Quotients of Mori Dream Spaces

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CONTENTS

Introduction 5

Chapter 1. Basic Notation 13
1.1. The Cox Ring 13
1.2. Geometry of the Cox Construction 15
1.3. The toric Cox Construction 17
1.4. Bunched Rings 19
1.5. The GKZ-Decomposition 22

Chapter 2. Quotients 27
2.1. Variation of GIT-Quotients 27
2.2. The GIT-Limit and the Limit Quotient 31
2.3. The Chow Quotient 37
2.4. Comparing Chow and Limit Quotient 39
2.5. The non-reductive GIT-Limit 43

Chapter 3. Cox Rings and Good Quotients 49
3.1. Good Quotients of Mori Dream Spaces 49
3.2. Proof of Theorem 3.1.2 50
3.3. Proof of Theorem 3.1.1 52

Chapter 4. Cox Rings and Blow-ups 57
4.1. Toric ambient Modifications 57
4.2. On blowing up the Diagonal 62
4.3. Proofs of Theorems 4.2.1 and 4.2.2 63

Chapter 5. Chow Quotients of Quadrics 75
5.1. The Cox Ring of the Chow Quotient 75
5.2. Proof of Theorem 5.1.3 77

5.3. Proof of Theorem 5.1.2 83

Chapter 6. Point Configurations and Translations 91
6.1. A Compactification of the non-reductive Limit Quotient 91
6.2. Point Configurations on $\mathbb{P}_1$ and Translations 93
6.3. Combinatorial Blow-ups 101
6.4. Stellar Subdivisions and Blow-ups 108
6.5. The Limit Quotient as Blow-up 110

Index 124

Bibliography 129

INTRODUCTION

In the present thesis we study quotient spaces arising from group actions on algebraic varieties. In general, it is not evident how to assign such a quotient to the action of an algebraic group. There are different approaches to accomplishing this in a canonical manner including the Mumford quotients, the GIT-limit, the closely related limit quotient and the Chow quotient. We examine these quotients with regard to their Cox rings and we enquire how they arise from simpler varieties by blowing up a sequence of subspaces.

We consider the action of an affine-algebraic group G on the normal algebraic variety X. In the 1960s Mumford studied the concept of *good quotients*, see [**55**]. As their name suggests these quotients have various neat properties, for example they parameterise the collection of closed orbits. Also, for affine X and reductive G they always exist. To obtain quotients for projective X it is, however, more reasonable to pass to the open G-invariant subsets of X. In general, there exists a multitude of such sets admitting a good quotient, yet none of them is canonical in any way. This drawback is overcome by the construction of the GIT-limit. A certain subcollection of the Mumford quotients forms an inverse sytem and the *GIT-limit* is the inverse limit of this system, see [**31**]. It possesses a canonical component, the *limit quotient*.

The Chow quotient constitutes an entirely different approach of devising a canonical quotient. It was introduced for toric varieties by Kapranov, Sturmfels and Zelevinsky in [**49**] and more generally by Kapranov in [**50**]. One first considers the Chow variety, which parameterises certain subvarieties of X. The closures of the general G-orbits then define a certain subset of the Chow variety. Taking the closure of this set yields the *Chow quotient*. This construction is more canonical and essentially independent of any choices, however, the Chow variety and thereby the Chow quotient are fairly hard to access.

In general, the Chow quotient and the GIT-limit do not coincide [**50**]. But for subtorus actions on toric varieties the Chow quotient and the limit quotient share a common normalisation which was shown by Kapranov, Sturmfels and Zelevinsky in [**49**], see also the papers of Craw, Maclagan and Hu [**23, 45**] for generalisations. As a first result we extend this theorem to the case of a torus action on non-toric varieties, see Theorem 2.4.2 and Corollary 2.4.3.

One approach to understand the geometry of a certain space is to determine its Cox ring. For toric varieties the *Cox ring* was introduced by Cox, see [**22**]. Later it was generalised to non-toric varieties, see [**13, 46**]. Once the Cox ring of a projective variety is known, many of the geometric invariants can be read off. Moreover, it provides a canonical embedding into a toric variety, which allows explicit computations. All these methods work well in the case of a finitely generated Cox ring. Thus it is interesting which of the quotient spaces have this property. As a first result in this direction we give a positive answer in the case of good quotients, see Theorem 3.1.1 and [**10**].

Theorem. *Suppose that G is reductive and X has a finitely generated Cox ring. If an open subset $U \subseteq X$ admits a good quotient $U \to U /\!\!/ G$, then also $U /\!\!/ G$ has a finitely generated Cox ring.*

The Chow quotient possesses proper, birational morphisms onto the Mumford quotients of maximal dimension [**45**], hence it arises as a blow-up of these. Our first task is to give a precise description of these blow-ups. As we are also interested in the Cox ring, it is now a natural thing to ask how the Cox ring changes in this process. There does not seem to be an easy answer to this question; it is even hard to decide in which cases finite generation of the Cox ring is preserved.

Interestingly, the first counterexamples were constructed as invariant algebras of unipotent groups. Nagata showed in [**56**] that there exists an action of the unipotent group $\mathbb{G}_a^{13}$ on $\mathbb{K}^{32}$ such that the invariant algebra is not finitely generated. Later, Mukai related this invariant algebra to the Cox ring of the blow-up of $\mathbb{P}_2$ in a certain point configuration [**54**]. He also provided a criterion under which the Cox ring of such a blow-up remains finitely generated. The results were further strengthened by Castravet and Tevelev in [**20**], who determined how many points in general position in a product of projective spaces could be blown-up without losing finite generation of the Cox ring. Other results in this direction can be found in [**19, 20, 32, 40, 41, 58**].

In order to compute the Cox rings of certain blow-ups, we suitably refine the technique of *toric ambient modifications*, which was proposed in [**39**]. As an application we consider torus actions on smooth projective quadrics. Note that by our Reduction Theorem 2.4.5 an essential step is understanding

$\mathbb{K}^*$-actions. For these we obtain the following results, see Theorems 5.1.2 and 5.1.3.

Theorem. *If $\mathbb{K}^*$ acts on a smooth, projective quadric X, then the normalised Chow quotient $X \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} \mathbb{K}^*$ has a finitely generated Cox ring.*

For a number of cases we give an explicit description of the Cox ring in terms of generators and relations. After applying a suitable linear transformation, the smooth projective quadric X is of the following shape:

$$X \;=\; V(g_1) \;\subseteq\; \mathbb{P}_r, \qquad g_1 \;=\; \begin{cases} T_0T_1 + \ldots + T_{r-1}T_r, & r \text{ odd},\\ T_0T_1 + \ldots + T_{r-2}T_{r-1} + T_r^2, & r \text{ even},\end{cases}$$

where the $\mathbb{K}^*$-action is diagonal with weights $\zeta_0, \ldots, \zeta_r$ and the defining equation is of degree zero. Consider an integral matrix P such that

$$Q \cdot P^t \;=\; 0, \qquad \text{where} \qquad Q \;:=\; \begin{bmatrix} \zeta_0 & \ldots & \zeta_r \\ 1 & \ldots & 1 \end{bmatrix}.$$

We set Σ as the Gelfand-Kapranov-Zelevinsky decomposition associated to P and put the primitive generators $b_1, \ldots, b_l$ of Σ differing from the columns of P as columns into a matrix B. Then there is an integral matrix A such that $B = P \cdot A$ holds. We define shifted row sums

$$\eta_i := A_{i\,*} + A_{i+1\,*} + \mu \quad \text{for } i = 0, 2, \ldots; \qquad \eta_r := 2A_{r\,*} + \mu, \quad \text{if } r \text{ is even},$$

where μ is the componentwise minimal vector such that the entries of the η_i are all nonnegative. Then our result reads as follows.

Theorem. *In the above setting, assume that any r columns of Q generate $\mathbb{Z}^2$ and that for odd (even) r there are at least four (three) weights ζ_i of minimal absolute value. Then the normalised Chow quotient $X \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} \mathbb{K}^*$ has Cox ring*

$$\mathcal{R}(X \mathbin{\tilde{/\!\!/}_{\mathrm{CQ}}} \mathbb{K}^*) \;=\; \mathbb{K}[T_0, \ldots, T_r, S_1, \ldots, S_l] \;/\; \langle g_2 \rangle$$

with

$$g_2 \;:=\; \begin{cases} T_0T_1S^{\eta_0} + T_2T_3S^{\eta_2} + \ldots + T_{r-1}T_rS^{\eta_{r-1}}, & r \text{ odd},\\ T_0T_1S^{\eta_0} + \ldots + T_{r-2}T_{r-1}S^{\eta_{r-2}} + T_r^2S^{\eta_r}, & r \text{ even}\end{cases}$$

graded by $\mathbb{Z}^{l+2}$ via assigning to the i-th variable the i-th column of a Gale dual of the block matrix $[P, B]$.

As an independent second application we consider the blow-up of the (generalised) diagonal in a product of projectives spaces. For this let $X' = \mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_{\mathbf{r}}}$ be a such a product and set $\Delta_X \subseteq X := X' \times X'$ as the diagonal. Then we obtain the following Theorem, see 4.2.1.

Theorem. *The Cox ring $\mathcal{R}(\mathrm{Bl}_{\Delta_X}(X))$ of the blow-up $\mathrm{Bl}_{\Delta_X}(X)$ is isomorphic to the $\mathbb{Z}^{\mathbf{r}} \times \mathbb{Z}^{\mathbf{r}} \times \mathbb{Z}$-graded factor algebra R_X/I_X where*

$$\begin{aligned} R_X &:= \mathbb{K}[T_\infty,\ {}_rT_{ij};\quad r = 1,\ldots,\mathbf{r},\quad 0 \le i < j \le n_r + 2,\quad i \le n_r], \\ I_X &:= I(1) + \ldots + I(\mathbf{r}), \end{aligned}$$

for every $r = 1,\ldots,\mathbf{r}$ the ideal $I(r)$ is generated by the twisted Plücker relations

$$\begin{aligned} &{}_rT_{ij}\ T_\infty \ -\ {}_rT_{ik}\ {}_rT_{jk}\ +\ {}_rT_{il}\ {}_rT_{jk}; && 0 \le i < j \le n_r,\ k = n_r + 1,\ l = n_r + 2, \\ &{}_rT_{ij}\ {}_rT_{kl} \ -\ {}_rT_{ik}\ {}_rT_{jk}\ +\ {}_rT_{il}\ {}_rT_{jk}; && 0 \le i < j < k < l \le n + 2,\quad k \le n_r \end{aligned}$$

and the grading of R_X/I_X is given by

$$\deg(T_\infty) \ =\ (0,0,1), \qquad \deg({}_rT_{ij}) \ =\ \begin{cases} (e_r,0,0) & \textit{if } j = n_r + 1, \\ (0,e_r,0) & \textit{if } j = n_r + 2, \\ (e_r,e_r,-1) & \textit{else.} \end{cases}$$

As a similar class of examples we treat the blow-up of the variety $Y := \mathbb{P}_1^n$ in the generalised diagonal $\Delta_Y := \{(x,\ldots,x);\ x \in \mathbb{P}_1\} \subseteq Y$. Again we prove that the Cox ring of $\mathrm{Bl}_{\Delta_Y}(Y)$ is finitely generated and we give an explicit presentation.

Theorem. *The Cox ring $\mathcal{R}(\mathrm{Bl}_{\Delta_Y}(Y))$ of the blow-up $\mathrm{Bl}_{\Delta_Y}(Y)$ is isomorphic to the $\mathbb{Z}^{n+1}$-graded factor algebra R_Y/I_Y where*

$$\begin{aligned} R_Y &:= \mathbb{K}[S_{ij};\ 1 \le i < j \le n+2] \\ I_Y &:= \langle\, S_{ij}S_{kl} \ -\ S_{ik}S_{jl}\ +\ S_{il}S_{jk};\quad 1 \le i < j < k < l \le n+2\,\rangle, \end{aligned}$$

and the grading of R_Y/I_Y is given by

$$\deg(S_{ij}) \ =\ \begin{cases} e_i & \textit{if } i \le n,\ j = n+1,\ n+2, \\ e_{n+1} & \textit{if } i = n+1,\ j = n+2, \\ e_i + e_j - e_{n+1} & \textit{else.} \end{cases}$$

Probably the best known examples of Chow quotients are the Grothdendieck-Knudsen and Losev-Manin moduli spaces $\overline{M}_{0,n}$ and $\overline{L}_n$. They can be thought of as canonical compactifications of the spaces of point configurations on $\mathbb{P}_1$ up to the action of the full automorphism group $\mathrm{SL}(2)$ and the maximal subtorus $T \subseteq \mathrm{SL}(2)$ respectively. It was shown that both of them have a description as an iterated blow-up of some projective spaces, see [**50, 53**].

We examine the space of point configurations up to translation, i.e. up to the action of the maximal unipotent group $\mathbb{G}_a \subseteq \mathrm{SL}(2,\mathbb{K})$ on $\mathbb{P}_1^n$. In order to define a compactification analagous to the cases just discussed, we introduce the *non-reductive limit quotient* $\mathbb{P}_1^n /\!\!/_{\mathrm{LQ}}\, \mathbb{G}_a$. Similar to the results of Kapranov, Losev and Manin in the cases of $\overline{M}_{0,n}$ and $\overline{L}_n$ this quotient space essentially arises by blowing up a sequence of subspaces in product of projective lines.

Theorem. *Denoting by $T_2, S_2, \dots, T_n, S_n$ the homogeneous coordinates on $\mathbb{P}_1^{n-1}$ we consider for every $A \subseteq \{2, \dots, n\}$ the subschemes X_A on $\mathbb{P}_1^{n-1}$ given by the ideals*

$$\langle\, T_i^2,\ T_jS_k - T_kS_j;\ i,j,k \in A,\ j < k \,\rangle.$$

Let $\tilde{\mathrm{Bl}}(\mathbb{P}_1^{n-1})$ be the normalised blow-up of $\mathbb{P}_1^{n-1}$ in all these subschemes. If we write $\mathbb{P}_1^n \tilde{/\!\!/}_{\mathrm{LQ}} \mathbb{G}_a$ for the normalisation of the limit quotient, then we have an open embedding

$$\mathbb{P}_1^n \tilde{/\!\!/}_{\mathrm{LQ}} \mathbb{G}_a \quad \subseteq \quad \tilde{\mathrm{Bl}}(\mathbb{P}_1^{n-1}).$$

For details on this we refer to Theorems 6.5.1 and 6.5.2. Since $\overline{L}_n$ is a toric variety, its geometry and also its Cox ring are well known. On the contrary, the computation of the Cox ring $\mathcal{R}(\overline{M}_{0,n})$ has long been an open problem, see e.g. [**15, 18, 30, 46**]. Only very recently it was proved that for $n \geq 134$ the Cox ring is not finitely generated, see [**19**]. The limit quotient $\mathbb{P}_1^3 \tilde{/\!\!/}_{\mathrm{LQ}} \mathbb{G}_a$ essentially arises as a good quotient of the affine cone over the Grassmannian $\mathrm{Gr}(2,4)$ by a (submaximal) torus action. Similar to $\overline{M}_{0,5}$ this already determines the Cox ring of $\mathbb{P}_1^3 \tilde{/\!\!/}_{\mathrm{LQ}} \mathbb{G}_a$; it is given by Plücker relations. For $n \geq 4$ it would be interesting to know whether the Cox rings of $\overline{M}_{0,n+2}$ and $\mathbb{P}_1^n \tilde{/\!\!/}_{\mathrm{LQ}} \mathbb{G}_a$ are related in some way.

This thesis consists of six chapters, we give a brief summary for each of them.

In *Chapter 1* we introduce the basic notations and concepts. We give an overview of Cox rings and related geometric constructions. For toric varieties we discuss the Cox construction with respect to the convex geometric aspects. Moreover, we treat bunched rings, which are a method of constructing varieties with prescribed Cox ring, and their canonical toric embeddings. In a final section we deal with the GKZ-decomposition of a vector configuration.

In *Chapter 2* we summarise the different notions of quotients. We treat the Mumford quotients, the GIT-limit, the limit quotient and the Chow quotient. We show that for tori the latter two constructions essentially coincide and discuss further properties. In the last section of this chapter we extend the construction of the GIT-limit to certain non-reductive groups.

In *Chapter 3* we will prove that for a reductive group any good quotient of a Mori Dream Space is a Mori Dream Space itself. This was known for certain GIT-quotients only and expected by Hu and Keel in [**46**].

In *Chapter 4* we refine the technique of toric ambient modifications and give a criterion to determine whether a certain candidate for the Cox ring of the blow-up is in fact the Cox ring. In the remaining two sections of this chapter we apply this method to two classes of examples and compute explicit presentations for the Cox rings of $\mathrm{Bl}_{\Delta}(X' \times X')$ and $\mathrm{Bl}_{\Delta}(\mathbb{P}_1^n)$ where Δ is the (generalised) diagonal and X' is a product of projective spaces.

In *Chapter 5* we discuss Chow quotients of quadrics arising from an action of the algebraic torus $\mathbb{K}^*$. Using some tropical geometry and a result from [**42**] we show that the Cox ring of the Chow quotient always is finitely generated. Moreover, in certain cases we can also compute an explicit presentation in terms of generators and relations.

In *Chapter 6* we construct a compactification of the space of point configurations on $\mathbb{P}_1$ up to translation similar to $\overline{M}_{0,n}$ and $\overline{L}_n$. For this we use the methods provided in Chapter 2 and we show how this compactification arises from a product of projective lines by blowing up a sequence of subschemes.

CHAPTER

ONE

BASIC NOTATION

In this chapter we provide the basic notations and concepts needed throughout the present thesis. All of this chapter's content is well known, much of it is summarised in our main source [**4**].

1.1. The Cox Ring

Let X be a normal algebraic variety over an algebraically closed field $\mathbb{K}$ of characteristic zero. We denote by $\mathrm{WDiv}(X)$, $\mathrm{PDiv}(X)$ and $\mathrm{CaDiv}(X)$ the groups of *Weil divisors*, *principal divisors* and *Cartier divisors* respectively. To any f in the field of rational functions $\mathbb{K}(X)$ we denote the associated principal divisor by $\mathrm{div}(f)$. Furthermore, the *divisor class group* and *Picard group* of X are given by

$$\mathrm{Cl}(X) \; = \; \mathrm{WDiv}(X) \,/\, \mathrm{PDiv}(X), \qquad \mathrm{Pic}(X) \; = \; \mathrm{CaDiv}(X) \,/\, \mathrm{PDiv}(X),$$

respectively. For two Weil divisors $D := \sum a_P P$, $D' := \sum a'_P P$ we write $D \geq D'$ if and only if $a_P \geq a'_P$ holds for all coefficients. A divisor D is called *effective* if $D \geq 0$ holds. To a Weil divisor $D \in \mathrm{WDiv}(X)$ and an open subset $U \subseteq X$ we associate the vector space

$$\Gamma(U, D) \; := \; \left\{ f \in \mathbb{K}(X)\, ; \quad \mathrm{div}(f_{|U}) \, + \, D_{|U} \, \geq \, 0 \right\}.$$

Any subgroup $K \subseteq \mathrm{WDiv}(X)$ of Weil divisors then gives rise to a K-graded *sheaf of divisorial $\mathcal{O}_X$-algebras*

$$\mathcal{S}_K(U) \; := \; \bigoplus_{D \in K} \Gamma(U, D).$$

Generalising the idea of the homogeneous coordinate ring of toric varieties (cf. [**22**]) one can associate a Cox ring to any normal irreducible variety X with finitely generated divisor class group and only constant invertible global functions. Note that the latter condition is always satisfied for complete varieties.

We fix a finitely generated subgroup K of Weil divisors such that the projection $c\colon K \to \mathrm{Cl}\,(X)$ is surjective with kernel K^0. We now associate to K the sheaf of divisorial $\mathcal{O}_X$-algebras $\mathcal{S}$. In order to identify the isomorphic homogeneous components of $\mathcal{S}$ we fix a character $\chi\colon K^0 \to \mathbb{K}(X)^*$ such that $\mathrm{div}\,(\chi(E)) = E$ holds for every $E \in K^0$ and consider the sheaf of ideals $\mathcal{I}$ locally generated by the sections $1 - \chi(E)$ where E runs through K^0 and $\chi(E)$ is homogeneous of degree $-E$.

Definition 1.1.1. The *Cox sheaf* of X is the sheaf $\mathcal{R} := \mathcal{S}/\mathcal{I}$ together with the $\mathrm{Cl}\,(X)$-grading

$$\mathcal{R} = \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{R}_{[D]}, \qquad \mathcal{R}_{[D]} := p\left(\bigoplus_{D'\in c^{-1}([D])} \mathcal{S}_{D'}\right),$$

where $p : \mathcal{S} \to \mathcal{R}$ denotes the projection. The algebra of global sections $\mathcal{R}(X)$ is called the *Cox ring* of X.

The Cox ring is - up to isomorphy - independent of the choices of K and χ, see [**4**, I, Section 4.3]. For later use, note that by [**4**, I, Lemma 3.3.5] for any open set $U \subseteq X$ we have

$$\Gamma(U,\mathcal{R}) \;\cong\; \Gamma(U,\mathcal{S})\,/\,\Gamma(U,\mathcal{I}).$$

Moreover, from [**4**, Lemma 4.2.2] we infer that the Cox ring does not change when passing to a big open subset, i.e. an open subset whose complement is of codimension at least two. In particular, by normality of X the two algebras $\mathcal{R}(X^{\mathrm{reg}})$ and $\mathcal{R}(X)$ are equal, where X^{reg} denotes the set of regular points of the variety X.

Definition 1.1.2 ([4, I, Definition 5.3.1]). Let K be an abelian group and R a K-graded integral $\mathbb{K}$-algebra.

(i) A non-zero non-unit $0 \neq p \in R\backslash R^*$ is *K-prime*, if p is homogeneous and $p|ab$ for homogeneous $a, b \in R$ implies $p|a$ or $p|b$.
(ii) The algebra R is *factorially K-graded* (or short *K-factorial*) if every homogeneous non-zero non-unit $0 \neq f \in R \setminus R^*$ is a product of K-prime elements.

Theorem 1.1.3 ([4, I, Proposition 5.2.5, Theorems 3.3.3, 5.1.1]). *Let X be a normal variety with finitely generated divisor class group, only constant* *invertible regular functions and* $\mathrm{Cl}(X)$*-graded Cox ring* $\mathcal{R}(X)$*. Then the following assertions hold.*

(i) *The ring* $\mathcal{R}(X)$ *is normal, integral and factorially* $\mathrm{Cl}(X)$*-graded.*
(ii) *If* $\mathcal{O}(X) = \mathbb{K}$ *holds, then* $\mathcal{R}(X)^*$ *equals* $\mathbb{K}^*$*.*
(iii) *If* $\mathrm{Cl}(X)$ *is free, then* $\mathcal{R}(X)$ *is factorial.*

In general, $\mathcal{R}(X)$ does not need to be finitely generated. For example, this happens when blowing up at least nine points in general position on $\mathbb{P}_2$, see [**20**].

Definition 1.1.4. Let X be a normal, irreducible variety with finitely generated divisor class group and only constant invertible functions. If its Cox ring $\mathcal{R}(X)$ is finitely generated, then X is called *Mori Dream Space*, or short *MDS*.

1.2. Geometry of the Cox Construction

In this section we deal with the geometric aspects of the previously discussed Cox ring construction. For this let X be a Mori Dream Space with Cox sheaf $\mathcal{R}$. Since $\mathcal{R}(X)$ is finitely generated, $\mathcal{R}$ is locally of finite type (cf. [**4**, I, 3.2.2]), i.e. for every $x \in X$ there exists an affine neighbourhood U such that $\mathcal{R}(U)$ is finitely generated. With this we obtain the relative spectrum $\hat{X} := \mathrm{Spec}_X(\mathcal{R})$ by gluing together the affine pieces $\mathrm{Spec}(\mathcal{R}(U))$. The relative spectrum $\hat{X}$ is called *characteristic space* of X. If X is of affine intersection, i.e. for any two open affine subsets their intersection is affine again, then $\hat{X}$ is a quasiaffine variety. It comes with a canonical open embedding into $\overline{X} := \mathrm{Spec}(\mathcal{R}(X))$, which we call the *total coordinate space* of X.

In order to describe how to obtain X back from its characteristic space $\hat{X}$ we briefly recall the connection between graded algebras and quasitorus actions. A *quasitorus* H is an affine-algebraic group which is isomorphic to some $(\mathbb{K}^*)^r \times C$ where C is a finite abelian group; a *torus* is a connected quasitorus. A *character* of an algebraic group G is a morphism (of algebraic groups) $\chi\colon G \to \mathbb{K}^*$. The characters again form a group, denoted by $\mathbb{X}(G)$. The categories of quasitori and finitely generated abelian groups are equivalent via the essentially inverse functors

$$H \mapsto \mathbb{X}(H), \qquad K \mapsto \mathrm{Spec}(\mathbb{K}[K]).$$

Note that under these functors the subcategory of tori corresponds to the free groups. We now turn to the correspondence between quasitori actions and gradings by finitely generated abelian groups. For this let K be a finitely generated abelian group and A a K-graded affine $\mathbb{K}$-algebra. Then there exists a $\mathrm{Spec}(\mathbb{K}[K])$-action on $\mathrm{Spec}(A)$ which is defined by its comorphism

$$A \to A \otimes \mathbb{K}[K], \quad A_w \ni a \mapsto a \otimes \chi^w.$$

Vice versa the action of a quasitorus H on some affine variety X defines a grading of the regular functions $\mathcal{O}(X)$ by

$$\mathcal{O}(X) = \bigoplus_{w \in K} \mathcal{O}(X)_w, \quad \mathcal{O}(X)_w := \{f \in \mathcal{O}(X);\ f(t \cdot x) = \chi^w(t) f(x)\}.$$

These correspondences give rise to an equivalence of the categories of affine K-graded $\mathbb{K}$-algebras and affine H-varieties, i.e. affine varieties with an H-action.

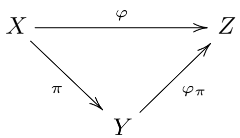
Now consider the action of an affine-algebraic group G on an algebraic variety X. If X is affine this gives rise to a linear G-action on the regular functions $\mathcal{O}(X)$ defined by

$$(g \cdot f)(x) \quad := \quad f(g^{-1} \cdot x).$$

The collection of invariant functions, i.e. the functions with $g \cdot f = f$ is a subalgebra of $\mathcal{O}(X)$, we denote it by $\mathcal{O}(X)^G$. By Hilbert's Finiteness Theorem this algebra is finitely generated if G is reductive and X is affine.

For a not necessarily affine X an affine morphism $\pi\colon X \to Y$ is called *good quotient* for the G-action if there exists an open, affine cover $(U_i)_i$ of Y such that $\mathcal{O}(U_i) \to \mathcal{O}(\pi^{-1}(U_i))^G$ is an isomorphism for all i. A good quotient is *geometric*, if each fibre is a single orbit.

The good quotient $\pi\colon X \to Y$ enjoys the following universal property: For any G-invariant morphism $\varphi\colon X \to Z$ there exists a unique morphism $\varphi_\pi\colon Y \to Z$ such that the following diagram commutes.



In particular, if a good quotient $\pi\colon X \to Y$ exists, then it is unique up to isomorphy. In this case we write $X /\!\!/ G$ for the quotient space. Clearly, Hilbert's Finiteness Theorem guarantees existence of a good quotient in the case where X is affine and G is reductive, e.g. a quasitorus. We are now ready to interpret the Cox ring geometrically.

The $\mathrm{Cl}(X)$-grading of the Cox ring $\mathcal{R}(X)$ gives rise to an action of the quasitorus $H := \mathrm{Spec}(\mathbb{K}[\mathrm{Cl}(X)])$ on $\overline{X}$. The characteristic space $\hat{X}$ is an invariant open subset of $\overline{X}$ admitting a good quotient for this action

$q_X\colon \hat{X} \to \hat{X}/\!\!/H$ and the quotient space is isomorphic to the variety X. The situation fits into the diagram.

$$\begin{array}{ccccccc} \mathrm{Spec}_X(\mathcal{R}) & = & \hat{X} & \longrightarrow & \overline{X} & = & \mathrm{Spec}(\mathcal{R}(X)) \\ & & \Big\downarrow{\scriptstyle q_X} & & & & \\ & & X & & & & \end{array}$$

1.3. The toric Cox Construction

While for arbitrary varieties their Cox rings are fairly hard to calculate, it is well known that for toric varieties the Cox ring - as abstract ring - is isomorphic to a polynomial ring. The grading can be derived from the structure of the corresponding fan.

We will recall some basic notions from convex geometry and their link to toric geometry. For this let N and M be mutually dual lattices and $N_{\mathbb{Q}}$, $M_{\mathbb{Q}}$ the corresponding rational vector spaces.

By a *cone* σ in $N_{\mathbb{Q}}$ we always mean a convex polyhedral cone, its dual cone $\sigma^\vee$ is the (convex) set of linear forms $l \in M_{\mathbb{Q}}$ for which $l_{|\sigma} \geq 0$ holds. A cone is said to be *pointed* if it does not contain a line and a *face* τ of σ is a convex subset of σ for which there exists an $l \in \sigma^\vee$ such that $l_{|\tau} = 0$ holds. The *dimension* of a cone is the dimension of the vector space generated by it and the *facets* of σ are its 1-codimensional faces. To the 1-dimensional faces of σ we refer as its *(extremal) rays* and write $\sigma^{(1)}$. A *lattice cone* is a pair (N, σ) where N is a lattice and σ is a cone in $N_{\mathbb{Q}}$.

By a *toric variety* we mean a normal, irreducible variety Z together with an action of an algebraic torus T_Z and a base point $z_0 \in Z$ such that T_Z is openly embedded into Z via the morphism $T_Z \to Z,\ t \mapsto t \cdot z_0$.

The categories of lattice cones and affine toric varieties are covariantly equivalent with a functor mapping a lattice cone (N, σ) onto the affine toric variety $Z(\sigma) := \mathrm{Spec}(\mathbb{K}[\sigma^\vee \cap M])$ with dense torus $T_Z := \mathrm{Spec}(\mathbb{K}[M])$.

This correspondence extends to non-affine toric varieties. A *quasifan* Σ in $N_{\mathbb{Q}}$ is a finite collection of convex, polyhedral cones such that for any cone $\sigma \in \Sigma$ all of its faces are members of Σ and for any two cones $\sigma_1, \sigma_2 \in \Sigma$ their intersection $\sigma_1 \cap \sigma_2$ is a face of both cones. A quasifan is called a *fan* if all its cones are pointed. The *support* $|\Sigma|$ of a fan is the union of its cones and Σ is said to be *complete*, if $|\Sigma| = N_{\mathbb{Q}}$ holds. By a *lattice fan* we mean the pair (N, Σ).

By a result of Sumihiro [**64**] any toric variety with dense torus T is covered by affine T-invariant toric varieties and the gluing data is reflected in the fan structure of the associated collections of lattice cones. More precisely, the categories of lattice fans and toric varieties are equivalent.

We now turn to the toric Cox construction , see [**22**]. For this let Z be a toric variety and Σ its corresponding fan in $N_{\mathbb{Q}}$. We assume Z to have only constant invertible functions which is equivalent to Σ not being contained in a proper vector subspace of $N_{\mathbb{Q}}$. Set $F := \mathbb{Z}^r$ where r is the number of rays of Σ and denote by $P\colon F \to N$ the homomorphism taking the canonical basis vectors f_i to the primitive generators v_i of the rays of Σ. We now consider the following fan in the vector space $F_{\mathbb{Q}}$

$$\hat{\Sigma} \quad := \quad \{\hat{\sigma} \preccurlyeq \delta; \quad P(\hat{\sigma}) \subseteq \sigma \text{ for some } \sigma \in \Sigma\},$$

where $\delta \subseteq F_{\mathbb{Q}}$ is the positive orthant. Clearly, $\hat{\Sigma}$ is a subfan of the fan $\overline{\Sigma}$ defined by the single maximal cone δ. This inclusion gives rise to an open embedding of the corresponding toric varieties $\hat{Z} \subseteq \overline{Z} = \mathbb{K}^r$.

If $E := F^*$ denotes the dual lattice, then the regular functions of $\overline{Z}$ are given by $\mathbb{K}[\delta^\vee \cap E]$. Moreover, setting $Q\colon E \to K := E \,/\, P^*(M)$ as the projection, this algebra is K-graded by $\deg(\chi^e) := Q(e)$. The K-grading then gives rise to an action of the quasitorus $H := \mathrm{Spec}(\mathbb{K}[K])$ on $\overline{Z}$.

Theorem 1.3.1 ([**4**, II, Theorem 1.3.2])**.** *Let the notation be as above, then the following assertions hold.*

(i) *The groups K and $\mathrm{Cl}(Z)$ are isomorphic.*
(ii) *The Cox ring of Z is the K-graded ring $\mathcal{O}(\overline{Z})$.*
(iii) *The space $\hat{Z}$ is a characteristic space and the space $\overline{Z}$ is a total coordinate space for Z.*
(iv) *The toric morphism $p_Z\colon \hat{Z} \to Z$ arising from the morphism of fans $P\colon \hat{\Sigma} \to \Sigma$ is a good quotient for the H-action on $\hat{Z}$.*

An advantage of the Cox ring is that it facilitates explicit computations with a (toric) variety. In particular, for the remainder of this section we will discuss how a graded $\mathcal{R}(Z)$-module gives rise to a sheaf on Z, for details on this see [**1, 22**]. We set

$$\mathcal{R} \;:=\; \mathcal{R}(Z) \;=\; \mathbb{K}[T_1,\dots,T_r]$$

and consider a K-graded $\mathcal{R}$-module M. From this we obtain a sheaf on Z in the following way. First, for a cone $\sigma \in \Sigma$ we consider the localisations

$$\mathcal{R}_\sigma \;:=\; \mathcal{R}_{T_\sigma}, \qquad M_\sigma \;:=\; M_{T_\sigma}, \qquad \text{where} \qquad T_\sigma \;:=\; \prod_{P(f_i)\notin\sigma^{(1)}} T_i$$

Then M_σ is a K-graded $\mathcal{R}_\sigma$-module and by taking the respective homogeneous components in degree zero we obtain the $(\mathcal{R}_\sigma)_0$-module $(\mathcal{M}_\sigma)_0$. Note that Z is covered by the affine pieces $Z(\sigma) := \mathrm{Spec}(\mathcal{R}_\sigma)_0$. On $Z(\sigma)$ the module $(M_\sigma)_0$ gives rise to a sheaf, for this see e.g. [**37**, II, Section 5]. These sheaves patch together and form a sheaf $\mathcal{M}$ on Z.

Let $\mathcal{I}$ be a sheaf of $\mathcal{O}_Z$-modules (or short $\mathcal{O}_Z$-module) on Z. The sheaf $\mathcal{I}$ is a *sheaf of ideals*, if for every open $U \subseteq Z$ the sections $\Gamma(\mathcal{I}, U)$ constitute an ideal in $\mathcal{O}_Z(U)$. A closed subscheme $\iota\colon Y \to Z$ is characterised by its ideal sheaf $\mathcal{I}_Y$, i.e. the kernel of $\mathcal{O}_Z \to \iota_*\mathcal{O}_Y$.

Theorem 1.3.2 ([**22**, Proposition 2.4, Theorem 3.2, Corollary 3.9])**.** *Let Z be a simplicial toric variety. Then the following assertions hold.*

(i) *Every quasicoherent sheaf on Z is of the form $\mathcal{M}$ for some graded $\mathcal{R}$-module M.*
(ii) *If I is a graded ideal of $\mathcal{R}$, then $\mathcal{I}$ is a sheaf of ideals on Z. Vice versa, for every closed subscheme X of Z there exists a graded ideal $I \subseteq \mathcal{R}$ such that $\mathcal{I}$ gives rise to the subscheme X.*
(iii) *If I is a graded radical ideal in $\mathcal{R}$, then the subscheme corresponding to $\mathcal{I}$ is a variety.*

If certain restrictions are imposed on the graded module, then the above assertions can be strengthend to give one-to-one correspondences, see [**22**] for details.

1.4. Bunched Rings

Bunched rings are an answer to the problem of constructing varieties with prescribed Cox ring. However, this answer is not unique. In general, even if two varieties have isomorphic (graded) Cox rings they need not be isomorphic, they rather are isomorphic in codimension two. Bunched rings essentially consist of a Cox ring (with a choice of generators) and an additional combinatorial datum, which fixes the isomorphy type of the variety.

Let K be a finitely generated abelian group and R an integral, normal, affine, K-graded $\mathbb{K}$-algebra. Consider a system $\mathcal{F} = (f_1, \ldots, f_r)$ of homogeneous generators of R and let $Q\colon E \to K$ be the map taking the i-th canonical basis vector $e_i \in E := \mathbb{Z}^r$ to $w_i := \deg(f_i) \in K$. The grading gives rise to a quasitorus action of $H := \operatorname{Spec}(\mathbb{K}[K])$ on $\overline{X} := \operatorname{Spec}(R)$. Moreover, we obtain a closed embedding

$$\bar{\imath}\colon \overline{X} \to \mathbb{K}^r; \qquad x \mapsto (f_1(x), \ldots, f_r(x)).$$

The H-action on $\overline{X}$ extends to a diagonal action on $\mathbb{K}^r$ given by the characters $\chi^{w_1}, \ldots, \chi^{w_r}$, i.e.

$$h \cdot z \quad = \quad (\chi^{w_1}(h)z_1, \ldots, \chi^{w_r}(h)z_r),$$

turning the above embedding into an H-equivariant morphism. A face $\gamma_0 \preccurlyeq \gamma$ of the positive orthant γ in $E_{\mathbb{Q}}$ is called $\mathfrak{F}$*-face* , if there exists some $x \in \overline{X}$ such that

$$x_i \neq 0 \qquad \Longleftrightarrow \qquad e_i \in \gamma_0.$$

We say that the K-grading of R is *almost free* , if any $r-1$ of the weights $w_1, \dots, w_r$ generate K as an abelian group. Denoting by $\Omega_{\mathfrak{F}} := \{Q(\gamma_0);\ \gamma_0 \preccurlyeq \gamma \text{ an } \mathfrak{F}\text{-face}\}$ the set of projected $\mathfrak{F}$-faces we call a subset $\Phi \subseteq \Omega_{\mathfrak{F}}$ thereof an $\mathfrak{F}$-*bunch*, if the following two conditions are satisfied

- If τ_1, τ_2 lie in Φ, then $\tau_1^\circ \cap \tau_2^\circ$ is non-empty.
- If we have $\tau_1 \in \Phi$ and $\tau \in \Omega$ such that $\tau_1^\circ \subseteq \tau^\circ$, then $\tau \in \Phi$ holds.

An $\mathfrak{F}$-bunch is said to be *true* , if for every (one-codimensional) facet $\gamma_0 \preccurlyeq \gamma$ we have $Q(\gamma_0) \in \mathfrak{F}$. We are now ready for the definition of bunched rings. A *bunched ring* is a triple $(R, \mathfrak{F}, \Phi)$ where

- R is an almost freely, factorially K-graded affine $\mathbb{K}$-algebra,
- $\mathfrak{F}$ is a family of homogeneous generators of R and
- Φ is a true $\mathfrak{F}$-bunch.

From these three pieces of data, we now construct a variety having R as Cox ring. For this consider the set of *relevant faces*

$$\operatorname{rlv}(\Phi) \ := \ \{\gamma_0 \preccurlyeq \gamma;\ \gamma_0 \text{ an } \mathfrak{F}\text{-face and } \mathrm{Q}(\gamma_0) \in \Phi\}.$$

In order to shorten the notation we set for every $\gamma_0 \preccurlyeq \gamma$

$$\overline{X}_{\gamma_0} \ := \ \overline{X}_{f_1^{u_1} \cdots f_r^{u_r}} \quad \text{with an arbitrary } u \in \gamma_0^\circ.$$

One easily sees that $\overline{X}_{\gamma_0}$ is independent of the choice of u and we set

$$\begin{aligned} \hat{X} \quad &:= \quad \hat{X}(R, \mathfrak{F}, \Phi) \quad := \quad \bigcup_{\gamma_0 \in \operatorname{rlv}(\Phi)} \overline{X}_{\gamma_0}, \\ X \quad &:= \quad X(R, \mathfrak{F}, \Phi) \quad := \quad \hat{X} /\!\!/ H. \end{aligned}$$

Theorem 1.4.1 ([4, III, Theorem 2.1.9]). *Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring and X, $\hat{X}$ and $\overline{X}$ defined as above. Then the following assertions hold.*

(i) *The variety X is normal and its divisor class group is isomorphic to the abelian group K.*
(ii) *All invertible regular functions on X are constant and the Cox ring $\mathcal{R}(X)$ is isomorphic to R (as K-graded ring).*
(iii) *The dimension of X is given by $\dim(\overline{X}) - \dim(K_{\mathbb{Q}})$.*

Proposition 1.4.2 ([4, III, Corollary 2.1.11]). *Every projective Mori Dream Space arises from a bunched ring.*

Any variety arising from a bunched ring comes with a canonical embedding into a toric variety in the sense that the embedding defines an isomorphism on the level of divisor class groups. First note that with $M := \ker(Q)$ we obtain the following exact sequences.

$$0 \longrightarrow L \longrightarrow F \xrightarrow{\;P\;} N$$

$$0 \longleftarrow K \xleftarrow{\;Q\;} E \longleftarrow M \longleftarrow 0$$

Now we set $\delta := \gamma^\vee \subset F_\mathbb{Q}$ as the dual cone of the positive orthant $\gamma \subseteq E_\mathbb{Q}$ and for any $\gamma_0 \preccurlyeq \gamma$ we denote its corresponding face by $\gamma_0^* := \delta \cap \gamma_0^\perp$. We define the *enveloping collection* and the following fans in $F_\mathbb{Q}$ and $N_\mathbb{Q}$ respectively

$$\begin{aligned}
\mathrm{Env}(\Phi) &:= \{\gamma_0 \preccurlyeq \gamma;\ \text{there exists } \mathrm{rlv}(\Phi) \ni \gamma_1 \preccurlyeq \gamma_0 \text{ with } \mathrm{Q}(\gamma_1)^\circ \subseteq \mathrm{Q}(\gamma_0)^\circ\},\\
\hat{\Sigma} &:= \{\delta_0 \preccurlyeq \delta;\ \text{there exists } \gamma_0 \in \mathrm{Env}(\Phi) \text{ with } \delta_0 \preccurlyeq \gamma_0^*\},\\
\Sigma &:= \{P(\gamma_0^*);\ \gamma_0 \in \mathrm{Env}(\Phi)\}.
\end{aligned}$$

Clearly, $\hat{\Sigma}$ is a subfan of the fan $\overline{\Sigma}$ consisting of the positive orthant δ and all its faces. Hence, there is an open embedding of the corresponding toric varieties $\hat{Z} \subseteq \overline{Z} := \mathbb{K}^r$. The subset $\hat{Z}$ is invariant under the H-action and admits a good quotient $p_Z\colon \hat{Z} \to Z := \hat{Z} /\!\!/ H$. The quotient space Z is toric again and its fan is given by Σ.

We turn to the embedded spaces; recall that $\overline{X}$ is embedded into $\overline{Z}$ via $\bar{\imath}$. This embedding restricts to a closed embedding $\hat{\imath}\colon \hat{X} \to \hat{Z}$ of the characteristic spaces and then descends to a closed embedding $\iota\colon X \to Z$ of the respective quotient spaces. This situation fits into the following commutative diagram.

$$\begin{array}{ccc}
\overline{X} & \xrightarrow{\;\bar{\imath}\;} & \overline{Z}\\
\uparrow & & \uparrow\\
\hat{X} & \xrightarrow{\;\hat{\imath}\;} & \hat{Z}\\
\downarrow{\scriptstyle /\!\!/ H} & & \downarrow{\scriptstyle /\!\!/ H}\\
X & \xrightarrow{\;\iota\;} & Z
\end{array}$$

Proposition 1.4.3 ([4, III, Proposition 2.5.4]). *The embedding $\iota\colon X \to Z$ has the following properties.*

(i) *The embedding is neat, i.e. the inverse images $\iota^{-1}(D_Z^i)$ of the toric prime divisors D_Z^i are pairwise distinct, irreducible hypersurfaces of X and ι induces an isomorpism $\iota^*\colon \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$ on the level of divisor class groups.*

(ii) *The maximal cones of* Σ *are* $\Sigma^{\max} = \{P(\gamma_0^*);\ \gamma_0 \in \mathrm{rlv}(\Phi)$ *minimal*$\}$.
(iii) *The image* $\iota(X)$ *intersects every closed toric orbit of* Z *non-trivially.*

In general, even if X is complete, Z need not be. There exists however a not necessarily unique toric completion, for details see [**4**, III, Construction 2.5.7]. All these completions share Z as minimal subvariety containing X.

Let us look closer at the general question which toric orbits are intersected non-trivially by X. For this let $T_{Z'}$ be the dense torus of a toric variety Z' with fan Σ' in $N_{\mathbb{Q}}$. Recall that for every Laurent polynomial $f \in \mathcal{O}(T_{Z'})$ its *Newton polytope* is given as

$$\mathrm{New}(f) \;:=\; \mathrm{conv}(\nu;\ a_\nu \neq 0) \;\subseteq\; N_{\mathbb{Q}}, \qquad \text{where} \qquad f = \sum_{\nu \in N} a_\nu T^\nu.$$

We consider a non-empty closed subset $X_T \subseteq T_{Z'}$ and define the *tropical variety* of X_T as

$$\mathrm{trop}(X_T) \quad := \quad \bigcap_{f \in I(X_T)} \left| \mathcal{N}(\mathrm{New}(f))^{(n-1)} \right|,$$

where $\mathcal{N}$ denotes the normal fan. Note that although finitely many Laurent polynomials suffice for this intersection, in general we cannot replace $I(X_T)$ by an arbitrary set of ideal generators. We ask the question which orbits of Z' are intersected non-trivially by the closure $\overline{X_T} \subseteq Z'$. An answer to this is the following result by Tevelev.

Theorem 1.4.4 ([**65**, Proposition 2.8])**.** *Let* $T_{Z'}$ *be the dense torus of a toric variety* Z' *with corresponding fan* Σ'. *Moreover, let* $X_T \subseteq T_{Z'}$ *be a closed subset of the torus and* σ *a cone in* Σ'. *If* $T_Z \cdot z_\sigma$ *denotes the corresponding torus orbit, then* $\overline{X_T} \cap (T_Z \cdot z_\sigma)$ *is non-empty if and only if* $\mathrm{trop}(X_T) \cap \sigma^\circ$ *is.*

Let the notation be as above, Σ' the fan of Z' and set

$$\Sigma^{\mathrm{trop}(X_T)} \;:=\; \{\sigma \in \Sigma';\ \text{there ex. } \sigma \preccurlyeq \tau \in \Sigma' \text{ s. that } \tau^\circ \cap \mathrm{trop}(X_T) \neq \emptyset\}.$$

We return to our original situation where X arises from a bunched ring and is canonically embedded into the toric variety Z with corresponding fan Σ.

Corollary 1.4.5. *Let* $Z \subseteq Z_1$ *be a toric completion corresponding to the completion* $\Sigma \subseteq \Sigma_1$. *Then* $\Sigma_1^{\mathrm{trop}(X)} = \Sigma$ *holds.*

1.5. The GKZ-Decomposition

Let $\mathcal{V} := (v_1, \ldots, v_r)$ be a family of vectors in the rational vector space $N_{\mathbb{Q}}$. By a $\mathcal{V}$*-cone* we mean a cone in $N_{\mathbb{Q}}$ with rays generated by elements of $\mathcal{V}$.

Analogously we define the terms $\mathcal{V}$-*(quasi)fan.* For a collection of $\mathcal{V}$-quasifans $\Sigma_1, \ldots, \Sigma_r$ the *coarsest common refinement* is given as the fan

$$\Sigma_1 \sqcap \ldots \sqcap \Sigma_r \quad := \quad \{\sigma_1 \cap \ldots \cap \sigma_r;\ \sigma_i \in \Sigma_i\}\,.$$

The special case where the collection of cones consists of all possible $\mathcal{V}$-quasifans yields the *Gelfand-Kapranov-Zelevinsky-decomposition (GKZ-decomposition).* By [**4**, II, Theorem 2.2.3] it is equal to the fan

$$\mathrm{GKZ}(\mathcal{V}) \quad := \quad \left\{ \bigcap_{\sigma \text{ a } \mathcal{V}\text{-cone}} \sigma \right\}.$$

For a given family $\mathcal{V}$ we are interested in the structure of its GKZ-decomposition, in particular the newly occuring rays. To this end we introduce the notion of Gale duality. If $\mathcal{W} := (w_1, \ldots, w_r)$ is a family of vectors in the rational vector space $K_{\mathbb{Q}}$ we call $\mathcal{V}$ and $\mathcal{W}$ *Gale dual* (to each other) if for any tuple $(a_1, \ldots, a_r) \in \mathbb{Q}^r$ the following conditions are equivalant.

(i) $a_1 w_1 + \ldots + a_r w_r = 0$
(ii) There exists a linear form $u \in \mathrm{Hom}(N_{\mathbb{Q}}, \mathbb{Q})$ such that $u(v_i) = a_i$ holds for all $i = 1, \ldots, r$.

In order to construct Gale dual vector configurations we follow [**4**, II, Construction 2.1.3]. Consider a pair of mutually dual exact sequences of finite dimensional rational vector spaces.

$$0 \longrightarrow L_{\mathbb{Q}} \xrightarrow{Q^*} F_{\mathbb{Q}} \xrightarrow{P} N_{\mathbb{Q}} \longrightarrow 0$$

$$0 \longleftarrow K_{\mathbb{Q}} \xleftarrow{Q} E_{\mathbb{Q}} \xleftarrow{P^*} M_{\mathbb{Q}} \longleftarrow 0$$

If $(f_1, \ldots, f_r)$ and $(e_1, \ldots, e_r)$ are mutually dual bases of $F_{\mathbb{Q}}$ and $E_{\mathbb{Q}}$ respectively, then the following two collections in $N_{\mathbb{Q}}$ and $K_{\mathbb{Q}}$ respectively are Gale dual

$$\mathcal{V} := (P(f_1), \ldots, P(f_r)) \qquad \text{and} \qquad \mathcal{W} := (Q(e_1), \ldots, Q(e_r)).$$

As before let $\gamma \subseteq E_{\mathbb{Q}}$ be the positive orthant. We define a γ-*collection* to be a set $\mathfrak{B}$ of faces of γ such that any two $\gamma_1, \gamma_2 \in \mathfrak{B}$ admit an $M_{\mathbb{Q}}$-*invariant separating linear form* $f \in F_{\mathbb{Q}}$ in the sense that

$$f_{|M_{\mathbb{Q}}} = 0, \qquad f_{|\gamma_1} \geq 0, \qquad f_{|\gamma_2} \leq 0, \qquad \ker(f) \cap \gamma_i = \gamma_1 \cap \gamma_2.$$

For two γ-collections $\mathfrak{B}_1$ and $\mathfrak{B}_2$ we write $\mathfrak{B}_1 \leq \mathfrak{B}_2$ if for every $\gamma_1 \in \mathfrak{B}_1$ there is a $\gamma_2 \in \mathfrak{B}_2$ with $\gamma_1 \subseteq \gamma_2$. Moreover, a γ-collection $\mathfrak{B}$ is said to be *normal* if it cannot be enlarged as a γ-collection and the images $Q(\gamma_0)$, where $\gamma_0 \in \mathfrak{B}$, form the normal fan of a polyhedron.

Recall that for a face $\gamma_0 \preceq \gamma$, we denote by $\gamma_0^* = \gamma_0^\perp \cap \gamma^\vee$ the corresponding face of the dual cone $\gamma^\vee$. Now suppose that $\mathcal{V}$ consists of pairwise linearly independent, non-zero vectors. Then [**4**, II, Section 2] provides us with an order-reversing bijection

$$\{\text{normal } \gamma\text{-collections}\} \ \rightarrow \ \mathrm{GKZ}(\mathcal{V}), \qquad \mathfrak{B} \ \mapsto \ \bigcap_{\gamma_0 \in \mathfrak{B}} P(\gamma_0^*).$$

In particular, the rays of $\mathrm{GKZ}(\mathcal{V})$ correspond to the *submaximal collections* in the sense that they are dominated only by the collection $\langle \gamma \rangle$ of faces which are invariantly separable from γ.

CHAPTER

TWO

QUOTIENTS

In this chapter we will discuss different notions of quotients which can be assigned to the action of a linear algebraic group G on a normal variety X. In general, it is not evident how to assign such a quotient to the action of an algebraic group. We will introduce the concepts of the GIT-limit, the closely related limit quotient and the Chow quotient. We show that for torus actions the normalisations of the limit quotient and the Chow quotient coincide.

Section 2.1 contains an overview of the variation of GIT-quotients, our main sources for this are [**4, 7, 14, 38**]. In Sections 2.2 and 2.3 we discuss the GIT-limit, the limit quotient and the Chow quotient, in Section 2.4 we prove that various properties of these quotients. Small parts of Sections 2.2 and 2.3 and with minor modifications the entire Section 2.4 have already been published in our paper *'On Chow quotients of torus actions'* (joint work with Jürgen Hausen and Simon Keicher, see [**11**]). In Section 2.5 we introduce the GIT-limit for the action of a non-reductive group. This section is part of the author's paper *'Point Configurations and Translations'*, see [**9**].

2.1. Variation of GIT-Quotients

The conditions for the existence of good quotients are quite restrictive. In fact, if the linear algebraic group G acts on a complete variety X such that there exist a point $x \in X$ with finite isotropy group and a good quotient $X \to X /\!\!/ G$, then G is finite. However, the situtation looks better if we pass to open G-invariant subsets. By a theorem of Rosenlicht [**60**], for every irreducible G-variety there exists an open G-invariant subset with a good geometric quotient. Naturally the question arises how one can obtain all these subsets in systematic manner. This problem is open in general. Mumford showed in [**55**] that those open subsets admitting a quasiprojective good quotient can be obtained from linearisations of (ample) line bundles. This approach was generalised in [**38**] to work for Weil divisors. Moreover, there exists a combinatorial description for all quotients spaces with the A_2-property, see [**7**].

Let a reductive affine-algebraic group G act on a normal variety X. By a *good G-set U* we mean a G-invariant, open subset $U \subseteq X$ admitting a good quotient $\pi\colon U \to U/\!\!/G$. A subset U' of a good G-set U is said to be *saturated*, if the set U' coincides with $\pi^{-1}(\pi(U'))$. This is the case if and only if for every $u' \in U'$ the orbit closure $\overline{G \cdot u'} \subseteq U$ is contained in U'.

For a saturated subset $U' \subseteq U$ of a good G-set the quotient morphism $\pi\colon U \to U/\!\!/G$ restricts to a good quotient $\pi\colon U' \to \pi(U')$ with an open embedding of the quotient spaces $\pi(U') \subseteq U/\!\!/G$. This means that it suffices to describe good G-sets which are maximal with respect to saturated inclusion.

A good G-set $U \subseteq X$ is called *qp-maximal* if its quotient space $U/\!\!/G$ is quasiprojective and U is maximal with respect to saturated inclusion among those good G-sets with quasiprojective quotient spaces. We now show how these qp-maximal sets are contructed.

As before let X be a normal variety with the action of a reductive group G. A *G-linearisation* of a Weil divisor $D \in \mathrm{WDiv}(X)$ is an extension of the G-action to the relative spectrum

$$X(D) := \mathrm{Spec}_X(\mathcal{A}), \qquad \mathcal{A} := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Gamma(X, \mathcal{O}(nD)),$$

commuting with the canonical $\mathbb{K}^*$-acting and making the projection equivariant. To such a linearised divisor we associate a set of semistable points in the following way. A point $x \in X$ is called *semistable* (with respect to this particular linearised divisor) if there exists a G-invariant global section of some positive multiple nD such that x is not contained in its zero set and the complement of the zero set is affine. The set of *semistable points* is denoted $X^{\mathrm{ss}}(D)$. The quotient spaces correspond to the sets of semistable points in the following way.

Proposition 2.1.1 ([**38**, Proposition 3.3], [**16**, Main Theorem]). *Let G be a reductive group and X a normal G-variety. Then the following assertions hold.*

(i) *For every linearised divisor D on X the set $X^{\mathrm{ss}}(D)$ admits a good quotient $X^{\mathrm{ss}}(D) \to X^{\mathrm{ss}}(D)/\!\!/G$ with a quasiprojective quotient space.*

(ii) *If $U \subseteq X$ is qp-maximal, then there exists a linearised divisor D on X such that $U = X^{\mathrm{ss}}(D)$ holds.*
(iii) *The number of good G-sets of X which are maximal with respect to saturated inclusion is finite.*

Note that for any two linearised divisors D_1 and D_2 their sum $D_1 + D_2$ comes with a canonical linearisation, see [**14**, Section 1]. Moreover, D_1 and D_2 are said to be isomorphic, if there exists a $G\times\mathbb{K}^*$-equivariant isomorphism $X(D_1) \to X(D_2)$ over X. By [**14**, Proposition 1.10] the isomorphism classes of linearised divisors form the group $\mathrm{Cl}_G(X)$ of linearised Weil divisors. It comes with a canonical homomorphisms $\mathrm{Cl}_G(X) \to \mathrm{Cl}(X)$ forgetting about the linearisation. Furthermore, the set of semistable points $X^{\mathrm{ss}}(D)$ only depends on the class of D in $\mathrm{Cl}_G(X)$.

In the case of a principal linearised divisor D every linearisation of the corresponding trivial bundle $X(D) = X\times\mathbb{K} \to X$ is given by a character $w \in \mathbb{X}(G)$ of G, see [**14**, Lemma 2.7]:

$$G \times (X \times \mathbb{K}) \to X \times \mathbb{K}; \qquad g \cdot (x, k) \mapsto (g \cdot x,\ w(g)k). \tag{2.1.1}$$

We now show how to treat the collection of qp-maximal subsets combinatorially. In a first step we consider different linearisations of a principal divisor on an affine variety. For our purposes it suffices to restrict to quasitorus actions, although most of the results also hold for reductive groups, see [**7**]. Let K be a finitely generated, abelian group and A a K-graded, affine $\mathbb{K}$-algebra

$$A = \bigoplus_{w\in K} A_w.$$

Its spectrum $\overline{X} := \mathrm{Spec}(A)$ then comes with the action of the quasitorus $H := \mathrm{Spec}(\mathbb{K}[K])$. Now, let D be a linearised principal divisor on $\overline{X}$. Then the linearisation is uniquely determined by some $w \in \mathbb{X}(H) = K$ as in formula 2.1.1. The corresponding set of semistable points is explicitly given by

$$\overline{X}^{\mathrm{ss}}(w) := \overline{X}^{\mathrm{ss}}(D) = \{x \in \overline{X};\ f(x) \neq 0 \text{ for some } f \in A_{nw} \text{ with } n \geq 1\}.$$

We now discuss which elements of K yield the same sets of semistable points. For this we set $K_{\mathbb{Q}} := K \otimes \mathbb{Q}$ and identify w and $w\otimes 1$ for an element $w \in K$. We then define for any $x \in \overline{X}$ the *orbit cone*

$$\omega_H(x) := \mathrm{cone}(w \in K_{\mathbb{Q}};\ \text{there exists } f \in A_w \text{ with } f(x) \neq 0).$$

The collection Ω_X of all orbit cones is finite. The *GIT-fan* is the following quasifan in $K_{\mathbb{Q}}$

$$\Lambda_H(\overline{X}) := \{\lambda(w);\ w \in K_{\mathbb{Q}}\}, \qquad \lambda(w) := \bigcap_{w\in\omega_H(x)} \omega_H(x) \subseteq K_{\mathbb{Q}}.$$

Its support is the the *weight cone* $\omega_{\overline{X}} := \mathrm{cone}(w \in K_{\mathbb{Q}};\ A_w \neq \{0\})$. It turns out that for a cone $\lambda \in \Lambda_H(\overline{X})$ and any two $w_1, w_2 \in \lambda^\circ$ the sets of semistable points coincide. By [**4**, III, Lemma 1.2.7] for any $w \in \lambda^\circ$ we can write

$$\overline{X}^{\mathrm{ss}}(\lambda) \ := \ \overline{X}^{\mathrm{ss}}(w) \ = \ \{x \in \overline{X};\ w \in \omega_H(x)\} \ = \ \{x \in \overline{X};\ \lambda(w) \subseteq \omega_H(x)\}.$$

Theorem 2.1.2 ([**4**, III, Theorems 1.2.8 and 1.4.3])**.** *Let K be a finitely generated, abelian group, A a K-graded, affine $\mathbb{K}$-algebra and $\overline{X} = \mathrm{Spec}(A)$ its spectrum. Then there exists an order reversing bijection between the GIT-fan $\Lambda_H(\overline{X})$ and the sets of semistable points of $\overline{X}$ arising from a principal linearised divisor*

$$\begin{array}{rcl} \Lambda_H(\overline{X}) & \to & \{\overline{X}^{\mathrm{ss}}(w);\ w \in K\}, \\ \lambda & \mapsto & \overline{X}^{\mathrm{ss}}(\lambda). \end{array}$$

In particular, for any two cones $\lambda_1, \lambda_2 \in \Lambda_H(\overline{X})$ in the GIT-fan we have

$$\begin{array}{rclcrcl} \overline{X}^{\mathrm{ss}}(\lambda_1) & \subseteq & \overline{X}^{\mathrm{ss}}(\lambda_2) & \iff & \lambda_1 & \supseteq & \lambda_2, \\ \overline{X}^{\mathrm{ss}}(\lambda_1) & = & \overline{X}^{\mathrm{ss}}(\lambda_2) & \iff & \lambda_1 & = & \lambda_2. \end{array}$$

If moreover A is factorially K-graded, then every set of semistable points stems from a principal linearised divisor, i.e. we have

$$\{\overline{X}^{\mathrm{ss}}(w);\ w \in K\} \ = \ \{\textit{qp-maximal subsets of } \overline{X}\}.$$

We now turn to the non-affine case. For this let X be a normal variety with finitely generated divisor class group $K := \mathrm{Cl}(X)$, only constant invertible functions and finitely generated Cox ring $\mathcal{R}(X)$. Let $p_X\colon \hat{X} \to X$ be the corresponding Cox construction. Suppose moreover that X comes with the action of a torus $T \times X \to X$. By [**14**, Proposition 3.1(iv)] this action lifts to an action of T on $\overline{X} := \mathrm{Spec}(\mathcal{R}(X))$ leaving $\hat{X}$ invariant, commuting with the H-action and turning p_X into a T-equivariant morphism.

Since $\mathcal{R}(X)$ is K-factorial, all sets of $H \times T$-semistable sets stem from characters in $\mathbb{X}(H \times T) \cong K \times M$ where $M := \mathbb{X}(T)$. We now want to relate these sets of semistable points to the sets of T-semistable points of X.

It is not true that good $H \times T$-sets of $\overline{X}$ are in one-to-one correspondence with the good T-sets of X. In fact, if $U \subseteq \overline{X}$ is a good $H \times T$ set, then its image $p_X(U)$ may even fail to be a good T-set at all. However, there is a way to surjectively map the good $H \times T$ sets of $\overline{X}$ onto the good T-sets of X. For details on this we refer to [**7**, Theorem 4.5].

Let us discuss a setting in which the situation looks significantly better. In addition to the assumptions made so far let X be projective. We want to relate the sets of semistable points arising from ample divisor classes to a partial fan of the GIT-fan $\Lambda_{H\times T}(\overline{X})$. For this let $\kappa^\circ \subseteq K_{\mathbb{Q}}$ denote the (open) cone of ample divisor classes of X. By the open *T-ample cone* we mean $\kappa^\circ \times M_{\mathbb{Q}} \subseteq K_{\mathbb{Q}} \times M_{\mathbb{Q}}$ and we call the partial fan

$$\Lambda^{\mathrm{am}}_{H\times T}(X) \;:=\; \{\lambda \cap (\kappa^\circ \times M_{\mathbb{Q}});\quad \lambda \in \Lambda_{H\times T}(\overline{X})\}$$

the *ample GIT fan* of X. It describes the sets of semistable points arising from ample linearised divisor classes in the following sense.

Proposition 2.1.3 ([**7**, Proposition 6.1]). *Let X be a projective Mori Dream Space and the notation be as before. Then we have an order-reversing bijection*

$$\Lambda^{\mathrm{am}}_{H\times T}(X) \quad\longrightarrow\quad \left\{ \begin{array}{c} \textit{sets of semistable points} \\ X^{\mathrm{ss}}(D) \textit{ with } D \textit{ ample} \end{array} \right\},$$

$$\lambda \quad\mapsto\quad \overline{X}^{\mathrm{ss}}(\lambda)/\!\!/H.$$

Note that this decomposition of the T-ample cone was originally already considered in [**24, 66**]. However, it was looked at from a different point of view, for the connection see [**14**]. Also it was clear to the authors that this decomposition gives rise to the GIT-limit which we will discuss in the next section.

2.2. The GIT-Limit and the Limit Quotient

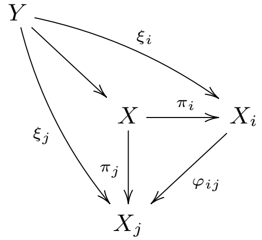
In this section we discuss the construction of the GIT-limit and the limit quotient. As we have seen in the preceding section the quotient spaces depend on the choice of a linearised divisor. However, we would like to define a canonical quotient space. The GIT-limit is a method of constructing such a space from the collection of quotient spaces stemming from different semistable sets. Let us recall the notion of an inverse system, its limit and universal property.

Definition 2.2.1. Let $(I, >)$ be a partially ordered set and $\mathcal{X} := \{X_i;\ i \in I\}$ a collection of objects in a category $\mathfrak{C}$. Assume that for any $i, j \in I$ with $i \geq j$ there exists a morphism $\varphi_{ij} \colon X_i \to X_j$. Then the pair $(\mathcal{X}, \{\varphi_{ij};\ i \geq j\})$ is an *inverse system* if it satisfies the following two conditions.

(i) $\varphi_{ii} = \mathrm{id}_{X_i}$ holds for every $i \in I$.
(ii) For any $i \geq k \geq j$ the equation $\varphi_{ij} = \varphi_{kj} \circ \varphi_{ik}$ holds.

Definition 2.2.2. Let I be a partially ordered set and $\mathcal{S} := (\mathcal{X}, \{\varphi_{ij};\ i \geq j\})$ be an inverse system. An object $X \in \mathrm{Ob}(\mathfrak{C})$ in a category $\mathfrak{C}$ together with morphisms $\pi_i \colon X \to X_i$ satisfying $\pi_i = \varphi_{ij} \circ \pi_j$ is called its *inverse limit* of $\mathcal{S}$ if it has the following universal property.

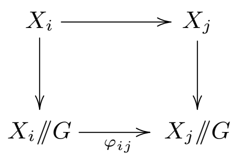
For every object Y in $\mathfrak{C}$ with morphisms $\xi_i \colon Y \to X_i$ satisfying $\xi_i = \varphi_{ij} \circ \xi_j$ there exists a unique morphism $Y \to X$ making the following diagram commutative.



Remark 2.2.3. The inverse limit need not exists in a specific category; but if it does it is unique up to isomorphy. In this case we write $\varprojlim(\mathcal{S})$. Moreover, in the case of algebraic varieties and a finite inverse system $\mathcal{S}$ the limit has the following explicit form

$$\varprojlim(\mathcal{S}) = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i;\ x_j = \varphi_{ij}(x_i) \text{ for any } i \geq j \right\}.$$

Construction 2.2.4. Suppose that G is a reductive affine-algebraic group and X is a normal G-variety. Let $X_1, \ldots, X_r \subseteq X$ be the open sets of semistable points arising from G-linearised ample divisor classes on X. Then, whenever $X_i \subseteq X_j$ holds, the universal property of good quotients gives rise to a commutative diagram



where the induced map $\varphi_{ij} \colon X_i /\!\!/ G \to X_j /\!\!/ G$ of quotients is a dominant projective morphism. This turns the quotient spaces into an inverse system, the *(ample) GIT-system.*

Definition 2.2.5. Let $\mathcal{S}$ be the (ample) GIT-system for the action of G on X. Then the *GIT-limit* is the inverse limit

$$X \overset{\text{GIT}}{\underset{\text{Lim}}{/}} G \ := \ \varprojlim(\mathcal{S}).$$

Although each quotient of a set of semistables points of a normal irreducible varity is normal and irreducible again, their limit need not be. For this consider the following counterexample.

Example 2.2.6. Consider the action of $T := \mathbb{K}^*$ on the affine toric variety $X := V(T_1T_2 - T_3T_4) \subseteq \mathbb{K}^4$ given by

$$t \cdot (x_1\,,\, x_2\,,\, x_3\,,\, x_4) \;=\; (tx_1\,,\, t^{-1}x_2\,,\, t \cdot x_3\,,\, t^{-1}x_4).$$

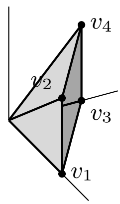
We will now show that the GIT-limit $X \overset{\text{GIT}}{\underset{\text{Lim}}{/}} H$ is reducible. For this we will explicitly compute the quotient spaces of the sets of semistable points arising from different linearisations of a principal divisor. Note that this might not yield the GIT-limit itself but rather a partial limit. Still, there exists a surjective morphism from the full limit onto this partial limit which preserves reducibility.

The idea is to view X as toric variety and deal with its sets of semistable points and their respective quotients in terms of lattices and fans. For this consider the action of the algebraic torus $T_X := (\mathbb{K}^*)^3$ on X by

$$(t_1, t_2, t_3) \cdot (x_1, x_2, x_3, x_4) \quad := \quad (t_1x_1\,,\, t_2x_2\,,\, t_3x_3\,,\, t_1t_2t_3^{-1}x_4).$$

This fixes an open embedding $T_X \subseteq X$. The cone of convergent one-parameter subgroups has four extremal rays in $\mathbb{Q}^3$ generated by

$$v_1 = (1,0,0), \qquad v_2 = (1,0,1),$$
$$v_3 = (0,1,0), \qquad v_4 = (0,1,1).$$



The T-action on X gives rise to an inclusion $T \subseteq T_X$ and hence an injection of the respective lattices of one-parameter subgroups $\mathbb{Z} \to \mathbb{Z}^3$. Explicitly this homomorphism is given by

$$Q^* \colon \mathbb{Z} \to \mathbb{Z}^3; \qquad \nu \mapsto (\nu, -\nu, \nu).$$

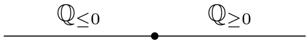
It fits into an exact sequence which allows us to compute the (toric) quotient space of the sets of semistable points. For details on this see [**4**, II, Section 3.1].

$$0 \longrightarrow \mathbb{Z} \xrightarrow{Q^*} \mathbb{Z}^3 \xrightarrow{P} \mathbb{Z}^2 \longrightarrow 0$$

The matrix P contains as rows a basis for $\mathbb{Z}^3/\mathrm{Im}(Q^*)$ and can be chosen as

$$P = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

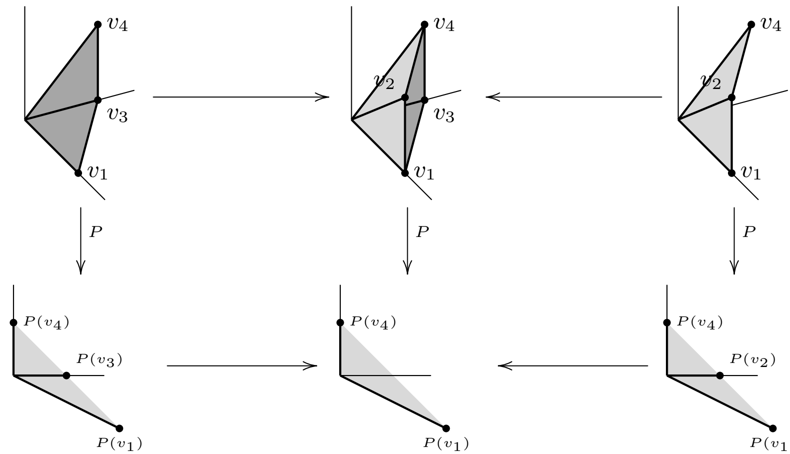
Let us determine the sets of semistable points of X arising from the possible linearisations of a principal divisor. For this note that the GIT-fan for the T-action on X is the (unique) fan in $\mathbb{Q}$ with two maximal cones.



To each of the three cones there corresponds a set of semistable points

$$X^{\mathrm{ss}}(-) = X_{T_2} \cup X_{T_4}, \qquad X^{\mathrm{ss}}(0) = X, \qquad X^{\mathrm{ss}}(+) = X_{T_1} \cup X_{T_3}.$$

The complements of $X^{\mathrm{ss}}(-)$ and $X^{\mathrm{ss}}(+)$ are precisely the toric divisors which correspond to the rays generated by v_2 and v_3 respectively. We then obtain a commutative diagram where the vertical arrows are the good quotients given by the matrix P. Note that all three fans in the top row are projectible in the sense of [**4**, II, Definition 3.1.3], however in the second case not all the faces contribute to the quotient space, see [**4**, II, Construction 3.1.5].



The two maps of the quotient spaces each contract a toric divisor isomorphic to $\mathbb{P}_1$, hence the partial limit of these quotients (i.e. the fibre product) has two irreducible components. The first one is isomorphic to either one of the outer quotients, the second is $\mathbb{P}_1 \times \mathbb{P}_1$. They intersect in a $\mathbb{P}_1$ and the universal property of the fibre product yields a morphism from the GIT-limit onto the fibre product. This shows that the GIT-limit cannot be irreducible.

Although the GIT-limit is in general not irreducible, it has a canonical irreducible component. Again let X_i, $i = 1, \ldots, r$ be the sets of semistable points arising from ample linearised divisor classes on the normal projective variety X. Note that the GIT-limit $Y := X \overset{\mathrm{GIT}}{\underset{\mathrm{Lim}}{/}} G$ comes with a canonical morphism

$$U \; := \; \bigcap_{i=1}^{r} X_i \quad \to \quad Y.$$

Definition 2.2.7. The closure of the image of $U \to Y$ is denoted by $X /\!\!/_{\mathrm{LQ}} G$ and is called the *limit quotient* (of X with respect to G). Its normalisation $X \tilde{/\!\!/}_{\mathrm{LQ}} G$ is the *normalised limit quotient*.

There are canonical proper birational morphisms onto the GIT quotients:

$$\pi_i \colon X /\!\!/_{\mathrm{LQ}} G \ \to \ X_i /\!\!/ G.$$

Suitably shrinking the open set $U \subseteq X$, we obtain a commutative diagram involving the normalisation map:

$$\begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ X \tilde{/\!\!/}_{\mathrm{LQ}} G & \longrightarrow & X /\!\!/_{\mathrm{LQ}} G. \end{array}$$

Note that, in the literature, $X /\!\!/_{\mathrm{LQ}} G$ is called also the 'canonical component' of the GIT-limit, or even shortly the 'GIT-limit'. Similar to the full inverse limit, the limit quotient $X /\!\!/_{\mathrm{LQ}} G$ enjoys a universal property.

Remark 2.2.8. Given an irreducible variety W and a collection of dominant morphisms $\psi_i \colon W \to X_i /\!\!/ G$ with $\psi_j = \varphi_{ij} \circ \psi_i$ for all i, j, there is a unique morphism $\psi \colon W \to X /\!\!/_{\mathrm{LQ}} G$ with $\psi_i = \pi_i \circ \psi$ for all i.

For toric varieties with the action of a subtorus of the dense torus there is a very convenient way to compute the (normalisation) of the limit quotient. For this we consider the following setting.

Setting 2.2.9. Let Z be a quasiprojective toric variety with acting torus T_Z and consider the action of a subtorus $T \subseteq T_Z$. The toric variety Z arises from a fan Σ in some $\mathbb{Z}^r$ and $T \subseteq T_Z$ corresponds to an embedding $\mathbb{Z}^k \subseteq \mathbb{Z}^r$ of a sublattice. Let $P \colon \mathbb{Z}^r \to \mathbb{Z}^{r-k}$ the projection. The *quotient fan* of Σ with respect to P is the fan in $\mathbb{Z}^{r-k}$ with the cones

$$\tau(v) \ := \ \bigcap_{\sigma \in \Sigma, v \in P(\sigma)} P(\sigma), \qquad v \in \mathbb{Q}^{r-k}.$$

Proposition 2.2.10. *See* [**23**]. *Consider the Setting 2.2.9, let Σ' be the quotient fan in $\mathbb{Z}^{r-k}$ with respect to $\mathbb{Z}^r \to \mathbb{Z}^{r-k}$ and let Z' the associated toric variety. Then Z' is isomorphic to the normalised limit quotient $Z \tilde{/\!\!/}_{\mathrm{LQ}} T$.*

Example 2.2.11. We consider two examples.

(i) We return to Example 2.2.6 and again consider the action of $T := \mathbb{K}^*$ on the affine toric variety $X := V(T_1T_2 - T_3T_4)$ given by

$$t \cdot (x_1\,,\, x_2\,,\, x_3\,,\, x_4) \ := \ (tx_1\,,\, t^{-1}x_2\,,\, tx_3\,,\, t^{-1}x_4).$$

The fan of X has the single maximal cone generated by

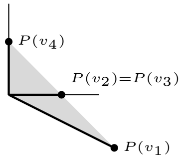
$$v_1 = (1,0,0), \qquad v_2 = (1,0,1),$$
$$v_3 = (0,1,0), \qquad v_4 = (0,1,1).$$

The inclusion $T \subseteq T_X$ corresponds to an inclusion of the respective lattices of one-parameter subgroups

$$\mathbb{Z}_T \to \mathbb{Z}^3_X; \qquad \nu \mapsto (\nu, -\nu, \nu).$$

Then the projection $P\colon \mathbb{Z}^3_X \to \mathbb{Z}^3_X/\mathbb{Z}_T = \mathbb{Z}^2$ is given by the matrix P and from this we can compute the quotient fan.

$$P = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \qquad \Sigma' =$$



(ii) Consider the action of $\mathbb{K}^*$ on the the projective space $\mathbb{P}_4$ given by

$$t \cdot [x_0 : x_1 : x_2 : x_3 : x_4] := [tx_0 : t^{-1}x_1 : tx_2 : t^{-1}x_3 : x_4].$$

The fan of $\mathbb{P}_4$ is the complete fan with rays generated by

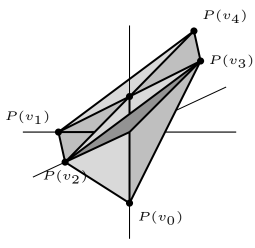
$$v_0 = (-1,-1,-1,-1), \qquad v_1 = (1,0,0,0), \qquad v_2 = (0,1,0,0),$$
$$v_3 = (0,0,1,0), \qquad v_4 = (0,0,0,1).$$

The inclusion $T \subseteq T_X$ corresponds to an inclusion of the respective lattices of one-parameter subgroups

$$\mathbb{Z} \to \mathbb{Z}^4; \qquad \nu \mapsto (\nu, -\nu, \nu, -\nu).$$

Then the projection $P\colon \mathbb{Z}^4 \to \mathbb{Z}^4/\mathbb{Z} = \mathbb{Z}^3$ is given by the matrix P and from this we can compute the quotient fan.

$$P = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \Sigma' =$$



In this example the additional ray generated by $(1,0,0)$ appears. It corresponds to the blow-up of one of the GIT-quotients.

2.3. The Chow Quotient

We consider the action of an affine-algebraic group G on a normal variety X. The GIT-limit and the limit quotient have the advantage that they come with a combinatorial description which makes them at least partially accessible to computations. However, there are two main drawback of the classic GIT-constructions. Firstly, the GIT-limit lacks being canonical insofar as it depends on a choice of quotient spaces (namely those stemming from ample linearised divisors). But there are examples of so-called exotic orbit spaces which cannot be constructed from ample classes, see [**14**]. Secondly, classic invariant geometry heavily relies on Hilbert's Finiteness Theorem stating that the algebra of invariants is finitely generated if G is reductive. Although there are examples of non-reductive groups with finitely generated invariant algebras, this fails in general, even for relatively simple algebraic groups, see [**54, 56, 59**].

Before we introduce a method to extend the notion of the GIT-limit to certain unipotent groups, we discuss an alternative approach, namely the Chow quotient. The main idea is to view the orbits (more precisely their closures) of some action as points in a variety parametrizing subvarieties of X, its Chow variety. The construction behaves better than the GIT-limit concerning the two mentioned aspects. Neither does one have to make any (relevant) choices nor is this method restricted to a certain class of groups. However, the Chow variety and thereby also the Chow quotient are quite hard to access. Even for relatively simple examples the Chow variety is unknown. Surprisingly, for torus actions the Chow quotient and the limit quotient are closely related, in fact they share a common normalisation, for details on this we refer to the next Section 2.4.

First let us discuss the Chow variety. It is a classical construction and was originally introduced by Chow and van der Waerden, see [**21**]. Our main source for this section is [**36**], but see also [**51, 61**]. Let $\tilde{\mathbb{P}}_n := \mathrm{Gr}(n-1,n) \cong \mathbb{P}_n$ be the variety parametrizing hyperplanes in $\mathbb{P}_n$. For a k-dimensional, irreducible subvariety $X \subseteq \mathbb{P}_n$ its *degree* is the number of points in $E \cap X$ where E is a generic point in the Grassmannian $\mathrm{Gr}(n-k,n)$.

Construction 2.3.1 (Chow variety of $\mathbb{P}_n$). Let $X \subseteq \mathbb{P}_n$ be a purely k-dimensional subvariety of degree d. We first consider the set

$$\Gamma \quad := \quad \left\{(x, H_0, \ldots, H_k);\ x \in H_i \text{ for } i = 0, \ldots, k\right\} \quad \subseteq \quad X \times \tilde{\mathbb{P}}_n^{k+1}.$$

It is closed, purely $(n(k+1)-1)$-dimensional and it has as many irreducible components as does X. Now let $p\colon \Gamma \to \tilde{\mathbb{P}}_n^{k+1}$ be the projection. It turns out that p is birational and its image $p(\Gamma)$ is a hypersurface in $\tilde{\mathbb{P}}_n^{k+1}$. As such it is the zero set of the so-called *Chow form* of X, a $\mathbb{Z}^{k+1}$-homogeneous polynomial

$$F_X \in W := \mathbb{K}[T_{ij},\ i=0,\dots,n,\ j=0,\dots,k]_{\mathbf{d}},$$

where $\mathbf{d} := (d,\dots,d) \in \mathbb{Z}^{k+1}$ and the grading ist given by $\deg(T_{ij}) := e_j$. Since F_X is unique up to multiplication with a scalar, there exists a well-defined map

$$\xi\colon \left\{ \begin{array}{c} \text{subvarieties } X \subseteq \mathbb{P}_n \\ \text{of pure dimension } k \\ \text{and degree } d \end{array} \right\} \to \mathbb{P}(W), \qquad X \mapsto [F_X].$$

This map is injective and $\xi(X)$ is called the *Chow point* of X. The image $\tilde{\mathcal{C}}(\mathbb{P}_n,k,d)$ of ξ is a locally closed subset of $\mathbb{P}(W)$ and its closure $\mathcal{C}(\mathbb{P}_n,k,d)$ is the *Chow variety* of $\mathbb{P}_n$ for the parameters k and d.

We now want to generalise this idea to an arbitrary projective variety Z. To this end we choose an embedding of Z into some $\mathbb{P}_n$. It turns out that the Construction of the Chow variety is in fact independent of the choice of this embedding.

Construction 2.3.2 (Chow variety)**.** Let $Z \subseteq \mathbb{P}_n$ be a projective variety. Then we set

$$\tilde{\mathcal{C}}(Z,k,d) := \left\{[F_X] \in \tilde{\mathcal{C}}(\mathbb{P}_n,k,d);\ X \subseteq Z\right\}.$$

This is a subvariety of $\tilde{\mathcal{C}}(\mathbb{P}_n,k,d)$ and its closure $\mathcal{C}(Z,k,d)$ in $\mathcal{C}(\mathbb{P}_n,k,d)$ is called the *Chow variety* of Z with respect to k and n.

Remark 2.3.3. Let $Y \subseteq Z \subseteq \mathbb{P}_n$ be two projective varieties. Then $\mathcal{C}(Y,k,d) \subseteq \mathcal{C}(Z,k,d)$ holds for all possible choices of k and d.

Example 2.3.4. Compare [**29**, Chapter 4, Examples 1.2, 1.3] and [**26**, Theorem 1].

(i) The Chow variety parametrising subvarieties of $\mathbb{P}_n$ with degree 1 and dimension k is the Grassmannian variety $\mathrm{Gr}(k,n)$. It is irreducible and smooth and can be described by the Plücker relations.
(ii) The Chow variety of 1-dimensional subvarieties and degree 2 in $\mathbb{P}_3$ has two irreducible components of dimension 8. The first component describes the subvarieties consisting of two lines, the second parametrises quadrics.

In [**49**] Kapranov, Sturmfels and Zelevinsky introduced the notion of the Chow quotient in order to obtain a somewhat canonical quotient of a group action. As its contruction relies on the Chow variety again the Chow quotient does not depend on the embedding.

Construction 2.3.5 (Chow quotient)**.** Let Z be an algebraic variety with the action of an algebraic group G. Then on a sufficienly small open subset of Z the closures of the G-orbits have a common dimension k and degree d. The collection of orbit closures corresponds to a certain subset of $\mathcal{C}(\mathbb{P}_n, k, d)$. Its closure $Z /\!\!/_{\mathrm{CQ}} G$ is called the *Chow quotient* of Z with respect to the G-action.

Construction 2.3.6. By the *normalised Chow quotient* $X \,\widetilde{/\!\!/}_{\mathrm{CQ}}\, G$ we mean the normalisation of $X /\!\!/_{\mathrm{CQ}} G$. With a suitably small chosen $U \subseteq X$, one obtains a commutative diagram of morphisms involving the normalisation map:

$$\begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ X \,\widetilde{/\!\!/}_{\mathrm{CQ}}\, G & \longrightarrow & X /\!\!/_{\mathrm{CQ}} G. \end{array}$$

Consider the Setting 2.2.9 and assume that in addition Z is projective. In this situation the normalised Chow quotient is a toric variety.

Proposition 2.3.7 ([**49**])**.** *Consider the Setting 2.2.9 and assume that Z is projective. Let Σ' be the quotient fan in $\mathbb{Z}^{r-k}$ with respect to $\mathbb{Z}^r \to \mathbb{Z}^{r-k}$ and let Z' the associated toric variety. Then Z' is isomorphic to the normalised Chow quotient $Z \,\widetilde{/\!\!/}_{\mathrm{CQ}}\, T$.*

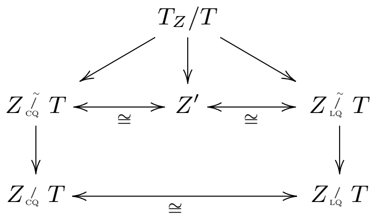
2.4. Comparing Chow and Limit Quotient

With minor modifications this section has already been published in the paper *'On Chow quotients of torus actions'* ([**11**]), which is a joint work with Jürgen Hausen and Simon Keicher.

The limit quotient arises from the variation of Mumford's GIT quotients [**55**]. Its construction relies on finiteness of the number of possible sets of semistable points [**24, 66**].

For a general reductive group action, the (normalised) Chow quotient and the (normalised) limit quotient need not coincide. For torus actions, however, they do. This statement seems to have folklore status; a proof under a certain hypothesis can be found in [**45**, Thm. 3.8]. Let us indicate how to deduce it from the corresponding statement in the case of subtorus actions on projective toric varieties obtained in [**23, 49**]. For this we consider again the Setting 2.2.9.

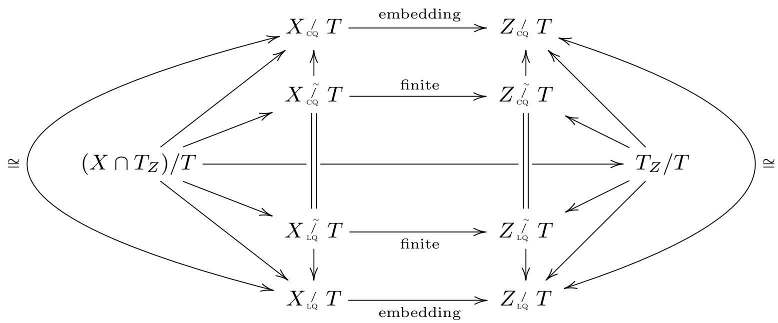
Proposition 2.4.1. *See* [**23, 49**]. *Consider the projective toric variety Z arising from a fan Σ in $\mathbb{Z}^r$ and the action of a subtorus $T \subseteq T_Z$ corresponding to a sublattice $\mathbb{Z}^k \subseteq \mathbb{Z}^r$. Let Σ' be the quotient fan in $\mathbb{Z}^{r-k}$ with* *respect to* $\mathbb{Z}^r \to \mathbb{Z}^{r-k}$ *and let* Z' *the associated toric variety. Then we have a commutative diagram*



In particular, the (normalised) Chow quotient and the (normalised) limit quotient of the T*-action on* Z *are isomorphic to each other.*

We turn to the general case. The result is formulated for a projective variety X which is equivariantly embedded into a toric variety Z. Note that for a normal projective X, this can always be achieved, even with a projective space Z.

Proposition 2.4.2. *Let* Z *be a projective toric variety,* $T \subseteq T_Z$ *a subtorus of the big torus and* $X \subseteq Z$ *a closed* T*-invariant subvariety intersecting* T_Z*. Then there is a commutative diagram*



where $X \mathbin{\widetilde{/}}_{\mathrm{CQ}} T \to Z \mathbin{\widetilde{/}}_{\mathrm{CQ}} T$ *and* $X \mathbin{\widetilde{/}}_{\mathrm{LQ}} T \to Z \mathbin{\widetilde{/}}_{\mathrm{LQ}} T$ *normalise the closures of the images of* $(X \cap T_Z)/T$ *under the canonical open embeddings of* T_Z/T*.*

Proof of Proposition 2.4.2, version 1. The right part of the diagram is Proposition 2.4.1. The closed embedding $X \mathbin{/}_{\mathrm{CQ}} T \to Z \mathbin{/}_{\mathrm{CQ}} T$ exists by the construction of the Chow quotient; compare also [**30**, Thm. 3.2].

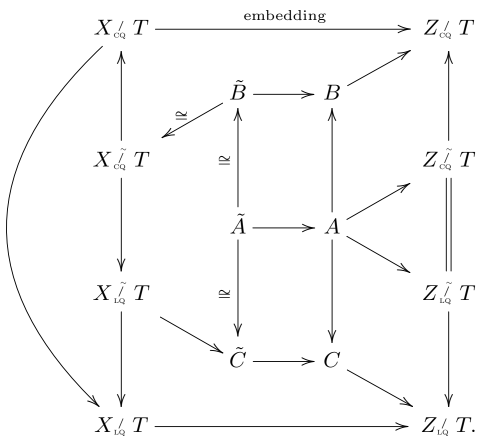
To obtain a morphism $X \mathbin{/}_{\mathrm{LQ}} T \to Z \mathbin{/}_{\mathrm{LQ}} T$, consider the sets of semistable points $V_1, \ldots, V_s \subseteq Z$ defined by T-linearised ample line bundles on Z. Then the sets $U_i := X \cap V_i$ are sets of semistable points of the respective pullback bundles, see [**55**, Thm. 1.19] and we have induced morphisms $U_i /\!\!/ T \to V_i /\!\!/ T$.

Since the $U_i /\!\!/ T$ form a subsystem of the full GIT-system of X, the universal property 2.2.8 yields a morphism of the limit quotients sending $X /_{\mathrm{LQ}} T$ birationally onto the closure of $(X \cap T_Z)/T$.

Now look at the canonical morphism $X /_{\mathrm{CQ}} T \to X /_{\mathrm{LQ}} T$ provided by [**50, 66**]. It fits into the diagram established so far which in turn implies that $X /_{\mathrm{CQ}} T \to X /_{\mathrm{LQ}} T$ is an isomorphism and $X /_{\mathrm{LQ}} T \to Z /_{\mathrm{LQ}} T$ is an embedding. Finally, the respective normalisations fit into the diagram via their universal properties. □

Note that we will only use the part of Proposition 2.4.2 concerning the normalisations of the Chow and limit quotients. We provide another alternative proof using similar arguments as above but not the isomorphism $Z /_{\mathrm{CQ}} T \to Z /_{\mathrm{LQ}} T$ of Proposition 2.4.1.

Proof of Proposition 2.4.2, version 2. By the definition of the Chow quotient, there is a canonical closed embedding $X /_{\mathrm{CQ}} T \to Z /_{\mathrm{CQ}} T$ and the image is the closure $B \subseteq Z /_{\mathrm{CQ}} T$ of $(X \cap T_Z)/T$; see also [**30**, Thm. 3.2] The universal property of the normalisation $\tilde{B} \to B$ provides a morphism $\tilde{B} \to X \tilde{/}_{\mathrm{CQ}} T$ which turns out to be birational and finite and hence is an isomorphism. The closure $A \subseteq Z \tilde{/}_{\mathrm{CQ}} T$ of $(X \cap T_Z)/T$ is mapped onto B under $Z \tilde{/}_{\mathrm{CQ}} T \to Z /_{\mathrm{CQ}} T$ and for the normalisation $\tilde{A} \to A$ we obtain an induced isomorphism $\tilde{A} \to \tilde{B}$. Together, this gives the upper half of the following commutative diagram:



The morphism $X /_{\mathrm{CQ}} T \to X /_{\mathrm{LQ}} T$ from the Chow quotient onto the limit quotient was established in [**45**]. It respects the canonical embedding of $(X \cap T_Z)/T$, is birational and lifts to a morphism $X \tilde{/}_{\mathrm{CQ}} T \to X \tilde{/}_{\mathrm{LQ}} T$ of the normalisations. The canonical isomorphism $Z \tilde{/}_{\mathrm{CQ}} T \to Z \tilde{/}_{\mathrm{LQ}} T$ is due to Proposition 2.4.1.

For the remaining part of the diagram, consider the sets of semistable points $V_1, \dots, V_s \subseteq Z \tilde{/}_{\mathrm{CQ}} T_0$ defined by T-linearised ample line bundles on Z. Then the sets $U_i := X \cap V_i$ are sets of semistable points of the respective pullback bundles, see [**55**, Thm. 1.19] and we have induced morphisms $U_i /\!\!/ T \to V_i /\!\!/ T$. Since the $U_i /\!\!/ T$ form a subsystem of the full GIT-system of X, the universal property 2.2.8 yields a morphism of the limit quotients sending $X /_{\mathrm{LQ}} T$ birationally onto the closure $C \subseteq Z /_{\mathrm{LQ}} T$ of $(X \cap T_Z)/T$. For the normalisation $\tilde{C} \to C$ we obtain an isomorphism $\tilde{A} \to \tilde{C}$ and a morphism $X \tilde{/}_{\mathrm{LQ}} T \to \tilde{C}$. We conclude that $X \tilde{/}_{\mathrm{CQ}} T \to X \tilde{/}_{\mathrm{LQ}} T$ is an isomorphism. □

Corollary 2.4.3. *Let $T \times X \to X$ be the action of a torus T on a normal projective variety X. Then the normalised Chow quotient $X \tilde{/}_{\mathrm{CQ}} T$ and the normalised limit quotient $X \tilde{/}_{\mathrm{LQ}} T$ are isomorphic to each other.*

The following corollary shows that for torus actions, the limit quotient is up to normalisation already determined by the possible linearisations of a single ample bundle; a statement which fails in general for other reductive groups, compare also [**50**, Remark 0.4.10].

Corollary 2.4.4. *Let $T \times X \to X$ be the action of a torus T on a normal projective variety X. Then the subsystem of GIT quotients arising from the possible T-linearisations of a given ample line bundle $\mathcal{L}$ has the same normalised limit quotient as the full system of GIT quotients.*

Proof. Fix a T-linearisation of $\mathcal{L}$ and consider the T-equivariant embedding $X \to \mathbb{P}_r$ defined by the a suitable power of $\mathcal{L}$. Then the subsystem of the GIT quotients on X arising from other linearisations of $\mathcal{L}$ is induced from the full GIT system on $\mathbb{P}_r$. Now apply Proposition 2.4.2. □

We now prove the reduction theorem. It says in particular, that the Chow quotient of a torus action is birationally dominated by an iterated Chow quotient with respect to $\mathbb{K}^*$-actions.

Theorem 2.4.5. *Let $T \times X \to X$ be the action of a torus T on a normal projective variety X. Fix a subtorus $T_0 \subseteq T$ and set $T_1 := T/T_0$. Then we have canonical proper birational morphisms*

$$(X \tilde{/}_{\mathrm{CQ}} T_0) \tilde{/}_{\mathrm{CQ}} T_1 \ \to\ X \tilde{/}_{\mathrm{CQ}} T, \qquad\qquad (X \tilde{/}_{\mathrm{LQ}} T_0) \tilde{/}_{\mathrm{LQ}} T_1 \ \to\ X \tilde{/}_{\mathrm{LQ}} T.$$

Proof. First consider the case that T is a subtorus of the big torus T_Z of a toric variety Z. Then the maps $T_Z \to T_Z/T_0 \to T_Z/T$ correspond to lattice homomorphisms $\mathbb{Z}^r \to \mathbb{Z}^{r-k_0} \to \mathbb{Z}^{r-k}$. The fan Σ of Z lives in $\mathbb{Z}^r$ and we have the quotient fan Σ_0 of Σ with respect to $\mathbb{Z}^r \to \mathbb{Z}^{r-k_0}$. The quotient fan of Σ_0 with respect to $\mathbb{Z}^{r-k_0} \to \mathbb{Z}^{r-k}$ refines the quotient fan of Σ with respect to $\mathbb{Z}^r \to \mathbb{Z}^{r-k}$. Translated to toric varieties, this means that we have the desired maps

$$(Z \tilde{/\!/}_{\mathrm{CQ}} T_0) \tilde{/\!/}_{\mathrm{CQ}} T_1 \ \to\ Z \tilde{/\!/}_{\mathrm{CQ}} T, \qquad\qquad (Z \tilde{/\!/}_{\mathrm{LQ}} T_0) \tilde{/\!/}_{\mathrm{LQ}} T_1 \ \to\ Z \tilde{/\!/}_{\mathrm{LQ}} T.$$

We turn to the general case. Suitably embedding X, we can arrange the setup of Proposition 2.4.2. Then we have a finite T_1-equivariant map $\nu\colon X \tilde{/\!/}_{\mathrm{CQ}} T_0 \to Z \tilde{/\!/}_{\mathrm{CQ}} T_0$. We consider the normalised limit quotient of the T_1-action on $X \tilde{/\!/}_{\mathrm{CQ}} T_0$. In a first step, we establish a commutative diagram

$$\begin{array}{ccc} & (X \cap T_Z)/T_0 & \\ \swarrow & & \searrow \\ (X \tilde{/\!/}_{\mathrm{CQ}} T_0) /\!/_{\mathrm{LQ}} T_1 & \longrightarrow & (Z \tilde{/\!/}_{\mathrm{CQ}} T_0) /\!/_{\mathrm{LQ}} T_1 \end{array}$$

For this, let $V_1, \ldots, V_s \subseteq Z \tilde{/\!/}_{\mathrm{CQ}} T_0$ be the sets of semistable points arising from T_1-linearised ample line bundles. Then the inverse images $\nu^{-1}(V_i) \subseteq X \tilde{/\!/}_{\mathrm{CQ}} T_0$ are sets of semistable points of the respective pullback bundles, see [**55**, Thm. 1.19]. Note that we have canonical induced maps

$$\nu^{-1}(V_i)/\!\!/T_1 \ \to\ V_i/\!\!/T_1.$$

Consequently, the limit quotient of the system of the quotients $\nu^{-1}(V_i)/\!\!/T_1$ maps to the limit quotient $(Z \tilde{/\!/}_{\mathrm{CQ}} T_0) /\!/_{\mathrm{LQ}} T_1$. Since the $\nu^{-1}(V_i)/\!\!/T_1$ form a subsystem of the full GIT system of $X \tilde{/\!/}_{\mathrm{CQ}} T_0$, this gives rise to a morphism

$$(X \tilde{/\!/}_{\mathrm{CQ}} T_0) /\!/_{\mathrm{LQ}} T_1 \ \to\ (Z \tilde{/\!/}_{\mathrm{CQ}} T_0) /\!/_{\mathrm{LQ}} T_1$$

as needed for the above commutative diagram. As in the proof of Proposition 2.4.2, we may pass to the normalisations and thus obtain a morphism

$$(X \tilde{/\!/}_{\mathrm{CQ}} T_0) \tilde{/\!/}_{\mathrm{CQ}} T_1 \ \to\ (Z \tilde{/\!/}_{\mathrm{CQ}} T_0) \tilde{/\!/}_{\mathrm{CQ}} T_1.$$

Now, by the toric case, we have a proper birational morphism from the toric variety on the right hand side onto $Z \tilde{/\!/}_{\mathrm{CQ}} T$. Using once more Proposition 2.4.2, the assertion follows. □

2.5. The non-reductive GIT-Limit

This section has already been published in the author's paper *'Point configurations and Translations'*, see [**9**].

In this section we deal with the problem of assigning a canonical quotient to the action of a unipotent group G on a $\mathbb{Q}$-factorial, projective Mori Dream Space X. For reductive groups an answer to this problem is the GIT-limit, i.e. the limit of the inverse system consisting of the Mumford quotients $X^{\mathrm{ss}}(D)/\!\!/G$. However, this method relies on Hilbert's Finiteness Theorem which guarantees, that for a linear action of a reductive group G on any affine algebra the invariant algebra is affine again. So we make a further finiteness assumption on certain G-invariants which for example holds when $G = \mathbb{G}_a$.

In [**25**, Definition 4.2.6] Doran and Kirwan introduce the notion of *finitely generated semistable sets* for the action of a unipotent group, namely the sets $X^{\mathrm{ss}}_{\mathrm{fg}}(D) := \bigcup X_f$ where D is some ample divisor, $f \in \mathcal{O}_{nD}(X)^G$ is an invariant section for some $n > 0$ and $\mathcal{O}(X_f)^G$ is finitely generated. These sets possess *enveloped quotients*

$$r\colon X^{\mathrm{ss}}_{\mathrm{fg}}(D) \;\to\; r\left(X^{\mathrm{ss}}_{\mathrm{fg}}(D)\right) \subseteq X/\!\!/_D G$$

where the *enveloping quotient* $X/\!\!/_D G$ is obtained by gluing together the affine pieces $\mathrm{Spec}(\mathcal{O}(X_f)^G)$. Using a Gelfand-MacPherson type correspondence described in [**6**] we now turn this collection of enveloped quotients into an inverse system.

Consider the action of an affine-algebraic, simply connected group G with trivial character group $\mathbb{X}(G)$ on the normal, projective variety X. Let $K \subseteq \mathrm{WDiv}(X)$ be a free and finitely generated group of Weil divisors mapping isomorphically onto the divisor class group $\mathrm{Cl}(X)$. We then associate to X a sheaf of graded algebras

$$\mathcal{R} \quad := \bigoplus_{D \in K} \mathcal{O}(D).$$

We suppose that the algebra of global sections $\mathcal{R}(X)$, i.e. the Cox ring of X, is finitely generated. The K-grading yields an action of the torus $H := \mathrm{Spec}(\mathbb{K}[K])$ on the relative spectrum $\hat{X} := \mathrm{Spec}_X(\mathcal{R})$ and the canonical morphism $p\colon \hat{X} \to X$ is a good quotient for this action. By linearisation the G-action on X lifts to a unique action of G on the total coordinate space $\overline{X} := \mathrm{Spec}(\mathcal{R}(X))$ which commutes with the H-action and turns p into an equivariant morphism, see [**38**, Section 1].

Now suppose that the algebra of invariants $\mathcal{R}(X)^G$ is finitely generated as well and let $\overline{Y}$ be its spectrum. The inclusion of the invariants gives rise to a morphism $\kappa\colon \overline{X} \to \overline{Y}$. Since κ is not necessarily surjective, it need not have the universal property of quotients. However, passing to the category of constructible spaces we obtain a categorical quotient $\kappa\colon \overline{X} \to \overline{Y}' := \kappa(\overline{X})$, see [**6**] for details.

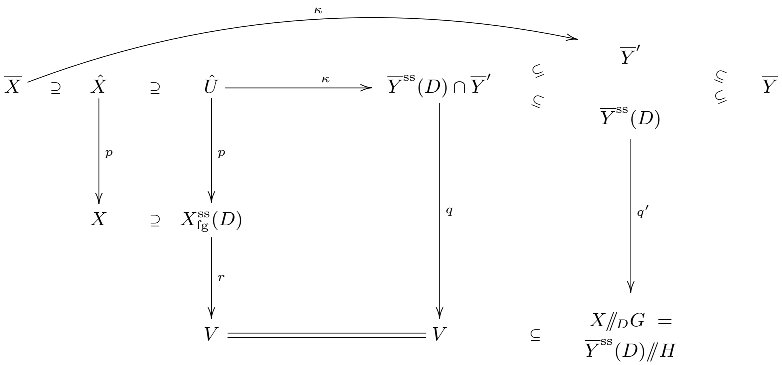
For every ample $D \in K$ standard geometric invariant theory provides us with a set of semistable points

$$\overline{Y}^{\mathrm{ss}}(D) := \bigcup \overline{Y}_f \quad \text{where} \quad f \in \mathcal{R}(X)^G_{nD} \quad \text{and} \quad n > 0.$$

These sets admit good quotients for the H-action which are isomorphic to the enveloping quotient $X/\!\!/_D G$ in the sense of Doran and Kirwan. The set of finitely generated semistable points $X^{\mathrm{ss}}_{\mathrm{fg}}(D)$ can be retrieved from $\overline{Y}^{\mathrm{ss}}(D)$ by

$$X^{\mathrm{ss}}_{\mathrm{fg}}(D) = p(\hat{U}) \qquad \text{where} \qquad \hat{U} := \kappa^{-1}(\overline{Y}^{\mathrm{ss}}(D)).$$

The situation fits into the following commutative diagram:



In this setting [**6**, Corollary 5.3] answers the question whether the morphisms q and r are categorical quotients.

Proposition 2.5.1 ([6]). *If for every $v \in V$ the closed H-orbit lying in $q'^{-1}(v)$ is contained in $\overline{Y}'$ (e.g. q' is geometric), then q and r are categorical quotients for the H- and G-actions respectively.*

In order to define a canonical quotient for the action of G on X we first recall the respective methods in reductive geometric invariant theory. For the affine variety $\overline{Y}$ let $\overline{Y}_1, \ldots, \overline{Y}_r$ be the sets of semistable points arising from ample divisors. Whenever we have $\overline{Y}_i \subseteq \overline{Y}_j$ for two of these set we obtain a commutative diagram.

$$\begin{array}{ccc} \overline{Y}_i & \longrightarrow & \overline{Y}_j \\ \downarrow & & \downarrow \\ \overline{Y}_i/\!\!/H & \xrightarrow{\varphi_{ij}} & \overline{Y}_j/\!\!/H \end{array}$$

The morphisms $\varphi_{ij} \colon \overline{Y}_i/\!\!/H \to \overline{Y}_j/\!\!/H$ turn the collection of quotients into an inverse system, the *GIT-system*. Its inverse limit $\overline{Y} \overset{\mathrm{GIT}}{\underset{\mathrm{Lim}}{/}} H$ is called *GIT-limit*. There exists a canonical morphism

$$\bigcap \overline{Y}_i \quad \to \quad \overline{Y} \overset{\mathrm{GIT}}{\underset{\mathrm{Lim}}{/}} H$$

and the closure of its image is the *limit quotient* $\overline{Y} /\!\!/_{\text{LQ}} H$ of $\overline{Y}$ with respect to H. Note that in the literature this space is also called 'canonical component' or 'GIT-limit'. In general, the limit quotient need not be normal; its normalisation is the *normalised limit quotient* $\overline{Y} \tilde{/\!\!/}_{\text{LQ}} H$.

We now turn to the non-reductive case. As constructible subsets of $\overline{Y}_i/\!\!/H$ the corresponding enveloped quotients V_i inherit the above morphisms φ_{ij}, and again form an inverse system.

Definition 2.5.2. The *(non-reductive) GIT-limit* $X \overset{\text{GIT}}{/\!\!/}_{\text{Lim}} G$ of X with respect to the G-action is the limit of the inverse system of enveloped quotients.

The non-reductive GIT-limit $X \overset{\text{GIT}}{/\!\!/}_{\text{Lim}} G$ is a constructible subset of the reductive GIT-limit $\overline{Y} \overset{\text{GIT}}{/\!\!/}_{\text{Lim}} H$. Analogously, we obtain a canonical morphism into the (non-reductive) GIT-limit $X \overset{\text{GIT}}{/\!\!/}_{\text{Lim}} G$

$$\bigcap (\overline{Y}' \cap \overline{Y}_i) \quad \to \quad X \overset{\text{GIT}}{/\!\!/}_{\text{Lim}} G.$$

Definition 2.5.3. The *(non-reductive) limit quotient* $X /\!\!/_{\text{LQ}} G$ of X with respect to the G-action is the closure of the image of the above morphism. Its normalisation is the *normalised limit quotient* $X \tilde{/\!\!/}_{\text{LQ}} G$.

The limit quotient in general appears to be relatively hard to access. However, if $\overline{Y}$ is factorial we can realise it up to normalisation as a certain closed subset of a toric variety as follows. For this consider homogeneous generators $f_1, \ldots, f_r$ of the K-graded algebra $\mathcal{O}(\overline{Y})$. With $\deg(T_i) := \deg(f_i)$ we obtain a graded epimorphism

$$\mathbb{K}[T_1, \ldots, T_r] \ \to \ \mathcal{O}(\overline{Y}); \qquad T_i \ \mapsto \ f_i.$$

This gives rise to an equivariant closed embedding of $\overline{Y}$ into $\mathbb{K}^r$. We denote by Q the the matrix recording the weights $\deg(f_i)$ as columns and fix a Gale dual matrix P, i.e. a matrix with $PQ^t = 0$. The *Gelfand-Kapranov-Zelevinsky-decomposition (GKZ-decomposition)* of P is the fan

$$\Sigma \ := \ \{\sigma(v);\ v \in \mathbb{Q}^{r-\mathrm{rk}(K)}\}, \qquad \sigma(v) \ := \ \bigcap_{v \in \tau^\circ} \tau$$

where τ is a cone generated by some of the columns of P. It is known that the normalised limit quotient $\mathbb{K}^r \tilde{/\!\!/}_{\text{LQ}} H$ is a toric variety with corresponding fan Σ. Now suppose that $\overline{Y}$ is factorial. Then every set of semistable points of $\overline{Y}$ arises as intersection of $\overline{Y}$ with a set of semistable points on $\mathbb{K}^r$. In this situation we obtain a closed embedding of the GIT-limits $\overline{Y} \overset{\text{GIT}}{/\!\!/}_{\text{Lim}} H \to \mathbb{K}^r \overset{\text{GIT}}{/\!\!/}_{\text{Lim}} H$ and hence of the respective limit quotients. The inverse image of $\overline{Y} /\!\!/_{\text{LQ}} H$ under the normalisation map $\nu\colon \mathbb{K}^r \tilde{/\!\!/}_{\text{LQ}} H \to \mathbb{K}^r /\!\!/_{\text{LQ}} H$ is in general not normal. However, its normalisation conincides with the normalised limit quotient $\overline{Y} \,\tilde{/\!\!/}_{\mathrm{LQ}}\, H$. The situation fits into the following commutative diagram.

$$\begin{array}{ccccc}
\overline{Y} \,\tilde{/\!\!/}_{\mathrm{LQ}}\, H & \longrightarrow & \nu^{-1}(\overline{Y} /\!\!/_{\mathrm{LQ}} H) & \longrightarrow & \mathbb{K}^r \,\tilde{/\!\!/}_{\mathrm{LQ}}\, H \\
& \searrow & \big\downarrow \nu & & \big\downarrow \nu \\
& & \overline{Y} /\!\!/_{\mathrm{LQ}} H & \longrightarrow & \mathbb{K}^r /\!\!/_{\mathrm{LQ}} H
\end{array}$$

Finally, if T is the dense torus in $\mathbb{K}^r$, then $\nu^{-1}(\overline{Y} /\!\!/_{\mathrm{LQ}} H)$ coincides with the closure of $(\overline{Y} \cap T)/H$ in $\mathbb{K}^r \,\tilde{/\!\!/}_{\mathrm{LQ}}\, H$. Hence we obtain a normalisation map

$$\overline{Y} \,\tilde{/\!\!/}_{\mathrm{LQ}}\, H \;\to\; \overline{\big((\overline{Y} \cap T) \,/\, H\big)^{\Sigma}}.$$

CHAPTER

THREE

COX RINGS AND GOOD QUOTIENTS

With minor modifications this entire chapter was first published in *'Good quotients of Mori Dream spaces'* in *Proc. Amer. Math. Soc., 139(9), 2011* published by the American Mathematical Society, see [**10**].

3.1. Good Quotients of Mori Dream Spaces

Let X be a normal variety over some algebraically closed field $\mathbb{K}$ of characteristic zero. If X has finitely generated divisor class group and only constant invertible global functions then one can associate to X a *Cox ring*; this is the graded $\mathbb{K}$-algebra

$$\mathcal{R}(X) := \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

In the case of torsion in $\mathrm{Cl}(X)$ the precise definition requires a little care; see Section 2 for a reminder and [**4**] for details. We ask whether finite generation of the Cox ring is preserved when passing to the quotient by a group action. More precisely, for an action of a reductive affine algebraic group G on X we consider *good quotients*; by definition these are affine morphisms $\pi\colon U \to V$ with $\mathcal{O}_V = (\pi_* \mathcal{O}_U)^G$ where $U \subseteq X$ may be any open G-invariant subset.

Theorem 3.1.1. *Let a reductive affine algebraic group G act on a normal variety X with finitely generated Cox ring $\mathcal{R}(X)$, and let $U \subseteq X$ be an open invariant subset admitting a good quotient $\pi\colon U \to U /\!\!/ G$ such that $U /\!\!/ G$ has only constant invertible global functions. Then the Cox ring $\mathcal{R}(U /\!\!/ G)$ is finitely generated as well.*

Note that this statement was proven in [**46**, Theorem 2.3] for the case that X is affine with finite divisor class group and $U /\!\!/ G$ is a GIT-quotient. Moreover, in [**46**, Remark 2.3.1] it was expected that (geometric) GIT-quotients of Mori dream spaces, i.e. $\mathbb{Q}$-factorial, projective varieties with finitely generated Cox ring, are again Mori dream spaces, which is a direct consequence of Theorem 3.1.1.

The following result is a step in the proof of Theorem 3.1.1 but it also might be of independent interest. Let $K \subseteq \mathrm{WDiv}\,(X)$ be a finitely generated subgroup of Weil divisors. By the *sheaf of divisorial algebras* associated to K we mean the sheaf of $\mathcal{O}_X$-algebras

$$\mathcal{S} := \bigoplus_{D \in K} \mathcal{S}_D, \qquad \mathcal{S}_D := \mathcal{O}_X\,(D)\,.$$

Theorem 3.1.2. *Let X be a normal variety with finitely generated Cox ring $\mathcal{R}(X)$. Then, for any finitely generated subgroup $K \subseteq \mathrm{WDiv}\,(X)$ and any open subset $U \subseteq X$, the algebra of sections $\Gamma(U, \mathcal{S})$ of the sheaf of divisorial algebras $\mathcal{S}$ associated to K is finitely generated.*

In particular, if X has finitely generated Cox ring, then for every open subset $U \subseteq X$ the algebra of regular functions $\Gamma(U, \mathcal{O})$ is finitely generated; note that even for affine varieties this fails in general, compare Example 3.2.2.

3.2. Proof of Theorem 3.1.2

Let us recall the construction of the Cox ring of a normal irreducible variety X with finitely generated divisor class group and only constant invertible global functions. Fixing a finitely generated subgroup K of the Weil divisors such that the projection $c\colon K \to \mathrm{Cl}\,(X)$ is surjective with kernel K^0, we can associate to K the sheaf of divisorial $\mathcal{O}_X$-algebras $\mathcal{S}$. In order to identify the isomorphic homogeneous components of $\mathcal{S}$ we fix a character $\chi\colon K^0 \to \mathbb{K}(X)^*$ such that $\mathrm{div}\,(\chi(E)) = E$ holds for every $E \in K^0$ and consider the sheaf of ideals $\mathcal{I}$ locally generated by the sections $1 - \chi(E)$ where E runs through K^0 and $\chi(E)$ is homogeneous of degree $-E$. The *Cox sheaf* is the sheaf $\mathcal{R} := \mathcal{S}/\mathcal{I}$ together with the $\mathrm{Cl}\,(X)$-grading

$$\mathcal{R} = \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{R}_{[D]}, \qquad \mathcal{R}_{[D]} := p\left(\bigoplus_{D' \in c^{-1}([D])} \mathcal{S}_{D'}\right),$$

where $p : \mathcal{S} \to \mathcal{R}$ denotes the projection. The algebra of global sections is called the *Cox ring* of X, which is - up to isomorphy - independent of the choices of K and χ. For later use, note that by [**4**, I, Lemma 4.3.5] for any open set $U \subseteq X$ we have

$$\Gamma(U, \mathcal{R}) \quad \cong \quad \Gamma(U, \mathcal{S})/\Gamma(U, \mathcal{I}).$$

Moreover, from [**4**, I, Lemma 5.1.2] we infer that the Cox ring is invariant when passing to a big open subset, i.e. an open subset whose complement is of codimension at least two. In particular, the two algebras $\Gamma(X^{\mathrm{reg}}\mathcal{R})$ and $\Gamma(X,\mathcal{R})$ are equal, where X^{reg} denotes the set of regular points of X.

Proof of Theorem 3.1.2. For the first part of the proof we proceed as in [**5**, Proposition 5.1.4]. First assume that K projects onto $\mathrm{Cl}\,(X)$. By K^0 we denote the subgroup of K consisting of principal divisors, i.e., the kernel of the projection $c\colon K \to \mathrm{Cl}\,(X)$, and fix a basis $D_1,\dots,D_s$ for K, such that K^0 is generated by $a_1D_1,\dots,a_kD_k$ with certain $a_i \in \mathbb{Z}_{\geq 0}$. Moreover, let $K^1 \subseteq K$ be the subgroup generated by $D_{k+1},\dots,D_s$ and set $K' := K^0 \oplus K^1$. We then have the associated Veronese subsheaves

$$\mathcal{S}^0 := \bigoplus_{D\in K^0} \mathcal{S}_D, \qquad \mathcal{S}^1 := \bigoplus_{D\in K^1} \mathcal{S}_D, \qquad \mathcal{S}' := \bigoplus_{D\in K'} \mathcal{S}_D.$$

We claim that $\Gamma(X,\mathcal{S})$ is finitely generated. First note, that $\mathcal{S}_D \to \mathcal{R}_{[D]}$ is an isomorphism by [**4**, I, Lemma 4.3.4]. Since $K^1 \cong c\left(K^1\right)$ holds, these isomorphisms fit together to an isomorphism of sheaves

$$\mathcal{S}^1 \;=\; \bigoplus_{D\in K^1} \mathcal{S}_D \;\to\; \bigoplus_{D\in c\left(K^1\right)} \mathcal{R}_{[D]} \;=:\; \mathcal{R}^1.$$

Since $\Gamma(X,\mathcal{R})$ is finitely generated, the Veronese subalgebra $\Gamma(X,\mathcal{R}^1)$ of the Cox ring is as well finitely generated (cf. [**4**, I, Proposition 1.2.1]) which gives finite generation of $\Gamma(X,\mathcal{S}^1)$. Every homogeneous function $f \in \Gamma(X,\mathcal{S}'_{E_0+E_1})$, where $E_i \in K^i$, is a product of a homogeneous section in $\Gamma(X,\mathcal{S}_{E_1})$ and an invertible section $g \in \Gamma(X,\mathcal{S}_{E_0})$, which itself is the product of certain $g_i^{\alpha_i}$ with $\mathrm{div}\,(g_i) = a_iD_i$. Consequently, $\Gamma(X,\mathcal{S}')$ is generated by the functions g_i and generators of $\Gamma(X,\mathcal{S}^1)$; and thus is finitely generated. Since K' is of finite index in K the algebra $\Gamma(X,\mathcal{S})$ inherits finite generation from $\Gamma(X,\mathcal{S}')$ by [**2**, Proposition 4.4].

Now, let $U \subsetneq X$ be an arbitrary open subset. Then the complement $X\backslash U$ can be written as a union of the support of an effective divisor D' and a closed subset of codimension at least two. Let $D \in K$ be a divisor which is linearly equivalent to D', i.e. $D' = D + \mathrm{div}\,(f)$ with a suitable rational function f. Then f is contained in $\Gamma(X,\mathcal{S}_D)$ and [**4**, I, Remark 3.1.7] shows that $\Gamma(U,\mathcal{S}) = \Gamma(X,\mathcal{S})_f$ is finitely generated.

Finally, if $K \subseteq \mathrm{WDiv}\,(X)$ does not project onto $\mathrm{Cl}\,(X)$, then we take any finitely generated group $\tilde{K} \subseteq \mathrm{WDiv}\,(X)$ with $K \subseteq \tilde{K}$ projecting onto $\mathrm{Cl}\,(X)$ and obtain finite generation of $\Gamma(U,\tilde{\mathcal{S}})$ for the associated sheaf $\tilde{\mathcal{S}}$ of divisorial algebras. This gives finite generation for the Veronese subalgebra $\Gamma(U,\mathcal{S}) \subseteq \Gamma(U,\tilde{\mathcal{S}})$ corresponding to $K \subseteq \tilde{K}$. □

Corollary 3.2.1. *Let X be normal variety with finitely generated Cox ring. Then for every open subset $U \subseteq X$ the algebra $\Gamma(U, \mathcal{O})$ is finitely generated.*

This observation allows us to construct normal affine varieties with non-finitely generated Cox ring.

Example 3.2.2. Let G be a connected semi-simple algebraic group and H a unipotent subgroup such that the ring of invariants

$$\Gamma(G, \mathcal{O})^H \quad = \quad \Gamma(G/H, \mathcal{O})$$

is not finitely generated. By [**34**, Corollary 2.8] there is an open G-equivariant embedding $G/H \subseteq X$ into a normal affine variety X. Since G is semi-simple, X has only constant invertible global functions and by the exact sequence

$$0 \longrightarrow \operatorname{Pic}(G/H) \longrightarrow \operatorname{Pic}(G)$$

in [**52**, Proposition 3.2] the divisor class group of G/H is finitely generated. Consequently, $\operatorname{Cl}(X)$ is finitely generated as well but by Corollary 3.2.1 the Cox ring $\mathcal{R}(X)$ is not finitely generated. We consider the following explicit example of Nagata [**56**], see also [**3, 20**]. Let $\mathbb{G}_{16}$ act on $Z := \mathbb{K}^{16} \oplus \mathbb{K}^{16}$ by

$$k \cdot (x, y) \ := \ (x, \ y_1 + k_1 x_1, \ \ldots, \ y_{16} + k_{16} x_{16})$$

Moreover, let $H \subseteq \mathbb{G}_{16}$ be a general 3-codimensional linear subspace. Then the algebra of invariants $\mathcal{O}(Z)^H$ of the induced H-action on Z is not finitely generated. Since H can be viewed as a subgroup of $G := \mathrm{SL}(32)$, we infer from [**3, 33**] that also $\mathcal{O}(G/H)$ is not finitely generated. Hence, the Cox ring of any affine variety X with $\mathrm{SL}(32)/H \subseteq X$ has a non-finitely generated Cox ring.

3.3. Proof of Theorem 3.1.1

We consider a smooth irreducible algebraic variety X. Fix a finitely generated subgroup $K \subseteq \operatorname{WDiv}(X)$. By smoothness of X, the associated sheaf of divisorial algebras $\mathcal{S}$ is locally of finite type. This allows us to consider its relative spectrum over X which we will denote by $\hat{X} := \operatorname{Spec}_X(\mathcal{S})$. Note that the regular functions on $\hat{X}$ are precisely the global sections $\Gamma(X, \mathcal{S})$. Since $\mathcal{S}$ is K-graded, $\hat{X}$ comes with the action of the torus $H := \operatorname{Spec}(\mathbb{K}[K])$ and the canonical morphism $p\colon \hat{X} \to X$ is a good quotient for this action.

Now let an affine algebraic reductive group G act on X. By a *G-linearisation* of the group K we mean a lifting of the G-action to the relative spectrum $\hat{X}$ commuting with the H-action and making the projection p equivariant. Any such G-linearization yields a G-representation on the regular functions of $\hat{X}$ via $g \cdot f(\hat{x}) = f(g^{-1} \cdot \hat{x})$ and thereby induces a G-representation on $\Gamma(X, \mathcal{S})$. In the special case where K is a group of G-invariant divisors, [**38**, Propositions 1.3 and 1.7] show that K is canonically G-linearized and the induced representation on the global sections $\Gamma(X, \mathcal{S})$ coincides with the action of G on the rational functions of X given by $g \cdot f(x) = f(g^{-1} \cdot x)$.

Lemma 3.3.1. *Let an affine algebraic reductive group G act on the normal variety X and let $U \subseteq X$ be an open G-invariant subset which admits a good quotient $\pi\colon U \to U/\!\!/G$. If $\mathrm{Cl}\,(X)$ is finitely generated then $\mathrm{Cl}\,(U/\!\!/G)$ is finitely generated as well.*

Proof. Without loss of generality we assume X and $U/\!\!/G$ to be smooth. From [**52**, Proposition 4.2] we infer that the pullback homomorphism

$$\pi^*\colon\ \mathrm{Pic}\,(U/\!\!/G) \quad\to\quad \mathrm{Pic}_G(U)$$

into the classes of G-linearised line bundles is injective. It therefore suffices to show that $\mathrm{Pic}_G(U)$ is finitely generated. By [**52**, Lemma 2.2] the following sequence is exact

$$\mathrm{H}^1_{\mathrm{alg}}(G, \mathcal{O}(U)^*) \longrightarrow \mathrm{Pic}_G(U) \longrightarrow \mathrm{Pic}(U).$$

Note that the group of algebraic cocycles $\mathrm{H}^1_{\mathrm{alg}}\,(G, \mathcal{O}(U)^*)$ is finitely generated by the exact sequence in [**52**, Proposition 2.3]

$$\mathbb{X}(G) \longrightarrow \mathrm{H}^1_{\mathrm{alg}}\,(G, \mathcal{O}(U)^*) \longrightarrow \mathrm{H}^1\left(G/G^0, E\,(U)\right),$$

where G/G^0 is finite and $E\,(U) = \mathcal{O}(U)^*/\mathbb{K}^*$ is finitely generated by [**52**, Proposition 1.3]. □

Proof of Theorem 3.1.1. Without loss of generality we assume X and $U/\!\!/G$ to be smooth. By Lemma 3.3.1 we can choose a finitely generated group K of Weil divisors on the quotient space $U/\!\!/G$ projecting surjectively onto the divisor class group $\mathrm{Cl}\,(U/\!\!/G)$. With $\mathcal{S}$ denoting the sheaf of divisorial algebras associated to K, the Cox ring $\mathcal{R}\,(U/\!\!/G)$ is the quotient of $\Gamma(U/\!\!/G, \mathcal{S})$ by the ideal $\Gamma(U/\!\!/G, \mathcal{I})$. Thus it suffices to show that the algebra of global sections $\Gamma(U/\!\!/G, \mathcal{S})$ is finitely generated.

The pullback group π^*K consists of invariant Weil divisors on U. It is therefore canonically G-linearized and we have the corresponding G-representation on the algebra $\Gamma(U, \mathcal{T})$ where $\mathcal{T}$ denotes the sheaf of divisorial algebras associated to the group π^*K. We claim that we have a pullback homomorphism mapping $\Gamma(U/\!\!/G, \mathcal{S})$ injectively onto the algebra $\Gamma(U, \mathcal{T})^G$ of invariant sections of $\Gamma(U, \mathcal{T})$:

$$\pi^*\colon \Gamma(U/\!\!/G, \mathcal{S}) \ \to\ \Gamma(U, \mathcal{T})^G, \qquad \Gamma(U/\!\!/G, \mathcal{S}_D) \ni f \ \mapsto\ \pi^*f \in \Gamma(U, \mathcal{T}_{\pi^*D}).$$

We first note that every pullback section $\pi^*f \in \Gamma(U, \mathcal{T}_{\pi^*D})$ is indeed G-invariant because π^*K is canonically G-linearized and π^*f is G-invariant as a rational function on U. On each homogeneous component of $\Gamma(U/\!\!/G, \mathcal{S})$ the map π^* is injective because it is the pullback with respect to the surjective morphism $\pi\colon U \to U /\!\!/ G$. Since π^* is graded this yields injectivity of π^* as an algebra homomorphism. For surjectivity it suffices to show that every homogeneous G-invariant section is a pullback section because the actions of G and H commute and, thus, $\Gamma(U, \mathcal{T})^G$ is a graded subalgebra of $\Gamma(U, \mathcal{T})$. Consider a G-invariant homogeneous section $f \in \Gamma(U, \mathcal{T}_{\pi^* D})$. Since f is invariant as a rational function in $\mathbb{K}(U)$ and it is regular on $U' := U \backslash \pi^{-1}(\mathrm{Supp}\,(D))$, it descends to a regular function $\tilde{f}$ on $\pi(U')$ which is an open subset of $U /\!\!/ G$. Observe that we have

$$\pi^*(\mathrm{div}(\tilde{f}) + D) \;=\; \mathrm{div}\,(f) + \pi^* D \;\geq\; 0.$$

In particular, we obtain that the divisor $\mathrm{div}(\tilde{f}) + D$ is effective and thus $\tilde{f}$ is a section in $\Gamma(U /\!\!/ G, \mathcal{S}_D)$. By construction f equals the pullback $\pi^* \tilde{f}$; hence our claim follows.

Thus the algebras $\Gamma(U /\!\!/ G, \mathcal{S})$ and $\Gamma(U, \mathcal{T})^G$ are isomorphic. The algebra $\Gamma(U, \mathcal{T})$ is finitely generated by Theorem 3.1.2. Hilbert's Finiteness Theorem then shows that the invariant algebra $\Gamma(U, \mathcal{T})^G$ is finitely generated as well. □

CHAPTER

FOUR

COX RINGS AND BLOW-UPS

With minor modifications all sections of this chapter have already been published. Section 4.1 is the third section of the paper *'On Chow quotients of torus actions'* ([**11**]), which is a joint work with Jürgen Hausen and Simon Keicher. The remaining sections are published as the author's paper *'On the Cox ring of blowing up the diagonal'*, see [**8**].

Let Z be a Mori Dreams Space with Cox construction $p\colon \hat{Z} \to Z$ and $\pi\colon Z' \to Z$ the blow-up in a subscheme. In general it is not true that Z' is a Mori Dream Space again. However, if Z and the center of the blow-up are toric, then so is Z'. This clearly preserves finite generation of the Cox ring. In this case on the level of total coordinate spaces the blow-up is given by a morphism of affine spaces

$$\overline{\pi}\colon\ \mathbb{K}^{r+1} = \overline{Z}' \quad\longrightarrow\quad \overline{Z} = \mathbb{K}^r.$$

Now consider an embedding of a Mori Dream Space X into the toric varity Z such that $\overline{X} := \overline{p^{-1}(X)} \subseteq \overline{Z}$ is a total coordinate space for X. Then we ask whether $\overline{\pi}^{-1}(\overline{X}) \subseteq \overline{Z}'$ is a total coordinate space for the proper transform X' of X under the toric blow-up $Z' \to Z$. In general this fails. In the upcoming chapter we provide a criterion in which cases this is true and perform these *ambient modifications* for two classes of examples.

4.1. Toric ambient Modifications

In this section, we provide a general machinery to study the effect of modifications on the Cox ring. Similar to [**39**], we use toric embeddings. In contrast to the geometric criteria given there, our approach here is purely algebraic, based on results of [**12**]. The heart is a construction of factorially graded rings out of given ones.

We begin with recalling the necessary algebraic concepts. Let K be a finitely generated abelian group and R a finitely generated integral K-graded $\mathbb{K}$-algebra. A homogeneous nonzero nonunit $f \in R$ is called *K-prime* if $f \mid gh$ with homogeneous $g, h \in R$ always implies $f \mid g$ or $f \mid h$. The algebra R is called *factorially K-graded* if every homogeneous nonzero nonunit $f \in R$ is a product of K-primes.

We enter the construction of factorially graded rings. Consider a grading of the polynomial ring $\mathbb{K}[T_1, \ldots, T_{r_1}]$ by a finitely generated abelian group K_1 such that the variables T_i are homogeneous. Then we have a pair of exact sequences

$$0 \longrightarrow \mathbb{Z}^{k_1} \xrightarrow{Q_1^*} \mathbb{Z}^{r_1} \xrightarrow{P_1} \mathbb{Z}^{n}$$

$$0 \longleftarrow K_1 \xleftarrow[Q_1]{} \mathbb{Z}^{r_1} \xleftarrow[P_1^*]{} \mathbb{Z}^{n} \longleftarrow 0$$

where $Q_1 \colon \mathbb{Z}^{r_1} \to K_1$ is the degree map sending the i-th canonical basis vector e_i to $\deg(T_i) \in K_1$. We enlarge P_1 to a $n \times r_2$ matrix P_2 by concatenating further $r_2 - r_1$ columns. This gives a new pair of exact sequences

$$0 \longrightarrow \mathbb{Z}^{k_2} \xrightarrow{Q_2^*} \mathbb{Z}^{r_2} \xrightarrow{P_2} \mathbb{Z}^{n}$$

$$0 \longleftarrow K_2 \xleftarrow[Q_2]{} \mathbb{Z}^{r_2} \xleftarrow[P_2^*]{} \mathbb{Z}^{n} \longleftarrow 0$$

Construction 4.1.1. Given a K_1-homogeneous ideal $I_1 \subseteq \mathbb{K}[T_1, \ldots, T_{r_1}]$, we transfer it to a K_2-homogeneous ideal $I_2 \subseteq \mathbb{K}[T_1, \ldots, T_{r_2}]$ by taking extensions and contractions according to the scheme

$$\begin{array}{ccccc}
\mathbb{K}[T_1, \ldots, T_{r_2}] & & & & \mathbb{K}[T_1, \ldots, T_{r_1}] \\
\downarrow \imath_2 & & & & \downarrow \imath_1 \\
\mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}] & \xleftarrow[\pi_2^*]{} & \mathbb{K}[S_1^{\pm 1}, \ldots, S_n^{\pm 1}] & \xrightarrow[\pi_1^*]{} & \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_1}^{\pm 1}]
\end{array}$$

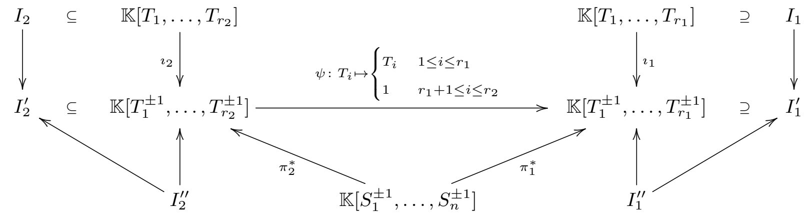
where $\imath_1, \imath_2$ are the canonical embeddings and π_i^* are the homomorphisms of group algebras defined by $P_i^* \colon \mathbb{Z}^n \to \mathbb{Z}^{r_i}$.

Now let $I_1 \subseteq \mathbb{K}[T_1, \ldots, T_{r_1}]$ be a K_1-homogeneous ideal and $I_2 \subseteq \mathbb{K}[T_1, \ldots, T_{r_2}]$ the transferred K_2-homogeneous ideal. Our result relates factoriality properties of the algebras $R_1 := \mathbb{K}[T_1, \ldots, T_{r_1}]/I_1$ and $R_2 := \mathbb{K}[T_1, \ldots, T_{r_2}]/I_2$ to each other.

Theorem 4.1.2. *Assume R_1, R_2 are integral, $T_1, \ldots, T_{r_1}$ define K_1-primes in R_1 and $T_1, \ldots, T_{r_2}$ define K_2-primes in R_2. Then the following statements are equivalent.*

(i) *The algebra R_1 is factorially K_1-graded.*
(ii) *The algebra R_2 is factorially K_2-graded.*

Proof. First observe that the homomorphisms π_j^* embed $\mathbb{K}[S_1^{\pm 1}, \ldots, S_n^{\pm 1}]$ as the degree zero part of the respective K_j-grading and fit into a commutative diagram



The factor ring R_1' of the extension $I_1' := \langle \imath_1(I_1) \rangle$ is obtained from R_1 by localization with respect to K_1-primes $T_1, \ldots, T_{r_1}$:

$$R_1' \;:=\; \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_1}^{\pm 1}]/I_1' \;\cong\; (R_1)_{T_1 \cdots T_{r_1}}.$$

The ideal I_1'' is the degree zero part of I_1'. Thus, its factor algebra is the degree zero part of R_1':

$$R_1'' \;:=\; \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_1}^{\pm 1}]_0/I_1'' \;\cong\; (R_1')_0.$$

Note that $\mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_1}^{\pm 1}]$ and hence R_1' admit units in every degree. Thus, [**12**, Thm. 1.2] yields that R_1 is a factorially K_1-graded if and only if R_1'' is a UFD.

The homomorphism ψ restricts to an isomorphism ψ_0 of the respective degree zero parts. Thus, the shifted ideal $I_2'' := \psi_0^{-1}(I_1'')$ defines an algebra R_2'' isomorphic to R_1'':

$$R_2'' \;:=\; \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}]_0/I_2'' \;\cong\; R_1''.$$

The ideal $I_2' := \left\langle \pi_2^*((\pi_0^*)^{-1}(I_1')) \right\rangle$ has I_2'' as its degree zero part and $\mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}]$ admits units in every degree. The associated K_2-graded algebra

$$R_2' \;:=\; \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}]/I_2'$$

is the localization of R_2 by the K_2-primes $T_1, \dots, T_{r_2}$. Again by [**12**, Thm. 1.2] we obtain that R_2'' is a UFD if and only if R_2 is factorially K_2-graded. □

The following observation is intended for practical purposes; it reduces, for example, the number of necessary primality tests.

Proposition 4.1.3. *Assume that R_1 is integral and the canonical map $K_2 \to K_1$ admits a section (e.g. K_1 is free).*

(i) *Let $T_1, \dots, T_{r_1}$ define K_1-primes in R_1 and $T_{r_1+1}, \dots, T_{r_2}$ define K_2-primes in R_2. If no T_j with $j \geq r_1 + 1$ divides a T_i with $i \leq r_1$, then also $T_1, \dots, T_{r_1}$ define K_2-primes in R_2.*

(ii) *The ring R_2 is integral. Moreover, if R_1 is normal and $T_{r_1+1}, \dots, T_{r_2}$ define primes in R_2 (e.g. they are K_2-prime and K_2 is free), then R_2 is normal.*

Proof. The exact sequences involving the grading groups K_1 and K_2 fit into a commutative diagram where the upwards sequences are exact and $\mathbb{Z}^{r_2 - r_1} \to K_2'$ is an isomorphism:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & K_1 & \xleftarrow{Q_1} & \mathbb{Z}^{r_1} & \xleftarrow{P_1^*} & \mathbb{Z}^n & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & K_2 & \xleftarrow{Q_2} & \mathbb{Z}^{r_2} & \xleftarrow{P_2^*} & \mathbb{Z}^n & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & K_2' & \longleftarrow & \mathbb{Z}^{r_2 - r_1} & \longleftarrow & 0 & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Moreover, denoting by $K_1' \subseteq K_2$ the image of the section $K_1 \to K_2$, there is a splitting $K_2 = K_2' \oplus K_1'$. As $K_2' \subseteq K_2$ is the subgroup generated by the degrees of $T_{r_1+1}, \dots, T_{r_2}$, we obtain a commutative diagram

$$
\begin{array}{ccc}
\mathbb{K}[T_1, \dots, T_{r_2}] & & \\
\downarrow{\scriptstyle \imath_2} & & \\
\mathbb{K}[T_1, \dots, T_{r_1}, T_{r_1+1}^{\pm 1}, \dots, T_{r_2}^{\pm 1}] & \xrightarrow{\psi\colon T_i \mapsto \begin{cases} T_i & 1 \leq i \leq r_1 \\ 1 & r_1+1 \leq i \leq r_2 \end{cases}} & \mathbb{K}[T_1, \dots, T_{r_1}] \\
\uparrow{\scriptstyle \mu} & \nearrow{\scriptstyle \cong} & \\
\mathbb{K}[T_1, \dots, T_{r_1}, T_{r_1+1}^{\pm 1}, \dots, T_{r_2}^{\pm 1}]_0 & &
\end{array}
$$

where the map μ denotes the embedding of the degree zero part with respect to the K_2'-grading. By the splitting $K_2 = K_2' \oplus K_1'$, the image of μ is precisely the Veronese subalgebra associated to the subgroup $K_1' \subseteq K_2$. For the factor rings R_2 and R_1 by the ideals I_2 and I_1, the above diagram leads to the following situation

$$
\begin{array}{ccc}
R_2 & & \\
\downarrow{\scriptstyle \imath_2} & & \\
(R_2)_{T_{r_1+1}\cdots T_{r_2}} & \xrightarrow{\ \psi\ } & R_1 \\
\uparrow{\scriptstyle \mu} & \nearrow{\scriptstyle \cong} & \\
\Big((R_2)_{T_{r_1+1}\cdots T_{r_2}}\Big)_0 & &
\end{array}
$$

To prove (i), consider a variable T_i with $1 \le i \le r_1$. We have to show that T_i defines a K_2-prime element in R_2. By the above diagram, T_i defines a K_1'-prime element in $((R_2)_{T_{r_1+1}\cdots T_{r_2}})_0$, the Veronese subalgebra of R_2 defined by $K_1' \subseteq K_2$. Since every K_2-homogeneous element of $(R_2)_{T_{r_1+1}\cdots T_{r_2}}$ can be shifted by a homogeneous unit into $((R_2)_{T_{r_1+1}\cdots T_{r_2}})_0$, we see that T_i defines a K_2-prime in $(R_2)_{T_{r_1+1}\cdots T_{r_2}}$. By assumption, $T_{r_1+1}, \ldots, T_{r_2}$ define K_2-primes in R_2 and are all coprime to T_i. It follows that T_i defines a K_2-prime in R_2.

We turn to assertion (ii). As just observed, the degree zero part $((R_2)_{T_{r_1+1}\cdots T_{r_2}})_0$ of the K_2'-grading is isomorphic to R_1 and thus integral (normal if R_1 is so). Moreover, the K_2'-grading is free in the sense that the associated torus $\mathrm{Spec}\,(\mathbb{K}[K_2'])$ acts freely on $\mathrm{Spec}\,((R_2)_{T_{r_1+1}\cdots T_{r_2}})$. It follows that $(R_2)_{T_{r_1+1}\cdots T_{r_2}}$ is integral (normal if R_1 is so). Construction 4.1.1 gives that R_2 is integral. Moreover, if $T_{r_1+1}, \ldots, T_{r_2}$ define primes in R_2, we can conclude that R_2 is normal. □

Let us apply the results to Cox rings. We first briefly recall the basic definitions and facts; for details we refer to [**4**]. For a normal variety X with finitely generated divisor class group $\mathrm{Cl}(X)$ and $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$, one defines its Cox ring as the graded ring

$$\mathcal{R}(X) \quad := \quad \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)).$$

This ring is factorially $\mathrm{Cl}(X)$-graded. Moreover, if $\mathcal{R}(X)$ is finitely generated, then one can reconstruct X from $\mathcal{R}(X)$ as a good quotient of an open subset of $\mathrm{Spec}\,(\mathcal{R}(X))$ by the action of $\mathrm{Spec}\,(\mathbb{K}[\mathrm{Cl}(X)])$.

Now return to the setting fixed at the beginning of the section and assume in addition that the columns of P_2 are pairwise different primitive vectors in

$\mathbb{Z}^n$ and those of P_1 generate $\mathbb{Q}^n$ as a convex cone. Suppose we have toric Cox constructions $\pi_i\colon \hat{Z}_i \to Z_i$, where $\hat{Z}_i \subseteq \mathbb{K}^{r_i}$ are open toric subvarieties and π_i are toric morphisms defined by P_i, see [**22**]. Then the canonical map $Z_2 \to Z_1$ is a toric modification. Consider the ideal I_1 as discussed before and the geometric data

$$\overline{X}_1 \ := \ V(I_1) \subseteq \mathbb{K}^{r_1}, \qquad \hat{X}_1 \ := \ \overline{X}_1 \cap \hat{Z}_1, \qquad X_1 \ := \ \pi_1(\hat{X}_1) \subseteq Z_1.$$

Assume that R_1 is factorially K_1-graded and $T_1, \ldots, T_{r_1}$ define pairwise nonassociated prime elements in R_1. Then R_1 is the Cox ring of X_1, see [**4**]. Our statement concerns the Cox ring of the proper transform $X_2 \subseteq Z_2$ of $X_1 \subseteq Z_1$ with respect to $Z_2 \to Z_1$.

Corollary 4.1.4. *In the above setting, assume that R_2 is normal and the variables $T_1, \ldots, T_{r_2}$ define pairwise nonassociated K_2-prime elements in R_2. Then the K_2-graded ring R_2 is the Cox ring of X_2.*

Proof. According to Theorem 4.1.2, the ring R_2 is factorially K_2-graded. Moreover, with the toric Cox construction $\pi_2\colon \hat{Z}_2 \to Z_2$, we obtain that R_2 is the algebra of functions of the closure $\hat{X}_2 \subseteq \hat{Z}_2$ of $\pi_2^{-1}(X_2 \cap \mathbb{T}^{r_2})$. Thus, [**4**] yields that R_2 is the Cox ring of X_2. □

Example 4.1.5. We start with the UFD $R_1 = \mathbb{K}[T_1, \ldots, T_8]/I_1$, where the ideal I_1 is defined as

$$I_1 \ = \ \langle T_1T_2 + T_3T_4 + T_5T_6 + T_7T_8 \rangle.$$

The ideal I_1 is homogeneous with respect to the standard grading given by $Q_1 = [1, \ldots, 1]$. A Gale dual is $P_1 = [e_0, e_1, \ldots, e_7]$, where $e_0 = -e_1 - \ldots - e_7$ and e_i are the canonical basis vectors. Concatenating $e_1 + e_3$ gives a matrix P_2. The resulting UFD is $R_2 = \mathbb{K}[T_1, \ldots, T_9]/I_2$ with

$$I_2 \ = \ \langle T_1T_2T_9 + T_3T_4T_9 + T_5T_6 + T_7T_8 \rangle.$$

4.2. On blowing up the Diagonal

In recent literature it has been discussed how the Cox ring behaves under blow-ups. In particular, it is of interest whether finite generation is preserved in this process, and, if so what a presentation in terms of generators and relations looks like, see for example [**19, 20, 32, 40, 41, 58**].

In this section we employ the techniques of toric ambient modifications developed in the preceding section and [**11, 40**] to compute the Cox rings of the following blow-ups. Let $X' := \mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_r}$ be a product of projective spaces and denote by $\Delta_X \subseteq X := X' \times X'$ the diagonal. The variety X is spherical and $\mathrm{Bl}_{\Delta_X}(X)$ inherits this property. Hence, it is known that the

Cox ring $\mathcal{R}(\mathrm{Bl}_{\Delta_X}(X))$ is finitely generated, see [**4, 17**]. Our first result is an explicit presentation.

Theorem 4.2.1. *The Cox ring $\mathcal{R}(\mathrm{Bl}_{\Delta_X}(X))$ of the blow-up $\mathrm{Bl}_{\Delta_X}(X)$ is isomorphic to the $\mathbb{Z}^{\mathbf{r}} \times \mathbb{Z}^{\mathbf{r}} \times \mathbb{Z}$-graded factor algebra R_X/I_X where*

$$\begin{aligned} R_X &:= \mathbb{K}[T_\infty,\ {}_rT_{ij};\quad r = 1,\ldots,\mathbf{r},\quad 0 \le i < j \le n_r + 2,\quad i \le n_r], \\ I_X &:= I(1) + \ldots + I(\mathbf{r}), \end{aligned}$$

for every $r = 1,\ldots,\mathbf{r}$ the ideal $I(r)$ is generated by the twisted Plücker relations

$$\begin{aligned} &{}_rT_{ij}\ T_\infty \ -\ {}_rT_{ik}\ {}_rT_{jk}\ +\ {}_rT_{il}\ {}_rT_{jk}; && 0 \le i < j \le n_r,\ k = n_r + 1,\ l = n_r + 2, \\ &{}_rT_{ij}\ {}_rT_{kl} \ -\ {}_rT_{ik}\ {}_rT_{jk}\ +\ {}_rT_{il}\ {}_rT_{jk}; && 0 \le i < j < k < l \le n_r + 2,\quad k \le n_r \end{aligned}$$

and the grading of R_X/I_X is given by

$$\deg(T_\infty) \ =\ (0,0,1), \qquad \deg({}_rT_{ij}) \ =\ \begin{cases} (e_r, 0, 0) & \text{if } j = n_r + 1, \\ (0, e_r, 0) & \text{if } j = n_r + 2, \\ (e_r, e_r, -1) & \text{else.} \end{cases}$$

In particular, the spectrum of the Cox ring $\mathcal{R}(\mathrm{Bl}_{\Delta_X}(X))$ is the intersection of a product of affine Grassmannian varieties (w.r.t. the Plücker embedding) with a linear subspace.

As a second class of examples we treat the (non-spherical) blow-up of the variety $Y := \mathbb{P}_1^n$ in the generalised diagonal $\Delta_Y := \{(x,\ldots,x);\ x \in \mathbb{P}_1\} \subseteq Y$. Again we prove that the Cox ring of $\mathrm{Bl}_{\Delta_Y}(Y)$ is finitely generated and we give an explicit presentation.

Theorem 4.2.2. *The Cox ring $\mathcal{R}(\mathrm{Bl}_{\Delta_Y}(Y))$ of the blow-up $\mathrm{Bl}_{\Delta_Y}(Y)$ is isomorphic to the $\mathbb{Z}^{n+1}$-graded factor algebra R_Y/I_Y where*

$$\begin{aligned} R_Y &:= \mathbb{K}[S_{ij};\ 1 \le i < j \le n+2] \\ I_Y &:= \langle\, S_{ij}S_{kl} \ -\ S_{ik}S_{jl}\ +\ S_{il}S_{jk};\quad 1 \le i < j < k < l \le n+2\,\rangle, \end{aligned}$$

and the grading of R_Y/I_Y is given by

$$\deg(S_{ij}) \ =\ \begin{cases} e_i & \text{if } i \le n,\ j = n+1,\ n+2, \\ e_{n+1} & \text{if } i = n+1,\ j = n+2, \\ e_i + e_j - e_{n+1} & \text{else.} \end{cases}$$

4.3. Proofs of Theorems 4.2.1 and 4.2.2

Let us recall some definitions, for details see [**4**]. Let Z be a normal variety with free and finitely generated divisor class group $K := \mathrm{Cl}(Z)$ and only constant invertible regular functions. Then we define its *Cox ring* as the K-graded $\mathbb{K}$-algebra

$$\mathcal{R}(Z) \quad := \quad \bigoplus_K \Gamma(Z, \mathcal{O}(D)).$$

If the Cox ring $\mathcal{R}(Z)$ is finitely generated, we call its spectrum $\overline{Z} := \operatorname{Spec}(\mathcal{R}(Z))$ the *total coordinate space* of Z. The K-grading of $\mathcal{R}(Z)$ gives rise to an action of the quasitorus $H_Z := \operatorname{Spec}(\mathbb{K}[K])$ on $\overline{Z}$. Moreover, there exists an open invariant subset, the *characteristic space*, $\hat{Z} \subseteq \overline{Z}$ admitting a good quotient $p_Z \colon \hat{Z} \to \hat{Z} /\!\!/ H_Z \cong Z$ for this action.

Before we enter the proofs we will sketch the methods developed in [**11, 40**] but see also Chapter 6, Section 6.4. Let Z be a toric variety with Cox construction $p_Z \colon \hat{Z} \to Z$, total coordinate space $\overline{Z} = \mathbb{K}^r$ and an ample class $w \in K$ in the divisor class group.

Now let $\mathfrak{A}$ be a subscheme of Z; we ask for the Cox ring of the blow-up $\operatorname{Bl}_{\mathfrak{A}}(Z)$ of Z in $\mathfrak{A}$. By Cox' construction [**22**] the subscheme $\mathfrak{A}$ arises from a homogeneous ideal $\mathfrak{a} = \langle f_1, \ldots, f_l \rangle$ in the K-graded Cox ring $\mathcal{R}(Z)$. For this consider the associated K-graded sheaf $\tilde{\mathfrak{a}}$ on $\overline{Z}$; then $\mathfrak{A}$ is given by $({p_Z}_* \tilde{\mathfrak{a}})_0$. As a first step we embedd Z into a larger toric variety Z_1 such that the blow-up can be dealt with using methods from toric geometry. For this we consider the closed embedding

$$\overline{\pi} \colon \mathbb{K}^r \quad \to \quad \mathbb{K}^{r_1}; \qquad z \quad \mapsto \quad (z, f_1(z), \ldots, f_l(z)),$$

where $r_1 := r + l$. We endow $\mathbb{K}[T_1, \ldots, T_{r_1}]$ with a grading of $K_1 := K$ by assigning to $T_1, \ldots, T_r$ the original K-degrees and setting $\deg(T_{r+i}) := \deg f_i$ for the remaining variables. Then the quasitorus $H_{Z_1} := \operatorname{Spec}(\mathbb{K}[K_1])$ acts on the affine space $\overline{Z}_1 := \mathbb{K}^{r_1}$ and this makes $\overline{\pi}$ equivariant. The class $w \in K_1$ gives rise to an open subset $\hat{Z}_1 \subseteq \overline{Z}_1$ and a toric variety $Z_1 := \hat{Z}_1 /\!\!/ H_{Z_1}$.

The closed embedding $\overline{\pi}$ restricts to a closed embedding $\hat{\pi} \colon \hat{Z} \to \hat{Z}_1$ of the corresponding characteristic spaces and then descends to a closed embedding $\pi \colon Z \to Z_1$ of the respective quotients. The setting fits into the following commutative diagram.

$$\begin{array}{ccc}
\overline{Z} & \xrightarrow{\overline{\pi}} & \overline{Z}_1 \\
\uparrow & & \uparrow \\
\hat{Z} & \xrightarrow{\hat{\pi}} & \hat{Z}_1 \\
\downarrow {\scriptstyle /\!\!/ H_Z} & & \downarrow {\scriptstyle /\!\!/ H_{Z_1}} \\
Z & \xrightarrow{\pi} & Z_1
\end{array}$$

The idea is to compute the Cox ring of the proper transform Z' of $Z \subseteq Z_1$ with respect to a toric blow-up of Z_1. The following lemma relates Z' to the blow-up $\mathrm{Bl}_{\mathfrak{A}}(Z)$. Although the result was to be expected, we do not know of a reference and provide a proof.

Lemma 4.3.1. *Let $\mathfrak{b} \subseteq \mathcal{O}(\overline{Z}_1)$ be a K_1-homogeneous ideal and let $\mathfrak{B}$ be the corresponding subscheme of Z_1. Then the proper transform of $Z \subseteq Z_1$ under the blow-up $\mathrm{Bl}_{\mathfrak{B}}(Z_1) \to Z_1$ is isomorphic to the blow-up of Z in the subscheme of Z associated to the K-homogeneous ideal $\overline{\pi}^*\mathfrak{b} \subseteq \mathcal{O}(\overline{Z})$.*

Remark 4.3.2. If we apply Lemma 4.3.1 in the case $\mathfrak{b} := \langle T_{r+1}, \dots, T_{r_1} \rangle$, then we obtain $\mathfrak{a} = \overline{\pi}^*\mathfrak{b}$ and the proper transform Z' of $Z \subseteq Z_1$ is the blow-up of Z in $\mathfrak{A}$. Moreover, if $\mathfrak{a}$ is prime, then the associated subscheme $\mathfrak{A}$ is the subvariety $p_Z(V(\mathfrak{a}))$ and Z' is the ordinary blow-up of Z in $p_Z(V(\mathfrak{a}))$.

Proof of Lemma 4.3.1. First blow-ups are determined locally. We consider a suitable partial open cover of $\hat{Z}_1$ and of the characteristic space $\hat{Z}$. Let $w \in K = K_1$ be an ample class of Z as above. We set

$$\Gamma := \left\{\gamma \in \{0,1\}^{r_1}; \quad w \ \in \ \mathrm{relint}\,(\,\mathrm{cone}\,(\deg T_i;\ \text{where } \gamma_i = 1)\,)\right\}.$$

Then $\hat{Z}_1$ is covered by the H_{Z_1}-invariant sets $\overline{Z}_{1\gamma} := \overline{Z}_1 \setminus V(T^\gamma)$ where $\gamma \in \Gamma$. We now determine a partial cover which already contains $\overline{\pi}(\hat{Z})$. For this we consider the subset

$$\Gamma' \ := \ \Gamma \cap (\{0,1\}^r \times \{0\}^l) \ \subseteq \ \Gamma.$$

Then the corresponding open subvarieties cover the image of $\overline{\pi}$. More precisely, if we set $\overline{Z}_\gamma := \overline{\pi}^{-1}(\overline{Z}_{1\gamma})$, then we have

$$\hat{Z} = \bigcup_{\gamma \in \Gamma'} \overline{Z}_\gamma \quad \text{and hence} \quad \overline{\pi}(\hat{Z}) \subseteq \bigcup_{\gamma \in \Gamma'} \overline{Z}_{1\gamma}.$$

Moreover, we denote by $Z_\gamma := \overline{Z}_\gamma /\!\!/ H_Z$ and $Z_{1\gamma} := \overline{Z}_{1\gamma} /\!\!/ H_{Z_1}$ the respective quotient spaces and fix some $\gamma \in \Gamma'$. If we set $I_1 \subseteq \mathbb{K}[T_1, \dots, T_{r_1}]$ as the ideal generated by all the $T_{r+i} - f_i$, then the image of $\overline{\pi}$ is given by $V(I_1)$. The morphism $\overline{\pi}$ factors into an isomorphism $\overline{\pi}' \colon \overline{Z} \to V(I_1)$ and a closed embedding $\iota \colon V(I_1) \to \overline{Z}_1$.

On the algebraic side we set $A := \mathcal{O}(\overline{Z})$ and $B := \mathcal{O}(\overline{Z}_1)$. We write B_γ, $(B/I)_\gamma$ and A_γ for the localised algebras and $B_{(0)}$, $(B/I)_{(0)}$ and $A_{(0)}$

for their respective homogeneous components of degree zero. Then the situation fits into the following commutative diagrams.

$$
\begin{array}{ccccc}
\overline{Z} & \xrightarrow{\overline{\pi}'} & V(I_1) & \xrightarrow{\overline{\iota}} & \overline{Z}_1 \\
\uparrow & & \uparrow & & \uparrow \\
\overline{Z}_\gamma & \xrightarrow{\overline{\pi}'_\gamma} & V(I_1) \cap \overline{Z}_{1\gamma} & \xrightarrow{\iota_\gamma} & \overline{Z_1}_\gamma \\
\downarrow & & \downarrow & & \downarrow \\
Z_\gamma & \xrightarrow{\pi'} & Z_\gamma & \xrightarrow{\iota} & Z_{1\gamma}
\end{array}
\qquad
\begin{array}{ccccc}
A & \xleftrightarrow{\overline{\pi}'^*} & B/I_1 & \twoheadleftarrow{\scriptstyle\overline{\iota}^*} & B \\
\downarrow & & \downarrow & & \downarrow \\
A_\gamma & \xleftrightarrow{\pi'^*_\gamma} & (B/I_1)_\gamma & \twoheadleftarrow{\scriptstyle\iota^*_\gamma} & B_\gamma \\
\uparrow & & \uparrow & & \uparrow \\
A_{(0)} & \xleftrightarrow{\pi'^*} & (B/I_1)_{(0)} & \twoheadleftarrow{\scriptstyle\iota^*} & B_{(0)}
\end{array}
$$

The proper transform of $Z_\gamma \subseteq Z_{1\gamma}$ is the blow-up of Z_γ with center given by the affine scheme associated to the ideal $\iota^*\mathfrak{b}_{(0)} \subseteq (B/I_1)_{(0)}$. Our assertion then follows from the fact that in $A_{(0)}$ the ideals $\pi'^*(\iota^*\mathfrak{b}_{(0)})$ and $(\overline{\pi}^*\mathfrak{b})_{(0)}$ coincide. □

Construction 4.3.3. Let $\mathfrak{b}$ be the ideal $\langle T_{r+1}, \ldots, T_{r_1} \rangle$ and $Z' \to Z$ the proper transform of $Z \subseteq Z_1$ with respect to the toric blow-up $\mathrm{Bl}_{\mathfrak{B}}(Z_1) \to Z_1$. We turn to the problem of determining the Cox ring $\mathcal{R}(Z')$. For this we set $r_2 := r_1 + 1$ and consider the $r_1 \times r_2$-matrix

$$A := [E_{r_1}, \mathbf{1}_l], \qquad \text{where} \quad \mathbf{1}_l := (\underbrace{0, \ldots, 0}_{r}, \underbrace{1 \ldots, 1}_{l})^t.$$

The dual map $A^* \colon \mathbb{Z}^{r_1} \to \mathbb{Z}^{r_2}$ yields a homomorphism α^* of group algebras and a morphism $\alpha \colon (\mathbb{K}^*)^{r_2} \to (\mathbb{K}^*)^{r_1}$. Together with the canonical embeddings ι_1^* and ι_2^* we now have to transfer the ideal

$$I_1 \; := \; \langle T_{r+i} - f_i;\ i = 1, \ldots, l \rangle \quad \subseteq \quad \mathbb{K}[T_1, \ldots, T_{r_1}]$$

by taking extensions and contractions via the construction

$$
\begin{array}{ccccccc}
I_1 & & I_1' := \langle \iota_1^* I_1 \rangle & & I_2' := \langle \alpha^* I_1' \rangle & & I_2 := \iota_2^{*-1} I_2' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{K}[T_1, \ldots, T_{r_1}] & \xrightarrow{\iota_1^*} & \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_1}^{\pm 1}] & \xrightarrow{\alpha^*} & \mathbb{K}[T_1^{\pm 1}, \ldots, T_{r_2}^{\pm 1}] & \xleftarrow{\iota_2^*} & \mathbb{K}[T_1, \ldots, T_{r_2}]
\end{array}
$$

and call the resulting ideal I_2. If we endow $\mathbb{K}[T_1, \ldots, T_{r_2}]$ with the grading of $K_2 := K_1 \times \mathbb{Z}$ given by

$$\deg(T_i) := \begin{cases} (\deg_{K_1}(T_i), 0) & \text{for } 1 \le i \le r, \\ (\deg_{K_1}(T_i), -1) & \text{for } r+1 \le i \le r_1, \\ (0, 1) & \text{for } i = r_2, \end{cases}$$

then I_2 is K_2-homogeneous and the following Proposition provides us with a criterion to show that I_2 defines the desired Cox ring.

Proposition 4.3.4 (Proposition 4.1.3 and Corollary 4.1.4). *If in the K_2-graded ring $R_2 := \mathbb{K}[T_1, \dots, T_{r_2}]/I_2$ the variable T_{r_2} is prime and does not divide a T_i with $1 \le i \le r_1$, then R_2 is Cox ring of the proper transform Z'.*

We return to our two cases of $X = X' \times X'$ and $Y = \mathbb{P}_1^n$. Both of them are toric varieties, their respective Cox rings are polynomial rings and the total coordinate spaces are

$$\overline{X} \;=\; \bigoplus_{r=1}^{\mathbf{r}} (\mathbb{K}^{n_r+1} \oplus \mathbb{K}^{n_r+1}) \qquad \text{and} \qquad \overline{Y} \;=\; \underbrace{\mathbb{K}^2 \oplus \dots \oplus \mathbb{K}^2}_{n}.$$

On $\overline{X}$ we will label the coordinates of the r-th factor with ${}_rT_{ij}$ where $i = 0, \dots, n_r$ and $j = n_r+1, n_r+2$. On $\overline{Y}$ we will use the notation S_{ij} for the coordinates where similarly $i = 1, \dots, n$ and $j = n+1, n+2$.

The first step is to determine generators for the vanishing ideals of the generalised diagonals Δ_X and Δ_Y in the respective Cox rings, i.e. the ideals

$$\mathfrak{a}_X \;:=\; I(p_X^{-1}(\Delta_X)) \;\subseteq\; \mathcal{O}(\overline{X}) \qquad \text{and} \qquad \mathfrak{a}_Y \;:=\; I(p_Y^{-1}(\Delta_Y)) \;\subseteq\; \mathcal{O}(\overline{Y}).$$

Lemma 4.3.5. *As above let $\mathfrak{a}_X$ and $\mathfrak{a}_Y$ be the ideals of the generalised diagonals Δ_X and Δ_Y in the respective Cox rings. Both of them are prime and they are generated by the following elements.*

(i) *The ideal $\mathfrak{a}_X$ is generated by the 2×2-minors of the matrices*

$$\begin{bmatrix} {}_rT_{0,n_r+1} & {}_rT_{1,n_r+1} & \cdots & {}_rT_{n_r,n_r+1} \\ {}_rT_{0,n_r+2} & {}_rT_{1,n_r+2} & \cdots & {}_rT_{n_r,n_r+2} \end{bmatrix}, \quad r = 1, \dots, \mathbf{r}.$$

(ii) *The ideal $\mathfrak{a}_Y$ is generated by the 2×2-minors of the matrix*

$$\begin{bmatrix} S_{1,n+1} & S_{2,n+1} & \cdots & S_{n,n+1} \\ S_{1,n+2} & S_{2,n+2} & \cdots & S_{n,n+2} \end{bmatrix}.$$

The idea of the proof is to execute the computations on the respective tori. For future reference let us make the following remark.

Remark 4.3.6. Let $\iota\colon (\mathbb{K}^*)^n \to \mathbb{K}^n$ be the canonical open embedding and ι^* its comorphism. If $I \subseteq \mathbb{K}[T_1, \dots, T_n]$ a prime ideal not containing any of the variables T_i, then $(\iota^*)^{-1}\langle \iota^*(I) \rangle = I$ holds.

Proof. The affine variety $X := V(J)$ is irreducible and it intersects $(\mathbb{K}^*)^n$. Hence we have $I(X) = I(X \cap (\mathbb{K}^*)^n)$ and this implies

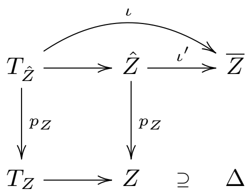
$$X \;=\; \overline{X} \;=\; V(I(X)) \;=\; V(I((\mathbb{K}^*)^n \cap X)) \;=\; \overline{(\mathbb{K}^*)^n \cap X}.$$

Since J is prime, the claim follow from

$$J = I(X) = I(\overline{(\mathbb{K}^*)^n \cap X}) = I(\overline{\iota(\iota^{-1}(X))})$$
$$= I(\overline{\iota(V(\iota^* J))}) = I(V((\iota^*)^{-1}\iota^* J)).$$

□

Proof of Lemma 4.3.5. Let $p_Z \colon \hat{Z} \to Z$ be the Cox construction of a toric variety Z and $\overline{Z}$ its total coordinate space. We view the toric morphism p_Z as a mophism $T_{\hat{Z}} \to T_Z$ of the openly embedded dense tori. Moreover, we denote by $\Delta \subseteq Z$ a subvariety with $\Delta = \overline{\Delta \cap T_Z}$ and write $\iota' \colon \hat{Z} \to \overline{Z}$ and $\iota \colon T_{\hat{Z}} \to \overline{Z}$ for the canonical open embeddings.



Let $\mathfrak{d} \subseteq \mathcal{O}(T_Z)$ be the vanishing ideal of $\Delta \cap T_Z$. For the vanishing ideal of Δ in the Cox ring we obtain

$$I(\iota'(p_Z^{-1}(\Delta))) \;=\; I(\overline{\iota(p_Z^{-1}(\Delta \cap T_Z))}) \;=\; \sqrt{(\iota^*)^{-1}(p_Z^* \mathfrak{d})}.$$

We turn to i) and label the coordinates of

$$T_X \;=\; ((\mathbb{K}^*)^{n_1} \times (\mathbb{K}^*)^{n_1}) \times \ldots \times ((\mathbb{K}^*)^{n_{\mathbf{r}}} \times (\mathbb{K}^*)^{n_{\mathbf{r}}})$$

by ${}_rU_{ij}$ where $r = 1, \ldots, \mathbf{r}$, $i = 1, \ldots, n_r$ and $j = n_r + 1, n_r + 2$. Then the comorphism p_X^* of the corresponding Laurent polynomial rings is given as

$$p_X^* \colon\; \mathbb{K}[{}_rU_{ij}^{\pm}] \;\to\; \mathbb{K}[{}_rT_{ij}^{\pm}]; \qquad {}_rU_{ij} \;\mapsto\; {}_rT_{ij}\, {}_rT_{0j}^{-1}.$$

The vanishing ideal of $\Delta_X \cap T_X$ is generated by

$${}_rU_{i,n_r+1} \;-\; {}_rU_{i,n_r+2} \qquad \text{where} \quad r = 1, \ldots, \mathbf{r}, \text{ and } i = 1, \ldots, n_r.$$

Note that for any $r = 1, \ldots, \mathbf{r}$ and $i, i' = 1, \ldots, n_r$ this ideal also contains the elements

$${}_rU_{i,n_r+1}\; {}_rU_{i',n_r+1}^{-1} \;-\; {}_rU_{i,n_r+2}\; {}_rU_{i',n_r+2}^{-1}.$$

Pulling back all these equation via p_X^* yields the ideal $\iota_X^*(\mathfrak{a}_X)$ in the Laurent polynomial ring $\mathcal{O}(T_{\hat{X}})$. Since $\mathfrak{a}_X$ is an ideal of 2×2-minors, it is prime (in fact, it is the vanishing ideal of the Segre embedding). Hence Remark 4.3.6 gives our assertion.

We turn to ii) and proceed analogously. Here the coordinates of the dense torus $T_Y = (\mathbb{K}^*)^n$ will be labeled U_i with $i = 1, \dots, n$. The comorphism p_Y^* is given by

$$p_Y^*\colon\ \mathbb{K}[U_i^{\pm}] \quad \to \quad \mathbb{K}[T_{ij}^{\pm}]; \qquad U_i \quad \mapsto \quad T_{i,n+1}\, T_{i,n+2}^{-1}.$$

In $\mathcal{O}(T_Y)$ the ideal of $\Delta_Y \cap T_Y$ is generated by the relations $U_i - U_j$ for $1 \le i < j \le n$. Pulling them back via p_Y^* yields the ideal $\iota^*\mathfrak{a}_Y$ and the same argument as in i) yields the assertion. □

We denote the functions from Lemma 4.3.5 by ${}_rf_{ij} \in \mathcal{O}(\overline{X})$ and $g_{ij} \in \mathcal{O}(\overline{Y})$ where r corresponds to the r-th matrix and in both cases i, j define the respective columns. These functions ${}_rf_{ij}$ and g_{ij} give rise to the stretched embeddings

$$\begin{aligned} \overline{\pi}_X\colon\ \bigoplus_{r=1}^{\mathbf{r}} \mathbb{K}^{2(n_r+1)} \quad &\to \quad \bigoplus_{r=1}^{\mathbf{r}} \left(\mathbb{K}^{2(n_r+1)} \oplus \mathbb{K}^{\binom{n_r+1}{2}}\right) \\ (x_1, \dots, x_{\mathbf{r}}) \quad &\mapsto \quad (\,(x_1,\ {}_1f_{ij}(x_1))\,, \ \dots\ , (x_{\mathbf{r}},\ {}_{\mathbf{r}}f_{ij}(x_{\mathbf{r}}))\,) \end{aligned}$$

$$\begin{aligned} \overline{\pi}_Y\colon\ \mathbb{K}^{2n} \quad &\to \quad \mathbb{K}^{2n} \oplus \mathbb{K}^{\binom{n}{2}} \\ y \quad &\mapsto \quad (y,\ g_{ij}(y)). \end{aligned}$$

The vanishing ideals of the images are given by

$$\begin{aligned} I_{X,1} \ &:= \ \langle\, {}_rT_{ij} \ - \ {}_rf_{ij};\ r = 1, \dots, \mathbf{r},\ 0 \le i < j \le n_r + 2,\ i \le n_r \,\rangle, \\ I_{Y,1} \ &:= \ \langle\, S_{ij} \ - \ g_{ij};\ 1 \le i < j \le n + 2,\ i \le n \,\rangle. \end{aligned}$$

We denote by $\iota_{X,1}^*,\ \iota_{X,2}^*,\ \alpha_X^*$ and $\iota_{Y,1}^*,\ \iota_{Y,2}^*,\ \alpha_Y^*$ the respective morphisms from Construction 4.3.3. The new Laurent polynomial rings are then given by

$$\mathbb{K}[T_\infty^{\pm},\ {}_rT_{ij}^{\pm};\ r = 1, \dots, \mathbf{r},\ 0 \le i < j \le n_r + 2,\ i \le n_r],$$
$$\mathbb{K}[S_{ij};\ 1 \le i < j \le n + 2],$$

where the additional variables are T_∞ and $S_{n+1,n+2}$ respectively. We transfer the above ideals according to Construction 4.3.3, i.e. we set

$$\begin{aligned} I'_{X,2} :&= \langle\, \alpha_X^*\ \left(\iota_{X,1}^*(I_{X,1})\right)\,\rangle \\ &= \langle\, {}_rT_{ij}T_\infty \ - \ {}_rf_{ij};\ r = 1, \dots, \mathbf{r},\ 0 \le i < j \le n_r + 2,\ i \le n_r \,\rangle, \end{aligned}$$

$$\begin{aligned} I'_{Y,2} :&= \langle\, \alpha_Y^*\ \left(\iota_{Y,1}^*(I_{Y,1})\right)\,\rangle \\ &= \langle\, S_{ij}S_{n+1,n+2} \ - \ g_{ij};\ 1 \le i < j \le n + 2,\ i \le n \,\rangle. \end{aligned}$$

We first have to compute their preimages under $\iota_{X,2}^*$ and $\iota_{Y,2}^*$, we then show that T_∞ and $S_{n+1,n+2}$ define prime elements and divide none of the remaining variables. Since the resulting relations are very closely related to the Plücker relations, we introduce some new notation. For this let $0 \le i,j,k,l \le n$ be distinct integers. Then we denote by $q(i,j,k,l)$ the corresponding Plücker relation; i.e. if $i<j<k<l$ holds, then we set

$$q(i,j,k,l) \quad := \quad T_{ij}T_{kl} - T_{ik}T_{jl} + T_{il}T_{jk} \ \in \ \mathbb{K}[T_{ij};\ 0 \le i < j \le n].$$

Lemma 4.3.7. *Let $0 \le i_0, j_0 \le n$ be distinct integers. In the Laurent polynomial ring $\mathbb{K}[T_{ij}^{\pm};\ 0 \le i < j \le n]$ consider the ideal*

$$I \ := \ \left\langle q\,(i_0,j_0,k,l); \quad 0 \le k,l \le n, \quad i_0,j_0,k,l \text{ pairwise distinct} \right\rangle.$$

Then for any pairwise distinct $0 \le i,j,k,l \le n$ we have $q(i,j,k,l) \in I$.

Proof. We first claim that for distinct $0 \le i,j,k,l,m \le n$ we have

$$(*) \quad q(i,j,k,l),\ q(i,j,k,m),\ q(i,j,l,m) \in I \quad \Longrightarrow \quad q(i,k,l,m) \in I.$$

For this we assume without loss of generality that $i<j<k<l<m$ holds. The claim then follows from the relation

$$q(i,k,l,m) \ = \ \frac{T_{jk}}{T_{ij}}\,q(i,j,l,m) \ - \ \frac{T_{jl}}{T_{ij}}\,q(i,j,k,m) \ + \ \frac{T_{jm}}{T_{ij}}\,q(i,j,k,l) \ \in \ I.$$

Now consider distinct $0 \le \alpha,\beta,\gamma,\delta \le n$. If $\{\alpha,\beta,\gamma,\delta\} \cap \{i_0,j_0\} \ne \emptyset$ holds, then $q(\alpha,\beta,\gamma,\delta) \in I$ follows from the above claim $(*)$. So assume that $\{\alpha,\beta,\gamma,\delta\}$ and $\{i_0,j_0\}$ are disjoint. Applying $(*)$ to the three collections of indices

$$i_0,j_0,\alpha,\beta,\gamma; \qquad i_0,j_0,\alpha,\beta,\delta; \qquad i_0,j_0,\alpha,\gamma,\delta$$

shows that $q(i_0,\alpha,\beta,\gamma)$, $q(i_0,\alpha,\beta,\delta)$ and $q(i_0,\alpha,\gamma,\delta)$ lie in I. Another application of $(*)$ then proves $q(\alpha,\beta,\gamma,\delta) \in I$. □

We are now ready to prove Theorem 4.2.2, for Theorem 4.2.1 we require some further preparations.

Proof of Theorem 4.2.2. Using Lemma 4.3.7 we see that the ideals $\langle\, \iota_{Y,2}^* I_Y \,\rangle$ and $I'_{Y,2}$ coincide. Since I_Y is prime from Remark 4.3.6 we infer that

$$(\iota_{Y,2}^*)^{-1} I'_{Y,2} \ = \ (\iota_{Y,2}^*)^{-1} \langle\, \iota_{Y,2}^* I_Y \,\rangle \ = \ I_Y.$$

Since I_Y is the ideal of Plücker relations, $S_{n+1,n+2}$ is prime and does not divide any of the remaining variables. We determine the grading of the Cox ring. The ring $\mathcal{O}(\overline{Y}) = \mathbb{K}[S_{ij};\ i=1,\ldots,n,\ j=n+1,n+2]$ is $\mathbb{Z}^n$-graded by $\deg(S_{ij}) = e_i$. Under the stretched embedding the new variables S_{ij} where $1 \le i < j \le n$ are assigned the degrees $\deg(S_{ij}) = \deg(f_{ij}) = e_i + e_j$. Finally, under the blow-up the weights are modified according to 4.3.3 to give the asserted grading. □

We turn to the remaining case of $X = X' \times X'$.

Lemma 4.3.8. *Let $R := \mathbb{K}[T_\infty, T_1, \ldots, T_n]$ be graded by $\mathbb{Z}_{\geq 0}$ and let $I \subseteq R$ be a homogeneous ideal. Suppose that $T_\infty \notin \sqrt{I}$ and $\deg(T_\infty) > 0$ hold. If the ideals $I + \langle T_\infty \rangle$ and $\sqrt{I}$ are prime, then so is I.*

Lemma 4.3.9 (Graded version of Nakayama's lemma). *Let R be a $\mathbb{Z}_{\geq 0}$-graded ring and M a $\mathbb{Z}$-graded R-module such that there exists some $d_0 \in \mathbb{Z}$ with $M_d = 0$ for $d < d_0$. If $I \subseteq R_{>0}$ is an ideal contained in the irrelevant ideal of R with $I\,M = M$, then $M = 0$ holds.*

Proof of Lemma 4.3.8. Compare also [**41**, Proof of Theorem 1]. Since $I + \langle T_\infty \rangle$ is a radical ideal, we have $\sqrt{I} \subseteq I + \langle T_\infty \rangle$. With this we obtain

$$\sqrt{I} \quad = \quad (I + \langle T_\infty \rangle) \cap \sqrt{I} \quad = \quad I + \langle T_\infty \rangle \sqrt{I}.$$

Note that for the second equality we used that $\sqrt{I}$ is prime and $T_\infty \notin \sqrt{I}$ holds. Let $\pi\colon R \to R/I$ denote the canonical projection of $\mathbb{Z}_{\geq 0}$-graded algebras. Then we have $\pi(\sqrt{I}) = \pi(\langle T_\infty \rangle \sqrt{I})$ and $\deg(\pi(T_\infty)) > 0$. The assertion follows from the graded version of Nakayama's Lemma. □

Lemma 4.3.10 ([**41**, Proposition 4]). *Let $1 \leq c \leq n$ be an integer. Then in the polynomial ring $\mathbb{K}[T_{ij};\ 0 \leq i < j \leq n+2]$ the following relations generate a prime ideal*

$$\begin{array}{ll} -T_{ik}T_{jk} + T_{il}T_{jk}; & 0 \leq i < j \leq c < k < l \leq n+2, \\ T_{ij}T_{kl} - T_{ik}T_{jk} + T_{il}T_{jk}; & 0 \leq i < j \qquad < k < l \leq n+2 \quad \textit{other than above.} \end{array}$$

Proof of Theorem 4.2.2. First we claim that the ideal I_X is prime. For this note that the ideal $\langle T_\infty \rangle + I_X$ is generated by T_∞ and the equations

$$\begin{array}{ll} -\ {}_rT_{ik}\ {}_rT_{jk}\ +\ {}_rT_{il}\ {}_rT_{jk}; & 0 \leq i < j \leq n_r, \quad k = n_r + 1,\ l = n_r + 2, \\ {}_rT_{ij}\ {}_rT_{kl}\ -\ {}_rT_{ik}\ {}_rT_{jk}\ +\ {}_rT_{il}\ {}_rT_{jk}; & 0 \leq i < j < k < l \leq n+2 \quad \text{oth. t. above} \end{array}$$

where $r = 1, \ldots, \mathbf{r}$. From Lemma 4.3.10 we infer that $\langle T_\infty \rangle + I_X$ is prime; we check the remaining assumptions of Lemma 4.3.8. Consider the classical grading of R_X, then I_X is homogeneous and $\deg T_\infty > 0$ holds. We only have to verify that $V(I_X)$ is irreducible. For this recall that we transferred the ideal $I_{X,1}$ via

$$I'_{X,1} \;=\; \langle\, \iota^*_{X,1} I_{X,1} \,\rangle \qquad \text{and} \qquad I'_{X,2} \;=\; \langle\, \alpha^*_X I'_{X,1} \,\rangle.$$

Treating the index ∞ as $n_r + 1, n_r + 2$ in Lemma 4.3.7 we see that the latter ideal is given by $I'_{X,2} = \langle \iota^*_{X,2}(I_X) \rangle$. We track the respective zero sets.

$$V(I_X) \;=\; \overline{V(I'_{X,2})} \;=\; \overline{\alpha_X^{-1} V(I'_{X,1})} \;=\; \overline{\alpha_X^{-1}(\iota_{X,1}^{-1}(V(I_{X,1}))}$$

Since α_X has connected kernel and $V(I_{X,1})$ is the graph of $\overline{X}$ and as such irreducible, so is $V(I_X)$. This then implies that I_X is prime.

By Remark 4.3.6 this means that $I_X = (\iota^*_{X,2})^{-1}\langle \iota^*_{X,2} I_X\rangle = (\iota^*_{X,2})^{-1}(I'_{X,2})$ holds. By Proposition 4.3.4 the only thing left to verify is that T_∞ does not divide any of the remaining variables. For this we compute the grading of the Cox ring; for reasons of degree it is then impossible for T_∞ to divide any other variable. The $\mathbb{Z}^{\mathbf{r}} \times \mathbb{Z}^{\mathbf{r}}$-grading of

$$\mathcal{O}(\overline{X}) = \mathbb{K}[{}_rT_{ij};\ r = 1, \ldots, \mathbf{r},\ i = 0, \ldots, n_r,\ j = n_r + 1, n_r + 2]$$

is given by

$$\deg({}_rT_{ij}) \;=\; \begin{cases} (e_r, 0) & \text{if } j = n_r + 1, \\ (0, e_r) & \text{if } j = n_r + 2. \end{cases}$$

When stretching the embedding we add for every $r = 1, \ldots, \mathbf{r}$ the variables ${}_rT_{ij}$ where $0 \leq i < j \leq n_r$. These are assigned the degrees $\deg({}_rT_{ij}) = \deg({}_rg_{ij}) = (e_r, e_r)$. Finally under the blow-up the degrees are modified according to Construction 4.3.3 to give the asserted grading. $\square$

CHAPTER

FIVE

CHOW QUOTIENTS OF QUADRICS

With only minor modifications this entire chapter has already been published in the paper *'On Chow quotients of torus actions'*, which is a joint work with Jürgen Hausen and Simon Keicher, see [**11**].

5.1. The Cox Ring of the Chow Quotient

Consider an action $G \times X \to X$ of a connected linear algebraic group G on a projective variety X defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. The Chow quotient is an answer to the problem of associating in a canonical way a quotient to this action: it is defined as the closure of the set of general G-orbit closures viewed as points in the Chow variety, see Chapter 2, Section 2.3 for more background. The Chow quotient always exists but, in general, its geometry appears to be not easily accessible.

The perhaps most prominent example is the Grothendieck-Knudsen moduli space $\overline{M}_{0,n}$ of stable n-pointed curves of genus zero. Kapranov [**50**] showed that it arises as the Chow quotient of the maximal torus action on the Grassmannian $G(2,n)$. While the Cox ring of $\overline{M}_{0,5}$ is easily computed and finite generation was proven by Castravet in [**18**], the spaces $\overline{M}_{0,n}$ are not Mori Dream for $n \geq 134$, see [**19**]. Motivated by this example, we formulate the following.

Problem 5.1.1. Consider the action $T \times X \to X$ of an algebraic torus T on a Mori dream space X and the normalization Y of the associated Chow quotient.

(i) Is Y a Mori dream space?
(ii) If Y is a Mori dream space, describe its Cox ring.

The situation is well understood in the case of subtorus actions on toric varieties [**49, 23**]. There, the normalized Chow quotient is again toric and hence a Mori dream space. Moreover, the corresponding fan can be computed and thus the Cox ring of the normalized Chow quotient is accessible as well. Similarly, one may treat subtorus actions on rational varieties with a complexity one torus action using their recent combinatorial description. In this chapter we provide tools for a study of the general case. For example, our methods allow a complete answer to 5.1.1 (i) in the case of $\mathbb{K}^*$-actions on smooth projective quadrics:

Theorem 5.1.2. *Let $\mathbb{K}^*$ act on a smooth projective quadric X. Then the associated normalized Chow quotient is a Mori dream space.*

Note that a positive answer to the question 5.1.1 (i) in the case of $\mathbb{K}^*$-actions on arbitrary Mori dream spaces will imply a positive answer for all torus actions on Mori dream spaces: as we show in Theorem 2.4.5, the normalized Chow quotient of a torus action is birationally dominated by the space obtained via stepwise taking normalized Chow quotients by subtori and thus, if the latter space has finitely generated Cox ring, then the normalized Chow quotient does so.

We turn to Problem 5.1.1 (ii). The motivation to describe the Cox ring is that this leads to a systematic approach to the geometry of the Chow quotient. Let us present the results in the case of $\mathbb{K}^*$-actions on quadrics. After equivariantly embedding into a projective space and applying a suitable linear transformation, the smooth projective quadric X is of the following shape:

$$X \;=\; V(g_1) \;\subseteq\; \mathbb{P}_r, \qquad g_1 \;=\; \begin{cases} T_0T_1 + \ldots + T_{r-1}T_r, & r \text{ odd}, \\ T_0T_1 + \ldots + T_{r-2}T_{r-1} + T_r^2, & r \text{ even}, \end{cases}$$

where the $\mathbb{K}^*$-action is diagonal with weights $\zeta_0, \ldots, \zeta_r$ and the defining equation is of degree zero. In order to write down the Cox ring of the Chow quotient, consider the extended weight matrix

$$Q \;:=\; \begin{bmatrix} \zeta_0 & \ldots & \zeta_r \\ 1 & \ldots & 1 \end{bmatrix}$$

where we assume that the columns of Q generate $\mathbb{Z}^2$. Let P be an integral Gale dual, i.e. an $r-1$ by $r+1$ matrix with the row space of Q as kernel. Determine the Gelfand-Kapranov-Zelevinsky decomposition Σ associated to P and put the primitive generators $b_1, \ldots, b_l$ of Σ differing from the columns of P as columns into a matrix B. Then there is an integral matrix A such that $B = P \cdot A$ holds. Define shifted row sums

$$\eta_i := A_{i\,*} + A_{i+1\,*} + \mu \quad \text{for } i = 0, 2, \ldots; \qquad \eta_r := 2A_{r\,*} + \mu, \quad \text{if } r \text{ is even.}$$

where μ is the componentwise minimal vector such that the entries of the η_i are all nonnegative. Then our result reads as follows.

Theorem 5.1.3. *In the above setting, assume that any r columns of Q generate $\mathbb{Z}^2$ and that for odd (even) r there are at least four (three) weights ζ_i of minimal absolute value. Then the normalized Chow quotient Y of the $\mathbb{K}^*$-action on X has Cox ring*

$$\mathcal{R}(Y) \quad = \quad \mathbb{K}[T_0, \ldots, T_r, S_1, \ldots, S_l] \;/\; \langle g_2 \rangle$$

with

$$g_2 \quad := \quad \begin{cases} T_0T_1S^{\eta_0} + T_2T_3S^{\eta_2} + \ldots + T_{r-1}T_rS^{\eta_{r-1}}, & r \text{ odd,} \\ T_0T_1S^{\eta_0} + \ldots + T_{r-2}T_{r-1}S^{\eta_{r-2}} + T_r^2S^{\eta_r}, & r \text{ even} \end{cases}$$

graded by $\mathbb{Z}^{l+2}$ via assigning to the i-th variable the i-th column of a Gale dual of the block matrix $[P, B]$.

The proof of Theorem 5.1.3 is performed in Section 5.2. Besides the explicit description of the rays of the Gelfand-Kapranov-Zelevinsky decomposition provided in Proposition 5.2.1, it requires controlling the behaviour of the Cox ring under certain modifications. This technique is of independent interest and developed in full generality in Section 4.1. The proof of Theorem 5.1.2, given in Section 5.3, uses moreover methods from tropical geometry: we consider a 'weak tropical resolution' of the Chow quotient, see Construction 5.3.3, and provide a reduction principle to divide out intrinsic torus symmetry, see Proposition 5.3.6.

5.2. Proof of Theorem 5.1.3

We approach the Chow quotient via toric embeddings. The idea then is to obtain the Cox ring via toric ambient modifications. An essential step for this is the explicit description of the rays of certain Gelfand-Kapranov-Zelevinsky decompositions given in Proposition 5.2.1; note that in the setting of polytopes related statements implicitly occur in literature, e.g. [**44, 43**].

Recall that the Gelfand-Kapranov-Zelevinsky decomposition associated to a matrix $P \in \mathrm{Mat}(n, r{+}1; \mathbb{Z})$; is the fan Σ in $\mathbb{Q}^n$ with the cones $\sigma(v) = \cap_{v \in \tau^\circ} \tau$, where $v \in \mathbb{Q}^n$ and τ runs through the P-cones, i.e., the cones generated by some of the columns $p_0, \ldots, p_r$ of P. Fix a Gale dual matrix $Q \in \mathrm{Mat}(k, r +$

1; $\mathbb{Z}$), where $r+1 = k+n$, and denote the columns of Q by $q_0, \ldots, q_r$. Then we have mutually dual exact sequences of rational vector spaces

$$0 \longrightarrow \mathbb{Q}^k \xrightarrow{Q^*} \mathbb{Q}^{r+1} \xrightarrow{P} \mathbb{Q}^n \longrightarrow 0$$

$$0 \longleftarrow \mathbb{Q}^k \xleftarrow[Q]{} \mathbb{Q}^{r+1} \xleftarrow[P^*]{} \mathbb{Q}^n \longleftarrow 0.$$

By a *Q-hyperplane* we mean a linear hyperplane in $\mathbb{Q}^k$ generated by some of the columns $q_0, \ldots, q_r$. Given a Q-hyperplane we write it as the kernel $u^\perp$ of a linear form u and associate to it a ray in $\mathbb{Q}^n$ as follows:

$$\varrho(u) \quad := \quad \mathrm{cone}\left(\sum_{u(q_i)>0} u(q_i) p_i \right).$$

It turns out that $\varrho(u) = \varrho(-u)$ holds and thus the ray is well defined. We say that a column q_i of Q is *extremal* if it does not belong to the relative interior of the "movable cone" $\cap_i \mathrm{cone}(q_j;\ j \neq i)$.

Proposition 5.2.1. *Let Q and P be Gale dual matrices as before, assume that the columns of P are pairwise linearly independent nonzero vectors generating $\mathbb{Q}^n$ as a cone and let Σ be the Gelfand-Kapranov-Zelevinsky decomposition associated to P.*

(i) *If a ray $\varrho \in \Sigma$ is the intersection of two P-cones, then $\varrho = \varrho(u)$ holds with a Q-hyperplane $u^\perp$.*
(ii) *If $k = 2$ holds, then every ray of Σ can be obtained as an intersection of two P-cones.*
(iii) *Assume $k = 2$ and fix nonzero linear forms u_i with $u_i \perp q_i$. Then the rays of Σ are* $\mathrm{cone}(p_0), \ldots, \mathrm{cone}(p_r)$ *and the $\varrho(u_i)$ with q_i not extremal.*

The proof relies on the fact that Σ describes the lifts of regular Q-subdivisions. We adapt the precise formulation of this statement to our needs. Let $\gamma \subseteq \mathbb{Q}^{r+1}$ be the positive orthant and define a *γ-collection* to be a set $\mathfrak{B}$ of faces of γ such that any two $\gamma_1, \gamma_2 \in \mathfrak{B}$ admit an *invariant separating linear form* f in the sense that

$$P^*(\mathbb{Q}^n) \subseteq f^\perp, \qquad f_{|\gamma_1} \geq 0, \qquad f_{|\gamma_2} \leq 0, \qquad f^\perp \cap \gamma_i = \gamma_1 \cap \gamma_2.$$

Write $\mathfrak{B}_1 \leq \mathfrak{B}_2$ if for every $\gamma_1 \in \mathfrak{B}_1$ there is a $\gamma_2 \in \mathfrak{B}_2$ with $\gamma_1 \subseteq \gamma_2$. Moreover, call a γ-collection $\mathfrak{B}$ *normal* if it cannot be enlarged as a γ-collection and the images $Q(\gamma_0)$, where $\gamma_0 \in \mathfrak{B}$, form the normal fan of a polyhedron.

For a face $\gamma_0 \preceq \gamma$, we denote by $\gamma_0^* = \gamma_0^\perp \cap \gamma^\vee$ the corresponding face of the dual cone $\gamma^\vee$.

Now assume that the columns of P are pairwise different nonzero vectors. Then [**4**, Sec. II.2] provides us with an order-reversing bijection

$$\{\text{normal } \gamma\text{-collections}\} \;\to\; \Sigma, \qquad\qquad \mathfrak{B} \;\mapsto\; \bigcap_{\gamma_0 \in \mathfrak{B}} P(\gamma_0^*).$$

Proof of Proposition 5.2.1. We prove (i). Let $\varrho = P(\gamma_1^*) \cap P(\gamma_2^*)$ with $\gamma_1, \gamma_2 \preceq \gamma$. We may assume that the relative interiors $P(\gamma_1^*)^\circ$ and $P(\gamma_2^*)^\circ$ intersect nontrivially. Then γ_1 and γ_2 admit an invariant separating linear form $f = Q^*(u)$ with a linear form u on $\mathbb{Q}^k$. In terms of the components of $f_i = u(q_i)$ of f, we have

$$\gamma_1 \;=\; \operatorname{cone}(e_i;\ f_i \geq 0), \qquad\qquad \gamma_2 \;=\; \operatorname{cone}(e_i;\ f_i \leq 0).$$

Write $f = f^+ - f^-$ with the unique vectors $f^+, f^- \in \mathbb{Q}^{r+1}$ having only nonnegative components. Then $P(f) = 0$ gives $P(f^+) = P(f^-)$. We conclude $\varrho = \operatorname{cone}(P(f^+))$ and the assertion follows.

We prove (ii) and (iii). The rays of Σ arise from normal γ-collections which are submaximal with respect to "$\leq$" in the sense that the only dominating γ-collection is the trivial collection $\langle\gamma\rangle$ consisting of all faces $\gamma_0 \preceq \gamma$ which are invariantly separable from γ. There are precisely two types of such submaximal collections:

- the normal γ-collections $\mathfrak{B} = \langle\gamma_0\rangle$, where $\gamma_0 \precneqq \gamma$ is a facet satisfying $Q(\gamma_0) = Q(\gamma)$,
- the normal γ-collections $\mathfrak{B} = \langle\gamma_1, \gamma_2\rangle$, where $\gamma_1, \gamma_2 \precneqq \gamma$ are invariantly separable from each other and satisfy

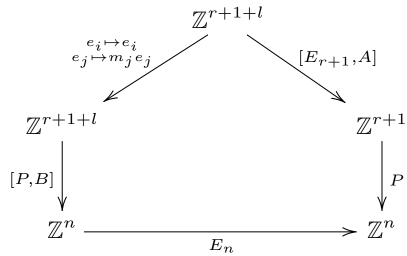
$$\gamma_i \;=\; Q^{-1}(Q(\gamma_i)) \cap \gamma, \qquad Q(\gamma) \;=\; Q(\gamma_1) \cup Q(\gamma_2).$$

The submaximal γ-collections of the first type give the rays $\operatorname{cone}(p_i) \in \Sigma$ with q_i not extremal. If q_i is extremal, then the (unique) γ-collection of the second type with $Q(\gamma_1) = \operatorname{cone}(q_j;\ j \neq i)$ defines the ray $\operatorname{cone}(p_i)$. The remaining rays of Σ are of the form $\varrho = P(\gamma_1^*) \cap P(\gamma_2)^*$ with the remaining collections of the second type. □

Remark 5.2.2. Statements (ii) and (iii) of Proposition 5.2.1 hold as well for pairs P, Q, where the columns of Q generate the cone over a so called *totally-2-splittable* polytope; these have been studied in [**44, 43**].

As a further preparation of the proof of Theorem 5.1.3 we have to specialize the discussion of Section 4.1 to the case of a single defining equation. The following notion will be used for an explicit description of the transferred ideal.

Definition 5.2.3. Consider an $n \times (r+1)$ matrix P and an $n \times l$ matrix B, both integral. A *weak B-lifting (with respect to P)* is an integral $(r+1) \times l$ matrix A allowing a commutative diagram



where the e_i are the first $r+1$, the e_j the last l canonical basis vectors of $\mathbb{Z}^{r+1+l}$, the m_j are positive integers and E_n, E_{r+1} denote the unit matrices of size $n, r+1$ respectively.

Note that weak B-liftings A always exist. Given such A, consider the following homomorphism of Laurent polynomial rings:

$$\psi_A \colon \mathbb{K}[T_0^{\pm 1}, \ldots, T_r^{\pm 1}] \;\to\; \mathbb{K}[T_0^{\pm 1}, \ldots, T_r^{\pm 1}, S_1^{\pm 1}, \ldots, S_l^{\pm 1}],$$
$$\sum \alpha_\nu T^\nu \;\mapsto\; \sum \alpha_\nu T^\nu S^{A^t \cdot \nu}.$$

Set $K_1 := \mathbb{Z}^{r+1}/P^*(\mathbb{Z}^n)$. Then the left hand side algebra is K_1-graded by assigning to the i-th variable the class of e_i in K_1.

Lemma 5.2.4. *In the above notation, let $g_1 \in \mathbb{K}[T_0^{\pm 1}, \ldots, T_r^{\pm 1}]$ be a K_1-homogeneous polynomial.*

(i) *We have $T^\nu S^\mu \psi_A(g_1) = g_2'$ with $\nu \in \mathbb{Z}^{r+1}$, $\mu \in \mathbb{Z}^l$ and a unique monomial free $g_2' \in \mathbb{K}[T_0, \ldots, T_r, S_1, \ldots, S_l]$.*

(ii) *The polynomial g_2' is of the form $g_2' = g_2(T_0, \ldots, T_{r+1}, S_1^{m_1}, \ldots, S_l^{m_1})$ with a $g_2 \in \mathbb{K}[T_0, \ldots, T_r, S_1, \ldots, S_l]$ not depending on the choice of A.*

(iii) *If, in the setting of Construction 4.1.1, we have $I_1 = \langle g_1 \rangle$, then the transferred ideal is given by $I_2 = \langle g_2 \rangle$.*

(iv) *The variable T_i defines a prime element in $\mathbb{K}[T_0, \ldots, T_{r+l+1}]/\langle g_2 \rangle$ if and only if the polynomial $g_2(T_1, \ldots, T_{i-1}, 0, T_{i+1}, \ldots, T_{r+l+1})$ is irreducible.*

Proof. Consider the commutative diagram of group algebras corresponding to the dualized diagram 5.2.3. There, ψ_A occurs as the homomorphism of group algebras defined by the transpose $[E_{r+1}, A]^*$. Let T^κ be any monomial of g_1. Then $g_1' := T^{-\kappa} g_1$ gives rise to the same g_2, but g_1' is of K_1-degree zero and hence a pullback $g_1' = \psi_{P^*}(h)$. The latter allows to use commutativity of the diagram which gives (i) and (ii). Assertions (iii) and (iv) are clear. $\square$

Proof of Theorem 5.1.3. Recall that we consider the quadric $X = V(g_1) \subseteq \mathbb{P}_r$ with $g_1 = T_0T_1 + \ldots + T_{r-1}T_r$, where we replace the last term with T_r^2 in the case of an even r, and a $\mathbb{K}^*$-action on $\mathbb{P}_r$, given by weights $\zeta_0, \ldots, \zeta_r$ such that g_0 is of degree zero and, in particular, X is invariant.

In a first step, we construct a suitable GIT quotient X_1 of the $\mathbb{K}^*$-action on X. Lifting the above data to $\mathbb{K}^{r+1}$ gives $\bar{X} := V(g_1) \subseteq \mathbb{K}^{r+1}$ which is invariant under the action of $\mathbb{T}^2 = \mathbb{K}^* \times \mathbb{K}^*$ on $\mathbb{K}^{r+1}$ given by the weight matrix

$$Q \quad := \quad \begin{bmatrix} \zeta_0 & \ldots & \zeta_r \\ 1 & \ldots & 1 \end{bmatrix}$$

Consider the weight $w = (0,1)$ of $\mathbb{T}^2$ and the associated set of semistable points $\hat{Z}_1 \subseteq \mathbb{K}^{r+1}$, that means the union of all localizations $\mathbb{K}^{r+1}_f$, where f is homogeneous with respect to some positive multiple of w. Then $\hat{Z}_1$ is a toric open subset, and with $\hat{X}_1 := \bar{X} \cap \hat{Z}_1$ we obtain a commutative diagram

$$\begin{array}{ccc} \hat{X}_1 & \subseteq & \hat{Z}_1 \\ {\scriptstyle /\!\!/\mathbb{T}^2}\downarrow & & \downarrow{\scriptstyle /\!\!/\mathbb{T}^2} \\ X_1 & \longrightarrow & Z_1 \end{array}$$

where the induced map $X_1 \to Z_1$ of quotients is a closed embedding. We are in the setting presented before Corollary 4.1.4. In particular, $\hat{Z}_1 \to Z_1$ is a toric Cox construction with a Gale dual P of Q as describing matrix; note that the columns of P generate $\mathbb{Z}^{r-1}$ as a lattice. Moreover, the Cox ring of X_1 is the $\mathbb{Z}^2$-graded ring

$$R_1 \quad = \quad \mathbb{K}[T_0, \ldots, T_r] \;/\; \langle g_1 \rangle.$$

Observe that X_1 is as well the $\mathbb{K}^*$-quotient of the image of $\hat{X}_1$ in X which in turn is the set of semistable points of a suitable linearization of $\mathcal{O}(1)$.

Set $n := r - 1$ and consider the Gelfand-Kapranov-Zelevinsky decomposition Σ associated to P. Then, according to Proposition 2.4.1, the toric variety Z_2 determined by Σ is the normalized Chow quotient of the $\mathbb{K}^*$-action on $\mathbb{P}_r$. Moreover, let $X_2 \subseteq Z_2$ denote the proper transform of $X_1 \subseteq Z_1$ under the toric morphism $Z_2 \to Z_1$. Then Proposition 2.4.2 tells us that X_2 and the Chow quotient $X /\!\!/_{\mathrm{CQ}} \mathbb{K}^*$ share the same normalization.

We will now show that X_2 is in fact normal and that its Cox ring is as claimed in the Theorem. As before, put the primitive generators $b_1, \ldots, b_l$ of rays of Σ differing from columns of P into a matrix B and choose a weak B-lifting

A with respect to P; using the fact that the columns of P generate $\mathbb{Z}^n$, we can choose the numbers m_j all equal to one. With the shifted row sums $\eta_0, \eta_2, \ldots, \eta_{r-1}$ we set

$$g_2 \quad := \quad \begin{cases} T_0T_1S^{\eta_0} + T_2T_3S^{\eta_2} + \ldots + T_{r-1}T_rS^{\eta_{r-1}}, & r \text{ odd}, \\ T_0T_1S^{\eta_0} + \ldots + T_{r-2}T_{r-1}S^{\eta_{r-2}} + T_r^2S^{\eta_r}, & r \text{ even}. \end{cases}$$

Lemma 5.2.4 then ensures that $I_2 := \langle g_2 \rangle$ is the transferred ideal of $I_1 := \langle g_1 \rangle$ in the sense of Construction 4.1.1; define $P_1 := P$ and $P_2 := [P, B]$ to adapt the settings. Consider the ring

$$R_2 \quad = \quad \mathbb{K}[T_0, \ldots, T_r, S_1, \ldots, S_l] \ / \ \langle g_2 \rangle.$$

Our task is to show that the variables $S_1, \ldots, S_l$ define prime elements in R_2. Then Proposition 4.1.3 tells us that R_2 and thus X_2 are normal and Corollary 4.1.4 yields that the Cox ring of X_2 is R_2 together with the $\mathbb{Z}^{2+l}$-grading defined by a Gale dual Q_2 of $P_2 = [P, B]$.

Suitably renumbering the variables T_i, we achieve that $|\zeta_{r-3}|, \ldots, |\zeta_r|$ are minimal among all $|\zeta_i|$ in the case of odd r and, similarly, in the case of even r, we have $\zeta_{r-3} = \zeta_{r-2} = \zeta_{r-1} = 0$. In order to see that the S_j define primes, it suffices to show that, according to odd and even r,

$$g_2 \;=\; T_{r-3}T_{r-2} + T_{r-1}T_r + h, \qquad \text{or} \qquad g_2 \;=\; T_{r-2}T_{r-1} + T_r^2 + h,$$

holds with a polynomial $h \in \mathbb{K}[T_0, \ldots, T_r, S_1, \ldots, S_l]$ not depending on the last four (three) T_i, see Lemma 5.2.4 (iv). This in turn is seen by constructing a suitable weak B-lifting via the description of the rays through $b_1, \ldots, b_l$ provided by Proposition 5.2.1. Each b_j (or a suitable integral multiple) stems from a Q-hyperplane and the u_j can be chosen to be nonpositive on the last four (three) q_i. Putting $\max(0, u_j(q_i))$ into a matrix A' gives a weak B-lifting A' with $A'_{i*} = 0$ for the last four (three) rows. By Lemma 5.2.4, the weak B-lifting A' yields the same g_2 which now has the desired form. □

Example 5.2.5. Consider the quadric $X = V(T_0T_1 + T_2T_3 + T_4T_5 + T_6^2) \subseteq \mathbb{P}_6$ and the action of $\mathbb{K}^*$ on $\mathbb{P}_6$ given by

$$t \cdot [x_0, \ldots, x_6] \quad := \quad [t^{-2}x_0, t^2x_1, t^{-1}x_2, t^1x_3, x_4, x_5, x_6].$$

An integral Gale dual P of the extended weight matrix Q is of size 5×7 and explicitly given as

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

Computing the associated Gelfand-Kapranov-Zelevinsky decomposition we see that it comes with one new ray, namely

$$b_1 = (-1, 0, -1, 1, 0) = 2p_0 + p_2,$$

where $p_0, \ldots, p_6$ are the columns of P. The Cox ring of the normalized Chow quotient $X \tilde{/\!\!/}_{cQ} \mathbb{K}^*$ is the ring

$$\mathcal{R}(X \tilde{/\!\!/}_{cQ} \mathbb{K}^*) = \mathbb{K}[T_0, \ldots, T_6, S_1] \,/\, \langle T_0T_1S_1^2 + T_2T_3S_1 + T_4T_5 + T_6^2 \rangle$$

together with the grading by $\mathrm{Cl}(X) = \mathbb{Z}^3$ via a Gale dual of $[p_0, \ldots, p_6, b_1]$, i.e. the degrees of the variable are the columns of

$$\begin{bmatrix} -2 & 2 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Remark 5.2.6. The setting of Theorem 5.1.3 can also be interpreted in terms of Mori theory. There are (up to isomorphism) finitely many normal projective varieties $Y_1, \ldots, Y_s$ sharing as their Cox ring a given $R_1 = \mathbb{K}[T_0, \ldots, T_n]/\langle g_1 \rangle$ with its $\mathbb{Z}^2$-grading coming from the extended weight matrix Q. Each Y_i is a GIT-quotient of the induced $\mathbb{K}^*$-action on the quadric $X = V(g_1) \subseteq \mathbb{P}_r$ and thus dominated in universal manner by the normalized Chow quotient $Y = X \tilde{/\!\!/}_{cQ} \mathbb{K}^*$. Thus, Y is the "Mori master space" controlling the whole class of small birational relatives Y_i. This picture obviously extends to all Mori dream spaces, and it is a natural desire to study the geometry of the Mori master spaces.

5.3. Proof of Theorem 5.1.2

The main idea of the proof is to consider instead of the Chow quotient its "weak tropical resolution" and to use intrinsic symmetry of the latter space. This approach applies also to problems beyond $\mathbb{K}^*$-actions on quadrics; we therefore develop it in sufficient generality. We begin with recalling the necessary concepts from tropical geometry.

Let f be a Laurent polynomial in n variables. The Newton polytope $B_f \subseteq \mathbb{Q}^n$ is the convex hull over the exponent vectors of f. The tropical variety $\mathrm{trop}(V(f))$ of the zero set $V(f) \subseteq \mathbb{T}^n$ lives in $\mathbb{Q}^n$ and is defined to be the union of all $(n-1)$-dimensional cones of the normal fan of B_f. The tropical variety of an arbitrary closed subset $Y \subseteq \mathbb{T}_n$ is the intersection $\mathrm{trop}(Y)$ over all $\mathrm{trop}(V(f))$, where f runs through the ideal of Y. It turns out that $\mathrm{trop}(Y)$ is the support of an (in general not unique and not pointed) fan in $\mathbb{Q}^n$.

Definition 5.3.1. Consider a toric variety Z defined by a fan Σ in $\mathbb{Q}^n$ and an irreducible subvariety $Y \subseteq Z$ intersecting the big torus $\mathbb{T}^n \subseteq Z$ nontrivially.

We call the embedding $Y \subseteq Z$ *weakly tropical* if the support $|\Sigma| \subseteq \mathbb{Q}^n$ equals the tropical variety $\mathrm{trop}(Y \cap \mathbb{T}^n) \subseteq \mathbb{Q}^n$.

Remark 5.3.2. Any tropical embedding in the sense of Tevelev [**65**] is weakly tropical. If $Y \subseteq Z$ is a weakly tropical subvariety of a toric variety Z, then, by [**35**, Sec. 14], for any toric orbit $\mathbb{T}^n \cdot z \subseteq Z$ intersecting Y nontrivially, we have

$$\dim(Z) \;-\; \dim(\mathbb{T}^n \cdot z) \;=\; \dim(Y) \;-\; \dim(\mathbb{T}^n \cdot z \cap Y).$$

Construction 5.3.3 (Weak tropical resolution)**.** Let Z be a complete toric variety arising from a fan Σ in $\mathbb{Q}^n$ and $Y \subseteq Z$ an irreducible subvariety intersecting the big torus $\mathbb{T}^n \subseteq Z$ nontrivially. Fix a fan structure Σ_Y carried on the tropical variety $\mathrm{trop}(Y \cap \mathbb{T}^n) \subseteq \mathbb{Q}^n$ for $Y \cap \mathbb{T}^n$ and consider the coarsest common refinement

$$\Sigma' \;:=\; \Sigma \sqcap \Sigma_Y \;=\; \{\tau \cap \sigma;\ \sigma \in \Sigma,\ \tau \in \Sigma_Y\}$$

of the fans Σ and Σ_Y. Then the canonical map of fans $\Sigma' \to \Sigma$ defines a birational toric morphism $Z' \to Z$ of the associated toric varieties. With the proper transform $Y' \subseteq Z'$ of $Y \subseteq Z$, we obtain a proper birational map $Y' \to Y$ which we call a *weak tropical resolution* of $Y \subseteq Z$.

Proof. The only thing to show is properness of the morphism $Y' \to Y$. But this follows directly from Tevelev's criterion [**65**, Prop. 2.3]. □

The use of passing to the weak tropical resolution in our context is that it enables us to divide out torus symmetries in a controlled manner. This leads to an explicit version of [**42**, Thm. 1.2] relating the Mori dream space property of a variety to the Mori dream space property of a certain quotient.

Construction 5.3.4. Consider a toric variety Z arising from a fan Σ in $\mathbb{Q}^r$, and a weakly tropical embedded subvariety $Y \subseteq Z$. Suppose that Y is invariant under the action of a subtorus $T \subseteq \mathbb{T}^r$. Set

$$Z_0 \;:=\; \{z \in Z;\ \dim(\mathbb{T}^r \cdot z) \geq r-1,\ T_z \text{ finite}\}, \qquad\qquad Y_0 \;:=\; Y \cap Z_0.$$

Then $Z_0 \subseteq Z$ is an open toric subset corresponding to a subfan $\Sigma_0 \preceq \Sigma$ with certain rays $\varrho_1, \ldots, \varrho_s$ of Σ as its maximal cones. Let the matrix $P \in \mathrm{Mat}(n, r; \mathbb{Z})$ describe an epimorphism $\pi\colon \mathbb{T}^r \to \mathbb{T}^n$ with $\ker(\pi) = T$ and consider the following fan in $\mathbb{Z}^n$:

$$\Delta_0 \;:=\; \{0, P(\varrho_1), \ldots, P(\varrho_s)\}.$$

Note that $\varrho_1, \ldots, \varrho_s$ are precisely the rays of Σ which are not contained in $\ker(P)$. The matrix P determines a toric morphism $Z_0 \to Z /_{\circ} T$ onto the toric variety associated to Δ_0. We define $Y /_{\circ} T \subseteq Z /_{\circ} T$ to be the closure of the image $\pi(Y \cap \mathbb{T}^r)$.

Remark 5.3.5. The tropical variety $\mathrm{trop}(Y /_{\circ} T \cap \mathbb{T}^n)$ contains all rays $P(\varrho_1), \ldots, P(\varrho_s)$ of the fan Δ_0. If there is a fan Δ in $\mathbb{Z}^n$ having $\mathrm{trop}(Y /_{\circ} T \cap \mathbb{T}^n)$ as its support and $P(\varrho_1), \ldots, P(\varrho_s)$ as its rays, then $Y /_{\circ} T$ admits a weakly tropical completion with boundary of codimension at least two.

Proposition 5.3.6. *Consider a toric variety Z, a weakly tropical subvariety $Y \subseteq Z$ and suppose that Y is invariant under the action of a subtorus $T \subseteq \mathbb{T}^r$. Then the following statements are equivalent.*

(i) *The normalization of Y has finitely generated Cox ring.*
(ii) *The normalization of $Y /_{\circ} T$ has finitely generated Cox ring.*

Proof. Let $\nu \colon \tilde{Y} \to Y$ be the normalization map. By $W \subseteq Y$ we denote the open T-invariant subset consisting of all points $y \in Y$ having a finite isotropy group T_y. The fact that $Y \subseteq Z$ is tropically embedded ensures that $Y_0 \subseteq W$ has a complement of codimension at least two in W. This property is preserved when passing to the respective normalizations $\tilde{W} := \nu^{-1}(W)$ and $\tilde{Y}_0 := \nu^{-1}(Y_0)$. In particular the separations in the sense of [**42**, p. 978] of the corresponding quotients $\tilde{W}/T$ and $\tilde{Y}_0/T$ have the same Cox rings. Since normalizing commutes with taking quotients and separating, the latter space is isomorphic to the normalization of $Y /_{\circ} T$. Thus the assertion follows from [**42**, Theorem 1.2]. □

Proposition 5.3.7. *Let Z be a toric variety, $Y \subseteq Z$ a complete subvariety which is invariant under a subtorus T of the big torus of Z and $Y' \to Y$ be a weak tropical resolution. If the normalization of $Y' /_{\circ} T$ has finitely generated Cox ring, then the normalization $\tilde{Y}$ of Y is a Mori dream space.*

Proof. Since the normalization of $Y' /_{\circ} T$ has finitely generated Cox ring, Proposition 5.3.6 shows that the normalization Y'' of Y' has finitely generated Cox ring and thus is a Mori dream space. The canonical morphism $\pi \colon Y'' \to \tilde{Y}$ is proper and birational. In order to see that $\tilde{Y}$ is a Mori dream space, we may apply the general [**57**, Thm. 10.4], or look at a suitable sheaf $\mathcal{S} = \oplus_K \mathcal{O}_Y(D)$ of divisorial algebras on Y mapping onto the Cox sheaf $\mathcal{R}$ of Y. By properness of π, we obtain $\mathcal{S} = \pi_* \mathcal{S}''$ over the set $W \subseteq \tilde{Y}$ of regular points for $\mathcal{S}'' = \oplus_K \mathcal{O}_X(\pi^*(D))$. Since Y'' is a Mori dream space, $\Gamma(\pi^{-1}(W), \mathcal{S}'')$ is finitely generated. This implies finite generation of the Cox ring $\mathcal{R}(\tilde{Y}) = \Gamma(W, \mathcal{R})$. □

A second preparation of the proof of Theorem 5.1.2 concerns toric ambient modification. We will always write $e_1, \ldots, e_n \in \mathbb{Z}^n$ for the canonical basis vectors and set $e_0 := -e_1 - \ldots - e_n$. Moreover, we denote by $\Delta(n)$ the fan in $\mathbb{Z}^n$ consisting of all cones spanned by at most n of the vectors $e_0, \ldots, e_n$ and by $\Delta'(n) \subseteq \Delta(n)$ the subfan consisting of all cone of dimension at most $n-1$.

Lemma 5.3.8. *Consider nonzero vectors $v_1, \ldots, v_l \in \mathbb{Q}^n$ contained in a maximal cone $\tau \in \Delta(n)$, a cone $\sigma \subseteq \mathbb{Q}^n$ generated by some of the vectors $e_0, \ldots, e_n, v_1, \ldots, v_l$ and a cone $\delta \in \Delta'(n)$. Suppose that $\varrho := \delta \cap \sigma$ is one-dimensional and $\varrho \notin \Delta'(n)$. Then ϱ is contained in some facet of τ.*

Proof. We may assume that $\tau = \mathrm{cone}(e_1, \ldots, e_n)$ holds. Replacing δ and σ with suitable faces, we may assume $\varrho^\circ = \delta^\circ \cap \sigma^\circ$. The proof uses Gale duality and we work in the notation of Section 5.2. Consider the matrix $P := [e_0, \ldots, e_n, v_1, \ldots, v_l]$ and its Gale dual

$$Q \;:=\; [q_0, \ldots, q_{n+l}] \;:=\; \begin{bmatrix} 0 & v_{11} & \cdots & v_{1n} & -1 & & 0 \\ \vdots & \vdots & & \vdots & & \ddots & \\ 0 & v_{l1} & \cdots & v_{ln} & 0 & & -1 \\ 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Set $r := n + l$, let $e'_0, \ldots, e'_r$ denote the canonical basis vectors of $\mathbb{Z}^{r+1}$ and $\gamma := \mathbb{Q}^{r+1}_{\geq 0}$ the positive orthant. Then there are faces $\gamma_1, \gamma_2 \preceq \gamma$ such that for the corresponding dual faces γ_i^* we have

$$P(\gamma_1^*) \;=\; \delta, \qquad P(\gamma_2^*) \;=\; \sigma, \qquad P(\gamma_1^*)^\circ \cap P(\gamma_2^*)^\circ \;\neq\; \emptyset.$$

For some $n + 1 \leq j \leq r$ we have $e'_j \in \gamma_2^*$ and we may assume that γ_1^* is generated by at most $n - 1$ of the vectors $e'_0, \ldots, e'_n$. The latter implies $e'_{n+1}, \ldots, e'_{n+l} \in \gamma_1$. Let $f = Q^*(u)$ be a separating linear form for γ_1 and γ_2. Then $f_{|\gamma_1} \geq 0$ implies

$$u(q_{n+1}), \ldots, u(q_{n+l}) \;\geq\; 0, \qquad u(q_0) \;\geq\; u(q_1), \ldots, u(q_n).$$

Note that we must have $f(e'_j) = u(q_j) > 0$, because e'_j does not lie in γ_2. Let $\tau_1, \tau_2 \preceq \gamma$ be the maximal faces with $f_{|\tau_1} \geq 0$ and $f_{|\tau_2} \leq 0$. Then f separates τ_1, τ_2 and $\tau_i^* \subseteq \gamma_i^*$ holds. We conclude

$$\emptyset \;\neq\; P(\tau_1^*)^\circ \cap P(\tau_2^*)^\circ \;\subseteq\; P(\tau_1^*) \cap P(\tau_2^*) \;\subseteq\; P(\gamma_1^*) \cap P(\gamma_2^*) \;=\; \varrho.$$

Since $e'_j \notin \tau_2$ holds, we obtain $\tau_2^* \neq \{0\}$ and thus $0 \notin P(\tau_2^*)^\circ$. Together with the displayed line this gives $P(\tau_1^*) \cap P(\tau_2^*) = \varrho$. Since at least two of $e'_0, \ldots, e'_n$ lie in γ_1, we obtain $e'_0 \in \tau_1$ and thus

$$\varrho \;\subseteq\; P(\tau_1^*) \;\subseteq\; \mathrm{cone}(e_1, \ldots, e_n).$$

□

Lemma 5.3.9. *For $n \in \mathbb{Z}_{\geq 1}$ consider $\Delta'(n)$ and let $b_1, \ldots, b_l \in \mathbb{Q}^n$ be pairwise different primitive vectors lying on the support of $\Delta'(n)$ but not on its rays. Denote by $\sigma_j \in \Delta'(n)$ the minimal cone with $b_j \in \sigma_j$ and write*

$$b_j \;=\; a_{0j}e_0 + \ldots + a_{nj}e_n, \quad \text{where } a_{ij} > 0 \text{ if } e_i \in \sigma_j,\; a_{ij} = 0 \text{ if } e_i \notin \sigma_j.$$

Then, for $P := [e_0, \dots, e_n]$ and $B := [b_1, \dots, b_l]$, the matrix $A := (a_{ji})$ is a weak B-lifting with respect to P. The lift of $h_1 = T_0 + \dots + T_n$ in the sense of Lemma 5.2.4 is given by

$$h_2 \;=\; T_0 S_1^{a_{01}} \cdots S_l^{a_{0l}} \;+\; \dots \;+\; T_n S_1^{a_{n1}} \cdots S_l^{a_{nl}}.$$

Moreover, the variables $T_0, \dots T_n, S_1, \dots, S_l$ define pairwise nonassociated prime elements in $\mathbb{K}[T_0, \dots T_n, S_1, \dots, S_l]/\langle h_2 \rangle$ if and only if the vectors $b_1, \dots, b_l$ lie in a common cone of $\Delta(n)$.

Proof. Only the last sentence needs some explanation. The fact that $b_1, \dots, b_l$ lie in a common cone of $\Delta(n)$ is equivalent to the fact that there is a term of h_2 not depending on $S_1, \dots, S_l$, and, moreover, for every k there is a further is terms of h_2 not depending on S_k. Now, Lemma 5.2.4 (iv) gives the desired characterization. □

Proof of Theorem 5.1.2. We may assume that $X = V(g_1) \subseteq \mathbb{P}_r$ holds with a polynomial $g_1 = T_0T_1 + \dots + T_{r-1}T_r$, where we replace the last term with T_r^2 in the case of an even r, and $\mathbb{K}^*$ acts linearly with weights $\zeta_0, \dots, \zeta_r$, where $|a_r|$ is minimal among all $|\zeta_i|$, see [**4**, Prop. III.2.4.7].

The first step is to determine the normalized Chow quotient of the $\mathbb{K}^*$-action on X. As observed in Proposition 2.4.2, the Chow quotient $X /\!\!/_{\mathrm{CQ}} \mathbb{K}^*$ is canonically embedded into the Chow quotient of $\mathbb{P}_r$ by the $\mathbb{K}^*$-action. To determine the latter, consider the extended weight matrix

$$Q \;:=\; \begin{bmatrix} \zeta_0 & \dots & \zeta_r \\ 1 & \dots & 1 \end{bmatrix}$$

and let P be a Gale dual matrix. Then, according to Proposition 2.4.1, the normalized Chow quotient of the $\mathbb{K}^*$-action on $\mathbb{P}_r$ is the toric variety Z having the Gelfand-Kapranov-Zelevinsky-decomposition Σ defined by the columns of P as its fan. Moreover, by Proposition 2.4.2, the Chow quotient of the $\mathbb{K}^*$-action on X has the same normalization as the closure

$$Y \;=\; \overline{(X \cap \mathbb{T}^r)\,/\,\mathbb{K}^*} \;\subseteq\; Z.$$

The second step is to determine a weak tropical resolution of $Y \subseteq Z$. For this we first need $\mathrm{trop}(Y \cap \mathbb{T}_Z)$. Let $\mu_0, \dots, \mu_n \in \mathbb{Z}^{r+1}$ be the vertices of the Newton polytope g_1 and consider the matrix P_{gr} with the rows $\mu_i - \mu_0$, where $i = 1, \dots, n$. Then we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Q}^2 & \longrightarrow & \mathbb{Q}^{r+1} & \xrightarrow{\;P\;} & \mathbb{Q}^{r-1} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle \Pi} & & \\
0 & \longrightarrow & \mathbb{Q}^{r+1-n} & \longrightarrow & \mathbb{Q}^{r-1} & \xrightarrow[P_{\mathrm{gr}}]{} & \mathbb{Q}^{n} & \longrightarrow & 0
\end{array}$$

Note that g_1 equals T^{μ_0} times the pullback of the polynomial $h_1 := 1 + S_1 + \ldots + S_n$ under the homomorphism of tori $\mathbb{T}^r \to \mathbb{T}^n$ defined by P_{gr}. The tropical variety of $V(h_1) \subseteq \mathbb{T}^n$ is the support of the fan $\Delta'(n)$ and thus we have

$$\mathrm{trop}(Y \cap \mathbb{T}_Z) \;=\; \Pi^{-1}(\mathrm{trop}(V(h_1))) \;=\; \Pi^{-1}(|\Delta'(n)|).$$

We endow $\mathrm{trop}(Y \cap \mathbb{T}_Z)$ with the natural fan structure lifting $\Delta'(n)$; note that the cones are in general not pointed. By definition, the weak tropical resolution Y' of Y is the closure of $Y \cap \mathbb{T}_Z$ in the toric variety Z' with the coarsest common refinement $\Sigma' := \Sigma \sqcap \mathrm{trop}(Y \cap \mathbb{T}_Z)$ as its fan.

In the third step, we pass to $Y' /\!\!/_0 T_{Y'}$, where $T_{Y'}$ is the kernel of the homomorphism of tori $\mathbb{T}_Z \to \mathbb{T}^n$ defined by Π. By Construction 5.3.4, the quotient $Y' /\!\!/_0 T_{Y'}$ is the closure of the image of $Y \cap \mathbb{T}_Z$ under $\mathbb{T}_Z \to \mathbb{T}^n$ in the toric variety $Z' /\!\!/_0 T_{Y'}$ associated to the describing fan in $\mathbb{Z}^n$ having as maximal cones the rays $\Pi(\varrho)$, where ϱ runs through the rays of Σ'.

Claim. For every ray $\varrho \in \Sigma'$ there is a facet of $\mathrm{cone}(e_0, \ldots, e_{n-1})$ containing the image $b := \Pi(\varrho) \in \mathbb{Q}^n$.

Indeed, since every cone of $\mathrm{trop}(Y \cap \mathbb{T}_Z)$ is saturated with respect to Π, we have $\Pi(\varrho) = \Pi(\sigma) \cap \delta$ for some $\sigma \in \Sigma$ and $\delta \in \Delta'(n)$. The image $\Pi(\sigma)$ is a cone spanned by some e_i and some images $v_j := \Pi(\nu_j)$, where ν_j are the primitive generators of the rays of Σ different from columns p_i of P. Proposition 5.2.1 yields presentations

$$\nu_j \;=\; \sum_{i=0}^{r-1} \alpha_{ij} p_i \qquad \text{with certain } \alpha_{ij} \geq 0.$$

Hence we obtain $v_j \in \mathrm{cone}(e_0, \ldots, e_{n-1})$. Lemma 5.3.8 then shows that $\Pi(\varrho)$ lies in some facet of $\mathrm{cone}(e_0, \ldots e_{n-1})$ and the claim is verified.

Finally, in the fourth step, we show that $Y' /\!\!/_0 T_{Y'}$ is normal and has finitely generated Cox ring; by Proposition 5.3.7 this will complete the proof. First note that we have the toric modification $Z' /\!\!/_0 T_{Y'} \to W$, where $W \subseteq \mathbb{P}_n$ is the open toric subset corresponding to the subfan $\Delta'(n)$ of $\Delta(n)$. Moreover, $Y' /\!\!/_0 T_{Y'}$ is the proper transform under $Z' /\!\!/_0 T_{Y'} \to W$ of the closure of $V(h_1) \subseteq \mathbb{T}^n$ in W. The claim just verified and Lemma 5.3.9 ensure that we may apply Proposition 4.1.3 and Corollary 4.1.4. In particular, we see that $Y' /\!\!/_0 T_{Y'}$ is normal with finitely generated Cox ring. □

Example 5.3.10. Consider the quadric $X = V(T_0T_1 + \ldots + T_6T_7) \subseteq \mathbb{P}_7$ and the action of $\mathbb{K}^*$ on $\mathbb{P}_7$ given by

$$t \cdot [x_0, \ldots, x_7] \;\;:=\;\; [t^{-3}x_0, t^3x_1, t^{-3}x_2, t^3x_3, t^{-2}x_4, t^2x_5, t^{-1}x_6, tx_7].$$

Theorem 5.1.3 and its proof do not apply to this case, because only two weights ζ_i have minimal absolute value. The way through the weak toric resolution Y' as gone in the proof of Theorem 5.1.2 produces a quotient $Y' /_{0} T_{Y'}$ embedded into the toric variety with fan obtained by subdividing $\Delta(3)$ at $(0, -1, -1)$.

CHAPTER

SIX

POINT CONFIGURATIONS AND TRANSLATIONS

With only minor modifications this entire chapter has already been published in the author's paper *'Point Configurations and Translations'*, see [**9**].

6.1. A Compactification of the non-reductive Limit Quotient

In this chapter we examine point configurations on the projective line up to translations. In general, let us consider n distinct points on $\mathbb{P}_1$. Then the open subset $U \subseteq \mathbb{P}_1^n$ consisting of pairwise different coordinates is the space of possible configurations. For an algebraic group G acting on $\mathbb{P}_1$ the question arises what the resulting equivalence classes of configurations are, i.e. we ask for a quotient U/G of the diagonal action and a possible canonical compactification.

In the case of the full automorphism group $G = \mathrm{SL}(2, \mathbb{K})$ this problem has been thoroughly studied. The space of configuration classes is canonically compactified by the famous Grothendieck-Knudsen moduli space $\overline{M}_{0,n}$, i.e. we have

$$M_{0,n} \;=\; U \,/\, \mathrm{SL}(2, \mathbb{K}) \;\subseteq\; \overline{M}_{0,n}.$$

Originally introduced as moduli space of certain marked curves Kapranov shows in [**50**] that $\overline{M}_{0,n}$ has (among others) the following two equivalent descriptions. Firstly it arises as the GIT-limit of $\mathbb{P}_1^n$ with respect to the G-action, i.e. the limit of the inverse system of Mumford quotients. Secondly, it can be viewed as the blow-up of $\mathbb{P}_{n-3}$ in $n-1$ general points and all the linear subspaces of dimension at most $n-5$ spanned by them.

Later this setting has been studied in the case where the full automorphism group was replaced by its maximal torus $\mathbb{K}^* \subseteq \mathrm{Sl}(2,\mathbb{K})$. Similarly, it turns out that the Losev-Manin moduli space $\overline{L}_n$ coincides with the the GIT-limit, which in this case is the toric variety associated to the permutahedron. Again, the GIT-limit arises in a sequence of (toric) blow-ups from projective space, see [**29, 49, 53**].

In this chapter we treat point configurations on $\mathbb{P}_1$ up to the action of the maximal connected unipotent subgroup $\mathbb{G}_a \subseteq \mathrm{SL}(2,\mathbb{K})$. It consists of upper triangular matrices with diagonal elements equal to $1_{\mathbb{K}}$ and can be thought of as group of translations. Since $\mathbb{G}_a$ is not reductive, we are faced with the additional problem of first finding a suitable replacement for the GIT-limit, i.e. assigning a canonical quotient to this action. Recall that we overcame this problem in the following manner, see Section 2.5.

Doran and Kirwan introduced in [**25**] the notion of finitely generated semistable points admitting so-called enveloped quotients. Moreover, in [**6**] Arzhantsev, Hausen and Celik proposed a Gelfand-MacPherson type construction which allowed to apply methods from reductive GIT to obtain these enveloped quotients. Building on this work we obtained again an inverse system and the corresponding GIT-limit. Note that in general the enveloped quotients are not projective, hence one cannot expect the GIT-limit to be so.

We then show that (up to nomalisation) the limit quotient, i.e. a canonical component of the GIT-limit, is canonically compactified by an iterated blow-up of $\mathbb{P}_1^{n-1}$. To make this a little more precise consider a subset $A \subseteq \{2,\dots,n\}$. Denoting by $T_2, S_2, \dots, T_n, S_n$ the homogeneous coordinates on $\mathbb{P}_1^{n-1}$ we consider the subschemes X_A on $\mathbb{P}_1^{n-1}$ given by the ideals

$$\left\langle\, T_i^2,\ T_jS_k - T_kS_j;\ i,j,k \in A,\ j<k \,\right\rangle.$$

The scheme-theoretic inclusions give rise to a partial order of these subschemes. Let $\mathrm{Bl}(\mathbb{P}_1^{n-1})$ denote the blow-up of $\mathbb{P}_1^{n-1}$ in all these subschemes in non-descending order.

Theorem. *If $\mathbb{P}_1 \mathbin{\tilde{/\!\!/}_{\mathrm{LQ}}} \mathbb{G}_a$ and $\tilde{\mathrm{Bl}}(\mathbb{P}_1^{n-1})$ denote the normalisations of the limit quotient and the above blow-up of $\mathbb{P}_1^{n-1}$ respectively, then we have open embeddings*

$$U/\mathbb{G}_a \quad\subseteq\quad \mathbb{P}_1^n \mathbin{\tilde{/\!\!/}_{\mathrm{LQ}}} \mathbb{G}_a \quad\subseteq\quad \tilde{\mathrm{Bl}}(\mathbb{P}_1^{n-1}).$$

For a precise formulation of the main results see Section 6.5.

In the case of two distinct points, i.e. $n=2$, the latter space is simply $\mathbb{P}_1$. If $n=3$ holds, then the compatification $\tilde{\mathrm{Bl}}(\mathbb{P}_1 \times \mathbb{P}_1)$ is the unique non-toric, Gorenstein, log del Pezzo $\mathbb{K}^*$-surface of Picard number 3 with a singularity of type A_1. Similar to $\overline{M}_{0,5}$ which arises as a single Mumford quotient of the cone over the Grassmannian $\mathrm{Gr}(2,5)$, this surface is the Mumford quotient of the cone over the Grassmannian Gr(2, 4). For higher n an analogous Mumford quotient needs to be blown up as will be described in Section 6.5.

The chapter is organised as follows. In Section 6.2 we apply these constructions proposed in Section 2.5 to the action of $\mathbb{G}_a$ on $\mathbb{P}_1^n$. We discuss explicitly the GIT-fan which contains the combinatorial data needed to make the limit quotient accessible. The blow-ups of $\mathbb{P}_1^{n-1}$ will be dealt with in a mostly combinatorial way, i.e. as proper transforms with respect to toric blow-ups. For this we prove a result on combinatorial blow-ups in the spirit of Feichtner and Kozlov, see [**28**]. This will be carried out in Section 6.3. In the short Section 6.4 we will deal with the connection between stellar subdivisions and toric blow-ups. The final Section 6.5 then is dedicated to the proof of the main theorems.

6.2. Point Configurations on $\mathbb{P}_1$ and Translations

In this section we examine point configurations on $\mathbb{P}_1^n$ up to translations. For this we consider the diagonal action of $\mathbb{G}_a$ on $\mathbb{P}_1^n$ and explicitly perform the Gelfand-MacPherson type construction introduced in the preceding section. We determine the GIT-fan describing the variation of quotients and show that it is closely related to the well known GIT-fan stemming from the action of the full automorphism group $\mathrm{SL}(2, \mathbb{K})$ on $\mathbb{P}_1^n$.

For this we consider the unipotent group

$$\mathbb{G}_a = \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix};\ k \in \mathbb{K} \right\} \subseteq \mathrm{SL}(2, \mathbb{K}),$$

and its action on $\overline{X} := (\mathbb{K}^n)^2$ given by

$$A \cdot \begin{bmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{bmatrix} := \left[A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \dots, A \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right].$$

Viewing $[x_i, y_i]$ as homogeneous coordinates of the factors in $\mathbb{P}_1^n$ this gives rise to an induced action on $X := \mathbb{P}_1^n$. Note that the Cox ring of X is

$$\mathcal{R}(X) = \mathcal{O}(\overline{X}) = \mathbb{K}[T_1, \dots, T_n, S_1, \dots, S_n]$$

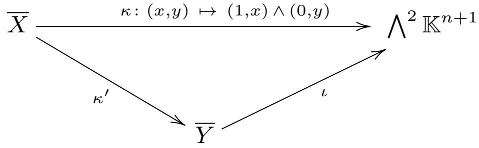
together with a $\mathrm{Cl}(X)$-grading defined by $\deg(T_i) = \deg(S_i) = e_i \in \mathbb{Z}^n = \mathrm{Cl}(X)$. A first Propositions concerns the algebra of invariants in $\mathcal{R}(X)$ and its spectrum.

Proposition 6.2.1. *Consider the above $\mathbb{G}_a$-action on $\overline{X}$.*

(i) *The subalgebra $\mathcal{O}(\overline{X})^{\mathbb{G}_a} \subseteq \mathcal{O}(\overline{X})$ is generated by*

$$S_1, \dots, S_n, \qquad T_j S_k - T_k S_j, \text{ with } 1 \le j < k \le n.$$

(ii) *The canonical morphism* $\kappa' \colon \overline{X} \to \overline{Y}$ *where* $\overline{Y} := \operatorname{Spec}(\mathcal{O}(\overline{X})^{\mathbb{G}_a})$ *fits into a commutative diagram*



where ι *is a closed embedding and its image* $\iota(\overline{Y})$ *is the affine cone over the Grassmannian* $\operatorname{Gr}(2, n+1)$*. Its vanishing ideal is generated by the Plücker relations*

$$T_{ij}T_{kl} - T_{ik}T_{jl} + T_{il}T_{jk}, \qquad \textit{with} \qquad 0 \le i < j < k < l \le n,$$

where $T_{ij} = (e_i \wedge e_j)^*$ *are the dual basis vectors of the standard basis.*

Proof. The invariants have been described by Shmelkin, see [**62**, Theorem 1.1]. For (ii) we define ι by its comorphism

$$\iota^* \colon T_{0i} \mapsto S_i, \quad T_{jk} \mapsto T_j S_k - T_k S_j \quad \text{where } 1 \le i \le n,\ 1 \le j < k \le n.$$

Clearly, ι^* is surjective, hence ι is an embedding. Moreover, the pullback of the Plücker relations with ι^* gives the zero ideal. Thus $\overline{Y}$ lies in the affine cone $C(\operatorname{Gr}(2, n+1))$. It now suffices to show that $\operatorname{Im}(\kappa')$ has dimension $2n - 1$.

For this consider two points (x, y), (x', y') with only non-zero coefficients. If they have distinct orbits, then the orbits are separated by the invariants: If $y \neq y'$ holds, then there exists a separating S_i. Otherwise we can choose a separating $T_i S_j - T_j S_i$. Hence, over an open set the fibres of κ' are one-dimensional and thus the image of κ' is $(2n-1)$-dimensional. □

While for reductive groups the quotient morphism κ' is surjective, this fails in general, also see [**67**]. We provide a description of the image of

$$\kappa \colon \overline{X} = (\mathbb{K}^n)^2 \to \bigwedge^2 \mathbb{K}^{n+1}; \quad (x, y) \mapsto (1, x) \wedge (0, y).$$

Via the embedding of the preceeding proposition we view $\overline{Y}$ as subset of $\bigwedge^2 \mathbb{K}^{n+1}$. Observe that $\overline{Y}$ contains the affine cone $\overline{Y}^\star$ of the smaller Grassmannian $\operatorname{Gr}(2, n)$ in the following canonical manner:

$$\overline{Y}^\star = \{(0, x) \wedge (0, y);\ x, y \in \mathbb{K}^n\} \subseteq \overline{Y}.$$

Proposition 6.2.2. *The image of* κ *is* $\kappa(\overline{X}) = (\overline{Y} \setminus \overline{Y}^\star) \cup \{0\}$.

Lemma 6.2.3. *Let* V *be an* n*-dimensional vector space,* $0 \neq v_1 \in V$ *and consider the linear map* $\varphi_{v_1} \colon \bigwedge^{k-1} V \to \bigwedge^k V; x \mapsto x \wedge v_1$*. Then the rank of* φ_{v_1} *is* $\binom{n-1}{k-1}$.

Proof. Fix some basis $(v_1, v_2, \ldots, v_n)$ of V. From this we then obtain a basis $(v_{i_1} \wedge \ldots \wedge v_{i_{k-1}};\ 1 \le i_1 < \ldots < i_k \le n)$ of $\bigwedge^{k-1} V$ and $\varphi_{v_1}(x) = 0$ holds if and only if x lies in $W := \operatorname{Lin}\big(v_{i_1} \wedge \ldots \wedge v_{i_{k-1}};\ i_1 = 1\big)$. This means that the rank of φ_{v_1} is given by

$$\operatorname{rk}(\varphi_{v_1}) = \dim \bigwedge^{k-1} V - \dim W = \binom{n}{k-1} - \binom{n-1}{k-2} = \binom{n-1}{k-1}.$$

□

Proof of 6.2.2. From the definition of the morphism κ it follows that its image is contained in $(\overline{Y} \setminus \overline{Y}^\star) \cup \{0\}$. For the reverse inclusion consider

$$z = \sum z_{ij} e_i \wedge e_j \ \in\ \overline{Y} \setminus \overline{Y}^\star.$$

We define $y := (z_{01}, \ldots, z_{0n}) \in \mathbb{K}^n$; note that $y \neq 0$ holds. With the identification $\mathbb{K}^n = \{0\} \times \mathbb{K}^n \subseteq \mathbb{K}^{n+1}$ we obtain an affine subspace W_y by

$$W_y \ := \ e_0 \wedge y + \bigwedge^2 \mathbb{K}^n \quad \subseteq \quad \Big(\mathbb{K}e_0 \bigwedge \mathbb{K}^n\Big) \oplus \bigwedge^2 \mathbb{K}^n \quad = \quad \bigwedge^2 \mathbb{K}^{n+1}.$$

Since z lies in $W_y \cap \overline{Y}$, it suffices to show that $\kappa(\,\cdot\,, y)$ maps $\mathbb{K}^n$ onto $W_y \cap \overline{Y}$. Clearly, by definition of κ, the image of $\kappa(\,\cdot\,, y)$ lies in $W_y \cap \overline{Y}$. To show surjectivity we regard W_y as a vector space with origin $e_0 \wedge y$. Then there is a linear map

$$\varphi\colon W_y \to \bigwedge^3 \mathbb{K}^n; \quad e_0 \wedge y + u \wedge v \ \mapsto\ u \wedge v \wedge y.$$

Observe that we have inclusions $\operatorname{Im}(\kappa(\,\cdot\,, y)) \subseteq Z_y \subseteq \ker(\varphi)$. We claim that equality holds in both cases. Since by Lemma 6.2.3 $\kappa(\,\cdot\,, y)$ is linear of rank $n-1$, the claim follows from

$$\dim(\ker(\varphi)) \ = \ \dim(W_y) - \operatorname{rank}(\varphi) \ = \ \binom{n}{2} - \binom{n-1}{2} \ = \ n-1.$$

□

We recall from [**14**, Section 2] the definition of the GIT-fan. Let the algebraic torus $H := (\mathbb{K}^*)^n$ act diagonally on $\mathbb{K}^r$ via the characters $\chi^{w_1}, \ldots, \chi^{w_r}$, $w_i \in \mathbb{Z}^n$, i.e.

$$h \cdot z \ := \ (\chi^{w_1}(h)\, z_1, \ldots, \chi^{w_r}(h)\, z_r)$$

and suppose that $Y \subseteq \mathbb{K}^r$ is invariant under this action. Then the *GIT-fan* is defined as the collection of cones

$$\Lambda_H(Y) \ := \ \{\lambda(w);\ w \in \mathbb{Q}^n\}; \qquad \lambda(w) \ := \ \bigcap_{w \in \omega_I} \omega_I \ \subseteq\ \mathbb{Q}^n,$$

where $\omega_I := \operatorname{cone}(w_i;\ i \in I)$ is the cone associated to a *Y-set* I, i.e. a subset $I \subseteq \{1, \ldots, r\}$ for which the corresponding stratum $\{y \in Y;\ y_i \neq 0 \iff i \in I\}$ is non-empty.

We turn back to our setting. The $\mathrm{Cl}(X)$-grading of the Cox ring $\mathcal{R}(X) = \mathcal{O}(\overline{X})$ yields a diagonal action of the algebraic torus $H := (\mathbb{K}^*)^n = \mathrm{Spec}(\mathbb{K}[\mathrm{Cl}(X)])$ on $\overline{X} = (\mathbb{K}^n)^2$ where

$$h \cdot (x, y) \;=\; (h_1 x_1\,, \dots,\, h_n x_n\,,\, h_1 y_1\,, \dots,\, h_n y_n).$$

Since the subalgebra $\mathcal{O}(\overline{X})^{\mathbb{G}_a}$ inherits the $\mathrm{Cl}(X)$-grading, the H-action descends to its spectrum $\overline{Y} \subseteq \bigwedge^2 \mathbb{K}^{n+1}$, turning κ into an equivariant morphism. Here the action is explicitly described by

$$h \cdot e_0 \wedge e_j = h_j \; e_0 \wedge e_j, \qquad h \cdot e_i \wedge e_j = h_i h_j \; e_i \wedge e_j.$$

Note that this action differs from the well known maximal torus action. It rather is a submaximal action, with some connection to the maximal one, see Proposition 6.2.7.

In order to obtain the GIT-fan $\Lambda_H(\overline{Y})$ we consider the *two-block partitions* of $N := \{1, \dots, n\}$, i.e. partitions where N is a union of two disjoint subsets A, A^c. To each such partition $R = \{A, A^c\}$ we associate the hyperplane

$$\mathcal{H}_R := \left\{ x \in \mathbb{Q}^n;\ \sum_{i \in A} x_i = \sum_{i \in A^c} x_i \right\}.$$

Theorem 6.2.4. *Consider the above H-action on the affine cone $\overline{Y}$ over the Grassmann variety $\mathrm{Gr}(2, n+1)$ and set $\Omega := \mathbb{Q}^n_{\geq 0}$. The GIT-fan $\Lambda_H(\overline{Y})$ is the fan supported on Ω with walls given by the intersections $\mathcal{H}_R \cap \Omega$ where R runs through the two-block partitions of N.*

The key step of the proof is relate our submaximal H-action on $\overline{Y}$ to the maximal torus action on the smaller Grassmannian cone $\overline{Y}^\star$, see Proposition 6.2.7. The latter action is well understood, in particular the GIT-fan was described in [**24**, Example 3.3.21].

The first step, however, is to provide a description of the $\overline{Y}$- and $\overline{Y}^\star$-sets. We need some further notation:

$$N := \{1, \dots, n\} \qquad \mathbf{N} := \{\{i, j\};\ 1 \leq i < j \leq n\}$$

$$N_0 := \{0, \dots, n\} \qquad \mathbf{N}_0 := \{\{i, j\};\ 0 \leq i < j \leq n\}$$

Recall that the cones over the Grassmannians lie in the wedge products $\overline{Y}^\star \subseteq \bigwedge^2 \mathbb{K}^n$, $\overline{Y} \subseteq \bigwedge^2 \mathbb{K}^{n+1}$. We use the above index sets $\mathbf{N}$ and $\mathbf{N}_0$ to refer to the coordinate indices where $\{i, j\}$ labels $e_i \wedge e_j$.

Proposition 6.2.5. *A subset $I \subseteq \mathbf{N}_0$ is a $\overline{Y}$-set if and only if I satisfies the following condition*

$$(*) \qquad \{i, j\}, \{k, l\} \in I \quad \Longrightarrow \quad \{j, l\},\ \{i, k\} \in I \quad \text{or} \quad \{j, k\},\ \{i, l\} \in I.$$

Proof. It follows from the nature of the Plücker relations that a $\overline{Y}$-set I has in fact the property $(*)$. We prove that a subset of N satisfying $(*)$ is a $\overline{Y}$-set by induction on n. For this recall that we have commutative diagram of closed embeddings

$$\begin{array}{ccccccc} C(\mathrm{Gr}(2,n+1)) & = & \overline{Y} & \longrightarrow & (\mathbb{K}e_0 \bigwedge \mathbb{K}^n) \oplus \bigwedge^2 \mathbb{K}^n & = & \bigwedge^2 \mathbb{K}^{n+1} \\ \uparrow & & \uparrow & & \uparrow & & \\ C(\mathrm{Gr}(2,n)) & = & \overline{Y}^{\star} & \longrightarrow & \bigwedge^2 \mathbb{K}^n & & \end{array}$$

where the embedding of the surrounding wedge products is reflected by the inclusion $\mathbf{N} \subseteq \mathbf{N}_0$. Let $I \subseteq \mathbf{N}_0$ be a set with the property $(*)$. If $I \subseteq \mathbf{N}$ holds, then the assertion follows from the induction hypothesis. We turn to the case where there exists $k \in N$ such that $\{0,k\}$ lies in I. We will explicitly construct an element $z \in \overline{Y}$ for which z_{ij} vanishes if and only if $\{i,j\}$ does not lie in I. For this we introduce two graph structures on N by $\mathcal{G}_{12} := (N, \mathcal{E}_1 \cup \mathcal{E}_2)$ and $\mathcal{G}_2 := (N, \mathcal{E}_2)$, where $\mathcal{E}_1$, $\mathcal{E}_2$ are sets of edges on N defined by

$$\begin{aligned} \mathcal{E}_1 &:= \Big\{\{i,j\} \in I \cap \mathbf{N};\ \{0,i\} \in I \text{ or } \{0,j\} \in I\Big\}, \\ \mathcal{E}_2 &:= \Big\{\{i,j\} \in \mathbf{N} \setminus I;\ \{0,i\},\{0,j\} \in I\Big\}. \end{aligned}$$

From the definition of the edge sets of the respective graphs we know that if $\{i\}$ is a connected component of $\mathcal{G}_{12}$, then it also is a connected component of $\mathcal{G}_2$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_q$ be the connected components of $\mathcal{G}_2$ different from a component $\{i\}$ of $\mathcal{G}_{12}$. We define a vector $x \in \mathbb{K}^n$ by

$$x_i := \begin{cases} 0 & \text{if } \{i\} \text{ is a component of } \mathcal{G}_{12}, \\ p & \text{if } \{i\} \subseteq \mathcal{F}_p \text{ holds.} \end{cases}$$

Moreover, we define $y \in \mathbb{K}^n$ by $y_j := 1$ if $\{0,j\} \in I$ and $y_j := 0$ if $\{0,j\} \notin I$. We then claim that $z := (1,x) \wedge (0,y)$ has the property

$$z_{ij} \neq 0 \quad \Longleftrightarrow \quad \{i,j\} \in I.$$

Since $z_{0j} = y_j$ holds, it is clear that the claim is true for the components of this type. For $0 \neq i < j$ the components of z can be written as

$$z_{ij} = x_i\, y_j - x_j\, y_i = \begin{cases} 0 & \text{if } \{0,i\},\{0,j\} \notin I, \\ \pm\, x_i & \text{if } \{0,i\} \notin I, \{0,j\} \in I, \\ x_i - x_j & \text{if } \{0,i\},\{0,j\} \in I. \end{cases}$$

We now go through these three cases and verify for each that $\{i,j\}$ lies in I if and only if $z_{ij} \neq 0$ holds.

Assume that $\{0,i\},\{0,j\} \notin I$ holds and recall that there exists a $k \in N$ with $\{0,k\} \in I$. It follows from $(*)$ applied to $\{0,k\}$, $\{i,j\}$ that $\{i,j\}$ does not lie in I.

For the second case suppose that $\{0,i\} \notin I$ and $\{0,j\} \in I$ hold. We then have

$$\begin{aligned} x_i \neq 0 &\iff \text{there exists } l \in N \text{ such that } \{i,l\} \in \mathcal{E}_1 \text{ or } \{i,l\} \in \mathcal{E}_2 \\ &\iff \text{there exists } l \in N \text{ such that } \{i,l\} \in \mathcal{E}_1 \\ &\iff \{i,j\} \in I \end{aligned}$$

For the second equivalence note that $\{0,i\} \notin I$ holds which implies $\{i,l\} \notin \mathcal{E}_2$. The third equivalence is due to an application of $(*)$ to $\{0,j\}$ $\{i,l\}$.

In the last case where $\{0,i\},\{0,j\} \in I$ holds we obtain

$$\begin{aligned} x_i = x_j &\iff \quad i,j \text{ lie in the same connected component of } \mathcal{G}_2 \\ &\qquad\quad \text{or} \quad \{i\},\{j\} \text{ are connected components of } \mathcal{G}_{12} \\ &\iff \{i,j\} \in \mathcal{E}_2 \text{ or } \{i\},\{j\} \text{ are connected components of } \mathcal{G}_{12} \\ &\iff \{i,j\} \notin I \end{aligned}$$

For the second equivalence we use that each connected component of $\mathcal{G}_2$ is a complete graph, which follows from $(*)$. □

Remark 6.2.6. The affine cone $\overline{Y}^{\star}$ over the smaller Grassmannian $\mathrm{Gr}(2,n)$ is invariant under the H-action. The corresponding GIT-fan $\Lambda_H(\overline{Y}^{\star})$ of this restricted action is well known, it was described in terms of walls in [**24**, Example 3.3.21] and [**7**, Example 8.5] as follows: Set

$$\Omega^{\star} := \mathrm{cone}(e_i + e_j;\ 1 \leq i < j \leq n) \subseteq \mathbb{Q}^n_{\geq 0}.$$

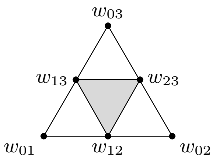
Then the GIT fan $\Lambda_H(\overline{Y}^{\star})$ is the fan supported on $\Omega^{\star}$ with walls given by the intersections of $\Omega^{\star}$ with the above hyperplanes $\mathcal{H}_R$.

Proposition 6.2.7. *The GIT-fan $\Lambda_H(\overline{Y}^{\star})$ is a subfan of $\Lambda_H(\overline{Y})$.*

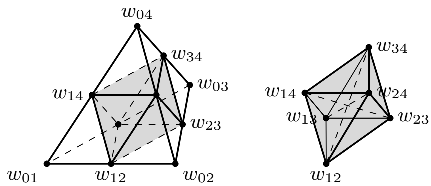
Example 6.2.8. Consider the weights of the coordinates of the H-action on $\bigwedge^2 \mathbb{K}^{n+1}$

$$w_{01} := e_1, \quad \ldots, \quad w_{0n} := e_n, \qquad w_{jk} := e_j + e_k,\ 1 \leq j < k \leq n.$$

The following pictures of polytopal complexes arise from intersecting the GIT-fan $\Lambda_H(\overline{Y})$ with the hyperplane given by $1 = x_1 + \ldots + x_n$ in the cases $n = 3, 4$. The shaded area indicates the support $\Omega^{\star}$ of $\Lambda_H(\overline{Y}^{\star})$.



$n = 3$



$n = 4$

In the case $n = 3$ the three walls of the GIT-fan are generated by two of the vectors w_{12}, w_{13}, w_{23} and correspond to the two-block partitions

$$\{\{1\}, \{2, 3\}\}, \qquad \{\{2\}, \{1, 3\}\} \qquad \text{and} \qquad \{\{3\}, \{1, 2\}\}.$$

In the case $n = 4$ again the hyperplanes separating $\Omega^\star$ from the remaining 4 cones correspond to the partitions of the type $\{\{i\}, \{j, k, l\}\}$. The dotted lines in the right picture indicate the fan structure inside $\Lambda_H(\overline{Y}^\star)$. There are eight maximal cones arising from 3 hyperplanes of the form $\{\{i, j\}, \{k, l\}\}$.

Proof of Proposition 6.2.7. Recall that the weights of the coordinates of the H-action are

$$w_{01} := e_1, \quad \ldots, \quad w_{0n} := e_n, \qquad w_{jk} := e_j + e_k, \ 1 \le j < k \le n.$$

The GIT-fans $\Lambda_H(\overline{Y})$ and $\Lambda_H(\overline{Y}^\star)$ are the collections of cones which arise as intersections of cones $\omega_I = \operatorname{cone}(w_{ij};\ \{i, j\} \in I)$ associated to $\overline{Y}$- or $\overline{Y}^\star$-sets respectively. From Proposition 6.2.5 we know that every $\overline{Y}^\star$-set is also a $\overline{Y}$-set. This means we only have to show that for every $\overline{Y}$-set $I \subseteq \mathbf{N}_0$ there exists a $\overline{Y}^\star$-set $J \subseteq \mathbf{N}$ such that $\omega_I \cap \Omega^\star = \omega_J$ holds. For a $\overline{Y}$-set $I \subseteq \mathbf{N}_0$ we set

$$J := J_1 \cup J_2, \qquad J_1 := I \cap \mathbf{N}, \qquad J_2 := \{\{i, j\};\ \{0, i\}, \{0, j\} \in I\}$$

and prove that J has the required properties. We first claim that J is an $\overline{Y}^\star$-set. For this we check that the condition of Proposition 6.2.5 applies to any two elements of J. If these two elements lie either both in J_1 or J_2 then the claim follows from I being a $\overline{Y}$-set or the construction of J_2 respectively. For the remaining case consider $\{j, k\} \in J_1$ and $\{i_1, i_2\} \in J_2$. Since both $\{0, i_1\}$ and $\{j, k\}$ lie in I, we can without loss of generality assume that also $\{i_1, j\}$ and $\{0, k\}$ lie in I. Finally with $\{0, i_2\} \in I$ we conclude that $\{i_2, j\}, \{i_1, k\}$ are elements of J. This shows that J is a $\overline{Y}^\star$-set.

We now prove $\omega_I \cap \Omega^\star = \omega_J$. It is easy to see that ω_J is in fact contained in $\omega_I \cap \Omega^\star$; we turn to the reverse inclusion. With non-negative a_i, a_{jk} let

$$x \ := \ \sum_{I \setminus \mathbf{N}} a_i w_{0i} \ + \ \sum_{I \cap \mathbf{N}} a_{jk} w_{jk}$$

lie in $\omega_I \cap \Omega^\star$. We show that x is a non-negative linear combination of elements w_η, $\eta \in J$. Let a_{i_1} be minimal among all a_i with $\{0,i\} \in I$. For an arbitrary $\{0,i_2\} \in I$ we then replace in the above sum

$$a_{i_1} w_{0i_1} \; + \; a_{i_2} w_{0i_2} \qquad \text{by} \qquad (a_{i_2} - a_{i_1}) w_{0i_2} \; + \; a_{i_1} w_{i_1 i_2}.$$

Note that now $\{i_1, i_2\}$ lies in J_2. Iterating this process we see that there exists some $\{0,i\} \in I$ such that x has the form

$$(**) \qquad\qquad x \;=\; b_i w_{0i} \;+\; \sum_{J_1 \cup J_2} b_{jk} w_{jk}.$$

Without loss of generality we assume that $i = 1$ holds. The condition $x \in \Omega^\star$ implies $x_1 \le x_2 + \ldots + x_n$, hence we have

$$b_1 \;\le\; 2 \sum_{\substack{\{j,k\} \in J \\ j,k \neq 1}} b_{jk} \qquad \text{and} \qquad b_1 \;=\; 2 \sum_{\substack{\{j,k\} \in J \\ j,k \neq 1}} b'_{jk}$$

for certain $0 \le b'_{jk} \le b_{jk}$. Plugging $w_{01} = \;^1\!/_2(w_{1j} + w_{1k} - w_{jk})$ into $(**)$ we obtain a non-negative linear combination

$$x \;=\; \sum_{\substack{\{j,k\} \in J \\ j,k \neq 1}} \big((b_{1j} + b'_{jk}) w_{1j} + (b_{1k} + b'_{jk}) w_{1k} + (b_{jk} - b'_{jk}) w_{jk}\big) + \sum_{\{1,k\} \in J} b_{1k} w_{1k}.$$

The last step to show is that for $\{j,k\} \in J$ both $\{1,j\}$ and $\{1,k\}$ lie in J. Recall that we have $\{0,1\} \in I$. If $\{j,k\}$ lies in J_2, then this follows directly from construction of J_2. Otherwise we can without loss of generality assume that $\{0,j\}, \{1,k\}$ lie in I. The claim again follows from the construction of J_2. □

Proof of Theorem 6.2.4. As before we denote the weights of coordinates with respect to the H-action by $w_{0i} = e_i$, $w_{jk} = e_j + e_k$. From Proposition 6.2.7 we know that $\Lambda_H(\overline{Y})$ has the asserted form on $\Omega^\star$. Note that the remaining support $\Omega \setminus \mathrm{relint}(\Omega^\star)$ is the union of the cones

$$\sigma_i \;:=\; \mathrm{cone}(w_{ij};\; j \in N \setminus \{i\}), \qquad i = 1, \ldots, n.$$

None of the hyperplanes $\mathcal{H}_R$ intersect σ_i in its relative interior. This means that we have to prove that σ_i is a cone in the GIT-fan $\Lambda_H(\overline{Y})$, i.e. the intersection of cones ω_I associated to $\overline{Y}$-sets. Note that σ_i itself is a cone associated to a $\overline{Y}$-set. Hence, it suffices to show that for any $\overline{Y}$-set $I \subseteq \mathbf{N}_0$ the intersection $\omega_I \cap \sigma_i$ is a face of σ_i. Without loss of generality we assume that i equals 1 and set $\sigma := \sigma_1$. We now claim that $\omega_I \cap \sigma = \omega_J$ holds where

$$J := J_1 \cup J_2; \qquad J_1 := I \cap \{\{1,j\};\; j \in N_0 \setminus \{1\}\}, \quad J_2 := \{\{1,j\};\; \{0,j\} \in I\}.$$

To prove $\omega_J \subseteq \omega_I \cap \sigma$ note that any w_{1j} with $\{1,j\} \in J_1$ clearly lies in $\omega_I \cap \sigma$. Hence, it suffices to show that for w_{1j} with $\{0,j\} \in I$ the same holds. In case $\{0,1\} \in I$ this follows from $w_{1j} = w_{01} + w_{0j} \in \omega_I \cap \sigma$. Otherwise there must exist $\{1,l\} \in I$ and from Proposition 6.2.5 we know $\{0,l\}, \{1,j\} \in I$. This implies $w_{1j} \in \omega_I \cap \sigma$.

For the reverse inclusion $\omega_I \cap \sigma \subseteq \omega_J$ consider the non-negative linear combination

$$x \;:=\; a_{01}w_{01} \;+ \sum_{\substack{\{1,j\}\,\in\, I\\ j\neq 0}} a_{1j}w_{1j} \;+ \sum_{\substack{\{0,j\}\,\in\, I\\ j\neq 1}} a_{0j}w_{0j} \;+ \sum_{\substack{\{j,k\}\,\in\, I\\ j,k\neq 0,1}} a_{jk}w_{jk} \;\in\; \omega_I$$

Since x lies in σ, we have $x_1 \geq x_2 + \ldots + x_n$ and this amounts to

$$a_{01} \;\geq \sum_{\substack{\{0,j\}\,\in\, I\\ j\neq 1}} a_{0j} \;+ 2\sum_{\substack{\{j,k\}\,\in\, I\\ j,k\neq 0,1}} a_{jk}.$$

If $\{0,1\} \notin I$ holds, i.e. $a_{01} = 0$, then x lies in the cone generated by the w_{1j}, $\{1,j\} \in J_1$. Otherwise with $w_{0j} = w_{1j} - w_{01}$ and $w_{jk} = w_{1j} + w_{1k} - 2w_{01}$ we get a non-negative linear combination

$$\begin{aligned} x = &\sum_{\substack{\{1,j\}\,\in\, I\\ j\neq 0}} a_{1j}w_{1j} \;+ \sum_{\substack{\{0,j\}\,\in\, I\\ j\neq 1}} a_{0j}w_{1j} \;+ \sum_{\substack{\{j,k\}\,\in\, I\\ j,k\neq 0,1}} a_{jk}(w_{1j}+w_{1k}) \\ &+ \left(a_{01} \;- \sum_{\substack{\{0,j\}\,\in\, I\\ j\neq 1}} a_{0j} \;- 2\sum_{\substack{\{j,k\}\,\in\, I\\ j,k\neq 0,1}} a_{jk} \right) w_{01}. \end{aligned}$$

The last thing to check is that all the above w_{ij} lie in ω_J. For this suppose that $\{j,k\} \in I$ holds. Since $\{0,1\}$ is contained in I, it follows from the construction of J that both $\{1,j\}$ and $\{1,k\}$ lie in J. □

6.3. Combinatorial Blow-ups

In this section we will provide a criterion whether a given cone lies in the iterated stellar subdivision of a simplicial fan. In [**28**] Feichtner and Kozlov deal with this problem in the more general setting of semilattices and give a nice characterisation in the case where the collection of subdivided cones forms a building set. We approach the issue of blowing up non-building sets, see Theorem 6.3.10.

Let us recall the definition of stellar subdivisions, for details see e.g. [**39**, Definition 5.1]. For a fan Σ_0 in a vector space $N_{\mathbb{Q}}$ and a cone $\sigma_0 \in \Sigma_0$ the *star* of σ_0 is given as

$$\operatorname{star}(\sigma_0) \;:=\; \big\{\sigma \in \Sigma_0; \quad \sigma_0 \preccurlyeq \sigma\big\}.$$

We insert a new ray into the fan Σ_0. For this let $\nu \in \sigma_0^\circ$ be some vector in the relative interior, then the *stellar subdivision* of Σ_0 at ν is

$$\mathrm{stSubDiv}_\nu(\Sigma_0) \quad := \quad \big(\Sigma_0 \setminus \mathrm{star}(\sigma_0)\big) \;\cup\; \big\{\tau + \mathrm{cone}(\nu); \quad \tau \precneqq \sigma \in \mathrm{star}(\sigma_0)\big\}.$$

We now iterate this process. For this let $\mathcal{V}$ be a family of rays in a vector space and consider a $\mathcal{V}$-fan Σ_0, i.e. a fan with rays given by $\mathcal{V}$. We then choose additional rays ν_i, $i = 1, \ldots, r$ lying in the relative interiors σ_i° of pairwise different cones $\sigma_i \in \Sigma_0$. Moreover, we assume that $\sigma_i \precneqq \sigma_j$ implies $j < i$, which means that the larger the cone the earlier it will be subdivided. Now the questions comes up what the cones of the fan Σ_r are which arises from Σ_0 by the subsequent stellar subdivisions in the rays ν_i.

We call a subset $\mathcal{S}'$ of $\mathcal{S} := \{\sigma_1, \ldots, \sigma_r\}$ *conjunct*, if the union $\bigcup_{\sigma \in \mathcal{S}'}(\sigma \setminus \{0\})$ is a connected subset in the usual sense and we set

$$\langle \mathcal{S} \rangle := \left\{ \sum_{\sigma \in \mathcal{S}'} \sigma\, ;\; \mathcal{S}' \subseteq \mathcal{S} \text{ conjunct} \right\}.$$

A collection $\mathcal{C} \subseteq \mathcal{V} \cup \mathcal{S}$ is called *geometrically nested*, if for any subset $\mathcal{H} \subseteq \mathcal{C}$ of pairwise incomparable cones with $|\mathcal{H}| \geq 2$ the following holds:

$$\sum_{\tau \in \mathcal{H}} \tau \;\in\; \Sigma_0 \setminus \langle \mathcal{S} \rangle.$$

Proposition 6.3.1. *Let Σ_0 be a simplicial fan and $\nu_i \in \sigma_i^\circ$ some rays in the relative interiors of pairwise different cones $\sigma_i \in \Sigma_0$. Assume that $\sigma_i \precneqq \sigma_j$ implies $j < i$ and let Σ_r be the iterated stellar subdivision of Σ_0 in the rays $\nu_1, \ldots, \nu_r$ in order of ascending indices. If in the above notation $\mathcal{C} \subseteq \mathcal{V} \cup \mathcal{S}$ is geometrically nested, then* $\mathrm{cone}(v, \nu_i;\ v \in \mathcal{C} \cap \mathcal{V},\ \sigma_i \in \mathcal{C} \cap \mathcal{S})$ *lies in Σ_r.*

We will prove this using the technique of combinatorially blowing up elements in a semilattice developed by Feichtner and Kozlov in [**28**].

Definition 6.3.2. A *meet-semilattice* is a partially ordered set $(\mathcal{L}, \leq)$ such that for any $\mathcal{A} \subseteq \mathcal{L}$ the set $\{z \in \mathcal{L};\ z \leq a \text{ for all } a \in \mathcal{A}\}$ posesses a greatest element $\bigwedge \mathcal{A}$ called *meet*. For the meet of $\mathcal{A} = \{a_1, \ldots, a_n\}$ we also write $a_1 \wedge \ldots \wedge a_n$.

It is well known that any such semilattice has a unique minimal element 0. Also, for a family or subset $\mathcal{A} \subseteq \mathcal{L}$ the set $\{z \in \mathcal{L};\ z \geq a \text{ for all } a \in \mathcal{A}\}$ is either empty or has a unique minimal element $\bigvee \mathcal{A}$ called *join*. If the join of $\mathcal{A} = \{a_1, \ldots, a_n\}$ exists, then for it we also write $a_1 \vee \ldots \vee a_n$. For $x, y \in \mathcal{L}$ we denote $\mathcal{A}_{\leq x} := \{a \in \mathcal{A};\ a \leq x\}$ and $\mathcal{A}_{<x}$ in the analog way.

We now turn to blow-ups of semilattices in the sense of [**28**, Definition 3.1].

Definition 6.3.3. The *blow-up* of $(\mathcal{L}, \geq)$ in an element $\alpha \in \mathcal{L}$ is the semilattice $\mathrm{Bl}_{(\alpha)}(\mathcal{L})$ consisting of the elements and pairs

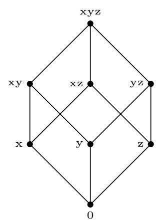
- $x \in \mathcal{L}$ with $x \not\geq \alpha$,
- (α, x) where $\mathcal{L} \ni x \not\geq \alpha$ and $x \vee \alpha$ exists.

with the order relation $\geq_{\mathrm{Bl}}$ given by

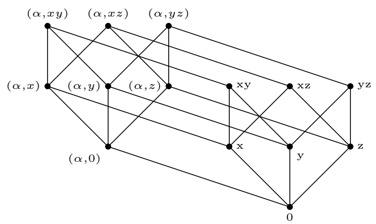
- $x >_{\mathrm{Bl}} y \quad \text{if} \quad x > y$,
- $(\alpha, x) >_{\mathrm{Bl}} (\alpha, y) \quad \text{if} \quad x > y$ and
- $(\alpha, x) >_{\mathrm{Bl}} y \quad \text{if} \quad x \geq y$.

where in all three cases $x, y \not\geq \alpha$ holds.

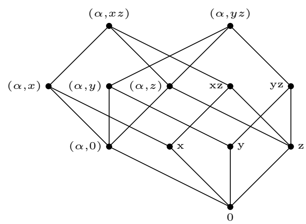
Example 6.3.4. Let $\mathcal{L}$ be the semilattice given by the upper diagram and set $\alpha := xyz$, $\beta := xy$. Then the blow-ups $\mathrm{Bl}_\alpha(\mathcal{L})$ and $\mathrm{Bl}_\beta(\mathcal{L})$ is given by the lower diagrams.



$\mathcal{L}$



$\mathrm{Bl}_\alpha(\mathcal{L})$



$\mathrm{Bl}_\beta(\mathcal{L})$

We now want to iterate the blow-up process. Let $\mathcal{G} = (\xi_1, \ldots, \xi_r)$ be a family of elements $\xi_i \in \mathcal{L}$. The blow-up of $\mathcal{L}$ in $\mathcal{G}$ is simply the subsequent blow-up of $\mathcal{L}$ in the elements ξ_i in order of ascending indices. When we speak of a subfamily $(\xi_{i_1}, \ldots \xi_{i_s})$ of $\mathcal{G}$ we always tacitly assume, that the order is preserved, i.e. that $j < k$ implies $i_j < i_k$. We call $\mathcal{G}$ *sorted* if $\xi_i > \xi_j$ implies $i < j$. Moreover, we denote the underlying set of the family $\mathcal{G}$ by $\mathcal{S}_\mathcal{G}$.

Definition 6.3.5. A subset $\mathcal{S} \subseteq \mathcal{L}\backslash\{0\}$ is called *building set* for $\mathcal{L}$, if for any element $0 \neq x \in \mathcal{L}$ and $\max(\mathcal{S}_{\geq x}) = \{x_1, \ldots, x_r\}$ there exists an isomorphism of patially ordered sets

$$\varphi_x \colon \prod_{i=1}^{r} [0, x_i] \;\to\; [0, x]$$

where for every $j = 1, \ldots, r$ the element $(0, \ldots, 0, x_j, 0, \ldots, 0)$ maps to x_j.

Proposition 6.3.6 ([**28**, Proposition 2.3])**.** *The set $\mathcal{S} \subseteq \mathcal{L} \setminus \{0\}$ is a building set for $\mathcal{L}$ if and only if $\mathcal{S}$ generates $\mathcal{L}$ by $\vee$ and for any $x \in \mathcal{L}$, $\{x_q, \ldots, x_s\} \subseteq \max(\mathcal{S}_{\geq x})$ and $z < y \in \mathcal{L}$ the following conditions hold:*

(i) $\mathcal{A}_{\leq y} \cap \mathcal{S}_{\leq x_1 \vee \ldots \vee x_s} = \emptyset$,
(ii) $z \vee x_1 \vee \ldots \vee x_s \;<\; y \vee x_1 \ldots \vee x_s$.

Definition 6.3.7 ([**28**, Definition 2.2])**.** A subset $\mathcal{C}$ of a building set $\mathcal{S}$ is called *nested* if for any subset $\mathcal{H} \subseteq \mathcal{C}$ of pairwise incomparable elements and $|\mathcal{H}| \geq 2$ the join $\bigvee \mathcal{H}$ exists and is not an element in $\mathcal{S}$.

Remark 6.3.8. Note that the collection of nested sets forms an abstract simplicial complex $\mathfrak{C}(\mathcal{S})$ with vertex set $\mathcal{S}$.

Theorem 6.3.9 ([**28**, Theorem 3.4])**.** *Assume that $\mathcal{G}$ is a sorted familiy in the semilattice $\mathcal{L}$ such that the underlying set $\mathcal{S}_\mathcal{G}$ is a building set. Then we have an isomorphism of posets*

$$\mathfrak{C}(\mathcal{S}_\mathcal{G}) \to \mathrm{Bl}_\mathcal{G}(\mathcal{L}); \qquad \mathcal{C} \mapsto \bigvee_{\xi \in \mathcal{C}} (\xi, 0).$$

We now describe a suffient criterion to test whether an element lies in $\mathrm{Bl}_\mathcal{F}(\mathcal{L})$ in the case where $\mathcal{S}_\mathcal{F}$ is not a building set.

Theorem 6.3.10. *Let $\mathcal{F}$ be a sorted family in $\mathcal{L}$ and consider a subset $\mathcal{C}$ of the underlying set $\mathcal{S}_\mathcal{F}$. If there exists a building set $\mathcal{S}$ of $\mathcal{L}$ with $\mathcal{S}_\mathcal{F} \subseteq \mathcal{S}$ such that $\mathcal{C}$ is nested in $\mathcal{S}$, then $\bigvee_{\xi \in \mathcal{C}} (\xi, 0)$ exists in the blow-up $\mathrm{Bl}_\mathcal{F}(\mathcal{L})$.*

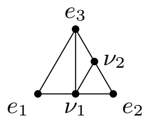
Before we enter the proof of the Theorem we consider an example. Furthermore, for distributive $\mathcal{L}$ we provide an explicit construction of such a building set in the case where $\mathcal{S}_\mathcal{F}$ generates $\mathcal{L}$ by $\vee$, see Construction 6.3.13, Lemma 6.3.15.

Example 6.3.11. The face poset of a polyhedral fan is a semilattice in which the stellar subdivision in a ray $\nu \in \sigma^\circ$ corresponds to the blow-up of the element σ, see [**28**, Proposition 4.9]. Viewing the positive orthant $\Sigma := \mathbb{Q}^3_{\geq 0}$ as a fan, we ask for the combinatoric structure of its stellar subdivisions Σ_1

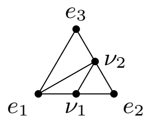
and Σ_2 in the sorted families

$$\mathcal{G}_1 := (\nu_1, \nu_2, e_1, e_2, e_3), \qquad \mathcal{G}_2 := (\nu_2, \nu_1, e_1, e_2, e_3),$$
$$\text{where} \quad \nu_1 := (1,1,0), \quad \nu_2 := (0,1,1).$$

If $\mathcal{G}_1$ and $\mathcal{G}_2$ were building sets, then Theorem 6.3.9 would imply that Σ_1 equals Σ_2. Clearly, this is not the case.



Σ_1

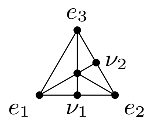


Σ_2

We now add to $\mathcal{G}_1$ and $\mathcal{G}_2$ a ray lying in the relative interior of the join of the faces cone(e_1, e_2), cone(e_2, e_3), e.g. $\nu_0 = (1,1,1)$. This yields two building sets

$$\mathcal{G}_{1a} := (\nu_0, \nu_1, \nu_2, e_1, e_2, e_3), \qquad \mathcal{G}_{2a} := (\nu_0, \nu_2, \nu_1, e_1, e_2, e_3).$$

Both families give rise to the same subdivided fan. Note that the faces of $\Sigma_{1a} = \Sigma_{2a}$ not having ν_0 as a ray lie in both Σ_1 and Σ_2. This is essentially the idea of the proof of Proposition 6.3.1.



$\Sigma_{1a} = \Sigma_{2a}$

Definition 6.3.12. Let $\mathcal{S} = \{\xi_1, \ldots, \xi_r\}$ be a subset of the semilattice $\mathcal{L}$. We call a (non-ordered) pair $\{\xi_i, \xi_j\}$ *harmonious* (with respect to $\mathcal{S}$) if at least one of the following conditions is satisfied:

$$\xi_i \wedge \xi_j = 0 \qquad \text{or} \qquad \xi_i \vee \xi_j \text{ does not exist} \qquad \text{or} \qquad \xi_i \vee \xi_j \in \mathcal{S}.$$

Construction 6.3.13. Let $\mathcal{S} = \{\xi_1, \ldots, \xi_r\}$ be a subset of $\mathcal{L}$. For all pairs of non-harmonious elements $\{\xi_i, \xi_j\}$ we add to $\mathcal{S}$ the element $\xi_i \vee \xi_j$:

$$\mathcal{S}' \quad := \quad \mathcal{S} \,\cup\, \{\xi_i \vee \xi_j;\ \{\xi_i, \xi_j\} \text{ non-harmonious with respect to } \mathcal{S}\}.$$

We continue this process with the new set $\mathcal{S}'$ instead of $\mathcal{S}$ until all pairs are harmonious and denote the final set by $\langle\!\langle \mathcal{S} \rangle\!\rangle$. Since $\mathcal{L}$ is finite, clearly this process terminates after finitely many steps.

Definition 6.3.14. A semilattice $\mathcal{L}$ is called *distributive* if for any $x, y, z \in \mathcal{L}$ the following equation holds

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Lemma 6.3.15. *Assume that $\mathcal{L}$ is distributive and a subset $\mathcal{S} \subseteq \mathcal{L} \setminus \{0\}$ generates it by $\vee$. Then the following assertions hold.*

(i) *If for any $x \in \mathcal{L}$ and distinct $\xi_i, \xi_j \in \max(\mathcal{S}_{\leq x})$ their meet $\xi_i \wedge \xi_j$ equals 0, then $\mathcal{S}$ is a building set for $\mathcal{L}$.*
(ii) *The set $\langle\!\langle \mathcal{S} \rangle\!\rangle$ is a building set for $\mathcal{L}$.*

Proof. For the proof of (i) we check the two conditions of [**28**, Proposition 2.3 (4)]. Fix an $x \in \mathcal{L}$ and a subset $\{y, y_1, \ldots, y_t\} \subseteq \max(\mathcal{S}_{\leq x})$. By assumption we have

$$0 = (y \wedge y_1) \vee \ldots \vee (y \wedge y_t) = y \wedge (y_1 \vee \ldots \vee y_t).$$

Since $0 \notin \mathcal{S}$ holds, this implies $\mathcal{S}_{\leq y} \cap \mathcal{S}_{\leq y_1 \vee \ldots \vee y_t} = \emptyset$. For the second condition let $z < y$. Clearly $z \vee y_1 \vee \ldots \vee y_t \leq y \vee y_1 \vee \ldots \vee y_t$ holds. If they were equal, so would be the respective meets with y and this would imply $z = y$.

We now prove the second assertion (ii). By construction of $\langle\!\langle \mathcal{S} \rangle\!\rangle$, for any $x \in \mathcal{L}$ and $\xi_i, \xi_j \in \max(\langle\!\langle \mathcal{S} \rangle\!\rangle_{\leq x})$ the pair $\{\xi_i, \xi_j\}$ is harmonious (with respect to $\langle\!\langle \mathcal{S} \rangle\!\rangle$). Its join exists but - by maximality of ξ_i and ξ_j - does not lie in $\langle\!\langle \mathcal{S} \rangle\!\rangle$. This implies that $\xi_i \wedge \xi_j = 0$ holds and the assertion follows from (i). $\square$

Proof of Theorem 6.3.10. Before we enter the proof let us recall the join rules of blow-ups from [**28**, Lemma 3.2]. Let x, y, ξ lie in the semilattice $\mathcal{L}$ and consider the blow-up $\mathcal{L}'$ of $\mathcal{L}$ in ξ. Then the join $(\xi, x) \vee_{\mathcal{L}'} y$ exists if and only if $x \vee_{\mathcal{L}} y$ exists and $x \vee y \not\geq \xi$ holds. The join $x \vee_{\mathcal{L}'} y$ exist if and only if $x \vee_{\mathcal{L}} y$ exists. In case the joins exist the following formulae hold

$$(\xi, x) \vee_{\mathcal{L}'} y \;=\; (\xi, x \vee_{\mathcal{L}} y), \qquad\qquad x \vee_{\mathcal{L}'} y \;=\; x \vee_{\mathcal{L}} y.$$

We turn back to our case and fix some notation. We write $\mathcal{F} = (\xi_1, \ldots, \xi_r)$ and denote the elements lying in $\mathcal{C}$ by $\xi_{i_j},\ j = 1, \ldots, s$ where we assume that the order is preserved, i.e. $j < j'$ is equivalent to $i_j < i_{j'}$. Moreover, for $k = 1, \ldots, r$ let $\mathcal{L}_k$ be the blow-up of $\mathcal{L}$ in $(\xi_1, \ldots, \xi_k)$and for consistency we set $\mathcal{L}_0 := \mathcal{L}$. In $\mathcal{L}_k$ we consider the following (a priori non-existent) join

$$\bigvee_{j=1}^{j(k)} (\xi_{i_j}, 0) \;\vee \bigvee_{j=j(k)+1}^{s} \xi_{i_j}, \quad \text{where} \quad j(k) := \max(\{0\} \cup \{j;\ i_j \leq k\}).$$

In case this join does exist, we denote it by z_k. Note that $i_{j(k)}$ is the smallest index, such that $\xi_{i_1}, \ldots, \xi_{i_{j(k)}}$ are among the $\xi_1, \ldots, \xi_k$. Since $\mathcal{C}$ is nested, it is clear that $z_0 = \bigvee \mathcal{C}$ does exist in $\mathcal{L}_0$. We prove the existence of $z_r = \bigvee_{\xi \in \mathcal{C}} (\xi, 0)$ by induction on k. For this assume that $z_k \in \mathcal{L}_k$ exists. We discriminate two possible cases: In the first case ξ_{k+1} does not lie in $\mathcal{C}$ in the second case it does.

Assume that $\xi_{k+1} \notin \mathcal{C}$ holds and note that this is equivalent to $j(k) = j(k+1)$. Hence $z_{k+1} = z_k$ holds in $\mathcal{L}_k$ and the only thing to check is that we have $z_k \not\geq \xi_{k+1}$. For this note that the iterated application of the above join rules shows that

$$z_k = (\xi_{i_{j(k)}}, 0) \vee \left(\bigvee_{j=1}^{j(k)-1} (\xi_{i_j}, 0) \vee \zeta_k \right) = \left(\xi_{i_{j(k)}}, \bigvee_{j=1}^{j(k)-1} (\xi_{i_j}, 0) \vee \zeta_k \right)$$

$$= \ldots = \Big(\xi_{i_{j(k)}}, (\ldots(\xi_{i_1}, \zeta_k)\ldots) \Big) \qquad \text{where } \zeta_k := \bigvee_{j=j(k)+1}^{s} \xi_{i_j}.$$

If we had $z_k \geq \xi_{k+1}$, then this would mean $(\xi_{i_{j(k)-}}, (\ldots(\xi_{i_1}, \zeta_k)\ldots)) \geq \xi_{k+1}$. Iterating this argument we would get $\zeta_k \geq \xi_{k+1}$ in $\mathcal{L}_0$ which would imply $\xi_{k+1} \in \mathcal{S}_{\leq \zeta_k}$. Since $\mathcal{S}$ is a building set, by [**28**, Proposition 2.8 (2)]

$$\max(\mathcal{S}_{\leq \zeta_k}) = \max(\xi_{i_j},\ j = j(k)+1, \ldots, s)$$

holds. Hence there must exist $j_0 \geq j(k)+1$ with $\xi_{k+1} \leq \xi_{i_{j_0}}$. Since $\xi_{k+1} \notin \mathcal{C}$ holds, we have $\xi_{k+1} \neq \xi_{j_0}$. In particular, this implies $k > i_{j_0} - 1 \geq i_{j(k)+1} - 1$. However, from the definition of $j(k)$ we easily see that $k \leq i_{j(k)+1} - 1$ holds, a contradiction.

We turn to the second case where $\xi_{k+1} \in \mathcal{C}$ holds which is equivalent to $j(k)+1 = j(k+1)$. In $\mathcal{L}_k$ we consider the element

$$y_k := \bigvee_{j=1}^{j(k)} (\xi_{i_j}, 0) \ \vee \bigvee_{j=j(k)+2}^{s} \xi_{i_j}.$$

Since z_k exists, it follows that also y_k and the join $\xi_{k+1} \vee y_k$ exist. Then the last thing to show is that $y_k \not\geq \xi_{k+1}$ holds. This follows from the same argument as above with y_k instead of z_k. □

Proof of Proposition 6.3.1. First note that since Σ_0 is simplicial so is the iterated stellar subdivision Σ_r. In particular, the further application of stellar subdivisions in the original rays $\mathcal{V}$ leaves Σ_r unchanged. From [**28**, Proposition 4.9] we know that a stellar subdivision in a ray $\nu \in \sigma^\circ$ corresponds to the blow-up of the face poset of the original fan in σ. More precisely, as posets Σ_r and $\mathrm{Bl}_{\mathcal{F}}(\Sigma_0)$ are isomorphic, where

$$\mathcal{F} := (\sigma_1, \ldots, \sigma_r, v_1, \ldots, v_t), \qquad \mathcal{V} = \{v_1, \ldots, v_t\}.$$

For the proof of the Proposition we now check the assumptions of Theorem 6.3.10. First note that Σ_0 is simplicial, hence it is distributive as a semilattice. Its joins and meets can be computed by taking convex geometric sums and intersections respectively. Also, with $\mathcal{S} = \{\sigma_1, \ldots, \sigma_r\}$ it is clear that $\mathcal{V} \cup \mathcal{S}$, the underlying set of $\mathcal{F}$, generates $\Sigma_0 \setminus \{0\}$ by $+$. In particular, from Lemma 6.3.15 we infer that $\langle\!\langle \mathcal{V} \cup \mathcal{S} \rangle\!\rangle$ is a building set for Σ_0.

Now note that $\langle\!\langle \mathcal{V} \cup \mathcal{S} \rangle\!\rangle \setminus \mathcal{V}$ equals $\langle\!\langle \mathcal{S} \rangle\!\rangle$ and from the respective constructions it follows that $\langle\!\langle \mathcal{S} \rangle\!\rangle \subseteq \langle \mathcal{S} \rangle$ holds. Together this means

$$\Sigma_0 \setminus \langle \mathcal{S} \rangle \;\subseteq\; \Sigma_0 \setminus \langle\!\langle \mathcal{S} \rangle\!\rangle \;=\; \Sigma_0 \setminus (\langle\!\langle \mathcal{V} \cup \mathcal{S} \rangle\!\rangle \setminus \mathcal{V}) \;=\; (\Sigma_0 \setminus \langle\!\langle \mathcal{V} \cup \mathcal{S} \rangle\!\rangle) \cup \mathcal{V}.$$

Since $\mathcal{C} \subseteq \mathcal{V} \cup \mathcal{S}$ is geometrically nested, it follows that it is also nested in $\langle\!\langle \mathcal{V} \cup \mathcal{S} \rangle\!\rangle$ in the sense of semilattices. From Theorem 6.3.10 we now know that $\bigvee_{c \in \mathcal{C}} (c, 0)$ lies in $\mathrm{Bl}_{\mathcal{F}}(\Sigma_0)$. Under the above isomorphy $\Sigma_r \cong \mathrm{Bl}_{\mathcal{F}}(\Sigma_0)$ this means

$$\mathrm{cone}(v, \nu_i;\; v \in \mathcal{C} \cap \mathcal{V},\, \sigma_i \in \mathcal{C} \cap \mathcal{S}) \in \Sigma_r.$$

□

6.4. Stellar Subdivisions and Blow-ups

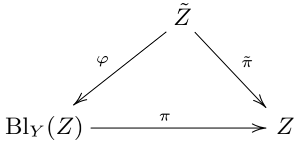
In this section we relate the toric morphism arising from a stellar subdivision to classical blow-ups. We recall some of the basic notation, see [**27, 37**].

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Z$-modules (or short $\mathcal{O}_Z$-module) on the normal variety Z. We call $\mathcal{F}$ *invertible* if it is locally free of rank 1. Moreover, $\mathcal{F}$ is a *sheaf of ideals*, if for every open $U \subseteq Z$ the sections $\Gamma(\mathcal{F}, U)$ constitute an ideal in $\mathcal{O}_Z(U)$. Consider a morphism of varieties $\iota\colon Y \to Z$ and an $\mathcal{O}_Y$-module $\mathcal{G}$. Then $\iota_* \mathcal{G}$ and $\iota^{-1}\mathcal{F}\mathcal{O}_Y$ are the *direct image sheaf* and *inverse image sheaf* respectively.

Let $\iota\colon Y \to Z$ be the embedding of a closed subscheme. Then Y is characterised by its ideal sheaf $\mathcal{I}_Y$, i.e. the kernel of $\mathcal{O}_Z \to \iota_* \mathcal{O}_Y$. The *blow-up* of Z along Y is given as

$$\mathrm{Bl}_Y(Z) \;:=\; \mathrm{Proj}(\mathcal{A}), \qquad \text{where} \qquad \mathcal{A} \;:=\; \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{I}_Y^n$$

and $\mathcal{I}_Y^0 = \mathcal{O}_Z$. It comes with a morphism $\pi\colon \mathrm{Bl}_Y(Z) \to Z$ and satisfies the following universal property, see [**37**, Proposition 7.14]. If there exists a morphism of varieties $\tilde{\pi}\colon \tilde{Z} \to Z$ such that the inverse image sheaf $\tilde{\pi}^{-1}\mathcal{I}_Y \mathcal{O}_{\tilde{Z}}$ is an invertible sheaf of ideals on $\tilde{Z}$, then there exists a unique morphism $\varphi\colon \tilde{Z} \to \mathrm{Bl}_Y(Z)$ such that the following diagram commutes.



Now consider a morphism of varieties $\varphi\colon X \to Z$ and let $\iota\colon Y \to Z$ be a closed subscheme. Then the sheaf of ideals corresponding to the preimage $\varphi^{-1}(Y) := X \times_Z Y \to X$ is given by the inverse image sheaf $\varphi^{-1}\mathcal{I}_Y \mathcal{O}_X$.

Moreover, there exists a unique morphism $\tilde{\varphi}$ making the following diagram commutative, see [**37**, Corollary 7.15].

$$\begin{array}{ccc} \mathrm{Bl}_{\varphi^{-1}(Y)}(X) & \xrightarrow{\quad\tilde{\varphi}\quad} & \mathrm{Bl}_Y(Z) \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\quad\varphi\quad} & Z \end{array}$$

If φ is a closed embedding, then so is $\tilde{\varphi}$ and the image of the latter coincides with the *proper transform* of X, i.e. the closure $\overline{\pi^{-1}(X \setminus Y)} \subseteq \mathrm{Bl}_Y(Z)$.

Suppose that Z is affine; then there is a one-to-one correspondence between the subschemes of Z and the ideals of $\mathcal{O}(Z)$. So let $I_Y \subseteq \mathcal{O}(Z)$ be the ideal corresponding to Y and suppose that $\varphi\colon X \to Z$ is a closed embedding. Then the blow-ups of Z in Y and of X in $\varphi^{-1}(Y)$ are

$$\mathrm{Bl}_Y(Z) \;=\; \mathrm{Proj}\left(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} I_Y^n\right), \qquad \mathrm{Bl}_{\varphi^{-1}(Y)}(X) \;=\; \mathrm{Proj}\left(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} (\varphi^* I_Y)^n\right).$$

We now turn to the following question. Let $\pi\colon Z_1 \to Z_0$ be the toric morphism arising from a stellar subdivision $\Sigma_1 \to \Sigma_0$. Describe a homogeneous ideal I in the graded Cox ring $\mathcal{R}(Z_0)$ such that Z_1 is isomorphic to the blow-up of Z_0 in the subscheme associated to the ideal sheaf $\mathcal{I}$ on Z_0 in the sense of Cox, [**22**, Section 3].

We fix some notation. Let Σ_0 be a simplicial lattice fan in $N_{\mathbb{Q}}$ and suppose that $v_1, \ldots, v_r \in N$ are primitive lattice vectors in the rays of Σ_0. Set P as the homomorphism mapping the standard basis vectors $f_1, \ldots, f_r$ of $F := \mathbb{Z}^r$ to the v_i. Then the Cox ring of Z_0 is $\mathbb{K}[\gamma \cap E]$ where $E := F^*$ is the dual lattice of F and γ is the positive orthant in $E_{\mathbb{Q}} := E \otimes \mathbb{Q}$.

Let ν be a lattice vector in the support of Σ_0. Then there exists a unique subset $A \subseteq \{1, \ldots, r\}$ and a minimal positive integer m such that

$$m\nu \;=\; \sum_{i \in A} \alpha_i v_i, \qquad \alpha_i \in \mathbb{Z}_{\geq 1}$$

is a linear combination with only positive integer coefficients. By $(e_1, \ldots, e_r)$ we denote the dual basis of $(f_1, \ldots, f_r)$ and set

$$E_A := \mathrm{cone}(e_i;\ i \in A), \qquad f := \sum_{i \in A} \alpha_i f_i \in F, \qquad c := \mathrm{lcm}(\alpha_i;\ i \in A).$$

and obtain a homogeneous ideal in the Cox ring of Z_0 and a subscheme Y of Z_0 in the sense of [**22**].

$$I \;:=\; \langle \chi^e;\ e \in E_A,\ \langle e, f \rangle = c \rangle \;\subseteq\; \mathbb{K}[\gamma \cap E].$$

Proposition 6.4.1. *Let $\Sigma_1 \to \Sigma_0$ be the stellar subdivision of the simplicial fan Σ_0 in* $\mathrm{cone}(\nu)$*. If Z_1, Z_0 are the toric varieties arising from Σ_1, Σ_0 respectively, then Z_1 is isomorpic to the normalised blow-up of Z_0 in the subscheme Y.*

Lemma 6.4.2. *Let Z be an affine toric variety with corresponding lattice cone σ in N. Consider a subset $\mathbf{M} \subseteq \sigma^\vee \cap N^*$ and the subscheme Y of Z corresponding to the ideal*

$$\langle \chi^y;\ y \in \mathbf{M} \rangle \quad \subseteq \quad \mathbb{K}[\sigma^\vee \cap N^*].$$

Then the toric variety corresponding to the normal fan $\mathcal{N}(\sigma^\vee + \mathrm{conv}(\mathbf{M}))$ is isomorphic to the normalisation of $\mathrm{Bl}_Y(Z)$.

Proof of Proposition 6.4.1. Let $\sigma \in \Sigma_0$ be a cone and denote by $B \subseteq \{1, \ldots, r\}$ the indices with the property $\mathrm{cone}(v_j) \in \sigma^{(1)}$. We denote by $\mathcal{R}$ the Cox ring of Z_0 and consider the localisation $\mathcal{R}_\sigma$ at the element $\prod_{j \notin B} T_j$. Then the regular functions on the affine chart $Z(\sigma)$ are given by the degree zero part $(\mathcal{R}_\sigma)_0$. Moreover, on this chart the subscheme Y is defined by the respective degree zero part of the localised ideal $(I_\sigma)_0$. It is explicitly given by

$$(I_\sigma)_0 \;=\; \left\langle \chi^{e-k};\ e \in E_A,\ k \in E_{A^c},\ \langle e, f \rangle = c,\ e - k \in \mathrm{Im}(P^*) \right\rangle$$

If $A \subseteq B$ holds, which is equivalent to $\nu \in \sigma$, then this ideal is equal to $\langle 1 \rangle$. For the case $A \not\subseteq B$, which holds if and only if $\nu \notin \sigma$, consider the isomorphism of affine algebras

$$\varphi\colon \mathbb{K}[\sigma^\vee \cap N^*] \to (\mathcal{R}_\sigma)_0; \qquad \chi^m \mapsto T^{P^*(m)}.$$

Under this isomorphism the preimage of $(I_\sigma)_0$ is given by

$$\varphi^{-1}((I_\sigma)_0) \;=\; \{m \in \sigma^\vee \cap N^*;\ \langle m, \nu \rangle = c\} \;=:\; \mathbf{M}.$$

Blow-ups are determined locally, so by Lemma 6.4.2 our assertion follows from

$$\mathrm{stSubDiv}_\nu(\sigma) \;=\; \sigma \sqcap \mathcal{N}(\mathrm{conv}(\mathbf{M})) \;=\; \mathcal{N}(\sigma + \mathrm{conv}(\mathbf{M})).$$

□

Remark 6.4.3. In the affine case a blow-up is normal if and only if the ideal corresponding to the center of the blow-up is integrally closed. Criteria for this are provided in [**48**, Proposition 1.4.6].

6.5. The Limit Quotient as Blow-up

This section is devoted to the main result and its proof. As before, let $\overline{Y} \subseteq \bigwedge^2 \mathbb{K}^{n+1}$ be the affine cone over the Grassmannian $\mathrm{Gr}(2, n+1)$ and consider the torus $H = (\mathbb{K}^*)^n$ acting on $\overline{Y}$ by

$$h \cdot e_0 \wedge e_i \;=\; h_i e_0 \wedge e_i, \qquad h \cdot e_i \wedge e_j \;=\; h_i h_j e_i \wedge e_j.$$

We assert that the normalised limit quotient $\overline{Y} \,\tilde{/}_{\mathrm{LQ}}\, H$ normalises the following iterated blow-up of $\mathbb{P}_1^{n-1}$. We set $N_2 := \{2, \dots, n\}$ and consider a subset $A \subseteq N_2$ with at least two elements. Labeling by $T_2, S_2, \dots, T_n, S_n$ the homogeneous coordinates of $\mathbb{P}_1^{n-1}$ we associate to A the subscheme of $\mathbb{P}_1^{n-1}$ given by the ideal

$$\left\langle\, T_i^2,\; T_j S_k \,-\, T_k S_j;\;\; i,\, j,\, k \in A,\; j < k \,\right\rangle$$

The collection $\mathcal{X}$ of corresponding subschemes X_A comes with a partial order given by the schme-theorectic inclusions with X_{N_2} being the minimal element. A *linear extension* of this partial order is a total order on $\mathcal{X}$ which is compatible with the partial order.

Theorem 6.5.1. *Fix a linear extension of the partial order on $\mathcal{X}$. Then the normalised limit quotient $\overline{Y} \,\tilde{/}_{\mathrm{LQ}}\, H$ normalises the blow-up of $\mathbb{P}_1^{n-1}$ in all the subschemes X_A (i.e. their respective proper transforms) in ascending order.*

Recall that the above action stems from the action of $\mathbb{G}_a$ on $X = \mathbb{P}_1^n$ as shown in Sections 2.5 and 6.2. Moreover, keep in mind that the enveloped quotients V_i of X are only subsets of the Mumford quotients of $\overline{Y}$. Hence the non-reductive limit quotient $X /_{\mathrm{LQ}} \mathbb{G}_a$ in general only is a subset of the reductive limit quotient. This is reflected in the second step of the following procedure to obtain $X \,\tilde{/}_{\mathrm{LQ}}\, \mathbb{G}_a$.

Theorem 6.5.2. *The normalised limit quotient $X \,\tilde{/}_{\mathrm{LQ}}\, \mathbb{G}_a$ can be obtained by the following procedure.*

- (i) *Let X_1 be the blow-up of $\mathbb{P}_1^{n-1}$ in the subscheme X_{N_2}.*
- (ii) *Let $X_1' := X_1 \setminus E$ be the quasiprojective subvariety of X_1 where E is the intersection of the proper transform of $V(T_2, \dots, T_n) \subseteq \mathbb{P}_1^{n-1}$ with the exeptional divisor in X_1.*
- (iii) *Fix a linear extension of the partial order on $\mathcal{X}$ and blow up X_1' in the respective proper transforms of the remaining subschemes X_A, $A \subsetneq N_2$ in ascending order.*
- (iv) *Normalise the resulting space.*

We briefly outline the structure of our proof. For this consider E_n the identity matrix and

$$Q := (E_n, D_n), \qquad \text{where} \qquad D_n := (e_j \,+\, e_k)_{1 \le j < k \le n}.$$

Note that Q is the matrix recording the weights of the coordinates of the above H-action. We denote the first n columns of Q by w_{0i} and the remaining ones by w_{jk}. Furthermore, we fix a Gale dual matrix P of Q, i.e. a matrix with $PQ^t = 0$, and analogously write v_{0i}, v_{jk} for its columns. Denoting by T the dense algebraic torus of $\bigwedge^2 \mathbb{K}^{n+1}$ we recall from Section 2.5 that there is a normalisation map

$$\overline{Y} \tilde{/\!\!/}_{\mathrm{LQ}} H \;\to\; \overline{\left((\overline{Y} \cap T)/H\right)^{\Sigma}},$$

where the latter is the closure in the toric variety associated to the fan $\Sigma := \mathrm{GKZ}(P)$. With this the proof of Theorem 6.5.1 will be split into two parts. As a first step we will prove that the blow-up of $\mathbb{P}_1^{n-1}$ in the subscheme X_{N_2} yields one of the Mumford quotients X_1 of $\overline{Y}$. This quotient comes with a canonical embeddeding into a simplicial toric variety Z_1, which arises from a simplicial fan Σ_1 with rays generated by the columns of P. Finally we show that the iterated stellar subdivision of Σ_1 and the fan Σ share a sufficiently large subfan. This implies that the proper transform of X_1 under the corresponding toric blow-ups and the limit quotient $\overline{Y} /\!\!/_{\mathrm{LQ}} H$ share a common normalisation.

In the case $n = 2$ the normalised limit quotient is the projective line. If we consider three distinct points the resulting normalised limit quotient is the unique non-toric, Gorenstein, log del Pezzo $\mathbb{K}^*$-surface of Picard number 3 and a singularity of type A_1, see [**47**, Theorem 5.27]. The standard construction of this surface is the blow-up of three points on $\mathbb{P}_2$ followed by the contraction of a (-2)-curve. However, we realise it as a single (weighted) blow-up of $\mathbb{P}_1 \times \mathbb{P}_1$ in the subscheme $V(T_2^2,\ T_3^2,\ T_2S_3 - T_3S_2)$ where T_2, S_2, T_3, S_3 are the homogeneous coordinates on $\mathbb{P}_1 \times \mathbb{P}_1$. Similar to $\overline{M}_{0,5}$ which is isomorphic to a single Mumford quotient of the cone over the Grassmannian $\mathrm{Gr}(2,5)$, this surface arises as Mumford quotient of the cone over the Grassmannian $\mathrm{Gr}(2,4)$. For higher n an analogous Mumford quotient needs to be blown up as described above to obtain the limit quotient.

Step 1. Recall that each chamber in the GIT-fan $\Lambda_H(\overline{Y})$ gives rise to a set of semistable points admitting a Mumford quotient. We define two particular chambers and look at their respective quotients. For this consider the following linear forms on $\mathbb{Q}^n$:

$$f_1 := e_1^* - \sum_{i \neq 1} e_i^*; \qquad f_{1j} := e_1^* + e_j^* - \sum_{i \neq 1,j} e_i^*.$$

The zero sets of these linear forms are precisely the walls arising from the partitions $\{\{1\}, N \setminus \{1\}\}$ and $\{\{1,j\}, N \setminus \{1,j\}\}$ of $N = \{1, \ldots, n\}$ in the sense of Section 6.2. We define the following two full dimensional cones in the GIT-fan

$$\lambda_0 \;:=\; \Omega \cap \{w \in \mathbb{Q}^n;\ f_1(w) \geq 0\},$$
$$\lambda_1 \;:=\; \Omega \cap \{w \in \mathbb{Q}^n;\ f_1(w) \leq 0,\ f_{1j}(w) \geq 0 \text{ for } j = 2, \ldots, n\}.$$

where Ω is the support of $\Lambda_H(\overline{Y})$. While λ_1 lies inside $\Omega^\star = \mathrm{supp}(\Lambda_H(\overline{Y}^\star))$ the cone λ_0 does not. The two cones are adjacent in the sense that they share a common facet, namely $\Omega \cap \ker(f_1)$. Now consider the corresponding Mumford quotients $X_i := \overline{Y}^{\mathrm{ss}}(\lambda_i) /\!\!/ H$ with $i = 0, 1$.

Proposition 6.5.3. *In the above notation X_0 is isomorphic to $\mathbb{P}_1^{n-1}$. Moreover, X_1 is isomorphic to the blow-up of X_0 in the subscheme X_{N_2}.*

Recall that $\lambda_1 \in \Lambda_H(\overline{Y})$ gives rise to the enveloped quotient V_1 which is the image of the restricted Mumford quotient $\overline{Y}^{\mathrm{ss}}(\lambda_1) \cap \overline{Y}' \to X_1$.

Proposition 6.5.4. *Let E denote the intersection of the exceptional divisor of $X_1 \to X_0$ with the proper transform of $V(T_2, \ldots, T_n)$. Then the enveloped quotient V_1 is given by $X_1 \setminus E$. In particular, it is quasiprojective.*

Proposition 6.5.5. *Let $A \subseteq N_2$ be a subset with at least two elements. Then the cone* $\mathrm{cone}(v_\eta;\ \eta \subseteq A \cup \{0\})$ *lies in Σ_0. Moreover, consider the ray*

$$\nu \quad := \quad \mathrm{cone}\left(\sum_{i \in A} v_{0i} + 2 \sum_{\eta \subseteq A} v_\eta \right)$$

in the relative interior of the above cone. Let X' be the proper transform of X_0 under the blow-up corresponding to the stellar subdivision of Σ_0 in ν. Then X' is isomorphic to the blow-up of X_0 in the subscheme of $\mathbb{P}_1^{n-1}$ given by

$$\left\langle T_i^2,\ T_j S_k - T_k S_j\ ; \quad i,\, j,\, k \in A,\ j < k \right\rangle.$$

We prove Propositions 6.5.3, 6.5.4 and 6.5.5 using the method of ambient modifications, see [**39**, Proposition 6.7]. For this note that X_0 and X_1 come with canonical embeddings into simplicial toric varieties. We provide an explicit construction, for the general case see [**4**, Chapter III, Section 2.5]. For the index sets we use the same notation as in Section 6.2:

$$\mathbf{N} = \{\{i,j\},\ 1 \le i < j \le n\}, \qquad \mathbf{N}_0 = \{\{i,j\},\ 0 \le i < j \le n\}.$$

Viewing $\bigwedge^2 \mathbb{K}^{n+1}$ as the toric variety arising from the positive orthant δ in $\bigwedge^2 \mathbb{Q}^{n+1}$ we define a subset as follows. We set

$$\mathrm{envs}(\lambda_i) \;=\; \{I \subseteq \mathbf{N}_0;\ J \subseteq I,\ \lambda_i^\circ \subseteq \omega_J^\circ \subseteq \omega_I^\circ \text{ for some } \overline{Y}\text{-set } J\}$$

as the collection of *enveloping sets*. Denoting by f_η with $\eta \in \mathbf{N}_0$ the standard basis vector in $\bigwedge^2 \mathbb{Q}^{n+1}$ we consider the subfan of δ

$$\hat{\Sigma}_i \;:=\; \{\mathrm{cone}(f_\eta;\ \eta \in J);\ J \subseteq \mathbf{N}_0 \setminus I \text{ for some } I \in \mathrm{envs}(\lambda_i)\}$$

and the corresponding toric variety $\hat{Z}_i \subseteq \bigwedge^2 \mathbb{K}^{n+1}$. Then $\hat{Z}_i$ admits a good quotient $\hat{Z}_i \to Z_i$; the quotient space is toric again and the quotient morphism corresponds to the lattice homomorphim $P\colon \mathbb{Z}^{\binom{n+1}{2}} \to \mathbb{Z}^{\binom{n}{2}}$. The fan of Z_i is given by

$$\Sigma_i \;=\; \{\mathrm{cone}(v_\eta;\ \eta \in \mathbf{N}_0 \setminus I);\ I \in \mathrm{envs}(\lambda_i)\}$$

We now turn to the embedded spaces. Starting with the embedding $\overline{Y} \subseteq \bigwedge^2 \mathbb{K}^{n+1}$ we have $\overline{Y} \cap \hat{Z}_i = \overline{Y}^{\mathrm{ss}}(\lambda_i)$ and the quotient $\hat{Z}_i \to Z_i$ restricts to the good quotient $\overline{Y}^{\mathrm{ss}}(\lambda_i) \to X_i$. The situation fits into the following commutative diagram where the vertical arrows are closed embeddings.

$$\begin{array}{ccccc}
\bigwedge^2 \mathbb{K}^{n+1} & \longleftarrow & \hat{Z}_i & \longrightarrow & Z_i \\
\uparrow & & \uparrow & & \uparrow \\
\overline{Y} & \longleftarrow & \overline{Y}^{\mathrm{ss}}(\lambda_i) & \longrightarrow & X_i
\end{array}$$

Proofs of Proposition 6.5.3, 6.5.4 and 6.5.5. We prove the first part of Proposition 6.5.5. For this we set $J := \mathbf{N}_0 \setminus \{\eta;\ \eta \subseteq A \cup \{0\}\}$. With Proposition 6.2.5 it is easy to see, that J is a $\overline{Y}$-set. Moreover, $\lambda_0^\circ \subseteq \omega_J^\circ$ holds. By definition of Σ_0 it is now clear that it contains $\mathrm{cone}(v_\eta;\ \eta \subseteq A \cup \{0\})$.

We now perform the ambient modification. For this note that the weight w_{01} is extremal in $\Lambda_H(\overline{Y})$, hence we can contract v_{01}. It can be written as a non-negative linear combination

$$v_{01} = \sum_{\eta \in \mathbf{N}_0} \alpha_\eta v_\eta, \quad \text{where} \quad \alpha_\eta = \begin{cases} 0 & \text{if } 1 \in \eta \\ 1 & \text{if } 0 \in \eta, 1 \notin \eta \\ 2 & \text{else} \end{cases}.$$

In particular, it lies in the above cone $\mathrm{cone}(v_\eta;\ \eta \subseteq \{0\} \cup N_2\})$. The total coordinate spaces of the embedding toric varieties Z_0 and Z_1 are affine spaces, they are given by

$$\overline{Z}_0 = \mathbb{K}^{\mathbf{N}_0 \setminus \{0,1\}} \quad \text{and} \quad \overline{Z}_1 = \bigwedge^2 \mathbb{K}^{n+1} = \mathbb{K}^{\mathbf{N}_0}.$$

Furthermore, the ambient modification $\Sigma_1 \to \Sigma_0$ gives rise to a morphism of the total coordinate spaces of the respective toric varieties

$$c\colon \overline{Z}_1 \to \overline{Z}_0; \qquad (x_\eta)_{\eta \in \mathbf{N}_0} \;\mapsto\; (x_{01}^{\alpha_\eta}\, x_\eta)_{\eta \in \mathbf{N}_0 \setminus \{0,1\}}.$$

We label the variables of the total coordinate space $\overline{Z}_0$ by S_η where η runs through $\mathbf{N}_0 \setminus \{0,1\}$. Recall that we have a closed embedding $\overline{Y} \subseteq \overline{Z}_1$. The vanishing ideal of the image $\overline{X}_0 := c(\overline{Y})$ in the Cox ring is given as

$$\langle\, S_{ij} \,-\, S_{0i}S_{1j} \,+\, S_{0j}S_{1i}\ ;\ \ 2 \le i < j \le n \,\rangle \quad \subseteq \quad \mathcal{R}(Z_0).$$

It turns out that $\overline{X}_0$ is in fact isomorphic to the affine space via

$$\iota\colon \mathbb{K}^{n-1} \times \mathbb{K}^{n-1} \to \overline{Z}_0 \qquad (x,y) \mapsto (x, y, (x_i y_j - x_j y_i)_{i<j}).$$

The original H-action on $\overline{Y}$ descends via $\iota^{-1} \circ c$ to $\mathbb{K}^{n-1} \times \mathbb{K}^{n-1}$ and is explicitly given by the weight matrix $Q_0 = [E_{n-1}, E_{n-1}]$ where E_{n-1} is the identity matrix. This shows that X_0 is isomorphic to $\mathbb{P}_1^{n-1}$. For convenience we summarise the situation in the following commutative diagram.

$$\begin{array}{ccccc} \overline{Z}_1 & \xrightarrow{c} & \overline{Z}_0 & & \\ \uparrow & & \uparrow & \nwarrow^{\iota} & \\ \overline{Y} & \xrightarrow{c} & \overline{X}_0 & \xleftarrow{\iota} & \mathbb{K}^{2(n-1)} \end{array}$$

The next step of the proof is the second half of Proposition 6.5.5. From Proposition 6.4.1 we infer that the ideal in $\mathcal{O}(\overline{Z}_0) = \mathcal{R}(Z_0)$ yielding the center of the blow-up is given by

$$\langle\ S_{0i}^2,\ S_\eta;\ \ i \in A,\ \eta \subseteq A\ \rangle.$$

If we pullback this ideal via ι^*, then in homogeneous coordinates over $\mathbb{P}_1^{n-1}$ we obtain

$$\langle\ T_i^2,\ T_j S_k - T_k S_j;\ \ i,j,k \in A,\ j < k\rangle,$$

see 4.3.1. In the case of the ambient modification of Proposition 6.5.3 we set $A = N_2$ to obtain the assertion. Finally, we turn to Proposition 6.5.4 and determine the enveloped quotient. For this recall that the image of the categorical quotient in Section 6.2 was given by $\overline{Y}' = (\overline{Y} \setminus \overline{Y}^\star) \cup \{0\}$, see Proposition 6.2.2. This means that the enveloped quotient $V_1 \subseteq X_1$ is given as the image of

$$\pi \colon \overline{Y}^{\mathrm{ss}}(\lambda_1) \setminus D \quad \to \quad X_1,$$

where $D := V(S_{0i};\ i = 1, \ldots, n) \subseteq \bigwedge^2 \mathbb{K}^{n+1}$. The quotient is geometric, hence the enveloped quotient is $V_1 = X_1 \setminus \pi(D)$. Now consider the subvariety $V(T_2, \ldots, T_n) \subseteq \mathbb{P}_1^{n-1}$. Transferring it via ι and then taking the proper transform we obtain the subvariety of X_1 given by $\langle\, S_{02}, \ldots, S_{0n}\, \rangle$ in the Cox ring $\mathcal{R}(Z_1)$. The intersection with the exceptional divisor is precisely the set $E = \pi(D)$.

□

Step 2. In this step we show that the remaining blow-ups lead to the limit quotient $\overline{Y} /\!\!/_{\mathrm{LQ}} H$. As before, $Q = (E_n, D_n)$ is the matrix recording the weights of the coordinates of the H-action and we label its columns by w_η with $\eta \in \mathbf{N}_0$ and $\mathbf{N}_0 = \{\{i,j\};\ 0 \le i < j \le n\}$. We then have the Gale dual matrix P with columns denoted by v_η. Moreover, Σ_1 is the simplicial fan in $\mathbb{Z}^{\binom{n}{2}}$ from the preceeding step and we recall that $X_1 = \overline{Y}^{\mathrm{ss}}(\lambda_1) /\!\!/ H$ is embedded into the corresponding toric variety Z_1.

Now let $R = \{A_1, A_2\}$ be a *true* two-block partition of N, i.e. a partition with $|A_1|, |A_2| \geq 2$. To every such partition we associate a ray

$$\begin{aligned}\nu_R \ &:= \ \operatorname{cone}\left(\sum_{i\in A_1} v_{0i} \ + \ 2\sum_{j<k\in A_1} v_{jk}\right)\\ &= \ \operatorname{cone}\left(\sum_{i\in A_2} v_{0i} \ + \ 2\sum_{j<k\in A_2} v_{jk}\right).\end{aligned}$$

Clearly, there exists $A_R \in \{A_1, A_2\}$ with $1 \notin A_R$. From Proposition 6.5.5 we now infer that the cone $\sigma_R := \operatorname{cone}(v_\eta;\ \eta \subseteq \{0\} \cup A_R)$ containing ν_R in its relative interior lies in Σ_1.

Note that no two rays lie in the relative interior of the same cone of Σ_1. The above defined collection of rays hence comes with a natural partial order inherited from the fan Σ_1:

$$\nu_R \leq \nu_S :\Longleftrightarrow \ \sigma_R \preccurlyeq \sigma_S \ \Longleftrightarrow \ A_R \subseteq A_S.$$

We choose a linear extension of this partial order. Beginning with the maximal ray we then consider the iterated stellar subdivision of Σ_1 in all the rays in descending order. The resulting fan we denote by Σ_r.

While it is not true that Σ_r coincides with the GKZ-decomposition $\Sigma = \operatorname{GKZ}(P)$, both fans share a sufficiently large subfan. To make this precise let T be the dense torus of $\bigwedge^2 \mathbb{K}^{n+1}$. To $\overline{Y} \cap T$ we can associate its *tropical variety* $\operatorname{Trop}(\overline{Y} \cap T)$, which is the support of a quasifan in $\bigwedge^2 \mathbb{Q}^{n+1}$. For a detailed description of this space see [**63**]. For our purposes it suffices to know that the image $\Delta := P(\operatorname{Trop}(\overline{Y} \cap T))$ intersects the relative interior $\operatorname{cone}(v_\eta;\ \eta \in J)^\circ$ of a cone if and only if $\mathbf{N}_0 \setminus J$ is a $\overline{Y}$-set, see [**65**, Proposition 2.3]. We now define the Δ-*reduction* of Σ as the fan

$$\Sigma^\Delta := \{\sigma;\ \sigma \preccurlyeq \tau \in \Sigma \text{ for some } \tau \text{ with } \tau^\circ \cap \Delta \neq \emptyset\}.$$

Note that the relative interiors of all maximal cones of Σ^Δ intersect Δ. Moreover, by [**65**, Proposition 2.3] the closure of $(\overline{Y} \cap T)/H$ in the toric variety corresponding to Σ is already contained in the toric subvariety defined by $\Sigma^\Delta \subseteq \Sigma$.

Proposition 6.5.6. *The Δ-reduction Σ^Δ is a subfan of Σ_r.*

Corollary 6.5.7. *The proper transform of the Mumford quotient $X_1 \subseteq Z_1$ under the toric morphism arising from $\Sigma_r \to \Sigma_1$ and the limit quotient $\overline{Y} /\!\!/_{\mathrm{LQ}} H$ share a common normalisation.*

Proof. For this just note that the following closures coincide and the first morphism is the normalisation map.

$$\overline{Y} \,\tilde{/}_{\scriptscriptstyle LQ}\, H \;\to\; \overline{\left((\overline{Y} \cap T) \,/\, H\right)^{\Sigma}} \;=\; \overline{\left((\overline{Y} \cap T) \,/\, H\right)^{\Sigma\Delta}} \;=\; \overline{\left((\overline{Y} \cap T) \,/\, H\right)^{\Sigma_r}}.$$

□

Remark 6.5.8. In fact, with only minor modifications the Step 2 works for every Mumford quotient of $\overline{Y}$ which arises from a fulldimensional chamber λ lying in $\Omega^\star$.

The idea of the proof of Proposition 6.5.6 is to give a combinatorial description of the cones in Σ^Δ and to show that these are geometrically nested in the sense of Section 6.3.

For the moment let $Q \in \mathrm{Mat}(k, r; \mathbb{Z})$ and $P \in \mathrm{Mat}(n, r; \mathbb{Z})$ be arbitrary Gale dual matrices. We set $R := \{1, \ldots, r\}$. For a subset $I \subseteq R$ we denote by $\gamma_I \subseteq \mathbb{Q}^n$ the cone generated by the $e_i,\ i \in I$ and by $\omega_I := Q(\gamma_I)$ its image under Q. Moreover, if $v_i,\ i \in R$ are the columns of P we set $\sigma_J := \mathrm{cone}(v_j;\ j \in J)$. A system $\mathfrak{B}$ of subsets of R is a *separated R-collection* if any two $I_1, I_2 \in \mathfrak{B}$ admit an invariant separating linear form f, in the sense that

$$P^*(\mathbb{Q}^n) \subseteq \ker(f), \quad f_{|\gamma_{I_1}} \geq 0, \quad f_{|\gamma_{I_2}} \leq 0, \quad \ker(f) \cap \gamma_{I_i} = \gamma_{I_1} \cap \gamma_{I_2}.$$

The separated R-collections come with a partial order; for two R-collections $\mathfrak{B}_1, \mathfrak{B}_2$ we write $\mathfrak{B}_1 \leq \mathfrak{B}_2$ if for every $I_1 \in \mathfrak{B}_1$ there exists $I_2 \in \mathfrak{B}_2$ such that $I_1 \subseteq I_2$ holds. A separated R-collection $\mathfrak{B}$ will be called *normal* if it cannot be enlarged as an R-collection and the cones $\omega_I,\ I \in \mathfrak{B}$ form the normal fan of a polyhedron. With respect to the above partial order there exists a unique maximal normal R-collection, namely $\langle R \rangle$ which consists of all subsets which are invariantly separable from R. By $\mathcal{M}$ we denote the submaximal normal R-collections in the sense, that $\langle R \rangle$ is the only dominating normal R-collection. Finally, for a fixed normal R-collection $\mathfrak{B}$ let $\mathcal{M}(\mathfrak{B})$ consist of those collections of $\mathcal{M}$ lying above $\mathfrak{B}$.

If P consists of pairwise linearly independent columns, then by [**4**, Section II.2] there is an order reversing bijection

$$\{\text{normal } R\text{-collections}\} \;\to\; \Sigma; \qquad \mathfrak{B} \;\mapsto\; \bigcap_{I \in \mathfrak{B}} \sigma_{R \setminus I}.$$

where again $\Sigma = \mathrm{GKZ}(P)$ is the GKZ-decomposition. It is clear that each maximal R-collection $\mathfrak{A} \in \mathcal{M}$ gives rise to a ray $\nu_{\mathfrak{A}} = \bigcap_{\mathfrak{A}} \sigma_{R \setminus I}$ of Σ.

Proposition 6.5.9. *Let $\mathfrak{B}$ be a normal R-collection. Then the cone corresponding to $\mathfrak{B}$ can be written as*

$$\bigcap_{I \in \mathfrak{B}} \sigma_{R \setminus I} \;=\; \mathrm{cone}\left(\nu_{\mathfrak{A}};\ \mathfrak{A} \in \mathcal{M}(\mathfrak{B})\right).$$

Proof. From the order reversing property of the above bijection it is clear, that every ray $\nu_{\mathfrak{A}}$ with $\mathfrak{A} \in \mathcal{M}(\mathfrak{B})$ lies in $\sigma := \bigcap_{\mathfrak{B}} \sigma_{R\setminus I}$. Moreover, there must exists a set of maximal γ-collections $\mathcal{N} \subseteq \mathcal{M}$ such that the extremal rays of σ are precisely the $\nu_{\mathfrak{A}}$ with $\mathfrak{A} \in \mathcal{N}$. Again from the above bijection we know that this means $\mathfrak{A} \geq \mathfrak{B}$. The assertion then follows from the maximality of $\mathfrak{A}$. □

We now return to our special case where $Q = (E_n, D_n)$ holds and the index set R equals $\mathbf{N}_0$. We are interested in a description of the submaximal collections $\mathcal{M}(\mathfrak{B})$ where $\mathfrak{B}$ consists of $\overline{Y}$-sets. The reason is the following Proposition.

Proposition 6.5.10. *Let $\mathfrak{B}$ be a normal $\mathbf{N}_0$-collection and suppose that its associated cone $\bigcap_{I\in\mathfrak{B}} \sigma_{\mathbf{N}_0\setminus I}$ is a maximal cone in Σ^Δ. Then $\mathfrak{B}$ is a collection of $\overline{Y}$-sets.*

Proof. Since $(\bigcap_{\mathfrak{B}} \sigma_{\mathbf{N}_0\setminus I})^\circ \cap \Delta \neq \emptyset$ holds the same is true for every $\sigma^\circ_{\mathbf{N}_0\setminus I}$ with $I \in \mathfrak{B}$. By [**65**, Proposition 2.3] this implies that $\mathfrak{B}$ is a collection of $\mathfrak{F}$-faces. □

Proposition 6.5.11. *Suppose that $\mathfrak{B}$ is a normal $\mathbf{N}_0$-collection of $\overline{Y}$-sets and $\mathfrak{A} \in \mathcal{M}(\mathfrak{B})$ is a submaximal collection dominating it. Then $\mathfrak{A}$ is of either one of the following types.*

(i) *The collections $\langle I\rangle$ where $I := \mathbf{N}_0 \setminus \{\eta\}$ for some $\eta \in \mathbf{N}$.*
(ii) *The collections $\langle I_1, I_2\rangle$ where $I_i := \{\eta;\ \eta\cap A_i \neq \emptyset\}$ for a two-block partition $R = \{A_1, A_2\}$ of N.*

Moreover, if a collection of the second type lies over $\mathfrak{B}$, then $\mathfrak{B}$ contains the set $J_0 := \{\eta;\ A_i \cap \eta \neq \emptyset \text{ for } i = 1, 2\}$.

Since every collection of the first type is uniquely determined by the element η, we write it as $\mathfrak{A}_\eta$. The ray ϱ_η of Σ corresponding to this submaximal collection is generated by v_η.

If a submaximal collection is of the second type, then it is characterised by the partition R of N; for it we write $\mathfrak{A}_R$. Moreover, the associated ray arises as intersection of $\sigma_{\mathbf{N}_0\setminus I_1}$ and $\sigma_{\mathbf{N}_0\setminus I_2}$. We now have to discriminate two cases. If the partition R is of the form $[i] := \{\{i\}, N\setminus\{i\}\}$, then the corresponding ray $\varrho_{[i]} = \varrho_{0i}$ is generated by v_{0i}. Otherwise, if R is a true two-block partition, by [**11**, Proposition 4.1] we know that this ray is precisely ν_R, which was defined at the beginning of Step 2.

Proof of Proposition 6.5.11. Consider an $I \in \mathfrak{A}$ such that ω_I is full dimensional. We now discrimnate two cases. For the first case assume that $\omega_I = \Omega$ holds. Since $\mathfrak{A}$ is submaximal, $I = \mathbf{N}_0 \setminus \{\eta\}$ for some $\eta \in \mathbf{N}_0$. If we had $0 \in \eta$, then ω_I would be a proper subset of Ω.

We turn to the second case where $\omega_I \subsetneq \Omega$ holds. Then there exists an $I' \in \mathfrak{A}$ such that $\omega_{I'}$ is a facet of ω_I and $\omega_{I'}^\circ \cap \Omega^\circ$ is non-empty. Since $\mathfrak{B}$ cannot be enlarged as $\mathbf{N}_0$-collection, there moreover exist $J, J' \in \mathfrak{B}$ such that

$$\omega_J^\circ \subseteq \omega_I^\circ, \qquad \omega_{J'} \text{ is a facet of } \omega_J, \qquad \omega_{I'}^\circ \cap \omega_{J'}^\circ \neq \emptyset.$$

Now $\omega_{J'}$ is a subset of one of the walls of $\Lambda_H(\overline{Y})$. Thus, from Theorem 6.2.4 we know that there exists some partition $\{A_1, A_2\}$ of N such that J' is a subset of $J_0 := \{\eta;\ A_i \cap \eta \neq \emptyset \text{ for } i = 1, 2\}$. We now claim that J' equals J_0.

For this let $i_1 \in A_1, i_2 \in A_2$ be two indices. Since $\omega_{J'}$ is of dimension $n-1$, there exist $i_1' \in A_1,\ i_2' \in A_2$ such that $\{i_1, i_2'\}$ and $\{i_2, i_1'\}$ lie in J'. From the inclusion $J' \subseteq J_0$ we know that $\{i_1, i_1'\}$ does not lie in J', hence from the characterisation of $\overline{Y}$-sets in Proposition 6.2.5 it follows that $\{i_1, i_2\}$ lies in J'. This proves our claim.

Now let $\mathfrak{A}'$ be the normal R-collection consisting of all faces which are invariantly separable from

$$\{\eta;\ \eta \cap A_1 \neq \emptyset\} \qquad \text{and} \qquad \{\eta;\ \eta \cap A_2 \neq \emptyset\}.$$

Then $\mathfrak{A}'$ is submaximal and the assertion follows if we show that $\mathfrak{A} \le \mathfrak{A}'$ holds. For this note that $\omega_{J'}$ is the intersection of Ω with the zero set of

$$l := \sum_{i \in A_1} e_i^* - \sum_{i \in A_2} e_i^*.$$

Since the collection $\{\omega_K;\ K \in \mathfrak{A}\}$ forms a fan with support Ω, for every cone $\omega_K,\ K \in \mathfrak{A}$ we have $l_{|\omega_K} \ge 0$ or $l_{|\omega_K} \le 0$. This implies that $\mathfrak{A} \le \mathfrak{A}'$ holds.

□

Recall that we want show that the (maximal) cones of Σ^Δ are geometrically nested in the sense of Section 6.3 and hence lie in Σ_r. The relevant property of the corresponding $\mathbf{N}_0$-collections shall be discusses in the sequel.

Let $R = \{A_1, A_2\}$ and $S = \{B_1, B_2\}$ be two-block partitions of N and $\eta \in \mathbf{N}_0$. We then call the pair $\{\eta, R\}$ *compatible* if η lies in A_1 or in A_2. Moreover, we call $\{R, S\}$ *compatible*, if there exist $i, j \in \{1, 2\}$ such that $A_i \subseteq B_j$ holds. The pairs of submaximal collections $\{\mathfrak{A}_\eta, \mathfrak{A}_R\}$ and $\{\mathfrak{A}_R, \mathfrak{A}_S\}$ are *compatible*, if the corresponding pairs $\{\eta, R\}$ and $\{R, S\}$ are compatible.

Proposition 6.5.12. *Let $\mathfrak{B}$ be a normal $\mathbf{N}_0$-collection of $\overline{Y}$-sets. Then the submaximal collections in $\mathcal{M}(\mathfrak{B})$ are pairwise compatible.*

Proof. Let $\mathfrak{A}_\eta, \mathfrak{A}_R \ge \mathfrak{B}$ be two submaximal collections with $R = \{A_1, A_2\}$. Then $\{i, j\} := \eta$ is contained in no $I \in \mathfrak{B}$. However, the cones $\omega_I,\ I \in \mathfrak{B}$ cover Ω. Since $w_{ij} = w_{0i} + w_{0j}$ is the only positive linear combination of w_{ij}, the sets $\{0, i\},\ \{0, j\}$ must lie in a common $I \in \mathfrak{B}$. From the characterisation in Proposition 6.5.11(ii) we can now infer that without loss of generality $i, j \in A_1$ holds and this implies compatibility of η with R.

Suppose we have $\mathfrak{A}_R, \mathfrak{A}_S \geq \mathfrak{B}$ with $R = \{A_1, A_2\}$ and $S = \{B_1, B_2\}$. From Proposition 6.5.11 we infer that the set $J_0 = \{\eta;\ \eta \cap A_i \neq \emptyset \text{ for } i = 1, 2\}$ lies in $\mathfrak{B}$. This means that J_0 lies in one of the maximal sets of $\mathfrak{A}_S$. In other words, there exists j such that

$$\eta \cap A_1 \neq \emptyset \quad \text{and} \quad \eta \cap A_2 \neq \emptyset \qquad \Longrightarrow \qquad \eta \cap B_j \neq \emptyset.$$

This implies that there exists i such that $A_i \subseteq B_j$ holds and hence $\{R, S\}$ is compatible. □

The final thing we show is that the cones defined by compatible submaximal collections are geometrically nested in the sense of Section 6.3. For this we define $\mathbf{S}$ as the collection of two-block partitions of N and set $\mathbf{S}_{\geq 2}$ as the subcollection of true two-block partitions, i.e. the partitions $\{A_1, A_2\}$ with $|A_1|, |A_2| \geq 2$.

We set $\mathcal{V} = \{\varrho_\eta;\ \eta \in \mathbf{N}_0\}$ as the set of rays of Σ_1. Keep in mind that the rays ϱ_{0i} stem from the partitions $[i] = \{\{i\}, N \setminus \{i\}\}$, hence we have $\varrho_{0i} = \varrho_{[i]}$.

Moreover, we define $\mathcal{S} := \{\sigma_R;\ R \in \mathbf{S}_{\geq 2}\}$ as the collection of cones in Σ_1 associated to true two-block partitions. This is precisely the collection of cones containing the rays ν_R in their relative interiors.

Lemma 6.5.13. *Consider the collection of cones*

$$\mathcal{C} := \{\varrho_\eta, \sigma_R;\ \mathfrak{A}_\eta, \mathfrak{A}_R \in \mathcal{N}\} \qquad \textit{for some} \qquad \mathcal{N} \subseteq \{\mathfrak{A}_\eta, \mathfrak{A}_R;\ \eta \in \mathbf{N}, R \in \mathbf{S}\}.$$

If any pair in $\mathcal{N}$ is compatible, then $\mathcal{C}$ is geometrically nested in $\mathcal{V} \cup \mathcal{S}$.

Proof. Consider a subset $\mathcal{H} \subseteq \mathcal{C}$ of imcomparable elements with $|\mathcal{H}| \geq 2$. Moreover, take $\mathcal{S}' \subseteq \mathcal{S}$ to be a non-empty conjunct subset. Assuming that

$$\sigma := \sum_{\tau \in S'} \tau \ = \ \sum_{\tau \in \mathcal{H}} \tau \in \Sigma_1$$

holds we have to show that there exist an incompatible pair in $\mathcal{N}$.

Recall that the cones of $\mathcal{S} = \{\sigma_R;\ R \in \mathbf{S}_{\geq 2}\}$ have the form

$$\sigma_R = \operatorname{cone}(v_\eta;\ \eta \subseteq \{0\} \cup A_R) \qquad \text{where} \qquad 1 \notin A_R \in R.$$

Consider two cones $\sigma_1, \sigma_2 \in \Sigma_1$ such that their sum lies in Σ_1 as well. Since Σ_1 is simplicial, the rays of $\sigma_1 + \sigma_2$ are precisely given by the union of the rays of σ_1 and σ_2. In particular, if ϱ is a ray of some $\tau \in \mathcal{H}$, then there exists $\tau' \in \mathcal{S}'$ such that ϱ is a ray of τ'. Clearly, the same is true with $\mathcal{H}$ and $\mathcal{S}'$ exchanged.

If $|\mathcal{H} \cap \mathcal{S}| = 0$ holds, i.e. $\mathcal{H}$ is a subset of $\mathcal{V}$, then one easily sees that there exist $\varrho_{[i]}$, $\varrho_{ij} \in \mathcal{H}$. Clearly, $[i]$ and $\{i, j\}$ are incompatible, hence $\mathfrak{A}_{[i]}$, $\mathfrak{A}_{\{i,j\}} \in \mathcal{N}$ are the incompatible partitions.

We consider the case $|\mathcal{H} \cap \mathcal{S}| = 1$ and denote the single cone in $\mathcal{H} \cap \mathcal{S}$ by σ_R. Since $|\mathcal{H}| \geq 2$ holds there exists an element $\varrho \in \mathcal{H} \cap \mathcal{V}$. We distinguish two subcases.

In the first case let this ray be of the form $\varrho = \varrho_{[i]}$. Then we find $\sigma_S \in \mathcal{S}'$ with $\varrho \preccurlyeq \sigma_S$. From the special form of the cone σ_R we know that there also exists $j \in N$ with $\varrho_{ij} \preccurlyeq \sigma_S$. By the assumption made on $\mathcal{H}$ we have $\varrho_{[i]} \not\preccurlyeq \sigma_R$; and the special form of σ_R then means that also $\varrho_{ij} \not\preccurlyeq \sigma_R$ holds. Hence ϱ_{ij} lies in $\mathcal{H}$ and $\mathfrak{A}_{[i]}$, $\mathfrak{A}_{\{i,j\}} \in \mathcal{N}$ are the incompatible collections.

In the second case where $\varrho = \varrho_{ij}$ holds we again find $\sigma_S \in \mathcal{S}'$ with $\varrho_{ij} \preccurlyeq \sigma_S$. From the special form of the cone σ_S we know that both $\varrho_{[i]}$ and $\varrho_{[j]}$ are rays of σ_S. Since $\varrho_{ij} \not\preccurlyeq \sigma_R$ holds, at least one of the rays $\varrho_{[i]}$, $\varrho_{[j]}$ is not a ray of σ_R. Without loss of generality this implies that again $\varrho_{[i]}$ lies in $\mathcal{H}$ and $\mathfrak{A}_{[i]}$, $\mathfrak{A}_{\{i,j\}} \in \mathcal{N}$ are the incompatible collections.

Now we assume that $|\mathcal{H} \cap \mathcal{S}| \geq 2$ holds. Then there exist σ_R, $\sigma_S \in \mathcal{H} \cap \mathcal{S}$. For $\eta, \zeta \in \mathbf{N}$ let $\varrho_\eta \preccurlyeq \sigma_R$ and $\varrho_\zeta \preccurlyeq \sigma_S$ be rays such that $\varrho_\eta \not\preccurlyeq \sigma_S$ and $\varrho_\zeta \not\preccurlyeq \sigma_R$ hold. Since $\mathcal{S}'$ is conjunct, we find $\xi_1, \ldots, \xi_r \in \mathbf{N}$ with

$$\xi_1 = \eta, \qquad \xi_r = \zeta, \qquad \varrho_{\xi_i} \preccurlyeq \sigma \qquad \text{and} \qquad \xi_i \cap \xi_{i+1} \neq \emptyset.$$

Let i' be the smallest index, for which $\varrho_{\xi_{i'}}$ is not a ray of σ_R. If $\varrho_{\xi_{i'}}$ lies in $\mathcal{H}$, then we know $\varrho_{\xi_{i'}} \not\preccurlyeq \sigma_R$ holds. This means that $\xi_{i'}$ and R are incompatible. If $\varrho_{\xi_{i'}}$ does not lie in $\mathcal{H}$, then there exists an $\sigma_{R'} \in \mathcal{H}$ such that $\varrho_{\xi_{i'}}$ is a ray of $\sigma_{R'}$. This implies that R and R' are incompatible. □

Proof of Proposition 6.5.6. Consider the cone $\sigma \in \Sigma^\Delta$. In order to show show that σ lies in Σ_r we can without loss of generality assume that σ is maximal. Let $\mathfrak{B}$ be the associated normal $\mathbf{N}_0$-collection with

$$\sigma \;=\; \bigcap_{I \in \mathfrak{B}} \sigma_{\mathbf{N}_0 \setminus I}.$$

By Propositions 6.5.10, 6.5.12 we know that $\mathcal{M}(\mathfrak{B})$ is a set of compatible normal $\mathbf{N}_0$-collections. Furthermore, by Proposition 6.5.9

$$\sigma \;=\; \operatorname{cone}(\nu_{\mathfrak{A}};\; \mathfrak{A} \in \mathcal{M}(\mathfrak{B})) \;=\; \operatorname{cone}(\varrho,\, \nu_R;\; \varrho \in \mathcal{V} \cap \mathcal{C},\; \sigma_R \in \mathcal{S} \cap \mathcal{C})$$

holds. Lemma 6.5.13 shows that $\mathcal{C} = \{\varrho_\eta,\, \sigma_R;\; \mathfrak{A}_\eta, \mathfrak{A}_R \in \mathcal{M}(\mathfrak{B})\}$ is geometrically nested in $\mathcal{V} \cup \mathcal{S}$. And finally, from Proposition 6.3.1 we infer that σ lies in Σ_r. □

Proof of Theorem 6.5.1. The Theorem now follows directly from Proposition 6.5.3 and Corollary 6.5.7. □

Proof of Theorem 6.5.2. As in the reductive case we performed the first blow-up in Proposition 6.5.3. In Proposition 6.5.4 determined the subset of X_1 that has to be removed due to the fact that the morphism $\kappa\colon \mathbb{K}^{2n} \to \bigwedge^2 \mathbb{K}^{n+1}$ is not surjective. Finally the remaining blow-ups are performed as in the reductive case, see Corollary 6.5.7. $\square$

INDEX

γ-collection, 18
$\mathcal{V}$-cone, 17
$\mathcal{V}$-fan, 17
$\mathfrak{F}$-bunch, 15
  true, 15
$\mathfrak{F}$-face, 15, 76

algebra
  sheaf of divisorial, 9
almost free, 15
ambient modification, 43
ample cone, 22

blow-up, 48, 71, 80, 81, 83, 86
building set, 82–84
bunched ring, 14, 15

canonical embedding, 16
character, 11
characteristic space, 12
Chow form, 28
Chow point, 28
Chow quotient, 29, 30, 32, 58
Chow variety, 28
coarsest common refinement, 17
collection, 94
  γ, 18, 60
  enveloping, 16
  normal, 18, 60, 94
  separated, 94
  submaximal, 18, 60, 95, 96
compatible, 96, 97
complete, 13
cone, 12
  lattice, 12
conjunct, 81
contraction, 44, 51
Cox construction, 13, 43
Cox ring, 10, 34, 37, 48, 49, 51
cox ring, 48

degree, 28
diagonal, 48
distributive, 84
divisor
  ample, 22
  Cartier, 9
  class group, 9
  effective, 9
  principal, 9, 21
  Weil, 9

enveloped quotient, 33
enveloping collection, 16
enveloping quotient, 34
extension, 44, 51

face, 12
facet, 12
factorially graded, 10, 22, 43, 44
fan, 13
  GIT, 75
  lattice, 13

Gale dual, 18
GIT-fan, 21, 75
  ample, 22
GIT-limit, 24, 33, 35
GIT-system, 24
GKZ-decomposition, 17, 36, 59
good G-set, 20
good quotient, 11
graded
  ideal, 14
Grassmannian, 28, 48, 57, 73, 75, 89
group
  affine-algebraic, 11
  character, 34
  divisor class, 9
  free, 11
  non-reductive, 34
  Picard, 9
  reductive, 11

harmonious, 84
Hilbert's Finiteness Theorem, 11, 12, 41
Hilbert's fourteenth problem, 40
hyperplane, 59

invariant algebra, 73
invariants, 11
inverse limit, 23
inverse system, 23, 34

join, 81

lattice cone, 12
lattice fan, 13
limit quotient, 25, 30, 32, 35, 36, 89
linear extension, 89
linearisation, 19–21, 40

meet, 81
Mori Dream Space, 10, 15, 22, 37, 43, 57

Nakayama Lemma, 55
neat, 16
nested, 82, 86, 96, 97
  geometrically, 81
Newton polytope, 17
non-reductive, 33
normal collection, 18

orbit cone, 21

partition, 96
  true, 95, 97
  two-block, 95, 97
Picard group, 9
Plücker relations, 54
point configuration, 71
pointed, 12
prime, 45, 51
proper transform, 50, 51, 87, 94

qp-maximal, 20, 22
quadric, 58
quasicoherent, 14
quasitorus, 11
quotient
  categorical, 35
  Chow, 29, 30, 32, 58
  enveloped, 33
  enveloping, 34
  good, 11, 37
  limit, 30, 32, 35, 36, 89
quotient fan, 26, 29, 30

relative spectrum, 11
relevant faces, 15
resolution
  weakly tropical, 65
ring
  Cox, 10

saturated, 20
semilattice, 81
semistable, 20, 22
  finitely generated, 33
sheaf, 49
  Cox, 10, 38
  direct image, 86
  inverse image, 86
  invertible, 86
  of divisor $\mathcal{O}_X$-algebras, 38
  of divisorial algebras, 9
  of ideals, 86
  quasicoherent, 14
star, 80
stellar subdivision, 80, 83, 88
strict transform, 50, 51, 87
submaximal, 18
support, 13

toric variety, 12
torus, 11
total coordinate space, 11
tropical variety, 17

universal property
  inverse limit, 23
  limit quotient, 26
  of blow-ups, 86
  of good quotients, 12

variety
  Chow, 28
  toric, 12
  tropical, 17, 93

weak lifting, 61
weakly tropical, 64
weakly tropical resolution, 65
weight cone, 21

BIBLIOGRAPHY

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