Fano Varieties with Torus Action of Complexity One

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Introduction

The subject of the present thesis are varieties with a torus action of complexity one, i.e. algebraic varieties X with an algebraic torus T acting effectively on them, where $\dim(T) = \dim(X) - 1$. These varieties are shortly called complexity-one T-varieties. A first aim is to provide a combinatorial description for such varieties which generalizes the convex geometrical description of toric varieties by lattice fans. The main focus lies on the application of this theory to classification problems on complexity-one T-varieties, where special attention is paid to Fano varieties, i.e. projective varieties with ample anticanonical divisor.

In algebraic geometry, toric varieties are a well known example for the use of combinatorial methods. The first formal definition of a (smooth) toric variety was given in 1970 by Demazure [18]. This paper already includes a convex geometrical description of toric varieties by fans. From the end of the seventies on, the theory of toric varieties expanded rapidly, see for example Danilov [17], Oda [42, 43], Fulton [22] and Cox/Little/Schenk [16]. Combinatorial methods were also successfully developed for larger classes of varieties. Kempf, Knudsen, Mumford and Saint-Donat studied in [36] toroidal varieties and extended the convex geometrical language to this more general setup; in this book complexity-one T-varieties appeared as special cases. This was the first time they were described by combinatorial data. Besides [36], the work [45] of Orlik and Wagreich is one of the first publications about complexity-one T-varieties. They discussed the special case of $\mathbb{K}^*$-surfaces and developed a combinatorial description of their structure by weighted graphs. More recently, Altmann and Hausen studied in [3] varieties with torus action by polyhedral divisors, which give especially in case of complexity-one T-varieties a simple description of these varieties. The approach of the present thesis relies on the Cox ring. Hausen and Süß determined the Cox ring of a given rational normal complete variety with a torus action of complexity one in terms of the action, see [29]. Such Cox rings are finitely generated and admit a simple presentation by trinomial relations. This provides new aspects and possibilities for a combinatorial approach to complexity-one T-varieties and is the starting point of this thesis. We present a systematical description of rational complexity-one T-varieties in terms of certain matrices A and P and a collection Φ of polyhedral cones. Parts of these results have been published in [28, Section 1] and [27].

Motivated by the classification of toric Fano varieties, initiated by Batyrev [8], we apply our combinatorial approach to Fano varieties with a torus action of complexity one. The main focus lies on effective bounds and concrete classifications.

A first example class are Fano $\mathbb{K}^*$-surfaces, so called del Pezzo $\mathbb{K}^*$-surfaces. Our combinatorial approach differs from the work of Alekseev/Nikulin [2] and Nakayama [40] based on classical surface geometry. For Gorenstein log del Pezzo $\mathbb{K}^*$-surfaces we obtain a complete classification, see Theorems 5.25, 5.26, 5.27 and 5.28. Moreover, due to the methods, we list the Cox rings of all these surfaces. This complements results of Derenthal [19] for the case of hypersurface Cox rings and Hausen/Süß [51, 29], who settled the case of Picard number one and two by other methods.

An essential case are Fano varieties of Picard number one. Nadel provided in [39] a general effective bound on the degree $(-K_x)^n$ of smooth Fano varieties depending on the dimension n. Once a degree bound is known, a result of Kóllar [37] gives effective bounds for the number of different deformation types of smooth Fano varieties. In [35] Kasprzyk studies toric varieties with Picard number one, so called (fake) weighted projective spaces and provided boundedness conditions for the weights in the terminal and canonical case.

For $\mathbb{Q}$-factorial complexity-one T-varieties of Picard number one, we obtain in Theorem 6.10 explicit bounds for the number of possible deformation types depending on the dimension and the Picard index, which is the index of the Picard group in the divisor class group. As a consequence, Theorem 6.12 provides the following results on the asymptotical behavior of the number $\delta(d,\mu)$ of different deformation types of $\mathbb{Q}$-factorial d-dimensional complexity-one Fano varieties with Picard number one and Picard index μ. For fixed $d_0 \in \mathbb{Z}_{>0}$ and fixed $\mu_0 \in \mathbb{Z}_{>0}$, we have

$$\delta(d_0,\mu) \ \sim \ \mu^{A\mu^2} \quad \text{and} \quad \delta(d,\mu_0) \ \sim \ d^{Bd},$$

with constants $A > 1$ and $B > 3$ arbitrarily small. The explicit bounds are used to produce classifications for fixed dimension and Picard index. In the Theorems 6.18, 6.23, 6.24 and 6.26 we exemplarily list surfaces up to Picard index 6, threefolds for Picard index 1 and 2 and fourfolds with Picard index 1. In all cases we list the Cox rings explicitly. These results are published in [28, Sections 2 and 3] and [30].

In 1970 Demazure studied the automorphism group of smooth complete toric varieties and described the roots in terms of fans, see [18]. Later Cox generalized the results in [15] to the simplicial case. In [41] Nill provided effective combinatorial criteria for the automorphism group of a complete toric variety to be reductive. More recently, Arzhantsev, Hausen, Liendo and myself described the automorphism group of a complexity-one T-variety by combinatorial data, see [6]. We use this approach for the study of almost homogeneous complexity-one T-varieties, i.e. their automorphism group acts with an open orbit. In Proposition 7.7 almost homogeneous $\mathbb{K}^*$-surfaces are described explicitly. As a consequence, we classify in Corollary 7.12 all log-terminal non-toric almost homogeneous $\mathbb{K}^*$-surfaces with exactly one singularity and Picard number one up to Gorenstein index 5. It turns out, that all of them are Fano. These results are published in [6, Sections 6]. In Theorem 7.22 we determine all almost homogeneous complexity-one T-varieties of dimension three with Picard number one and reductive automorphism group. All these varieties turn out to be Fano. These results are published in [6, Section 8].

The present thesis is divided into seven chapters. We now give a brief summary of each chapter.

The first chapter is a short summary of basic notations and statements about Cox rings and bunched rings given in [9] and [25], see also [5] and [26]. Each bunched ring is the Cox ring of a $\mathbb{Q}$-factorial normal variety that can be obtained by a standard construction as a good quotient of an open subset of the spectrum of this ring. A short overview of the geometrical properties of such varieties in terms of their Cox ring and their convex geometrical meaning is given.

Chapter 2 is dedicated to complexity-one T-varieties, i.e. algebraic varieties X with an effective action of an algebraic torus T of dimension $\dim(X) - 1$, and their Cox rings which are factorially graded rings of complexity one. We describe factorially graded rings in terms of generators and relations and determine the Cox rings among them. The combinatorial language of P-matrices will be introduced. It is comparable to the convex geometrical description of toric varieties by fans. Parts of this chapter are already published in [27] and [28, Section 1].

Chapter 3 is dedicated to the resolution of singularities of complexity-one T-varieties. A canonical resolution of singularities is discussed. We make use of the fact that complexity-one T-varieties come canonically embedded into toric varieties. This allows working with toric ambient modifications, see [25]. A similar construction based on polyhedral divisors was introduced by Liendo and Süß in [38]. The behavior of the anticanonical class $-K_X$ of a complexity-one T-variety under toric ambient modifications is discussed.

In chapter 4 we discuss complexity-one T-varieties of dimension two, in other words $\mathbb{K}^*$-surfaces. We give a survey of their geometry and determine all types of Cox rings of combinatorially minimal $\mathbb{K}^*$-surfaces, i.e. $\mathbb{K}^*$-surfaces without contractible prime divisors. Furthermore, we compute intersection numbers and affiliate conditions for $\mathbb{K}^*$-surfaces to be Fano. Finally, we introduce the anticanonical complex for log-terminal $\mathbb{K}^*$-surfaces, a convex geometrical object which is comparable to the lattice polytope describing toric Fano varieties. The anticanonical complex is used to describe the singularities and the Gorenstein index of log-terminal Fano $\mathbb{K}^*$-surfaces.

Chapter 5 is dedicated to log del Pezzo $\mathbb{K}^*$-surfaces, i.e. log-terminal Fano $\mathbb{K}^*$-surfaces. The main result is a complete classification list of all non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surfaces. In order to achieve this aim we describe the Gorenstein index of a $\mathbb{K}^*$-surface combinatorially in terms of their P-matrix and anticanonical complex and consider the equivariant geometry of del Pezzo $\mathbb{K}^*$-surfaces. As a consequence, we obtain explicit bounds, which enables us to classify all non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surfaces by indicating their Cox rings and $\mathrm{Cl}(X)$-gradings.

In chapter 6 we provide effective bounds and classification results for rational $\mathbb{Q}$-factorial Fano varieties with a complexity-one torus action and Picard number one depending on the invariants dimension and Picard index. Concretely, we list all surfaces up to Picard index 6, threefolds for Picard index 1 and 2 and fourfolds with Picard index 1. Most of the results of this chapter is already published in [28] and [30].

Chapter 7 is dedicated to classification problems on almost homogeneous complexity-one T-varieties, i.e. their automorphism group $\mathrm{Aut}(X)$ acts with an open orbit. By introducing Demazure P-roots we obtain a combinatorial approach for the automorphism group of such varieties, describing the roots of $\mathrm{Aut}(X)$. The Demazure P-roots turn out to be lattice points of certain polytopes. This convex geometrical description is used for classification problems on almost homogeneous complexity-one T-varieties of dimension two and three. Concretely, we provide a complete list of all log-terminal non-toric almost homogeneous $\mathbb{K}^*$-surfaces with exactly one singularity and Picard number one up to

Gorenstein index 5. Furthermore, we determine almost homogeneous complexity-one threefolds with Picard number one and reductive automorphism group. These results are published in [6].

1 Cox rings and bunched ring formalism

This chapter is a short summary of basic notations and statements about Cox rings and bunched rings given in [9] and [25], see also [5] and [26].

Throughout the whole thesis $\mathbb{K}$ is an algebraically closed field of characteristic zero.

1.1 Cox rings and factorially graded rings

First, we recall basic definitions and notions on divisors of normal algebraic varieties and divisorial sheaves. A *prime divisor* in X is an irreducible hypersurface $D \subseteq X$. The prime divisors generate a free abelian group $\mathrm{WDiv}(X)$, the *group of Weil divisors* on X. We call a divisor $D \in \mathrm{WDiv}(X)$ *effective* if $D = a_1D_1 + \ldots + a_nD_n$ with prime divisors D_i and $a_i \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq n$ and we write $D \geq 0$. Let $\mathbb{K}(X)$ be the field of rational functions on X. To every $f \in \mathbb{K}(X)^*$ we define a *principal divisor*

$$\mathrm{div}(f) \ := \ \sum_{D \text{ prime}} \mathrm{ord}_D(f)D \ \in \ \mathrm{WDiv}(X),$$

where $\mathrm{ord}_D(f)$ is the vanishing order of f at D. The *group of principal divisors* is denoted by $\mathrm{PDiv}(X)$ and the group of divisors being principal near a point $x \in X$ is denoted by $\mathrm{PDiv}(X, x)$. The *divisor class group* $\mathrm{Cl}(X)$ is defined as the quotient of $\mathrm{WDiv}(X)$ modulo principal divisors $\mathrm{PDiv}(X)$. Analogously, we define the *local divisor class group* $\mathrm{Cl}(X, x)$ of X in x as the quotient of $\mathrm{WDiv}(X)$ modulo $\mathrm{PDiv}(X, x)$. Note that there is a canonical map $\pi_x\colon \mathrm{Cl}(X) \to \mathrm{Cl}(X, x)$. Divisors being locally principal for every $x \in X$ are called *Cartier divisors* and the group of these divisors is denoted by $\mathrm{CDiv}(X)$. The *Picard group* $\mathrm{Pic}(X)$ of X is the quotient of $\mathrm{CDiv}(X)$ modulo principal divisors. It is given as

$$\mathrm{Pic}(X) \ := \ \bigcap_{x \in X} \ker(\pi_x) \ \subseteq \ \mathrm{Cl}(X).$$

A point $x \in X$ is called $\mathbb{Q}$-*factorial*, if near x for every Weil divisor some multiple is principal and $x \in X$ is called *factorial*, if near x every Weil divisor is principal. The variety X is called ($\mathbb{Q}$-)factorial if all points $x \in X$ are ($\mathbb{Q}$-)factorial. For any open subset $U \subseteq X$, we define a restriction map

$$\mathrm{WDiv}(X) \to \mathrm{WDiv}(U), \ D \mapsto D_{|U} \ := \ \begin{cases} D \cap U & \text{if } D \cap U \neq \emptyset \, , \\ 0 & \text{else.} \end{cases}$$

To every Weil divisor D on X we associate a divisorial sheaf $\mathcal{O}_X(D)$ of $\mathcal{O}_X$-modules. For any open $U \subseteq X$, we set

$$\Gamma(U, \mathcal{O}_X(D)) \ := \ \left\{ f \in \mathbb{K}(X)^*; \ (\mathrm{div}(f) + D)_{|U} \geq 0 \right\} \ \cup \ \{0\}.$$

The *sheaf of divisorial algebras* associated to a finitely generated subgroup $K \subseteq \mathrm{WDiv}(X)$ is defined as

$$\mathcal{S} := \bigoplus_{D\in K} \mathcal{S}_D, \qquad \mathcal{S}_D := \mathcal{O}_X(D).$$

The multiplication in $\mathcal{S}$ is given by multiplying homogeneous sections in the field of rational functions $\mathbb{K}(X)$. Note that for $f_1 \in \Gamma(U, \mathcal{O}_X(D_1))$ and $f_2 \in \Gamma(U, \mathcal{O}_X(D_2))$ we have $f_1 \cdot f_2 \in \Gamma(U, \mathcal{O}_X(D_1 + D_2))$.

Now, we turn to Cox sheaves and Cox rings. We assume X to be a normal variety with only constant globally invertible functions, i.e. $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$, and finitely generated divisor class group $\mathrm{Cl}(X)$. Note that $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ is satisfied, for example, if X is complete. Choose a subgroup $K \leq \mathrm{WDiv}(X)$ such that the canonical map $c\colon K \to \mathrm{Cl}(X),\ D \mapsto [D]$ is surjective, and let $K^0 \subseteq K$ be the kernel of c, i.e. $K^0 = K \cap \mathrm{PDiv}(X)$. The idea is to identify $\mathcal{O}_X(D)$ and $\mathcal{O}_X(D')$ if $D' = D + \mathrm{div}(h)$, i.e. $D' - D \in K^0$. For this purpose we choose a character $\chi\colon K^0 \to \mathbb{K}(X)^*$ with $\mathrm{div}(\chi(E)) = E$ for all $E \in K^0$. We consider the sheaf of divisorial algebras $\mathcal{S}$ associated to K. Let $\mathcal{I}$ be the sheaf of ideals of $\mathcal{S}$ locally generated by the sections $1 - \chi(E)$, where 1 is locally of degree zero, E runs through K^0 and $\chi(E)$ is homogeneous of degree $-E$, i.e. for every open $U \subseteq X$ we have $1 - \chi(E) \in \Gamma(U, \mathcal{S})$ and $\chi(E) \in \Gamma(U, \mathcal{S}_{-E})$. This sheaf of ideals is given by

$$\Gamma(U, \mathcal{I}) := \left\{ f \in \Gamma(U, \mathcal{S});\ f = \sum_{E\in K^0} h_E(1 - \chi(E)) \text{ locally, where } h_E \in \Gamma(U, \mathcal{S}) \right\}.$$

Definition 1.1. The Cox sheaf associated to K and χ is the quotient sheaf $\mathcal{R} := \mathcal{S}/\mathcal{I}$ together with the $\mathrm{Cl}(X)$-grading

$$\mathcal{R} := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{R}_{[D]}, \qquad \mathcal{R}_{[D]} := \pi\left(\bigoplus_{D'\in c^{-1}([D])} \mathcal{S}_{D'} \right),$$

where $\pi\colon \mathcal{S} \to \mathcal{R}$ denotes the projection. Note that $\mathcal{R}$ is a quasicoherent sheaf of $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebras. The *Cox ring*

$$R(X) := \Gamma(X, \mathcal{R}) = \bigoplus_{[D]\in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D))$$

is the ring of global sections of the Cox sheaf $\mathcal{R}$. Note that if $\mathrm{Cl}(X)$ is torsion free, then the Cox sheaf can be defined in a simpler way by setting $\mathcal{R}_{[D]} := S_D = \mathcal{O}_X(D)$. If the Cox ring $R(X)$ is finitely generated, the variety X is called a *Mori Dream Space (MDS).*

The next step is to recall the relation between quasitorus actions and graded algebras. A *quasitorus*, also called a *diagonalizable group*, is an affine algebraic group H whose algebra of regular functions $\Gamma(H, \mathcal{O})$ is generated as a $\mathbb{K}$-vector space by the characters $\chi \in \mathbb{X}(H)$, where a *character* of H is a morphism $\chi\colon H \to \mathbb{K}^*$. A *torus* is a connected quasitorus. We denote by $\mathbb{T}^n := (\mathbb{K}^*)^n$ the *standard n-torus*. Each quasitorus is isomorphic to a direct product of a torus and a finite abelian group. There is a one-to-one correspondence between quasitori and finitely generated abelian groups given by $H \mapsto \mathbb{X}(H)$ and $K \mapsto \mathrm{Spec}(\mathbb{K}[K])$, respectively. Furthermore, there is a contravariant functor being essentially inverse from the category of finitely generated K-graded affine algebras to the category of affine varieties with a quasitorus action. We shortly recall the basic constructions needed for this correspondence. Let H be a quasitorus acting on an affine variety X. Then the algebra $\Gamma(X, \mathcal{O})$ becomes $\mathbb{X}(H)$-graded by

$$\Gamma(X,\mathcal{O}) \;=\; \bigoplus_{\chi \in \mathbb{X}(H)} \Gamma(X,\mathcal{O})_\chi, \quad \Gamma(X,\mathcal{O})_\chi \;:=\; \{f \in \Gamma(X,\mathcal{O});\; f(h \cdot x) = \chi(h) f(x)\}.$$

Conversely, let K be a finitely generated abelian group and let R be a finitely generated K-graded affine algebra. Set $\overline{X} := \mathrm{Spec}(R)$ and let $f_1, \ldots, f_r$ be generators of R with $f_i \in R_{w_i}$, i.e. $\deg(f_i) = w_i$ and $f_i(h \cdot x) = \chi^{w_i}(h) f_i(x)$ for all $h \in H$, $x \in X$. Then we have a closed embedding

$$\overline{X} \to \mathbb{K}^r, \quad x \mapsto (f_1(x), \ldots, f_r(x)),$$

and $\overline{X} \subseteq \mathbb{K}^r$ is invariant under the *diagonal action* of the quasitorus $H = \mathrm{Spec}(\mathbb{K}[K])$ given by the characters χ^{w_i}, i.e.

$$h \cdot x \;:=\; (\chi^{w_1}(h)x_1, \ldots, \chi^{w_n}(h)x_n).$$

Now, let H be a quasitorus acting on a prevariety X. Then one defines the ring of *invariants*

$$\mathcal{O}(X)^H \;:=\; \{f \in \mathcal{O}(X);\; f(h \cdot x) = f(x) \text{ for all } x \in X,\, h \in H\}.$$

A morphism $\pi\colon X \to Y$ is called a *good quotient* for the H-action if the following holds:

(i) $\pi\colon X \to Y$ is affine and H-invariant, i.e. π is constant along the orbits.

(ii) The pullback $\pi^*\colon \mathcal{O}(Y) \to (\pi_* \mathcal{O}(X))^H$ is an isomorphism.

A good quotient $\pi\colon X \to Y$ is called a *geometric quotient* if it separates the orbits, i.e. the fibers coincide exactly with the H-orbits.

Good quotients map closed invariant subsets to closed sets and separate disjoint closed invariant sets. Moreover, in each fiber, there is exactly one closed H-orbit and each orbit which is contained in the fiber has this closed orbit in its closure. In particular, the quotient space is unique up to isomorphy. We denote it by $X /\!\!/ H$.

In the next part of this chapter we consider the geometrical object that corresponds to the algebraic concept of a Cox sheaf. Let X be a $\mathbb{Q}$-factorial normal variety or a Mori

Dream Space with Cox sheaf $\mathcal{R}$. Then the Cox sheaf $\mathcal{R}$ is locally of finite type and we have the following situation:

$$\begin{array}{ccccc} \mathrm{Spec}_X(\mathcal{R}) =: & \widehat{X} & \subseteq & \overline{X} & := \mathrm{Spec}(\mathcal{R}(X)) \\ & \big\downarrow {\scriptstyle /\!\!/ H} & & & \\ & X & & & \end{array}$$

The affine variety $\overline{X}$ comes with the action of the quasitorus $H := \mathrm{Spec}(\mathbb{K}[\mathrm{Cl}(X)])$ which is given by the $\mathrm{Cl}(X)$-grading of $\mathcal{R}$. The relative spectrum $\widehat{X} = \mathrm{Spec}_X(\mathcal{R})$ is an open H-invariant subset of $\overline{X}$, i.e. quasiaffine, and the map $p_X \colon \widehat{X} \to X$ defined by the H-action is a good quotient. Note, that if X is $\mathbb{Q}$-factorial, the quotient is always geometric.

Definition 1.2. In the situation above we call $\overline{X}$ the *total coordinate space*, H the *characteristic quasitorus* and $\widehat{X}$ the *characteristic space*.

All varieties sharing the same divisor class group K and finitely generated Cox ring R occur as good quotients of suitable open subsets of $\overline{X} = \mathrm{Spec}(R)$ by the action of $H = \mathrm{Spec}(\mathbb{K}[K])$. An open invariant subset $U \subseteq X$ is called a *good H-set* if it admits a good quotient $U \to U /\!\!/ H$.

An important concept used in this work is the following homogeneous version of a unique factorization domain.

Definition 1.3. Let K be an abelian group and let R be a finitely generated normal algebra $R = \bigoplus_{w \in K} R_w$. A homogeneous non-zero non-unit $f \in R$ is *K-prime* if $f \mid gh$ with homogeneous $g, h \in R$ implies $f \mid g$ or $f \mid h$. Furthermore, R is called *factorially (K-)graded* if every homogeneous non-zero non-unit of R is a product of K-primes.

For torsion free K, the properties factorial and factorially graded are equivalent [4], but for a K with torsion the latter is more general, as we will see in Example 2.9. The main reason for introducing such rings in this work is that Cox rings of complete normal varieties are factorially graded, compare [5].

Theorem 1.4. *Let X be a complete normal variety. Then its Cox ring $R(X)$ is factorially $\mathrm{Cl}(X)$-graded. Moreover, if $\mathrm{Cl}(X)$ is torsion free, then $R(X)$ is factorial.*

1.2 Bunched rings

The content of this section is a short summary of the theory of bunched rings given in [25]. Bunched rings are essential to combine the algebraic situation described in the previous section with convex geometrical methods.

The setting is as follows. Let K be a finitely generated abelian group and consider a finitely generated factorially K-graded affine $\mathbb{K}$-algebra

$$R = \bigoplus_{w \in K} R_w$$

with K-prime homogeneous generators $f_1, \ldots, f_r \in R$. Then $H = \operatorname{Spec}(\mathbb{K}[K])$ acts on the affine variety $\overline{X} = \operatorname{Spec}(R)$. For each abelian group L we define the rational vector space $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$ and for $w \in L$ we write again w for the element $w \otimes 1 \in L_{\mathbb{Q}}$.

Definition 1.5. The *weight cone* of a K-graded algebra R is the convex polyhedral cone

$$\omega_{\overline{X}} \;:=\; \omega(R) \;:=\; \operatorname{cone}(w \in K; R_w \neq \{0\}) \;\subseteq\; K_{\mathbb{Q}}.$$

Furthermore, for every $x \in \overline{X} := \operatorname{Spec}(R)$ we define its *orbit cone* as the convex polyhedral cone

$$\omega_x \;:=\; \operatorname{cone}(w \in K;\; f(x) \neq 0 \text{ for some } f \in R_w) \;\subseteq\; \omega_{\overline{X}}.$$

In particular, each orbit cone ω_x is generated by the degrees of those generators f_i satisfying $f_i(x) \neq 0$.

Definition 1.6. Let K be a finitely generated abelian group and let R be a factorially K-graded affine algebra with $R^* = \mathbb{K}^*$. Moreover, let $\mathfrak{F} = (f_1, \ldots, f_r)$ be a system of pairwise non-associated K-prime generators for R.

(i) The *projected cone* associated to $\mathfrak{F}$ is $(E \xrightarrow{Q} K, \gamma)$, where $E := \mathbb{Z}^r$, the homomorphism $Q\colon E \to K$ sends the canonical basis vector e_i to $w_i := \deg(f_i)$ and $\gamma \subseteq E_{\mathbb{Q}}$ is the convex cone generated by $e_1, \ldots, e_r$.

(ii) The K-grading of R is called *almost free* if for every facet $\gamma_0 \preceq \gamma$ the image $Q(\gamma_0 \cap E)$ generates the abelian group K.

(iii) We call $\gamma_0 \preceq \gamma$ an $\mathfrak{F}$*-face* if the product over all f_i with $e_i \in \gamma_0$ is not contained in $\sqrt{\langle f_j;\; e_j \notin \gamma_0 \rangle} \subseteq R$. Geometrically, this means that there is an element $x \in \overline{X}$ such that $e_i \in \gamma_0$ if and only if $f_i(x) \neq 0$.

(iv) Let $\Gamma_{\mathfrak{F}} := \{Q(\gamma_0);\; \gamma_0 \preceq \gamma\ \mathfrak{F}\text{-face}\}$ denote the collection of projected $\mathfrak{F}$-faces. An $\mathfrak{F}$*-bunch* is a non-empty subset $\Phi \subseteq \Gamma_{\mathfrak{F}}$ such that

   (a) $\tau_1^\circ \cap \tau_2^\circ \neq \emptyset$ holds for any two $\tau_1, \tau_2 \in \Phi$,

   (b) if $\tau_1^\circ \subseteq \tau^\circ$ holds for $\tau_1 \in \Phi$ and $\tau \in \Gamma_{\mathfrak{F}}$, then $\tau \in \Phi$ holds,

   (c) for every facet $\gamma_0 \in \gamma$, the image $Q(\gamma_0)$ belongs to Φ.

Note that Φ consists of orbit cones of the action of $H = \operatorname{Spec}(\mathbb{K}[K])$ on $\overline{X} = \operatorname{Spec}(R)$. Moreover, the generator system $\mathfrak{F}$ gives rise to a diagonal H-action on $\mathbb{K}^r$ defined by the characters $\chi^{w_1}, \ldots, \chi^{w_r}$. This action induces an H-equivariant closed embedding $\overline{X} = \operatorname{Spec}(R) \subseteq \mathbb{K}^r$. We have $\mathbb{K}[T_1, \ldots, T_r] \cong \mathbb{K}[E \cap \gamma]$ and $\mathbb{K}^r$ is the affine toric variety corresponding to the cone $\delta := \gamma^\vee$.

A *toric variety* is a normal variety Z with a torus action $T \times Z \to Z$ and a base point $z_0 \in Z$ such that the orbit map $t \mapsto t \cdot z_0$ is an open embedding. There is a combinatorial description of toric varieties via fans. A *fan* is a finite collection Λ of pointed polyhedral cones in a rational vector space such that any two $\lambda_1, \lambda_2 \in \Lambda$ intersect in a common face and for $\lambda \in \Lambda$ also every face of λ belongs to Λ. If we do not require the cones to be pointed, then Λ is called a *quasifan*. If Z is an affine toric variety, the corresponding fan simplifies to a convex polyhedral cone. For a detailed background on toric varieties, see [16] for example.

Remark 1.7. For any face $\gamma_0 \preceq \gamma$, we define $\gamma_0^* := \gamma_0^\perp \cap \delta$, where $\delta = \gamma^\vee$. Then γ_0 is an $\mathfrak{F}$-face if and only if the toric orbit $\mathbb{T}^r \cdot z_{\delta_0} \subseteq \mathbb{K}^r$ corresponding to the face $\delta_0 \preceq \delta$ meets $\overline{X}$.

Definition 1.8. A *bunched ring* is a triple $(R, \mathfrak{F}, \Phi)$, where R is an almost freely factorially K-graded affine $\mathbb{K}$-algebra such that $R^* = \mathbb{K}^*$ holds, $\mathfrak{F}$ is a system of pairwise non-associated K-prime generators for R and Φ is an $\mathfrak{F}$-bunch.

Construction 1.9. Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring and let $(E \stackrel{Q}{\to} K, \gamma)$ be its projected cone. We define the *collection of relevant $\mathfrak{F}$-faces* and the *covering collection* as

$$\begin{aligned} \mathrm{rlv}(\Phi) &:= \{\gamma_0 \preceq \gamma;\ \gamma_0\ \mathfrak{F}\text{-face with } Q(\gamma_0) \in \Phi\}, \\ \mathrm{cov}(\Phi) &:= \{\gamma_0 \in \mathrm{rlv}(\Phi);\ \gamma_0 \text{ minimal}\}. \end{aligned}$$

The projected cone $(E \stackrel{Q}{\to} K, \gamma)$ defines a grading on R given by $\deg(f_i) = Q(e_i)$. Consider the associated action of $H := \mathrm{Spec}(\mathbb{K}[K])$ on $\overline{X} := \mathrm{Spec}(R)$. For an $\mathfrak{F}$-face γ_0, we define

$$\overline{X}_{\gamma_0} := \overline{X}_{f_1^{u_1} \cdots f_r^{u_r}} \quad \text{for some } u = (u_1, \ldots, u_r) \in \gamma_0^\circ.$$

We define the open subset

$$\widehat{X} := \widehat{X}(R, \mathfrak{F}, \Phi) := \bigcup_{\gamma_0 \in \mathrm{rlv}(\Phi)} \overline{X}_{\gamma_0} = \bigcup_{\gamma_0 \in \mathrm{cov}(\Phi)} \overline{X}_{\gamma_0} = \widehat{X}(\Phi).$$

The H-action on $\widehat{X}$ admits a good quotient

$$X := X(R, \mathfrak{F}, \Phi) := \widehat{X}(R, \mathfrak{F}, \Phi) /\!\!/ H.$$

We denote the quotient map by $p\colon \widehat{X} \to X$. The quotient X is a normal variety of dimension $\dim(X) = \dim(R) - \dim(K_{\mathbb{Q}})$ with

$$\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*, \quad \mathrm{Cl}(X) \cong K \quad \text{and Cox ring} \quad R(X) \cong R.$$

In this situation, the affine variety $\overline{X}$ is the total coordinate space and $\widehat{X}$ the characteristic space. Furthermore, the affine open subsets $\overline{X}_{\gamma_0} \subseteq \widehat{X}$ for $\gamma_0 \in \mathrm{rlv}(\Phi)$ are H-saturated and their images $X_{\gamma_0} := p(\overline{X}_{\gamma_0}) \subseteq X$ form an affine cover of X. Note that every member $f_i \in \mathfrak{F}$ defines a prime divisor $D_X^i := p(V(\widehat{X}, f_i))$ on X.

By means of the following two constructions, we assign to every normal variety $X = X(R, \mathfrak{F}, \Phi)$ a toric variety Z such that X is naturally embedded into Z.

Construction 1.10 (Cox construction)**.** Let Z be a toric variety arising from a fan Σ in a lattice N and let the primitive generators $v_1, \dots, v_r \in N$ of the rays of Σ generate $N_{\mathbb{Q}}$ as a vector space. Set $F := \mathbb{Z}^r$ and consider $P\colon F \to N,\ f_i \to v_i$, where $f_1, \dots, f_r$ are the canonical basis vectors of F. We define a fan $\widehat{\Sigma}$ in $F_{\mathbb{Q}}$, consisting of faces of the positive orthant $\delta \subseteq F$, by

$$\widehat{\Sigma} \ := \ \{\widehat{\sigma} \preceq \delta;\ P(\widehat{\sigma}) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

The fan $\widehat{\Sigma}$ defines an open toric subvariety $\widehat{Z}$ of $\overline{Z} = \operatorname{Spec}(\mathbb{K}[\delta^\vee \cap E])$, where $E := F^*$. Note that all rays $\operatorname{cone}(f_1), \dots, \operatorname{cone}(f_r)$ of the positive orthant belong to $\widehat{\Sigma}$. Hence, we have

$$\Gamma(\widehat{Z}, \mathcal{O}) = \Gamma(\overline{Z}, \mathcal{O}) = \mathbb{K}[\delta^\vee \cap E]$$

since $\widehat{Z}$ has a small complement in $\overline{Z}$. The map P sends cones of $\widehat{\Sigma}$ onto cones of Σ and consequently induces a morphism $p\colon \widehat{Z} \to Z$ of toric varieties, a so called *Cox construction.*

Construction 1.11 (Canonical toric embedding)**.** Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring with homogeneous generators $\mathfrak{F} := (f_1, \dots, f_r)$ and consider the associated projected cone $(E \xrightarrow{Q} K, \gamma)$. Set $M := \ker(Q)$. Then we have the following exact sequences,

$$0 \longrightarrow L \xrightarrow{Q^*} F \xrightarrow{P} N$$

$$0 \longleftarrow K \xleftarrow{Q} E \xleftarrow{P^*} M \longleftarrow 0$$

where $L := \ker(P)$, $K := E/\operatorname{im}(P^*)$ and P^* is the dual map of P. Furthermore, we can define a polynomial bunched ring $(R', \mathfrak{F}', \Phi')$ where

$$R' \ := \ \mathbb{K}[T_1, \dots, T_r], \quad \deg(T_i) \ := \ \deg(f_i), \quad \mathfrak{F}' \ := \ (T_1, \dots, T_r),$$

and Φ' consists of all projected faces $Q(\gamma_0)$ with $\tau^\circ \subseteq Q(\gamma_0)^\circ$ for some $\tau \in \Phi$. This induces a commutative diagram,

$$\begin{array}{ccccccc} \overline{X} & \supseteq & \widehat{X} & \longrightarrow & \widehat{Z} & \subseteq & \overline{Z} \\ & & {\scriptstyle /\!\!/ H}\downarrow & & \downarrow{\scriptstyle /\!\!/ H} & & \\ & & X & \xrightarrow[\iota]{} & Z & & \end{array}$$

where $\overline{X} := \operatorname{Spec}(R)$, $\overline{Z} := \operatorname{Spec}(R') \cong \mathbb{K}^r$ and $\iota\colon X \to Z$ is a closed embedding of the varieties X and Z associated to the bunched rings $(R, \mathfrak{F}, \Phi)$ and $(R', \mathfrak{F}', \Phi')$, respectively. The fans $\widehat{\Sigma}$ and Σ in $F_{\mathbb{Q}}$ corresponding to the toric varieties $\widehat{Z}$ and Z are given by

$$\widehat{\Sigma} \ := \ \{\delta_0 \preceq \delta;\ \delta_0 \preceq \gamma_0^* \text{ for some } \gamma_0 \in \operatorname{Env}(\Phi)\}, \quad \Sigma \ := \ \{P(\gamma_0^*);\ \gamma_0 \in \operatorname{Env}(\Phi)\},$$

where

$$\operatorname{Env}(\Phi) := \{\gamma_0 \preceq \gamma;\ \gamma_1 \preceq \gamma_0 \text{ and } Q(\gamma_1)^\circ \subseteq Q(\gamma_0)^\circ \text{ for some } \gamma_1 \in \operatorname{rlv}(\Phi)\}$$

and $\gamma_0^* = \gamma_0^\perp \cap \delta$ for $\delta = \gamma^\vee$. Consequently, Σ is the quotient fan of $\widehat{\Sigma}$ and $p\colon \widehat{Z} \to Z$, arising from $P\colon F \to N$, defines a Cox construction. Let D_Z^i be the T_Z-invariant prime divisors of Z, where T_Z denotes the big torus of Z. Then we have by construction $\iota^*(D_Z^i) = D_X^i = p(V(\widehat{X}, f_i))$ and furthermore $\iota^*\colon \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$ is an isomorphism. The toric orbits $Z_\sigma = \overline{T_Z \cdot z_\sigma}$ of Z intersecting X are precisely the orbits corresponding to the cones $\sigma := P(\gamma_0^*)$ for $\gamma_0 \in \mathrm{rlv}(\Phi)$, i.e. there is a canonical bijection between the relevant faces of X and the orbits of the toric variety Z. This induces a decomposition into locally closed strata

$$X \;=\; \bigcup_{\gamma_0 \in \mathrm{rlv}(\Phi)} X_{\gamma_0}, \qquad X_{\gamma_0} \;:=\; X_\sigma \;:=\; X \cap Z_\sigma = \iota^{-1}(Z_{P(\gamma_0^*)}).$$

We call Z the *minimal toric ambient variety of* X. Note that, in general, the toric variety Z is not complete, even if X is. We also say that Φ is *Gale dual* to Σ.

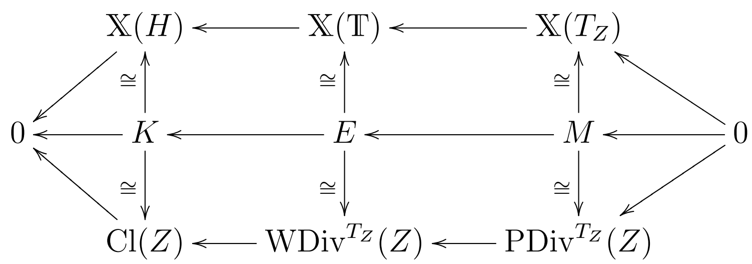
We give a short overview about how the sequence

$$0 \longrightarrow M \xrightarrow{P^*} E \xrightarrow{Q} K \longrightarrow 0$$

can be interpreted geometrically. By construction, M is the lattice of characters $\mathbb{X}(T_Z)$ where T_Z is the torus corresponding to the toric ambient variety Z. The characters χ^u are the rational functions on Z. That means that we have $M \cong \mathrm{PDiv}(Z)$. Along the open toric orbit, all Weil divisors of Z are principal. Hence, every Weil divisor is linearly equivalent to a T_Z-invariant one. This gives $E \cong \mathrm{WDiv}(Z) \cong \mathrm{WDiv}^{T_Z}(Z)$, where $\mathrm{WDiv}^{T_Z}(Z)$ denotes the T_Z-invariant Weil divisors. The isomorphism is explicitly given by

$$e \mapsto \langle e, f_1\rangle D_1 + \ldots + \langle e, f_r\rangle D_r$$

with $D_i := \overline{T_Z \cdot z_{\varrho_i}}$ and $\varrho_i := \mathrm{cone}(v_i)$ for the columns v_i of P. Furthermore, we obtain $M \cong \mathrm{PDiv}^{T_Z}(Z)$ and $K \cong \mathrm{Cl}(Z)$. On the other hand we have by construction $K \cong \mathbb{X}(H)$ where $H := \mathrm{Spec}(\mathbb{K}[K])$ and $E \cong \mathbb{X}(\mathbb{T})$ where $\mathbb{T} := \mathrm{Spec}(\mathbb{K}[E])$ is the big torus corresponding to the toric varieties $\overline{Z} = \mathbb{K}^{n+m}$ and $\widehat{Z}$, respectively.



The last part of this section is dedicated to normal projective varieties $X(R, \mathfrak{F}, \Phi)$ arising from bunched rings.

Definition 1.12. Let R be a finitely generated K-graded algebra, $H := \mathrm{Spec}(\mathbb{K}[K])$ and $\overline{X} := \mathrm{Spec}(R)$. For $w \in \omega_{\overline{X}}$ we define the *GIT-cone* of w as

$$\lambda(w) \ := \ \bigcap_{x \in \overline{X}, w \in \omega_x} \omega_x,$$

where ω_x denotes the orbit cone of $x \in \overline{X}$. Note that $\lambda(w)$ is always polyhedral. The collection $\Lambda(\overline{X}, H) = \{\lambda(w);\ w \in \omega_{\overline{X}}\}$ of all GIT-cones is a quasifan in $K_{\mathbb{Q}}$ having $\omega_{\overline{X}}$ as its support. It is called the *GIT-(quasi)fan* of $\overline{X}$.

Each $w \in \omega_{\overline{X}}$ defines an associated set $X^{\mathrm{ss}}(w) \subseteq \overline{X}$ of *semistable points* which is given by

$$\begin{aligned} X^{\mathrm{ss}}(w) \ &:= \ \{x \in \overline{X};\ w \in \omega_x\} \\ &= \ \{x \in \overline{X};\ \lambda(w) \subseteq \omega_x\} \\ &= \ \{x \in \overline{X};\ f(x) \neq 0 \text{ for some } f \in R_{nw} \text{ where } n > 0\} \\ &= \ \bigcup_{0 \neq f \in R_{nw}} \overline{X}_f. \end{aligned}$$

Given a GIT-cone $\lambda \in \Lambda(\overline{X}, H)$ and any weight $w \in \lambda^{\circ}$ we define $X^{\mathrm{ss}}(\lambda) := X^{\mathrm{ss}}(w)$. Note that $X^{\mathrm{ss}}(w)$ is the set of semistable points associated to the linearization of the trivial bundle given by the character χ^w. For every $\lambda \in \Lambda(\overline{X}, H)$ there is a good quotient $X^{\mathrm{ss}}(w) \to X^{\mathrm{ss}}(w) /\!\!/ H$ for the action of H on the open set $X^{\mathrm{ss}}(w) \subseteq \overline{X}$.

Construction 1.13. Let K be a finitely generated abelian group and let R be a factorially K-graded affine algebra with $R^* = \mathbb{K}^*$. Further let $\mathfrak{F} = (f_1, \ldots, f_r)$ be a system of pairwise non-associated K-prime generators of R. Consider the GIT-fan $\Lambda(\overline{X}, H) = \{\lambda(w);\ w \in \omega_{\overline{X}}\}$ of R. Every GIT-chamber $\lambda = \lambda(w) \in \Lambda(\overline{X}, H)$ defines a bunched ring $(R, \mathfrak{F}, \Phi)$, where Φ is given by

$$\Phi \ = \ \Phi(\lambda) \ := \ \Phi(w) \ := \ \{\omega_x;\ \lambda^{\circ} \subseteq \omega_x^{\circ}\}.$$

Consequently, every GIT-cone λ defines a variety

$$X(R, \mathfrak{F}, w) \ := \ X(R, \mathfrak{F}, \Phi(w)) \ = \ X(R, \mathfrak{F}, \Phi(\lambda)) \ =: \ X(R, \mathfrak{F}, \lambda),$$

and $X(R, \mathfrak{F}, \lambda)$ is given as the good quotient $X^{\mathrm{ss}}(\lambda) /\!\!/ H$.

Theorem 1.14. *Each variety $X = X(R, \mathfrak{F}, \lambda)$ is a normal projective variety with*

$$\dim(X) = \dim(R) - \dim(K_{\mathbb{Q}}), \qquad \Gamma(X, \mathcal{O}^*) = \mathbb{K}^*,$$

and there is an isomorphism $\mathrm{Cl}(X) \to K$ sending $[D_X^i]$ to $\deg(f_i)$. The quotient map $p \colon \widehat{X} \to X$ is a characteristic space and the Cox ring $R(X)$ is isomorphic to R.

Definition 1.15. Let R be a factorially K-graded algebra and let $(E \stackrel{Q}{\to} K, \gamma)$ be its projected cone. We define the *moving cone* of R and $\overline{X} = \mathrm{Spec}(R)$ respectively as

$$\mathrm{Mov}(R) \;:=\; \mathrm{Mov}(\overline{X}) \;:=\; \bigcap_{\gamma_0 \text{ facet of } \gamma} Q(\gamma_0).$$

Theorem 1.16. *All normal projective varieties with a finitely generated Cox ring are isomorphic to some $X(R, \mathfrak{F}, \lambda)$ with $\lambda \cap \mathrm{Mov}(R)^\circ \neq \emptyset$ and $\lambda \in \Lambda(\overline{X}, H)$.*

In general the minimal toric ambient variety Z of a variety $X(R, \mathfrak{F}, \Phi)$ is not complete. The toric GIT-fan $\Lambda(\overline{Z}, H)$ refines $\Lambda(\overline{X}, H)$ and every $\eta \in \Lambda(\overline{Z}, H)$ with $\eta^\circ \subseteq \lambda^\circ$ defines a projective completion $Z(R', \mathfrak{F}', \eta)$ of Z, where $R' = \mathbb{K}[T_1, \dots, T_r]$ and $\mathfrak{F}' = (T_1, \dots, T_r)$. Recall that a toric variety is complete if and only if the associated fan Σ in the lattice N is complete, i.e. the support of Σ is the whole vector space $N_{\mathbb{Q}}$.

1.3 Geometry

In this section we will give a short overview about geometrical properties of a variety $X(R, \mathfrak{F}, \Phi)$ and their convex geometrical meaning, compare [9, Section 7] and [25, Section 4]. We describe the Picard group as well as effective, movable, semiample and ample divisors.

Theorem 1.17. *Consider a relevant $\mathfrak{F}$-face $\gamma_0 \in \mathrm{rlv}(\Phi)$ and a point $x \in X_{\gamma_0}$. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Cl}(X) & \longrightarrow & \mathrm{Cl}(X, x) \\ \cong \updownarrow & & \updownarrow \cong \\ K & \longrightarrow & K/Q(\mathrm{lin}(\gamma_0) \cap E)\,. \end{array}$$

In particular, the local divisor class groups are constant along X_{γ_0} where $\gamma_0 \in \mathrm{rlv}(\Phi)$. Moreover, the Picard group of X is given by

$$\mathrm{Pic}(X) \;:=\; \bigcap_{\gamma_0 \in \mathrm{cov}(\Phi)} Q(\mathrm{lin}(\gamma_0) \cap E).$$

Theorem 1.18. *Consider a relevant $\mathfrak{F}$-face $\gamma_0 \in \mathrm{rlv}(\Phi)$ and a point $x \in X_{\gamma_0}$. Then the following statements hold.*

(i) The point x is factorial if and only if $Q(\mathrm{lin}(\gamma_0) \cap E) = K$.

(ii) The point x is $\mathbb{Q}$-factorial if and only if $Q(\gamma_0)$ is of full dimension.

In particular, X is $\mathbb{Q}$-factorial if and only if Φ consists of only fulldimensional cones. If $\widehat{X}$ is smooth, then every factorial point of X is smooth.

Now we want to describe effective, movable, semiample and ample divisors in terms of the Cox ring. Let X be a normal complete variety with finitely generated Cox ring $R(X)$. A divisor $D \in \mathrm{WDiv}(X)$ is called *effective* if its multiplicities are all non-negative, i.e. $D = a_1D_1 + \ldots a_rD_r$ with primitive divisors D_i and $a_i \geq 0$. All classes $[D]$ of effective divisors build a convex cone, the so called *effective cone* $\mathrm{Eff}(X)$. We have $[D] \in \mathrm{Eff}(X)$ if and only if $[D] \in \omega_{\overline{X}}$, i.e. there is a $0 \neq f \in \Gamma(X, \mathcal{O}_X(nD))$ for some $n > 0$. If $\mathfrak{F} = (f_1, \ldots, f_r)$ is a system of homogeneous generators of the Cox ring $R(X)$, then the effective cone is generated by the degrees $\deg(f_i)$.

The *support* of a Weil divisor $D \in \mathrm{WDiv}(X)$ is defined as

$$\mathrm{supp}(D) \ = \ \mathrm{supp}\left(\sum_{D_i \text{ prime}} a_iD_i\right) \ := \ \bigcup_{a_i \neq 0} D_i.$$

For a divisor $D \in \mathrm{WDiv}(X)$ and a section $f \in \Gamma(X, \mathcal{O}_X(D))$ we define the *D-divisor* as $\mathrm{div}_D(f) := \mathrm{div}(f) + D$. The *base locus* and the *stable base locus* of a divisor D are defined as

$$\mathrm{Bs}(D) \ := \bigcap_{f \in \Gamma(X, \mathcal{O}_X(D))} \mathrm{supp}(\mathrm{div}_D(f)), \qquad \mathrm{sBs}(D) \ := \bigcap_{n \in \mathbb{Z}_{\geq 1}} \mathrm{Bs}(nD).$$

A divisor $D \in \mathrm{WDiv}(X)$ is called *movable* if its stable base locus is of codimension at least two in X. The *moving cone* $\mathrm{Mov}(X)$ is the convex cone generated by the classes of movable divisors. A divisor $D \in \mathrm{WDiv}(X)$ is called *semiample* if its stable base locus is empty and a divisor $D \in \mathrm{WDiv}(X)$ is called *ample* if X is covered by affine sets

$$X_{nD,f} := X \setminus \mathrm{supp}(nD + \mathrm{div}(f))$$

for some $n \in \mathbb{Z}_{\geq 1}$, where $f \in \Gamma(X, \mathcal{O}(nD))$. The *semiample cone* $\mathrm{SAmple}(X)$ and *ample cone* $\mathrm{Ample}(X)$ are generated by all classes of semiample and ample divisors, respectively.

Theorem 1.19. *In the divisor class group $K = \mathrm{Cl}(X)$ we have the following descriptions of effective, movable, semiample and ample divisors:*

$$\mathrm{Eff}(X) \ := \ Q(\gamma), \qquad \mathrm{Mov}(X) \ := \bigcap_{\gamma_0 \text{ facet of } \gamma} Q(\gamma_0),$$

$$\mathrm{SAmple}(X) \ := \ \bigcap_{\tau \in \Phi} \tau, \qquad \mathrm{Ample}(X) \ := \ \bigcap_{\tau \in \Phi} \tau^\circ.$$

2 Complexity-one T-varieties

This chapter is dedicated to complexity-one T-varieties, i.e. varieties X with an effective action of a torus T of dimension $\dim(X) - 1$, and their Cox rings which are factorially graded rings of complexity one. A combinatorial language will be introduced comparable to the convex geometrical description of toric varieties by fans. Parts of this chapter are already published in [27] and [28, Section 1].

2.1 Factorially graded rings of complexity one

In this section we present a construction of factorially K-graded algebras of complexity one. We describe these algebras in terms of generators and relations.

Definition 2.1. Let K be a finitely generated abelian group and $R = \bigoplus_{w \in K} R_w$ a finitely generated normal graded K-algebra.

(i) We say that R has an *effective K-grading of complexity one* if all the $w \in K$ with $R_w \neq 0$ generate K and K is of rank $\dim(R) - 1$.

(ii) We call the K-grading *pointed* if $R_0 = \mathbb{K}$ holds.

(iii) Let $\{f_1, \ldots, f_r\}$ be a system of homogeneous generators of R. Then we call the K-grading *almost free* if K is generated by any $r - 1$ of the degrees of the generators $\deg(f_i)$.

Construction 2.2. For $r \geq 1$, we fix integers $n_0, \ldots, n_r \in \mathbb{Z}_{>0}$, $m \in \mathbb{Z}_{\geq 0}$ and set $n := n_0 + \ldots + n_r$. Let $A = (a_0, \ldots, a_r)$ be a sequence of vectors $a_i = (b_i, c_i)$ in $\mathbb{K}^2$ such that any pair (a_i, a_k) with $k \neq i$ is linearly independent and let $l_i = (l_{i1}, \ldots, l_{in_i}) \in {\mathbb{Z}_{>0}}^{n_i}$, $0 \leq i \leq r$, be tuples of positive integers. These data define a $(r \times (n + m))$-matrix

$$P_0 \; = \; (L_0, 0), \qquad \text{where} \;\; L_0 \; = \; \begin{pmatrix} -l_0 & l_1 & 0 & \ldots & 0 \\ -l_0 & 0 & l_2 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -l_0 & 0 & 0 & \ldots & l_r \end{pmatrix}.$$

We consider the polynomial ring $\mathbb{K}[T_{ij}, S_k]$ in the variables T_{ij} and S_k where $0 \leq i \leq r$, $1 \leq j \leq n_i$ and $1 \leq k \leq m$. For every $0 \leq i \leq r$, define a monomial

$$T_i^{l_i} \; := \; T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \; \in \; \mathbb{K}[T_{ij}, S_k].$$

Moreover, for any two indices $0 \leq i, j \leq r$, set $\alpha_{ij} := \det(a_i, a_j) = b_i c_j - b_j c_i$ and for any three indices $0 \leq i < j < k \leq r$ define a trinomial

$$g_{i,j,k} \; := \; \alpha_{jk} T_i^{l_i} \; + \; \alpha_{ki} T_j^{l_j} \; + \; \alpha_{ij} T_k^{l_k} \; \in \; \mathbb{K}[T_{ij}, S_k].$$

We define a grading of $\mathbb{K}[T_{ij}, S_k]$ by an abelian group K_0 such that all the $g_{i,j,k}$ become homogeneous of the same degree. For this, consider the linear map $P_0\colon F \to N$, given by the matrix $P_0 = (L_0, 0)$ and let $P_0{}^*$ be the dual map, given by the transpose of P_0. Set $K_0 := \mathbb{Z}^{n+m}/\mathrm{im}(P_0{}^*)$ and let $Q_0\colon \mathbb{Z}^{n+m} \to K_0$ be the projection. This defines a K_0-grading on $\mathbb{K}[T_{ij}, S_k]$ given by

$$\deg(T_{ij}) \;:=\; Q_0(e_{ij}), \quad \deg(S_k) \;:=\; Q_0(e_k),$$

where $e_{ij},\ e_k \in \mathbb{Z}^{n+m}$ are the canonical basis vectors. By construction all $g_{i,j,k}$ are homogeneous of the same degree. Hence, we obtain a K_0-graded factor algebra

$$R(A, P_0) \;:=\; \mathbb{K}[T_{ij}, S_k] \;/\; \langle g_{i,j,k};\ 0 \le i < j < k \le r \rangle.$$

We say that (A, P_0) is *sincere* if $r \ge 2$ and $n_i l_{ij} > 1$ for all i, j hold. This ensures that there exist in fact relations $g_{i,j,k}$ and none of these relations contains a linear term. Note that for $r = 1$ we obtain the diagonal complexity-one gradings of the polynomial ring $\mathbb{K}[T_{ij}, S_k]$. Furthermore, the matrix P_0 is called *gradiator matrix.*

Lemma 2.3. *In the setting of Construction 2.2, the identities*

$$g_{i,k,l} \;=\; \alpha_{kl} \cdot g_{i,j,k} + \alpha_{ik} \cdot g_{j,k,l} \qquad \text{and} \qquad g_{i,j,l} \;=\; \alpha_{jl} \cdot g_{i,j,k} + \alpha_{ij} \cdot g_{j,k,l}$$

hold for any $0 \le i < j < k < l \le r$. *In particular, every trinomial* $g_{i,j,k}$, *where* $0 \le i < j < k \le r$ *is contained in the ideal* $\langle g_{i,i+1,i+2};\ 0 \le i \le r-2 \rangle$.

Proof. The identities are easily obtained by direct computation. We may assume $a_j = (1, 0)$ and $a_k = (0, 1)$. The other points are given by $a_i = (a_{i1}, a_{i2})$ and $a_l = (a_{l1}, a_{l2})$. Consequently, we obtain $\alpha_{jk} = 1$, $\alpha_{ik} = a_{i1}$, $\alpha_{ij} = -a_{i2}$, $\alpha_{kl} = -a_{l1}$ and $\alpha_{jl} = a_{l2}$. This gives

$$\begin{aligned}
\alpha_{kl} \cdot g_{i,j,k} + \alpha_{ik} \cdot g_{j,k,l} &= \alpha_{kl}(\alpha_{jk} T_i^{l_i} + \alpha_{ki} T_j^{l_j} + \alpha_{ij} T_k^{l_k}) + \alpha_{ik}(\alpha_{kl} T_j^{l_j} + \alpha_{lj} T_k^{l_k} + \alpha_{jk} T_l^{l_l}) \\
&= \alpha_{kl} T_i^{l_i} + (\alpha_{kl}\alpha_{ij} + \alpha_{ik}\alpha_{lj}) T_k^{l_k} + \alpha_{ik} T_l^{l_l} \\
&= \alpha_{kl} T_i^{l_i} + (a_{l1} a_{i2} - a_{l2} a_{i1}) T_k^{l_k} + \alpha_{ik} T_l^{l_l} \\
&= g_{i,k,l}, \\
\alpha_{jl} \cdot g_{i,j,k} + \alpha_{ij} \cdot g_{j,k,l} &= \alpha_{jl}(\alpha_{jk} T_i^{l_i} + \alpha_{ki} T_j^{l_j} + \alpha_{ij} T_k^{l_k}) + \alpha_{ij}(\alpha_{kl} T_j^{l_j} + \alpha_{lj} T_k^{l_k} + \alpha_{jk} T_l^{l_l}) \\
&= \alpha_{jl} T_i^{l_i} + (\alpha_{jl}\alpha_{ki} + \alpha_{ij}\alpha_{kl}) T_j^{l_j} + \alpha_{ij} T_l^{l_l} \\
&= \alpha_{jl} T_i^{l_i} + (a_{l1} a_{i2} - a_{l2} a_{i1}) T_k^{l_k} + \alpha_{ij} T_l^{l_l} \\
&= g_{i,j,l}.
\end{aligned}$$

The supplement then follows by repeated application of these identities. □

Lemma 2.3 allows us to present the rings $R(A, P_0)$ in the following form:

$$R(A, P_0) \;:=\; \mathbb{K}[T_{ij}, S_k] \;/\; \langle g_{i,i+1,i+2};\ 0 \le i \le r-2 \rangle$$

We denote the points of $\mathbb{K}^{n+m}$ as tuples $z = (z_{ij}, z_k)$ according to the variables T_{ij} and S_k and we set

$$\overline{X} \;:=\; V(\mathbb{K}^{n+m};\; g_0, \ldots, g_{r-2}).$$

Once we know that $R(A, P_0)$ is reduced we have $\overline{X} = \mathrm{Spec}(R(A, P_0))$. Furthermore, set $t_i^{l_i} := t_{i1}^{l_{i1}} \cdots t_{in_i}^{l_{in_i}}$ and consider the homomorphism

$$\mathbb{T}^{n+m} \;\to\; \mathbb{T}^r, \qquad (t_{ij}, t_k) \;\mapsto\; \left(\frac{t_1^{l_1}}{t_0^{l_0}}, \ldots, \frac{t_r^{l_r}}{t_0^{l_0}}\right)$$

and its kernel H_0 which is isomorphic to $\mathrm{Spec}(\mathbb{K}[K_0])$. This quasitorus acts as a subgroup of $\mathbb{T}^{n+m}$ on $\mathbb{K}^{n+m}$ and $\overline{X}$ is invariant under this action by construction.

Proposition 2.4. *For every pair (A, P_0) as in Construction 2.2, the ring $R(A, P_0)$ is an integral normal complete intersection of dimension*

$$\dim(R(A, P_0)) \;=\; n + m - r + 1, \qquad\qquad n \;:=\; n_0 + \ldots + n_r.$$

Furthermore the K_0-grading is pointed, effective and of complexity one.

Lemma 2.5. *In the notation of Construction 2.2 and Proposition 2.4 let $z \in \overline{X}$, where $\overline{X} := V(\mathbb{K}^{n+m};\; g_0, \ldots, g_{r-2})$. If $T_i^{l_i}(z) = T_j^{l_j}(z) = 0$ for $0 \le i < j \le r$, then $T_k^{l_k}(z) = 0$ holds for $0 \le k \le r$.*

Proof. If $i < k < j$ holds, then, according to Lemma 2.3, we have $g_{i,k,j}(z) = 0$, which implies $T_k^{l_k}(z) = 0$. The cases $k < i$ and $j < k$ are obtained similarly. □

Proof of Proposition 2.4. Set $\overline{X} := V(\mathbb{K}^{n+m}; g_0, \ldots, g_{r-2})$, where $g_i := g_{i,i+1,i+2}$. Then we have to show that $\overline{X}$ is a connected complete intersection with at most normal singularities. In order to see that $\overline{X}$ is connected, set $\ell := \prod n_i \prod l_{ij}$ and $\zeta_{ij} := \ell n_i^{-1} l_{ij}^{-1}$. Then $\overline{X} \subseteq \mathbb{K}^{n+m}$ is invariant under the $\mathbb{K}^*$-action given by

$$t \cdot z \;\;:=\;\; (t^{\zeta_{ij}} z_{ij}, z_1, \ldots, z_m)$$

and the point $0 \in \mathbb{K}^{n+m}$ lies in the closure of any orbit $\mathbb{K}^* \cdot z \subseteq \overline{X}$, $z \in \overline{X}$, which implies connectedness. To proceed, consider the Jacobian J_g of $g := (g_0, \ldots, g_{r-2})$. According to Serre's criterion (see [20, Section 11]), we have to show that the set of points of $z \in X$, with $J_g(z)$ not of full rank, is of codimension at least two in X. Note that the Jacobian is of the shape $(J_g, 0)$ with a zero block of size $(r-1) \times m$ corresponding to the variables $S_1, \ldots, S_k$ and

$$J_g = \begin{pmatrix} \delta_{0\,0} & \delta_{0\,1} & \delta_{0\,2} & 0 & & & & & & 0 \\ 0 & \delta_{1\,1} & \delta_{1\,2} & \delta_{1\,3} & 0 & & & & & \\ & & & & \vdots & & & & & \\ & & & & & 0 & \delta_{r-3\,r-3} & \delta_{r-3\,r-2} & \delta_{r-3\,r-1} & 0 \\ 0 & & & & & & 0 & \delta_{r-2\,r-2} & \delta_{r-2\,r-1} & \delta_{r-2\,r} \end{pmatrix},$$

where δ_{ti} is a nonzero multiple of the gradient $\delta_i := \operatorname{grad} T_i^{l_i}$. Consider $z \in \overline{X}$ with $J_g(z)$ not of full rank. Then $\delta_i(z) = 0 = \delta_k(z)$ holds with some $0 \le i < k \le r$. This implies $z_{ij} = 0 = z_{kl}$ for some $1 \le j \le n_i$ and $1 \le l \le n_k$. Thus, we have $T_i^{l_i}(z) = 0 = T_k^{l_k}(z)$. Lemma 2.5 gives $T_s^{l_s}(z) = 0$, for all $0 \le s \le r$. Thus, some coordinate z_{st} must vanish for every $0 \le s \le r$. This shows that z belongs to a closed subset of $\overline{X}$ having codimension at least two in $\overline{X}$. Hence, $R(A, P_0)$ is an integral normal complete intersection with

$$\dim(R(A, P_0)) \;=\; n + m - r + 1 \;=\; \dim(\ker(P_0)) + 1.$$

Effectivity of the K_0-grading is given by construction, because the degrees $\deg(T_{ij}) = Q_0(e_{ij})$ generate K_0. This implies that the K_0-grading is of complexity one. Now consider the action of the quasitorus $H_0 := \operatorname{Spec} \mathbb{K}[K_0]$ on $\mathbb{K}^{n+m}$ given by the K_0-grading. Note that $H_0 \subseteq \mathbb{T}^{n+m}$ is the kernel of the homomorphism of tori

$$\mathbb{T}^{n+m} \;\to\; \mathbb{T}^r, \qquad (t_{ij}, t_k) \;\mapsto\; \left(\frac{t_1^{l_1}}{t_0^{l_0}}, \ldots, \frac{t_r^{l_r}}{t_0^{l_0}}\right).$$

The set $\overline{X} \subseteq \mathbb{K}^{n+m}$ of common zeros of all the $g_{i,i+1,i+2}$ is H_0-invariant and thus it is invariant under the one-parameter subgroup of H_0 given by

$$\mathbb{K}^* \;\to H_0, \qquad t \;\mapsto\; (t^{\zeta_{ij}}, t, \ldots, t), \qquad\qquad \zeta_{ij} \;:=\; n_i^{-1} l_{ij}^{-1} \prod_k n_k \prod_{k,m} l_{km}.$$

Since all ζ_{ij} are positive, any orbit of this one-parameter subgroup in $\mathbb{K}^{n+m}$ has the origin in its closure. Consequently, every H_0-invariant function on $\overline{X}$ is constant. This shows $R(A, P_0)_0 = \mathbb{K}$. Hence, the grading is pointed. □

Lemma 2.6. *In the situation of Construction 2.2, the variable T_{ij} defines a prime ideal in $R(A, P_0)$ if and only if the numbers $\gcd(l_{k1}, \ldots, l_{kn_k})$, where $k \neq i$, are pairwise coprime.*

Proof. We treat exemplarily T_{01}. Using Lemma 2.3, we see that the ideal of relations of $R(A, P_0)$ can be presented as follows:

$$\langle g_{s,s+1,s+2};\ 0 \le s \le r-2 \rangle \;=\; \langle g_{0,s,s+1};\ 1 \le s \le r-1 \rangle$$

Thus, the ideal $\langle T_{01} \rangle \subseteq R(A, P_0)$ is prime if and only if the following binomial ideal is prime:

$$\mathfrak{a} \;:=\; \langle \alpha_{s+1\,0} T_s^{l_s} + \alpha_{0s} T_{s+1}^{l_{s+1}};\ 1 \le s \le r-1 \rangle \;\subseteq\; \mathbb{K}[T_{ij};\ (i,j) \neq (0,1)]$$

Set $l_i := (l_{i1}, \ldots, l_{in_i})$. Then the ideal $\mathfrak{a}$ is prime if and only if the following family can be complemented to a lattice basis

$$(l_1, -l_2, 0, \ldots, 0),\ \ldots,\ (0, \ldots, 0, l_{r-1}, -l_r).$$

This in turn is equivalent to the statement that the numbers $\gcd(l_{k1}, \ldots, l_{kn_k})$, where $1 \le k \le r$, are pairwise coprime. □

We say that a Weil divisor on $\overline{X}$ is H_0-*prime* if it is non-zero, has only multiplicities zero or one and the group H_0 permutes transitively the prime components with multiplicity one. Note that the divisor $\mathrm{div}(f)$ of a homogeneous function $f \in R(A, P_0)$ on $\overline{X}$ is H_0-prime if and only if f is K_0-prime [25, Prop. 3.2].

Proposition 2.7. *Let the variables T_{ij} be regarded as regular functions on the affine variety $\overline{X} = \mathrm{Spec}(R(A, P_0))$.*

(i) The divisors of the T_{ij} on $\overline{X}$ are H_0-prime and pairwise different. In particular, the T_{ij} define pairwise non-associated K_0-prime elements in $R(A, P_0)$.

(ii) If the ring $R(A, P_0)$ is factorial and $n_i l_{ij} > 1$ holds, then the divisor of T_{ij} on $\overline{X}$ is even prime.

Proof. For (i), we exemplarily show that the divisor of T_{01} is H_0-prime. First note that by Lemma 2.3 the zero set $V(\overline{X}; T_{01})$ is described in $\mathbb{K}^{n+m}$ by the equations

$$T_{01} \;=\; 0, \qquad \alpha_{s+1\,0} T_s^{l_s} + \alpha_{0s} T_{s+1}^{l_{s+1}} \;=\; 0, \quad 1 \le s \le r-1. \tag{1}$$

Let $h \in S$ denote the product of all T_{ij} with $(i, j) \neq (0, 1)$. Then, in $\mathbb{K}_h^{n+m}$, the above equations are equivalent to

$$T_{01} \;=\; 0, \qquad -\frac{\alpha_{s+1\,0} T_s^{l_s}}{\alpha_{0s} T_{s+1}^{l_{s+1}}} \;=\; 1, \quad 1 \le s \le r-1.$$

Now, choose a point $z \in \mathbb{K}_h^{n+m}$ satisfying these equations. Then z_{01} is the only vanishing coordinate of z. Any other such point is of the form

$$(0,\, t_{02} z_{02},\, \ldots,\, t_{rn_r} z_{rn_r}), \qquad t_{ij} \in \mathbb{K}^*,\ t_s^{l_s} = t_{s+1}^{l_{s+1}},\ 1 \le s \le r-1.$$

Setting $t_{01} := t_{02}^{-l_{02}} \cdots t_{0n_0}^{-l_{0n_0}} t_1^{l_1}$, we obtain an element $t = (t_{ij}) \in H_0$ such that the above point equals $t \cdot z$. This consideration shows

$$V(\overline{X}_h; T_{01}) \;=\; H_0 \cdot z.$$

Using Lemma 2.5, we see that $V(\overline{X}; T_{01}, T_{ij})$ is of codimension at least two in $\overline{X}$ whenever $(i, j) \neq (0, 1)$. This allows to conclude

$$V(\overline{X}; T_{01}) \;=\; \overline{H_0 \cdot z}.$$

Thus, to obtain that T_{01} defines an H_0-prime divisor on $\overline{X}$, we only need that the equations (1) define a radical ideal. This in turn follows from the fact that their Jacobian at the point $z \in V(\overline{X}; T_{01})$ is of full rank.

To verify (ii), let $R(A, P_0)$ be factorial. Assume that the divisor of T_{ij} is not prime. Then we have $T_{ij} = h_1 \cdots h_s$ with prime elements $h_l \in R(A, P_0)$. Consider their decomposition into homogeneous parts

$$h_l \;=\; \sum_{w \in K} h_{l,w}.$$

Plugging this into the product $h_1 \cdots h_s$ shows that $\deg(T_{ij})$ is a positive combination of some $\deg(T_{kl})$ with $(k,l) \neq (i,j)$. Thus, there is a vector $(c_{kl}) \in \ker(Q_0) \subseteq \mathbb{Z}^{n+m}$ with $c_{ij} = 1$ and $c_{kl} \leq 0$ whenever $(k,l) \neq (i,j)$. Since, $\ker(Q_0)$ is spanned by the rows of P_0, we must have $n_i = 1$ and $l_{ij} = 1$, a contradiction to our assumptions. □

Theorem 2.8. *Let (A, P_0) be as introduced in Construction 2.2. Then the following statements hold.*

(i) The algebra $R(A, P_0)$ is factorially K_0-graded and the K_0-grading is effective, pointed and of complexity one.

(ii) The variables T_{ij} and S_k define a system of pairwise non-associated K_0-prime generators of $R(A, P_0)$.

(iii) Suppose that (A, P_0) is sincere. Then $R(A, P_0)$ is factorial if and only if the group K_0 is torsion free, i.e. the numbers $\ell_i := \gcd(l_{i1}, \ldots, l_{in_i})$ are pairwise coprime.

Proof. The first two assertions are almost proven by Proposition 2.4 and Proposition 2.7. The K_0-factoriality still has to be proven. In the next section we will realize $R(A, P_0)$ as Cox ring, see Proposition 2.16. Hence, $R(A, P_0)$ is factorially graded, compare [25].

We prove (iii). If K_0 is torsion free, then K_0-factoriality of $R(A, P_0)$ implies factoriality. Conversely, if $R(A, P_0)$ is factorial, then the generators T_{ij} are prime by Proposition 2.7. Furthermore, by Lemma 2.6 the numbers $\gcd(l_{i1}, \ldots, l_{in_i})$ are pairwise coprime. This implies that the rows of P_0 generate a primitive sublattice of $\mathbb{Z}^{n+m}$ and thus K_0 is torsion free. □

In the following example we consider a factorially graded algebra which is not factorial. Note that K_0 is torsion free if and only if the numbers $\ell_i := \gcd(l_{i1}, \ldots, l_{in_i})$ are pairwise coprime.

Example 2.9. Set $r = 2$, $n_0 = n_1 = n_2 = 1$, $l_{01} = l_{11} = l_{21} = 2$ and let A consist of the vectors $(1,0)$, $(1,1)$ and $(0,1)$. Then the matrix

$$\begin{pmatrix} -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

describes the map $P_0 \colon \mathbb{Z}^3 \to \mathbb{Z}^2$. Thus, the grading group is $K_0 = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Concretely, this grading can be realized as

$$\deg(T_{01}) = (1, \overline{0}, \overline{0}), \qquad \deg(T_{11}) = (1, \overline{1}, \overline{0}), \qquad \deg(T_{21}) = (1, \overline{0}, \overline{1}).$$

The associated algebra $R(A, P_0)$ is factorially K_0-graded but not factorial. It is explicitly given by

$$R(A, P_0) \quad = \quad \mathbb{K}[T_{01}, T_{11}, T_{21}] / \langle T_{01}^2 - T_{11}^2 + T_{21}^2 \rangle.$$

Now we will consider an extended version of Construction 2.2 by coarsening the K_0-grading as follows.

Construction 2.10. For $r \geq 1$, we fix integers $n_0, \ldots, n_r \in \mathbb{Z}_{>0}$, $m \in \mathbb{Z}_{\geq 0}$ and $0 < s \leq n + m + r$, where $n := n_0 + \ldots + n_r$. As input data we have a sequence $A = (a_0, \ldots, a_r)$ of vectors $a_i = (b_i, c_i)$ in $\mathbb{K}^2$ such that any pair (a_i, a_k) with $k \neq i$ is linearly independent and an integral block matrix P of size $(r + s) \times (n + m)$ whose columns are pairwise different primitive vectors generating the vector space $\mathbb{Q}^{r+s}$ as a cone:

$$P \;=\; \begin{pmatrix} L_0 & 0 \\ d & d' \end{pmatrix},$$

where L_0 is defined by tuples $l_i = (l_{i1}, \ldots, l_{in_i})$ as in Construction 2.2, d is an integral $(s \times n)$-matrix, and d' is an integral $(s \times m)$-matrix. Let P^* denote the transpose of P and define $K := \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$. We consider the projection $Q\colon \mathbb{Z}^{n+m} \to K$ and obtain a K-grading of $\mathbb{K}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) \;:=\; w_{ij} \;:=\; Q(e_{ij}), \quad \deg(S_k) \;:=\; u_k \;:=\; Q(e_k).$$

By construction the trinomials g_i are K-homogeneous of the same degree

$$\gamma \;=\; l_{01}w_{01} + \ldots + l_{0n_0}w_{0n_0} \;=\; \ldots \;=\; l_{r1}w_{r1} + \ldots + l_{rn_r}w_{rn_r}.$$

Furthermore, we obtain the following K-graded factor ring

$$R(A, P) \;:=\; \mathbb{K}[T_{ij}, S_k]/\langle g_i; 0 \leq i \leq r-2\rangle.$$

The rings $R(A, P)$ and $R(A, P_0)$ coincide as rings, but they are not isomorphic as graded rings since the K_0-grading is finer than the K-grading. Consider the down grading map $K \to K_0$ which is the canonical inclusion. We have the following commutative diagram with exact sequences:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & 0 & & & & \mathbb{Z}^s & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}^r & \xrightarrow{P_0^*} & \mathbb{Z}^{n+m} & \xrightarrow{Q_0} & K_0 & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}^{r+s} & \xrightarrow[P^*]{} & \mathbb{Z}^{n+m} & \xrightarrow[Q]{} & K & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & \mathbb{Z}^s & & & & 0 & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

By the snake lemma we can identify the direct factor $\mathbb{Z}^s$ of $\mathbb{Z}^{r+s}$ with the kernel of the downgrading map $K \to K_0$. Furthermore, let T, H_0 and H denote the quasitori of the abelian groups $\mathbb{Z}^s$, K_0 and K. Then we have $T = H_0/H$.

Remark 2.11. The condition that the numbers $\ell_i = \gcd(l_{i1}, \ldots, l_{in_i})$ are pairwise coprime is necessary for K to be torsion free. But it is not sufficient. See for example

$$P = \begin{pmatrix} -7 & -1 & 5 & 0 \\ -7 & -1 & 0 & 2 \\ -4 & -1 & 1 & 1 \end{pmatrix}.$$

The group K is torsion free if and only if $P\colon \mathbb{Z}^{n+m} \to \mathbb{Z}^{r+s}$ is surjective.

Definition 2.12. If (A, P) is sincere, i.e. $r \geq 2$ holds and for every $0 \leq i \leq r$ and $1 \leq j \leq n_i$ we have $n_i l_{ij} > 1$, then the ring $R(A, P)$ is called *minimally represented.*

This property ensures that the relations are really counting which means that there are no linear terms (ommitting redundant generators) and that the relations are trinomials. In particular, if (A, P) is sincere, then $R(A, P)$ is not polynomial.

2.2 Varieties with a complexity-one torus action

In this chapter we will construct varieties whose Cox rings are of the form $R(A, P)$. It turns out that these rings describe complexity-one T-varieties.

Definition 2.13. Let X be a variety with an effective action of a torus T of complexity one, i.e. $\dim(T) = \dim(X) - 1$. Then X is called a *complexity-one T-variety.*

Assume that X is a rational $\mathbb{Q}$-factorial complete normal complexity-one T-variety. Note that in case of a complexity-one T-variety, the property of X being rational is equivalent to the condition that the divisor class group $\mathrm{Cl}(X)$ is finitely generated, see [29, Section 5]. We will obtain X as a subvariety of a toric variety Z and the construction of Z is performed in terms of fans.

Construction 2.14. Let (A, P) be data as in Construction 2.10. Consider the lattice

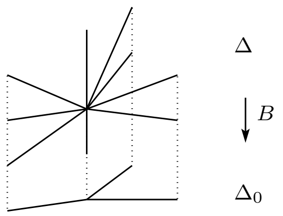
$$F \;:=\; \bigoplus_{i=0}^{r} \bigoplus_{j=1}^{n_i} \mathbb{Z} \cdot f_{ij} \oplus \mathbb{Z} \cdot f_1 \oplus \ldots \oplus \mathbb{Z} \cdot f_m \;\cong\; \mathbb{Z}^{n+m}.$$

Let $\widehat{\Delta}$ be the fan in F having the rays $\widehat{\varrho}_{ij}$ and $\widehat{\varrho}_k$ through the basis vectors f_{ij} and f_k as its maximal cones. Let P be a matrix as defined in Construction 2.10 with P_0 as defined in Construction 2.2 and suppose that the columns of P are primitive, pairwise different and generate $N_{\mathbb{Q}}$ as a cone, where $N := \mathbb{Z}^{r+s}$. With $N_0 := \mathbb{Z}^r$, we have the projection $B\colon N \to N_0$ onto the first r coordinates and the matrices P and P_0 define linear maps $P\colon F \to N$ and $P_0\colon F \to N_0$, respectively.

Let Δ be the fan in N with the rays $\varrho_{ij} := P(\widehat{\varrho}_{ij})$ and $\varrho_k := P(\widehat{\varrho}_k)$ as its maximal cones. The ray $\widetilde{\varrho}_{ij}$ through the ij-th column of P_0 is given in terms of the canonical basis vectors $v_1, \ldots, v_r$ in $N_0 = \mathbb{Z}^r$ as

$$\widetilde{\varrho}_{ij} \;=\; \mathbb{Q}_{\geq 0} v_i, \quad 1 \leq i \leq r, \qquad \widetilde{\varrho}_{0j} \;=\; -\mathbb{Q}_{\geq 0}(v_1 + \ldots + v_r).$$

For fixed i, all $\widetilde{\varrho}_{ij}$ are equal to each other. We list them nevertheless all separately in a system of fans $\widetilde{\Delta}$ having the zero cone as the common gluing data; see [1] for the formal definition of this concept. Finally, we have the fan Δ_0 in $\mathbb{Z}^r$ with the rays $\mathbb{Q}_{\geq 0} v_i$ and $-\mathbb{Q}_{\geq 0}(v_1 + \ldots + v_r)$ as its maximal cones.



The toric variety $\widehat{Y}$ associated to $\widehat{\Delta}$ has $\operatorname{Spec} \mathbb{K}[E] \cong \mathbb{T}^{n+m}$ as its acting torus, where E is the dual lattice of F. The fan Δ in N defines a toric variety Y and the system of fans $\widetilde{\Delta}$ defines a toric prevariety $\widetilde{Y}$. The toric prime divisors corresponding to the rays $\widehat{\varrho}_{ij}, \widehat{\varrho}_k \in \widehat{\Delta}$, $\varrho_{ij}, \varrho_k \in \Delta$ and $\widetilde{\varrho}_{ij} \in \widetilde{\Delta}$, are denoted as

$$\widehat{D}_{ij}, \widehat{D}_k \subseteq \widehat{Y}, \qquad D_{ij}, D_k \subseteq Y, \qquad \widetilde{D}_{ij} \subseteq \widetilde{Y}.$$

The toric variety associated to Δ_0 is the open subset $\mathbb{P}_r^{(1)} \subseteq \mathbb{P}_r$ of the projective space obtained by removing all toric orbits of codimension at least two. The maps P and P_0 define toric morphisms $\pi\colon \widehat{Y} \to Y$ and $\pi_0\colon \widehat{Y} \to \widetilde{Y}$. Moreover, $B\colon N \to N_0$ defines a toric morphism $\beta\colon Y \to \widetilde{Y}$ and the identity $\mathbb{Z}^r \to \mathbb{Z}^r$ defines a toric morphism $\kappa\colon \widetilde{Y} \to \mathbb{P}_r^{(1)}$. These morphisms fit into the commutative diagram

$$\begin{array}{ccc} \widehat{Y} & \xrightarrow{\ \pi\ } & Y \\ & {\scriptstyle \pi_0} \searrow \quad \swarrow {\scriptstyle \beta} & \\ & \widetilde{Y} & \\ & \downarrow {\scriptstyle \kappa} & \\ & \mathbb{P}_r^{(1)} \ . & \end{array}$$

Note that $\kappa\colon \widetilde{Y} \to \mathbb{P}_r^{(1)}$ is a local isomorphism which, for fixed i, identifies all the divisors $\widetilde{D}_{ij}$ with $1 \leq j \leq n_i$. Let $H \subseteq \mathbb{T}^{n+m}$ and $H_0 \subseteq \mathbb{T}^{n+m}$ be the kernels of the toric morphisms $\pi\colon \widehat{Y} \to Y$ and $\pi_0\colon \widehat{Y} \to \widetilde{Y}$, respectively.

Proposition 2.15. *In the above notation, the following statements hold.*

(i) With $\widehat{Y}_0 := \widehat{Y} \setminus (\widehat{D}_1 \cup \ldots \cup \widehat{D}_m)$, the restriction $\pi_0\colon \widehat{Y}_0 \to \widetilde{Y}$ is a geometric quotient for the action of H_0 on $\widehat{Y}_0$.

(ii) The quasitorus H acts freely on $\widehat{Y}$ and $\pi\colon \widehat{Y} \to Y$ is the geometric quotient for this action.

(iii) The factor group $T := H_0/H$ is isomorphic to $\mathbb{T}^s$ and it acts canonically on Y.

(iv) The T-action on Y has infinite isotropy groups along $D_1, \dots, D_m$ and isotropy groups of order l_{ij} along D_{ij}.

(v) With $Y_0 := Y \setminus (D_1 \cup \ldots \cup D_m)$, the restriction $\beta\colon Y_0 \to \widetilde{Y}$ is a geometric quotient for the action of T on Y.

Proof. The fact that $\pi_0\colon \widehat{Y}_0 \to \widetilde{Y}$ and $\pi\colon \widehat{Y} \to Y$ are geometric quotients is due to known characterizations of these notions in terms of (systems of) fans, see e.g. [1]. As a consequence, also $\beta\colon Y_0 \to \widetilde{Y}$ is a geometric quotient for the induced action of $T = H_0/H$.

We verify the remaining part of (ii). According to [5, Prop. II.1.4.2], the isotropy group of $H = \ker(\pi)$ at a distinguished point $y_{\widehat{\varrho}} \in \widehat{Y}$ has character group isomorphic to

$$\ker(P) \cap \mathrm{lin}_{\mathbb{Q}}(\widehat{\varrho}) \ \oplus \ (P(\mathrm{lin}_{\mathbb{Q}}(\widehat{\varrho})) \cap N)/P(\mathrm{lin}_{\mathbb{Q}}(\widehat{\varrho}) \cap F).$$

By the choice of d and d', the map P sends the primitive generators of the rays of $\widehat{\Delta}$ to the primitive generators of the rays of Δ. Thus we obtain that the isotropy of $y_{\widehat{\varrho}_{ij}}$ and $y_{\widehat{\varrho}_k}$ are all trivial.

We turn to (iii). With the dual lattices M of N and M_0 of N_0, we obtain the character groups of H and H_0 and the factor group H_0/H as

$$\mathbb{X}(H) \ = \ E/P^*(M), \qquad \mathbb{X}(H_0) \ = \ E/P^*(M_0), \qquad \mathbb{X}(H_0/H) \ = \ P^*(M)/P^*(M_0).$$

By definition of the matrices P and P_0, we have $P^*(M)/P^*(M) \cong \mathbb{Z}^s$. This implies $T \cong \mathbb{T}^s$ as claimed.

To see (iv), first note that the group T equals $\ker(\beta)$ and hence corresponds to the sublattice $\ker(B) \subseteq \mathbb{Z}^{r+s}$. Thus, the isotropy group T_{y_ϱ} for the distinguished point $y_\varrho \in Y$ corresponding to $\varrho \in \Delta$ has character group isomorphic to

$$\ker(B) \cap \mathrm{lin}_{\mathbb{Q}}(\varrho) \ \oplus \ (B(\mathrm{lin}_{\mathbb{Q}}(\varrho)) \cap N_0)/B(\mathrm{lin}_{\mathbb{Q}}(\varrho) \cap N).$$

Consequently, for $\varrho = \varrho_k$ the isotropy group T_{y_ϱ} is infinite and for $\varrho = \varrho_{ij}$ it is of order l_{ij}. □

Now we come to the construction of the embedded variety. Let $\delta \subseteq F_{\mathbb{Q}}$ be the orthant generated by the basis vectors f_{ij} and f_k. The associated affine toric variety $\overline{Z} \cong \mathbb{K}^{n+m}$ is the spectrum of the polynomial ring

$$\mathbb{K}[E \cap \delta^\vee] \ = \ \mathbb{K}[T_{ij}, S_k;\ 0 \le i \le r,\ 1 \le j \le n_i,\ 1 \le k \le m].$$

Moreover, $\overline{Z}$ contains $\widehat{Y}$ as an open toric subvariety and the complement $\overline{Z} \setminus \widehat{Y}$ is the union of all toric orbits of codimension at least two. We obtain an H_0-invariant subvariety

$$\overline{X} \ := \ V(g_{i,i+1,i+2};\ 0 \le i \le r-2) \ \subseteq \ \overline{Z}.$$

Proposition 2.16. *Set $\widehat{X} := \overline{X} \cap \widehat{Y}$. Consider the images $X' := \pi(\widehat{X}) \subseteq Y$ and $C := \beta(X') \subseteq \widetilde{Y}$.*

(i) $X' \subseteq Y$ is a normal closed T-invariant $s+1$ dimensional variety, $C \subseteq \widetilde{Y}$ is a closed non-separated curve and $\kappa(C) \subseteq \mathbb{P}_r$ is a line.

(ii) The intersection $C_{ij} := X' \cap D_{ij}$ with the toric divisor $D_{ij} \subseteq Y$ is a single T-orbit with isotropy group of order l_{ij}.

(iii) The intersection $C_k := X' \cap D_k$ with the toric divisor $D_k \subseteq Y$ is a smooth rational prime divisor consisting of points with infinite T-isotropy.

(iv) For every point $x \in X'$ not belonging to some C_{ij} or to some C_k, the isotropy group T_x is trivial.

(v) The variety X' satisfies $\Gamma(X', \mathcal{O}) = \mathbb{K}$, its divisor class group and Cox ring are given by

$$\mathrm{Cl}(X') \;\cong\; K, \qquad\qquad \mathcal{R}(X') \;\cong\; R(A, P).$$

Furthermore, the variables T_{ij} and S_k define pairwise non-associated K-prime elements in $R(A, P)$.

(vi) There is a T-equivariant completion $X' \subseteq X$ with a $\mathbb{Q}$-factorial projective variety X such that $\mathcal{R}(X) = \mathcal{R}(X')$ holds.

Proof. By the definition of P_0 and H_0, the closed subvariety $\overline{X} \subseteq \overline{Z}$ is invariant under the action of H_0. In particular, $\widehat{X}$ is H-invariant and thus the image $X' := \pi(\widehat{X})$ under the quotient map is closed as well. Moreover, the dimension of X' equals $\dim(\widehat{X}/H) = s+1$. Analogously we obtain closedness of $C = \pi_0(\widehat{X})$. The image $\kappa(C) = \kappa(\pi_0(\widehat{X}))$ is given in $\mathbb{P}_r$ by the equations

$$\alpha_{jk}U_i \;+\; \alpha_{ki}U_j \;+\; \alpha_{ij}U_k \;=\; 0$$

with the variables $U_0, \ldots, U_r$ on $\mathbb{P}_r$ corresponding to the toric divisors given by the rays $\mathbb{Q}_{\geq 0}v_i$ and $-\mathbb{Q}_{\geq 0}(v_0 + \ldots + v_{r-1})$ of $\widetilde{\Delta}$. To see this, use that pulling back the above equations via $\kappa \circ \pi_0$ gives the defining equations for $\widehat{X}$. Consequently, $\kappa(C)$ is a projective line. This shows (i).

We turn to (ii). According to Proposition 2.7, the intersection $\widehat{X} \cap \widehat{D}_{ij}$ is a single H_0-orbit. Since $\pi \colon \widehat{X} \to X'$ is a geometric quotient for the H-action, we conclude that $C_{ij} = \pi(\widehat{D}_{ij})$ is a single T-orbit. Moreover, since H acts freely, the isotropy group of $G = H_0/H$ along C_{ij} equals that of H_0 along $\widehat{D}_{ij}$ which, by Proposition 2.15 (iv), is of order l_{ij}.

For (iii) first note that the restrictions $\beta \colon D_k \to \widetilde{Y}$ are isomorphisms onto the acting torus of $\widetilde{Y}$. Moreover, the restricting κ gives an isomorphism of the acting tori of $\widetilde{Y}$ and $\mathbb{P}_r$. Consequently, β maps C_k isomorphically onto the intersection of the line C with the acting torus of $\mathbb{P}_r$. Thus, C_k is a smooth rational curve. Proposition 2.15 (iv) ensures that C_k consists of fixed points. Assertion (iv) is clear.

We prove (v). From Proposition 2.4 we infer $\Gamma(\widehat{X}, \mathcal{O})^H = \mathbb{K}$ which implies $\Gamma(X', \mathcal{O}) = \mathbb{K}$. The next step is to establish a surjection $K \to \mathrm{Cl}(X')$, where $K := E/P^*(M)$ is the character group of H. Consider the push forward π_* from the H-invariant Weil divisors on $\widehat{X}$ to the Weil divisors on X' sending $\widehat{D}$ to $\pi(\widehat{D})$. For every $w \in K$, we fix a w-homogeneous rational function $f_w \in \mathbb{K}(X')$ and define a map

$$\mu\colon K \to \mathrm{Cl}(X'), \qquad w \mapsto [\pi_* \mathrm{div}(f_w)].$$

One directly checks that this does not depend on the choice of the f_w and thus is a well defined homomorphism. In order to see that it is surjective, note that due to Proposition 2.4, we obtain C_{ij} as $\pi_*\mathrm{div}(T_{ij})$ and C_k as $\pi_*\mathrm{div}(T_k)$. The claim then follows from the observation that removing all C_{ij} and C_k from X' leaves the set $X' \cap \mathbb{T}^{r+s}$ which is isomorphic to $V \times \mathbb{T}^r$ with a proper open subset $V \subseteq \kappa(C)$ and hence has trivial divisor class group.

Now [29, Theorem 1.3] shows that the Cox ring of X' is $R(A, P)$ with the $\mathrm{Cl}(X')$-grading given by $\deg(T_{ij}) = [C_{ij}]$ and $\deg(S_k) = [C_k]$. Consequently, $R(A, P)$ is factorially $\mathrm{Cl}(X')$-graded and thus also the finer K-grading is factorial. Since H acts freely on $\widehat{X}$, we can conclude $\mathrm{Cl}(X') = K$.

Finally, we construct a completion of $X' \subseteq X$ as wanted in (vi). Choose any simplicial projective fan Σ in N having the same rays as Δ, see [44, Corollary 3.8]. The associated toric variety Z is projective and it is the good quotient of an open toric subset $\widehat{Z} \subseteq \overline{Z}$ by the action of H. The closure X of X' in Z is projective and, as the good quotient of the normal variety $\overline{X} \cap \widehat{Z}$, it is normal. By Proposition 2.7, the complement $X \setminus X'$ is of codimension at least two, which gives $\mathcal{R}(X) = \mathcal{R}(X')$. From [25, Cor. 4.13] we infer that X is $\mathbb{Q}$-factorial. □

Remark 2.17. We may realize any given $R(A, P)$ as a subring of the Cox ring of a surface: For every $l_i = (l_{i1}, \ldots, l_{in_i})$ choose a tuple $d_i = (d_{i1}, \ldots, d_{in_i})$ of positive integers with $\gcd(l_{ij}, d_{ij}) = 1$ and $d_{i1}/l_{i1} < \ldots < d_{in_i}/l_{in_i}$. Then take

$$P := \begin{pmatrix} L_0 & 0 & 0 \\ d & 1 & -1 \end{pmatrix}.$$

Theorem 2.18. *In the notation of Construction 2.10, the following holds.*

(a) *The algebra $R(A, P)$ is factorially K-graded and the K-grading is almost free, effective and pointed. Moreover, T_{ij}, S_k define pairwise non-associated K-prime generators.*

(b) *The K-graded algebra $R(A, P)$ is the Cox ring of a $\mathbb{Q}$-factorial rational projective variety with a complexity one torus action.*

Proof. These statements are a direct consequence of Proposition 2.16. □

Note that, if K is torsion free, then K-factoriality of $R(A, P)$ implies factoriality, see [4, Theorem 4.2]. The converse, however, is not true. There are K-graded factorial algebras $R(A, P)$ with non torsion free K, see Remark 2.11.

Theorem 2.19. *Let X be a normal rational complete variety with a torus action of complexity one. Then the Cox ring of X is isomorphic as a graded ring to some $R(A, P)$ with a K-grading as in Construction 2.10.*

Proof. According to [29, Theorem 1.3], the Cox ring $\mathcal{R}(X)$ is isomorphic to a ring $R(A, P)$ with a grading by $K := \mathrm{Cl}(X)$ such that the variables T_{ij} and S_k are homogeneous. In particular, X is the quotient by the action of $H = \operatorname{Spec} \mathbb{K}[K]$ on an open subset $\widehat{X}$ of

$$\overline{X} \;=\; V(g_{i,i+1,i+2};\ 0 \le i \le r-2) \;\subseteq\; \overline{Z}.$$

For $r < 2$, the variety X is toric. We may assume that T acts as a subtorus of the big torus and the assertion follows by standard toric geometry. So, let $r \geq 2$. By construction, the K_0-grading of $R(A, P)$ and $R(A, P_0)$ respectively, is the finest possible such that all variables T_{ij} and S_k are homogeneous. Consequently, we have exact sequences of abelian groups fitting into a commutative diagram.

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & \longrightarrow & 0 & \longrightarrow & \widetilde{K} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_0 & \xrightarrow{P_0^*} & E & \longrightarrow & K_0 & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & M/M_0 & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
\tag{2}
$$

In particular we extract from this the following two commutative triangles, where the second one is obtained by dualizing the first one.

$$
\begin{array}{ccc}
E & \longleftarrow & M \\
{\scriptstyle P_0^*}\nwarrow & & \nearrow \\
 & M_0 &
\end{array}
\qquad\qquad
\begin{array}{ccc}
F & \xrightarrow{P} & N \\
{\scriptstyle P_0}\searrow & & \swarrow \\
 & N_0 &
\end{array}
\tag{3}
$$

We claim that the kernel $\widetilde{K}$ is free. Consider $H_0 := \mathrm{Spec}(\mathbb{K}[K_0])$ and the isotropy group ${H_0}^{ij} \subseteq H_0$ of a general point $\widehat{x}(i,j) \in \widehat{X} \cap V(T_{ij})$. Then we have exact sequences

$$1 \longleftarrow H_0/{H_0}^{ij} \longleftarrow H_0 \longleftarrow {H_0}^{ij} \longleftarrow 1\ ,$$

$$0 \longrightarrow K_0(i,j) \longrightarrow K_0 \longrightarrow K_0/K_0(i,j) \longrightarrow 0\ ,$$

where the second one arises from the first one by passing to the character groups. Note that the subgroup $K_0(i,j) \subseteq K_0$ is given by

$$K_0(i,j) \ = \ \mathrm{lin}_{\mathbb{Z}}(\deg T_{kl};\ (k,l) \neq (i,j)) + \mathrm{lin}_{\mathbb{Z}}(\deg T_p;\ 1 \leq p \leq m). \tag{4}$$

Now [29, Theorem 1.3] tells us that each variable T_{ij} defines a K-prime element in $\mathcal{R}(X)$ and thus its divisor is H-prime. Consequently, $H_0/H{H_0}^{ij}$ is connected and has a free character group

$$\mathbb{X}(H_0/H{H_0}^{ij}) \ = \ \widetilde{K}(i,j) \ := \ \widetilde{K} \cap K_0(i,j).$$

Mimicking equation (4), we define a subgroup $K(i,j) \subseteq K$ fitting into a commutative net of exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{K}(i,j) & \longrightarrow & \widetilde{K} & \longrightarrow & \widetilde{K}/\widetilde{K}(i,j) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0(i,j) & \longrightarrow & K_0 & \longrightarrow & K_0/K_0(i,j) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K(i,j) & \longrightarrow & K & \longrightarrow & K/K(i,j) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5}$$

By general properties of Cox rings [25, Prop. 2.2] we must have $K/K(i,j) = 0$ and thus we conclude

$$\widetilde{K}/\widetilde{K}(i,j) \ = \ K_0/K_0(i,j) \ \cong \ \mathbb{Z}/l_{ij}\mathbb{Z}. \tag{6}$$

Consider again $\widehat{x}(i,j) \in V(T_{ij}) \cap \widehat{X}$, set $x(i,j) := p_X(\widehat{x}(i,j))$ and let T denote the torus acting on X. Then [29, Prop. 2.6] and its proof provides a commutative diagram

$$\begin{array}{ccc}
\widetilde{H} & \supseteq & \widetilde{H}_{ij} \\
\uparrow & & \uparrow \cong \\
T & \supseteq & T_{x(i,j)}
\end{array}$$

where $\widetilde{H} = H_0/\mathrm{H}$ and $\widetilde{H}_{ij} = H_0/HH_0{}^{ij}$. Using (6) and passing to the character groups we arrive at a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{K}(i,j) & \longrightarrow & \widetilde{K} & \longrightarrow & \mathbb{Z}/l_{ij}\mathbb{Z} & \longrightarrow & 0 \\ & & & & \downarrow & & \uparrow \cong & & \\ & & & & \mathbb{X}(T) & \longrightarrow & \mathbb{Z}/l_{ij}\mathbb{Z} & \longrightarrow & 0 \end{array}$$

with exact rows. As seen before, the group $\widetilde{K}(i,j)$ is free abelian. Consequently, also $\widetilde{K}$ must be free abelian.

Now the snake lemma tells us that M/M_0 is free as well. In particular, the first vertical sequence of (2) splits. Thus, we obtain the desired matrix presentation of P from rewriting the second commutative triangle of (3) as

$$\begin{array}{ccc} F & \xrightarrow{\quad P \quad} & N_0 \oplus N/M_0^{\perp} \\ & {}_{P_0}\searrow \quad \swarrow & \\ & N_0 \, . & \end{array}$$

□

Construction 2.20. Let (A, P) be data as in Construction 2.10 and consider the associated K-graded ring $R := R(A, P)$. Let $\mathfrak{F}$ be a system of generators of R consisting of the variables T_{ij} and S_k. This data defines a projected cone $(E \stackrel{Q}{\to} K, \gamma)$, where γ is the positive orthant in $E := \mathbb{Z}^{n+m}$ generated by the canonical basis vectors e_{ij}, e_k and $K := E/\mathrm{im}(P^*)$. The map $Q\colon E \to K$ is the projection. Every $\mathfrak{F}$-bunch Φ defines a bunched ring $(R, \mathfrak{F}, \Phi)$. The K-grading defines an $H := \mathrm{Spec}(\mathbb{K}[K])$-action on $\overline{X} := \overline{X}(A, P) := \mathrm{Spec}(R)$ and according to Construction 1.9 the varieties

$$\widehat{X} \;:=\; \widehat{X}(A, P, \Phi) \;:=\; X(R, \mathfrak{F}, \Phi) \quad \text{and} \quad X \;:=\; X(A, P, \Phi) \;:=\; X(R, \mathfrak{F}, \Phi),$$

where $X = \widehat{X}/H$. The action of $H_0 := \mathrm{Spec}(\mathbb{K}[K_0])$ on $\overline{X}$ leaves $\widehat{X}$ invariant and induces consequently an effective complexity-one action of the torus $T := H_0/H = \mathrm{Spec}(\mathbb{K}[\mathbb{Z}^s])$ on X. Note that T is the stabilizer of X under the action of H_0. Since every chamber λ of the GIT-fan is defining an $\mathfrak{F}$-bunch $\Phi(\lambda)$, we can also define

$$X(A, P, \lambda) \;:=\; X(A, P, \Phi(\lambda)).$$

The variety $X := X(A, P, \lambda)$ is a normal projective variety with $\dim(X) = \dim(R) - \dim(K_{\mathbb{Q}}) = s + 1$ and $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$. By construction, R is the Cox ring of X and $\pi\colon \widehat{X} \to X$ is a characteristic space for X. There is an isomorphism $K \to \mathrm{Cl}(X)$ sending the variables T_{ij} and S_k to the divisors $D_X^{ij} = \pi(V(\widehat{X}, T_{ij}))$ and $D_X^k = \pi(V(\widehat{X}, T_k))$ in X.

Definition 2.21. Let $X = X(A, P, \Phi)$ a complexity-one T-variety. Then P is called the *P-matrix* of X.

Theorem 2.22. *Let X be an n-dimensional projective normal rational variety with an effective action of an $(n-1)$-dimensional torus T. Then X is equivariantly isomorphic to a complexity-one T-variety arising from data (A, P) as in Construction 2.20.*

Proof. This theorem is a direct consequence of Theorem 2.19. □

2.3 Normal form of complexity-one T-varieties

This section describes isomorphisms of complexity-one T-varieties $X = X(A, P, \Phi)$ in terms of the defining data (A, P, Φ). For this purpose we introduce normal forms for the defining matrix P as well as for the graded ring $R(A, P)$.

Definition 2.23. We call an elementary row or column operation of the matrix P *admissible* if it is one of the following:

(i) Switch two columns inside a block $v_{i1}, \ldots v_{in_i}$.

(ii) Switch two whole column blocks $v_{i1}, \ldots, v_{in_i}$ and $v_{j1}, \ldots, v_{jn_j}$.

(iii) Add multiples of the upper r rows to one of the last s rows.

(iv) Any elementary row operation among the last s rows.

(v) Switch two columns inside the d' block.

We will see that operations of type (iii) and (iv) do not change the ring $R(A, P)$ whereas (i), (ii) and (v) cause switches of the involved variables that do not affect the isomorphy type of $R(A, P)$.

Definition 2.24. Two pairs (A, P) and (A', P') as introduced in Construction 2.10 are said to be *isomorphic* if

$$A' = B \cdot A \cdot D \qquad \text{and} \qquad P' = S \cdot P \cdot U$$

with a matrix $B \in \mathrm{GL}(2, \mathbb{K})$, a diagonal matrix $D \in \mathrm{GL}(r+1, \mathbb{K})$, a unimodular matrix S causing admissible matrix operations of type (iii) and (iv) and a permutation matrix U built from permutation blocks of sizes $n_0, \ldots, n_r, m$ causing admissible matrix operations of type (i), (ii) and (v). We call two matrices A and A' *isomorphic* if $A' = B \cdot A \cdot D$ as above.

Remark 2.25. The vectors $a_i \in \mathbb{K}^2 \setminus \{0\}$ define points in $\mathbb{P}_1$ with fixed given coordinates. Consequently, applying the matrix D on A is kind of a scaling and just means that the coordinates of the points a_i do change. The matrix B represents an automorphism of $\mathbb{K}^2 \setminus \{0\}$ which can be interpreted as automorphism of $\mathbb{P}_1$ and can hence even be chosen out of $\mathrm{SL}(2, \mathbb{K})$. Note that we need the images of three points of $\mathbb{P}_1$ to fix an automorphism of $\mathbb{P}_1$.

Definition 2.26. In the situation of Construction 2.10 the matrix A is called *standard* if the relations of a ring $R(A,P)$ have the following form,

$$
\begin{aligned}
g_0 &= T_0^{l_0} + T_1^{l_1} + T_2^{l_2} \\
g_1 &= \lambda_1 T_1^{l_1} + T_2^{l_2} + T_3^{l_3} \\
g_2 &= \lambda_2 T_2^{l_2} + T_3^{l_3} + \lambda_1 T_4^{l_4} \\
g_3 &= \lambda_3 T_3^{l_3} + T_4^{l_4} + \lambda_2 T_5^{l_5} \\
&\vdots \\
g_{r-3} &= \lambda_{r-3} T_{r-3}^{l_{r-3}} + T_{r-2}^{l_{r-2}} + \lambda_{r-4} T_{r-1}^{l_{r-1}} \\
g_{r-2} &= \lambda_{r-2} T_{r-2}^{l_{r-2}} + T_{r-1}^{l_{r-1}} + \lambda_{r-3} T_r^{l_r}
\end{aligned}
$$

where $1, \lambda_1, \ldots, \lambda_{r-2} \in \mathbb{K}^*$ are pairwisely different. Note that $\lambda_i \neq 1$ is due to the fact that (a_i, a_k) is linearly independent whenever $i \neq k$.

Lemma 2.27. *Every matrix A is isomorphic to a unique standard matrix.*

Proof. By applying a suitable matrix B, the first three points a_0, a_1 and a_2 can be mapped to scalar multiples of the points $(1,0)$, $(0,1)$ and $(-1,-1)$. By scaling these coordinates by an appropriate diagonal matrix D we can achieve $a_0' = (1,0)$, $a_1' = (0,1)$ and $a_2' = (-1,-1)$. Furthermore, we can choose D in such a way that the points $a_3, \ldots, a_r$ are send to points $a_3', \ldots, a_r'$ satisfying $\det(a_{i+2}, a_i) = 1$ for all $3 \leq i \leq r-2$, i.e. the coefficient of the second monomial of each relation equals one. □

Corollary 2.28. *Two matrices A and A' are isomorphic if and only if they have the same standard matrix.*

Proposition 2.29. *Let $R(A,P)$ be a graded ring and let $1, \lambda_1, \ldots, \lambda_{r-2} \in \mathbb{K}^*$ be pairwisely different. Then $R(A,P)$ is isomorphic to a ring $\mathbb{K}[T_{ij}, S_k]/\langle g_0, \ldots, g_{r-2}\rangle$ with relations of the following form:*

$$
\begin{aligned}
g_0 &= T_0^{l_0} + T_1^{l_1} + T_2^{l_2} \\
g_1 &= \lambda_1 T_1^{l_1} + T_2^{l_2} + T_3^{l_3} \\
g_2 &= \lambda_2 T_2^{l_2} + T_3^{l_3} + T_4^{l_4} \\
&\vdots \\
g_{r-2} &= \lambda_{r-2} T_{r-2}^{l_{r-2}} + T_{r-1}^{l_{r-1}} + T_r^{l_r}.
\end{aligned}
$$

Proof. By Lemma 2.27 we may assume A to be standard. Applying ring homomorphisms we have two further possibilities to simplify the coefficients of the relations $g_0, \ldots, g_{r-2}$. Firstly, we can send variables T_{ij} to scalar multiples $c_{ij} T_{ij}$ with $c_{ij} \in \mathbb{K}^*$. Secondly, we can multiply a whole relation with an element of $\mathbb{K}^*$ which is not changing the ideal generated by the relations. These operations are sufficient to obtain the desired coefficients. □

Remark 2.30. In general, the ring $\mathbb{K}[T_{ij}, S_k]/\langle g_0, \dots, g_{r-2}\rangle$ defined in Proposition 2.29 is not an $R(A,P)$-ring, but for $r \leq 3$ it is.

Example 2.31. Consider the points $a_0 = (1,1)$, $a_1 = (0,1)$, $a_2 = (-1,2)$ $a_3 = (1,3)$, $a_4 = (1,2)$ and $a_5 = (1,-1)$. Then we obtain

$$
\begin{aligned}
B \cdot A \cdot D &= \begin{pmatrix} -3 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -1 & 1 & -\frac{3}{4} & -\frac{3}{4} \\ 1 & 1 & -1 & -\frac{2}{3} & \frac{1}{4} & -\frac{1}{2} \end{pmatrix} = A',
\end{aligned}
$$

where the matrix A' is standard. Then the relations are of the form

$$
\begin{aligned}
g_0 &= T_0^{l_0} + T_1^{l_1} + T_2^{l_2} \\
g_1 &= \frac{5}{3}T_1^{l_1} + T_2^{l_2} + T_3^{l_3} \\
g_2 &= -\frac{1}{4}T_2^{l_2} + T_3^{l_3} + \frac{5}{3}T_4^{l_4} \\
g_3 &= \frac{9}{16}T_3^{l_3} + T_4^{l_4} - \frac{1}{4}T_5^{l_5}.
\end{aligned}
$$

Now apply the ring homomorphism sending $T_4 \mapsto \sqrt[l_4]{\frac{3}{5}} \cdot T_4$ and $T_5 \mapsto \sqrt[l_5]{-\frac{12}{5}} \cdot T_5$ and multiply the last relation with $\frac{5}{3}$. Then we have

$$
\begin{aligned}
g_0 &= T_0^{l_0} + T_1^{l_1} + T_2^{l_2} \\
g_1 &= \frac{5}{3}T_1^{l_1} + T_2^{l_2} + T_3^{l_3} \\
g_2 &= -\frac{1}{4}T_2^{l_2} + T_3^{l_3} + T_4^{l_4} \\
g_3 &= \frac{15}{16}T_3^{l_3} + T_4^{l_4} + T_5^{l_5}.
\end{aligned}
$$

We recall the notion of a graded ring homomorphism. Let R and R' be graded rings with grading groups K and K' respectively. Then $(\varphi, \widetilde{\varphi})$ is called a *graded ring homomorphism* if $\varphi\colon R \to R'$ is a ring homomorphism and $\widetilde{\varphi}\colon K \to K'$ is a group homomorphism such that $\varphi(R_w) \subseteq R'_{\widetilde{\varphi}(w)}$ is satisfied for all $w \in K$.

Proposition 2.32. *Let (A,P) and (A',P') be given as in Construction 2.10 and consider the associated graded rings $R(A,P)$ and $R(A',P')$. Then (A,P) is isomorphic to (A',P') if and only if $R(A,P)$ and $R(A',P')$ are isomorphic as graded rings.*

Proof. Let us assume that (A, P) is isomorphic to (A', P'). Then we have $A' = B \cdot A \cdot D$. Applying the matrix B on A maps each point a_i to $B(a_i)$. The coefficients of the relations of $R(A, P)$ are given by $\alpha_{ij} = \det(a_i, a_j)$ and the coefficients of the relations of $R(B \cdot A, P)$ are given by $\beta_{ij} = (B(a_i), B(a_j)) = \det(B) \cdot \alpha_{ij}$. Consequently, we obtain

$$\begin{aligned} R(A,P) &= \mathbb{K}[T_{ij}, S_k]/\langle g_i;\ 0 \leq i \leq r-2\rangle \\ &= \mathbb{K}[T_{ij}, S_k]/\langle \det(B) \cdot g_i;\ 0 \leq i \leq r-2\rangle \\ &= R(B \cdot A, P). \end{aligned}$$

The scaling matrix $D = \operatorname{diag}(d_0, \ldots, d_r)$ maps each point a_i to $d_i \cdot a_i$. Hence, the coefficients of the relations of $R(A \cdot D, P)$ are given by $\delta_{ij} = d_i d_j \alpha_{ij}$ and the matrix D induces a ring isomorphism

$$R(A,P) \to R(A \cdot D, P),\ T_{ij} \mapsto \begin{cases} d_{i-1} d_{i+1} d_{i+2} T_{i1} & \text{for } j = 1 \\ T_{ij} & \text{for } j \neq 1 \end{cases},$$

where $d_{-1} := 1$. Consequently, $R(A, P)$ is isomorphic to $R(A', P)$ for $A' = B \cdot A \cdot D$ and it is sufficient to consider the rings $R(A, P)$ and $R(A, P')$ with a standard matrix A. The condition $P' = S \cdot P \cdot U$ induces an automorphism $\mathbb{K}[T_{ij}, S_k] \to \mathbb{K}[T'_{ij}, S'_k]$ sending the variables T_{ij}, S_k to T'_{ij}, S'_k according to the rules given in Definition 2.24. The matrix U permutes the blocks indexed from 0 to r of the matrix P and allows that variables within one block are permuted. The unimodular matrix S satisfying the conditions of Definition 2.24 leaves the ring invariant and respects the grading of $R(A, P)$. Consequently, we obtain

$$\begin{aligned} R(A,P) &= \mathbb{K}[T_{ij}, S_k]/\langle g_i, 0 \leq i \leq r-2\rangle \\ &\cong \mathbb{K}[T'_{ij}, S'_k]/\langle g'_i, 0 \leq i \leq r-2\rangle \\ &= R(A, P'). \end{aligned}$$

Now let us assume that $\varphi^*\colon R(A', P') \to R(A, P)$ is an isomorphism with A' and A in standard form and set $\overline{X} := \operatorname{Spec}(R(A, P))$ as well as $\overline{X}' := \operatorname{Spec}(R(A', P'))$. Then the maximal K_0- and K'_0-gradings of $R(A, P)$ and $R(A', P')$ (defined by the gradiator matrices P_0 and P'_0) induce maximal torus actions on $\overline{X}$ and $\overline{X}'$ by the tori $T_0 := \operatorname{Spec}(\mathbb{K}[K_0])$ and $T'_0 := \operatorname{Spec}(\mathbb{K}[K'_0])$, respectively. The isomorphism φ^* defines an affine equivariant isomorphism $\varphi\colon \overline{X} \to \overline{X}'$. Consider an open set $\widehat{X} := \overline{X}^{\mathrm{ss}}(w) \subseteq \overline{X}$ where $w \in \operatorname{Mov}(\overline{X})$. Let $\widehat{X}_0$ denote the open subset of $\widehat{X}$ consisting of all points $x \in \widehat{X}$ with finite isotropy, i.e. $\dim(T_{0x}) = 0$. Since φ is an isomorphism, the set $\widehat{X}'_0 := \varphi(\widehat{X}_0)$ contains all points $x' \in \widehat{X}'$ with finite isotropy. The variables T_{ij}, S_k represent prime divisors $\overline{D}_{ij}, \overline{E}_k$ on $\overline{X}$. By choice of $\widehat{X}$ we have prime divisors $\widehat{D}_{ij} = \widehat{X} \cap \overline{D}_{ij}$ and $\widehat{E}_k = \widehat{X} \cap \overline{E}_k$. Analogously, we have prime divisors $\overline{D}'_{ij}, \overline{E}'_k$ of $\overline{X}'$ and prime divisors $\widehat{D}'_{ij}, \widehat{E}'_k$ of $\widehat{X}'$. Furthermore, we obtain the following commutative diagram:

$$\begin{array}{ccc} \widehat{X}_0 & \xrightarrow{\varphi} & \widehat{X}'_0 \\ \big\downarrow{\scriptstyle /\!\!/ T_0} & & \big\downarrow{\scriptstyle /\!\!/ T'_0} \\ \widehat{X}_0/T_0 & \xrightarrow{\cong} & \widehat{X}'_0/T'_0 \\ \big\downarrow & & \big\downarrow \\ \mathbb{P}_1 & \xrightarrow{\cong} & \mathbb{P}_1 \end{array}$$

The lower downward arrows are separation maps sending multiplied points to one point and the relations g_i and g'_i to linear relations on $\mathbb{P}_1$. Consequently, we have a linear automorphism $\mu\colon \mathbb{P}_1 \to \mathbb{P}_1$, i.e $\mu \in \mathrm{GL}(2,\mathbb{K})/\mathbb{K}^* = \mathrm{PGL}(1) = \mathrm{PSL}(1)$. Every such μ comes from a matrix $B \in \mathrm{GL}(2,\mathbb{K})$ defining an automorphism of $\mathbb{K}^2$. It sends A to A' such that $\mu(a_i) = a'_j$. This implies $n_i = n'_j$. Consequently, the map φ sends the set of prime divisors $\{\widehat{D}_{ij}, \widehat{E}_k\}$ on $\widehat{X}$ and $\{\widehat{D}'_{ij}, \widehat{E}'_k\}$ on $\widehat{X}'$ bijectively into each other in such a way that any tower $\widehat{D}_{i*}$ is sent to some tower $\widehat{D}_{j*}$. Since $l_{ij} = |T_{0\widehat{D}_{ij}}|$ holds, we conclude that, inside one tower, $\widehat{D}_{ij}$ is mapped to $\widehat{D}'_{i'j'}$ if and only if $l_{ij} = l'_{i'j'}$ holds. The prime divisors with infinite isotropy $\widehat{E}_k$ are sent to $\widehat{E}'_k$. These conditions correspond exactly to matrices U as described in Definition 2.24. Note that applying D from the right side to A does not change the points a_i in $\mathbb{P}_1$ since the homogeneous coordinates of a point a_i are only multiplied by a scalar of $\mathbb{K}^*$. Moreover, applying S from the left side to P does not change the ring $R(A,P)$. □

Corollary 2.33. *Let $R(A,P)$ and $R(A',P')$ be the Cox rings of two non toric complexity-one varieties $X = X(A,P,\lambda)$ and $X' = X(A',P',\lambda')$ with acting tori T and T' and let K and K' denote the associated grading groups. Then the following three statements are equivalent:*

(i) $R(A,P) \cong R(A',P')$ *and there is a graded isomorphism* $(\varphi,\widetilde{\varphi})$ *such that* $\widetilde{\varphi}(\lambda) = \lambda'$.

(ii) $X(A,P,\lambda) \cong X(A',P',\lambda')$ *as complexity-one varieties.*

(iii) $X(A,P,\lambda) \cong X(A',P',\lambda')$ *as algebraic varieties.*

Proof. The first and the second statement are equivalent by [9, Corollary 6.8]. From [6, Theorem 5.5] we infer that the automorphism groups $\mathrm{Aut}(X)$ and $\mathrm{Aut}(X')$ of X and X' are linear algebraic groups with maximal torus T and T', respectively. Let $\varphi\colon X \to X'$ be an isomorphism. Then the T-action on X defines a $\varphi(T)$-action on X'. Since $\varphi(T)$ is conjugated to T' this implies that X and X' are even equivariantly isomorphic (as complexity-one varieties). □

Given data (A,P) fixes $X = X(A,P,\Phi)$ up to small birational equivalence, depending on the $\mathfrak{F}$-bunch Φ. In order to fix X up to isomorphy one has to fix Φ, which means fixing an interior GIT-chamber of the moving cone in K. The anticanonical class $w_X := [-K_X] \in K$ is not affected by small birational modifications. Since $R(A,P)$ is a complete intersection we infer from [9, Proposition 8.5] that

$$w_X \;=\; Q(e_{w_X}) \;=\; \sum_{i=0}^{r}\sum_{j=1}^{n_i} Q(e_{ij}) + \sum_{k=1}^{m} Q(e_k) - \sum_{i=0}^{r-2} \deg(g_i).$$

Anyway, if X satisfies some special properties, then (A,P) defines a unique variety X.

Definition 2.34. We call X a *Fano variety* if its anticanonical class w_X is ample. Furthermore, X is of *Picard number* $n \in \mathbb{Z}_{>0}$ if $\mathrm{rk}(\mathrm{Pic}(X)) = n$.

If X is a Fano variety, then (A,P) fixes X already up to isomorphy, since the anticanonical chamber is uniquely determined and given by $\lambda(w_X)$ and we have $X = X(A,P,\lambda(w_X))$. Furthermore, if X has Picard number one, then there is only one fulldimensional chamber in the moving cone, which is the moving cone itself, i.e. the positive orthant in $K_{\mathbb{Q}} = \mathbb{Q}$. Consequently, in this case the variety X is also uniquely determined, Fano or not. If X is a projective surface with finitely generated Cox ring, then $\mathrm{Ample}(X) = \mathrm{Mov}(X)^\circ$ holds. In particular, we have $X = X(A,P,\lambda(w))$ for an arbitrary $w \in \mathrm{Mov}(X)^\circ$. Consequently, for varieties satisfying one of these properties the notation $X(A,P)$ is justified.

Corollary 2.35. *Let (A,P) and (A',P') be data defining projective varieties $X(A,P)$ and $X(A',P')$ that are of dimension two or Fano varieties or varieties with Picard number one. Then the following statements are equivalent.*

(i) $X(A,P) \cong X(A',P')$ *as complexity-one varieties.*

(ii) $X(A,P) \cong X(A',P')$ *as algebraic varieties.*

(iii) $R(A,P) \cong R(A',P')$ *as graded rings.*

Corollary 2.36. *Let $X = X(A,P)$ be a $\mathbb{K}^*$-surface, i.e. $s = 1$. Then we can find a matrix P' such that $X(A,P)$ is isomorphic to $X(A,P')$ and P' satisfies*

$$n_0 \geq n_1 \geq \ldots \geq n_r, \quad l_{in_i} > d_{in_i} \geq 0 \quad \textit{and} \quad \frac{d_{ij}}{l_{ij}} < \frac{d_{ij+1}}{l_{ij+1}}.$$

3 Resolution of complexity-one T-varieties

This chapter is dedicated to the resolution of singularities of complexity-one T-varieties. We will discuss a canonical way of resolving a complexity-one T-variety X by toric modifications of its minimal toric ambient variety Z, so-called toric ambient modifications. The last section of this chapter is dedicated to discrepancies of such modifications.

3.1 Toric ambient modifications

As seen in Construction 1.11 each variety $X = X(R, \mathfrak{F}, \Phi)$ comes with a natural embedding into a toric ambient variety. Consequently, it suggests itself to use toric resolution theory to resolve their singularities. We briefly recall the concept of *toric ambient modifications*, see [25, Sections 5 and 6].

Consider a complexity-one T-variety $X = X(A, P, \Phi)$ and its canonical toric embedding $X \subseteq Z$ and denote the fan associated to Z by Σ. Let $\sigma \in \Sigma$ be generated by some columns of the matrix P. We consider a primitive lattice vector $v_\infty \in \sigma^\circ$. Then we can find non-negative integers a_{ij}, a_k and $m_\infty \in \mathbb{Z}_{>0}$ with $\gcd(a_{ij}, a_k, m_\infty) = 1$ such that

$$m_\infty v_\infty \;=\; \sum_{i=0}^{r} \sum_{j=i}^{n_i} a_{ij} v_{ij} + \sum_{k=1}^{m} a_k v_k,$$

and $\sigma = \operatorname{cone}(v_{ij}, v_k;\; a_{ij} \neq 0, a_k \neq 0)$. The ray $\mathbb{Q}_{>0} \cdot v_\infty$ subdivides the cone σ. We call m_∞ the *index* of this subdivision. Let Σ' be the fan that we obtain by the stellar subdivision of σ at v_∞ and denote the associated toric variety by Z'. Consider the Cox constructions $P' \colon \widehat{\Sigma}' \to \Sigma'$ and $P \colon \widehat{\Sigma} \to \Sigma$. The fans $\widehat{\Sigma}$ and $\widehat{\Sigma}'$ lie in the lattices

$$F = \bigoplus_{i=0}^{r} \bigoplus_{j=1}^{n_i} \mathbb{Z} e_{ij} \oplus \bigoplus_{i=1}^{k} \mathbb{Z} e_k \quad \text{and} \quad F' = F \oplus \mathbb{Z} e_\infty,$$

and consist of faces of the positive orthants in F and F' respectively. The projection maps are given by

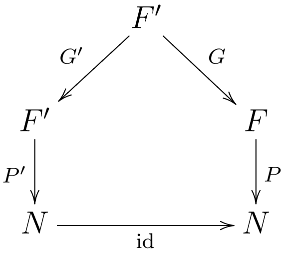
$$P \colon F \to N,\; e_{ij} \mapsto v_{ij},\; e_k \mapsto v_k, \qquad P' \colon F' \to N,\; e_{ij} \mapsto v_{ij},\; e_k \mapsto v_k,\; e_\infty \mapsto v_\infty.$$

Furthermore, we have the following lattice homomorphisms:

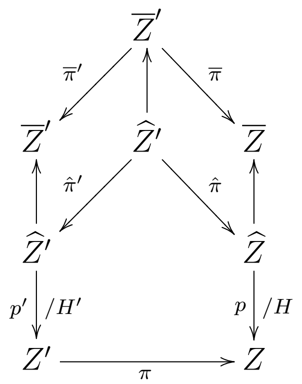
$$G \colon F' \to F, \quad e_{ij} \mapsto e_{ij},\; e_k \mapsto e_k,\; e_\infty \mapsto \sum_{i=0}^{r} \sum_{j=1}^{n_i} a_{ij} e_{ij} + \sum_{k=1}^{m} a_k e_k,$$

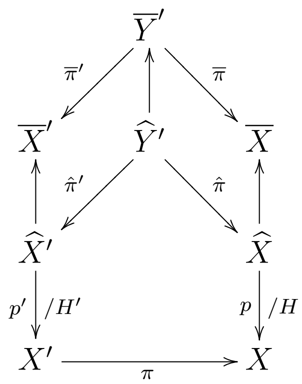
$$G' \colon F' \to F', \quad e_{ij} \mapsto e_{ij},\; e_k \mapsto e_k,\; e_\infty \mapsto m_\infty e_\infty,$$

fitting in a commutative diagram:



The homomorphism G defines maps of fans $\widehat{\Sigma}' \to \widehat{\Sigma}$ and $\overline{\Sigma}' \to \overline{\Sigma}$, where $\overline{\Sigma}'$ and $\overline{\Sigma}$ are the fans of faces of the positive orthants in F' and F respectively. The stellar subdivision $\Sigma' \to \Sigma$ defines a toric modification $\pi\colon Z' \to Z$. Let $E \subseteq Z'$ be the exceptional divisor and denote the strict transform of X under this modification by $X' := \pi^{-1}(X)$. We call $\pi\colon Z' \to Z$ a *neat ambient modification* for $X' \subseteq Z'$ and $X \subseteq Z$ if $X \cap \pi(E)$ is of codimension at least two in X. If this is the case, then we set $\overline{Y}' := \overline{\pi}'^{-1}(\overline{X}')$ and we have commutative diagrams





where $\pi\colon Z' \to Z$ properly contracts an invariant prime divisor, p and p' are geometric quotients of quasitorus actions, where $H := \ker(p)$ and $H' = \ker(p')$, $\overline{\pi}\colon \overline{Z}' \to \overline{Z}$ is the quotient for a $\mathbb{K}^*$-action and $\overline{\pi}'\colon \overline{Z}' \to \overline{Z}'$ is the quotient of an action of the group C_{m_∞} of m_∞-th roots.

Let X be a $\mathbb{Q}$-factorial projective variety with finitely generated Cox ring. We call a class $[D] \in \mathrm{Cl}(X)$ *combinatorially contractible* if it generates an extremal ray of the effective cone of X and, for some representative D and all $n > 0$, the vector spaces $\Gamma(X, \mathcal{O}(nD))$ are of dimension one. Furthermore, X is called *combinatorially minimal* if it has no combinatorially contractible divisor classes. By [25, Theorem 6.2] all $\mathbb{Q}$-factorial projective varieties with finitely generated Cox ring arise from combinatorially minimal ones by toric ambient modifications and small birational transformations. Furthermore, [25, Corollary 6.8] states that all $\mathbb{Q}$-factorial projective varieties with finitely generated Cox ring are combinatorially minimal if and only if $\mathrm{Eff}(X) = \mathrm{Mov}(X)$ holds.

Let $X' = X(A, P, \Phi)$ be a complexity-one T-variety with minimal toric ambient variety Z' and consider the weights $w_{ij} := \deg(T_{ij})$ and $u_k := \deg(S_k)$. We call a weight $w \in$

$\{w_{ij}, u_k\}$ *exceptional*, if $\mathbb{Q}_{\geq 0}w$ is an extremal ray of $\mathrm{Eff}(X')$ and there is no other weight $w_{i,j}$, u_k contained in $\mathbb{Q}_{\geq 0}w$. Consider the prime divisors D_{ij}, E_k of X' corresponding to the variables T_{ij} and S_k and let D_∞ be one of them. Furthermore, let w_∞ be the degree of the corresponding variable in $K \cong \mathrm{Cl}(X')$. In this setting we can formulate the Gale dual description of toric ambient modifications in K. The following statements are equivalent:

(i) There is a neat toric ambient modification $\pi\colon Z' \to Z$ for $X' \subseteq Z'$ and $X \subseteq Z$ contracting the divisor D_∞, where $X = \pi(X')$.

(ii) The weight $w_\infty \in K$ is exceptional and $w_\infty^0 \in \lambda$ holds for all fulldimensional chambers $\lambda \in \Lambda(\overline{X})$ having a common facet with $\lambda' := \mathrm{SAmple}(X')$. Here w_∞^0 denotes the class of w_∞ in $K^0 = K/K^t$, where K^t is the torsion part of K.

3.2 Resolution via weak tropicalisation

In this section we introduce a canonical resolution for complexity-one T-varieties. It uses the concept of weak tropical resolutions as defined in [7].

Let X be a complexity-one T-variety. A *resolution* of singularities for X is a morphism $\pi\colon X' \to X$ such that X' is smooth, π is proper, and the restriction $\pi\colon \pi^{-1}(X_{\mathrm{reg}}) \to X_{\mathrm{reg}}$ is an isomorphism, where X_{reg} denotes the set of non-singular points.

Consider a complexity-one T-variety $X = X(A, P, \Phi)$ and its minimal toric ambient variety Z with associated fan Σ. There are two reasons for X having singularities. Firstly, X inherits singularities from its minimal toric ambient variety Z. Consider a point $x \in X_{\gamma_0} = X_\sigma$ with $\gamma_0 \in \mathrm{rlv}(\Phi)$ and $\sigma = P(\gamma_0^*) \in \Sigma$. If σ is not regular, then Z is singular along Z_σ and X has a quotient singularity in x. Note that we have $Q(\mathrm{lin}(\gamma_0) \cap E) \neq K$ in this situation. Secondly, we obtain X as good quotient $p\colon \widehat{X} \to X$ with an open subset $\widehat{X} \subseteq \overline{X}$. Every $\overline{x} \in \widehat{X}$ defines a point $x \in X$ by $x := p(\overline{x})$. We call X *quasismooth* if $\widehat{X}$ is smooth. If X is not quasismooth, then there is a singular point $\overline{x} \in \overline{X}_{\gamma_0}$ defining a singular point $x \in X_{\gamma_0}$ and $\overline{X}_{\gamma_0} := p^{-1}(X_{\gamma_0})$ describes the singular locus above x. These singularities can be detected by the Jacobian matrix of the relations $g_0, \ldots, g_{r-2}$. Note, that these singularities can be factorial. Certainly, there are also singularities existing because of both reasons.

Construction 3.1. Let $X = X(A, P, \Phi)$ be a variety with complexity-one torus action and let Z be the minimal toric ambient variety of X and Σ its fan. Set

$$\varrho_0 := \mathrm{cone}(-e_1 - \ldots - e_r), \qquad \varrho_i := \mathrm{cone}(e_i) \subseteq \mathbb{Z}^r, \quad 1 \leq i \leq r\ ,$$

where $e_1, \ldots, e_r$ are the canonical basis vectors in $\mathbb{Z}^r$. Let $\mathrm{trop}(X)$ be the quasifan consisting of all cones of the form $\varrho_i \times \mathbb{Q}^s$ living in $\mathbb{Z}^{r+s}$. These cones are also called *arms* of $\mathrm{trop}(X)$. The lineality space of $\mathrm{trop}(X)$, which is denoted by $\mathrm{lin}(\mathrm{trop}(X))$, is exactly $\{0\} \times \mathbb{Q}^s$. We define

$$\Sigma' \ := \ \Sigma'(A, P, \Phi) \ := \ \Sigma \cap \mathrm{trop}(X) \ = \ \{\sigma \cap \tau;\ \sigma \in \Sigma, \tau \in \mathrm{trop}(X)\}$$

as the coarsest common refinement of the two fans Σ and $\mathrm{trop}(X)$. In particular Σ' is a refinement of Σ and consequently defines a toric morphism $Z' \to Z$ arising from a map of fans $\Sigma' \to \Sigma$. Let X' be the proper transform of X under this modification, i.e. the closure of $X \cap \mathbb{T}^{r+s}$ in Z'. The restriction $X' \to X$ is called the *weak tropical resolution* of X. And we call X *weakly tropical* if $\Sigma' = \Sigma$ holds. We obtain $X' = X(A, P', \Phi')$ where the columns of P' are the primitive generators of Σ' and Φ' is dual to Σ'.

Remark 3.2. The quasi-fan $\mathrm{trop}(X)$ is the tropical variety of the closed subvariety $X \cap T_Z$ in the sense of [52], where T_Z denotes the torus of the minimal toric ambient variety Z.

Definition 3.3. Let $X = X(A, P, \Phi)$ be a complexity-one T-variety. Then we have two types of $\mathfrak{F}$-faces $\gamma_0 \preceq \gamma$ corresponding to cones $\sigma = P(\gamma_0^*) \in \Sigma$ where $\gamma_0^* = \gamma_0^\perp \cap \gamma^\vee$.

(i) The basis vectors $e_{ij} \notin \gamma_0$ do all belong to one block, i.e. they have all the same index $0 \le i \le r$. In this case, the cone $P(\gamma_0^*)$ is of the form

$$\mathrm{cone}(v_{ij_1}, \dots, v_{ij_s}, v_{k_1}, \dots, v_{k_t})$$

and we call $\sigma = P(\gamma_0^*)$ a *tower cone*. Note that $v_1, \dots, v_m$ are contained in $\mathrm{lin}(\mathrm{trop}(X))$ by definition.

(ii) For each $0 \le i \le r$ there is at least one $1 \le j_i \le n_i$ such that $e_{ij_i} \notin \gamma_0$. This means that for each $0 \le i \le r$ there is at least one $1 \le j_i \le n_i$ such that $v_{ij_i} \in P(\gamma_0^*)$. In this situation $\sigma = P(\gamma_0^*)$ is called a *big cone*. Furthermore, we call a cone $\sigma \in \Sigma$ *elementary big* if for each $0 \le i \le r$ there is exactly one $1 \le j_i \le n_i$ such that $v_{ij_i} \in \sigma$, i.e. it is of the form $\mathrm{cone}(v_{0j_0}, \dots, v_{rj_r})$ for one choice $(j_0, \dots, j_r) \in \mathbb{Z}^{r+1}$ where $1 \le j_i \le n_i$.

Remark 3.4. Elementary big cones are exactly those cones such that its rays are the only faces being tower cones. In particular, the fan Σ' of Construction 3.1 does not contain big cones because it is supported by $\mathrm{trop}(X)$.

Lemma 3.5. *Let $\varrho' \in \Sigma' \setminus \Sigma$. Then $\varrho' \subseteq \mathrm{lin}(\mathrm{trop}(X))$ and ϱ' is contained in the relative interior of an elementary big cone.*

Proof. By definition of $\mathrm{trop}(X)$ and the matrix P a cone $\sigma = P(\gamma_0^*)$ of Σ is supported by $\mathrm{trop}(X)$, i.e. $\sigma \subseteq |\mathrm{trop}(X)|$, if and only if it is a tower cone. To be more precise each arm $\varrho_i \times \mathbb{Q}^s$ of $\mathrm{trop}(X)$ supports precisely those cones of Σ which satisfy $\sigma = P(\gamma_0^*) = \mathrm{cone}(v_{ij_1}, \dots, v_{ij_s})$. This gives $\varrho' \subseteq \mathrm{lin}(\mathrm{trop}(X))$. The only cones that intersect $\mathrm{lin}(\mathrm{trop}(X))$ non-trivially are big cones $P(\gamma_0^*)$. By Definition 3.3 all rays $\varrho' \in \Sigma' \setminus \Sigma$ are contained in the relative interior of an elementary big cone. □

Lemma 3.6. *Let $\sigma = \mathrm{cone}(v_{0j_0}, \dots, v_{rj_r})$ be an elementary big cone. Then σ contains exactly one primitive lattice vector $v \in \mathrm{lin}(\mathrm{trop}(X))$ in its relative interior. Moreover, the primitve vector v generates a ray $\varrho' \in \Sigma' \setminus \Sigma$.*

Proof. Assume $v \in \sigma^\circ$. Then we have $v = a_0 v_{0j_0} + \ldots + a_r v_{rj_r}$ with $a_i \in \mathbb{Q}_{>0}$. The first r coordinates of v have to be zero. This fixes the coefficients a_i up to a common scalar multiple. □

Lemma 3.7. *In the situation of Construction 3.1 the variety X' is a complexity-one T-variety $X(A, P', \Phi')$ such that $P' = (P, d')$ holds, where d' consists of vectors lying in the lineality space of* $\operatorname{trop}(X)$.

Proof. We denote the primitive vectors of all rays $\varrho' \in \Sigma' \setminus \Sigma$ by $v_{\varrho'}$. From Lemma 3.5 we infer that $v_{\varrho'} \in \operatorname{lin}(\operatorname{trop}(X))$ holds. Hence, adding all vectors $v_{\varrho'}$ to P we obtain a matrix P' defining a complexity-one T-variety $X' = X(A, P', \Phi')$ where Φ' is dual to Σ'. □

Lemma 3.8. *Let $X = X(A, P, \Phi)$ be a complexity-one T-variety with minimal toric ambient variety Z and associated fan Σ which does not contain big cones. Then the following statements hold.*

(i) X is quasismooth.

(ii) X is smooth if and only if Z is smooth.

Proof. To verify assertion (i) we have to consider the Jacobian J_g of $g := (g_0, \ldots, g_{r-2})$. It is given by

$$J_g = \begin{pmatrix} \delta_{0\,0} & \delta_{0\,1} & \delta_{0\,2} & 0 & & & & & 0 \\ 0 & \delta_{1\,1} & \delta_{1\,2} & \delta_{1\,3} & 0 & & & & \\ & & & & \vdots & & & & \\ & & & & & 0 & \delta_{r-3\,r-3} & \delta_{r-3\,r-2} & \delta_{r-3\,r-1} & 0 \\ 0 & & & & & & 0 & \delta_{r-2\,r-2} & \delta_{r-2\,r-1} & \delta_{r-2\,r} \end{pmatrix},$$

where δ_{ti} is a nonzero multiple of the gradient $\delta_i := \operatorname{grad} T_i^{l_i}$. Let us assume there is a $\widehat{x} \in \widehat{X}$ with $J_g(\widehat{x})$ not of full rank. Then $\delta_i(\widehat{x}) = 0 = \delta_k(\widehat{x})$ holds with some $0 \le i < k \le r$. This implies $\widehat{x}_{ij} = 0 = \widehat{x}_{kl}$ for some $1 \le j \le n_i$ and $1 \le l \le n_k$. Thus, we have $T_i^{l_i}(\widehat{x}) = 0 = T_k^{l_k}(\widehat{x})$. Lemma 2.5 gives $T_s^{l_s}(\widehat{x}) = 0$, for all $0 \le s \le r$. Thus, some coordinate $\widehat{x}_{st}$ must vanish for every $0 \le s \le r$. Since there are no big cones in Σ there are no points $\widehat{x} \in \widehat{X}$ of this shape. Hence, $\widehat{X}$ is smooth. Assertion (ii) follows from (i) and [25, Corollary 4.13]. □

Lemma 3.9. *Let $X = X(A, P, \Phi)$ be a complexity-one T-variety and Z its minimal toric ambient variety with associated fan Σ. Consider a cone $\sigma \in \Sigma$ which is supported by* $\operatorname{trop}(X)$*, that means $\sigma \subseteq |\operatorname{trop}(X)|$. Then $X_\sigma = Z_\sigma \cap X$ is an open subset of the affine toric variety $Z(\sigma)$ corresponding to the lattice cone $\sigma \subseteq \operatorname{lin}(\sigma) \cap N$.*

Proof. Assume that σ is supported by the r-th arm of $\operatorname{trop}(X)$. The projection

$$P_1\colon \mathbb{Z}^{r+s} \to \mathbb{Z}^r$$

onto the first r coordinates maps the cone σ in $\mathbb{Z}^{r+s}$ onto the ray ϱ_r in $\mathbb{Z}^r$. This induces a toric morphism $\pi_1\colon Z_\sigma \to Y_r$ where $Y_r \subseteq \mathbb{P}_r$ denotes the affine toric variety corresponding to the ray ϱ_r in $\mathbb{Z}^r$, i.e. $Y_r \cong \mathbb{T}^{r-1} \times \mathbb{K}$. Under this projection X_σ is sent to $C \cap Y_r$, where C is a projective line intersecting each toric prime divisor $V(U_i)$ of $\mathbb{P}_r$ in exactly one point. Note, that $U_0, \dots, U_r$ denote the homogeneous coordinates of $\mathbb{P}_r$. The curve C is parametrized on Y_r by

$$c\colon \mathbb{K} \to \mathbb{T}^{r-1} \times \mathbb{K}, \qquad t \mapsto (c_1(t), \dots, c_{r-1}(t), t)$$

with suitably chosen $c_i \in \mathbb{K}[T]$. Consider the T-equivariant morphism

$$\varphi\colon X_\sigma \to Z_\sigma, \qquad x \mapsto \left(\frac{1}{c_1(t(x))}, \dots, \frac{1}{c_{r-1}(t(x))}, t(x)\right) \cdot x.$$

Let $F_r := \overline{\lambda(\mathbb{K}^*)}$ be the closure of the image of the one-parameter group $\lambda_r\colon \mathbb{K}^* \to \mathbb{T}^r$ corresponding to the primitive generator of the ray ϱ_r. Then $\pi_1(\varphi(x)) = \lambda_r(t(x))$ holds and we have $\varphi(X_\sigma) = \pi_1^{-1}(F_r')$ where F_r' is obtained by removing all points $\lambda_r(t) \in V(U_i)$, $i \neq r$, off F_r. All in all , we have a commutative diagram

$$\begin{array}{ccccc}
X_\sigma & \xrightarrow{\cong} & \varphi(X_\sigma) & \subseteq & Z(\sigma) \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\
C \cap Y_r & \xrightarrow{\cong} & F_r' & \subseteq & F_r \ ,
\end{array}$$

and $\varphi(X_\sigma) = \pi_1^{-1}(F_r')$ is a π_1-saturated open subset of the affine toric variety $Z(\sigma) = \pi_1^{-1}(F_r)$ corresponding to the convex lattice cone σ in $\operatorname{lin}(\sigma) \cap N$. □

Theorem 3.10. *Let $X = X(A, P, \Phi)$ be a complexity-one T-variety and let Z be the minimal toric ambient variety with associated fan Σ. Then a T-invariant desingularization $X'' \to X$ is obtained as follows:*

(a) *Determine the fan $\Sigma' := \Sigma'(A, P, \Phi)$ and compute a regular subdivision Σ'' of Σ'. This leads to a map of fans $\Sigma'' \to \Sigma$.*

(b) *Let $Z'' \to Z$ be the toric morphism defined by $\Sigma'' \to \Sigma$ and let X'' be the closure of $X \cap \mathbb{T}^{r+s}$ in Z''. Then the restriction $X'' \to X$ is the searched desingularization.*

In particular X'' is smooth and of the form $X'' = X(A, P'', \Phi'')$ where the columns of P'' are the primitive generators of Σ'' and Φ'' is Gale dual to Σ''.

Proof. The first step is given by the weak tropical resolution $X' \to X$. Afterwards by Lemma 3.9 the variety X' is locally toric, i.e it is covered by open subsets of toric varieties X'_i with fans Σ'_i. These fans live in the i-th arm of $\mathrm{trop}(X)$ which is defined by $\varrho_i \times \mathbb{Q}^s$. Any regular subdivision of Σ' provides a regular subdivision of Σ'_i. Consequently, X'' is smooth. □

Remark 3.11. The resolution procedure of Theorem 3.10 provides an easy way for computing the Cox ring of the resolution X'' of a complexity-one T-variety $X = X(A, P, \Phi)$. In order to obtain the matrix P'' the primitive generators of all rays in $\Sigma'' \setminus \Sigma$ are added to the matrix P in such a way that the special form of the matrix is maintained, i.e.

$$P'' = \begin{pmatrix} L_0 & 0 \\ d & d' \end{pmatrix}$$

for appropriate matrices L_0, d and d' (compare 2.10). In particular, the Cox ring $R(X'')$ of the resolution X'' is given by $R(A, P'')$.

The relations of the Cox ring are not affected by the weak tropicalisation $X' \to X$. We just add some variables S_k corresponding to invariant prime divisor with infinite isotropy. Note that $\mathrm{lin}(\mathrm{trop}(X))$ equals the vector subspace generated by the lattice of the one-parameter subgroups of the torus T. Furthermore, the weak tropicalisation eliminates all factorial singularities, i.e. X' has only quotient singularities coming from the toric ambient variety Z'.

Example 3.12. Let $X = X(A, P)$ be the $\mathbb{K}^*$-surface arising from the data

$$P = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -1 & 0 & -1 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Then the Cox ring of X is given by $\mathbb{K}[T_{ij}]/\langle T_{01}T_{02} + T_{11}T_{12} + T_{21}^2 \rangle$ and we have

$$\begin{aligned} \Sigma^{\max} &= \{\mathrm{cone}(v_{01}, v_{11}, v_{21}), \mathrm{cone}(v_{02}, v_{12}, v_{21}), \mathrm{cone}(v_{i1}, v_{i2});\ 0 \le i \le 2\}, \\ \mathrm{trop}(X) &= \mathrm{cone}(e_1, \pm e_3) \cup \mathrm{cone}(e_2, \pm e_3) \cup \mathrm{cone}(-e_1 - e_2, \pm e_3). \end{aligned}$$

We obtain the weak tropicalisation X' by drawing in rays along $v^+ = (0, 0, 1)$ and $v^- = (0, 0, -1)$. Consequently, we have

$$\Sigma'^{\max} = \Sigma^{\max} \cap \mathrm{trop}(X) = \{\mathrm{cone}(v_{i1}, v^-), \mathrm{cone}(v_{i2}, v^+), \mathrm{cone}(v_{i1}, v_{i2});\ 0 \le i \le 2\}.$$

Resolving the two singular cones $\mathrm{cone}(v_{21}, v^+)$ and $\mathrm{cone}(v_{21}, v^-)$ of Σ' by drawing in rays along the elements of the Hilbert basis $v_{22} = (0, 1, 1)$ and $v_{23} = (0, 1, 0)$ we obtain the resolution $X'' = X(A, P'')$ with the following P-matrix and Cox ring:

$$\begin{aligned} P &= \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 1 & 0 & 1 & -1 \end{pmatrix}, \\ R(X'') &= \mathbb{K}[T_{ij}, S_1, S_2]/\langle T_{01}T_{02} + T_{11}T_{12} + T_{21}^2 T_{22} T_{23} \rangle. \end{aligned}$$

3.3 Discrepancies

In this chapter we analyze the behavior of the anticanonical class $-K_X$ of a complexity-one T-variety under toric ambient modifications.

Definition 3.13. Let X be a normal $\mathbb{Q}$-factorial variety, and consider a resolution $\pi\colon X' \to X$ of X. We write

$$K_{X'} = \pi^*(K_X) + \sum_i a_i E_i,$$

where E_i are the exceptional divisors. Then the coefficients $a_i \in \mathbb{Q}$ are called the *discrepancies* of π.

Lemma 3.14. *Consider a toric ambient modification $\pi\colon X' \to X$ for complexity-one T- and T'-varieties $X \subseteq Z$ and $X' \subseteq Z'$ with minimal toric ambient varieties Z and Z' and let $R(X)$ and $R(X')$ be their K and K'-graded Cox rings, respectively. Let D_∞ be the exceptional divisor represented by w'_∞ in the divisor class group $\mathrm{Cl}(X') \cong K'$. Then we have*

$$K_{X'} - \pi^*(K_X) \;=\; d \cdot w'_\infty + \sum_{i=0}^{r-2} \deg_{K'}(g'_i) - \pi^*(\sum_{i=0}^{r-2} \deg_K(g_i)),$$

where $g_0, \ldots, g_{r-2}$ are the relations of the Cox ring $R(X)$ and $g'_0, \ldots, g'_{r-2}$ are the relations of the Cox ring $R(X')$ and d denotes the toric discrepancy of π, i.e. $d \cdot w'_\infty = K_{Z'} - \pi^*(K_Z)$.

Proof. By Proposition 2.4 the Cox rings $R(X)$ and $R(X')$ are complete intersections. Hence, by using the concrete formula for the anticanonical divisor proven in [9, Proposition 8.5], we obtain by an easy computation

$$\begin{aligned} K_{X'} &= K_{Z'} + \sum_{i=0}^{r-2} \deg_{K'}(g'_i) \\ &= d \cdot w'_\infty + \pi^*(K_Z) + \sum_{i=0}^{r-2} \deg_{K'}(g'_i) \\ &= d \cdot w'_\infty + \pi^*(K_X - \sum_{i=0}^{r-2} \deg_K(g_i)) + \sum_{i=0}^{r-2} \deg_{K'}(g'_i) \\ &= d \cdot w'_\infty + \pi^*(K_X) - \pi^*(\sum_{i=0}^{r-2} \deg_K(g_i)) + \sum_{i=0}^{r-2} \deg_{K'}(g'_i). \end{aligned}$$

□

Proposition 3.15. *In the situation of Lemma 3.14 consider the exceptional divisor D_∞ and its divisor class w'_∞ in $\mathrm{Cl}(X')$ as well as the associated primitive vector v_∞ where*

$$m_\infty v_\infty \;=\; \sum_{i=0}^{r} \sum_{j=1}^{n_i} a_{ij} v_{ij} + \sum_{k=1}^{m} a_k u_k,$$

with non-negative integers a_{ij}, a_k *satisfying* $\gcd(a_{ij}, a_k) = 1$, $m_\infty \in \mathbb{Z}_{>0}$ *and* $v_\infty \in \operatorname{cone}(v_{ij}, u_k;\ a_{ij} \neq 0, a_k \neq 0)^\circ$. *Then we have*

$$K_{Z'} - \pi^*(K_Z) \;=\; \frac{\sum_{i=0}^{r}\sum_{j=1}^{n_i} a_{ij} + \sum_{k=1}^{m} a_k - m_\infty}{m_\infty} \cdot w'_\infty$$

and

$$K_{X'} - \pi^*(K_X) \;=\; \frac{\sum_{i=0}^{r}\sum_{j=1}^{n_i} a_{ij} + \sum_{k=1}^{m} a_k - m_\infty - \sum_{i=0}^{r-2} k_{0i}}{m_\infty} \cdot w'_\infty\ ,$$

where k_{0i} *denotes the minimal degree of the decomposition* $g_i := g_{k_{0i}} + \ldots + g_{k_{mi}}$ *in homogeneous components concerning the grading given by the modification* π, *i.e.* $\deg_\pi(T_{ij}) = a_{ij}$, $\deg_\pi(S_k) = a_k$.

Proof. The modification π defined by the exceptional divisor D_∞ induces the following lattice homomorphisms

$$\begin{aligned} G^*\colon E \to E', &\quad e_{ij} \mapsto e_{ij} + a_{ij}e_\infty,\ e_k \mapsto e_k + a_k e_\infty,\\ G'^*\colon E' \to E', &\quad e_{ij} \mapsto e_{ij},\ e_k \mapsto e_k,\ e_\infty \mapsto m_\infty e_\infty. \end{aligned}$$

In particular, π induces a pullback map $\pi^*\colon E \to E'$ where $G^* = G'^* \circ \pi^*$. Note that $E \cong \operatorname{WDiv}^T(Z)$ and $E' \cong \operatorname{WDiv}^T(Z')$. The map π^* defines a map $\operatorname{Cl}(Z) \to \operatorname{Cl}(Z')$ which is also denoted by π^*. We extend these maps to the corresponding rational vector spaces and obtain

$$\begin{aligned} \pi^*(Q(e_{ij})) &= \pi^*(w_{ij}) = Q'(e_{ij}) + \frac{a_{ij}}{m_\infty}Q'(e_\infty) = w'_{ij} + \frac{a_{ij}}{m_\infty}w'_\infty,\\ \pi^*(Q(e_k)) &= \pi^*(u_k) = Q'(e_k) + \frac{a_k}{m_\infty}Q'(e_\infty) = u'_k + \frac{a_k}{m_\infty}w'_\infty. \end{aligned}$$

This gives

$$\begin{aligned} K_{Z'} - \pi^*(K_Z) &= -\sum_{i=0}^{r}\sum_{j=1}^{n_i} w'_{ij} - \sum_{k=1}^{m} u'_k - w'_\infty - \pi^*\left(-\sum_{i=0}^{r}\sum_{j=1}^{n_i} w_{ij} - \sum_{k=1}^{m} u_k\right)\\ &= -\sum_{i=0}^{r}\sum_{j=1}^{n_i} w'_{ij} - \sum_{k=1}^{m} u'_k - w'_\infty\\ &\quad - \left(-\sum_{i=0}^{r}\sum_{j=1}^{n_i} w'_{ij} - \sum_{k=1}^{m} u'_k - \left(\sum_{i=0}^{r}\sum_{j=1}^{n_i}\frac{a_{ij}}{m_\infty} + \sum_{k=1}^{m}\frac{a_k}{m_\infty}\right)w'_\infty\right)\\ &= -w'_\infty + \left(\sum_{i=0}^{r}\sum_{j=1}^{n_i}\frac{a_{ij}}{m_\infty} + \sum_{k=1}^{m}\frac{a_k}{m_\infty}\right)w'_\infty\\ &= \frac{\sum_{i=0}^{r}\sum_{j=1}^{n_i} a_{ij} + \sum_{k=1}^{m} a_k - m_\infty}{m_\infty} \cdot w'_\infty. \end{aligned}$$

For the second part of the statement we first note that the $g_{k_{0i}}$-parts of the polynomials g_i remain untouched under the modification, i.e. $g_{k_{0i}} = g'_{k_{0i}}$. Furthermore we have $\deg_K(g_i) = \deg_K(g_{k_{0i}})$ since $g_{k_{0i}}$ consists of monomials of g_i. This gives

$$\begin{aligned}\deg_{K'}(g'_i) - \pi^*(\deg_K(g_i)) &= \deg_{K'}(g'_{k_{0i}}) - \pi^*(\deg_K(g_{k_{0i}})) \\ &= \deg_{K'}(g'_{k_{0i}}) - \deg_{K'}(g'_{k_{0i}}) - \frac{k_{0i}}{m_\infty} w'_\infty \\ &= -\frac{k_{0i}}{m_\infty} w'_\infty .\end{aligned}$$

The previous statement together with Lemma 3.14 proves the assertion. Note that, since $\mathrm{Cl}(X) \cong \mathrm{Cl}(Z)$ holds, we can naturally consider the pull back map $\pi^*\colon \mathrm{Cl}(X) \to \mathrm{Cl}(X')$. □

Corollary 3.16. *The toric discrepancy $K_{Z'} - \pi^*(K_Z)$ is always greater or equal to $K_{X'} - \pi^*(K_X)$.*

Definition 3.17. Let X be a normal ($\mathbb{Q}$-factorial) variety, the canonical class K_X $\mathbb{Q}$-Cartier and $\varphi\colon X' \to X$ a resolution of X. We consider

$$K_{X'} = \varphi^*(K_X) + \sum_i a_i E_i,$$

where E_i are the exceptional divisors and $a_i \in \mathbb{Q}$. The singularities of X are called

- *terminal*, if $a_i > 0$ for all i,
- *canonical*, if $a_i \geq 0$ for all i,
- *log-terminal*, if $a_i > -1$ for all i,
- *log-canonical*, if $a_i \geq -1$ for all i,
- *ε-log-terminal*, if $a_i > -1 + \varepsilon$ for all i, where $0 < \varepsilon < 1$,
- *ε-log-canonical*, if $a_i \geq -1 + \varepsilon$ for all i, where $0 < \varepsilon < 1$.

We call the variety X terminal (canonical, (ε-)log-terminal, (ε-)log-canonical) if all singularities are so.

Corollary 3.18. *Let X be a complexity-one T-variety and $X \subseteq Z$ its minimal toric embedding. If X is terminal/canonical/log-terminal/ε-log-terminal then Z is terminal/canonical/log-terminal/ε-log-terminal.*

Proposition 3.19. *Let $X = X(A, P, \Phi)$ be a log-terminal complexity-one T-variety and X' its weak tropicalisation. Then, for each choice $(j_0, \ldots, j_r)$, $1 \leq j_i \leq n_i$, such that there is a ray $\varrho' \in \Sigma' \setminus \Sigma$ with $v_{\varrho'} \in \mathrm{cone}(v_{0j_0}, \ldots, v_{rj_r})^\circ$ the following inequality holds:*

$$\sum_{i=0}^{r} \frac{1}{l_{j_i}} \; > \; r - 1.$$

Proof. We resolve the complexity-one T-variety X as described in Theorem 3.10. Therefore consider the fan

$$\Sigma' \;=\; \Sigma \cap \operatorname{trop}(X) \;=\; \{\sigma \cap \operatorname{trop}(X);\ \sigma \in \Sigma\}$$

and a tuple $(j_0, \dots, j_r)$ with $1 \le j_i \le n_i$, such that there is a ray $\varrho' \in \Sigma' \setminus \Sigma$ with primitive generator $v_{\varrho'} \in \operatorname{cone}(v_{0j_0}, \dots, v_{rj_r})^\circ$. Then $v_{\varrho'}$ can be represented as positive combination

$$m_{\varrho'} v_{\varrho'} \;=\; \sum_{i=0}^{r} a_{ij_i} v_{ij_i}\,,$$

where $m_{\varrho'}$ is chosen such that $a_{ij_0}, \dots, a_{ij_r}$ are integers and $\gcd(m_{\varrho'}, a_{0,j_0}, \dots a_{r,j_r}) = 1$. The ray ϱ' corresponds to a variable S_k not occurring in the Cox ring relations of X'. Consequently, we obtain

$$a_{ij_i} \;=\; \frac{\operatorname{lcm}(l_{0j_0}, \dots, l_{rj_r})}{l_{ij_i}}.$$

Using Proposition 3.15, the discrepancy of the modification defined by the stellar subdivision along ϱ' can be calculated explicitly by

$$\frac{1}{m_{\varrho'}} \cdot \left(\sum_{i=0}^{r} \frac{\operatorname{lcm}(l_{0j_0}, \dots, l_{rj_r})}{l_{ij_i}} - m_{\varrho'} - (r-2)\operatorname{lcm}(l_{0j_0}, \dots, l_{rj_r}) \right).$$

If X is log-terminal, this expression has to be greater than -1 which is equivalent to the condition

$$\sum_{i=0}^{r} \frac{1}{l_{j_i}} \;>\; r-1.$$

□

Example 3.20. Let $X = X(A,P)$ be a log-terminal complexity-one T-variety satisfying $r = 2$. Then $l_{0j_0} l_{1j_1} + l_{0j_0} l_{2j_2} + l_{1j_1} l_{2j_2} \;>\; l_{0j_0} l_{1j_1} l_{2j_2}$ or equivalently

$$\frac{1}{l_{0j_0}} + \frac{1}{l_{1j_1}} + \frac{1}{l_{2j_2}} \;>\; 1$$

holds for all choices (j_0, j_1, j_2) with $\operatorname{cone}(v_{0j_0}, v_{1j_1}, v_{2j_2}) \in \Sigma$ and $(l_{0i_0}, l_{1i_1}, l_{2i_2})$ is a *platonic triple*, i.e. a triple of the following form:

- $(1, x, y)$, where $x, y \ge 1$
- $(2, 2, x)$, where $x \ge 2$
- $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$

Corollary 3.21. *Let $X = X(A,P)$ be a log-terminal complexity-one T-variety with $r \ge 2$ and $\operatorname{cone}(v_{0j_0}, \dots, v_{rj_r}) \in \Sigma$ such that $l_{0j_0} \ge \dots \ge l_{rj_r}$. Then $l_{3j_3} = \dots = l_{rj_r} = 1$ holds and $(l_{0j_0}, l_{1j_1}, l_{2j_2})$ is a platonic triple.*

Proof. Assume $l_{ij_i} \neq 1$ for all $0 \leq i \leq r$. Then $1/l_{ij_i} \leq 1/2$ holds and consequently by Proposition 3.19 we obtain

$$\frac{r+1}{2} \;\geq\; \sum_{i=0}^{r} \frac{1}{l_{ij_i}} \;>\; r-1.$$

This implies $r+1 > 2r-2$ and hence $r < 3$. □

Let $X_0 = X(A_0, P_0, \Phi_0)$ be a $\mathbb{Q}$-factorial complexity-one T-variety and Z_0 its minimal toric ambient variety with fan Σ_0. Let $\Sigma_1, \ldots, \Sigma_n$ be refinements of Σ_0 arising from Σ_0 by doing stellar subdivisions of a cone $\sigma_0 \in \Sigma_0$ successively. That means, that we have rays $\varrho_1, \ldots, \varrho_n$ with primitive generators v_{ϱ_i} and cones $\sigma_i \in \Sigma_i$ such that $v_{\varrho_{i+1}} \in \sigma_i^\circ$ and $\sigma_{i+1} \subseteq \sigma_i$ for all $0 \leq i \leq n-1$. Then we have toric ambient modifications $\pi_i \colon X_{i+1} \to X_i$ with $\mathbb{Q}$-factorial complexity-one T-varieties $X_1, \ldots, X_n$ such that

$$X_n \stackrel{\pi_{n-1}}{\to} X_{n-1} \to \ldots \to X_1 \stackrel{\pi_0}{\to} X_0.$$

We set $\kappa_i := \pi_0 \circ \ldots \circ \pi_i$, $0 \leq i \leq n-1$. Let K_{X_0} be the canonical divisor class of X_0. Then there is an $n \in \mathbb{N}$ and a linear form $u \in M = N^* \cong \mathbb{Z}^{r+s}$ such that $n \cdot (K_{X_0}) = \chi^u$ on $X_{0\sigma_0} = X_0 \cap Z_{0\sigma_0}$ where $\sigma_0 = P_0(\gamma_0^*)$. Hence, we can find an element $u_0 \in M_{\mathbb{Q}} = M \otimes \mathbb{Q}$ such that

$$K_{X_0} \;=\; \sum_{i,j} \langle u_0, v_{ij} \rangle D_{ij} + \sum_k \langle u_0, v_k \rangle E_k.$$

Since X_0 is $\mathbb{Q}$-factorial, so are $X_1, \ldots, X_n$. We consider the relevant $\mathfrak{F}$-faces $\gamma_1, \ldots, \gamma_n$ with $\sigma_i = P_i(\gamma_i)$. Then we can find $u_1, \ldots, u_n$ such that, locally on $X_{i\gamma_i} = X_{i\sigma_i}$, we have

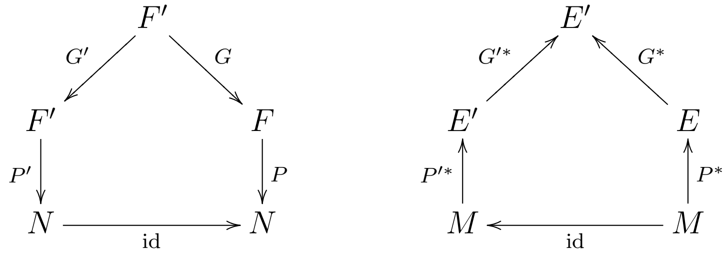
$$K_{X_s} \;=\; \sum_{i,j} \langle u_s, v_{ij} \rangle D_{ij} + \sum_k \langle u_s, v_k \rangle E_k, \qquad 1 \leq s \leq n\ .$$

Now we describe the behavior of the anticanonical divisor $-K_{X_0}$ under the toric ambient modifications κ_i in terms of discrepancies.

Proposition 3.22. *In the situation above we have for* $1 \leq i \leq n$

$$K_{X_i} - \kappa_i^*(K_{X_0}) \;=\; (\langle u_1, v_{\varrho_1} \rangle - \langle u_0, v_{\varrho_1} \rangle) D_{\varrho_1} + \ldots + (\langle u_i, v_{\varrho_i} \rangle - \langle u_0, v_{\varrho_i} \rangle) D_{\varrho_i}.$$

Proof. For an arbitrary toric ambient modification $\pi \colon X' \to X$ we have commutative diagrams



where the second one is obtained by dualizing the first one; compare section 1 of this chapter. These diagrams extend to diagrams of the corresponding vector spaces. Hence, the toric modifications κ_i leave $M_{\mathbb{Q}}$ untouched and we have $\kappa_{i*}(u_0) = u_0$. In particular, $\kappa^*(K_{X_0})$ is still represented by u_0 on X_{σ_i} for each $\sigma_i \subseteq \sigma_0$. For each v_{ij}, v_k that is contained in both σ_i and σ_0 we have $\langle u_i, v_{ij}\rangle = \langle u_0, v_{ij}\rangle$ and $\langle u_i, v_k\rangle = \langle u_0, v_k\rangle$ respectively. This states the assertion. □

Example 3.23 (Example 3.12 continued)**.** We consider the two elementary big cones $\sigma^+ = \operatorname{cone}(v_{02}, v_{12}, v_{22})$ and $\sigma^- = \operatorname{cone}(v_{01}, v_{11}, v_{21})$. Locally on X_{σ^+} and X_{σ^-} the anticanonical divisor can be represented by the linear forms $u^+ = (1, -1, 3)$ and $u^- = (0, 1, -1)$ respectively. Note that we have

$$\begin{aligned} \langle u^+, v_{02}\rangle = 0 = l_{02} - 1, \quad \langle u^+, v_{12}\rangle = 1, \quad \langle u^+, v_{21}\rangle = 1, \\ \langle u^-, v_{01}\rangle = 0 = l_{01} - 1, \quad \langle u^-, v_{11}\rangle = 1, \quad \langle u^-, v_{21}\rangle = 1. \end{aligned}$$

We denote the exceptional divisors corresponding to the primitive vectors $v^\pm = (0, 0, \pm 1)$, $v_{22} = (0, 1, 1)$ and $v_{23} = (0, 1, 0)$ by $D^\pm$, D_{22} and D_{23} respectively. Hence, the discrepancies $K_{X''} - \pi^*(K_X)$ of the resolution $\pi\colon X'' \to X$ are given by

$$\begin{aligned} &(\langle u^+, v^+\rangle - 1)D^+ + (\langle u^+, v_{22}\rangle - 1)D_{22} + (\langle u^-, v^-\rangle - 1)D^- + (\langle u^-, v_{32}\rangle - 1)D_{23} \\ &= 2D^+ + D_{22} + 0 \cdot D^- + 0 \cdot D_{23}. \end{aligned}$$

In particular, X is canonical. Note that, locally on X''_σ with $\sigma \in \Sigma''$, the anticanonical divisor $K_{X''}$ is given by a linear form u_σ having value 1 for all primitive column vectors of P'' generating σ.

4 $\mathbb{K}^*$-surfaces

In this chapter we investigate complexity-one T-varieties of dimension two, so called $\mathbb{K}^*$-surfaces. We will give a survey of their geometry and determine all types of Cox rings of combinatorially minimal $\mathbb{K}^*$-surfaces, i.e. surfaces without contractible prime divisors. Furthermore, we compute intersection numbers and affiliate conditions for $\mathbb{K}^*$-surfaces to be Fano. Finally, we introduce the anticanonical complex for log-terminal $\mathbb{K}^*$-surfaces, a convex geometrical tool which can be used to describe their singularities.

4.1 P-Matrices for $\mathbb{K}^*$-surfaces

In this chapter we are concerned with $\mathbb{K}^*$-surfaces. The special case of dimension two simplifies the approach for varieties with torus action of complexity one considerably. It is not necessary to work with the concept of bunches, as presented in chapter 1. There is only one representative of a small birational class since there is only one single ample chamber $\mathrm{SAmple}(X) = \mathrm{Mov}(X)$ defining one single bunch and hence one single unique surface. Consequently, in case of surfaces, we can use the notation $X = X(A, P)$. We briefly recall the basic steps needed for the construction of $X(A, P)$.

Construction 4.1. We start with a set A of $r + 1$ pairwise linearly independent points $a_i \in \mathbb{K}^2$ and an integer $(n + m) \times (r + 1)$-matrix of the form

$$P = \begin{pmatrix} -l_0 & l_1 & 0 & \dots & 0 & 0 \\ -l_0 & 0 & l_2 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ -l_0 & 0 & 0 & & l_r & 0 \\ d_0 & d_1 & d_2 & \dots & d_r & d' \end{pmatrix},$$

where $n := n_1 + \ldots + n_r$ and $0 \le m \le 2$. The entries of the matrix P are vectors $l_i := (l_{i1}, \ldots, l_{in_i})$, $d_i := (d_{i1}, \ldots, d_{in_i})$ and d' is either empty or equals 1, -1 or $(1, -1)$. Recall that the columns v_{ij}, v_k, where $0 \le i \le r$, $1 \le j \le n_i$ and $1 \le k \le m$, are primitive, pairwise different and generate $\mathbb{Q}^{r+1}$ as a cone. We denote $(0, \ldots, 0, 1)$ by v^+ and $(0, \ldots, 0, -1)$ by v^-. Furthermore, the columns of P are ordered in such a way that the following conditions hold for all $0 \le i \le r$ (compare Corollary 2.36):

$$\frac{d_{in_i}}{l_{in_i}} > \ldots > \frac{d_{i1}}{l_{i1}} \qquad \text{and} \qquad l_{in_i} > d_{in_i} \ge 0.$$

We construct the minimal toric ambient variety Z by defining the maximal cones of the corresponding fan Σ in $N := \mathbb{Z}^{r+1}$. Furthermore, we define a fan $\widehat{\Sigma}$ in

$$F := \bigoplus_{i=0}^{r} \bigoplus_{j=1}^{n_i} \mathbb{Z} \cdot e_{ij} \oplus \mathbb{Z} \cdot e^+ \oplus \mathbb{Z} \cdot e^- \cong \mathbb{Z}^{n+m},$$

such that $P\colon \mathbb{Z}^{n+m} \to \mathbb{Z}^{r+1},\ e_{ij} \mapsto v_{ij}, e^{\pm} \mapsto v^{\pm}$ maps $\widehat{\Sigma}$ to Σ. According to the form of d' we distinguish four types of P-matrices.

Type 1: Assume, that d' is empty, i.e. $m = 0$. Then the maximal cones of Σ are

$$\begin{aligned}\sigma^+ &:= \operatorname{cone}(v_{1n_1}, \dots, v_{rn_r}),\\ \sigma^- &:= \operatorname{cone}(v_{11}, \dots, v_{r1}),\\ \tau_{ij} &:= \operatorname{cone}(v_{ij}, v_{ij+1}) \quad \text{for} \quad 0 \le i \le r,\ 1 \le j < n_i.\end{aligned}$$

The maximal cones of $\widehat{\Sigma}$ are

$$\begin{aligned}\widehat{\sigma}^+ &:= \operatorname{cone}(e_{1n_1}, \dots, e_{rn_r}),\\ \widehat{\sigma}^- &:= \operatorname{cone}(e_{11}, \dots, e_{r1}),\\ \widehat{\tau}_{ij} &:= \operatorname{cone}(e_{ij}, e_{ij+1}) \quad \text{for} \quad 0 \le i \le r,\ 1 \le j < n_i.\end{aligned}$$

Type 2: Assume, that $m = 1$ and $d' = 1$. Then the maximal cones of Σ are

$$\begin{aligned}\sigma^- &:= \operatorname{cone}(v_{11}, \dots, v_{r1}),\\ \sigma_i^+ &:= \operatorname{cone}(v^+, v_{in_i}) \quad \text{for} \quad 0 \le i \le r,\\ \tau_{ij} &:= \operatorname{cone}(v_{ij}, v_{ij+1}) \quad \text{for} \quad 0 \le i \le r,\ 1 \le j < n_i.\end{aligned}$$

The maximal cones of Σ' are

$$\begin{aligned}\widehat{\sigma}^- &:= \operatorname{cone}(e_{11}, \dots, e_{r1}),\\ \widehat{\sigma}_i^+ &:= \operatorname{cone}(e^+, e_{in_i}) \quad \text{for} \quad 0 \le i \le r,\\ \widehat{\tau}_{ij} &:= \operatorname{cone}(e_{ij}, e_{ij+1}) \quad \text{for} \quad 0 \le i \le r,\ 1 \le j < n_i.\end{aligned}$$

Type 3: Assume that $m = 1$ and $d' = -1$. Then the maximal cones of Σ are

$$\begin{aligned}\sigma^+ &:= \operatorname{cone}(v_{1n_1}, \dots, v_{rn_r}),\\ \sigma_i^- &:= \operatorname{cone}(v^-, v_{i1}) \quad \text{for} \quad 0 \le i \le r,\\ \tau_{ij} &:= \operatorname{cone}(v_{ij}, v_{ij+1}) \quad \text{for} \quad 0 \le i \le r,\ 1 \le j < n_i.\end{aligned}$$

The maximal cones of Σ' are

$$\begin{aligned}\widehat{\sigma}^+ &:= \operatorname{cone}(e_{1n_1}, \dots, e_{rn_r}),\\ \widehat{\sigma}_i^- &:= \operatorname{cone}(e^-, e_{i1}) \quad \text{for} \quad 0 \le i \le r,\\ \widehat{\tau}_{ij} &:= \operatorname{cone}(e_{ij}, e_{ij+1}) \quad \text{for} \quad 0 \le i \le r,\ 1 \le j < n_i.\end{aligned}$$

Type 4: Assume that $d' = (1, -1)$, i.e. $m = 2$. Then the maximal cones of Σ are

$$\begin{aligned}\sigma_i^+ &:= \operatorname{cone}(v^+, v_{in_i}),\\ \sigma_i^- &:= \operatorname{cone}(v^-, v_{i1}) \quad \text{for} \quad 0 \le i \le r,\ 1 \le k \le m,\\ \tau_{ij} &:= \operatorname{cone}(v_{ij}, v_{ij+1}) \quad \text{for} \quad 0 \le i \le r,\ 1 \le j < n_i.\end{aligned}$$

The maximal cones of Σ' are

$$\begin{aligned}
\widehat{\sigma}_i^+ &:= \operatorname{cone}(e^+, e_{in_i}),\\
\widehat{\sigma}_i^- &:= \operatorname{cone}(e^-, e_{i1}) \quad \text{for} \quad 0 \le i \le r,\ 1 \le k \le m,\\
\widehat{\tau}_{ij} &:= \operatorname{cone}(e_{ij}, e_{ij+1}) \quad \text{for} \quad 0 \le i \le r,\ 1 \le j < n_i.
\end{aligned}$$

We denote the minimal toric ambient variety corresponding to the fan Σ by Z and the toric variety corresponding to $\widehat{\Sigma}$ by $\widehat{Z}$. Then P induces a toric morphism $\pi\colon \widehat{Z} \to Z$ representing the Cox Construction.

Let $P^*\colon M \to E$ be the dual map of $P\colon F \to N$ and consider the exact sequence

$$0 \longleftarrow K \xleftarrow{\;Q\;} E \xleftarrow{\;P^*\;} M \longleftarrow 0\,,$$

where $E := F^*$, $M := N^*$, $K := E/\operatorname{im}(P^*)$ and Q is given by the kernel of P^*. For $a_i, a_j \in A$ we define $\alpha_{ij} := \det(a_i, a_j)$. Then the data (A, P) defines trinomials

$$g_i \;:=\; \alpha_{i+1,i+2} T_i^{l_i} + \alpha_{i+2,i} T_{i+1}^{l_{i+1}} + \alpha_{i,i+1} T_{i+2}^{l_{i+2}},$$

where $T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}$. Consider the polynomial ring

$$\mathbb{K}[T_{ij}, S_k;\ 0 \le i \le r,\ 1 \le j \le n_i,\ 1 \le k \le m].$$

By setting $\deg(T_{ij}) := Q(e_{ij})$, $\deg(S_k) := Q(e_k)$ we obtain a grading of this ring such that the trinomials g_i are homogeneous. This grading defines an action of the quasitorus

$$H := \operatorname{Spec}(\mathbb{K}[K]) = \ker(\pi)$$

on $\mathbb{K}^{n+m}$ leaving the vanishing set $\overline{X} := V(g_i;\ 0 \le i \le r-2)$ invariant. The map $\pi\colon \widehat{Z} \to Z$ is the geometric quotient of this action and the restriction on $\widehat{X} := \widehat{Z} \cap \overline{X}$ defines also a geometric quotient $\widehat{X} \to X$.

$$\begin{array}{ccc}
\widehat{X} & \hookrightarrow & \widehat{Z}\\
{\scriptstyle /\!\!/ H}\downarrow & & \downarrow{\scriptstyle /\!\!/ H}\\
X & \hookrightarrow & Z
\end{array}$$

We obtain a projective surface $X := \widehat{X} /\!\!/ H$ neatly embedded into the (minimal) toric variety Z with divisor class group and Cox ring

$$\operatorname{Cl}(X) \;=\; K, \qquad R(X) \;=\; \mathbb{K}[T_{ij}, S_k]/\langle g_i;\ 0 \le i \le r-2\rangle.$$

By construction, there is an effective $\mathbb{K}^*$-action on Z, given by the one-parameter subgroup $(0, \ldots, 0, 1) \in N \cong \mathbb{Z}^{r+1}$, which leaves $X = X(A, P)$ invariant. This induces a $\mathbb{K}^*$-action on X.

Proof. The general construction of complexity-one T-varieties has already been discussed in chapter 2 and 3. We show how the $\mathbb{K}^*$-action is established in the special situation of $\mathbb{K}^*$-surfaces. The trinomials $g_0, \ldots, g_{r-2}$ are homogeneous concerning an even finer grading as that one given by Q. Consider the gradiator matrix

$$P_0 \ = \ \begin{pmatrix} -l_0 & l_1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ -l_0 & 0 & & l_r & 0 \end{pmatrix},$$

set $K_0 := E/\mathrm{im}(P_0^*)$ and let Q_0 be the kernel of P_0. Then the K_0-grading given by $\deg(T_{ij}) = Q_0(e_{ij})$, $\deg(S_k) = Q_0(e_k)$ defines an $H_0 := \mathrm{Spec}(\mathbb{K}[K_0])$-action on $\mathbb{K}^{n+m}$ leaving $\overline{X}$ invariant. Taking the quotient by the $H = \mathrm{Spec}(\mathbb{K}[K])$-action establishes an $H_0/H \cong \mathbb{K}^*$-action on $\mathbb{K}^{n+m}$. Set $l^+ := l_{0n_0} \cdots l_{rn_r}$. Then the $\mathbb{K}^*$-action is given by the one-parameter subgroup corresponding to the vector

$$\overline{x}^+ \ := \ (0, \ldots, 0, \frac{l^+}{l_{0n_0}}, \ \ldots \ , 0, \ldots, 0, \frac{l^+}{l_{rn_r}}, 0, \ldots, 0) \in \mathbb{Z}^{n+m}$$

having the last entry of each block equal to l^+/l_{in_i} for $0 \leq i \leq r$ and all other entries equal to zero. Applying the matrix P gives

$$P(\overline{x}^+) = (0, \ldots, 0, a), \qquad \text{where} \quad a \ = \ \sum_{i=0}^{r} \frac{l^+ d_{in_i}}{l_{in_i}}.$$

This vector defines the one-parameter subgroup

$$\lambda \colon \mathbb{K}^* \to \mathbb{T}^{r+1}, \, t \mapsto (1, \ldots, 1, t^a)$$

inducing a $\mathbb{K}^*$-action on the last coordinate $0 \times \ldots \times 0 \times \mathbb{Z}$ in $N \cong \mathbb{Z}^{r+1}$. Note that we obtain effectivity by considering the primitive lattice vector $(0, \ldots, 0, 1)$. That means that the acting torus $\mathbb{K}^*$ is represented by the lattice generated by $(0, \ldots, 0, 1) \in N$. □

Example 4.2. Consider the $\mathbb{K}^*$-surface $X = X(A, P)$ given by the following data:

$$P \ = \ \begin{pmatrix} -3 & -1 & 3 & 0 \\ -3 & -1 & 0 & 2 \\ -4 & -1 & 2 & 1 \end{pmatrix}, \qquad A \ = \ \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Then we have $m = 0$, $n_0 = 2$, $n_1 = n_2 = 1$ and the minimal toric ambient variety Z of the resulting $\mathbb{K}^*$-surface X is given by a fan of type 1. The one-parameter groups defining the $\mathbb{K}^*$-action on $\mathbb{K}^4$ and X are given by

$$\overline{x}^+ \ = \ (0, 6, 2, 3) \quad \text{and} \quad P(\overline{x}^+) = (0, 0, 1), \quad \text{respectively.}$$

Now we will discuss some geometrical aspects of $\mathbb{K}^*$-surfaces. Therefore, the following definition is needed.

Definition 4.3. A fixed point of a normal $\mathbb{K}^*$-surface is called

- *elliptic* if it is isolated and lies in the closure of infinitely many orbits,
- *hyperbolic* if it is isolated and lies in the closure of two orbits,
- *parabolic* if it belongs to a fixed point curve and lies in the closure of exactly one orbit.

Example 4.4. Consider the $\mathbb{K}^*$-surface $X = X(A, P)$ given by the following data:

$$P = \begin{pmatrix} -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -2 & -1 & 1 & 1 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix},$$

i.e. we have $m = 1$, $n_0 = 2$ and $n_1 = n_2 = 1$. The $\mathbb{K}^*$-surface X has one elliptic fixed point corresponding to the cone $\sigma^- = \mathrm{cone}(v_{01}, v_{11}, v_{21})$, one hyperbolic fixed point corresponding to the cone $\tau_{01} = \mathrm{cone}(v_{01}, v_{02})$ and a parabolic fixed point curve corresponding to $v^+ = (0, 0, 1)$.

These three types of fixed points are the only possible fixed points that can occur in case of $\mathbb{K}^*$-surfaces. Furthermore, every normal $\mathbb{K}^*$-surface X has got a *sink* F^+ and a *source* F^-. They are defined by the general orbits of the $\mathbb{K}^*$-surface in the following way: There is an open subset $U \subseteq X$ such that

$$\lim_{t\to\infty} t \cdot x \in F^+ \quad \text{and} \quad \lim_{t\to 0} t \cdot x \in F^-$$

for all $x \in U$. Both, sink and source can either be an elliptic fixed point or a curve of parabolic fixed points isomorphic to $\mathbb{P}_1$. The latter case is equivalent to the existence of a divisor with infinite isotropy corresponding to a variable S_k not occurring in the Cox ring relations.

Proposition 4.5. *According to the type of P there are four possibilities concerning the geometry of the sink and the source:*

(1) If P is of type 1, then both the source F^- and the sink F^+ are elliptic fixed points. We call this kind of surface of type (ell,ell).

(2) If P is of type 2, then the source F^- is an elliptic fixed point and the sink F^+ is a parabolic fixed point curve isomorphic to $\mathbb{P}_1$. We call this kind of surface of type (par,ell).

(3) If P is of type 3, then the sink F^+ is an elliptic fixed point and the source F^- is a parabolic fixed point curve isomorphic to $\mathbb{P}_1$. We call this kind of surface of type (ell,par).

(4) If P is of type 4, then both the source F^- and the sink F^+ are parabolic fixed point curves isomorphic to $\mathbb{P}_1$. We call this kind of surface of type (par,par).

Besides F^+ and F^- and the general orbits there are special orbits corresponding to the $\mathbb{K}^*$-invariant prime divisors of X. These are the orbits with non trivial finite isotropy and they correspond to the rays ϱ_{ij}, $\varrho^\pm$ generated by the columns $v_{ij}, v^\pm$ of the matrix P. The next proposition summarizes some geometrical aspects of these orbits. For this purpose we briefly recall the notion of cotangent representations.

Let $\mathbb{K}^*_x$ be the isotropy group of $x \in X$ under the $\mathbb{K}^*$-action of X. We consider the tangent space T_xX of x. Furthermore, let $T_x\varphi_t$ be the differential of

$$\varphi_t \colon X \to X,\ x' \mapsto t \cdot x'.$$

Then we can define a representation of $\mathbb{K}^*_x$ on T_xX by

$$t \cdot v \ := \ T_x\varphi_t \ .$$

This representation is called the *tangent representation* of $\mathbb{K}^*_x$ on T_xX. The dual representation is called the *cotangent representation*.

Proposition 4.6. *Let $X = X(A, P)$ be a $\mathbb{K}^*$-surface and let D_{ij} and $E^\pm$ be the invariant prime divisors corresponding to the rays ϱ_{ij} and $\varrho^\pm$ generated by the columns v_{ij}, $v^\pm$. Then the following statements hold:*

(i) The divisors $E^\pm$ have infinite isotropy and the exponents l_{ij} are the orders of the isotropy groups $\mathbb{K}^*_{D_{ij}}$ for all points in D_{ij}.

(ii) The pair $(d_{ij}, 0)$ is representing the weight of the cotangent representation of the isotropy group $\mathbb{K}^*_{ij}$ at $x_{ij} \in D_{ij}$.

Proof. Assertion (i) is a special case of Proposition 2.15. We prove (ii). Let D_{ij} be a $\mathbb{K}^*$-invariant divisor of X and consider the associated ray $\varrho_{ij} = \mathbb{Q}_{\geq 0} \cdot v_{ij}$. We can simplify the situation by a locally toric consideration within the i-th block, compare Lemma 3.9. Set $\widehat{v}_{ij} := (l_{ij}, d_{ij})$ and $\widehat{v}^+ := (0, 1)$. Then $\widehat{v}^+$ represents the one-parameter group which induces the $\mathbb{K}^*$-action. Since l_{ij} and d_{ij} are coprime, we find $a, b \in \mathbb{Z}$ with $al_{ij} + bd_{ij} = 1$. Consequently, by applying the matrix

$$B \ = \ \begin{pmatrix} a & b \\ -d_{ij} & l_{ij} \end{pmatrix}$$

we obtain $B\widehat{v}_{ij} = (1, 0)$ and the $\mathbb{K}^*$-action corresponds to the vector (b, l_{ij}). This means, the $\mathbb{K}^*$-action on $\mathbb{K}^2$ is now given by

$$t \cdot (z, w) = (t^b z, t^{l_{ij}} w).$$

and we have $D_{ij} = 0 \times \mathbb{K}$. In particular, the order of the isotropy group of D_{ij} is given by l_{ij} since the l_{ij}-th elementary units act trivially on the second component. The action of the isotropy group transversal to the tangent space is given by

$$\zeta \cdot (z, w) = (\zeta^b \cdot z, w) \ .$$

The cotangent representation is given by

$$\zeta \cdot (z, w) = (\zeta^{-b} \cdot z, w) \ .$$

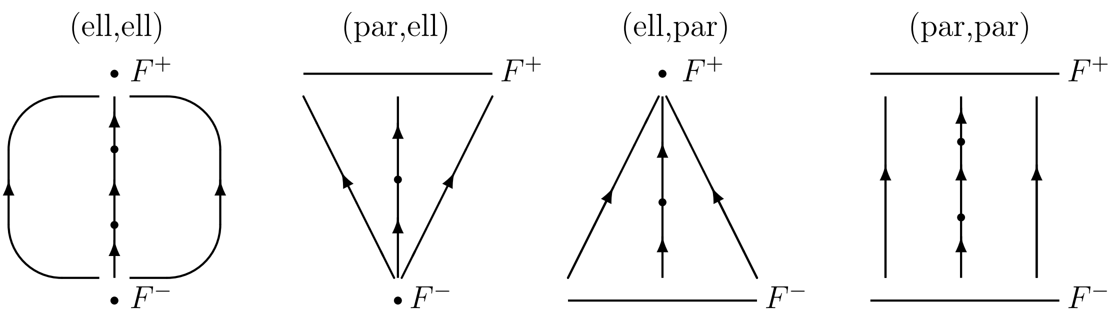
In particular, we have $b = d_{ij}^{-1} \pmod{l_{ij}}$ and $(d_{ij}, 0)$ is the weight of the cotangent representation. □

Any other fixed point besides the sink and the source is hyperbolic. Such fixed points occur within one block. For $x \in X_{\gamma_0}$ the isotropy group is given by

$$\mathbb{K}^*_x = \mathrm{lin}(\sigma) \cap N \subseteq N \ ,$$

where $\sigma := P(\gamma_0^*) \in \Sigma$. In particular, x is a fixed point of the $\mathbb{K}^*$-action if and only if $\mathbb{Q} \cdot v^+ \subseteq \mathrm{lin}(\sigma)$.

All in all, we have the following four possibilities for the orbit decomposition of $\mathbb{K}^*$-surfaces:



Now we consider the resolution of singularities of $\mathbb{K}^*$-surfaces. There are two reasons for the existence of singularities. First, there can be (factorial) singularities coming from the total coordinate space $\overline{X}$ and surviving the quotient process, i.e. the locus $\overline{X}_\sigma$ lying above the singularity $x \in X_\sigma$ is not regular. Note that these singularities can easily be found by computing the Jacobian matrix of the Cox ring relations. Second, there can be singularities coming from the toric ambient variety Z, e.g. there is a cone $\sigma \in \Sigma$ that is not regular and X inherits the corresponding singularity from Z. It is also possible that a singularity exists because of both reasons.

Proposition 4.7. *Let X be a $\mathbb{K}^*$-surface arising from a matrix P and suppose that d' is empty or equals -1. Then the following statements are equivalent:*

(i) The upper elliptic fixed point is smooth.

(ii) The following two conditions hold:

(a) $\det(\sigma^+) = \det(v_{0n_0}, \ldots, v_{rn_r}) = \pm 1$,

(b) $l_{in_i} = 1$ *for some* $0 \leq i \leq r$.

Note that the analogous statement holds for a lower elliptic fixed point.

Proof. The conditions for X not having a singularity in the upper elliptic fixed point coming from the toric ambient variety is equivalent to the condition that σ^+ is regular which in turn is equivalent to the condition that the determinant of the generators of σ^+ is ± 1. Computing the Jacobian of the defining relations gives

$$J = \begin{pmatrix} * \; l_{01} T_{01}^{l_{01}-1} \prod_{j \neq 1} T_{0j}^{l_{0j}} & & 0 \\ & \ddots & \\ 0 & & * \; l_{rn_r} T_{rn_r}^{l_{rn_r}-1} \prod_{j \neq n_j} T_{rj}^{l_{rj}} \end{pmatrix}.$$

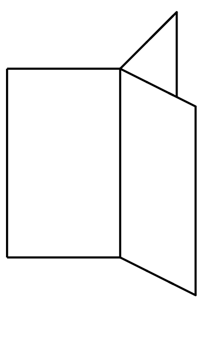
The point $\overline{x} \in \overline{X}_{\sigma^+} \subseteq \overline{X}$ having all in_i-coordinates T_{in_i} equal to 0 and all other coordinates equal to 1 defines the elliptic fixed point $x \in X$ corresponding to the cone σ^+, i.e. $p(\overline{x}) = x$. This point x is a singularity coming from the singular locus $\overline{X}_{\sigma^+}$ if and only if T_{in_i} occurs within every non trivial entry of the Jacobian. This on the other hand is equivalent to the condition that every exponent l_{in_i} satisfies $l_{in_i} \neq 1$. □

Remark 4.8. The determinants of the two elementary big cones $\sigma^\pm$ satisfy

$$(-1)^r \det(\sigma^+) \; = \; \sum_{i=0}^{r} \frac{l^+ d_{in_i}}{l_{in_i}} \qquad \text{and} \qquad (-1)^r \det(\sigma^-) \; = \; \sum_{i=0}^{r} \frac{l^- d_{i1}}{l_{i1}},$$

where $l^+ = l_{0n_0} \cdots l_{rn_r}$ and $l^- = l_{01} \cdots l_{r1}$. Furthermore, the columns of P are oriented in such a way that $(-1)^r \det(\sigma^+)$ is always positive and $(-1)^r \det(\sigma^-)$ is always negative.

There is a canonical way of resolving singularities of $\mathbb{K}^*$-surfaces (compare 3.10). The maximal cones of $\mathrm{trop}(X)$ are given by the cones $\varrho_i := \mathrm{cone}(e_i, \pm e_{r+1})$ for $1 \leq i \leq r$ and $\varrho_0 := \mathrm{cone}(e_0, \pm e_{r+1})$ where $e_0 := -e_1 - \ldots - e_r$ in $\mathbb{Z}^{r+1}$. In particular, the lineality space of X is generated by $\pm e_{r+1}$.



For the weak tropicalisation we have to blow up the two elliptic fixed points (if they exist). That means, we subdivide the cones σ^+ and σ^- by drawing lines along $\pm e_{r+1}$. This subdivision corresponds to a refinement of the fan Σ which is given by $\Sigma' := \Sigma \cap \operatorname{trop}(X)$. After this step we are in a locally toric situation, i.e. there are only singularities left coming from the toric ambient variety Z' corresponding to the fan Σ'. Consequently, we can resolve all singular cones by adding rays along the elements of the Hilbert basis.

Example 4.9. We continue Example 4.4. This $\mathbb{K}^*$-surface has the Cox ring

$$R(X) \;=\; \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}, S_1]/\langle T_{01}T_{02} + T_{11}^2 + T_{21}^2\rangle$$

and two singularities corresponding to the cones σ^- and $\sigma_2^+ = \operatorname{cone}(v^+, v_{21})$. Following the canonical resolution process we end up with the smooth surface $\widetilde{X}$ having P-matrix and Cox ring

$$\widetilde{P} \;=\; \begin{pmatrix} -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 & 1 & 1 & 0 & 0 \\ -2 & -1 & 1 & 1 & 0 & 1 & 1 & -1 \end{pmatrix},$$

$$R(\widetilde{X}) \;=\; \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}, T_{22}, T_{23}, S_1, S_2]/\langle T_{01}T_{02} + T_{11}^2 + T_{21}^2 T_{22} T_{23}\rangle.$$

Example 4.10. Consider the $\mathbb{K}^*$-surface $X = X(A, P)$ given by the data

$$P \;=\; \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \\ -1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}, \qquad A \;=\; \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & c \end{pmatrix},$$

where $c \in \mathbb{K}^* \setminus \{-1\}$. Then $R(X)$ is given by $\mathbb{K}[T_{01}, T_{02}, T_{11}, T_{12}, T_{21}, T_{31}]/\langle g_0, g_1\rangle$ with

$$g_0 \;=\; T_{01}T_{02} + T_{11}T_{12} + T_{21}^2 \qquad \text{and} \qquad g_1 \;=\; \lambda T_{11}T_{12} + T_{21}^2 + T_{31}^2,$$

where $\lambda := -1 - c$. Following the canonical resolution process we end up with the smooth surface $\widetilde{X}$ having P-matrix and Cox ring

$$\widetilde{P} \;=\; \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 \end{pmatrix},$$

$$R(\widetilde{X}) \;=\; \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{12}, T_{21}, T_{22}, T_{23}, T_{31}, T_{32}, T_{33}, S_1, S_2]/\langle g_0, g_1\rangle\;,$$

where $g_0 = T_{01}T_{02} + T_{11}T_{12} + T_{21}^2 T_{22} T_{23}$ and $g_1 = \lambda T_{11}T_{12} + T_{21}^2 T_{22} T_{23} + T_{31}^2 T_{32} T_{33}$.

This canonical resolution is in general not the minimal resolution of the $\mathbb{K}^*$-surface. There are possibly (-1)-curves, e.g. curves with self-intersection number -1, that can be smoothly contracted (Castel-Nuovo).

Example 4.11. Consider the $\mathbb{K}^*$-surface $X = X(A, P)$ given by the data

$$P = \begin{pmatrix} -1 & -2 & 1 & 2 & 0 \\ -1 & -2 & 0 & 0 & 3 \\ -1 & -1 & 0 & 1 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

This surface has two singularities corresponding to the two elementary big cones σ^+ and σ^-. The canonical resolution leads to the P-matrix

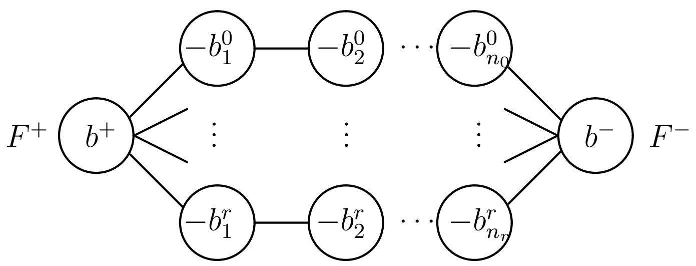
$$\widetilde{P} = \begin{pmatrix} -1 & -2 & -1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & 0 & 0 & 0 & 3 & 2 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & -1 \end{pmatrix}.$$

In this case the curve corresponding to $v^- = (0, 0, -1)$ can be smoothly contracted. Hence, by deleting the last column of $\widetilde{P}$ we end up with the minimal resolution X' of X with Cox ring

$$R(X') = \mathbb{K}[T_{01}, T_{02}, T_{03}, T_{11}, T_{12}, T_{13}, T_{21}, T_{22}, T_{23}, T_{24}, S_1]/\langle g_0 \rangle,$$

where $g_0 = T_{01}T_{02}^2T_{03} + T_{11}T_{12}^2T_{13} + T_{21}^3T_{22}^2T_{23}T_{24}$.

The canonical resolution of singularities of $\mathbb{K}^*$-surfaces leads exactly to the resolution graph that was introduced by Orlik and Wagreich, see [45]. To each smooth $\mathbb{K}^*$-surface without elliptic fixed points they relate a graph of the following form:



Thereby F^+ and F^- represent parabolic fixed point curves. The other circles describe invariant prime divisors D_{ij} that can be contracted. Two of these divisors are connected by an edge if and only if they intersect and have a common fixed point. The numbers $-b_j^i$ are the self-intersection numbers of the invariant prime divisors. Note that one can read off the isotropy orders l_{ij} of this graph. They are given as the numerators of the corresponding canceled continued fraction

$$b_1^i - \cfrac{1}{b_2^i - \cfrac{1}{\dots - \cfrac{1}{b_{j-1}^i}}}.$$

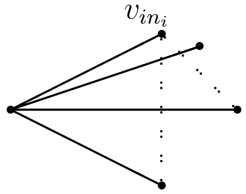
Furthermore, set $X_0 := X \setminus \{F^+, F^-\}$ and $a_i := F^+ \cap D_{i1}$. Then we get a canonical morphism $\varphi\colon X_0/\mathbb{K}^* \to F^+$ such that $\varphi^{-1}(a_i) = \{a_{i1}, \dots, a_{in_i}\}$, where a_{ij} denotes the non trivial $\mathbb{K}^*$-orbit of D_{ij}. Compare [29, Section 5].

4.2 Combinatorially minimal $\mathbb{K}^*$-surfaces

This section is dedicated to combinatorially minimal $\mathbb{K}^*$-surfaces, i.e. $\mathbb{K}^*$-surfaces, that do not have contractible invariant prime divisors. We prove that there are only three types of relations that can occur in the Cox ring of a combinatorially minimal non-toric $\mathbb{K}^*$-surface. Note that in the toric case the only combinatorially minimal surfaces are fake weighted projective spaces of dimension two, corresponding to fans with three rays, and $\mathbb{P}_1 \times \mathbb{P}_1$.

Lemma 4.12. *Let X be a non-toric $\mathbb{K}^*$-surface with two elliptic fixed points. Then X can be contracted to a $\mathbb{K}^*$-surface satisfying $n_i \leq 2$ for all $0 \leq i \leq r$.*

Proof. It is sufficient to consider the situation within one block. All invariant divisors D_{ij} corresponding to rays $\varrho_{ij} = \mathbb{Q}_{\geq 0} v_{ij}$ that fulfill $1 \neq j \neq n_i$ can obviously be contracted.



□

Lemma 4.13. *Let X be a non-toric $\mathbb{K}^*$-surface with only one elliptic fixed point. Then X can be contracted to a $\mathbb{K}^*$-surface satisfying $n_i = 1$ for all $0 \leq i \leq r$.*

Proof. Let X be a $\mathbb{K}^*$-surface with a parabolic fixed point curve F^+. Then all invariant divisors D_{ij} corresponding to rays $\varrho_{ij} = \mathbb{Q}_{\geq 0} v_{ij}$ with $j \neq 1$ can be contracted. □

Lemma 4.14. *Let X be a non-toric $\mathbb{K}^*$-surface with two elliptic fixed points satisfying $n_i = 2$ for all $0 \leq i \leq r$. Then X is contractible.*

Proof. For the given situation we have $\mathrm{rk}(\mathrm{Cl}(X)) = 2(r+1) - (r-1) - 2 = r+1$ and $n = 2(r+1)$. Furthermore, the effective cone has to be fulldimensional and thus has at least $r+1$ extremal rays. Suppose the surface is not contractible, that is all weights $w_{ij} \in K$ are not exceptional. Then there are exactly two weights lying on each of these extremal rays. The weight $\deg(g_0)$ is contained in the relative interior of every cone $\mathrm{cone}(w_{i1}, w_{i2})$. Consequently, for each $0 \leq i \leq r$ we find a $0 \leq j \leq r$ with $i \neq j$ such that $\mathrm{cone}(w_{i1}, w_{i2}) = \mathrm{cone}(w_{j1}, w_{j2})$ holds. In particular, these four weights lie in the same plane. Hence, we conclude

$$\mathrm{rk}(\mathrm{Cl}(X)) \;=\; 2 + \frac{2(r+1)-4}{4} \;=\; \frac{r+3}{2},$$

a contradiction to $\mathrm{rk}(\mathrm{Cl}(X)) = r+1$ for $r \geq 2$. □

Lemma 4.15. *Let X be a non-toric $\mathbb{K}^*$-surface with two elliptic fixed points satisfying $n_j = 2$ for $j \leq s$ and $n_j = 1$ for $j > s$ where $3 \leq s \leq r$. Then X is contractible.*

Proof. If $n_i = 2$ holds, then the weights w_{i1}, w_{i2} generate a cone that contains $\deg(g_0)$ in its relative interior. Furthermore, all weights w_{j1} with $n_j = 1$ are lying on the ray $\mathbb{Q}_{\geq 0} \cdot \deg(g_0)$. We have $\mathrm{rk}(\mathrm{Cl}(X)) = 2(s+1) + r - s - (r-1) - 2 = s + 1$. Thus, analogously to the proof of Lemma 4.14 we obtain the assertion. □

Definition 4.16. We call a $\mathbb{K}^*$-surface X *combinatorially minimal* if there is no invariant prime divisor that can be contracted.

Remark 4.17. A $\mathbb{Q}$-factorial projective surface with finitely generated Cox ring is combinatorially minimal if and only if its effective cone and its moving cone coincide, see [25, Corollary 6.8/6.9].

Proposition 4.18. *Let X be a non-toric combinatorially minimal $\mathbb{K}^*$-surface. Then its Cox ring has one of the following forms:*

- $\mathbb{K}[T_{01}, T_{11}, T_{21}, \ldots, T_{r1}, S_1]/\langle g_0, \ldots, g_r \rangle$, *where* $\mathrm{rk}(\mathrm{Cl}(X)) = 1$ *and*

$$g_0 = T_{01}^{l_{01}} + T_{11}^{l_{11}} + T_{21}^{l_{21}}, \qquad g_i = *T_{i1}^{l_{i1}} + *T_{i+1,1}^{l_{i+1,1}} + *T_{i+2,1}^{l_{i+2,1}} \quad \text{for} \quad 1 \leq i \leq r-2.$$

- $\mathbb{K}[T_{01}, T_{02}, T_{11}, \ldots, T_{r1}]/\langle g_0, \ldots, g_r \rangle$, *where* $\mathrm{rk}(\mathrm{Cl}(X)) = 1$ *and*

$$g_0 = T_{01}^{l_{01}} T_{02}^{l_{02}} + T_{11}^{l_{11}} + T_{21}^{l_{21}}, \qquad g_i = *T_{i1}^{l_{i1}} + *T_{i+1,1}^{l_{i+1,1}} + *T_{i+2,1}^{l_{i+2,1}} \quad \text{for} \quad 1 \leq i \leq r-2.$$

- $\mathbb{K}[T_{01}, T_{02}, T_{11}, T_{12}, T_{21}, \ldots, T_{r1}]/\langle g_0, \ldots, g_r \rangle$, *where* $\mathrm{rk}(\mathrm{Cl}(X)) = 2$ *and*

$$g_0 = T_{01}^{l_{01}} T_{02}^{l_{02}} + T_{11}^{l_{11}} T_{12}^{l_{12}} + T_{21}^{l_{21}}, \qquad g_1 = *T_{11}^{l_{11}} T_{12}^{l_{12}} + *T_{21}^{l_{21}} + *T_{31}^{l_{31}},$$
$$g_i = *T_{i1}^{l_{i1}} + *T_{i+1,1}^{l_{i+1,1}} + *T_{i+2,1}^{l_{i+2,1}} \quad \text{for} \quad 2 \leq i \leq r-2.$$

Proof. If $\mathrm{rk}(\mathrm{Cl}(X)) = 1$ holds, then there are no contractible divisors. The statements of Lemma 4.12, 4.13, 4.14, and 4.15 complete the proof. □

Example 4.19 shows that the third case in Proposition 4.18 really occurs.

Example 4.19. Let X be the Fano $\mathbb{K}^*$-surface with Cox ring $R(X) = \mathbb{K}[T]/\langle g_0 \rangle$, where $g_0 = T_{01}^2 T_{02} + T_{11}^2 T_{12} + T_{21}^2$, whose P-matrix and grading matrix Q are given by

$$P = \begin{pmatrix} -2 & -1 & 2 & 1 & 0 \\ -2 & -1 & 0 & 0 & 2 \\ -3 & -1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 & 1 \end{pmatrix}.$$

Then X is not contractible, since the effective cone coincides with the moving cone.

4.3 Intersection Theory

In this section we want to apply toric intersection theory on complexity-one T-varieties and we will give concrete formulas for the intersection numbers of two invariant prime divisors of a $\mathbb{K}^*$-surface.

First we shortly recall some basic facts about toric intersection theory (compare [5, Prop 1.2.8]). Let Z be a toric variety (of dimension $r+1$). Then the intersection number of $r+1$ pairwise different invariant prime divisors $D_Z^1, \dots, D_Z^{r+1}$ of Z can be computed in terms of the associated fan Σ. Let $\varrho_1, \dots, \varrho_{r+1}$ be the rays corresponding to the invariant prime divisors and $v_1, \dots, v_r$ their primitive generators. Consider the cone $\sigma = \mathrm{cone}(v_1, \dots, v_r)$. Then the intersection number of these $r+1$ divisors in Z is given as

$$D_Z^1 \cdots D_Z^{r+1} \;=\; \begin{cases} \frac{1}{|\det(\sigma)|} & \text{if } \sigma \in \Sigma \\ 0 & \text{if } \sigma \notin \Sigma \end{cases},$$

where $\det(\sigma) = \det(v_1, \dots, v_r)$. Note that the absolute value of the determinant $\det(\sigma)$ is the index of the sublattice spanned by the generators $v_1, \dots, v_r$ in the lattice $N \cap \mathrm{lin}(v_1, \dots, v_r)$.

Let $X = X(A, P, \Phi)$ be a complexity-one T-variety of dimension d and Z its minimal toric ambient variety. Since X is a complete intersection, toric intersection theory suffices to calculate intersection numbers of X. The intersection number of d invariant prime divisors $D_X^1 \cdots D_X^d$ of X is given by the toric intersection number of

$$D_Z^1 \cdots D_Z^d \cdot D_{\deg(g_0)} \cdots D_{\deg(g_{r-2})},$$

where $D_X^i = D_Z^i \cap X$, for $i = 1, \dots, d$. Note that [5, Proposition 4.2.11] provides a possibility to calculate intersection numbers in the divisor class group $K = \mathrm{Cl}(X)$ by computing the index of the sublattice which is given by the weights which are "complementary" with regard to the generators of σ.

Now, consider a $\mathbb{K}^*$-surface $X = X(A, P)$ arising from a matrix P, where P is given as introduced in 4.1. Let $D_{ij} \subseteq X$ be the prime divisors corresponding to the rays ϱ_{ij} generated by the columns v_{ij} of the matrix P. Analogously, we denote the corresponding toric prime divisors of the minimal toric ambient variety Z as D_{ij}^Z, where $D_{ij} = D_{ij}^Z \cap X$. Furthermore, let F^+ and F^- be the divisors corresponding to parabolic fixed point curves (if existing).

Proposition 4.20. *The intersection number of two different prime divisors D_{ij} and D_{kl} of X can be computed as follows:*

(i) Let $D_{ij} \cap D_{kl} = \emptyset$. Then $D_{ij} \cdot D_{kl} = 0$.

(ii) Let D_{ij} and D_{ij+1} be two adjacent divisors lying in the same block. Then they intersect in a hyperbolic fixed point and

$$D_{ij} \cdot D_{ij+1} = \frac{1}{l_{ij} d_{ij+1} - l_{ij+1} d_{ij}}.$$

(iii) Two divisors D_{ij} *and* D_{kl} *that lie in different blocks and intersect in an elliptic fixed point satisfy either* $j = l = 1$ *or* $j = n_i$ *and* $l = n_k$.

(a) If $n_i \neq 1$ *or* $n_k \neq 1$ *or if* X *has only one elliptic fixed point then*

$$D_{i1} \cdot D_{k1} \;=\; -\frac{1}{l_{i1}l_{k1}\sum_{j=0}^{r}\frac{d_{j1}}{l_{j1}}} \qquad \text{and} \qquad D_{in_i} \cdot D_{kn_k} \;=\; \frac{1}{l_{in_i}l_{kn_k}\sum_{j=0}^{r}\frac{d_{jn_j}}{l_{jn_j}}}.$$

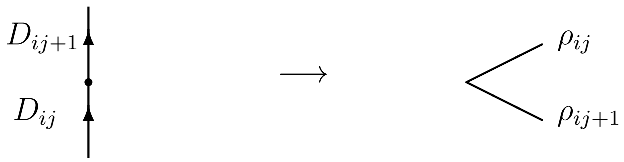
(b) If $n_i = n_k = 1$ *holds and* X *has two elliptic fixed points then*

$$D_{i1} \cdot D_{k1} \;=\; D_{in_i} \cdot D_{kn_k} \;=\; \frac{1}{l_{in_i}l_{kn_k}\sum_{j=0}^{r}\frac{d_{jn_j}}{l_{jn_j}}} - \frac{1}{l_{i1}l_{k1}\sum_{j=0}^{r}\frac{d_{j1}}{l_{j1}}}.$$

(iv) A divisor D_{ij} *that intersects a parabolic fixed point curve* F^+ *or* F^- *satisfies* $j = 1$ *or* $j = n_i$ *and*

$$D_{i1} \cdot F^- = \frac{1}{l_{i1}} \qquad \text{and} \qquad D_{in_i} \cdot F^+ = \frac{1}{l_{in_i}}.$$

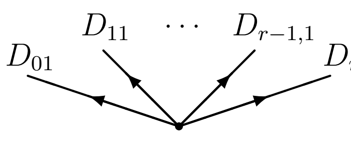
Proof. Case (i) is obvious. If we are in situation (ii), then we can locally restrict to the toric situation and use toric intersection theory (see 3.9). Two adjacent divisors D_{ij} and D_{ij+1} intersect in a hyperbolic fixed point corresponding to the cone $\mathrm{cone}(v_{ij}, v_{ij+1})$ generated by the corresponding rays $\rho_{ij} = \mathbb{Q}_{\geq 0} \cdot v_{ij}$ and $\rho_{ij+1} = \mathbb{Q}_{\geq 0} \cdot v_{ij+1}$.



Note that we chose P such that $l_{ij}d_{ij+1} - l_{ij+1}d_{ij} > 0$ holds. By setting $\widehat{v}_{ij} = (l_{ij}, d_{ij})$ and $\widehat{v}_{ij+1} = (l_{ij+1}, d_{ij+1})$ we obtain

$$D_{ij} \cdot D_{ij+1} \;=\; \frac{1}{|\det(\widehat{v}_{ij}, \widehat{v}_{ij+1})|} \;=\; \frac{1}{l_{ij}d_{ij+1} - l_{ij+1}d_{ij}}.$$

For case (iii) we assume that D_{i1} and D_{k1} intersect in an elliptic fixed point. Then all divisors D_{j1} with $0 \leq j \leq r$ intersect in this elliptic fixed point.



Since all relations of the Cox ring are homogeneous with the same degree we have

$$\deg(g_i) \;=\; l_{01}D_{01} + \ldots + l_{0n_0}D_{0n_0} \;=\; \ldots \;=\; l_{r1}D_{r1} + \ldots + l_{rn_r}D_{rn_r}$$

for $0 \leq i \leq r-2$. Using the fact that the intersection number of two divisors that do not intersect is zero we obtain the following intersection numbers:

(a) If $n_i \neq 1$ or $n_k \neq 1$ or if X has only one elliptic fixed point, then

$$\begin{aligned} D_{i1} \cdot D_{k1} &= D_{i1}^Z \cdot D_{k1}^Z \cdot D_{\deg(g_0)} \cdots D_{\deg(g_{r-2})} \\ &= D_{i1}^Z \cdot D_{k1}^Z \cdot \prod_{j \neq i,k} l_{j1} D_{j1}^Z + D_{i1}^Z \cdot D_{k1}^Z \cdot \prod_{j \neq i,k} l_{jn_j} D_{jn_j}^Z \\ &= \prod_{j \neq i,k} l_{j1} \prod_{j=0}^{r} D_{j1}^Z \end{aligned}$$

.

(b) If $n_i = n_k = 1$ holds and X has two elliptic fixed points, then

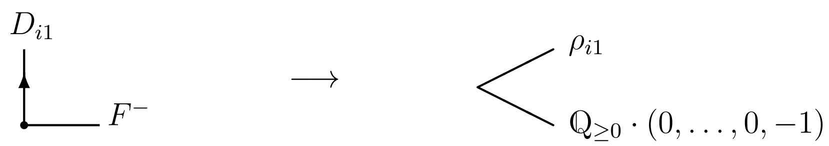
$$\begin{aligned} D_{i1} \cdot D_{k1} &= D_{i1}^Z \cdot D_{k1}^Z \cdot D_{\deg(g_0)} \cdots D_{\deg(g_{r-2})} \\ &= D_{i1}^Z \cdot D_{k1}^Z \cdot \prod_{j \neq i,k} l_{j1} D_{j1}^Z + D_{i1}^Z \cdot D_{k1}^Z \cdot \prod_{j \neq i,k} l_{jn_j} D_{jn_j}^Z \\ &= \prod_{j \neq i,k} l_{j1} \prod_{j=0}^{r} D_{j1}^Z + \prod_{j \neq i,k} l_{jn_j} \prod_{j=0}^{r} D_{jn_j}^Z. \end{aligned}$$

We set $l^- := l_{01} \cdots l_{r1}$. The toric intersection number $D_{01}^Z \cdots D_{r1}^Z$ is given by the absolute value of the inverse determinant of the cone σ^- corresponding to the lower elliptic fixed point. By 4.8 we obtain

$$D_{01}^Z \cdots D_{r1}^Z = \left(- \sum_{j=0}^{r} \frac{d_{j1} l^-}{l_{j1}} \right)^{-1},$$

which implies the assertion. Analogously, we proceed to obtain the intersection number $D_{in_i} \cdot D_{kn_k}$ by considering the cone σ^+ and $l^+ := l_{0n_0} \cdots l_{rn_r}$.

To prove assertion (iv) we locally restrict once more to the toric situation. Exemplarily, we compute the intersection number $D_{i1} \cdot F^-$.



By setting $\widehat{v}_{i1} := (l_{i1}, d_{i1})$ and $\widehat{v}^- := (0, -1)$ we obtain

$$D_{i1} \cdot F^- = \frac{1}{|\det(\widehat{v}_{i1}, \widehat{v}^-)|} = \frac{1}{l_{i1}}.$$

□

The precedent proposition allows us now to compute the self-intersection numbers of the invariant divisors D_{ij} and $F^{\pm}$.

Proposition 4.21. *Let X be a $\mathbb{K}^*$-surface. The self-intersection numbers of the invariant prime divisors can be computed as follows:*

(i) If $n_i = 1$, then

$$D_{i1}^2 \;=\; D_{in_i}^2 \;=\; \begin{cases} -\frac{1}{l_{i1}^2 \sum_{j=0}^r \frac{d_{j1}}{l_{j1}}} & \text{if } X \text{ is of type (par,ell)},\\ \frac{1}{l_{i1}^2 \sum_{j=0}^r \frac{d_{jn_j}}{l_{jn_j}}} & \text{if } X \text{ is of type (ell,par)},\\ \frac{1}{l_{i1}^2 \sum_{j=0}^r \frac{d_{jn_j}}{l_{jn_j}}} - \frac{1}{l_{i1}^2 \sum_{j=0}^r \frac{d_{j1}}{l_{j1}}} & \text{if } X \text{ is of type (ell,ell)},\\ 0 & \text{if } X \text{ is of type (par,par)}. \end{cases}$$

(ii) If $n_i \neq 1$ and $1 < j < n_i$, then

$$D_{ij}^2 \;=\; -\frac{l_{ij-1}d_{ij+1} - l_{ij+1}d_{ij-1}}{(l_{ij-1}d_{ij} - l_{ij}d_{ij-1})(l_{ij}d_{ij+1} - l_{ij+1}d_{ij})} \,.$$

(iii) If $n_i \neq 1$ and X is of type (ell,ell) or (par,ell), then

$$D_{i1}^2 \;=\; -\frac{1}{l_{i1}^2 \sum_{j=0}^r \frac{d_{j1}}{l_{j1}}} - \frac{l_{i2}}{l_{i1}(l_{i1}d_{i2} - l_{i2}d_{i1})} \,.$$

If $n_i \neq 1$ and X is of type (ell,par) or (par,par), then

$$D_{i1}^2 \;=\; -\frac{l_{i2}}{l_{i1}(l_{i1}d_{i2} - l_{i2}d_{i1})} \,.$$

(iv) If $n_i \neq 1$ and X is of type (ell,ell) or (ell,par), then

$$D_{in_i}^2 \;=\; \frac{1}{l_{in_i}^2 \sum_{j=0}^r \frac{d_{jn_j}}{l_{jn_j}}} - \frac{l_{i,n_i-1}}{l_{in_i}(-l_{in_i}d_{i,n_i-1} + l_{i,n_i-1}d_{in_i})} \,.$$

If $n_i \neq 1$ and X is of type (par,ell) or (par,par), then

$$D_{in_i}^2 \;=\; -\frac{l_{i,n_i-1}}{l_{in_i}(-l_{in_i}d_{i,n_i-1} + l_{i,n_i-1}d_{in_i})} \,.$$

(v) If F^+ resp. F^- is a parabolic fixed point curve, then

$$F^+ \cdot F^+ = -\sum_{i=0}^r \frac{d_{in_i}l^+}{l_{in_i}} \qquad \text{resp.} \qquad F^- \cdot F^- = \sum_{i=0}^r \frac{d_{i1}l^-}{l_{i1}} \,.$$

Proof. Using the homogeneity conditions for the Cox ring relations we obtain

$$D_{ij} \;=\; \frac{1}{l_{ij}}\left(\sum_{m=1}^{n_k} l_{km}D_{km} - \sum_{\substack{s=1\\ s\neq j}}^{n_i} l_{is}D_{is}\right)$$

for an arbitrary $k \neq i$. In situation (i) we have $n_i = 1$. Hence, we may assume that there is always one $n_k \neq 1$ whenever X is of type (ell,ell). This leads to the following cases:

$$\begin{aligned} D_{i1}^2 \;=\; D_{in_i}^2 \;&=\; D_{i1}\cdot\frac{1}{l_{i1}}\left(\sum_{m=1}^{n_k} l_{km}D_{km}\right) \\ &= \begin{cases} \frac{l_{k1}}{l_{i1}}D_{i1}\cdot D_{k1} & \text{if } X \text{ is of type (par,ell)}\,, \\ \frac{l_{kn_k}}{l_{i1}}D_{i1}\cdot D_{kn_k} & \text{if } X \text{ is of type (ell,par)}\,, \\ \frac{l_{k1}}{l_{i1}}D_{i1}\cdot D_{k1} + \frac{l_{kn_k}}{l_{i1}}D_{i1}\cdot D_{kn_k} & \text{if } X \text{ is of type (ell,ell)}\,, \\ 0 & \text{if } X \text{ is of type (par,par)}\,. \end{cases} \end{aligned}$$

Using the results of Proposition 4.20 gives the assertions.

For assertions (ii)-(vi) we assume $n_i \neq 1$. We first consider the case $1 < j < n_i$. The homogeneity condition together with Proposition 4.20 gives

$$\begin{aligned} D_{ij}^2 &= D_{ij}\cdot\frac{1}{l_{ij}}\Big(\sum_{m=1}^{n_k} l_{km}D_{km} - \sum_{\substack{s=1\\ s\neq j}}^{n_i} l_{is}D_{is}\Big) \\ &= \frac{1}{l_{ij}}(D_{ij}\cdot(-l_{ij-1}D_{ij-1} - l_{ij+1}D_{ij+1})) \\ &= -\frac{l_{ij-1}}{l_{ij}}D_{ij}\cdot D_{ij-1} - \frac{l_{ij+1}}{l_{ij}}D_{ij}\cdot D_{ij+1} \\ &= -\frac{l_{ij-1}}{l_{ij}(l_{ij-1}d_{ij} - l_{ij}d_{ij-1})} - \frac{l_{ij+1}}{l_{ij}(l_{ij}d_{ij+1} - l_{ij+1}d_{ij})} \\ &= -\frac{l_{ij-1}d_{ij+1} - l_{ij+1}d_{ij-1}}{(l_{ij-1}d_{ij} - l_{ij}d_{ij-1})(l_{ij}d_{ij+1} - l_{ij+1}d_{ij})}. \end{aligned}$$

Now, we compute the self-intersection numbers of D_{i1} and D_{in_i}. As before we can use the homogeneity conditions and then apply Proposition 4.20:

$$\begin{aligned} D_{i1}^2 &= D_{i1}\cdot\frac{1}{l_{i1}}\Big(\sum_{m=1}^{n_k} l_{km}D_{km} - \sum_{s=2}^{n_i} l_{is}D_{is}\Big) \\ &= \begin{cases} \frac{l_{k1}}{l_{i1}}D_{i1}\cdot D_{k1} - \frac{l_{i2}}{l_{i1}}D_{i1}D_{i2} & \text{if } X \text{ is of type (par,ell) or (ell,ell)} \\ -\frac{l_{i2}}{l_{i1}}D_{i1}\cdot D_{i2} & \text{if } X \text{ is of type (ell,par) or (par,par)} \end{cases} \end{aligned}$$

$$
\begin{aligned}
D_{in_i}^2 &= D_{in_i} \cdot \frac{1}{l_{in_i}} (\sum_{m=1}^{n_k} l_{km} D_{km} - \sum_{s=1}^{n_i-1} l_{is} D_{is}) \\
&= \begin{cases} \frac{l_{kn_k}}{l_{in_i}} D_{in_i} \cdot D_{kn_k} - \frac{l_{i,n_i-1}}{l_{in_i}} D_{in_i} D_{i,n_i-1} & \text{if } X \text{ is of type (ell,par) or (ell,ell)} \\ -\frac{l_{i,n_i-1}}{l_{in_i}} D_{in_i} \cdot D_{i,n_i-1} & \text{if } X \text{ is of type (par,ell) or (par,par)} \end{cases}
\end{aligned}
$$

The last statement can be easily derived from the fact that $\mathrm{Cl}(X) \cong \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$ and the degrees of the variables T_{ij} and S_k represent the kernel of P. Thus, taking the corresponding divisor classes and multiplying them with the last row of P we obtain

$$
\sum_{i=0}^{r} \sum_{j=1}^{n_i} d_{ij} D_{ij} - F^- + F^+ = 0.
$$

Note that F^- and F^+ do not intersect. Now, we can compute the self-intersection numbers directly in the divisor class group by using the "complementary indices", see [5, Proposition 4.2.11].

We obtain

$$
\begin{aligned}
F^- \cdot F^- &= d_{01} D_{01} \cdot F^- + \ldots + d_{r1} D_{r1} \cdot F^- \\
&= \frac{d_{01}}{\prod_{i \neq 0} l_{i1}} + \ldots + \frac{d_{r1}}{\prod_{i \neq r} l_{i1}} \\
&= \sum_{i=0}^{r} \frac{d_{i1} l^-}{l_{i1}} \, ,
\end{aligned}
$$

and analogously

$$
\begin{aligned}
F^+ \cdot F^+ &= -\left(d_{0n_0} D_{0n_0} \cdot F^+ + \ldots + d_{rn_r} D_{rn_r} \cdot F^+\right) \\
&= \frac{d_{0n_0}}{\prod_{i \neq 0} l_{in_i}} + \ldots + \frac{d_{rn_r}}{\prod_{i \neq r} l_{in_i}} \\
&= -\sum_{i=0}^{r} \frac{d_{in_i} l^+}{l_{in_i}} \, .
\end{aligned}
$$

□

Example 4.22 (Example 4.11 continued)**.** Having all these formulas we can compute the self-intersection numbers of the exceptional divisors of the resolution in Example 4.11. One easily checks, that $(F^-)^2 = -1$ holds for the canonical resolution. Contracting this (-1)-curve we obtain the minimal resolution and the following self-intersection numbers:

$$
D_{03}^2 = D_{12}^2 = D_{22}^2 = D_{23}^2 = D_{24}^2 = (F^+)^2 \;=\; -2.
$$

4.4 Kleiman condition for ampleness

In this chapter we will give concrete formulas for the intersection numbers of the anti-canonical divisor with the prime divisors D_{ij} and $F^{\pm}$ of a $\mathbb{K}^*$-surface X. These intersection numbers can be used as basis for concrete conditions for X to be Fano.

Remark 4.23 (Kleiman's criteria for ampleness)**.** Let D be a divisor of a normal complete variety of dimension two. Then D is ample if and only if $D \cdot C > 0$ for all effective curves C.

Proposition 4.24. *Let $-K_X$ be the anticanonical divisor of a (non-toric) $\mathbb{K}^*$-surface $X = X(A,P)$ and set $\widehat{v}_{ij} := (l_{ij}, d_{ij})$ and $\widehat{v}^{\pm} := (0, \pm 1)$. Then the following statements hold:*

(i) Let $F^{\pm}$ be parabolic fixed point curves. Then $-K_X \cdot F^+ > 0$ resp. $-K_X \cdot F^- > 0$ holds if and only if

$$\sum_{i=0}^{r} \frac{d_{i1}}{l_{i1}} \;>\; (r-1) - \sum_{i=0}^{r} \frac{1}{l_{i1}} \quad \textit{resp.} \quad -\sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}} \;>\; (r-1) - \sum_{i=0}^{r} \frac{1}{l_{in_i}}.$$

(ii) Let $n_i \neq 1$ and $1 < j < n_i$. Then $-K_X \cdot D_{ij} > 0$ if and only if

$$\det(\widehat{v}_{i,j-1}, \widehat{v}_{i,j}) + \det(\widehat{v}_{i,j}, \widehat{v}_{i,j+1}) \;>\; \det(\widehat{v}_{i,j-1}, \widehat{v}_{i,j+1}).$$

(iii) Let $n_i \neq 1$ and $j = 1$. If X is of type (ell,ell) or (par,ell), then $-K_X \cdot D_{ij} > 0$ if and only if

$$\frac{l_{i1} - l_{i2}}{l_{i1}d_{i2} - l_{i2}d_{i1}} \cdot \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}} \;<\; \sum_{j=0}^{r} \frac{1}{l_{j1}} - (r-1)\ ,$$

and if X is of type (ell,par) or (par,par), then $-K_X \cdot D_{ij} > 0$ if and only if

$$\det(\widehat{v}^{-}, \widehat{v}_{i1}) + \det(\widehat{v}_{i1}, \widehat{v}_{i2}) \;>\; \det(\widehat{v}^{-}, \widehat{v}_{i2})\ .$$

(iv) Let $n_i \neq 1$ and $j = n_i$. If X is of type (ell,ell) or (ell,par), then $-K_X \cdot D_{ij} > 0$ if and only if

$$\frac{l_{in_i} - l_{in_i-1}}{l_{in_i-1}d_{in_i} - l_{in_i}d_{in_i-1}} \cdot \sum_{j=0}^{r} \frac{d_{jn_j}}{l_{jn_j}} \;>\; (r-1) - \sum_{j=0}^{r} \frac{1}{l_{jn_j}}\ ,$$

and if X is of type (par,ell) or (par,par), then $-K_X \cdot D_{ij} > 0$ if and only if

$$\det(\widehat{v}_{in_i}, \widehat{v}^{+}) + \det(\widehat{v}_{i,n_i-1}, \widehat{v}_{in_i}) \;>\; \det(\widehat{v}_{i,n_i-1}, \widehat{v}^{+})\ .$$

(v) Let $n_i = 1$. If X is of type (par,ell), then $-K_X \cdot D_{i1} > 0$ if and only if

$$\sum_{i=0}^{r} \frac{d_{i1}}{l_{i1}} \;<\; -(r-1) + \sum_{i=0}^{r} \frac{1}{l_{i1}} \,.$$

If X is of type (ell,par), then $-K_X \cdot D_{i1} > 0$ if and only if

$$\sum_{j=0}^{r} \frac{d_{jn_j}}{l_{jn_j}} \;>\; -\sum_{k=1}^{r} \frac{1}{l_{kn_k}} + (r-1) \,.$$

If X is of type (ell,ell), then $-K_X \cdot D_{i1} > 0$ if and only if

$$\sum_{j=0}^{r} \frac{d_{jn_j}}{l_{jn_j}} \left(\sum_{k=0}^{r} \frac{1}{l_{k1}} - (r-1) \right) \;>\; \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}} \left(\sum_{k=0}^{r} \frac{1}{l_{kn_k}} - (r-1) \right) \,.$$

If X is of type (par,par), then $-K_X \cdot D_{i1} > 0$ is always satisfied.

Proof. Using the results of the last section we can prove the statements above. For assertion (i) we have

$$\begin{aligned}
-K_X \cdot F^- &= \left(\sum_{i=0}^{r} \sum_{j=1}^{n_i} D_{ij} + F^+ + F^- - (r-1) \sum_{j=1}^{n_0} l_{0j} D_{0j} \right) \cdot F^- \\
&= (F^-)^2 + \sum_{i=0}^{r} D_{i1} \cdot F^- - (r-1) l_{01} D_{01} \cdot F^- \\
&= \sum_{i=0}^{r} \frac{d_{i1}}{l_{i1}} + \sum_{i=0}^{r} \frac{1}{l_{i1}} - (r-1) \,, \\
-K_X \cdot F^+ &= -\sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}} + \sum_{i=0}^{r} \frac{1}{l_{in_i}} - (r-1) \,.
\end{aligned}$$

(ii) If $n_i \neq 1$ and $1 < j < n_i$, then the intersection number $-K_X \cdot D_{ij}$ is given by the following term (where $s \neq i$):

$$\begin{aligned}
&\left(\sum_{i=0}^{r} \sum_{j=1}^{n_i} D_{ij} + F^+ + F^- - (r-1) \sum_{j=1}^{n_s} l_{sj} D_{sj} \right) \cdot D_{ij} \\
&= D_{ij-1} \cdot D_{ij} + D_{ij}^2 + D_{ij+1} D_{ij} \\
&= \frac{1}{l_{ij-1} d_{ij} - l_{ij} d_{ij-1}} - \frac{l_{ij-1}}{l_{ij}(l_{ij-1} d_{ij} - l_{ij} d_{ij-1})} \\
&\quad - \frac{l_{ij+1}}{l_{ij}(l_{ij} d_{ij+1} - l_{ij+1} d_{ij})} + \frac{1}{l_{ij} d_{ij+1} - l_{ij+1} d_{ij}} \\
&= \frac{l_{ij} d_{ij+1} - l_{ij+1} d_{ij} + l_{ij-1} d_{ij} - l_{ij} d_{ij-1} + l_{ij+1} d_{ij-1} - l_{ij-1} d_{ij+1}}{(l_{ij-1} d_{ij} - l_{ij} d_{ij-1})(l_{ij} d_{ij+1} - l_{ij+1} d_{ij})} \\
&= \frac{\det(\widehat{v}_{i,j-1}, \widehat{v}_{i,j}) + \det(\widehat{v}_{i,j}, \widehat{v}_{i,j+1}) - \det(\widehat{v}_{i,j-1}, \widehat{v}_{i,j+1})}{\det(\widehat{v}_{i,j-1}, \widehat{v}_{i,j}) \cdot \det(\widehat{v}_{i,j}, \widehat{v}_{i,j+1})} .
\end{aligned}$$

Note that all these determinants are positive. Thus, $-K_X \cdot D_{ij} > 0$ gives the following inequality:

$$\det(\widehat{v}_{i,j-1}, \widehat{v}_{i,j}) + \det(\widehat{v}_{i,j}, \widehat{v}_{i,j+1}) \;>\; \det(\widehat{v}_{i,j-1}, \widehat{v}_{i,j+1})$$

Now we will prove assertions (iii) and (iv). Assume $n_i \neq 1$ and $j = 1$ or $j = n_i$. If X is of type (ell,ell) or (par,ell), then (for one $s \neq k$) the intersection number $-K_X \cdot D_{i1}$ is given by

$$\sum_{k=0}^{r} D_{i1}D_{k1} + D_{i1}D_{i2} - (r-1)l_{s1}D_{s1}D_{i1}.$$

If X is of type (ell,par) or(par,par) we obtain

$$-K_X \cdot D_{i1} \;=\; D_{i1} \cdot D_{i1} + D_{i1}D_{i2} + F^- \cdot D_{i1}.$$

Analogously, the intersection number $-K_X \cdot D_{in_i}$ is given by

$$\sum_{k=0}^{r} D_{in_i}D_{kn_k} + D_{in_i}D_{in_i-1} - (r-1)l_{sn_s}D_{sn_s}D_{in_i}$$

if X is of type (ell,par) or (ell,ell) and if X is of type (par,ell) or (par,par), we obtain

$$D_{in_i} \cdot D_{in_i} + D_{in_i}D_{in_i-1} + F^+ \cdot D_{in_i}.$$

Now assume that X is of type (ell,ell) or (par,ell). Then we have

$$\begin{aligned}
-K_X \cdot D_{i1} \;&=\; \sum_{k=0}^{r} D_{i1}D_{k1} + D_{i1}D_{i2} - (r-1)l_{s1}D_{s1}D_{i1} \\
&=\; -\frac{1}{l_{i1}^2 \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}}} - \frac{l_{i2}}{l_{i1}(l_{i1}d_{i2} - l_{i2}d_{i1})} - \sum_{k \neq i} \frac{1}{l_{i1}l_{k1} \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}}} + \frac{1}{l_{i1}d_{i2} - l_{i2}d_{i1}} \\
&\quad + (r-1)l_{s1} \cdot \frac{1}{l_{s1}l_{i1} \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}}} \\
&=\; \frac{(-\sum_{k=0}^{r} \frac{1}{l_{k1}} + (r-1))(l_{i1}d_{i2} - l_{i2}d_{i1}) + (l_{i1} - l_{i2}) \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}}}{l_{i1} \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}} (l_{i1}d_{i2} - l_{i2}d_{i1})}.
\end{aligned}$$

If X is of type (ell,par) or (ell,ell) we obtain analogously

$$-K_X \cdot D_{in_i} \;=\; \frac{(\sum_{k=0}^{r} \frac{1}{l_{kn_k}} - (r-1))(l_{in_i-1}d_{in_i} - l_{in_i}d_{in_i-1}) + (l_{in_i} - l_{in_i-1}) \sum_{j=0}^{r} \frac{d_{jn_j}}{l_{jn_j}}}{l_{in_i} \sum_{j=0}^{r} \frac{d_{jn_j}}{l_{jn_j}} (l_{in_i-1}d_{in_i} - l_{in_i}d_{in_i-1})}.$$

Since $\det(\widehat{v}_{i1}, \widehat{v}_{i2}) > 0, \det(\widehat{v}_{i,n_i-1}, \widehat{v}_{i,n_i}) > 0$, $(-1)^r\det(\sigma^+) > 0$, $(-1)^r\det(\sigma^-) < 0$ and $l_{i1}, l_{in_i} > 0$ hold, assertion (ii) follows.

Now assume that X is of type (ell,par) or (par,par). Then we have

$$\begin{aligned}-K_X \cdot D_{i1} &= D_{i1} \cdot D_{i1} + D_{i1}D_{i2} + F^- \cdot D_{i1} \\ &= -\frac{l_{i2}}{l_{i1}(l_{i1}d_{i2} - l_{i2}d_{i1})} + \frac{1}{l_{i1}d_{i2} - l_{i2}d_{i1}} + \frac{1}{l_{i1}} \\ &= \frac{-l_{i2} + l_{i1} + l_{i1}d_{i2} - l_{i2}d_{i1}}{l_{i1}(l_{i1}d_{i2} - l_{i2}d_{i1})} \\ &= \frac{\det(\widehat{v}^-, \widehat{v}_{i1}) + \det(\widehat{v}_{i1}, \widehat{v}_{i2}) - \det(\widehat{v}^-, \widehat{v}_{i2})}{\det(\widehat{v}^-, \widehat{v}_{i1}) \cdot \det(\widehat{v}_{i1}, \widehat{v}_{i2})}.\end{aligned}$$

Analogously, we obtain

$$\begin{aligned}-K_X \cdot D_{in_i} &= \frac{-l_{in_i-1} + l_{in_i} + l_{in_i-1}d_{in_i} - l_{in_i}d_{in_i-1}}{l_{in_i-1}(l_{in_i-1}d_{in_i} - l_{in_i}d_{in_i-1})} \\ &= \frac{\det(\widehat{v}_{in_i}, \widehat{v}^+) + \det(\widehat{v}_{i,n_i-1}, \widehat{v}_{in_i}) - \det(\widehat{v}_{i,n_i-1}, \widehat{v}^+)}{\det(\widehat{v}_{in_i}, \widehat{v}^+) \cdot \det(\widehat{v}_{i,n_i-1}, \widehat{v}_{in_i})}.\end{aligned}$$

Since all these determinants are positive, we obtain the inequalities

$$\det(\widehat{v}^-, \widehat{v}_{i1}) + \det(\widehat{v}_{i1}, \widehat{v}_{i2}) \;>\; \det(\widehat{v}^-, \widehat{v}_{i2})$$

and

$$\det(\widehat{v}_{in_i}, \widehat{v}^+) + \det(\widehat{v}_{i,n_i-1}, \widehat{v}_{in_i}) \;>\; \det(\widehat{v}_{i,n_i-1}, \widehat{v}^+).$$

(v) Assume $n_i = 1$ and hence $j = 1 = n_i$. Then $-K_X \cdot D_{i1}$ is given by the following formulas (where $s \neq i$). If X is of type (par,ell), then

$$-K_X \cdot D_{i1} \;=\; \sum_{k=0}^{r} D_{i1}D_{k1} + F^+ \cdot D_{i1} - (r-1)l_{s1}D_{s1}D_{i1},$$

and if X is of type (ell,par), then

$$-K_X \cdot D_{i1} \;=\; \sum_{k=0}^{r} D_{i1}D_{kn_k} + F^- \cdot D_{i1} - (r-1)l_{sn_s}D_{sn_s}D_{i1}.$$

If X is of type (ell,ell), then we can always choose $n_s \neq 1$ and we obtain

$$-K_X \cdot D_{i1} \;=\; \sum_{k=0}^{r} D_{i1}D_{k1} + \sum_{n_k \neq 1} D_{i1}D_{kn_k} - (r-1)(l_{s1}D_{s1}D_{i1} + l_{sn_s}D_{sn_s}D_{i1}).$$

If X is of type (par,par), then

$$-K_X \cdot D_{i1} \;=\; D_{i1} \cdot D_{i1} + F^- \cdot D_{i1} + F^+ \cdot D_{i1}.$$

Assume that X is of type (par,ell). Then we obtain

$$\begin{aligned}
-K_X \cdot D_{i1} &= \sum_{k=0}^{r} D_{i1}D_{k1} + F^+ \cdot D_{i1} - (r-1)l_{s1}D_{s1}D_{i1} \\
&= -\frac{1}{l_{i1}^2 \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}}} - \sum_{k \neq i} \frac{1}{l_{i1}l_{k1} \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}}} + \frac{1}{l_{i1}} + (r-1) \cdot \frac{1}{l_{i1} \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}}} \\
&= \frac{-\sum_{k=0}^{r} \frac{1}{l_{k1}} + (r-1) + \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}}}{l_{i1} \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}}} \\
&= -K_X \cdot D_{in_i},
\end{aligned}$$

and since $l_{i1} > 0$ and $(-1)^r \det(\sigma^-) < 0$, the assertion follows.
Assume that X is of type (ell,par). Then we obtain analogously to the previous case

$$\begin{aligned}
-K_X \cdot D_{i1} &= \sum_{k=0}^{r} D_{i1}D_{kn_k} + F^- \cdot D_{i1} - (r-1)l_{sn_s}D_{sn_s}D_{i1} \\
&= \frac{1}{l_{i1}^2 \sum_{j=0}^{r} \frac{d_{jn_j}}{l_{jn_j}}} - \sum_{k \neq i} \frac{1}{l_{i1}l_{k1} \sum_{j=0}^{r} \frac{d_{jn_j}}{l_{jn_j}}} + \frac{1}{l_{i1}} - (r-1) \cdot \frac{1}{l_{i1} \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}}} \\
&= \frac{\sum_{k=1}^{r} \frac{1}{l_{kn_k}} - (r-1) + \sum_{j=0}^{r} \frac{d_{jn_j}}{l_{jn_j}}}{l_{i1} \sum_{j=0}^{r} \frac{d_{jn_j}}{l_{jn_j}}} \\
&= -K_X \cdot D_{in_i}.
\end{aligned}$$

Assume that X is of type (ell,ell). Then

$$\begin{aligned}
-K_X \cdot D_{i1} &= \sum_{n_k=1} D_{i1}D_{k1} + \sum_{n_k \neq 1} D_{i1}D_{k1} + \sum_{n_k \neq 1} D_{i1}D_{kn_k} \\
&\quad - (r-1)(l_{s1}D_{s1}D_{i1} + l_{sn_s}D_{sn_s}D_{i1}) \\
&= -\sum_{k=0}^{r} \frac{1}{l_{i1}l_{k1} \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}}} + \sum_{k=0}^{r} \frac{1}{l_{in_i}l_{kn_k} \sum_{j=0}^{r} \frac{d_{jn_j}}{l_{jn_j}}} \\
&\quad - (r-1)\left(-\frac{1}{l_{i1}l_{k1} \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}}} + \frac{1}{l_{in_i}l_{kn_k} \sum_{j=0}^{r} \frac{d_{jn_j}}{l_{jn_j}}}\right) \\
&= \frac{1}{l_{in_i} \sum_{j=0}^{r} \frac{d_{jn_j}}{l_{jn_j}}} \left(\sum_{k=0}^{r} \frac{1}{l_{kn_k}} - (r-1)\right) - \frac{1}{l_{i1} \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}}} \left(\sum_{k=0}^{r} \frac{1}{l_{k1}} - (r-1)\right) \\
&= -K_X \cdot D_{in_i},
\end{aligned}$$

and since $(-1)^r\det(\sigma^+) > 0$, $(-1)^r\det(\sigma^-) < 0$ and $l_{i1} = l_{in_i}$, we obtain the inequality

$$\sum_{j=0}^{r} \frac{d_{jn_j}}{l_{jn_j}} \left(\sum_{k=0}^{r} \frac{1}{l_{k1}} - (r-1) \right) > \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}} \left(\sum_{k=0}^{r} \frac{1}{l_{kn_k}} - (r-1) \right).$$

Finally, assume that X is of type (par,par). Then

$$\begin{aligned} -K_X \cdot D_{i1} &= D_{i1} \cdot D_{i1} + F^- \cdot D_{i1} + F^+ \cdot D_{i1} = \frac{2}{l_{i1}} \qquad \text{and} \\ -K_X \cdot D_{in_i} &= D_{in_i} \cdot D_{in_i} + F^- \cdot D_{in_i} + F^+ \cdot D_{in_i} = \frac{2}{l_{in_i}}. \end{aligned}$$

In particular, we always have $-K_X \cdot D_{i1} > 0$ and $-K_X \cdot D_{in_i} > 0$ since $l_{i1}, l_{in_i} > 0$. □

Corollary 4.25. *If X is a Fano $\mathbb{K}^*$-surface having an index $0 \le i \le r$ such that $l_{ij} = 1$ holds for all $1 \le j \le n_i$, then $n_i \le 2$ holds. If X is non-toric, then $n_i = 2$ holds.*

Proof. This follows directly from Proposition 4.24, since for $1 < j < n_i$ we have

$$-K_X \cdot D_{ij} = \frac{d_{ij+1} - d_{ij} + d_{ij} - d_{ij-1} + d_{ij-1} - d_{ij+1}}{(d_{ij} - d_{ij-1})(d_{ij+1} - d_{ij})} = 0.$$

□

Proposition 4.24 can be used to describe Fano $\mathbb{K}^*$-surfaces $X = X(A, P)$. The inequalities give concrete conditions for the P-matrix of the surface. Note that it is sufficient to check these conditions only for prime divisors whose divisor classes define extremal rays of the effective cone in $\mathrm{Cl}(X)$. If X is a $\mathbb{K}^*$-surface satisfying $\mathrm{rk}(\mathrm{Cl}(X)) = 1$ then it even suffices to check the condition for only one divisor.

4.5 The anticanonical complex for $\mathbb{K}^*$-surfaces

For toric varieties, there is a one-to-one correspondence between toric Fano varieties and convex lattice polytopes, i.e. polytopes whose vertices are lattice points. The polytope is given by the convex hull of the prime generators of the rays of the fan associated to the toric variety. We will call this polytope the *toric anticanonical polytope*. This convex geometrical approach can also be used to describe singularity types of toric Fano varieties.

The following correspondences hold:

$$\begin{array}{rcl} \{\text{ toric Fano varieties }\} & \longleftrightarrow & \{\text{ convex lattice polytopes }\} \\ \{\text{ terminal toric Fano varieties }\} & \longleftrightarrow & \{\text{ convex lattice polytopes,} \\ & & \text{ s.th. zero and its vertices} \\ & & \text{ are the only lattice points in it }\} \\ \{\text{ canonical toric Fano varieties }\} & \longleftrightarrow & \{\text{ convex lattice polytopes,} \\ & & \text{s.th. zero is the only interior} \\ & & \text{lattice point in it}\} \\ \{\ \varepsilon\text{-log-terminal toric Fano varieties }\} & \longleftrightarrow & \{\text{ convex lattice polytopes } C, \\ & & \text{ s.th. zero is the only interior} \\ & & \text{ lattice point in } \varepsilon \cdot C\ \} \end{array}$$

The aim for this chapter is to find a similar convex geometrical object for log-terminal $\mathbb{K}^*$-surfaces that are Fano.

Proposition 4.26. *Let $X = X(A, P, \Phi)$ be a Fano variety with complexity-one torus action and* $\mathrm{SAmple}(X) = \mathrm{Mov}(X)$*. Then the minimal toric ambient variety Z of X has a toric Fano completion $\widehat{Z}$.*

Proof. Let X be a Fano variety with complexity-one torus action and $\mathrm{SAmple}(X) = \mathrm{Mov}(X)$. Then $\Phi = \Phi(\mathrm{Mov}(X))$ holds and the Cox ring $\mathcal{R}(X)$ has the form

$$\mathcal{R}(X) \ \cong\ \mathbb{K}[T_{ij};\ 0 \le i \le r,\ 1 \le j \le n_i][S_1, \dots, S_m]\ /\ \langle g_{i,i+1,i+2};\ 0 \le i \le r-2\rangle.$$

Let the $\mathrm{Cl}(X)$-grading of $\mathcal{R}(X)$ be given by the degrees $w_{ij} := \deg(T_{ij})$ for $0 \le i \le r,\ 1 \le j \le n_i$ and $u_k := \deg(S_k)$ for $1 \le k \le m$. Then the relations $g_0, \dots, g_{r-2}$ have all the same degree $\deg(g_0)$ concerning this grading. Furthermore, we consider the homomorphism

$$Q\colon \mathbb{Z}^{n+m} \to \mathrm{Cl}(X), \quad e_{ij} \mapsto w_{ij},\ e_k \mapsto u_k,$$

where $n = n_0 + \ldots + n_r$. By Theorem 1.19 the moving cone of X is given by

$$\mathrm{Mov}(X) = Q(\gamma_{01}) \cap \ldots \cap Q(\gamma_{rn_r}) \cap Q(\gamma_1) \cap \ldots \cap Q(\gamma_k),$$

where

$$\begin{array}{rcll} \gamma_{ij} & = & \mathrm{cone}(e_{01}, \dots, \widehat{e}_{ij}, \dots, e_{rn_r}, e_1, \dots, e_m) & \quad\text{and} \\ \gamma_k & = & \mathrm{cone}(e_{01}, \dots, e_{rn_r}, e_1, \dots, \widehat{e}_k, \dots, e_m) & \end{array}$$

denote the facets of the positive orthant γ in $\mathbb{Q}^{n+m}$. Let Z be the minimal toric ambient variety of X. Then the anticanonical divisor class of Z is given by

$$w_Z \ =\ \sum_{i=0}^{r} \sum_{J=1}^{n_i} w_{ij} + \sum_{k=1}^{m} u_k.$$

By [9, Proposition 8.5], the anticanonical divisor class of X is given by $w_X = w_Z - (r-1)\deg(g_0)$ which gives

$$w_Z \;=\; w_X + (r-1)\deg(g_0).$$

Since X is Fano, we have $w_X \in \mathrm{Ample}(X) = \mathrm{SAmple}(X)^\circ$. We will show that $\deg(g_0)$ lies in $\mathrm{SAmple}(X)$. Therefore we consider the first relation

$$g_0 = T_{01}^{l_{01}} \cdot \ldots \cdot T_{0n_0}^{l_{0n_0}} + T_{11}^{l_{11}} \cdot \ldots \cdot T_{1n_1}^{l_{1n_1}} + T_{21}^{l_{21}} \cdot \ldots \cdot T_{2n_2}^{l_{2n_2}}.$$

Then we have

$$\sum_{j=1}^{n_0} l_{0j} w_{0j} = \deg(g_0) \quad \text{and} \quad \deg(g_0) \in \mathrm{cone}(w_{01}, \ldots, w_{0n_0}) \;=:\; \sigma_0.$$

Since $\sigma_0 \subseteq \gamma_{ij}$ for $1 \le i \le r,\ 1 \le j \le n_i$ and $\sigma_0 \subseteq \gamma_k$ for $1 \le k \le m$, we know that $\deg(g_0)$ is contained in each γ_{ij} with $i \neq 0$ and γ_k respectively. Now consider γ_{0j} for $1 \le j \le n_0$. Then for the second monomial we get analogously $\deg(g_0) \in \mathrm{cone}(w_{11}, \ldots, w_{1n_1}) := \sigma_1$. Since $\sigma_1 \subseteq \gamma_{0j}$, we have $\deg(g_0) \in \gamma_{0j}$ for $1 \le j \le n_0$. All in all this implies that $\deg(g_0)$ lies in $\mathrm{SAmple}(X)$. In particular we obtain

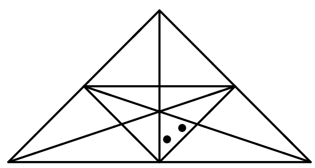
$$w_Z \;=\; w_X + (r-1)\deg(g_0) \in \mathrm{Ample}(X) = \mathrm{Ample}(Z).$$

Now let $\widehat{Z}$ be the completion of Z satisfying $\mathrm{Ample}(X) = \mathrm{Ample}(\widehat{Z})$. Then we have $w_Z = w_{\widehat{Z}}$. In particular $\widehat{Z}$ is Fano. □

Corollary 4.27. *Let X be a Fano $\mathbb{K}^*$-surface arising from data (A, P) and let Z be its minimal toric ambient variety. Then Z admits a small Fano completion.*

Proof. The Cox ring of a normal complete surface X with finitely generated divisor class group is finitely generated if and only if $\mathrm{Mov}(X) = \mathrm{SAmple}(X)$ holds and this cone is polytopal. In our situation, X is $\mathbb{Q}$-factorial and projective and has particularly finitely generated Cox ring. Thus, the assertion follows by Proposition 4.26. □

Concretely, in terms of the GIT fan, the existence of a toric ambient Fano variety $\widehat{Z}$ means that the anticanonical divisor class of $w_{\widehat{Z}}$ lies in the same chamber of the moving cone as the anticanonical divisor class of X, namely in the ample chamber of X.



The following example shows that for higher dimensions the analogous statement of 4.27 does not hold.

Example 4.28. Let $X = X(A, P, \Phi)$ be a three-dimensional complexity-one T-variety with Cox ring

$$\mathcal{R}(X) = \mathbb{K}[T_{01}, T_{11}, T_{21}, S_1, S_2, S_3]/\langle T_{01}^5 + T_{11}^3 + T_{21}^2\rangle$$

and corresponding P- and grading matrix

$$P = \begin{pmatrix} -5 & 3 & 0 & 0 & 0 & 0 \\ -5 & 0 & 2 & 0 & 0 & 0 \\ -4 & 1 & 1 & -1 & 0 & 1 \\ -5 & 3 & 0 & 0 & 1 & -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 6 & 10 & 15 & 0 & -1 & -1 \end{pmatrix}.$$

Furthermore, let $\Phi = \Phi(\lambda)$ with $\lambda = \mathrm{cone}(u_1, u_2)$ be the associated bunch. Then we have

$$w_X = w_{01} + w_{11} + w_{21} + u_1 + u_2 + u_3 - \deg(f) = \begin{pmatrix} 3 \\ 29 \end{pmatrix} - \begin{pmatrix} 0 \\ 30 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

and thus $w_X \in \mathrm{Ample}(X) = \mathrm{cone}(u_1, u_2)^\circ$. Anyway, for the toric anticanonical divisor class of any toric ambient variety Z we have

$$w_Z = w_{01} + w_{11} + w_{21} + u_1 + u_2 + u_3 = \begin{pmatrix} 3 \\ 29 \end{pmatrix},$$

and consequently $w_Z \notin \mathrm{Ample}(X)$. Since $\mathrm{Ample}(Z) \subseteq \mathrm{Ample}(X)$ holds, the given variety X can not be embedded into a toric ambient variety that is Fano.

Definition 4.29. Let X be a $\mathbb{Q}$-factorial Fano $\mathbb{K}^*$-surface arising from data (A, P). Then we call X a *del Pezzo surface*. A *log del Pezzo surface* is a del Pezzo surface X having only log-terminal singularities.

By Corollary 4.27 we find for every del Pezzo surface $X = X(A, P)$ a three-dimensional toric ambient variety that is Fano. In this situation the toric anticanonical polytope is given by the convex hull of the primitive column vectors v_{ij} and $v^\pm$ of P, where $v^\pm = (0, \ldots, 0, \pm 1)$.

Now we want to introduce a convex geometrical object for $\mathbb{K}^*$-surfaces which is comparable to the toric anticanonical polytope. For this purpose some preparation is needed.

Lemma 4.30. *Consider the two elementary big cones* $\sigma^+ = \mathrm{cone}(v_{0n_0}, \ldots, v_{rn_r})$ *and* $\sigma^- = \mathrm{cone}(v_{01}, \ldots, v_{r1})$ *and set* $l^+ := l_{0n_0} \cdots l_{rn_r}$, $l^- := l_{01} \cdots l_{r1}$. *For* $0 \le i \le r$ *let* $u_i^+ = (u_{i1}^+, \ldots, u_{ir+1}^+) \in M_\mathbb{Q} \cong \mathbb{Q}^{r+1}$ *and* $u_i^- = (u_{i1}^-, \ldots, u_{ir+1}^-) \in M_\mathbb{Q} \cong \mathbb{Q}^{r+1}$ *be the linear forms satisfying*

$$\langle u_i^+, v_{kn_k}\rangle = \begin{cases} 1 & \text{if } i \neq k \\ 1 - (r-1)l_{kn_k} & \text{if } i = k \end{cases}, \qquad \langle u_i^-, v_{k1}\rangle = \begin{cases} 1 & \text{if } i \neq k \\ 1 - (r-1)l_{k1} & \text{if } i = k \end{cases}.$$

Then for $1 \le k \le r$ the linear forms are given by

$$u_{0k}^+ = \frac{1}{\sum_{i=0}^r \frac{l^+ d_{in_i}}{l_{in_i}}} \cdot \left(\sum_{\substack{j=0 \\ j \neq k}}^r \frac{l^+}{l_{kn_k} l_{jn_j}} (d_{jn_j} - d_{kn_k}) + (r-1) \frac{l^+ d_{kn_k}}{l_{kn_k}} \right)$$

$$u_{ik}^+ = \begin{cases} \frac{1}{\sum_{i=0}^r \frac{l^+ d_{in_i}}{l_{in_i}}} \cdot \left(\sum_{\substack{j=0 \\ j \neq k}}^r \frac{l^+}{l_{kn_k} l_{jn_j}} (d_{jn_j} - d_{kn_k}) - (r-1) \sum_{\substack{j=0 \\ j \neq k}}^r \frac{l^+ d_{jn_j}}{l_{jn_j}} \right) & \text{for } i = k \\ \frac{1}{\sum_{i=0}^r \frac{l^+ d_{in_i}}{l_{in_i}}} \cdot \left(\sum_{\substack{j=0 \\ j \neq k}}^r \frac{l^+}{l_{kn_k} l_{jn_j}} (d_{jn_j} - d_{kn_k}) + (r-1) \frac{l^+ d_{kn_k}}{l_{kn_k}} \right) & \text{for } i \neq k \end{cases}$$

$$u_{ir+1}^+ = \frac{1}{\sum_{i=0}^r \frac{l^+ d_{in_i}}{l_{in_i}}} \cdot \left(\sum_{k=0}^r \frac{l^+}{l_{kn_k}} - (r-1) l^+ \right),$$

and analogously

$$u_{0k}^- = \frac{1}{\sum_{i=0}^r \frac{l^- d_{i1}}{l_{i1}}} \cdot \left(\sum_{\substack{j=0 \\ j \neq k}}^r \frac{l^-}{l_{k1} l_{j1}} (d_{j1} - d_{k1}) + (r-1) \frac{l^- d_{k1}}{l_{k1}} \right)$$

$$u_{ik}^- = \begin{cases} \frac{1}{\sum_{i=0}^r \frac{l^- d_{i1}}{l_{i1}}} \cdot \left(\sum_{\substack{j=0 \\ j \neq k}}^r \frac{l^-}{l_{k1} l_{j1}} (d_{j1} - d_{k1}) - (r-1) \sum_{\substack{j=0 \\ j \neq k}}^r \frac{l^- d_{j1}}{l_{j1}} \right) & \text{for } i = k \\ \frac{1}{\sum_{i=0}^r \frac{l^- d_{i1}}{l_{i1}}} \cdot \left(\sum_{\substack{j=0 \\ j \neq k}}^r \frac{l^-}{l_{k1} l_{j1}} (d_{j1} - d_{k1}) + (r-1) \frac{l^+ d_{k1}}{l_{k1}} \right) & \text{for } i \neq k \end{cases}$$

$$u_{ir+1}^- = \frac{1}{\sum_{i=0}^r \frac{l^- d_{i1}}{l_{i1}}} \cdot \left(\sum_{k=0}^r \frac{l^-}{l_{k1}} - (r-1) l^- \right).$$

Furthermore, the linear forms $t^+ = (t_1^+, \ldots, t_{r+1}^+)$ and $t^- = (t_1^-, \ldots, t_{r+1}^-)$ in $M_{\mathbb{Q}}$ satisfying $\langle t^+, v_{in_i} \rangle = 1$ and $\langle t^-, v_{i1} \rangle = 1$ for all $0 \le i \le r$ are given by

$$t_k^+ = \frac{1}{\sum_{i=0}^r \frac{l^+ d_{in_i}}{l_{in_i}}} \cdot \left(\sum_{\substack{j=0 \\ j \neq k}}^r \frac{l^+}{l_{kn_k} l_{jn_j}} (d_{jn_j} - d_{kn_k}) \right) \quad \text{for } 1 \le k \le r,$$

$$t_{r+1}^+ = \frac{1}{\sum_{i=0}^r \frac{l^+ d_{in_i}}{l_{in_i}}} \cdot \sum_{i=0}^r \frac{l^+}{l_{in_i}},$$

and

$$t_k^- = \frac{1}{\sum_{i=0}^r \frac{l^- d_{i1}}{l_{i1}}} \cdot \left(\sum_{\substack{j=0 \\ j \neq k}}^r \frac{l^-}{l_{k1} l_{j1}} (d_{j1} - d_{k1}) \right) \quad \text{for } 1 \le k < r+1,$$

$$t_{r+1}^- = \frac{1}{\sum_{i=0}^r \frac{l^- d_{i1}}{l_{i1}}} \cdot \sum_{i=0}^r \frac{l^-}{l_{i1}}.$$

Proof. We will prove the assertion exemplarily for u_i^- and we will write v_i, l_i and d_i instead of v_{i1}, l_{i1} and d_{i1}. Set $l^- := l_0 \cdots l_r$. First we will prove $\langle u_0^-, v_0\rangle = 1 - l_0$.

$$
\begin{aligned}
&\sum_{i=0}^{r} \frac{l^- d_i}{l_i} \cdot \langle u_0^-, v_0\rangle \\
&= -\sum_{k=1}^{r}\left(\sum_{\substack{j=0\\ j\neq k}}^{r} \frac{l^- l_0}{l_k l_j}(d_j - d_k) - (r-1)\frac{l^- l_0 d_k}{l_k}\right) + \sum_{j=0}^{r}\frac{l^- d_0}{l_j} - (r-1)l^- d_0 \\
&= -\sum_{k=1}^{r}\left(\sum_{\substack{j=1\\ j\neq k}}^{r} \frac{l^- l_0}{l_k l_j}(d_j - d_k)\right) - \sum_{k=1}^{r}\frac{l^-}{l_k}(d_0 - d_k) - (r-1)\sum_{k=1}^{r}\frac{l^- l_0 d_k}{l_k} \\
&\qquad + \sum_{j=0}^{r}\frac{l^- d_0}{l_j} - (r-1)\frac{l^- l_0 d_0}{l_0} \\
&= 0 - \sum_{k=1}^{r}\frac{l^- d_0}{l_k} + \sum_{k=1}^{r}\frac{l^- d_k}{l_k} - (r-1)l_0\sum_{k=0}^{r}\frac{l^- d_k}{l_k} + \sum_{j=1}^{r}\frac{l^- d_0}{l_j} + \frac{l^- d_0}{l_0} \\
&= (1-(r-1)l_0)\sum_{k=1}^{r}\frac{l^- d_k}{l_k}
\end{aligned}
$$

Now we will show that $\langle u_0^-, v_i\rangle = 1$ holds for all $i \neq 0$. Without loss of generality we may assume $i = 1$.

$$
\begin{aligned}
\sum_{i=0}^{r} \frac{l^- d_i}{l_i} \cdot \langle u_0^-, v_1\rangle &= \sum_{\substack{j=0\\ j\neq 1}}^{r} \frac{l^- l_1}{l_1 l_j}(d_j - d_1) + (r-1)\frac{l^- l_1 d_1}{l_1} + \sum_{j=0}^{r}\frac{l^- d_1}{l_j} - (r-1)l^- d_1 \\
&= \sum_{\substack{j=0\\ j\neq 1}}^{r}\frac{l^- d_j}{l_j} - \sum_{\substack{j=0\\ j\neq 1}}^{r}\frac{l^- d_1}{l_j} + \sum_{j=0}^{r}\frac{l^- d_1}{l_j} \\
&= \sum_{j=1}^{r}\frac{l^- d_j}{l_j}
\end{aligned}
$$

The next step is to show that $\langle u_i^-, v_i\rangle = 1 - l_i$ holds for all $i \neq 0$. Without loss of generality we may assume $i = 1$.

$$
\begin{aligned}
\sum_{i=0}^{r} \frac{l^- d_i}{l_i} \cdot \langle u_1^-, v_1 \rangle &= \sum_{\substack{j=0 \\ j\neq 1}}^{r} \frac{l^- l_1}{l_1 l_j}(d_j - d_1) - (r-1) \sum_{\substack{j=0 \\ j\neq 1}}^{r} \frac{l^- l_1 d_j}{l_j} + \sum_{j=0}^{r} \frac{l^- d_1}{l_j} - (r-1) l^- d_1 \\
&= \sum_{\substack{j=0 \\ j\neq 1}}^{r} \frac{l^- d_j}{l_j} - \sum_{\substack{j=0 \\ j\neq 1}}^{r} \frac{l^- d_1}{l_j} + \sum_{j=0}^{r} \frac{l^- d_1}{l_j} - (r-1) l_1 \sum_{j=0}^{r} \frac{l^- d_j}{l_j} \\
&= (1 - (r-1) l_1) \sum_{j=1}^{r} \frac{l^- d_j}{l_j}
\end{aligned}
$$

Now we prove that $\langle u_i^-, v_j \rangle = 1$ holds for $i, j \neq 0$ and $i \neq j$. We assume $i = 1$ and $j = 2$.

$$
\begin{aligned}
\sum_{i=0}^{r} \frac{l^- d_i}{l_i} \cdot \langle u_1^-, v_2 \rangle &= \sum_{\substack{j=0 \\ j\neq 2}}^{r} \frac{l^- l_2}{l_2 l_j}(d_j - d_2) + (r-1) \frac{l^- l_2 d_2}{l_2} + \sum_{j=0}^{r} \frac{l^- d_2}{l_j} - (r-1) l^- d_2 \\
&= \sum_{\substack{j=0 \\ j\neq 2}}^{r} \frac{l^- d_j}{l_j} - \sum_{\substack{j=0 \\ j\neq 2}}^{r} \frac{l^- d_2}{l_j} + \sum_{j=0}^{r} \frac{l^- d_2}{l_j} \\
&= \sum_{j=1}^{r} \frac{l^- d_j}{l_j}
\end{aligned}
$$

Finally we complete the proof by showing that $\langle u_i^-, v_0 \rangle = 1$ holds, where we once more may assume $i = 1$.

$$
\begin{aligned}
&\sum_{i=0}^{r} \frac{l^- d_i}{l_i} \cdot \langle u_1^-, v_0 \rangle \\
= \; & -\sum_{k=1}^{r} \left(\sum_{\substack{j=0 \\ j\neq k}}^{r} \frac{l^- l_0}{l_k l_j}(d_j - d_k) \right) + (r-1) \sum_{\substack{j=0 \\ j\neq 1}}^{r} \frac{l^- l_0 d_j}{l_j} - (r-1) \sum_{k=2}^{r} \frac{l^- l_0 d_k}{l_k} \\
& \quad + \sum_{j=0}^{r} \frac{l^- d_0}{l_j} - (r-1) l^- d_0 \\
= \; & -\sum_{k=1}^{r} \frac{l^-}{l_k}(d_0 - d_k) - \sum_{k=1}^{r} \left(\sum_{\substack{j=1 \\ j\neq k}}^{r} \frac{l^- l_0}{l_k l_j}(d_j - d_k) \right) + \sum_{j=0}^{r} \frac{l^- d_0}{l_j} \\
= \; & -\sum_{k=1}^{r} \frac{l^- d_0}{l_k} + \sum_{k=1}^{r} \frac{l^- d_k}{l_k} - 0 + \sum_{j=0}^{r} \frac{l^- d_0}{l_j} \\
= \; & \sum_{k=0}^{r} \frac{l^- d_k}{l_k}
\end{aligned}
$$

So far, we have proven the assertions for $u_i^{\pm}$. To prove the remaining parts of the lemma we restrict again exemplarily to t^-. First we show that $\langle t^-, v_0\rangle = 1$ holds.

$$\begin{aligned}
\sum_{i=0}^{r} \frac{l^- d_i}{l_i} \cdot \langle t^-, v_0\rangle &= -\sum_{k=1}^{r}\left(\sum_{\substack{j=0\\ j\neq k}}^{r} \frac{l^- l_0}{l_k l_j}(d_j - d_k)\right) + \sum_{i=0}^{r}\frac{l^- d_0}{l_j} \\
&= -\sum_{k=1}^{r} \frac{l^- d_j l_0}{l_0 l_j}(d_0 - d_k) - 0 + \sum_{i=0}^{r}\frac{l^- d_0}{l_j} \\
&= \sum_{k=0}^{r}\frac{l^- d_k}{l_k}
\end{aligned}$$

The last thing to show is that we have $\langle t^-, v_i\rangle = 1$ for all $i \neq 0$. We may assume $i = 1$.

$$\begin{aligned}
\sum_{i=0}^{r} \frac{l^- d_i}{l_i} \cdot \langle t^-, v_1\rangle &= \sum_{\substack{j=0\\ j\neq 1}}^{r} \frac{l^- l_1}{l_1 l_j}(d_j - d_1) + \sum_{j=0}^{r}\frac{l^- d_1}{l_j} \\
&= \sum_{\substack{j=0\\ j\neq 1}}^{r} \frac{l^- d_j}{l_j} - \sum_{\substack{j=0\\ j\neq 1}}^{r} \frac{l^- d_1}{l_j} + \sum_{j=0}^{r}\frac{l^- d_1}{l_j} \\
&= \sum_{j=0}^{r}\frac{l^- d_j}{l_j}.
\end{aligned}$$

□

The vectors $v_{01}, \ldots, v_{r1}$ and $v_{0n_0}, \ldots, v_{rn_r}$ respectively are linearly independent. Consequently, each linear form $u_i^{\pm} \in M_{\mathbb{Q}} \cong \mathbb{Q}^{r+1}$ defines a unique affine hypersurface $H_{u_i^{\pm}}$ in $N_{\mathbb{Q}} = M_{\mathbb{Q}}^*$ given by the equation $\langle u_i^{\pm}, x\rangle - 1 = 0$. Note that $v_{k1} \in H_{u_i^-}$ and $v_{kn_k} \in H_{u_i^+}$ if and only if $k \neq i$. Furthermore, the linear forms t^+ and t^- define affine hypersurfaces H^+ and H^-, which are given by the equations $\langle t^{\pm}, x\rangle - 1 = 0$. The $r+1$ points v_{i1} and v_{in_i} are contained in H^+ and H^- respectively, they even generate them.

Lemma 4.31. *The intersection point of the hypersurface H^+ defined by t^+ with the ray $\mathbb{Q}_{\geq 0} \cdot e_{r+1}$ and the intersection point of the hypersurface H^- defined by t^- with the ray $\mathbb{Q}_{\geq 0} \cdot (-e_{r+1})$ are given by*

$$v_t^+ = \left(0, \ldots, 0, \frac{\sum_{i=0}^{r} \frac{l^+ d_{in_i}}{l_{in_i}}}{\sum_{i=0}^{r} \frac{l^+}{l_{in_i}}}\right) \quad \textit{and} \quad v_t^- = \left(0, \ldots, 0, \frac{\sum_{i=0}^{r} \frac{l^- d_{i1}}{l_{i1}}}{\sum_{i=0}^{r} \frac{l^-}{l_{i1}}}\right).$$

Proof. This follows directly from Lemma 4.30. □

Now we want to determine the intersection of the $r+1$ affine hypersurfaces $H_{u_i^+}$ and $H_{u_i^-}$ respectively. For this purpose, we set

$$v_c^+ := \left(0,\ldots,0,\frac{\sum_{i=0}^r \frac{l^+ d_{in_i}}{l_{in_i}}}{\sum_{i=0}^r \frac{l^+}{l_{in_i}} - (r-1)l^+}\right) \quad \text{and} \quad v_c^- := \left(0,\ldots,0,\frac{\sum_{i=0}^r \frac{l^- d_{i1}}{l_{i1}}}{\sum_{i=0}^r \frac{l^-}{l_{i1}} - (r-1)l^-}\right).$$

Lemma 4.32. *If X is log-terminal, then $v_c^\pm$ is not contained in $H^\pm$ and the intersection of the $r+1$ affine hypersurfaces $H_{u_i^+}$ and $H_{u_i^-}$ is given by*

$$\bigcap H_{u_i^+} \;=\; \{v_c^+\} \quad \textit{and} \quad \bigcap H_{u_i^-} \;=\; \{v_c^-\}, \quad \textit{respectively.}$$

Proof. We consider exemplarily the $r+1$ hypersurfaces $H_{u_i^-}$. By Lemma 4.30 the point v_c^- is contained in each hypersurface $H_{u_i^-}$ and thus, it is also contained in their intersection. If X is log-terminal, then by Proposition 3.19 the following inequality holds:

$$\sum_{i=0}^r \frac{l^-}{l_{i1}} - (r+1)l^- > 0$$

Consequently, v_c^- is not contained in the affine space generated by $v_{01},\ldots,v_{r1}$, which is denoted by H^-, and we can write $H_{u_i^-} = v_c^- + \mathrm{lin}(v_{j1} - v_c^-, j \neq i)$ for all $0 \leq i \leq r$. Since $v_{01},\ldots,v_{r1}$ and v_c^- are in general position and $v_{i1} \notin H_{u_i^-}$ we conclude that the hypersurfaces $H_{u_i^-}$ intersect in exactly one point, namely v_c^-. □

Definition 4.33. Let $X = X(A,P)$ be a log-terminal $\mathbb{K}^*$-surface. Then we define the *anticanonical polytope* A_X *of* X as the convex hull of all columns of P and $v_c^\pm$ if $F^\pm$ is an elliptic fixed point. Furthermore we define the *anticanonical complex* of X as

$$A_X^c \;:=\; A_X \cap \mathrm{trop}(X).$$

Notation 4.34. The anticanonical complex of a $\mathbb{K}^*$-surface consists of $r+1$ purely two-dimensional arms corresponding to the blocks of the matrix P and the arms of $\mathrm{trop}(X)$ respectively. We denote these arms by $A_{X_i}^c$, $0 \leq i \leq r$. If we restrict our considerations to one single arm $A_{X_i}^c$, the situation can be simplified by considering the projected arm $\widehat{A}_{X_i}^c := \mathrm{pr}_i(A_{X_i}^c)$, where

$$\mathrm{pr}_i\colon \mathbb{Z}^{r+1} \to \mathbb{Z}^2\colon (x_1,\ldots,x_r,x_{r+1}) \mapsto (x_i,x_{r+1}).$$

Remark 4.35. If X is a log-terminal $\mathbb{K}^*$-surface with an upper or lower elliptic fixed point $F^\pm$ corresponding to $\sigma^\pm$, then the intersection point $v_c^\pm$ is a vertex of the anticanonical complex.

Example 4.36. Consider $R := \mathbb{K}[T_{01},T_{02},T_{11},T_{21}]/\langle T_{01}T_{02}^3 + T_{11}^3 + T_{21}^2\rangle$ with the $\mathbb{Z}$-grading given by the weights $3,1,2,3$. Then R is the Cox ring of a (unique) surface $X \subseteq \mathbb{P}_{1,3,2,3}$. Furthermore, $\mathrm{trop}(X)$ and the anticanonical polytope are given in $\mathbb{Q}^3$ by

$$\mathrm{trop}(X) \;=\; \mathrm{cone}(e_1,\pm e_3) \cup \mathrm{cone}(e_2,\pm e_3) \cup \mathrm{cone}(-e_1-e_2,\pm e_3),$$

$$A_X \;=\; \mathrm{conv}((-1,-1,-1),\ (-3,-3,-2),\ (3,0,1),\ (0,2,1),\ (0,0,1),\ (0,0,-1/5)),$$

where $e_i \in \mathbb{Q}^3$ is the i-th canonical basis vector and all points listed in the description of A_X are in fact vertices. The anticanonical complex $A_X^c = A_X \cap \mathrm{trop}(X)$ is supported by $\mathrm{trop}(X)$ and thus it is two-dimensional.

Example 4.37. Let $X = X(A,P)$ be the $\mathbb{K}^*$-surface with

$$P \;=\; \begin{pmatrix} -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -2 & -1 & 1 & 1 & 1 \end{pmatrix}, \qquad R(X) := \mathbb{K}[T_{01},T_{02},T_{11},T_{21},S_1]/\langle T_{01}T_{02} + T_{11}^2 + T_{21}^2 \rangle.$$

Then X has a parabolic fixed point curve F^+ and an elliptic fixed point F^-. The anticanonical polytope is given by

$$A_X \;=\; \mathrm{conv}((-1,-1,-1),\ (-1,-1,-2),\ (2,0,1),\ (0,2,1),\ (0,0,1),\ (0,0,-1)).$$

Example 4.38. Let X be the $\mathbb{K}^*$-surface with

$$P \;=\; \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 & -1 & 0 & 1 \end{pmatrix},$$

$$R(X) = \mathbb{K}[T_{01},T_{02},T_{11},T_{12},T_{21},T_{22},T_{31}]/\langle g_0, g_1 \rangle,$$

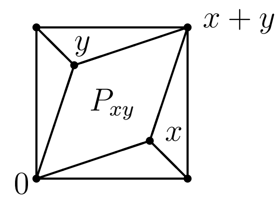
with $g_0 = T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}$ and $g_1 = \lambda T_{11}T_{12} + T_{21}T_{22} + T_{31}^2$ for one $\lambda \in \mathbb{K}^*$. Then we have

$$\mathrm{trop}(X) \;=\; \mathrm{cone}(e_1, \pm e_4) \cup \mathrm{cone}(e_2, \pm e_4) \cup \mathrm{cone}(e_3, \pm e_4) \cup \mathrm{cone}(-e_1 - e_2 - e_3, \pm e_4),$$

$$\begin{aligned} A_X \;=\; & \mathrm{conv}((-1,-1,-1,0),\ (-1,-1,-1,1),\ (3,0,1),\ (0,1,0,0),\ (0,1,0,-1), \\ & (0,0,1,0),\ (0,0,1,-1),\ (0,0,0,-1), (0,0,0,1)). \end{aligned}$$

Lemma 4.39. *Let $x = (x_1,x_2)$ and $y = (y_1,y_2)$ be two linear independent vectors in $\mathbb{Q}^2$. Then the area of the parallelogram P_{xy} spanned by zero and these two vectors is given by $|\det(x,y)|$.*

Proof. We consider the rectangle with the vertices $(x_1+y_1,0)$, $(0,x_2+y_2)$, (x_1+y_1,x_2+y_2) and zero.

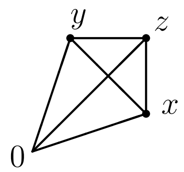


Without loss of generality we may assume that the angle $\measuredangle(x, y)$ is positive oriented. Then $\det(x, y) > 0$ holds and the area of the parallelogram P_{xy} can be easily computed by

$$\begin{aligned} A_{P_{xy}} &= (x_1 + y_1)(x_2 + y_2) - 2 \cdot \frac{1}{2}(x_1 + y_1)x_2 - 2 \cdot \frac{1}{2}(x_2 + y_2)y_1 \\ &= x_1y_2 - x_2y_1 \;=\; \det(x, y). \end{aligned}$$

□

Lemma 4.40. *Let x and y be two linear independent vectors in $\mathbb{Q}^2$ and $z \in \operatorname{cone}(x, y)^\circ$. Denote the polytope generated by these three points and zero by P_{xyz}. Then x, y, z and zero are vertices of P_{xyz} if and only if $|\det(x, z)| + |\det(y, z)| > |\det(x, y)|$.*



Proof. It is obvious that x, y and zero are vertices of P_{xyz}. Furthermore, z is a vertex of P_{xyz} if and only if $A_{P_{xz}} + A_{P_{zy}} > A_{P_{xy}}$. By Lemma 4.39 the assertion follows. □

Lemma 4.41. *Let $X = X(A, P)$ be a log del Pezzo $\mathbb{K}^*$-surface. Then all primitive column vectors v_{ij} and $v^\pm$ (if existing) of P are vertices of the anticanonical complex A^c_X.*

Proof. Let $X = X(A, P)$ be a log del Pezzo $\mathbb{K}^*$-surface. Then all conditions of Proposition 4.24 have to be satisfied. By Lemma 4.40 condition 4.24(ii) is equivalent to the condition that all v_{ij} for $j \neq i$ and $j \neq n_i$ are vertices of A^c_X. First assume that F^+ resp. F^- is an elliptic fixed point. We treat exemplarily the case of F^-. Then by 4.24(iii) the following inequality holds:

$$\frac{l_{i1} - l_{i2}}{l_{i1}d_{i2} - l_{i2}d_{i1}} \cdot \sum_{j=0}^{r} \frac{d_{j1}}{l_{j1}} \;<\; \sum_{j=0}^{r} \frac{1}{l_{j1}} - (r - 1)$$

We claim that this condition is equivalent to the condition that v_{i1} is a vertex of A^c_X. Consider the vectors $\widehat{v}_{i1} = (l_{i1}, d_{i1})$, $\widehat{v}_{i2} = (l_{i2}, d_{i2})$ and $\widehat{v}_c^- = \operatorname{pr}_i(v_c^-)$. By applying Lemma 4.40 we obtain that $\widehat{v}_{i1}$ is a vertex of $\widehat{A}^c_{X_i}$ if and only if $\det(\widehat{v}_c^-, \widehat{v}_{i1}) + \det(\widehat{v}_{i1}, \widehat{v}_{i1}) > \det(\widehat{v}_c^-, \widehat{v}_{i2})$. Concretely, we obtain the inequality

$$-l_{i1} \cdot \frac{\sum_{j=0}^{r} \frac{l^- d_{j1}}{l_{j1}}}{\sum_{j=0}^{r} \frac{l^-}{l_{j1}} - (r-1)l^-} + l_{i1}d_{i2} - l_{i2}d_{i1} \;>\; -l_{i2} \cdot \frac{\sum_{j=0}^{r} \frac{l^- d_{j1}}{l_{j1}}}{\sum_{j=0}^{r} \frac{l^-}{l_{j1}} - (r-1)l^-}.$$

Since X is log-terminal, the denominators are positive. Hence, we end up with condition 4.24(iii). Analogously, we can proceed with F^+ and v_{in_i} using 4.24(iv). Now assume that F^+ or F^- is a parabolic fixed point curve. (Note that not both of them can be parabolic fixed point curves.) By Proposition 4.26 the log del Pezzo $\mathbb{K}^*$-surface X can be embedded into a toric Fano variety. The vertices of the corresponding toric Fano polytope are the columns of the P-matrix P of X. Hence the columns v^+ resp. v^- as well as v_{i1} and v_{in_i} are vertices of the anticanonical complex A_X^c. Note that the conditions 4.24(iii) and (iv) yield also directly that v_{i1} and v_{in_i} are vertices of A_X^c in this case. □

Corollary 4.42. *Let X be a log del Pezzo $\mathbb{K}^*$-surface. Then each arm $A_{X_i}^c$ as well as the projected arm $\widehat{A}_{X_i}^c$ is a convex polytope with vertices v_{ij}, $v_c^\pm$, $v^\pm$ and $\mathrm{pr}_i(v_{ij})$, $\mathrm{pr}_i(v_c^\pm)$, $\mathrm{pr}_i(v^\pm)$ respectively.*

Remark 4.43. If X is a del Pezzo $\mathbb{K}^*$-surface, then the anticanonical complex is locally bounded by hypersurfaces defined by the anticanonical divisor $-K_X$. For each two-dimensional tower cone $\sigma \in \Sigma$, we find a linear form $u_\sigma \in M_\mathbb{Q}$ such that $-K_X$ is locally represented by u_σ. Let H_{u_σ} be the hypersurface in $N_\mathbb{Q}$ defined by $\langle u_\sigma, x\rangle - 1 = 0$ and let $\widehat{H}_{u_\sigma}$ be the half space defined by the inequation $\langle u_\sigma, x\rangle - 1 < 0$. Then $\sigma \cap \mathrm{trop}(X) \cap \widehat{H}_{u_\sigma}$ defines a polytope of A_X^c. Note that in this situation the linear form u_σ is not uniquely determined. For the two elementary big cones $\sigma^\pm$ we have $r+1$ unique linear forms $u_0^\pm, \ldots, u_r^\pm$ representing locally $-K_X$, compare 4.30. Let $H_{u_i^\pm}$ be the associated hypersurfaces and $\widehat{H}_{u_i^\pm}$ the corresponding half spaces in $N_\mathbb{Q}$ defined by $\langle u_i^\pm, x\rangle - 1 < 0$. Then by $\sigma \cap \mathrm{trop}(X) \cap \widehat{H}_\sigma$ we obtain $r+1$ polytopes of A_X^c. In particular, there is a one-to-one correspondence between the cones of $\Sigma \cap \mathrm{trop}(X)$ and the polytopes of A_X^c.

Example 4.44 (D_4). Consider the $\mathbb{K}^*$-surface given by the P-matrix

$$P = \begin{pmatrix} -1 & -2 & 1 & 2 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 2 \\ -1 & -1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

with Cox ring $\mathbb{K}[T_{01}, T_{02}, T_{11}, T_{12}, T_{21}, T_{22}]/\langle T_{01}T_{02}^2 + T_{11}T_{12}^2 + T_{21}T_{22}^2\rangle$. The fan Σ of the minimal toric ambient variety is given by the maximal cones

$$\Sigma^{\max} = \{\tau_0 := \mathrm{cone}(v_{01}, v_{02}), \tau_1 := \mathrm{cone}(v_{11}, v_{12}), \tau_2 := \mathrm{cone}(v_{21}, v_{22}), \sigma^+, \sigma^-\}.$$

The following linear forms define bounding hypersurfaces for the anticanonical polytope:

$$\begin{aligned} u_{\tau_0} &= (1,-1,-1) & u_0^- &= (1,1,-2) & u_0^+ &= (0,0,1) \\ u_{\tau_1} &= (1,1,-1) & u_1^- &= (0,1,-2) & u_1^+ &= (-1,0,1) \\ u_{\tau_2} &= (-1,1,-1) & u_2^- &= (1,0,-2) & u_2^+ &= (0,-1,1) \end{aligned}$$

The anticanonical complex is given by $\mathrm{trop}(X) \cap A_X$ where

$$A_X = \mathrm{conv}\left(v_{01}, v_{02}, v_{11}, v_{12}, v_{21}, v_{22}, (0,0,1), (0,0,-\frac{1}{2})\right).$$

The following example shows that neither the Fano property does imply log-terminality nor the other way around. Hence, both properties have to be required independently.

Example 4.45. Consider the two P-matrices

$$P_1 = \begin{pmatrix} -7 & -1 & 5 & 0 \\ -7 & -1 & 0 & 2 \\ -8 & -1 & 3 & 1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} -3 & -1 & -2 & 5 & 0 \\ -3 & -1 & -2 & 0 & 2 \\ -4 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The $\mathbb{K}^*$-surface defined by P_1 is not log-terminal but Fano (see 6.18) and the $\mathbb{K}^*$-surface defined by P_2 is log-terminal but not Fano. The grading matrix of P_2 is given by

$$Q_2 = \begin{pmatrix} 2 & 2 & 1 & 2 & 5 \\ 1 & -11 & 4 & 0 & 0 \end{pmatrix}.$$

The anticanonical divisor class $w_{X_2} = (2, -6)$ is obviously not contained in the ample cone $\mathrm{Ample}(X_2) = \mathrm{cone}((1,0),(2,1))^\circ$.

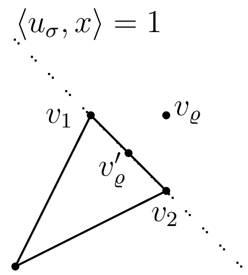
Lemma 4.46. *Let X be a $\mathbb{K}^*$-surface and consider a stellar subdivision of a cone $\sigma \in \Sigma$, occurring within the canonical resolution of X, given by the exceptional ray ϱ and its primitive generator $v_\varrho \in \sigma^\circ$. Then the discrepancy of the associated modification is given by $\langle u_\sigma, v_\varrho \rangle - 1$.*

Proof. This follows directly from Proposition 3.22. □

Lemma 4.47. *In the situation of Lemma 4.46 let v'_ϱ be the intersection point of $\mathbb{Q}_{\geq 0} \cdot v_\varrho$ with the hypersurface H_{u_σ} defined by $\langle u_\sigma, x \rangle - 1 = 0$. Then*

$$\langle u_\sigma, v_\varrho \rangle - 1 = \frac{\|v_\varrho\|}{\|v'_\varrho\|} - 1.$$

Proof. By assumption we have $\langle u_\sigma, v'_\varrho \rangle = 1$. Since u_σ is a linear form, the assertion follows from Lemma 4.46 and the theorem on intersecting lines. □



The anticanonical complex gives information about the canonical resolution of singular log del Pezzo $\mathbb{K}^*$-surfaces. We can determine the singularity type of X by means of the anticanonical complex.

Theorem 4.48. *Let X be a log del Pezzo $\mathbb{K}^*$-surface. Then the following statements hold:*

(a) *X has at most ε-log terminal singularities if and only if 0 is the only lattice point in the relative interior of $\varepsilon \cdot A_X^c$.*

(b) *X has at most canonical singularities if and only if 0 is the only lattice point in the relative interior of A_X^c.*

(c) *X has at most terminal singularities if and only if 0 and the primitive generators are the only lattice points of A_X^c.*

Proof. The assertions follow by Lemma 4.47 and Lemma 4.46. □

The subsequent construction of two-dimensional P-complexes is a first and raw trial for the definition of a category of convex geometrical objects being in one-to-one correspondence to log del Pezzo $\mathbb{K}^*$-surfaces.

Construction 4.49 (Two-dimensional P-complex)**.** Consider the cones

$$\begin{aligned} \tau_0 &:= \operatorname{cone}(-e_1 - \ldots - e_r, \pm e_{r+1}), \\ \tau_1 &:= \operatorname{cone}(e_1, \pm e_{r+1}), \\ &\vdots \\ \tau_r &:= \operatorname{cone}(e_r, \pm e_{r+1}), \end{aligned}$$

and let Δ be the fan in $\mathbb{Q}^{r+1}$ generated by the maximal cones $\tau_0, \ldots, \tau_r$. A polytopal complex C is called a two-dimensional *P-complex* if the following conditions are satisfied:

(i) C is supported by Δ, i.e. $C \subseteq |\Delta|$.

(ii) C is complete in the sense that it cannot be enlarged without adding new vertices.

(iii) The vertices of C coincide with the vertices of the anticanonical complex A_X^c of a log del Pezzo $\mathbb{K}^*$-surface $X = X(A, P)$.

Corollary 4.50. *For the category of log del Pezzo $\mathbb{K}^*$-surfaces the following correspondences hold:*

{*log del Pezzo* $\mathbb{K}^*$-*surfaces*}	$\longleftrightarrow$	{*2-dim. P-complexes*}
{*terminal del Pezzo* $\mathbb{K}^*$-*surfaces*}	$\longleftrightarrow$	{*2-dim. P-complexes, s.th. zero and its vertices are the only lattice points in it*}
{*canonical del Pezzo* $\mathbb{K}^*$-*surfaces*}	$\longleftrightarrow$	{*2-dim. P-complexes, s.th. zero is the only interior lattice point*}
{*ε-log-terminal del Pezzo* $\mathbb{K}^*$-*surfaces*}	$\longleftrightarrow$	{*2-dim. P-complexes C, s.th. zero is the only interior lattice point in $\varepsilon \cdot C$*}

Remark 4.51. The concept of the anticanonical complex can be generalized to complexity-one Fano T-varieties in general. This is work in progress together with Jürgen Hausen and Benjamin Bechtold.

5 Del Pezzo $\mathbb{K}^*$-surfaces

This chapter is dedicated to del Pezzo $\mathbb{K}^*$-surfaces, i.e. $\mathbb{K}^*$-surfaces that are Fano. The main result is a complete classification list of all non-toric log-terminal Gorenstein del Pezzo $\mathbb{K}^*$-surfaces. To achieve this aim we describe the Gorenstein index of a $\mathbb{K}^*$-surface in terms of their P-matrix and anticanonical complex and consider the special geometry of del Pezzo $\mathbb{K}^*$-surfaces. Finally, we will give explicit bounds needed for the classification results.

5.1 Del Pezzo and Gorenstein $\mathbb{K}^*$-surfaces

In this section we will describe the Gorenstein index of a del Pezzo $\mathbb{K}^*$-surface in terms of its P-matrix and its anticanonical complex. Furthermore, we will prove some basic statements about the geometry of del Pezzo $\mathbb{K}^*$-surfaces.

Recall that a *del Pezzo surface* is a complete algebraic surface X over $\mathbb{K}$ such that the anticanonical divisor class $-K_X$ is ample. A *log del Pezzo surface* is a del Pezzo surface X having only log-terminal singularities. Furthermore, the *degree* of a del Pezzo surface is defined as the self-intersection number of the anticanonical divisor $d_X = (-K_X)^2$.

Definition 5.1. A variety X has *Gorenstein index* $\iota(X) = a$ if a is the minimal positive integer such that $a \cdot (-K_X) \in \mathrm{Pic}(X)$. Furthermore, X is said to be *Gorenstein* if the Gorenstein index is one, i.e. the anticanonical divisor is Cartier.

We want to describe the Gorenstein index by means of P-matrices. It turns out that the Gorenstein index requires some divisibility conditions to be satisfied:

Proposition 5.2. *Let $X = X(A, P)$ be a $\mathbb{K}^*$-surface and let Z be its minimal toric ambient variety with associated fan Σ. Then X has Gorenstein index a if and only if a is the smallest integer such that the following conditions are satisfied depending on the cones $\sigma \in \Sigma$:*

(i) Let $\sigma = \tau_{ij}$. Then

$$l_{ij}d_{ij+1} - l_{ij+1}d_{ij} \ \mid \ a(d_{ij+1} - d_{ij}) \qquad \text{and} \qquad l_{ij}d_{ij+1} - l_{ij+1}d_{ij} \ \mid \ a(l_{ij} - l_{ij+1}).$$

(ii) Let $\sigma = \sigma_i^+$ or $\sigma = \sigma_i^-$. Then

$$l_{in_i} \ \mid \ a(d_{in_i} - 1) \qquad \text{and} \qquad l_{i1} \ \mid \ a(d_{i1} + 1) \quad \text{respectively.}$$

(iii) Let $\sigma = \sigma^{\pm}$. Then

$$\sum_{i=0}^{r} \frac{l}{l_i} d_i \ \left| \ a\left((r-1)l - \sum_{i=0}^{r} \frac{l}{l_i}\right)\right. ,$$

$$\sum_{i=0}^{r} \frac{l}{l_i} d_i \ \left| \ a\left((r-1)\frac{l}{l_k} d_k + \sum_{i \neq k} \frac{l}{l_k l_i}(d_i - d_k)\right)\right. \quad \text{for } k = 1, \ldots, r \ ,$$

where (l, l_i, d_i) is either (l^-, l_{i1}, d_{i1}) or $(l^+, l_{in_i}, d_{in_i})$ for $l^- := l_{01} \cdots l_{r1}$ and $l^+ := l_{0n_0} \cdots l_{rn_r}$.

Proof. Let X be a $\mathbb{K}^*$-surface arising from a matrix P and let Z be its minimal toric ambient variety with corresponding fan Σ. For each cone $\sigma \in \Sigma$, the anticanonical divisor is locally given by a linear form $u_\sigma \in M_\mathbb{Q}$. Let the primitive generators of σ be the columns of a matrix A_σ. Then A_σ is a submatrix of P satisfying

$$A_\sigma^T \cdot u_\sigma \;= e_X \;:=\; \begin{cases} (-(r-1)l_0 + 1, 1, \ldots, 1) & \text{if } \sigma = \sigma^\pm \\ (1, \ldots, 1) & \text{if } \sigma = \tau_{ij} \text{ or } \sigma = \sigma_i^\pm \end{cases}.$$

The $\mathbb{K}^*$-surface X has Gorenstein index a if and only if a is the smallest integer such that $a \cdot u_\sigma$ is an integer linear form for all $\sigma \in \Sigma$.

We can explicitly compute the linear form by $u_\sigma = (A_\sigma^T)^{-1} e_X$. For the tower cones $\tau_{ij} = \operatorname{cone}(v_{ij}, v_{ij+1})$ we can locally restrict to the two-dimensional toric variety corresponding to the cone τ_{ij} (compare Lemma 3.9). By considering the projected generators $\widehat{v}_{ij} = (l_{ij}, d_{ij})$ and $\widehat{v}_{ij+1} = (l_{ij+1}, d_{ij+1})$ in $\mathbb{Q}^2$ we obtain

$$\frac{1}{l_{ij}d_{ij+1} - l_{ij+1}d_{ij}} \cdot \begin{pmatrix} d_{ij+1} & -d_{ij} \\ -l_{ij+1} & l_{ij} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;=\; \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Hence, the linear form $a \cdot u_{\tau_{ij}}$ is integral if and only if

$$l_{ij}d_{ij+1} - l_{ij+1}d_{ij} \;\mid\; a \cdot (d_{ij+1} - d_{ij}) \qquad \text{and} \qquad l_{ij}d_{ij+1} - l_{ij+1}d_{ij} \;\mid\; a \cdot (l_{ij} - l_{ij+1}).$$

For the cones $\sigma_i^\pm$ we can proceed analogously with $\widehat{v}^\pm = (0, \pm 1)$. Hence, we obtain the conditions

$$l_{in_i} \;\mid\; a(d_{in_i} - 1) \qquad \text{and} \qquad l_{i1} \;\mid\; a(d_{i1} + 1).$$

Now we will have a look at the elementary big cones $\sigma^\pm$ corresponding to elliptic fixed points. Let A_σ be the submatrix of P having all generators of $\sigma = \sigma^\pm$ as its columns. Let (l, l_i, d_i) be (l^-, l_{i1}, d_{i1}) or $(l^+, l_{in_i}, d_{in_i})$. We compute a general formula for the inverse of this matrix. Then

$$A_\sigma^{-1} \;=\; \frac{1}{\sum_{i=0}^r \frac{l}{l_i} d_i} \cdot \begin{pmatrix} -\frac{l}{l_0 l_1} d_1 & -\frac{l}{l_0 l_2} d_2 & \cdots & -\frac{l}{l_0 l_r} d_r & \frac{l}{l_0} \\ \sum_{i \neq 1} \frac{l}{l_i l_1} d_i & -\frac{l}{l_1 l_2} d_2 & \cdots & -\frac{l}{l_1 l_r} d_r & \frac{l}{l_1} \\ -\frac{l}{l_2 l_1} d_1 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ -\frac{l}{l_{r-1} l_1} d_1 & & \ddots & -\frac{l}{l_{r-1} l_r} d_r & \frac{l}{l_{r-1}} \\ -\frac{l}{l_r l_1} d_1 & \cdots & -\frac{l}{l_r l_{r-1}} d_{r-1} & \sum_{i \neq r} \frac{l}{l_i l_r} d_i & \frac{l}{l_r} \end{pmatrix}.$$

The linear form $a \cdot u_\sigma$ is integral if and only if the following conditions are satisfied:

$$\det(A_\sigma) = \sum_{i=0}^r \frac{l}{l_i} d_i \;\middle|\; a \left((r-1)l - \sum_{i=0}^r \frac{l}{l_i} \right)$$

$$\det(A_\sigma) = \sum_{i=0}^r \frac{l}{l_i} d_i \;\middle|\; a \left((r-1)\frac{l}{l_k} d_k + \sum_{i \neq k} \frac{l}{l_k l_i}(d_i - d_k) \right) \quad \text{for } k = 1, \ldots, r.$$

□

Remark 5.3. Comparing the Gorenstein conditions to the linear forms u_σ given in 4.30 one easily detects that X has Gorenstein index a if and only if $a \cdot u_\sigma$ is an integer linear form.

Lemma 5.4. *Assume that X is a log-terminal complexity-one Fano variety of Gorenstein index a. Then X is $\frac{1}{a}$-log-canonical, i.e. the discrepancies are greater or equal to $-1 + \frac{1}{a}$.*

Proof. Let X be a log-terminal Fano variety of Gorenstein index a, i.e. a is the smallest integer such that $a \cdot K_X$ is Cartier. Consider a toric ambient modification $\pi\colon \widetilde{X} \to X$ with exceptional divisor D and the associated pullback $\pi^*\colon \mathrm{Cl}(X) \to \mathrm{Cl}(\widetilde{X})$. Then $\pi^*(a \cdot K_X) = a \cdot \pi^*(K_X)$ is integral. The discrepancy is given by $K_{\widetilde{X}} - \pi^*(K_X) = \alpha \cdot D$. This implies $a \cdot \alpha \in \mathbb{Z}$. In particular, α has to be greater or equal to $-1 + \frac{1}{a}$. Consequently, X is $\frac{1}{a}$-log-canonical. □

Corollary 5.5. *Let X be a Gorenstein log del Pezzo $\mathbb{K}^*$-surface. Then X is canonical.*

Corollary 5.6. *Let $X = X(A, P)$ be a log del Pezzo $\mathbb{K}^*$-surface of Gorenstein index $\iota(X) = a$. Then the anticanonical complex A_X^c satisfies*

$$\frac{1}{a} A_X^{c\,\circ} \cap N \ = \ \{0\}.$$

Lemma 5.7. *Let $X = X(A, P)$ be a non-toric $\mathbb{K}^*$-surface. Then the following inequalities cannot be satisfied simultaneously:*

$$\sum_{i=0}^{r} \frac{d_{i1}}{l_{i1}} \ > \ (r-1) - \sum_{i=0}^{r} \frac{1}{l_{i1}} \qquad \text{and} \qquad \sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}} \ < \ -(r-1) + \sum_{i=0}^{r} \frac{1}{l_{in_i}}.$$

Proof. First note, that P is given in such a form that $d_{ij} l_{ij+1} < d_{ij+1} l_{ij}$ holds for all $1 \le j \le n_i$. In particular we have $d_{i1} l_{in_i} < d_{in_i} l_{i1}$. Now consider the second inequality which implies

$$-(r-1) \ > \ \sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}} - \sum_{i=0}^{r} \frac{1}{l_{in_i}}$$

and hence, by adding $r+1$ on both sides, we obtain

$$2 \ > \ \sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}} + \sum_{i=0}^{r} \frac{l_{in_i} - 1}{l_{in_i}} \ .$$

Now we will prove that

$$\sum_{i=0}^{r} \left\lceil \frac{d_{in_i}}{l_{in_i}} \right\rceil \ \le \ \sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}} + \sum_{i=0}^{r} \frac{l_{in_i} - 1}{l_{in_i}}$$

holds, where $\lceil x \rceil$ denotes the smallest integer bigger than the rational number x. We may write $\frac{d_{in_i}}{l_{in_i}} = a_i + \frac{b_i}{l_{in_i}}$ where a_i is an integer and $0 < b_i < l_{in_i}$. Consequently, we have

$$\frac{b_i}{l_{in_i}} + \frac{l_{in_i} - 1}{l_{in_i}} \;=\; 1 + \frac{b_i - 1}{l_{in_i}} \;\geq\; 1,$$

which gives the assertion. Analogously, one obtains for the first inequality

$$-2 \;<\; \sum_{i=0}^{r} \frac{d_{i1}}{l_{i1}} + \sum_{i=0}^{r} \frac{1 - l_{i1}}{l_{i1}}$$

and

$$\sum_{i=0}^{r} \left\lfloor \frac{d_{i1}}{l_{i1}} \right\rfloor \;\geq\; \sum_{i=0}^{r} \frac{d_{i1}}{l_{i1}} + \sum_{i=0}^{r} \frac{l_{i1} - 1}{l_{i1}}.$$

Using the fact that we always have $\frac{d_{in_i}}{l_{in_i}} \geq \frac{d_{i1}}{l_{i1}}$, we obtain all in all

$$\begin{aligned} 2 \;>\; \sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}} + \sum_{i=0}^{r} \frac{l_{in_i} - 1}{l_{in_i}} \;&\geq\; \sum_{i=0}^{r} \left\lceil \frac{d_{in_i}}{l_{in_i}} \right\rceil \;\geq\; \sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}} \\ \geq\; \sum_{i=0}^{r} \frac{d_{i1}}{l_{i1}} \;\geq\; \sum_{i=0}^{r} \left\lfloor \frac{d_{i1}}{l_{i1}} \right\rfloor \;&\geq\; \sum_{i=0}^{r} \frac{d_{i1}}{l_{i1}} + \sum_{i=0}^{r} \frac{l_{i1} - 1}{l_{i1}} \;>\; -2. \end{aligned}$$

Since the surface X is not toric, we have $r \geq 2$. Furthermore, we know that the columns v_{ij} of P are pairwisely different and primitive, which implies $\gcd(l_{i1}, d_{i1}) = 1$ as well as $\gcd(l_{in_i}, d_{in_i}) = 1$. This gives

$$\sum_{i=0}^{r} \left(\left\lceil \frac{d_{in_i}}{l_{in_i}} \right\rceil - \left\lfloor \frac{d_{i1}}{l_{i1}} \right\rfloor \right) \;\geq\; r + 1 \;\geq\; 3.$$

Together with the integer property this is a contradiction to

$$2 \;>\; \sum_{i=0}^{r} \left\lceil \frac{d_{in_i}}{l_{in_i}} \right\rceil \;\geq\; \sum_{i=0}^{r} \left\lfloor \frac{d_{i1}}{l_{i1}} \right\rfloor \;>\; -2.$$

□

Lemma 5.8. *Let X be a non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface with two elliptic fixed points. Then the following inequalities hold:*

$$\sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}} \leq \sum_{i=0}^{r} \frac{1}{l_{in_i}} - (r-1)\,, \qquad\qquad \sum_{i=0}^{r} \frac{d_{i1}}{l_{i1}} \geq -\sum_{i=0}^{r} \frac{1}{l_{i1}} + (r-1).$$

Furthermore, at least one of them holds equality, i.e. $v_c^+ = v^+$ or $v_c^- = v^-$.

Proof. Let X be a $\mathbb{K}^*$-surface and consider the elementary big cones $\sigma^\pm$ associated to the elliptic fixed points. We subdivide these two cones by drawing in rays along $v^+ = e_{r+1}$ and $v^- = -e_{r+1}$ respectively. Set $k^+ := \operatorname{lcm}(l_{0n_0}, \ldots, l_{rn_r})$ and $k^- = \operatorname{lcm}(l_{01}, \ldots, l_{r1})$. Then, by Proposition 3.15, the discrepancies of the associated toric ambient modifications are given by

$$\mathrm{Disc}_{v^+} = \frac{\sum_{i=0}^r \frac{k}{l_{in_i}} - k \cdot \sum_{i=0}^r \frac{d_{in_i}}{l_{in_i}} - (r-1)\cdot k}{k \cdot \sum_{i=0}^r \frac{d_{in_i}}{l_{in_i}}} = \frac{\sum_{i=0}^r \frac{1}{l_{in_i}} - \sum_{i=0}^r \frac{d_{in_i}}{l_{in_i}} - (r-1)}{\sum_{i=0}^r \frac{d_{in_i}}{l_{in_i}}}$$

and

$$\mathrm{Disc}_{v^-} = \frac{\sum_{i=0}^r \frac{k}{l_{i1}} + k \cdot \sum_{i=0}^r \frac{d_{i1}}{l_{i1}} - (r-1)\cdot k}{-k \cdot \sum_{i=0}^r \frac{d_{i1}}{l_{i1}}} = \frac{\sum_{i=0}^r \frac{1}{l_{i1}} + \sum_{i=0}^r \frac{d_{i1}}{l_{i1}} - (r-1)}{-\sum_{i=0}^r \frac{d_{i1}}{l_{i1}}}.$$

From Corollary 5.5 we infer that X is canonical. Hence, these discrepancies have to be greater or equal to zero. Since the denominators are positive in both cases, we obtain the following inequalities:

$$\sum_{i=0}^r \frac{1}{l_{in_i}} - \sum_{i=0}^r \frac{d_{in_i}}{l_{in_i}} - (r-1) \ \geq \ 0 \qquad \text{and} \qquad \sum_{i=0}^r \frac{1}{l_{i1}} + \sum_{i=0}^r \frac{d_{i1}}{l_{i1}} - (r-1) \ \geq \ 0.$$

As seen in Lemma 5.7 the strict inequality of both terms leads to a contradiction. Thus, we must have equality in one case. This implies $v_c^+ = v^+$ or $v_c^- = v^-$. □

Proposition 5.9. *Every non-toric del Pezzo $\mathbb{K}^*$-surface has at least one elliptic fixed point.*

Proof. Assume that X is a non-toric Fano $\mathbb{K}^*$-surface with two parabolic fixed point curves F^+ and F^-. Then Kleiman's condition for ampleness of the anticanonical divisor class gives $-K_X \cdot F^+ > 0$ and $-K_X \cdot F^- > 0$. With Proposition 4.24 we obtain the two inequalities

$$\sum_{i=0}^r \frac{d_{i1}}{l_{i1}} \ > \ (r-1) - \sum_{i=0}^r \frac{1}{l_{i1}} \qquad \text{and} \qquad \sum_{i=0}^r \frac{d_{in_i}}{l_{in_i}} \ < \ -(r-1) + \sum_{i=0}^r \frac{1}{l_{in_i}}\ .$$

By Lemma 5.7 these two inequalities cannot be satisfied simultaneously. □

Proposition 5.10. *Every non-toric del Pezzo $\mathbb{K}^*$-surface has a singular elliptic fixed point. In particular, there are no smooth non-toric del Pezzo $\mathbb{K}^*$-surfaces.*

Proof. Let X be a smooth non-toric Fano $\mathbb{K}^*$-surface. Then the discrepancies of every modification are greater than zero. In particular this holds for the discrepancies occurring for the modification that we obtain by subdividing an elementary big cone $\sigma^\pm$ via $v^\pm$. Hence, if X has two elliptic fixed points, we obtain the following inequalities:

$$\sum_{i=0}^r \frac{d_{i1}}{l_{i1}} l^- > (r-1)l^- - \sum_{i=0}^r \frac{l^-}{l_{i1}}, \qquad \sum_{i=0}^r \frac{d_{in_i}}{l_{in_i}} l^+ < -(r-1)l^+ + \sum_{i=0}^r \frac{l^+}{l_{in_i}}.$$

If X has only one elliptic fixed point, say F^+ and one parabolic fixed point curve F^-, then the first inequality can be replaced by the inequality given by the Fano condition $F^- \cdot (-K_X) > 0$. We infer from Proposition 4.24 that it is also given by

$$\sum_{i=0}^{r} \frac{d_{i1}}{l_{i1}} l^- > (r-1)l^- - \sum_{i=0}^{r} \frac{l^-}{l_{i1}}.$$

Hence, we end up with the same inequalities. The case that X has two parabolic fixed point curves cannot occur, see Proposition 5.9. By Lemma 5.8 we know that these two inequalities cannot be satisfied simultaneously, a contradiction to the smoothness of X. Furthermore, we can conclude that there is always one elliptic fixed point which is singular. □

Observation 5.11. In case of dimension two, Gorenstein and terminality imply smoothness. Consequently, non-toric Gorenstein del Pezzo $\mathbb{K}^*$-surfaces cannot be terminal.

5.2 Effective bounds for Gorenstein del Pezzo $\mathbb{K}^*$-surfaces

We intend to classify Gorenstein log del Pezzo surfaces with a $\mathbb{K}^*$-action. For this purpose we will give explicit bounds for the number of Cox ring relations, the Picard number, i.e. the rank of the divisor class group, and finally for all entries l_{ij}, d_{ij} of possible P-matrices.

Proposition 5.12. *Let $X = X(A, P)$ be a non-toric log-terminal $\mathbb{K}^*$-surface of Gorenstein index $\iota(X) = a$ with two elliptic fixed points. Then $r \leq 4 \cdot a - 1$. In particular, the number of relations of the Cox ring is bounded by $4 \cdot a - 2$.*

Proof. By Lemma 5.4 we know that the discrepancies of the two elementary big cones $\sigma^\pm$ have to be greater or equal to $-1 + \frac{1}{a}$. Consequently, for σ^+ we obtain

$$\sum_{i=0}^{r} \frac{1}{l_{in_i}} - \sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}} - (r-1) \ \geq \ (-1 + \frac{1}{a}) \sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}}$$

and thus

$$-(r-1) \ \geq \ \frac{1}{a} \sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}} - \sum_{i=0}^{r} \frac{1}{l_{in_i}}.$$

For σ^- we obtain analogously

$$-(r-1) \ \geq \ -\frac{1}{a} \sum_{i=0}^{r} \frac{d_{i1}}{l_{i1}} - \sum_{i=0}^{r} \frac{1}{l_{i1}}.$$

For $x \in \mathbb{Q}$ let $\lceil x \rceil$ be the smallest integer greater than x and $\lfloor x \rfloor$ the greatest integer smaller than x. Adding $r+1$ to both sides of the inequalities yields

$$\begin{aligned} 2 \;\geq\; & \frac{1}{a}\sum_{i=0}^{r}\frac{d_{in_i}}{l_{in_i}} + \sum_{i=0}^{r}\frac{l_{in_i}-1}{l_{in_i}} \;\geq\; \frac{1}{a}\left(\sum_{i=0}^{r}\frac{d_{in_i}}{l_{in_i}} + \sum_{i=0}^{r}\frac{l_{in_i}-1}{l_{in_i}}\right) \\ \geq\; & \frac{1}{a}\sum_{i=0}^{r}\left\lceil\frac{d_{in_i}}{l_{in_i}}\right\rceil \;\geq\; \frac{1}{a}\sum_{i=0}^{r}\frac{d_{in_i}}{l_{in_i}} \;\geq\; \frac{1}{a}\sum_{i=0}^{r}\frac{d_{i1}}{l_{i1}} \;\geq\; \frac{1}{a}\sum_{i=0}^{r}\left\lfloor\frac{d_{i1}}{l_{i1}}\right\rfloor \\ \geq\; & \frac{1}{a}\left(\sum_{i=0}^{r}\frac{d_{i1}}{l_{i1}} + \sum_{i=0}^{r}\frac{1-l_{i1}}{l_{i1}}\right) \;\geq\; \frac{1}{a}\sum_{i=0}^{r}\frac{d_{i1}}{l_{i1}} + \sum_{i=0}^{r}\frac{1-l_{i1}}{l_{i1}} \;\geq\; -2. \end{aligned}$$

With the above estimate we conclude

$$\frac{1}{a}(r+1) \;\leq\; \frac{1}{a}\sum_{i=0}^{r}\left(\left\lceil\frac{d_{in_i}}{l_{in_i}}\right\rceil - \left\lfloor\frac{d_{i1}}{l_{i1}}\right\rfloor\right) \;\leq\; 4$$

and thus $r \leq 4a-1$ as claimed. □

Proposition 5.13. *Let $X = X(A,P)$ be a non-toric log del Pezzo $\mathbb{K}^*$-surface of Gorenstein index $\iota(X) = a$ with one parabolic fixed point curve. Then $r \leq 4\cdot a-1$. In particular, the number of relations of the Cox ring is bounded by $4\cdot a-2$.*

Proof. Assume that X has a parabolic fixed point curve F^+. The argument is similar to the argument used in the proof of Proposition 5.12. We just replace the Gorenstein condition for the upper elliptic fixed point by the Fano condition $(-K_X)\cdot F^+ > 0$, which is given by

$$\sum_{i=0}^{r}\frac{1}{l_{in_i}} - \sum_{i=0}^{r}\frac{d_{in_i}}{l_{in_i}} - (r-1) \;>\; 0,$$

compare Proposition 4.24. In particular, for every $a \in \mathbb{Z}_{>0}$ we have

$$-(r-1) \;>\; -\sum_{i=0}^{r}\frac{1}{l_{in_i}} + \sum_{i=0}^{r}\frac{d_{in_i}}{l_{in_i}} \;\geq\; -\sum_{i=0}^{r}\frac{1}{l_{in_i}} + \frac{1}{a}\sum_{i=0}^{r}\frac{d_{in_i}}{l_{in_i}}\;.$$

Hence, we can use exactly the same estimate as in the proof of Proposition 5.12 which verifies the assertion. □

Corollary 5.14. *Let $X = X(A,P)$ be a non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface. Then its Cox ring has at most two relations.*

An important convex geometrical result which turned out to be useful for log-terminal Gorenstein del Pezzo $\mathbb{K}^*$-surfaces is the following theorem of Schicho, [49, Theorem 2]:

Theorem 5.15. (Schicho' s Theorem). *Let B be a two-dimensional lattice polytope with no interior lattice points. Then, up to unimodular transformation, B is one of the following:*

$$\mathrm{conv}((0,0),(n,0),(m,1),(0,1)), \qquad \mathrm{conv}((0,0),(2,0),(0,2)).$$

Proposition 5.16. *Let $X = X(A, P)$ be a non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface. Then $n_i \leq 2$ holds for all $0 \leq i \leq r$ and the rank $\mathrm{rk}(\mathrm{Cl}(X))$ of the divisor class group of X is bounded by 5. Furthermore, the number of singularities is bounded by 10.*

Proof. By Lemma 5.8 we know that one of the vertices v_c^+, v_c^- of the anticanonical complex is a lattice point, i.e. we have $v_c^+ = v^+ = e_{r+1}$ or $v_c^- = v^- = -e_{r+1}$. Since X is Fano and Gorenstein, all the primitive generators v_{ij} are vertices of the anticanonical complex A_X^c and there are no lattice points in the relative interior of A_X^c. By Schicho's Theorem we know that each arm of A_X^c lying within one block can have maximal four vertices. Without loss of generality we assume one of them to be v^-. Furthermore, we may assume that one vertex is either v^+ or zero. Consequently, we have $n_i \leq 2$. Moreover, by Corollary 5.14, the number of Cox ring relations is bounded by two. The rank of the class group is given by

$$\mathrm{rk}(\mathrm{Cl}(X)) \;=\; n + m - (r-1) - 2 \leq \begin{cases} 4, & \text{if } X \text{ has two elliptic fixed points,} \\ 5, & \text{if } X \text{ has one elliptic fixed point.} \end{cases}$$

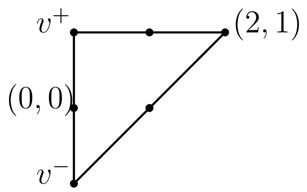
Note that $m \leq 1$ holds since X is Fano. Since there are at most two maximal tower cones for each $0 \leq i \leq r \leq 3$ and at most two big cones $\sigma^{\pm}$, the number of relevant maximal cones in the fan Σ of the toric ambient variety Z is bounded by 10. □

Proposition 5.17. *Let $X = X(A, P)$ be a non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface with a parabolic fixed point curve F^+ and an elliptic fixed point F^-. Then ($n_i = 1$ and $l_{i1} = 2$) or ($n_i = 2$ and $l_{i1} = l_{i2} = 1$) hold for each $0 \leq i \leq r$.*

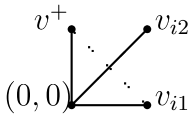
Proof. First note that Proposition 5.16 gives the restriction $n_i = 1$ or $n_i = 2$. By Proposition 4.24 we obtain for F^+ the Fano condition

$$\sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}} < \sum_{i=0}^{r} \frac{1}{l_{in_i}} - (r-1) \quad \text{and consequently} \quad \sum_{i=0}^{r} \frac{d_{i1}}{l_{i1}} = -\sum_{i=0}^{r} \frac{1}{l_{i1}} + (r-1).$$

Note that the last equality follows from Corollary 5.8 and the fact that log-terminality and Gorenstein property imply canonical singularities. According to the proof of Proposition 5.10 we have a (canonical) singularity in the lower elliptic fixed point. Consequently, we have to subdivide the lower elementary big cone σ^- along v^-. Furthermore, we conclude that $v_c^- = v^-$ is an integral vertex of A_X^c. Note that $v_c^+ = v^+$ is an integral vertex since F^+ is a parabolic fixed point curve. If X is Gorenstein, then the lattice polytope $\mathrm{conv}(v^-, v^+, v_{i1})$ does not contain inner lattice points. First we consider the case $n_i = 1$. Without loss of generality we can assume $l_{i1} \geq 2$ and $l_{i1} > d_{i1} \geq 0$.



Consequently, $\operatorname{conv}((0,1),(0,-1),(2,1))$ is the only possibility for a polytope without inner lattice points. Otherwise $(1,1)$ or $(1,0)$ would be contained in the interior. Now we have a look at the second case $n_i = 2$. We restrict ourselves again to the situation within one arm. The upper ray v_{i2} is contractible.



Since X is Gorenstein, there are no points on the line between v^+ and v_{i1} after the contraction of v_{i2}. In dimension two, terminality and Gorenstein imply smoothness. Consequently, we obtain $\det(v^+, v_{i1}) = 1$ and thus $l_{i1} = 1$ and $d_{i1} \leq 0$ since

$$d_{i1} \leq \frac{d_{i2}}{l_{i2}} < 1.$$

Now consider the second ray. The Gorenstein property gives $l_{i2} \mid d_{i2} - 1$ where $l_{i2} > d_{i2} \geq 0$. Thus, we obtain $l_{i2} = 1$ (and $d_{i2} = 0$) or $d_{i2} = 1$. Assume $d_{i2} = 1$. The Gorenstein property for the cone $\operatorname{cone}(v_{i1}, v_{i2})$ gives the condition

$$l_{i1}d_{i2} - l_{i2}d_{i1} \;=\; 1 - l_{i2}d_{i1} \mid 1 - d_{i1} = d_{i2} - d_{i1}.$$

Since $d_{i1} \leq 0$, we have $1 - l_{i2}d_{i1} \geq 1 - d_{i1} > 0$ and the divisibility condition can only be satisfied for $l_{12} = 1$ or for $d_{i1} = 0$. In the latter case Schicho's Theorem gives $l_{i2} \leq 2$. But for $l_{i2} = 2$ the primitive vector $v_{i1} = (l_{i1}, d_{i1})$ cannot be a vertex of A_X^c, which contradicts the Fano condition. □

Lemma 5.18. *Let $X = X(A,P)$ be a non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface with two elliptic fixed points such that P has standard form. If X has a singularity in the lower elliptic fixed point F^- and satisfies $n_i = 2$ for an index $0 \leq i \leq r$, then $l_{i2} = 1$ holds.*

Proof. Without loss of generality we may assume $i \neq 0$ and we can consider the primitive vectors $\widehat{v}_{i1} = (l_{i1}, d_{i1})$ and $\widehat{v}_{i2} = (l_{i2}, d_{i2})$ such that $l_{i1}d_{i2} < l_{i2}d_{i1}$ and $l_{21} > d_{21} \geq 0$. Then $\widehat{v}_{i2}$ is contained in $\operatorname{cone}((1,1),(1,0))$. Assume that $l_{i2} \geq 2$ holds and recall that the singularity in F^- implies that $(0,-1)$ is a vertex of the subcomplex $\widehat{A}^c_{X_i}$ of the anticanonical complex that we obtain by restricting on the projected i-th arm. If $l_{i2} < d_{i2} - 1$, then $(1,0)$ is an interior point of $\widehat{A}^c_X$, a contradiction to the Gorenstein condition. If $l_{i2} = d_{i2} - 1$, then we must set $l_{i1} := d_{i1} - 1$ to avoid that $(1,0)$ is an interior point of $\widehat{A}^c_X$. But this is a contradiction to the (Fano) condition that $\widehat{v}_{i1}$ is a vertex of $\widehat{A}^c_{X_i}$. □

Lemma 5.19. *Let $X = X(A,P)$ be a non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface with two elliptic fixed points and a singularity in the lower elliptic fixed point F^- and let P be in standard form. If $n_i = 1$, then $l_{i1} = d_{i1} + 1$ holds for all $i \neq 0$ and if $n_i = 2$, then $l_{i1} = 1$ or $d_{i1} = -1$ holds for all $i \neq 0$.*

Proof. We have $\widehat{v}_c^- = (0,-1)$, since F^- is singular. Consider $\widehat{v}_{i1} = (l_{i1}, d_{i1})$ with $l_{i1} > d_{i1} \geq 0$. The assumption $n_i = 1$ implies $l_{i1} \geq 2$ and $d_{i1} > 0$. For all pairs (l_{i1}, d_{i1}) with $l_{i1} \neq d_{i1} + 1$, the point $(1,0)$ is contained in the relative interior of $\widehat{A}^c_{X_i}$, a contradiction to the Gorenstein condition. Now assume $n_i = 2$. Then by Lemma 5.18 we have $l_{i2} = 1$. Since $(1,-1)$ may not be an interior point of $\widehat{A}^c_{X_i}$, the assertion follows. □

Proposition 5.20. *Let $X = X(A,P)$ be a non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface with two elliptic fixed points. Then the exponents l_{ij} are bounded by* 6.

Proof. Since X is Gorenstein, we may assume $r \leq 3$, compare Corollary 5.14. The Fano property of X allows us to work with the anticanonical complex. Without loss of generality we can assume $v_c^- = -e_{r+1}$, i.e. X has a singularity in the lower elliptic fixed point F^-. Furthermore, we know that

$$\sum_{i=0}^{r} \frac{l^+ d_{in_i}}{l_{in_i}} > 0 \qquad \text{and} \qquad \sum_{i=0}^{r} \frac{l^- d_{i1}}{l_{i1}} < 0$$

hold and that the log-terminality condition for $\sigma^\pm$ gives

$$\sum_{i=0}^{r} \frac{l^+}{l_{in_i}} - (r-1)l^+ > 0 \qquad \text{and} \qquad \sum_{i=0}^{r} \frac{l^-}{l_{i1}} - (r-1)l^- > 0.$$

The last coordinate of the upper vertex v_c^+ of the anticanonical complex is given by

$$0 \;\leq\; \frac{\sum_{i=0}^{r} \frac{l^+ d_{in_i}}{l_{in_i}}}{\sum_{i=0}^{r} \frac{l^+}{l_{in_i}} - (r-1)l^+} \;\leq 1.$$

By Proposition 5.16 we may also assume that $n_i \leq 2$ holds for all $0 \leq i \leq r$. And since we have two elliptic fixed points, there is at least one block with $n_i = 2$. We go through all possible cases:

(1): Let $r = 2$ or $r = 3$ and $n_i = 2$ for all $0 \leq i \leq r$. Then $l_{i2} = 1$ for $0 \leq i \leq r$ and $d_{i2} = 0$ for $1 \leq i \leq r$. The $(r+1)$-th coordinate of the vertex v_c^+ is positive but not greater than 1. This gives $0 \leq d_{02}/2 \leq 1$ and thus $d_{02} = 1$ or $d_{02} = 2$. In particular, $v_c^+ = e_{r+1}$ or $v_c^+ = \frac{1}{2}e_{r+1}$. This in turn implies $l_{i1} < 3$ for $0 \leq i \leq 3$, otherwise $(1,0)$ resp. $(-1,0)$ or $(-1,1)$ would be interior points of $\widehat{A}^c_{X_i}$.

(2): Let $n_0 = 1$ and $n_i = 2$ for $1 \leq i \leq r$. Then $l_{i2} = 1$ and $d_{i2} = 0$ for $1 \leq i \leq r$ and $l_{01} \geq 2$. Furthermore, we know that $(-1)^r \det(\sigma^+) = d_{01} > 0$. For the $(r+1)$-th coordinate of v_c^+ we obtain

$$\frac{d_{01}}{l_{01}+1} \;\leq\; 1.$$

Furthermore, the Gorenstein condition for the upper elliptic fixed point says that d_{01} is a divisor of $l_{01}+1$, thus $d_{01} \leq l_{01}+1$. Since $v_c^- = -e_{r+1}$, we know that $d_{01} \geq l_{01} - 1$ holds.

Since d_{01} and l_{01} are coprime, there are two possibilities left. If $d_{01} = l_{01}+1$, then $l_{01} \geq 2$ since otherwise $(-1,1)$ would be an interior point of $\widehat{A}^c_{X_0}$. For $l_{01} = 2$ and $d_{01} = 3$ we obtain $v_c^+ = -e_{r+1}$. If $d_{01} = l_{01}-1$, then $l_{01}-1 \mid l_{01}+1$ which implies $l_{01} \leq 3$. If $l_{01} = 2$ and $d_{01} = 1$, then $v_c^+ = \frac{1}{3}e_{r+1}$ and if $l_{01} = 3$ and $d_{01} = 2$, then $v_c^+ = \frac{1}{2}e_{r+1}$. All in all this implies $l_{i1} \leq 3$ for $1 \leq i \leq r$.

(3): Let $n_0 = n_1 = 1$ and $n_i = 2$ for $2 \leq i \leq r$. Then $l_{i2} = 1$ and $d_{i2} = 0$ for $2 \leq i \leq r$ and $l_{11} = d_{11}+1$. Furthermore $d_{01}l_{11} + l_{01}l_{11} - l_{01} > 0$ is a divisor of $l_{01} + l_{11}$. In particular, we have

$$d_{01}l_{11} + l_{01}l_{11} - l_{01} \ \leq \ l_{01} + l_{11}.$$

This implies $d_{01} \leq 1$. Since $l_{01} \geq 2$, we can conclude ($d_{01} = 1$ and $l_{01} = 2$) or ($d_{01} = -1$). Otherwise $(-1,0)$ and $(-1,-1)$ respectively would be an interior point of $\widehat{A}^c_{X_0}$. In the first case we obtain directly $l_{11} \leq 2$. So, assume $d_{01} = -1$. We want to avoid that $(-1,0)$ is contained in the relative interior of the polytope spanned by $(l_{01},-1)$, $\widehat{v}_c^+$ and $\widehat{v}_c^- = (0,-1)$. For that purpose we compute the intersection point of the line spanned by $(l_{01},-1)$ and $\widehat{v}_c^+$ with the horizontal 0-level and require it to be greater than -1. The slope of this line can be computed by

$$\left(\frac{-l_{11} + l_{01}l_{11} - l_{01}}{l_{01}+l_{11}} + 1\right) \cdot \frac{1}{l_{01}} \ = \ \frac{l_{01}l_{11}}{l_{01}+l_{11}} \cdot \frac{1}{l_{01}} \ = \ \frac{l_{11}}{l_{01}+l_{11}}.$$

Hence, the intersection point is given by

$$\frac{-l_{11} + l_{01}l_{11} - l_{01}}{l_{01}+l_{11}} + \frac{l_{11}}{l_{01}+l_{11}}x \ = \ 0,$$

and we require

$$x \ = \ \frac{l_{11} + l_{01} - l_{01}l_{11}}{l_{11}} \ \geq \ -1$$

which is equivalent to $2l_{11} + l_{01} - l_{01}l_{11} \geq 0$. Let $l_{01} = 2$. Then this inequality is always fulfilled. But since $-l_{11} + l_{01}(l_{11}-1) = -l_{11} + 2(l_{11}-1) = l_{11}-2$ is a divisor of $l_{11} + l_{01} = l_{11}+2$, we obtain $l_{11} \in \{3,4,6\}$. For $l_{01} = 3$ the inequality gives $l_{11} \leq 3$. For $l_{01} = 4$ we obtain $l_{11} \leq 2$ and thus $l_{01} \geq 5$ can be excluded since this would imply $l_{11} \leq 1$. The smallest value for the last coordinate of v_c^+ is given by $1/5$. Consequently, we obtain $l_{i1} \leq 5$ for $2 \leq i \leq r$.

(4): For $r = 3$, let $n_i = 1$ for $0 \leq i \leq 2$ and $n_3 = 2$ and assume $l_{01} \geq l_{11} \geq l_{21} \geq 2$. Then we have $l_{32} = 1$ and $d_{32} = 0$ as well as $l_{11} = d_{11}+1$ and $l_{21} = d_{21}+1$. Putting this in the condition

$$d_{01}l_{11}l_{21}l_{32} + d_{11}l_{01}l_{21}l_{32} + d_{21}l_{01}l_{11}l_{32} + d_{32}l_{01}l_{11}l_{21} \ > \ 0$$

we obtain

$$d_{01}l_{11}l_{21} + 2l_{01}l_{11}l_{21} - l_{01}l_{21} - l_{01}l_{11} \ > \ 0.$$

The log-terminality condition gives

$$\begin{aligned} & -2l_{01}l_{11}l_{21}l_{32} + l_{01}l_{11}l_{21} + l_{01}l_{11}l_{32} + l_{01}l_{21}l_{32} + l_{11}l_{21}l_{32} \\ = \ & -l_{01}l_{11}l_{21} + l_{01}l_{11} + l_{01}l_{21} + l_{11}l_{21} \\ > \ & 0. \end{aligned}$$

This condition is fulfilled if and only if (l_{01}, l_{11}, l_{21}) is a platonic triple, i.e. it is of the form $(l_{01}, 2, 2)$, $(3, 3, 2)$, $(4, 3, 2)$ or $(5, 3, 2)$. Since X is Gorenstein the following divisibility condition has to be satisfied:

$$d_{01}l_{11}l_{21} + 2l_{01}l_{11}l_{21} - l_{01}l_{21} - l_{01}l_{11} \ \mid \ -l_{01}l_{11}l_{21} + l_{01}l_{11} + l_{01}l_{21} + l_{11}l_{21}$$

Hence we have to go through all the cases mentioned above:

If $(l_{01}, l_{11}, l_{21}) = (l_{01}, 2, 2)$, then $4d_{01} + 8l_{01} - 4l_{01} = 4d_{01} + 4l_{01} > 0$ has to be a divisor of $-4l_{01} + 4 + 4l_{01} = 4$. We can conclude $d_{01} + l_{01} = 1$ and hence $l_{01} = 2$ and $d_{01} = -1$, otherwise $(-1, -1)$ would be an interior point of $\widehat{A}^c_{X_0}$. Since $(4 \cdot (-1) + 4 \cdot 2)/4 = 1$ we conclude $\widehat{v}^+_c = (0, 1)$. Now we consider the lower elliptic fixed point. Since $v^-_c = -e_{r+1}$, we obtain the condition

$$\begin{aligned} & \frac{d_{01}l_{11}l_{21}l_{31} + d_{11}l_{01}l_{21}l_{31} + d_{21}l_{01}l_{11}l_{31} + d_{31}l_{01}l_{11}l_{21}}{-2l_{01}l_{11}l_{21}l_{31} + l_{01}l_{11}l_{21} + l_{01}l_{11}l_{31} + l_{01}l_{21}l_{31} + l_{11}l_{21}l_{31}} \\ = & \frac{-4l_{31} + 4l_{31} + 4l_{31} + 8d_{31}}{-16l_{31} + 8 + 12l_{31}} \\ = & \frac{4l_{31} + 8d_{31}}{-4l_{31} + 8} \ = \ -1. \end{aligned}$$

Consequently, we have $d_{31} = -l_{31}$ which implies $d_{31} = -1$ and $l_{31} = 1$, since l_{31} and d_{31} are coprime.

If $(l_{01}, l_{11}, l_{21}) = (3, 3, 2)$, then $6d_{01} + 36 - 6 - 9 = 6d_{01} + 21 > 0$ has to be a divisor of $-18 + 9 + 6 + 6 = 3$. Since $6d_{01} + 21 \neq 1$ we conclude $6d_{01} + 21 = 3$ and hence $d_{01} = -3$, a contradiction to the condition $\gcd(l_{01}, d_{01}) = 1$.

If $(l_{01}, l_{11}, l_{21}) = (4, 3, 2)$, then $6d_{01} + 48 - 8 - 12 = 6d_{01} + 28 > 0$ has to be a divisor of $-24 + 12 + 6 + 8 = 2$, which is not possible.

If $(l_{01}, l_{11}, l_{21}) = (5, 3, 2)$, then $6d_{01} + 60 - 10 - 15 = 6d_{01} + 35 > 0$ has to be a divisor of $-30 + 15 + 10 + 6 = 1$, which is not possible. □

Corollary 5.21. *Let $X = X(A, P)$ be a non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface. Then all exponents are bounded by $l_{ij} < 6$.*

Remark 5.22. We shortly recall some bounding statements that are used in the proof of the next proposition. Let $X = X(A, P)$ be a non-toric log del Pezzo $\mathbb{K}^*$-surface. Then the following inequality is always satisfied (see proof of Lemma 5.7):

$$2 \ \geq \ \sum_{i=0}^{r} \left\lceil \frac{d_{in_i}}{l_{in_i}} \right\rceil \ \geq \ \sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}} \ \geq \ \sum_{i=0}^{r} \frac{d_{i1}}{l_{i1}} \ \geq \ \sum_{i=0}^{r} \left\lfloor \frac{d_{i1}}{l_{i1}} \right\rfloor \ \geq \ -2.$$

Furthermore, we may assume that F^- is a singular elliptic fixed point which implies

$$-\sum_{i=0}^{r} \frac{l^-}{l_{in_i}} + (r-1)l^- \ = \ \sum_{i=0}^{r} \frac{d_{i1}}{l_{i1}} \ < \ 0 \, .$$

If F^+ is an elliptic fixed point, then the following inequality holds and if F^+ is an parabolic fixed point curve, then equality is even excluded.

$$\sum_{i=0}^{r} \frac{l^+}{l_{in_i}} - (r-1)l^+ \ \geq \ \sum_{i=0}^{r} \frac{d_{in_i}}{l_{in_i}} \ > \ 0$$

Proposition 5.23. *Let $X = X(A, P)$ be a non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface. Then the entries d_{ij} of the matrix P are bounded by $|d_{ij}| \leq 6$.*

Proof. We have to check all possible cases and use the statements of Remark 5.22. Set

$$D^+ := \sum_{i=0}^{r} \frac{d_{in_i} l^+}{l_{in_i}}, \qquad D^- := \sum_{i=0}^{r} \frac{d_{i1} l^-}{l_{i1}} \ .$$

First we assume that X has a parabolic fixed point curve F^+ and an elliptic fixed point F^-. Then we have $l_{ij} = 2$ for $n_i = 1$ and $l_{ij} = 1$ for $n_i = 2$. Hence, we have the following cases subject to $n = (n_0, \ldots, n_r)$:

- $n = (1, 1, 1)$:
  Then we have $l_{i1} = 2$ for $i = 0, 1, 2$ and $d_{i1} = 1$ for $i = 1, 2$. Using Remark 5.22 we obtain $D^- = 4d_{01} + 4 + 4 = 4d_{01} + 8 = 8 - 4 - 4 - 4 = -4$ and thus $d_{01} = -3$.

- $n = (1, 1, 1, 1)$:
  Then we have $l_{i1} = 2$ for $i = 0, 1, 2, 3$ and $d_{i1} = 1$ for $i = 1, 2, 3$ and consequently $D^- = 8d_{01} + 8 + 8 + 8 = 2 \cdot 16 - 8 - 8 - 8 - 8 = 0$, a contradiction.

- $n = (2, 1, 1)$:
  Then $l_{01} = l_{02} = 1$ and $l_{i1} = 2$, $d_{i1} = 1$ for $i = 1, 2$ and $D^- = 4d_{01} + 4 = -4$ which implies $d_{01} = -2$ and consequently $-2 < d_{02} < 0$ since $D^+ = 4d_{02} + 4 < 4$.

- $n = (2, 1, 1, 1)$:
  Then $l_{01} = l_{02} = 1$ and $l_{i1} = 2$ for $i = 1, 2$, $d_{i1} = 1$. Hence, we have $D^- = 8d_{01} + 4 + 4 + 4 = 2 \cdot 8 - 8 - 4 - 4 - 4 = -4$ which implies $d_{01} = -4$ and thus $d_{02} > -4$. $D^+ = 8d_{02} + 4 + 4 + 4 < 4$ gives $d_{02} < -1$.

- $n = (2, 2, 1)$:
  Then $l_{i1} = l_{i2} = 1$ for $i = 0, 1$ and $l_{21} = 2$ as well as $d_{12} = 0$ and $d_{21} = 1$. So we have $D^- = 2d_{01} + 2d_{11} + 1 = -3$ and $D^+ = 2d_{02} + 1 < 3$ which implies $d_{02} < 1$. Since $d_{01} + d_{11} = -2$ and $d_{01}, d_{11} \leq 0$, we obtain $-2 \leq d_{01}, d_{11} \leq 0$.

- $n = (2, 2, 1, 1)$:
  Then $l_{31} = l_{21} = 2$, $d_{31} = d_{21} = 1$ as well as $l_{01} = l_{02} = l_{11} = l_{12} = 1$ and $d_{12} = 0$, $d_{11} < 0$. We have $D^+ = 4d_{02} + 2 + 2 < 4$ and thus $d_{02} \leq 0$ and $d_{01} < 0$. Since $D^- = 4d_{01} + 4d_{11} + 2 + 2 = -4$, we have $d_{01} + d_{11} = -2$ and thus $-2 \leq d_{01}, d_{11} < 0$.

- $n = (2,2,2)$:
  Then we have $l_{i1} = l_{i2} = 1$ and $d_{12} = d_{22} = 0$ as well as $D^+ = d_{02} < 2$ and $D^- = d_{01} + d_{11} + d_{21} = -2$. This implies $0 \geq d_{01}, d_{11}, d_{21} \geq -2$.

- $n = (2,2,2,1)$:
  Then $l_{i1} = l_{i2} = 1$ for $i = 0,1,2$ and $l_{31} = 2$ which implies $d_{31} = 1$ and $d_{22} = d_{12} = 0$ and thus $d_{11}, d_{21} < 0$. Since $D^+ = 2d_{02} + 1 < 3$, we obtain $d_{02} \leq 0$. Furthermore, we have $2d_{01} + 2d_{11} + 2d_{21} + 1 = -3$ which implies $d_{01} + d_{11} + d_{21} = -2$ where $d_{01}, d_{11}, d_{21} < 0$, a contradiction.

- $n = (2,2,2,2)$:
  Then we obtain analogously to the previous case $d_{12} = d_{22} = d_{32} = 0$ and $0 < d_{02} < 2$ and thus $d_{01} \leq 0$ and $d_{11}, d_{21}, d_{21} < 0$ which is a contradiction to $d_{01}+d_{11}+d_{21}+d_{21} = -2$.

Now we consider the case of X having two elliptic fixed points. Analogously, we have to go through all cases as before using the inequalities of Remark 5.22. Note that for $i \neq 0$ we always have

$$0 \ \leq \ d_{i2} \ < \ 6, \qquad \left\lceil \frac{d_{in_i}}{l_{in_i}} \right\rceil \ = \ 1 \quad \text{and} \quad \left\lfloor \frac{d_{in_i}}{l_{in_i}} \right\rfloor \ = \ 0.$$

Furthermore, we will use the fact, that if $n_i = 2$, we can assume $l_{i2} = 1$ and that if $d_{i1} \neq -1$ holds for $i \neq 0$, we have $l_{i1} = 1$, compare Lemma 5.18.

- $n = (2,1,1)$:
  Then we have $2 \geq d_{02} + 1 + 1 > 0$ and thus we obtain $-2 < d_{02} \leq 0$. Furthermore, we have
  $$-2 \leq \left\lfloor \frac{d_{01}}{l_{01}} \right\rfloor + 0 + 0 \ < \ 0.$$
  If $l_{01} = 1$ holds, we obtain $-2 \leq d_{01} < 0$ and if $l_{01} > 1$ we obtain $0 > d_{01} \geq -l_{01} - 1 > -7$ since $l_{01} < 6$ and the polytope $\operatorname{conv}(v_{01}, v_{02}, v^-)$ does not have inner points.

- $n = (2,1,1,1)$:
  Analogously to the case before, we have $2 \geq d_{02} + 1 + 1 + 1 > 0$ and thus $-3 < d_{02} \leq -1$ as well as $-2 \leq d_{01} + 0 + 0 + 0 < 0$ if $l_{01} = 1$ and $0 > d_{01} \geq -l_{01} - 1 > -7$ otherwise.

- $n = (2,2,1)$:
  Here we have $2 \geq d_{02} + 0 + 1 > 0$ and thus $-1 < d_{02} \leq 1$. We infer from Remark 5.22 that
  $$0 \ > \ \left\lfloor \frac{d_{01}}{l_{01}} \right\rfloor + \left\lfloor \frac{d_{11}}{l_{11}} \right\rfloor \geq -2$$

holds. Furthermore, we have $\frac{d_{01}}{l_{01}} < \frac{d_{02}}{l_{02}} \leq 1$ which gives $d_{01} < 5$. Since $d_{11} < 0$ and $\left\lfloor \frac{d_{01}}{l_{01}} \right\rfloor \leq 0$, we have the possibilities

$$\left\lfloor \frac{d_{01}}{l_{01}} \right\rfloor = 0 \text{ or } \left\lfloor \frac{d_{01}}{l_{01}} \right\rfloor = -1 \quad \text{which implies } \left\lfloor \frac{d_{11}}{l_{11}} \right\rfloor = -1 \text{ or } \left\lfloor \frac{d_{11}}{l_{11}} \right\rfloor = -2$$

and $d_{01} > -5$. If $l_{11} \neq 1$, then $d_{11} = -1$ and if $l_{11} = 1$, we obtain $d_{11} \geq -2$.

- $n = (2, 2, 1, 1)$:
  Analogously to the case before, we obtain $2 \geq d_{02} + 0 + 1 + 1 > 0$ and thus $-2 < d_{02} \leq 0$ which implies $d_{01} > -5$ and $d_{11} \geq -2$.

- $n = (2, 2, 2)$:
  Here we have $2 \geq d_{02} > 0$ and

$$0 \;>\; \left\lfloor \frac{d_{01}}{l_{01}} \right\rfloor + \left\lfloor \frac{d_{11}}{l_{11}} \right\rfloor + \left\lfloor \frac{d_{21}}{l_{21}} \right\rfloor \;\geq\; -2.$$

  Since $d_{11}, d_{21} < 0$ holds, we have

$$\left\lfloor \frac{d_{11}}{l_{11}} \right\rfloor \;\leq\; -1 \quad \text{and} \quad \left\lfloor \frac{d_{21}}{l_{21}} \right\rfloor \;\leq\; -1$$

  which implies $\lfloor d_{01}/l_{01} \rfloor \geq 0$ and in particular $d_{01} \geq 0$. Furthermore, we have $d_{01}/l_{01} < 2$ and thus

$$0 \leq \left\lfloor \frac{d_{01}}{l_{01}} \right\rfloor \;\leq\; 1 \text{ which implies } \left\lfloor \frac{d_{i1}}{l_{i1}} \right\rfloor \;\geq\; -2$$

  for $i = 1, 2$ and consequently $d_{11} \geq -2$ and $d_{21} \geq -2$ for $l_{11} = 1$ and $l_{21} = 1$ respectively (otherwise we have $d_{i1} = -1$ anyway). Now consider the point $(-1, 1)$ which is not contained in the relative interior of $\mathrm{conv}(v_{01}, v_{02}, v^-)$. This implies $d_{01}/l_{01} \leq 1$ and thus $d_{01} \leq l_{01} < 6$.

- $n = (2, 2, 2, 1)$:
  Here we have $2 \geq d_{02} + 1 > 0$ and thus $1 \geq d_{02} > -1$ which implies $\frac{d_{01}}{l_{01}} < 1$. All other arguments run analogously since the following condition still holds:

$$0 \;>\; \left\lfloor \frac{d_{01}}{l_{01}} \right\rfloor + \left\lfloor \frac{d_{11}}{l_{11}} \right\rfloor + \left\lfloor \frac{d_{21}}{l_{21}} \right\rfloor \;\geq\; -2.$$

- $n = (2, 2, 2, 2)$:
  Here we have $2 \geq d_{02} > 0$ and

$$0 \;>\; \left\lfloor \frac{d_{01}}{l_{01}} \right\rfloor + \left\lfloor \frac{d_{11}}{l_{11}} \right\rfloor + \left\lfloor \frac{d_{21}}{l_{21}} \right\rfloor + \left\lfloor \frac{d_{31}}{l_{31}} \right\rfloor \;\geq\; -2$$

  where $\lfloor d_{i1}/l_{i1} \rfloor < 0$ for $i = 1, 2, 3$. This implies $\lfloor d_{01}/l_{01} \rfloor \geq 1$ and in particular $d_{01} > 0$. On the other hand we know that $d_{01}/l_{01} < 2$, so equality has to be satisfied. Consider again the point $(-1, 1)$. This gives $d_{01}/l_{01} \leq 1$ and thus $d_{01} \leq 5$. Since $\lfloor d_{i1}/l_{i1} \rfloor \geq -1$ for $i = 1, 2, 3$ we obtain $d_{i1} > -5$.

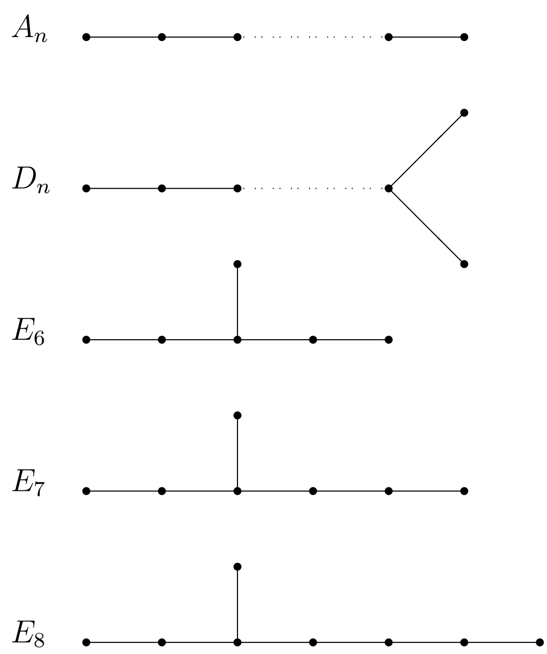
□

5.3 Classification results for Gorenstein log del Pezzo $\mathbb{K}^*$-surfaces

In this section we state the complete list of all non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surfaces by indicating their Cox rings and $\mathrm{Cl}(X)$-gradings. Furthermore, we list their P-matrices as well as their degree $(-K_X)^2$ and their singularity type. Note that Hendrik Süss classified them already up to Picard number two by other methods, see [51] and his dissertation.

By Corollary 5.5 we know that Gorenstein log del Pezzo $\mathbb{K}^*$-surfaces are canonical. In dimension two the canonical singularities are exactly the so called ADE-singularities which are defined as follows.

Definition 5.24. The *ADE-classification* is the complete list of *simply laced Dynkin diagrams*, which is the following:



We call a singularity an *ADE-singularity* if its resolution-curve is an *ADE-curve*. This means that its irreducible components are all (-2)-curves. Two of these components can intersect only once and they intersect transversally. The intersection graph of such a curve corresponds to one of the Dynkin-graphs above, where the points are the irreducible components. They share a common edge if and only if they intersect.

Now we will use the explicit bounds on the number of relations r and the parameters l_{ij} and d_{ij} that we found in the last section to generate the complete list of all non-toric Gorenstein del Pezzo $\mathbb{K}^*$surfaces. We will write them down by means of their Cox rings. Note that a $\mathbb{K}^*$-surface is uniquely determined by its Cox ring. Furthermore, the singularity type $\mathrm{S}(X)$, the degree $d_X = (-K_X)^2$ and the Picard index $\mu = [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ of the surfaces are specified.

Theorem 5.25. *Let X be a non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface of Picard number one. Then its Cox ring $R(X)$ is one of the graded rings in the following table.*

$\mathcal{R}(X)$	$(w_1, \ldots, w_r)$	$\mathrm{Cl}(X)$	$\mathrm{S}(X)$	d_X	μ
$\mathbb{K}[T_1, T_2, T_3, S_1] \ / \ \langle T_1^2 + T_2^2 + T_3^2 \rangle$	$\left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{smallmatrix}\right)$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$D_4 3A_1$	2	4
$\mathbb{K}[T_1, \ldots, T_4] \ / \ \langle T_1T_2 + T_3^2 + T_4^2 \rangle$	$\left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ \bar{1} & \bar{3} & \bar{2} & \bar{0} \end{smallmatrix}\right)$	$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	$2A_3A_1$	2	4
$\mathbb{K}[T_1, \ldots, T_4] \ / \ \langle T_1T_2 + T_3^2 + T_4^3 \rangle$	$(1\ 5\ 3\ 2)$	$\mathbb{Z}$	A_4	5	5
$\mathbb{K}[T_1, \ldots, T_4] \ / \ \langle T_1T_2 + T_3^2 + T_4^4 \rangle$	$\left(\begin{smallmatrix} 1 & 3 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{smallmatrix}\right)$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	A_5A_1	3	6
$\mathbb{K}[T_1, \ldots, T_4] \ / \ \langle T_1T_2 + T_3^3 + T_4^3 \rangle$	$\left(\begin{smallmatrix} 2 & 1 & 1 & 1 \\ \bar{2} & \bar{1} & \bar{2} & \bar{0} \end{smallmatrix}\right)$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	A_5A_2	2	6
$\mathbb{K}[T_1, \ldots, T_4] \ / \ \langle T_1^2T_2 + T_3^2 + T_4^3 \rangle$	$(1\ 4\ 3\ 2)$	$\mathbb{Z}$	D_5	4	4
$\mathbb{K}[T_1, \ldots, T_4] \ / \ \langle T_1^2T_2 + T_3^2 + T_4^4 \rangle$	$\left(\begin{smallmatrix} 1 & 2 & 2 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{smallmatrix}\right)$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	D_6A_1	2	4
$\mathbb{K}[T_1, \ldots, T_4] \ / \ \langle T_1^2T_2 + T_3^3 + T_4^3 \rangle$	$\left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{2} & \bar{0} \end{smallmatrix}\right)$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	E_6A_2	1	3
$\mathbb{K}[T_1, \ldots, T_4] \ / \ \langle T_1^3T_2 + T_3^2 + T_4^3 \rangle$	$(1\ 3\ 3\ 2)$	$\mathbb{Z}$	E_6	3	3
$\mathbb{K}[T_1, \ldots, T_4] \ / \ \langle T_1^3T_2 + T_3^2 + T_4^4 \rangle$	$\left(\begin{smallmatrix} 1 & 1 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{smallmatrix}\right)$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	E_7A_1	1	2
$\mathbb{K}[T_1, \ldots, T_4] \ / \ \langle T_1^4T_2 + T_3^2 + T_4^3 \rangle$	$(1\ 2\ 3\ 2)$	$\mathbb{Z}$	E_7	2	2
$\mathbb{K}[T_1, \ldots, T_4] \ / \ \langle T_1^5T_2 + T_3^2 + T_4^3 \rangle$	$(1\ 1\ 3\ 2)$	$\mathbb{Z}$	E_8	1	1
$\mathbb{K}[T_1, \ldots, T_5] \ / \ \left\langle \begin{smallmatrix} T_1T_2+T_3^2+T_4^2, \\ \lambda T_3^2+T_4^2+T_5^2 \end{smallmatrix} \right\rangle$	$\left(\begin{smallmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{smallmatrix}\right)$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$2D_4$	1	4

Possible P-matrices for the Gorenstein del Pezzo surfaces with Picard number one are

$$P_{D_4 3A_1} = \begin{pmatrix} -2 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -3 & 1 & 1 & 1 \end{pmatrix}, \quad P_{2A_3A_1} = \begin{pmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ -2 & 0 & 1 & 1 \end{pmatrix}, \quad P_{A_4} = \begin{pmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 3 \\ -2 & -1 & 1 & 1 \end{pmatrix},$$

$$P_{A_5A_1} = \begin{pmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 4 \\ -2 & -1 & 1 & 3 \end{pmatrix}, \quad P_{A_5A_2} = \begin{pmatrix} -1 & -1 & 3 & 0 \\ -1 & -1 & 0 & 3 \\ -1 & 0 & 1 & 1 \end{pmatrix}, \quad P_{D_5} = \begin{pmatrix} -2 & -1 & 2 & 0 \\ -2 & -1 & 0 & 3 \\ -3 & -1 & 1 & 2 \end{pmatrix},$$

$$P_{D_6A_1} = \begin{pmatrix} -2 & -1 & 2 & 0 \\ -2 & -1 & 0 & 4 \\ -3 & -1 & 1 & 3 \end{pmatrix}, \quad P_{E_6A_2} = \begin{pmatrix} -2 & -1 & 3 & 0 \\ -2 & -1 & 0 & 3 \\ -3 & -1 & 2 & 2 \end{pmatrix}, \quad P_{E_6} = \begin{pmatrix} -3 & -1 & 2 & 0 \\ -3 & -1 & 0 & 3 \\ -4 & -1 & 1 & 2 \end{pmatrix},$$

$$P_{E_7A_1} = \begin{pmatrix} -3 & -1 & 2 & 0 \\ -3 & -1 & 0 & 4 \\ -4 & -1 & 1 & 3 \end{pmatrix}, \quad P_{E_7} = \begin{pmatrix} -4 & -1 & 2 & 0 \\ -4 & -1 & 0 & 3 \\ -5 & -1 & 1 & 2 \end{pmatrix}, \quad P_{E_8} = \begin{pmatrix} -5 & -1 & 2 & 0 \\ -5 & -1 & 0 & 3 \\ -6 & -1 & 1 & 2 \end{pmatrix},$$

$$P_{2D_4} = \begin{pmatrix} -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -2 & -1 & 1 & 1 & 1 \end{pmatrix}.$$

Theorem 5.26. *Let X be a non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface of Picard number two. Then its Cox ring $R(X)$ is one of the graded rings in the following table.*

$\mathcal{R}(X)$	$(w_1, \dots, w_r)$	$\mathrm{Cl}(X)$	$\mathrm{S}(X)$	d_X
$\mathbb{K}[T_1,\dots,T_4,S_1] \;/\; \langle T_1T_2 + T_3^2 + T_4^2 \rangle$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{pmatrix}$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$	$A_3 2A_1$	3
$\mathbb{K}[T_1,\dots,T_5] \;/\; \langle T_1T_2 + T_3T_4 + T_5^2 \rangle$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & -2 & 0 \end{pmatrix}$	$\mathbb{Z}^2$	$2A_2A_1$	3
$\mathbb{K}[T_1,\dots,T_5] \;/\; \langle T_1T_2 + T_3T_4 + T_5^2 \rangle$	$\begin{pmatrix} 1 & 3 & 1 & 3 & 2 \\ 1 & 1 & 0 & 2 & 1 \end{pmatrix}$	$\mathbb{Z}^2$	A_2	6
$\mathbb{K}[T_1,\dots,T_5] \;/\; \langle T_1T_2 + T_3T_4 + T_5^3 \rangle$	$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix}$	$\mathbb{Z}^2$	A_1A_3	4
$\mathbb{K}[T_1,\dots,T_5] \;/\; \langle T_1T_2 + T_3^2T_4 + T_4^2 \rangle$	$\begin{pmatrix} 1 & 3 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 & 1 \end{pmatrix}$	$\mathbb{Z}^2$	A_3	5
$\mathbb{K}[T_1,\dots,T_5] \;/\; \langle T_1T_2 + T_3^2T_4 + T_5^3 \rangle$	$\begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ -1 & 1 & 1 & -2 & 0 \end{pmatrix}$	$\mathbb{Z}^2$	A_4A_1	3
$\mathbb{K}[T_1,\dots,T_5] \;/\; \langle T_1T_2 + T_3^3T_4 + T_5^2 \rangle$	$\begin{pmatrix} 1 & 3 & 1 & 1 & 2 \\ -1 & -1 & 0 & -2 & -1 \end{pmatrix}$	$\mathbb{Z}^2$	A_4	4
$\mathbb{K}[T_1,\dots,T_5] \;/\; \langle T_1^2T_2 + T_3^2T_4 + T_5^2 \rangle$	$\begin{pmatrix} 1 & 2 & 1 & 2 & 2 \\ -1 & 0 & 0 & -2 & -1 \end{pmatrix}$	$\mathbb{Z}^2$	D_4	4
$\mathbb{K}[T_1,\dots,T_5] \;/\; \langle T_1^2T_2 + T_3^2T_4 + T_5^3 \rangle$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 2 & 1 & -2 & 0 \end{pmatrix}$	$\mathbb{Z}^2$	D_5A_1	2
$\mathbb{K}[T_1,\dots,T_5] \;/\; \langle T_1^3T_2 + T_3^2T_4 + T_5^2 \rangle$	$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 \\ -1 & 1 & 0 & -2 & -1 \end{pmatrix}$	$\mathbb{Z}^2$	D_5	3
$\mathbb{K}[T_1,\dots,T_5] \;/\; \langle T_1^3T_2 + T_3^3T_4 + T_5^2 \rangle$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ -1 & 1 & 0 & -2 & -1 \end{pmatrix}$	$\mathbb{Z}^2$	E_6	2
$\mathbb{K}[T_1,\dots,T_6] \;/\; \left\langle \begin{smallmatrix} T_1T_2+T_3T_4+T_5^2, \\ \lambda T_3T_4+T_5^2+T_6^2 \end{smallmatrix} \right\rangle$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$	$2A_3$	2

Possible P-matrices for the Gorenstein del Pezzo surfaces with Picard number two are

$$P_{A_3 2A_1} = \begin{pmatrix} -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -2 & -1 & 1 & 1 & 1 \end{pmatrix}, \qquad P_{2A_2A_1} = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -1 & 1 & -1 & 0 & 1 \end{pmatrix},$$

$$P_{A_2} = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -1 & 0 & -1 & 0 & 1 \end{pmatrix}, \qquad P_{A_1A_3} = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 3 \\ -1 & 0 & -1 & 0 & 2 \end{pmatrix},$$

$$P_{A_3} = \begin{pmatrix} -1 & -1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -1 & 0 & -1 & 0 & 1 \end{pmatrix}, \qquad P_{A_4A_1} = \begin{pmatrix} -1 & -1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 & 3 \\ -1 & 0 & -1 & 0 & 2 \end{pmatrix},$$

$$P_{A_4} = \begin{pmatrix} -1 & -1 & 3 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -1 & 0 & -1 & 0 & 1 \end{pmatrix}, \qquad P_{D_4} = \begin{pmatrix} -2 & -1 & 2 & 1 & 0 \\ -2 & -1 & 0 & 0 & 2 \\ -1 & 0 & -1 & 0 & 1 \end{pmatrix},$$

$$P_{D_5A_1} = \begin{pmatrix} -2 & -1 & 2 & 1 & 0 \\ -2 & -1 & 0 & 0 & 3 \\ -1 & 0 & -1 & 0 & 2 \end{pmatrix}, \qquad P_{D_5} = \begin{pmatrix} -3 & -1 & 2 & 1 & 0 \\ -3 & -1 & 0 & 0 & 2 \\ -1 & 0 & -1 & 0 & 1 \end{pmatrix},$$

$$P_{E_6} = \begin{pmatrix} -3 & -1 & 3 & 1 & 0 \\ -3 & -1 & 0 & 0 & 2 \\ -1 & 0 & -1 & 0 & 1 \end{pmatrix}, \qquad P_{2A_3} = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \\ -1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}.$$

Theorem 5.27. *Let X be a non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface of Picard number three. Then its Cox ring $R(X)$ is one of the graded rings in the following table.*

$\mathcal{R}(X)$	$(w_1, \dots, w_r)$	$\mathrm{Cl}(X)$	$\mathrm{S}(X)$	d_X
$\mathbb{K}[T_1, \dots, T_5, S_1] \ / \ \langle T_1T_2 + T_3T_4 + T_5^2 \rangle$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}^3$	A_1A_2	4
$\mathbb{K}[T_1, \dots, T_6] \ / \ \langle T_1T_2 + T_3T_4 + T_5T_6 \rangle$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$	$\mathbb{Z}^3$	$3A_1$	4
$\mathbb{K}[T_1, \dots, T_6] \ / \ \langle T_1T_2 + T_3T_4 + T_5T_6 \rangle$	$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}^3$	A_1	6
$\mathbb{K}[T_1, \dots, T_6] \ / \ \langle T_1T_2 + T_3T_4 + T_5^2T_6 \rangle$	$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}^3$	A_2	5
$\mathbb{K}[T_1, \dots, T_6] \ / \ \langle T_1T_2 + T_3^2T_4 + T_5^2T_6 \rangle$	$\begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{pmatrix}$	$\mathbb{Z}^3$	A_3	4
$\mathbb{K}[T_1, \dots, T_6] \ / \ \langle T_1^2T_2 + T_3^2T_4 + T_5^2T_6 \rangle$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{pmatrix}$	$\mathbb{Z}^3$	D_4	3
$\mathbb{K}[T_1, \dots, T_7] \ / \ \left\langle \begin{smallmatrix} T_1T_2+T_3T_4+T_5T_6, \\ \lambda T_3T_4+T_5T_6+T_7^2 \end{smallmatrix} \right\rangle$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}^3$	$2A_2$	3

Possible P-matrices for the Gorenstein del Pezzo surfaces with Picard number three are

$$P_{A_1A_2} = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}, \qquad P_{3A_1} = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 & -1 & 0 \end{pmatrix},$$

$$P_{A_1} = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \end{pmatrix}, \qquad P_{A_2} = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \end{pmatrix},$$

$$P_{A_3} = \begin{pmatrix} -1 & -1 & 2 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \end{pmatrix}, \qquad P_{D_4} = \begin{pmatrix} -2 & -1 & 2 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 & 2 & 1 \\ 1 & 1 & -1 & 0 & -1 & 0 \end{pmatrix},$$

$$P_{2A_2} = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

Theorem 5.28. *Let X be a non-toric Gorenstein log del Pezzo $\mathbb{K}^*$-surface of Picard number four. Then its Cox ring $R(X)$ is one of the graded rings in the following table.*

$\mathcal{R}(X)$	$(w_1, \dots, w_r)$	$\mathrm{Cl}(X)$	$\mathrm{S}(X)$	d_X
$\mathbb{K}[T_1, \dots, T_6, S_1] \ / \ \langle T_1T_2 + T_3T_4 + T_5T_6 \rangle$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}^4$	A_1	5
$\mathbb{K}[T_1, \dots, T_8] \ / \ \left\langle \begin{smallmatrix} T_1T_2+T_3T_4+T_5T_6, \\ \lambda T_3T_4+T_5T_6+T_7T_8 \end{smallmatrix} \right\rangle$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}^4$	$2A_1$	4

Possible P-matrices for the Gorenstein del Pezzo surfaces with Picard number four are

$$P_{A_1} = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 1 \end{pmatrix}, \quad P_{2A_1} = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & -1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}.$$

Remark 5.29. Toric Gorenstein del Pezzo surfaces are completely classified by reflexive polytopes in $\mathbb{Q}^2$, i.e. lattice polytopes containing only 0 in their relative interior. Up to unimodular transformations, there are exactly 16 such polytopes, see [48].

Example 5.30. We will resolve exemplarily the D_4-surface $X = X(A, P)$ of Picard number two given by

$$P_{D_4} = \begin{pmatrix} -2 & -1 & 2 & 1 & 0 \\ -2 & -1 & 0 & 0 & 2 \\ -1 & 0 & -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad R(X) = \mathbb{K}[T_{ij}]/\langle T_{01}^2T_{02} + T_{11}^2T_{12} + T_{21}^2\rangle.$$

In order to obtain the weak tropicalisation we have two add rays along $(0, 0, \pm 1)$. Then we resolve locally the remaining (toric) singularities by adding rays along $(-1, -1, -1)$, $(1, 0, -1)$ and $(0, 1, 0)$. This resolution is not minimal since the ray $\mathbb{Q} \cdot (0, 0, 1)$ corresponds to the (-1)-curve F^+. Contracting this curve we obtain the minimal resolution $X' = X(A, P')$ with Cox ring $\mathbb{K}[T_{ij}, S_1]/\langle T_{01}^2T_{02}T_{03} + T_{11}^2T_{12}T_{13} + T_{21}^2T_{22}\rangle$ and P-matrix

$$P' = \begin{pmatrix} -2 & -1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & 0 & 0 & 0 & 2 & 1 & 0 \\ -1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & -1 \end{pmatrix}.$$

Remark 5.31. Consider the two D_4-surfaces with Picard number two and three. By reordering the variables and applying admissible transformations we obtain the Cox rings

$$\begin{aligned} R(X_1) &= \mathbb{K}[T_1, \dots, T_6]/\langle T_1T_2^2 + T_3T_4^2 + T_5T_6^2\rangle, \\ R(X_2) &= \mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2^2 + T_3T_4^2 + T_5^2\rangle, \end{aligned}$$

and the following two P-matrices:

$$P_1 = \begin{pmatrix} -1 & -2 & 1 & 2 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 2 \\ -1 & -1 & 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} -1 & -2 & 1 & 2 & 0 \\ -1 & -2 & 0 & 0 & 2 \\ -1 & -1 & 0 & 1 & 1 \end{pmatrix}.$$

Then we obtain X_2 out of X_1 by contracting the column v_{21}:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

In particular, the Fano condition is respected by this modification. Note that we can also find such representations in case of the A_1, A_2, A_3 and E_6-surfaces which occur with different Picard numbers.

We conclude this chapter with two infinite series of log-terminal del Pezzo $\mathbb{K}^*$-surfaces with arbitrary high Gorenstein index and Picard number.

Example 5.32. Let $m = 0$ and $r \geq 2$ as well as $n_i = 2$ and $l_{ij} = 1$ for $0 \leq i \leq r$, $1 \leq j \leq n_i$, and consider the following $((r+1) \times 2(r+1))$-matrix:

$$P = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 & \dots & & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & & 0 & 0 \\ -1 & -1 & 0 & 0 & \dots & & 0 & 0 & 1 & 1 \\ 0 & r & -1 & 0 & \dots & & -1 & 0 & -1 & 0 \end{pmatrix}.$$

This matrix defines a log-terminal $\mathbb{K}^*$-surface X with two elliptic fixed points whose Cox ring has $n = 2(r+1)$ variables and $r-1$ relations $g_0, \dots, g_{r-2}$, where $g_i = *T_{i1}T_{i2} + *T_{i+1,1}T_{i+1,2} + *T_{i+2,1}T_{i+2,2}$. Consequently, we have

$$\mathrm{rk}(\mathrm{Cl}(X)) \; = \; n - (r-1) - 2 \; = \; 2(r+1) - (r-1) - 2 \; = \; r+1.$$

The anticanonical complex A_X^c is given by the intersection $\mathrm{trop}(X) \cap A_X$, where

$$A_X = \mathrm{conv}(v_{i1}, v_{i2}, v_c^+, v_c^-;\; 0 \leq i \leq r) \quad \text{and} \quad v_c^{\pm} = \left(0, \dots, 0, \pm\frac{r}{2}\right).$$

In particular, all vectors v_{i1}, v_{i2} are vertices of A_X^c, which implies that X is Fano. The surface X has three singularities corresponding to the two elementary big cones $\sigma^{\pm}$ and $\tau = (v_{01}, v_{02})$. By Proposition 5.2, we know that X has Gorenstein index $\iota(X) = a$ if and only if r divides $a \cdot 2$. Hence we have $\iota(X) = r$ if r is odd and $\iota(X) = r/2$ if r is even. And indeed for $r = 2$ we obtain the only possible Gorenstein surface of that type, compare Theorem 5.27.

Example 5.33. Let $m = 1$ and $r \geq 2$ as well as $n_i = 2$ and $l_{ij} = 1$ for $0 \leq i \leq r$, $1 \leq j \leq n_i$, and consider the following $((r+1) \times (2(r+1)+1))$-matrix:

$$P = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 & \dots & & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & \dots & & 0 & 0 & 1 & 1 & 0 \\ 0 & r-1 & -1 & 0 & \dots & & -1 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

This matrix defines a log-terminal $\mathbb{K}^*$-surface X with one elliptic fixed point and one parabolic fixed point curve whose Cox ring has $n + m = 2(r+1) + 1$ variables and $r - 1$ relations $g_0, \dots, g_{r-2}$, where $g_i = *T_{i1}T_{i2} + *T_{i+1,1}T_{i+1,2} + *T_{i+2,1}T_{i+2,2}$. Consequently, we have

$$\mathrm{rk}(\mathrm{Cl}(X)) \; = \; n + m - (r-1) - 2 \; = \; 2(r+1) + 1 - (r-1) - 2 \; = \; r+2.$$

The anticanonical complex A_X^c is given by the intersection $\mathrm{trop}(X) \cap A_X$, where

$$A_X = \mathrm{conv}(v_{i1}, v_{i2}, v^+, v_c^-;\ 0 \le i \le r) \quad \text{and} \quad v_c^- = (0, \dots, 0, -r).$$

In particular, all vectors v_{i1}, v_{i2} are vertices of A_X^c, which implies that X is Fano. By Proposition 5.2, we know that X has Gorenstein index $\iota(X) = a$ if and only if r divides $a \cdot 2$. Hence we have $\iota(X) = r$ if r is odd and $\iota(X) = r/2$ if r is even. And indeed for $r = 2$ we obtain the only possible Gorenstein surface of that type, compare Theorem 5.28.

6 Complexity-one Fano T-varieties with Picard number one

In this chapter we provide effective bounds and classification results for rational $\mathbb{Q}$-factorial Fano varieties with a complexity-one torus action and Picard number one depending on the invariants dimension and Picard index. The results of this chapter are already published in [28, Sections 2 and 3] and [30].

6.1 Divisor class group and Picard group

Let X be a variety with a complexity-one torus action and let $\mathrm{Pic}(X)$ be its Picard group. The *Picard index* of X is defined as the index $\mu := [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ of the Picard group in the divisor class group $\mathrm{Cl}(X)$ of X. Furthermore, the *Picard number* denotes the rank of the Picard group Pic(X). Note that in case of $\mathbb{Q}$-factorial (rational) varieties this is always the rank of the divisor class group $\mathrm{Cl}(X)$.

For this chapter we assume X to have Picard number one. Then the divisor class group is of the form

$$\mathrm{Cl}(X) \cong \mathrm{Cl}(X)^0 \oplus \mathrm{Cl}(X)^t \cong \mathbb{Z} \oplus \mathrm{Cl}(X)^t,$$

where $\mathrm{Cl}(X)^t$ denotes the torsion part of $\mathrm{Cl}(X)$ and $\mathrm{Cl}(X)^0 = \mathrm{Cl}(X)/\mathrm{Cl}(X)^t$. We briefly recall the situation of the constructions 2.10 and 2.20 in this special case. There is a matrix P and a sequence A satisfying all assumptions of 2.10 such that $\mathrm{Cl}(X) \cong \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$ and the positively $\mathrm{Cl}(X)$-graded ring $R(A,P) = \mathbb{K}[T_{ij}, S_k]/\langle g_i;\ 0 \le i \le r-2\rangle$ is the Cox ring of X. The grading of $R(A,P)$ is given by

$$\deg(T_{ij}) =: w_{ij}, \quad 0 \le i \le r, 1 \le j \le n_i, \qquad \deg(S_k) =: u_k, \quad 1 \le k \le m,$$

where $w_{ij},\ u_k \in \mathbb{Z}_{>0} \oplus \mathrm{Cl}(X)^t$. Note that any $n+m-1$ of these degrees generate $\mathrm{Cl}(X)$ as a group. The $\mathrm{Cl}(X)$-grading defines a diagonal action of $H := \mathrm{Spec}\ \mathbb{K}[K]$ on $\mathbb{K}^{n+m}$. By construction

$$\overline{X} \ := \ V(g_i; 0 \le i \le r-2) \ = \ \mathrm{Spec}\, R(A,P)$$

is invariant under this H-action. The open set $\mathbb{K}^{n+m} \setminus \{0\}$ allows a geometric quotient of this H-action which is denoted by $p\colon \mathbb{K}^{n+m} \setminus \{0\} \to Z'$, where the toric variety Z' is a fake weighted projective space in the sense of [35]. In the special case of Picard number one, each $\mathfrak{F}$-face is relevant. Hence, we have a geometric quotient $p\colon \widehat{X} \to X$ of the embedded open subset $\widehat{X} := \overline{X} \setminus \{0\}$ on X.

$$\begin{array}{ccc} \widehat{X} & \hookrightarrow & \mathbb{K}^{n+m} \setminus \{0\} \\ \downarrow p & & \downarrow p \\ X & \hookrightarrow & Z' \end{array}$$

In this setting X has dimension $\dim(X) = n+m-r$ and the torus of X is given by the stabilizer of X under the action of the maximal torus T' of Z'.

Furthermore, X is uniquely determined by its Cox ring (as $\mathrm{Cl}(X)$-graded ring), since $\Phi = \{\mathbb{Q}_{\geq 0}\}$ is the only possible $\mathfrak{F}$-bunch. This justifies the notion $X = X(A, P)$.

As already mentioned, in the toric situation, these varieties correspond to the fake weighted projective spaces as defined in [35] and the Cox ring is polynomial. In general, X is a well-formed complete intersection in a fake weighted projective space. If the divisor class group $\mathrm{Cl}(X)$ is torsion free then X is a well-formed complete intersection in a weighted projective space in the sense of [31].

Every element $w \in \mathrm{Cl}(X)$ can be written as $w = w^0 + w^t$ where $w^0 \in \mathbb{Z}$ and $w^t \in \mathrm{Cl}(X)^t$. Furthermore, every $\overline{x} = (\overline{x}_{ij}, \overline{x}_k) \in \widehat{X} \subseteq \mathbb{K}^{n+m}$ defines a point $x \in X$ by $x := p(\overline{x})$; the points $\overline{x} \in \widehat{X}$ are called *Cox coordinates* of x. We denote the set of all weights corresponding to a non-zero coordinate of $\overline{x}$ by

$$W_{\overline{x}} := \{w_{ij};\ \overline{x}_{ij} \neq 0\} \cup \{u_k;\ \overline{x}_k \neq 0\}.$$

Proposition 6.1. *Let $X = X(A, P)$ be a $\mathbb{Q}$-factorial complete normal variety of dimension d with complexity-one torus action and Picard number one and set $\gamma_i := \deg(g_i)$, $0 \leq i \leq r$. Then the following statements hold:*

(i) For any $\overline{x} \in \widehat{X}$, the local divisor class group $\mathrm{Cl}(X, x)$ of $x := p(\overline{x})$ is finite and $\gcd(w^0;\ w \in W_{\overline{x}})$ always divides the order of this group. If $\mathrm{Cl}(X) \cong \mathbb{Z}$, then $|\mathrm{Cl}(X, x)| = \gcd(w; w \in W_{\overline{x}})$ holds.

(ii) The Picard group $\mathrm{Pic}(X)$ is free and the Picard index is given by

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \quad = \quad \mathrm{lcm}_{x \in X}(\gcd(w^0;\ w \in W_{\overline{x}})) \cdot |\mathrm{Cl}(X)^t|.$$

In particular $|\mathrm{Cl}(X)^t|$ is a divisor of $[\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ and we have $|\mathrm{Cl}(X)^t| \leq [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$.

(iii) Let $-K_X \in \mathrm{Cl}(X)$ be the anticanonical class and $d_X := (-K_X)^d$ its self-intersection number. Then

$$-K_X = \sum_{i=0}^{r}\sum_{j=1}^{n_i} w_{ij} + \sum_{k=1}^{m} u_k - \sum_{i=0}^{r-2} \gamma_i,$$

$$d_X = \left(\sum_{i=0}^{r}\sum_{j=1}^{n_i} w_{ij}^0 + \sum_{k=1}^{m} u_k^0 - \sum_{i=0}^{r-2} \gamma_i^0\right)^d \frac{\gamma_0^0 \cdots \gamma_{r-2}^0}{\prod_{i=0}^{r}\prod_{j=1}^{n_i} w_{ij}^0 \prod_{k=1}^{m} u_k^0 \cdot |\mathrm{Cl}(X)^t|}.$$

(iv) The variety X is Fano if and only if the following inequality holds:

$$(r-1)\deg(g_0)^0 \;=\; \sum_{i=0}^{r-2} \deg(g_i)^0 \;<\; \sum_{i=0}^{r}\sum_{j=1}^{n_i} w_{ij}^0 + \sum_{i=1}^{m} u_k^0.$$

Proof. Let $\overline{x}(i,j)$ resp. $\overline{x}(k)$ be a point in $\widehat{X}$, where the ij-th resp. $(n+k)$-th entry equals 1 and all others are 0. Consider the minimal toric ambient variety Z satisfying $\mathrm{Cl}(X) \cong \mathrm{Cl}(Z)$ and $\mathrm{Pic}(X) \cong \mathrm{Pic}(Z)$. By choice $\overline{x}(i,j)$ resp. $\overline{x}(k)$ is a toric fixed point which is equivalent to the existence of a full dimensional cone in the fan Σ_Z. Consequently, by [21, Theorem VII. 2.16] the Picard group $\mathrm{Pic}(Z)$ is free, and so is $\mathrm{Pic}(X)$. According to 1.17, see also [25, Corollary 4.9], we obtain

$$\mathrm{Pic}(X) \;=\; \bigcap_{\overline{x}\in\widehat{X}} \langle w;\ w \in W_{\overline{x}}\rangle \;\cong\; \bigcap_{\overline{x}\in\widehat{X}} \langle w^0;\ w \in W_{\overline{x}}\rangle,$$

where the last isomorphy follows from the fact that $\mathrm{Pic}(X)$ is free. This proves assertions (i) and (ii). The formula for $-K_X$ as well as statement (iv) are special cases of [25, Proposition 4.15 and Corollary 4.16]. Since X is embedded into a toric variety Z of dimension $n+m-1$ with $\mathrm{Cl}(Z) \cong \mathrm{Cl}(X)$ we will use toric intersection theory to determine $(-K_X)^d = (-K_X)^{n+m-r}$. Fixing a pair (s,t) with $0 \le s \le r$, $1 \le t \le n_s$ we first compute the self-intersection number of the $\mathbb{1}$-class. By [5, Construction III 3.3.4] we obtain

$$\begin{aligned}\prod_{(i,j)\neq(s,t)} w_{ij} \prod_{k=1}^{m} u_k \;&=\; \frac{1}{[\mathrm{Cl}(Z) : \langle w_{st}\rangle]}\\ &=\; \frac{1}{w_{st}^0 \cdot |\mathrm{Cl}(Z)^t|}\\ &=\; \prod_{(i,j)\neq(s,t)} w_{ij}^0 \prod_{k=1}^{m} u_k^0 \cdot \mathbb{1}^{n+m-1}.\end{aligned}$$

This implies

$$\mathbb{1}^{n+m-1} \;=\; \frac{1}{\prod_{i=0}^{r}\prod_{j=1}^{n_i} w_{ij}^0 \prod_{k=1}^{m} u_k^0 \cdot |\mathrm{Cl}(Z)^t|}.$$

With this result we can compute $(-K_X)^{n+m-r}$ by using again toric intersection theory.

$$\begin{aligned}(-K_X)^{n+m-r} \;&=\; \left(\sum_{i=0}^{r}\sum_{j=1}^{n_i} w_{ij} + \sum_{k=1}^{m} u_k - \sum_{i=0}^{r-2} \gamma_i\right)^{n+m-r} \cdot \gamma_0 \cdots \gamma_{r-2}\\ &=\; \left(\sum_{i=0}^{r}\sum_{j=1}^{n_i} w_{ij}^0 + \sum_{k=1}^{m} u_k^0 - \sum_{i=0}^{r-2} \gamma_i^0\right)^{n+m-r} \cdot \gamma_0^0 \cdots \gamma_{r-2}^0 \cdot \mathbb{1}^{n+m-1}\\ &=\; \left(\sum_{i=0}^{r}\sum_{j=1}^{n_i} w_{ij}^0 + \sum_{k=1}^{m} u_k^0 - \sum_{i=0}^{r-2} \gamma_i^0\right)^{d} \frac{\gamma_0^0 \cdots \gamma_{r-2}^0}{\prod_{i=0}^{r}\prod_{j=1}^{n_i} w_{ij}^0 \cdot \prod_{k=1}^{m} u_k^0 \cdot |\mathrm{Cl}(Z)^t|}\end{aligned}$$

□

Corollary 6.2. *Let X be a $\mathbb{Q}$-factorial complete normal variety with complexity-one torus action and Picard number one. If X is locally factorial, then the divisor class group* $\mathrm{Cl}(X)$ *is free.*

The following example shows that one can use Proposition 6.1(iv) to create series of Fano varieties by altering the torsion part of the divisor class group $\mathrm{Cl}(X)$:

Example 6.3. Set $l_{01} = 7$, $l_{02} = 1$, $l_{11} = 5$ and $l_{21} = 2$ as well as $w_{01}^0 = 1$, $w_{02}^0 = 3$, $w_{11}^0 = 2$ and $w_{21}^0 = 5$. These data define one single Cox ring relation of the form $g_0 = T_{01}^7 T_{02} + T_{11}^5 + T_{21}^2$. Since we have

$$w_{01}^0 + w_{02}^0 + w_{11}^0 + w_{21}^0 \;=\; 11 \;>\; 10 \;=\; \deg(g_0)^0,$$

one can use these data to create Cox rings of Fano varieties. We provide some possible $\mathrm{Cl}(X)$-gradings, given by the matrices Q_i and the associated P-matrices P_i, defining del Pezzo $\mathbb{K}^*$-surfaces with fixed grading in the free part of the divisor class group and varying torsion part of the class group $\mathrm{Cl}(X)^t$:

$$Q_1 \;=\; \begin{pmatrix} 1 & 3 & 2 & 5 \end{pmatrix}, \qquad P_1 \;=\; \begin{pmatrix} -7 & -1 & 5 & 0 \\ -7 & -1 & 0 & 2 \\ -8 & -1 & 3 & 1 \end{pmatrix}, \qquad \mathrm{Cl}(X_1) = \mathbb{Z};$$

$$Q_2 \;=\; \begin{pmatrix} 1 & 3 & 2 & 5 \\ \overline{0} & \overline{2} & \overline{1} & \overline{1} \end{pmatrix}, \qquad P_2 \;=\; \begin{pmatrix} -7 & -1 & 5 & 0 \\ -7 & -1 & 0 & 2 \\ -10 & -1 & 4 & 1 \end{pmatrix}, \qquad \mathrm{Cl}(X_2) = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z};$$

$$Q_3 \;=\; \begin{pmatrix} 1 & 3 & 2 & 5 \\ \overline{2} & \overline{1} & \overline{3} & \overline{3} \end{pmatrix}, \qquad P_3 \;=\; \begin{pmatrix} -7 & -1 & 5 & 0 \\ -7 & -1 & 0 & 2 \\ -9 & 0 & 2 & 1 \end{pmatrix}, \qquad \mathrm{Cl}(X_3) = \mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z};$$

$$Q_4 \;=\; \begin{pmatrix} 1 & 3 & 2 & 5 \\ \overline{0} & \overline{1} & \overline{9} & \overline{6} \end{pmatrix}, \qquad P_4 \;=\; \begin{pmatrix} -7 & -1 & 5 & 0 \\ -7 & -1 & 0 & 2 \\ -11 & 0 & 3 & 1 \end{pmatrix}, \qquad \mathrm{Cl}(X_4) = \mathbb{Z} \oplus \mathbb{Z}/11\mathbb{Z};$$

$$Q_5 \;=\; \begin{pmatrix} 1 & 3 & 2 & 5 \\ \overline{0} & \overline{3} & \overline{11} & \overline{8} \end{pmatrix}, \qquad P_5 \;=\; \begin{pmatrix} -7 & -1 & 5 & 0 \\ -7 & -1 & 0 & 2 \\ -13 & 0 & 4 & 1 \end{pmatrix}, \qquad \mathrm{Cl}(X_5) = \mathbb{Z} \oplus \mathbb{Z}/13\mathbb{Z};$$

$$Q_6 \;=\; \begin{pmatrix} 1 & 3 & 2 & 5 \\ \overline{0} & \overline{7} & \overline{15} & \overline{12} \end{pmatrix}, \qquad P_6 \;=\; \begin{pmatrix} -7 & -1 & 5 & 0 \\ -7 & -1 & 0 & 2 \\ -4 & -3 & 4 & 1 \end{pmatrix}, \qquad \mathrm{Cl}(X_6) = \mathbb{Z} \oplus \mathbb{Z}/17\mathbb{Z}.$$

Note that in this situation not every group of the form $\mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$, $k \in \mathbb{N}_{>0}$, can be realized as a divisor class group.

In Example 6.3 the numbers $\ell_i := \gcd(l_{i1}, \ldots, l_{in_i})$ are pairwise coprime, namely $\ell_0 = 1$, $\ell_1 = 2$ and $\ell_2 = 5$. This in an essential requirement for varieties with a free divisor class group $\mathrm{Cl}(X)$. Example 6.3 also shows that this is not sufficient to ensure that $\mathrm{Cl}(X)$ has no torsion. If the numbers ℓ_i are not pairwise coprime, then there is always non trivial torsion in the divisor class group as the following lemma shows, whereas the reversed implication does not hold.

Lemma 6.4. *Set* $\ell_i := \gcd(l_{i1}, \ldots, l_{in_i})$. *Then all numbers* $\gcd(\ell_i, \ell_j)$, *where* $0 \le i \ne j \le r$, *divide* $|\mathrm{Cl}(X)^t|$ *and the Picard index* μ. *In particular this holds for* $\mathrm{lcm}_{j \ne i}(\gcd(\ell_i, \ell_j))$.

Proof. The divisor class group $\mathrm{Cl}(X)$ is isomorphic to $\mathbb{Z}^{n+m}/\mathrm{im}(P^*)$ where P^* is dual to $P\colon \mathbb{Z}^{n+m} \to \mathbb{Z}^{n+m-1}$ given by a matrix of the form

$$P = \begin{pmatrix} -l_0 & l_1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ -l_0 & 0 & \dots & l_r & 0 \\ d_0 & d_1 & \dots & d_r & d' \end{pmatrix},$$

with $l_i = (l_{i0}, \dots, l_{in_i})$ and some integral block matrices d_i and d'. Consequently, $|\mathrm{Cl}(X)^t|$ is the product of all elementary divisors of P which implies that $\gcd(\ell_0, \ell_j)$ divides $|\mathrm{Cl}(X)^t|$. By an elementary row transformation we obtain the analogous result for $\gcd(\ell_i, \ell_j)$, where $0 \le i, j \le r$, $i \ne j$. Since $|\mathrm{Cl}(X)^t|$ divides the Picard index μ, the assertion follows. □

Corollary 6.5. *Let $\mathrm{Cl}(X)$ be free. Then the numbers $\ell_i = \gcd(l_{i1}, \dots, l_{in_i})$ are coprime.*

Example 6.6. Consider the surface X with Cox ring $R(X) = \mathbb{K}[T_{01}, T_{01}, T_{11}, T_{12}]/\langle g\rangle$ where $g = T_{01}^2 T_{02} + T_{11}^3 + T_{21}^3$ (surface number 7 of 6.18). Then we have $\mu = 3$ and the P-matrix as well as the grading matrix are given by

$$P = \begin{pmatrix} -2 & -1 & 3 & 0 \\ -2 & -1 & 0 & 3 \\ -3 & -1 & 2 & 2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{2} & \bar{0} \end{pmatrix},$$

with $\mathrm{Cl}(X) = \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. For $x \in X_\tau$ with $\tau = \mathrm{cone}(v_{01}, v_{02})$ we have a trivial local class group. For $x \in X_{\sigma^\pm}$ with $\sigma^- = \mathrm{cone}(v_{01}, v_{11}, v_{21})$ or $\sigma^+ = \mathrm{cone}(v_{02}, v_{11}, v_{21})$, we have $\mathrm{Cl}(X, x) \cong \mathbb{Z}/3\mathbb{Z}$.

Remark 6.7. One can even prove that $\mathrm{lcm}_{0\le j\le r}(\prod_{i\ne j} \gcd(\ell_i, \ell_j))$ divides $|\mathrm{Cl}(X)^t|$. This is due to the fact that the product of the first n elementary divisors of P equals the greatest common divisor of all $(n \times n)$-minors of P. Consider for example the surface X with Cox ring $\mathbb{K}[T_{01}, T_{11}, T_{21}, S_1]/\langle g\rangle$ and $g = T_{01}^2 + T_{11}^2 + T_{21}^2$ (surface number 10 in 6.18). In this case we have $\mu = 4$ and the P-matrix as well as the grading matrix are given by

$$P = \begin{pmatrix} -2 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -3 & 1 & 1 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$$

and we obtain $\mathrm{Cl}(X, x) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if $x \in X_{\sigma^-}$ with $\sigma^- = \mathrm{cone}(v_{01}, v_{11}, v_{21})$. Note that $\mathrm{Cl}(X) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In particular, the torsion part of the divisor class group is not cyclic.

6.2 Effective bounds

Since normal complete rational Fano varieties with a complexity-one torus action are uniquely determined by their Cox rings, one can classify these varieties via their Cox rings. In this section we will state effective bounds on the parameters l_{ij}, w^0_{ij}, u^0_k and r for the special case of Picard number one, depending on the invariants dimension and Picard index. These can be used to compute concrete classification lists, which is done in the next section.

Toric varieties with Picard number one correspond to fake weighted projective spaces as defined in [35] and the Cox ring is polynomial. In case of a free divisor class group one gets the well known weighted projective spaces. Particularly, these toric varieties are all Fano. Hence, we concentrate on non-toric (Fano) varieties with complexity-one torus action and Picard number one and show that for fixed dimension d and fixed Picard index μ there are only finitely many possibilities for the corresponding Cox ring.

First we consider the case $n_0 = \ldots = n_r = 1$, that means each relation g_i of the Cox ring $\mathcal{R}(X)$ depends only on three variables. Then we have $n = r + 1$ and consequently $m = d - 1$. Furthermore, we may write T_i instead of T_{i1} and w_i instead of w_{i1}. In this setting, we obtain the following bounds for the numbers of possible varieties X (Fano or not).

Proposition 6.8. *For any pair $(d, \mu) \in \mathbb{Z}^2_{>0}$ there is, up to deformation equivalence, only a finite number of complete d-dimensional varieties with Picard number one, Picard index $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = \mu$ and Cox ring of the form*

$$\mathbb{K}[T_0, \ldots, T_r, S_1, \ldots, S_m] \ / \ \langle \alpha_i T_i^{l_i} + \alpha_{i+1} T_{i+1}^{l_{i+1}} + \alpha_{i+2} T_{i+2}^{l_{i+2}};\ 0 \le i \le r-2 \rangle.$$

In this situation we have $r < \mu + \xi(\mu) - 1$ where $\xi(\mu)$ denotes the number of primes smaller than μ. Moreover, for $w^0_i \in \mathbb{Z}_{>0}$ and $u^0_k \in \mathbb{Z}_{>0}$, where $0 \le i \le r$, $1 \le k \le m$, and the exponents l_i one has

$$l_i \le \mu, \qquad w^0_i \ \le \ \mu^r, \qquad u^0_k \ \le \ \mu.$$

Proof. Consider the total coordinate space $\overline{X} \subseteq \mathbb{K}^{r+1+m}$ and the quotient $p\colon \widehat{X} \to X$ as well as the points $\overline{x}(k) \in \widehat{X}$ having the $(r + k)$-th coordinate one and all others zero. Set $x(k) := p(\overline{x}(k))$. Then u^0_k divides the order of the local class group $\mathrm{Cl}(X, x(k))$. In particular, we have $u^0_k \le \mu$.
For each $0 \le i \le r$, fix a point $\overline{y}(i) = (\overline{y}_0, \ldots, \overline{y}_r, 0, \ldots, 0)$ in $\widehat{X}$ such that $\overline{y}_i = 0$ and $\overline{y}_j \neq 0$ for $i \neq j$, and set $y_i := p(\overline{y}(i))$. Then we obtain

$$\gcd(w^0_j, j \neq i) \ \Big| \ |\mathrm{Cl}(X, y(i))|.$$

By Lemma 6.4 we have $\mathrm{lcm}_{j \neq i}(\gcd(l_i, l_j)) \mid |\mathrm{Cl}(X)^t|$. Now consider l'_i such that $l_i = \mathrm{lcm}_{j \neq i}(\gcd(l_i, l_j)) \cdot l'_i$. Then the homogeneity condition $l_i w^0_i = l_j w^0_j$ gives $l'_i \mid w^0_j$ for all $j \neq i$ and consequently $l'_i \mid \gcd(w^0_j, j \neq i)$. Since $l_i = l'_i \cdot \mathrm{lcm}_{j \neq i}(\gcd(l_i, l_j))$ we can conclude $l_i \le \mu$ by using the formula

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \ = \ \mathrm{lcm}_{x \in X}(\gcd(w^0;\ w \in W_{\overline{x}})) \cdot |\mathrm{Cl}(X)^t|$$

of Proposition 6.1(ii). Since the l_i' are pairwise coprime, we obtain $l_0' \cdots l_r' \mid \gamma^0$ and $l_0' \cdots l_r' \mid \mu$, where $\gamma^0 := \deg(g_0)^0 = l_i w_i^0$. From $l_i w_i^0 = l_j w_j^0$ we deduce that

$$l_i = l_0 \frac{w_0^0}{w_i^0} = l_0 \frac{w_0^0 \cdots w_{i-1}^0}{w_1^0 \cdots w_i^0} = \eta_i \cdot \frac{\gcd(w_0^0, \ldots, w_{i-1}^0)}{\gcd(w_0^0, \ldots, w_i^0)} \leq \mu,$$

where $1 \leq \eta_i \leq \mu$. In particular, the last fraction is smaller than μ. All in all this gives

$$\begin{aligned} w_0^0 &= \frac{w_0^0}{\gcd(w_0^0, w_1^0)} \cdot \frac{\gcd(w_0^0, w_1^0)}{\gcd(w_0^0, w_1^0, w_2^0)} \cdot \ldots \cdot \frac{\gcd(w_0^0, \ldots, w_{r-2}^0)}{\gcd(w_0^0, \ldots, w_{r-1}^0)} \cdot \gcd(w_0^0, \ldots, w_{r-1}^0) \\ &\leq \mu^{r-1} \cdot \mu = \mu^r. \end{aligned}$$

Analogously, we obtain the boundedness for all w_i^0. Now let q be the number of all l_i' being greater than one. Since all l_i', $0 \leq i \leq r$, are coprime, q is bounded by $\xi(\mu)$, i.e. the number of primes smaller than μ. To avoid the toric case we assume $l_i \neq 1$ for all $0 \leq i \leq r$. Consequently, if $l_i' = 1$, then there is at least one $0 \leq j \leq r$ such that $\gcd(l_i, l_j) > 1$. Since $\gcd(l_i, l_j)$ divides μ, we get $r + 1 - q < \mu$ as a rough bound. All in all obtain get $r + 1 \; = \; r + 1 - q + q \; < \; \mu + \xi(\mu)$. ☐

Let X be a normal complete rational variety coming with a complexity-one torus action of T. Consider the T-invariant open subset X_0 consisting of all points $x \in X$ having finite isotropy group. According to [50, Corollary 3] there is a geometric quotient $q\colon X_0 \to X_0/T$ such that X_0/T is irreducible and normal, but possibly not separated. The property of the orbit space X_0/T being separated is reflected in the Cox ring relations by the condition that each monomial depends on only one variable, e.g. surface number 3 in Theorem 6.18; see [29, Theorem 1.2]. Geometrically, this means that every orbit is contained in the closure of either exactly one maximal orbit or of infinitely many maximal orbits. For such varieties we have the following general finiteness statement:

Theorem 6.9. *The number of d-dimensional normal complete rational varieties of Picard number one with a complexity-one torus action of T and Picard index μ such that X_0/T is separated is finite.*

Proof. Let X be a variety as required in the assertion. Then each monomial of the Cox ring relations depends on only one variable, i.e. $n_i = 1$ for $0 \leq i \leq r$; for details see [29, Theorem 1.2]. Consequently, Proposition 6.8 provides bounds for the discrete data such as the non torsion parts of the weights w_{ij}^0 and u_k^0, the exponents l_{ij} and the number of Cox ring relations r. Since $|\mathrm{Cl}(X)^t| \leq \mu$ holds, the number of possibilities for the torsion part of the grading is also restricted which implies the assertion. ☐

Theorem 6.10. *Let $X = X(A, P)$ be a complexity-one Fano variety with Picard number one. Fix the dimension $d = \dim(X) = m + n + r$ and the Picard index $\mu = [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$. Then the number of Cox ring relations r, the free part of the degree of the relations γ^0, the weights w_{ij}^0, u_k^0 and the exponents l_{ij}^0, where $0 \leq i \leq r$, $1 \leq j \leq n_i$ and $1 \leq k \leq m$, are bounded. In particular, the following effective bounds hold:*

$$u_k^0 \;\leq\; \mu \quad \text{for } 1 \leq k \leq m \quad \text{and} \quad |\mathrm{Cl}(X)^t| \leq \mu.$$

Moreover, the handling of the remaining data can be organized in five cases, where $\xi(x)$ denotes the number of primes smaller than x.

(i) *Let $r = 0$ or $r = 1$. Then $n + m \le d + 1$ holds and one has the bounds*

$$w_{ij}^0 \;\le\; \mu \quad \text{for } 0 \le i \le r \text{ and } 1 \le j \le n_i.$$

The Picard index is given by

$$\mu \;=\; \operatorname{lcm}(w_{ij}^0, u_k^0;\ 0 \le i \le r, 1 \le j \le n_i, 1 \le k \le m) \cdot |\mathrm{Cl}(X)^t|.$$

(ii) *Let $r \ge 2$ and $n_0 = 1$. Then $r \le \mu + \xi(\mu) - 1$, $n = r + 1$ and $m = d - 1$ hold and one has*

$$w_{i1}^0 \;\le\; \mu^r, \quad l_{i1} \;|\; \mu \qquad \text{for } 0 \le i \le r, \qquad \gamma^0 \;\le\; \mu^{r+1}.$$

The Picard index is given by

$$\mu \;=\; \operatorname{lcm}(\gcd_i(w_{j1}^0;\ i \ne j), u_k^0;\ 0 \le i \le r, 1 \le k \le m) \cdot |\mathrm{Cl}(X)^t|.$$

(iii) *Let $r \ge 2$, $n_0 > n_1 = 1$ and $l_{11} \ge \ldots \ge l_{r1} \ge 2$. Then $r \le \mu + \xi(6d\mu) - 1$ and $n_0 + m = d$ hold and one has the bounds*

$$w_{01}^0, \ldots, w_{0n_0}^0 \;\le\; \mu, \qquad l_{01}, \ldots, l_{0n_0} \;\le\; 6d\mu, \qquad \gamma^0 \;< 6d\mu,$$
$$w_{11}^0 \;<\; 2d\mu, \quad w_{21}^0 \;<\; 3d\mu, \qquad w_{i1}^0,\ l_{i1} \;<\; 6d\mu \quad \text{for } 1 \le i \le r.$$

The Picard index is given by

$$\mu \;=\; \operatorname{lcm}(w_{0j}^0, \gcd(w_{11}^0, \ldots, w_{r1}^0), u_k^0;\ 1 \le j \le n_0, 1 \le k \le m) \cdot |\mathrm{Cl}(X)^t|.$$

(iv) *Let $n_1 > n_2 = 1$ and $l_{21} \ge \ldots \ge l_{r1} \ge 2$. Then $r \le \mu + \xi(2(d+1)\mu) - 1$ and $n_0 + n_1 + m = d + 1$ hold and one has the bounds*

$$w_{ij}^0 \;\le\; \mu \quad \text{for } i = 0, 1 \text{ and } 1 \le j \le n_i, \qquad w_{21}^0 \;<\; (d+1)\mu,$$
$$\gamma^0,\ w_{ij}^0, l_{ij} \;<\; 2(d+1)\mu \quad \text{for } 0 \le i \le r, \text{ and } 1 \le j \le n_i.$$

The Picard index is given by

$$\mu \;=\; \operatorname{lcm}(w_{ij}^0, u_k^0;\ 0 \le i \le 1, 1 \le j \le n_i, 1 \le k \le m) \cdot |\mathrm{Cl}(X)^t|.$$

(v) *Let $n_2 > 1$ and let s be the maximal number with $n_s > 1$. Assume $l_{s+1,1} \ge \ldots \ge l_{r1} \ge 2$. Then we have $s \le d$, $r \le \mu + \xi((d+2)\mu) + d - 1$ and $n_0 + \ldots + n_s + m = d + s$ and the bounds*

$$w_{ij}^0 \;\le\; \mu, \quad \text{for } 0 \le i \le s, \qquad \gamma^0 \;<\; (d+2)\mu,$$
$$w_{ij}^0, l_{ij} \;<\; (d+2)\mu \quad \text{for } 0 \le i \le r \text{ and } 1 \le j \le n_i.$$

The Picard index is given by

$$\mu \;=\; \operatorname{lcm}(w_{ij}^0, u_k^0;\ 0 \le i \le s, 1 \le j \le n_i, 1 \le k \le m) \cdot |\mathrm{Cl}(X)^t|.$$

Note that assertion (i) and (ii) do not require the Fano condition.

The remaining part of this chapter is devoted to the proofs of the main statements of this chapter. To prove Theorem 6.10 we need the following essential lemma.

Lemma 6.11. *Consider the ring* $\mathbb{K}[T_{ij};\ 0 \le i \le 2,\ 1 \le j \le n_i][S_1, \dots, S_k]/\langle g \rangle$, *where* $n_0 \ge n_1 \ge n_2 \ge 1$ *holds and let* K *be a finitely generated abelian group of the form* $K = \mathbb{Z} \oplus K^t$ *with torsion part* K^t. *Suppose that* g *is homogeneous with respect to the* K*-grading of* $\mathbb{K}[T_{ij}, S_k]$ *given by* $\deg T_{ij} =: w_{ij} = w_{ij}^0 + w_{ij}^t \in K$ *with* $w_{ij}^0 \in \mathbb{Z}_{>0}$ *and* $\deg S_k =: u_k = u_k^0 + u_k^t \in K$ *with* $u_k^0 \in \mathbb{Z}_{>0}$, *and assume*

$$\deg(g)^0 \ < \ \sum_{i=0}^{2}\sum_{j=1}^{n_i} w_{ij}^0 \ + \ \sum_{i=1}^{m} u_i^0.$$

Let $\mu \in \mathbb{Z}_{>1}$, *assume* $w_{ij}^0 \le \mu$ *whenever* $n_i > 1$, $1 \le j \le n_i$ *and* $u_k^0 \le \mu$ *for* $1 \le k \le m$ *and set* $d := n_0 + n_1 + n_2 + m - 2$. *Depending on the shape of* g, *one obtains the following bounds.*

(i) *Suppose that* $g = \eta_0 T_{01}^{l_{01}} \cdots T_{0n_0}^{l_{0n_0}} + \eta_1 T_{11}^{l_{11}} + \eta_2 T_{21}^{l_{21}}$ *with* $n_0 > 1$ *and coefficients* $\eta_i \in \mathbb{K}^*$ *holds. If we have* $l_{11} > l_{21} \ge 2$ *and* $\gcd(l_{11}, l_{21}) \mid \mu$, *then*

$$w_{11}^0 \ < \ 2d\mu, \quad w_{21}^0 \ < \ 3d\mu, \qquad l_{22}, l_{21}, \deg(g)^0 \ < \ 6d\mu.$$

If $l_{11} = l_{21} \ge 2$, *then*

$$l_{11}\,, w_{11}^0\,, l_{21}\,, w_{21}^0\,, \deg(g)^0 \ \le \ \mu.$$

(ii) *Suppose that* $g = \eta_0 T_{01}^{l_{01}} \cdots T_{0n_0}^{l_{0n_0}} + \eta_1 T_{11}^{l_{11}} \cdots T_{1n_1}^{l_{1n_1}} + \eta_2 T_{21}^{l_{21}}$ *with* $n_1 > 1$ *and coefficients* $\eta_i \in \mathbb{K}^*$ *holds and we have* $l_{21} \ge 2$. *Then*

$$w_{21}^0 \ < \ (d+1)\mu, \qquad \deg(g)^0 \ < \ 2(d+1)\mu.$$

Proof. We prove (i). Set for short $c := (n_0 + m)\mu = d\mu$. Then, using homogeneity of g and the assumed inequality, we obtain

$$l_{11}w_{11}^0 \ = \ l_{21}w_{21}^0 \ = \ \deg(g)^0 \ < \ \sum_{i=0}^{2}\sum_{j=1}^{n_i} w_{ij}^0 + \sum_{i=1}^{m} u_i^0 \ \le \ c + w_{11}^0 + w_{21}^0.$$

First have a look at the case $l_{11} > l_{21} \ge 2$. Plugging this into the above inequalities, we arrive at $2w_{11}^0 < c + w_{21}^0$ and $w_{21}^0 < c + w_{11}^0$. We conclude $w_{11}^0 < 2c$ and $w_{21}^0 < 3c$. Consequently, we obtain

$$\deg(g)^0 \ < \ c + w_{11}^0 + w_{21}^0 \ < \ 6c = 6d\mu.$$

If we have $l_{11} = l_{21}$, the homogeneity condition $l_{11}w_{11}^0 = l_{21}w_{11}^0$ gives us $w_{11}^0 = w_{21}^0$. Thus we have $\gcd(w_{11}^0, w_{21}^0) = w_{11}^0 = w_{21}^0 \mid \mu$ and by assumption $\gcd(l_{11}, l_{21}) = l_{21} = l_{11} \mid \mu$. Consequently, $l_{11}, w_{11}^0, l_{21}, w_{21}^0, \deg(g)^0 \le \mu$ holds.

We prove (ii). Here we set $c := (n_0 + n_1 + m)\mu = (d+1)\mu$. Then the assumed inequality gives

$$l_{21}w_{21}^0 \;=\; \deg(g)^0 \;<\; \sum_{i=0}^{1}\sum_{j=1}^{n_i} w_{ij}^0 + \sum_{i=1}^{m} u_i^0 + w_{21}^0 \;\le\; c + w_{21}^0.$$

Since we assumed $l_{21} \ge 2$, we conclude $w_{21}^0 < c$. This in turn gives us $\deg(g)^0 < 2c$. □

Proof of Theorem 6.10. As before, we denote by $\overline{X} \subseteq \mathbb{K}^{n+m}$ the total coordinate space and we consider the quotient $p\colon \widehat{X} \to X$.

We first discuss the case that X is a toric variety. Then the Cox ring is a polynomial ring, $\mathcal{R}(X) = \mathbb{K}[S_1, \ldots, S_m]$. For each $1 \le k \le m$, consider the point $\overline{x}(k) \in \widehat{X}$ having the k-th coordinate one and all others zero and set $x(k) := p(\overline{x}(k))$. Then, by Proposition 6.1, the order of the local class group $\mathrm{Cl}(X, x(k))$ is divisible by u_k^0. Together with Proposition 6.1(ii) we obtain $u_k^0 \le \mu$ for $1 \le k \le m$ and $|\mathrm{Cl}(X)^t| \le \mu$ which settles assertion (i).

Now we treat the non-toric case, which means $r \ge 2$. Note that we have $n \ge 3$. The case $n_0 = 1$ is done in Proposition 6.8, which proves assertion (ii). Hence, we are left with $n_0 > 1$. For every i with $n_i > 1$ and every $1 \le j \le n_i$, there is the point $\overline{x}(i,j) \in \widehat{X}$ with ij-coordinate T_{ij} equal to one and all others equal to zero, and thus we have the point $x(i,j) := p(\overline{x}(i,j)) \in X$. Moreover, for every $1 \le k \le m$, we have the point $\overline{x}(k) \in \overline{X}$ having the k-coordinate S_k equal to one and all others zero; we set $x(k) := p(\overline{x}(k))$. Proposition 6.1 provides the bounds

$$w_{ij}^0 \;\le\; \mu, \quad u_k^0 \;\le\; \mu \qquad \text{for} \quad n_i > 1,\; 0 \le i \le r,\; 1 \le j \le n_i,\; 1 \le k \le m. \tag{7}$$

Let $0 \le s \le r$ be the maximal number with $n_s > 1$. Then g_{s-2} is the last polynomial such that each of its three monomials depends on more than one variable. For any $t \ge s$, we have the "cut ring"

$$R_t \;:=\; \mathbb{K}[T_{ij}, S_k] \;/\; \langle g_0, \ldots, g_{t-2}\rangle$$

where $0 \le i \le t$, $1 \le j \le n_i$, $1 \le k \le m$ and the relations g_i depend on only three variables as soon as $i > s$ holds. For the free part of the degree γ^0 of the relations we have

$$\begin{aligned}
(r-1)\gamma^0 \;&=\; (t-1)\gamma^0 \;+\; (r-t)\gamma^0 \\
&=\; (t-1)\gamma^0 \;+\; l_{t+1,1}w_{t+1,1}^0 + \ldots + l_{r1}w_{r1}^0 \\
&<\; \sum_{i=0}^{r}\sum_{j=1}^{n_i} w_{ij}^0 \;+\; \sum_{i=1}^{m} u_i^0 \\
&=\; \sum_{i=0}^{t}\sum_{j=1}^{n_i} w_{ij}^0 \;+\; w_{t+1,1}^0 + \ldots + w_{r1}^0 \;+\; \sum_{i=1}^{m} u_i^0.
\end{aligned}$$

Note that the inequality is derived from the Fano condition of Proposition 6.1(iv). Since $l_{i1}w_{i1}^0 > w_{i1}^0$ holds in particular for $t+1 \le i \le r$, we derive from this the inequality

$$\gamma^0 \;<\; \frac{1}{t-1}\left(\sum_{i=0}^{t}\sum_{j=1}^{n_i} w_{ij}^0 \;+\; \sum_{i=1}^{m} u_i^0\right). \tag{8}$$

To obtain the bounds in assertions (iii) and (iv), we consider the cut ring R_t with $t = 2$ and apply Lemma 6.11 and Proposition 6.1; note that we have $d = n_0+n_1+n_2+m-2$ for the dimension $d = \dim(X)$ and that $l_{21} \ge 2$ is due to the fact that X is non-toric. The bounds $w_{i1}^0, l_{i1} < 6d\mu$ for $3 \le i \le r$ in assertion (iii) follow from $\gamma^0 < 6d\mu$. Similarly $w_{ij}^0, l_{ij} < 2(d+1)\mu$ for $0 \le i \le r$, $1 \le j \le n_i$ in assertion (iv) follow from $\gamma^0 < 2(d+1)\mu$. We still have to prove the restriction for the number of relations, which means bounding r. Recall from Lemma 6.4 the definition $\ell_i := \gcd(l_{i1},\ldots,l_{in_i})$ and set $\ell_i = \mathrm{lcm}_{0\le j\ne i\le r}(\gcd(\ell_i,\ell_j))\cdot\ell_i'$. Then $\ell_0',\ldots,\ell_r'$ are coprime. For $i \ge 1$ we have $n_i = 1$. Thus, analogously to the proof of Proposition 6.8, we obtain $r+1 = r+1-q+q \le \mu + \xi(6d\mu)$, where q is the number of ℓ_i' that are greater than one and satisfy $n_i = 1$. For the bound in assertion (iv) the same argument yields $r+1 = r+1-q+q \le \mu+\xi(2(d+1)\mu)$.

To obtain the bounds in assertion (v), we consider the cut ring R_t with $t = s$. Using $n_i = 1$ for $i \ge t+1$ and applying the inequalities (7) and (8), we can derive an upper bound for the degree of the relation as follows:

$$\gamma^0 \;<\; \frac{(n_0+\ldots+n_t+m)\mu}{t-1} \;=\; \frac{(d+t)\mu}{t-1} \;\le\; (d+2)\mu.$$

We have $w_{ij}^0 l_{ij} \le \gamma^0$ for any $0 \le i \le r$ and any $1 \le j \le n_i$, which implies that all w_{ij}^0 and l_{ij} are bounded by $(d+2)\mu$. Since $n_0,\ldots,n_{s-1} > 1$ holds, the number s is bounded by $s = 2s-(s-1)-1 \le d$. Consequently, we obtain $r+1 = r+1-s-q+s+q \le \mu+\xi((d+2)\mu)+d$, where q is defined as above.

Finally, we have to express the Picard index μ in terms of the free part of the weights w_{ij}^0, u_k^0 and the torsion part $\mathrm{Cl}(X)^t$ as claimed in the assertions. This is a direct application of the formula of Proposition 6.1. □

As a consequence we obtain restricting statements about the number $\delta(d,\mu)$ of different deformation types of $\mathbb{Q}$-factorial d-dimensional Fano varieties with a complexity-one torus action, Picard number one and Picard index μ. In the toric situation $\delta(d,\mu)$ is bounded above by μ^{d^2}. For the non-toric case we get the following asymptotic results:

Theorem 6.12. *For fixed $d \in \mathbb{Z}_{>0}$, the number $\delta(d,\mu)$ is asymptotically bounded above by $\mu^{A\mu^2}$ for a constant $A > 1$ arbitrarily small, and for fixed $\mu \in \mathbb{Z}_{>0}$, it is asymptotically bounded above by d^{Bd} with a constant $B > 3$ arbitrarily small.*

Proof. Theorem 6.10 provides bounds for the exponents and the number of relations as well as for the free part of the weights and the torsion part of $\mathrm{Cl}(X)$. Since we have

$|\mathrm{Cl}(X)^t| \leq \mu$ the possibilities for the torsion part of the weights are also restricted. One computes that the number $\delta(d, \mu)$ of different deformation types is bounded above by

$$\mu^{\mu^2+3\mu+\xi(\mu)^2+\xi(6d\mu)+5d}(6d\mu)^{2\mu+2\xi(6d\mu)+3d-2}$$

which leads to the results of Theorem 6.12. □

We conclude the section with discussing some aspects of the not necessarily Fano varieties of Proposition 6.8. First we consider varieties with a free divisor class group satisfying $n_0 = \ldots = n_r = 1$ and thus rings R of the form

$$\mathbb{K}[T_0, \ldots, T_r, S_1, \ldots, S_m] \;/\; \langle \alpha_{i+1,i+2}T_i^{l_i} + \alpha_{i,i+2}T_{i+1}^{l_{i+1}} + \alpha_{i,i+1}T_{i+2}^{l_{i+2}};\; 0 \leq i \leq r-2 \rangle.$$

Since $\mathrm{Cl}(X)$ is free, we will write $w_{ij},\ u_k \in \mathrm{Cl}(X)$ instead of $w_{ij}^0,\ u_k^0$.

Proposition 6.13. *Suppose that the ring R as above is the Cox ring of a non-toric variety X with $\mathrm{Cl}(X) = \mathbb{Z}$. Then $m \geq 1$ and $\mu := [\mathrm{Cl}(X) : \mathrm{Pic}(X)] \geq 30$. Moreover, if X is a surface, then $m = 1$ and $w_i = l_i^{-1} l_0 \cdots l_r$.*

Proof. The homogeneity condition $l_i w_i = l_j w_j$ together with the condition $\gcd(l_i, l_j) = 1$ for $0 \leq i \neq j \leq r$, which ensures a free divisor class group, gives us $l_i \mid \gcd(w_j^0; j \neq i)$. Moreover, every set of $m + r$ weights w_i has to generate the class group $\mathbb{Z}$, so they must have greatest common divisor one. Since X is non-toric, $l_i \geq 2$ holds and we obtain $m \geq 1$. To proceed, we infer $l_0 \cdots l_r \mid \mu$ and $l_0 \cdots l_r \mid \deg g_i$ from Proposition 6.1. As a consequence, the minimal value for μ and $\deg g_i$ is obviously $2 \cdot 3 \cdot 5 = 30$. Note that if X is a surface we have $m = 1$ and $\gcd(w_i; 0 \leq i \leq r) = 1$. Thus, $l_i w_i = l_j w_j$ gives us $\deg g_i = l_0 \cdots l_r$ and $w_i = l_i^{-1} l_0 \cdots l_r$. □

The bound $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \geq 30$ given in the above proposition is even sharp; the surface discussed below realizes it.

Example 6.14. Consider X with $\mathcal{R}(X) = \mathbb{K}[T_0, T_1, T_2, S_1]/\langle g \rangle$ with $g = T_0^2 + T_1^3 + T_2^5$ and the grading

$$\deg(T_0) \;=\; 15, \quad \deg(T_1) \;=\; 10, \quad \deg(T_2) \;=\; 6, \quad \deg(S_1) \;=\; 1.$$

Then we have $\gcd(15, 10) = 5$, $\gcd(15, 6) = 3$ and $\gcd(10, 6) = 2$ and therefore $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 30$. Further X is Fano because of

$$\deg(g) \;=\; 30 \;<\; 32 \;=\; \deg(T_0) + \deg(T_1) + \deg(T_2) + \deg(S_1).$$

Finally, we present a couple of examples showing that there are also non-Fano varieties with a complexity one torus action having divisor class group $\mathbb{Z}$ and maximal orbit space $\mathbb{P}_1$.

Example 6.15. Consider X with $\mathcal{R}(X) = \mathbb{K}[T_0, T_1, T_2, S_1]/\langle g\rangle$ with $g = T_0^2 + T_1^3 + T_2^7$ and the grading

$$\deg(T_0) \ = \ 21, \quad \deg(T_1) \ = \ 14, \quad \deg(T_2) \ = \ 6, \quad \deg(S_1) \ = \ 1.$$

Then we have $\gcd(21, 14) = 7$, $\gcd(21, 6) = 3$ and $\gcd(14, 6) = 2$ and therefore $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 42$. Moreover, X is not Fano, because its canonical class $\mathcal{K}_X$ is trivial

$$\mathcal{K}_X \ = \ \deg(g) - \deg(T_0) - \deg(T_1) - \deg(T_2) - \deg(S_1) \ = \ 0.$$

Example 6.16. Consider X with $\mathcal{R}(X) = \mathbb{K}[T_0, T_1, T_2, S_1]/\langle g\rangle$ with $g = T_0^2 + T_1^3 + T_2^{11}$ and the grading

$$\deg(T_0) \ = \ 33, \quad \deg(T_1) \ = \ 22, \quad \deg(T_2) \ = \ 6, \quad \deg(S_1) \ = \ 1.$$

Then we have $\gcd(22, 33) = 11$, $\gcd(33, 6) = 3$ and $\gcd(22, 6) = 2$ and therefore $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 66$. The canonical class $\mathcal{K}_X$ of X is even ample:

$$\mathcal{K}_X \ = \ \deg(g) - \deg(T_0) - \deg(T_1) - \deg(T_2) - \deg(S_1) \ = \ 4.$$

The following example shows that the Fano assumption is essential for the finiteness results in Theorem 6.10.

Remark 6.17. For any pair p, q of coprime positive integers, we obtain a locally factorial $\mathbb{K}^*$-surface $X(p, q)$ with $\mathrm{Cl}(X) = \mathbb{Z}$ and Cox ring

$$\mathcal{R}(X(p,q)) \ = \ \mathbb{K}[T_{01}, T_{02}, T_1, T_2] \ / \ \langle g\rangle, \qquad\qquad g \ = \ T_{01}T_{02}^{pq-1} + T_1^q + T_2^p;$$

the $\mathrm{Cl}(X)$-grading is given by $\deg(T_{01}) = \deg(T_{02}) = 1$, $\deg(T_1) = p$ and $\deg(T_2) = q$. Note that $\deg(g) = pq$ holds and for $p, q \geq 3$, the canonical class $\mathcal{K}_X$ satisfies

$$\mathcal{K}_X \ = \ \deg(g) - \deg(T_{01}) - \deg(T_{02}) - \deg(T_1) - \deg(T_2) \ = \ pq - 2 - p - q \ \geq \ 0.$$

6.3 Classification results

In the subsequent theorems we list non-toric complexity-one Fano T-varieties with Picard number one. The Cox rings are described in terms of generators and relations and we specify the $\mathrm{Cl}(X)$-grading by giving the degrees of the generators. Additionally, we list the degree of the Fano varieties $d_X := (-K_X)^d$ and the Gorenstein index $\iota(X)$, i.e. the smallest positive integer such that $\iota(X) \cdot K_X$ is Cartier.

Theorem 6.18. *Let X be a non-toric Fano surface with an effective $\mathbb{K}^*$-action such that* $\operatorname{rk}(\operatorname{Cl}(X)) = 1$ *and* $[\operatorname{Cl}(X) : \operatorname{Pic}(X)] \le 6$ *hold. Then its Cox ring is precisely one of the following:*

$[\operatorname{Cl}(X) : \operatorname{Pic}(X)] = 1$

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_4)$	d_X	$\operatorname{Cl}(X)$	$\iota(X)$
1	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1T_2^5 + T_3^3 + T_4^2\rangle$	$(1, 1, 2, 3)$	1	$\mathbb{Z}$	1

$[\operatorname{Cl}(X) : \operatorname{Pic}(X)] = 2$

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_4)$	d_X	$\operatorname{Cl}(X)$	$\iota(X)$
2	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^4T_2 + T_3^3 + T_4^2\rangle$	$(1, 2, 2, 3)$	2	$\mathbb{Z}$	1
3	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1T_2^3 + T_3^4 + T_4^2\rangle$	$\left(\begin{smallmatrix} 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1

$[\operatorname{Cl}(X) : \operatorname{Pic}(X)] = 3$

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_4)$	d_X	$\operatorname{Cl}(X)$	$\iota(X)$
4	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^3T_2 + T_3^3 + T_4^2\rangle$	$(1, 3, 2, 3)$	3	$\mathbb{Z}$	1
5	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1T_2^3 + T_3^5 + T_4^2\rangle$	$(1, 3, 2, 5)$	$1/3$	$\mathbb{Z}$	3
6	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^7T_2 + T_3^5 + T_4^2\rangle$	$(1, 3, 2, 5)$	$1/3$	$\mathbb{Z}$	3
7	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1T_2^2 + T_3^3 + T_4^3\rangle$	$\left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{2} & \bar{0} \end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$	1

$[\operatorname{Cl}(X) : \operatorname{Pic}(X)] = 4$

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_4)$	d_X	$\operatorname{Cl}(X)$	$\iota(X)$
8	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^2T_2 + T_3^3 + T_4^2\rangle$	$(1, 4, 2, 3)$	4	$\mathbb{Z}$	1
9	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^6T_2 + T_3^5 + T_4^2\rangle$	$(1, 4, 2, 5)$	1	$\mathbb{Z}$	2
10	$\mathbb{K}[T_1, T_2, T_3, S_1]/\langle T_1^2 + T_2^2 + T_3^2\rangle$	$\left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{smallmatrix}\right)$	2	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
11	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1T_2 + T_3^2 + T_4^2\rangle$	$\left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ \bar{1} & \bar{3} & \bar{2} & \bar{0} \end{smallmatrix}\right)$	2	$\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$	1
12	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1^2T_2 + T_3^2 + T_4^4\rangle$	$\left(\begin{smallmatrix} 1 & 2 & 2 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{smallmatrix}\right)$	2	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
13	$\mathbb{K}[T_1, \ldots, T_4]/\langle T_1T_2^2 + T_3^6 + T_4^2\rangle$	$\left(\begin{smallmatrix} 2 & 2 & 1 & 3 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
14	$\mathbb{K}[T_1, \ldots, T_5]/\left\langle \begin{smallmatrix} T_1T_2 + T_3^2 + T_4^2, \\ \lambda T_3^2 + T_4^2 + T_5^2 \end{smallmatrix} \right\rangle$	$\left(\begin{smallmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1

$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 5$

No.	$\mathcal{R}(X)$	$(w_1, \dots, w_4)$	d_X	$\mathrm{Cl}(X)$	$\iota(X)$
15	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1T_2 + T_3^3 + T_4^2\rangle$	$(1, 5, 2, 3)$	5	$\mathbb{Z}$	1
16	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1^5T_2 + T_3^5 + T_4^2\rangle$	$(1, 5, 2, 5)$	9/5	$\mathbb{Z}$	5
17	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1^9T_2 + T_3^7 + T_4^2\rangle$	$(1, 5, 2, 7)$	1/5	$\mathbb{Z}$	5
18	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1^7T_2 + T_3^4 + T_4^3\rangle$	$(1, 5, 3, 4)$	1/5	$\mathbb{Z}$	5

$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 6$

No.	$\mathcal{R}(X)$	$(w_1, \dots, w_4)$	d_X	$\mathrm{Cl}(X)$	$\iota(X)$
19	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1^4T_2 + T_3^5 + T_4^2\rangle$	$(1, 6, 2, 5)$	8/3	$\mathbb{Z}$	3
20	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1^8T_2 + T_3^7 + T_4^2\rangle$	$(1, 6, 2, 7)$	2/3	$\mathbb{Z}$	3
21	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1^6T_2 + T_3^4 + T_4^3\rangle$	$(1, 6, 3, 4)$	2/3	$\mathbb{Z}$	3
22	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1^9T_2 + T_3^3 + T_4^2\rangle$	$(1, 3, 4, 6)$	2/3	$\mathbb{Z}$	3
23	$\mathbb{K}[T_1, T_2, T_3, S_1]/\langle T_1^3 + T_2^3 + T_3^2\rangle$	$\left(\begin{smallmatrix} 2 & 2 & 3 & 1 \\ \bar{1} & \bar{2} & \bar{0} & \bar{0} \end{smallmatrix}\right)$	2/3	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	3
24	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1T_2 + T_3^3 + T_4^3\rangle$	$\left(\begin{smallmatrix} 1 & 2 & 1 & 1 \\ \bar{1} & \bar{2} & \bar{2} & \bar{0} \end{smallmatrix}\right)$	2	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	1
25	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1T_2 + T_3^2 + T_4^4\rangle$	$\left(\begin{smallmatrix} 3 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{smallmatrix}\right)$	3	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	1
26	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1T_2^5 + T_3^2 + T_4^8\rangle$	$\left(\begin{smallmatrix} 3 & 1 & 4 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{smallmatrix}\right)$	1/3	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	3

where $\iota(X)$ denotes the Gorenstein index, $d_X = (-K_X)^2$ and the parameter λ occurring in the second relation of surface number 14 can be any element of $\mathbb{K}^* \setminus \{1\}$. Furthermore, the Cox rings listed above are pairwise non-isomorphic as graded rings.

Remark 6.19. Gorenstein surfaces are well known to have ADE-singularities which are in particular canonical. Consequently, the surfaces of number 1 to 4 and 10 to 12, as well as 7, 8, 14, 15, 24 and 25 are canonical. Furthermore, in [51] all log-terminal del Pezzo $\mathbb{K}^*$-surfaces of Gorenstein index up to 3 are classified. These are exactly those surfaces whose maximal exponents of the monomials form a platonic triple $(1, k, l)$, $(2, 2, k)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$. Comparing the surfaces listed in [51, Theorems 4.9, 4.10] with the table above shows that the numbers 6, 9, 16 to 22 and 26 are not log-terminal. The resolution of these surfaces can be explicitly computed by using the canonical resolution of 3.10.

The varieties listed so far might suggest that we always obtain only one relation in the Cox ring if $\mathrm{Cl}(X)$ is torsion free, i.e. $\mathrm{Cl}(X) \cong \mathbb{Z}$. We discuss now some examples, showing that for a Picard index big enough, we need in general more than one relation.

Example 6.20. This is an example of a Fano $\mathbb{K}^*$-surface X with $\mathrm{Cl}(X) = \mathbb{Z}$ such that the Cox ring $\mathcal{R}(X)$ needs two relations. Consider the $\mathbb{Z}$-graded ring

$$R \;=\; \mathbb{K}[T_{01}, T_{02}, T_1, T_2, T_3]/\langle g_0, g_1\rangle,$$

where the degrees of $T_{01}, T_{02}, T_1, T_2, T_3$ are $29, 1, 6, 10, 15$, respectively, and the relations g_0, g_1 are given by

$$g_0 \;:=\; T_{01}T_{02} + T_1^5 + T_2^3\,, \qquad\qquad g_1 \;:=\; \lambda T_1^5 + T_2^3 + T_3^2\,.$$

Then R is the Cox ring of a Fano $\mathbb{K}^*$-surface. Note that the Picard index is given by $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = \mathrm{lcm}(29, 1) = 29$.

Proposition 6.21. *Let X be a non-toric Fano surface with an effective $\mathbb{K}^*$-action such that $\mathrm{Cl}(X) \cong \mathbb{Z}$ and $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] < 29$ hold. Then the Cox ring of X is of the form*

$$\mathcal{R}(X) \;\cong\; \mathbb{K}[T_{01}, T_{02}, T_1, T_2]/\langle T_{01}^{l_{01}}T_{02}^{l_{02}} + T_1^{l_1} + T_2^{l_2}\rangle.$$

Proof. The Cox ring $\mathcal{R}(X)$ is given by a ring $R(A, P)$ as in 2.10 and, in the notation used there, we have $n_0 + \ldots + n_r + m = 2 + r$. This leaves us with the possibilities $n_0 = m = 1$ and $n_0 = 2$, $m = 0$. In the first case, Proposition 6.13 tells us that the Picard index of X is at least 30.

Consider the case $n_0 = 2$ and $m = 0$. Then, the Cox ring $\mathcal{R}(X)$ is $\mathbb{K}[T_{01}, T_{02}, T_1, \ldots, T_r]$ divided by relations

$$g_0 = T_{01}^{l_{01}}T_{02}^{l_{02}} + T_1^{l_1} + T_2^{l_2}, \quad g_i = \alpha_{i+1,i+2}T_i^{l_i} + \alpha_{i+2,i}T_{i+1}^{l_{i+1}} + \alpha_{i,i+1}T_{i+2}^{l_{i+2}},$$

where $1 \le i \le r-2$. We have to show that $r = 2$ holds. Set $\mu := [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ and let $\gamma \in \mathbb{Z}$ denote the degree of the relations. Then we have $\gamma = w_i l_i$ for $1 \le i \le r$, where $w_i := \deg T_i$. With $w_{0i} := \deg T_{0i}$, Proposition 6.1 gives us

$$(r-1)\gamma \;<\; w_{01} + w_{02} + w_1 + \ldots + w_r.$$

We claim that w_{01} and w_{02} are coprime. Otherwise they have a common prime divisor p. This p divides $\gamma = l_i w_i$. Since $l_1, \ldots, l_r$ are pairwise coprime, p divides at least $r-1$ of the weights $w_1, \ldots, w_r$. This contradicts the Cox ring condition that any $r+1$ of the $r+2$ weights generate the class group $\mathbb{Z}$. Thus, w_{01} and w_{02} are coprime and we obtain

$$\mu \;\ge\; \mathrm{lcm}(\mathrm{w}_{01}, \mathrm{w}_{02}) \;=\; \mathrm{w}_{01} \cdot \mathrm{w}_{02} \;\ge\; \mathrm{w}_{01} + \mathrm{w}_{02} - 1.$$

Now assume that $r \ge 3$ holds. Then we conclude

$$2\gamma \;<\; w_{01} + w_{02} + w_1 + w_2 + w_3 \;\le\; \mu + 1 + \gamma\left(\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3}\right).$$

Since the numbers l_i are pairwise coprime, we obtain $l_1 \ge 5$, $l_2 \ge 3$ and $l_3 \ge 2$. Moreover, $l_i w_i = l_j w_j$ implies $l_i \mid w_j$ and hence $l_1 l_2 l_3 \mid \gamma$. Thus, we have $\gamma \ge 30$. Plugging this in the above inequality gives

$$\mu \;\ge\; \gamma\left(2 - \frac{1}{l_1} - \frac{1}{l_2} - \frac{1}{l_3}\right) - 1 \;=\; 29.$$

□

The Fano assumption is essential in this result; if we omit it, we may even construct locally factorial surfaces with a Cox ring that needs more than one relation.

Example 6.22. This is an example of a locally factorial $\mathbb{K}^*$-surface X with $\mathrm{Cl}(X) = \mathbb{Z}$ such that the Cox ring $\mathcal{R}(X)$ needs two relations. Consider the $\mathbb{Z}$-graded ring

$$R \;=\; \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}, T_{31}]/\langle g_0, g_1\rangle,$$

where the degrees of $T_{01}, T_{02}, T_{11}, T_{21}, T_{31}$ are $1, 1, 6, 10, 15$, respectively, and the relations g_0, g_1 are given by

$$g_0 \;:=\; T_{01}^7 T_{02}^{23} + T_{11}^5 + T_{21}^3, \qquad\qquad g_1 \;:=\; \lambda T_{11}^5 + T_{21}^3 + T_{31}^2.$$

Then R is the Cox ring of a non Fano $\mathbb{K}^*$-surface X of Picard index one, i.e. X is locally factorial.

Theorem 6.23. *Let X be a three-dimensional locally factorial non-toric Fano variety with an effective two torus action such that $\mathrm{Cl}(X)$ is of rank one. Then its Cox ring is precisely one of the following.*

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_5)$	$(-K_X)^3$	$\mathrm{Cl}(X)$
1	$\mathbb{K}[T_1, \ldots, T_5] \;/\; \langle T_1T_2^5 + T_3^3 + T_4^2\rangle$	$(1, 1, 2, 3, 1)$	8	$\mathbb{Z}$
2	$\mathbb{K}[T_1, \ldots, T_5] \;/\; \langle T_1T_2T_3^4 + T_4^3 + T_5^2\rangle$	$(1, 1, 1, 2, 3)$	8	$\mathbb{Z}$
3	$\mathbb{K}[T_1, \ldots, T_5] \;/\; \langle T_1T_2^2T_3^3 + T_4^3 + T_5^2\rangle$	$(1, 1, 1, 2, 3)$	8	$\mathbb{Z}$
4	$\mathbb{K}[T_1, \ldots, T_5] \;/\; \langle T_1T_2 + T_3T_4 + T_5^2\rangle$	$(1, 1, 1, 1, 1)$	54	$\mathbb{Z}$
5	$\mathbb{K}[T_1, \ldots, T_5] \;/\; \langle T_1T_2^2 + T_3T_4^2 + T_5^3\rangle$	$(1, 1, 1, 1, 1)$	24	$\mathbb{Z}$
6	$\mathbb{K}[T_1, \ldots, T_5] \;/\; \langle T_1T_2^3 + T_3T_4^3 + T_5^4\rangle$	$(1, 1, 1, 1, 1)$	4	$\mathbb{Z}$
7	$\mathbb{K}[T_1, \ldots, T_5] \;/\; \langle T_1T_2^3 + T_3T_4^3 + T_5^2\rangle$	$(1, 1, 1, 1, 2)$	16	$\mathbb{Z}$
8	$\mathbb{K}[T_1, \ldots, T_5] \;/\; \langle T_1T_2^5 + T_3T_4^5 + T_5^2\rangle$	$(1, 1, 1, 1, 3)$	2	$\mathbb{Z}$
9	$\mathbb{K}[T_1, \ldots, T_5] \;/\; \langle T_1T_2^5 + T_3^3T_4^3 + T_5^2\rangle$	$(1, 1, 1, 1, 3)$	2	$\mathbb{Z}$

The singular threefolds listed in this theorem are rational degenerations of smooth Fano threefolds from [32]. The (smooth) general Fano threefolds of the corresponding families are non-rational see [24] for no. 1-3, [11] for no. 5, [34] for no. 6, [54, 53] for no. 7 and [33] for no. 8-9. Even if one allows certain mild singularities, one still has non-rationality in some cases, see [23], [12, 47], [13], [10].

Theorem 6.24. *Let X be a three-dimensional non-toric Fano variety with an effective two torus action such that $\mathrm{Cl}(X)$ is of rank one and $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 2$ holds. Then its Cox ring is precisely one of the following.*

No.	$\mathcal{R}(X)$	$(w_1,\dots,w_5)$	d_X	$\mathrm{Cl}(X)$	$\iota(X)$
1	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3^3+T_4^2\rangle$	$(1,2,2,3,1)$	$27/2$	$\mathbb{Z}$	2
2	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^3+T_3^5+T_4^2\rangle$	$(1,2,2,5,1)$	$1/2$	$\mathbb{Z}$	2
3	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3^5+T_4^2\rangle$	$(1,2,2,5,1)$	$1/2$	$\mathbb{Z}$	2
4	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3^3+T_4^2\rangle$	$(1,2,2,3,2)$	16	$\mathbb{Z}$	1
5	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^3+T_3^5+T_4^2\rangle$	$(1,2,2,5,2)$	2	$\mathbb{Z}$	1
6	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3^5+T_4^2\rangle$	$(1,2,2,5,2)$	2	$\mathbb{Z}$	1
7	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3^3+T_4^2\rangle$	$(1,1,2,3,2)$	$27/2$	$\mathbb{Z}$	2
8	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^9+T_3^5+T_4^2\rangle$	$(1,1,2,5,2)$	$1/2$	$\mathbb{Z}$	2
9	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^7+T_3^5+T_4^2\rangle$	$(1,1,2,5,2)$	$1/2$	$\mathbb{Z}$	2
10	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^{11}+T_3^3+T_4^2\rangle$	$(1,1,4,6,1)$	$1/2$	$\mathbb{Z}$	2
11	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^5T_2^7+T_3^3+T_4^2\rangle$	$(1,1,4,6,1)$	$1/2$	$\mathbb{Z}$	2
12	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^{11}+T_3^3+T_4^2\rangle$	$(1,1,4,6,2)$	2	$\mathbb{Z}$	1
13	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^5T_2^7+T_3^3+T_4^2\rangle$	$(1,1,4,6,2)$	2	$\mathbb{Z}$	1
14	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^5+T_3^3+T_4^2\rangle$	$(1,2,4,6,1)$	2	$\mathbb{Z}$	1
15	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^{10}T_2+T_3^3+T_4^2\rangle$	$(1,2,4,6,1)$	2	$\mathbb{Z}$	1
16	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2+T_3^3+T_4^2\rangle$	$(2,2,2,3,1)$	16	$\mathbb{Z}$	1
17	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^4+T_3^5+T_4^2\rangle$	$(2,2,2,5,1)$	2	$\mathbb{Z}$	1
18	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3+T_3^5+T_4^2\rangle$	$(2,2,2,5,1)$	2	$\mathbb{Z}$	1
19	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2+T_3T_4+T_5^3\rangle$	$(1,1,1,2,1)$	$81/2$	$\mathbb{Z}$	2
20	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^4+T_3T_4^2+T_5^5\rangle$	$(1,1,1,2,1)$	$5/2$	$\mathbb{Z}$	2
21	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3+T_3T_4^2+T_5^5\rangle$	$(1,1,1,2,1)$	$5/2$	$\mathbb{Z}$	2
22	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3^2T_4+T_5^4\rangle$	$(1,1,1,2,1)$	16	$\mathbb{Z}$	1
23	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^4+T_3^3T_4+T_5^5\rangle$	$(1,1,1,2,1)$	$5/2$	$\mathbb{Z}$	2
24	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3+T_3^3T_4+T_5^5\rangle$	$(1,1,1,2,1)$	$5/2$	$\mathbb{Z}$	2
25	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3^2T_4+T_5^2\rangle$	$(1,1,1,2,2)$	27	$\mathbb{Z}$	2
26	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3^2T_4^2+T_5^3\rangle$	$(1,1,1,2,2)$	$3/2$	$\mathbb{Z}$	2
27	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3^4T_4+T_5^3\rangle$	$(1,1,1,2,2)$	$3/2$	$\mathbb{Z}$	2

28	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4+T_3^4T_4+T_5^3\rangle$	$(1,1,1,2,2)$	$3/2$	$\mathbb{Z}$	2
29	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3^4T_4+T_5^2\rangle$	$(1,1,1,2,3)$	8	$\mathbb{Z}$	1
30	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^3+T_3^4T_4+T_5^2\rangle$	$(1,1,1,2,3)$	8	$\mathbb{Z}$	1
31	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^7+T_3^2T_4^3+T_5^2\rangle$	$(1,1,1,2,4)$	1	$\mathbb{Z}$	2
32	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^5+T_3^2T_4^3+T_5^2\rangle$	$(1,1,1,2,4)$	1	$\mathbb{Z}$	2
33	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^7+T_3^6T_4+T_5^2\rangle$	$(1,1,1,2,4)$	1	$\mathbb{Z}$	2
34	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^5+T_3^6T_4+T_5^2\rangle$	$(1,1,1,2,4)$	1	$\mathbb{Z}$	2
35	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3T_4+T_5^4\rangle$	$(1,1,2,2,1)$	27	$\mathbb{Z}$	2
36	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3T_4^2+T_5^6\rangle$	$(1,1,2,2,1)$	$3/2$	$\mathbb{Z}$	2
37	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3T_4+T_5^2\rangle$	$(1,1,2,2,2)$	16	$\mathbb{Z}$	1
38	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3T_4^2+T_5^3\rangle$	$(1,1,2,2,2)$	6	$\mathbb{Z}$	1
39	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4+T_3T_4^2+T_5^3\rangle$	$(1,1,2,2,2)$	6	$\mathbb{Z}$	1
40	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^3+T_3T_4^2+T_5^2\rangle$	$(1,1,2,2,3)$	$27/2$	$\mathbb{Z}$	1
41	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^5+T_3T_4^3+T_5^2\rangle$	$(1,1,2,2,4)$	32	$\mathbb{Z}$	1
42	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3T_4^2+T_5^2\rangle$	$(1,1,2,2,3)$	4	$\mathbb{Z}$	2
43	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^7+T_3T_4^3+T_5^2\rangle$	$(1,1,2,2,4)$	32	$\mathbb{Z}$	1
44	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^9+T_3T_4^4+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2
45	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^9+T_3^2T_4^3+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2
46	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^7+T_3T_4^4+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2
47	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^7+T_3^2T_4^3+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2
48	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^5T_2^5+T_3T_4^4+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2
49	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^5T_2^5+T_3^2T_4^3+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2
50	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3T_4+T_5^3\rangle$	$(1,2,1,2,1)$	48	$\mathbb{Z}$	1
51	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2+T_3^2T_4+T_5^4\rangle$	$(1,2,1,2,1)$	27	$\mathbb{Z}$	2
52	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2+T_3T_4^2+T_5^5\rangle$	$(1,2,1,2,1)$	10	$\mathbb{Z}$	1
53	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2+T_3^3T_4+T_5^5\rangle$	$(1,2,1,2,1)$	10	$\mathbb{Z}$	1
54	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2+T_3^3T_4+T_5^5\rangle$	$(1,2,1,2,1)$	10	$\mathbb{Z}$	1
55	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3^4T_4+T_5^6\rangle$	$(1,2,1,2,1)$	$3/2$	$\mathbb{Z}$	2
56	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2+T_3^2T_4+T_5^2\rangle$	$(1,2,1,2,2)$	32	$\mathbb{Z}$	1
57	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^2+T_3^4T_4+T_5^3\rangle$	$(1,2,1,2,2)$	6	$\mathbb{Z}$	1
58	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3^4T_4+T_5^3\rangle$	$(1,2,1,2,2)$	6	$\mathbb{Z}$	1

59	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3^4T_4+T_5^2\rangle$	$(1,2,1,2,3)$	$27/2$	$\mathbb{Z}$	2
60	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3+T_3^2T_4^3+T_5^2\rangle$	$(1,2,1,2,4)$	4	$\mathbb{Z}$	1
61	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3+T_3^6T_4+T_5^2\rangle$	$(1,2,1,2,4)$	4	$\mathbb{Z}$	1
62	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^6T_2+T_3^6T_4+T_5^2\rangle$	$(1,2,1,2,4)$	4	$\mathbb{Z}$	1
63	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^3+T_3^4T_4^3+T_5^2\rangle$	$(1,2,1,2,5)$	$1/2$	$\mathbb{Z}$	2
64	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3^4T_4^3+T_5^2\rangle$	$(1,2,1,2,5)$	$1/2$	$\mathbb{Z}$	2
65	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3^8T_4+T_5^2\rangle$	$(1,2,1,2,5)$	$1/2$	$\mathbb{Z}$	2
66	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2+T_3T_4+T_5^4\rangle$	$(1,2,2,2,1)$	32	$\mathbb{Z}$	1
67	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3T_4^2+T_5^6\rangle$	$(1,2,2,2,1)$	6	$\mathbb{Z}$	1
68	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2+T_3T_4^2+T_5^2\rangle$	$(1,2,2,2,3)$	16	$\mathbb{Z}$	1
69	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^3+T_3T_4^4+T_5^2\rangle$	$(1,2,2,2,5)$	2	$\mathbb{Z}$	1
70	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^4T_2^3+T_3^2T_4^3+T_5^2\rangle$	$(1,2,2,2,5)$	2	$\mathbb{Z}$	1
71	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3T_4^4+T_5^2\rangle$	$(1,2,2,2,5)$	2	$\mathbb{Z}$	1
72	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^8T_2+T_3^2T_4^3+T_5^2\rangle$	$(1,2,2,2,5)$	2	$\mathbb{Z}$	1
73	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^{10}+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$	$\mathbb{Z}$	2
74	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2T_3^9+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$	$\mathbb{Z}$	2
75	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3T_3^8+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$	$\mathbb{Z}$	2
76	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^4T_3^7+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$	$\mathbb{Z}$	2
77	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5T_3^6+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$	$\mathbb{Z}$	2
78	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^3T_3^7+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$	$\mathbb{Z}$	2
79	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^5T_3^5+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$	$\mathbb{Z}$	2
80	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^3T_2^4T_3^5+T_4^3+T_5^2\rangle$	$(1,1,1,4,6)$	$1/2$	$\mathbb{Z}$	2
81	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^2+T_4^3+T_5^2\rangle$	$(1,1,2,2,3)$	$27/2$	$\mathbb{Z}$	2
82	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3T_3+T_4^3+T_5^2\rangle$	$(1,1,2,2,3)$	$27/2$	$\mathbb{Z}$	2
83	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^2T_3+T_4^3+T_5^2\rangle$	$(1,1,2,2,3)$	$27/2$	$\mathbb{Z}$	2
84	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^4+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2
85	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3T_3^3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2
86	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5T_3^2+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2
87	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^7T_3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2
88	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^2T_3^3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2
89	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^6T_3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2

90	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^3T_2^3T_3^2+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2
91	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^3T_2^5T_3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2
92	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^4T_2^4T_3+T_4^5+T_5^2\rangle$	$(1,1,2,2,5)$	$1/2$	$\mathbb{Z}$	2
93	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2T_3^5+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2	$\mathbb{Z}$	1
94	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3T_3^4+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2	$\mathbb{Z}$	1
95	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^5T_3^3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2	$\mathbb{Z}$	1
96	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^7T_3^2+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2	$\mathbb{Z}$	1
97	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^9T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2	$\mathbb{Z}$	1
98	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^2T_2^4T_3^3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2	$\mathbb{Z}$	1
99	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^2T_2^8T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2	$\mathbb{Z}$	1
100	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^3T_2^5T_3^2+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2	$\mathbb{Z}$	1
101	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^3T_2^7T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2	$\mathbb{Z}$	1
102	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^4T_2^6T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2	$\mathbb{Z}$	1
103	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^5T_2^5T_3+T_4^3+T_5^2\rangle$	$(1,1,2,4,6)$	2	$\mathbb{Z}$	1
104	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^2T_2T_3+T_4^3+T_5^2\rangle$	$(1,2,2,2,3)$	16	$\mathbb{Z}$	1
105	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^2T_2T_3^3+T_4^5+T_5^2\rangle$	$(1,2,2,2,5)$	2	$\mathbb{Z}$	1
106	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^4T_2T_3^2+T_4^5+T_5^2\rangle$	$(1,2,2,2,5)$	2	$\mathbb{Z}$	1
107	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^6T_2T_3+T_4^5+T_5^2\rangle$	$(1,2,2,2,5)$	2	$\mathbb{Z}$	1
108	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2+T_3^2+T_4^2\rangle$	$\left(\frac{1}{1}\ \frac{1}{1}\ \frac{1}{1}\ \frac{1}{0}\ \frac{1}{0}\right)$	27	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
109	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3+T_3^2+T_4^4\rangle$	$\left(\frac{1}{0}\ \frac{1}{0}\ \frac{2}{1}\ \frac{1}{0}\ \frac{1}{1}\right)$	8	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
110	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3+T_3^2+T_4^4\rangle$	$\left(\frac{1}{0}\ \frac{1}{0}\ \frac{2}{1}\ \frac{1}{1}\ \frac{1}{0}\right)$	8	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
111	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3+T_3^2+T_4^4\rangle$	$\left(\frac{1}{0}\ \frac{1}{0}\ \frac{2}{1}\ \frac{1}{1}\ \frac{1}{1}\right)$	8	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
112	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^5+T_3^2+T_4^6\rangle$	$\left(\frac{1}{0}\ \frac{1}{0}\ \frac{3}{0}\ \frac{1}{1}\ \frac{1}{1}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
113	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^5+T_3^2+T_4^6\rangle$	$\left(\frac{1}{0}\ \frac{1}{0}\ \frac{3}{1}\ \frac{1}{0}\ \frac{1}{1}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
114	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^2T_2^4+T_3^2+T_4^3\rangle$	$\left(\frac{1}{0}\ \frac{1}{0}\ \frac{3}{1}\ \frac{2}{0}\ \frac{1}{1}\right)$	4	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
115	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1^2T_2^4+T_3^2+T_4^3\rangle$	$\left(\frac{1}{0}\ \frac{1}{1}\ \frac{3}{1}\ \frac{2}{0}\ \frac{1}{0}\right)$	4	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
116	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^5+T_3^3+T_4^2\rangle$	$\left(\frac{1}{1}\ \frac{1}{1}\ \frac{2}{0}\ \frac{3}{0}\ \frac{1}{0}\right)$	4	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
117	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3+T_3^2T_4^2+T_5^2\rangle$	$\left(\frac{1}{0}\ \frac{1}{0}\ \frac{1}{1}\ \frac{1}{1}\ \frac{2}{1}\right)$	8	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
118	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^3+T_3^2T_4^2+T_5^4\rangle$	$\left(\frac{1}{1}\ \frac{1}{1}\ \frac{1}{0}\ \frac{1}{1}\ \frac{1}{0}\right)$	2	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
119	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^5+T_3^2T_4^4+T_5^2\rangle$	$\left(\frac{1}{0}\ \frac{1}{0}\ \frac{1}{1}\ \frac{1}{1}\ \frac{3}{0}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
120	$\mathbb{K}[T_1,\ldots,T_5]/\langle T_1T_2^5+T_3^2T_4^4+T_5^2\rangle$	$\left(\frac{1}{0}\ \frac{1}{0}\ \frac{1}{0}\ \frac{1}{1}\ \frac{3}{1}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1

121	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4+T_3^3T_4^3+T_5^2\rangle$	$\left(\begin{smallmatrix}1&1&1&1&3\\0&0&1&1&1\end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
122	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4+T_3^3T_4^3+T_5^2\rangle$	$\left(\begin{smallmatrix}1&1&1&1&3\\0&1&0&0&1\end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
123	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4+T_3^5T_4+T_5^2\rangle$	$\left(\begin{smallmatrix}1&1&1&1&3\\0&0&1&1&1\end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
124	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4+T_3^5T_4+T_5^2\rangle$	$\left(\begin{smallmatrix}1&1&1&1&3\\0&1&0&0&1\end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
125	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3T_4+T_5^2\rangle$	$\left(\begin{smallmatrix}1&1&1&1&1\\0&0&1&1&1\end{smallmatrix}\right)$	27	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
126	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2+T_3T_4+T_5^2\rangle$	$\left(\begin{smallmatrix}1&1&1&1&1\\0&0&1&1&0\end{smallmatrix}\right)$	27	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
127	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2+T_3T_4^2+T_5^3\rangle$	$\left(\begin{smallmatrix}1&1&1&1&1\\0&1&0&1&0\end{smallmatrix}\right)$	12	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
128	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3T_4^3+T_5^4\rangle$	$\left(\begin{smallmatrix}1&1&1&1&1\\1&1&0&0&0\end{smallmatrix}\right)$	2	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
129	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3T_4^3+T_5^4\rangle$	$\left(\begin{smallmatrix}1&1&1&1&1\\0&0&1&1&0\end{smallmatrix}\right)$	2	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
130	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3T_4^3+T_5^2\rangle$	$\left(\begin{smallmatrix}1&1&1&1&2\\0&0&1&1&0\end{smallmatrix}\right)$	8	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
131	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3T_4^3+T_5^2\rangle$	$\left(\begin{smallmatrix}1&1&1&1&2\\0&0&1&1&1\end{smallmatrix}\right)$	8	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
132	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3T_4^5+T_5^2\rangle$	$\left(\begin{smallmatrix}1&1&1&1&3\\0&0&1&1&0\end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
133	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3T_4^5+T_5^2\rangle$	$\left(\begin{smallmatrix}1&1&1&1&3\\0&0&1&1&1\end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
134	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3^3T_4^3+T_5^2\rangle$	$\left(\begin{smallmatrix}1&1&1&1&3\\0&0&1&1&0\end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
135	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3^3T_4^3+T_5^2\rangle$	$\left(\begin{smallmatrix}1&1&1&1&3\\0&0&1&1&1\end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
136	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^2+T_4^2+T_5^4\rangle$	$\left(\begin{smallmatrix}1&1&1&2&1\\0&0&1&1&0\end{smallmatrix}\right)$	8	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
137	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^2+T_4^2+T_5^4\rangle$	$\left(\begin{smallmatrix}1&1&1&2&1\\0&0&0&1&1\end{smallmatrix}\right)$	8	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
138	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^2+T_4^2+T_5^4\rangle$	$\left(\begin{smallmatrix}1&1&1&2&1\\0&0&1&1&1\end{smallmatrix}\right)$	8	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
139	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^4+T_4^2+T_5^6\rangle$	$\left(\begin{smallmatrix}1&1&1&3&1\\0&0&1&1&0\end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
140	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^4+T_4^2+T_5^6\rangle$	$\left(\begin{smallmatrix}1&1&1&3&1\\0&0&1&0&1\end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
141	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2T_3^3+T_4^2+T_5^6\rangle$	$\left(\begin{smallmatrix}1&1&1&3&1\\0&1&0&1&0\end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
142	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2T_3^3+T_4^2+T_5^6\rangle$	$\left(\begin{smallmatrix}1&1&1&3&1\\0&1&0&0&1\end{smallmatrix}\right)$	1	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
143	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2T_3^3+T_4^3+T_5^2\rangle$	$\left(\begin{smallmatrix}1&1&1&2&3\\1&0&1&0&0\end{smallmatrix}\right)$	4	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	1
144	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^4+T_4^3+T_5^2\rangle$	$\left(\begin{smallmatrix}1&1&1&2&3\\0&0&1&0&1\end{smallmatrix}\right)$	4	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2
145	$\mathbb{K}[T_1,\dots,T_6]/\left\langle \begin{smallmatrix}T_1T_2+T_3T_4+T_5^2,\\ \lambda T_3T_4+T_5^2+T_6^2\end{smallmatrix}\right\rangle$	$\left(\begin{smallmatrix}1&1&1&1&1&1\\0&0&1&1&1&0\end{smallmatrix}\right)$	16	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	2

where $\iota(X)$ denotes the Gorenstein index, $d_X = (-K_X)^3$ and the parameter λ occuring in the second relation of surface number 145 can be any element of $\mathbb{K}^* \setminus \{1\}$. Furthermore, the Cox rings listed above are pairwise non-isomorphic as graded rings.

The varieties no. 2,3 and 25, 26 are rational degenerations of quasismooth varieties from the list in [31]. In [14] the non-rationality of a general (quasismooth) element of the corresponding family was proved.

For non-toric Fano threefolds X with an effective 2-torus action and $\mathrm{Cl}(X) \cong \mathbb{Z}$, the classifications 6.23 and 6.24 show that for Picard indices one and two we only obtain hypersurfaces as Cox rings. The following example shows that this stops at Picard index three.

Example 6.25. This is an example of a Fano threefold X with $\mathrm{Cl}(X) = \mathbb{Z}$ and a 2-torus action such that the Cox ring $\mathcal{R}(X)$ needs two relations. Consider

$$R \;=\; \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{12}, T_{21}, T_{31}]/\langle g_0, g_1\rangle$$

where the degrees of $T_{01}, T_{02}, T_{11}, T_{12}, T_{21}, T_{31}$ are $1, 1, 3, 3, 2, 3$, respectively, and the relations are given by

$$g_0 \;=\; T_{01}^5 T_{02} + T_{11}T_{12} + T_{21}^3, \qquad g_1 \;=\; \lambda T_{11}T_{12} + T_{21}^3 + T_{31}^2.$$

Then R is the Cox ring of a Fano threefold with a 2-torus action. Note that the Picard index is given by

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \;=\; \mathrm{lcm}(1, 1, 3, 3) \;=\; 3.$$

Finally, we turn to locally factorial Fano fourfolds. Here we observe more than one relation in the Cox ring even in the locally factorial case.

Theorem 6.26. *Let X be a four-dimensional locally factorial non-toric Fano variety with an effective three torus action. Then its Cox ring is precisely one of the following.*

No.	$\mathcal{R}(X)$	$(w_1, \ldots, w_6)$	$(-K_X)^4$
1	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^5 + T_3^3 + T_4^2\rangle$	$(1, 1, 2, 3, 1, 1)$	81
2	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^9 + T_3^2 + T_4^5\rangle$	$(1, 1, 2, 5, 1, 1)$	1
3	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1^3T_2^7 + T_3^2 + T_4^5\rangle$	$(1, 1, 2, 5, 1, 1)$	1
4	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2T_3^4 + T_4^3 + T_5^2\rangle$	$(1, 1, 1, 2, 3, 1)$	81
5	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^2T_3^3 + T_4^3 + T_5^2\rangle$	$(1, 1, 1, 2, 3, 1)$	81
6	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2T_3^8 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
7	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^2T_3^7 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
8	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^3T_3^6 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
9	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^4T_3^5 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
10	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1^2T_2^3T_3^5 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
11	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1^3T_2^3T_3^4 + T_4^5 + T_5^2\rangle$	$(1, 1, 1, 2, 5, 1)$	1
12	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle$	$(1, 1, 1, 1, 1, 1)$	512
13	$\mathbb{K}[T_1, \ldots, T_6]/\langle T_1T_2^2 + T_3T_4^2 + T_5^3\rangle$	$(1, 1, 1, 1, 1, 1)$	243

14	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^3+T_3T_4^3+T_5^4\rangle$	$(1,1,1,1,1,1)$	64
15	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^4+T_3T_4^4+T_5^5\rangle$	$(1,1,1,1,1,1)$	5
16	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^4+T_3^2T_4^3+T_5^5\rangle$	$(1,1,1,1,1,1)$	5
17	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1^2T_2^3+T_3^2T_4^3+T_5^5\rangle$	$(1,1,1,1,1,1)$	5
18	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^3+T_3T_4^3+T_5^2\rangle$	$(1,1,1,1,2,1)$	162
19	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^5+T_3T_4^5+T_5^3\rangle$	$(1,1,1,1,2,1)$	3
20	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^5+T_3^2T_4^4+T_5^3\rangle$	$(1,1,1,1,2,1)$	3
21	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^5+T_3T_4^5+T_5^2\rangle$	$(1,1,1,1,3,1)$	32
22	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^5+T_3^3T_4^3+T_5^2\rangle$	$(1,1,1,1,3,1)$	32
23	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^7+T_3T_4^7+T_5^2\rangle$	$(1,1,1,1,4,1)$	2
24	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^7+T_3^3T_4^5+T_5^2\rangle$	$(1,1,1,1,4,1)$	2
25	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1^3T_2^5+T_3^3T_4^5+T_5^2\rangle$	$(1,1,1,1,4,1)$	2
26	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3T_4^3+T_5^3+T_6^2\rangle$	$(1,1,1,1,2,3)$	81
27	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^2T_4^2+T_5^3+T_6^2\rangle$	$(1,1,1,1,2,3)$	81
28	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3T_4^7+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
29	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^2T_4^6+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
30	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^3T_4^5+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
31	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^4T_4^4+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
32	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^2T_4^5+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
33	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^3T_4^4+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
34	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^3T_3^3T_4^3+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
35	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1^2T_2^2T_3^3T_4^3+T_5^5+T_6^2\rangle$	$(1,1,1,1,2,5)$	1
36	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3+T_4T_5^2+T_6^3\rangle$	$(1,1,1,1,1,1)$	243
37	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^2+T_4T_5^3+T_6^4\rangle$	$(1,1,1,1,1,1)$	64
38	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^3+T_4T_5^4+T_6^5\rangle$	$(1,1,1,1,1,1)$	5
39	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^3+T_4^2T_5^3+T_6^5\rangle$	$(1,1,1,1,1,1)$	5
40	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^2+T_4T_5^4+T_6^5\rangle$	$(1,1,1,1,1,1)$	5
41	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2^2T_3^2+T_4^2T_5^3+T_6^5\rangle$	$(1,1,1,1,1,1)$	5
42	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^2+T_4T_5^3+T_6^2\rangle$	$(1,1,1,1,1,2)$	162
43	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^4+T_4T_5^5+T_6^3\rangle$	$(1,1,1,1,1,2)$	3
44	$\mathbb{K}[T_1,\dots,T_6]/\langle T_1T_2T_3^4+T_4^2T_5^4+T_6^3\rangle$	$(1,1,1,1,1,2)$	3

45	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2T_3^3+T_4T_5^5+T_6^3\rangle$	$(1,1,1,1,1,2)$	3
46	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2T_3^3+T_4^2T_5^4+T_6^3\rangle$	$(1,1,1,1,1,2)$	3
47	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1^2T_2^2T_3^2+T_4T_5^5+T_6^3\rangle$	$(1,1,1,1,1,2)$	3
48	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2T_3^3+T_4^3T_5^3+T_6^2\rangle$	$(1,1,1,1,1,3)$	32
49	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2T_3^3+T_4T_5^5+T_6^2\rangle$	$(1,1,1,1,1,3)$	32
50	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^4+T_4^3T_5^3+T_6^2\rangle$	$(1,1,1,1,1,3)$	32
51	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^4+T_4T_5^5+T_6^2\rangle$	$(1,1,1,1,1,3)$	32
52	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^6+T_4T_5^7+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
53	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2T_3^6+T_4^3T_5^5+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
54	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2T_3^5+T_4T_5^7+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
55	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2T_3^5+T_4^3T_5^5+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
56	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^3T_3^4+T_4T_5^7+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
57	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^3T_3^4+T_4^3T_5^5+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
58	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1^2T_2^3T_3^3+T_4T_5^7+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
59	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1^2T_2^3T_3^3+T_4^3T_5^5+T_6^2\rangle$	$(1,1,1,1,1,4)$	2
60	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2+T_3T_4+T_5T_6\rangle$	$(1,1,1,1,1,1)$	512
61	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle$	$(1,1,1,1,1,1)$	243
62	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^3+T_3T_4^3+T_5T_6^3\rangle$	$(1,1,1,1,1,1)$	64
63	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^3+T_3T_4^3+T_5^2T_6^2\rangle$	$(1,1,1,1,1,1)$	64
64	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^4+T_3T_4^4+T_5T_6^4\rangle$	$(1,1,1,1,1,1)$	5
65	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^4+T_3T_4^4+T_5^2T_6^3\rangle$	$(1,1,1,1,1,1)$	5
66	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1T_2^4+T_3^2T_4^3+T_5^2T_6^3\rangle$	$(1,1,1,1,1,1)$	5
67	$\mathbb{K}[T_1,\ldots,T_6]/\langle T_1^2T_2^3+T_3^2T_4^3+T_5^2T_6^3\rangle$	$(1,1,1,1,1,1)$	5
68	$\mathbb{K}[T_1,\ldots,T_7]/\left\langle \begin{smallmatrix} T_1T_2+T_3T_4+T_5T_6, \\ \lambda T_3T_4+T_5T_6+T_7^2 \end{smallmatrix} \right\rangle$	$(1,1,1,1,1,1,1)$	324
69	$\mathbb{K}[T_1,\ldots,T_7]/\left\langle \begin{smallmatrix} T_1T_2^2+T_3T_4^2+T_5T_6^2, \\ \lambda T_3T_4^2+T_5T_6^2+T_7^3 \end{smallmatrix} \right\rangle$	$(1,1,1,1,1,1,1)$	9

where in the last two rows of the table the parameter λ can be any element from $\mathbb{K}^* \setminus \{1\}$. Furthermore, the Cox rings listed above are pairwise non-isomorphic as graded rings.

By the result of [46], the singular quintics of this list are rational degenerations of smooth non-rational Fano fourfolds.

Proof of Theorems 6.18, 6.23, 6.24 and 6.26. For fixed d and μ Theorem 6.10 bounds the number of possible data l_{ij}, w_{ij}^0, u_k^0, belonging to Fano varieties. We identify all these constellations by a computer based algorithm. Since $|\mathrm{Cl}(X)^t| \leq \mu$ holds, there is only a finite number of possibilities for the torsion part of the weights that we have to check. By this procedure we receive the tables of 6.18, 6.23, 6.24 and 6.26.

We claim that any two of the listed Cox rings do not describe varieties that are isomorphic to each other. Two minimal systems of homogeneous generators of the Cox ring contain (up to reordering) the same free parts of generator degrees w_{ij}^0, $u_k^0 \in \mathbb{Z}$. Consequently, they are invariant under isomorphy. Furthermore the exponents $l_{ij} > 1$ represent the orders of all finite non-trivial isotropy groups of one-codimensional orbits of the action T on X; see [29, Theorem 1.3]. Moreover, since none of the listed Cox rings is polynomial, the varieties are all non-toric. This implies that every complexity-one action is maximal and consequently can be assigned to a maximal torus in $\mathrm{Aut}(X)$. Note that $\mathrm{Aut}(X)$ is also acting effectively on X. Since the maximal tori of $\mathrm{Aut}(X)$ are all conjugated, the varieties with complexity-one torus action are isomorphic if and only if they are T-equivariantly isomorphic. Thus, running through the exponents l_{ij} we see that any two of the varieties listed in Theorem 6.18, 6.23 and 6.26 are not isomorphic.

In case of Theorem 6.24 there is some more work to do. There are non-isomorphic threefolds varying only in the torsion part of the weights, see for example number 2, 3 and 4. In these cases, comparing the torsion parts of the gradings shows that it is not possible to install a $\mathrm{Cl}(X)$-graded ring isomorphism between the Cox rings of two different threefolds.

As an example we consider the threefolds number 2 and 3: Let D_2 be a prime divisor, representing $\deg(T_2) \in \mathrm{Cl}(X)$ and let E_1 be a prime divisor, representing $\deg(S_1) \in \mathrm{Cl}(X)$. Then D_2 has isotropy group of order $l_2 = 3$ and E_1 has infinite isotropy. In case of threefold number 2 the term $D_2 - E_1$ represents a non-trivial torsion element whereas in case of threefold number 3 it is the zero element in $\mathrm{Cl}(X)$. Thus, these two varieties are not isomorphic. Analogously, we proceed with all other cases to obtain finally the list of Theorem 6.24.

Finally, we apply [25, Corollary 4.9] to compute the Gorenstein index $\iota(X)$ for all listed varieties, i.e. we have to find the smallest integer $\iota(X)$ such that $\iota(X) \cdot K_X$ is contained in all local divisor class groups $\mathrm{Cl}(X, x)$; see also Proposition 6.1. □

7 Almost homogeneous complexity-one T-varieties

This chapter is dedicated to classification problems of almost homogeneous complexity-one T-varieties, i.e. complexity-one T-varieties X whose automorphism group $\mathrm{Aut}(X)$ acts with an open orbit. By introducing Demazure P-roots, we obtain a combinatorial approach to the automorphism group of such varieties, describing the roots of $\mathrm{Aut}(X)$. The Demazure P-roots turn out to be lattice points of certain polytopes. This convex geometrical approach will be used for classification problems on almost homogeneous complexity-one T-varieties of dimension two and three. Concretely, we provide a complete list of all log-terminal non-toric almost homogeneous $\mathbb{K}^*$-surfaces with exactly one singularity and Picard number one up to Gorenstein index five. Furthermore, we describe almost homogeneous complexity-one threefolds with Picard number one and reductive automorphism group. These results are published in [6, Sections 6 and 8]

7.1 The automorphism group of complexity-one T-varieties and Demazure P-roots

Round 1970 Demazure studied the automorphism group of smooth complete toric varieties. Later Cox generalized the results to the simplicial case. The aim of this section is to recall a description of the automorphism group of a complexity-one T-variety by combinatorial data from [6]. A useful notion in this context are Demazure P-roots. The statements of this section appeared in [6] where one can also find their proofs.

By [6, Theorem 5.5] the automorphism group $\mathrm{Aut}(X)$ of a normal complete rational (non-toric) variety X with an effective torus action of complexity one is a linear algebraic group. Thus, we first recall some basic definitions and facts for linear algebraic groups. Let G be a linear algebraic group with maximal torus T. The *adjoint representation* Ad of G is the representation of G in the tangent space $\mathrm{T}_e(G)$ (which is isomorphic to the Lie algebra $\mathcal{G}$ of G), given by

$$G \to \mathrm{GL}(\mathrm{T}_e(G)), \quad g \mapsto \mathrm{Ad}(g) = \mathrm{d}(\mathrm{Int}\ g),$$

where Int g is given by the inner automorphism $x \mapsto gxg^{-1}$. The group G is acting on $\mathrm{T}_e(G)$ by this representation and hence, so is T. We call an element $0 \neq v \in T_e(G)$ an eigenvector if there is a character $\chi^w \in \mathbb{X}(T)$ such that $t \cdot v = \chi^w(t)v$ for all $t \in T$. In this situation χ is called an eigenvalue of the adjoint representation. They are also called the *weights* of the representation. We have

$$\mathrm{T}_e(G) \;=\; \bigoplus V_w \;=\; \bigoplus_{\chi^w \in \mathbb{X}(T)} V_{\chi^w} \;=\; V_0 \oplus \bigoplus_{0 \neq w} V_w\ ,$$

where the sets $V_w = \{v \in \mathrm{T}_e(G);\ t \cdot v = \chi^w(t)v\ \forall t \in T\}$ denote the eigenspaces associated to the character defined by the weight $w \in \mathbb{X}(T)$ and $V_0 = \mathrm{T}_e(T)$. The non-zero weights of the adjoint representation Ad are called the *roots* of G. Linear algebraic groups are generated by their maximal tori and the one parameter subgroups corresponding to the roots. For each root χ^w we define $T_w = (\ker(\chi^w))^\circ$ and $Z_w = Z_G(T_w)$. Then U_w, the set of all unipotent elements of Z_w, is a connected T-stable (with respect to conjugation) subgroup and there is a one-parameter subgroup $\lambda_w \colon \mathbb{G}_a \to U_w \subseteq G$.

Definition 7.1. Let P be a matrix as defined in Construction 2.10. Furthermore, let $v_{ij}, v_k \in N = \mathbb{Z}^{r+s}$ denote the columns of P and let $M = N^* \cong \mathbb{Z}^{r+s}$ be the dual lattice of N.

(i) A *vertical Demazure P-root* is a tuple (u, k_0) with a linear form $u \in M$ and an index $1 \leq k_0 \leq m$ satisfying

$$\begin{aligned} \langle u, v_{ij} \rangle &\geq 0 \quad \text{for all } i, j \\ \langle u, v_k \rangle &\geq 0 \quad \text{for all } k \neq k_0 \\ \langle u, v_{k_0} \rangle &= -1 \end{aligned}$$

(ii) A *horizontal Demazure P-root* is a tuple (u, i_0, i_1, C) with a linear form $u \in M$, two indices $i_0 \neq i_1$ with $0 \leq i_0, i_1 \leq r$, and a sequence $C = (c_0, \ldots, c_r)$ with $1 \leq c_i \leq n_i$ such that

$$\begin{aligned} l_{ic_i} &= 1 \quad \text{for all } i \neq i_0, i_1 \\ \langle u, v_{ic_i} \rangle &= \begin{cases} 0, & i \neq i_0, i_1, \\ -1, & i = i_1, \end{cases} \\ \langle u, v_{ij} \rangle &\geq \begin{cases} l_{ij}, & i \neq i_0, i_1,\ j \neq c_i \\ 0, & i = i_0, i_1,\ j \neq c_i \\ 0, & i = i_0,\ j = c_i, \end{cases} \\ \langle u, v_k \rangle &\geq 0 \quad \text{for all } k. \end{aligned}$$

The *$\mathbb{Z}^s$-part* of a Demazure P-root $\kappa = (u, k_0)$ or $\kappa = (u, i_0, i_1, C)$ is the tuple α_κ of the last s coordinates of the linear form $u \in M$. We simply call it *P-root*.

Remark 7.2. Demazure P-roots are given by integral points of certain polytopes in $M_\mathbb{Q}$ defined by the equations and inequations of Definition 7.1. These polytopes are called *root polytopes*. In general they can be described as follows:

(i) For a given index $1 \leq k \leq m$ we consider the vector $\zeta = (\zeta_{ij}, \zeta_k) \in \mathbb{Z}^{n+m}$ satisfying

$$\zeta_{ij} := 0 \text{ for all i,j}, \quad \zeta_k := 0 \text{ for all k}, \quad \zeta_{k_0} := -1$$

and the affine subspace

$$\eta \ := \ \{u' \in M_\mathbb{Q} \mid \langle u', v_{k_0} \rangle = -1\} \ \subseteq \ M_\mathbb{Q}.$$

Then the vertical Demazure P-roots $\kappa = (u, k_0)$ are given by the integral points of the polytope

$$B(k_0) \ := \ \{u' \in \eta \mid P^* u' \geq \zeta\} \ \subseteq \ M_\mathbb{Q}.$$

(ii) For any two indices $1 \leq i_0, i_1 \leq r$, $i_0 \neq i_1$, and any sequence $C = (c_0, \dots, c_r)$ with $1 \leq c_i \leq n_i$ such that $l_{ic_i} = 1$ for all $i \neq i_0, i_1$ we set

$$\zeta_{ij} := \begin{cases} l_{ij}, & i \neq i_0, i_1,\ j \neq c_i \\ -1, & i = i_1,\ j = c_i \\ 0 & \text{else} \end{cases}$$
$$\zeta_k = 0 \quad \text{for all } 1 \leq k \leq m$$

and we define the affine subspace

$$\eta := \{u' \in M_{\mathbb{Q}};\ \langle u', v_{ic_i}\rangle = 0, \langle u', v_{i_1c_{i_1}}\rangle = -1\} \subseteq M_{\mathbb{Q}}.$$

Then the horizontal Demazure P-roots $\kappa = (u, i_0, i_1, C)$ are given by the integral points of the polytope

$$B(i_0, i_1, C) := \{u' \in \eta \mid P^* u' \geq \zeta\} \subseteq M_{\mathbb{Q}}.$$

Example 7.3 (Del Pezzo surface E_6)**.** Consider the E_6-singular del Pezzo surface X with Cox ring $R(X) = \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}]/\langle T_{01}T_{02}^3 + T_{11}^3 + T_{21}^2\rangle$ given by the P-matrix

$$P = \begin{pmatrix} -1 & -3 & 3 & 0 \\ -1 & -3 & 0 & 2 \\ -1 & -2 & 1 & 1 \end{pmatrix}.$$

There are no vertical Demazure P-roots since $m = 0$ holds. But there is a horizontal Demazure P-root $\kappa(u, i_0, i_1, C)$ given by

$$u = (-1, -2, 3), \quad i_0 = 1,\ i_1 = 2, \quad C = (1, 1, 1),$$

and it turns out that this is the only one. The P-root of κ is the last coordinate $u_3 = 3$ of u.

Theorem 7.4. (See [6, Theorem 5.5]). *Let X be a (non-toric) complexity-one T-variety arising from sincere data (A, P) as seen in Construction 2.20. Then the following statements hold:*

(i) The automorphism group $\mathrm{Aut}(X)$ *is a linear algebraic group with maximal torus T.*

(ii) Under the canonical identification $\mathbb{X}(T) = \mathbb{Z}^s$, the roots of $\mathrm{Aut}(X)$ *with respect to T are precisely the P-roots.*

Geometrically, the vertical P-roots correspond to those root subgroups whose orbits are contained in the closure of generic torus orbits. These P-roots are defined by generators S_k of the Cox ring not occurring in the Cox ring relations (as in the toric situation). The horizontal P-roots, on the other hand, correspond to those root subgroups whose orbits are transversal to generic torus orbits. In this context, the relations between the generators of the Cox ring do play an important role.

7.2 Almost homogeneous surfaces

This section is dedicated to rational $\mathbb{K}^*$-surfaces having horizontal Demazure P-roots and Picard number one. We will give general formulas for the Demazure P-roots. Furthermore, we concentrate on log-terminal $\mathbb{K}^*$-surfaces of that type having only one singularity. As a result, we list all such surfaces up to Gorenstein index five.

Definition 7.5. A variety is called *almost homogeneous* if its automorphism group acts with an open orbit.

Theorem 7.6. (See [6, Theorem 6.1]). *Let X be a non-toric normal complete rational complexity-one T-variety arising from sincere data (A, P) and let $\mathcal{R}(X) = R(A, P)$ be its Cox ring. Then the following statements are equivalent:*

(i) The variety X is almost homogeneous.

(ii) There exists a horizontal Demazure P-root.

Moreover, if one of these statements holds, then the number of relations of $R(A, P)$ is bounded by

$$r - 1 \ < \ \dim(X) + \mathrm{rk}(\mathrm{Cl}(X)) - m - 2.$$

Let X be a normal complete rational $\mathbb{K}^*$-surface. Then X is isomorphic to some $X(A, P)$ as in Construction 2.20. We assume that X has Picard number one. The following proposition determines possible Demazure P-roots in this setting.

Proposition 7.7. *Consider integers $l_{02} \geq 1$, $l_{11} \geq l_{21} \geq 2$ and d_{01}, d_{02}, d_{11}, d_{21} such that the following matrix has pairwise different primitive columns generating $\mathbb{Q}^3$ as a convex cone:*

$$P \ := \ \begin{bmatrix} -1 & -l_{02} & l_{11} & 0 \\ -1 & -l_{02} & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{21} \end{bmatrix}.$$

Moreover, assume that P is positive in the sense that $\det(P_{01}) > 0$ holds, where P_{01} is the 3×3-matrix obtained from P by deleting the first column. Then the possible horizontal Demazure P-roots are

(i) $\kappa = (u, 1, 2, (1, 1, 1))$, where $u = \left(d_{01}\alpha + \frac{d_{21}\alpha+1}{l_{21}}, -\frac{d_{21}\alpha+1}{l_{21}}, \alpha\right)$ with an integer α satisfying

$$l_{21} \mid d_{21}\alpha + 1, \qquad \frac{l_{02}}{d_{02} - l_{02}d_{01}} \ \leq \ \alpha \ \leq \ -\frac{l_{11}}{l_{21}d_{11} + l_{11}d_{21} + d_{01}l_{11}l_{21}},$$

(ii) if $l_{02} = 1$: $\kappa = (u, 1, 2, (2, 1, 1))$, where $u = \left(d_{02}\alpha + \frac{d_{21}\alpha+1}{l_{21}}, -\frac{d_{21}\alpha+1}{l_{21}}, \alpha\right)$ with an integer α satisfying

$$l_{21} \mid d_{21}\alpha + 1, \qquad -\frac{l_{11}}{l_{21}d_{11} + l_{11}d_{21} + d_{02}l_{11}l_{21}} \ \leq \ \alpha \ \leq \ \frac{1}{d_{01} - d_{02}},$$

(iii) $\kappa = (u, 2, 1, (1, 1, 1))$, *where* $u = \left(-\frac{d_{11}\alpha+1}{l_{11}}, d_{01}\alpha + \frac{d_{11}\alpha+1}{l_{11}}, \alpha\right)$ *with an integer* α *satisfying*

$$l_{11} \mid d_{11}\alpha + 1, \qquad \frac{l_{02}}{d_{02} - l_{02}d_{01}} \leq \alpha \leq -\frac{l_{21}}{l_{21}d_{11} + l_{11}d_{21} + d_{01}l_{11}l_{21}},$$

(iv) if $l_{02} = 1$*:* $\kappa = (u, 2, 1, (2, 1, 1))$, *where* $u = \left(-\frac{d_{11}\alpha+1}{l_{11}}, d_{02}\alpha + \frac{d_{11}\alpha+1}{l_{11}}, \alpha\right)$ *with an integer* α *satisfying*

$$l_{11} \mid d_{11}\alpha + 1, \qquad -\frac{l_{21}}{l_{21}d_{11} + l_{11}d_{21} + d_{02}l_{11}l_{21}} \leq \alpha \leq \frac{1}{d_{01} - d_{02}}.$$

Note that under these assumptions the P*-roots are always positive.*

Proof. In the situation of (i), evaluating the general linear form $u = (u_1, u_2, u_3)$ on the columns of P gives the following conditions for a Demazure P-root:

$$\langle u, v_{01}\rangle = -u_1 - u_2 + u_3d_{01} = 0, \qquad \langle u, v_{21}\rangle = u_2l_{21} + u_3d_{21} = -1,$$

$$\langle u, v_{02}\rangle = -u_1l_{02} - u_2l_{02} + u_3d_{02} \geq l_{02}, \qquad \langle u, v_{11}\rangle = u_1l_{11} + u_3d_{11} \geq 0.$$

Resolving the equations for u_1, u_2 gives

$$u_1 = \frac{1 + u_3d_{21}}{l_{21}} + u_3d_{01} \quad \text{and} \quad u_2 = -\frac{1 + u_3d_{21}}{l_{21}}.$$

Plugging these results into the inequalities one obtains the desired roots with $\alpha := u_3$. Note that the assumption $\det(P_{01}) > 0$ implies $d_{02} - l_{02}d_{01} > 0$ and $\det(P_{02}) = d_{01}l_{11}l_{21} + d_{21}l_{11} + d_{11}l_{21} < 0$. Furthermore, the condition $l_{21} \mid d_{21}\alpha + 1$ ensures that u_1 and u_2 are integers.

In case (ii), under the assumption $l_{02} = 1$, we get the following conditions for a Demazure P-root:

$$\langle u, v_{02}\rangle = -u_1 - u_2 + u_3d_{02} = 0, \qquad \langle u, v_{21}\rangle = u_2l_{21} + u_3d_{21} = -1,$$

$$\langle u, v_{01}\rangle = -u_1 - u_2 + u_3d_{01} \geq 1, \qquad \langle u, v_{11}\rangle = u_1l_{11} + u_3d_{11} \geq 0.$$

Resolving the equations for u_1, u_2 gives

$$u_1 = \frac{1 + u_3d_{21}}{l_{21}} + u_3d_{02} \quad \text{and} \quad u_2 = -\frac{1 + u_3d_{21}}{l_{21}}.$$

Plugging these results into the inequalities one analogously obtains the desired roots with $\alpha := u_3$. Once more we use the assumption $\det(P_{01}) = d_{02}l_{11}l_{21} + d_{21}l_{11} + d_{11}l_{21} > 0$ to transform the inequalities and the condition $l_{21} \mid d_{21}\alpha + 1$ to ensure that u_1 and u_2 are integers.

The other cases are treated analogously by switching i_0 and i_1 which means switching the roles of v_{11} and v_{21}. Consequently, in case (iii) we obtain the conditions

$$\langle u, v_{01}\rangle = -u_1 - u_2 + u_3 d_{01} = 0, \qquad \langle u, v_{21}\rangle = u_1 l_{11} + u_3 d_{11} = -1,$$

$$\langle u, v_{02}\rangle = -u_1 l_{02} - u_2 l_{02} + u_3 d_{02} \geq l_{02}, \qquad \langle u, v_{11}\rangle = u_2 l_{21} + u_3 d_{21} \geq 0,$$

which gives

$$u_1 \;=\; \frac{1 + u_3 d_{11}}{l_{11}} + u_3 d_{01} \quad \text{and} \quad u_2 \;=\; -\frac{1 + u_3 d_{11}}{l_{11}}$$

and the desired roots. In case (iv) we obtain

$$\langle u, v_{02}\rangle = -u_1 - u_2 + u_3 d_{02} = 0, \qquad \langle u, v_{11}\rangle = u_1 l_{11} + u_3 d_{11} = -1,$$

$$\langle u, v_{01}\rangle = -u_1 - u_2 + u_3 d_{01} \geq 1, \qquad \langle u, v_{21}\rangle = u_2 l_{21} + u_3 d_{21} \geq 0,$$

and consequently,

$$u_1 \;=\; \frac{1 + u_3 d_{11}}{l_{11}} + u_3 d_{02} \quad \text{and} \quad u_2 \;=\; -\frac{1 + u_3 d_{11}}{l_{11}},$$

which finally completes the proof. □

Corollary 7.8. *The non-toric almost homogeneous normal complete rational $\mathbb{K}^*$-surfaces X of Picard number one are precisely the ones arising from data*

$$A \;=\; \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \qquad\qquad P \;=\; \begin{bmatrix} -1 & -l_{02} & l_{11} & 0 \\ -1 & -l_{02} & 0 & l_{21} \\ d_{01} & d_{02} & d_{11} & d_{21} \end{bmatrix}$$

as in Proposition 7.7 allowing an integer α according to one of the Conditions (i) to (iv) of Proposition 7.7. In particular, the Cox ring of X is given as

$$\mathcal{R}(X) \;=\; \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}] \;/\; \langle T_{01} T_{02}^{l_{02}} + T_{11}^{l_{11}} + T_{21}^{l_{21}} \rangle$$

with the grading by $\mathbb{Z}^4/\mathrm{im}(P^*)$. Moreover, the anticanonical divisor of X is ample, i.e. X is a del Pezzo surface.

Proof. As any surface with finitely generated Cox ring, X is $\mathbb{Q}$-factorial. Since X has Picard number one, the divisor class group $\mathrm{Cl}(X)$ is of rank one. Now take a minimal presentation $\mathcal{R}(X) = R(A, P)$ of the Cox ring. Then, according to Theorem 7.6, we have $m = 0$ and there is exactly one relation in $R(A, P)$. Thus, P is a (3×4)- matrix. Moreover, Theorem 7.6 says that there is a horizontal Demazure P-root. Consequently, one of the exponents l_{01} and l_{02} must equal one, say l_{01}. Fixing a suitable order for the last two variables we ensure $l_{11} \geq l_{21}$. Passing to the $\mathbb{K}^*$-action $t^{-1} \cdot x$ instead of $t \cdot x$, if necessary, we achieve that P is positive in the sense of Proposition 7.7.

Let us see why X is a del Pezzo surface. Denote by P_{ij} the matrix obtained from P by deleting the column v_{ij}. Then, in $\mathrm{Cl}(X)^0 = \mathrm{Cl}(X)/\mathrm{Cl}(X)^t = \mathbb{Z}$, the weights w_{ij}^0 of T_{ij} are given up to a positive factor β as

$$\begin{aligned}
(w_{01}^0, w_{02}^0, w_{11}^0, w_{21}^0) &= \beta(\det(P_{01}), -\det(P_{02}), \det(P_{11}), -\det(P_{21})), \quad \text{where} \\
\det(P_{01}) &= d_{02}l_{11}l_{21} + d_{11}l_{02}l_{11} + d_{21}l_{02}l_{11}, \\
-\det(P_{02}) &= -(d_{01}l_{11}l_{21} + d_{11}l_{11} + d_{21}l_{11}), \\
\det(P_{11}) &= d_{01}l_{02}l_{21} - d_{02}l_{21}, \\
-\det(P_{21}) &= -(d_{01}l_{02}l_{11} - d_{02}l_{11}).
\end{aligned}$$

Note that $w_{01}^0, w_{02}^0, w_{11}^0, w_{21}^0 > 0$. According to [5, Prop. III.3.4.1], the class of the anticanonical divisor in $\mathrm{Cl}(X)^0$ is given as the sum over all w_{ij}^0 minus the degree of the relation

$$\begin{aligned}
-K_X &= \beta(\det(P_{01}) - \det(P_{02}) + \det(P_{11}) - \det(P_{21}) - (\det(P_{01}) - l_{02}\det(P_{02}))) \\
&= \beta(-\det(P_{02})(-l_{02} + 1) + \det(P_{11}) - \det(P_{21})).
\end{aligned}$$

The surface X is Fano if and only if $-K_X > 0$ holds. Consequently, the factor β can be omitted. Now we will use the inequalities on l_{ij}, d_{ij} implied by the existence of an integer α as in Proposition 7.7. In the cases (ii) and (iv) we have $l_{02} = 1$ which implies $-K_X = \det(P_{11}) - \det(P_{21}) > 0$. The inequalities in case (i) and (iii) give $l_{02} \le -\det(P_{21})$ and $l_{02} \le \det(P_{11})$ and hence $-K_X \ge -\det(P_{02}) + \det(P_{11}) > 0$ and $-K_X \ge -\det(P_{02}) - \det(P_{21}) > 0$, respectively. □

We turn to the case of X having precisely one singular point. Note that by Proposition 5.10 this singular point has to be an elliptic fixed point. The situation then is a lot simpler since the divisibility conditions (i) to (iv) of Proposition 7.7 disappear.

Construction 7.9 ($\mathbb{K}^*$-surfaces with one singularity)**.** Consider a triple (l_0, l_1, l_2) of integers satisfying the following conditions:

$$l_0 \ \ge\ 1, \qquad l_1 \ \ge\ l_2 \ \ge\ 2, \qquad l_0 \ <\ l_1l_2, \qquad \gcd(l_1, l_2) \ =\ 1.$$

Let (d_1, d_2) be the (unique) pair of integers with $d_1l_2 + d_2l_1 = -1$ and $0 \le d_2 < l_2$ and consider the data

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \qquad P = \begin{bmatrix} -1 & -l_0 & l_1 & 0 \\ -1 & -l_0 & 0 & l_2 \\ 0 & 1 & d_1 & d_2 \end{bmatrix}.$$

Then the associated ring $R(l_0, l_1, l_2) := R(A, P)$ is graded by $\mathbb{Z}^4/\mathrm{im}(P^*) \cong \mathbb{Z}$, and is explicitly given by

$$R(l_0, l_1, l_2) = \mathbb{K}[T_1, T_2, T_3, T_4]/\langle T_1T_2^{l_0} + T_3^{l_1} + T_4^{l_2}\rangle,$$

$$\deg(T_1) = l_1l_2 - l_0, \qquad \deg(T_2) = 1, \qquad \deg(T_3) = l_2, \qquad \deg(T_4) = l_1.$$

Proposition 7.10. *For the* $\mathbb{K}^*$*-surface* $X = X(l_0, l_1, l_2)$ *with Cox ring* $R(l_0, l_1, l_2)$*, the following statements hold:*

(i) X *is non-toric and we have* $\mathrm{Cl}(X) = \mathbb{Z}$*.*

(ii) X *comes with precisely one singularity.*

(iii) X *is a del Pezzo surface if and only if* $l_0 < l_1 + l_2 + 1$ *holds.*

(iv) X *is almost homogeneous if and only if* $l_0 \le l_1$ *holds.*

Moreover, any normal complete rational non-toric $\mathbb{K}^*$*-surface of Picard number one with precisely one singularity is isomorphic to some* $X(l_0, l_1, l_2)$*.*

Proof. First note that $X = X(l_0, l_1, l_2)$ is obtained as in Construction 7.9. The group $H = \mathbb{K}^*$ acts on $\mathbb{K}^4$ by

$$t \cdot z \;=\; (t^{l_1 l_2 - l_0} z_1, t z_2, t^{l_2} z_3, t^{l_1} z_4),$$

the total coordinate space $\overline{X} := V(T_1 T_2^{l_0} + T_3^{l_1} + T_4^{l_2})$ is invariant under this action and we have

$$\widehat{X} \;=\; \overline{X} \setminus \{0\}, \qquad\qquad X \;=\; \widehat{X}/\mathbb{K}^*.$$

Thus, $\mathrm{Cl}(X) = \mathbb{Z}$ holds and since the Cox ring $\mathcal{R}(X) = R(l_0, l_1, l_2)$ is not a polynomial ring, X is non-toric.

Using [5, Prop. III.3.1.5], we show that the set of singular points of X consists of the image $x_0 \in X$ of the point $(1, 0, 0, 0) \in \widehat{X}$ under the quotient map $\widehat{X} \to X$. If $l_1 l_2 - l_0 > 1$ holds, then the local divisor class group

$$\mathrm{Cl}(X, x_0) \;=\; \mathbb{Z}/(l_1 l_2 - l_0)\mathbb{Z}$$

is non-trivial and thus $x_0 \in X$ is singular. If $l_1 l_2 - l_0 = 1$ holds, then we have $l_0 > 1$ and therefore $(1, 0, 0, 0) \in \widehat{X}$. Hence, $x_0 \in X$ is singular. Since all other local divisor class groups of X are trivial and, moreover, all singular points of $\widehat{X}$ lie in the orbit $\mathbb{K}^* \cdot (1, 0, 0, 0)$, we conclude that $x_0 \in X$ is the only singular point.

According to [5, Prop. III.3.4.1], the anticanonical class of X is $l_1 + l_2 + 1 - l_0$. This proves (iii). Finally, for (iv), we infer from Proposition 7.7 that existence of a horizontal Demazure P-root is equivalent to existence of an integer α with $l_0 \le \alpha \le l_1$ which in turn is equivalent to $l_0 \le l_1$.

We come to the supplement. The surface X arises from a ring $R(A, P)$, where we may assume that $R(A, P)$ is minimally presented. The first task is to show that $n = 4$, $m = 0$ and $r = 2$ hold. We have

$$n + m - (r - 1) \;=\; \dim(X) + \mathrm{rk}(\mathrm{Cl}(X)) \;=\; 3.$$

Any relation g_i involving only three variables gives rise to a singularity in the source and a singularity in the sink of the $\mathbb{K}^*$-action. We conclude that at most two of the monomials occurring in the relations may depend only on one variable. Thus, the above equation shows that $n = 4$, $m = 0$ and $r = 2$ hold.

We may assume that the defining equation is of the form $T_{01}^{l_{01}} T_{02}^{l_{02}} + T_{11}^{l_{11}} + T_{21}^{l_{21}}$. Again, since one of the two elliptic fixed points must be smooth, we can conclude that one l_{0j} equals one, say l_{01}. Now it is a direct consequence of the description of the local divisor class groups given in [5, Prop. III.3.1.5] that a $\mathbb{K}^*$-surface with precisely one singularity arises from a matrix P as in the assertion. □

Now we are interested in the log-terminal varieties of the form $X(l_0, l_1, l_2)$. Recall, that a singularity is *log-terminal* if all its resolutions have discrepancies greater than -1. Over $\mathbb{C}$, the log-terminal surface singularities are precisely the quotient singularities by subgroups of $\mathrm{GL}_2(\mathbb{C})$. The *Gorenstein index* of X is the minimal positive integer $\imath(X)$ such that $\imath(X)$ times the canonical divisor $\mathcal{K}_X$ is Cartier.

Corollary 7.11. *Assume that $X = X(l_1, l_2, l_3)$ is log-terminal. Then we have the following three cases:*

(i) The surface X is almost homogeneous.

(ii) The singularity of X is of type E_7.

(iii) The singularity of X is of type E_8.

Moreover, for the almost homogeneous surfaces $X = X(l_1, l_2, l_3)$ of Gorenstein index $\imath(X) = a$, we have

(i) $(l_0, l_1, l_2) = (1, l_1, l_2)$ with the bounds $l_2 < l_1 < \frac{8}{3}a^2 + \frac{4}{3}a$,

(ii) $(l_0, l_1, l_2) = (2, l_1, 2)$ with the bound $l_1 < 4a$,

(iii) $(l_0, l_1, l_2) = (3, 3, 2), (2, 4, 3), (2, 5, 3), (3, 5, 2)$.

Proof. The surface $X(l_0, l_1, l_2)$ has only one singularity, occurring in the upper elliptic fixed point. We consider the canonical resolution of $X(l_0, l_1, l_2)$ as presented in Theorem 3.10. In the tropical resolution step we have discrepancy greater than -1 if and only if

$$l_0 l_1 l_2 \;<\; l_0 l_1 + l_0 l_2 + l_1 l_2.$$

Thus, the allowed (l_0, l_1, l_2) must be platonic triples (compare Example 3.20) and we are left with

$$(1, l_1, l_2),\; (2, l_1, 2),\; (3, 3, 2),\; (2, 4, 3),\; (2, 5, 3),\; (3, 5, 2),\; (4, 3, 2),\; (5, 3, 2).$$

The last two give the surfaces with singularities E_7, E_8 and in all other cases, the resulting surface is almost homogeneous by Proposition 7.10. Furthermore $X(l_0, l_1, l_2)$ has Gorenstein index a if and only if $a\mathcal{K}_X$ lies in the Picard group. According to [5, Cor. III.3.1.6], this is equivalent to the fact that $l_1 l_2 - l_0$ divides $a \cdot (l_1 + l_2 + 1 - l_0)$. The bounds then follow by the subsequent elementary estimations. First we consider the case $(l_0, l_1, l_2) = (1, l_1, l_2)$. Then the following equivalences hold with a suitable positive integer $b \in \mathbb{Z}_{>0}$:

$$\begin{aligned} l_1 l_2 - 1 \mid a \cdot (l_1 + l_2 + 1 - 1) \iff b(l_1 l_2 - 1) &= a(l_1 + l_2) \\ \iff b &= \frac{a}{l_2} + \frac{a}{l_1} + \frac{b}{l_1 l_2} \\ \iff b l_2 - a &= a\frac{l_2}{l_1} + \frac{b}{l_1}. \end{aligned}$$

In particular, $bl_2 - a$ is a positive integer. Since we assumed $l_1, l_2 > 1$ and $l_1 > l_2$, we have

$$\frac{3}{4} \leq b\frac{3}{4} = b(1 - \frac{1}{4}) < b(1 - \frac{1}{l_1 l_2}) \leq b - \frac{b}{l_1 l_2} = \frac{a}{l_1} + \frac{a}{l_2} < \frac{2a}{l_2} \leq a,$$

and hence $l_2 < \frac{8}{3}a$ as well as $b < \frac{4}{3}a$. All in all, we obtain

$$1 \leq bl_2 - a = \frac{al_2}{l_1} + \frac{b}{l_1} < \frac{1}{l_1} \cdot \left(\frac{8}{3}a^2 + \frac{4}{3}a\right),$$

and consequently

$$l_2 < l_1 < \frac{8}{3}a^2 + \frac{4}{3}a.$$

Now, we assume $(l_0, l_1, l_2) = (2, l_1, 2)$. Then we have the following equivalences for a positive integer $b \in \mathbb{Z}_{>0}$:

$$\begin{aligned} 2l_1 - 2 \mid a(l_1 + 1) \iff b(2l_1 - 2) &= a(l_1 + 1) \\ \iff 2bl_1 - al_1 &= 2b + a \\ \iff l_1 &= \frac{2b + a}{2b - a}. \end{aligned}$$

Since $2b + a \geq 0$ and $l_1 > 2$, we obtain $2b - a \geq 1$ and thus $2b \geq a + 1$ as well as $2b + a > 4b - 2a$, which gives $2b < 3a$. All in all, we can conclude $l_1 \leq 2b + a < 4a$. □

Corollary 7.12. *The following tables list the triples* (l_0, l_1, l_2) *together with roots of* $\operatorname{Aut}(X)$ *for the log-terminal almost homogeneous complete rational* $\mathbb{K}^*$*-surfaces* $X = X(l_0, l_1, l_2)$ *with precisely one singularity up to Gorenstein index* $\imath(X) = 5$.

$\imath(X) = 1$	$\imath(X) = 2$	$\imath(X) = 3$
$(1,3,2) : \{1,2,3\}$	$(1,7,3) : \{1,3,4,7\}$	$(2,7,2) : \{2,3,5,7\}$
$(2,3,2) : \{2,3\}$		$(1,13,4) : \{1,4,5,9,13\}$
$(3,3,2) : \{3\}$		$(1,8,5) : \{3,5,8\}$

$\imath(X) = 4$	$\imath(X) = 5$
$(2,5,2) : \{2,3,5\}$	$(2,11,2) : \{2,3,5,7,9,11\}$
$(1,21,5) : \{1,5,6,11,16,21\}$	$(1,13,7) : \{2,6,13\}$
	$(2,4,3) : \{3,4\}$
	$(1,17,3) : \{2,3,5,8,11,14,17\}$
	$(1,31,6) : \{1,6,7,13,19,25,31\}$
	$(1,18,7) : \{4,7,11,18\}$

We conclude this section with a series of almost homogeneous log-terminal $\mathbb{K}^*$-surfaces having one singularity with a D_5-like resolution graph.

Example 7.13. Let $p \geq 3$ be an odd element of $\mathbb{Z}$ and consider the following matrices:

$$P = \begin{pmatrix} -1 & -2 & p & 0 \\ -1 & -2 & 0 & 2 \\ 0 & 1 & \frac{p-1}{2} & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 2p-2 & 1 & 2 & p \end{pmatrix}.$$

They define a $\mathbb{K}^*$-surface X with Cox ring $R(X) = \mathbb{K}[T_{01}, T_{02}, T_{11}, T_{21}]/\langle T_{01}T_{02}^2 + T_{11}^p + T_{21}^2 \rangle$ and $\mathrm{Cl}(X) \cong \mathbb{Z}$ and the $\mathrm{Cl}(X)$-grading of $R(X)$ is given by Q. Note that the columns of P are always primitive since $\gcd(p, \frac{p-1}{2}) = 1$ holds. This can easily be seen by the following argument. Assume that there exists an element $a \in \mathbb{Z}_{>0}$ such that

$$p = ax \quad \text{and} \quad \frac{p-1}{2} = ay$$

for suitably chosen $x, y \in \mathbb{Z}_{>0}$. Then we have

$$\frac{p}{x} = \frac{p-1}{2y}$$

and thus $2py = px - x$ implies that p divides x. Consequently, we obtain $a = 1$. The $\mathbb{K}^*$-surface X has exactly one singularity in the upper elliptic fixed point corresponding to the cone $\sigma^+ = \mathrm{cone}(v_{02}, v_{11}, v_{21})$. Furthermore, we have

$$-K_X = 2p - 2 + 1 + 2 + p - 2p = p + 1 > 0.$$

In particular, X is Fano. For the degree of X, i.e. the self-intersection number of $-K_X$, one has

$$(-K_X)^2 = \frac{(p+1)^2}{2p(2p-2)} \cdot 2p = \frac{(p+1)^2}{2p-2}.$$

Now, we want to determine the Gorenstein index. Since X has only one singularity, only the upper elementary big cone σ^+ corresponding to this singularity is interesting. The determinant of this cone is $2p - 2$. Hence, the Gorenstein index is given by

$$\iota(X) = \frac{2p-2}{\gcd(2p-2, p+1)} = \mathrm{lcm}(2p-2, p+1).$$

In order to compute the horizontal Demazure P-roots of X we use the results (i) and (iii) of Proposition 7.7. According to case (i), all integers α satisfying $2 \leq \alpha \leq p$ and $2 \mid -\alpha+1$ are P-roots. Hence, all odd integers $3 \leq x \leq p$ give horizontal P-roots. According to case (iii), all integers α satisfying $2 \leq \alpha \leq p$ and $p \mid \frac{p-1}{2}\alpha + 1$ are P-roots. It is sufficient to consider all α that are even. Then α is a P-root if and only if p divides $1 - \frac{\alpha}{2}$, where $\alpha \leq p$. Consequently, we obtain exactly one more P-root, namely 2. Hence the set of horizontal Demazure P-roots is given by

$$\mathrm{Roots}(X) \;=\; \{2\} \cup \{3 \leq \alpha \leq p;\ \alpha \in \mathbb{Z}_{>0} \text{ odd}\}.$$

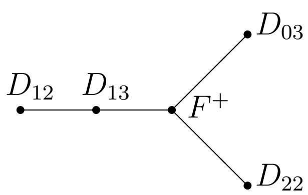
As already mentioned, the surface X has exactly one singularity corresponding to the upper elliptic fixed point. Following the canonical resolution procedure of Theorem 3.10 and contracting (-1)-curves afterwards we obtain a D_5-like resolution graph. The first resolution step is given by the stellar subdivision of σ^+ given by

$$(p-1) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = p \cdot \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} p \\ 0 \\ \frac{p-1}{2} \end{pmatrix} + p \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

Note that we can skip the stellar subdivision of the lower elementary cone σ^- corresponding to the lower elliptic fixed point since this is a smooth point. Using the formula for the discrepancy of Proposition 3.15 we obtain

$$\mathrm{Disc}_{(0,0,1)} = \frac{-(p-3)}{p-1} > -1.$$

In particular X, is log-terminal. The remaining toric modifications arise from stellar subdivisions by rays along $v_{03} = (-1,-1,1)$, $v_{22} = (0,1,0)$ and $v_{13} = (1,0,1)$, $v_{12} = (2,0,1)$. Note that they are independent of the choice of p.



We will give a short proof for that. Consider the cone $\sigma = \mathrm{cone}((0,1),(p,\frac{p-1}{2}))$. Then the Hilbert basis of σ is given by

$$H(\sigma) \;=\; \left\{(0,1),(1,1),(2,1),(p,\frac{p-1}{2})\right\}.$$

It is obvious that $(1,1)$ and $(2,1)$ are elements of $H(\sigma)$. Since

$$\det\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \;=\; 1, \quad \det\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = 1 \quad \text{and} \quad \det\begin{pmatrix} 2 & p \\ 1 & \frac{p-1}{2} \end{pmatrix} \;=\; -1,$$

the Hilbert basis has the asserted form. Consequently, the Cox ring $R(\widehat{X})$ of the resolution $\widehat{X}$ is given by $\mathbb{K}[T_{ij}, S_1]/\langle T_{01}T_{02}{}^2\mathrm{T}_{03} + T_{11}^p T_{12}^2 T_{13} + T_{21}^2 T_{22}\rangle$ and $\widehat{X} = X(A, \widehat{P})$ holds, where

$$\widehat{P} \;=\; \begin{pmatrix} -1 & -2 & -1 & p & 2 & 1 & 0 & 0 & 0 \\ -1 & -2 & -1 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & \frac{p-1}{2} & 1 & 1 & -1 & 0 & 1 \end{pmatrix}.$$

In case of $p = 3$ we have the canonical D_5-singularity. For $p > 3$ the resolution graph remains the same, but the intersection numbers of the exceptional divisors are growing. By using the formulas of Proposition 4.21 we obtain

$$(F^+)^2 = D_{13}^2 = D_{03}^2 = D_{22}^2 = -2 \quad \text{and} \quad D_{12}^2 = -\frac{p+1}{2}.$$

7.3 Semisimple P-roots

A linear algebraic group is called *semisimple* if it has only trivial closed connected solvable normal subgroups. Each linear algebraic group has a maximal connected solvable normal subgroup H such that G/H is semisimple. This quotient is the semisimple part of G and it is denoted by G^{ss}. The unit component $R(G) = H^\circ$ is called the *radical* of G. Furthermore, we define the *unipotent radical* $R_u(G)$ as the set of all unipotent elements of $R(G)$. Note that G^{ss} is uniquely determined up to conjugacy by elements of the unipotent radical $R_u(G)$. With these notions we can reformulate the definition of a semisimple group: A linear algebraic group is called semisimple if its radical $R(G)$ is trivial and a linear algebraic group is called *reductive* if its unipotent radical $R_u(G)$ is trivial.

If G is semisimple, then its roots $\Phi_G \subseteq \mathbb{X}_{\mathbb{R}}(T)$ with respect to a given maximal torus T form a root system. This means that for every $\alpha \in \Phi_G$ one has

$$\Phi_G \cap \mathbb{R}\alpha \;=\; \{\pm\alpha\}, \qquad s_\alpha(\Phi_G) = \Phi_G,$$

where $s_\alpha\colon \mathbb{X}_{\mathbb{R}}(T) \to \mathbb{X}_{\mathbb{R}}(T)$ denotes the reflection through the hyperplane $\alpha^\vee$ perpendicular to α with respect to a given scalar product on $\mathbb{X}_{\mathbb{R}}(T)$. For our purpose the following root systems are important:

$$\begin{aligned} A_n &:= \{e_i - e_j;\ 1 \le i, j \le n+1, i \ne j\} \;\subseteq\; \mathbb{R}^{n+1}, \\ B_2 &:= \{\pm e_1, \pm e_2, \pm(e_1 + e_2), \pm(e_1 - e_2)\} \;\subseteq\; \mathbb{R}^2. \end{aligned}$$

We turn to varieties with a complexity-one torus action. Consider data (A, P) as in Construction 2.10 and the resulting ring $R(A, P)$. Recall that $R(A, P)$ is equipped with a K_0-grading and a coarser K-grading. The grading group K_0 splits as

$$K_0 \;=\; K_0^{\mathrm{vert}} \oplus K_0^{\mathrm{hor}}, \quad \text{where} \quad K_0^{\mathrm{vert}} \;:=\; \langle \deg_{K_0}(S_k)\rangle, \quad K_0^{\mathrm{hor}} \;:=\; \langle \deg_{K_0}(T_{ij})\rangle,$$

and $K_0^{\mathrm{vert}} \cong \mathbb{Z}^m$ is freely generated by $\deg_{K_0}(S_1), \ldots, \deg_{K_0}(S_m)$. Moreover, the direct factor $\mathbb{Z}^s$ of the column space $\mathbb{Z}^{r+s}$ of P is identified via $Q_0 \circ P^*$ with the kernel of the downgrading map $K_0 \to K$, compare Construction 2.10.

Definition 7.14. Let (A, P) be data as in Construction 2.10 such that the associated ring $R(A, P)$ is minimally presented and write α_κ for the P-root, i.e. the $\mathbb{Z}^s$-part associated to the Demazure P-root κ.

(a) We call a P-root α_κ *semisimple* if $-\alpha_\kappa = \alpha_{\kappa'}$ holds for some Demazure P-root κ', i.e. $-\alpha_k$ is a P-root.

(b) We call a semisimple P-root α_κ *vertical* if $\alpha_\kappa \in K_0^{\mathrm{vert}}$ and *horizontal* if $\alpha_\kappa \in K_0^{\mathrm{hor}}$ holds.

(c) We write Φ_P^{ss}, Φ_P^{vert} and Φ_P^{hor} for the set of semisimple, vertical semisimple and horizontal semisimple P-roots in $\mathbb{R}^s$, respectively.

The main result about the semisimple roots of a complexity-one T-variety X given in [6] is the following theorem.

Theorem 7.15. (See [6, Theorem 7.2]). *Let A, P be as in Construction 2.10 such that $R(A, P)$ is minimally presented and let X be a (non-toric) variety with a complexity-one torus action $T \times X \to X$ arising from data (A, P) according to Construction 2.20. Then the following statements hold:*

(i) $\Phi_P^{\mathrm{vert}}, \Phi_P^{\mathrm{hor}}$ *and* Φ_P^{ss} *are root systems with* $\Phi_P^{\mathrm{ss}} = \Phi_P^{\mathrm{vert}} \oplus \Phi_P^{\mathrm{hor}}$ *and* Φ_P^{ss} *is the root system with respect to T of the semisimple part* $\mathrm{Aut}(X)^{\mathrm{ss}}$.

(ii) For $p \in K$ denote by m_p the number of variables S_k with $\deg_K(S_k) = p$. Then

$$\Phi_P^{\mathrm{vert}} \;\cong\; \bigoplus_{p\in K} A_{m_p-1}, \qquad\qquad \sum_{p\in K}(m_p - 1) \;<\; \dim(X) - 1.$$

(iii) Let $\Phi_P^{\mathrm{hor}} \neq \emptyset$. Then $r = 2$ holds, and, after suitably renumbering the variables one has

(a) $T_{01}T_{02} + T_{11}T_{12} + T_2^{l_2}$, $w_{01} = w_{11}$ *and* $w_{02} = w_{12}$,

(b) $T_{01}T_{02} + T_{11}^2 + T_2^{l_2}$, $w_{01} = w_{02} = w_{11}$,

for the defining relation of $R(A, P)$ and the degrees $w_{ij} = \deg_K(T_{ij})$ of the variables.

(iv) In the above case (iii)(a), there are the following possibilities for the root system Φ_P^{hor}*:*

- *If $l_{21} + \ldots + l_{2n_2} \geq 3$ holds, then*

$$\Phi_P^{\mathrm{hor}} \;=\; \begin{cases} A_1 \oplus A_1, & w_{01} = w_{02} = w_{11} = w_{12}, \\ A_1, & \textit{otherwise}. \end{cases}$$

- *If* $n_2 = 2$ *and* $l_{21} = l_{22} = 1$ *hold, then*

$$\Phi_P^{\mathrm{hor}} = \begin{cases} A_3, & w_{01} = w_{02} = w_{11} = w_{12} = w_{21} = w_{22}, \\ A_2, & w_{01} = w_{11} = w_{21}, w_{02} = w_{12} = w_{22}, w_{01} \neq w_{02}, \\ A_1 \oplus A_1, & w_{01} = w_{02} = w_{11} = w_{21}, w_{01} \neq w_{21}, w_{01} \neq w_{22}, \\ A_1, & \textit{otherwise.} \end{cases}$$

(v) *In the above case (iii)(b), there are the following possibilities for the root system* Φ_P^{hor}*:*

- *If* $l_{21} + \ldots + l_{2n_2} \geq 3$ *holds, then*

$$\Phi_P^{\mathrm{hor}} = A_1.$$

- *If* $n_2 = 1$ *and* $l_{21} = 2$ *hold, then*

$$\Phi_P^{\mathrm{hor}} = \begin{cases} A_1 \oplus A_1, & w_{01} = w_{02} = w_{11} = w_{21}, \\ A_1, & \textit{otherwise.} \end{cases}$$

- *If* $n_2 = 2$ *and* $l_{21} = l_{22} = 1$ *hold, then*

$$\Phi_P^{\mathrm{hor}} = \begin{cases} B_2, & w_{01} = w_{02} = w_{11} = w_{21} = w_{22}, \\ A_1, & \textit{otherwise.} \end{cases}$$

We will prove some lemmas that contribute to the proof of this theorem and which are needed in the next section for classification issues.

Let X be a complexity-one T-variety, arising from a matrix P, having a pair of semisimple roots $\alpha_\pm \in \Phi_P^{\mathrm{hor}}$. Then we infer from [6, Lemma 7.7] that $r = 2$ holds. After reordering l_0, l_1 and l_2 the following two cases can occur:

- We have $n_0 = n_1 = 2$ and $l_{01} = l_{02} = l_{11} = l_{12} = 1$ and for any pair of Demazure P-roots $u_\pm$ associated to $\alpha_\pm$ one has $i_0^+ = i_0^- = 2$.
- We have $n_0 = 1$, $l_{01} = 2$ and $n_1 = 2$, $l_{11} = l_{12} = 1$ and for any pair of Demazure P-roots $u_\pm$ associated to $\alpha_\pm$ one has $i_0^+ = i_0^- = 2$.

In particular, for a given pair $\alpha_\pm \in \Phi_P^{\mathrm{hor}}$, all associated pairs of Demazure P-roots share the same $i_0 = i_0^+ = i_0^-$. This allows us to speak about the *distinguished index* i_0 *of* $\alpha_\pm \in \Phi_P^{\mathrm{hor}}$.

We briefly recall which elementary row and column operations of the matrix P are called *admissible* (compare Definition 2.23):

(i) Switch two columns inside a block $v_{i1}, \ldots, v_{in_i}$.

(ii) Switch two whole column blocks $v_{i1},\dots v_{in_i}$ and $v_{j1},\dots,v_{jn_j}$.

(iii) Add multiples of the upper r rows the one of the last s rows.

(iv) Any elementary row operation among the last s rows.

(v) Switch two columns inside the d' block.

Lemma 7.16. *Let $n_0 = n_1 = 2$ and $l_{01} = l_{02} = l_{11} = l_{12} = 1$. If there exists a pair $\alpha_\pm \in \Phi_P^{\mathrm{hor}}$ with distinguished index $i_0 = 2$, then P can be transformed by admissible operations, without moving the n_2-block, into the form*

$$P \;=\; \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_2 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & d_{12}^* & d_2^* & d'_* \end{bmatrix}, \tag{9}$$

where the lower line is a matrix of size $(s-1)\times(n+m)$. Conversely, if P is of the above shape, then $\alpha_\pm = (\pm 1, 0) \in \Phi_P^{\mathrm{hor}}$ has distinguished index $i_0 = 2$. Moreover, up to admissible operations of type (iii) and (iv), situation (9) is equivalent to

$$\deg_K(T_{01}) \;=\; \deg_K(T_{12}), \qquad\qquad \deg_K(T_{02}) \;=\; \deg_K(T_{11}).$$

Proof. Fix an associated pair $\kappa_\pm = (u^\pm, 2, i_1^\pm, C_\pm)$ of Demazure P-roots. Renumbering the variables, we first achieve $i_1^+ = 1$ and $C_+ = (1,1,1)$. Adding suitable multiples of the top two rows of P to the lower s rows brings P into the form

$$P \;=\; \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_2 & 0 \\ 0 & d_{02} & 0 & d_{12} & d_2 & d' \end{bmatrix}.$$

Now we explicitly go through the defining conditions of the Demazure P-root κ_+ with

$$u^+ \;=\; (u_1^+, u_2^+, \alpha_+), \qquad \text{where} \quad u_i^\pm \in \mathbb{Z}, \quad i_1^+ \;=\; 1, \quad C_+ \;=\; (1,1,1).$$

This gives the following root conditions:

$$\begin{aligned}
\langle u^+, v_{01}\rangle &= u_1^+ = -1,\\
\langle u^+, v_{02}\rangle &= -u_1^+ - u_2^+ + \langle \alpha_+, d_{02}\rangle = \langle \alpha_+, d_{02}\rangle \ge 1,\\
\langle u^+, v_{11}\rangle &= u_1^+ + u_2^+ = 0,\\
\langle u^+, v_{12}\rangle &= u_1^+ + \langle \alpha_+, d_{12}\rangle = -1 + \langle \alpha_+, d_{12}\rangle \ge 0,\\
\langle u^+, v_{2j}\rangle &= l_{2j}u_2^+ + \langle \alpha_+, d_{2j}\rangle \ge 0,\\
\langle u^+, v_k\rangle &= \langle \alpha_+, d'_k\rangle \ge 0.
\end{aligned}$$

Consequently, $u_2^+ = 1$ and $\langle \alpha_+, d_{2j}\rangle \ge -l_{2j}$ hold. Since $\alpha_- = -\alpha_+$, we obtain $\langle \alpha_-, d_{02}\rangle \le -1$, $\langle \alpha_-, d_{12}\rangle \le -1$ and $\langle \alpha_-, d_{2j}\rangle \le l_{2j}$. For u^- we obtain independently of the choice of C_- the condition

$$0 \le \langle u^-, d_{2j}\rangle = l_{2j}u_2^- + \langle \alpha_-, d_{2j}\rangle \le l_{2j}u_2^- + l_{2j},$$

and consequently $u_2^- \geq -1$. Now, we have to go through all possible cases for i_1 and $C_- = (c_0, c_1, c_2)$.

We assume $i_1 = 1$. If $c_0 = 1$, then

$$\begin{aligned}\langle u^-, v_{01}\rangle &= -u_1^- - u_2^- = 0\\ \langle u^-, v_{02}\rangle &= -u_1^- - u_2^- + \langle \alpha_-, d_{02}\rangle = \langle \alpha_-, d_{02}\rangle \geq 1,\end{aligned}$$

a contradiction to $\langle \alpha_-, d_{02}\rangle \leq -1$. If $c_1 = 1$, then $\langle u^-, v_{11}\rangle = u_1^- = -1$ and $\langle u^-, v_{12}\rangle = u_1^- + \langle \alpha_-, d_{12}\rangle = -1 + \langle \alpha_-, d_{12}\rangle \geq 0$ must hold, a contradiction to $\langle \alpha_-, d_{12}\rangle \leq -1$. For $c_0 = c_1 = 2$ we have

$$\begin{aligned}\langle u^-, v_{01}\rangle &= -u_1^- - u_2^- \geq 1,\\ \langle u^-, v_{02}\rangle &= -u_1^- - u_2^- + \langle \alpha_-, d_{02}\rangle = 0,\\ \langle u^-, v_{11}\rangle &= u_1^- \geq 0,\\ \langle u^-, v_{12}\rangle &= u_1^- + \langle \alpha_-, d_{12}\rangle = -1.\end{aligned}$$

Adding $\langle u^-, v_{02}\rangle$ and $\langle u^-, v_{12}\rangle$ gives $-u_2^- - 2 \geq -u_2^- + \langle \alpha_-, d_{02}\rangle + \langle \alpha_-, d_{12}\rangle = -1$ and thus $u_2^- \leq -1$. Consequently, $u_2^- = -1$ holds. Putting this into the first inequality we obtain $u_1^- \leq 0$ and together with the third inequality $u_1^- = 0$.

Now, we assume $i_1 = 0$. If $c_1 = 1$, then $\langle u^-, v_{11}\rangle = u_1^- = 0$ and $\langle u^-, v_{12}\rangle = u_1^- + \langle \alpha_-, d_{12}\rangle = \langle \alpha_-, d_{12}\rangle \geq 1$, a contradiction to $\langle \alpha_-, d_{12}\rangle \leq -1$. If $c_0 = 1$, then $\langle u^-, v_{01}\rangle = -u_1^- - u_2^- = -1$ and $\langle u^-, v_{02}\rangle = -u_1^- - u_2^- + \langle \alpha_-, d_{02}\rangle = -1 + \langle \alpha_-, d_{12}\rangle \geq 0$, a contradiction to $\langle \alpha_-, d_{02}\rangle \leq -1$. For $c_0 = c_1 = 2$, we have

$$\begin{aligned}\langle u^-, v_{01}\rangle &= -u_1^- - u_2^- \geq 0,\\ \langle u^-, v_{02}\rangle &= -u_1^- - u_2^- + \langle \alpha_-, d_{02}\rangle = -1,\\ \langle u^-, v_{11}\rangle &= u_1^- \geq 1,\\ \langle u^-, v_{12}\rangle &= u_1^- + \langle \alpha_-, d_{12}\rangle = 0.\end{aligned}$$

Adding $\langle u^-, v_{02}\rangle$ and $\langle u^-, v_{12}\rangle$ gives $-u_2^- - 2 \geq -u_2^- + \langle \alpha_-, d_{02}\rangle + \langle \alpha_-, d_{12}\rangle = -1$ and thus $u_2^- \leq -1$. Consequently, $u_2^- = -1$ holds. Putting this into the first inequality we obtain $u_1^- \leq 1$ which implies together with the third inequality that $u_1^- = 1$ holds. Thus, we are left with the two possibilities

$$\begin{aligned}u^- &= (1, -1, -\alpha_+), & i_1^- &= 0, & C_- &= (2, 2, 1),\\ u^- &= (0, -1, -\alpha_+), & i_1^- &= 1, & C_- &= (2, 2, 1).\end{aligned}$$

In both cases, we obtain

$$\begin{aligned}&\langle \alpha_+, d_{02}\rangle = \langle \alpha_+, d_{12}\rangle = 1,\\ &\langle \alpha_+, d_{2j}\rangle = -l_{2j} \text{ for } j = 1, \ldots, n_2,\\ &\langle \alpha_+, d_k'\rangle = 0 \text{ for } j = 1, \ldots, m.\end{aligned}$$

Now choose any invertible $(s \times s)$-matrix with α_+ as its first row and apply it from the left to P. Then the third row of P looks as follows:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & -l_2 & 0 \end{bmatrix}$$

Adding suitable multiples of the third row to the last $s-1$ rows and adding the second to the third row brings P into the desired form. The remaining statements are directly checked. □

Lemma 7.17. *Let $n_0 = 1$, $l_{01} = 2$ and $n_2 = 2$, $l_{11} = l_{12} = 1$. Then there is at most one pair $\alpha_\pm \in \Phi_P^{\mathrm{hor}}$ with distinguished index $i_0 = 2$. If there is one, then P can be brought by admissible operations, without moving the n_2-block, into the form*

$$P \;=\; \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & l_2 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ d_{01}^* & 0 & 0 & d_2^* & d'_* \end{bmatrix}, \tag{10}$$

where the lower line is a matrix of size $(s-1) \times (n+m)$. Conversely, if P is of the above shape, then $\alpha_\pm = (\pm 1, 0) \in \Phi_P^{\mathrm{hor}}$ has distinguished index $i_0 = 2$. Moreover, up to admissible operations of type (iii) and (iv), situation (10) is equivalent to

$$\deg_K(T_{01}) \;=\; \deg_K(T_{11}) \;=\; \deg_K(T_{12}).$$

Proof. This is a similar computation as in the previous lemma. Clearly, we may assume $C_+ = (1,1,1)$ and by suitable row operations, we bring P into the form

$$P \;=\; \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & l_2 & 0 \\ d_{01} & 0 & d_{12} & d_2 & d' \end{bmatrix}.$$

Now enter the defining conditions of a Demazure P-root κ_+ with $u^+ = (u_1^+, u_2^+, \alpha_+)$. Since $i_0 = 2$ holds, we have $i_1^+ = i_1^- = 0$. Hence,

$$\begin{aligned}
\langle u^+, v_{01}\rangle &= -2u_1^+ - 2u_2^- + \langle \alpha_+, d_{01}\rangle = -1,\\
\langle u^+, v_{11}\rangle &= u_1^+ = 0,\\
\langle u^+, v_{12}\rangle &= u_1^+ + \langle \alpha_+, d_{12}\rangle \geq 1,\\
\langle u^+, v_{2j}\rangle &= u_2^+ l_{2j} + \langle \alpha_+, d_{2j}\rangle \geq 0,\\
\langle u^+, v_k\rangle &= \langle \alpha_+, d'_k\rangle \geq 0.
\end{aligned}$$

Note that the root conditions are independent of the choice of c_2. If $c_1 = 1$, we obtain the inequality $\langle u^-, v_{12}\rangle = \langle \alpha_-, d_{12}\rangle = -\langle u^+, v_{12}\rangle \geq 1$, a contradiction to $\langle \alpha_+, d_{12}\rangle \geq 1$. Consequently, $c_1 = 2$ must hold and we have the following root conditions.

$$\begin{aligned}
\langle u^-, v_{01}\rangle &= -2u_1^- - 2u_2^- + \langle \alpha_-, d_{01}\rangle = -1,\\
\langle u^-, v_{11}\rangle &= u_1^- \geq 1,\\
\langle u^-, v_{12}\rangle &= u_1^- + \langle \alpha_-, d_{12}\rangle = 0,\\
\langle u^-, v_{2j}\rangle &= u_2^- l_{2j} + \langle \alpha_-, d_{2j}\rangle \geq 0,\\
\langle u^-, v_k\rangle &= \langle \alpha_-, d'_k\rangle \geq 0.
\end{aligned}$$

Adding $\langle u^+, v_{01}\rangle$ and $\langle u^-, v_{01}\rangle$ we obtain $-2u_1^- - 2u_2^- - 2u_2^+ = -2$ and hence $u_1^- = -u_2^- - u_2^+ + 1 \geq 1$ which gives $u_2^- + u_2^+ \leq 0$. Adding $\langle u^+, v_{2j}\rangle$ and $\langle u^-, v_{2j}\rangle$ we obtain $(u_2^- + u_2^+)l_{2k} \geq 0$ and hence $u_2^- + u_2^+ \geq 0$. Consequently, $u_2^+ = u_2^-$ must hold which implies $u_1^- = 1$. All in all we end up with $u_1^+ = 0$ and

$$\begin{aligned}
\langle \alpha_+, d_{01}\rangle &= 2u_2^+ - 1, \\
\langle \alpha_+, d_{12}\rangle &= 1, \\
\langle \alpha_+, d_{2j}\rangle &= -u_2^+ l_{2j} \text{ for } j = 1, \ldots, n_2, \\
\langle \alpha_+, d'_k\rangle &= 0 \text{ for } j = 1, \ldots, m.
\end{aligned}$$

Analogously to the proof of Lemma 7.16, this enables us to bring P via suitable row operations into the desired form. Again, the remaining statements are directly seen. □

Lemma 7.18. *Let $1 \leq k_0^+ < k_0^- \leq m$ and denote by $f \in \mathbb{Z}^{n+m}$ the vector with $f_{n+k_0^\pm} = \mp 1$ and all other entries zero. Then the following statements are equivalent.*

(i) There exists a pair $\alpha^\pm$ of vertical semisimple roots corresponding to the indices $k_0^\pm$.

(ii) The vector f can be realized as the $(r+1)$-th row of P by applying only admissible operations of type (iii) and (iv).

(iii) The variables $S_{k_0^+}$ and $S_{k_0^-}$ have the same degree with respect to the $\mathrm{Cl}(X)$-grading.

Proof. To prove that (i) implies (ii), let $\kappa^\pm = (u^\pm, k_0^\pm)$ be a pair of Demazure P-roots associated to $\alpha^\pm$. Then we have $\langle u^\pm, v_{ij}\rangle \geq 0$ for all $0 \leq i \leq r, 1 \leq j \leq n_i$ and $\langle u^\pm, v_k\rangle \geq 0$ for all $1 \leq k \leq m,\ k \neq k_0^\pm$, as well as $\langle u^+, v_{k_0^-}\rangle \geq 0,\ \langle u^+, v_{k_0^+}\rangle = -1$ and $\langle u^-, v_{k_0^+}\rangle \geq 0,\ \langle u^-, v_{k_0^-}\rangle = -1$. We define $u := u^+ + u^-$ and conclude $\langle u, v_{ij}\rangle \geq 0$ for all $0 \leq i \leq r, 1 \leq j \leq n_i$ and $\langle u, v_k\rangle \geq 0$ for all $1 \leq k \leq m,\ k \neq k_0^\pm$. Since $\alpha^+ = \alpha^-$ and the first s coordinates of every column v_k are zero, we obtain

$$\langle u, v_{k_0^+}\rangle = \langle u^+, v_{k_0^+}\rangle + \langle u^-, v_{k_0^+}\rangle = \langle u^+, v_{k_0^+}\rangle - \langle u^+, v_{k_0^+}\rangle = 0$$

and analogously

$$\langle u, v_{k_0^-}\rangle = \langle u^+, v_{k_0^-}\rangle + \langle u^-, v_{k_0^-}\rangle = -\langle u^-, v_{k_0^-}\rangle + \langle u^-, v_{k_0^-}\rangle = 0,$$

which yields $\langle u, v_{ij}\rangle \geq 0$ for all $0 \leq i \leq r, 1 \leq j \leq n_i$ and $\langle u, v_k\rangle \geq 0$ for all $1 \leq k \leq m$. Furthermore, we know that the columns of P are generating $\mathbb{Q}^{r+s}$ as a cone. This implies $u = 0$ and thus $u^- = -u^+$. Consequently, we conclude

$$\langle u^+, v_{ij}\rangle = 0 \quad \text{for all } i, j, \quad \langle u^+, v_k\rangle = 0 \quad \text{for all } k \neq k_0^\pm,$$

$$\langle u^+, v_{k_0^+}\rangle = -1, \quad \langle u^+, v_{k_0^-}\rangle = 1.$$

Now, write $u^+ = (u_1^+, \alpha^+)$ with the $\mathbb{Z}^s$-part α^+ and let σ be a $((s-1) \times s)$-matrix complementing the row α^+ to a unimodular matrix. Then applying the block matrix

$$\begin{pmatrix} E_r & 0 \\ u_1^+ & \alpha^+ \\ 0 & \sigma \end{pmatrix},$$

where E_r is the r-dimensional identity matrix, from the left to P describes admissible operations of type (iii) and (iv) realizing the vector f as the $(r+1)$-th row of the resulting matrix P.

Now assume that f is the $(r+1)$-th row of P and consider $u^{\pm} \in \mathbb{Z}^s$ with $u^{\pm}_{r+1} = \pm 1$ and all other entries zero. Then the $\mathbb{Z}^s$-parts $\alpha^{\pm}$ of the vertical Demazure P-roots $(u^{\pm}, k_0^{\pm})$ are representing a pair of semisimple roots. This shows the implication from (ii) to (i).

Since P is the kernel of the grading matrix Q, it is obvious that (ii) implies (iii).

In the last step we will prove that (iii) implies (ii). Therefore assume that $S_{k_0^+}$ and $S_{k_0^-}$ have the same $\mathrm{Cl}(X)$-degree. This is equivalent to $Q(e_{k_0^+}) = Q(e_{k_0^-})$ and thus to $Q(e_{k_0^-} - e_{k_0^+}) = 0$ which means that $f = e_{k_0^-} - e_{k_0^+}$ is an element of the kernel of Q and consequently is contained in the lattice generated by the rows of P. Since f is a linear combination of the rows of P, there exists a linear form u such that

$$u(v_{ij}) = 0, \quad u(v_k) = 0 \text{ for } k \neq k_0^{\pm}, \quad u(v_{k_0^+}) = -1, \quad u(v_{k_0^-}) = 1.$$

Applying the block matrix of the same form as above from the left to P yields statement (ii). □

7.4 Almost homogeneous 3-folds with reductive automorphism group

A linear algebraic group G is called *reductive* if the radical of the connected unit component G° is an algebraic torus. Equivalently, one can require the unipotent radical of G° to be trivial. The automorphism group of a complexity-one T-variety is reductive if and only if its roots build a root system, i.e. X has only semisimple roots. Hence, one can use the description of semisimple roots to classify complexity-one T-varieties with reductive automorphism group. The aim of this chapter is to describe all three-dimensional almost homogeneous complexity-one T-varieties with reductive automorphism group and Picard number one.

Proposition 7.19. *Let X be a three-dimensional non-toric complete normal rational variety. Suppose that X is almost homogeneous under an action of a reductive group and there is an effective action of a two-dimensional torus on X. Then the Cox ring of X is given as $\mathcal{R}(X) = R(A, P)$ with a matrix P according to the following cases.*

$$\text{(i)} \qquad P \quad = \quad \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_2 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & d_{12}^* & d_2^* & d_*' \end{bmatrix},$$

$$\text{(ii)} \qquad P \quad = \quad \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & l_2 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ d_{01}^* & 0 & 0 & d_2^* & d_*' \end{bmatrix}.$$

In both cases, $m \leq 2$ holds; this means that the d_*'-part can be either empty, equal to ± 1 or equal to $(\pm 1, \mp 1)$.

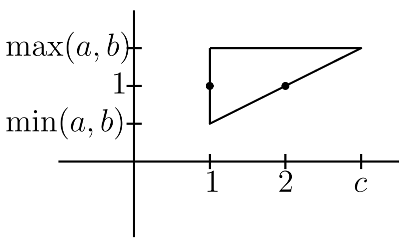
Proof. Clearly, we may assume that we are in the situation of Theorem 7.15. Since X is non-toric but almost homogeneous, there must be a semisimple horizontal P-root. Thus, Lemmas 7.16 and 7.17 show that after admissible operations, P is of the desired shape. □

Lemma 7.20. *Let $x = (1, a)$, $y = (1, b)$ and $z = (c, b)$ be points in $\mathbb{Q}^2$. Then the following statements are equivalent.*

(i) The simplex $\operatorname{conv}(x, y, z)$ contains an integral point.

(ii) There is an integer d with $\min(a, b) \leq d \leq \max(a, b)$.

Proof. Clearly, (ii) implies (i). So let $q = (q_1, q_2)$ be an integral point in $\operatorname{conv}(x, y, z)$. Then its second coordinate satisfies $\min(a, b) \leq q_2 \leq \max(a, b)$.



□

Lemma 7.21. *Let X be an almost homogeneous three-dimensional complexity-one T-variety with Picard number one and reductive automorphism group. Then X has no vertical semisimple P-roots.*

Proof. Since X is almost homogeneous, it has at least one pair of semisimple horizontal Demazure P-roots. Hence, by Theorem 7.15 we have $r = 2$ and $n_0 \geq 2$. Since $\operatorname{rk}(\operatorname{Cl}(X)) = 1$ holds, we obtain $m \leq 1$ which contradicts the existence of a pair of semisimple vertical Demazure P-roots. □

Theorem 7.22. *Let X be a $\mathbb{Q}$-factorial three-dimensional complete normal variety of Picard number one. Suppose that $\mathrm{Aut}(X)$ is reductive, has a maximal torus of dimension two and acts with an open orbit on X. Then X is a rational Fano variety and, up to isomorphy, X arises from a matrix P of one of the following. Additionally, we give the free part of the Cox ring grading up to a multiple $\beta \in \mathbb{Z}_{>0}$:*

$$(i) \quad P = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & l_{21} \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d_{12} & d_{21} \end{pmatrix}, \quad l_{21} > 1,\ d_{12} > 2,\ -\frac{d_{21}}{d_{12}-1} < l_{21} < -d_{21},$$

$$\beta Q^0 = (-d_{21}, d_{21} + l_{21}d_{12}, d_{21} + l_{21}d_{12}, -d_{21}, d_{12}).$$

$$(ii) \quad P = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & l_{21} & l_{22} \\ -1 & 0 & 1 & 0 & 0 \\ d_{01} & 0 & 0 & d_{21} & d_{22} \end{pmatrix}, \quad l_{21}, l_{22} > 1,\ 2d_{22} > -d_{01}l_{22},\ -2d_{21} > d_{01}l_{21},$$

$$\beta Q^0 = (l_{21}d_{22} - l_{22}d_{21}, l_{21}d_{22} - l_{22}d_{21}, l_{21}d_{22} - l_{22}d_{21}, 2d_{22} + d_{01}l_{22}, -2d_{21} - d_{01}l_{21}).$$

$$(iii) \quad P = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & l_{22} \\ -1 & 0 & 1 & 0 & 0 \\ d_{01} & 0 & 0 & d_{21} & d_{22} \end{pmatrix}, \quad \begin{array}{l} l_{22} > 1,\ d_{22} > d_{21}l_{22} + l_{22}, \\ 2d_{22} > -d_{01}l_{22},\ -2d_{21} > d_{01}, \end{array}$$

or

$$P = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & l_{22} \\ -1 & 0 & 1 & 0 & 0 \\ d_{01} & 0 & 0 & d_{21} & d_{22} \end{pmatrix}, \quad l_{22} > 1,\ 2d_{22} > -d_{01}l_{22},\ 1 - 2d_{21} > d_{01},$$

$$\beta Q^0 = (d_{22} - l_{22}d_{21}, d_{22} - l_{22}d_{21}, d_{22} - l_{22}d_{21}, 2d_{22} + d_{01}l_{22}, -2d_{21} - d_{01}).$$

$$(iv) \quad P = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad P = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

$$\beta Q^0 = (1, 1, 1, 1, 1).$$

$$(v) \quad P = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & l_{21} & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & d_{21} & 1 \end{pmatrix}, \quad 1 < l_{21} < -2d_{21} < 2l_{21},$$

$$\beta Q^0 = (l_{21}, l_{21}, l_{21}, 2, -2d_{21} - l_{21}).$$

Conversely, each of the above listed matrices defines a $\mathbb{Q}$-factorial rational almost homogeneous Fano variety with reductive automorphism group having a two-dimensional maximal torus.

Proof. First note that up to a common multiple $\beta \in \mathbb{Z}_{>0}$ the weight vectors w_{ij}^0, u_k^0 of Q^0 can be easily computed by minors of P. They are the determinants of the matrices P_{ij} and P_k resulting from P by deleting the column v_{ij} and v_k respectively. Without loss of generality, we may assume them to be positive. From Proposition 7.19 we infer the possible forms of the matrix P. In Lemma 7.16 and Lemma 7.17 we proved that every such variety X has at least one pair of semisimple roots, namely $\alpha_\pm = (\pm 1, 0)$. The Demazure P-roots $(u_1, u_2, \alpha_1, \alpha_2) \in \mathbb{Z}^4$ are the lattice points of the root polytope. It is given by the five conditions for Demazure P-roots, splitting into two equations and three inequations. We can resolve the equations for u_1 and u_2. Then the P-roots $\alpha = (\alpha_1, \alpha_2)$ are given by the lattice points of a polytope in $\mathbb{Z}^2$ defined by the three remaining inequations. With help of computer routines we determine the possible vertices of this polytope and conditions ensuring the polytope not to be empty. Thereby, we deduce conditions to exclude all roots that are not semisimple.

(i) First we consider the situation $l_{01} = l_{02} = l_{11} = l_{12} = 1$ and $l_{21} > 1$ with $d_{12} > 0$ and $0 < -d_{21} < l_{21}d_{12}$. Every choice of

$$(i_0, i_1) \in \{(0,2), (2,0), (1,2), (2,1)\} \quad \text{and} \quad C \in \{(1,1,1), (1,2,1), (1,1,2), (1,2,2)\}$$

can cause possible roots. If $(i_0, i_1) = (2,1)$ or $(i_0, i_1) = (2,0)$ and $C = (1,1,1)$, the root polytope coincides with the integral point $(1,0)$. If $(i_0, i_1) = (2,1)$ or $(i_0, i_1) = (2,0)$ and $C = (2,2,1)$, the root polytope coincides with the integral point $(-1,0)$. Hence, there are twelve remaining cases that have to be considered:

(1) For $(i_0, i_1) = (0,2)$ and $C = (1,1,1)$ we have the vertices

$$E_1 = \left(\frac{d_{21} + d_{12}}{d_{21} + l_{21}d_{12}}, \frac{l_{21} - 1}{d_{21} + l_{21}d_{12}}\right), \quad E_2 = \left(\frac{d_{21} + d_{12}}{d_{21}}, -\frac{1}{d_{21}}\right), \quad E_3 = \left(0, -\frac{1}{d_{21}}\right),$$

and the condition for a non-empty root polytope

$$d_{21} + d_{12} \geq 0.$$

This condition is equivalent to

$$0 < \frac{l_{21} - 1}{d_{21} + l_{21}d_{12}} \leq -\frac{1}{d_{21}} \leq 1.$$

In particular $(\pm 1, 0)$ is not contained in the root polytope. Since we want to refer to Lemma 7.20 we apply the unimodular transformation

$$U = \begin{pmatrix} 1 - d_{12} & 1 \\ 0 & 1 \end{pmatrix}.$$

The vertices E_1, E_2 and E_3 are sent to the coordinates

$$U(E_1) = \left(1, \frac{l_{21} - 1}{d_{21} + l_{21}d_{12}}\right), \quad U(E_2) = \left(1, -\frac{1}{d_{21}}\right), \quad U(E_3) = \left(-\frac{d_{12}}{d_{21}}, -\frac{1}{d_{21}}\right).$$

Consequently, there are integral points in the root polytope if and only if

$$-\frac{1}{d_{21}} = 1 \quad \text{and} \quad \frac{l_{21}-1}{d_{21}+l_{21}d_{12}} \leq 1.$$

The condition $d_{21} = -1$ implies the second condition. The integral points of the modified polytope are given by $(x, 1)$ with $d_{12} \geq x \geq 1$. Consequently, the lattice points of the root polytope lie between $(1-d_{12}, 1)$ and $(1, 1)$. Since the two root equations are given by $\langle u, v_{11}\rangle = u_1 = 0$ and $\langle u, v_{21}\rangle = u_2 l_{21} + \alpha_2 d_{21} = -1$, we obtain Demazure P-roots by setting $u_1 = u_2 = 0$. Consequently, the existence of Demazure P-roots is equivalent to the condition $d_{21} = -1$.

(2) For $(i_0, i_1) = (0, 2)$ and $C = (1, 2, 1)$ we have the vertices

$$E_1 = \left(-\frac{l_{21}d_{12}+d_{21}-d_{12}}{d_{21}+l_{21}d_{12}}, -\frac{1}{d_{21}+l_{21}d_{12}}\right),\ E_2 = \left(-\frac{l_{21}d_{12}+d_{21}-d_{12}}{d_{21}}, \frac{l_{21}-1}{d_{21}}\right),$$
$$E_3 = \left(0, -\frac{1}{d_{21}+l_{21}d_{12}}\right),$$

and the condition for a non-empty root polytope

$$l_{21}d_{12} + d_{21} - d_{12} \leq 0.$$

This condition is equivalent to

$$0 > \frac{l_{21}-1}{d_{21}} \geq -\frac{1}{d_{21}+l_{21}d_{12}} \geq -1.$$

In particular, $(\pm 1, 0)$ is not contained in the root polytope. Since we want to use Lemma 7.20, we first have to apply the unimodular transformation

$$U = \begin{pmatrix} 1 & d_{12} \\ 0 & 1 \end{pmatrix}$$

which then gives us the new vertices

$$U(E_1) = \left(-1, -\frac{1}{d_{21}+l_{21}d_{12}}\right), \quad U(E_2) = \left(-1, \frac{l_{21}-1}{d_{21}}\right),$$
$$U(E_3) = \left(-\frac{d_{12}}{d_{21}+l_{21}d_{12}}, -\frac{1}{d_{21}+l_{21}d_{12}}\right).$$

Consequently, the root polytope contains integral points if and only if

$$-\frac{1}{d_{21}+l_{21}d_{12}} = -1 \quad \text{and} \quad -1 \leq \frac{l_{21}-1}{d_{21}}.$$

Since the first condition already implies the second one, we can restrict to the single condition

$$d_{21} + l_{21}d_{12} = 1.$$

The lattice points in the modified polytope are given by $(-x,-1)$ with $1 \le x \le d_{12}$. Consequently, the lattice points in the root polytope lie between $(-1+d_{12},-1)$ and $(0,-1)$. The two root equations are given by $\langle u, v_{12}\rangle = u_1+\alpha_1+\alpha_2 d_{12} = 0$ and $\langle u, v_{21}\rangle = u_2 l_{21} + \alpha_2 d_{21} = -1$. Hence, (α_1,α_2) can be completed to Demazure P-roots by $u_2 = -d_{12}$. This implies that the existence of Demazure P-roots is equivalent to the condition $d_{21}+l_{21}d_{12} = 1$.

(3) For $(i_0,i_1) = (0,2)$ and $C = (2,1,1)$ we have the vertices

$$E_1 = \left(\frac{d_{21}+d_{12}}{d_{21}+l_{21}d_{12}}, \frac{l_{21}-1}{d_{21}+l_{21}d_{12}}\right),\ E_2 = \left(\frac{d_{21}+d_{12}}{d_{21}}, -\frac{1}{d_{21}}\right),\ E_3 = \left(0, -\frac{1}{d_{21}}\right),$$

and the condition for a non-empty root polytope

$$d_{21}+d_{12} \ \ge\ 0.$$

Thus, this case is equivalent to case (1).

(4) For $(i_0,i_1) = (0,2)$ and $C = (2,2,1)$ we have the vertices

$$E_1 = \left(-\frac{l_{21}d_{12}+d_{21}-d_{12}}{d_{21}+l_{21}d_{12}}, -\frac{1}{d_{21}+l_{21}d_{12}}\right),\ E_2 = \left(-\frac{l_{21}d_{12}+d_{21}-d_{12}}{d_{21}}, \frac{l_{21}-1}{d_{21}}\right),$$
$$E_3 = \left(0, -\frac{1}{d_{21}+l_{21}d_{12}}\right),$$

and the condition for a non-empty root polytope

$$l_{21}d_{12}+d_{21}-d_{12} \ \le\ 0.$$

Thus, this case is equivalent to case (2).

(5) For $(i_0,i_1) = (1,2)$ and $C = (1,1,1)$, we have the vertices

$$E_1 = \left(1, \frac{l_{21}-1}{d_{21}}\right), \quad E_2 = \left(1, -\frac{1}{d_{21}+l_{21}d_{12}}\right),$$
$$E_3 = \left(\frac{d_{12}}{d_{21}+l_{21}d_{12}}, -\frac{1}{d_{21}+l_{21}d_{12}}\right),$$

and the condition for a non-empty root polytope

$$l_{21}d_{12}+d_{21}-d_{12} \ \le\ 0.$$

Thus, this case is equivalent to case (2). Note that the two root equations are given by $\langle u, v_{01}\rangle = -u_1-u_2-\alpha_1 = 0$ and $\langle u, v_{21}\rangle = u_2 l_{21}+\alpha_2 d_{21} = -1$.

(6) For $(i_0,i_1) = (1,2)$ and $C = (1,2,1)$ we have the vertices

$$E_1 = \left(1, \frac{l_{21}-1}{d_{21}}\right), \quad E_2 = \left(1, -\frac{1}{d_{21}+l_{21}d_{12}}\right),$$
$$E_3 = \left(\frac{d_{12}}{d_{21}+l_{21}d_{12}}, -\frac{1}{d_{21}+l_{21}d_{12}}\right),$$

and the condition for a non-empty root polytope

$$l_{21}d_{12} + d_{21} - d_{12} \;\leq\; 0.$$

Thus, this case is equivalent to case (5) and (2).

(7) For $(i_0, i_1) = (1, 2)$ and $C = (2, 1, 1)$ we have the vertices

$$E_1 \;=\; \left(-1, -\frac{1}{d_{21}}\right), \quad E_2 \;=\; \left(-1, \frac{l_{21}-1}{d_{21}+l_{21}d_{12}}\right), \quad E_3 \;=\; \left(\frac{d_{12}}{d_{21}}, -\frac{1}{d_{21}}\right),$$

and the condition for a non-empty root polytope

$$d_{21} + d_{12} \;\geq\; 0.$$

Thus this case is equivalent to case (1). Note that the two root equations are given by $\langle u, v_{11}\rangle = -u_1 - u_2 = 0$ and $\langle u, v_{21}\rangle = u_2 l_{21} + \alpha_2 d_{21} = -1$.

(8) For $(i_0, i_1) = (1, 2)$ and $C = (2, 2, 1)$ we get the vertices

$$E_1 \;=\; \left(-1, -\frac{1}{d_{21}}\right), \quad E_2 \;=\; \left(-1, \frac{l_{21}-1}{d_{21}+l_{21}d_{12}}\right), \quad E_3 \;=\; \left(\frac{d_{12}}{d_{21}}, -\frac{1}{d_{21}}\right),$$

and the condition for a non-empty root polytope

$$d_{21} + d_{12} \;\geq\; 0.$$

Thus, this case is equivalent to case (7) and (1).

(9) For $(i_0, i_1) = (2, 0)$ and $C = (1, 2, 1)$ we have the vertices

$$E_1 = \left(-\frac{d_{21}}{d_{21}+l_{21}d_{12}}, -\frac{l_{21}}{d_{21}+l_{21}d_{12}}\right), \; E_2 = \left(1, -\frac{2}{d_{12}}\right), \; E_3 = \left(1, -\frac{l_{21}}{d_{21}+l_{21}d_{12}}\right),$$

and the condition for a non-empty root polytope

$$2d_{21} + l_{21}d_{12} \;\leq\; 0.$$

This condition is equivalent to

$$-\frac{2}{d_{12}} \;\geq\; -\frac{l_{21}}{d_{21}+l_{21}d_{12}}.$$

Consequently, there is an integral point contained in the root polytope if and only if there exists an integer x satisfying

$$-\frac{l_{21}}{d_{21}+l_{21}d_{12}} \;\leq\; x \;\leq\; -\frac{2}{d_{12}}.$$

Note that the two root equations are given by $\langle u, v_{01}\rangle = -u_1 - u_2 - u_3 = -1$ and $\langle u, v_{12}\rangle = u_1 + \alpha_1 + \alpha_2 d_{12} = 0$. Hence, lattice points of the root polytope can always be completed to Demazure P-roots. All in all, there is a P-root different from $(\pm 1, 0)$ if and only if one of the following two conditions are satisfied:

$$d_{12} = 1,\ l_{21} + 2d_{21} \le 0 \quad \text{or} \quad d_{12} \ge 2,\ l_{21}(d_{12} - 1) + d_{21} \le 0.$$

(10) For $(i_0, i_1) = (2, 0)$ and $C = (2, 1, 1)$ we have the vertices

$$E_1 = \left(\frac{d_{21} + l_{21}d_{12}}{d_{21}}, -\frac{l_{21}}{d_{21}}\right), \quad E_2 = \left(-1, \frac{2}{d_{12}}\right), \quad E_3 = \left(-1, -\frac{l_{21}}{d_{21}}\right),$$

and the condition for a non-empty root polytope

$$2d_{21} + l_{21}d_{12} \ \ge\ 0.$$

This condition is equivalent to

$$\frac{2}{d_{12}} \ \le\ -\frac{l_{21}}{d_{21}}.$$

Consequently, there are integral points in the root polytope if and only if there is an integer x satisfying

$$\frac{2}{d_{12}} \ \le\ x \ \le\ -\frac{l_{21}}{d_{21}}.$$

Note that the two root equations are given by $\langle u, v_{11}\rangle = u_1 = 0$ and $\langle u, v_{02}\rangle = -u_1 - u_2 = -1$. Hence, lattice points of the root polytope can always be completed to Demazure P-roots. All in all, there is a P-root different from $(\pm 1, 0)$ if and only if the following conditions are satisfied:

$$d_{12} = 1,\ 0 \le l_{21} + 2d_{21} \quad \text{or} \quad 2 \le d_{12},\ 0 \le d_{21} + l_{21}.$$

(11) For $(i_0, i_1) = (2, 1)$ and $C = (1, 2, 1)$ we have the vertices

$$E_1 = \left(1, -\frac{l_{21}}{d_{21} + l_{21}d_{12}}\right),\ E_2 = \left(1, -\frac{2}{d_{12}}\right),\ E_3 = \left(-\frac{d_{21}}{d_{21} + l_{21}d_{12}}, -\frac{l_{21}}{d_{21} + l_{21}d_{12}}\right),$$

and the condition for a non-empty root polytope

$$2d_{21} + l_{21}d_{12} \ \le\ 0.$$

Thus, this case is equivalent to case (9).

(12) For $(i_0, i_1) = (2, 1)$ and $C = (2, 1, 1)$ we get the vertices

$$E_1 = \left(-1, -\frac{l_{21}}{d_{21}}\right), \quad E_2 = \left(-1, \frac{2}{d_{12}}\right), \quad E_3 = \left(\frac{d_{21} + l_{21}d_{12}}{d_{21}}, -\frac{l_{21}}{d_{21}}\right),$$

and the condition for a non-empty root polytope

$$2d_{21} + l_{21}d_{12} \ \ge\ 0.$$

Thus, this case is equivalent to case (10). Note that the two root equations are given by $\langle u, v_{11}\rangle = u_1 = -1$ and $\langle u, v_{02}\rangle = -u_1 - u_2 = 0$.

Finally, we have to put all the conditions together. There is a vertical Demazure P-root not equal to $(\pm 1, 0)$ if and only if one of the following conditions is satisfied:

$$d_{21} = -1 \qquad (1)$$
$$d_{21} + l_{21}d_{12} = 1 \qquad (2)$$
$$d_{12} = 1, l_{21} + 2d_{21} \leq 0 \qquad (3)$$
$$d_{12} \geq 2,\ l_{21}(d_{12} - 1) + d_{21} \leq 0 \qquad (4)$$
$$d_{12} = 1, l_{21} + 2d_{21} \geq 0 \qquad (5)$$
$$d_{12} \geq 2,\ d_{21} + l_{21} \geq 0 \qquad (6)$$

Note that conditions (3) and (5) are equivalent to $d_{12} = 1$ and that conditions (4) and (6) can not be satisfied simultaneously since $-l_{21} \leq d_{21} \leq -l_{21}(d_{12} - 1)$ implies $d_{12} = 1$. Negating all the conditions and taking the positivity of the weights into account we obtain that to avoid vertical Demazure P-root not equal to $(\pm 1, 0)$ all of the following conditions have to be satisfied:

$$d_{21} < -1 \qquad (1')$$
$$d_{21} + l_{21}d_{12} > 1 \qquad (2')$$
$$d_{12} > 1 \qquad (3')$$
$$l_{21}(d_{12} - 1) + d_{21} > 0 \qquad (4')$$
$$d_{21} + l_{21} < 0 \qquad (5')$$

Note that (4') implies (2'). Finally, we can deduce the conditions of assertion (i).

So far, we described all varieties having no other roots but $(\pm 1, 0)$. The computations, that we made so far can also be used to describe those cases, where we have more than one pair of semistable roots. One easily checks that there is only one possible choice for the parameters, namely $d_{21} = -1$, $d_{12} = 1$ and $l_{21} = 2$. Then we have the P-roots $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm 1, \mp 1)$ and $(\pm 1, \mp 2)$, and the P-matrix is given by

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

(ii) Now, we assume $l_{11} = l_{12} = 1$, $l_{21}, l_{22} > 1$ and $l_{01} = 2$. First note that this implies $w_{11} = w_{12} = w_{01} > w_{21}, w_{22}$. So the conditions for all weights to be positive are

$$l_{21}d_{22} - l_{22}d_{21} \ > \ 2d_{22} + d_{01}l_{22} \ > \ 0 \quad \text{and} \quad l_{21}d_{22} - l_{22}d_{21} \ > \ -2d_{21} - d_{01}l_{21} \ > \ 0.$$

Since $l_{21}, l_{22} > 1$, we have to check eight cases given by all possible choices of $(i_0, i_1) \in \{(0, 2), (2, 0)\}$ and $C \in \{(1, 1, 1), (1, 2, 1), (1, 1, 2), (1, 2, 2)\}$. If $(i_0, i_1) = (2, 0)$ and $C =$

$(1,1,1)$ or $C = (1,1,2)$, the root polytope consists only of the integral point $(1,0)$. Furthermore, if $(i_0,i_1) = (2,0)$ and $C = (1,2,1)$ or $C = (1,2,2)$, the root polytope consists only of the integral point $(-1,0)$. So we are left with the following four cases:

(1) For $(i_0,i_1) = (0,2)$ and $C = (1,1,1)$, we have the vertices

$$\begin{aligned} E_1 &= \left(1, \frac{l_{21}-2}{2d_{21}-d_{01}l_{21}}\right), \quad E_2 = \left(1, \frac{l_{22}}{l_{21}d_{22}-l_{22}d_{21}}\right), \\ E_3 &= \left(\frac{2d_{22}+d_{01}l_{22}}{l_{21}d_{22}-l_{22}d_{21}}, \frac{l_{22}}{l_{21}d_{22}-l_{22}d_{21}}\right), \end{aligned}$$

and the condition for a non-empty root polytope

$$-l_{21}d_{22} + l_{22}d_{21} + 2d_{22} + d_{01}l_{22} \geq 0.$$

This condition is satisfied if and only if

$$\frac{l_{21}-2}{2d_{21}+d_{01}l_{21}} \geq \frac{l_{22}}{l_{21}d_{22}-l_{22}d_{21}} > 0,$$

which is only satisfied if $l_{21} = 1$, a contradiction to $l_{21} > 1$. Thus, in this case there are no P-roots.

(2) For $(i_0,i_1) = (0,2)$ and $C = (1,2,1)$ we have the vertices

$$\begin{aligned} E_1 &= \left(-1, \frac{l_{21}-2}{2d_{21}-d_{01}l_{21}}\right), \quad E_2 = \left(-1, \frac{l_{22}}{l_{21}d_{22}-l_{22}d_{21}}\right), \\ E_1 &= \left(-\frac{2d_{22}+d_{01}l_{22}}{l_{21}d_{22}-l_{22}d_{21}}, \frac{l_{22}}{l_{21}d_{22}-l_{22}d_{21}}\right), \end{aligned}$$

and the condition for a non-empty root polytope

$$-l_{21}d_{22} + l_{22}d_{21} + 2d_{22} + d_{01}l_{22} \geq 0.$$

Consequently, this case can be treated analogously to case (1).

(3) For $(i_0,i_1) = (0,2)$ and $C = (1,1,2)$ we have the vertices

$$\begin{aligned} E_1 &= \left(1, \frac{l_{22}-2}{2d_{22}+d_{01}l_{22}}\right), \quad E_2 = \left(1, \frac{l_{21}}{-l_{21}d_{22}+l_{22}d_{21}}\right), \\ E_1 &= \left(-\frac{2d_{21}+d_{01}l_{21}}{l_{21}d_{22}-l_{22}d_{21}}, -\frac{l_{21}}{l_{21}d_{22}-l_{22}d_{21}}\right), \end{aligned}$$

and the condition for a non-empty root polytope

$$l_{21}d_{22} - l_{22}d_{21} + 2d_{21} + d_{01}l_{21} \leq 0.$$

This condition is satisfied if and only if

$$\frac{l_{22}-2}{2d_{22}+d_{01}l_{22}} \leq -\frac{l_{21}}{l_{21}d_{22}-l_{22}d_{21}} < 0,$$

which is only satisfied, if $l_{22} = 1$, a contradiction to $l_{22} > 1$. Thus, in this case there are no P-roots.

(4) For $(i_0, i_1) = (0, 2)$ and $C = (1, 2, 2)$ we have the vertices

$$E_1 = \left(-1, \frac{l_{22}-2}{2d_{22}+d_{01}l_{22}}\right), \quad E_2 = \left(-1, \frac{l_{21}}{-l_{21}d_{22}+l_{22}d_{21}}\right),$$
$$E_1 = \left(\frac{2d_{21}+d_{01}l_{21}}{l_{21}d_{22}-l_{22}d_{21}}, -\frac{l_{21}}{l_{21}d_{22}-l_{22}d_{21}}\right),$$

and the condition for a non-empty root polytope

$$l_{21}d_{22} - l_{22}d_{21} + 2d_{21} + d_{01}l_{21} \leq 0.$$

Consequently, this case can be treated analogously to case (3).

Summarized, there are no P-roots but $(\pm 1, 0)$. Hence, we only have to ensure that the weights are positive, which gives the assertion.

(iii) Now, we assume that $l_{21} = 1$ and $l_{22} > 1$. The positivity conditions for the weights are

$$2d_{22} > -d_{01}l_{22}, \quad -2d_{21} > d_{01} \quad \text{and} \quad d_{22} > l_{22}d_{21}.$$

Note that the first two inequalities imply the last one. We have to go through all possibilities $(i_0, i_1) \in \{(0,1), (1,0), (0,2), (2,0)\}$ and $C \in \{(1,1,1), (1,2,1), (1,1,2), (1,2,2)\}$. So we have to check 16 cases. Note that the cases $(i_0, i_1) = (0,1), C = (1,1,2)$ and $(i_0, i_1) = (0,1)$, $C = (1,2,2)$ as well as $(i_0, i_1) = (1,0)$, $C = (1,1,2)$ and $(i_0, i_1) = (1,0)$, $C = (1,2,2)$ can not occur since we assumed $l_{22} > 1$. Furthermore, if $(i_0, i_1) = (2,0)$ and $C = (1,1,1)$ or $C = (1,1,2)$, the root polytope only consists of the integral point $(1,0)$ and if $(i_0, i_1) = (2,0)$ and $C = (1,2,1)$ or $C = (1,2,2)$ it consists only of the integral point $(-1,0)$. So we are left with the following eight cases:

(1) For $(i_0, i_1) = (0,1)$ and $C = (1,1,1)$, we have the vertices

$$E_1 = \left(\frac{2d_{22}+d_{01}l_{22}}{d_{22}-l_{22}d_{21}}, \frac{l_{22}}{d_{22}-l_{22}d_{21}}\right), E_2 = \left(1, \frac{l_{22}}{d_{22}-l_{22}d_{21}}\right), E_3 = \left(1, -\frac{1}{2d_{21}+d_{01}}\right),$$

and the conditions for a non-empty root polytope

$$d_{22} + l_{22}d_{21} + d_{01}l_{22} \geq 0.$$

The non-emptiness condition is fulfilled if and only if

$$-\frac{1}{2d_{21}+d_{01}} \geq \frac{l_{22}}{d_{22}-l_{22}d_{21}}.$$

Consequently by Lemma 7.20 there is an integral point inside the root polytope if and only if there is an integer x satisfying

$$0 \leq \frac{l_{22}}{d_{22}-l_{22}d_{21}} \leq x \leq -\frac{1}{2d_{21}+d_{01}} \leq 1.$$

The root equations are given by $\langle u, v_{21}\rangle = u_2 + \alpha_2 d_{21} = 0$ and $\langle u, v_{11}\rangle = u_1 = -1$. Thus, the existence of P-roots not equal to $(\pm 1, 0)$ is equivalent to the conditions

$$2d_{21} + d_{01} = -1,\ l_{22} \leq d_{22} - l_{22}d_{21}.$$

(2) For $(i_0, i_1) = (0, 1)$ and $C = (1, 2, 1)$ we have the vertices

$$\begin{aligned} E_1 &= \left(-\frac{2d_{22} + d_{01}l_{22}}{d_{22} - l_{22}d_{21}}, \frac{l_{22}}{d_{22} - l_{22}d_{21}}\right), E_2 = \left(-1, \frac{l_{22}}{d_{22} - l_{22}d_{21}}\right), \\ E_3 &= \left(-1, -\frac{1}{2d_{21} + d_{01}}\right), \end{aligned}$$

and the condition for a non-empty root polytope

$$d_{22} + l_{22}d_{21} + d_{01}l_{22} \geq 0.$$

Consequently, this case can be treated analogously to case (1).

(3) For $(i_0, i_1) = (0, 2)$ and $C = (1, 1, 1)$ we have the vertices

$$E_1 = \left(1, -\frac{1}{2d_{21} + d_{01}}\right), E_2 = \left(1, \frac{l_{22}}{d_{22} - l_{22}d_{21}}\right), E_3 = \left(\frac{2d_{22} + d_{01}l_{22}}{d_{22} - l_{22}d_{21}}, \frac{l_{22}}{d_{22} - l_{22}d_{21}}\right),$$

and the condition for a non-empty root polytope

$$d_{22} + l_{22}d_{21} + d_{01}l_{22} \geq 0.$$

Consequently, this case can be treated analogously to case (1).

(4) For $(i_0, i_1) = (0, 2)$ and $C = (1, 2, 1)$ we have the vertices

$$\begin{aligned} E_1 &= \left(-1, -\frac{1}{2d_{21} + d_{01}}\right) \quad E_2 = \left(-1, \frac{l_{22}}{d_{22} - l_{22}d_{21}}\right), \\ E_3 &= \left(-\frac{2d_{22} + d_{01}l_{22}}{d_{22} - l_{22}d_{21}}, \frac{l_{22}}{d_{22} - l_{22}d_{21}}\right), \end{aligned}$$

and the condition for a non-empty root polytope

$$d_{22} + l_{22}d_{21} + d_{01}l_{22} \geq 0.$$

Consequently, this case can be also treated analogously to case (1).

(5) For $(i_0, i_1) = (0, 2)$ and $C = (1, 1, 2)$ we have the vertices

$$\begin{aligned} E_1 &= \left(1, \frac{l_{22} - 2}{2d_{22} + d_{01}l_{22}}\right) \quad E_2 = \left(1, -\frac{1}{d_{22} - l_{22}d_{21}}\right), \\ E_3 &= \left(-\frac{2d_{21} + d_{01}}{d_{22} - l_{22}d_{21}}, -\frac{1}{d_{22} - l_{22}d_{21}}\right), \end{aligned}$$

and the condition for a non-empty root polytope

$$-d_{22} + l_{22}d_{21} - 2d_{21} - d_{01} \geq 0.$$

The non-emptiness condition is full-filled if and only if

$$-\frac{1}{d_{22} - l_{22}d_{21}} \geq \frac{l_{22} - 2}{2d_{22} + d_{01}l_{22}}.$$

Since $d_{22} - l_{22}d_{21} > 0$, this is only satisfied for $l_{22} = 1$ which we have excluded.
(6) For $(i_0, i_1) = (0, 2)$ and $C = (1, 2, 2)$ we have the vertices

$$\begin{aligned} E_1 &= \left(-1, \frac{l_{22} - 2}{2d_{22} + d_{01}l_{22}}\right) \quad E_2 = \left(-1, -\frac{1}{d_{22} - l_{22}d_{21}}\right), \\ E_3 &= \left(\frac{2d_{21} + d_{01}}{d_{22} - l_{22}d_{21}}, -\frac{1}{d_{22} - l_{22}d_{21}}\right), \end{aligned}$$

and the condition for a non-empty root polytope

$$-d_{22} + l_{22}d_{21} - 2d_{21} - d_{01} \geq 0.$$

Consequently, this case can be treated analogously to case (5).
(7) For $(i_0, i_1) = (1, 0)$ and $C = (1, 1, 1)$ we have the vertices

$$\begin{aligned} E_1 &= \left(\frac{d_{22} + l_{22}d_{21} + d_{01}l_{22}}{d_{22} - l_{22}d_{21}}, \frac{l_{22}}{d_{22} - l_{22}d_{21}}\right), \\ E_2 &= \left(-\frac{d_{22} + l_{22}d_{21} + d_{01}l_{22}}{d_{22} - l_{22}d_{21}}, \frac{l_{22}}{d_{22} - l_{22}d_{21}}\right), \\ E_3 &= \left(0, -\frac{1}{2d_{21} + d_{01}}\right), \end{aligned}$$

and the condition for a non-empty root polytope

$$d_{22} + l_{22}d_{21} + d_{01}l_{22} \geq 0.$$

The root polytope is not empty if and only if

$$\frac{l_{22}}{d_{22} - l_{22}d_{21}} \leq -\frac{1}{2d_{21} + d_{01}}.$$

Now, we are applying the following unimodular transformation

$$U = \begin{pmatrix} 1 & -2d_{21} - d_{01} \\ 0 & 1 \end{pmatrix}.$$

The vertices E_1, E_2 and E_3 are sent to

$$\begin{aligned} U(E_1) &= \left(1, \frac{l_{22}}{d_{22}-l_{22}d_{21}}\right), \quad U(E_2) = \left(-\frac{d_{22}+3l_{22}d_{21}+2d_{01}l_{22}}{d_{22}-l_{22}d_{21}}, \frac{l_{22}}{d_{22}-l_{22}d_{21}}\right), \\ U(E_3) &= \left(1, -\frac{1}{2d_{21}+d_{01}}\right). \end{aligned}$$

Hence, we use Lemma 7.20 which says that there is an integral point inside the root polytope different from $(\pm 1, 0)$ if and only if there is an integer x satisfying

$$\frac{l_{22}}{d_{22}-l_{22}d_{21}} \leq x \leq -\frac{1}{2d_{21}+d_{01}}.$$

If this is fulfilled, then the lattice point in the modified polytope is $(1,1)$ and the lattice point in the root polytope is $(\alpha_1, \alpha_2) = (1 + 2d_{21} + d_{01}, 1)$. Consequently, we obtain the same conditions as in case (1):

$$2d_{21} + d_{01} = -1 \qquad l_{22} \leq d_{22} - l_{22}d_{21}.$$

Anyway, here, the existence of a Demazure P-root additionally requires an integer condition to be satisfied since the two root equations are $\langle u, v_{21}\rangle = u_2 + \alpha_2 d_{21} = 0$, $\langle u, v_{01}\rangle = -2u_1 - 2u_2 - \alpha_1 + \alpha_2 d_{01} = -1$. The pair (α_1, α_2) can be completed to a Demazure P-root if and only if $\alpha_1 - \alpha_2 d_{01} - 1 = 1 + 2d_{01} + d_{01} + d_{01} - 1 = 2d_{01} + 2d_{01}$ is even, which is always the case.

(8) For $(i_0, i_1) = (1, 0)$ and $C = (1, 2, 1)$ we have the vertices

$$\begin{aligned} E_1 &= \left(\frac{d_{22}+l_{22}d_{21}+d_{01}l_{22}}{d_{22}-l_{22}d_{21}}, \frac{l_{22}}{d_{22}-l_{22}d_{21}}\right), \\ E_2 &= \left(-\frac{d_{22}+l_{22}d_{21}+d_{01}l_{22}}{d_{22}-l_{22}d_{21}}, \frac{l_{22}}{d_{22}-l_{22}d_{21}}\right), \\ E_3 &= \left(0, -\frac{1}{2d_{21}+d_{01}}\right), \end{aligned}$$

and the condition for a non-empty root polytope

$$d_{22} + l_{22}d_{21} + d_{01}l_{22} \geq 0.$$

Consequently, we get the same results as in case (7).

Finally we have to summarize all the cases and to negate the conditions for existence of a P-roots. There are no other P-roots but $(\pm 1, 0)$ if and only if

$$1 < -2d_{21} - d_{01},\ 0 < 2d_{22} + d_{01}l_{22}$$

or

$$d_{22} - l_{22}d_{21} - l_{22} < 0,\ 0 < 2d_{22} + d_{01}l_{22},\ 0 < -2d_{21} - d_{01}.$$

Furthermore, one easily checks that the only P-roots that could occur in this case are $(-1,1)$, $(1,1)$ and $(0,1)$. In particular, they are not semisimple.

(iv) If $l_{21} = 1$, then by adding suitably often the second row to the last row, we may assume $d_{21} = 0$. This implies $d_{01} = d_{22} = 1$ and hence we obtain the matrix

$$P = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

One easily computes that this threefold has exactly four pairs of semisimple P-roots, namely $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm 1, \pm 1)$ and $(\mp 1, \pm 1)$. Note that the two matrices of the assertion are defining isomorphic threefolds.

(v) Finally, we assume $m = 1$, $l_{01} = 2$, $l_{11} = l_{12} = 1$ and $l_{21} > 0$. To ensure positivity of the weights we require $-2d_{21} - l_{21}d_{01} > 0$. Since we have a free variable S_k there can both exist, vertical and horizontal Demazure roots. For the horizontal case there are only four cases to check. Consider $i = (2,0)$. For $C = (1,1,1)$ the root polytope consists only of the point $(1,0)$ and for $C = (1,2,1)$ it consists of $(-1,0)$. If we have $i = (0,2)$ and $C = (1,1,1)$, then the vertices are

$$E_1 = \left(1, \frac{l_{21} - 2}{2d_{21} + l_{21}d_{01}}\right), \quad E_2 = (1,0), \quad E_3 = \left(\frac{2}{l_{21}}, 0\right)$$

and the condition for a non-empty root polytope is given by

$$\frac{l_{21} - 2}{2d_{21} + l_{21}d_{01}} \geq 0.$$

Since $-2d_{21} - l_{21}d_{01} > 0$, this yields $l_{21} \leq 2$. Since X is not toric, we have $l_{21} \geq 2$ and thus $l_{21} = 2$. Consequently $(1,0)$ is the only possible root. In case of $i = (0,2)$ and $C = (1,2,1)$ analogous argumentation leads to the result that $(-1,0)$ is the only possible root.

Now we consider the situation for vertical Demazure roots. The conditions for a vertical Demazure root $z = [z_1, z_2, z_3, z_4]$ are as follows:

$$\begin{aligned} \langle z, v_1 \rangle &= z_4 = -1, \\ \langle z, v_{01} \rangle &= -2z_1 - 2z_2 - z_3 + z_4 d_{01} \geq 0, \\ \langle z, v_{11} \rangle &= z_1 \geq 0, \\ \langle z, v_{12} \rangle &= z_1 + z_3 \geq 0, \\ \langle z, v_{21} \rangle &= z_2 l_{21} + d_{21} z_4 \geq 0. \end{aligned}$$

First note that these conditions are fulfilled if and only if they are fulfilled by a vector $[0, z_2, 0, -1]$. Consequently, the conditions can be simplified to

$$\frac{d_{21}}{l_{21}} \leq z_2 \leq -\frac{1}{2}d_{01}.$$

Since the weight $\deg(S_1) = -2d_{21} - l_{21}d_{01}$ is positive by assumption, so is the difference of the two bounds which is given by

$$-\frac{1}{2}d_{01} - \frac{d_{21}}{l_{21}}.$$

Consequently, there is a vertical root if and only if d_{01} is even or if d_{01} is odd and

$$-\frac{1}{2}d_{01} - \frac{d_{21}}{l_{21}} \geq \frac{1}{2}. \quad (*)$$

Since we have $m = 1$ there are no semisimple vertical roots. Thus, we have to exclude the existence of vertical roots. In particular, d_{01} has to be odd. By suitably often adding the second row to the last row we can achieve $d_{01} = 1$. Furthermore, negating $(*)$ we obtain the condition $l_{21} > -d_{21}$. Together with the positivity condition $-2d_{21} - l_{21} > 0$ and the assumption $l_{21} > 1$ we receive the statement of (v). □

Corollary 7.23 (of Theorem 7.15)**.** *Let X be a non-toric complexity-one variety arising from data (A, P) as in Theorem 7.15. If X has Picard number one and satisfies $\Phi_P^{hor} \neq \emptyset$, then X is Fano. In particular, all varieties of Theorem 7.22 are Fano.*

Proof. In this situation the anticanonical class w_X in $\mathrm{Cl}(X)$ is given by

$$w_X = \sum_{i=0}^{r}\sum_{j=1}^{n_i} w_{ij} + \sum_{k=1}^{m} u_k - (r-1)\deg(g_0).$$

By Theorem 7.15(iii) $r = 2$ holds and we have to distinguish two cases:

(i) $g_0 = T_{01}T_{02} + T_{11}T_{12} + T_2^{l_2}$ with $w_{01} = w_{11}$ and $w_{02} = w_{12}$,

(ii) $g_0 = T_{01}T_{02} + T_{11}^2 + T_2^{l_2}$ with $w_{01} = w_{02} = w_{11}$.

Consequently, we obtain

$$\begin{aligned}
(i) \qquad w_X &= 2w_{01} + 2w_{02} + \sum_{j=1}^{n_2} w_{2j} + \sum_{k=1}^{m} u_k - w_{01} - w_{02} \\
&= w_{01} + w_{02} + \sum_{j=1}^{n_2} w_{2j} + \sum_{k=1}^{m} u_k > 0, \\
(ii) \qquad w_X &= 3w_{01} + \sum_{j=1}^{n_2} w_{2j} + \sum_{k=1}^{m} u_k - 2w_{01} = w_{01} + \sum_{j=1}^{n_2} w_{2j} + \sum_{k=1}^{m} u_k > 0,
\end{aligned}$$

which implies that X is Fano in both cases. □

Example 7.24. The following series of P-matrices defines non-toric almost homogeneous complexity-one threefolds with at least $d \geq 1$ different Demazure P-roots.

$$P_d := \begin{pmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d & -2d+1 \end{pmatrix}$$

For $C = (1, 2, 1)$ and $i_0 = 0$, $i_1 = 2$ we obtain the following root polytope whose integral points are P-roots:

$$A := \operatorname{conv}\left([d-1, -1], [0, -1], \left[\frac{d-1}{-2d+1}, \frac{1}{-2d+1} \right] \right).$$

In particular, we obtain at least d different roots

$$[0, -1], [1, -1], \ldots, [d-1, -1],$$

and the corresponding linear forms have the form

$$u = [u_1, u_2, u_3, u_4] = [d - u_3, -d, u_3, -1] \quad \text{with} \quad u_3 \in \{0, \ldots, d-1\}.$$

Explicitly, we have

$$\begin{aligned}
\langle u, v_{01} \rangle &= -d + u_3 + d_{12} - u_3 = 0 \geq 0, \\
\langle u, v_{02} \rangle &= u_3 \geq 0, \\
\langle u, v_{11} \rangle &= d - u_3 \geq 1, \\
\langle u, v_{12} \rangle &= 0, \\
\langle u, v_{21} \rangle &= -1.
\end{aligned}$$

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Deutsche Zusammenfassung

Das Thema der vorliegenden Arbeit sind Varietäten mit Toruswirkung der Komplexität 1, das heißt, algebraische Varietäten X mit einer effektiven Wirkung eines algebraischen Torus der Dimension $\dim(T) = \dim(X) - 1$. Diese Varietäten werden auch kurz Komplexität-Eins-T-Varietäten genannt. Es wird eine kombinatorische Beschreibung solcher Varietäten eingeführt, welche die konvexgeometrische Beschreibung torischer Varietäten durch Fächer verallgemeinert. Der Schwerpunkt dieser Arbeit liegt dabei in der Anwendung dieser Theorie auf Klassifikationsprobleme für Komplexität-Eins-T-Varietäten. Besondere Bedeutung kommt hierbei Fanovarietäten zu, das heißt Varietäten mit amplem antikanonischen Divisor.

In der algebraischen Geometrie sind torische Varietäten ein bekanntes Beispiel für den Einsatz kombinatorischer Methoden. Die Struktur dieser Varietäten wird durch deren Verbindung zu Gitterfächern auf anschauliche Art und Weise wiedergegeben. Im Jahr 1970 wurden (glatte) torische Varietäten zum ersten Mal von Demazure formal definiert [18]. Diese Arbeit beinhaltet bereits eine konvexgeometrische Beschreibung torischer Varietäten durch Fächer. Die Theorie torischer Varietäten entwickelte sich ab Ende der 70er Jahre rasch weiter. Als einige Beispiele seien an dieser Stelle Danilov [17], Oda [42, 43], Fulton [22] und Cox/Little/Schenk [16] genannt. Kombinatorische Methoden wurden auch für größere Klassen von Varietäten erfolgreich eingesetzt. Kempf, Knudsen, Mumford und Saint-Donat befassten sich in [36] mit toroidalen Varietäten, und erweiterten die konvexgeometrische Sprache auf diese allgemeinere Situation. In diesem Buch treten Komplexität-Eins-T-Varietäten als Spezialfälle auf und werden zum ersten Mal durch kombinatorische Daten beschrieben. Neben [36] ist die Arbeit [45] von Orlik und Wagreich eine der ersten Publikationen über Komplexität-Eins-T-Varietäten. Sie diskutierten den Spezialfall von $\mathbb{K}^*$-Flächen und entwickelten eine kombinatorische Beschreibung deren Struktur durch gewichtete Graphen. In neuerer Forschung beschrieben Altmann und Hausen in [3] Varietäten mit Toruswirkung durch polyedrische Divisoren. Dies liefert insbesondere im Fall von Komplexität-Eins-T-Varietäten eine recht einfache Beschreibung dieser Varietäten. Der Ansatz der vorliegenden Arbeit basiert auf Coxringen. Hausen und Süß bestimmten den Coxring einer gegebenen rationalen vollständigen Varietät mit Toruswirkung der Komplexität 1 mittels deren Wirkung, siehe [29]. Solche Coxringe sind endlich erzeugt und erlauben eine einfache Darstellung durch trinomiale Gleichungen. Dies liefert neue Ansätze für einen kombinatorischen Zugang zu Komplexität-Eins-T-Varietäten und ist der Ausgangspunkt dieser Doktorarbeit. Wir führen eine systematische Konstruktion für Komplexität-Eins-T-Varietäten mittels bestimmter ganzzahliger Matrizen A und P und einer Kollektion Φ von polyedrischen Kegeln ein. Diese Resultate wurden teilweise in [28, Kapitel 1] und [27] veröffentlicht.

Motiviert durch die Klassifikation torischer Fanovarietäten, deren Beginn auf Batyrev zurück geht [8], wenden wir den kombinatorischen Ansatz auf Fanovarietäten der Komplexität 1 an. Der Schwerpunkt der vorliegenden Arbeit liegt auf effektiven Schranken und konkreten Klassifikationen.

Eine erste Beispielklasse sind Fano-$\mathbb{K}^*$-Flächen, sogenannte del-Pezzo-$\mathbb{K}^*$-Flächen. Der in dieser Arbeit verwendete kombinatorische Ansatz unterscheidet sich von den Arbeiten von Alekseev/Nikulin [2] und Nakayama [40], welche auf klassischer Flächengeometrie basieren. In den Theoremen 5.25, 5.26, 5.27 und 5.28 erhalten wir eine vollständige Klassifikation von Gorenstein-log-del-Pezzo-$\mathbb{K}^*$-Flächen. Die verwendeten Methoden liefern darüber hinaus die Coxringe all dieser Flächen. Dies ergänzt Ergebnisse von Derenthal [19] für Coxringe im Hyperflächenfall und Hausen/Süß [27], welche die Fälle Picardzahl 1 und 2 mit anderen Methoden behandelten.

Fanovarietäten mit Picardzahl 1 sind von besonderer Bedeutung. Nadel gibt in [39] eine effektive Schranke für den Grad $(-K_x)^n$ einer glatten Fanovarietät an, welche von der Dimension n abhängig ist. Sind Gradschranken bekannt, so liefert Kóllar in [37] effektive Schranken für die Anzahl unterschiedlicher Deformationstypen glatter Fanovarietäten. In [35] studiert Kasprzyk torische Varietäten mit Picardzahl 1, so genannte „(fake) weighted projective spaces“ und liefert im terminalen und kanonischen Fall Beschränktheitsbedingungen für deren Gewichte.

In Theorem 6.10 erhalten wir explizite Schranken für die Anzahl möglicher Deformationstypen $\mathbb{Q}$-faktorieller Komplexität-Eins-T-Varietäten mit Picardzahl 1 in Abhängigkeit der Dimension und des Picardindex, das heißt des Index der Picardgruppe in der Divisorenklassengruppe. Als Konsequenz liefert Theorem 6.12 die folgenden Ergebnisse für das asymptotische Verhalten der Anzahl $\delta(d,\mu)$ unterschiedlicher Deformationstypen $\mathbb{Q}$-faktorieller d-dimensionaler Komplexität-Eins-T-Varietäten mit Picardzahl 1 und Picardindex μ. Für festes $d_0 \in \mathbb{Z}_{>0}$ und festes $\mu_0 \in \mathbb{Z}_{>0}$ erhalten wir

$$\delta(d_0,\mu) \ \sim \ \mu^{A\mu^2} \quad \text{und} \quad \delta(d,\mu_0) \ \sim \ d^{Bd}$$

mit beliebig kleinen Konstanten $A > 1$ und $B > 3$. beziehungsweise d_0 abhängig sind. Mittels der expliziten Schranken erhalten wir Klassifikationen für feste Dimension und festen Picardindex. In den Theoremen 6.18, 6.23, 6.24 und 6.26 geben wir exemplarisch alle zweidimensionalen Varietäten bis Picardindex 6, alle dreidimensionalen Varietäten mit Picardindex 1 und 2 und alle vierdimensionalen Varietäten mit Picardindex 1 an. In allen Fällen werden die Coxringe explizit aufgeführt. Diese Ergebnisse sind in [28, Kapitel 2 und 3] und [30] veröffentlicht.

Im Jahr 1970 studierte Demazure die Automorphismengruppen glatter vollständiger torischer Varietäten und beschrieb die Wurzeln mittels Fächer, siehe [18]. Später verallgemeinerte Cox diese Ergebnisse in [15] für den simplizialen Fall. In [41] liefert Nill effektive kombinatorische Kriterien für die Reduktivität von Automorphismengruppen vollständiger torischer Varietäten. In neuerer Forschung beschreiben Arzhantsev, Hausen, Liendo und Herppich die Automorphismengruppen von Komplexität-Eins-T-Varietäten durch kombinatorische Daten, siehe [6]. Wir verwenden diesen Ansatz für das Studium fast-homogener Komplexität-Eins-T-Varietäten, das heißt deren Automorphismengruppen wirken mit einer offenen Bahn. In Proposition 7.7 werden fast-homogene $\mathbb{K}^*$-Fläche explizit beschrieben. Als Folge davon klassifizieren wir in Korollar 7.12 alle log-terminalen nicht torischen fast-homogenen $\mathbb{K}^*$-Flächen mit exakt einer Singularität und Picardzahl

1 bis Gorensteinindex 5. Es stellt sich heraus, dass alle diese Flächen Fano sind. Diese Ergebnisse sind veröffentlicht in [6, Kapitel 6]. In Theorem 7.22 bestimmen wir alle dreidimensionalen fast-homogenen Komplexität-Eins-T-Varietäten mit Picardzahl 1 und reduktiver Automorphismengruppe. Alle diese Varietäten sind ebenfalls Fano. Diese Ergebnisse sind in [6, Kapitel 8] veröffentlicht.

Die vorliegende Arbeit hat sieben Kapitel, welche nun jeweils kurz zusammengefasst werden.

Das erste Kapitel ist eine kurze Zusammenfassung grundlegender Bezeichnungen und Aussagen über Coxringe und gestraußte Ringe, welche den Arbeiten [9] und [25] entnommen sind, siehe auch [5] und [26]. Jeder gestraußte Ring ist der Coxring einer $\mathbb{Q}$-factoriellen normalen Varietät, welche wir durch eine Standardkonstruktion als guten Quotienten einer offenen Menge des Spektrums des Rings erhalten. Darüber hinaus werden geometrische Eigenschaften solcher Varietäten mittels ihrer Coxringe formuliert und deren konvexgeometrische Bedeutung besprochen.

Kapitel 2 ist Komplexität-Eins-T-Varietäten gewidmet, das heißt algebraischen Varietäten X mit einer effektiven Wirkung eines Torus T der Dimension $\dim(X)-1$, sowie deren Coxringen, welche faktoriell graduierte Ringe der Komplexität 1 sind. Wir beschreiben die Coxringe mittels Erzeugern und Relationen und führen die kombinatorische Sprache der P-Matrizen ein. Diese ist vergleichbar mit der konvexgeometrischen Beschreibung für torische Varietäten durch Fächer. Teile dieses Kapitels sind bereits veröffentlicht in [27] und [28, Kapitel 1].

In Kapitel 3 widmen wir uns der Singularitätenauflösung von Komplexität-Eins-T-Varietäten. Wir diskutieren eine kanonische Weise, Singularitäten solcher Varietäten aufzulösen. Da Komplexität-Eins-T-Varietäten auf kanonische Weise in torische Varietäten eingebettet sind, werden torische umgebende Modifikationen für die Auflösung verwendet, siehe dazu [25]. Wir untersuchen das Verhalten der antikanonischen Klasse $-K_X$ einer Komplexität-Eins-T-Varietät unter torischen umgebenden Modifikationen. Eine ähnliche Konstruktion basierend auf polyedrischen Divisoren wird von Liendo und Süß in [38] vorgestellt.

In Kapitel 4 betrachten wir Komplexität-Eins-T-Varietäten der Dimension 2, so genannte $\mathbb{K}^*$-Flächen. Wir geben einen Überblick über ihre Geometrie und bestimmen alle Typen von Coxringen kombinatorisch minimaler $\mathbb{K}^*$-Flächen, das heißt $\mathbb{K}^*$-Flächen ohne kontrahierbare Primdivisoren. Des Weiteren berechnen wir Schnittzahlen invarianter Kurven und leiten daraus Fanobedingungen für $\mathbb{K}^*$-Flächen ab. Schließlich führen wir den antikanonischen Komplex für log-terminale $\mathbb{K}^*$-Flächen ein, ein konvexgeometrisches Objekt, das vergleichbar ist mit Gitterpolytopen, die torische Fanovarietäten beschreiben. Mit Hilfe des antikanonischen Komplexes lassen sich Singularitäten und Gorensteinindex log-terminaler Fano-$\mathbb{K}^*$-Flächen konvexgeometrisch beschreiben.

In Kapitel 5 befassen wir uns mit log-del-Pezzo-$\mathbb{K}^*$-Flächen, das heißt mit log-terminalen Fano-$\mathbb{K}^*$-Flächen. Das Hauptergebnis ist eine vollständige Klassifikation aller nicht torischen Gorenstein-del-Pezzo-$\mathbb{K}^*$-Flächen. Um diese zu erhalten, beschreiben wir den

Gorensteinindex einer $\mathbb{K}^*$-Fläche kombinatorisch mittels ihrer P-Matrix und ihres antikanonischen Komplexes und betrachten die spezielle Geometrie von del-Pezzo-$\mathbb{K}^*$-Flächen. Als Folgerung erhalten wir explizite Schranken, welche die Klassifikation aller nicht torischer log-del-Pezzo-$\mathbb{K}^*$-Flächen ermöglicht, wobei deren Coxringe und $\mathrm{Cl}(X)$-Graduierungen konkret angegeben werden.

In Kapiel 6 erhalten wir effektive Schranken und Klassifikationsergebnisse für rationale $\mathbb{Q}$-faktorielle Fanovarietäten mit einer Toruswirkung der Komplexität 1 und Picardzahl 1 in Abhängigkeit von den Invarianten Dimension und Picardindex. Konkret geben wir alle zweidimensionalen Varietäten bis Picardindex 6, alle dreidimensionalen Varietäten mit Picardindex 1 und 2 und alle vierdimensionalen Varietäten mit Picardindex 1 an. Die Ergebnisse dieses Kapitels sind bereits in in [28] und [30] veröffentlicht.

In Kapitel 7 behandeln wir Klassifikationsprobleme fast-homogener Komplexität-Eins-T-Varietäten, das heißt, deren Automorphismengruppe $\mathrm{Aut}(X)$ wirkt mit einer offenen Bahn. Durch das Einführen von Demazure-P-Wurzeln erhalten wir einen kombinatorischen Ansatz für die Automorphismengruppe solcher Varietäten, welcher die Wurzeln von $\mathrm{Aut}(X)$ beschreibt. Die Demazure-P-Wurzeln sind Gitterpunkte bestimmter Polytope. Diese konvexgeometrische Beschreibung wird für Klassifikationsprobleme fast-homogener Komplexität-Eins-T-Varietäten der Dimension 2 und 3 verwendet. Konkret werden vollständige Listen aller log-terminalen nicht torischen fast-homogenen $\mathbb{K}^*$-Flächen mit genau einer Singularität und Picardzahl 1 bis Gorensteinindex 5 angegeben. Außerdem bestimmen wir alle fast-homogenen dreidimensionalen Komplexität-Eins-T-Varietäten mit reduktiver Automorphismengruppe. Diese Ergebnisse sind in [6] veröffentlicht.

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