3+a_2+\dots+a_m=ma_m$	$\begin{bmatrix} 1 \\ a_m+1 \end{bmatrix}$	$m+2$
12	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 3$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 2c & a & b & c & 1 & 1 & \dots & 1 \end{array}\right]$ $0\leq a\leq c\leq b,\ a+b=2c,$ $m=3c$	$\begin{bmatrix} 1 \\ 2c+1 \end{bmatrix}$	$m+2$

Moreover, each of the listed data defines a smooth rational non-toric truly almost Fano variety of Picard number two coming with a torus action of complexity one.

Proof of Theorems 4.3.1, 4.3.2 and 4.3.3. The preceeding analysis of the cases of Proposition 4.1.14 shows that every smooth rational non-toric projective variety of Picard number two coming with a torus action of complexity one occurs in Theorem 4.3.1 and, among these, the Fano ones in Theorem 4.3.2 and the truly almost Fano ones in Theorem 4.3.3. Comparing the defining data, one directly verifies that any two different listed varieties are not isomorphic to each other. Finally, using Proposition 1.4.2 one explicitly checks that indeed all varieties listed in Theorem 4.3.1 are smooth. □

4.4 Duplicating free weights

Up to isomorphy, there are just two smooth non-toric projective varieties with a torus action of complexity one and Picard number one, namely the smooth projective quadrics in dimension three and four. In Picard number two we obtained examples in every dimension and this even holds when we restrict to the Fano case. Nevertheless, also in Picard number two we can observe a certain finiteness feature: each Fano variety listed in Theorem 4.3.2 arises from a smooth, but not necessarily Fano, variety of dimension at most seven via an iterated generalized cone construction. In terms of the Cox ring the generalized cone construction simply means *duplicating a free weight*.

For the precise treatment, the setting of bunched rings $(R, \mathfrak{F}, \Phi)$ is most appropriate. Recall from Section 1.3 that R is a normal factorially K-graded $\mathbb{K}$-algebra, $\mathfrak{F}$ a system of pairwise non-associated K-prime generators for R and Φ a certain collection of polyhedral cones in $K_{\mathbb{Q}}$ defining an open set $\widehat{X} \subseteq \overline{X} = \operatorname{Spec} R$ with a good quotient $X = \widehat{X} /\!\!/ H$ by the action of the quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$ on $\overline{X}$. Dimension, divisor class group and Cox ring of X are given by

$$\dim(X) \ = \ \dim(R) - \dim(K_{\mathbb{Q}}), \qquad \operatorname{Cl}(X) \ = \ K, \qquad \mathcal{R}(X) \ = \ R.$$

Following Costruction 1.3.4, we call $X := X(R, \mathfrak{F}, \Phi)$ the variety associated with the bunched ring $(R, \mathfrak{F}, \Phi)$.

Construction 4.4.1. Let $R = \mathbb{K}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_s \rangle$ a K-graded algebra presented by K-homogeneous generators T_i and relations $g_j \in \mathbb{K}[T_1, \ldots, T_{r-1}]$. By *duplicating the free weight* $\deg(T_r)$ we mean passing from R to the K-graded algebra

$$R' \ := \ \mathbb{K}[T_1, \ldots, T_r, T_{r+1}]/\langle g_1, \ldots, g_s \rangle, \qquad \deg(T_{r+1}) \ := \ \deg(T_r) \ \in \ K,$$

where $g_j \in \mathbb{K}[T_1, \ldots, T_{r-1}] \subseteq \mathbb{K}[T_1, \ldots, T_r, T_{r+1}]$. If $(R, \mathfrak{F}, \Phi)$ is a bunched ring with $\mathfrak{F} = (T_1, \ldots, T_r)$, then $(R', \mathfrak{F}', \Phi)$ is a bunched ring with $\mathfrak{F}' = (T_1, \ldots, T_r, T_{r+1})$.

Proof. The $\mathbb{K}$-algebra R' is normal and, by [7, Thm. 1.4], factorially K-graded. Obviously, the K-grading is almost free. Moreover, $(R, \mathfrak{F})$ and $(R', \mathfrak{F}')$ have the same sets of generator weights in the common grading group K and the collection of projected $\mathfrak{F}'$-faces equals the collection of projected $\mathfrak{F}$-faces. We conclude that Φ is a true $\mathfrak{F}'$-bunch and thus $(R', \mathfrak{F}', \Phi)$ is a bunched ring. □

Here are some basic features of the procedure of Construction 4.4.1.

Proposition 4.4.2. *Let $(R', \mathfrak{F}', \Phi)$ arise from the bunched ring $(R, \mathfrak{F}, \Phi)$ via Construction 4.4.1. Set $X' := X(R', \mathfrak{F}', \Phi)$ and $X := X(R, \mathfrak{F}, \Phi)$.*

(i) *We have* $\dim(X') = \dim(X) + 1$.
(ii) *The cones of semiample divisor classes satisfy* $\operatorname{SAmple}(X') = \operatorname{SAmple}(X)$.
(iii) *The variety X' is smooth if and only if X is smooth.*
(iv) *The ring R' is a complete intersection if and only if R is a complete intersection.*
(v) *If R is a complete intersection,* $\deg(T_r)$ *semiample and X Fano, then X' is Fano.*

Proof. By construction, $\dim(R') = \dim(R) + 1$ holds. Since R and R' have the same grading group K, we obtain (i). Moreover, R and R' have the same defining relations g_j, hence we have (iv). According to Proposition 1.4.3, the semiample cone is the intersection of all elements of Φ and thus (ii) holds.
To obtain the third assertion, we show first that $\widehat{X}'$ is smooth if and only if $\widehat{X}$ is smooth. For every relevant $\mathfrak{F}$-face $\gamma_0 \preceq \mathbb{Q}^r_{\geq 0}$ consider

$$\gamma_0' \ := \ \gamma_0 + \mathrm{cone}(e_{r+1}), \qquad \gamma_0'' \ := \ \mathrm{cone}(e_i;\ 1 \leq i < r,\ e_i \in \gamma_0) + \mathrm{cone}(e_{r+1}).$$

Then $\gamma_0, \gamma_0', \gamma_0'' \preceq \mathbb{Q}^{r+1}_{\geq 0}$ are relevant $\mathfrak{F}'$-faces and, in fact, all relevant $\mathfrak{F}'$-faces are of this form. Since the variables T_r and T_{r+1} do not appear in the relations g_j, we see that a stratum $\overline{X}(\gamma_0)$ is smooth if and only the strata $\overline{X}'(\gamma_0)$, $\overline{X}'(\gamma_0')$ and $\overline{X}'(\gamma_0'')$ are smooth. Now Proposition 1.4.2 gives (iii).
Finally, we show (v). As we have complete intersection Cox rings, Proposition 1.3.13 applies and we obtain

$$-\mathcal{K}_{X'} \ = \ \sum_{i=1}^{r+1} \deg(T_i) - \sum_{j=1}^{s} \deg(g_j) \ = \ -\mathcal{K}_X + \deg(T_{r+1}).$$

Since X and X' share the same ample cone, we conclude that ampleness of $-\mathcal{K}_X$ implies ampleness of $-\mathcal{K}_{X'}$, □

We interpret the duplication of free weights in terms of birational geometry: it turns out to be a composition of a contraction of fiber type, a series of flips and a birational divisorial contraction, where both contractions are elementary; see [14] for a detailed study of the latter type of maps in the context of general smooth Fano 4-folds.

Proposition 4.4.3. *Let $(R', \mathfrak{F}', \Phi)$ arise from the bunched ring $(R, \mathfrak{F}, \Phi)$ via Construction 4.4.1. Set $X' := X(R', \mathfrak{F}', \Phi)$ and $X := X(R, \mathfrak{F}, \Phi)$. Assume that X is $\mathbb{Q}$-factorial. Then there is a sequence*

$$X \ \longleftarrow \ \widetilde{X}_1 \ \dashrightarrow \ \ldots \ \dashrightarrow \ \widetilde{X}_t \ \longrightarrow \ X',$$

where $\widetilde{X}_1 \to X$ is a contraction of fiber type with fibers $\mathbb{P}_1$, every $\widetilde{X}_i \dashrightarrow \widetilde{X}_{i+1}$ is a flip and $\widetilde{X}_t \to X'$ is the contraction of a prime divisor. If $\deg(T_r) \in K$ is Cartier, then $\widetilde{X}_1 \to X$ is the $\mathbb{P}_1$-bundle associated with the divisor on X corresponding to T_r.

Proof. In order to define $\widetilde{X}_1$, we consider the minimal toric embedding $X \subseteq Z$ in the sense of Construction 1.3.8. Let Σ be the fan of Z and $P = [v_1, \ldots, v_r]$ be the matrix having the primitive generators $v_i \in \mathbb{Z}^n$ of the rays of Σ as its columns. Define a further matrix

$$\widetilde{P} \ := \ \left[\begin{array}{cccccc} v_1 & \ldots & v_{r-1} & v_r & 0 & 0 \\ 0 & \ldots & 0 & -1 & 1 & -1 \end{array} \right].$$

We denote the columns of $\widetilde{P}$ by $\widetilde{v}_1, \ldots, \widetilde{v}_r, \widetilde{v}_+, \widetilde{v}_- \in \mathbb{Z}^{n+1}$, write ϱ_+, ϱ_- for the rays through $\widetilde{v}_+$, $\widetilde{v}_-$ and define a fan

$$\widetilde{\Sigma}_1 \ := \ \{\widetilde{\sigma} + \varrho_+,\ \widetilde{\sigma} + \varrho_-,\ \widetilde{\sigma};\ \sigma \in \Sigma\}, \qquad\qquad \widetilde{\sigma} \ := \ \mathrm{cone}(\widetilde{v}_i; v_i \in \sigma).$$

The projection $\mathbb{Z}^{n+1} \to \mathbb{Z}^n$ is a map of fans $\widetilde{\Sigma}_1 \to \Sigma$. The associated toric morphism $\widetilde{Z}_1 \to Z$ has fibers $\mathbb{P}_1$. If the toric divisor D_r corresponding to the ray through v_r is Cartier, then $\widetilde{Z}_1 \to Z$ is the $\mathbb{P}_1$-bundle associated with D_r. We define $\widetilde{X}_1 \subseteq \widetilde{Z}_1$ to be the preimage of $X \subseteq Z$. Then $\widetilde{X}_1 \to X$ has fibers $\mathbb{P}_1$. If $\deg(T_r)$ is Cartier, then so is D_r and hence $\widetilde{X}_1 \to X$ inherits the $\mathbb{P}_1$-bundle structure.

Now we determine the Cox ring of the variety $\widetilde{X}_1$. For this, observe that the projection $\mathbb{Z}^{r+2} \to \mathbb{Z}^r$ defines a lift of $\widetilde{Z}_1 \to Z$ to the toric characteristic spaces and thus leads to the commutative diagram

$$\begin{array}{ccccccc} \widetilde{\pi}^\sharp(\widetilde{X}_1) & \subseteq & \widetilde{W}_1 & \longrightarrow & W & \supseteq & \pi^\sharp(X) \\ \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \widetilde{X}_1 & \subseteq & \widetilde{Z}_1 & \longrightarrow & Z & \supseteq & X \end{array}$$

where $\widetilde{\pi}^\sharp(\widetilde{X}_1)$ and $\pi^\sharp(X)$ denote the proper transforms with respect to the downwards toric morphisms. Pulling back the defining equations of $\pi^\sharp(X) \subseteq W$, we see that $\widetilde{\pi}^\sharp(\widetilde{X}_1) \subseteq \widetilde{W}_1$ has coordinate algebra $\widetilde{R} := R[S^+, S^-]$ graded by $\widetilde{K} := K \times \mathbb{Z}$ via

$$\deg(T_i) := (w_i, 0), \qquad w^+ := \deg(S^+) := (w_r, 1), \qquad w^- := \deg(S^-) := (0, 1),$$

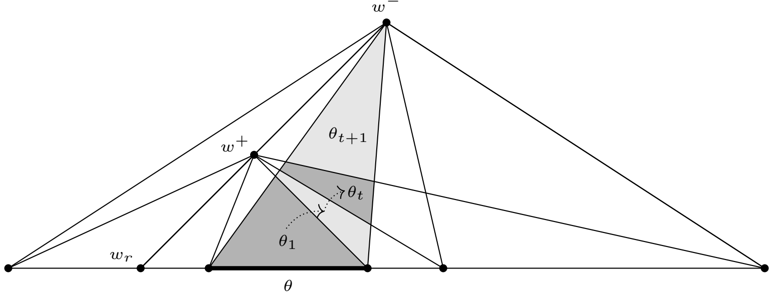
where $w_i := \deg(T_i) \in K$. The $\mathbb{K}$-algebra $\widetilde{R}$ is normal and, by [7, Thm. 1.4], factorially $\widetilde{K}$-graded. Moreover the $\widetilde{K}$-grading is almost free, as the K-grading of R has this property and $\widetilde{\mathfrak{F}} = (T_1, \ldots, T_r, S^+, S^-)$ is a system of pairwise non-associated $\widetilde{K}$-prime generators. We conclude that $\widetilde{R}$ is the Cox ring of $\widetilde{X}_1$.

Next we look for the defining bunch of cones for $\widetilde{X}_1$. Observe that K sits inside $\widetilde{K}$ as $K \times \{0\}$. With $\theta := \mathrm{SAmple}(X) \times \{0\}$ we obtain a GIT-cone $\theta_1 := \mathrm{cone}(\theta, w^+) \cap \mathrm{cone}(\theta, w^-)$ of the $\widetilde{K}$-graded ring $\widetilde{R}$. The associated bunch $\widetilde{\Phi}_1$ consists of all cones of the form

$$\widetilde{\tau} + \mathrm{cone}(w^+), \qquad \widetilde{\tau} + \mathrm{cone}(w^-), \qquad \widetilde{\tau} + \mathrm{cone}(w^+, w^-),$$

where $\widetilde{\tau} = \tau \times \{0\}$, $\tau \in \Phi$. Since Φ is a true bunch, so is $\widetilde{\Phi}_1$. Together we obtain a bunched ring $(\widetilde{R}, \widetilde{\mathfrak{F}}, \widetilde{\Phi}_1)$. By construction, the fan corresponding to $\widetilde{\Phi}_1$ via Gale duality is $\widetilde{\Sigma}_1$. We conclude that $\widetilde{X}_1$ is the variety associated with $(\widetilde{R}, \widetilde{F}, \widetilde{\Phi}_1)$ and $\widetilde{X}_1 \subseteq \widetilde{Z}_1$ is the canonical toric embedding.

Observe that $\widetilde{X}_1 \to X$ corresponds to the passage from the GIT-cone θ_1 to the facet θ. In particular, we see that $\widetilde{X}_1 \to X$ is a contraction of fiber type. To obtain the flips and the final divisorial contraction, we consider the full GIT-fan.



Important are the GIT-cones inside $\theta + \mathrm{cone}(w^-)$. There we have the facet θ and the semiample cone θ_1 of $\widetilde{X}_1$. Proceding in the direction of w^-, we come across other full-dimensional GIT-cones, say $\theta_2, \ldots, \theta_{t+1}$. This gives a sequence of flips $\widetilde{X}_1 \dashrightarrow \ldots \dashrightarrow \widetilde{X}_t$, where $\widetilde{X}_i$ is the variety with semiample cone θ_i. Passing from θ_t to θ_{t+1} gives a morphism $\widetilde{X}_t \to \widetilde{X}_{t+1}$ contracting the prime divisor corresponding to the variable S^- of the Cox ring $\widetilde{R}$ of $\widetilde{X}_t$. Note that $\widetilde{X}_{t+1}$ is $\mathbb{Q}$-factorial, as it is the GIT-quotient associated with a full-dimensional chamber.

We show $\widetilde{X}_{t+1} \cong X'$. Recall that X' arises from X by duplicating the weight $\deg(T_r)$. We have $\mathrm{Cl}(X') = K$ and the Cox ring $R' = R[T_{r+1}]$ of X' is K-graded via $\deg(T_i) = w_i$ for $i = 1, \ldots, r$ and $\deg(T_{r+1}) = w_r$. In particular, the fan of the canonical toric ambient variety of X' has as its primitive ray generators the columns of the matrix

$$P' \;=\; \begin{bmatrix} v_1 & \ldots & v_{r-1} & v_r & 0 \\ 0 & \ldots & 0 & -1 & 1 \end{bmatrix}.$$

On the other hand, the canonical toric ambient variety $\widetilde{Z}_{t+1}$ of $\widetilde{X}_{t+1}$ is obtained from $\widetilde{Z}_t$ by contracting the divisor corresponding to the ray ϱ_-. Hence P' is as well the primitive generator matrix for the fan of $\widetilde{Z}_{t+1}$. We conclude

$$\mathrm{Cl}(\widetilde{X}_{t+1}) \;=\; \mathbb{Z}^{r+1}/\, \mathrm{im}((P')^*) \;=\; \mathrm{Cl}(X') \;=\; K.$$

Similarly, we compare the Cox rings of $\widetilde{X}_{t+1}$ and X'. Let $\widetilde{Z}_t$ denote the canonical toric ambient variety of $\widetilde{X}_t$. Then the projection $\mathbb{Z}^{r+2} \to \mathbb{Z}^{r+1}$ defines a lift of $\widetilde{Z}_t \to \widetilde{Z}_{t+1}$ to the toric characteristic spaces and thus leads to the commutative diagram

$$\begin{array}{ccccccc} \widetilde{\pi}^\sharp(\widetilde{X}_t) & \subseteq & \widetilde{W}_t & \longrightarrow & \widetilde{W}_{t+1} & \supseteq & \pi^\sharp(\widetilde{X}_{t+1}) \\ \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \widetilde{X}_t & \subseteq & \widetilde{Z}_t & \longrightarrow & \widetilde{Z}_{t+1} & \supseteq & \widetilde{X}_{t+1} \end{array}$$

where the proper transforms $\widetilde{\pi}^\sharp(\widetilde{X}_t)$ and $\pi^\sharp(\widetilde{X}_{t+1})$ are the characteristic spaces of $\widetilde{X}_t$ and $\widetilde{X}_{t+1}$ respectively and the first is mapped onto the second one. We conclude that the Cox ring of $\widetilde{X}_{t+1}$ is $R[S^+]$ graded by $\deg(T_i) = w_i$ for $i = 1, \ldots, r$ and $\deg(S^+) = w_r$ and thus is isomorphic to the Cox ring R' of X'.

The final step is to compare the defining bunches of cones $\widetilde{\Phi}_{t+1}$ of $\widetilde{X}_{t+1}$ and Φ' of X'. For this, observe that the fan of the toric ambient variety $\widetilde{Z}_{t+1}$ contains the cones $\widetilde{\sigma} + \varrho_+$, where $\sigma \in \Sigma$. Thus, every $\tau \in \Phi'$ belongs to $\widetilde{\Phi}_{t+1}$. We conclude

$$\mathrm{SAmple}(\widetilde{X}_{t+1}) \;\subseteq\; \mathrm{SAmple}(X').$$

Since $\widetilde{X}_{t+1}$ is $\mathbb{Q}$-factorial, its semiample cone is of full dimension. Both cones belong to the GIT-fan, hence we see that the above inclusion is in fact an equality. Thus $\widetilde{\Phi}_{t+1}$ equals Φ'. □

We return to the Fano varieties of Theorem 4.3.2. We first list the (finitely many) examples which do not allow duplication of a free weight and then present the starting models for constructing the Fano varieties via duplication of weights.

Proposition 4.4.4. *The varieties of Theorem 4.3.2 containing no divisor with infinite general isotropy are precisely the following ones:*

No.	$\mathcal{R}(X)$	$[w_1, \ldots, w_r]$	$-\mathcal{K}_X$	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1&1 \\ 1&1&0&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$	4
2	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0&0&1&1&0&1&0 \\ 1&1&0&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$	4
3	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2T_3^2+T_4T_5+T_6^2\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1 \\ 1&1&0&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	3
4.A	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\begin{bmatrix} 0&1&0&1&0&1 \\ 1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	3
4.B	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$	$\begin{bmatrix} 0&1&1&1&1&1 \\ 1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$	3
4.C	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$	$\begin{bmatrix} 0&1&0&1&0&1 \\ 1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	3
13	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5T_6, \\ aT_3T_4+T_5T_6+T_7T_8 \end{array} \right\rangle}$ $a \in \mathbb{K}^* \setminus \{1\}$	$\begin{bmatrix} 1&0&1&0&1&0&1&0 \\ 0&1&0&1&0&1&0&1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	4

Proof. For a T-variety $X = X(A, P, u)$, the divisors having infinite general T-isotropy are precisely the vanishing sets of the variable S_k. Thus we just have to pick out the cases with $m = 0$ from Theorem 4.3.2. □

Theorem 4.4.5. *Let X be a smooth rational Fano variety with a torus action of complexity one and Picard number two. If there is a prime divisor with infinite general isotropy on X, then X arises via iterated duplication of the free weight w_r from one of the following varieties Y:*

No.	$\mathcal{R}(Y)$	$[w_1, \ldots, w_r]$	u	$\dim(Y)$
4.A	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0&1&0&1&0&1&0 \\ 1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	4
4.A	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|cc} 0&1&0&1&0&1&-1&0 \\ 1&0&1&0&1&0&1&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	5
4.B	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0&1&1&1&1&1&1 \\ 1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	4
4.C	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$	$\left[\begin{array}{cccccc\|c} 0&1&0&1&0&1&0 \\ 1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	4
5	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0&2a+1&a&1&a&1&1 \\ 1&1&1&0&1&0&0 \end{array}\right]$ $a \geq 0$	$\begin{bmatrix} 2a+2 \\ 1 \end{bmatrix}$	4
6	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0&2c+1&a&b&c&1&1 \\ 1&1&1&1&1&0&0 \end{array}\right]$ $a, b, c \geq 0, \quad a < b,$ $a + b = 2c + 1$	$\begin{bmatrix} 2c+2 \\ 1 \end{bmatrix}$	4
7	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0&0&0&0&-1&1&1 \\ 1&1&1&1&1&1&0 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	4
8	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|cc} 0&0&0&0&0&0&1&1 \\ 1&1&1&1&1&1&0&a \end{array}\right]$ $a \in \{1, 2, 3\}$	$\begin{bmatrix} 1 \\ a+1 \end{bmatrix}$	5
8	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,S_2,S_3]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|ccc} 0&0&0&0&0&0&1&1&1 \\ 1&1&1&1&1&1&0&a-1&a \end{array}\right]$ $a \in \{1, 2\}$	$\begin{bmatrix} 1 \\ a+1 \end{bmatrix}$	6

8	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_4]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|cccc} 0&0&0&0&0&0&1&1&1&1\\ 1&1&1&1&1&1&0&0&0&1 \end{array}\right]$	$\begin{bmatrix}1\\2\end{bmatrix}$	7
9	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccc\|cc} 0&a_2&\ldots&a_6&1&1\\ 1&1&\ldots&1&0&0 \end{array}\right]$ $0 \le a_3 \le a_5 \le a_6 \le a_4 \le a_2,$ $a_2 = a_3 + a_4 = a_5 + a_6$	$\begin{bmatrix}a_2+1\\1\end{bmatrix}$	5
10	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{ccccc\|c} 1&1&1&1&1&0\\ -1&1&0&0&0&1 \end{array}\right]$	$\begin{bmatrix}2\\1\end{bmatrix}$	3
11	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{ccccc\|cc} 1&1&1&1&1&0&a\\ 0&0&0&0&0&1&1 \end{array}\right]$ $a \in \{1,2\}$	$\begin{bmatrix}a+1\\1\end{bmatrix}$	4
11	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,S_2,S_3]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{ccccc\|ccc} 1&1&1&1&1&0&0&1\\ 0&0&0&0&0&1&1&1 \end{array}\right]$	$\begin{bmatrix}2\\1\end{bmatrix}$	5
12	$\frac{\mathbb{K}[T_1,\ldots,T_5,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{ccccc\|cc} 1&1&1&1&1&0&0\\ 0&2c&a&b&c&1&1 \end{array}\right]$ $0 \le a \le c \le b,\ a+b=2c$	$\begin{bmatrix}1\\2c+1\end{bmatrix}$	4

For Nos. 4, 8 and 11, the variety Y is Fano and any iterated duplication of w_r produces a Fano variety X. For the remaining cases, the following table tells which Y are Fano and gives the characterizing condition when an iterated duplication of w_r produces a Fano variety X:

No.	*5*	*6*	*7*	*9*	*10*	*12*
Y Fano	$a = 0$	$c = 0$	✓	$a_2 = 0$	✓	$c = 0$
X Fano	$m > 2a$	$m > 3c + 1$	$m \le 3$	$m > 2a_2$	$m \le 2$	$m > 3c$

Proof. A T-variety $X = X(A, P, u)$ has a divisor with infinite general T-isotropy if and only if $m \ge 1$ holds. In the cases 4.A, 4.B, 4.C, 5, 6, 7, 9, 10 and 12 we directly infer from Theorem 4.3.2 that the examples with higher m arise from those listed in the table above via iterated duplication of w_r.

We still have to consider Nos. 8 and 11. If X is a variety of type 8, then the condition for X to be a Fano variety is

$$4 + a_2 + \ldots, + a_m > m a_m,$$

where $a_m = 1, 2, 3$ and $0 \le a_2 \le \ldots \le a_m$. This is satisfied if and only if one of the following conditions holds:

(i) $a_2 = \ldots = a_m \in \{1, 2, 3\}$.
(ii) $a_2 + 1 = a_3 \ldots = a_m \in \{1, 2\}$, with $m \ge 3$.
(iii) $a_2 = a_3 = 0$ and $a_4 = \ldots = a_m = 1$, with $m \ge 4$.

Similarly for No. 11 the Fano condition in the table of Theorem 4.3.2 is equivalent to the fulfillment of one of the following:

(i) $a_2 = \ldots = a_m \in \{1, 2\}$.
(ii) $a_2 = 0$ and $a_3 = \ldots = a_m = 1$, with $m \ge 3$.

In both cases this explicit characterization makes clear that we are in the setting of the duplication of a free weight. □

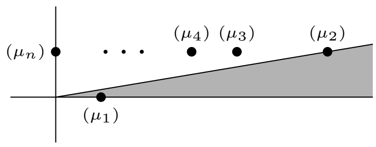
Remark 4.4.6. Consider iterated duplication of w_r for a variety $X = X(A, P, u)$ as in Theorem 4.4.5. Recall that the effective cone of X is decomposed as $\tau^+ \cup \tau_X \cup \tau^-$, where $\tau_X = \mathrm{Ample}(X)$. Lemma 4.1.10 (i) says $w_r \notin \tau_X$ and thus we have a unique $\kappa \in \{\tau^+, \tau^-\}$ with $w_r \notin \kappa$. Then the number of flips per duplication step equals

$$|\{\mathrm{cone}(w_{ij}), \mathrm{cone}(w_k);\ w_{ij}, w_k \in \kappa\}| - 1.$$

In particular, for Nos. 4.A, 4.B, 4.C, 8, 11, No. 9 with $a_i = 0$ and No. 12 with $b = 0$, the duplication steps require no flip.

Remark 4.4.7. For toric Fano varieties, there is no statement like Theorem 4.4.5. Recall from [9] that all smooth projective toric varieties Z with $\mathrm{Cl}(Z) = \mathbb{Z}^2$ admit a description via the following data:

- weight vectors $w_1 := (1, 0)$ and $w_i := (b_i, 1)$ with $0 = b_n < b_{n-1} < \ldots < b_2$,
- multiplicities $\mu_i := \mu(w_i) \geq 1$, where $\mu_1 \geq 2$ and $\mu_2 + \ldots + \mu_n \geq 2$.



The variety Z arises from the bunched ring $(R, \mathfrak{F}, \Phi)$, where R equals the polynomial ring $\mathbb{K}[S_{ij};\ 1 \leq i \leq n, 1 \leq j \leq \mu_i]$ with the system of generators $\mathfrak{F} = (S_{11}, \ldots, S_{n\mu_n})$ and the bunch $\Phi = \{\mathrm{cone}(w_1, w_i); i = 2, \ldots, n\}$. In this setting Z is Fano if and only if

$$b_2(\mu_3 + \ldots + \mu_n) < \mu_1 + \mu_3 b_3 + \ldots + \mu_{n-1} b_{n-1}.$$

For any $n \in \mathbb{Z}_{\geq 4}$ and $i = 2, \ldots, n$ set $\mu_i := 1$ and $w_i := (n - i, 1)$. Then, with $\mu_1 := 2$ we obtain a smooth (non-Fano) toric variety Z'_n of Picard number two and dimension $n - 1$. Moreover, for $\mu_1 := 1 + (n - 2)(n - 1)/2$ we obtain a smooth toric Fano variety Z_n of Picard number two that is Fano and is obtained from Z'_n via iterated duplication of w_1 but cannot be constructed from any lower dimensional smooth variety this way.

4.5 Geometry of the Fano varieties

We take a closer look at the Fano varieties X listed in Theorem 4.3.2 and describe explicitly their Mori fiber spaces and their divisorial contractions. The approach uses suitable toric ambient varieties. The following Remark can be found, at least partially, for example in [15, Section 7.3].

Remark 4.5.1. Let Z be a smooth projective toric variety of Picard number 2, given by weight vectors $w_1 := (1, 0)$ and $w_i := (b_i, 1)$ with $0 = b_n < b_{n-1} < \ldots < b_2$, and multiplicities $\mu_i := \mu(w_i) \geq 1$, where $\mu_1 \geq 2$ and $\mu_2 + \ldots + \mu_n \geq 2$ as in Remark 4.4.7.

Then the toric variety Z is a projectivized split vector bundle of rank r over a projective space $\mathbb{P}_s$, where $s := \mu_1 - 1$ and $r := \mu_2 + \ldots + \mu_n - 1$. More precisely, we have

$$Z \;\cong\; \mathbb{P}\left(\bigoplus_{i=1}^{\mu_n} \mathcal{O}_{\mathbb{P}_s} \oplus \bigoplus_{i=1}^{\mu_{n-1}} \mathcal{O}_{\mathbb{P}_s}(b_{n-1}) \oplus \ldots \oplus \bigoplus_{i=1}^{\mu_2} \mathcal{O}_{\mathbb{P}_s}(b_2)\right).$$

The bundle projection $Z \to \mathbb{P}_s$ is the elementary contraction associated to the divisor class $w_1 \in \mathbb{Z}^2 = \mathrm{Cl}(Z)$. If $n = 2$ holds, then we have $Z \cong \mathbb{P}_s \times \mathbb{P}_r$. If $n = 3$ and $\mu_3 = 1$ hold, then the class $w_3 \in \mathbb{Z}^2 = \mathrm{Cl}(Z)$ gives rise to a divisorial contraction onto a weighted projective space:

$$Z \;\to\; Z' \;:=\; \mathbb{P}(\underbrace{1, \ldots, 1}_{\mu_1}, \underbrace{b_2, \ldots, b_2}_{\mu_2}).$$

The exceptional divisor $E_Z \subseteq Z$ is isomorphic to $\mathbb{P}_s \times \mathbb{P}_{\mu_2 - 1}$ and the center $C(Z') \subseteq Z'$ of the contraction is isomorphic to $\mathbb{P}_{\mu_2 - 1}$. In particular, for $\mu_2 = 1$, we have $E_Z \cong \mathbb{P}_s$ and $C(Z')$ is a point.

From the explicit description of the Cox ring of our Fano variety X, we obtain via Construction 1.3.8 a closed embedding $X \to Z$ into a toric variety Z. As a byproduct of our classification, it turns out that, whenever X admits an elementary contraction, then X inherits all its elementary contractions from Z. Remark 4.5.1 together with the explicit equations for X in Z will then allow us to study the situation in detail. We now present the results. The cases are numbered according to the table of Theorem 4.3.2. Moreover, we denote by $Q_3 \subseteq \mathbb{P}_4$ and $Q_4 \subseteq \mathbb{P}_5$ the three and four-dimensional smooth projective quadrics and we write $\mathbb{P}(a_1^{\mu_1}, \ldots, a_r^{\mu_r})$ for the weighted projective space, where the superscript μ_i indicates that the weight a_i occurs μ_i times.

No. 1. The variety X is of dimension four and admits two elementary contractions, $Q_4 \leftarrow X \to \mathbb{P}_1$. The morphism $X \to Q_4$ is a divisorial contraction with exceptional divisor isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ and center isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. The morphism $X \to \mathbb{P}_1$ is a Mori fiber space with general fiber isomorphic to Q_3 and singular fibers over $[0,1]$ and $[1,0]$ each isomorphic to the singular quadric $V(T_2T_3 + T_4T_5) \subseteq \mathbb{P}_4$.

No. 2. The variety X is of dimension four and admits two elementary contractions, $Q_4 \leftarrow X \to \mathbb{P}_3$. The morphism $X \to Q_4$ is a divisorial contraction with exceptional divisor isomorphic to a hypersurface of bidegree $(1,1)$ in $\mathbb{P}_1 \times \mathbb{P}_3$ and center isomorphic to $\mathbb{P}_1$. The morphism $X \to \mathbb{P}_3$ is a Mori fiber space with fibers isomorphic to $\mathbb{P}_1$.

No. 3. The variety X is of dimension three and occurs as No. 2.29 in the Mori-Mukai classification [44]. Moreover, X admits two elementary contractions, $Q_3 \leftarrow X \to \mathbb{P}_1$. The morphism $X \to Q_3$ is a divisorial contraction with exceptional divisor isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$ and center isomorphic to $\mathbb{P}_1$. The morphism $X \to \mathbb{P}_1$ is a Mori fiber space with general fiber isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$ and singular fibers over $[0,1]$ and $[1,0]$ each isomorphic to $V(T_1T_2 + T_3^2) \subseteq \mathbb{P}_3$.

No. 4A. *Case 1:* we have $c = -1$. Then X admits two elementary contractions $Y \leftarrow X \rightarrow \mathbb{P}_2$, where $Y := V(T_1T_2 + T_3T_4 + T_5T_6) \subseteq \mathbb{P}_{m+4}$ is a terminal factorial Fano variety which is smooth if and only if $m = 1$ holds. The morphism $X \rightarrow Y$ is a divisorial contraction with exceptional divisor isomorphic to a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}_2 \times \mathbb{P}_{m+1}$ and center isomorphic to $\mathbb{P}_{m+1}$. The morphism $X \rightarrow \mathbb{P}_2$ is a Mori fiber space with fibers isomorphic to $\mathbb{P}_{m+1}$.

Case 2: we have $c = 0$. Then X is a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}_2 \times \mathbb{P}_{m+2}$. Moreover, X admits two Mori fiber spaces $\mathbb{P}_{m+2} \leftarrow X \rightarrow \mathbb{P}_2$. The Mori fiber space $X \rightarrow \mathbb{P}_2$ has fibers isomorphic to $\mathbb{P}_{m+1}$, whereas the Mori fiber space $X \rightarrow \mathbb{P}_{m+1}$ has general fiber isomorphic to $\mathbb{P}_1$ and special fibers over $V(T_1, T_2, T_3) \subseteq \mathbb{P}_{m+2}$ isomorphic to $\mathbb{P}_2$. For $m = 0$, we have $\dim(X) = 3$ and X is the variety No. 2.32 in [44].

No. 4B. The variety X admits two elementary contractions $Y \leftarrow X \rightarrow \mathbb{P}_2$, where $Y := V(T_1^2 + T_2T_3 + T_4T_5) \subseteq \mathbb{P}_{m+4}$ is a terminal factorial Fano variety. The variety Y is smooth if and only if $m = 0$ holds and in this case X occurs as No. 2.31 in [44]. The morphism $X \rightarrow Y$ is a divisorial contraction with exceptional divisor isomorphic to a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}_2 \times \mathbb{P}_{m+1}$ and center isomorphic to $\mathbb{P}_{m+1}$. The morphism $X \rightarrow \mathbb{P}_2$ is a Mori fiber space with fibers isomorphic to $\mathbb{P}_{m+1}$.

No. 4C. The variety X is a hypersurface of bidegree $(2, 1)$ in $\mathbb{P}_2 \times \mathbb{P}_{m+2}$; for $m = 0$ we have $\dim(X) = 3$ and X is No. 2.24 in [44]. Moreover, X admits two Mori fiber spaces $\mathbb{P}_{m+2} \leftarrow X \rightarrow \mathbb{P}_2$. The morphism $X \rightarrow \mathbb{P}_2$ has fibers isomorphic to $\mathbb{P}_{m+1}$. To describe the fibers of $\varphi\colon X \rightarrow \mathbb{P}_{m+2}$, set $Y_i := V_{\mathbb{P}_{m+2}}(T_i)$, $Y_{ij} := V_{\mathbb{P}_{m+2}}(T_i, T_j)$ and $Y_{123} := V_{\mathbb{P}_{m+2}}(T_1, T_2, T_3)$. Then we have

$$\varphi^{-1}(z) \;\cong\; \begin{cases} \mathbb{P}_2 & \text{if } z \in Y_{123}, \\ \mathbb{P}_1 & \text{if } z \in (Y_{12} \cup Y_{13} \cup Y_{23}) \setminus Y_{123}, \\ V_{\mathbb{P}_2}(T_1T_2) & \text{if } z \in (Y_1 \cup Y_2 \cup Y_3) \setminus (Y_{12} \cup Y_{13} \cup Y_{23}), \\ \mathbb{P}_1 & \text{otherwise.} \end{cases}$$

No. 5. The variety X admits a Mori fiber space $\varphi\colon X \rightarrow \mathbb{P}_{m+1}$, whose general fiber is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. More precisely, with $Y_1 := V_{\mathbb{P}_{m+1}}(T_1)$ and $Y_2 := V_{\mathbb{P}_{m+1}}(T_2)$, we have

$$\varphi^{-1}(z) \;\cong\; \begin{cases} V_{\mathbb{P}_3}(T_1T_2) & \text{if } z \in Y_1 \cap Y_2, \\ V_{\mathbb{P}_3}(T_1T_2 + T_3^2) & \text{if } z \in Y_1 \setminus Y_2 \text{ or } z \in Y_2 \setminus Y_1, \\ \mathbb{P}_1 \times \mathbb{P}_1 & \text{otherwise.} \end{cases}$$

No. 6. The variety X admits a Mori fiber space $X \rightarrow \mathbb{P}_m$, with general fiber isomorphic to Q_3 and singular fibers over $V(T_1) \subseteq \mathbb{P}_m$ each isomorphic to $V(T_1T_2 + T_3T_4) \subseteq \mathbb{P}_4$.

No. 7. The variety X admits a divisorial contraction $X \rightarrow \mathbb{P}_{m+3}$ with exceptional divisor isomorphic to the projectivized split bundle

$$\mathbb{P}\left(\bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(1, 1) \right)$$

and center isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. Moreover, if $m = 1$ holds, X admits a further divisorial contraction $X \to Q_4$ with exceptional divisor isomorphic to $\mathbb{P}_3$ and center a point.

No. 8. Here we have $X = \mathbb{P}(\mathcal{O}_{Q_4} \oplus \mathcal{O}_{Q_4}(a_2) \ldots \oplus \mathcal{O}_{Q_4}(a_m))$. Thus, there is a Mori fiber space $X \to Q_4$ with fibers isomorphic to $\mathbb{P}_{m-1}$. If $a_2 = \ldots = a_m > 0$ holds, then X admits in addition a divisorial contraction $X \to Y$, where $Y := V(T_1T_2 + T_3T_4 + T_5T_6) \subseteq \mathbb{P}(1^6, a_2^{m-1})$. The exceptional divisor is isomorphic to $Q_4 \times \mathbb{P}_{m-2}$ and the center to $\mathbb{P}_{m-2}$.

No. 9. The variety X is a bundle over $\mathbb{P}_{m-1}$ with fibers isomorphic to Q_4. In particular, if $a_i = 0$ holds for all $2 \le i \le 6$, then $X \cong Q_4 \times \mathbb{P}_{m-1}$.

No. 10. The variety X admits a divisorial contraction $X \to \mathbb{P}_{m+2}$ with exceptional divisor isomorphic to the projectivized split bundle

$$\mathbb{P}\left(\bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1)\right)$$

and center isomorphic to $\mathbb{P}_1$. For $m = 1$, we have $\dim(X) = 3$ and X is No. 2.30 from [44]; in this case it admits a further divisorial contraction $X \to Q_3$ with exceptional divisor isomorphic to $\mathbb{P}_2$ and center a point.

No. 11. Here $X = \mathbb{P}(\mathcal{O}_{Q_3} \oplus \mathcal{O}_{Q_3}(a_2) \ldots \oplus \mathcal{O}_{Q_3}(a_m))$ holds. Thus, there is a Mori fiber space $X \to Q_3$ with fibers isomorphic to $\mathbb{P}_{m-1}$. If $a_2 = \ldots = a_m > 0$ holds, then X admits a divisorial contraction $X \to Y$, where the variety Y equals $V(T_1T_2 + T_3T_4 + T_5^2) \subseteq \mathbb{P}(1^5, a_2^{m-1})$. The exceptional divisor is isomorphic to $Q_3 \times \mathbb{P}_{m-2}$ and the center to $\mathbb{P}_{m-2}$.

No. 12. The variety X is a bundle over $\mathbb{P}_{m-1}$ with fibers isomorphic to Q_3. In particular, if $a = b = c = 0$ holds, then $X \cong Q_3 \times \mathbb{P}_{m-1}$.

No. 13. This case presents a one-parameter family of varieties X_a, with parameter $a \in \mathbb{K}^* \setminus \{1\}$. They are generally non-isomorphic to each other, except for the pairs $X_a \cong X_{a^{-1}}$ for all a. The variety X_a is the intersection of two hypersurfaces

$$D_1 = V(T_1S_1 + T_2S_2 + T_3S_3), \qquad D_2 = V(aT_2S_2 + T_3S_3 + T_4S_4),$$

both of bidegree (1,1) in $\mathbb{P}_3 \times \mathbb{P}_3$, where the T_j are the coordinates of the first $\mathbb{P}_3$ and the S_j those of the second. Note that each D_i has an isolated singularity, which is not contained in the other hypersurface. Both D_1, D_2 are terminal and factorial. Moreover, X admits two Mori fiber spaces $\mathbb{P}_3 \leftarrow X \to \mathbb{P}_3$, both with typical fiber $\mathbb{P}_1$ and having four special fibers, all isomorphic to $\mathbb{P}_2$ and lying over the points $[1, 0, 0, 0]$, $[0, 1, 0, 0]$, $[0, 0, 1, 0]$ and $[0, 0, 0, 1]$.

Remark 4.5.2. In contrast to the toric case, a smooth projective variety of Picard number 2 with torus action of complexity one need not admit a non-trivial Mori fiber space. For example, in Theorem 4.3.2, this happens in precisely two cases, namely No. 7 and No. 10, both with $m = 1$.

Remark 4.5.3. In the list of Theorem 4.3.2 there are several examples where the effective cone coincides with the cone of movable divisor classes: No. 4A with $c = 0$, No. 4C, No. 5 with $a = 0$, No. 6 with $a = 0$, No. 8 with $a_2 = 0$, No. 9 with $a_3 = 0$, No. 11 with $a_2 = 0$, No. 12 with $a = 0$ and No. 13. Thus, these varieties admit no divisorial contraction. They are combinatorially minimal in the sense of Remark 3.1.1.

Remark 4.5.4. In Theorem 4.3.1 it is possible for non-isomorphic varieties to share the same Cox ring. In that case they differ from each other by a small quasimodification, i.e. only by the choice of the ample class. This happens precisely in the following cases:

(i) No. 4 with $_2 = l_4 = 2$, $l_6 = 1$, $a = 0$, $b = 1$, $c_i = 0$ for all $i = 1, \ldots, m$ has the same Cox ring as No. 5 with $a = 0$. Note that for $m = 0$ both varieties are truly almost Fano, whereas for $m \geq 1$ No. 5 is Fano.

(ii) For $m \geq 1$, No. 4 with $_2 = 2$, $l_4 = l_6 = 1$, $a = b = 1$, $c_i = 0$ for all $i = 1, \ldots, m$ has the same Cox ring as No. 6 with $a = c = 0$ and $b = 1$. Note that for $m = 1$ both varieties are truly almost Fano, whereas for $m \geq 2$ No. 6 is Fano.

(iii) For $m \geq 2$, No. 7 has the same Cox ring as No. 9 with $a_2 = 2$ and $a_3 = \ldots = a_6 = 1$. Note that for $m = 2, 3$ No. 7 is Fano, for $m = 4$ both varieties are truly almost Fano, whereas for $m \geq 5$ No. 9 is Fano.

(iv) For $m \geq 2$, No. 10 has the same Cox ring as No. 12 with $a = b = c = 1$. Note that for $m = 2$ No. 10 is Fano, for $m = 3$ both varieties are truly almost Fano, whereas for $m \geq 4$ No. 12 is Fano.

APPENDIX

A

COMPENDIUM OF THE CLASSIFIED FANO VARIETIES

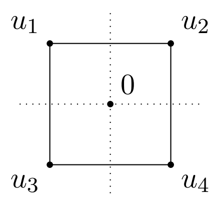
In this last part of the thesis we give detailed information about the varieties from Theorem 2.5.1 and Theorem 3.4.1. For each $X = X(A, P)$ we list

- the Cox ring $\mathcal{R}(X)$;
- the divisor class group $\mathrm{Cl}(X)$;
- the grading matrix Q;
- a possible defining matrix P;
- the anticanonical class $-\mathcal{K}_X \in \mathrm{Cl}(X)$;
- the selfintersection number $(-\mathcal{K}_X)^3$ (only for varieties from Theorem 2.5.1);
- the first terms of the Hilbert series $H(t)$ following Construction A.1.1.

Furthermore, we provide a sketch of the lineality part $A^c_{X,0}$ defined by the given matrix P. We give the vertices u_i explicitly as rational points in $\mathbb{Q}^2$. Hollow circles represent lattice points in the proximity of $A^c_{X,0}$.

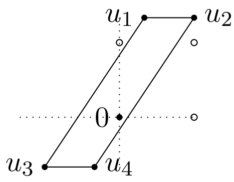
No. 1.01

$$
\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1, \ldots, T_5]/\langle g_0 \rangle \\
g_0 &= T_1T_2 + T_3T_4 + T_5^2 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \\
-\mathcal{K}_X &= 3, \qquad (-\mathcal{K}_X)^3 = 54 \\
H(t) &= 1 + 30t + 140t^2 + 385t^3 + 819t^4 + 1496t^5 + 2470t^6 + \ldots
\end{aligned}
$$



$u_1 = (-1/3, 1/3)$
$u_2 = (1/3, 1/3)$
$u_3 = (-1/3, -1/3)$
$u_4 = (1/3, -1/3)$

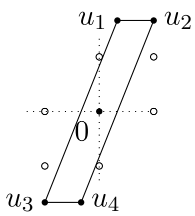
No. 1.02



$$
\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3T_4 + T_5^2 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 1 & 5 & 2 & 4 & 3 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 3 & 0 & -2 \end{bmatrix} \\
-\mathcal{K}_X &= 9, \qquad (-\mathcal{K}_X)^3 \;=\; 729/20 \\
H(t) &= 1 + 20t + 94t^2 + 259t^3 + 552t^4 + 1009t^5 + 1666t^6 + \dots
\end{aligned}
$$

$u_1 = (1/3, 4/3)$
$u_2 = (1, 4/3)$
$u_3 = (-1, -2/3)$
$u_4 = (-1/3, -2/3)$

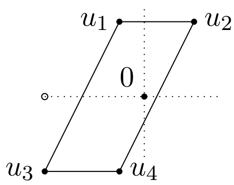
No. 1.03



$$
\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3T_4 + T_5^2 \\
\mathrm{Cl}(X) &= \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \\
Q &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{2} & \bar{3} & \bar{1} & \bar{4} & \bar{0} \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 5 & 0 & -5 \end{bmatrix} \\
-\mathcal{K}_X &= (3, \bar{0}), \qquad (-\mathcal{K}_X)^3 \;=\; 54/5 \\
H(t) &= 1 + 6t + 28t^2 + 77t^3 + 163t^4 + 300t^5 + 494t^6 + \dots
\end{aligned}
$$

$u_1 = (1/3, 5/3)$
$u_2 = (1, 5/3)$
$u_3 = (-1, -5/3)$
$u_4 = (-1/3, -5/3)$

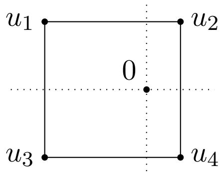
No. 1.04



$$
\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3T_4 + T_5^3 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 1 & 5 & 3 & 3 & 2 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -4 \\ 0 & 0 & 2 & 0 & -3 \end{bmatrix} \\
-\mathcal{K}_X &= 8, \qquad (-\mathcal{K}_X)^3 \;=\; 512/15 \\
H(t) &= 1 + 19t + 88t^2 + 243t^3 + 517t^4 + 945t^5 + 1561t^6 + \dots
\end{aligned}
$$

$u_1 = (-1/4, 3/4)$
$u_2 = (1/2, 3/4)$
$u_3 = (-1, -3/4)$
$u_4 = (-1/4, -3/4)$

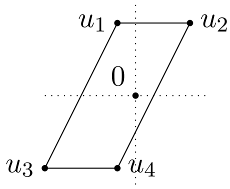
No. 1.05



$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\ldots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2+T_3T_4+T_5^4 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 1 & 3 & 2 & 2 & 1 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix} \\
-\mathcal{K}_X &= 5, \qquad (-\mathcal{K}_X)^3 \ = \ 125/3 \\
H(t) &= 1+23t+108t^2+297t^3+632t^4+1154t^5+1906t^6+\ldots
\end{aligned}$$

$u_1 = (-3/5, 2/5)$
$u_2 = (1/5, 2/5)$
$u_3 = (-3/5, -2/5)$
$u_4 = (1/5, -2/5)$

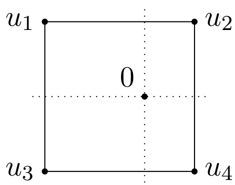
No. 1.06



$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\ldots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2+T_3T_4+T_5^4 \\
\mathrm{Cl}(X) &= \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} \\
Q &= \begin{bmatrix} 1 & 3 & 2 & 2 & 1 \\ \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{0} \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 0 & -5 \\ 0 & 0 & 2 & 0 & -4 \end{bmatrix} \\
-\mathcal{K}_X &= (5,\overline{0}), \qquad (-\mathcal{K}_X)^3 \ = \ 125/6 \\
H(t) &= 1+12t+54t^2+148t^3+316t^4+577t^5+953t^6+\ldots
\end{aligned}$$

$u_1 = (-1/5, 4/5)$
$u_2 = (3/5, 4/5)$
$u_3 = (-1, -4/5)$
$u_4 = (-1/5, -4/5)$

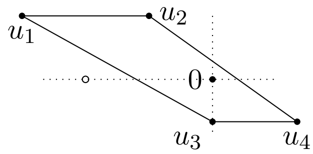
No. 1.07



$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\ldots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2+T_3T_4+T_5^6 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 2 & 4 & 3 & 3 & 1 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & -3 \end{bmatrix} \\
-\mathcal{K}_X &= 7, \qquad (-\mathcal{K}_X)^3 \ = \ 343/12 \\
H(t) &= 1+16t+74t^2+204t^3+434t^4+792t^5+1308t^6+\ldots
\end{aligned}$$

$u_1 = (-4/7, 3/7)$
$u_2 = (2/7, 3/7)$
$u_3 = (-4/7, -3/7)$
$u_4 = (2/7, -3/7)$

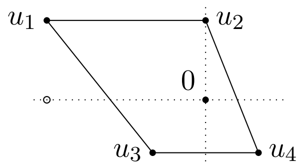
No. 1.08



$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\ldots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^2T_4 + T_5^2 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 1 & 3 & 1 & 2 & 2 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \\
-\mathcal{K}_X &= 5, \qquad (-\mathcal{K}_X)^3 = 125/3 \\
H(t) &= 1 + 23t + 108t^2 + 297t^3 + 632t^4 + 1154t^5 + 1906t^6 + \ldots
\end{aligned}$$

$u_1 = (-3/2, 1/2)$
$u_2 = (-1/2, 1/2)$
$u_3 = (0, -1/3)$
$u_4 = (2/3, -1/3)$

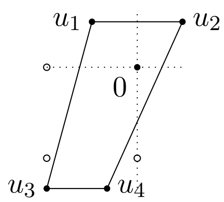
No. 1.09



$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\ldots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^2T_4 + T_5^2 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 1 & 5 & 2 & 2 & 3 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & -3 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \\
-\mathcal{K}_X &= 7, \qquad (-\mathcal{K}_X)^3 = 343/10 \\
H(t) &= 1 + 19t + 89t^2 + 244t^3 + 520t^4 + 950t^5 + 1569t^6 + \ldots
\end{aligned}$$

$u_1 = (-1, 1/2)$
$u_2 = (0, 1/2)$
$u_3 = (-1/3, -2/3)$
$u_4 = (1/3, -2/3)$

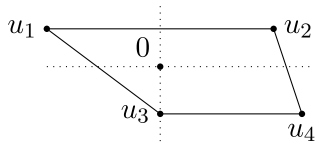
No. 1.10



$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\ldots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^2T_4 + T_5^2 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 3 & 7 & 4 & 2 & 5 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix} \\
-\mathcal{K}_X &= 11, \qquad (-\mathcal{K}_X)^3 = 1331/84 \\
H(t) &= 1 + 9t + 41t^2 + 113t^3 + 241t^4 + 439t^5 + 725t^6 + \ldots
\end{aligned}$$

$u_1 = (-1/2, 1/2)$
$u_2 = (1/2, 1/2)$
$u_3 = (-1, -4/3)$
$u_4 = (-1/3, -4/3)$

No. 1.11

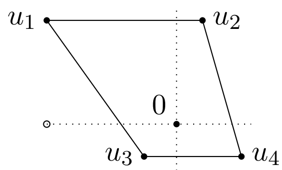


$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^2T_4 + T_5^3 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}
\end{aligned}$$

$u_1 = (-3/5, 1/5)$
$u_2 = (3/5, 1/5)$
$u_3 = (0, -1/4)$
$u_4 = (3/4, -1/4)$

$$\begin{aligned}
-\mathcal{K}_X &= 3, \qquad (-\mathcal{K}_X)^3 = 81/2 \\
H(t) &= 1 + 23t + 106t^2 + 290t^3 + 616t^4 + 1124t^5 + 1855t^6 + \dots
\end{aligned}$$

No. 1.12

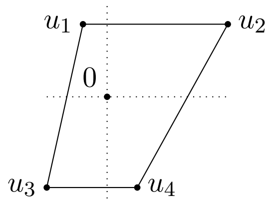


$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^2T_4 + T_5^3 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 3 & 3 & 1 & 4 & 2 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 \end{bmatrix}
\end{aligned}$$

$u_1 = (-1, 4/5)$
$u_2 = (1/5, 4/5)$
$u_3 = (-1/4, -1/4)$
$u_4 = (1/2, -1/4)$

$$\begin{aligned}
-\mathcal{K}_X &= 7, \qquad (-\mathcal{K}_X)^3 = 343/12 \\
H(t) &= 1 + 16t + 74t^2 + 204t^3 + 434t^4 + 792t^5 + 1308t^6 + \dots
\end{aligned}$$

No. 1.13

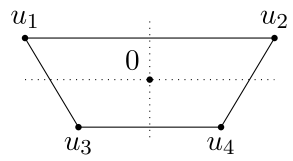


$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^2T_4 + T_5^3 \\
\mathrm{Cl}(X) &= \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \\
Q &= \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{2} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 & 3 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}
\end{aligned}$$

$u_1 = (-1/5, 3/5)$
$u_2 = (1, 3/5)$
$u_3 = (-1/2, -3/4)$
$u_4 = (1/4, -3/4)$

$$\begin{aligned}
-\mathcal{K}_X &= (3, \bar{2}), \qquad (-\mathcal{K}_X)^3 = 27/2 \\
H(t) &= 1 + 8t + 35t^2 + 97t^3 + 205t^4 + 374t^5 + 619t^6 + \dots
\end{aligned}$$

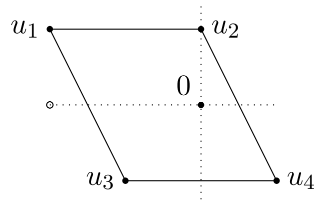
No. 1.14



$$\begin{array}{rcl}
\mathcal{R}(X) & = & \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 & = & T_1T_2 + T_3^2T_4 + T_5^6 \\
\mathrm{Cl}(X) & = & \mathbb{Z} \\
Q & = & \begin{bmatrix} 3 & 3 & 2 & 2 & 1 \end{bmatrix} \\
P & = & \begin{bmatrix} -1 & -1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -2 \end{bmatrix} \\
-\mathcal{K}_X & = & 5, \qquad (-\mathcal{K}_X)^3 \ = \ 125/6 \\
H(t) & = & 1 + 12t + 55t^2 + 150t^3 + 318t^4 + 579t^5 + 956t^6 + \dots
\end{array}$$

$u_1 = (-3/4, 1/4)$
$u_2 = (3/4, 1/4)$
$u_3 = (-3/7, -2/7)$
$u_4 = (3/7, -2/7)$

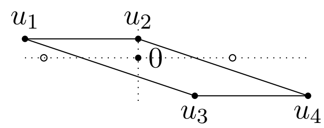
No. 1.15



$$\begin{array}{rcl}
\mathcal{R}(X) & = & \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 & = & T_1T_2 + T_3^2T_4^2 + T_5^2 \\
\mathrm{Cl}(X) & = & \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \\
Q & = & \begin{bmatrix} 1 & 3 & 1 & 1 & 2 \\ \overline{1} & \overline{1} & \overline{0} & \overline{0} & \overline{1} \end{bmatrix} \\
P & = & \begin{bmatrix} -1 & -1 & 2 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & -3 & -2 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \\
-\mathcal{K}_X & = & (4, \overline{1}), \qquad (-\mathcal{K}_X)^3 \ = \ 64/3 \\
H(t) & = & 1 + 12t + 56t^2 + 152t^3 + 324t^4 + 591t^5 + 977t^6 + \dots
\end{array}$$

$u_1 = (-1, 1/2)$
$u_2 = (0, 1/2)$
$u_3 = (-1/2, -1/2)$
$u_4 = (1/2, -1/2)$

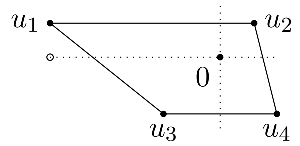
No. 1.16



$$\begin{array}{rcl}
\mathcal{R}(X) & = & \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 & = & T_1T_2 + T_3^2T_4^2 + T_5^3 \\
\mathrm{Cl}(X) & = & \mathbb{Z} \\
Q & = & \begin{bmatrix} 3 & 3 & 2 & 1 & 2 \end{bmatrix} \\
P & = & \begin{bmatrix} -1 & -1 & 2 & 2 & 0 \\ -1 & -1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -2 & 2 \end{bmatrix} \\
-\mathcal{K}_X & = & 5, \qquad (-\mathcal{K}_X)^3 \ = \ 125/6 \\
H(t) & = & 1 + 12t + 55t^2 + 150t^3 + 318t^4 + 579t^5 + 956t^6 + \dots
\end{array}$$

$u_1 = (-6/5, 1/5)$
$u_2 = (0, 1/5)$
$u_3 = (3/5, -2/5)$
$u_4 = (9/5, -2/5)$

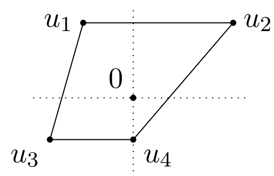
No. 1.17



$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^3T_4 + T_5^2 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 1 & 3 & 1 & 1 & 2 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 3 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} \\
-\mathcal{K}_X &= 4, \qquad (-\mathcal{K}_X)^3 \;=\; 128/3 \\
H(t) &= 1 + 24t + 111t^2 + 305t^3 + 648t^4 + 1183t^5 + 1953t^6 + \dots
\end{aligned}$$

$u_1 = (-1, 1/5)$
$u_2 = (1/5, 1/5)$
$u_3 = (-1/3, -1/3)$
$u_4 = (1/3, -1/3)$

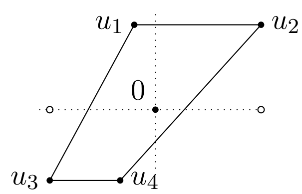
No. 1.18



$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^3T_4 + T_5^2 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 2 & 4 & 1 & 3 & 3 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 3 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \\
-\mathcal{K}_X &= 7, \qquad (-\mathcal{K}_X)^3 \;=\; 343/12 \\
H(t) &= 1 + 16t + 74t^2 + 204t^3 + 434t^4 + 792t^5 + 1308t^6 + \dots
\end{aligned}$$

$u_1 = (-2/5, 3/5)$
$u_2 = (4/5, 3/5)$
$u_3 = (-2/3, -1/3)$
$u_4 = (0, -1/3)$

No. 1.19



$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^3T_4 + T_5^2 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 3 & 7 & 2 & 4 & 5 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 3 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix} \\
-\mathcal{K}_X &= 9, \qquad (-\mathcal{K}_X)^3 \;=\; 1331/84 \\
H(t) &= 1 + 9t + 41t^2 + 113t^3 + 241t^4 + 439t^5 + 725t^6 + \dots
\end{aligned}$$

$u_1 = (-1/5, 4/5)$
$u_2 = (1, 4/5)$
$u_3 = (-1, -2/3)$
$u_4 = (-1/3, -2/3)$

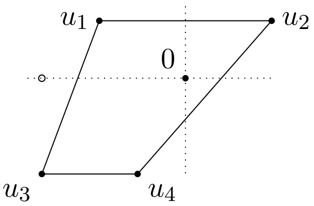
No. 1.20

$$\mathcal{R}(X) = \mathbb{K}[T_1, \dots, T_5]/\langle g_0 \rangle$$

$$g_0 = T_1T_2 + T_3^3T_4 + T_5^2$$

$$\mathrm{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

$$Q = \begin{bmatrix} 1 & 3 & 1 & 1 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$$



$$P = \begin{bmatrix} -1 & -1 & 3 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & -3 & -2 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

$u_1 = (-3/5, 2/5)$
$u_2 = (3/5, 2/5)$
$u_3 = (-1, -2/3)$
$u_4 = (-1/3, -2/3)$

$$-\mathcal{K}_X = (4, \bar{1}), \qquad (-\mathcal{K}_X)^3 = 64/3$$

$$H(t) = 1 + 12t + 56t^2 + 152t^3 + 324t^4 + 591t^5 + 977t^6 + \dots$$

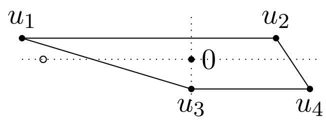
No. 1.21

$$\mathcal{R}(X) = \mathbb{K}[T_1, \dots, T_5]/\langle g_0 \rangle$$

$$g_0 = T_1T_2 + T_3^3T_4 + T_5^4$$

$$\mathrm{Cl}(X) = \mathbb{Z}$$

$$Q = \begin{bmatrix} 2 & 2 & 1 & 1 & 1 \end{bmatrix}$$



$$P = \begin{bmatrix} -1 & -1 & 3 & 1 & 0 \\ -1 & -1 & 0 & 0 & 4 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & -1 & 3 \end{bmatrix}$$

$u_1 = (-8/7, 1/7)$
$u_2 = (4/7, 1/7)$
$u_3 = (0, -1/5)$
$u_4 = (4/5, -1/5)$

$$-\mathcal{K}_X = 3, \qquad (-\mathcal{K}_X)^3 = 27$$

$$H(t) = 1 + 16t + 72t^2 + 195t^3 + 413t^4 + 752t^5 + 1240t^6 + \dots$$

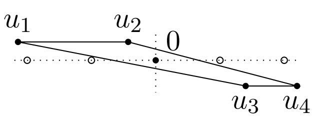
No. 1.22

$$\mathcal{R}(X) = \mathbb{K}[T_1, \dots, T_5]/\langle g_0 \rangle$$

$$g_0 = T_1T_2 + T_3^3T_4 + T_5^4$$

$$\mathrm{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

$$Q = \begin{bmatrix} 2 & 2 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{bmatrix}$$



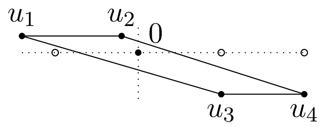
$$P = \begin{bmatrix} -1 & -1 & 3 & 1 & 0 \\ -1 & -1 & 0 & 0 & 4 \\ 0 & 1 & -3 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

$u_1 = (-15/7, 2/7)$
$u_2 = (-3/7, 2/7)$
$u_3 = (7/5, -2/5)$
$u_4 = (11/5, -2/5)$

$$-\mathcal{K}_X = (3, \bar{0}), \qquad (-\mathcal{K}_X)^3 = 27/2$$

$$H(t) = 1 + 8t + 36t^2 + 97t^3 + 207t^4 + 376t^5 + 620t^6 + \dots$$

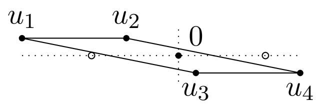
No. 1.23



$$
\begin{array}{rcl}
\mathcal{R}(X) & = & \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 & = & T_1T_2 + T_3^3T_4^2 + T_5^2 \\
\mathrm{Cl}(X) & = & \mathbb{Z} \\
Q & = & \begin{bmatrix} 3 & 5 & 2 & 1 & 4 \end{bmatrix} \\
P & = & \begin{bmatrix} -1 & -1 & 3 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & -2 & 3 & -1 \\ 0 & 0 & -1 & -2 & 1 \end{bmatrix} \\
-\mathcal{K}_X & = & 7, \qquad (-\mathcal{K}_X)^3 \ = \ 343/15 \\
H(t) & = & 1 + 13t + 60t^2 + 164t^3 + 348t^4 + 635t^5 + 1048t^6 + \dots
\end{array}
$$

$$
u_1 = (-7/5, 1/5), \quad u_2 = (-1/5, 1/5), \quad u_3 = (1, -1/2), \quad u_4 = (2, -1/2)
$$

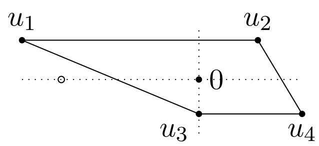
No. 1.24



$$
\begin{array}{rcl}
\mathcal{R}(X) & = & \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 & = & T_1T_2 + T_3^3T_4^3 + T_5^2 \\
\mathrm{Cl}(X) & = & \mathbb{Z} \\
Q & = & \begin{bmatrix} 2 & 4 & 1 & 1 & 3 \end{bmatrix} \\
P & = & \begin{bmatrix} -1 & -1 & 3 & 3 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & -3 & 2 & -1 \\ 0 & 0 & -1 & -2 & 1 \end{bmatrix} \\
-\mathcal{K}_X & = & 5, \qquad (-\mathcal{K}_X)^3 \ = \ 125/4 \\
H(t) & = & 1 + 18t + 82t^2 + 224t^3 + 476t^4 + 868t^5 + 1432t^6 + \dots
\end{array}
$$

$$
u_1 = (-9/5, 1/5), \quad u_2 = (-3/5, 1/5), \quad u_3 = (1/5, -1/5), \quad u_4 = (7/5, -1/5)
$$

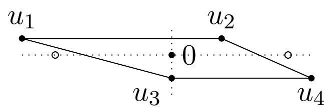
No. 1.25



$$
\begin{array}{rcl}
\mathcal{R}(X) & = & \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 & = & T_1T_2 + T_3^4T_4 + T_5^3 \\
\mathrm{Cl}(X) & = & \mathbb{Z} \\
Q & = & \begin{bmatrix} 3 & 3 & 1 & 2 & 2 \end{bmatrix} \\
P & = & \begin{bmatrix} -1 & -1 & 4 & 1 & 0 \\ -1 & -1 & 0 & 0 & 3 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & -2 & -1 & 2 \end{bmatrix} \\
-\mathcal{K}_X & = & 5, \qquad (-\mathcal{K}_X)^3 \ = \ 125/6 \\
H(t) & = & 1 + 12t + 55t^2 + 150t^3 + 318t^4 + 579t^5 + 956t^6 + \dots
\end{array}
$$

$$
u_1 = (-9/7, 2/7), \quad u_2 = (3/7, 2/7), \quad u_3 = (0, -1/4), \quad u_4 = (3/4, -1/4)
$$

No. 1.26

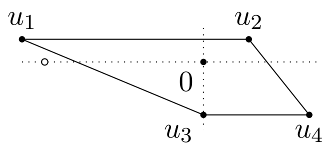


$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle$$
$$g_0 = T_1T_2 + T_3^4T_4^2 + T_5^3$$
$$\mathrm{Cl}(X) = \mathbb{Z}$$
$$Q = \begin{bmatrix} 3 & 3 & 1 & 1 & 2 \end{bmatrix}$$
$$P = \begin{bmatrix} -1 & -1 & 4 & 2 & 0 \\ -1 & -1 & 0 & 0 & 3 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$

$u_1 = (-9/7, 1/7)$
$u_2 = (3/7, 1/7)$
$u_3 = (0, -1/5)$
$u_4 = (6/5, -1/5)$

$$-\mathcal{K}_X = 4, \qquad (-\mathcal{K}_X)^3 = 64/3$$
$$H(t) = 1 + 13t + 57t^2 + 155t^3 + 327t^4 + 595t^5 + 981t^6 + \dots$$

No. 1.27

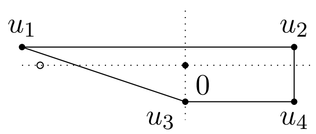


$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle$$
$$g_0 = T_1T_2 + T_3^5T_4 + T_5^2$$
$$\mathrm{Cl}(X) = \mathbb{Z}$$
$$Q = \begin{bmatrix} 2 & 4 & 1 & 1 & 3 \end{bmatrix}$$
$$P = \begin{bmatrix} -1 & -1 & 5 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 \end{bmatrix}$$

$u_1 = (-8/7, 1/7)$
$u_2 = (2/7, 1/7)$
$u_3 = (0, -1/3)$
$u_4 = (2/3, -1/3)$

$$-\mathcal{K}_X = 5, \qquad (-\mathcal{K}_X)^3 = 125/4$$
$$H(t) = 1 + 18t + 82t^2 + 224t^3 + 476t^4 + 868t^5 + 1432t^6 + \dots$$

No. 1.28

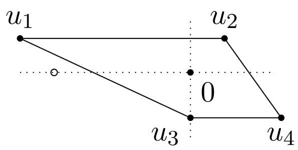


$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle$$
$$g_0 = T_1T_2 + T_3^5T_4 + T_5^3$$
$$\mathrm{Cl}(X) = \mathbb{Z}$$
$$Q = \begin{bmatrix} 3 & 3 & 1 & 1 & 2 \end{bmatrix}$$
$$P = \begin{bmatrix} -1 & -1 & 5 & 1 & 0 \\ -1 & -1 & 0 & 0 & 3 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix}$$

$u_1 = (-9/8, 1/8)$
$u_2 = (3/4, 1/8)$
$u_3 = (0, -1/4)$
$u_4 = (3/4, -1/4)$

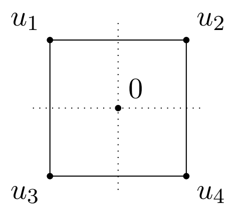
$$-\mathcal{K}_X = 4, \qquad (-\mathcal{K}_X)^3 = 64/3$$
$$H(t) = 1 + 12t + 55t^2 + 150t^3 + 318t^4 + 579t^5 + 956t^6 + \dots$$

No. 1.29



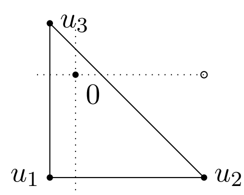
$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle$$
$$g_0 = T_1T_2 + T_3^6T_4 + T_5^2$$
$$\mathrm{Cl}(X) = \mathbb{Z}$$
$$Q = \begin{bmatrix} 3 & 5 & 1 & 2 & 4 \end{bmatrix}$$
$$P = \begin{bmatrix} -1 & -1 & 6 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 4 & 0 & -1 \end{bmatrix}$$
$u_1 = (-5/4, 1/4)$
$u_2 = (1/4, 1/4)$
$u_3 = (0, -1/3)$
$u_4 = (2/3, -1/3)$
$$-\mathcal{K}_X = 7, \qquad (-\mathcal{K}_X)^3 = 343/15$$
$$H(t) = 1 + 13t + 60t^2 + 164t^3 + 348t^4 + 635t^5 + 1048t^6 + \dots$$

No. 1.30



$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_6]/\langle g_0, g_1\rangle$$
$$g_0 = T_1T_2 + T_3T_4 + T_5^2$$
$$g_1 = aT_3T_4 + T_5^2 + T_6^2$$
$$\mathrm{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$
$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$$
$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{bmatrix}$$
$u_1 = (-1/2, 1/2)$
$u_2 = (1/2, 1/2)$
$u_3 = (-1/2, -1/2)$
$u_4 = (1/2, -1/2)$
$$-\mathcal{K}_X = (2, \bar{1}), \qquad (-\mathcal{K}_X)^3 = 16$$
$$H(t) = 1 + 9t + 43t^2 + 115t^3 + 245t^4 + 445t^5 + 735t^6 + \dots$$

No. 1.31



$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle$$
$$g_0 = T_1T_2T_3 + T_4^3 + T_5^2$$
$$\mathrm{Cl}(X) = \mathbb{Z}$$
$$Q = \begin{bmatrix} 1 & 1 & 4 & 2 & 3 \end{bmatrix}$$
$$P = \begin{bmatrix} -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & -1 & 0 & 2 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}$$
$u_1 = (-1/5, -4/5)$
$u_2 = (1, -4/5)$
$u_3 = (-1/5, 2/5)$
$$-\mathcal{K}_X = 5, \qquad (-\mathcal{K}_X)^3 = 125/4$$
$$H(t) = 1 + 18t + 82t^2 + 224t^3 + 476t^4 + 868t^5 + 1432t^6 + \dots$$

No. 1.32

$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle$$

$$g_0 = T_1T_2T_3 + T_4^3 + T_5^2$$

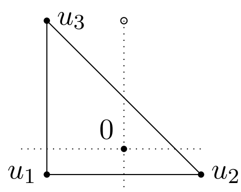
$$\mathrm{Cl}(X) = \mathbb{Z}$$

$$Q = \begin{bmatrix} 2 & 3 & 1 & 2 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & -1 & 0 & 2 \\ 0 & 1 & 0 & -3 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

$$-\mathcal{K}_X = 5, \qquad (-\mathcal{K}_X)^3 = 125/6$$

$$H(t) = 1 + 12t + 55t^2 + 150t^3 + 318t^4 + 579t^5 + 956t^6 + \dots$$



$u_1 = (-3/5, -1/5)$
$u_2 = (3/5, -1/5)$
$u_3 = (-3/5, 1)$

No. 1.33

$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle$$

$$g_0 = T_1T_2 + T_3^2 + T_4^2$$

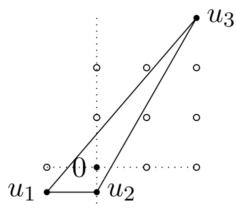
$$\mathrm{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

$$Q = \begin{bmatrix} 2 & 4 & 3 & 3 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 1 & 2 \\ 0 & 0 & -2 & 1 & 3 \end{bmatrix}$$

$$-\mathcal{K}_X = (7, \bar{1}), \qquad (-\mathcal{K}_X)^3 = 343/24$$

$$H(t) = 1 + 8t + 37t^2 + 102t^3 + 217t^4 + 396t^5 + 654t^6 + \dots$$



$u_1 = (-1, -1/2)$
$u_2 = (0, -1/2)$
$u_3 = (2, 3)$

No. 1.34

$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle$$

$$g_0 = T_1T_2 + T_3^3 + T_4^2$$

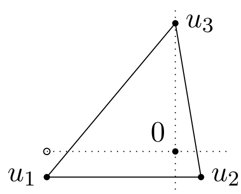
$$\mathrm{Cl}(X) = \mathbb{Z}$$

$$Q = \begin{bmatrix} 1 & 5 & 2 & 3 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & 3 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & -2 & 1 & 1 \end{bmatrix}$$

$$-\mathcal{K}_X = 6, \qquad (-\mathcal{K}_X)^3 = 216/5$$

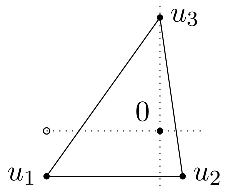
$$H(t) = 1 + 24t + 112t^2 + 308t^3 + 655t^4 + 1197t^5 + 1976t^6 + \dots$$



$u_1 = (-1, -1/5)$
$u_2 = (1/5, -1/5)$
$u_3 = (0, 1)$

No. 1.35

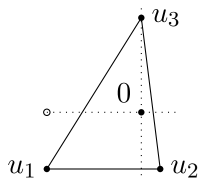
$$
\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^3 + T_4^2 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 1 & 5 & 2 & 3 & 2 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 3 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \\
-\mathcal{K}_X &= 7, \qquad (-\mathcal{K}_X)^3 \;=\; 343/10 \\
H(t) &= 1 + 19t + 89t^2 + 244t^3 + 520t^4 + 950t^5 + 1569t^6 + \dots
\end{aligned}
$$



$u_1 = (-1, -2/5)$
$u_2 = (1/5, -2/5)$
$u_3 = (0, 1)$

No. 1.36

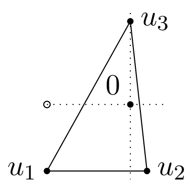
$$
\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^3 + T_4^2 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 1 & 5 & 2 & 3 & 3 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 3 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & -3 & 1 & 1 \end{bmatrix} \\
-\mathcal{K}_X &= 8, \qquad (-\mathcal{K}_X)^3 \;=\; 512/15 \\
H(t) &= 1 + 19t + 88t^2 + 243t^3 + 517t^4 + 945t^5 + 1561t^6 + \dots
\end{aligned}
$$



$u_1 = (-1, -3/5)$
$u_2 = (1/5, -3/5)$
$u_3 = (0, 1)$

No. 1.37

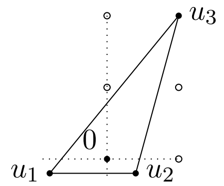
$$
\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^3 + T_4^2 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 1 & 5 & 2 & 3 & 4 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 3 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & -2 & 0 & 1 \end{bmatrix} \\
-\mathcal{K}_X &= 9, \qquad (-\mathcal{K}_X)^3 \;=\; 729/20 \\
H(t) &= 1 + 20t + 94t^2 + 259t^3 + 552t^4 + 1009t^5 + 1666t^6 + \dots
\end{aligned}
$$



$u_1 = (-1, -4/5)$
$u_2 = (1/5, -4/5)$
$u_3 = (0, 1)$

No. 1.38

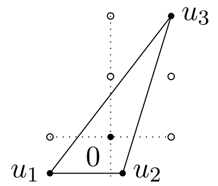
$$\mathcal{R}(X) = \mathbb{K}[T_1,\ldots,T_5]/\langle g_0\rangle$$
$$g_0 = T_1T_2 + T_3^3 + T_4^2$$
$$\mathrm{Cl}(X) = \mathbb{Z}$$
$$Q = \begin{bmatrix} 5 & 7 & 4 & 6 & 1 \end{bmatrix}$$
$$P = \begin{bmatrix} -1 & -1 & 3 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & -2 & 1 & 2 \end{bmatrix}$$
$$-\mathcal{K}_X = 11, \qquad (-\mathcal{K}_X)^3 = 1331/70$$
$$H(t) = 1 + 11t + 50t^2 + 136t^3 + 289t^4 + 528t^5 + 871t^6 + \ldots$$



$u_1 = (-4/5, -1/5)$
$u_2 = (2/5, -1/5)$
$u_3 = (1, 2)$

No. 1.39

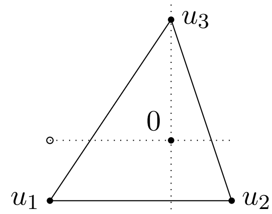
$$\mathcal{R}(X) = \mathbb{K}[T_1,\ldots,T_5]/\langle g_0\rangle$$
$$g_0 = T_1T_2 + T_3^3 + T_4^2$$
$$\mathrm{Cl}(X) = \mathbb{Z}$$
$$Q = \begin{bmatrix} 5 & 7 & 4 & 6 & 3 \end{bmatrix}$$
$$P = \begin{bmatrix} -1 & -1 & 3 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 0 & -3 & 1 & 2 \end{bmatrix}$$
$$-\mathcal{K}_X = 13, \qquad (-\mathcal{K}_X)^3 = 2197/210$$
$$H(t) = 1 + 6t + 27t^2 + 75t^3 + 159t^4 + 290t^5 + 479t^6 + \ldots$$



$u_1 = (-1, -3/5)$
$u_2 = (1/5, -3/5)$
$u_3 = (1, 2)$

No. 1.40

$$\mathcal{R}(X) = \mathbb{K}[T_1,\ldots,T_5]/\langle g_0\rangle$$
$$g_0 = T_1T_2 + T_3^3 + T_4^3$$
$$\mathrm{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$$
$$Q = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ \bar{1} & \bar{2} & \bar{2} & \bar{0} & \bar{0} \end{bmatrix}$$
$$P = \begin{bmatrix} -1 & -1 & 3 & 0 & 0 \\ -1 & -1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 0 & -3 & 2 & 1 \end{bmatrix}$$
$$-\mathcal{K}_X = (3, \bar{2}), \qquad (-\mathcal{K}_X)^3 = 27/2$$
$$H(t) = 1 + 8t + 35t^2 + 97t^3 + 205t^4 + 374t^5 + 619t^6 + \ldots$$



$u_1 = (-1, -1/2)$
$u_2 = (1/2, -1/2)$
$u_3 = (0, 1)$

No. 1.41

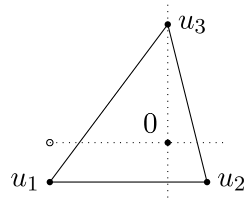
$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle$$

$$g_0 = T_1T_2 + T_3^4 + T_4^2$$

$$\mathrm{Cl}(X) = \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$$

$$Q = \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & 4 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -5 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$



$u_1 = (-1,-1/3)$
$u_2 = (1/3,-1/3)$
$u_3 = (0,1)$

$$-\mathcal{K}_X = (4,\bar{1}), \qquad (-\mathcal{K}_X)^3 = 64/3$$

$$H(t) = 1 + 12t + 56t^2 + 152t^3 + 324t^4 + 591t^5 + 977t^6 + \dots$$

No. 1.42

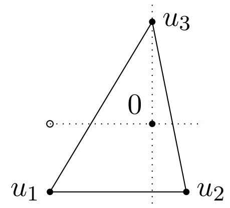
$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle$$

$$g_0 = T_1T_2 + T_3^4 + T_4^2$$

$$\mathrm{Cl}(X) = \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$$

$$Q = \begin{bmatrix} 1 & 3 & 1 & 2 & 2 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & 4 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -5 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 \end{bmatrix}$$



$u_1 = (-1,-2/3)$
$u_2 = (1/3,-2/3)$
$u_3 = (0,1)$

$$-\mathcal{K}_X = (5,\bar{0}), \qquad (-\mathcal{K}_X)^3 = 125/6$$

$$H(t) = 1 + 12t + 54t^2 + 148t^3 + 316t^4 + 577t^5 + 953t^6 + \dots$$

No. 1.43

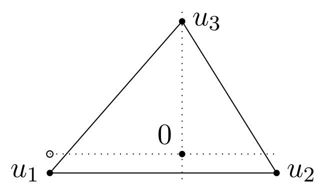
$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_5]/\langle g_0\rangle$$

$$g_0 = T_1T_2 + T_3^4 + T_4^3$$

$$\mathrm{Cl}(X) = \mathbb{Z}$$

$$Q = \begin{bmatrix} 5 & 7 & 3 & 4 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & 4 & 0 & 0 \\ -1 & -1 & 0 & 3 & 0 \\ 0 & 1 & -5 & 2 & 0 \\ 0 & 0 & -3 & 2 & 1 \end{bmatrix}$$

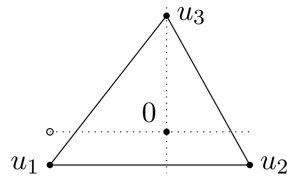


$u_1 = (-1,-1/7)$
$u_2 = (5/7,-1/7)$
$u_3 = (0,1)$

$$-\mathcal{K}_X = 8, \qquad (-\mathcal{K}_X)^3 = 512/35$$

$$H(t) = 1 + 9t + 39t^2 + 106t^3 + 224t^4 + 408t^5 + 672t^6 + \dots$$

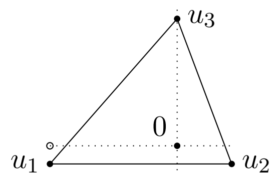
No. 1.44



$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\ldots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^4 + T_4^3 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 5 & 7 & 3 & 4 & 2 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 4 & 0 & 0 \\ -1 & -1 & 0 & 3 & 0 \\ 0 & 1 & -5 & 2 & 0 \\ 0 & 0 & -2 & 1 & 1 \end{bmatrix} \\
-\mathcal{K}_X &= 9, \qquad (-\mathcal{K}_X)^3 \;=\; 729/70 \\
H(t) &= 1 + 6t + 28t^2 + 75t^3 + 159t^4 + 290t^5 + 478t^6 + \ldots
\end{aligned}$$

$u_1 = (-1, -2/7)$
$u_2 = (5/7, -2/7)$
$u_3 = (0, 1)$

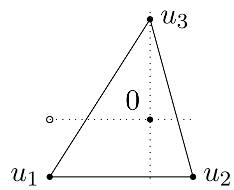
No. 1.45



$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\ldots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^5 + T_4^2 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 3 & 7 & 2 & 5 & 1 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 5 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -6 & 1 & 0 \\ 0 & 0 & -3 & 1 & 1 \end{bmatrix} \\
-\mathcal{K}_X &= 8, \qquad (-\mathcal{K}_X)^3 \;=\; 512/21 \\
H(t) &= 1 + 14t + 64t^2 + 175t^3 + 371t^4 + 677t^5 + 1117t^6 + \ldots
\end{aligned}$$

$u_1 = (-1, -1/7)$
$u_2 = (3/7, -1/7)$
$u_3 = (0, 1)$

No. 1.46



$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\ldots,T_5]/\langle g_0\rangle \\
g_0 &= T_1T_2 + T_3^5 + T_4^2 \\
\mathrm{Cl}(X) &= \mathbb{Z} \\
Q &= \begin{bmatrix} 3 & 7 & 2 & 5 & 4 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 5 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -6 & 1 & 0 \\ 0 & 0 & -2 & 0 & 1 \end{bmatrix} \\
-\mathcal{K}_X &= 11, \qquad (-\mathcal{K}_X)^3 \;=\; 1331/84 \\
H(t) &= 1 + 9t + 41t^2 + 113t^3 + 241t^4 + 439t^5 + 725t^6 + \ldots
\end{aligned}$$

$u_1 = (-1, -4/7)$
$u_2 = (3/7, -4/7)$
$u_3 = (0, 1)$

No. 1.47

$$\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_5]/\langle g_0 \rangle$$

$$g_0 = T_1T_2 + T_3^6 + T_4^2$$

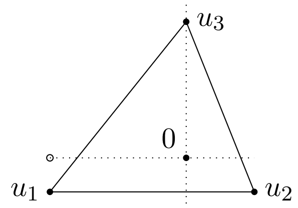
$$\mathrm{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

$$Q = \begin{bmatrix} 2 & 4 & 1 & 3 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & 6 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -7 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 \end{bmatrix}$$

$$-\mathcal{K}_X = (5, \bar{1}), \qquad (-\mathcal{K}_X)^3 = 125/8$$

$$H(t) = 1 + 9t + 41t^2 + 112t^3 + 238t^4 + 434t^5 + 716t^6 + \ldots$$



$u_1 = (-1, -1/4)$
$u_2 = (1/2, -1/4)$
$u_3 = (0, 1)$

No. 2.01

$$\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_6]/\langle g_0 \rangle$$

$$g_0 = T_1T_2 + T_3T_4 + T_5T_6$$

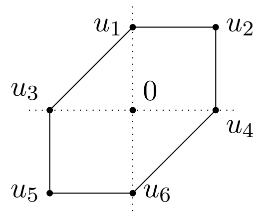
$$\mathrm{Cl}(X) = \mathbb{Z}^2$$

$$Q = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$-\mathcal{K}_X = (2, 2)$$

$$H(t) = 1 + 27t + 125t^2 + 343t^3 + 729t^4 + 1331t^5 + 2197t^6 + \ldots$$



$u_1 = (0, 1/2)$
$u_2 = (1/2, 1/2)$
$u_3 = (-1/2, 0)$
$u_4 = (1/2, 0)$
$u_5 = (-1/2, -1/2)$
$u_6 = (0, -1/2)$

No. 2.02

$$\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_6]/\langle g_0 \rangle$$

$$g_0 = T_1T_2^2 + T_3T_4^2 + T_5T_6^2$$

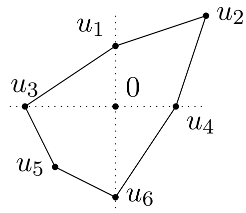
$$\mathrm{Cl}(X) = \mathbb{Z}^2$$

$$Q = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -2 & 1 & 2 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$-\mathcal{K}_X = (2, 1)$$

$$H(t) = 1 + 18t + 80t^2 + 217t^3 + 459t^4 + 836t^5 + 1378t^6 + \ldots$$



$u_1 = (0, 1/3)$
$u_2 = (1/2, 1/2)$
$u_3 = (-1/2, 0)$
$u_4 = (1/3, 0)$
$u_5 = (-1/3, -1/3)$
$u_6 = (0, -1/2)$

No. 2.03

$$\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_6]/\langle g_0 \rangle$$

$$g_0 = T_1T_2 + T_3T_4 + T_5T_6$$

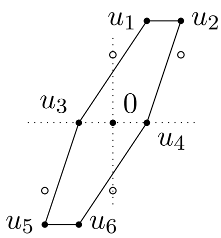
$$\mathrm{Cl}(X) = \mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$$

$$Q = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \overline{2} & \overline{1} & \overline{1} & \overline{2} & \overline{0} & \overline{0} \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & -3 & 0 & 3 \end{bmatrix}$$

$$-\mathcal{K}_X = (2, 2, \overline{0})$$

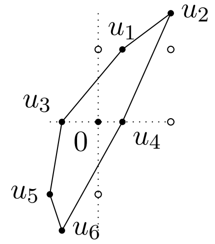
$$H(t) = 1 + 9t + 41t^2 + 115t^3 + 243t^4 + 443t^5 + 733t^6 + \ldots$$



$u_1 = (1/2, 3/2)$
$u_2 = (1, 3/2)$
$u_3 = (-1/2, 0)$
$u_4 = (1/2, 0)$
$u_5 = (-1, -3/2)$
$u_6 = (-1/2, -3/2)$

No. 2.04

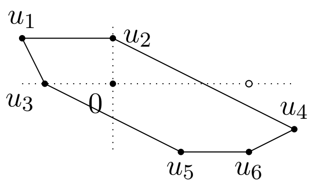
$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\dots,T_6]/\langle g_0\rangle \\
g_0 &= T_1T_2^2+T_3T_4^2+T_5T_6^2 \\
\mathrm{Cl}(X) &= \mathbb{Z}^2\oplus\mathbb{Z}/3\mathbb{Z} \\
Q &= \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{0} & \bar{0} \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -2 & 1 & 2 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 2 \\ 0 & -2 & 0 & 1 & 0 & 1 \\ 0 & -3 & 0 & 3 & 0 & 0 \end{bmatrix} \\
-\mathcal{K}_X &= (2,1,\bar{0}) \\
H(t) &= 1+6t+26t^2+73t^3+153t^4+278t^5+460t^6+\dots
\end{aligned}$$



$u_1=(1/3,1)$
$u_2=(1,3/2)$
$u_3=(-1/2,0)$
$u_4=(1/3,0)$
$u_5=(-2/3,-1)$
$u_6=(-1/2,-3/2)$

No. 2.05

$$\begin{aligned}
\mathcal{R}(X) &= \mathbb{K}[T_1,\dots,T_6]/\langle g_0\rangle \\
g_0 &= T_1T_2+T_3^2T_4+T_5T_6 \\
\mathrm{Cl}(X) &= \mathbb{Z}^2 \\
Q &= \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \\
P &= \begin{bmatrix} -1 & -1 & 2 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \\
-\mathcal{K}_X &= (3,2) \\
H(t) &= 1+23t+107t^2+293t^3+623t^4+1137t^5+1877t^6+\dots
\end{aligned}$$



$u_1=(-2/3,1/3)$
$u_2=(0,1/3)$
$u_3=(-1/2,0)$
$u_4=(4/3,-1/3)$
$u_5=(1/2,-1/2)$
$u_6=(1,-1/2)$

No. 2.06

$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_6]/\langle g_0\rangle$$

$$g_0 = T_1T_2 + T_3T_4^2 + T_5^2T_6^2$$

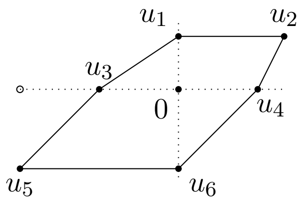
$$\mathrm{Cl}(X) = \mathbb{Z}^2$$

$$Q = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & 1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$-\mathcal{K}_X = (3,2)$$

$$H(t) = 1 + 16t + 74t^2 + 201t^3 + 427t^4 + 778t^5 + 1284t^6 + \dots$$



$u_1 = (0, 1/3)$
$u_2 = (2/3, 1/3)$
$u_3 = (-1/2, 0)$
$u_4 = (1/2, 0)$
$u_5 = (-1, -1/2)$
$u_6 = (0, -1/2)$

No. 2.07

$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_6]/\langle g_0\rangle$$

$$g_0 = T_1T_2T_3 + T_4T_5 + T_6^2$$

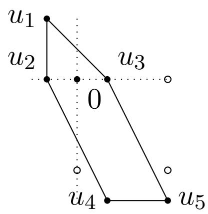
$$\mathrm{Cl}(X) = \mathbb{Z}^2$$

$$Q = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -2 & 0 \end{bmatrix}$$

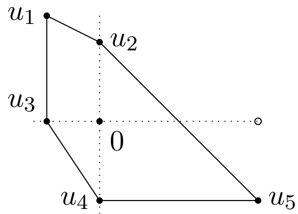
$$-\mathcal{K}_X = (3,3)$$

$$H(t) = 1 + 19t + 88t^2 + 241t^3 + 513t^4 + 936t^5 + 1545t^6 + \dots$$



$u_1 = (-1/3, 2/3)$
$u_2 = (-1/3, 0)$
$u_3 = (1/3, 0)$
$u_4 = (1/3, -4/3)$
$u_5 = (1, -4/3)$

No. 2.08



$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_6]/\langle g_0\rangle$$

$$g_0 = T_1T_2T_3 + T_4T_5^2 + T_6^2$$

$$\mathrm{Cl}(X) = \mathbb{Z}^2$$

$$Q = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

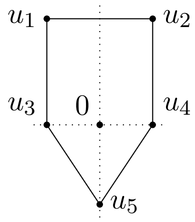
$$P = \begin{bmatrix} -1 & -1 & -1 & 1 & 2 & 0 \\ -1 & -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

$$-\mathcal{K}_X = (3,2)$$

$$H(t) = 1 + 16t + 74t^2 + 201t^3 + 427t^4 + 778t^5 + 1284t^6 + \dots$$

$u_1 = (-1/3, 2/3)$
$u_2 = (0, 1/2)$
$u_3 = (-1/3, 0)$
$u_4 = (0, -1/2)$
$u_5 = (1, -1/2)$

No. 2.09



$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_6]/\langle g_0\rangle$$

$$g_0 = T_1T_2T_3^2 + T_4T_5 + T_6^2$$

$$\mathrm{Cl}(X) = \mathbb{Z}^2$$

$$Q = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

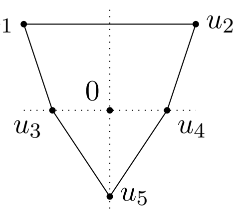
$$P = \begin{bmatrix} -1 & -1 & -2 & 1 & 1 & 0 \\ -1 & -1 & -2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

$$-\mathcal{K}_X = (3,2)$$

$$H(t) = 1 + 23t + 106t^2 + 290t^3 + 616t^4 + 1124t^5 + 1855t^6 + \dots$$

$u_1 = (-1/3, 2/3)$
$u_2 = (1/3, 2/3)$
$u_3 = (-1/3, 0)$
$u_4 = (1/3, 0)$
$u_5 = (0, -1/2)$

No. 2.10



$$\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_6]/\langle g_0\rangle$$

$$g_0 = T_1T_2T_3^2 + T_4T_5^2 + T_6^2$$

$$\mathrm{Cl}(X) = \mathbb{Z}^2$$

$$Q = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

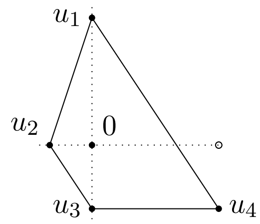
$$P = \begin{bmatrix} -1 & -1 & -2 & 1 & 2 & 0 \\ -1 & -1 & -2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

$$-\mathcal{K}_X = (3,1)$$

$$H(t) = 1 + 16t + 72t^2 + 195t^3 + 413t^4 + 752t^5 + 1240t^6 + \dots$$

$u_1 = (-1/2, 1/2)$
$u_2 = (1/2, 1/2)$
$u_3 = (-1/3, 0)$
$u_4 = (1/3, 0)$
$u_5 = (0, -1/2)$

No. 2.11



$$\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_6]/\langle g_0 \rangle$$

$$g_0 = T_1T_2 + T_3^2T_4 + T_5^2$$

$$\mathrm{Cl}(X) = \mathbb{Z}^2$$

$$Q = \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 0 \end{bmatrix}$$

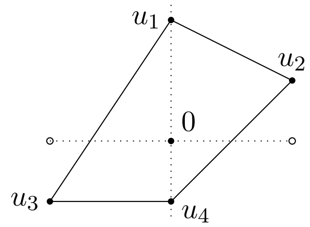
$$P = \begin{bmatrix} -1 & -1 & 2 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$u_1 = (0, 1)$
$u_2 = (-1/3, 0)$
$u_3 = (0, -1/2)$
$u_4 = (1, -1/2)$

$$-\mathcal{K}_X = (3, 3)$$

$$H(t) = 1 + 19t + 88t^2 + 241t^3 + 513t^4 + 936t^5 + 1545t^6 + \ldots$$

No. 2.12



$$\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_6]/\langle g_0 \rangle$$

$$g_0 = T_1T_2T_3 + T_4^2 + T_5^2$$

$$\mathrm{Cl}(X) = \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$$

$$Q = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & -1 & 0 & 2 & 0 \\ 0 & 2 & 1 & -3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$u_1 = (0, 1)$
$u_2 = (1, 1/2)$
$u_3 = (-1, -1/2)$
$u_4 = (0, -1/2)$

$$-\mathcal{K}_X = (2, 3, \bar{1})$$

$$H(t) = 1 + 8t + 37t^2 + 100t^3 + 214t^4 + 389t^5 + 642t^6 + \ldots$$

A.1 The Hilbert series of a Fano variety

Construction A.1.1. Let $R = \oplus_{w \in K} R_w$ be an integral affine $\mathbb{K}$-algebra graded by a finitely generated abelian group K. The *weight cone* of R is the convex cone $\omega(R) \subseteq K_{\mathbb{Q}}$ generated by all weights $w \in K$ with $R_w \neq 0$. We call the grading *pointed* if $R_0 = \mathbb{K}$ holds and $\omega(R)$ is a pointed cone.
In this setting we define the *Hilbert function* F as

$$F : K \to \mathbb{Z}_{\geq 0}, \qquad w \mapsto \dim_{\mathbb{K}}(R_w).$$

Fix a degree $w \in K$. The *w-truncated Hilbert series* of R is the formal series

$$H_w(t) := \sum_{n=0}^{\infty} F(nw)t^n.$$

Now consider a Fano variety X with finitely generated Cox ring and pointed grading. The *Hilbert series* $H(t)$ of X is defined as the $(-\mathcal{K}_X)$-truncated Hilbert series of the Cox ring $\mathcal{R}(X)$ of X, i.e.

$$H(t) := H_{-\mathcal{K}_X}(t) = \sum_{n=0}^{\infty} F(-n\mathcal{K}_X)t^n = \sum_{n=0}^{\infty} \dim(\mathcal{R}(X)_{-n\mathcal{K}_X})t^n.$$

Remark A.1.2. Some of the Fano varieties X listed in this Appendix share the same Hilbert series. Moreover, in some cases we retrieve the same Hilbert series of a toric Fano threefold; these can be found in the Graded Ring Database [13]. Here we give all cases of coinciding Hilbert series within our classifications and two examples of toric Fano threefolds that share their Hilbert series with one of our classified varieties:

- 1.02 and 1.37;
- 1.04 and 1.36;
- 1.05 and 1.08;
- 1.06 and 1.42;
- 1.07, 1.12 and 1.18;
- 1.09 and 1.35;
- 1.10, 1.19 and 1.46;
- 1.11 and 2.09;
- 1.13 and 1.40;
- 1.14, 1.16, 1.25 and 1.32;
- 1.15, 1.20 and 1.41;
- 1.21 and 2.10;
- 1.23 and 1.29;
- 1.24, 1.27 and 1.31;
- 1.26 and 1.28;
- 2.06 and 2.08;
- 2.07 and 2.11;
- 1.01 and $\mathbb{P}_1 \times \mathbb{P}_2$;
- 2.01 and $(\mathbb{P}_1)^3$.

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