# Logical Tomography

## Exposing the Structural Constituents of Logic

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**Michael Arndt**

aus Bisingen

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Dekan: Prof. Dr. Michael Diehl

1. Berichterstatter: Prof. Dr. Peter Schroeder-Heister

2. Berichterstatter: Prof. Dr. Reinhard Kahle

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# Chapter 1

# Introduction

The prominence of formal logic is justified by its many successes. These successes are more often related to logical syntax providing a concise means of communicating the content it is reasoned about and less often about the process of reasoning itself. This is especially the case for first-order logic, which is often employed as universal specification language. Applications of logic in this sense very often extend the logical language in order to obtain the descriptiveness, which is required to detail the subject matter at hand. More often than otherwise, the logical formalism is used as specificational shorthand only, the actual reasoning being performed outside of the formal system. Be that as it may, a significant side effect of such use of logical syntax has been the emergence of the opinion that logic is *about* logical syntax, that it is about expressing concrete content with the help of logical formalism.

Of course, such a conception is erroneous. Instead, logic is the study of reasoning and of delivering sound argument. The focus lies specifically on the *structure* of the argument and, ideally, the content of the argument can be disregarded entirely. The Aristotelean syllogisms must be considered as the first logical system in this sense.[1] Apart from numerous reviews and commentaries and a few independent observations by medieval and early modern scholars, Aristoteles observations outlasted 22 centuries without any need of elaboration, let alone modification.

From the mid 19th century onward, logic was being recast into a quite new discipline, however. This was almost always done in view of particular applications, especially investigations into the foundations of mathematics. These endeavours suggested the deployment of a detailed formalism, into which the existent mathematical formalism could easily be incorporated. However, just

[1]For example, the syllogism, which is called *modus barbara* admits the progression from the major premiss *"all $Y$ are $Z$"* and the minor premiss *"all $X$ are $Y$"* to the conclusion *"all $X$ are $Z$"*, regardless of the instantiation of $X$, $Y$ and $Z$.

as mathematical formalism is generally considered to be nothing but a means of expression for ideal entities, the newly introduced logical formalism has also fallen under the thrall of such a perspective. Logical formulae were assumed to be meaningful only in view of the abstract value entities "truth" and "falsity", which were easily absorbed into mathematical structures. Hence, formal logic enabled a shift from considering the structure of an argument towards considering the structure of a complex proposition, as expressed by a particular complex formula. However, the structure of a formula is merely a series of instructions that specifies, how abstrract values are to be manipulated.

In the year 1922, at the high time and in the immediate vicinity of Hilbert's formalist programme, the physicist Paul Hertz began advocating the revivification of syllogistic reasoning. His logical articles evidence a highly original mind, as he managed to create an entirely independent approach to the industriously investigated discipline of logic on the basis of the syllogistic rule *modus barbara* alone. Hertz studied the properties of *systems of sentences* of the form $(a_1, \ldots, a_n) \to b$, which express relations of elementary assumptions and elementary assertions,[2] in view of an inference rule, which is related to the syllogistic modus barbara. A decade later, Hertz directly argued against the constriction of logic to the realm of formal logic, which he by then prejudicially referred to as "logical conventionalism". Unfortunately, his ideas hardly received any recognition, until Gerhard Gentzen picked them up in the early 1930s. However, Gentzen only briefly worked with Hertz' pure logic, before extending it and fusing it with formal logic into his conception of logic, which is today called " structural proof theory". While Gentzen certainly retained important ideas of Hertz, such as reasoning with implicit or explicit relations of assumptions and assertions, at the same time he relegated them to a secondary status. In his logistic calculus of explicit relations of assumptions and assertions, which he renamed "*sequents*", Gentzen distinguishes *logical rules*, which are concerned with logical formalism, from *structural rules*, which govern issues related to the structure of the sequents. Simply by means of his terminology, the eminence of logical syntax was reemphasised. This was even further corroborated by Gentzen's main result, the *Hauptsatz*, which states that the main structural rule of *cut*, which is closely related to Hertz' syllogistic inference rule, is obsolete in the presence of the logical rules.

Far from marking the end point of the interest in structural reasoning, it was the success of Gentzen's structural proof theory, which paved the way for a wider recognition of the structural aspects of reasoning. However,

[2]Actually, Hertz also employs physical notions, such as *events*, *causes* and *effects*.

such interest rarely goes as far as to challenge the formalist paradigm altogether. In view of the swift fading of Paul Hertz' original ideas, the question, whether a purely structural approach in his sense is an adequate alternative to Gentzen's logistic calculus and its emphasis of logical language, or whether it was necessarily restricted to a supporting role, has been awaiting closer examination. It is this question that will be addressed in this investigation.

## The Objectives

The guiding theme of our examination of the logistic calculus is the shift of the emphasis from its syntactic aspects to its structural aspects. For this purpose we adopt a bottom-up perspective on derivations in a particular variant of the classical logistic calculus, called RK (for "reference calculus"). Instead of eliminating instances of the cut rule, we will use a plenitude of instances of this structural rule in order to separate every complex formula or subformula, which occurs in a sequent, from its context by means of a unique atomic formula. At the same time, very restricted instances of the logical rules shall be employed, which are only applicable to sequents, which result from such atomic cuts. By alternating applications of these rules, a sequent can be decomposed into a large number of *elementary structural sequents* (ESSs). As these elementary structural sequents do not contain any complex logical formulae, they correspond to the sentences of Hertz.

We will then argue that the collection of these ESSs, which we shall call the *explosion set* of the sequent, not only has the same expressiveness as the original sequent and all of the formulae it contains, but that it can indeed be considered as the meaning of the sequent and its formulae in the first place. This argument consists of two parts. Firstly, we will exhibit that each occurrence of a subformula in the sequent can be uniquely related to a particular subset of this collection of ESSs. Secondly, we will show that every branch of every possible derivation of that sequent in RK can be mapped onto a family of particular subsets, which are called *connexion sets*, which is linearly ordered by set inclusion, such that each of these subsets corresponds to a particular sequent of that branch. Such a correspondence is exhibited by employing the cut rule on the elementary structural sequents of the connexion set. In a certain sense, we will thereby have achieved a reversion to purely structural reasoning.

Another issue, which we wish to address, is that of a suitable interpretation of elementary structural sequents. Although these sequents do not contain logical syntax as far as logical connectives are concerned, they must nonetheless be considered as syntactic entities, as the logistic calculus, from which they were obtained, is a syntactic calculus. In Hertz' conception of structural reasoning, on the other hand, sentences were considered as expressions of particular relations. Indeed, Paul Hertz had already used graph representations for those special cases of sentences, which are of the form $a \to b$. We will extend this intuition and interpret explosion sets by a particular class of *directed hypergraphs*, which we will call *tomographs*.[3] An elementary structural sequent $a_1, \ldots, a_m \to b_1, \ldots, b_n$ and its relational interpretation $\big(\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_n\}\big)$ only differ in as far as the order of elements is relevant in the syntactic representation of the sequent. Thus the remaining logical meta-syntax of a sequent can be swiftly related to an abstract set-theoretic entity in a loss-free manner.[4]

## Significant Restrictions

There are a number of restrictions, which we had to impose on this investigation in order to keep it manageable. The first restriction concerns the fact that the calculus, which is employed in order to obtain the explosion set of a sequent, is intended to extract the structural constituents of sequents containing logical formulae. It is not a calculus, which establishes, whether a sequent is provable in the original logistic calculus or the variant RK, which we shall consider. We will briefly address this issue by indicating, how a procedure can be developed, which decides for any explosion set, whether it was obtained from a provable sequent or not.

The most important limitation is that we will only consider the propositional fragment of the logistic calculus. This is done for a number of reasons, to which we will only briefly allude. First and foremost, we wanted to consider a contraction-free calculus, in which the number of subformula occurrences cannot increase as a derivation is developed from the bottom-up. This will allow the relation of formula occurrences in the end sequent to particular elementary structural sequents and vice versa. As it is not possible to retain the full strength of the logistic predicate calculus in the absence of the contraction rule (or a reformulation of the rules for the quantifiers, which amounts to the same increase of subformula occurrences), we desisted from extending the investigation to the predicate calculus. Another issue, which has a bearing on the question of considering the predicate calculus, is the intended use of local logical rules. The variable condition of the quantifier rules

[3] The name expresses the fact that all of the internal vertices of a tomograph are cut vertices, i.e. removing any one of these vertices results in a directed hypergraph, which is disconnected. Thus, tomographs have a very frail structure.

[4] We have to introduce *occurrence instances* of sequents, in which different occurrences of the same formula are explicitly distinguished. Otherwise, the relational interpretation would identify such occurrences.

is non-local in the sense that we wish to adopt. While it might be possible to add quantifier scope to explosion sets and their relational interpretations, e.g. by considering particular subsets thereof, it would complicate matters beyond the limits of this investigation.

Another important restriction lies in disregarding the intuitionistic logistic calculus. This has, again, to do with our approach of employing local logical rules. It is widely known that some of the intuitionistic variants of the logical rules have a global effect on the state of a derivation in as far as they can effect the discharge of certain formulae. Just as in the case of the non-local effects of quantifier rules, a significantly more elaborate treatment could account for the involved intricacies. As it is subject to debate, whether the logistic calculus provides a natural framework for intuitionistic logic at all, this matter is avoided at this point.

## Outlining the Course of Action

In the first part, the context of this investigation, which has briefly been summed up above, will be properly set out. In chapter 2, Paul Hertz' structural logic is described in some detail and a discussion of its most distinguishing features is given. In chapter 3, Gerhard Gentzen's contribution to Hertz' logic is presented and the logistic calculus is properly introduced and compared to the purely structural logic. In chapter 4, we introduce the particular variant RK of the classical logistic propositional calculus, which is particularly well-suited for proof-search and will serve as a reference calculus for the following investigations. Moreover, the restricted variants of the inference rules, which are used in the explosion calculus, are also introduced and discussed at that point.

The second part is concerned with the explosion procedure and explosion sets. The three stages of the explosion procedure are presented in some detail in chapter 5. In chapter 6, a number of examples for explosion procedures are presented. Several properties of explosion sets are stated, the most important of which is the uniqueness up to renaming of new variables. It is demonstrated in chapter 7 that the meaning of formula occurrences as well as the base structure of a sequent is indeed represented by particular collections of elementary structural sequents of the explosion set obtained from that sequent. In chapter 8, we trace particular connections within explosion sets, based on RK-derivations of the corresponding sequent. For this purpose, the notions of connexion set and connexion tree are introduced. Based on these notions, in chapter 9 we will turn to the issue of deciding, whether an explosion set was derived from a provable sequent. For this purpose we introduce a modified notion of connexion set, which does not depend on any particular

RK-derivation. A simple refutation procedure is developed in detail, and a sketch is provided, how it can be extended to a decision procedure.

The third part is concerned with the relational interpretation of explosion sets. The required notions are introduced in chapter 10, including general definitions of directed hypergraphs and their components, the extension of the notions of traversals of graphs to corresponding partial notions and the required notion of a total traversal of a hypergraph. In chapter 11 the relational interpretation of explosion sets is presented, connexion sets are related to strands, and the notion of a logical tomograph, which is the immediate development of a directed hypergraph from a logical sequent, is finally introduced. The issues of refutability and decidability of a sequent based on its relational interpretation, are addressed in chapter 12. For this purpose, the connection between total traversals and connexion sets will be exploited in order to relate these issues to those regarding explosion sets. In chapter 13, a procedure for developing a relational interpretation of a sequent directly, i.e. without first generating its explosion set, is presented. The hypergraphs developed by that procedure are called logical tomographs. Chapter 14 concludes this investigation with a brief discussion.

# Part I

# The Elements of Structural Reasoning

# Chapter 2

# Paul Hertz – Satzsysteme

From the year 1912 until his emigration to the United States in the 1930s, Paul Hertz held positions at the Georg-August-Universität in Göttingen. Very likely due to the vicinity of Hilbert's industry, Hertz eventually developed a taste for matters of logic. It is very remarkable, then, that he managed to develop a highly original approach despite of this surely influential ambience. Thoroughly shunning the formalistic orientation of Hilbert's programme, Hertz' logic instead refers back to the syllogistic method of Aristoteles, particularly the modus barbara. Thereby, Hertz was able to avoid the particular focus of Hilbert's programme and formal logic in general of providing a formal framework for mathematics. His perspective on logic was not bound to any particular kind of content but that of an abstract interest in the process of reasoning itself. His entirely structural approach to logic turned out to be quite different to the understanding of his contemporaries, however. Only in the investigations of Gerhard Gentzen did Hertz' logic bear fruit, but not without the forfeit of one of Hertz' most estimated principles. For in his later works, he explicitly argued that logic is not about language, and that, therefore, any formal approach must misrepresent the essence of logic. A surprising aspect of Paul Hertz' logic is, as we shall see, that it bears a strong resemblance to and anticipates several notions of much later developments in informatics, most notably graph theory.[1]

## 2.1 The Conception of Structural Logic

The title of Paul Hertz' dissertation [Her04] reveals that, in the beginnig of his scientific career, starting in 1903, his field of expertise was electrodynam-

[1]This fact has also remained obscure, and certain important results and notions in graph theory were independently reformulated several decades after Hertz' publications.

ics. In the following years he made regular contributions to that field, up to his habilitation in 1908. Around that time his attention shifted to thermodynamics with a particular focus on statistical mechanics. Following the interruption caused by the Great War, Paul Hertz resumed his scientific work in that very same field, culminating in his 1922 article *Statistische Mechanik* [Her22a].

At this point, his interest shifted suddenly towards logic. In the same year, his first in a series of articles is published: *Über Axiomensysteme für beliebige Satzsysteme. I. Teil. Sätze ersten Grades* [Her22b]. In this, Hertz introduces his structural logic, which is based on the notion of *sentences* of the form $a \to b$, where $a$ and $b$ are taken from a set of *real* elements.[2] In view of his later articles, these sentences are called sentences of *first degree.* Hertz then gives a single rule of inference, which is simply called *inference* ("Schluß"):

$$\frac{\begin{array}{r}\text{I. } a \to b\\ \text{II. } b \to c\end{array}}{\text{III. } a \to c}$$

In this scheme, I. is called the *minor sentence*, II. is called the *major sentence*, and III. is called the *conclusion*; I. and II. are called the *premises.* A *proof* of a sentence $\mathfrak{e}$ is a system of inferences ending in $\mathfrak{e}$, where each minor or major sentence of an inference is either taken from a fixed set of sentences $\mathfrak{S}$ or obtained as conclusion of another inference. Furthermore, for a set of sentences $\mathfrak{S}$, that which Hertz calls a *closed system of sentences* ("abgeschlossenes Satzsystem")[3], an *independent axiom system* $\mathfrak{A}$ is a set of sentences, from which each sentence of $\mathfrak{S}$ can be obtained by repeated applications of the inference rule to sentences of $\mathfrak{A}$, whereas no sentence of $\mathfrak{A}$ can be inferred from other sentences of $\mathfrak{A}$ in this manner. Hertz then develops a sufficient condition for the case that a system of sentences has a unique axiom system. Moreover, he investigates, how the size of axiom systems can be reduced by the introduction of *ideal* elements. For example, consider the six sentences $a_1 \to b_1$, $a_1 \to b_2$, $a_1 \to b_3$ and $a_2 \to b_1$, $a_2 \to b_2$, $a_2 \to b_3$. Consider then an ideal element $p$, which does not occur in any sentence under consideration and the five new sentences $a_1 \to p$, $a_2 \to p$ and $p \to b_1$, $p \to b_2$,

[2]Here, "real" is understood as opposed to "ideal" in a very specific sense unrelated to sets of numbers.

[3]The past participle "closed" corresponds to the German "abgeschlossen", the past participle of "abschließen". The latter is obtained by prefixing "schließen", which is translated into English as "infer", with the prefix "ab-", which already indicates closure. In this sense, an "abgeschlossenes Satzsystem" is literally a "system of sentences closed *with regard to inference*".

$p \to b_3$. It is obvious that each one of the former sentences can be inferred by some pair of the latter sentences. Using ideal elements, Hertz established several results about *minimal systems of sentences*, which are too involved to go into at this point.

The first article on logic was succeeded in the following year by *Über Axiomensysteme für beliebige Satzsysteme. II. Teil. Sätze höheren Grades* [Her23], in which his investigations are extended to include sentences of the form $a_1, \ldots, a_n \to b$, so called sentences of *higher degree*. Utilising the extended structure of the antecedent, it should possible to weaken any sentence by adding additional elements to the antecedent. Hertz introduces a new rule, called *immediate inference*, which accounts for this possibility:

$$\frac{\text{I.} \qquad \|(a_1, a_2, \ldots) \;\to\; c}{\text{II.} \quad \|(b_1, b_2, \ldots a_1, a_2, \ldots) \;\to\; c}$$

The sole rule of inference, which was sufficient in the case of sentences of first degree, is extended and renamed "*syllogism*":

$$\begin{array}{ll} \text{I.} & \left\{ \begin{array}{l} (a_1^1, a_2^1, \ldots) \;\to\; b_1 \\ (a_1^2, a_2^2, \ldots) \;\to\; b_2 \\ \qquad\qquad \vdots \end{array} \right. \\ \text{II.} & \qquad\qquad \| \; (b_1, b_2, \ldots, a_1, a_2, \ldots) \;\to\; c \\ \hline \text{III.} & \left\| \left( \begin{array}{ll} a_1^1, a_2^1, \ldots & \\ a_1^2, a_2^2, \ldots & a_1, a_2, \ldots \\ \quad \vdots & \end{array} \right) \right. \;\to\; c \end{array}$$

In both rules the sentence or sentences above the line are the premiss or premises and the sentence under the line is the conclusion. In the syllogism rule, I. is called the *system of minor sentences* and II. is called the *major sentence*. In the antecedent of the major sentence, the elements $b_1, b_2, \ldots$ are called *main elements* and the $a_1, a_2, \ldots$ are called *accessory elements*. Note that despite of the use of dots, none of the antecedents in any of the rules may be an infinitary object. Repeated occurrences of the same element in an antecedent are to be discounted, a fact that is explicitly emphasised in the syllogism rule by the notational marker $\|$. The notion of proof is generalised over these new rules. Hertz stipulated that the set of elements, from which the succendent and the antecedent of a sentence are made up, be finite. His purpose was to be able to characterise *closed systems* of sentences, by which he understood finite sets of sentences that are closed under the rules of inference. This self-imposed limitation to finite sets of sentences necessitates the restriction to a finite set of elements, as any application of immediate inference can introduce a new element, which is not already present in any of the sentences of the system, into an arbitrarily long antecedent, if there were an infinite supply of them. Hence, a system of sentences cannot attain closure under immediate inference, unless the set of elements is finite. In addition to proposing the notion of a *normal proof* of a sentence, Hertz gave a sufficient condition for the uniqueness of an independent axiom system for a closed system of sentences. Furthermore, instead of resorting to ideal elements as he did in the first part, he introduced $(a_1, \ldots, a_n) \rightarrow [\, b_1, \ldots, b_m \,]$ as an abbreviation for the system of sentences ranging from $(a_1, \ldots, a_n) \rightarrow b_1$ to $(a_1, \ldots, a_n) \rightarrow b_m$.

In 1929, a third article *Über Axiomensysteme für beliebige Satzsysteme* [Her29] completed the series. Some of the results of the previous article were restated in a revised and technically more concise form and subsequently generalised to take into account what he called *macro sentences* of the form $\alpha_1(x_1, \ldots, x_{k_1}), \ldots, \alpha_n(x_{k_{n-1}+1}, \ldots, x_{k_n}) \rightarrow \beta(x_{k_n+1}, \ldots, x_{k_{n+1}})$ and infinite systems of sentences. In the context of that article, those sentences, which had been investigated in the two preceding publications, were called *micro sentences*. As part of the revision, Hertz introduced two types of normal proofs, each having a particular shape, the *Aristotelian normal proof* and the *Goclenian normal proof*, and showed that each sentence, which has a proof, also has a normal proof of each type.[4] For the treatment of macro sentences, which is covered in a relatively independent second part of the article, he gave up on the finitistic approach, which he had entertained in the first articles, and had the $x_i$ range over an enumerable domain of entities. He called these *subordinary members*, and from these the *ordinary members* of the form $\alpha(x_1, \ldots, x_k)$ are made up. In addition to immediate inference and syllogism, he introduced another rule, called *inference by binding*, which allows the replacement of all identical subordinary members of a sentence by some other subordinary member of the same sentence. Hertz gave the following example of binding, in which $x_3$ is bound to $x_1$:

$$\frac{\rho(x_1, x_2), \rho(x_2, x_3) \rightarrow \rho(x_1, x_3)}{\rho(x_1, x_2), \rho(x_2, x_1) \rightarrow \rho(x_1, x_1)}$$

Hertz briefly mentions that by modifying the notion of a normal proof, which takes binding into account, the existence of normal proofs for provable sentences holds even for macro sentences. As fourth and final rule, *formal inference* allows the replacement of arbitrary symbols occurring in a sentence

[4] This has been investigated in detail by Schroeder-Heister in [SH02].

by new symbols, as long as identical symbols in the sentence are replaced by the same new symbol. The rule is added as a convenience in order to "enhance the recognisability" of the conclusion of a sentence with regard to its premiss or premises, as Hertz elucidated. Without this explicit fourth rule, an application of binding, for example, might have to be expressed in the following form in the context of some concrete proof:

$$\frac{\rho(x_1, x_2), \rho(x_2, x_3) \to \rho(x_1, x_3)}{\rho(x_3, x_4), \rho(x_4, x_3) \to \rho(x_4, x_4)}$$

Although the bindings are adequately expressed in the conclusion, this is not as immediately obvious as it was in the frist example of this inference step. With the help of formal inference, changes in the names of subordinate members have to be performed explicitly. The inference step above has to be split into two separate steps:

$$\frac{\dfrac{\rho(x_1, x_2), \rho(x_2, x_3) \to \rho(x_1, x_3)}{\rho(x_1, x_2), \rho(x_2, x_1) \to \rho(x_1, x_1)}}{\rho(x_3, x_4), \rho(x_4, x_3) \to \rho(x_4, x_4)}$$

As premiss and conclusion of a formal inference represent the same structural content, Hertz made clear that this rule of inference is only added because of its utility in view of the sybolic representation of this content. Partial results were developed for questions regarding the existence of axiom systems for systems of macro sentences. Apart from these mostly technical results, Paul Hertz' most important insight is to be found on less than two pages in the initial part of this article concerning micro sentences, where he hinted at the possibility of a "considerable generalisation" of some of the most fundamental results for micro sentences in the following sense. Abbreviating sentences by $\mathfrak{a}_1, \ldots, \mathfrak{a}_n, \mathfrak{b}$, and abbrevating inferences or, more generally, *provability relations* by $(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \rightrightarrows \mathfrak{b}$, Hertz stated that the same two rules, which he had introduced for the purpose of reasoning over sentences, immediate inference and syllogism, were also suitable for reasoning over provability relations. This observation is highly relevant in view of Gentzen's suggested relation between his calculus of natural deduction and the logistic calculus.

Three more of Paul Hertz' contributions to logic, two of which are of a more general philosophical nature, should be mentioned at this point. In [Her28], he developed a semantics for micro sentences and addressed several particularities concerning sentences of the first degree. In [Her31] Hertz gave a very general justification of the *modus barbara* as the most important law of reasoning, which his rule of syllogism can be considered a generalisation of. Finally, the article [Her37b] is an argument against the, as Hertz called it, “logical conventionalism” of Carnap. His fundamental disagreement with Carnap lay in Hertz’ rejection of the reduction of logic to formal language and the corresponding reduction of reasoning to applications of admissible transformations of formulae. He argued that, contrary to rules of admissible transformation, which depend on the particulars of the chosen formal language, the syllogistic modus barbara is a law of logical reasoning in itself, and has, therefore, a more fundamental status. Even his late articles on physics, [Her30] and [Her37a], which are concerned with very gerenal notions such as *causality* and the *direction of time*, are rather epistemological in nature with a particular focus on matters of reasoning about causation. In the first of these articles, Hertz actually used the abbreviation $a \rightarrow b$ to express the relation between a cause $a$ and its effect $b$.

It should be apparent even from this very limited exposition of some of the elements of Hertz’ work that his conception of logic was very different from that, which was in his eyes a “logical conventionalism”. His results were at the same time groundbreaking, anticipating later developments in proof theory, as well as conceptionally novel and technically intricate. Unfortunately, several of his ideas were only briefly sketched out or hinted at but not thoroughly executed. All of these factors might have colluded to eclipse Hertz’ influence. Neither must it be forgotten that at the same time and even at the same locality, Hilbert’s massive and widely acknowledged endeavour was underway and being propagated. It could be speculated that Paul Hertz’ articles of 1922 and 1923 were inspired by Hilbert’s program and at the same time developed as a tentative alternative to the formalist paradigm employed therein. What is certain is that the particular structural nature of Paul Hertz’ logic was picked up and developed further by Gerhard Gentzen, albeit at the cost of abandoning Hertz’ epistomological misgivings regarding formalism.

## 2.2 Noteworthy Particularities

While Hertz put a particular emphasis on many technical issues, the fundamental principles, upon which his structural reasoning rested, were only alluded to in an often matter-of-factly manner with little or no explication. Any attempted return to a purely structural logic must begin with a revisitation of its fundamental principles. Specifically, to give a brief and perhaps acute summary, it shall be argued in the remainder of this chapter that Hertz’ structural logic rested upon three assumptions, namely that

1) logic is reasoning about the dynamics of systems;

2) finitistic logic is interesting in itself;

3) logic has nothing to do with language.

Assumption 1) summarises Hertz' understanding of a logical sentence as a particular causal relation between events, and a system of sentences as providing a specification for the entire dynamics of some abstract system. This assumption, especially in connection with 3) contrasts the conventional value-based, static logical semantics. Assumption 2), while it is not essential in itself, is nonetheless important in view of a contemporary understanding of Hertz' work, for, as we shall see, together with 1) it suggests a close correspondence between logic and graph theory. The most important of Hertz' assumptions by far is 3), for it suggests that any kind of formal logic, which is at all concerned with complex formulae made up of logical connectives, including proof theory, is at best a mere front-end for that, which is inherently logical, namely the process of reasoning.

## Logic as Descriptive Dynamics

Both the intuitive interpretation of sentences provided by Hertz in [Her29][5], whereby a sentence $(a_1, \ldots, a_n) \to b$ is taken as an expression of the entailment of event $b$ by events $a_1, a_2, \ldots, a_n$, and the usage of the very same notation for the relation of cause and effect in [Her30] indicate that for Paul Hertz logic was concerned with reasoning about chains of abstract events. A logical sentence then expresses a single abstract relation of one or more events, the causes, on one hand and a single event, the effect, on the other hand. We might envisage such a relation as a process, which occurs when all the causes coincide, thereby producing the effect. Since both events and elements, from which sentences are made up, are abstract entities, we shall not distinguish between them at this point. Likewise, sentences are to be considered as expressing the aforementioned kinds of relations. Given a system of causal relations, an inference step using the rule of syllogism is then a means of generating a more immediate relation of this kind by discovering effects in certain relations, which are at the same time contributing causes of another relation. Syllogism can then be used successively to deduce less and less "immediate" relations of this kind from a system of given relations. Conversely, the rule provides a means for constructing an independent system of such relations, which are at the same time the most elementary causal

[5] *"Dabei sind unter $a_1, a_2, \ldots, a_n$ etwa Ereignisse (abstrakt genommen; nicht konkrete Einzelereignisse) zu verstehen, deren Eintreten das Eintreten von b bedingt."*, p.459

relations of the system and yet account for all the possible relations that are consequences of the system, the axioms of the system. Immediate inference can be used to weaken the dependencies of effects on causes. Then, out of all possible configurations of simultaneous occurrences of events as causes, only those that are absolutely essential for particular events have to be included. Thereby, axioms can be kept as concise as possible as far as their causes are concerned.

A collection of events could be considered as the *state* of some abstract system, where the individual events are the variable attributes or *micro states* of the system. This terminology is borrowed from that used in statistical mechanics, the field Paul Hertz was working on before he turned his attention to logic. Of course, "state", "micro state" and "state transition" are no longer exclusively used in physics. Apparently, Hertz' concept of an "abstract event" is very much related to the abstract states of automata in informatics. Adopting this perspective, sentences represent individual tuples of relations of a (generalised) state transition system, i.e. the relations represent the possible dynamics of such a system. Reasoning is then the process of obtaining insights into the eventual or overall potential dynamics of a given system on the basis of its elementary potential dynamics, as expressed bz the axioms, without actually having the system undergo the transitions.

A dynamics of such systems in the sense of a very simple operational semantics could be developed along the lines given in [Her28]. In the context of establishing the result that every provable sentence has a normal proof, Hertz called an element $b$ *distinguished* with regard to distinguished elements $A$, if $(a_1, \ldots, a_n) \rightarrow b$ is an axiom and $\{a_1, \ldots, a_n\} \subseteq A$.[6] The set $A$ as subset of the set of all elements could be considered a state of the system, and the relation of being a distinguished element could be used to construct a simple operational semantics along the lines of $A \mapsto A \cup \{b\}$ or the slightly more involved $A \mapsto (A \setminus \{a_1, \ldots, a_n\}) \cup \{b\}$.

While Paul Hertz did not actually develop an operational semantics of any kind, the intuitive interpretation he provided for sentences clearly suggests the possibility. His logic is not merely an instrument for describing state transition systems, however. Instead, Hertz repeatedly suggested that logic is inherently about dynamics. Sentences express the causal dynamics of events, and reasoning is a dynamic process of inferring new sentences from systems of sentences. The entirety of Hertz' logic seems, therefore, aimed at a general descriptive dynamics. Such an interpretation is corroborated in [Her37b], where he argued that the attempt to capture complex logical

[6]This is a somewhat simplified account. In fact, Hertz inductively defines sets of $i$-distinguished elements.

correlations formally, i.e. by complex syntactic entities, which are inherently static, is fundamentally flawed.[7]

What lies at the core of a descriptive dynamics is the relational aspect of causes and effect, which is expressed in the logical sentences. Gentzen explicitly shifted from this physical terminology to a logical one, and obtained relations of *assumptions* and *assertions*. Regardless of the terminology, the central claim of Hertz was that reasoning is inextricably concerned with the manipulation of abstract relations, for which he chose descriptive dynamics as his prime example.

## Finitistic Logic

Hertz curious insistence in keeping the set of elements, from which sentences are made up, finite, seems unfertile in contrast to the mathematical presentiment that only infinite realms are worth investigating. The ostensible reason for it is the fact that he wished to investigate sets of finite sentences, which are closed under the application of the inference rules. In view of the rule of immediate inference, the presupposition of an infinite number of elements would make this scheme unattainable from the outset. This self-imposed restriction becomes only understandable, when we consider some of Hertz' results. For example, the characterisation of closed systems of sentences in view of the kind of axiom systems that they possess makes clear that even under this restriction there is a wealth of results to be uncovered. Often enough, Paul Hertz did not elaborate on his curtly presented results, but quickly moved on to address yet another aspect of his logic.

A detached vista on his logic, especially his first article [Her22b], in which he presented his investigations into sentences of first degree, i.e. of the kind $a \rightarrow b$, reveals an unexpected conceptional ingenuity and pioneering spirit. In order to see this, one has to step outside of the conceptional context of logic, into which Hertz embedded this investigation simply by using logical terminology. What Hertz called a "system of sentences", say $\mathfrak{S}$, made up from elements of a given "complex", which we shall call $E$, could be understood as a binary relation on a given set of elements. If this is written as $\langle E, \mathfrak{S} \rangle$, it becomes immediately obvious that Hertz' systems of sentences correspond to directed graphs. The condition of finiteness of $E$, curious from a traditionalistic logical point of view, becomes the default assumption of contemporary graph theory, where the richness of the study of finite graphs is

[7]There is a particular comment of Hertz, which seems to indicate otherwise. However, in the context of having giving only a very vague intuitive interpretation of the particular association of symbols $a \rightarrow b$, this remark can be conceived as an informal adduction for readers, who are already familiar with the Russell-Hilbert approach.

taken for granted, and where infinitary graphs lie at the periphery of the interest. In order to corroborate the claim that Hertz anticipated graph theory, the terminology of his 1922 article could be systematically related to graph theoretical notions. Unfortunately, due to the extremely technical nature of Hertz' atricle, this cannot be adequately condensed into a brief exposition. Moreover, due to his pre-orthodox approach, some of his most elementary notions correspond to somewhat specific graph theoretical notions, which makes an easily evident correlation difficult. What is obvious, however, is the fact that the sole inference rule ( *"Schluß"*) enriches a system of sentences by a single transitive step. A closed system of sentences on $E$, which is obtained from a system of sentences $\langle E, \mathfrak{S}\rangle$, is then simply the *transitively closed graph* $\langle E, \mathfrak{S}^*\rangle$. An independent axiom system $\langle E, \mathfrak{A}\rangle$ is a *minimal directed spanning graph* of the system of sentences $\langle E, \mathfrak{S}\rangle$. In contrast to the early precursors of graph theory, Paul Hertz developed an original systematic graph theory in the guise of a study of systems of sentences as an abstract model of the reasoning process.[8] He conceptionalised, investigated and solved questions concerning such systems of sentences. In doing so, he obtained several interesting results, such as the sufficient condition for the unique existence of a minimal directed spanning graph of a directed graph.

Unfortunately, the generalisation of his logic to sentences of higher degree, which have the richer general structure $(a_1, \ldots, a_n) \to b$, renders a simple graph theoretical analogy impossible. Instead, two equivalent generalisations are possible. The first one is a straightforward generalisation to relations between non-empty sets of elements and sigleton sets of elements. In this approach, the sentence above would represent the tuple $(\{a_1, \ldots, a_n\}, \{b\})$. Given a set of elements $E$, a system of sentences could then be considered as graph on $\mathbb{P}(E)$, or, equivalently, a hypergraph on $E$ itself. The second possibility involves the introduction of sentence designators. For example, calling the above sentence $\mathfrak{e}$, the set $\big\{(a_1, \mathfrak{e}), \ldots, (a_n, \mathfrak{e}), (\mathfrak{e}, b)\big\}$ could be considered as a bipartite graph on the set of elements and the set of sentence designators. Paul Hertz did not explicitly provide any formal interpretation for sentences of higher degree above the intuitive one. The manner, in which he considered "complexes" $(a_1, \ldots, a_n)$ of elements suggests that the first graph theoretical interpretation is more in spirit of Hertz than the second.[9]

[8] Leonhard Euler and Augustin Cauchy, who are generally considered as the originators of graph theory, were inspired by specific problems or the realisation that particular classes of graphs can be used as abstract representations of particular classes of concrete problems. However, there is no theory concerning these abstract representations themselves worth mentioning to be found in their works, certainly not in the sense of what are nowadays considered as results in general graph theory.

[9] Occasionally, Hertz does use designators for sentences, such as $\mathfrak{e} = (a_1, \ldots, a_n) \to b$.

Hertz' restriction to finite sets of elements, from which sentences are to be constructed, and finite systems of sentences expressing relations are only minor factors suggesting an interrelation between his structural logic and graph theory. The entire set up and terminology of his investigations, especially in [Her22b] and [Her23], bear a striking resemblance to later graph theoretical notions. Moreover, the focus of Hertz' finitistic approach lies in the investigation of the consequences of particular arrangements of individual sentences occurring in systems of sentences in view of overall properties of these systems, in particular the existence of independent axiom systems; the same problems, with appropriately modified terminology, are typical questions of graph theory. Although Hertz strove to generalise his investigations to infinite systems of sentences and macro sentences, this was envisioned as an extension of the techniques, which he had developed and applied to the finitistic investigations of micro sentences. Unfortunately, his remarks on macro sentences are only of a very general nature.

## Avoidance of Logical Language

Throughout his logical articles, Paul Hertz diligently evaded the use of logical connectives or even sentence connectors of a more general kind in connection with his sentences. Considering the historical context of his work, such an avoidance cannot be considered a mere excentricity. For Hertz, this must have been a deliberate commitment, but, unfortunately, he gave no reasons for his abandonment of the then fashionable approach of Russell, which had been picked up and advanced by Hilbert.

An explicit rejection of connectives can be found in [Her22b]. Having introduced ideal elements in order to minimise the size of axiom systems, Hertz remarks that by means of ideal elements the usage of the word "or" in the antecedent as well as the usage of the word "and" in the succedent can be avoided. For example, The six sentences $a_1 \rightarrow b_1$, $a_1 \rightarrow b_2$, $a_1 \rightarrow b_3$ and $a_2 \rightarrow b_1$, $a_2 \rightarrow b_2$, $a_2 \rightarrow b_3$ can be replaced by the new sentences $a_1 \rightarrow p$, $a_2 \rightarrow p$ and $p \rightarrow b_1$, $p \rightarrow b_2$, $p \rightarrow b_3$, where $p$ is an ideal element, which does not already occur in any other sentence. The meaning of Hertz' remark is not entirely clear, but it can be speculated that he suggested that with the help of connectives the six sentences could be expressed in a single sentence, such as $(a_1\ or\ a_2) \rightarrow (b_1\ and\ b_2\ and\ b_3)$.[10] It appears that the avoidance of logical connectives is achieved in passing. An explicit demonstration of sentences enriched with logical connectives occurs in [Her30], where he made

[10] Why such is avoided by means of the ideal element $p$, is not clear from this brief example, however. For it should still be possible to obtain both $(a_1\ or\ a_2) \rightarrow p$ and $p \rightarrow (b_1\ and\ b_2\ and\ b_3)$.

some cursory remarks on the validity of certain sentences, in which such connectives occur, with respect to sentences, in which only elements occur. The most impressive example, given on page 213, can be summarised as follows. Assuming that it is $a_1 \rightarrow b_1$, $b_1 \rightarrow a_1$, $a_2 \rightarrow b_2$ and $b_2 \rightarrow a_2$, then it is also $a_1 \rightarrow (b_1 \textit{ or } b_2)$ and $a_2 \rightarrow (b_1 \textit{ or } b_2)$, but neither $(b_1 \textit{ or } b_2) \rightarrow a_1$ nor $(b_1 \textit{ or } b_2) \rightarrow a_2$. This clearly shows that he was aware of how the well-known logical connectives could be incorporated into his logic.

In [Her37b], Paul Hertz explicitly argued against what he called logical conventionalism, which he indirectly attibuted to Carnap. He completely rejected the claim that logic has anything to do with language, including the logical connectives. His motivation for such a rejection is based on epistemological grounds, for Hertz assumed that a logical sentence in Carnap's understanding is not a *recognition* of a fundamental regularity, but merely a *stipulation*, an agreement, which has to be interpreted even in relation to the language, in which it is expressed. It appears that Hertz eschewed such a logical relativism, because it merely adds another layer, which requires justification and proper interpretation, without adding any real benefit. This confirms Hertz' understanding of logical sentences as expressing the most elementary relations between abstracta.

The issue that he himself uses a meta-notation for these relations, which incorporates both implicative aspects via the distinction between antecedent and succedent of sentences as well as conjunctive aspects in their complex antecedents, has no bearing on this position. For Hertz', a logical sentence was inherently about entailment and at the same time inherently about expressing possible joint dependencies, each sentence representing an elementary logical compound. The elements of an antecedent have the property of conjointly entailing the succedent. Therefore, the notion of conjunction is not (and cannot be) an isolated and static logical concept, but is inherently meaningful only in view of such a conjoint causation. In his brief remark on how the usage of "and" in the succedent can be avoided, Hertz indicates that two or more sentences, which have the same antecedent, can be considered to express a conjunction of the elements of the succedent. A conjoint effect is, then, simply an alternative of possible effects. Hence, although not integral to a single sentence but instead expressed by combinations of sentences, a conjunctive meaning in the succedent position is also possible. But if this structural logic already possesses this expressiveness, the introduction of explicit logical connectives is clearly unnecessary.[11]

[11]Paul Hertz considered relations of conjoint causes and individual effects, which suffice to give accounts of "and" in both antecedent and succedent. Dual to the representation of "and" in the succedent by more than one sentence sharing the same antecedent, the "or" in the antecedent is realised by two or more sentences sharing the same succedent. Since

Paul Hertz' strong sentiment against language as carrier of logical content is very striking when regarding later developments in what is now called proof theory, especially in view of the logistic calculi and their main results. What is performed as a proof is constructed in a logistic calculus is to entangle many elementary trivial relations in such a manner that a single relation between complex formulae can be concluded; the particular constitution of these formulae merely retains and expresses the particular kinds of entanglements. Hence, even structural proof theory allocates a large part of its resources to the investigation of logical language instead of focussing on what in Hertz' view was the inherently logical: the investigation of dependency relations between elementary abstract entities. Although Paul Hertz only voiced explicit objections in one of his late articles, the presentiment of the necessity of avoiding logical syntax permeated all of his logical works. His deliberate opposition to locigal conventionalism is, therefore, most prominent not in his explicit argument against it, but in his outright refusal to take logical connectives into consideration within his framework of logic.

## Summary

Paul Hertz' contributions to logic did not meet with a wide reception. His rejection of the paradigm of formalism, coupled with the unique and original approach of only considering sentences expressing causal relations, which are to be understood as relations between assumptions and assertions, placed him outside of the scope of contemporary interest. Had it not been for the effort of Gerhad Gentzen, who recognised the fundamental importance of the relational character of reasoning and used it as the foundational principle of his logical calculi, Hertz' logic might well have been forgotten.

It is one of the main motivations of this investigation to heed Hertz' peculiarities by demonstrating that the elements of formal logic, which Gentzen later introduced into Paul Hertz' structural logic, can be easily removed in such a manner that finite systems of sentences are obtained, which can be thought of as expressing causal relations.

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Hertz' notion of causation is deterministic, a fact reflected in the restriction to a single element in the succedent, the dual to the conjoint causation by elements in the antecedent, an *alternative* of effects, is not represented in his logic. The suitable generalisation was later introduced by Gentzen in the logistic calculus for classical logic. Unfortunately, Paul Hertz gave no indication whatsoever how "not" might be represented in his logic.

# Chapter 3

# Gerhard Gentzen – The Logistic Calculus

Between 1928 and 1933, Gerhard Gentzen studied mathematics at various universities, beginning with Greifswald, then Göttingen, Munich and Berlin. Having returned to Göttingen, he began his dissertation under Hermann Weyl, who was Hilbert's successor. By that time the full extent of Hilbert's programme had already been shown impossible to accomplish. Rather than thwarting the interest in formal logic, Gödel's negative result liberated the discipline from the strict regime of Hilbert's ambitions. In the wake of this ruin, intuitionistic and generally constructivist ideas, formerly utterly deprecated by Hilbert, began to rise in value. It was this fertile state of logic, upon which Gentzen's seminal *Untersuchungen über das logische Schließen* [Gen35] throve. He completed his dissertation in 1933 and one year later, after Weyl's departure from Germany, Gerhard Gentzen became the assistant of Hilbert, who held the empty chair of mathematics as emeritus. It can be assumed that Gentzen's perspective on logic had been considerably influenced by the formalist stance of Hilbert. At the same time, Gentzen had already encountered and taken an interest in Hertz' conception of logic. It was a concurrence of disparate lines of thought, from which a new and fruitful discipline came into existence: structural proof theory.

## 3.1 The Union of Formal and Structural Logic

Before the publication of his renowned dissertation in 1934, Gerhard Gentzen had already had an article with the title *Über die Existenz unabhängiger Axiomensysteme zu unendlichen Satzsystemen* [Gen33] published. This investigation was entirely based upon Hertz' conception of logic. In the first part of the article, Gentzen recast Hertz' concepts into a somewhat modified terminology. The first change is merely a convention. Using capital letters for *complexes* of elements, such as $K = (u_1, \ldots, u_m)$ and $L = (v_1, \ldots, v_n)$, he abbreviated sentences $(u_1, \ldots, u_m) \rightarrow w$ to $K \rightarrow w$ and $(u_1, \ldots, u_m, v_1, \ldots, v_n) \rightarrow w$ to $KL \rightarrow w$. With this simplification, he came up with a more concise account of Hertz' rules of inference, which are also given new names along the way. Immediate inference became *thinning* ("Verdünnung") and was defined thus:

$$\frac{L \rightarrow v}{ML \rightarrow v}$$

Hertz' syllogism was restricted to the case of one minor sentence and renamed *cut* ("Schnitt"):

$$\frac{L \rightarrow u \qquad Mu \rightarrow v}{LM \rightarrow v}$$

Gentzen showed how Hertz' syllogism can be disassembled into combinations of cuts and thinnings, particularised the notion of *proof* to his system and gave a stricter definition of a *normal proof*. He established a completeness result along the lines of that of Hertz. In the second and third parts of the article, Gerntzen considered questions regarding independent axiom systems for infinite systems of sentences. He described an infinite system of sentences, which does not have an independent axiom system, and provided a method for obtaining an independent axiom system for a particular class of infinite systems of sentences.[1]

In his dissertation, which was published as *Untersuchungen über das logische Schließen*, Gerhard Gentzen distanced himself somewhat from the more technical and exclusively structural aspects of Hertz' investigations, while retaining the essential aspect of Hertz' intuition that reasoning is concerned with the dynamics of cause and effect. Gentzen provided a more logical terminology for this crucial recognition of Hertz by proposing means of reasoning, in which *assertions* ("Aussagen") are explicitly *dependent* ("abhängig") from *assumptions* ("Annahmen"). Certainly in this regard, one of the particularities of Hertz became the cornerstone of Gentzen's approach. In doing so he, he argued, he provided a formulation of logical deduction, which was

[1]The method is applicable to *linear* (enumerably) infinite systems of sentences, where "linear" is Gentzen's term for what Hertz had called "first order". A linear system of sentences may contain sentences, which are not linear, if it contains a linear sentence for each of them, from which it can be inferred.

closer to the real reasoning process ("dem wirklichen Schließen") than the formalisations of Russell and Hilbert.[2] Bringing together structural logic and the formal logic, he opened up the possibility of investigating both at the same time and brought Hertz' solitary endeavour to the attention of mathematicians and logicians, who had followed the more prominent and vigorous course of Hilbert. For this purpose, Gentzen introduced two different types of calculi: the calculus of natural deduction, in which the dependencies between assumptions and assertions are implicit, and the logistic calculus, in which these relations are explicit.[3] As the latter type of calculus is closer to the structural calculus of Hertz, the former will not be discussed futher. It shall suffice to recall that a *derivation* in the calculus of natural deduction expresses the assertion of a formula $B$, which may depend on assumptions $A_1, \ldots, A_n$, which are also formulae, or may be independent of assumptions. Following the formalist paradigm, the notion of *formula* is taken for granted as the primary mode of expression of logical content throughout Gentzen's dissertation. Complex formulae are made up from propositional variables by means of symbols $\neg$, &, $\vee$ and $\supset$ in the usual manner.[4] Reasoning is presented as a means of transforming formulae by either introducing additional logical connectives to a given formula or eliminating the most accessible connective.

The step from the calculus of natural deduction to the logistic calculus consists in making the implicit relations between assumptions and assertions explicit. Gentzen remarked that such a dependency could, in turn, be expressed by the formula $A_1 \,\&\, \ldots \,\&\, A_n \supset B$, but that such a representation was impracticable in view of the fact that this would require additional rules for the connectives & and $\supset$, which would disturb the rigorous regime of introduction and elimination rules employed in the calculus of natural deduction for these connectives. Instead, Gentzen suggested the introduction of the notion of a *sequent*, which should express dependencies of assumptions and assertions as $A_1, \ldots, A_n \rightarrow B$. Thus, the notion of a formula, expressing primary logical content, and that of a sequent, expressing meta-logical content, cannot interfere. This introduces a distinction between logical content,

[2]Even in Hilbert-type deduction the dependency of assertions from assumptions can be construed, but it is not an integral characteristic of that formalisation of reasoning.

[3]It should be mentioned that Gerhard Gentzen's dissertation was reconciliatory in another regard. For not only did he provide two different frameworks for reasoning from assumptions, he also addressed the even greater divide between the classical and the intuitionistic conceptions by giving classical and intuitionistic variants of each one of his calculi.

[4]In view of the investigation, which is to follow, we restict ourselves to the propositional part even in this exposition of Gentzen's logic.

which corresponds to linguistic content in the sense of what Hertz would a few years later call "logical conventionalism", and meta-logical content, which was the only proper logical content for Hertz.

A very gainful insight of Gentzen was the realisation that a generalisation of sequents in such a manner that more than one formula may occur in the succedent, as in e.g. $A_1, \ldots, A_m \to B_1, \ldots, B_n$, gives the logistic calculus the expressiveness of classical logic. It must be noted that a sequent of this kind is not an abbreviation for sequents $A_1, \ldots, A_m \to B_1$ to $A_1, \ldots, A_m \to B_n$, as it was the case with a similar abbreviation of Hertz. Instead, the $B_1, \ldots, B_n$ must be considered as *alternative* assertions, that is from assumptions $A_1, \ldots, A_m$ at least one of the formulae $B_1, \ldots, B_n$ can be asserted.[5] By allowing this kind of symmetry in the shape of sequents, "or" became an intergral part of the meta-logic of the logistic calculus. Consequently, in each sequent there are simultaneously conjunctive, implicative and disjunctive properties inherent.[6]

Gentzen departs from Hertz' presetting in another aspect of the logistic calculus. While Hertz had investigated inferences from given systems of sentences, which were non-trivial in general, and had showed that trivial sentences of the kind $a \to a$ do not have to be specifically considered, Gentzen only considered *derivations* from sequents of the kind $A \to A$, which he calls *initial sequents*, where $A$ can be any formula. An initial sequent expresses the trivial relation that any assumed $A$ can be asserted. Although arbitrary systems of sequents, which could be used as initial sequents, could be considered, the introduction of logical syntax enriches the logistic calculus enough to render such considerations unrewarding.

Inferences in the logistic calculus are governed by an array of *inference rules*, which are to be instantiated from *inference schemes*.[7] Each one of the inference schemes has one or two *premises* and a *conclusion*. In addition to the *logical inference schemes*, i.e. schemes governing the logical connectives (see figure 3.1), Gentzen provides *structural inference schemes*, which are concerned exclusively with meta-reasoning, i.e. with operations on the structure of sequents (see figure 3.2). The inference schemes, apart from the structural rule (Cut), are distinguished by "A" or "S", depending on that part of the

[5]It is interesting to note that the dependency relations, which are expressible in this manner, cannot be realised in the calculus of natural deduction, as derivations in that calculus always yield a single assertion.

[6]Not all of these properties are apparent in all sequents. For instance, consider the sequent $A \to B$, which lacks conjunctive and disjunctive features, whereas the sequent $\to A, B$ only exhibits disjunctive features.

[7]As was mentioned before, in view of the scope of this investigation, only the inference schemes for the *propositional* fragment of the *classical* logistic calculus LK are given.

$$\frac{A, \Gamma \to \Delta}{A \mathbin{\&} B, \Gamma \to \Delta}\ (\&\mathrm{A}_1) \qquad \frac{\Gamma \to \Delta, A \qquad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \mathbin{\&} B}\ (\&\mathrm{S})$$

$$\frac{B, \Gamma \to \Delta}{A \mathbin{\&} B, \Gamma \to \Delta}\ (\&\mathrm{A}_2)$$

$$\frac{\Gamma \to \Delta, A}{\Gamma \to \Delta, A \vee B}\ (\vee\mathrm{S}_1)$$

$$\frac{A, \Gamma \to \Delta \qquad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\ (\vee\mathrm{A})$$

$$\frac{\Gamma \to \Delta, B}{\Gamma \to \Delta, A \vee B}\ (\vee\mathrm{S}_2)$$

$$\frac{\Gamma \to \Delta, A \qquad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\ (\supset\mathrm{A}) \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}\ (\supset\mathrm{S})$$

$$\frac{\Gamma \to \Delta, A}{\neg A, \Gamma \to \Delta}\ (\neg\mathrm{A}) \qquad \frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \neg A}\ (\neg\mathrm{S})$$

Figure 3.1: Logical inference schemes of LK

sequent, which is the focus of the scheme in its conclusion. All of the logical inference schemes are introduction schemes in the sense that each premiss of the rule has an occurrence of a unique formula, called *side formula*, from which a new formula, called the *main formula*, is obtained in the conclusion. In all the logical schemes, apart from ($\&\mathrm{A}_1$), ($\&\mathrm{A}_2$), ($\vee\mathrm{S}_1$) and ($\vee\mathrm{S}_2$), the main formula contains the side formulae and the additional logical connective corresponding to the scheme. In the cases of the exceptions, an arbitrary formula, which does not have to occur in the premiss, is connected to the single side formula by the new logical connective corresponding to the scheme. In contrast to the logical inference schemes, the structural ones are not about the introduction of logical connectives. Instead, they recapture the properties of Hertz' inference system. The inference scheme (WA), *weakening* of the antecedent, corresponds to what Gentzen had already called "Verdünnung" in [Gen33], i.e. Hertz' immediate inference, although this variant only allows the addition of a single formula. Correspondingly, (WS) weakens the succendent by adding another alternative assertion. The *contraction* inference schemes (CA) and (CS) ("Zusammenziehung") allow the fusing of multiple occurrences of the same formula in either the antecedent or the succedent. Hertz' calculus had provided for this by means of the symbol "‖", which

$$\frac{\Gamma_1 \to \Delta_1, A \qquad A, \Gamma_2 \to \Delta_2}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2} \text{(Cut)}$$

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta} \text{(WA)} \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A} \text{(WS)}$$

$$\frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta} \text{(CA)} \qquad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A} \text{(CS)}$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \to \Delta}{\Gamma_1, B, A, \Gamma_2 \to \Delta} \text{(XA)} \qquad \frac{\Gamma \to \Delta_1, A, B, \Delta_2}{\Gamma \to \Delta_1, B, A, \Delta_2} \text{(XS)}$$

Figure 3.2: Structural inference schemes of LK

can be found in several places in his rules of inference. In contrast to such an implicit treatment, Gentzen made every manipulation of symbols explicit by these separate rules. The same must be said for the *exchange* inference schemes (XA) and (XS) ("Vertauschung"), by which pairs of formulae can change positions within the antecedent or succedent. Hertz had not been specific about the position, at which an element occurs, and it can be assumed that the order, in which elements were listed in the antecedent, was of no consequence; again, Gentzen makes this an explicit issue. Eventually, the inference scheme (Cut) is a generalisation of the one Gentzen had given in [Gen33]. The difference lies in the fact that the *cut formula* $A$ can occur as an alternative to other formulae in $\Delta_1$ in the generalised sequents. The extension is done in a manner, which retains the symmetry of the calculus. In all the inference schemes with the exception of (XA), (XS) and (Cut), $\Gamma$ and $\Delta$ represent sequences formulae, which are to occur adjacent to the arrow in the premiss or premises and the conclusion; either of these may be empty. In those exceptions, antecedent or succedent are split up further into $\Gamma_1$ and $\Gamma_2$ or $\Delta_1$ and $\Delta_2$, where, again, either may be an empty sequence of formulae. The side formula or side formulae as well as the main formula of logical inference schemes are attached to the left of an antecedent or the right of a succedent. *Derivations* are obtained by connecting instances of inference schemes in such a manner that the premiss of each one of these instances is either an initial sequent or the conclusion of some other instance of an inference scheme. The unique sequent in a derivation, which is the conclusion of an instance of an inference scheme, but not the premiss of another, is called the *end sequent* of the derivation, and the derivation itself is called the derivation *of* this sequent.

Gentzen's main result of his dissertation is the *Hauptsatz*. It states that every derivation in a logistic calculus can be transformed into a derivation having the same end sequent, in which no instance of (Cut) occurs. The lengthy proof of this result makes up the second half of the first published part of his dissertation. In a second part, he gives applications of the Hauptsatz and proves the equivalence of the calculus of natural deduction, the logistic calculus and a calculus corresponding to Hilbert's formalism, for both the classical and the intuitionistic cases. This concludes the brief review of Gentzen's dissertation.

In his following works, Gerhard Gentzen turned to particular problems regarding the foundations of mathematics, such as the consistency of arithmetic [Gen36a] under particular consideration of the transfinite induction principle [Gen43]. Although he employed the formalism of the logistic calculus in these articles, it was used as a notational means, not as an object of philosophical investigation. Therefore, we shall not go into any detail of these later works at this point.

## 3.2 A Comparison with Hertz' Logic

It is not necessary to emphasise the importance of Gentzen's dissertation, which is generally acknowledged as the beginning of structural proof theory. What is not as widely known is that many of his ideas were not only inspired by but almost directly taken from the structural logic of Paul Hertz. The remainder of this chapter shall serve as an elaboration of the influence of Hertz' logic on Gentzen's dissertation.

### The Relational Nature of Logic

One of the most important elements of Gentzen's logic in contrast to that of Hilbert is the fact that the dependency of assertions on assumptions is intrinsic in the formulation of reasoning. Hilbert-style reasoning is a generation of increasingly complex assertions from formula instances of axiom schemes, which can be immediately asserted, by means of a single rule of inference.[8] Although the dependency on assumptions can be worked into Hilbert-style reasoning, this is achieved only by explicitly allowing additional formulae as immediately assertible. In contrast to this, any formula can be assumed in Gentzen's calculus of natural deduction. Every step in the process of generating a deduction expresses a particular relation of zero or more assumed formulae and a single asserted formula. Certain rules allow the discharge

[8] At least in the propositional case, only a single rule of inference is required.

of one or more assumptions, and the goal of the reasoning process is to obtain a particular assertion, which is independent of any assumption. This is indeed a more "natural" manner of reasoning than the static generation of increasingly complex assertions of Hilbert-style reasoning.

Clearly, the idea that reasoning is about relations of assumptions and assertions, particularly that of one or more assumptions and a single assertion, can already be found in the very general remarks of Hertz. His sentences $a_1, \ldots, a_n \rightarrow b$ are exactly expressions of such dependency relations. While Hertz did not use the commonplace logical terminology, instead mentioning causal relationships between events, this could perhaps be attributed to the association of the notions of "assumption" and "assertion" with the formal approach, which he had rejected. Moreover, Hertz had not introduced "unconditional" events into his logic, although he had specifically mentioned such a possibility in [Her29].[9] If this possibility had been developed by Hertz, such unconditional elements might have been presented in the manner of sentences of the form $\rightarrow b$ or simply by explicitly listing all continual elements independently of the considered system of sentences. Gerhard Gentzen picked up and fleshed out the idea and used an explicit extension of the relational conception in such a manner that assertions are allowed, which are independent of any assumptions. In fact, in his conception of reasoning it is the goal to obtain a deduction, which expresses such a relation. The *proof* of a formula $B$ is a deduction, which does not have any uncancelled assumptions.

Regardless of this detailed implementation, it must not be forgotten that the backdrop for Gentzen's calculus of natural deduction is a calculus for reasoning over relations of assumptions and assertions, as it had been provided in great detail, albeit in a very general manner, by Paul Hertz. Of ocurse, this influence is unmistakably recognisable in Gentzen's logistic calculus. Gentzen himself remarks that this calculus is employed particularly in view of its property that all such dependencies are explicit in every sequent. It is this property, which made the proof of the Hauptsatz manageable for Gentzen. Despite of this, Gentzen apparently gave this fundamental property a mostly technical recognition.

## The Meaning of the Logical Connectives

The structural paradigm of Hertz was given up by Gentzen in favour of the formalist paradigm, which Hertz himself clearly held in disregard. It is, then, particularly instructive to consider how logical syntax is introduced. In both

[9] On page 468, he states in footnote 19): "[...] unless one were to introduce distinguished elements (which might denote continual states)."

the calculus of natural deduction and the logistic calculus, the introduction of a logical connective is justified by a particular inference scheme, especially the relation of certain formulae occurring in its premiss or premises. For the calculus of natural deduction, some of these schemes not only refer to the premises, which are recently asserted formulae, but also to assumptions, which may occur at a much earlier stage of the derivation. Due to these long-distance dependencies, the inference schemes of the calculus of natural deduction are less immediate than those of the logistic calculus. In the logistic calculus, all of the inference schemes are immediate in the sense that each sequent, which is the instance of a conclusion of such an inference scheme, is only related to either a single premiss or to two premises, and to nothing else.

The relation of the conclusion of a logical inference scheme and its premiss or premises lies in the fact that the conclusion contains an additional logical connective than the premiss or either one of the premises. Consider, for example, the inference scheme (&S):

$$\frac{\Gamma \rightarrow \Delta, A \qquad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \,\&\, B}\ (\&\mathrm{S})$$

The conclusion differes from its premises in the rightmost formula of the succedent. Where the premises contain formulae $A$ and $B$, the conclusion replaces these by the formula $A \,\&\, B$. Sequences $\Gamma$ and $\Delta$ have to be the same in both premises and the conclusion. The scheme is to read as follows: if both $A$ and $B$ can be asserted independently of one another, based the same assumptions $\Gamma$ and in the context of the same alternative assertions $\Delta$, then $A \,\&\, B$ can be asserted from those assumptions in the context of these alternative assertions. Considering the special case that $\Delta$ is empty, the scheme is given as follows:

$$\frac{\Gamma \rightarrow A \qquad \Gamma \rightarrow B}{\Gamma \rightarrow A \,\&\, B}\ (\&\mathrm{S})$$

In the case that $A$, $B$ and the (single) formulae in $\Gamma$ are propositional variables, i.e. elementary formulae, the premises have the shape of Hertz' sentences. Moreover, the logical connective is introduced in the conclusion exactly in such a manner, that it corresponds to the remark in [Her22b], where Paul Hertz briefly comments on how "and" is to be *avoided* in the succedent (see page 27). The same remark in view of the avoidance of "or" in the antecedent has a bearing on the following special case of ($\vee$A):

$$\frac{A \rightarrow \Delta \qquad B \rightarrow \Delta}{A \vee B \rightarrow \Delta}\ (\vee\mathrm{A})$$

Similarly, the brief remark in [Her30], where Hertz states that with $a_1 \rightarrow b_1$ and $a_2 \rightarrow b_2$ it is also $a_1 \rightarrow (b_1 \textit{ or } b_2)$ and $a_2 \rightarrow (b_1 \textit{ or } b_2)$, is reflected in these special cases of Gentzen's logical inference schemes governing the connective $\vee$ in the succedent:

$$\frac{\Gamma \rightarrow A}{\Gamma \rightarrow A \vee B} \, (\vee S_1) \qquad\qquad \frac{\Gamma \rightarrow B}{\Gamma \rightarrow A \vee B} \, (\vee S_2)$$

Gentzen's formulations of the logical rules concerning & and $\vee$ were just suitable generalisations of Hertz' remarks, compiled into a unified framework. Of course, this approach completely disregarded Hertz' opinion that logical connectives should be avoided in the first place. As far as the negation is concered, Gentzen found formulations for the intuition that $\neg$ toggles formulae between the roles of assumptions and assertions. The inference schemes for $\supset$ can be considered meta-rules obtained by the appropriate combinations of schemes for $\neg$ and $\vee$, interpreting $A \supset B$ as $\neg A \vee B$.[10]

In the justificational perspective of the inference schemes, which considers the premises after the conclusion, each step in the justification process results in less complex sequents in the sense that they contain less complex formulae. If atomic formulae, i.e. propositional variables, were to be equated with Hertz' elements, it could be said that each step of a justification process results in sequents, which are closer to the shape of Hertz' sentences.[11] In fact, all of the premises in Gentzen's logical inference schemes have the shape of such generalised sentences, as all of the premises only contain formula variables $A$ or $B$, and variables $\Gamma$ and $\Delta$, which represent sequences of formulae. If each formula variable is considered to be an element in the sense of Hertz, then each inference scheme can be read as relating Hertz-style sentences of the premises or premiss to a conclusion containing a single complex formula. This formula is justified by some particular occurrence of an element in the premiss sentence or the arrangement of two elements in the premiss sentences.

In Gentzen's logistic calculus, the introduction of a logical connective into a sequent is justified by the fact that one or two sequents have already been obtained, which have particular properties as far as the role or roles of the single side formula or the two side formulae is concerned. All the other formulae of the antecedents and succedents form a common context in all of the sequents, which are involved: conclusion and premiss or premises. An inference scheme, then, specifies in the premiss or premises, which formulae

[10] We are restricting ourselves to the classical setting. This seems to be the more natural choice for the logistic calculus for the reason that the inference schemes stand as they are stated, without having to consider restrictions on the succedent.

[11] In this we make allowance for the generalisation that the succedent may contain an arbitrary number of elements.

constitute the fixed and, if applicable, shared context, and which formula or formulae justify by the *position* within the premiss or premises the introduction of the complex formula associated with the scheme.[12] In other words, the introduction of a formula in a particular position in the sequent and the manner, in which it connects the one side formula or the two side formulae occurring in its premiss or premises, is justified by the particular position of the side formula or side formulae.

Summarising these observations, it can be concluded that Gentzen executed in the logical rules of his logistic calculus ideas, which can partially already be found in Hertz' works. However, whereas Paul Hertz decidedly avoided the accomodation of logical syntax in favour of a pure structural logic, Gentzen took exactly this step. It must be said that Gentzen provided a suitable generalisation of Hertz' sentences, which extends to assertions of a disjunctive character, and he considered the case of negation, which Hertz had not mentioned at all.

## The Subordination of Structural Reasoning

Meta-reasoning had already been suggested by Hertz in the sense of provability relations, which he had expressed by $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \rightrightarrows \mathfrak{b}$, where $\mathfrak{a}_1, \ldots, \mathfrak{a}_n, \mathfrak{b}$ are themselves sentences, in contrast to sentences of the elementary level, which are of the form $a_1, \ldots, a_n \rightarrow b$. A similar relation between reasoning on an elementary level and meta-reasoning is certainly established by Gentzen in the relation of the calculus of natural deduction and the logistic calculus. However, if Hertz' elementary level, generalised in a manner that admits as elements arbitrary logical formulae, is related to the logistic calculus, then the calculus of natural deduction undercuts the principles of structural reasoning, as it is set at a level below what Hertz would consider the domain of logic in the first place.

Another threat to the idea of a purely structural reasoning is formulated in the Hauptsatz: The cut scheme, which is the remainder of the structural reasoning rule of syllogism, Hertz' main rule of inference, is no longer required in a calculus that has is both formal and structural aspects. Reasoning in the way evisioned by Hertz, namely as the process of deducing ever more immediate causal relations from a set of givens or, alternatively, discovering particularly elegant sets of such relations, from which all of the givens can be deduced, is more or less relinquished. Instead, formal reasoning in a structural framework of the logistic calculus is the discipline of justifying the

[12]The position of a formula is one out of two roles a formula can have in a sequent: it can be either an assumption or an assertion.

occurrence of a complex formula in a sequent by reverting to less complex sequents, the most fundamental of which, the initial sequents, correspond to Hertz' trivial sentences. Hence, logical reasoning in the sense of understanding the consequences of relations between assumptions and assertions, as it had been envisioned by Hertz, was replaced by a method of justifying logical syntax by successively reducing it to instances of the trivial relations between a single formula occurring both as assumption and assertion. The Hauptsatz fortifies the formal paradigm, because it makes the most important structural reasoning principle obsolete.

In the calculus of natural deduction, the structural aspect is merely implicit. Hertz' vision of reasoning principles, which hold not only on the elementary level of reasoning, but also on the meta-level, are certainly not realised as far as structural aspects are concerned. But even as far as the syntactic aspect of reasoning is concerned, while there is some correspondence between the schemes for natural deduction and the logistic calculus as far as introduction schemes and logical schemes for the succendent are concerned, a similar correspondence does not hold for elimination schemes and logical schemes for the antecedent. In summary, Gentzen put his focus on the justificational character of formal reasoning.

## Conclusion

When reviewing Gentzen's logical calculi against the background of Hertz' structural framework, it is obvious that Gentzen drew heavily from that framework, while simultaneously immolating Hertz' philosophical convictions by enhancing – or diluting, depending on the perspective – a purely structural logic with what Hertz had called conventionalist elements. Of course, this is what was exactly the factor, which made the structural aspects of logic widely accessible and appreciated by the mathematical community in the first place. Unfortunately, this success throve on the abandonment of a purely structural logic, which is most prominently demonstrated in Gentzens main logical result. The *Hauptsatz* established that the cut inference rule, which corresponds to Hertz' main inference rule of syllogism, is obsolete in the logistic calculus.[13]

However, it has already been argued here and will become more evident in the course of this investigation that the entire justificational mechanism of the logistic calculus is inherently structural. The task is, therefore, to bring this mechanism to visible appearance.

[13] So striking is this result that the opinion has been voiced that a sequent-style calculus, which does not have this property, must be fundamentally flawed.

# Chapter 4

# The Calculus RK

Gentzen's shift of focus towards the formal paradgim has cast Paul Hertz' structural approach into a supporting role. At the same time, it is a supporting role of fundamental importance, as the logical inference schemes of Gentzen's logistic calculus can be read as justifications of individual logical connectives on the basis of relations of assumptions and assertions in the vein of Hertz' sentences. Hence, at least in the case of the logistic calculus, formal reasoning is justified on the basis of structural reasoning.[1]

In this chapter, after a proper introduction of the propositional fragment of the logistic calculus for classical logic, several preparatory modifications to the calculus with the purpose of making the primacy of structural reasoning explicit will be undertaken. This will result in the calculus RK, which will be the basis for the subsequent investigation.

From now on, we will adopt the terminology *rule of inference* or just *rule* for both rules of inference in the sense of Hertz as well as inference schemes and instances thereof in the sense of Gentzen.

## 4.1 Basic Definitions

Rather than resorting to the modern formalism, we shall adhere to the symbols that were used by Gentzen. The *logical symbols* are these: & for "and", $\vee$ for "or", $\supset$ for "if ... then ..." and $\neg$ for "not"; the *structural symbols* are $\rightarrow$ for the relation between assumptions and assertions and the comma as separator for assumptions and assertions; the *figurative symbol* ——— serves as a separator between individual steps in the reasoning process. Further, an

[1]The same could be argued in the case of the calculus of natural deduction. In that case it is not as obvious due to the complications, which arise due to the fact that the dependency relations of assertions on assumptions are not explicit.

enumerable set $\mathcal{A}$ of *propositional variables* is required, and the elements of $\mathcal{A}$ are named $a, b, c, d, \ldots$; capital letters $A, B, C, \ldots$ will be used to denote formulae[2]. From these elements, all that is required can be defined.

**Definition 4.1** *The* formulae *over $\mathcal{A}$ are given as follows:*

- *Every $a \in \mathcal{A}$ is a formula, called an* elementary *formula;*
- *if $A$ is a formula, so is $\neg A$;*
- *if $A$ and $B$ are formulae, so are $A \,\&\, B$, $A \vee B$, $A \supset B$.*

*The function* $\mathbf{cmx}(A)$ *of the* complexity *of a formula $A$ is given as follows:*

- $\mathbf{cmx}(a) \stackrel{def}{=} 0$;
- $\mathbf{cmx}(\neg A) \stackrel{def}{=} \mathbf{cmx}(A) + 1$;
- $\mathbf{cmx}(A \star B) \stackrel{def}{=} \mathbf{cmx}(A) + \mathbf{cmx}(B) + 1$ *for* $\star \in \{\&, \vee, \supset\}$;

*The set* $\mathbf{sub}(A)$ *of all* subformulae *of a formula $A$ is given as follows:*

- $\mathbf{sub}(a) \stackrel{def}{=} \{a\}$;
- $\mathbf{sub}(\neg A) \stackrel{def}{=} \mathbf{sub}(A) \cup \{\neg A\}$;
- $\mathbf{sub}(A \star B) \stackrel{def}{=} \mathbf{sub}(A) \cup \mathbf{sub}(B) \cup \{A \star B\}$ *for* $\star \in \{\&, \vee, \supset\}$;

*Every formula is its own (improper) subformula. For a formula $A \star B$ with $\star \in \{\&, \vee, \supset\}$, formulae $A$ and $B$ are called its* immediate *subformulae; for a formula $\neg A$, the formula $A$ is called its* immediate *subformula.*

In contemporary terminlology, elementary formulae are usually called *atomic* formulae. When speaking of composing and decomposing formulae, we shall use the latter term, and reserve the use of the former for sequents (see the following definition). A formula that is not atomic is called a *complex* formula. The complexity of $A$ is simply the number of logical connectives that occur in $A$.

[2]Gentzen used capitals for propositional variables and Gothic capitals for formulae. We shall employ Gothic capitals otherwise, and, hence, we have to diverge from Gentzen in this detail.

**Definition 4.2** *A* sequent *is an expression of the form*

$$A_1, \ldots, A_m \to B_1, \ldots, B_n,$$

*where* $m \geq 0, n \geq 0$ *and* $A_1, \ldots, A_m, B_1, \ldots, B_n$ *are arbitrary* formula occurrences*. The* negative *formula occurrences* $A_1, \ldots, A_m$ *make up the* antecedent *and the* positive *formula occurrences* $B_1, \ldots, B_n$ *make up the* succedent *of the sequent. A sequent is* elementary*, if all of its formula occurrences are atomic.*

A formula occurrence is a formula indexed by its position withing the antecedent or succedent. The terminology of a polarity of formula occurrences can be extended to include subformulae; this is done in appendix A. It is useful to consider groups of formula occurrences as lists of formulae. Greek capitals $\Gamma, \Delta, \ldots$ will be used to denote lists of formulae of the antecedent and succedent of a sequent, occasionally indexed with natural numbers. This notation is employed in the usual lax manner; for instance, $\Gamma_1, A, B, \Gamma_2$ shall denote a list of formulae. Let $\mathbf{len}(\Gamma)$ be the number of formulae occurring in a list of formulae $\Gamma$. Capital letters in script style $\mathcal{S}, \mathcal{T}, \ldots$ will be used to denote sequents, and sequents will occasionally be set in square brackets as delimiters. We extend the notion of complexity to sequents as follows:

$$\mathbf{cmx}([A_1, \ldots, A_m \to B_1, \ldots, B_n]) = \sum_{i=1}^{m} \mathbf{cmx}(A_i) + \sum_{i=1}^{n} \mathbf{cmx}(B_i)$$

**Definition 4.3** *For a list of formulae* $\Gamma = A_1, \ldots, A_m$*, its* legth *is given by* $\mathbf{len}(\Gamma) \stackrel{def}{=} m$*, and its formulae are collected in the set* $\{\!\{\Gamma\}\!\} \stackrel{def}{=} \{A_1, \ldots, A_m\}$*. For a sequent* $\mathcal{S} = [A_1, \ldots, A_m \to B_1, \ldots, B_n]$*, its formulae are collected in the set* $\{\!\{\mathcal{S}\}\!\} \stackrel{def}{=} \{A_1, \ldots, A_m, B_1, \ldots, B_n\}$*.*

The notation $\{\!\{\cdot\}\!\}$ identifies multiple occurrences of the same formula in a list or sequent. It simply collects all of the formulae that occur in some list or sequent at all, regardless of their position.

**Definition 4.4** *An* inference rule *is a figure of the form*

$$\frac{\mathcal{S}_1 \quad \cdots \quad \mathcal{S}_n}{\mathcal{T}},$$

*where* $(n \geq 0)$ *and* $\mathcal{S}_1, \ldots, \mathcal{S}_n, \mathcal{T}$ *are sequents. The sequents* $\mathcal{S}_1, \ldots, \mathcal{S}_n$ *are called the* premises *and* $\mathcal{T}$ *is called the* conclusion*. If* $n = 0$*, then the inference rule is called* improper *inference rule or* axiom*.*

For logical inference rules, the notions of a *main formula* occurring in the conclusion of the rule, into which a particular logical connective has been introduced, and the corresponding *side formulae*, which are always immediate subformulae of the main formula that occurs in the premises of the rule, are retained as it has been described in the preceding chapter.

**Definition 4.5** *A* derivation *is a figure, which consists of one or more sequents, which contribute to inference rules in the following manner:*

- *Each sequent is conclusion of of at most one inference rule.*
- *Each sequent, except for a single one, is the premiss of exactly one inference rule.*

*Sequents, which are not conclusions of an inference rule, are called* initial sequents *of the derivation, and the single sequent, which is not the premiss of an inference rule, is called the* end sequent *of the derivation. A derivation, which has no initial sequents, is called a* proof *of the end sequent. A* subderivation *of a derivation is a figure, which consists of one or more sequents of the original sequent, which themselves constitute a derivation.*

Sometimes, when we consider very short derivations consisting of only a few inference rules, we will call the initial sequents of that derivation its premises and the end sequent its conclusion, as if the entire derivation were itself an inference rule.

In the following point we disgress from Gentzen's terminology. Where we distinguish a proof from a derivation, Gentzen had only considered derivations. In his understanding, every derivation had as its initial sequents sequents of the form $A \to A$. We want to be able to consider derivations, which are *partial* in the sense that their initial sequents need not necessarily be of that form. In order to recapture the notion of derivation in the sense of Gentzen, we introduce the notion of proof and add the following improper rule:

$$\frac{}{A \to A}\ (\text{Ax})$$

The label (Ax) represents the contemporary name of this improper rule: *axiom*. Since every derivation in the sense of Gentzen always has initial sequents of the form $A \to A$, all of them can be endued with an instance of (Ax), and since (Ax) has no premises, no initial sequents remain in the modified derivation, and, hence, the derivation is a proof in our sense.

Greek capitals $\Pi, \Xi, \ldots$ will be used to denote derivations; taking the context into account, it will not be possible to confuse derivations with sequences of formulae.

**Definition 4.6** *A* path *in a derivation* $\Pi$ *is a sequence* $(\mathcal{S}_0, \ldots, \mathcal{S}_n)$ *with* $n \geq 0$*, such that* $\mathcal{S}_0$ *is the end sequent of* $\Pi$ *and every* $\mathcal{S}_{i+1}$ *is the premiss in an inference rule of* $\Pi$*, which has* $\mathcal{S}_i$ *as conclusion. A* branch *is a path, for which* $\mathcal{S}_n$ *is either an initial sequent of* $\Pi$ *or the conclusion of an instance of* (Ax) *in* $\Pi$*; in either case,* $\mathcal{S}_n$ *is called a* leaf *of* $\Pi$*.*

A branch is roughly what Gentzen called a *thread*, although a thread lists seuqents in reverse order. As we will adopt the bottom-up perspective of *proof search*, i.e. consider some sequent and construct derivations in an attempt to obtain a proof of it, it is more useful to have branches of derivations begin with the respective end sequents. As every path contains the end sequent of a proof – the only sequent, which does not occur as the premiss of a rule – the end sequent is given the index 0. Thereby, the index of a sequent $\mathcal{S}_i$ of a path is also a measure of how many inference rules lie between it and the end sequent.

From now on, we shall employ the bottom-up perspective, which corresponds to a proof search. Derivations will be constructed from the bottom up, starting with an arbitrary sequent, in an attempt to obtain initial sequents, which can then be sealed by instances of (Ax). In particular, rules will be read in the manner of an attempted justification: If we want to write down the conclusion of a rule, we have to be allowed to write down the premiss or all of the premises of the rule. In the bottom-up perspective, logical rules are employed to eliminate logical connectives, and the structural rules of weakening and contraction rather have the function of considering a strengthened premiss or, respectively, a premiss, in which a particular formula occurrence has been expanded into two occurrences.

## 4.2 From LK to RK

The calculus LK has several properties, which are undesirable for this investigation. The most important one is the fact that it includes rules for existentially and universally quantified formulae, i.e. LK is a predicate calculus. Unfortunately, the rules governing the quantifiers do not have the subformula property in the *literal* sense. It will become obvious in the following chapter, that we aim at taking every aspect of the logical language literally, which will even lead to making different occurrences of propositional variables explicit by using *occurrence instances* of these variables.[3] For this reason, it is impracticable to extend these investigations to the predicate calculus at this

[3]Even a sequent $a \to a$ shall be rendered by the *occurrence instance* $a_1 \to a_2$, in which every occurrence of a propositional variable is replaced by a unqiue *occurrence variable*.

time. Fortunately, removing the four quantifier rules yields the propositional calculus. The other undesireable properties will be briefly addressed and remedied in turn in the remainder of this section.[4] The changes all have to with the fact that we shall be mostly interested in the perspective of proof-search, i.e. the generation of derivations from the bottom-up, for which it is useful to accumulate and retain formulae in the sequents as the derivation is constructed and to avoid choices. This will eventually lead to the calculus RK, the calculus on the basis of which the following investigations shall be executed.

## Atomic Initial Sequents

Inital sequents in Gentzen's logistic calculus may be of the form $A \to A$ for an aribtrary formula $A$. In our variant of the calculus, this is represented by the improper rule (Ax):

$$\frac{}{A \to A}\,(\mathrm{Ax})$$

From a naive justificational perspective, it is curious that justifications for the logical connectives in the two occurrences of formulae $A$ may be avoided, when the entire logistic calculus is nothing but a means of providing justifications for logical connectives occurring in formulae. Hence, it would have been more in the spirit of the logistic calculus, if Gentzen had only allowed elementary sequents of the form $a \to a$, where $a \in \mathcal{A}$ is a propositional variable, as initial sequents. Certainly, a more immidiate relation of assumption and assertion than that expressed by such sequents is not possible.

Of course, it is easy to see that every sequent of the form $A \to A$, where $A$ is a complex formula, can be further justified by the usual inference schemes, down to elementary sequents of the form $a \to a$. Let us briefly consider two particular cases of Gentzen's initial sequents, which contain occurences of a complex formula $A$, and see how they can be reduced to less complex formulae:

- $A = B \vee C$:

$$\dfrac{\dfrac{\dfrac{B \to B}{B \to B, C}\,(\mathrm{WS}) \qquad \dfrac{\dfrac{C \to C}{C \to C, B}\,(\mathrm{WS})}{C \to B, C}\,(\mathrm{XS})}{B \vee C \to B, C}\,(\vee\mathrm{A})}{B \vee C \to B \vee C}\,(\vee\mathrm{S})$$

[4] For detailed descriptions of the feasability of these remedies, see e.g. [TS96].

- $A = \neg B$:

$$\cfrac{\cfrac{\cfrac{B \to B}{\neg B, B \to}\ (\neg \mathrm{A})}{B, \neg B \to}\ (\mathrm{XA})}{\neg B \to \neg B}\ (\neg \mathrm{S})$$

The case of $A = B \supset C$ is similar to the first case, and the case of $A = B\&C$ is dual to it. If either $B$ or $C$ is still a complex formula, it can be further reduced to less complex formulae. Since every formula can only contain finitely many logical connectives, this procedure terminates and results in a derivation for $A \to A$, whose initial sequents have the form $a \to a$ for $a \in \mathcal{A}$. Hence, every derivation having initial sequents of the form $A \to A$ for complex formulae $A$ can be transformed into a somewhat larger derivation, in which all of these sequents are end sequents of subderivations, which are constructed according to the indicated procedure.

For this reason, we can replace the improper rule (Ax) by the more specific improper rule (ax), which is only applicable to elementary sequents:

$$\cfrac{}{a \to a}\ (\mathrm{ax})$$

Of course, proofs using (ax) instead of (Ax) are larger in general. However, all of the logical connectives occurring in the end sequent are properly justified by elementary sequents.

## Absorbing Weakening into the Axiom

It is well-known that every application of weakening can be pushed upward in the derivation, as the formula introduced by weakening can be passed downward as one of the context formulae. Consider the following derivation of $A, \Gamma \to \Delta$, which has initial sequents $\Gamma_i \to \Delta_i$ for $1 \leq i \leq n$, and in which the bottommost rule application is (WA):

$$\begin{array}{ccc} \Gamma_1 \to \Delta_1 & \cdots & \Gamma_n \to \Delta_n \\ & \ddots \quad \cdot^{\cdot^{\cdot}} & \\ & \cfrac{\Gamma \to \Delta}{A, \Gamma \to \Delta}\ (\mathrm{WA}) & \end{array}$$

Instead of introducing $A$ with the bottommost rule application, all of the $n$ initial sequents can be weakened by introducing the formula $A$, and then $A$ can be moved to the rightmost position of the respective antecedent by $\mathbf{len}(\Gamma_i)$ applications of (XA) (indicated by the double lines):

$$
\begin{array}{ccc}
\text{(WA)}\,\dfrac{\dfrac{\Gamma_1 \to \Delta_1}{A, \Gamma_1 \to \Delta_1}}{\Gamma_1, A \to \Delta_1} & \dots & \dfrac{\dfrac{\Gamma_n \to \Delta_n}{A, \Gamma_n \to \Delta_n}}{\Gamma_n, A \to \Delta_n}\,\text{(WA)} \\
 & \ddots \quad \cdot^{\cdot^{\cdot}} & \\
 & \dfrac{\Gamma, A \to \Delta}{A, \Gamma \to \Delta} &
\end{array}
$$

In the rightmost position of the antecedent, the formula $A$ does not interfere with the following rule applications, which can be carried over from the original dervation. Eventually, $A$ is moved to the leftmost position of the antecedent by $\mathbf{len}(\Gamma)$ applications of (XA). The result is a derivation of the same end sequent as before, in which the application of weakening has been pushed upward into $n$ topmost applications of weakening. The same procedure can be repeated for every application of weakening occurring anywhere within a given derivation, and the same can be performed for applications of weakening in the succedent. This results in a derivation, in which all applications of weakening rules occur at the very top, i.e. in which no application of a weakening rule occurs underneath an application of any other rule, except for exchange rules.

As a result of these two observations, we can replace (ax) by the improper rule (ax*):[5]

$$\frac{}{\Gamma_1, a, \Gamma_2 \to \Delta_1, a, \Delta_2}\,(\text{ax}^*)$$

When employing this new axiom, we can dispense of the two structural rules (WA) and (WS). Any formula, which would to be introduced by applications of (WA) and (WS) can be introduced at the top of the derivation by appropriate instances of (ax*).

In this point, it might appear that the taken direction is heading somewhat against the intuition of Hertz, in whose logic the rule of immediate inference, which corresponds to (WA), is one of the two only rules. In [Gen33], Gerhard Gentzen showed that in Hertz' logic, every proof can be transformed into a normal proof, in which there occurs only one application of immediate inference ("thinning" in Gentzen's terminology) at the very bottom. Thereby, he had already established the mobility of this inference rule and its relative insignificance compared to the cut rule. It is obvious that all of

[5] From the conclusion of this rule, a sequent of the form $a \to a$ could be obtained in a bottom-up manner by $\mathbf{len}(\Gamma_2)$ applications of (XA) in order to move $a$ to the rightmost position of the antecedent, followed by $\mathbf{len}(\Gamma_1) + \mathbf{len}(\Gamma_2)$ applications of (WA), and appropriate numbers of applications of (XS) and (WS).

the extra elements, which are introduced into a sentence by a final application of immediate inference do not contribute to the proof in an essential manner. But then, all of these elements could already have been introduced at the very top of the proof and passed through all applications of cut in the proof unaffectedly. So even in Hertz' logic, the rule of immediate inference is of relatively low relevance. Recall that the rearranging of elements and the contraction of multiple occurrences of the same element in the antecedent of a sentence were integrated into the symbol $\|$, which occurs in the two inference rules.

The question, at which point of a derivation weakening (immediate inference) should be available, very much depends on the manner, in which the calculus is to be employed. For proof construction, i.e. building derivations in a top-down manner, it is useful to start with sequents, which are as lean as possible, and to introduce formulae by weakening only as they are needed. Such a course of action keeps derivations as concise as possible. For proof search, on the other hand, it is much more effective to simply retain all the formulae that are introduced into the respective sequents rather than deciding at every step of the construction of a derivation, whether it is safe to remove a formula, which has just been extracted from a complex formula, from the context by weakening. As we shall employ this latter perspective, we do not require weakening rules for the derivations. Instead of delaying the removal of superfluous formulae to the top of the derivation, we simply modify the axiom in a manner that allows contextual formulae.

Moreover, in view of the preceding argument, namely that no complex formula should occur in an axiom, the variant (ax$^*$) seems counter-intuitive, as the formulae occurring in $\Gamma_1, \Gamma_2, \Delta_1$ and $\Delta_2$ are not restricted. For instance, the introduction of $A \mathbin{\&} B$ by weakening in

$$\frac{\Gamma \to \Delta}{A \mathbin{\&} B, \Gamma \to \Delta}\,(\mathrm{WA})$$

can be modified to independent weakenings by formulae $A$ and $B$:

$$\frac{\dfrac{\Gamma \to \Delta}{A, \Gamma \to \Delta}\,(\mathrm{WA}) \qquad \dfrac{\Gamma \to \Delta}{B, \Gamma \to \Delta}\,(\mathrm{WA})}{A \mathbin{\&} B, \Gamma \to \Delta}\,(\&\mathrm{A})$$

Of course, corresponding modifications are possible for other logical connectives. Hence, weakening can be restricted to atomic formulae, i.e. we can demand that sequences $\Gamma_1, \Gamma_2, \Delta_1$ and $\Delta_2$ in (ax$^*$) only contain atomic formulae. We represent this by using lower case letters $\gamma_1, \gamma_2, \delta_1$ and $\delta_2$ for the context. The final version of the axiom is then:

$$\frac{}{\gamma_1, a, \gamma_2 \to \delta_1, a, \delta_2}\,(\mathrm{ax}^*)$$

## Propositional Contraction

The two sets of logical rules (&A$_1$) and (&A$_2$) as well as ($\vee$S$_1$) and ($\vee$S$_2$) have particularly undesirable properties. In each one of these rules, only one of the immediate subformulae of the main formula occurs in the premiss.

$$\frac{A, \Gamma \rightarrow \Delta}{A \,\&\, B, \Gamma \rightarrow \Delta}\ (\&\mathrm{A}_1) \qquad \frac{\Gamma \rightarrow \Delta, A}{\Gamma \rightarrow \Delta, A \vee B}\ (\vee\mathrm{S}_1)$$

$$\frac{B, \Gamma \rightarrow \Delta}{A \,\&\, B, \Gamma \rightarrow \Delta}\ (\&\mathrm{A}_2) \qquad \frac{\Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \vee B}\ (\vee\mathrm{S}_2)$$

Not only are arbitrary formulae introduced in the conclusion of the rule, which do not occur in the premiss, $B$ in the case of (&A$_1$) and ($\vee$S$_2$) and $A$ in the case of (&A$_2$) and ($\vee$S$_1$). From the perspective of proof construction, these rules require a choice to be made. As soon as a disjunction $A \vee B$ in the succedent of a sequent is to be treated, it has to be decided, which one of the rules ($\vee$S$_1$) and ($\vee$S$_2$) should be employed for the purpose. This choice is crucial, for only one of the subformulae $A$ and $B$ can be retained in the premiss, but at that point of the proof search, it might not be immediately obvious, which one can be safely abandoned without forfeiting the possibility of obtaining a proof.

Fortunately, the contraction rules remedy this predicament. The rule (CS) can be used in combination with the two rules for the disjunction in the succedent to construct a short derivation, which has as topmost sequent a premiss, in which both subformulae of the disjunction occur:

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma \rightarrow \Delta, A, B}{\Gamma \rightarrow \Delta, A, A \vee B}\ (\vee\mathrm{S}_2)}{\Gamma \rightarrow \Delta, A \vee B, A}\ (\mathrm{XS})}{\Gamma \rightarrow \Delta, A \vee B, A \vee B}\ (\vee\mathrm{S}_1)}{\Gamma \rightarrow \Delta, A \vee B}\ (\mathrm{CS})$$

We can, therefore, introduce the following new rule ($\vee$S), in which both subformulae of a disjunction in succedent position occur in the premiss:

$$\frac{\Gamma \rightarrow \Delta, A, B}{\Gamma \rightarrow \Delta, A \vee B}\ (\vee\mathrm{S})$$

In the presence of (WS), the original premises of ($\vee$S$_1$) and ($\vee$S$_2$) could easily be recovered. However, as we are expressly interested in accumulating all subformulae, this will not be necessary at any point. As desired, both subformulae are retained in the premiss of this new rule.

$$\frac{}{\gamma_1, a, \gamma_2 \to \delta_1, a, \delta_2}\ (\text{ax}^*)$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \to \Delta}{\Gamma_1, B, A, \Gamma_2 \to \Delta}\ (\text{XA}) \qquad \frac{\Gamma \to \Delta_1, A, B, \Delta_2}{\Gamma \to \Delta_1, B, A, \Delta_2}\ (\text{XS})$$

Figure 4.1: Structural rules of RK

Dually, we can obtain the new rule (&A), replacing (&A$_1$) and (∨S$_2$). This new rule retains both subformulae of a conjunction in the antecedent of its premiss:

$$\frac{A, B, \Gamma \to \Delta}{A \,\&\, B, \Gamma \to \Delta}\ (\&\text{A})$$

As a consequence, the contraction rules have become obsolete for our purpose, which is to successively remove all logical connectives in a proof search until all the topmost sequents in a derivation are atomic. Applications of contraction in the presence of (&A) and (∨S) would only generate more work in general, as the premiss of a contraction rule contains an additional occurrence of some particular formula.

## The calculus RK

We can now compile the calculus RK, which will be the basis for the investigations in parts II and III.[6] Apart from giving up (&A$_1$) and (&A$_2$) in favour of (&A) and replacing (∨S$_1$) and (∨S$_2$) by (∨S), the logical rules remain unchanged. Since (CA) and (CS) are no longer necessary in view of the new rules (&A) and (∨S), they are removed. We also give up (Cut), as we only want to consider cut-free derivations in RK, as we wish to avoid the possibility of introducing new complex formulae in a proof search.[7] Moreover, the improper rule (ax$^*$) is added to the structural rules, which allows the closure of initial sequents and at the same time the removal of the structural rules (WA) and (WS). We retain the structural rules (XA) and (XS) and will use the abbreviation

$$\frac{\Gamma \to \Delta}{\overline{\Gamma' \to \Delta'}}$$

[6]This calculus roughly corresponds to the propositional fragment of **G3c** of [TS96].

[7]A very restricted instance of cut, the *atomic cut* or *proxy cut* will be used in the explosion calculus that will be introduced in the next chapter.

$$\frac{A, B, \Gamma \to \Delta}{A \,\&\, B, \Gamma \to \Delta}\,(\&\text{A}) \qquad \frac{\Gamma \to \Delta, A \qquad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \,\&\, B}\,(\&\text{S})$$

$$\frac{A, \Gamma \to \Delta \qquad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\,(\vee\text{A}) \qquad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}\,(\vee\text{S})$$

$$\frac{\Gamma \to \Delta, A \qquad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\,(\supset\text{A}) \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}\,(\supset\text{S})$$

$$\frac{\Gamma \to \Delta, A}{\neg A, \Gamma \to \Delta}\,(\neg\text{A}) \qquad \frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \neg A}\,(\neg\text{S})$$

Figure 4.2: Logical rules of RK

for zero or more applications of either exchange rule; the antecedent $\Gamma'$ of the conclusion is then some permutation of the antecedent $\Gamma$ of the premiss, and the same relation holds for the succedents. The structural rules, including (ax*) are given in figure 4.1, and the logical rules of RK are collected in figure 4.2.

The calculus RK is most useful for proof search. When constructing a derivation starting from an arbitrary sequent, its set of rules guarantees that, although each application of a logical rule will leave its premiss or premises with one less logical connective, all of the formulae that occur as context in the conclusion of the rule are still present in its premiss or premises, and, moreover, all of the immediate subformulae of the main formula occur as side formulae in the premiss or premises. The overall effect is that the calculus allows the construction of derivations, in which all the information, which is relevant for the justification of the logical connectives occurring in the end sequent, is accumulated in the premises of the rules as the derivations are developed towards their eventual initial sequents.

At the same time, a proof of a sequent in RK can be developed in such a manner that all of its topmost sequents, i.e. conclusions of (ax*), only contain atomic formulae. As sequents of this form, which we have called "elementary sequents", have the form of a generalised sentence in the sense of Hertz[8], all of the topmost sequents of a RK-proof constitute a system of sentences in the sense of Hertz, i.e. an entirely structural justification of the end sequent. Unfortunately, these sentences are nothing but relations of assumptions of

[8]This requires considering a generalisation in as far as succedents may contain any number of elements.

propositional variables and assertions of other propositonal variables, and it is not possible in general to understand by just considering the topmost sequents of a derivation, what the end sequent looks like. Consider the following set of elementary sequents:

$$\big\{[a, b \to a, a], [a, b, c \to a], [a, b \to c, a], [a, b, c \to c]\big\}$$

Even in this rather concise example, it is hardly possible to survey these sequents and design the end sequent of a proof, in which these sequents occur at the top of its branches. The following proof is an example for a proof, which has these elementary sequents as leaves:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a, b \to a, a}\,(\text{ax}^*) \quad \dfrac{}{a, b, c \to a}\,(\text{ax}^*)}{a \supset c, a, b, \to a}\,(\supset\text{A})}{a, b, a \supset c \to a} \qquad \dfrac{\dfrac{\dfrac{}{a, b \to c, a}\,(\text{ax}^*) \quad \dfrac{}{a, b, c \to c}\,(\text{ax}^*)}{a \supset c, a, b \to c}\,(\supset\text{A})}{a, b, a \supset c \to c}}{a, b, a \supset c \to a \,\&\, c}\,(\&\text{S})}{a \,\&\, b, a \supset c \to a \,\&\, c}\,(\&\text{A})$$

The relation of the elementary sequents to the sequent $a \,\&\, b, a \supset c \to a \,\&\, c$ has to be facilitated by means of this proof.

It will be the purpose of the following part of this investigation to provide some means of interrelating the leaves of particular RK-derivations, so called *explosion derivations*, in such a manner, that it will be possible to understand the meaning of the end sequent, from which these elementary sequents have been generated, immediately on the basis of the collection of these leaves, i.e. without having to consider or reconstruct that end sequent. It will become apparent that these collections of leaves correspond to systems of axioms in a somewhat generalised sense of Paul Hertz. For this purpose, particular variants of the logical rules are required, which will be introduced forthwith.

## 4.3 From General Rules to Local Rules

We have argued that constructing a derivation of a sequent corresponds to giving an entirely structural justification for the relation of assumptive and assertive formulae therein. This perspective is immediately apparent, if the logical inference rules are indeed regarded as schemes, i.e if all the formula variables occurring in the premiss or premises are considered "elementary" in the sense of Hertz. In each of these justification schemes, the context, consisting of the assumptive and assertive formulae, which are not relevant to the complex formula introduced in the conclusion, can be momentarily disregarded. This is respresented by collecting all contextual formulae into context variables Γ and Δ, which occur without modification in the premiss or premises and in the conclusion.

Unfortunately, the complex formulae occuring as context are in need of justification themselves, and, hence, have to be considered in turn. Due to the fact that some logical connectives are justified on the basis of the particular interaction of two structural sentences, in the corresponding inference scheme of two premises, the context formulae in Γ and Δ have to be considered twice, once in each premiss. Therefore, the justification for a complex formula occurring in Γ or Δ has to be given twice, even if the same justification can be give in both cases. Of course, this problem engenders an explonential blowup as other complex formulae may also have to be justified by two sequents.

It is obvious that this blowup results from the specific formulation of the inference rules (∨A), (⊃A) and (&S). All of these rules have two premises, each of which must have the same context. For instance, consider the inference rule (&S):

$$\frac{\Gamma \rightarrow \Delta, A \qquad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \,\&\, B}\ (\&\mathrm{S})$$

The premises agree in the entirety of their antecedents and almost all of their succedents, except for the side formulae. As justification of the assertive formula $A \,\&\, B$, an assertion of $A$ and an assertion of $B$ is required.[9] In the logistic calculus, assertions cannot occur as isolated formulae, however, and (&S) simply accounts for the (unrestricted) posibilities of relations, in which these assertions of $A, B$ and $A \,\&\, B$ can be found. The rule then demands that the contexts, in which these formulae occur, must be identical.[10] For the purpose of proof construction, this is a reasonable demand in the sense that the required strict observation of the two contexts of the side formulae puts the focus on the actual justification of the logical connective, which is

[9]This is akin to the introduction rule for the conjunction in the calculus of natural deduction.

[10]Another approach is possible, in which a mixing of different contexts is permitted.

$$\frac{\Gamma_1 \rightarrow \Delta_1, A \qquad \Gamma_2 \rightarrow \Delta_2, B}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, A \,\&\, B}\ (\&\mathrm{S'})$$

When all the structural rules are freely available, this rule is equivalent to (&S). Whenever structural rules *are* restricted or particular structural rules are entirely unavailable, exchanging one rule for the other will result in a change of provable sequents. In linear logic, where structural rules are restricted, the two different justifications are considered concurrently, resulting in two different types of conjunction. In that context, the standard justification introduces the additive conjunction, whereas (&S') is used to introduce the multiplicative conjunction.

introduced. That is, logical connectives are to be introduced only under controlled circumstances. From the perspective of proof search, the duplication of the context will eventually amount to a considerable effort, especially in the calculus RK, in which all the side formulae are retained and weakening is worked into the axiom. Of course, this effort is absolutely necessary, for simply omitting parts of the context will gnerally render the possibility of obtaining a proof impossible. In what is to follow, we intend to separate the derivation stage, in which the structural justification for all the logical connectives in the end sequent is provided, from the decision stage, in which it is checked, whether the provability of the end sequent can be decided on the basis of the leaves of the derivation.

Hence, we can focus on extracting the justificational aspect of the general logical rules with as little contextual burden as possible. By this we mean that we want to express the justification a complex formula on the basis of its immediate subformulae or subformula, and also retain the possibility of relating all of these components to some context, but without explicitly specifying an arbitrarily complex context in the rule itself. This is achieved by moving from logical rules, in which context is explicitly represented, to rules allowing as context at most a single propositional variable. The former kind of rules shall henceforth be called *general rules*, the latter *local rules*. The propositional variable represents as elementary a context as is required in view of the relational nature of sequents. Via the intermediate step of this propositional variable, any context can be connected to the relevant formulae by an application of (Cut). For instance, compare the general rule (&S)

$$\frac{\Gamma \to \Delta, A \qquad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \,\&\, B}\,(\&\mathrm{S})$$

to the following short derivation, in which the propositional variable $p$ must not occur anywhere in $A$ nor $B$ nor in any formula of $\Gamma$ and $\Delta$:

$$\frac{\Gamma \to \Delta, p \qquad \dfrac{p \to A \qquad p \to B}{p \to A \,\&\, B}\,(\&\mathrm{S})}{\Gamma \to \Delta, A \,\&\, B}\,(\mathrm{Cut})$$

From a botom-up perspective, a new propositional variable $p$ is introduced by means of the cut rule in a first step, thereby disconnecting the main formula $A \& B$ from the context. Note that the right premiss still has the form required for the application of the local rule (&S). The new propositional variable $p$ is can be seen as representing the specific occurrence of the formula $A \,\&\, B$ in the end sequent. Even if $\Delta$ should contain another occurrence of $A \,\&\, B$, the variable $p$ only serves as proxy for relating the very rightmost occurrence of $A \,\&\, B$ to the succedent.[11]

The connection between the formula and its original occurrence in the sequent is represented by a unique variable, which is the cut formula. Such a variable can, therefore, be viewed as representing an *occurrence* of a formula. In the above example $p$ can be considered an occurrence of the formula $A \,\&\, B$. Dually, it can be considered as the *location* at which some formula occurs, e.g. in the sequent $\Gamma \rightarrow \Delta, p$ the propositional variable $p$ marks the location, where some formula may occur. Both occurrence and location refer to an abstract entity relating a formula to a sequent or, more generally, an immediate subformula to a formula. While it is common use to speak of occurrences of formulae, the converse view of considering some complex entity, sequent or formula, as providing locations for formulae is usually not required. Instead, "occurrence" is used to simultaneously address both a particular instance of a formula as well as the location, where it occurs. While we are aware of the distinction, we shall not employ it. From here on, the notions *occurrence*, *location* will be used interchangeably for propositional variables that are introduced by applications of (Cut).[12]

Returning to the short derivation, the left premiss of (Cut) shall be called the *context premiss*. Via the variable $p$, the context sequent $\Gamma \rightarrow \Delta, p$ is shared by both of the *logical premises*, the premises of (&S):

$$\frac{p \rightarrow A \qquad p \rightarrow B}{p \rightarrow A \,\&\, B}\ (\&\mathrm{S})$$

This particular instance of (&S) has the intended property that it does not justify the conjunction in an arbitrary context, but only relative to its occurrence, which is represented by the propositional variable $p$. Both premises of this rule instance, as well as the conclusion express very simple relations of the single assumption $p$ and the respective single assertion. This instance of (&S) therefore expresses the desired localised perspective on the justification of an assertive conjunctive formula in the sense that only the immediately relevant information is represented in the formulation of the logical rule.

In accordance with this example, we will introduce the remaining local logical rules. These local variants of the logical rules will replace their general

[11] In order to disconnect the second occurrence of $A \& B$ from the succedent, another new propositional variable, say $q$, will have to be introduced by another application of (Cut) to the left premiss, after that occurrence has been moved into the required position by one or more applications of (XS). By repeated application of this procedure, every complex formula can be detached from the succedent, and the same is possible for the antecedent.

[12] In the following part, we will explicitly introduce *proxy variables*, which will only be used for this purpose.

counterparts in the explosion calculus, which will be introduced in the second part of this investigation.

## Conjunction and Disjunction

The local rule ($\&S_l$), which is to be used instead of ($\&S$), is simply the general rule under the restriction of the antecedent to a single propositional variable and of the succedent to the main formula in the conclusion or, respectively, the side formulae in the premises.

$$\frac{p \to A \qquad p \to B}{p \to A \,\&\, B}\,(\&\mathrm{S}_l)$$

Dually, the local rule ($\vee A_l$) can be given as follows:

$$\frac{A \to p \qquad B \to p}{A \vee B \to p}\,(\vee\mathrm{A}_l)$$

In this case, the main formula and the side formulae occur in the antecedent, i.e. as assumptions. Hence, the propositional variable $p$, which represents the occurrence of $A \vee B$, should be in the succedents of the involved sequents. For both local rules, the premises can be understood as sentences of first degree of Hertz' structural logic. Applying this pattern to ($\&A$) yields the following local variant of that rule:

$$\frac{A, B \to p}{A \,\&\, B \to p}\,(\&\mathrm{A}_l)$$

The difference to the preceding two cases is the fact that ($\&A_l$) has a single premiss, in which both side formulae occur. As the general rule has only a single premiss, this local variant is not required in view of the aforementioned issue of duplicating complex formulae. However, a local variant of every logical rule is required for the explosion calculus, as it will systematically introduce variables representing occurrences of complex formulae. By taking $A$ and $B$ as elementary, the premiss $A, B \to p$ can be interpreted as an elementary justification in the form of a sentence of Hertz' structural logic. The dual local rule ($\vee S_l$) provides an elementary justification for the introduction of an assertive disjunction:

$$\frac{p \to A, B}{p \to A \vee B}\,(\vee\mathrm{S}_l)$$

In this case, the premiss only resembles a sentence of Hertz' structural logic in the sense that we allow an extension thereof along the lines of Gentzen's logistic calculus.

What we have obtained are local rules justifying the introduction of conjunctive and disjunctive formulae in both assumptive and assertive roles, which can be considered as elementary justifications in the sense that there is no context of possibly complex formulae to consider in this formulation. Moreover, a further restriction of these rules by removing the propositional variable would render them useless. For instance, consider the following candidate for ($\vee\mathrm{S}_l$):

$$\frac{\rightarrow A, B}{\rightarrow A \vee B}$$

Although disjunction obtained in this manner could be used to either generate more complex formulae, it could never be set into any kind of structural context of other formulae. The crucial feature of the logistic calculus, that of allowing the reasoning about relations of assertions and assumptions, would be lost. The single propositional variables make it possible to retain this feature. Instead, every formula occurrence is countered with a reference point, and is meaningful only in relation to it. The importance of such a reference point is the reason why we will later introduce a particular kind of variables, which are specific for this purpose.

## Negation and Implication

Applying the pattern from the previous cases to the general negation rule ($\neg$A) yields this local rule:

$$\frac{\rightarrow p, A}{\neg A \rightarrow p}\ (\neg\mathrm{A}_l)$$

As in the general variant, the assumption of a negated formula is justified by the assertion of the formula itself, that is, the main formula occurs in the antecedent, whereas the side formula occurs in the succedent. While the conclusion of the local variant represents a relation between assumption and assertion, this is no longer the case in the premiss. Instead, the justification of the conclusion is an absolute assertion of the alternative of the formula $A$ and its occurrence, represented by the variable $p$. In contrast to the example given above for a minimal justification of an assertive disjunction, the conclusion contains a propositional variable, by which the negated formula can be joined to any context. It is obvious that the analogy to Hertz' sentences breaks down at this point, unless we take his remark seriously that unconditional effects could also be considered. As far as the logistic calculus is concerned, this local rule is straightforward, and, as we shall see, it will serve its purpose. Similar remarks apply to the local variant of (¬S):

$$\frac{A, p \rightarrow}{p \rightarrow \neg A}\ (\neg\mathrm{S}_l)$$

The justification for the assertion of a negation finally lies well outside of that, which Hertz had envisioned. Of course, Hertz had not taken negation into consideration at all. Within the regularity of RK, this local rule is perfectly acceptable, however.

The fact that a formula $A \supset B$ is equivalent to the formula $\neg A \vee B$ in the classical setting is reflected in the fact that in RK the premises of the general rule

$$\frac{\Gamma \rightarrow \Delta A \qquad B, \Gamma \rightarrow \Delta}{A \supset B, \Gamma \rightarrow \Delta}\ (\supset\mathrm{A})$$

correspond to the two topmost sequents of this short derivation:

$$\frac{\dfrac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta}\ (\neg\mathrm{A}) \qquad B, \Gamma \rightarrow \Delta}{\neg A \vee B, \Gamma \rightarrow \Delta}\ (\vee\mathrm{A})$$

Likewise, the premises of the local rule ($\supset\mathrm{A}_l$) resemble that of ($\neg\mathrm{A}_l$) and the right premiss of ($\vee\mathrm{A}_l$):

$$\frac{\rightarrow p, A \qquad B \rightarrow p}{A \supset B \rightarrow p}\ (\supset\mathrm{A}_l)$$

Apart from the particular shapes of its premises, the local variant of (⊃A) is otherwise unremarkable. Unfortunately, the similitude of the premises will have consequences in the further proceedings in as far as certain case analyses will become more involved. E.g. a sequent of the form $\rightarrow p, a$ cannot be brought in relation to an instance of ($\supset\mathrm{A}_l$), as it could also be related to ($\neg\mathrm{A}_l$). The same can be said for a sequent of the form $b \rightarrow p$, which can either be the premiss of an instance of ($\supset\mathrm{A}_l$) or ($\vee\mathrm{A}_l$). This problem underscores the question, whether implication is a useful logical connective in the classical context at all. It is particularly accentuated, as it is our intention to emphasise importance of the structural relation →. The justification of assertive implications makes this most evident, even in its local variant:

$$\frac{A, B, \rightarrow p}{A \,\&\, B, \rightarrow p}\ (\&\text{A}) \qquad \frac{p \rightarrow, A \qquad p \rightarrow, B}{p \rightarrow, A \,\&\, B}\ (\&\text{S})$$

$$\frac{A, \rightarrow p \qquad B, \rightarrow p}{A \vee B, \rightarrow p}\ (\vee\text{A}) \qquad \frac{p \rightarrow, A, B}{p \rightarrow, A \vee B}\ (\vee\text{S})$$

$$\frac{p \rightarrow, A \qquad B, \rightarrow p}{A \supset B, \rightarrow p}\ (\supset\text{A}) \qquad \frac{A, \rightarrow, B}{p \rightarrow, A \supset B}\ (\supset\text{S})$$

$$\frac{p \rightarrow, A}{\neg A, \rightarrow p}\ (\neg\text{A}) \qquad \frac{A, \rightarrow p}{p \rightarrow, \neg A}\ (\neg\text{S})$$

Figure 4.3: Local logical rules

$$\frac{A, p \rightarrow B}{p \rightarrow A \supset B}\ (\supset\text{S}_l)$$

An assertion of a logical implication $A \supset B$ is just the formal expression of the structural relation between an assumption $A$ and an assertion $B$. In case of ($\supset$S$_l$), the propositional variable $p$ is an additional assumption, by which a particular instance of the relation is chosen. However, the justification does have the shape of a sentence in the sense of Hertz.

Although the local logical rules for negation and ($\supset$A$_l$) lack the elegance of their counterparts for conjunction and disjunction, especially in view of a possible interpretation as structural justifications for the respective connectives, it is straightforward to obtain them by removing or reducing the contexts from the general rules of RK.

## Structural Rules

As the structural rules do not introduce any logical connectives, they have no justificational character that is to be isolated. Hence, there are no local variants of the structural rules. However, in a bottom-up perspective, in order to obtain a sequent, to which a local logical rule can to be employed, it must be possible to detach a single complex formula from either the antecedent or succedent position by means of the cut rule. As the cut rule is not a rule of RK, a means has to be provided, with which a new atomic cut formula can be introduced. These atomic cut formulae act as proxies between the complex formula and its context. Therefore, we will introduce the new rule (Prx), called *proxy cut* or *atomic cut*, which ist the restriction of (Cut) to atomic cut formulae:

$$\frac{\Gamma_1 \rightarrow \Delta_1, p \qquad p, \Gamma_2 \rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2} \text{(Prx)}$$

The propositional variable $p$ has the restriction that it must not occur as cut formula in any other application of (Prx) within the same derivation or as subformula in the end sequent.

In fact, (Prx) is the most general formulation of the rule. From what we have indicated of the explosion calculus so far, the following two variants would be sufficient:

$$\frac{A \rightarrow p \qquad p, \Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta} \text{(Prx}_1\text{)} \qquad\qquad \frac{\Gamma \rightarrow \Delta, p \qquad p \rightarrow A}{\Gamma \rightarrow \Delta, A} \text{(Prx}_2\text{)}$$

The rule ($\text{Prx}_1$) introduces a proxy for a formula occurring in the leftmost position in the antecedent, and ($\text{Prx}_2$) does the same for the rightmost formula of the succedent. Hence, the left premiss of ($\text{Prx}_1$) can be the conclusion of any local logical rule ($\star$A), whereas the right premiss of ($\text{Prx}_2$) can serve as conclusion of any local logical rule ($\star$S), where $\star \in \{\&, \vee, \supset, \neg\}$. In fact, (Prx) will be used most of the time in one of these two manners. However, another special case of (Prx) will be required:

$$\frac{\Gamma \rightarrow p \qquad p \rightarrow \Delta}{\Gamma \rightarrow \Delta} \text{(Prx}_3\text{)}$$

The necessity of this final variant will become apparent shortly. As three specific instances of (Prx) would be necessary for our purposes, we will be satisfied with the single rule (Prx) and not explicitly use these variants. However, it will be useful to keep in mind that the rule will never be used in another manner than given by one of these variants.

# Part II

# The Explosion Calculus

# Chapter 5

# The Explosion Procedure

The aim of the explosion calculus is to reduce a sequent to a collection of a particular kind of purely structural sequents. In contrast to leaves of RK-derivations, from which it is near impossible to understand the end sequent, from which they have been obtained by a proof search, the leaves of an explosion derivation will make such understanding possible.[1]

The means by which this is to be achieved is the introduction of extra atomic cut formulae of a particular sort, called *proxy variables*. In some sense, they can be related to Hertz' ideal elements, which also had the purpose of structuring systems of sentences. Each formula occurrence of the sequent will be inextricably connected to a unique proxy variable, i.e. an atomic cut formula. Moreover, all the propositional variables occurring in a sequent will be replaced by *proper occurrence variables* in order to distinguish multiple occurrences of the same variable. In view of the above comparison, occurrence variables have to be likeneded to Hertz' real elements. Both kinds of variables are called *occurrence variables*. Due to the introduction of occurrence variables, an explosion derivation will not result in any axiom leaves whatsoever. Hence, the task of determining whether the structural elements extracted from a given sequent correspond to a proof of the sequent or not, is not performed by the explosion calculus.[2]

Instead, the explosion calculus mechanically decomposes any given sequent into its most elementary structural sequents. It is exactly this decomposition, by which the problem of duplicating the formula context of a main formula is avoided. Hence, each occurrence of a complex formula of a sequent, including all the complex subformulae, will be subject to exactly two rule applications in the explosion procedure.

[1]In chapter 7, it will be argued that these collections of leaves constitute the meaning of sequents.

[2]A decision procedure for this purpose will be presented in chapter 9.

The explosion procedure consists of three steps, each of which will be described at length. Briefly, the three steps are:

1) initialisation of the sequent,

2) detachment of a complex formula,

3) application of local logical rule.

Throughout this desciption, we will employ the bottom-up perspective, as we will decompose an arbtrary sequent $\Gamma \rightarrow \Delta$ into its elementary structural sequents. The initialisation step is performed exactly once for a given sequent. It separates the antecedent from the succedent by the introduction of a proxy variable, which is called the *watershed proxy*. After the initialisation, alternations of detachments by (Prx) and applications of an appropriate local logical rule follow. The exact number of alterations of 2) and 3) is determined by the number of logical connectives, which occur in $\Gamma$ and $\Delta$, which is the number of occurrences of complex subformulae in $\Gamma$ and $\Delta$. The derivation obtained by this procedure for a given sequent is called the *explosion derivation* of that sequent.

## 5.1 Elementary Structural Sequents

Up to this point, we have considered formulae, which were generated inductively on the set of propositional variables $\mathcal{A}$. Formulae over $\mathcal{A}$ will not be suitable for the purposes of the following investigation, however. In order to illustrate this, consider some sequent $\Gamma \rightarrow \Delta_1, a \vee \neg a, \Delta_2$, where $a$ also appears in some or all of the sequences $\Gamma, \Delta_1, \Delta_2$. The formula $a \vee \neg a$ contains two occurrences of the propositional variable $a$. In a proof search, starting from this sequent, we might eventually arrive at a sequent $\Gamma_1', a, \Gamma_2' \rightarrow \Delta_1', a, \Delta_2'$. In this latter sequent, it is possible, but not certain, that the $a$ occurring in the antecedent was obtained from the right subformula $\neg a$ of the disjunction and the $a$ accuring in the succedent is the $a$ obtained from the left subformula of the disjunction. While it is possible to establish whether this is the case or not by following the branch of derivation that connects the two sequents, it is not obvious by simply looking at the two sequents. In addition to this, logistic calculi distinguish different occurrences of the same propositional variable only by their position within the antecedent or succedent, which is variable even under the exchange rules.

As we separate checking for provability from the extraction of elementary structural sequents, there is no need to use the same propositional variable to represent different occurrences thereof. Quite to the contrary, since our first interest lies in the development of the pure structure inherent in a sequent, the identification of the atomic formulae would lead to problems.[3] The solution to this problem is to employ *occurrence variables* instead of propositional variables. Instead of using some propositional variable, say $a$, repeatedly in a sequent, each one of the $m$ occurrences of $a$ is represented by a unique occurrence variable, say $a_1, a_2, \ldots, a_m$. The first sequent in the example above would then be represented by $\Gamma \rightarrow \Delta_1, a_k \vee \neg a_{k+1}, \Delta_2$ for suitable $k$. If the proof search of this sequent leads to $\Gamma'_1, a_{k+1}, \Gamma'_2 \rightarrow \Delta'_1, a_k, \Delta'_2$, we would be able to immediately relate these occurrence variables to those in the former. The obvious property of occurrence variables that different occurrences thereof can never be identical results in the fact that a sequent made up from occurrence variables cannot have a proof in RK.[4]

We introduce an enumerable set of *proper occurrence variables* $\mathcal{O}$ and explain, how to replace propositional variables by occurrence variables and how to reverse the process.

**Definition 5.1** *A sequent which contains only propositional variables, i.e. no occurrence variables, is called a* standard *sequent. For every standard sequent $\mathcal{S}$, the* occurrence instance $\widetilde{\mathcal{S}}$ *of $\mathcal{S}$ is the sequent that is obtained by replacing each occurrence of a propositional variable in $\mathcal{S}$ by a new occurrence variable. A* restoration function *is a function $\rho : \mathcal{O} \rightarrow \mathcal{A}$. For a sequent $\widetilde{\mathcal{T}}$ containing only occurrence variables and a restoration function $\rho$, the* restoration *of $\widetilde{\mathcal{T}}$ under $\rho$, written $\widetilde{\mathcal{T}}\rho$, is the sequent that is obtained by replacing every occurrence variable $a$ in $\widetilde{\mathcal{T}}$ by the propositional variable $\rho(a)$.*

The restoration function complements the generation of an occurrence instance in the sense that for all sequents $\mathcal{S}$, it is $\widetilde{\mathcal{S}}\rho = \mathcal{S}$.[5] The restoration function retains the information about identity of atomic content. Technically, we obtain an occurrence instance of a sequent $\mathcal{S}$ and the correspondence restoration function as follows: we begin with the everywhere undefinded function as restoration function and scan $\mathcal{S}$ from left to right, replacing each propositional variable $a$ encountered by an occurrence variable $x$ that is not

[3]It will be outlined in detail in the following where those problem would occur. Let it suffice to state at this point that the result of the explosion procedure is a set of elementary structural sequents, which would not adequately represent certain crucial multiplicities.

[4]However, if we were to consider all the leaves of such a derivation in turn and restore all propositional variables therein, that is, replace each occurrence variable by the propositional variable that originally occured there, we could quickly establish whether the leaves are conclusions of (ax*), and, hence, wether the derivation could be turned into a proof.

[5]Any given $\rho$ is assumed to be inductively extended over formulae and sequents.

yet in the image of $\rho$, updating $\rho$ by the assignment $\{x \mapsto a\}$ and then proceeding to the next symbol in $\mathcal{S}$. In view of this procedure, an obvious choice for $\mathcal{O}$ is the cartesian product $\mathcal{A} \times \mathbb{N}$ and represent occurrence variables by $a_1, a_2, \ldots, a'_1, a'_2, \ldots, b_1, b_2, \ldots$, when $a, a', \ldots, b, \ldots$ are used as symbols for propositional variables. Note that we will, especially in later chapters, occasionally refer to explosion derivations of some arbitrary RK-sequent, especially when comparing a derivation in RK to the explosion procedure. In those cases, we take it to be understood that the explosion procedure is *always* applied to an occurrence instance of the sequent in question, not the sequent itself.

For the explosion procedure we need additional variables that will be introduced in order to detach complex formulae from their contexts. It is useful to emphasize the difference between occurrences of propositional variables themselves, which are the atomic constituents of formulae, and occurrences of complex formulae. These variables are called *proxy variables*, because the explosion procedure removes every complex formula from its context by introducing such a proxy variable, and afterwards the formula can only be related to its former context via this proxy variable.[6] In contrast to the proper occurrence variables, proxy variables will always occur as atomic formulae only. Since we do not want to run out of proxy variables during the explosion procedure, we require an enumerable set of them, which we call $\mathcal{P}$. Letters $p, q, r, s, t, \ldots$ will be used for proxy variables, and we attach natural numbers as indices in order to generate additional variables whenever required, e.g. $p_1, q_3$ etc.

The set of *occurrence variables* $\mathcal{V}$ is the disjoint union of the set of propoer occurrence variables and the set of proxy variables, i.e. $\mathcal{V} = \mathcal{O} \uplus \mathcal{P}$. We will use letters $x, y, \ldots$ to denote arbitrary occurrence variables, attaching natural number indices to the letters whenever required. A restoration function $\rho$ in the sense of definition 5.1 is extended over $\mathcal{V}$ in the trivial manner, i.e. $\rho(p)$ is undefined for every $p \in \mathcal{P}$.

Whenever a sequence of formulae contains no complex formulae, i.e. it consists of only occurrence variables, proper or proxy, lower case greek letters will be used to emphasize this fact; that is, we shall use $\gamma$ instead of $\Gamma$, $\delta$ instead of $\Delta$ etc.

With the new sets of variables established, we can proceed to characterise the specific kind of sequents, which we want to obtain as result of the explosion procedure.

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[6] As we have argued, proxy variables can be considered as occurrences of complex formulae. This is the reason for the name *proper* occurrence variables for those variables replacing propositional variables.

**Definition 5.2** *An* elementary structural sequent (ESS) *is a sequent* $\gamma \rightarrow \delta$*, for which the following holds:*

*1) All the formulae of* $\gamma$ *and* $\delta$ *are occurrence variables.*

*2) At least one of the formulae in* $\gamma$ *and* $\delta$ *is a proxy variable.*

*3) If* $\mathbf{len}(\gamma) \cdot \mathbf{len}(\delta) = 0$*, then* $1 \leq \mathbf{len}(\gamma) + \mathbf{len}(\delta) \leq 2$.

*4) If* $\mathbf{len}(\gamma) \cdot \mathbf{len}(\delta) \neq 0$*, then* $\mathbf{len}(\gamma) = 1$ *or* $\mathbf{len}(\delta) = 1$.

Condition 1) states that an ESS must not contain any complex formulae whatsoever. All formula occurrences in $\gamma \rightarrow \delta$ are atomic, as it is already indicated by the use of lower case greek letters. Condition 2) might be a surprising restriction, because it clearly prohibits even occurrence instances of the simplest axiom sequents, such as $a_1 \rightarrow a_2$, as elementary structural sequents. However, we will see shortly that as a result of the explosion procedure, every ESS must contain at least one proxy variable. Condition 3) demands that ESSs, which have an empty antecedent or an empty succedent, must have either one or two variables in their other part. Finally, condition 4) forbids sequents that have more than one variable on both sides. If antecedent and succedent both consist of one or more formulae, at least one of them must consist of exactly one variable. This condition disallows sequents that have more than one formula in both antecedent and succedent. Here are some examples for elementary structural sequents:

$$\begin{aligned} a_1 &\rightarrow p \\ t &\rightarrow q_1, q_2, q_3 \\ a_1, r &\rightarrow \\ s &\rightarrow a_3, b_2, c_1 \\ &\rightarrow p, q \\ a_2, p, t &\rightarrow b_4 \\ t &\rightarrow \end{aligned}$$

On the other hand, none of the following sequents is an EES:

| | |
|---|---|
| $a_2 \,\&\, b_3 \rightarrow p$ | violates condition 1) |
| $a_3, b_2 \rightarrow c_1$ | violates condition 2) |
| $a_1 \rightarrow$ | violates condition 2) |
| $\rightarrow q_1, q_2, q_3$ | violates condition 3) |
| $\rightarrow$ | violates condition 3) |
| $p, q \rightarrow r, s$ | violates condition 4) |

We will see shortly, how iterateded alternating applications of the atomic cut rule and local logical rules will decompose every sequent into a set of elementary structural sequents.

## 5.2 Initialisation of the Sequent

For an arbitrary sequent $\Gamma \rightarrow \Delta$, both $\Gamma$ and $\Delta$ can contain any number of formulae. In order to emphasise the separation of those formulae in antecedent and succedent, a proxy variable $w$, called the *watershed proxy* is introduced by application of (Prx). This proxy variable explicitly delimits the antecedent from the succedent.

$$\frac{\Gamma \rightarrow w \qquad w \rightarrow \Delta}{\Gamma \rightarrow \Delta}\ (\text{Prx})$$

Intuitively, instead of being able to move attention back and forth between antecedent and succedent within a single sequent, the introduction of a watershed proxy variable demands that attention is instead shifted from one sequent to another sequent. For instance, the antecedent $\Gamma$ of the original sequent is found in the leftmost sequent of the two premises of (Prx). Moving the attention to $\Delta$ now involves the intermediate step of discovering $w$ instead of $\Delta$ in the succedent of that first sequent. What has to be done next is to investigate all the other available sequents in order to find the one that contains the proxy $w$ in the antecedent. This task is obviously trivial at this stage, since there is only one other sequent, and $\Delta$ is its succedent.

What has been achieved by this separation step is that in any derivation, which is constructed on top of this initialisation step, no sequent will contain formulae or subformulae that originate from occurrences in both $\Gamma$ *and* $\Delta$.[7] Any explosion derivation that can be built on top of the left premiss can only consist of sequents containing formulae and their subformulae that already occur in $\Gamma$ and, in addition to that, proxy variables (including $w$). Consider the occurrence instance $a_1 \rightarrow a_2$ of the RK-axiom $a \rightarrow a$. Even this trivial sequent has to be initialised in the explosion calculus.

$$\frac{a_1 \rightarrow w \qquad w \rightarrow a_2}{a_1 \rightarrow a_2}\ (\text{Prx})$$

[7] Assuming that neither $\Gamma$ nor $\Delta$ are empty, it should be obvious that in general a RK-proof of $\Gamma \rightarrow \Delta$ could not be obtained after the initialisation step. This already establishes that the decision procedure for provability is not entwined with the justificational character of the explosion calculus as is the case with the conventional lofistic calculi.

Since this might seem rather perplexing at this point, we shall give a very brief informal description of the decision procedure that is to be be applied to the set of EESs obtained from some given sequent by the explosion calculus. All paths connecting EESs via proxies are considered in turn. All the proper occurrence variables occurring in the antecedents of ESSs connected by such a path are collected in the set of antecedent variables, and all the proper occurrence variables occurring in the succedents of ESSs connected are collected in the set of succedent variables. Then the restoration function is applied to these two sets. If the intersection of the result of the restoration is not empty, such a path is closed. If all possible paths are closed, the given sequent is provable in RK. In the example above there is only one path connecting the ESS $a_1 \rightarrow w$ to the ESS $w \rightarrow a_2$. Accumulating all proper occurrence variables occurring in antecedents along this path yields the set $\{a_1\}$, and doing the same for the succedents gives us $\{a_2\}$. The sequent was obtained from an instance of $a \rightarrow a$, so the restoration yields the sets $\{a\}$ and $\{a\}$. Since the intersection of these sets is non-empty, $a \rightarrow a$ is (in this case trivially) RK-provable. Of course, the execution of the decision procedure is trivial in the case of an elementary instance of the axiom.

From this intuitive description of the decision procedure, it is already apparent that the intitialisation step serves the important purpose of introducing a single proxy variable, which will occur in all possible proxy paths. In other words, all the paths connecting ESSs obtained by the explosion procedure will pass through the watershed proxy $w$. In this sense, $w$ can be thought of as the unique representation of the entire set of ESSs that will be obtained by exploding the original sequent $\Gamma \rightarrow \Delta$, and hence, as the *occurrence* of the entire sequent.

## 5.3 Detachment of a Complex Formula

Having made the separation of antecedent and succedent explicit by the introduction of the watershed proxy, the next step addresses the task of transforming sequents into a form, to which the local logical rules can be applied. Since local logical rules are applied to sequents that contain exactly one complex formula in either the antecedent or the succedent and one proxy variable in the complementary part of the sequent, the detachment step is intended to produce sequents fulfilling this specification. The detachment step can only be employed, if there occurs at least one leaf in the explosion derivation, which contains at least one complex formula in its antecedent or succedent. If this is no longer the case, all the leaves of the derivation are already elementary structural sequents, and the explosion procedure terminates. For example, this is the case right after the intialisation step of $a_1 \rightarrow a_2$, as we saw above, since neither one of the premises $a_1 \rightarrow w$ and $w \rightarrow a_2$ contains a complex formula.

Assuming there is a suitable complex formula in one of the sequents occurring as leaves of the current explosion derivation, detachment begins with a number of applications of (XA) or (XS), depending on whether the formula occurs in the antecedent or the succedent, to move it into cut position. This is followed by an application of (Prx) to introduce a new proxy variable, which separates the complex formula from its context. Finally, (XA) or (XS) is applied to the context premiss the same number of times in order to move the proxy variable into that position within the sequent that was previously occupied by the complex formula. This last step is not necessary. It is included, however, to have, as result of detachment, an unmistakably recognisable replacement of the complex formula in the sequent by its proxy.

The procedure is exemplified for an occurrence of $A \vee B$ in the antecedent of a sequent, say $\Gamma_1, A \vee B, \Gamma_2 \rightarrow w$, where $w$ is the watershed proxy. The complex formula $A \vee B$ is to be isolated from its context for a later application of the appropriate local rule.

$$\dfrac{\dfrac{A \vee B \rightarrow p \qquad \dfrac{\Gamma_1, p, \Gamma_2 \rightarrow w}{p, \Gamma_1, \Gamma_2 \rightarrow w}}{A \vee B, \Gamma_1, \Gamma_2 \rightarrow w}\;(\mathrm{Prx})}{\Gamma_1, A \vee B, \Gamma_2 \rightarrow w}$$

The double lines correspond to $\mathbf{len}(\Gamma_1)$ applications of the structural rule (XA). First, $A \vee B$ is moved into cut position, where it is detached from the context by means of a proxy $p$, which is then moved back into the previous position of $A \vee B$. Since the instances of (XA) are not particularly illuminating, we will from now on use the following abbreviation for derivations, which are similar to the one above:

$$\dfrac{A \vee B \rightarrow p \qquad \Gamma_1, p, \Gamma_2 \rightarrow w}{\Gamma_1, A \vee B, \Gamma_2 \rightarrow w}\;(\mathrm{Prx})$$

The result of the detachment step is that the sequent, to which it is applied, is transformed into two sequents, which occur as the two premises of that short derivation. One premiss containing a proxy variable instead of one of its original complex formulae is obtained as one sequent, the *context premiss*, in which the proxy variable marks the former occurrence of the complex formula. In this example this is the right premiss of (Prx). The second premiss contains only said formula in either the antecedent or the succedent, depending on its occurrence in the original sequent, and the proxy variable in the opposite position of the sequent. In this premiss, the proxy variable represents a specific occurrence of the complex formula, and hence this premiss is called the *occurrence premiss*. It is this premiss, to which a local logical rule will be applied in the following step of the explosion procedure. In the example, this is the left premiss of (Prx). The proxy variable both uniqely identifies some location within a specific context and an occurrence of a formula. Thereby the relative position of the detached formula is retained via this proxy.

As a second example, consider the sequent $q \to \Delta_1, \neg A, \Delta_2$, where $\neg A$ is to be detached. What follows is the full derivation that performs the detachment operation.

$$\dfrac{\dfrac{\dfrac{q \to \Delta_1, r, \Delta_2}{q \to \Delta_1, \Delta_2, r} \qquad r \to \neg A}{q \to \Delta_1, \Delta_2, \neg A}\ (\text{Prx})}{q \to \Delta_1, \neg A, \Delta_2}$$

Since $\neg A$ occurs in the succedent, the structural rule (XS) has to be employed $\mathbf{len}(\Delta_2)$ times to move the formula into cut position, and again as many times to move the introduced proxy variable $r$ back to the former position of $\neg A$ within the succedent after the proxy cut. The procedure, by which this derivation is obtained, is symmetric to the previous case, in which the complex formula was located in the antecedent. Instead of structural rule (XA), the rule (XS) is required; moreover, the order of the leaves of the derivation is exchanged, i.e. the left premiss is the context premiss and the right premiss of (Prx) contains the occurrence of the complex formula. In the simplified notation, the derivation can be given as follows:

$$\dfrac{q \to \Delta_1, r, \Delta_2 \qquad r \to \neg A}{q \to \Delta_1, \neg A, \Delta_2}\ (\text{Prx})$$

Although a single detachment step is very simple in itself and has been sufficently introduced by the two examples, it is still necessary to have a formal description thereof. We assume that some explosion derivation is given, and that a complex formula occurs in the antecedent of some leaf or in the succedent of some leaf. For simplicity, the procedure is described for the detachment of a complex formula occurring in the antecedent; the instructions for a formula occurring in the succedent are symmetric. If there is no leaf anywhere in the explosion derivation, which has a complex formula in either the antecedent or the succedent, the explosion procedure terminates.[8]

1) Decide on a leaf of the given explosion derivation containing at least one occurrence of a complex formula in its antecedent.

2) Pick one complex formula and mark its location within the sequent. For instance, for a complex formula $C$, divide a sequent into $\Gamma_1, C, \Gamma_2 \rightarrow \Delta$.

3) Apply (XA) $\mathbf{len}(\Gamma_1)$ times, each time moving $C$ one place to the left.

4) Apply (Prx), introducing a new proxy variable $r$.

5) Apply (XA) $\mathbf{len}(\Gamma_1)$ times, beginning with the right premiss of (Prx), each time moving $r$ one place to the right.

This procedure is expressed by the following derivation fragment:

$$\frac{C \rightarrow r \qquad \Gamma_1, r, \Gamma_2 \rightarrow \Delta}{\Gamma_1, C, \Gamma_2 \rightarrow \Delta}\ (\text{Prx})$$

The left premiss is the *occurrence premiss*, which relates the formula $C$ to a unique proxy $r$. Therefore $r$ can be understood as representing the specific occurrence formula $C$. The detachment step therefore corresponds to the preparation of the formula $C$ for the application of a local logical rule by detaching it from its context by means of the proxy. Similarly, all complex formulae occurring in $\Gamma_1$ and $\Gamma_2$ and all their complex subformulae will be related to a unique proxy variable by some later detachment step of the explosion procedure. This is especially important in view of the application of a local logical rule that immediately follows the detachment of a complex formula; for even though the connective will no longer be present in the premiss or premises of that rule, the proxy variable can never disappear. For instance, the sequent $C \rightarrow r$ has the required form for the local logical rule applicable to $C$, and although only the immediate subformulae of $C$ will reappear in its premiss or premises, $r$ will continue to represent the complex formula $C$ therein. The right premiss consists of the *context premiss*, in which the location of the formula $C$ is retained by the same proxy $r$. The context sequent $\Gamma_1, r, \Gamma_2 \rightarrow \Delta$ has the following obvious properties:

[8] An important result, which will be established later on is the following: The choice, which complex formula is detached at any given point during the explosion procedure, does not have any effect on the ESSs, which are yielded as the leaves of the entire explosion derivation when the procedure has terminated. For the time being, we choose arbitrarily at any stage the occurrence of a complex formula that is to be detached next and the corresponding leaf of the derivation.

1) It contains exactly one complex formula less than the sequent occurring in the conclusion of the cut.

2) If $\Gamma_1, \Gamma_2$ and $\Delta$ consist entirely of occurrence variables only, then the leaf $\Gamma_1, r, \Gamma_2 \rightarrow \Delta$ is an elementary structural sequent and requires no further treatment; otherwise, for some complex $D$ occurring in either $\Gamma_1$ or $\Gamma_2$, this leaf must be revisited at a future point for another detachment step.

It is important to stress the importance of adhering to the procedure as it has been presented this far. An example will serve to illustrate the possible problems arising from negligence. For sequent $A \,\&\, B \rightarrow A$, the initialisation step yields:

$$\frac{A \,\&\, B \rightarrow w \qquad w \rightarrow A}{A \,\&\, B \rightarrow A}\ (\text{Prx})$$

Without knowing more about the syntactic structure of $A$ and $B$, nothing can be done with the right premiss. However, since $A \,\&\, B$ is a complex formula, we can proceed to develop the left premiss by detachment as follows:

$$\dfrac{\dfrac{A \,\&\, B \rightarrow p \qquad p \rightarrow w}{A \,\&\, B \rightarrow w}\ (\text{Prx}) \qquad w \rightarrow A}{A \,\&\, B \rightarrow A}\ (\text{Prx})$$

This step might appear redundant, since $A \,\&\, B \rightarrow w$ already has the form that allows the application of the local logical rule ($\&\text{A}_l$). Furthermore, the sequent $p \rightarrow w$ might appear redundant at this point of the presentation. While it is undoubtedly an elementary structural sequent, its relevance is perhaps not obvious. There are very specific intended meanings for $p$ and $w$, however, and it is because of these meanings that the detachment step *must not* be skipped. The watershed proxy $w$ is introduced in the initialisation step and is closely related to the sequent as a whole. The proxy $p$, on the other hand, is only related to $A \,\&\, B$, marking both its occurrence and its location within a certain context. Here, we are simply facing the special case that the antecedent of the end sequent consists of a single formula. The elementary structural sequent $p \rightarrow w$ therefore relates this singleton antecedent, consisting of the complex formula whose occurence is $p$, to the sequent as a whole, represented by $w$. Thus, omitting the detachment step would fuse the proxy related to the sequent as a whole to the occurrence of $A \,\&\, B$. Such a conceptual overlap must be avoided. A related problem is that of fusing the occurrence of a formula and its subformulas. This will be explained in the next section. It is important to remember that the detachment step must be performed, even if the sequent under consideration already has a shape that would allow to skip it and proceed immediately to the application of a local logical rule.

## 5.4 Application of the Local Logical Rule

In this step, the appropriate local logical rule is applied to the occurrence leaf that was generated in the preceding detachment step. In view of the application of a local logical rule, the important result of the detachment step was to obtain either a sequent $C \to p$ or a sequent $p \to C$, where $C$ is some complex formula. The application step then consists of the application of the local logical rule appropriate for $C$, given as follows:

$$\begin{array}{rcl}A \vee B \to p & : & (\vee \mathrm{A}_l) \\ A \,\&\, B \to p & : & (\&\mathrm{A}_l) \\ \neg A \to p & : & (\neg \mathrm{A}_l) \\ A \supset B \to p & : & (\supset \mathrm{A}_l)\end{array} \qquad \begin{array}{lcl}p \to A \vee B & : & (\vee \mathrm{S}_l) \\ p \to A \,\&\, B & : & (\&\mathrm{S}_l) \\ p \to \neg A & : & (\neg \mathrm{S}_l) \\ p \to A \supset B & : & (\supset \mathrm{S}_l)\end{array}$$

The actual derivations obtained in this step are exactly instances of the local logical rules. Even though this step is quite straightforward, let us consider the examples from the previous sections. Recall the first example we gave for detachment:

$$\frac{A \vee B \to p \qquad \Gamma_1, p, \Gamma_2 \to w}{\Gamma_1, A \vee B, \Gamma_2 \to w}\ (\mathrm{Prx})$$

After the detachment step, the left premiss $A \vee B \to p$ has the form required by the local logical rule $(\vee \mathrm{A}_l)$.

$$\frac{A \to p \qquad B \to p}{A \vee B \to p}\ (\vee \mathrm{A}_l)$$

This yields the following composite derivation of both the decomposition step, followed by the application of the local logical rule:

$$\frac{\dfrac{A \to p \qquad B \to p}{A \vee B \to p}\ (\vee \mathrm{A}_l) \qquad \Gamma_1, p, \Gamma_2 \to w}{\Gamma_1, A \vee B, \Gamma_2 \to w}\ (\mathrm{Prx})$$

This is the derivation fragment, in which all the steps directly relating to the proxy $p$ are accomodated. It is introduced by (Prx), and immediately afterwards it enables the application of the local logical rule to its occurrence premiss. Its role in the application of the local logical rule, $(\vee \mathrm{A}_l)$ in this example, is that of providing the only permitted context for such an application, namely that of a single proxy variable in that part of the sequent opposite to the one in which the complex locial formula occurs. In the premiss or premises of the local locical rule, it is the proxy variable that retains the connection of the side formulae to the former context of the main formula. This context is given in the context premiss of the preceding proxy cut.

We can express the detachment step, followed by the application step, by a meta rule, which combines all the necessary rule applications. Such a meta rule has a single context premiss, resulting from the atomic cut, and as many logical premises as the corresponding local logical rule. The meta rule corresponding to the example above is:

$$\frac{A \to p \qquad B \to p \qquad \Gamma_1, p, \Gamma_2 \to \Delta}{\Gamma_1, A \vee B, \Gamma_2 \to \Delta}\ (\vee \mathrm{A}_m)$$

The comparison of the meta rule $(\vee \mathrm{A}_m)$ to the traditional logical rule $(\vee \mathrm{A})$[9] demonstrates how duplication of the context is avoided by the introduction of the proxy variable:

$$\frac{\Gamma_1, A, \Gamma_2 \to \Delta \qquad \Gamma_1, B, \Gamma_2 \to \Delta}{\Gamma_1, A \vee B, \Gamma_2 \to \Delta}\ (\vee \mathrm{A}) \quad \text{(double line)}$$

In the premises of $(\vee \mathrm{A})$, there are two occurrences of each of the sequences of formulae $\Gamma_1, \Gamma_2$ and $\Delta$, whereas in $(\vee \mathrm{A}_m)$ each of those occurs exactly once.

Recall the second example of the previous section: $q \to \Delta_1, \neg A, \Delta_2$. The detachment of $\neg A$ from the context yieldeded:

$$\frac{q \to \Delta_1, r, \Delta_2 \qquad r \to \neg A}{q \to \Delta_1, \neg A, \Delta_2}\ (\mathrm{Prx}) \quad \text{(double line)}$$

The appropriate local logical rule for dealing with the right premiss is $(\neg \mathrm{S}_l)$.

$$\frac{A, r \to}{r \to \neg A}\ (\neg \mathrm{S}_l)$$

[9]As before, the double line indicate the required $2 \cdot \mathbf{len}(\Gamma_1)$ applications of (XA).

This yields the following derivation as the result of the detachment step, followed by the application of the local logical rule:

$$\dfrac{q \rightarrow \Delta_1, r, \Delta_2 \qquad \dfrac{A, r \rightarrow}{r \rightarrow \neg A}\,(\neg\mathrm{S}_l)}{q \rightarrow \Delta_1, \neg A, \Delta_2}\,(\mathrm{Prx})$$

As in the previous example, we can express this derivation by a meta rule of the following form:

$$\dfrac{\Gamma \rightarrow \Delta_1, r, \Delta_2 \qquad A, r \rightarrow}{\Gamma \rightarrow \Delta_1, \neg A, \Delta_2}\,(\neg\mathrm{S}_m)$$

The traditional rule $(\neg\mathrm{S})$, including $2 \cdot \mathbf{len}(\Delta_2)$ applications of the exchange rule (XS), has the following form:

$$\dfrac{A, \Gamma \rightarrow \Delta_1, \Delta_2}{\Gamma \rightarrow \Delta_1, \neg A, \Delta_2}\,(\neg\mathrm{S})$$

In the rule $(\neg\mathrm{S})$ there is no duplication of the context. However, in contrast to the single premiss of this rule, the meta rule $(\neg\mathrm{S}_m)$ expresses the context by a separate context premiss and captures the purely logical part by a second, the local logical premiss.

The more extensive example from the previous section was this short derivation:

$$\dfrac{\dfrac{A \,\&\, B \rightarrow p \qquad p \rightarrow w}{A \,\&\, B \rightarrow w}\,(\mathrm{Prx}) \qquad w \rightarrow A}{A \,\&\, B \rightarrow A}\,(\mathrm{Prx})$$

The left premiss of the upmost (Prx) purports an application of $(\&\mathrm{A}_l)$, which results in the following derivation:

$$\dfrac{\dfrac{\dfrac{A, B \rightarrow p}{A \,\&\, B \rightarrow p}\,(\&\mathrm{A}_l) \qquad p \rightarrow w}{A \,\&\, B \rightarrow w}\,(\mathrm{Prx}) \qquad w \rightarrow A}{A \,\&\, B \rightarrow A}\,(\mathrm{Prx})$$

Using the meta rule $(\&\mathrm{A}_m)$ corresponding to this case, the derivation can be expressed as follows:

$$\dfrac{\dfrac{A, B \rightarrow p \qquad p \rightarrow w}{A \,\&\, B \rightarrow w}\,(\&\mathrm{A}_m) \qquad w \rightarrow A}{A \,\&\, B \rightarrow A}\,(\mathrm{Prx})$$

The pivotal role of the proxy variable can be fully appreciated now that both working steps of the explosion calculus, detachment and application of the local logical rule, have been presented. Its role in the detachment step is to retain the connection of the detached formula to its context. In the application of the local logical rule, this connection is surrendered to the side formulae or side formula by means of the proxy variable. This necessitates that the two working steps of the explosion calculus are always employed conjointly in the sense that the proxy variable that is introduced in the detachment step is the one that relates to the complex formula which is resolved by the local logical rule.

## Logical Meta Rules

A local logical rule requires a preceding introduction of a proxy variable by a proxy cut. Since the premiss of such a cut, which contains the detached formula and this proxy variable, can only be treated by the local logical rule corresponding to the formula, it is convenient and advisable to bundle these two derivation steps into meta rules.

All the possible combinations of (Prx) and one of the local logical rules are represented by the meta rules in figure 5.1. To be exact, each meta rule represents a derivation consisting of (from the bottom up):

1) a number of instances of the exchange rule required to move the formula under consideration into cut position;

2) an application of (Prx), by which the proxy variable is introduced;

3) the same number of exchange rules applied to that premiss of (Prx), which retains the context; thereby the proxy is returned to the starting position of the now detached formula;

4) the appropriate local logical rule, applied to the other premiss of (Prx).

Thus, each one of the meta rules has exactly one premiss more than the corresponding local logical rule or RK rule. This additional premiss retains the context, in which the complex logical formula originally occured. We will maintain the terminology that was introduced previously and call this premiss the context premiss. Likewise, any premiss of a meta rule, which is not the context premiss, is a *logical premiss*.

From now on, we will present the explosion procedure using the meta rules. This emphasizes the fact that a detachment step and an application of a local logical rule always go hand in hand.

$$\frac{A, B \to p \qquad \Gamma_1, p, \Gamma_2 \to \Delta}{\Gamma_1, A \,\&\, B, \Gamma_2 \to \Delta}\ (\&\mathrm{A}_m)$$

$$\frac{\Gamma \to \Delta_1, p, \Delta_2 \qquad p \to A \qquad p \to B}{\Gamma \to \Delta_1, A \,\&\, B, \Delta_2}\ (\&\mathrm{S}_m)$$

$$\frac{A \to p \qquad B \to p \qquad \Gamma_1, p, \Gamma_2 \to \Delta}{\Gamma_1, A \vee B, \Gamma_2 \to \Delta}\ (\vee\mathrm{A}_m)$$

$$\frac{\Gamma \to \Delta_1, p, \Delta_2 \qquad p \to A, B}{\Gamma \to \Delta_1, A \vee B, \Delta_2}\ (\vee\mathrm{S}_m)$$

$$\frac{\to p, A \qquad B \to p \qquad \Gamma_1, p, \Gamma_2 \to \Delta}{\Gamma_1, A \supset B, \Gamma_2 \to \Delta}\ (\supset\mathrm{A}_m)$$

$$\frac{\Gamma \to \Delta_1, p, \Delta_2 \qquad A, p \to B}{\Gamma \to \Delta_1, A \supset B, \Delta_2}\ (\supset\mathrm{S}_m)$$

$$\frac{\to p, A \qquad \Gamma_1, p, \Gamma_2 \to \Delta}{\Gamma_1, \neg A, \Gamma_2 \to \Delta}\ (\neg\mathrm{A}_m)$$

$$\frac{\Gamma \to \Delta_1, p, \Delta_2 \qquad A, p \to}{\Gamma \to \Delta_1, \neg A, \Delta_2}\ (\neg\mathrm{S}_m)$$

Figure 5.1: Logical meta rules

# Chapter 6

# Properties of the Explosion Procedure

The purpose of the explosion procedure is to obtain as leafs of the derivation elementary structural sequents only. The collection of all the ESSs obtained by this procedure can be considered to be the *structural skeleton* of the sequent. While all the logical connectives have been removed locally, the meaning of each one is retained by certain collections of structural sequents. Unlike in RK, this meaning is not only represented at the exact position in the derivation, at which the corresponding logical rule is applied. In addition to that, and more importantly, the meaning is also retained in certain collections of ESSs that are obtained by the procedure, even if they are considered independently of the derivation itself.

To gain an intuitive understanding of this idea, consider this continuation of the first example, written down explicitly without the meta rules, and assume that both $A$ and $B$ are complex formulae:

$$
\dfrac{\dfrac{\dfrac{\begin{array}{l}\Lambda_A\\ A \to q \quad q \to p\end{array}}{A \to p}\,(\mathrm{Prx}) \quad \dfrac{\begin{array}{l}\Lambda_B\\ B \to r \quad r \to p\end{array}}{B \to p}\,(\mathrm{Prx})}{A \lor B \to p}\,(\lor\mathrm{A}_m) \qquad \Gamma_1, p, \Gamma_2 \to w}{\Gamma_1, A \lor B, \Gamma_2 \to w}\,(\mathrm{Prx})
$$

On top of the left premiss of $(\lor\mathrm{A}_m)$, $A \to p$, we execute another detachment step, followed by the application of the local logical rule that is appropriate for $A$. That second step is merely indicated by $\Lambda_A$. Furthermore, we execute yet another detachment step on the right premiss $B \to p$, which, again, has to be followed by the local logical rule corresponding to $B$; this is abbreviated by $\Lambda_B$. Note that the two right premises of the topmost detachments, $q \to p$

and $r \rightarrow p$, are already elementary structural sequents. Hence, they are leafs of the explosion derivation, since neither of them contains a complex formula. Therefore, neither of them can be the conclusion of another detachment step.[1] There is another sequent, in which the proxy $p$ occurs, however, namely in the right premiss of the bottom (Prx), $\Gamma_1, p, \Gamma_2 \rightarrow w$. Since $\Gamma_1$ and $\Gamma_2$ may still contain complex formulae, the sequent is not necessarily an ESS. Assume that $\Gamma_1$ contains $m \leq \mathbf{len}(\Gamma_1)$ complex formulae and that $\Gamma_2$ contains $n \leq \mathbf{len}(\Gamma_2)$ such formulae. Then, these formulae can be replaced by proxy variables in $m + n$ steps.[2] This leaves an ESS $\gamma_1, p, \gamma_2 \rightarrow w$, where both $\gamma_1$ and $\gamma_2$ contain only occurrence variables (proxy or proper). Furthermore, since it is an ESS, no rule is applicable, and thus it is also a leaf of the explosion derivation. Since $p$ is a unique proxy variable, it can occur in no other leaf of the explosion derivation than those that have been mentioned. Even without the context of the derivation, the following information can be obtained from the three elementary structural sequents $q \rightarrow p$, $r \rightarrow p$ and $\gamma_1, p, \gamma_2 \rightarrow w$:

1) The ESSs $q \rightarrow p$ and $r \rightarrow p$ retain the structural information of the premises of the local logical rule. The proxy variables $q$ and $r$ mark the location of the side formulae $A$ and $B$, respectively, of the original disjunctive formula, whereas the proxy $p$ represents the occurrence of the disjunction itself.

2) The antecedent of the third ESS, $\gamma_1, p, \gamma_2$, gives the context, in which the original disjunction $A \vee B$ occured. The context is represented by the proxies, which were introduced by detachments of the complex formulae in $\Gamma_1$ and $\Gamma_1$, and the proper occurrence variables that were not detached. The proxy $p$ itself marks the location, at which the disjunction originally resided.

In summary, a proxy variable $p$ occurs as the single formula in the succedent of two ESSs and in some context $\gamma_1, \cdot, \gamma_2$ in the antecedent of a third ESS. Among the meta rules, this exact configuration can be found in the form of premises of $(\vee \mathrm{A}_m)$, although there, the side formulae $A$ and $B$ in the antecedents are not yet detached from the succedent $p$ in the logcial premises, and the context premiss is given in its general form. There is no other possibility for deriving such a configuration of ESSs sharing the proxy $p$. For recall that $p$ is introduced in order to detach $A \vee B$ from its context. In

[1] The sequents, from which they were obtained, $A \rightarrow p$ and $B \rightarrow p$, each contained exactly one complex formula, each of which was detached by an application of (Prx).

[2] Atomic formulae are never detached by the explosion procedure.

the context premiss, this proxy is not further treated, since it is already an atomic formula. It is simply retained from one context sequent to the next.

These observations convey the information that $p$ represents the location of a disjuntion in the ESS $\gamma_1, p, \gamma_2 \to w$, and that the other two ESSs are the structural representation of this disjunction.

## 6.1 Explosion Sets

Without the aid of a visual representation, which will be introduced in the final part of this investigation, formal tools are required to state properties about certain collections of ESSs. We will consider only explosion derivations that have been *completely developed*. This means that the explosion procedure has terminated, because there is no leaf in the derivation that still contains a complex formula. The main result of this chapter will be the proposition stating that all possible completely developed explosion derivations of a sequent yield the same set of elementary structural sequents (up to the names of the proxy variables).

**Definition 6.1** *For an explosion derivation* $\Pi$ *of a sequent* $\mathcal{S}$ *with watershed proxy* $w_\Pi$ *and the set* $\mathfrak{E}_\Pi$ *of all elementary structural sequents, which occur as leaves of* $\Pi$*, the tuple* $\langle \mathfrak{E}_\Pi, w_\Pi \rangle$ *is called the* explosion set *of* $\Pi$.

The watershep proxy $w_\Pi$, which is introduced in the first step of any explosion derivation $\Pi$, is a useful point of reference with regard to the set of leaves of $\Pi$. Hence, we will never consider just the set $\mathfrak{E}_\Pi$ alone, but instead the tuple $\langle \mathfrak{E}_\Pi, w_\Pi \rangle$ as explosion set of $\Pi$, although we shall occasionally also address $\mathfrak{E}_\Pi$ as explosion set, when $w_\Pi$ is not relevant. This overloading of the definiens is tolerated at this point in order to avoid an unnecessary terminological fabrication. In what follows, we will always assume that, whenever a set of ESSs is given, the watershed proxy is understood from this set. As a notational convenience, we will write sequents $\Gamma \to \Delta$ in square brackets $[\Gamma \to \Delta]$, whenever several sequents have to be separated by commas, in order to facilitate readability. This will be employed whenever ESSs are accumulated in sets.

It will prove very useful for the characterisation of ESSs to keep track of the order in which proxies are introduced. However, the exact order of application of the rules, by which the proxies were introduced, is not of interest. Instead, the relative distance of a proxy to the watershed proxy is the important measure. This relative distance is measured by the number of meta rules that lie between the one introducing said proxy and the intitial atomic cut, which introduces the watershed proxy. To gain an intuitive understanding, which cuts are relevant for this measure, consider the sequent from a previous example: $\Gamma_1, A \vee B, \Gamma_2 \rightarrow w$, where $w$ is the watershed proxy, $\mathbf{len}(\Gamma_1) = m$ and $\mathbf{len}(\Gamma_2) = n$. Since $A \vee B$ is a complex formula, the explosion procedure will eventually detach it from its context $\Gamma_1$ and $\Gamma_2$. However, since the choice of which formula to detach is arbitrary, there can occur up to $m + n$ applications of meta rules in an explosion derivation, by which complex formulae are removed from $\Gamma_1$ or $\Gamma_2$. This yields as rightmost premiss of such a derivation the sequent $\Gamma'_1, A \vee B, \Gamma'_2 \rightarrow w$ for suitable $\Gamma'_1$ and $\Gamma'_2$. If $A \vee B$ is detached from this sequent by the introduction of a proxy, say $p$, we obtain as context premiss the sequent $\Gamma'_1, p, \Gamma'_2 \rightarrow w$. After at most $m + n + 1$ applications of meta rules, we obtain the ESS $\gamma_1, p, \gamma_2 \rightarrow w$. All the proxy variables, which occur in the antecedent of this sequent, have been introduced by these rules applications which all treat the same context up to detachment of complex formulae, and hence they share the property that they were introduced in this sense *immediately after* the watershed proxy $w$. According to the explosion procedure, they could not have been introduced before $w$, since the first step is initialisation.

$$\dfrac{\vdots \qquad \dfrac{\vdots \qquad \dfrac{A \rightarrow p \qquad B \rightarrow p \qquad \dfrac{\vdots \qquad \dfrac{\vdots \qquad \gamma_1, p, \gamma_2 \rightarrow w}{\vdots}\,(\star\mathrm{A}_m)}{\Gamma'_1, p, \Gamma'_2 \rightarrow w}\,(\star\mathrm{A}_m)}{\Gamma'_1, A \vee B, \Gamma'_2 \rightarrow w}\,(\vee\mathrm{A}_m)}{\vdots}\,(\star\mathrm{A}_m)}{\Gamma_1, A \vee B, \Gamma_2 \rightarrow w}\,(\star\mathrm{A}_m)$$

Now consider the sequents $A \rightarrow p$ and $B \rightarrow p$, the logical premises of rule $(\vee\mathrm{A}_m)$. Assuming that both $A$ and $B$ are complex formulae, they will be detached from $p$ and handled by the explosion procedure by means of new proxy variables, say $q$ and $r$. The context premises of the two required meta rule applications will be $q \rightarrow p$ and $r \rightarrow p$.

$$\dfrac{\cdots \qquad q \rightarrow p}{A \rightarrow p}\,(\star\mathrm{A}_m) \qquad\qquad \dfrac{\cdots \qquad r \rightarrow p}{B \rightarrow p}\,(\star\mathrm{A}_m)$$

Hence, proxies $q$ and $r$ are introduced *after* the proxy $p$. They could not be introduced before $p$, because they are introduced to detach the side formulae $A$ and $B$ of $A \vee B$; but in order to access $A$ and $B$, the formula $A \vee B$ must already have been detached and handled.

The example shows that the right premiss of the initialisation has a sequence of succeeding context premises above it, which share common features. The same is true for every logical premiss that occurs in an explosion derivation. This is captured by the following notion.

**Definition 6.2** *A* context branch *in an explosion derivation* $\Pi$ *is a sequence of sequents* $(\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n)$ *with* $n \in \mathbb{N}$*, in which* $\mathcal{S}_0$ *is a logical premiss of a meta rule or a premiss of the initialisation step in* $\Pi$*, and every* $\mathcal{S}_{i+1}$ *is the context premiss of the meta rule, the conclusion of which is* $\mathcal{S}_i$ *in* $\Pi$*. The sequent* $\mathcal{S}_0$ *is called the* base *of the branch.*

For each $i > 0$, the sequent $\mathcal{S}_i$ in a context branch is similar to the base sequent $\mathcal{S}_0$ in the sense that it is the sequent $\mathcal{S}_0$, with $i$ complex formulae replaced by proxy variables.

**Definition 6.3** *For some sequent* $\mathcal{S}$ *containing complex formulae, the sequent* $\mathcal{S}'$*, which is obtained from* $\mathcal{S}$ *by replacing any number of complex formulae by proxy variables, is called a* variant *of* $\mathcal{S}$*. It is called an* immediate variant *of* $\mathcal{S}$*, if it is obtained from* $\mathcal{S}$ *by replacing exactly one complex formula by a single proxy variable.*

As a limit case, it is useful to consider any sequent to be an (improper) variant of itself.[3] Every context premiss of a meta rule is an immediate variant of the conclusion, since exactly one complex formula is replaced by a proxy variable. A context branch is a sequence, in which every element, apart from the first, is an immediate variant of the preceding element.

**Lemma 6.4** *For every sequent* $\mathcal{S}$ *containing* $n$ *complex formulae, which occurs as a logical premiss or as a premiss of the initial atomic cut in a completely developed explosion derivation* $\Pi$*, the following holds:*

1) *The context branch of* $\Pi$*, whose base is* $\mathcal{S}$*, has length* $n+1$*, and its last element is an elementary structural sequent, which is a variant of* $\mathcal{S}$ *and contains* $n+1$ *proxy variables.*

2) *If* $n > 0$*, then each* $\mathcal{S}_{i+1}$ *in the context branch* $(\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n)$ *with* $\mathcal{S}_0 = \mathcal{S}$ *is an immediate variant of* $\mathcal{S}_i$*.*

Proof: Note that $\mathcal{S}$ is not a context premiss. Hence, it cannot contain any other proxy variables than the single one, which was introduced by the meta rule or initialization step, of which $\mathcal{S}$ is a premiss. For either $\mathcal{S}$ is a logical premiss, in which case it can only contain the new proxy variable and one or two side formulae, each of which is either a complex formulae or a proper occurrence variable, or it is a premiss of the initialization step, in which case it can only contain the watershed proxy and the formulae of the antecedent or succedent of the original sequent, each of which is, again, either a complex

[3]Thus, the notion of variant induces a partial order on the set of sequents.

formulae or a proper occurrence variable. Hence, none of these formulae can be, much less contain, any proxy variables. There are two cases to distinguish. In the case that $\mathcal{S}$ contains no complex formulae, it is already an elementary structural sequent. All formulae occuring in the sequent, apart from the single proxy variable, must be proper occurrence variables. Furthermore, the branch, which has the ESS $\mathcal{S}$ as its base, contains only $\mathcal{S}$, and hence it is a branch of length 1. In the case that $\mathcal{S}$ contains $n > 0$ complex formulae, these formulae are detached in $\Pi$ by means of the appropriate meta rules. This requires $n$ applications of meta rules, each of which retains as context premiss an immediate variant of the conclusion, which is also a variant of $\mathcal{S}$. After $n$ applications of appropriate meta rules, the topmost context sequent in this branch is that variant of the sequent $\mathcal{S}$, in which all complex formulae have been replaced by proxy variables. Hence, this context formula is an ESS, and is, therefore, a leaf of the explosion derivation. Moreover, $n$ rules applications generate $n$ context premises, of which each one has exactly one proxy variable more than the previous one. Beginning with $\mathcal{S}$, which already contains exactly one proxy variable, this results in a branch of length $n+1$, where the leaf, which is the last element of the branch, contains $n+1$ proxy variables. □

Apart from the original sequent, every sequent that occurs in the explosion derivation belongs to exactly one of the branches that are described by lemma 6.4. For such a sequent $\mathcal{S}$ is either a logical premiss or a premiss of the initialisation step, in which case a new and possibly trivial branch begins at $\mathcal{S}$; otherwise, $\mathcal{S}$ is a context premiss, in which case it belongs to some branch beginning with a logical premiss or a premiss of the initialisation step further down in the derivation. Because of this, the following corollary holds.

**Corollary 6.5** *For every ESS $\mathcal{S}$ containing $n+1$ proxy variables, which occurs as leaf in an explosion derivation using the meta rules, the following holds: The context branch ending with $\mathcal{S}$ has length $n+1$ and begins with either a logical premiss or a premiss of the initial atomic cut containing $n$ complex formulae. Furthermore, $\mathcal{S}$ is a variant of that premiss.*

Lemma 6.4 and corollary 6.5 characterise the iterative development of sequents along a context branch from the usual bottom up perspective and the view from an ESS leaf down to the base of the context branch.

The example demonstrates that proxies, which occur in a sequent $\mathcal{S}$, are of one of two kinds. There is always exactly one proxy which relates $\mathcal{S}$ to a context[4]. In addition to it, there can be zero or more proxies, which relate

[4]In the exceptional case of the watershed proxy $w$, this "context" is just the sequent (or its variant) that was detached in the initialisation step.

other sequents to $\mathcal{S}$ and its variants.

**Definition 6.6** *For a sequent $\mathcal{S}$ occuring in an explosion derivation, the* base proxy *of $\mathcal{S}$ is the single proxy, which occurs in the base of the context branch containing $\mathcal{S}$. All the other proxies occuring in $\mathcal{S}$ are called* branching proxies*.*

A branching proxy is one which replaces a complex formula in a sequent. It is important to realise that any proxy, apart from the watershed proxy, occurs as both a branching proxy in some sequents of an explosion derivation and as base proxy in others. A proxy $p$ is a branching proxy in all the sequents of the single context branch, in which $p$ does not already occur as the single proxy variable in the base of the branch. In the context branch or those context branches, in which $p$ already occurs in the base sequent, it is a base proxy. Since the watershed proxy already appears in the base of two context branches, it is always a base proxy.

Since all the branching proxies in sequents of a context branch are introduced after the base proxy, and because they occur, in turn, as base proxies of other sequents, there is a genealogy of proxy variables, beginning with the watershed variable as the common ancestor.

**Definition 6.7** *The* immediate predecessor relation *on $\mathcal{P}$ induced by $\Pi$ is given by:*

$$p \prec_{\Pi} q :\Leftrightarrow \begin{cases} \textit{there is some } \mathcal{S} \in \mathfrak{E}_{\Pi}, \textit{ such that } p \textit{ is the base} \\ \textit{proxy of } \mathcal{S} \textit{ and } q \textit{ is a branching proxy of } \mathcal{S} \end{cases}$$

*The* predecessor relation $\preccurlyeq_{\Pi}$ *on $\mathcal{P}$ induced by $\Pi$ is the transitive reflexive closure $\prec_{\Pi}^{*}$ of $\prec_{\Pi}$.*

We will usually omit the index $\Pi$, when the explosion derivation inducing the order is understood from the context. The following lemma establishes that $\preccurlyeq_{\Pi}$ has the desired property.

**Lemma 6.8** *For every explosion derivation $\Pi$ with watershed proxy $w$, the predecessor relation $\preccurlyeq_{\Pi}$ is a partial order with minimal element $w$.*

PROOF: According to the definition, $\prec_{\Pi}^{*}$ is already a preorder. Recall that $\prec_{\Pi}^{*} = \prec_{\Pi}^{+} \cup \prec_{\Pi}^{0}$, where $\prec_{\Pi}^{+}$ is the transitive closure of $\prec_{\Pi}$, and $\prec_{\Pi}^{0}$ is the reflexive closure of $\prec_{\Pi}$. Whenever $p \prec_{\Pi}^{+} q$, then it is either $p \prec_{\Pi} q$ or there is a proxy $v$ such that $p \prec_{\Pi} v$ and $v \prec_{\Pi}^{+} q$. In the former case, the proxy $q$ is introduced further up in that context branch of $\Pi$, which begins with the base sequent that has $p$ as base proxy. In the latter case, $q$ is introduced in some other context branch of $\Pi$ that is part of the subderivation of $\Pi$ beginning with the same base sequent. That is, the proxy $q$ is introduced further up in $\Pi$ than $p$. Since every proxy is introduced exactly once, $p$ cannot be introduced again after $q$. Hence, the reverse $q \prec^{+}_{\Pi} p$ cannot hold. Furthermore, it is $\prec^{0}_{\Pi} \cap \prec^{+}_{\Pi} = \emptyset$, since $\prec_{\Pi}$ is not reflexive, and $p \not\prec^{+}_{\Pi} p$. Therefore, whenever there is both $p \prec^{*}_{\Pi} q$ and $p \prec^{*}_{\Pi} q$, it must be $p \prec^{0}_{\Pi} q$ and $q \prec^{0}_{\Pi} p$, i.e. $p = q$. Hence, the relation $\prec^{*}_{\Pi}$ is antisymmetric.

Finally, $w$ is the first proxy variable that is introduced in $\Pi$, and, hence, it is $w \preccurlyeq_{\Pi} p$ for all proxy variables $p$ that occur in $\Pi$. All those elements of $\mathcal{P}$, which do not occur in $\Pi$, are not related to any other elements of $\mathcal{P}$ under $\prec_{\Pi}$ and only trivially to themselves under $\preccurlyeq_{\Pi}$. Therefore there is no $p \in \mathcal{P}$ such that $p \preccurlyeq_{\Pi} w$ and $w$ is a minimal element. □

Of course, if we consider the set of proxy variables that actually occur in $\Pi$, then $w$ is the least element of that set.

For an explosion derivation $\Pi$ with watershed proxy $w$ of sequent $\Gamma \rightarrow \Delta$, we write $\Pi_A$ for the subderivation of the detached antecedent $\Gamma \rightarrow w$ and $\Pi_S$ for the subderivation of the detached succedent $w \rightarrow \Delta$.

$$\dfrac{\begin{array}{cc}\Pi_A & \Pi_S \\ \Gamma \rightarrow w & w \rightarrow \Delta\end{array}}{\Gamma \rightarrow \Delta}\ (\text{Prx})$$

Following definition 6.7, relations $\prec_{\Pi_A}$ on the set $\mathcal{P}_A$ of proxy variables occuring in $\Pi_A$ and $\prec_{\Pi_S}$ on the set $\mathcal{P}_S$ of proxy variables occuring in $\Pi_S$ can be defined. Since $\mathcal{P}_A \cap \mathcal{P}_S = \{w\}$ and neither $\prec_{\Pi_A}$ nor $\prec_{\Pi_S}$ is reflexive, we have $\prec_{\Pi_A} \cap \prec_{\Pi_S} = \emptyset$, i.e. the immediate predecessor relation of proxies occuring in $\Pi_A$ and that of proxies occuring in $\Pi_S$ are independent of one another, as expected. This separation property can be strengthened as follows.

**Lemma 6.9** *Let $\Pi$ be an explosion derivation. Then, for all proxy variables $p, q, r$ with $q \neq r$ such that $p \prec_{\Pi} q$ and $p \prec_{\Pi} r$, there is no proxy variable $s$ such that $q \preccurlyeq_{\Pi} s$ and $r \preccurlyeq_{\Pi} s$.*

Proof: As every ESS has only a single base proxy, the ESSs, in which $q$ and $r$ are base proxies, are not identical. Hence, in $\Pi$ there is a subderivation $\Pi_q$ ending in a sequent, in which $q$ is the only proxy variable, and all the proxy variables $s'$ with $q \preccurlyeq_{\Pi} s'$ are introduced further up in $\Pi_q$. Likewise, there is a different subderivation of $\Pi_r$ ending in a sequent, in which $r$ is the only proxy variable, and all the proxy variables $s''$ with $q \preccurlyeq_{\Pi} s''$ are introduced further up in $\Pi_r$. As $\Pi$ is a tree, $\Pi_q$ and $\Pi_r$ cannot share any proxy variables that were introduced further up in $\Pi_q$ and $\Pi_r$, and, hence, there is no proxy variable $s$ such that $q \preccurlyeq_\Pi s$ and $r \preccurlyeq_\Pi s$. $\square$

When considering the explosion set $\langle \mathfrak{E}_\Pi, w_\Pi \rangle$ of an explosion derivation $\Pi$, we will henceforth implicitly assume that, the base proxy of every ESS can be identified. Formally, we could introduce a function $\mathfrak{E}_\Pi \to \mathcal{P}$, which performs this identification, based on the inspection of the $\Pi$, and include it in the definition of an explosion set. Of course, the immediate predecessor relation can be restored from $\mathfrak{E}_\Pi$ and $w_\Pi$ alone. As $w$ is the base proxy of two ESSs, the branching proxies thereof must be base proxies of certain other ESSs. A method related to this procedure we will be employed in chapter 9 for the purpose of deciding, whether some explosion set corresponds to a provable sequent.

## 6.2 An Extensive Example

Consider the following schematic explosion derivation, using the meta rules, of the sequent $\mathcal{L}^w = [(b_1 \supset c_1) \,\&\, \neg(c_2 \supset a_1), \neg(b_2 \vee d_1), d_2, a_2 \,\&\, b_3 \to w]$, which is the left premiss of some initialisation step.
Every sequent that occurs in the derivation is either a logical premiss or a context premiss of some meta rule. The former sequents are labelled by $\mathcal{L}$, indexed with the proxy variable that was introduced by the rule.[5] A possible lower index differentiates two logical premises for certain rules. The context premises, represented by $\mathcal{C}$, are doubly indexed; the upper index retains the upper index of the conclusion of the rule, while the lower index accumulates all the proxy variables, by which complex formulae have been detached in the context branch, to which the sequent belongs.

$$\dfrac{\dfrac{\mathcal{L}^q_1 \quad \mathcal{L}^q_2 \quad \dfrac{\dfrac{\mathcal{C}^r_s \quad \mathcal{L}^s}{\mathcal{L}^r}\,(\supset\mathrm{S}_m) \quad \mathcal{C}^p_{q,r}}{\mathcal{C}^p_q}\,(\neg\mathrm{A}_m)}{\mathcal{L}^p}\,(\supset\mathrm{A}_m) \quad \dfrac{\dfrac{\mathcal{C}^t_u \quad \mathcal{L}^u}{\mathcal{L}^t}\,(\vee\mathrm{S}_m) \quad \dfrac{\mathcal{L}^v \quad \mathcal{C}^w_{p,t,v}}{\mathcal{C}^w_{p,t}}\,(\&\mathrm{A}_m)}{\mathcal{C}^w_p}\,(\neg\mathrm{A}_m)}{\mathcal{L}^w}\,(\&\mathrm{A}_m)$$

All the sequents that occur in the derivation are listed below.

[5] In this example, the premiss of the intialisation is called $\mathcal{L}^w$, although it is, of course, not a logical premiss, but the left premiss of the initial atomic cut.

$\cdot\, \mathcal{L}^w = [(b_1 \supset c_1) \,\&\, \neg(c_2 \supset a_1), \neg(b_2 \vee d_1), d_2, a_2 \,\&\, b_3 \to w]$

$\cdot\, \mathcal{L}^p = [b_1 \supset c_1, \neg(c_2 \supset a_1) \to p]$

$\cdot\, \mathcal{L}^q_1 = [\to q, b_1]$ ESS

$\cdot\, \mathcal{L}^q_2 = [c_1 \to q]$ ESS

$\cdot\, \mathcal{C}^p_q = [q, \neg(c_2 \supset a_1) \to p]$

$\cdot\, \mathcal{L}^r = [\to r, c_2 \supset a_1]$

$\cdot\, \mathcal{C}^r_s = [\to r, s]$ ESS

$\cdot\, \mathcal{L}^s = [c_2, s \to a_1]$ ESS

$\cdot\, \mathcal{C}^p_{q,r} = [q, r \to p]$ ESS

$\cdot\, \mathcal{C}^w_p = [p, \neg(b_2 \vee d_1), d_2, a_2 \,\&\, b_3 \to w]$

$\cdot\, \mathcal{L}^t = [\to t, b_2 \vee d_1]$

$\cdot\, \mathcal{C}^t_u = [\to t, u]$ ESS

$\cdot\, \mathcal{L}^u = [u \to b_2, d_1]$ ESS

$\cdot\, \mathcal{C}^w_{p,t} = [p, t, d_2, a_2 \,\&\, b_3 \to w]$

$\cdot\, \mathcal{L}^v = [a_2, b_3 \to v]$ ESS

$\cdot\, \mathcal{C}^w_{p,t,v} = [p, t, d_2, v \to w]$ ESS

The explosion set of this derivation is the following set:

$$\begin{aligned}
\mathfrak{E} &= \{\mathcal{L}^q_1, \mathcal{L}^q_2, \mathcal{C}^r_s, \mathcal{L}^s, \mathcal{C}^p_{q,r}, \mathcal{C}^t_u, \mathcal{L}^u, \mathcal{L}^v, \mathcal{C}^w_{p,t,v}\} \\
&= \left\{ \begin{array}{l} [\to q, b_1], [c_1 \to q], [\to r, s], [c_2, s \to a_1], [q, r \to p], \\ [\to t, u], [u \to b_2, d_1], [a_2, b_3 \to v], [p, t, d_2, v \to w] \end{array} \right\}
\end{aligned}$$

According to the remark following definition 6.1, we can also consider the pair $\langle \mathfrak{E}, w \rangle$ as the explosion set of this derivation, if we want to emphasise the watershed proxy.

Consider the rightmost context branch, beginning with the sequent $\mathcal{L}^w$. The first meta step detaches by proxy $p$ and subsequently detaches the formula $(b_1 \supset c_1) \,\&\, \neg(c_2 \supset a_1)$ from $\mathcal{L}^w$. This step yields the context premiss $\mathcal{C}^w_p = [p, \neg(b_2 \vee d_1), d_2, a_2 \,\&\, b_3 \to w]$. We further obtain the sequent $\mathcal{C}^w_{p,t} = [p, t, d, a_2 \,\&\, b_3 \to w]$ by detaching $\neg(b_2 \vee d_1)$ and, finally, $\mathcal{C}^w_{p,t,v} = [p, t, d_2, v \to w]$, which is an ESS, by detachment of $a_2 \,\&\, b_3$. The variable $d_2$ is a proper occurrence variable, hence it is *not* detached from its context. In each rule application, the context premiss differs from the conclusion only in the proxy variable, which replaces some complex formula. At the same time, any sequent only contains a finite number of complex formulae. Hence, there is a context branch consisting of variants of $\mathcal{L}^w$ in the derivation, beginning with $\mathcal{L}^w$ and continued through the context premises $\mathcal{C}^w_p$, $\mathcal{C}^w_{p,t}$ and $\mathcal{C}^w_{p,t,u}$. The last sequent contains no more complex formulae, i.e. it is an ESS and a leaf of the explosion derivation. Hence, the branch $(\mathcal{L}^w, \mathcal{C}^w_p, \mathcal{C}^w_{p,t}, \mathcal{C}^w_{p,t,v})$ is a sequence of variants of $\mathcal{L}^w$, where each sequent is an immediate variant of the preceding sequent. The other context branches in this example are $(\mathcal{L}^p, \mathcal{C}^p_q, \mathcal{C}^p_{q,r})$, $(\mathcal{L}^r, \mathcal{C}^r_s)$, $(\mathcal{L}^t, \mathcal{C}^t_u)$ and all the trivial branches that consist only of logical premises that are already ESSs: $(\mathcal{L}^q_1)$, $(\mathcal{L}^q_2)$, $(\mathcal{L}^s)$, $(\mathcal{L}^u)$ and $(\mathcal{L}^v)$.[6] Note that the first entry of these sequences, the base of the branch, is always either a logical premiss of a meta rule or the left premiss of the initialisation step. Because of this, there is always a single new proxy variable present in the corresponding sequent, and this single proxy is given as the upper index. As we progress upwards through a branch, branching proxies are introduced by successive detachments of complex formulae from the sequent. These proxies are accumulated in the lower index. A branch always begins with a sequent with empty lower index. Every successive context premiss contains an additional proxy variable. For example, consider the branch $(\mathcal{L}^p, \mathcal{C}^p_q, \mathcal{C}^p_{q,r})$. The element $\mathcal{L}^p$ of the branch is the sequent $b_1 \supset c_1, \neg(c_2 \supset a_1) \rightarrow p$, which contains only the base proxy $p$, by which the formula $(b_1 \supset c_1) \wedge \neg(c_2 \supset a_1)$ was detached from its context. The following application of $(\supset\mathrm{A}_m)$ yields the context premiss $q, \neg(c_2 \supset a_1) \rightarrow p$, the application of $(\neg\mathrm{A}_m)$ finally yields as context premiss the ESS $q, r \rightarrow p$. The base of the branch contains only the single base proxy variable; every succeeding sequent of the branch has one complex formula less and one branching proxy variable more than the preceding one. The final element of the branch, $\mathcal{C}^p_{q,r}$, contains three proxy variables: the base proxy $p$ and branching proxies $q$ and $r$.

To continue the example, recall the explosion set of the explosion derivation:

$$\mathfrak{E} = \left\{ \begin{array}{l} [\rightarrow q, b_1], [c_1 \rightarrow q], [\rightarrow r, s], [c_2, s \rightarrow a_1], [q, r \rightarrow p], \\ [\rightarrow t, u], [u \rightarrow b_2, d_1], [a_2, b_3 \rightarrow v], [p, t, d_2, v \rightarrow w] \end{array} \right\}$$

This set yields the following immediate predecessor relation on $\mathcal{P}$:

[6] In the case of $(\mathcal{L}^q_1)$ and $(\mathcal{L}^q_2)$, the lower indices do not refer to proxy variables, but merely serve to distinguish the two logical premises of $(\supset\mathrm{A}_m)$. If any one of those premises would have to be further exploded, the number indices could be dropped, since any context premises occuring in the respective branches would be uniquely distinguished by the new proxy variables that are introduced in the process.

$$\prec = \{(w,p),(w,t),(w,v),(p,q),(p,r),(t,u),(r,s)\}$$

The pairs $(w,p),(w,t)$ and $(w,v)$ are obtained from the elementary structural sequent $\mathcal{C}^w_{p,t,v} = [p,t,d_2,v \to w]$, where $w$ occurs as base proxy and $p,t,v$ occur as branching proxies. For the next wave of pairs, all ESSs must considered, in which $p,t,v$ occur as base proxies. Those are $\mathcal{C}^p_{q,r} = [q,r \to p]$ and $\mathcal{C}^t_u = [\to t,u]$ and $\mathcal{L}^v = [a_2,b_3 \to v]$. The latter contains, apart from the base proxy $v$, only proper occurrence variables, which do not fall under the relation $\prec$. Hence, we obtain pairs $(p,q),(p,r)$ and $(t,u)$. Among the second elements of these pairs, only $r$ occurs as base proxy in an ESS containing another proxy variable: $\mathcal{C}^r_s = [\to r,s]$. This yields the final pair $(r,s)$, since $s$ only occurs as base proxy in the ESS $\mathcal{L}^s = [c_2,s \to a_1]$, which contains no branching proxies.

According to the definition of $\prec$, for every pair $(x,y)$ with $x \prec y$, there is an ESS $\mathcal{C}^x_l$, such that $y$ occurs in $l$. In the preceding example, these are the following ESSs:

$$\mathcal{C}^w_{p,t,v}, \mathcal{C}^p_{q,r}, \mathcal{C}^t_u, \mathcal{C}^r_s$$

Moreover, for every pair $(x,y)$ with $x \prec y$, there is a context premiss $\mathcal{C}^x_{l,y}$, in which $y$ appears for the first time in the context path, when reading the derivation from the bottom up. It is obvious that all context premises that occur in the explosion derivation are represented by such a pair, because in each of those context premises, some $y$ was introduced, replacing some complex formula. The example contained the following context sequents, each of which corresponds to one of the pairs of proxies of the immediate predecessor relation:

$$\mathcal{C}^w_p, \mathcal{C}^w_{p,t}, \mathcal{C}^w_{p,t,v}, \mathcal{C}^p_q, \mathcal{C}^p_{q,r}, \mathcal{C}^t_u, \mathcal{C}^r_s$$

However, the predecessor relation relies on the occurrence of branching proxies in the ESSs of the explosion set, and does not accomodate the order, in which the branching proxies of some ESS were introduced in the explosion derivation. This already points towards the most important result regarding the explosion procedure, which will be presented in the following section.

## 6.3 Uniqueness of the Explosion Set

The explosion procedure successively detaches and handles all the complex formulae occuring in a sequent. The application of a meta rule yields a variant of the consequence as context premiss, whereas new base sequents arise as logical premises. There are no intelligent choices required in the execution of the procedure. The only choices regard the order, in which complex formula are detached from the antecedent or the succedent of a sequent, if there do indeed occur more than just a single one. Consider the following possible explosion derivations of $p \to A \,\&\, B, \neg C$.

$$\dfrac{\dfrac{p \to q, r \qquad \begin{matrix}\vdots \\ q \to A\end{matrix} \qquad \begin{matrix}\vdots \\ q \to B\end{matrix}}{p \to A \,\&\, B, r}\,(\&\mathrm{S}_m) \qquad \begin{matrix}\vdots \\ C, r \to\end{matrix}}{p \to A \,\&\, B, \neg C}\,(\neg\mathrm{S}_m)$$

$$\dfrac{\dfrac{p \to s, t \qquad \begin{matrix}\vdots \\ C, t \to\end{matrix}}{p \to s, \neg C}\,(\neg\mathrm{S}_m) \qquad \begin{matrix}\vdots \\ s \to A\end{matrix} \qquad \begin{matrix}\vdots \\ s \to B\end{matrix}}{p \to A \,\&\, B, \neg C}\,(\&\mathrm{S}_m)$$

In the second derivation the complex formulae are detached in the opposite order from the first one. It is obvious that the two derivations have the same premises, apart from the names of the proxy variables. However, after the renaming $\{s \mapsto q, t \mapsto r\}$ of proxy variables, both derivations have the same premises.

**Definition 6.10** *A* renaming *of proxy variables is a permutation $\sigma : \mathcal{P} \to \mathcal{P}$. The* renaming of proxy variables in a sequent $\mathcal{S}$*, written as $\mathcal{S}\sigma$, is obtained by simultaneous replacement of every occurrence of a proxy $p$ in $\mathcal{S}$ by $\sigma(p)$. The* renaming of proxy variables in a derivation $\Pi$*, written as $\Pi\sigma$, is obtained by simultaneous replacement of every occurrence of a proxy $p$ in every sequent of $\Pi$ by $\sigma(p)$. The* renaming of proxy variables in a set of elementary structural sequents $\mathfrak{A}$*, written as $\mathfrak{A}\sigma$, is obtained by simultaneous replacement of every occurrence of a proxy $p$ in every sequent of $\mathfrak{A}$ by $\sigma(p)$.*

Renaming of a single proxy variable $p$ by $q$ will be written in the mapping notation: $\{p \mapsto q\}$. In every sequent and every explosion derivation, there occur only finitely many proxy variables. Hence, relevant renaming operations can always be written as $\{p_1 \mapsto q_1, \ldots, p_n \mapsto q_n\}$ for some $n \in \mathbb{N}$. For example, consider the following renaming of a set of ESSs:

$$\{[w \to p], [a_i, p \to b_j]\}\{w \mapsto p, p \mapsto q\} \quad = \quad \{[p \to q], [a_i, q \to b_j]\}$$

With this notion established, we can formulate the following generalisation of the property indicated by the example above.

**Lemma 6.11** *The meta rules of the explosion calculus commute over their context premises up to renaming of proxy variables.*

PROOF: Consider the case that both $(R_1)$ and $(R_2)$ have one logical premiss.

$$\dfrac{\dfrac{\mathcal{S}_2 \qquad \mathcal{L}^{p_2}}{\mathcal{S}_1}\,(R_2) \qquad \mathcal{L}^{p_1}}{\mathcal{S}_0}\,(R_1) \qquad\qquad \dfrac{\dfrac{\mathcal{S}_2' \qquad \mathcal{L}^{q_1}}{\mathcal{S}_1'}\,(R_1) \qquad \mathcal{L}^{q_2}}{\mathcal{S}_0}\,(R_2)$$

The sequent $\mathcal{S}_0$ is the conclusion of meta rule $(R_1)$, which has context premiss $\mathcal{S}_1$ and logical premiss $\mathcal{L}^{p_1}$. At the same time, $\mathcal{S}_1$ is conclusion of rule $(R_2)$, which has context premiss $\mathcal{S}_2$ and logical premiss $\mathcal{L}^{p_2}$. As context premiss of $\mathcal{S}_0$, $\mathcal{S}_1$ is an immediate variant of it, that is, exactly one occurrence complex formula $C_1$ in $\mathcal{S}_1$ is replaced by a proxy variable $p_1$. Similarly, $\mathcal{S}_2$ is an immediate variant of $\mathcal{S}_1$, i.e. an occurrence of complex formula $C_2$ is replaced by proxy $p_2$. Commuting the rules yields context sequents $\mathcal{S}_1'$ of $(R_2)$ and $\mathcal{S}_2'$ of $(R_1)$. In the former, the same occurrence of $C_2$ is replaced by some proxy variable $q_2$. The latter is a variant of the former, in which the same occurrence of $C_1$ as before is replaced by a proxy variable $q_1$. Hence, both $\mathcal{S}_2$ and $\mathcal{S}_2'$ are variants of $\mathcal{S}_0$, which are obtained by replacing the same occurrences of $C_1$ and $C_2$ by proxy variables. Thus, it is $\mathcal{S}_2\{p_1 \mapsto q_1, p_2 \mapsto q_2\} = \mathcal{S}_2'$. Also we have obviously both $\mathcal{L}^{p_1}\{p_1 \mapsto q_1, p_2 \mapsto q_2\} = \mathcal{L}^{q_1}$ and $\mathcal{L}^{p_2}\{p_1 \mapsto q_1, p_2 \mapsto q_2\} = \mathcal{L}^{q_2}$. After renaming, both derivations have the same premises and the same conclusion. Therefore, $(R_1)$ and $(R_2)$ commute over their context premises up to renaming of the proxy variables that are introduced by the rules.[7]

The cases, in which one or both of the rules have two logical premises, can be treated accordingly, since every one of the two logical premises receives the same proxy variable. □

We can now state the main result about the uniqueness of explosion sets up to the names of proxy variables.

**Proposition 6.12** *Let* $\Pi_1$ *and* $\Pi_2$ *be two explosion derivations of some sequent with explosion sets* $\langle \mathfrak{E}_1, w_1\rangle$ *and* $\langle \mathfrak{E}_2, w_2\rangle$ *Then there is a renaming* $\sigma$*, such that* $\langle \mathfrak{E}_2, w_2\rangle = \langle \mathfrak{E}_1\sigma, \sigma(w_1)\rangle$.

PROOF: The explosion set of a derivation is just the set of sequents that occur as leaves in it. If $\Pi_1$ and $\Pi_2$ are two structurally different explosion derivations of the same sequent, then they can only differ in the order, in which

[7]It is obvious that a strict observance of which proxy variable is introduced by which rule in both derivations would enable us to use $p_1 = q_1$ and $p_2 = q_2$ throughout.

certain rules are applied to detach complex formulae from their respective contexts. Since the derivations are finite, lemma 6.11 can be applied finitely often to rearrange $\Pi_2$ into a derivation $\Pi_2'$, which matches the structure of $\Pi_1$. Each application of the lemma leaves the structure of the premises unchanged, hence $\Pi_2'$ can only differ from $\Pi_1$ in the names of the occurring proxy variables. Therefore there is a renaming $\sigma$, such that $\Pi_2'\sigma = \Pi_1$. But then it is particularly the case, that the leaves of $\Pi_2'\sigma$ and those of $\Pi_1$ are identical, and hence it is $\mathfrak{E}_1\sigma = \mathfrak{E}_2$. In addition to this, if $\Pi_1$ has watershed $w_1$ and $\Pi_2$ has watershed $w_2$, then $\sigma(w_1) = w_2$. □

The uniqueness of the explosion set of a sequent $\mathcal{S}$ is an important result, because it justifies the claim that the explosion set of $\mathcal{S}$, the structural skeleton of $\mathcal{S}$, constitutes its meaning. This will be elaborated in the next chapter. However, rather than relating explosion sets to specific explosion derivations, the result allows us to relate them to sequents. Henceforth we will write $\langle \mathfrak{E}_\mathcal{S}, w_\mathcal{S} \rangle$, if the particular explosion derivation for $\mathcal{S}$ is not relevant. While we do not have a unique explosion derivation of $\mathcal{S}$ either, at least the relevant information contained in explosion sets obtained by different explosion derivations of $\mathcal{S}$ is invariant, such as the number of ESSs contained and their relation to one another. It is therefore reasonable give an estimation about the size of an explosion set $\mathfrak{E}_\mathcal{S}$.

**Lemma 6.13** *For every sequent $\mathcal{S}$, the following holds:*

$$\mathbf{cmx}(\mathcal{S}) + 2 \leq |\mathfrak{E}_\mathcal{S}| \leq 2 \cdot \mathbf{cmx}(\mathcal{S}) + 2$$

Proof: For every sequent $\mathcal{S}$, there are two ESSs $\gamma \to w$ and $w \to \delta$ representing antecedent and succedent of $\mathcal{S}$. These ESSs are those two elements of $\mathfrak{E}_\mathcal{S}$, which are the leaves of the context branches beginning with the premises of the initialisation step. Additionaly, every logical connective occuring in $\mathcal{S}$ is resolved by an application of a meta rule. Each of these applications retains as context premiss a variant of the conclusion and introduces one or two logical premises. Since every logical premiss is the base sequent of another context branch, and since there are as many ESSs as there are context branches in the derivation of $\mathcal{S}$, there are exactly as many ESSs, as there are logical premises in the explosion derivation plus the two initial ESSs. The sequent $\mathcal{S}$ contains $\mathbf{cmx}(\mathcal{S})$ logical connectives, and each of these engenders one or two logical premises. This yields in the desired estimation. □

It is possible to determine the exact number of ESSs that make up the explosion set of a sequent $\mathcal{S}$. The meta rules for the binary logical connectives have one or two logical premises, depending on whether the connective occurs in the antecedent or in the succedent of a sequent. However, when such a logical connective occurs as subformula of a negation or as subformula of the hypothesis of an implication, the corresponding subformula changes position during the derivation. In order to determine the exact number of ESSs, a rigorous recursive regime regarding the polarity of subformulae has to be established. This is done in appendix A. Let $\mathbf{C}^+_\mathcal{S}$ be the number of positive occurrences of conjunctions in $\mathcal{S}$ and $\mathbf{C}^-_\mathcal{S}$ be the number of negative occurrences of conjunctions; let $\mathbf{D}^+_\mathcal{S}$, $\mathbf{D}^-_\mathcal{S}$ and $\mathbf{I}^+_\mathcal{S}$, $\mathbf{I}^-_\mathcal{S}$ give the same for disjunctions and implications respectively, and let $\mathbf{N}_\mathcal{S}$ be the number of occurrences of negations. Using these measures, we obtain the exact size of the explosion set.

**Proposition 6.14** *For every sequent $\mathcal{S}$, the following holds:*

$$|\mathfrak{E}_\mathcal{S}| = 2 \cdot (\mathbf{C}^+_\mathcal{S} + \mathbf{D}^-_\mathcal{S} + \mathbf{I}^-_\mathcal{S}) + \mathbf{C}^-_\mathcal{S} + \mathbf{D}^+_\mathcal{S} + \mathbf{I}^+_\mathcal{S} + \mathbf{N}_\mathcal{S} + 2$$

Proof: Every explosion set contains the two ESSs $\gamma \to w$ and $w \to \delta$, which represent the structure of the antecedent and succedent. Now consider all occurrences of subformulae of $\mathcal{S}$. Every positive occurence of a conjunctive subformula in $\mathcal{S}$ will be resolved by the explosion procedure by means of an application of (&$\mathrm{S}_m$), which introduces two new logical premises in addition to retaining a variant of the conclusion in the context premiss. Each one of the logical premises is the base sequents of a new context branch, which ends in an ESS, which is a variant of this premiss. Hence, each positive occurrence of a conjunctive subformula results in two additional ESSs. The same argument gives two ESSs for every negative occurrence of an disjunctive or implicative subformula in $\mathcal{S}$, which are resolved by the rules ($\vee\mathrm{A}_m$) and ($\supset\mathrm{A}_m$), each of which also has two logical premises. All other occurrences of complex subformulae are resolved by rules, which have a single logical premiss. Hence, a single logical premiss engendering another ESS is added by every application of the corresponding meta rule. □

The aspects of this argument, which relate to how the number of ESSs correspond to an occurrence of a complex formula, will be revisited in much greater detail in section structcomplex, especially in proposition 7.4 and its corollaries.

## 6.4 Further Examples

Consider the explosion derivation of the sequent $\mathcal{S} = [(a \vee b) \vee c \to a \vee (b \vee c)]$, the occurrence instance of which is $\widetilde{\mathcal{S}} = [(a_1 \vee b_1) \vee c_1 \to a_2 \vee (b_2 \vee c_2)]$. The explosion derivation for $\widetilde{\mathcal{S}}$ is

$$\dfrac{\begin{matrix}\Pi_A\\ (a_1 \vee b_1) \vee c_1 \to w\end{matrix} \qquad \begin{matrix}\Pi_S\\ w \to a_2 \vee (b_2 \vee c_2)\end{matrix}}{(a_1 \vee b_1) \vee c_1 \to a_2 \vee (b_2 \vee c_2)}\,(\mathrm{Prx})$$

where $\Pi_A$ is the derivation

$$\dfrac{\dfrac{a_1 \to q \qquad b_1 \to q \qquad q \to p}{a_1 \vee b_1 \to p}\,(\vee\mathrm{A}_m) \qquad c_1 \to p \qquad p \to w}{(a_1 \vee b_1) \vee c_1 \to w}\,(\vee\mathrm{A}_m)$$

and $\Pi_S$ is the following derivation:

$$\dfrac{w \to r \qquad \dfrac{r \to a_2, s \qquad s \to b_2, c_2}{r \to a_2, (b_2 \vee c_2)}\,(\vee\mathrm{S}_m)}{w \to a_2 \vee (b_2 \vee c_2)}\,(\vee\mathrm{S}_m)$$

The explosion set of $\widetilde{\mathcal{S}}$ is given by:

$$\mathfrak{E}_{\widetilde{\mathcal{S}}} = \left\{ \begin{array}{l} [a_1 \to q], [b_1 \to q], [q \to p], [c_1 \to p], \\ [p \to w], [w \to r], [r \to a_2, s], [s \to b_2, c_2] \end{array} \right\}$$

The structure of the antecedent is given by the ESS $p \to w$ and that of the succedent by the ESS $w \to r$. The antecedent formula $(a_1 \vee b_1) \vee c_1$ is represented by the ESSs $q \to p$ and $c_1 \to p$ for the outer disjunction and the ESSs $a_1 \to q$ and $b_1 \to q$ for the inner disjunction, both of which are negative occurrences in $\mathcal{S}$. The succedent formula $a_2 \vee (b_2 \vee c_2)$ is represented by the ESSs $r \to a_2, s$ for the outer disjunction and $s \to b_2, c_2$ for the inner disjunction. Both of these disjunctions have positive occurrences in $\mathcal{S}$. Hence, it is $|\mathfrak{E}_{\widetilde{\mathcal{S}}}| = 2 \cdot \mathbf{D}^-_{\widetilde{\mathcal{S}}} + \mathbf{D}^+_{\widetilde{\mathcal{S}}} + 2 = 2 \cdot 2 + 2 + 2 = 8$.

As a second example, consider the explosion derivation of the sequent $\mathcal{T} = [\to b \vee \neg b]$. Its occurrence instance is $\widetilde{\mathcal{T}} = [\to b_1 \vee \neg b_2]$. For this, we obtain the following explosion derivation:

$$\dfrac{\to w \qquad \dfrac{w \to p \qquad \dfrac{p \to b_1, q \qquad b_2, q \to}{p \to b_1, \neg b_2}\,(\neg\mathrm{S}_m)}{w \to b_1 \vee \neg b_2}\,(\vee\mathrm{S}_m)}{\to b_1 \vee \neg b_2}\,(\mathrm{Prx})$$

The explosion set of $\widetilde{\mathcal{T}}$, taken off the leaves of the explosion derivation, is:

$$\mathfrak{E}_{\widetilde{\mathcal{T}}} = \{[\to w], [w \to p], [p \to b_2, q], [b_1, q \to]\}$$

Antecedent and succedent are represented by the two ESSs $\rightarrow w$ and $w \rightarrow p$. The formula $b_2 \vee \neg b$ is represented by the ESSs $p \rightarrow b_1, q$ for the disjunction and $b_1, q \rightarrow$ for the negation; the occurrence of the disjunction is positive in $\mathcal{T}$, hence, it is $|\mathfrak{E}_{\widetilde{\mathcal{T}}}| = \mathbf{D}^+_{\widetilde{\mathcal{T}}} + \mathbf{N}_{\widetilde{\mathcal{T}}} + 2 = 1 + 1 + 2 = 4$.

Judging by these examples, one might assume that, contrary to the claim, this approach obfuscates meaning instead of emphasising it. An occurrence instance of a RK-sequent is taken apart, and all the meaning is diluted into inobvious arrangements of a large number of elementary structural sequents via proxy variables. The main reason for the inaccessibility is the fact that we have not introduced a reasonable representation for explosion sets. We will delay this necessary task to the third part. Before, we will have a closer look at very small and, hence, easily seizable subcollections of ESSs and the question, how they relate to the structural meaning of the coarse structure of the sequent as a whole and individual occurrences of the complex formulae.

# Chapter 7

# Structural Representation of Meaning

The explosion procedure is desiged to generate the localised justifications for all the logical connectives that occur in an occurrence instance $\widetilde{\mathcal{S}}$. This results in a set of elementary structural sequents, the explosion set $\mathfrak{E}_{\widetilde{\mathcal{S}}}$, which has been shown to be unique up to the renaming of proxy variables. This collection of ESSs contains thereby all of the structural information that is syntactically expressed in the sequent itself.[1] Moreover, the structure of the explosion derivation, which consists of alternated applications of atomic cuts and local logical rules, is still retained by certain subcollections of ESSs. These facts provide a solid basis for the claim that $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ should be regarded as the meaning of $\widetilde{\mathcal{S}}$. This means that not the structure of any concrete derivation is important, be it in a logistic calculus, for example RK, or the explosion calculus, but the interlocking and connectedness of elementary structural sequents, which comprise the leaves of every explosion derivation. These sequents are the elementary structural constituents of $\widetilde{\mathcal{S}}$, and the explosion procedure extracts and reveals them.

The explosion calculus puts the focus on the fine details, at the cost of efficiency. Applications of (Prx), a restriction of the RK cut rule, introduce single variables in order to separate complex formulae from their context, logical rules are replaced by their local variants. The entire purpose of the explosion calculus is to expose the inner structure of the sequent $\widetilde{\mathcal{S}}$ to its fullest extent. The proxy variables, which are introduced in the process and occur in the ESSs of the explosion set, are the links, by which these elemen-

[1] However, bear in mind that the information about the idetity of atomic content is not expressed in the ESSs. It is retained in the restoration function, which is completely independent of the explosion set.

tary structural sequents are connected to one another. The names of the proxies, which are used to express these connections, are only circumstantial, of course, as was established in the preceding chapter. They are chosen ad-hoc, the only condition being that the same proxy variable does not occur at some other place in the derivation to avoid unjustified connections.

In this chapter, we shall demonstrate in some detail, how meaning of the sequent can be thought of as represented by particular elementary structural sequents of the explosion set.

## 7.1 The Base Structure of a Sequent

While certain configurations of ESSs in the explosion set of a sequent $\widetilde{\mathcal{S}}$ must represent the meaning of the logical connectives of the formulae occuring in $\widetilde{\mathcal{S}}$, others represent the meaning of the *base structure* of the sequent itself. By this we mean, roughly speaking, the following:

1) the fact that the antecedent is related to the succedent and vice versa and a means to access both;

2) the number and nature of the formulae occurring in antecedent and succedent, divided as follows:

    (a) the number of complex formulae occurring in the antecedent and the succedent and the means by which these formulae can be accessed,

    (b) all the proper occurrence variables of the antecedent and the succedent.

It has to be clarified what we mean by the term *access*. Within the traditional framework, the accessing of the formulae occurring in the antecedent or succedent of a sequent takes places implicitly by looking at the sequent. It is difficult to look at a sequent, e.g. $\rightarrow \neg b, a, b \vee (a \supset c)$, without explicitly accessing, that is, taking in and registering, the complex formulae occurring in it. It is more of an effort to *not* access those formulae while looking at the sequent than to do so. When we consider collections of ESSs, on the other hand, nothing is easily accessible in this sense. Of course, it is exactly the point of our course of action to demand that those tasks, which are performed implicitly in traditional proof-theoretic endeavours, have to be performed explicitly. In this sense, to access a sequent or formula means to focus on its occurrence and take in the information contained therein.

Let us properly formulate what has only been suggested informally so far. For this purpose, it is necessary to relate occurrences of complex formulae in an explosion derivation to proxy variables.

**Definition 7.1** *Let $\widetilde{\mathcal{S}}$ be an occurrence instance of a sequent and $\Pi_{\widetilde{\mathcal{S}}}$ be its explosion derivation. Moreover, let $\mathcal{F}_{\widetilde{\mathcal{S}}}$ be the set of all subformulae occurring in $\widetilde{\mathcal{S}}$ and $\mathcal{V}_{\Pi_{\widetilde{\mathcal{S}}}}$ be the set of occurrence variables in $\Pi_{\widetilde{\mathcal{S}}}$. The* occurrence mapping *of $\Pi_{\widetilde{\mathcal{S}}}$ is the function $\pi_{\widetilde{\mathcal{S}}} : \mathcal{F}_{\widetilde{\mathcal{S}}} \rightarrow \mathcal{V}_{\Pi_{\widetilde{\mathcal{S}}}}$, which is given as follows:*

1) *$\pi_{\widetilde{\mathcal{S}}}(a) = a$ for every proper occurrence variable.*

2) *$\pi_{\widetilde{\mathcal{S}}}(C) = p$, if $C$ is a complex formula and there is an application of a meta rule in $\Pi_{\widetilde{\mathcal{S}}}$, which has $C$ as main formula and $p$ as the new proxy variable that is introduced.*

This definition is sensible, because $\widetilde{\mathcal{S}}$ is an occurrence instance of a sequent, in which every subformula of $\widetilde{\mathcal{S}}$ occurs exactly once. Conversely, every proper occurrence variable occurs exactly once in $\widetilde{\mathcal{S}}$ and every proxy variable in $\Pi_{\widetilde{\mathcal{S}}}$, apart from the watershed proxy $w$, is newly introduced in the premises of the rule, by which some subformula of $\widetilde{\mathcal{S}}$ is decomposed. Hence, with the exception of $w$, every occurrence variable of $\Pi_{\widetilde{\mathcal{S}}}$ relates to exactly one subformula of $\widetilde{\mathcal{S}}$. Now, the intuitive notion of *accessing* a formula that was used above is formalised by the inverse $\pi^{-1}_{\widetilde{\mathcal{S}}}$ of an occurrence mapping $\pi_{\widetilde{\mathcal{S}}}$. Note, however, that no formula of of $\widetilde{\mathcal{S}}$ corresponds to the watershed proxy $w$. Since the inverse mapping $\pi^{-1}_{\widetilde{\mathcal{S}}}$ is a useful notion, this defect is remedied by considering the occurrence mapping $\pi_{\widetilde{\mathcal{S}}}$ over the lifted domain $\mathcal{F}^{\perp}_{\widetilde{\mathcal{S}}}$ with $\pi(\perp) = w$, which is a bijection. We will omit the index $\widetilde{\mathcal{S}}$, when the sequent is understood from the context. Moreover, an occurrence mapping is extended naturally to sequences of formulae, as they occur in antecedents and succedents of sequents.

**Proposition 7.2** *Let $\widetilde{\mathcal{S}} = [\Delta \rightarrow \Gamma]$ be an occurrence instance of a sequent and let $\Pi_{\widetilde{\mathcal{S}}}$ be its explosion derivation with occurrence mapping $\pi$, and let $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ be its explosion set with watershed proxy $w$. Then $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ contains ESSs $\gamma \rightarrow w$ and $w \rightarrow \delta$, such that $\pi(\Gamma) = \gamma$ and $\pi(\Delta) = \delta$.*

Proof: The derivation $\Pi_{\widetilde{\mathcal{S}}}$ contains the following initialisation step:

$$\dfrac{\begin{matrix}\Pi_A \\ \Gamma \rightarrow w\end{matrix} \qquad \begin{matrix}\Pi_S \\ w \rightarrow \Delta\end{matrix}}{\Gamma \rightarrow \Delta}\ (\text{Prx})$$

The initialisation divides antecedent and succedent of a sequent into two sequents by the introduction of the watershed proxy $w$. Both premises $\Gamma \rightarrow w$ and $w \rightarrow \Delta$ are bottommost sequents of subderivations $\Pi_A$ for the antecedent part of the explosion derivation and $\Pi_S$ for its succedent part.

Consider the right premiss of the initialisation step, $w \rightarrow \Delta$. The subderivation $\Pi_S$ contains a context branch with that sequent as a base, which makes $w$ the base proxy of this sequent. According to lemma 6.4, by $\mathbf{len}(\Delta)$ applications of succedent meta rules ($*\mathrm{S}_m$), the ESS $w \rightarrow \delta$ is obtained as the leaf of this context branch, where the succedent $\delta$ contains only occurrence variables, proper or proxy. This ESS is an element of the explosion set $\mathfrak{E}_{\widetilde{\mathcal{S}}}$. Now, the ESS $w \rightarrow \delta$ is the result of having replaced every complex formula that occured in $\Delta$ by a proxy variable. Each rule application replaces a complex formula $C$ of $\Delta$ by a proxy variable $\pi(C)$. Furthermore, proper occurrence variables in $\Delta$ remain unchanged throughout and also occur in $\delta$. Hence, we have $\pi(\Delta) = \delta$.

A corresponding argument for the context branch in $\Pi_A$, whose base is $\Gamma \rightarrow w$ yields that there is an ESS $\gamma \rightarrow w$ such that $\pi(\Gamma) = \gamma$. □

Since $w \rightarrow \Delta$ retains the entire succedent $\Delta$ of the original sequent after the initialization step, its variant $w \rightarrow \delta$ represents the base structure of $\Delta$ in the following sense:

1) The base proxy $w$ is an abstract representation of the sequent as a whole and, in this case, it is presented as relating to the succedent, possessing a succedent, if you will.

2) As for $\delta$, we can state the following:

   (a) Every branching proxy, which occurs in $\delta$, relates to one of the complex formulae occuring in $\Delta$, and hence represents the occurrence of that formula in $\Delta$. By means of the branching proxy, the corresponding complex formula can be accessed.

   (b) Every proper occurrence variable, which occurs in $\delta$, represents its own original occurrence in $\Delta$.

As an example, consider the occurrence instance of the sequent that was mentioned above: $\rightarrow \neg b_1, a_1, b_2 \vee (a_2 \supset c_1)$. An explosion derivation of this sequent is:

$$
\dfrac{\rightarrow w \qquad \dfrac{\dfrac{w \rightarrow p, a_1, q \qquad \dfrac{q \rightarrow b_2, r \qquad a_2, r \rightarrow c_1}{q \rightarrow b_2, (a_2 \supset c_1)}\,(\supset\mathrm{S}_m)}{w \rightarrow p, a_1, b_2 \vee (a_2 \supset c_1)}\,(\vee\mathrm{S}_m) \qquad b_1, p \rightarrow}{w \rightarrow \neg b_1, a_1, b_2 \vee (a_2 \supset c_1)}\,(\neg\mathrm{S}_m)}{\rightarrow \neg b_1, a_1, b_2 \vee (a_2 \supset c_1)}\,(\mathrm{Prx})
$$

The ESS $w \rightarrow p, a_1, q$ is the leaf of the context branch that has the sequent $w \rightarrow \neg b_1, a_1, b_2 \vee (a_2 \supset c_1)$ as its base. This elementary structural sequent $w \rightarrow p, a_1, q$ has the following properties: by its base proxy $w$, it refers to the antecedent, its branching proxies $p$ and $q$ relate to the complex formulae $\neg b_1$ and $b_2 \vee (a_2 \supset c_1)$, whereas the proper occurrence variable $a_1$ represents itself. This is a fine grained representaion of the role of the succedent in the original sequent $\rightarrow \neg b_1, a_1, b_2 \vee (a_2 \supset c_1)$; namely, that it is the second premiss of an occurrence sequent sequent, and that it provides a list of three formulae, the first and third of which are complex formulae, whereas the second one is immediately provided.

The sequent $w \rightarrow \delta$ is thus the unique ESS in $\mathfrak{E}_{\Pi_S}$, which represents the structure of the succedent. Note that there cannot be another sequent $w \rightarrow \alpha$ in $\mathfrak{E}_{\Pi_S}$, because $w$ is base proxy of the context branch, whose base is $w \rightarrow \Delta$. Accordingly, the sequent $\gamma \rightarrow w$, which is obtained from $\Gamma \rightarrow w$, is the unique ESS in $\mathfrak{E}_{\Pi_A}$, which represents the structure of the antecedent. These two ESSs make up the base structure of the sequent. They both relate to one another by means of the watershed proxy, which thereby uniquely identifies the sequent itself, $\gamma$ represents the locations of the complex formulae in the antecedent $\Gamma$ as well as the atomic occurrence variables occuring therein, and finally, $\delta$ fulfills the corresponding task for the succedent $\Delta$. Thus, the ESSs $\gamma \rightarrow w$ and $w \rightarrow \delta$ together represent the base structure of the sequent as a whole. Note that both $\gamma$ and $\delta$ can be empty lists, which results in ESSs $\rightarrow w$, as in the preceding example, or $w \rightarrow$.

## 7.2 The Structure of Complex Formulae

The next step is to explain, what it means to access a complex formula when we consider collections of ESSs. Clearly, formulae themselves, syntactic expressions, are no longer present in any such collection. Instead, when accessing a complex formula, we focus our attention to its structure. We understand as its structure the following items of information:

1) the unique identification of the occurrence of the formula,

2) the number and nature of the formulae occurring in antecedent and succedent, divided as follows:

(a) the number of immediate complex subformulae occurring in the formula and the means by which these formulae can be accessed,

(b) the proper occurrence variables that are immediate subformulae,

3) the specific mode, by which the immediate subformulae are referred to.

While items 1) and 2) are comparable to those we porposed for the base structure of a sequent, item 3) did not occur there. The reason for this is that there is no need to specify the mode, in which the abstract representation of the sequent, the watershed proxy, refers to its antecedent and succedent, because all the formulae that occur in them are integral to the sequent. The sequent expresses that all of the formulae that occur in the antecedent are to be jointly considered and that, equally, all of the formulae that occur in the succedent are to be jointly considered. This is not necessarily the case when we explode the structure of a complex formula. Recall that in the logical rules of RK, which have two premises, each premiss retains exactly one of the two side formulae. Hence, those formulae are independently relevant to the meaning of the corresponding connective. In logical rules having a single premiss, there is no such independent relevance. The side formulae are retained within a single sequent, and are given in a manner that is more immediately integral to the sequent.

What does this mean for the explosion procedure? All complex formulae are detached and treated locally in an explosion derivation, and all the immediate complex subformulae are detached in turn. The intended structural representation of the meaning of a complex formulae is provided by what is the logical premiss or premises of the meta rules. While a logical premiss of an instance of a meta rule is not necessarily already an ESS within an explosion derivation, Lemma 6.4 states, that such a premiss is the base of a context branch extending to a leaf, which is both a variant of this premiss and an elementary structural sequent. Hence, there is an ESS for every sequent which occurs as logical premiss anywhere within an explosion derivation, and it is the ESS or ESSs originating in the logical premises of a meta rule, which have to be considered as structural representations of the corresponding main formula $C$. This means that there are two cases to be distinguished, depending on the number of logical premises of the rule that treats $C$.

(i) If $C$ is decomposed by means of $(\&\mathrm{A}_m)$, $(\vee\mathrm{S}_m)$ or $(\supset\mathrm{S}_m)$, $(\neg\mathrm{A}_m)$ or $(\neg\mathrm{S}_m)$, its structural representation is provided by that ESS, which occurs as leaf in the context branch of the explosion derivation, extending from the logical premiss of the rule. In each of these cases, the occurrences of the immediate subformulae of $C$ become integral to the ESS, trivially so in the cases that $C$ is a negation.

(ii) If $C$ is decomposed by means of (&$\mathrm{S}_m$), ($\vee\mathrm{A}_m$), ($\supset\mathrm{A}_m$), its structural representation is provided by those two ESSs, which occur as the leaves in those context branches of the explosion derivation, whose bases are the two logical premises of that rule. In each of these cases, the occurrences of the two immediate subformulae of $C$ do *not* become integral to a single ESS. Instead, the occurrences are distributed into two independent ESSs.

Recall that each ESS in question is a variant of such a logical premiss, and, hence, the number of formulae occurring in its antecedent and succedent, exactly matches the number of formulae occurring in the logical premiss itself. The base proxy and proper occurrence variables are the same in the logical premiss and the ESS, the difference lies in the fact that every occurrence of a complex formula in the logical premiss is replaced by a branching proxy in the elementary structural sequent.

It is necessary to establish a relation between the polarity of the occurrence of a formula and the position this formula has as a main formula of a sequent in an explosion derivation.

**Lemma 7.3** *Let $\widetilde{\mathcal{S}}$ be an occurrence instance of a sequent and let $\Pi_{\widetilde{\mathcal{S}}}$ be its explosion derivation.*

1) *For every negative occurrence of a formula $C$ in $\widetilde{\mathcal{S}}$, $\Pi_{\widetilde{\mathcal{S}}}$ contains a sequent $\Gamma_1, C, \Gamma_2 \rightarrow \Delta$.*

2) *For every positive occurrence of a formula $C$ in $\widetilde{\mathcal{S}}$, $\Pi_{\widetilde{\mathcal{S}}}$ contains a sequent $\Gamma \rightarrow \Delta_1, C, \Delta_2$.*

PROOF: We have to establish 1) and 2) simultaneously. If $C$ is a formula occurrence in the antecedent or succedent of $\widetilde{\mathcal{S}}$, one of the statements holds for $C$. If $C$ is the immediate subformula of some formula occurrence $D$ in $\widetilde{\mathcal{S}}$, then we assume that one of the statements already holds for $D$. There are eight cases to distinguish for $D$, out of which we consider three.

- If $D = A \,\&\, B$ has a negative occurrence in $\widetilde{\mathcal{S}}$, then $\Pi_{\widetilde{\mathcal{S}}}$ contains the sequent $\Gamma_1, A \,\&\, B, \Gamma_2 \rightarrow \Delta$. But then, an application of (&$\mathrm{A}_m$) yields:

$$\frac{A, B \rightarrow p \qquad \Gamma_1, p, \Gamma_2 \rightarrow \Delta}{\Gamma_1, A \,\&\, B, \Gamma_2 \rightarrow \Delta}\,(\&\mathrm{A}_m)$$

Both $A$ and $B$ have negative occurrences in $A, B \to p$, and as $C = A$ or $C = B$, statement 1) holds for $C$.

- If $D = A \vee B$ has a negative occurrence in $\widetilde{\mathcal{S}}$, then $\Pi_{\widetilde{\mathcal{S}}}$ contains the sequent $\Gamma_1, A \vee B, \Gamma_2 \to \Delta$. But then, an application of $(\vee\mathrm{A}_m)$ yields:

$$\frac{A \to p \qquad B \to p \qquad \Gamma_1, p, \Gamma_2 \to \Delta}{\Gamma_1, A \vee B, \Gamma_2 \to \Delta}\ (\vee\mathrm{A}_m)$$

  Hence, $A$ has a negative occurrence in $A \to p$, and $B$ has a negative occurrence in $B \to p$ Since $C = A$ or $C = B$, statement 1) holds for $C$.

- If $D = A \supset B$ has a positive occurrence in $\widetilde{\mathcal{S}}$, then $\Pi_{\widetilde{\mathcal{S}}}$ contains the sequent $\Gamma \to \Delta_1, A \supset B, \Delta_2$. But then, an application of $(\supset\mathrm{A}_m)$ yields:

$$\frac{\Gamma \to \Delta_1, p, \Delta_2 \qquad A, p \to B}{\Gamma \to \Delta_1, A \supset B, \Delta_2}\ (\supset\mathrm{A}_m)$$

  Both $A$ has a negative occurrence in $A, p \to B$, and $B$ has a positive occurrence therein. Either then statement 1) holds for $C = A$, or statement 2) holds for $C = B$.

□

Having established all the necessary tools, we can easily relate elementary structural sequents of an explosion derivation to subformulae of the of the original occurrence instance.

**Proposition 7.4** *Let $\widetilde{\mathcal{S}}$ be an occurrence instance of a sequent and let $\Pi_{\widetilde{\mathcal{S}}}$ be its explosion derivation with occurrence mapping $\pi$, and let $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ be its explosion set. Then the following holds:*

*1) For every negative occurrence of a subformula $A \,\&\, B$ in $\widetilde{\mathcal{S}}$, $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ contains a unique ESS $\pi(A), \pi(B) \to \pi(A \,\&\, B)$.*

*2) For every positive occurrence of a subformula $A \,\&\, B$ in $\widetilde{\mathcal{S}}$, $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ contains two unique ESSs $\pi(A \,\&\, B) \to \pi(A)$ and $\pi(A \,\&\, B) \to \pi(B)$.*

Proof:

1) If $A \,\&\, B$ has a negative occurrence in $\widetilde{\mathcal{S}}$, then somewhere in $\Pi_{\widetilde{\mathcal{S}}}$ there is a sequent $\Gamma_1, A \,\&\, B, \Gamma_2 \to \Delta$, which is the conclusion of $(\&\mathrm{A}_m)$, i.e. $\Pi_{\widetilde{\mathcal{S}}}$ contains the following subderivation:

$$\dfrac{\begin{matrix}\Pi_1 & \Pi_2 \\ A, B \rightarrow p & \Gamma_1, p, \Gamma_2 \rightarrow \Delta\end{matrix}}{\Gamma_1, A \,\&\, B, \Gamma_2 \rightarrow \Delta}\;(\&\mathrm{A}_m)$$

The proxy variable $p$ represents the occurrence of $A \,\&\, B$ in the context sequent. Following definition 7.1, it is $\pi(A \,\&\, B) = p$. The logical premiss $A, B \rightarrow p$ may or may not be the conclusion of other rules applications in $\Pi_{\widetilde{\mathcal{S}}}$. There are four cases to distinguish:

(a) Both $A$ and $B$ are already proper occurrence variables. Hence, it is both $\pi(A) = \pi(a_i) = a_i$ for some index $i$ and $\pi(B) = \pi(b_j) = b_j$ for some $j$, and the logical premiss of $(\&\mathrm{A}_m)$ can be written as $\pi(A), \pi(B) \rightarrow \pi(A \,\&\, B)$, which is an ESS. Here, $\Pi_1$ is empty.

(b) Only $B$ is already a proper occurrence variable and $A$ is a complex formula. In this case, the logical premiss is not yet of the desired form. At this point we only have $A, \pi(B) \rightarrow \pi(A \,\&\, B)$. However, since $A$ is the only complex formula in this sequent, it must be the conclusion of a rule application in $\Pi_{\widetilde{\mathcal{S}}}$, which introduces the main connective of $A$. If the corresponding meta rule has a single logical premiss, the subderivation $\Pi_1$ has the following form:

$$\dfrac{\begin{matrix}\Pi_1' & \\ \Theta \rightarrow q, \Xi & q, b_j \rightarrow p\end{matrix}}{A, b_j \rightarrow p}\;(\star\mathrm{A}_m)$$

Thus we have $\pi(A) = q$, and the context premiss of this rule application has the desired form $\pi(A), \pi(B) \rightarrow \pi(A \,\&\, B)$.

If the corresponding meta rule has two logical premises, the subderivation $\Pi_1$ has a different form, but the context premiss is identical to the case with one logical premiss:

$$\dfrac{\begin{matrix}\Pi_1' & \Pi_1'' & \\ \Theta_1 \rightarrow q, \Xi_1 & \Theta_2 \rightarrow q, \Xi_2 & q, b_j \rightarrow p\end{matrix}}{A, b_j \rightarrow p}\;(\star\mathrm{A}_m)$$

(c) The case that $A$ is already a proper occurrence variable and $B$ is a complex formula is treated correspondingly to the previous case.

(d) If both $A$ and $B$ are complex formulae, then both formulae are detached from $A, B \rightarrow p$ in $\Pi_{\widetilde{\mathcal{S}}}$, which requires exactly two successive rule applications. These must occur in one out of two possible configurations.[2]

$$\dfrac{\dots \quad \dfrac{\dots \quad q, r \to p}{q, B \to p}\,(\star_2 \mathrm{A}_m)}{A, B \to p}\,(\star_1 \mathrm{A}_m)$$

$$\dfrac{\dots \quad \dfrac{\dots \quad q, r \to p}{A, r \to p}\,(\star_1 \mathrm{A}_m)}{A, B \to p}\,(\star_2 \mathrm{A}_m)$$

In both cases the leaf of the context branch beginning with the logical premiss $A, B \to p$ is the ESS $q, r \to p$. With $\pi(A) = q$ and $\pi(B) = r$, this ESS is of the form $\pi(A), \pi(B) \to \pi(A \,\&\, B)$, as required.

This concludes the four possible cases arising from the dependent possibilities for the complexity of $A$ and $B$.

2) If $A \,\&\, B$ has a positive occurrence in $\widetilde{\mathcal{S}}$, then somewhere in $\Pi_{\widetilde{\mathcal{S}}}$ there is a sequent $\Gamma \to \Delta_1, A \,\&\, B, \Delta_2$, which is the conclusion of ($\&\mathrm{S}_m$).

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \to \Delta_1, p, \Delta_2\end{array} \qquad \begin{array}{c}\Pi_2\\ p \to A\end{array} \qquad \begin{array}{c}\Pi_3\\ p \to B\end{array}}{\Gamma \to \Delta_1, A \,\&\, B, \Delta_2}\,(\&\mathrm{S}_m)$$

As before, it is $\pi(A\&B) = p$. In this case, however, two logical premises $p \to A$ and $p \to B$ can be considered independently of one another. We treat the sequent $p \to A$ exemplarily and consider two cases:

(a) The formula $A$ is a proper occurrence variable $a_i$ for some index $i$. Then the sequent is the ESS $p \to a_i$. With $\pi(A) = \pi(a_i) = a_i$, the ESS has the desired form $\pi(A \,\&\, B) \to \pi(A)$. The subderivation $\Pi_2$ is empty in this case.

(b) The formula $A$ is a complex formula. At the same time, $A$ is the only complex formula occurring in $p \to A$. Hence, this sequent must be the conclusion of the rule introducing $A$. Depending on the connective of $A$, this rule is either

[2]Depending on $A$ and $B$, each rule application can have either one or two logical premises. As we saw in (b), the number of logical premises does not influence the argument. Therefore, we will only give the relevant context premises.

$$\dfrac{p \rightarrow q \qquad \begin{matrix}\Pi_2' \\ \Theta, q \rightarrow \Xi\end{matrix}}{p \rightarrow A}\,(\star\mathrm{S}_m)$$

or

$$\dfrac{p \rightarrow q \qquad \begin{matrix}\Pi_2' \\ \Theta_1, q \rightarrow \Xi_1\end{matrix} \qquad \begin{matrix}\Pi_2'' \\ \Theta_2, q \rightarrow \Xi_2\end{matrix}}{p \rightarrow A}\,(\star\mathrm{S}_m)$$

In either case, the context premiss is $p \rightarrow q$, an ESS. Due to this rule application, which introduces $q$, we further have $\pi(A) = q$. Hence, there is an ESS $\pi(A \,\&\, B) \rightarrow \pi(A)$ contained in $\mathfrak{E}_{\widetilde{S}}$.

Independently, we obtain from the second logical premiss $p \rightarrow B$ that $\pi(A \,\&\, B) \rightarrow \pi(B)$ is contained in $\mathfrak{E}_{\widetilde{S}}$. Since each of $\pi(A)$ and $\pi(B)$ is either a unique proper occurrence variable or a new proxy variable, the two ESSs are distinct.

□

Note that the argument depends on the fact that we use occurrence instances of sequents. Consider the case of a positive occurrence of $A \,\&\, B$, in which both $A$ and $B$ are proper occurrence variables, say $a_i$ and $a_{i+1}$. In this case, the meta rule $(\&\mathrm{S}_m)$ yields:

$$\dfrac{\Gamma \rightarrow \Delta_1, p, \Delta_2 \qquad p \rightarrow a_i \qquad p \rightarrow a_{i+1}}{\Gamma \rightarrow \Delta_1, a_i \,\&\, a_{i+1}, \Delta_2}\,(\&\mathrm{S}_m)$$

The logical premises of $(\&\mathrm{S}_m)$ are the elementary structural sequents $p \rightarrow a_i$ and $p \rightarrow a_{i+1}$. If we were to use propositional variables instead of proper occurrence variables, the two logical premises of $(\&\mathrm{S}_m)$ would be, $p \rightarrow a$ and $p \rightarrow a$, which are identical. Hence, they would collapse in the explosion set, violating the desired property that $\mathfrak{E}_{\widetilde{S}}$ contains two unique ESSs. This violation would pose a serious problem to the claim that the explosion set represents the structural skeleton of a sequent, because these most simple conjunctions could not be adequately represented.[3] This problem would not arise in the case of a more complex conjunctive formula, in which the propositional variable does occur twice, but not as immediate subformulae of a conjunction. For instance, for the sequent $p \rightarrow a \,\&\, (a \,\&\, b)$ the explosion procedure produces the four ESSs $[p \rightarrow a]$, $[p \rightarrow q]$, $[q \rightarrow a]$ and $[q \rightarrow b]$, each of which is unique.

[3]The dual problem would arise for disjunctive formulae $a \vee a$ occurring in the antecedent.

However, it was not merely this technical problem, which motivated the utilisation of occurrence variables. Instead, the intention was to effect a clear separation between purely structural considerations and those of the identity of atomic content. This example demonstrates that the former can only be properly accomodated by using occurrence instances of sequents. The latter is treated by means of the corresponding restoration functions, as we will see shortly in the chapter on the decision procedure.

Similar properties to those stated in proposition 7.4 hold for the remaining logical connectives.

**Corollary 7.5** *Let $\widetilde{\mathcal{S}}$, $\Pi_{\widetilde{\mathcal{S}}}$, $\pi$ and $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ be as before.Then the following holds:*

1) *For every negative occurrence of a subformula $A \vee B$ in $\widetilde{\mathcal{S}}$, $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ contains two unique ESSs $\pi(A) \rightarrow \pi(A \vee B)$ and $\pi(B) \rightarrow \pi(A \vee B)$.*

2) *For every positive occurrence of a subformula $A \vee B$ in $\widetilde{\mathcal{S}}$, $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ contains a unique ESS $\pi(A \vee B) \rightarrow \pi(A), \pi(B)$.*

**Corollary 7.6** *Let $\widetilde{\mathcal{S}}$, $\Pi_{\widetilde{\mathcal{S}}}$, $\pi$ and $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ be as before. Then the following holds:*

1) *For every negative occurrence of a subformula $A \supset B$ in $\widetilde{\mathcal{S}}$, $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ contains two unique ESSs $\rightarrow \pi(A \supset B), \pi(A)$ and $\pi(B) \rightarrow \pi(A \supset B)$.*

2) *For every positive occurrence of a subformula $A \supset B$ in $\widetilde{\mathcal{S}}$, $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ contains a unique ESS $\pi(A), \pi(A \supset B) \rightarrow \pi(B)$.*

**Corollary 7.7** *Let $\widetilde{\mathcal{S}}$, $\Pi_{\widetilde{\mathcal{S}}}$, $\pi$ and $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ be as before. Then the following holds:*

1) *For every negative occurrence of a subformula $\neg A$ in $\widetilde{\mathcal{S}}$, $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ contains a unique ESS $\rightarrow \pi(\neg A), \pi(A)$.*

2) *For every positive occurrence of a subformula $\neg A$ in $\widetilde{\mathcal{S}}$, $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ contains a unique ESS $\pi(A), \pi(\neg A) \rightarrow$.*

The proofs of corollaries 7.5 and 7.6 are dual to those of proposition 7.4. For corollary 7.7, there is just a single subformula to consider in both cases, which are dual to one another.

Figure 7.1 gives a summary of the structural configurations that have been detailed in the preceding lemma and corollaries for complex formulae $C$. In every instance, $\pi(C)$ is the base proxy, which was introduced to detach the complex formula. The variables $\pi(A)$ and, if appropriate, $\pi(B)$ are either

| | negative occurrence | positive occurrence |
|---|---|---|
| $\&$ | $\{\, \pi(A), \pi(B) \to \pi(A \,\&\, B) \,\}$ | $\left\{ \begin{array}{c} \pi(A \,\&\, B) \to \pi(A) \\ \pi(A \,\&\, B) \to \pi(B) \end{array} \right\}$ |
| $\vee$ | $\left\{ \begin{array}{c} \pi(A) \to \pi(A \vee B) \\ \pi(B) \to \pi(A \vee B) \end{array} \right\}$ | $\{\, \pi(A \vee B) \to \pi(A), \pi(B) \,\}$ |
| $\supset$ | $\left\{ \begin{array}{c} \to \pi(A \supset B), \pi(A) \\ \pi(B) \to \pi(A \supset B) \end{array} \right\}$ | $\{\, \pi(A), \pi(A \supset B) \to \pi(B) \,\}$ |
| $\neg$ | $\{ \to \pi(\neg A), \pi(A) \,\}$ | $\{\, \pi(A), \pi(\neg A) \to \,\}$ |

Figure 7.1: The ESS or ESSs representing the meaning of a connective

proper occurrence variables or proxy variables, depending on whether the side formula or side formulae are already proper occurrence variables or, in turn, complex formulae. What is important are the different distributions of base proxy and the other two occurrence variables into antecedent and succedent of the single ESS or the two ESSs in question. The pattern of this distribution is unique for each type of occurrence, positive or negative, of each logical connective.

In order to relate the intuitive analysis that introduced this section to the technical results we obtained, we can state that the structure of a complex formula $C$ occurring somewhere within a sequent $\widetilde{\mathcal{S}}$ is given by a collection of ESSs $\mathfrak{C}_C \subset \mathfrak{C}_{\widetilde{\mathcal{S}}}$, which contains the following items of information:

1) The base proxy $\pi(C)$ occurring in all of the ESSs $\mathfrak{C}_C$ is an abstract representation of the logical connective of the complex formula. The same $\pi(C)$ occurs as a branching proxy in some context branch of the explosion derivation.

2) The following holds for the occurrence variables $\pi(A)$ and, if appropriate[4], $\pi(B)$ occurring in $\mathfrak{C}_C$:

    (a) Every branching proxy, which occurs in $\mathfrak{C}_C$, refers to a complex subformula of $C$, and hence represents the occurrence of that subformula in $C$. By means of this branching proxy, the corresponding complex formula can be accessed.

    (b) Every proper occurrence variable, which can be found in ESSs in $\mathfrak{C}_C$, represents its own original occurrence in $C$.

[4]If $C$ is a negation formula, there is only one occurrence variable $\pi(A)$.

3) The number of ESSs that comprise $\mathfrak{C}_C$ reflects the mode, by which the subformulae are accessed. In the case that there is only a single ESS, which contains the base proxy and both branching proxies, we can think of an *internal* mode of access. This internal mode corresponds to the observation made for the ESSs representing the antecedent and succedent of the sequent itself in the preceding section. In the case that there are two ESS, each of which contains exactly one of the occurrence variables $\pi(A)$ and $\pi(B)$, we can think of an *external* or independent mode of access.

For example, consider the case of a positive occurrence of formulae $A \mathbin{\&} B$. Then, according to proposition 7.4, $A \mathbin{\&} B$ is the main formula of some application of ($\&\mathrm{S}_m$), and the structural meaning of the connective is given by the collection of sequents $\{[\pi(A \mathbin{\&} B) \to \pi(A)], [\pi(A \mathbin{\&} B) \to \pi(B)]\}$, which, for $\pi(A \mathbin{\&} B) = p$ and $\pi(A) = x$ and $\pi(B) = y$, is the set of ESSs $\{[p \to x], [p \to y]\}$. In both of these sequents and only in those, $p$ occurs as base proxy. By inspecting the collection of ESSs that have $x$ or $y$ as base proxies, the structural meaning of the subformulae $A$ or $B$ can be obtained. In the case that $x$ or $y$ is already an occurrence variable, there is no further structural meaning to the corresponding subformula. Instead, $p$ refers to a unique and atomic occurrence of a propositional variable, say $a$ or $b$, which is represented by a unique occurrence variable $a_i$ or $b_j$. Dually, we obtain as meaning of a negative occurrence of $A \vee B$ the set of ESSs $\{[x \to p], [y \to p]\}$ and as the meaning of a negative occurrence of $A \supset B$, as it is to be expected in the classical setting, the set of ESSs $\{[\to p, x], [y \to p]\}$. As above, $x$ and $y$ refer to the structural meaning of formulae $A$ and $B$ or are proper occurrence variables. Note that in all three cases it is two elementary structural sequents which constitute the structural meaning of the specific position of the connective of the complex formula. In any one of these cases, $x$ and $y$ are accessed from $p$ independently of one another. In other words, when accessing $p$ itself and retrieving the structural representation of the complex formula, whose occurrence the proxy $p$ represents, two ESSs are retrieved. At this point, a choice has to be made, which one of the subformulae, $A$ or $B$, should be further accessed by means of $\pi(A)$ or $\pi(B)$. Every access of either one of them has to be performed strictly independently of the other. In all of the other cases, it is always a single ESS, which constitutes the meaning of the corresponding connective. Within that ESS, all the relevant occurrence variables are distributed in some configuration, which is specific to the particular logical connective and its occurrence position in question. When accessing the base proxy of that elementary structural sequent, all of the occurrence variables are retrieved at the same time. For a negation formula, this is trivially the case, since this involves only a single occurrence variable apart from the base proxy, but in the other cases, both of the occurrence variables $\pi(A)$ and $\pi(B)$ are integral to the single ESS that constitutes the structural meaning of the complex formula. This requres that whenever $A$ or $B$ is accessed by means of the corresponding occurrence variable, the other occurrence variable is not deposited in the process, but retained for a possible future access. These two different modes of access will be studied in detail in the following chapter.

We conclude that the ESSs resulting from the premises of some meta rule do indeed represent the structural justification, and thereby the meaning, of the connective of the main formula of the corresponding rule application. The proof theoretic paradigm that the meaning of a logical connective is given by the structure of the rule, which is used to introduce it, is retained in a certain sense, albeit with a significant modification. In our teatment, the stripped-down elementary variants of the premises of the *local* logical rules, which might occur further up in a context branch starting at these premises, are considered as structural meaning. Thereby, we consider as meaning only that absolutely minimal part of the structure of a derivation that has a bearing on the connective in question. Contexts have no bearing on the meaning of a connective whatsoever, neither in some kind of structural representation let alone in the form of the formulae themselves, and consequently they do not contribute in any way to the ESSs that make up the structural meaning. Even the immediate subformulae of a complex formula make no immediate contribution to the meaning of the connective, and hence they do not appear explicitly in the ESSs. Only by means of the branching proxies or proper occurrence variables is its structural meaning related to that of the logical connectives or propositional variables, to which it is connected.[5] Hence, the meaning of a logical connective is represented entirely through a (often singleton) collection of *elementary structural sequents*. In contrast to this, the meaning of a logical connective is traditionally represented by a particular substructure of the derivation itself, namely that part consisting of the rule that introduces said connective, that rule's premises and conclusion. Thereby the meaning of a logical connective is unneccesarily syntactically emcumbered with the subformulae it connects, the sequents' formula contexts, and the whole context of the derivation. None of this is the case in our approach. The meaning of a logical connective can be considered independently of such distractions.

[5]The propositional variables themselves are not part of the structural meaning, but their occurrence instances are. Only by an application of the restoration function can the propositional variables themselves be considered.

# Chapter 8

# Explosion Sets and RK-Derivations

After having investigated the properties of the explosion procedure and the resulting explosion set in the previous chapters, it is now necessary to establish a relation between these results and derivations and proofs in RK. This will be done in several steps.

The first step is to facilitate such a relation by supplying a stepping stone between RK-derivations of regular sequents, i.e. sequents that do not contain any occurrence variables, and explosion sets, which are derived from occurrence instances of sequents. This will be accomplished by generalising the notion of occurrence instance from sequents to derivations.

The second step is to establish how the various sequents that occur in an occurrence instance of a RK-derivation can be related to particular subsets of the explosion set, which is derived from the base sequent of that derivation. These subsets will be called *connexion sets*, as they possess a particular connection property. It will be shown, how successive sequents occurring in some branch in a RK-derivation form a linearly ordered family of such connexion sets.

In the third step we will demonstrate, how the different explosion sets that can be obtained from the sequents of an occurrence instance of a RK-derivation relate to one another. That is, instead of relating every sequent to some subset of the explosion set of the end sequent of the derivation as in the second step, one explosion set will be developed for every sequent of the derivation. This perspective will prove to be complementary to that established in the second step in a certain sense.

Eventually, these developments will allow us to ascertain, in which manner certain connections that can be traced between occurrence variables of an explosion set can be related to the generalised axioms of a RK-derivation.

These connections will become especially interesting in view of the decision procedures in the following chapter.

## 8.1 Revisiting RK-Derivations

Explosion sets are derived from occurrence instances of sequents and therefore contain occurrence variables instead of propositional variables. For this reason, it would be very difficult to immediately relate a selection of ESSs taken from the explosion set to some sequent occurring in a RK-derivation of a standard sequent. In order to mediate the relation of explosion sets to RK-derivations, we will have to consider RK-derivations of occurrence instances of sequents instead of the derivations of the standard sequents.

**Definition 8.1** *For a RK-derivation $\Xi$ of a sequent $\mathcal{S}$, its* occurrence instance $\widetilde{\widetilde{\Xi}}$ *is is obtained from $\Xi$ by replacing every sequent occurring in $\Xi$ by the appropriate occurrence instance thereof and removing all applications of* (ax*). *The appropriate replacement is obtained by first replacing $\mathcal{S}$ by an occurrence instance $\widetilde{\mathcal{S}}$ and in the remaining replacements propagating the occurrence variables introduced therein upwards in such a manner that the rules of $\Xi$ are respected. In the context of occurence instances of RK-derivations, a RK-derivation, which does not contain any occurrence variables, is called a* standard RK-derivation*.*

An occurrence instance of a derivation $\Xi$ is best obtained by traversing the derivation from bottom-up and constructing a derivation for $\widetilde{\mathcal{S}}$.[1]

The claim that $\widetilde{\widetilde{\Xi}}$ is a RK-derivation has to be established formally. Moreover, it is easy to show that no occurrence instance of any RK-derivation can be a closed derivation.

**Lemma 8.2** *For a RK-derivation $\Xi$ of a sequent $\mathcal{S}$ and any occurrence instance $\widetilde{\widetilde{\Xi}}$ thereof the following hold:*

*1) $\widetilde{\widetilde{\Xi}}$ is a RK-derivation of the sequent $\widetilde{\mathcal{S}}$.*

*2) $\widetilde{\widetilde{\Xi}}$ is no RK-proof of the sequent $\widetilde{\mathcal{S}}$.*

[1] If $\widetilde{\widetilde{\Xi}}$ is to be constructed in a top-down manner instead, all initial sequents of $\Xi$ have to be replaced by occurrence instances of these initial sequents, in which every occurrence variable not only occurs uniquely in its own initial sequent, but uniquely in all of the occurrence instances of these initial sequents. If this provision is not made, then the construction could result in a derivation, in which the conclusion is not an occurrence instance of $\mathcal{S}$, because this conclusion contains one or more occurrence variables more than once.

Proof: For 1), consider the construction of $\widetilde{\Xi}$. The conclusion of $\Xi$, the sequent $S$, is replaced by an occurrence instance $\widetilde{\mathcal{S}}$. This is the initialisation of the construction of $\widetilde{\Xi}$. If a sequent $\mathcal{T}$ occurring as conclusion of a rule in $\Xi$ is replaced by $\widetilde{\mathcal{T}}$, then the premiss or premises of this rule are replaced as follows:

- If the rule is (&A), has conclusion $\mathcal{T} = [\Gamma_1, A \,\&\, B, \Gamma_2 \to \Delta]$ and premiss $\mathcal{U} = [\Gamma_1, A, B, \Gamma_2 \to \Delta]$, then T has been replaced in $\widetilde{\Xi}$ by the occurrence instance $\widetilde{\mathcal{T}} = [\Gamma_1, \widetilde{A \,\&\, B}, \Gamma_2 \to \Delta]$. Since $\widetilde{T}$ is an occurrence instance of $\mathcal{T}$, all occurrence variables thereof are unique and independent of one another, especially those occurring in $A \,\&\, B$. Hence, all occurence variables of $A$ and $B$ are also unique and independent of one another and of all occurrence variables of $\Gamma_1, \Gamma_2$ and $\Delta$, and therefore $\widetilde{\mathcal{U}} = [\Gamma_1, \widetilde{A, B}, \Gamma_2 \to \Delta]$ is indeed an occurrence instance of the premiss $\mathcal{U}$. Moreover, it is premiss of (&A) with regard to the conclusion $\widetilde{\mathcal{T}}$. Consequently, $\mathcal{U}$ is replaced by $\widetilde{\mathcal{U}}$ in $\widetilde{\Xi}$.

- If the rule is (&S), having conclusion $\mathcal{T} = [\Gamma \to \Delta_1, A \,\&\, B, \Delta_2]$ and premises $\mathcal{U}_1 = [\Gamma \to \Delta_1, A, \Delta_2]$ and $\mathcal{U}_2 = [\Gamma \to \Delta_1, B, \Delta_2]$, then T has been replaced in $\widetilde{\Xi}$ by the occurrence instance $\widetilde{\mathcal{T}} = [\Gamma \to \Delta_1, \widetilde{A \,\&\, B}, \Delta_2]$. Since $\widetilde{T}$ is an occurrence instance of $\mathcal{T}$, all occurrence variables thereof are unique and independent of one another, especially those in $A \,\&\, B$. Hence, all occurence variables of $A$ and $B$ are also unique and independent of one another and of all occurrence variables of $\Gamma_1, \Gamma_2$ and $\Delta$, and therefore $\widetilde{\mathcal{U}}_1 = [\Gamma \to \Delta_1, \widetilde{A}, \Delta_2]$ is indeed an occurrence instance of the premiss $\mathcal{U}_1$, and $\widetilde{\mathcal{U}}_2 = [\Gamma \to \Delta_1, \widetilde{B}, \Delta_2]$ is an occurrence instance of $\mathcal{U}_2$. Moreover, $\mathcal{U}_1$ and $\mathcal{U}_2$ are the premises of (&S) with regard to the conclusion $\widetilde{\mathcal{T}}$. Consequently, $\mathcal{U}_1$ is replaced by $\widetilde{\mathcal{U}}_1$ in $\widetilde{\Xi}$, and $\mathcal{U}_2$ is replaced by $\widetilde{\mathcal{U}}_2$.

The remaining logical rules as well as (XA) and (XS) are treated accordingly. For the case that the top of an open branch is encountered, nothing is added to $\widetilde{\Xi}$, and the construction, as far as that branch is concerned, terminates. Since all of the rules of $\Xi$, with the exception of the improper rule (ax*), which only occurs at the leaves of $\Xi$, are respected in the construction of $\widetilde{\Xi}$, it follows that $\widetilde{\Xi}$ is indeed a RK-derivation of the sequent $\widetilde{\mathcal{S}}$.

For 2), we only need to observe that, according to the construction, all of its initial sequents are occurrence instances of the initial sequents of $\Xi$. Since all atomic formulae of any occurrence instance occur uniquely, no formula can occur both in the antecedent and the succedent. Hence, no initial sequent of $\widetilde{\Xi}$ can be an instance of the axiom schema, and therefore $\widetilde{\Xi}$ cannot be a proof. □

As an example for the construction of occurrence instances of RK-derivations, consider the following RK-proof $\Xi$ of the sequent $(a \vee b) \vee c \rightarrow a \vee (b \vee c)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \rightarrow a, b, c}\,(\text{ax}^*)}{a \rightarrow a, b \vee c}\,(\vee\text{S}) \qquad \dfrac{\dfrac{}{b \rightarrow a, b, c}\,(\text{ax}^*)}{b \rightarrow a, b \vee c}\,(\vee\text{S})}{a \vee b \rightarrow a, b \vee c}\,(\vee\text{A}) \qquad \dfrac{\dfrac{}{c \rightarrow a, b, c}\,(\text{ax}^*)}{c \rightarrow a, b \vee c}\,(\vee\text{S})}{(a \vee b) \vee c \rightarrow a, b \vee c}\,(\vee\text{A})}{(a \vee b) \vee c \rightarrow a \vee (b \vee c)}\,(\vee\text{S})$$

Its occurrence instance $\widetilde{\Xi}$ is the RK-derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{a_1 \rightarrow a_2, b_2, c_2}{a_1 \rightarrow a_2, b_2 \vee c_2}\,(\vee\text{S}) \qquad \dfrac{b_1 \rightarrow a_2, b_2, c_2}{b_1 \rightarrow a_2, b_2 \vee c_2}\,(\vee\text{S})}{a_1 \vee b_1 \rightarrow a_2, b_2 \vee c_2}\,(\vee\text{A}) \qquad \dfrac{c_1 \rightarrow a_2, b_2, c_2}{c_1 \rightarrow a_2, b_2 \vee c_2}\,(\vee\text{S})}{(a_1 \vee b_1) \vee c_1 \rightarrow a_2, b_2 \vee c_2}\,(\vee\text{A})}{(a_1 \vee b_1) \vee c_1 \rightarrow a_2 \vee (b_2 \vee c_2)}\,(\vee\text{S})$$

All the sequents in $\widetilde{\Xi}$ are occurrence instances of their corresponding sequents in $\Xi$. What is more interesting is the fact that while $\Xi$ is a proof, $\widetilde{\Xi}$ is not. Strictly speaking, this makes $\widetilde{\Xi}$ structurally different from $\Xi$. However, due to the fact that this structural difference only concerns the closure of the leaves, this does not pose a problem. Recall that at this point, we are not concerned with whether a derivation is in fact a proof, but only with providing a method relating the sequents that occur in a RK-derivation to subsets of the explosion set of its occurrence instance. For this purpose, it is sufficient to relate all the sequents to their respective occurrence instances.

We are now in the position of being able to provide the occurrence instance $\widetilde{\mathcal{T}}$ of a sequent $\mathcal{T}$ that occur anywhere within a standard RK-derivation of a sequent $\mathcal{S}$, in which the occurrence variables are not simply introduced ad-hoc, but by refering to the corresponding sequent in the occurrence instance of the original derivation. Thereby, the occurrence variables of $\widetilde{\mathcal{T}}$ correspond to those in $\widetilde{\mathcal{S}}$.

In any branch of a RK-derivation (standard or occurrence instance), sequents further up are made up of certain subformulae of those formulae that comprise the sequents occuring further down in the branch. Although this property is obvious, it is worth stating it in a lemma, because it is very important for what we will develop in the remainder section.

**Lemma 8.3** *Let $\Xi$ be a RK-derivation and $(\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_k)$ some branch in $\Xi$. Let $\mathcal{F}_{\mathcal{S}_i}$ be the set of all subformulae of $\mathcal{S}_i$. Then, for every $0 \leq i < j \leq k$, it is $\mathcal{F}_{\mathcal{S}_i} \supseteq \mathcal{F}_{\mathcal{S}_j}$.*

Proof: For every pair $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$, the latter is the premiss of some rule application, which has the former as a conclusion. Since $\Xi$ is a RK-derivation, we have to consider exchange rules and logical rules. If the rule is an exchange rule, then $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$ contain exactly the same formulae, i.e. $\mathcal{F}_{\mathcal{S}_i} = \mathcal{F}_{\mathcal{S}_{i+1}}$. If the rule is an exchange rule, then $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$ contain exactly the same context formulae; however, $\mathcal{S}_i$ contains the main formula $C$ of the rule application, whereas all we can generally assert about $\mathcal{S}_{i+1}$ is that, depending on the rule, it contains one or all of its side formulae. Hence, we have $\mathcal{F}_{\mathcal{S}_i} \supseteq \mathcal{F}_{\mathcal{S}_{i+1}}$. Transitivity of $\supseteq$ yields the desired result. □

Note that we require $\supseteq$ instead of $\supset$ for two reasons. If $\mathcal{S}_i$ is the conclusion of one of the rules (XA) and (XS), we have $\mathcal{F}_{\mathcal{S}_i} = \mathcal{F}_{\mathcal{S}_{i+1}}$. The second reason is less obvious. For if $\mathcal{S}_i$ is the conclusion of a logical rule, we might expect $\mathcal{F}_{\mathcal{S}_i} \supset \mathcal{F}_{\mathcal{S}_{i+1}}$. But this only holds, if the main formula does not occur as (subformula of) any context formula. However, for occurrence instances of RK-derivations, we can state the following lemma:

**Lemma 8.4** *Let $\widetilde{\Xi}$ be an occurrence instance of a RK-derivation $\Xi$ and $\widetilde{\mathcal{T}}$ be the conclusion and $\widetilde{\mathcal{U}}$ be a premiss of some logical rule in $\widetilde{\Xi}$, which has $C$ as main formula. Let $\mathcal{F}_{\widetilde{\mathcal{T}}}$ be the set of all subformulae of $\widetilde{\mathcal{T}}$ and $\mathcal{F}_{\widetilde{\mathcal{U}}}$ be the set of all subformulae of $\widetilde{\mathcal{U}}$. Then one the following cases holds:*

1) *If the logical rule has a single premiss, then $\mathcal{F}_{\widetilde{\mathcal{T}}} \setminus \mathcal{F}_{\widetilde{\mathcal{U}}} = \{C\}$.*

2) *If the logical rule has two premises, then $C = A \circ B$ for formulae $A$, $B$ and logical connective $\circ$, and $\widetilde{\mathcal{U}}$ contains $A$, but not $B$. Then $\mathcal{F}_{\widetilde{\mathcal{T}}} \setminus \mathcal{F}_{\widetilde{\mathcal{U}}} = \{C\} \cup \mathcal{F}_B$, where $\mathcal{F}_B$ is the set of all subformulae of $B$.*

3) *If the logical rule has two premises, then $C = A \circ B$ for formulae $A$, $B$ and logical connective $\circ$, and $\widetilde{\mathcal{U}}$ contains $B$, but not $A$. Then $\mathcal{F}_{\widetilde{\mathcal{T}}} \setminus \mathcal{F}_{\widetilde{\mathcal{U}}} = \{C\} \cup \mathcal{F}_A$, where $\mathcal{F}_A$ is the set of all subformulae of $A$.*

*In any case, it is $\mathcal{F}_{\widetilde{\mathcal{T}}} \supset \mathcal{F}_{\widetilde{\mathcal{U}}}$.*

Proof: Since $\widetilde{\Xi}$ is an occurrence instance of a RK-derivation, all the sequents of $\widetilde{\Xi}$ are occurrence instances. Since all the occurrence variables of such a sequent are unique, every subformula occurring in such a sequent must be unique. Hence, the main formula $C$ of $\widetilde{\mathcal{T}}$ cannot also occur as (subformula of) any context formula in $\widetilde{\mathcal{T}}$. Therefore, $C$ cannot occur in $\widetilde{\mathcal{U}}$, which already establishes the strict inclusion $\mathcal{F}_{\widetilde{\mathcal{T}}} \supset \mathcal{F}_{\widetilde{\mathcal{U}}}$.

The three cases sum up the case analysis for the different logical rules of RK. We will exemplarily check the cases (&A), (¬S) and (⊃A), keeping in mind that $\widetilde{\Xi}$ is an occurrence instance.

- If $\widetilde{\mathcal{U}}$ is the premiss and $\widetilde{\mathcal{T}}$ is the conclusion of an instance of the single premiss logical rule (&A) in $\widetilde{\Xi}$, then $A \,\&\, B$ does not occur anywhere in $\Gamma$ or $\Delta$. Moreover, it is $\mathcal{F}_{A \,\&\, B} = \mathcal{F}_A \cup \mathcal{F}_B \cup \{A \,\&\, B\}$. Since $A \,\&\, B$ does not occur in $\widetilde{\mathcal{U}}$, it is $\mathcal{F}_{\widetilde{\mathcal{T}}} \setminus \mathcal{F}_{\widetilde{\mathcal{U}}} = \{A \,\&\, B\}$. This is an instance of case 1).

- If $\widetilde{\mathcal{U}}$ is the premiss and $\widetilde{\mathcal{T}}$ is the conclusion of an instance of the single premiss logical rule (¬S) in $\widetilde{\Xi}$, then $\neg A$ does not occur anywhere in $\Gamma$ or $\Delta$. Moreover, it is $\mathcal{F}_{\neg A} = \mathcal{F}_A \cup \{\neg A\}$. Since $\neg A$ does not occur in $\widetilde{\mathcal{U}}$, it is $\mathcal{F}_{\widetilde{\mathcal{T}}} \setminus \mathcal{F}_{\widetilde{\mathcal{U}}} = \{\neg A\}$. This is another instance of case 1).

- If $\widetilde{\mathcal{U}}$ is the left premiss and $\widetilde{\mathcal{T}}$ is the conclusion of an instance of the two premiss logical rule (⊃A) in $\widetilde{\Xi}$, then $A \supset B$ does not occur anywhere in $\Gamma$ or $\Delta$. Moreover, it is $\mathcal{F}_{A \supset B} = \mathcal{F}_A \cup \mathcal{F}_B \cup \{A \supset B\}$. Neither $A \supset B$ nor $B$ (and, hence, no subformulae of $B$) occur in $\widetilde{\mathcal{U}}$, and, therefore, we have $\mathcal{F}_{\widetilde{\mathcal{T}}} \setminus \mathcal{F}_{\widetilde{\mathcal{U}}} = \{A \supset B\} \cup \mathcal{F}_B$. This is an instance of case 2).

- We obtain an instance of 3) if we consider the right premiss and side formula $B$ as in the previous case: $\mathcal{F}_{\widetilde{\mathcal{T}}} \setminus \mathcal{F}_{\widetilde{\mathcal{U}}} = \{A \supset B\} \cup \mathcal{F}_A$.

The remaining cases are treated accordingly. □

For those rules, which have only one premiss, both side formulae (or the single side formula) are retained in the premiss. Consequently, in those cases, we have always $\mathcal{F}_{\widetilde{\mathcal{T}}} \setminus \mathcal{F}_{\widetilde{\mathcal{U}}} = C$, where $C$ is the main formula. In the cases, where there are two premises, more than a single formula disappears in the comparison between the conclusion and a single premiss, because the second side formula only occurs in the other premiss, which is not considered. We generalise this result to the following corollary of lemma 8.3.

**Corollary 8.5** *Let* $\widetilde{\Xi}$ *be an occurrence instance of a RK-derivation* $\Xi$ *and* $(\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1, \ldots, \widetilde{\mathcal{S}}_k)$ *some branch in* $\widetilde{\Xi}$. *Let* $\mathcal{F}_{\widetilde{\mathcal{S}}_i}$ *be the set of all subformulae of* $\widetilde{\mathcal{S}}_i$. *Then, for every* $0 \leq i < k$, *the following holds:*

- *if* $\widetilde{\mathcal{S}}_i$ *is the conclusion of (XA) or (XS), then* $\mathcal{F}_{\widetilde{\mathcal{S}}_i} = \mathcal{F}_{\widetilde{\mathcal{S}}_{i+1}}$,

- *if* $\widetilde{\mathcal{S}}_i$ *is the conclusion of a logical rule, then* $\mathcal{F}_{\widetilde{\mathcal{S}}_i} \supset \mathcal{F}_{\widetilde{\mathcal{S}}_{i+1}}$.

The next task is to demonstrate, how this property is reflected in view of subsets of the explosion set derived from the end sequent of a RK-derivation. However, it would be a futile attempt to consider any subset of the explosion set. Instead, only those subsets of an explosion set have to be considered, which correspond to some sequent ocurring in the RK-derivation under consideration. However, we have to establish that there is such a correspondence in the first place.

Up to this point, we have only considered explosion sets of (occurrence instances of) individual sequents. From now on, we will consider some RK-derivation $\Xi$ of a standard sequent $\mathcal{S}$ and its occurrence instance $\widetilde{\Xi}$, which has the occurrence instance $\widetilde{\mathcal{S}}$ of $\mathcal{S}$ as end sequent. Next, we generate the explosion set $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ for $\widetilde{\mathcal{S}}$ by constructing an explosion derivation $\Pi_{\widetilde{\mathcal{S}}}$. According to definition 7.1, $\Pi_{\widetilde{\mathcal{S}}}$ engenders an occurrence mapping $\pi_{\widetilde{\mathcal{S}}} : \mathcal{F}_{\widetilde{\mathcal{S}}} \to \mathcal{V}_{\Pi_{\widetilde{\mathcal{S}}}}$, which relates every subformula $A$ occurring in $\widetilde{\mathcal{S}}$ to some proxy variable $\pi_{\widetilde{\mathcal{S}}}(A)$. With the preceding lemma we know that every for sequent $\widetilde{\mathcal{T}}$, which occurs somewhere in $\widetilde{\Xi}$, it is $\mathcal{F}_{\widetilde{\mathcal{T}}} \subseteq \mathcal{F}_{\widetilde{\mathcal{S}}}$. Hence, the occurrence mapping $\pi_{\widetilde{\mathcal{S}}}$ not only relates to formulae occuring in the end sequent $\widetilde{\mathcal{S}}$, but also to those formulae that occur in every sequent $\widetilde{\mathcal{T}}$ of $\widetilde{\Xi}$. For this reason, we will omit the index of the occurrence mapping from now. Since $\pi$ relates formulae of $\widetilde{\mathcal{S}}$ to the occurrence variables of the explosion set, we can reformulate corollary 8.5.

**Corollary 8.6** *Let $\widetilde{\Xi}$ be an occurrence instance of a RK-derivation $\Xi$, and let $\widetilde{\mathcal{S}}$ be the end sequent of $\widetilde{\Xi}$. Let $\pi$ be the occurrence mapping obtained from an explosion derivation $\Pi$ of $\widetilde{\mathcal{S}}$. Let $(\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1, \ldots, \widetilde{\mathcal{S}}_k)$ be some branch of $\widetilde{\Xi}$, i.e. $\widetilde{\mathcal{S}}_0 = \widetilde{\mathcal{S}}$. Let $\mathcal{F}_{\widetilde{\mathcal{S}}_i}$ be the set of all subformulae of $\widetilde{\mathcal{S}}_i$. Then, for every $0 \leq i < k$, the following holds:*

- *if $\widetilde{\mathcal{S}}_i$ is the conclusion of (XA) or (XS), then $\pi(\mathcal{F}_{\widetilde{\mathcal{S}}_i}) = \pi(\mathcal{F}_{\widetilde{\mathcal{S}}_{i+1}})$,*
- *if $\widetilde{\mathcal{S}}_i$ is the conclusion of a logical rule, then $\pi(\mathcal{F}_{\widetilde{\mathcal{S}}_i}) \supset \pi(\mathcal{F}_{\widetilde{\mathcal{S}}_{i+1}})$.*

Just as each move from a conlusion of a locigal rule to a premiss removes formulae from the set of subformulae occuring therein, the proxy variables relating to these formulae are removed from the image of the set of these subformulae under $\pi$.

## 8.2 Connexion Sets

We will begin by establishing a complementary view on branches of a RK-dervation to that of the preceding section. Instead of having shrinking sets of formulae or proxy variables as a branch is traversed from the bottom up, we shall obtain a perspective, under which the set of proxy variables grows as a branch is traversed upwards. As a consequence of such a view, we will be able to relate a particular subset of the explosion set with each sequent occurring in the derivation, and demonstrate that the subsets corresponding to the sequents of a single branch yield an ascending family of such subsets. Therefore, the resulting inclusion relations of these subsets will reflect the tree structure of the derivation. For the purpose of demonstrating this property, some auxiliary notions are required.

**Definition 8.7** *Let $\widetilde{\Xi}$ be an occurrence instance of a RK-derivation $\Xi$, and let $\widetilde{\mathcal{S}}$ be the end sequent of $\widetilde{\Xi}$. Let $\pi$ be the occurrence mapping obtained from an explosion derivation $\Pi$ of $\widetilde{\mathcal{S}}$ with occurrence variables $\mathcal{V}_\Pi$. Let $\widetilde{\mathcal{T}}$ be some sequent occurring in $\widetilde{\Xi}$. Then the* occurrence set *of this sequent is the set of occurrence variables $V_{\widetilde{\mathcal{T}}} \stackrel{def}{=} \pi(\{\!\!\{\widetilde{\mathcal{T}}\}\!\!\})$.*

The occurrence set of a sequent is the image of the set of all formulae occurring immediately, i.e. not as subformulae, in the antecedent and succedent under the occurrence mapping. Recall that the occurrence mapping $\pi$ assigns a proxy variable to every complex subformula occurring in the sequent $\widetilde{\mathcal{S}}$, whereas all the proper occurrence variables of $\widetilde{\mathcal{S}}$ are mapped onto themselves. Since all formulae that occur in a sequent $\widetilde{\mathcal{T}}$ somewhere in $\widetilde{\Xi}$ are subformulae of those occurring in $\widetilde{\mathcal{S}}$, it is $V_{\widetilde{\mathcal{T}}} \subseteq \mathcal{V}_\Pi$. The occurrence set of a sequent $\widetilde{\mathcal{T}}$ provides the representations of these subformulae with respect to the explosion derivation.

As well as the collection of occurrence variables that relate to some sequent $\widetilde{\mathcal{T}}$ in an occurrence instance $\widetilde{\Xi}$, it is interesting to keep track of the proxy variables that correspond to the main formulae of those rules applications of $\widetilde{\Xi}$, which connect $\widetilde{\mathcal{T}}$ to the end sequent $\widetilde{\mathcal{S}}$.

**Definition 8.8** *Let $\widetilde{\Xi}$ be an occurrence instance of a RK-derivation $\Xi$, and let $\widetilde{\mathcal{S}}$ be the end sequent of $\widetilde{\Xi}$ and $\widetilde{\mathcal{T}}$ be some sequent of $\widetilde{\Xi}$. Let $\pi$ be the occurrence mapping obtained from an explosion derivation $\Pi$ of $\widetilde{\mathcal{S}}$ with explosion set $\langle \mathfrak{E}_{\widetilde{\mathcal{S}}}, w\rangle$ and occurrence variables $\mathcal{V}_\Pi$. Let $(\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1, \ldots, \widetilde{\mathcal{S}}_k)$ be that path of $\widetilde{\Xi}$ with $\widetilde{\mathcal{S}}_0 = \widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}}_k = \widetilde{\mathcal{T}}$. Then the* base proxy set *of $\widetilde{\mathcal{T}}$ with regard to $\langle \mathfrak{E}_{\widetilde{\mathcal{S}}}, w\rangle$ is the set of proxy variables $B_{\widetilde{\mathcal{T}}} \stackrel{def}{=} B_{\widetilde{\mathcal{T}},k}$, where the $B_{\widetilde{\mathcal{T}},i} \subseteq \mathcal{V}_\Pi$ $(0 \le i \le k)$ are defined as follows:*

- $B_{\widetilde{\mathcal{T}},0} \stackrel{def}{=} \{w\}$;
- $B_{\widetilde{\mathcal{T}},i+1} \stackrel{def}{=} B_{\widetilde{\mathcal{T}},i}$, *if $\widetilde{\mathcal{S}}_i$ is the conclusion of an exchange rule;*

- $B_{\widetilde{\mathcal{T}},i+1} \stackrel{def}{=} B_{\widetilde{\mathcal{T}},i} \cup \{\pi(C_i)\}$, *if* $C_i$ *is the main formula of* $\widetilde{\mathcal{S}}_i$*.*

In order to illustrate the motivation behind the construction, we consider a path $(\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1, \ldots, \widetilde{\mathcal{S}}_k)$. The initial step is to associate with $\widetilde{\mathcal{S}}_0$ the singleton set $\{w\}$, which consists of the watershed proxy of the explosion derivation $\Pi$. In the perspective of the preceding section, $\widetilde{\mathcal{S}}_0$ was associated with the set $\pi(\mathcal{F}_{\widetilde{\mathcal{S}}_0})$, the set of occurrence variables corresponding to all the subformulae occurring in $\widetilde{\mathcal{S}}_0$. Recall from section 7.2 that if $\mathcal{V}_\Pi$ is the set of occurrence variables in $\Pi$, then $\mathcal{V}_\Pi \setminus \pi(\mathcal{F}_{\widetilde{\mathcal{S}}_0}) = \{w\}$, since the watershed proxy $w$ is the only proxy variable, which does not correspond to some specific subformula occurrence in $\widetilde{\mathcal{S}}_0$. Hence, considering $\{w\}$ as representation of $\widetilde{\mathcal{S}}_0$ does certainly correspond to the suggested intuition of a complementary view. Next, consider a transition from $\widetilde{\mathcal{S}}_i$ to $\widetilde{\mathcal{S}}_{i+1}$ in $\mathfrak{S}$, which is not due to a exchange rule. In such a case, $\widetilde{\mathcal{S}}_i$ contains the main formula $C_i$ of the logical rule, which relates $\widetilde{\mathcal{S}}_i$ and $\widetilde{\mathcal{S}}_{i+1}$. From $C_i$ we immediately obtain a proxy variable $\pi(C_i)$, and, consequently, we can define $B_{\widetilde{\mathcal{T}},i+1}$ by $B_{\widetilde{\mathcal{T}},i} \cup \{\pi(C_i)\}$. If $\widetilde{\mathcal{S}}_i$ is the conclusion of an exchange rule, there is no main formula, and we can define $B_{\widetilde{\mathcal{T}},i+1}$ to be the same as $B_{\widetilde{\mathcal{T}},i}$. Iterating this addition of base proxies as the path is traversed, we obtain the set $B_{\widetilde{\mathcal{T}}}$ of all the base proxies corresponding to the main formulae of $\widetilde{\Xi}$ between $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{T}}$.

The goal is to relate each sequent of a derivation with a subset of the explosion set $\mathfrak{E}_{\widetilde{\mathcal{S}}}$. Recall that the meaning of a complex formula is given by one or two ESSs. The following definition assigns to each main formula of a path one such elementary structural sequent and accumulates these ESSs in a set.

**Definition 8.9** *Let* $\widetilde{\Xi}$ *be an occurrence instance of a RK-derivation* $\Xi$*, and let* $\widetilde{\mathcal{S}}$ *be the end sequent of* $\widetilde{\Xi}$ *and* $\widetilde{\mathcal{T}}$ *be some sequent of* $\widetilde{\Xi}$*. Let* $\pi$ *be the occurrence mapping obtained from an explosion derivation* $\Pi$ *of* $\widetilde{\mathcal{S}}$ *with explosion set* $\langle \mathfrak{E}_{\widetilde{\mathcal{S}}}, w\rangle$*. Let* $(\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1, \ldots, \widetilde{\mathcal{S}}_k)$ *be that path of* $\widetilde{\Xi}$ *with* $\widetilde{\mathcal{S}}_0 = \widetilde{\mathcal{S}}$ *and* $\widetilde{\mathcal{S}}_k = \widetilde{\mathcal{T}}$*. Then the* connexion set *of* $\widetilde{\mathcal{T}}$ *with regard to* $\langle \mathfrak{E}_{\widetilde{\mathcal{S}}}, w\rangle$ *is the set of ESSs* $\mathfrak{X}_{\widetilde{\mathcal{T}}} = \mathfrak{X}_{\widetilde{\mathcal{T}},k}$*, where the* $\mathfrak{X}_{\widetilde{\mathcal{T}},i} \subseteq \mathfrak{E}_{\widetilde{\mathcal{S}}}$ $(0 \leq i \leq k)$ *are defined as follows:*

- $\mathfrak{X}_{\widetilde{\mathcal{T}},0} = \{[\gamma \to w], [w \to \delta]\}$*;*
- *if* $\widetilde{\mathcal{S}}_i$ *is the conclusion of an exchange rule , then* $\mathfrak{X}_{\widetilde{\mathcal{T}},i+1} = \mathfrak{X}_{\widetilde{\mathcal{T}},i}$*;*
- *if* $C_i$ *is the main formula of* $\widetilde{\mathcal{S}}_i$ *and* $D_i$ *is a side formula of* $\widetilde{\mathcal{S}}_{i+1}$*, then*

$$\mathfrak{X}_{\widetilde{\mathcal{T}},i+1} = \mathfrak{X}_{\widetilde{\mathcal{T}},i} \cup \left\{ \mathcal{E} \in \mathfrak{E}_{\widetilde{\mathcal{S}}} \;\middle|\; \begin{array}{l} \mathcal{E} \textit{ contains } \pi(C_i) \textit{ as base proxy} \\ \textit{and } \pi(D_i) \textit{ as occurrence variable} \end{array} \right\}.$$

The definition provides a construction, which paralleles that of the base proxy set of a branch. Recall from section 7.1 that the main structure of the sequent $\widetilde{S} = \widetilde{S}_0 = [\Gamma \to \Delta]$ is represented by the two elementary structural sequents $\gamma \to w$ and $w \to \delta$, where $\gamma$ and $\delta$ are obtained from $\Gamma$ and $\Delta$ in the explosion procedure, i.e. $\pi(\{\!\!\{\Gamma\}\!\!\} \cup \{\!\!\{\Delta\}\!\!\}) = \{\!\!\{\gamma\}\!\!\} \cup \{\!\!\{\delta\}\!\!\}$. Since both of these ESSs have $w$ as the base proxy, there is a relation between the set $B_{\widetilde{\mathcal{T}},0} = \{w\}$ and $\mathfrak{X}_{\widetilde{\mathcal{T}},0} = \{[\gamma \to w], [w \to \delta]\}$. Unfortunately, for a transition from $\widetilde{\mathcal{S}}_i$ to $\widetilde{\mathcal{S}}_{i+1}$ in the path, which is not due to a exchange rule, such a relation is somewhat more difficult to establish. Simply adding all of those ESSs of $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ to $\mathfrak{X}_{\widetilde{\mathcal{T}},0}$, which contain the proxy $\pi(C_i)$ as base proxy, where $C_i$ be the main formula of $\widetilde{\mathcal{S}}_i$, does not produce the desired result in all cases. For if $C_i$ is a conjunction occuring in the succedent or a disjunction or implication occurring in the antecedent, proposition 7.4, corollary 7.5 or 7.6, respectively, state that the explosion set contains two ESSs having the base proxy $\pi(C_i)$. In each of these cases, this is due to the fact that the corresponding meta rule of the explosion calculus has two logical premises, each one containing a single one of the two side formulae in addition to the base proxy. In the derivation $\widetilde{\Xi}$, on the other hand, the path $(\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1, \ldots, \widetilde{\mathcal{S}}_k)$ contains only one of the logical premises of the logical rule of RK corresponding to that meta rule, namely $\widetilde{\mathcal{S}}_{i+1}$, and this $\widetilde{\mathcal{S}}_{i+1}$ contains only one of the two side formulae, called $D_i$ in the definition. Hence, of the two elementary structural sequents of the explosion set, which have $\pi(C_i)$ as base proxy, only that one must be considered for $\mathfrak{X}_{\widetilde{\mathcal{T}},i+1}$, which also contains $\pi(D_i)$. Note that $\pi(D_i)$ is either a branching proxy, namely if $D_i$ is a complex formula, or otherwise a proper occurrence variable. For all other possible cases for the main formula $C_i$, the lemma and corollaries mentioned above and corollary 7.7 state that there is only a single ESS which has $\pi(C_i)$ as base proxy. Therefore, the clause of the definition

$$\mathfrak{X}_{\widetilde{\mathcal{T}},i+1} = \mathfrak{X}_{\widetilde{\mathcal{T}},i} \cup \left\{ \mathcal{E} \in \mathfrak{E}_{\widetilde{\mathcal{S}}} \;\middle|\; \begin{array}{l} \mathcal{E} \text{ contains } \pi(C_i) \text{ as base proxy} \\ \text{and } \pi(D_i) \text{ as occurrence variable} \end{array} \right\}$$

yields the desired addition to $\mathfrak{X}_{\widetilde{\mathcal{T}},i}$ in all possible cases. Therefore, this definition retains the relation to $B_{\widetilde{\mathcal{T}},i+1}$ as far as the relevance of the base proxy $\pi(C_i)$ is concerned.

In view of the above complication, it appears that the base proxy set $B_{\widetilde{\mathcal{T}}}$ is not a sufficient representation of $\widetilde{\mathcal{T}}$, because its definition does not immediately involve the matter of logical rules having two premises. However, the following important result establishes that $\mathfrak{X}_{\widetilde{\mathcal{T}}}$ can be constructed simply by selecting such ESSs from the explosion set such that all occurrence variables thereof are contained in the base proxy set $B_{\widetilde{\mathcal{T}}}$ and the occurrence set $V_{\widetilde{\mathcal{T}}}$ corresponding to the sequent under consideration.

**Proposition 8.10** *Let $\widetilde{\Xi}$ be an occurrence instance of a RK-derivation $\Xi$, and let $\widetilde{\mathcal{S}}$ be the end sequent of $\widetilde{\Xi}$. Let $\pi$ be the occurrence mapping obtained from an explosion derivation $\Pi$ of $\widetilde{\mathcal{S}}$ with explosion set $\langle \mathfrak{E}_{\widetilde{\mathcal{S}}}, w \rangle$. Let further $(\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1, \ldots, \widetilde{\mathcal{S}}_k)$ be some path of $\widetilde{\Xi}$ with $\widetilde{\mathcal{S}}_0 = \widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}}_k = \widetilde{\mathcal{T}}$. Let $B_{\widetilde{\mathcal{T}}}$ be the base proxy set, $V_{\widetilde{\mathcal{T}}}$ be the occurrence set and $\mathfrak{X}_{\widetilde{\mathcal{T}}}$ the connexion set of $\widetilde{\mathcal{T}}$. Then it is*

$$\mathfrak{X}_{\widetilde{\mathcal{T}}} = \{\mathcal{E} \in \mathfrak{E}_{\widetilde{\mathcal{S}}} \,|\, \{\!\!\{\mathcal{E}\}\!\!\} \subseteq B_{\widetilde{\mathcal{T}}} \cup V_{\widetilde{\mathcal{T}}}\}.$$

PROOF: We shall establish "$\subseteq$" first. It is $\mathfrak{X}_{\widetilde{\mathcal{T}}} = \bigcup_{0 \le i \le k} \mathfrak{X}_{\widetilde{\mathcal{T}},i}$, and hence it will suffice to follow this finite construction. For $\mathfrak{X}_{\widetilde{\mathcal{T}},0} = \{[\gamma \to w], [w \to \delta]\}$ we observe that $w \in B_{\widetilde{\mathcal{T}}}$. Moreover, $\gamma$ and $\delta$ contain both those proper occurrence variables, which are already present in $\widetilde{\mathcal{S}}$, and all proxy variables corresponding to complex formulae of $\widetilde{\mathcal{S}}$, because those are detached by means of such a proxy variable in the explosion derivation. Each of the latter formulae is either still present in $\widetilde{\mathcal{T}}$, or it must be the main formula of some rule application somewhere in the path connecting $\widetilde{\mathcal{S}}$ to $\widetilde{\mathcal{T}}$. Since occurrence variables corresponding to other formulae cannot be involved in this path, it must be $\{\!\!\{\gamma\}\!\!\} \cup \{\!\!\{\delta\}\!\!\} \subseteq B_{\widetilde{\mathcal{T}}} \cup V_{\widetilde{\mathcal{T}}}$, which yields $\mathfrak{X}_{\widetilde{\mathcal{T}},0} \subseteq \{\mathcal{E} \in \mathfrak{E}_{\widetilde{\mathcal{S}}} \,|\, \{\!\!\{\mathcal{E}\}\!\!\} \subseteq B_{\widetilde{\mathcal{T}}} \cup V_{\widetilde{\mathcal{T}}}\}$. For $\mathfrak{X}_{\widetilde{\mathcal{T}},i+1}$ we only have to consider the case that $\mathfrak{X}_{\widetilde{\mathcal{T}},i+1} \supset \mathfrak{X}_{\widetilde{\mathcal{T}},i}$. According to definition 8.9, $\mathfrak{X}_{\widetilde{\mathcal{T}},i+1}$ contains that ESS $\mathcal{E} \in \mathfrak{E}_{\widetilde{\mathcal{S}}}$ in addition to those of $\mathfrak{X}_{\widetilde{\mathcal{T}},i}$, where $\mathcal{E}$ contains $\pi(C_i)$ as base proxy, and hence $\pi(C_i) \in B_{\widetilde{\mathcal{T}}}$, and $\pi(D_i)$ as occurrence variable, and where $C_i$ and $D_i$ are main formula and side formula of the rule, of which sequents $\widetilde{S}_i$ is the conclusion and $\widetilde{S}_{i+1}$ is a premiss. The formula $D_i$ is either still present in $\widetilde{\mathcal{T}}$, or it becomes the main formula of a rule application further up in the path. In the former case, it is $\pi(D_i) \in V_{\widetilde{\mathcal{T}}}$, in the latter case, it is $\pi(D_i) \in B_{\widetilde{\mathcal{T}}}$. Since we know that $C_i$ is a complex formula, we can draw on proposition 7.4 and its corollaries to obtain the exact structure of $\mathcal{E}$. In the cases that $C_i$ is a negation, a negative occurrence of an implication or a disjunction, or a positive occurrence of a conjunction, the elementary structural sequent $\mathcal{E}$ contains only these two occurrence variables, i.e. it is $\{\!\!\{\mathcal{E}\}\!\!\} = \{\pi(C_i), \pi(D_i)\} \subseteq B_{\widetilde{\mathcal{T}}} \cup V_{\widetilde{\mathcal{T}}}$. In the other three cases, the ESS also contains $\pi(E_i)$, where $E_i$ is the other immediate subformula of $C_i$. But then $\widetilde{\mathcal{S}}_{i+1}$ must also contain $E_i$ in addition to $D_i$, since it is the premiss of the corresponding logical rule of RK. With the same argument that we used for $D_i$, we obtain that $\pi(E_i) \in B_{\widetilde{\mathcal{T}}} \cup V_{\widetilde{\mathcal{T}}}$, and thus it is $\{\!\!\{\mathcal{E}\}\!\!\} = \{\pi(C_i), \pi(D_i), \pi(E_i)\} \subseteq B_{\widetilde{\mathcal{T}}} \cup V_{\widetilde{\mathcal{T}}}$.

For "$\supseteq$" we recall that every ESS of $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ is unique. Now, each $\mathcal{E} \in \mathfrak{E}_{\widetilde{\mathcal{S}}}$ such that $\{\!\!\{\mathcal{E}\}\!\!\} \subseteq B_{\widetilde{\mathcal{T}}} \cup V_{\widetilde{\mathcal{T}}}$ has a base proxy $\pi(C)$ for some complex formula $C$, and it must be $\pi(C) \in B_{\widetilde{\mathcal{T}}}$. For if $\pi(C)$ were an element of $V_{\widetilde{\mathcal{T}}}$ instead, then $C$ would not be a main formula anywhere in the path under consideration, but would occur as complex formula in $\widetilde{\mathcal{T}}$. Hence, for every immediate subformula $D$ of $C$, we would have $\pi(D) \notin B_{\widetilde{\mathcal{T}}} \cup V_{\widetilde{\mathcal{T}}}$, since such a $D$ could neither occur as a main formula in the path, nor would it occur as a formula of $\widetilde{\mathcal{T}}$; and, therefore, $\{\pi(C), \pi(D)\} \subseteq \{\!\{\mathcal{E}\}\!\} \nsubseteq B_{\widetilde{\mathcal{T}}} \cup V_{\widetilde{\mathcal{T}}}$. But if $\pi(C) \in B_{\widetilde{\mathcal{T}}}$, then, according to definition 8.9, there is some $1 \leq i \leq k$ such that the ESS $\mathcal{E} \in \mathfrak{X}_{\widetilde{\mathcal{T}},i} \subseteq \mathfrak{X}_{\widetilde{\mathcal{T}}}$. This concludes the argument. □

The above is the main result regarding the relation of explosion sets and RK-derivations. In summary, it states that for every sequent occurring anywhere in an occurrence instance of some RK-derivation, there is a corresponding subset of the explosion set of the end sequent; the selection of ESSs to be included in this subset depends both on the formulae which occur in the sequent and the main formulae of the path in the derivation leading up to the sequent. We can immediately infer from the definition of the connexion set and the above proposition that, for every branch $\mathfrak{S} = (\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1, \ldots, \widetilde{\mathcal{S}}_k)$ of such a derivation, there is a corresponding family $\{\mathfrak{X}_{\widetilde{S}_i}\}_{0 \leq i \leq k}$ of subsets of $\mathfrak{E}_{\widetilde{S}_0}$, where $\mathfrak{X}_{\widetilde{S}_i} \subseteq \mathfrak{X}_{\widetilde{S}_{i+1}}$ for all $0 \leq i < k$. This construction can be performed for each branch of a derivation and, hence, for the entire derivation itself.

We shall write $\{\mathfrak{X}_i\}_{\mathfrak{S}}$ to denote such a family, whenever the individual sequents of $\mathfrak{S}$ are not given explicitly, and write $\mathfrak{X}_{\mathfrak{S},i}$ for that element of that family, which has index $i$, and $\mathfrak{X}_{\mathfrak{S}}$ for the largest element of that family, which obviously corresponds to the topmost sequent of the branch. Now, for any two branches $\mathfrak{S}$ and $\mathfrak{T}$ of a derivation, there is always an initial segment $(\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1, \ldots, \widetilde{\mathcal{S}}_g)$, i.e. a path, which is shared by both branches. At the least this is the trivial branch $(\widetilde{\mathcal{S}}_0)$, since all branches of the derivation converge in the end sequent. Families of connexion sets corresponding to two branches have noteworthy properties.

**Corollary 8.11** *Let $\widetilde{\Xi}$ be an occurrence instance of a RK-derivation $\Xi$, and let $\widetilde{\mathcal{S}}$ be the end sequent of $\widetilde{\Xi}$. Let $\langle \mathfrak{E}_{\widetilde{\mathcal{S}}}, w\rangle$ be the explosion set obtained from an explosion derivation of $\widetilde{\mathcal{S}}$. Let further $\mathfrak{S} = (\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1, \ldots, \widetilde{\mathcal{S}}_k)$ and $\mathfrak{T} = (\widetilde{\mathcal{T}}_0, \widetilde{\mathcal{T}}_1, \ldots, \widetilde{\mathcal{T}}_l)$ be two different branches of $\widetilde{\Xi}$ with $\widetilde{\mathcal{S}}_0 = \widetilde{\mathcal{T}}_0 = \widetilde{\mathcal{S}}$. Let $(\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1, \ldots, \widetilde{\mathcal{S}}_g) = (\widetilde{\mathcal{T}}_0, \widetilde{\mathcal{T}}_1, \ldots, \widetilde{\mathcal{T}}_g)$, be the longest path, which is shared by $\mathfrak{S}$ and $\mathfrak{T}$. Then the following holds:*

*1) for every $0 < i \leq k$ and $0 < j \leq l$, it is*

$$\mathfrak{X}_{\mathfrak{S},i} \cap \mathfrak{X}_{\mathfrak{T},j} = \{\mathcal{E} \in \mathfrak{E}_{\widetilde{\mathcal{S}}} \mid \{\!\{\mathcal{E}\}\!\} \subseteq (B_{\mathfrak{S},i} \cup V_{\mathfrak{S},i}) \cap (B_{\mathfrak{T},j} \cup V_{\mathfrak{T},j})\};$$

*2) for every $0 \leq i \leq g$, it is $\mathfrak{X}_{\mathfrak{S},i} = \mathfrak{X}_{\mathfrak{T},i}$;*

*3) for every $g < i \leq k$ and $g < j \leq l$, it is $\mathfrak{X}_{\mathfrak{S},i} \neq \mathfrak{X}_{\mathfrak{T},j}$.*

PROOF: 1) is simply a logical consequence of the preceding proposition. Moreover, 2) follows immediately from 1), because, for $0 \leq i \leq g$, the $i$-th sequent of $\mathfrak{S}$ and the $i$-th sequent of $\mathfrak{T}$ are identical. For 3), note that it must be $h < min(k, l)$, since $\mathfrak{S}$ and $\mathfrak{T}$ are different branches. If it were the case that $h = min(k, l)$, then one of the two branches would be contained in the other one and, therefore, would not be a proper branch at all, contradicting the stipulation. Hence, there are distinct sequents $\widetilde{\mathcal{S}}_{h+1}$ and $\widetilde{\mathcal{T}}_{h+1}$. Both sequents must be premises of some rule application of $\widetilde{\Xi}$, of which $\widetilde{\mathcal{S}}_g = \widetilde{\mathcal{T}}_g$ is the conclusion, and, therefore, this must be the application of a logical rule having two premises. Then $\widetilde{\mathcal{S}}_g = \widetilde{\mathcal{T}}_g$ has as its main formula $C_g$ either a disjunction or an implication occurring in the antecedent or a conjunction occurring in the succedent, and each of $\widetilde{\mathcal{S}}_{h+1}$ and $\widetilde{\mathcal{T}}_{h+1}$ contains one of its subformulae, say $D_g$ in the case of $\widetilde{\mathcal{S}}_{h+1}$ and $E_g$ in the case of $\widetilde{\mathcal{T}}_{h+1}$ as side formula; it is exactly these side formulae, which distinguish the two sequents. It then follows from definition 8.9, that $\mathfrak{X}_{\mathfrak{S},h+1} = \mathfrak{X}_{\mathfrak{S},h} \cup \{\mathcal{E}\}$, where $\mathcal{E} \in \mathfrak{E}_{\widetilde{\mathcal{S}}}$ is that elementary structural sequent, which has $\pi(C_g)$ as its base proxy and $\pi(D_g)$ as occurrence variable, and that $\mathfrak{X}_{\mathfrak{T},h+1} = \mathfrak{X}_{\mathfrak{T},h} \cup \{\mathcal{F}\}$, where $\mathcal{F} \in \mathfrak{E}_{\widetilde{\mathcal{S}}}$ is that elementary structural sequent, which has $\pi(C_g)$ as base proxy and $\pi(E_g)$ as occurrence variable. Depending on the logical connective of $C_g$, one of proposition 7.4 2) or corollaries 7.5 1) and 7.6 1) applies. In any case it is $\mathcal{E} \neq \mathcal{F}$, and thus it is $\mathfrak{X}_{\mathfrak{S},h+1} \neq \mathfrak{X}_{\mathfrak{T},h+1}$. Since it is $\mathfrak{X}_{\mathfrak{S},i+1} \supseteq \mathfrak{X}_{\mathfrak{S},i}$ for all $0 \leq i < k$, and, likewise, $\mathfrak{X}_{\mathfrak{T},j+1} \supseteq \mathfrak{X}_{\mathfrak{T},j}$ for all $0 \leq j < l$, the inequality carries over to all $\mathfrak{X}_{\mathfrak{S},i}$ and $\mathfrak{X}_{\mathfrak{T},j}$ with $h < i \leq k$ and $h < j \leq l$. □

Properties 2) and 3) state that, when tracing two branches from the bottom upward, the connexion sets corresponding to the encountered sequents are identical as long as the branches coincide and begin to differ as soon as the branches diverge and that this difference is retained thereon. These properties come up as no surprise. Property 1) might seem even more obvious, because it is a trivial logical consequence of proposition 8.10. However, this seeming triviality hides a very interesting property of connexion sets. Consider a case, in which the topmost sequent in a path shared by two branches $\mathfrak{S}$ and $\mathfrak{T}$ contains two complex formulae $C_1$ and $C_2$. Out of the two formulae $C_1$ is the main formula of the rule application, which causes the diverge of the two branches, i.e. one premiss is contained in $\mathfrak{S}$, whereas the other is contained in $\mathfrak{T}$. Since $C_2$ is part of the context, it is simply copied into both premises, and, henceforth, occurs at least in one sequent of both branches above the one having $C_1$ as main formula. Now, for $h < i, j$ assume that $C_2$ is the $i$-th and $j$-th main formula of rule applications in $\mathfrak{S}$ and, respectively, in $\mathfrak{T}$. According to the definition, the connexion set $\mathfrak{X}_{\mathfrak{S},i}$ contains an ESS $\mathcal{E}$, which has $\pi(C_2)$ as its base proxy. Similarly, the connexion set

$\mathfrak{X}_{\mathfrak{T},j}$ must contain an ESS $\mathcal{F}$, which also has $\pi(C_2)$ as its base proxy. If the rule applications, which have $C_2$ as their main formula, have either a single premiss, or if they have two premises, but the same side formula occurs in the premises, which are contained in both branches $\mathfrak{S}$ and $\mathfrak{T}$, then it must be $\mathcal{E} = \mathcal{F}$.[2] So although property 3) guarantees $\mathfrak{X}_{\mathfrak{S},i} \neq \mathfrak{X}_{\mathfrak{T},j}$, property 1) and proposition 8.10 yield $\mathcal{E} \in \mathfrak{X}_{\mathfrak{S},i} \cap \mathfrak{X}_{\mathfrak{T},j}$. This is an extremely important property.

It is commonplace for a derivation in RK to have several rule applications for different copies of the very same formula. This is an artifact of the duplication of context formulae into two premises, that is intrinsic to logical rules of two premises. All the complex formulae, which occur as context formulae in the conclusion of an application of one of these rules, and which are subsequently duplicated into two sequents, can, independently of other copies of the same formula in other branches,[3] become the main formula of an appropriate rule application. In other words, each copy of the same formula must be treated by a separate application of the corresponding rule, and this enlarges the total number of required rule applications.[4]

On the other hand, all connexion sets and families of connexion sets, which can be constructed for any given derivation, are already contained in the explosion set of its end sequent. Moreover, the intersection of connexion sets corresponding to sequents, which occur in different branches, do not only contain the ESSs, which are added for the sequents shared by the branches, but, in addition to that, all the ESSs, which are due to rules applications independently acting on two copies of the corresponding main formula in the two branches, as long as the same side formulae occur in that premiss, which is part of the respective branch.[5] Thus, the relative locality of ESSs has the

[2]According to definition 8.9, the condition for $\mathcal{E} \neq \mathcal{F}$ is that the rule applications, which have $C_2$ as their main formula, have two premises, but different side formulae occur in those premises, which are contained in $\mathfrak{S}$ and $\mathfrak{T}$, respectively.

[3]Further applications of context-duplicating rules could generate additional copies of the formula.

[4]The same phenomenon occurs in derivations in LK. In that case, however, applications of weakening can remove one or more copies of a duplicated formula from some branches of a derivation.

[5]Actually, tying an ESS to a rule application of an RK-derivation is just a manner of speaking. Definition 8.9 states that an ESS is really related the main formula of a conclusion of a rule application and the corresponding side formula in a single premiss of that rule application. So it is only adequate to speak of a relation in the above mentioned sense in the cases of single-premiss rules. For rules having two premises, there are two "related" ESSs. For such a relation to be one-to-one, we must instead of an entire rule application consider pairs consisting of the conclusion of a rule application and a single one of its premises. Obviously, for rule applications having a single premiss, there is only one such pair, whereas there are two of them in the cases of rule applications having two

interesting effect of obtaining shared ESS in connexion sets, not only if they correspond to the same rule application in a shared path of two branches, but even if they correspond to different, but related rule applications occurring in the distinct parts of the branches.

Several examples shall demonstrate this property. Recall the occurrence instance $\widetilde{\Xi}$ of the proof of the sequent $(a \vee b) \vee c \to a \vee (b \vee c)$, which was an earlier example of this section.

$$\dfrac{\dfrac{\dfrac{\dfrac{a_1 \to a_2, b_2, c_2}{a_1 \to a_2, b_2 \vee c_2}\,(\vee\mathrm{S}) \quad \dfrac{b_1 \to a_2, b_2, c_2}{b_1 \to a_2, b_2 \vee c_2}\,(\vee\mathrm{S})}{a_1 \vee b_1 \to a_2, b_2 \vee c_2}\,(\vee\mathrm{A}) \quad \dfrac{c_1 \to a_2, b_2, c_2}{c_1 \to a_2, b_2 \vee c_2}\,(\vee\mathrm{S})}{(a_1 \vee b_1) \vee c_1 \to a_2, b_2 \vee c_2}\,(\vee\mathrm{A})}{(a_1 \vee b_1) \vee c_1 \to a_2 \vee (b_2 \vee c_2)}\,(\vee\mathrm{S})$$

The explosion set obtained for the end sequent is

$$\mathfrak{E} = \left\{ \begin{array}{l} [a_1 \to r], [b_1 \to r], [r \to q], [c_1 \to q], \\ [q \to w], [w \to p], [p \to a_2, s], [s \to b_2, c_2] \end{array} \right\},$$

with the occurrence mapping $\pi$ given by the following table, where the trivial cases of the form $a \mapsto a$ are omitted:

$$\begin{array}{rcl} a_1 \vee b_1 & \mapsto & r \\ (a_1 \vee b_1) \vee c_1 & \mapsto & q \\ a_2 \vee (b_2 \vee c_2) & \mapsto & p \\ b_2 \vee c_2 & \mapsto & s \end{array}$$

Let the three branches of $\widetilde{\Xi}$ be called, from left to right, $\mathfrak{S}$, $\mathfrak{T}$ and $\mathfrak{U}$. The construction of the families of connexion sets for each of these brances begins with the two ESSs representing the structure of the sequent: $q \to w$ and $w \to p$. This gives $\mathfrak{X}_{\mathfrak{S},0} = \mathfrak{X}_{\mathfrak{T},0} = \mathfrak{X}_{\mathfrak{U},0} = \{[q \to w], [w \to p]\}$ as initial connexion sets of the branches. For the next group of connexion sets, we must consider the main formula of the bottommost rule application $(\vee\mathrm{S})$, which is $a_2 \vee (b_2 \vee c_2)$. The elementary structural sequents corresponding to this rule application must have $\pi(a_2 \vee (b_2 \vee c_2)) = p$ as their base proxy. The explosion set contains a single ESS having $p$ as base proxy, namely $p \to a_2, s$. This correspond with the fact that $(\vee\mathrm{S})$ is a single premiss rule. It follows that $\mathfrak{X}_{\mathfrak{S},1} = \mathfrak{X}_{\mathfrak{T},1} = \mathfrak{X}_{\mathfrak{U},1} = \{[q \to w], [w \to p], [p \to a_2, s]\}$. As the next

premises.

It remains to remark that a connexion set can also be related to a sequent in the sense of proposition 8.10. In this case, the sequent is understood in the role of the topmost premiss of a path or branch.

rule, ($\vee$A), has two premises, the connexion sets for the corresponding step can no longer be equal. To be more precise, the branches $\mathfrak{S}$ and $\mathfrak{T}$ still share the left premiss of ($\vee$A), so it must still be $\mathfrak{X}_{\mathfrak{S},2} = \mathfrak{X}_{\mathfrak{T},2}$. The main formula of this rule application is $(a_1 \vee b_1) \vee c_1$, and, hence, the ESSs having, $\pi((a_1 \vee b_1) \vee c_1) = q$ as base proxy have to be considered. In this case there are indeed two such sequents: $r \to q$ and $c_1 \to q$. Since $\pi(a_1 \vee b_1) = r$, the former corresponds to the premiss, in which $a_1 \vee b_1$ occurs as side formula, i.e. the left premiss, whereas the latter corresponds to the right premiss, in which $c_1$ is the side formula; note that $\pi(c_1) = c_1$, since the proper occurrence variable $c_1$ represents its own occurrence. For the construction along the left branch we obtain $\mathfrak{X}_{\mathfrak{S},2} = \mathfrak{X}_{\mathfrak{T},2} = \{[r \to q], [q \to w], [w \to p], [p \to a_2, s]\}$, whereas $\mathfrak{X}_{\mathfrak{U},2} = \{[c_1 \to q], [q \to w], [w \to p], [p \to a_2, s]\}$ is the connextion set for the premiss occurring in the right branch. In the left branch, the next main formula is $a_1 \vee b_1$. The two ESSs with base proxy $\pi(a_1 \vee b_1) = r$ are $a_1 \to r$ and $b_1 \to r$, again corresponding to the fact that ($\vee$A) has two premises. We obtain $\mathfrak{X}_{\mathfrak{S},3} = \{[a_1 \to r], [r \to q], [q \to w], [w \to p], [p \to a_2, s]\}$ and $\mathfrak{X}_{\mathfrak{T},3} = \{[b_1 \to r], [r \to q], [q \to w], [w \to p], [p \to a_2, s]\}$ as connexion sets. In the three branches, the final rule applications all have $b_2 \vee c_2$ as main formula. The single ESS having $\pi(b_2 \vee c_2) = s$ as base proxy is $s \to b_2, c_2$, and this is added as final elementary structural sequent to $\mathfrak{X}_{\mathfrak{S},3}$, $\mathfrak{X}_{\mathfrak{T},3}$ and $\mathfrak{X}_{\mathfrak{U},2}$, resulting in these connexion sets:

$$\begin{aligned}
\mathfrak{X}_{\mathfrak{S},4} &= \{[a_1 \to r], [r \to q], [q \to w], [w \to p], [p \to a_2, s], [s \to b_2, c_2]\} \\
\mathfrak{X}_{\mathfrak{T},4} &= \{[b_1 \to r], [r \to q], [q \to w], [w \to p], [p \to a_2, s], [s \to b_2, c_2]\} \\
\mathfrak{X}_{\mathfrak{U},3} &= \{[c_1 \to q], [q \to w], [w \to p], [p \to a_2, s], [s \to b_2, c_2]\}
\end{aligned}$$

These connexion sets are also the largest sets of their corresponding family, i.e. it is $\mathfrak{X}_{\mathfrak{S}} = \mathfrak{X}_{\mathfrak{S},4}$ and $\mathfrak{X}_{\mathfrak{T}} = \mathfrak{X}_{\mathfrak{T},4}$ and $\mathfrak{X}_{\mathfrak{U}} = \mathfrak{X}_{\mathfrak{U},3}$.

## 8.3 Connexion Trees

Handling connexion sets in this manner, is extremely unwieldy, however. Fortunately, the incremental growth of the connexion sets up along the various paths of a derivation can be depicted in a tree, which has the same structure as the derivation, but is labelled by sets of ESSs instead of sequents.

**Definition 8.12** *Let $\widetilde{\Xi}$ be an occurrence instance of a RK-derivation $\Xi$, and let $\widetilde{\mathcal{S}}$ be the end sequent of $\widetilde{\Xi}$. Let $\langle \mathfrak{E}_{\widetilde{\mathcal{S}}}, w\rangle$ be the explosion set of $\widetilde{\mathcal{S}}$. The* connexion tree *with regard to $\langle \mathfrak{E}_{\widetilde{\mathcal{S}}}, w\rangle$ is the tree which is obtained from $\widetilde{\Xi}$ by replacing each sequent $\widetilde{\mathcal{T}}$ by its connexion set $\mathfrak{X}_{\widetilde{\mathcal{T}}}$.*

Due to the fact that, for every a branch $\mathfrak{S}$ of a derivation, it is always $\mathfrak{X}_{\mathfrak{S},i} \subseteq \mathfrak{X}_{\mathfrak{S},i+1}$ for all $0 \leq i < k$, where $k$ is the length of $\mathfrak{S}$, the following convention provides for a much more economic presentation: For each sequent $\widetilde{T}$ of the derivation, the corresponding node of the connexion tree displays only that ESS, which is added in the last step of the construction of the connexion set corresponding to $\widetilde{T}$. The entire connexion set corresponding to $\widetilde{T}$ is obtained by forming the union of all sets of ESSs that occur in the path of the connexion tree, which leads from the base of the tree and to that node corresponding to $\widetilde{T}$. Whenever we speak of connexion trees, it shall refer to this convention.

For the current example, compare the tree of the RK-derivation

$$\dfrac{\dfrac{\dfrac{a_1 \to a_2, b_2, c_2}{a_1 \to a_2, b_2 \vee c_2}(\vee\mathrm{S}) \qquad \dfrac{b_1 \to a_2, b_2, c_2}{b_1 \to a_2, b_2 \vee c_2}(\vee\mathrm{S})}{a_1 \vee b_1 \to a_2, b_2 \vee c_2}(\vee\mathrm{A}) \qquad \dfrac{c_1 \to a_2, b_2, c_2}{c_1 \to a_2, b_2 \vee c_2}(\vee\mathrm{S})}{\dfrac{(a_1 \vee b_1) \vee c_1 \to a_2, b_2 \vee c_2}{(a_1 \vee b_1) \vee c_1 \to a_2 \vee (b_2 \vee c_2)}(\vee\mathrm{S})}(\vee\mathrm{A})$$

to its connexion tree:

$$\dfrac{\dfrac{\dfrac{\{s \to b_2, c_2\}}{\{a_1 \to r\}}(\vee\mathrm{S}) \qquad \dfrac{\{s \to b_2, c_2\}}{\{b_1 \to r\}}(\vee\mathrm{S})}{\{r \to q\}}(\vee\mathrm{A}) \qquad \dfrac{\{s \to b_2, c_2\}}{\{c_1 \to q\}}(\vee\mathrm{S})}{\dfrac{\{p \to a_2, s\}}{\{q \to w\} \cup \{w \to p\}}(\vee\mathrm{S})}(\vee\mathrm{A})$$

The connexion set for the sequent $c_1 \to a_2, b_2, c_2$, the topmost sequent in the rightmost path $\mathfrak{U}$, is obtained by forming the union of all sets of ESSs, which occur in the rightmost branch:

$$\{q \to w\} \cup \{w \to p\} \cup \{p \to a_2, s\} \cup \{c_1 \to q\} \cup \{s \to b_2, c_2\}$$

This is, of course, exactly the set $\mathfrak{X}_{\mathfrak{U},3}$, i.e. the connexion set corresponding to the third sequent of the RK-derivation (not counting the end sequent).

A note on the labels of rule applications is in order. The labels of rules applications in a RK-derivation pertain to both conclusion and premiss or premises. In a connexion tree, on the other hand, only those ESSs in premiss positions are associated to the leftover labels of the rule application. For this reason, the labels of the rules applications should not be understood as relating premiss or premises to conclusions or vice versa. Instead, they merely signify the rule application in the derivation tree, which corresponds to the addition of the ESS, which occurs in the respective premiss position or one of the two respective premiss positions, in the construction of a connexion set. For example, the leftmost, topmost node of connexion tree contains the ESS $s \to b_2, c_2$, where $\pi(b_2 \vee c_2) = s$. Hence, both main formula and side formulae of the corresponding rule application ($\vee$S) are given by that ESS alone. The ESS $a_1 \to r$, which occurs immediately underneath in the connexion tree, is entirely unrelated to this rule application; it is related to the main formula of an application of ($\vee$A) with the main formula $a_1 \vee b_1$, for which it is $\pi(a_1 \vee b_1) = r$. Since the labels of rule applications are somewhat ill-placed in the connexion trees, we will usually simply omit them.

Returning to the example, observe the following: The ESS $p \to a_2, s$ is the first that is added to the initial two ESSs $q \to w$ and $w \to p$. Note that out of the variables $p$, $a_2$ and $s$ only one already occurs in the inital ESSs, whereas the other two are new. Similarly, in each of the two cases that would be considered next, $r \to q$ or $c_1 \to q$, only one of the two variables, namely $q$ already occurs in those ESSs that have already been accumulated, whereas the respective other one is new. The ESS $s \to b_2, c_2$ contains new variables $b_2$ and $c_2$, whereas $s$ already occurs in an ESS further down in the same branch. This observation is generalised in the following lemma.

**Lemma 8.13** *Let $X$ be a connexion tree for some occurrence instance $\widetilde{\Xi}$ of a derivation $\Xi$. Let $\{\mathcal{G}\}$ be a node of $X$, which is not the base node. Let $(M_0, M_1, \ldots, M_g)$ be that path in $X$ with $M_g = \{\mathcal{G}\}$, and let $\mathcal{M} = \bigcup_{0 \leq i < g} M_i$ be the set of all ESSs that occur in any set in the path leading up to $\{\mathcal{G}\}$. Then the following holds:*

1) $\left|\left(\bigcup_{\mathcal{E} \in \mathcal{M}} \{\!\{\mathcal{E}\}\!\}\right) \cap \{\!\{\mathcal{G}\}\!\}\right| = 1.$

2) *There is exactly one* $\mathcal{E} \in \mathcal{M}$ *such that* $\left|\{\!\{\mathcal{E}\}\!\} \cap \{\!\{\mathcal{G}\}\!\}\right| = 1.$

Proof: For 1), observe how the ESS $\mathcal{G}$ is added at some step during the construction of a connexion set. According to definition 8.9, $\mathcal{G}$ contains some base proxy $\pi(C)$, where $C$ is the main formula of the rule corresponding to that step, and some $\pi(D)$ as occurrence variable, where $D$ is a side formula. In certain cases for $C$, the ESS also contains some $\pi(E)$ for a second side formula $E$, as we know from proposition 7.4 1) and corollaries 7.5 2) and 7.6 2). That is, we have either $\{\!\{\mathcal{G}\}\!\} = \{\pi(C), \pi(D)\}$ or $\{\!\{\mathcal{G}\}\!\} = \{\pi(C), \pi(D), \pi(E)\}$. In the sequent $\mathcal{T}$ of $\widetilde{\Xi}$ corresponding to $\{\mathcal{G}\}$ in $X$, side formuala $D$ occurs for the first time as proper formula, i.e. not merely as subformula; the same is the case for $E$, where this is relevant. Hence, neither $\pi(D)$ nor $\pi(E)$ can occur anywhere in the branch leading up to $\mathcal{G}$. This leaves only $\pi(C)$ as candidate. Now, $\bigcup_{\mathcal{E} \in \mathcal{M}} \{\!\{\mathcal{E}\}\!\}$ contains all the occurrence variables that make up those ESSs leading up to $\mathcal{G}$ in $X$. Since $C$ is the main formula of the rule application in $\widetilde{\Xi}$, which has $\mathcal{T}$ as a premiss, $C$ must either be one of the formulae in the antecedent or succedent of the end sequent of $\widetilde{\Xi}$, or it must be the side formula of some rule application along the path connecting the end sequent with $\mathcal{T}$, except the one having $\mathcal{T}$ as a premiss. But then there is some $\mathcal{E} \in \mathcal{M}$ such that $\mathcal{E}$ contains $\pi(C)$. For it is either $\pi(C) \in \{\!\!\{\gamma\}\!\!\} \cup \{\!\!\{\delta\}\!\!\}$, where $\gamma \to w$ and $w \to \delta$ are the elementary structural sequents corresponding to the end sequent, or $\pi(C)$ is the branching proxy of some ESS with base proxy $B$, where $B$ is the main formula of the rule application having $C$ as side formula. Either way, it is $\pi(C) \in \bigcup_{\mathcal{E} \in \mathcal{M}} \{\!\!\{\mathcal{E}\}\!\!\}$, and hence $\left|\left(\bigcup_{\mathcal{E} \in \mathcal{M}} \{\!\!\{\mathcal{E}\}\!\!\}\right) \cap \{\!\!\{\mathcal{G}\}\!\!\}\right| = 1$.

For 2), recall that proxy variables are introduced as new variables in the explosion derivation. Therefore, depending on the logical meta rule introducing $\pi(C)$, there are either two or three ESSs containing this proxy variable: one ESS, in which it occurs as branching proxy, and one or two, in which it occurs as base proxy. The variable $\pi(C)$ is the base proxy of $\mathcal{G}$, meaning that $\mathcal{G}$ is added, because $C$ is the main formula of a rule application in $\widetilde{\Xi}$. Since no formula can be the main formula of more than one rule application in any given branch of $\widetilde{\Xi}$,[6] there cannot be another ESS in the path of the connexion tree leading up to $\{\mathcal{G}\}$, which has $\pi(C)$ as a base proxy.[7] This leaves only the single ESS $\mathcal{E}$ described above, which has $\pi(C)$ as branching proxy. But, of course, $\mathcal{E}$ itself can only be added once in the construction of the connexion set corresponding to this path in the connexion tree. For it is either one of $\gamma \to w$ and $w \to \delta$, which constitute the initial connexion set, or it is some ESS with base proxy $\pi(B)$, which is included due to a rule application with main formula $B$. Again, $B$ can only be the main formula of one rule application in any given branch of $\widetilde{\Xi}$. Hence, there is exactly one $\mathcal{E} \in \mathcal{M}$ such that $\left|\{\!\!\{\mathcal{E}\}\!\!\} \cap \{\!\!\{\mathcal{G}\}\!\!\}\right| = 1$. □

Simply put, every ESS occurring in one of the sets above the very base of the connexion tree contains as base proxy a variable, which already occurs in an ESS further down in the path leading up to that set as a branching proxy. Thus, connexion trees are not merely artificial and arbitrary constructions, but they express very succinctly the idea of *connection*. When moving upward within a branch of the connexion tree, each ESS that is added extends the collection of ESSs, which have already been selected from the explosion set, in such a way that it connects to this collection in exactly one shared proxy variable. This means that ESSs are not added in an arbitrary manner,

[6] We are dealing with occurrence instances, which account for individual occurrences.

[7] We already know that in those cases, in which there is a second ESS having $\pi(C)$ as base proxy, it occurs in the parallel premiss position to $\{\mathcal{G}\}$ of the connexion tree.

but the succession of their additions retains and reflects the successive development of the RK-derivation. In contrast to the derivation, which focusses on individual, unrelated formulae, the connexion tree emphasises the interconnectedness of the ESSs related to some particular branch of the derivation. Moreover, in the entirety of its branches, it demonstrates the different directions, in which the common set of ESSs $\{[\gamma \to w], [w \to \delta]\}$ extends within the explosion set. It is also interesting to realise that the explosion set already extends over all of the ESSs, which are contained in a connexion tree. In this sense, the connexion tree corresponding to some particular derivation merely provides instructions on some possibility of traversal through and ordering of the explosion set of the end sequent of the derivation. In other words, the explosion set already contains all relevant modes of connectedness that exist among the elementary structural sequents; a connexion tree merely exemplifies some particular mode of explicitly bringing these connections forward one after the other.

But what about different derivations of the same sequent and their respective connexion trees? Let us consider different derivations for the occurrence instance $(a_1 \vee b_1) \vee c_1 \to a_2 \vee (b_2 \vee c_2)$ and inspect the different connexion trees that are obtained for them, beginning with the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{a_1 \to a_2, b_2, c_2 \qquad b_1 \to a_2, b_2, c_2}{a_1 \vee b_1 \to a_2, b_2, c_2}\,(\vee\text{A}) \qquad c_1 \to a_2, b_2, c_2}{(a_1 \vee b_1) \vee c_1 \to a_2, b_2, c_2}\,(\vee\text{A})}{(a_1 \vee b_1) \vee c_1 \to a_2, b_2 \vee c_2}\,(\vee\text{S})}{(a_1 \vee b_1) \vee c_1 \to a_2 \vee (b_2 \vee c_2)}\,(\vee\text{S})$$

Its connexion tree is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\{a_1 \to r\} \qquad \{b_1 \to r\}}{\{r \to q\}}\,(\vee\text{A}) \qquad \{c_1 \to q\}}{\{s \to b_2, c_2\}}\,(\vee\text{A})}{\{p \to a_2, s\}}\,(\vee\text{S})}{\{q \to w\} \cup \{w \to p\}}\,(\vee\text{S})$$

Note that although the ESS $s \to b_2, c_2$ only occurs once in this tree, whereas it occurs three times in the connexion tree corresponding to the previous derivation, it is not the case that this ESS is only added once in here and three times there. Recall that the connexion tree is to be understood as a bundling of the different developments of connextion sets corresponding to a given derivation. This means that a connexion tree of three branches represents an abbreviated representation of three separate developments of connexion sets. The ESS $s \to b_2, c_2$ occurs in all three branches in both connexion trees. In the former one, it had to be listed three times, because it was added as the last ESS in all three cases, whereas in the last tree, it is added before the three branches separate. Still, it represents the fact that this ESS has to be added in each of the three connexion sets.

Lemma 8.13 necessitates that $s \to b_2, c_2$ can only be added in a branch, in which the ESS $p \to a_2, s$ has already been added. For example, since it is $(\{\!\{q \to w\}\!\} \cup \{\!\{w \to p\}\!\}) \cap \{\!\{s \to b_2, c_2\}\!\} = \emptyset$, no connexion tree can have the following initial segment:

$$\frac{\begin{array}{c}\vdots\\ \{s \to b_2, c_2\}\end{array}}{\{q \to w\} \cup \{w \to p\}}\,(\vee\mathrm{S})$$

In fact, lemma 8.13 exerts specific limitations on how a connexion tree for a given explosion set can possibly be structured. Recall the explosion set for the sequent $(a_1 \vee b_1) \vee c_1 \to a_2 \vee (b_2 \vee c_2)$:

$$\mathfrak{E} = \left\{ \begin{array}{l} [a_1 \to r], [b_1 \to r], [r \to q], [c_1 \to q], \\ [q \to w], [w \to p], [p \to a_2, s], [s \to b_2, c_2] \end{array} \right\}$$

It is $\{\!\{q \to w\}\!\} \cup \{\!\{w \to p\}\!\} = \{q, w, p\}$, and since $w$ occurs in no other ESS of $\mathfrak{E}$, the possible candidates for the extension of $\{[q \to w], [w \to p]\}$ must contain $p$ or $q$ as base proxies. Since in both previous examples the elementary structural sequent $p \to a_2, s$ was chosen, let us develop a third example based on the alternative choice. There are two ESSs, which has $q$ as base proxy, however, and, hence, there are two possible ways to extend this connexion set. We obtain the following initial fragment of a tree representation:

$$\frac{\begin{array}{c}\vdots\\ \{r \to q\}\end{array} \qquad \begin{array}{c}\vdots\\ \{c_1 \to q\}\end{array}}{\{q \to w\} \cup \{w \to p\}}\,(\vee\mathrm{A})$$

The label indicating some logical rule is obtained by investigating the kind of occurrence of the formula corresponding to the base proxy $q$, i.e. the formula $\pi^{-1}(q)$. It is $\pi^{-1}(q) = (a_1 \vee b_1) \vee c_1$, which has a negative occurrence in the sequent, and, hence, can only be the main formula of an application of $(\vee\mathrm{A})$. The extension of the connexion set for the right branch is straightforward, for it is $\{\!\{q \to w\}\!\} \cup \{\!\{w \to p\}\!\} \cup \{\!\{c_1 \to q\}\!\} = \{q, w, p, c_1\}$, an ESS containing $q$ has already been added, and $c_1$ is a proper occurrence variable, which occurs in no other ESS of $\mathfrak{E}$, leaving only $p \to a_2, s$. A new proxy $s$ has been added to those, which are possible candidates for base proxies of ESSs to be included for the following extension of the connexion set, whereas both $p$ and $q$ have already been used to extend the connexion set. Hence, only the sequent $s \to b_2, c_2$ remains to be added in this branch. This yields the following fragment:

$$\dfrac{\begin{array}{c}\vdots\\ \{r \to q\}\end{array} \qquad \dfrac{\dfrac{\{s \to b_2, c_2\}}{\{p \to a_2, s\}}\,(\vee\mathrm{S})}{\{c_1 \to q\}}\,(\vee\mathrm{S})}{\{q \to w\} \cup \{w \to p\}}\,(\vee\mathrm{A})$$

Again, the labels correspond to the logical rules required for treating the main formula corresponding to the base proxies introduced in the respective ESS.[8] For the left branch, it is $\{\!\{q \to w\}\!\} \cup \{\!\{w \to p\}\!\} \cup \{\!\{r \to q\}\!\} = \{q, w, p, r\}$, and since only $w$ and $q$ have already been used up, there is a choice of whether to proceed with $p$ or $r$ as base proxy of the next addition. Choosing the proxy $p$ results in the addition of $p \to a_2, s$, which leaves $s$ and $r$ as base proxy for the following addition of an ESS. After having chosen $s$ as base proxy, the tree looks thus:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \{s \to b_2, c_2\}\end{array}}{\{p \to a_2, s\}}\,(\vee\mathrm{S})}{\{r \to q\}}\,(\vee\mathrm{S}) \qquad \dfrac{\dfrac{\{s \to b_2, c_2\}}{\{p \to a_2, s\}}\,(\vee\mathrm{S})}{\{c_1 \to q\}}\,(\vee\mathrm{S})}{\{q \to w\} \cup \{w \to p\}}\,(\vee\mathrm{A})$$

This leaves $r$ as only choice for the completion of the left branch of the connexion tree. Since $r$ is the base proy of two ESSs, the left branch has to be subdivided into two branches. After having added $a_1 \to r$ and $b_1 \to r$, respectvely, all of the proxies that occur at all in the explosion set do occur twice in each one of the branches, wherefore no further extensions of the connexion sets are possible in any branch. This is the resulting connexion tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\{a_1 \to r\} \qquad \{b_1 \to r\}}{\{s \to b_2, c_2\}}\,(\vee\mathrm{A})}{\{p \to a_2, s\}}\,(\vee\mathrm{S})}{\{r \to q\}}\,(\vee\mathrm{S}) \qquad \dfrac{\dfrac{\{s \to b_2, c_2\}}{\{p \to a_2, s\}}\,(\vee\mathrm{S})}{\{c_1 \to q\}}\,(\vee\mathrm{S})}{\{q \to w\} \cup \{w \to p\}}\,(\vee\mathrm{A})$$

[8] This refers to the ESS in premiss position relative to the position of the label.

For this connexion tree, a RK-derivation can be constructed as follows. The initial step is to replace $\{q \to w\} \cup \{w \to p\}$ by the sequent, from which the explosion set was generated. Proceeding upwards along the branches, each set containing an ESS with base proxy $p$ and occurrence variables $x$ and $y$, where applicable, is replaced by that sequent, which is obtained from the one immediately below the current position in the tree by replacing in it the main formula $\pi^{-1}(p)$ by the side formula $\pi^{-1}(x)$, whenever the ESS contains only two variables, or $\pi^{-1}(x), \pi^{-1}(y)$, whenever it contains three variables. The application of this procedure yields the following tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{a_1 \to a_2, b_2, c_2 \qquad b_1 \to a_2, b_2, c_2}{a_1 \vee b_1 \to a_2, b_2, c_2}(\vee\mathrm{A})}{a_1 \vee b_1 \to a_2, b_2 \vee c_2}(\vee\mathrm{S})}{a_1 \vee b_1 \to a_2 \vee (b_2 \vee c_2)}(\vee\mathrm{S}) \qquad \dfrac{\dfrac{c_1 \to a_2, b_2, c_2}{c_1 \to a_2, b_2 \vee c_2}(\vee\mathrm{S})}{c_1 \to a_2 \vee (b_2 \vee c_2)}(\vee\mathrm{S})}{(a_1 \vee b_1) \vee c_1 \to a_2 \vee (b_2 \vee c_2)}(\vee\mathrm{A})$$

It is easily verified that this is, indeed, a RK-derivation.

A brief look at another example shall serve to illustrate the effect of the exchange rule on connexion trees. Consider the sequent $a, a \supset b \to a \,\&\, b$ and its occurrence instance $a_1, a_2 \supset b_1 \to a_3 \,\&\, b_2$. This is a RK-derivation of the occurrence instance:

$$\dfrac{\dfrac{\dfrac{a_1 \to a_3, a_2 \qquad b_1, a_1 \to a_3}{a_2 \supset b_1, a_1 \to a_3}(\supset\mathrm{A})}{a_1, a_2 \supset b_1 \to a_3}(\mathrm{XA}) \qquad \dfrac{\dfrac{a_1 \to b_2, a_2 \qquad b_1, a_1 \to b_2}{a_2 \supset b_1, a_1 \to b_2}(\supset\mathrm{A})}{a_1, a_2 \supset b_1 \to b_2}(\mathrm{XA})}{a_1, a_2 \supset b_1 \to a_3 \,\&\, b_2}(\&\mathrm{S})$$

The explosion of the end sequent

$$\dfrac{\dfrac{\to q, a_2 \qquad b_1 \to q \qquad a_1, q \to w}{a_1, a_2 \supset b_1 \to w}(\supset\mathrm{A}_m) \qquad \dfrac{w \to p \qquad p \to a_3 \qquad p \to b_2}{w \to a_3 \,\&\, b_2}(\&\mathrm{A}_m)}{a_1, a_2 \supset b_1 \to a_3 \,\&\, b_2}(\mathrm{Prx})$$

yields the explosion set

$$\mathfrak{F} = \left\{ \begin{array}{l} [\to q, a_2], [b_1 \to q], [a_1, q \to w], \\ [w \to p], [p \to a_3], [p \to b_2] \end{array} \right\}.$$

For each one of the four branches of the derivation, we can construct a family of connexion sets. These are jointly displayed in this connextion tree:

$$\dfrac{\dfrac{\dfrac{\{\to q, a_2\} \qquad \{b_1 \to q\}}{\{\,\}}\,(\supset\mathrm{A})}{\{p \to a_3\}}\,(\mathrm{XA}) \qquad \dfrac{\dfrac{\{\to q, a_2\} \qquad \{b_1 \to q\}}{\{\,\}}\,(\supset\mathrm{A})}{\{p \to b_2\}}\,(\mathrm{XA})}{\{a_1, q \to w\} \cup \{w \to p\}}\,(\&\mathrm{S})$$

Consider the premiss positions of the applications of (XA). According to the definition 8.9, no ESS is added to a connexion set in a step, which does not correspond to a logical rule in the underlying RK-derivation. Therefore, the node of the connexion tree corresponding to the premiss of an application of the exchange rule must have the empty set as its label.

While this does not come unexpected, it makes the construction of a connexion tree from an explosion set, which has an underlying RK-derivation, difficult, albeit not impossible. The reason for this is the following: Due to the manner, in which the logical meta rules of the explosion calculus are given, each ESS retains in the position of an occurrence variable $p$ the original position of the first proper position of the formula $\pi^{-1}(p)$ in a sequent of the explosion derivation. This is the first occurrence of this formula as a proper formula of the antecedent or succedent of a sequent. For example, in the explosion derivation of the sequent $a_1, a_2 \supset b_1 \to a_3 \& b_2$, the ESS representing the antecedent is $a_1, q \to w$, and the implication $a_2 \supset b_1$ is represented by the two ESSs $\to q, a_2$ and $b_1 \to q$. The extraction of the implication from $a_1, a_2 \supset b_1 \to w$ occurs in this single meta step of the explosion derivation:

$$\dfrac{\to q, a_2 \qquad b_1 \to q \qquad a_1, q \to w}{a_1, a_2 \supset b_1 \to w}\,(\supset\mathrm{A}_m)$$

Recall that a meta rule comprises a local logical rule, an atomic cut and the number of exchange rules required to move the main formula into cut position and back into its previous position, which really represents the individual steps

$$\dfrac{\dfrac{\dfrac{\to q, a_2 \qquad b_1 \to q}{a_2 \supset b_1 \to q}\,(\supset\mathrm{A}_l) \qquad \dfrac{a_1, q \to w}{q, a_1 \to w}\,(\mathrm{XA})}{a_2 \supset b_1, a_1 \to w}\,(\mathrm{Prx})}{a_1, a_2 \supset b_1 \to w}\,(\mathrm{XA})$$

In other words, in order to facilitate the relating of ESSs to the sequent from which they are derived, the explosion calculus generously inserts exchange rules whenever it serves this purpose. Hence, the order of occurrence variables in the ESSs of the explosion set follows the order of the first occurrence of the corresponding formulae in an explosion derivation. On the other hand, in an

RK-derivation of the exploded sequent the order of formulae can be changed arbitrarily by applications of exchange rules. Since these applications are explicit in a derivation tree, the are retained in the corresponding connexion tree.

The difficulty of creating a connexion tree from scratch, i.e. from a given explosion set instead of from an underlying RK-derivation, lies in the fact that the insertions of empty sets into the connexion tree due to necessary applications of the exchange rules in a RK-derivation, which would correspond to the tree being constructed, cannot be imposed without adding some bookkeeping regime on the position of proxy variables in ESSs. Rather than following this tedious path, it is more in the spirit of this investigation to relax the notion of a RK-derivation in the sense that exchange rules are no longer given explicitly. Consider the following construction of a connexion tree from the explosion set $\mathfrak{F}$:

$$\dfrac{\dfrac{\{p \to a_3\} \qquad \{p \to b_2\}}{\{\to q, a_2\}}\,(\&\mathrm{S}) \qquad \dfrac{\{p \to a_3\} \qquad \{p \to b_2\}}{\{b_1 \to q\}}\,(\&\mathrm{S})}{\{a_1, q \to w\} \cup \{w \to p\}}\,(\supset\mathrm{A})$$

This is the corresponding tree using logical rules of RK:

$$\dfrac{\dfrac{a_1 \to a_3, a_2 \qquad a_1 \to b_2, a_2}{a_1 \to a_3 \,\&\, b_2, a_2}\,(\&\mathrm{S}) \qquad \dfrac{b_1, a_1 \to a_3 \qquad b_1, a_1 \to b_2}{b_1, a_1 \to a_3 \,\&\, b_2}\,(\&\mathrm{S})}{a_1, a_2 \supset b_1 \to a_3 \,\&\, b_2}\,(\supset\mathrm{A})$$

While the above tree is no RK-derivation, by means of the notational convenience of having double lines in a rule application represent zero or more applications of either exchange rule before or after the explicity listed logical rule, we obtain the following tree:

$$\dfrac{\dfrac{a_1 \to a_3, a_2 \qquad a_1 \to b_2, a_2}{a_1 \to a_3 \,\&\, b_2, a_2}\,(\&\mathrm{S}) \qquad \dfrac{b_1, a_1 \to a_3 \qquad b_1, a_1 \to b_2}{b_1, a_1 \to a_3 \,\&\, b_2}\,(\&\mathrm{S})}{a_1, a_2 \supset b_1 \to a_3 \,\&\, b_2}\,(\supset\mathrm{A})$$

This tree is, of course, a proper RK-derivation.

In summary, this chapter investigated the relation between occurrence instances of RK-derivations and the explosion set of its end sequent. We began by pointing out the fact that in branches of occurrence instances of RK-derivations the sets of subformulae contained in the sequents of each branch get smaller as the branch is traversed from the root upward. This property is due to the absence of naive contraction rules, which could retain copies of formulae and their subformulae. The next step was to proceed from sets of subformulae to sets of occurrence variables, which especially encompasses the transition from complex formulae to proxy variables. However, rather than just transferring the aforementioned property from sets of subformulae to sets of occurrence variables, we aimed to find some kind of incremental correspondence. Accumulating the proxy variables corresponding to the main formulae occurring in a branch proved to be the proper approach. Relating these proxy variables to base proxies of the ESSs of the explosion set was the means, by which incremental families of subsets of the explosion set, the connexion sets, could be related to each branch of a given RK-derivation. It turned out that the connexion set corresponding to some sequent in the derivation encompasses exactly those elementary structural sequents, which are made up of the proxies of all the main formulae that occur in the path up to that sequentand the occurrence variables of all the formulae that occur in the sequent. This characterisation of connexion sets is the main result of this chapter, because it relates each sequent occurring in a RK-derivation to some subset of the explosion set. And since this characterisation holds for every possible derivation of an end sequent and its explosion set, the proposition establishes the explosion set of a sequent as the structural skeleton underlying *all* of its RK-derivations. Specifically, a particular family of connexion sets corresponds to each branch of a derivation, where these connexion sets are ordered by inclusion along the branch as it proceeds from base to leaf. Hence, the largest connexion set, which is also the union of all the sets of the family, corresponds to the leaf of the branch, and, thereby, the entire branch can be represented by this largest connexion set. For this reason, each RK-derivation can be represented by some set of connexion sets, i.e. a set of subsets of the explosion set of its end sequent, each connexion set corresponding to a branch of the derivation. In the last section of this chapter, a tree notation for the connexion sets corresponding to a RK-derivation was introduced. The tree structure of this notation is taken from the derivation itself, and the nodes, i.e. the sequents, of the tree are labelled by sets of ESSs.

# Chapter 9

# Decision Procedures on Explosion Sets

The explosion procedure for an occurrence instance $\widetilde{\mathcal{S}}$ of some sequent $\mathcal{S}$ yields an explosion set $\mathfrak{E}_{\widetilde{\mathcal{S}}}$, regardless of whether $\mathcal{S}$ is provable in RK or not. Hence, it is necessary to have a decision procedure for determining from the resulting explosion set, whether the original sequent $\mathcal{S}$ is provable in RK or not. Several such decision procedures will be described in this chapter, each of which corresponds to two conceptually different approaches. The proof-theoretic version provides the insight as to what a decision procedure has to achieve in the first place. The procedural version provides a method, which operates directly on explosion sets without the proof-theoretic details, although this method clearly mimics those details. In the section on the procedural approach, a probabilistic semi-decision procedure will be exhibited, which refutes explosion sets of non-provable sequents. Moreover, a proper decision procedure will be outlined, based on the refutation method.

What has to be achieved by a decision procedure is to reverse the complete decomposition of the sequents into ESSs to a certain degree. In LK, sequents $A \rightarrow A$ containing a single instance of a formula in the antecedent and another instance of the same formula in the succedent and no other formula beside those are the initial sequents, which close branches of derivations. An initial sequent, in which $A$ is a propositional variable, i.e. $a \rightarrow a$, expresses the immediate relation of the propositional variable $a$ to itself. In a static view, sequents of this kind would be understood as expressing the relation of *identity*. A dynamic reading would take it to express the possibility of an elementary movement from $a$ onto itself, which might be called *recurrence*, when the focus is put on the movement, or *selfhood*, when the focus lies upon the atom. Within the framework of the logistic calculi, there is no more immediate relation expressible than this. It is this property, which is captured by the axiom of RK under further consideration of contextual atomic formulae:

$$\frac{}{\gamma_1, a, \gamma_2 \rightarrow \delta_1, a, \delta_2}\ (\text{ax}^*)$$

Sequents that are instances of (ax*) are of the kind $\gamma \rightarrow \delta$ where there is at least one propositional variable $a$, such that $a \in \{\!\!\{\gamma\}\!\!\}$ and $a \in \{\!\!\{\delta\}\!\!\}$. Contrasting the intuition of considering these immediate relations as elementary, the rules of the explosion calculus are designed to emphasise distinctness and dissimilarity. This emphasis is fundamentally enforced by the use of occurrence instances of sequents, which serve to account for different occurrences of the same propositional variable, and it is further implemented by the cuts embedded in the meta rules, which introduce a new proxy variable for each occurrence of a complex formula. The explosion procedure yields two ESSs even for an initial sequent $a \rightarrow a$, as the following explosion derivation demonstrates:

$$\frac{a_1 \rightarrow w \qquad w \rightarrow a_2}{a_1 \rightarrow a_2}\ (\text{Prx})$$

Hence, as a result of applying the explosion procedure to a sequent, there will not be even a single leaf that is an instance of the RK axiom. The immediacy of checking, whether a given sequent has the required form, is lost. However, connections between occurrence variable in the antecedent of an ESS and other occurrence variables in the succedent of another ESS can still be traced. Such connections are mediated by sequences of proxy variables, which connect these two ESSs, in general via yet other elementary structural sequents.

Intuitively, the new proxy variables that are introduced in the explosion procedure scatter and spread the proper occurrence variables, which have to be compared in order to determine provability of the original sequent, over several different ESSs. Any decision procedure therefore has to either remove these proxy variables, thereby undoing part of the explosion procedure, or follow the connections between certain ESSs that are implicitly given through the proxies. However, since the explosion set is derived from an occurrence instance of a sequent, all the proper occurrence variables in the explosion set are distinct. For this reason, the restoration function of definition 5.1 has to be employed. The application of the restoration function to one or more sequents produces such sequents, which have all their proper occurrence variables replaced by propositional variables.

A decision procedure must establish that, for all possible connections of ESSs, some proper propositional variable, which occurs in the antecedent of one of these ESS, matches a proper propositional variable occurring in the succedent of another ESS.

## 9.1 Cut-Actions on Connexion Sets

A simple example shall serve to illustrate the idea of what a decision procedure has to accomplish. Consider the sequent $\mathcal{S} = [a, b \to a \,\&\, b]$ and an explosion derivation of its occurrence instance $\widetilde{\mathcal{S}} = [a_1, b_1 \to a_2 \,\&\, b_2]$. The restoration function is the mapping $\rho : \{a_1 \mapsto a, a_2 \mapsto a, b_1 \mapsto b, b_2 \mapsto b\}$. Sequent $\mathcal{S}$ has the following proof in RK:

$$\dfrac{\dfrac{}{a, b \to a}\,(\text{ax}^*) \qquad \dfrac{}{a, b \to b}\,(\text{ax}^*)}{a, b \to a \,\&\, b}\,(\&\text{S})$$

This is an explosion derivation of $\widetilde{\mathcal{S}}$:

$$\dfrac{a_1, b_1 \to w \qquad \dfrac{w \to p \qquad p \to a_2 \qquad p \to b_2}{w \to a_2 \,\&\, b_2}\,(\&\text{S}_m)}{a_1, b_1 \to a_2 \,\&\, b_2}\,(\text{Prx})$$

The corresponding explosion set is:

$$\mathfrak{E}_{\widetilde{\mathcal{S}}} = \big\{[a_1, b_1 \to w], [w \to p], [p \to a_2], [p \to b_2]\big\}$$

Notice that it is possible to trace a connection between $a_1$ occurring in the antecedent of $a_1, b_1 \to w$ and $a_2$ occuring in the succedent of another ESS by passing through the sequent $w \to p$ by means of proxy $w$, arriving at $p \to a_2$ through proxy $p$. Secondly, by passing through $w$ and $p$, but selecting the alternative sequent $p \to b_2$ instead of $p \to a_2$, a connection can be traced between $b_1$ occuring in $a_1, b_1 \to w$ and $b_2$ occuring in $p \to b_2$. The two subsets of $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ corresponding to these intuitive connections are:

$$\mathfrak{E}_{\widetilde{\mathcal{S}},1} = \big\{[a_1, b_1 \to w], [w \to p], [p \to a_2]\big\}$$
$$\mathfrak{E}_{\widetilde{\mathcal{S}},2} = \big\{[a_1, b_1 \to w], [w \to p], [p \to b_2]\big\}$$

Of course, these sets are just the two largest connexion sets arising from this derivation of $\widetilde{\mathcal{S}}$ in RK:

$$\dfrac{a_1, b_1 \to a_2 \qquad a_1, b_1 \to b_2}{a_1, b_1 \to a_2 \,\&\, b_2}\,(\&\text{S})$$

It is possible to construct a top-down derivation from the ESSs contained in the connexion set $\mathfrak{C}_{\widetilde{\mathcal{S}},1}$ using only (Prx):[1]

$$\dfrac{\dfrac{a_1, b_1 \to w \qquad w \to p}{a_1, b_1 \to p}\,(\mathrm{Prx}) \qquad p \to a_2}{a_1, b_1 \to a_2}\,(\mathrm{Prx})$$

Observe how applying the restoration function to the conclusion of this derivation yields $\rho([a_1, b_1 \to a_2]) = [a, b \to a]$. The latter is the sequent, which occurs as the left leaf of the RK proof of $\mathcal{S}$. Accordingly, the following derivation can be constructed from the elementary structural sequents that constitute $\mathfrak{C}_{\widetilde{\mathcal{S}},2}$:

$$\dfrac{\dfrac{a_1, b_1 \to w \qquad w \to p}{a_1, b_1 \to p}\,(\mathrm{Prx}) \qquad p \to b_2}{a_1, b_1 \to b_2}\,(\mathrm{Prx})$$

The restoration of the conclusion yields $\rho([a_1, b_1 \to b_2]) = [a, b \to b]$, which is the right leaf in the proof of $\mathcal{S}$. Removing the proxy variables that were introduced by the proxy cuts of the explosion procedure and restoring the conclusion of these operations, results in the two sequents, which are the generalized axioms of the RK proof.

The example underscores the fact that, in view of RK-provability, the explosion calculus accomplishes too much. Since we are interested in a decision procedure, some of that extra work has to be undone in order to establish the elementary relation expressed in the axiom of LK. The following proposition gives a generalised account of this observation.

**Proposition 9.1** *Let $\widetilde{\widetilde{\Xi}}$ be an occurrence instance of a RK-derivation $\Xi$ with restoration function $\rho$ and let $\langle \mathfrak{E}, w \rangle$ be the explosion set of the end sequent of $\widetilde{\widetilde{\Xi}}$ with occurrence mapping $\pi$. Let further $\mathcal{T}$ be some sequent occurring in $\Xi$, let $\widetilde{\mathcal{T}}$ be the corresponding sequent occurring in $\widetilde{\widetilde{\Xi}}$, and let $\mathfrak{X}_{\widetilde{\mathcal{T}}} \subseteq \mathfrak{E}$ be the connexion set of $\widetilde{\mathcal{T}}$. Then there is a top-down derivation $\Theta$, whose leaves are the elementary structural sequents of $\mathfrak{X}_{\widetilde{\mathcal{T}}}$, and which uses only the exchange rules and $|\mathfrak{X}_{\widetilde{\mathcal{T}}}| - 1$ applications of the proxy cut rule, of a sequent $\widetilde{\mathcal{U}}$, such that $\rho(\pi^{-1}(\widetilde{\mathcal{U}})) = \rho(\widetilde{\mathcal{T}}) = \mathcal{T}$.*

[1] Due to the nature of the logical meta rules of the explosion procedure, applications of exchange rules are generally necessary in order to move proxy variables into cut position.

PROOF: The sequent $\widetilde{\mathcal{U}}$ is constructed along the construction of $\mathfrak{X}_{\widetilde{\mathcal{T}}} = \mathfrak{X}_{\widetilde{\mathcal{T}},k}$ for some $k$. It is $\mathfrak{X}_{\widetilde{\mathcal{T}},0} = \{[\gamma \to w], [w \to \delta]\}$, where $w$ is the watershed proxy. Let $\gamma_0 = \gamma$ and $\delta_0 = \delta$. We initialise the derivation by $\Theta_0$, which is the following application of (Prx):

$$\frac{\gamma_0 \to w \qquad w \to \delta_0}{\gamma_0 \to \delta_0}\,(\mathrm{Prx})$$

The sequent $\pi^{-1}(\gamma_0 \to \delta_0)$ is just the end sequent $\widetilde{\mathcal{S}}_0$ of $\widetilde{\Xi}$, and, hence, the restoration $\rho(\pi^{-1}(\gamma_0 \to \delta_0))$ is the end sequent $\mathcal{S}_0$ of $\Xi$. This initialisation uses one application of (Prx).

Next, assume that the derivation $\Theta_i$ of $\gamma_i \to \delta_i$ has already been constructed, and that $\widetilde{\mathcal{S}}_i = \pi^{-1}([\gamma_i \to \delta_i])$ and $\mathcal{S}_i = \rho(\widetilde{\mathcal{S}}_i)$. According to definition 8.9, the difference $\mathfrak{D}_i = \mathfrak{X}_{\widetilde{\mathcal{T}},i+1} \setminus \mathfrak{X}_{\widetilde{\mathcal{T}},i}$ is either empty, or it contains a single ESS $\mathcal{E}_{i+1}$. If $\mathfrak{D}_i = \emptyset$, then there is an application of an exchange rule in $\widetilde{\Xi}$ corresponding to this step. Hence, let $\Theta_{i+1}$ be a derivation

$$\begin{array}{c}\Theta_i\\ \gamma_i \to \delta_i\\ \hline\hline \gamma_{i+1} \to \delta_{i+1}\end{array}$$

such that $\widetilde{\mathcal{S}}_{i+1} = \pi^{-1}([\gamma_{i+1} \to \delta_{i+1}])$ is the premiss of the exchange rule with conclusion $\widetilde{\mathcal{S}}_i = \pi^{-1}([\gamma_i \to \delta_i])$ in $\widetilde{\Xi}$. Moreover, $\rho(\widetilde{\mathcal{S}}_{i+1}) = \mathcal{S}_{i+1}$ is the premiss of the corresponding exchange rule in $\Xi$. If $\mathfrak{D}_i = \{\mathcal{E}_{i+1}\}$, then, according to definition 8.9, $\mathcal{E}_{i+1}$ has the base proxy $p_i = \pi(C_i)$, where $C_i$ is the main formula of $\widetilde{\mathcal{S}}_i$. Moreover, since $\pi(\widetilde{\mathcal{S}}_i) = [\gamma_i \to \delta_i]$, it is either $p_i \in \{\!\!\{\gamma_i\}\!\!\}$ or $p_i \in \{\!\!\{\delta_i\}\!\!\}$. While it is unproblematic to give the derivation $\Theta_{i+1}$, the reconstruction of $\widetilde{\mathcal{S}}_{i+1}$ does not only depend on $\mathcal{E}_{i+1}$, but also on whether $\mathfrak{E}$ contains other ESSs, which have $p_i$ as base proxy. Out of the eleven cases, one for each possibility for how $\mathcal{E}_{i+1}$ is obtained, we will cover five in detail.

1) If $\mathcal{E}_{i+1} = [\to p_i, x_i]$ and $\mathfrak{E}$ contains another ESS with the same base proxy $p_i$, then $\widetilde{\mathcal{S}}_i$ contains the main formula $C_i = D_i \supset E_i$ in its antecedent, where $\pi(D_i) = x_i$. Since $C_i$, as the main formula of $\widetilde{\mathcal{S}}_i$, is the first formula of its antecedent, $p_i$ must be the first formula in the antecedent of $\gamma_i \to \delta_i$, i.e. for a suitable $\epsilon_i$ it is $\gamma_i = p_i, \epsilon_i$. Then the following is the desired derivation $\Theta_{i+1}$:

$$\dfrac{\dfrac{\to p_i, x_i}{\to x_i, p_i}\,(\mathrm{XS}) \qquad \begin{array}{c}\Theta_i\\ p_i, \epsilon_i \to \delta_i\end{array}}{\begin{array}{c}\epsilon_i \to x_i, \delta_i\\ \hline\hline \epsilon_i \to \delta_i, x_i\end{array}}\,(\mathrm{Prx})$$

We let $[\gamma_{i+1} \rightarrow \delta_{i+1}] = [\epsilon_i \rightarrow \delta_i, x_i]$. Clearly, $\widetilde{\mathcal{S}}_{i+1} = \pi^{-1}([\epsilon_i \rightarrow \delta_i, x_i])$ is the sequent, which is obtained from $\widetilde{\mathcal{S}_i}$ by removing $C_i$ from the antecedent and adding $D_i$ as last formula to the succedent. But this is exactly the left premiss of the application of ($\supset$A) for the conclusion $\widetilde{\mathcal{S}_i}$ in $\widetilde{\Xi}$, from which $\mathcal{E}_i$ was derived in the first place. Of course, under the restoration function $\rho$, this observation carries over to the original derivation $\Xi$.

2) If $\mathcal{E}_{i+1} = [\rightarrow p_i, x_i]$ and $\mathfrak{E}$ does *not* contain another ESS with base proxy $p_i$, then $\widetilde{\mathcal{S}_i}$ contains the main formula $C_i = \neg D_i$ in its antecedent, where $\pi(D_i) = x_i$. Since $C_i$, as the main formula of $\widetilde{\mathcal{S}_i}$, is the first formula of its antecedent, $p_i$ must be the first formula in the antecedent of $\gamma_i \rightarrow \delta_i$, i.e. for a suitable $\epsilon_i$ it is $\gamma_i = p_i, \epsilon_i$. Then the following is the desired derivation $\Theta_{i+1}$:

$$\cfrac{\cfrac{\rightarrow p_i, x_i}{\rightarrow x_i, p_i}\,(\text{XS}) \qquad \cfrac{\Theta_i}{p_i, \epsilon_i \rightarrow \delta_i}}{\cfrac{\epsilon_i \rightarrow x_i, \delta_i}{\epsilon_i \rightarrow \delta_i, x_i}}\,(\text{Prx})$$

The remainder of the argument for this case is the same as in the previous one. We let $[\gamma_{i+1} \rightarrow \delta_{i+1}] = [\epsilon_i \rightarrow \delta_i, x_i]$. Once more, the sequent $\widetilde{\mathcal{S}}_{i+1} = \pi^{-1}([\epsilon_i \rightarrow \delta_i, x_i])$ is obtained from $\widetilde{\mathcal{S}_i}$ by removing $C_i$ from the antecedent and adding $D_i$ as last formula to the succedent. This is the premiss of the application of ($\neg$A) in $\widetilde{\Xi}$ with the conclusion $\widetilde{\mathcal{S}_i}$, from which $\mathcal{E}_i$ was derived. Again, under the restoration function $\rho$, this observation carries over to the original derivation $\Xi$.

3) If $\mathcal{E}_{i+1} = [y_i \rightarrow p_i]$ and $\mathfrak{E}$ contains an ESS $[\rightarrow p_i, x_i]$ with the same base proxy $p_i$, then $\widetilde{\mathcal{S}_i}$ contains the main formula $C_i = D_i \supset E_i$ in its antecedent, where $\pi(E_i) = y_i$. Since $C_i$, as the main formula of $\widetilde{\mathcal{S}_i}$, is the first formula of its antecedent, $p_i$ must be the first formula in the antecedent of $\gamma_i \rightarrow \delta_i$, i.e. for a suitable $\epsilon_i$ it is $\gamma_i = p_i, \epsilon_i$. Then the following is the desired derivation $\Theta_{i+1}$:

$$\cfrac{y_i \rightarrow p_i \qquad \cfrac{\Theta_i}{p_i, \epsilon_i \rightarrow \delta_i}}{y_i, \epsilon_i \rightarrow \delta_i}\,(\text{Prx})$$

In this case, we let $[\gamma_{i+1} \rightarrow \delta_{i+1}] = [y_i, \epsilon_i \rightarrow \delta_i]$. Then the sequent $\widetilde{\mathcal{S}}_{i+1} = \pi^{-1}([y_i, \epsilon_i \rightarrow \delta_i])$ can be obtained from $\widetilde{\mathcal{S}}_i$ by replacing $C_i$ by $E_i$ in the antecedent. This is just the right premiss of the application of ($\supset$A) with conclusion $\widetilde{\mathcal{S}}_i$ in $\widetilde{\Xi}$, from which $\mathcal{E}_i$ was derived. Under the restoration function $\rho$, this observation carries over to the original derivation $\Xi$.

4) If $\mathcal{E}_{i+1} = [y_i \rightarrow p_i]$ and $\mathfrak{E}$ contains an ESS $[x_i \rightarrow p_i]$ with the same base proxy $p_i$, then $\widetilde{\mathcal{S}}_i$ contains the main formula $C_i = D_i \vee E_i$ in its antecedent, where $\pi(E_i) = y_i$. Since $C_i$, as the main formula of $\widetilde{\mathcal{S}}_i$, is the first formula of its antecedent, $p_i$ must be the first formula in the antecedent of $\gamma_i \rightarrow \delta_i$, i.e. for a suitable $\epsilon_i$ it is $\gamma_i = p_i, \epsilon_i$. Then the following is the desired derivation $\Theta_{i+1}$:

$$\dfrac{y_i \rightarrow p_i \qquad \begin{matrix}\Theta_i \\ p_i, \epsilon_i \rightarrow \delta_i\end{matrix}}{y_i, \epsilon_i \rightarrow \delta_i}\ (\text{Prx})$$

Identically to the preceding case, we let $[\gamma_{i+1} \rightarrow \delta_{i+1}] = [y_i, \epsilon_i \rightarrow \delta_i]$. Then the sequent $\widetilde{\mathcal{S}}_{i+1} = \pi^{-1}([y_i, \epsilon_i \rightarrow \delta_i])$ can be obtained from $\widetilde{\mathcal{S}}_i$ by replacing $C_i$ by $E_i$ in the antecedent. This is just the right premiss of the application of ($\vee$A) with conclusion $\widetilde{\mathcal{S}}_i$ in $\widetilde{\Xi}$, from which $\mathcal{E}_i$ was derived. Of course, under the restoration function $\rho$, this observation carries over to the original derivation $\Xi$.

5) If $\mathcal{E}_{i+1} = [p_i \rightarrow x_i, y_i]$, then $\widetilde{\mathcal{S}}_i$ contains the main formula $C_i = D_i \vee E_i$ in its succedent, where $\pi(D_i) = x_i$ and $\pi(E_i) = y_i$. Since $C_i$, as the main formula of $\widetilde{\mathcal{S}}_i$, is the last formula of its succedent, $p_i$ must be the last formula in the succedent of $\gamma_i \rightarrow \delta_i$, i.e. for a suitable $\epsilon_i$ it is $\delta_i = \epsilon_i, p_i$. Then the following is the desired derivation $\Theta_{i+1}$:

$$\dfrac{\begin{matrix}\Theta_i \\ \gamma_i \rightarrow \epsilon_i, p_i\end{matrix} \qquad p_i \rightarrow x_i, y_i}{\gamma_i \rightarrow \epsilon_i, x_i, y_i}\ (\text{Prx})$$

We let $[\gamma_{i+1} \rightarrow \delta_{i+1}] = [\gamma_i \rightarrow \epsilon_i, x_i, y_i]$. In this case, the sequent $\widetilde{\mathcal{S}}_{i+1} = \pi^{-1}([\gamma_i \rightarrow \epsilon_i, x_i, y_i])$ can be obtained from $\widetilde{\mathcal{S}}_i$ by replacing $C_i$ by $D_i, E_i$ in the succedent. But this is the premiss of the application of ($\vee$S) with conclusion $\widetilde{\mathcal{S}}_i$ in $\widetilde{\Xi}$, from which $\mathcal{E}_i$ was derived. Again, under the restoration function $\rho$, this observation carries over to the original derivation $\Xi$ and the relevant sequents therein.

The remaining six cases are analogous. Note that in each of the cases (apart from the initialisation), in which an ESS from the connexion set is used, one application of (Prx) is required. In the initialisation step, which uses another application of (Prx), two of the ESSs of the connexion set are involved. Thus, a derivation $\Theta_k$ with end sequent $\gamma_k \to \delta_k$ containing $|\mathfrak{X}_{\widetilde{\mathcal{T}}}| - 1$ applications of (Prx) and the required applications of (XA) and (XS) is obtained after $k$ steps, and we set $\Theta = \Theta_k$ and $\widetilde{\mathcal{U}} = [\gamma_k \to \delta_k]$. According to the construction, it is $\pi^{-1}(\widetilde{\mathcal{U}}) = \widetilde{\mathcal{S}}_k$, where $\widetilde{\mathcal{S}}_k = \widetilde{\mathcal{T}}$. Of course, the restoration of $\widetilde{\mathcal{T}}$ is just $\rho(\widetilde{\mathcal{T}}) = \mathcal{T}$.

Moreover, every ESS contained in $\mathfrak{X}_{\widetilde{\mathcal{T}}}$ is used exactly once as a leaf in the construction of $\widetilde{\mathcal{U}}$, and, according to lemma 8.13, there is exactly one proxy variable in every nontrivial step of the construction, which can be used as cut formula. Therefore, $\Theta$ and $\widetilde{\mathcal{U}}$ are unique for any given $\mathcal{T}$. □

The proposition has established a procedure, which retrieves from a connexion set that sequent, from which it was constructed in the first place. We can generalise this notion by abstracting from the sequent in question, thereby arriving at a useful tool for working with connexion sets, as follows.

**Definition 9.2** *Let $\langle \mathfrak{E}, w \rangle$ be an explosion set and $\mathfrak{X} \subseteq \mathfrak{E}$ be some connextion set. Then the* cut-action $\mathcal{C}(\mathfrak{X})$ *is a derivation containing only applications of the proxy cut rule and the exchange rules and having as leaves all of the elementary structural sequents of $\mathfrak{X}$.*

A cut-action removes some or all of the proxy variables occurring in a connexion set, which were introduced in the explosion procedure. Although applications of the exchange rules are required to move the proxies into cut position, a cut-action is essentially an iterated application of the proxy cut. In order to be able to write $\mathcal{C}(\mathfrak{X})$, we will consider cut-actions to be constructed deterministically following the proof of the preceding proposition. It is obvious that various derivations observing the conditions mentioned in the definition could be built from a connexion set.

The proposition was stated for an arbitrary sequent $\mathcal{T}$ occurring in some RK-derivation. However, for the question of decidability, it is the leaves of RK-derivations, which are of particular interest. Hence, we will narrow our focus on a trivial consequence of the proposition.

**Corollary 9.3** *Let $\widetilde{\widetilde{\Xi}}$ be an occurrence instance of a RK-derivation $\Xi$ with restoration function $\rho$ and let $\langle \mathfrak{E}, w \rangle$ be the explosion set of the end sequent of $\widetilde{\widetilde{\Xi}}$. Let further $\mathfrak{X}_{\mathfrak{S}} \subseteq \mathfrak{E}$ be the connexion set of some branch $\mathfrak{S}$ occurring in $\widetilde{\widetilde{\Xi}}$. Then the cut-action $\mathcal{C}(\mathfrak{X})$ has the conclusion $\widetilde{\mathcal{U}}$, such that $\pi^{-1}(\widetilde{\mathcal{U}}) = \widetilde{\widetilde{\mathcal{U}}}$ is the leaf of $\mathfrak{S}$ and $\rho(\widetilde{\mathcal{U}})$ is the corresponding leaf of $\Xi$.*

PROOF: To the largest part, this follwos from the preceding proposition, when the sequent $\widetilde{\mathcal{T}}$ of the proposition is specifically the leaf of the branch $\mathfrak{S}$. However, the proposition only states that there is some $\widetilde{\mathcal{U}}$ to be constructed by means of a cut-action such that $\pi^{-1}(\widetilde{\mathcal{U}}) = \widetilde{\mathcal{T}}$, whereas here we have the stronger assertion that $\pi^{-1}(\widetilde{\mathcal{U}}) = \widetilde{\mathcal{U}}$. The difference between $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{U}}$ in the proposition is that the former is some sequent occurring in $\widetilde{\Xi}$, possibly containing complex formulae, whereas the latter is the end sequent of the cut-action, which is obtained from elementary structural sequents only, and which can therefore only contain occurrence variables. In this case, the sequent under consideration, is a leaf of an occurrence instance of a RK-derivation, which does not contain any complex formulae, only proper occurrence variables. But $\pi$ acts as identity on the set of proper occurrence variables, and hence, no complex formula can be reinstated by $\pi^{-1}(\widetilde{\mathcal{U}})$. Therefore, it is $\pi^{-1}(\widetilde{\mathcal{U}}) = \widetilde{\mathcal{U}}$. Trivially, the restoration of $\widetilde{\mathcal{U}}$, i.e. $\rho(\widetilde{\mathcal{U}})$, yields a leaf of $\Xi$. □

Not only have we established cut-actions as a means to obtain sequents from arbitrary connexion sets, which correspond to those sequents, from which those connexion sets were constructed. The corollary shows that the cut-action on a connexion set corresponding to a leaf of a derivation yields the leaf itself. These observations are the basis for the decision procedures that will be developed.

## 9.2 A Simple Refutation Procedure

Until now, connexion sets have been constructed for specific RK-derivations and their occurrence instances. Hence, for a given sequent, we first determine its explosion set, then we have to provide a RK-derivation, and only then is it possible to construct the connexion sets based on this derivation. In view of an efficient decision procedure operating on explosion sets, this is obviously an untenable detour. In fact, for the purpose of deciding whether a given sequent is provable or not, the construction of the explosion set and connexion sets would be unnecessary, if a RK-derivation had to be constructed at the same time. What is required, then, is a means to find connections directly within an explosion set. Fortunately, it is very simple to find just any connexion set, as we shall see shortly. Somewhat more effort will have to be put into establishing sufficient sets of connexion sets, which can be put in correspondence with RK-derivations, although the essential ideas are closely related to the notion of connexion trees, and some ideas, which have a bearing on this section, have already been mentioned in that context.

In the preceding section we saw that every cut-action on a connexion set is initialised by undoing that cut of the explosion derivation, which introduced the watershed proxy, i.e. the initialisation step of the explosion procedure. To be more precise, the initialisation of the explosion procedure for a sequent $\Gamma \to \Delta$ will yield sequents $\Gamma \to w$ and $w \to \Delta$, which will, in general, be different from the ESSs $\gamma \to w$ and $w \to \delta$, which are contained in the explosion set. However, we certainly have $\pi^{-1}([\gamma \to w]) = [\Gamma \to w]$ and $\pi^{-1}([w \to \delta]) = [w \to \Delta]$, and it is in this sense that the initialisation of a cut-action can be thought of undoing the initialisation of the explosion procedure. We put this idea to good use by taking the watershed proxy $w$ of an explosion set $\langle \mathfrak{E}, w \rangle$ to be the germ, from which every connexion set can be constructed. Based on this image, we shall call the result of the following construction a *germinated connexion set.* The difference between cut-actions and the intended construction lies in the set of elementary structural sequents, which are available. In the former, the connexion set is already given, and the cut-action can only draw from the ESSs contained therein. Moreover, the manner, in which the connexion set is constructed according to a given derivation, determines the order, in which these ESSs are added to the cut-derivation. In the latter, the entire explosion set is available, and the germinated connexion set has to be constructed from this in a meaningful manner without the guidance of a given derivation. The following definition describes the construction procedure, which will accomplish this feat.

**Definition 9.4** *Let $\langle \mathfrak{E}, w \rangle$ be an explosion set. A* germinated connexion set *$\mathfrak{X}$ is generated by the following procedure:*

- *Initialisation: Let*

$$\mathfrak{X}_0 = \big\{[\gamma \to w], [w \to \delta]\big\},$$
$$G_0 = \{\!\{\gamma\}\!\} \cup \{\!\{\delta\}\!\}.$$

- *Iteration: While $G_i \cap \mathcal{P} \neq \emptyset$, select some ESS $\mathcal{E} \in \mathfrak{E} \setminus \mathfrak{X}_i$ such that $G_i \cap \{\!\{\mathcal{E}\}\!\} \neq \emptyset$ and let*

$$\mathfrak{X}_{i+1} = \mathfrak{X}_i \cup \{\mathcal{E}\},$$
$$G_{i+1} = (G_i \cup \{\!\{\mathcal{E}\}\!\}) \setminus (G_i \cap \{\!\{\mathcal{E}\}\!\}).$$

*If $k$ is the number with $G_k \cap \mathcal{P} = \emptyset$, let $\mathfrak{X} = \mathfrak{X}_k$.*

The procedure constructs, in an interleaving manner, a family of sets of *germs* and a family of connexion sets. The watershed proxy is implicitly assumed to be the common germ for all germinated connexion sets.[2]

The construction begins by setting $\mathfrak{X}_0 = \{[\gamma \rightarrow w], [w \rightarrow \delta]\}$, thereby adding the only ESSs, which share the common germ $w$. Then, the set of germs is initialised by collecting the occurrence variables of $\gamma$ and $\delta$ as $G_0 = \{\!\!\{\gamma\}\!\!\} \cup \{\!\!\{\delta\}\!\!\}$. Keep in mind that the construction is intended to follow the principles underlying the generation of a cut-action. A cut-action using the ESSs of $\mathfrak{X}_0$ results in the sequent $\gamma \rightarrow \delta$. Any ESS $\mathcal{E}$ that is to be added to those of $\mathfrak{X}_0$ in the first iteration has to fulfil the condition that it must be possible to perform a cut on the sequents $\gamma \rightarrow \delta$ and $\mathcal{E}$. The only possible cut formulae are the proxy variables that occur in $\gamma$ and $\delta$. Since no proper occurrence variable can ever occur in more than one ESS and, hence, can never be cut formulae, it is of no ill consequence to collect them as well. If we were to remove the proper occurrence variables, the initialisation would have to be $G_0 = (\{\!\!\{\gamma\}\!\!\} \cup \{\!\!\{\delta\}\!\!\}) \cap \mathcal{P}$. But then the iteration step would become somewhat unwieldy. As no harm is done in keeping the proper occurrence variables, we shall do so. Note that all the proxies that are included in $G_0$ are branching proxies, since $w$ is the base proxy in both ESSs of $\mathfrak{X}_0$.

In the iteration step, an ESS is chosen, which has one of the germs in $G_i$ as one of its proxy variables. There must be such an ESS for every proxy variable of $G_i$. For any proxy variable, say $p$, in the germ set is a branching proxy, i.e. in the explosion derivation, from which $\mathfrak{E}$ was obtained, $p$ must have been introduced by the application of (Prx), and, thereby, there must be one or two ESSs contained in $\mathfrak{E}$, which have the same $p$ as base proxy. What is more, $p$ as a branching proxy and $p$ as a base proxy always occur in complementary positions of the two respective ESSs. Hence, for each $p \in G_i$, there must be at least one ESS $\mathcal{E}$ in $\mathfrak{E} \setminus \mathfrak{X}_i$ having $p$ as base proxy. Note that, since ESSs are chosen from $\mathfrak{E} \setminus \mathfrak{X}_i$ by virtue of their base proxies, and since every ESS contains exactly one base proxy, those that are already contained in $\mathfrak{X}_i$ should not be rechosen. Adding an ESS to a germinating connexion set means that the cut-action is extended to include this ESS, and, thereby, that proxy variable, which is the cut formula, is removed. This means that every proxy variable in the germ set must only be used once. This accounts for the instructions of the iteration step. One ESS $\mathcal{E}$ is arbitrarily selected, which contains one of the germs. Since the usable germs are branching proxies, $\mathcal{E}$ must contain that variable as base proxy. The clause $\mathfrak{X}_{i+1} = \mathfrak{X}_i \cup \{\mathcal{E}\}$ simply

[2]In order to account for the motivation of $w$ as being the germ for all connexion sets, we could try to add a pre-initial step with $\mathfrak{X}_{-1} = \emptyset$ and $G_{-1} = \{w\}$, but then the iteration step would not produce the correct set $\mathfrak{X}_0$. For the sake of elegance, we leave the guiding idea of having connexion sets germinate from $w$ represented only implicitly in the initialisation step of the construction. This irregularity is due to the fact that $w$ occurs as base proxy in both $\gamma \rightarrow w$ and $w \rightarrow \delta$.

adds $\mathcal{E}$ to the connexion set. The modification of the set of germs, given by the clause $G_{i+1} = (G_i \cup \{\!|\mathcal{E}|\!\}) \setminus (G_i \cap \{\!|\mathcal{E}|\!\})$, is more interesting. First, all the occurrence variables of $\mathcal{E}$ are added to $G_i$. Note that $\{\!|\mathcal{E}|\!\}$ contains, apart from the base proxy, proper occurrence variables and branching proxies. Then, the base proxy of $\mathcal{E}$, which is also a germ of $G_i$, is removed from that set. This corresponds to the cut-action having performed the proxy cut on this proxy variable. Since the base proxy of $\mathcal{E}$ is removed, the set $G_{i+1}$, again, only contains proper occurrence variables and branching proxy variables.

Obviously, the first thing, which has to be done in the iteration step, is to inspect, whether the set of germs still contains a proxy variable at all. For if it does not, no ESS can be found satisfying the condition $G_i \cap \{\!|\mathcal{E}|\!\} \neq \emptyset$. The procedure will terminate eventually, because the explosion procedure introduces pairs of base proxies and branching proxies only finitely often, and, hence, the cut-action underlying this construction, accordingly removing them in pairs, will eventually use up all the (branching) proxy variables of the germ set.

The following two properties are obvious consequences of the construction procedure, but it is worth stating them explicitly, because they demonstrate important relations between a germinating connexion set the explosion set, from which it is derived.

**Lemma 9.5** *Let $\widetilde{\mathcal{S}}$ be the occurrence instance of a sequent and let $\langle \mathfrak{E}, w\rangle$ be the explosion set of $\widetilde{\mathcal{S}}$ with occurrence mapping $\pi$. Let $\mathfrak{X}$ be a germinated connexion set.*

1) *If $\mathfrak{X}$ contains an ESS with two branching proxies $q$ and $r$, then $\mathfrak{X}$ contains an ESS with base proxy $q$ and also an ESS with base proxy $r$.*

2) *If $\mathfrak{E}$ contains two ESS having the same base proxy $p$, then $\mathfrak{X}$ contains at most one of these.*

Proof: For 1), a review of the initialisation step and the iteration step reveals that all the branching proxies of the ESSs, which are added to the connexion set in that step, are added to the germ set. Proxies can only be removed from the germ set by adding an ESS, which contains it as base proxy. Since, all proxies that are added as germs have to be removed again in that manner, this is especially true for two proxy variables, which are added through the addition of a single ESS. For 2), recall that every proxy variable $p$ of $\mathfrak{E}$ (apart from the watershed proxy $w$) occurs as a branching proxy in exactly one ESS. Therefore, if the ESS containing $p$ as branching proxy is ever added to the connexion set, $p$ is added as a germ. When $p$ is removed from the germ set, one of the ESSs having $p$ as a base proxy is added to the connexion set. But since the ESS containing $p$ as branching proxy is already a member of the connexion set, $p$ cannot be added as a germ again. Hence, the second ESS containing $p$ as a base proxy cannot be added to the connexion set. □

The two properties are quite relevant for the understanding of the different manners of connectedness that are to be found within a connexion sets.

The first property, which we shall henceforth call *internal branching*, suggests that if an explosion set contains three ESSs $\mathcal{E}$, $\mathcal{F}_1$ and $\mathcal{F}_2$, which are connected by the sharing of two proxy variables in the manner that they both occur as branching proxies in $\mathcal{E}$, and that each one of $\mathcal{F}_1$ and $\mathcal{F}_2$ contains one of the two as base proxy, then every connexion set containing $\mathcal{E}$ must also contain $\mathcal{F}_1$ and $\mathcal{F}_2$. An example of this property is, according to proposition 7.4, item 1), given by the ESS $\pi(A), \pi(B) \to \pi(A \,\&\, B)$, where $A$ and $B$ are complex formulae[3], and the two ESSs having $\pi(A)$ and $\pi(B)$ as base proxies in their succedent. Other possible cases are given by corollaries 7.5 2) and 7.6 2), but also by the two ESSs containing the watershed proxy $w$; for in $\gamma \to w$ and $w \to \delta$, either one of $\gamma$ and $\delta$ can contain more than one branching proxy each, all of which are germs and have to be followed up and connected to other ESSs.

The second property will be called *external branching*, since it is concerned with the fact that two particular ESSs cannot be members of a connexion set at the same time. Consider three ESSs $\mathcal{E}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ of an explosion set, which are connected by a single proxy variable occurring as branching proxy in $\mathcal{E}$ and as base proxy in both $\mathcal{F}_1$ and $\mathcal{F}_2$. Then any connexion set containing $\mathcal{E}$ will contain either $\mathcal{F}_1$ or $\mathcal{F}_2$, but never both. An example of this property corresponding to the one given above is due to proposition 7.4 2). It comprises some ESS having the branching proxy $\pi(A \& B)$ in the succedent and the two ESSs $\pi(A \,\&\, B) \to \pi(A)$ and $\pi(A \,\&\, B) \to \pi(B)$. For the other cases see corollaries 7.5 1) and 7.6 1). This property is related to logical rules having two premises. As a connexion set is always related to a single path or branch of some derivation, it can only contain ESSs related to one of the premises of applications of two premiss rules.

Note that the two properties can overlap, in a manner of speaking, as we shall see in the following example for the generation of a connexion set. Consider the sequent $a \,\&\, c, a \supset b \to b$ and its explosion set

$$\mathfrak{E} = \big\{[a_1, c_1 \to p], [\to q, a_2], [b_1 \to q], [p, q \to w], [w \to b_2]\big\}.$$

Any germinated connexion set must contain the ESSs $p, q \to w$ and $w \to b_2$,

[3] If $A$ or $B$ is not a complex formula, then $\pi(A)$ or $\pi(B)$ is not a proxy variable, and, hence, the precondition of the property is not fulfilled.

and we initialise the procedure by

$$\mathfrak{X}_0 = \{[p, q \to w], [w \to b_2]\},$$
$$G_0 = \{p, q, b_2\}.$$

The ESS $p, q \to w$ contains two branching proxies $p$ and $q$. Due to property 1) of the preceding lemma, two ESSs will have to be selected from $\mathfrak{E}$ for any connexion set, one of which contains $p$ as base proxy and one of which contains $q$ as base proxy. For the former there is only one choice, and we choose this for the next construction step, which yields

$$\mathfrak{X}_1 = \{[a_1, c_1 \to p], [p, q \to w], [w \to b_2]\},$$
$$G_1 = \{a_1, c_1, q, b_2\}.$$

The only proxy in the germ set is $q$. The ESSs $\to q, a_2$ and $b_1 \to q$ both contain $q$ as a base proxy and are candidates for addition. They are also the only ESSs available in $\mathfrak{E} \setminus \mathfrak{X}_1$. According to property 2) of lemma 9.5, only one of them can be added. We select the former and obtain

$$\mathfrak{X}_2 = \{[\to q, a_2], [a_1, c_1 \to p], [p, q \to w], [w \to b_2]\},$$
$$G_2 = \{a_2, a_1, c_1, b_2\}.$$

Since it is $G_2 \cap \mathcal{P} = \emptyset$, the procedure terminates. The overlapping of the two properties occurs in the proxy variable $q$. It is one of the two branching proxies of $p, q \to w$, which, according to property 1), require the addition of two ESSs, one with base proxy $p$ and the other with base proxy $q$. But the explosion set contains two candidates fulfilling the latter, which is the condition of property 2). Consequently, only one of the two can be added in the construction.

Incidentally, the same connexion set could be constructed along the left branch of this RK-proof of $a \,\&\, c, a \supset b \to b$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a, c \to b, a}\,(\text{ax}^*) \qquad \dfrac{}{b, a, c \to b}\,(\text{ax}^*)}{a \supset b, a, c \to b}\,(\supset\text{A})}{a, a \supset b, c \to b}\,(\text{XA})}{a, c, a \supset b \to b}\,(\text{XA})}{a \,\&\, c, a \supset b \to b}\,(\&\text{A})$$

Since a germinated connexion set is constructed from the explosion set alone, it is not related to any particular RK-derivation. Of course, this is quite in the spirit of the claim that the explosion set is a more fundamental representation of the meaning of a sequent than a derivation. However, some relation between germinated connexion sets and RK-derivations has to be established in order to have properties of such sets reflect back to derivations.[4] The following lemma establishes this relation.

**Proposition 9.6** *Let $\widetilde{\mathcal{S}}$ be the occurrence instance of a sequent and let $\langle \mathfrak{E}, w \rangle$ be the explosion set of $\widetilde{\mathcal{S}}$ with occurrence mapping $\pi$. Then, for every germinated connexion set $\mathfrak{X}$, there is a RK-derivation $\widetilde{\Xi}$ with end sequent $\widetilde{\mathcal{S}}$ such that $\widetilde{\Xi}$ contains a branch $\mathfrak{S}$, whose leaf is the end sequent of the cut-action $\mathcal{C}(\mathfrak{X})$.*

Proof: We construct $\mathfrak{S}$ following the construction of $\mathfrak{X}$ and the corresponding cut-action. Note that the explosion derivation yields an occurrence mapping $\pi$, the inverse of which will be used to generate sequents of the desired branch from the end sequents of cut-actions. Let $\{\mathfrak{X}_i\}_{0 \leq i \leq k}$ be the family of connexion sets leading up to $\mathfrak{X} = \mathfrak{X}_k$. The cut-action on $\mathfrak{X}_0$ yields a sequent $\gamma_0 \rightarrow \delta_0$, and $\widetilde{\mathcal{S}}_0 = \pi^{-1}([\gamma_0 \rightarrow \delta_0]) = \widetilde{\mathcal{S}}$, which is the base of the branch. For $\mathfrak{X}_{i+1}$ ($i < k$), we consider the ESS $\mathcal{E}_{i+1}$, given by $\mathfrak{X}_{i+1} \setminus \mathfrak{X}_i = \{\mathcal{E}_{i+1}\}$. The sequent $\mathcal{E}_{i+1}$ contains a base proxy, which also occurs in the end sequent $\gamma_i \rightarrow \delta_i$ of the cut-action on $\mathfrak{X}_i$. However, the cut-action on $\mathfrak{X}_{i+1}$, resulting in $\gamma_{i+1} \rightarrow \delta_{i+1}$, no longer contains that proxy, but all the remaining occurrence variables of $\mathcal{E}_{i+1}$. According to the proof of proposition 9.1, the logical rule and, in the appropriate cases, the choice of one of two premises of that logical rule and its main and side formula or side formulae can be inferred from $\mathcal{E}_{i+1}$ and an inspection of $\mathfrak{E}$. Then $\widetilde{\mathcal{S}}_{i+1} = \pi^{-1}([\gamma_{i+1} \rightarrow \delta_{i+1}])$ is the corresponding premiss of that rule with regard to the conclusion $\widetilde{\mathcal{S}}_i = \pi^{-1}([\gamma_i \rightarrow \delta_i])$ up to the applications of exchange rules required to move the appropriate main formula into the required position. Then, clearly, for a RK-derivation, in which applications of exchange rules are abstracted from, with a path $(\widetilde{\mathcal{S}}_0, \ldots, \widetilde{\mathcal{S}}_i)$ already given, $(\widetilde{\mathcal{S}}_0, \ldots, \widetilde{\mathcal{S}}_i, \widetilde{\mathcal{S}}_{i+1})$ must also be a path of that derivation. The construction terminates after $k$ steps, yielding $(\widetilde{\mathcal{S}}_0, \ldots, \widetilde{\mathcal{S}}_k)$. Moreover, the germ set $G_k$ does not contain any proxy variables. Therefore, $\gamma_k \rightarrow \delta_k$ cannot contain any proxy variable, and, hence, $\widetilde{\mathcal{S}}_k$ cannnot contain any complex formula and must be a leaf of that derivation. Finally, a branch $\mathfrak{S}$ of a tree, in which the applications of the exchange rules are explicit, can swiftly be constructed from $(\widetilde{\mathcal{S}}_0, \ldots, \widetilde{\mathcal{S}}_k)$ by inspecting pairs $(\widetilde{\mathcal{S}}_i, \widetilde{\mathcal{S}}_{i+1})$ for $0 \leq i < k$ and inserting the required intermediate sequents, which are obtained by appropriate applications of exchange rules. □

An easy corollary of this proposition will finally provide the desired refutation procedure.

[4]This will be exploited for the refutation procedure.

**Corollary 9.7** *Let $\widetilde{\mathcal{S}}$ be the occurrence instance of a sequent $\mathcal{S}$ and let $\rho$ be the restoration function. Let $\langle \mathfrak{E}, w \rangle$ be the explosion set of $\widetilde{\mathcal{S}}$ with occurrence mapping $\pi$. Let $\mathfrak{X} \subseteq \mathfrak{E}$ be any germinated connexion set, and let $\widetilde{\mathcal{U}}$ be the end sequent of the cut-action $\mathcal{C}(\mathfrak{X})$. If the restoration $\rho(\widetilde{\mathcal{U}})$ is not an instance of* (ax$^*$)*, then $\mathcal{S}$ does not have a RK-proof.*

Proof: According to the preceding lemma, $\widetilde{\mathcal{U}} = \widetilde{\mathcal{S}}_k$ is the leaf of branch $(\widetilde{\mathcal{S}}_0, \ldots, \widetilde{\mathcal{S}}_k)$ of some RK-derivation $\widetilde{\Xi}$. Then $\rho(\widetilde{\mathcal{U}})$ is the leaf of a derivation $\Xi$, which is obtained by applying the restoration function $\rho$ to every sequent occurring in $\widetilde{\Xi}$. Clearly, if $\rho(\widetilde{\mathcal{U}})$ is not an instance of (ax$^*$), then $\Xi$ is not a RK-proof.

It remains to be shown that under these circumstances there can be no RK-proof of $\mathcal{S}$. First of all, no leaf of any RK-derivation can contain any complex formula, every complex formula occurring anywhere in the branch must eventually be the main formula of a rule application. Moreover, each sequent $\widetilde{\mathcal{S}}_i$ with $i \neq k$ of the branch $(\widetilde{\mathcal{S}}_0, \ldots, \widetilde{\mathcal{S}}_k)$, which is not the conclusion of an exchange rule, contains a main formula $C_i$, of which side formula $D_i$ and, where applicable, side formula $E_i$ are identifiable. We collect these main formulae in the set $\mathcal{M} = \{C_i \mid 0 \leq i < k \text{ and } C_i \text{ is main formula of } \widetilde{\mathcal{S}}_i\}$. In developing the branch from $\widetilde{\mathcal{S}}_0$ to $\widetilde{\mathcal{S}}_k$, each one of these main formulae is addressed by an application of the corresponding rule. Of course, if $D_i$ is a complex formula, then, for some $i < j < k$, it is $D_i = C_j$, as side formula $D_i$ becomes the main formula of another rule application. The same goes for a second side formula $E_i$. Hence, any other RK-derivation of $\widetilde{\mathcal{S}}$ must contain a branch having the same set $\mathcal{M}$ of main formulae, and, hence, have $\widetilde{\mathcal{U}}$ as one of its leaves, thereby rendering the branch open. □

Therefore, in order to refute the provability of a sequent $\mathcal{S}$, given the explosion set of its occurrence instance, some germinated connexion set $\mathfrak{X}$ has to be found such that, for the end sequent $\widetilde{\mathcal{U}} = [\gamma \to \delta]$ of the cut-action $\mathcal{C}(\mathfrak{X})$, it is $\rho(\{\!\!\{\gamma\}\!\!\}) \cap \rho(\{\!\!\{\delta\}\!\!\}) = \emptyset$. This property can be exploited for a refutation procedure. For this purpose, two additional sets are constructed, which accumulate the proper occurrence variables of antecedent and succedent positions of the selected ESSs. On the other hand, the accumulation of the ESSs themselves is, strictly speaking, no longer required, although it can be retained if the demonstration of a counterexample in the form of a germinated connexion set is required.

**Definition 9.8** *Let $\langle \mathfrak{E}, w \rangle$ be the explosion set of an occurrece instance $\widetilde{\mathcal{S}}$ of a sequent $\mathcal{S}$, and let $\rho$ be the restoration function. The* refutation procedure *for $\mathcal{S}$ is given by the following algorithm.*

- *Initialisation: Let*

$$\begin{aligned}\mathfrak{X}_0 &= \big\{[\gamma \to w], [w \to \delta]\big\},\\ G_0 &= \{\!\!\{\gamma\}\!\!\} \cup \{\!\!\{\delta\}\!\!\},\\ A_0 &= \{\!\!\{\gamma\}\!\!\} \cap \mathcal{O},\\ S_0 &= \{\!\!\{\delta\}\!\!\} \cap \mathcal{O}.\end{aligned}$$

- *Iteration: While* $G_i \cap \mathcal{P} \neq \emptyset$*, select ESS* $\mathcal{E} = [\eta \to \theta] \in \mathfrak{E} \setminus \mathfrak{X}_i$ *such that* $G_i \cap \{\!\!\{\mathcal{E}\}\!\!\} \neq \emptyset$ *and let*

$$\begin{aligned}\mathfrak{X}_{i+1} &= \mathfrak{X}_i \cup \{\mathcal{E}\},\\ G_{i+1} &= (G_i \cup \{\!\!\{\mathcal{E}\}\!\!\}) \setminus (G_i \cap \{\!\!\{\mathcal{E}\}\!\!\}),\\ A_{i+1} &= A_i \cup (\{\!\!\{\eta\}\!\!\} \cap \mathcal{O}),\\ S_{i+1} &= S_i \cup (\{\!\!\{\theta\}\!\!\} \cap \mathcal{O}).\end{aligned}$$

*Let* $k$ *be the natural number such that* $G_k \cap \mathcal{P} = \emptyset$*. If* $\rho(A_k) \cap \rho(S_k) = \emptyset$*, then reject* $\mathcal{S}$ *on the basis of the counterexample* $\mathfrak{X}_k$*, otherwise restart with the initialisation step.*

The refutation of a sequent $\mathcal{S}$ succeeds, if some germinated connexion set can be constructed, which has the desired property stated in corollary 9.7. A connexion set, which does not have this property, can be the result of a wrong choice in one of one out of two ESS. In this case, this simple refutation procedure simply restarts the construction. Obviously, the procedure presupposes a random element in the selection of ESSs in the iteration step, such that the probability of obtaining any particular germinating connexion set is non-zeo. It is also obvious that a procedure, which instead deterministically generates all the possible choices, is a decision procedure.

The particularities of constructing the intersection are important. While it would be somewhat more efficient to perform the test for non-emptiness of the intersection in an interleaved manner with the construction of the sets rather than after the iteration has terminated, it would result in an incomplete connexion set, which could not serve as a counterexample.[5]

In order to illustrate the refutation procedure, we shall consider the sequent $a \vee b \to a \,\&\, b$. We yield the following explosion set from its occurrence instance $a_1 \vee b_1 \to a_2 \,\&\, b_2$ with restoration function $\rho$:

$$\mathfrak{E} = \left\{ \begin{array}{l} [a_1 \to p], [b_1 \to p], [p \to w] \\ [w \to q], [q \to a_2], [q \to b_2] \end{array} \right\}$$

[5]Of course, counterexamples could be generated from that set by continuing the original construction procedure for germinated connexion sets.

Initialising the procedure, beginning at the watershed proxy $w$, gives

$$\begin{aligned}
\mathfrak{X}_0 &= \{[p \to w], [w \to q]\}, \\
G_0 &= \{p, q\}, \\
A_0 &= \emptyset, \\
S_0 &= \emptyset.
\end{aligned}$$

Since $G_0 \cap \mathcal{P} = \{p, q\}$, some ESS other than those already in $\mathfrak{X}_0$ containing either $p$ or $q$ has to be added. In this case, any one of the remaining ESSs fulfils the requirement. We choose the sequent $a_1 \to p$ and obtain

$$\begin{aligned}
\mathfrak{X}_1 &= \{[a_1 \to p], [p \to w], [w \to q]\}, \\
G_1 &= \{a_1, q\}, \\
A_1 &= \{a_1\}, \\
S_1 &= \emptyset.
\end{aligned}$$

With $G_1 \cap \mathcal{P} = \{q\}$, one of the sequents $q \to a_2$ and $q \to b_2$ have to be chosen. Selecting the former yields

$$\begin{aligned}
\mathfrak{X}_2 &= \{[a_1 \to p], [p \to w], [w \to q], [q \to a_2]\}, \\
G_2 &= \{a_1, a_2\}, \\
A_2 &= \{a_1\}, \\
S_2 &= \{a_2\}.
\end{aligned}$$

Since $G_2 \cap \mathcal{P} = \emptyset$, the iteration terminates. With $\rho(A_2) \cap \rho(S_2) = \{a\}$, there is no basis for rejecting $a \vee b \to a \,\&\, b$, however. Hence, the procedure restarts with the initialisation. Assume that the second run yields sets $\mathfrak{X}'_1 = \mathfrak{X}_1$, $G'_1 = G_1$, $A'_1 = A_1$ and $S'_1 = S_1$. This time, we choose the other ESS having $q$ as a base proxy, $q \to b_2$, which was not selected in the first course, and thereby obtain

$$\begin{aligned}
\mathfrak{X}_2 &= \{[a_1 \to p], [p \to w], [w \to q], [q \to b_2]\}, \\
G_2 &= \{a_1, b_2\}, \\
A_2 &= \{a_1\}, \\
S_2 &= \{b_2\}.
\end{aligned}$$

Again it is $G'_2 \cap \mathcal{P} = \emptyset$, and the iteration terminates. Since $\rho(A'_2) \cap \rho(S'_2) = \emptyset$, the sequent is eventually rejected. The end sequent of the cut-action $\mathcal{C}(\mathfrak{X}'_2)$ is the sequent $a_1 \to b_2$. Its restoration $a \to b$ can be found as a leaf in both RK-derivations of the sequent $a \vee b \to a \,\&\, b$, neither of which constitues a proof:

$$\dfrac{\dfrac{\dfrac{}{a \to a}\,(\mathrm{ax}^*) \qquad a \to b}{a \to a \mathbin{\&} b}\,(\&\mathrm{S}) \qquad \dfrac{b \to a \qquad \dfrac{}{b \to b}\,(\mathrm{ax}^*)}{b \to a \mathbin{\&} b}\,(\&\mathrm{S})}{a \lor b \to a \mathbin{\&} b}\,(\lor\mathrm{A})$$

$$\dfrac{\dfrac{\dfrac{}{a \to a}\,(\mathrm{ax}^*) \qquad b \to a}{a \lor b \to a}\,(\lor\mathrm{A}) \qquad \dfrac{a \to b \qquad \dfrac{}{b \to b}\,(\mathrm{ax}^*)}{a \lor b \to b}\,(\lor\mathrm{A})}{a \lor b \to a \mathbin{\&} b}\,(\&\mathrm{S})$$

The refutation procedure terminates for any unprovable sequent and provides a connexion set, from which a sequent can be constructed by means of a cut-action. The restoration of that sequent occurs as leaf in every RK-derivation of that sequent, and it is *not* an instance of ($\mathrm{ax}^*$). In this sense, the connexion set constitutes a counterexample to the provability of the sequent. However, if the sequent happens to be provable, the procedure does not terminate.

## 9.3 Remarks on a Decision Procedure

It is evident that the refutation procedure can be utilised for a decision procedure by checking every possible germinated connexion set. However, we will merely provide an estimate of the number of connexion sets that have to be considered for this purpose.

We begin by investigating an example. Consider the provable sequent $\mathcal{S} = [a, a \supset b \to a \mathbin{\&} b]$. The following is the explosion derivation of $\widetilde{\mathcal{S}}$:

$$\dfrac{\dfrac{\to p, a_2 \qquad b_1 \to p \qquad a_1, p \to w}{a_1, a_2 \supset b_1 \to w}\,(\supset\mathrm{A}_m) \qquad \dfrac{w \to q \qquad q \to a_3 \qquad q \to b_2}{w \to a_3 \mathbin{\&} b_2}\,(\&\mathrm{S}_m)}{a_1, a_2 \supset b_1 \to a_3 \mathbin{\&} b_2}\,(\mathrm{Prx})$$

The explosion set of $\widetilde{\mathcal{S}}$ is then

$$\mathfrak{E}_{\widetilde{\mathcal{S}}} = \big\{[\to p, a_2], [b_1 \to p], [a_1, p \to w], [w \to q], [q \to a_3], [q \to b_2]\big\}.$$

When constructing germinated connexion sets there are two choices to be made: one for the proxy $p$ and one for the proxy $q$. For after the initialisation step, both $p$ and $q$ are germs, and for each of them two ESSs remain to be chosen as additions to the connexion set. Since either $p$ or $q$ can be selected as germs for the first iteration step, these choices of ESSs are independent of one another. Hence, they result in four possible connexion sets:

$$\begin{aligned}
\mathfrak{X}_{\widetilde{\mathcal{S}},1} &= \big\{[\to p, a_2], [a_1, p \to w], [w \to q], [q \to a_3]\big\}\\
\mathfrak{X}_{\widetilde{\mathcal{S}},2} &= \big\{[b_1 \to p], [a_1, p \to w], [w \to q], [q \to a_3]\big\}\\
\mathfrak{X}_{\widetilde{\mathcal{S}},3} &= \big\{[\to p, a_2], [a_1, p \to w], [w \to q], [q \to b_2]\big\}\\
\mathfrak{X}_{\widetilde{\mathcal{S}},4} &= \big\{[b_1 \to p], [a_1, p \to w], [w \to q], [q \to b_2]\big\}
\end{aligned}$$

As the example shows, the number of different connexion sets, which can be developed from any given explosion set, depends on external branchings occurring therein. In fact, we can derive a limit to the number of possible connexion sets from the number of external branchings.

**Lemma 9.9** *Let* $\mathfrak{E}$ *be an explosion set and let* $n$ *be the number of external branchings that occur in* $\mathfrak{E}$*. Then* $\mathfrak{E}$ *contains at most* $2^n$ *different germinated connexion sets.*

PROOF: Consider all cases of an ESS $\mathcal{E}$, in which some proxy variable $p$ occurs as branching proxy, and two ESSs $\mathcal{F}_1$ and $\mathcal{F}_2$, in which the same proxy occurs as base proxy in the complementary position of the sequent. According to definition 9.4, if such a configurations is encountered in the construction of a germinated connexion set, i.e. if $\mathcal{E}$ is already contained in the connexion set that is being generated, then eventually a choice has to be made, which one of $\mathcal{F}_1$ and $\mathcal{F}_2$ should be included in the connexion set. Since with property 9.5, item 2) the ESS, which is not chosen, cannot be added to the connexion set in a later step of the construction, it is clear that at this choice is indeed essential. The worst case is that, in which the choices are independent of one another, i.e. in which every possible choice actually has to be made in each possible construction of a germinated connexion set. □

Of course, the number of different connexion sets contained in any given explosion set, can also be determined by looking into how the different kinds of ESSs arise in the explosion procedure.

**Corollary 9.10** *The explosion set* $\mathfrak{E}_{\widetilde{\mathcal{S}}}$ *of an occurrence instance* $\widetilde{\mathcal{S}}$ *has at most* $2^{\mathbf{C}^+_{\widetilde{\mathcal{S}}}+\mathbf{D}^-_{\widetilde{\mathcal{S}}}+\mathbf{I}^-_{\widetilde{\mathcal{S}}}}$ *different connexion sets.*

PROOF: A proxy $p$ of an explosion derivation, which occurs in two logical premises and a context premiss, is introduced due to a conjunctive main formula in the succedent or a disjunctive or implicative main formula in the antecedent of a sequent. These two logical premises are the base sequents of two context branches, whose leaves are two ESSs of the explosion set containing $p$ as a base proxy. It is these ESSs, which give rise to an external branching. The ESS, which is taken from the leaf of the context branch, contains $p$ as branching proxy. When we consider a sequent $\mathcal{S}$, we must take into consideration the fact that only those conjunctive subformulae of $\mathcal{S}$, which have positive occurrences, and only those disjunctive or implicative subformulae, which have negative occurences, will lead to such configurations in the explosion derivation. □

An example of a worst case has been given in the sequent $\mathcal{S}$ above. A simple general scheme for a worst case is a sequent $A_1, \ldots, A_m \to B_1, \ldots, B_m$, where each $A_i$ is a disjunctive or implicative formula, whose immediate subformulae are proper occurrence variables, and each $B_j$ is a conjunctive formula, whose immediate subformulae are also proper occurrence variables. According to the preceding lemma, the explosion set of such a sequent contains $2^{m+n}$ connexion sets.

As another example, in which the maximal number of germinated connexion sets is not reached, consider the sequent $\mathcal{T}$: $(a \supset b) \vee c \to b \vee c$. There is a negative occurrence of an implication occurring underneath a negative occurrence of a disjunction, and we can expect a dependent choice in this case. The explosion set of $\widetilde{\mathcal{T}}$ is:

$$\mathfrak{E}_{\widetilde{\mathcal{T}}} = \left\{ \begin{array}{l} [\to q, a_1], [b_1 \to q], [q \to p], [c_1 \to p], \\ [p \to w], [w \to r], [r \to b_2, c_2] \end{array} \right\}$$

For all possible connexion sets of $\widetilde{\mathcal{T}}$, the initialisation is the set containing the two ESSs representing the structures of antecedent and succedent, i.e.

$$\begin{aligned} \mathfrak{X}_0 &= \big\{[p \to w], [w \to r]\big\}, \\ G_0 &= \{p, r\}. \end{aligned}$$

Out of the two germs $p$ and $r$, the latter leads to the sequent $r \to b_2, c_2$, which must therefore be part of every connexion set. Adding it in the first iteration, we obtain

$$\begin{aligned} \mathfrak{X}_1 &= \big\{[p \to w], [w \to r], [r \to b_2, c_2]\big\}, \\ G_1 &= \{p, b_2, c_2\}. \end{aligned}$$

The other proxy $p$ leads to the external branching of the ESSs $q \to p$ and $c_1 \to p$, only one of which can be added to any connexion set. However, if the latter is added, this yields

$$\begin{aligned} \mathfrak{X}_2 &= \big\{[c_1 \to p], [p \to w], [w \to r], [r \to b_2, c_2]\big\}, \\ G_2 &= \{c_1, b_2, c_2\}. \end{aligned}$$

With $G_2 \cap \mathcal{P} = \emptyset$, the construction is complete. The ESSs, which are not part of the connexion set, are $\mathfrak{E}_{\widetilde{\mathcal{T}}} \setminus \mathfrak{X}_2 = \big\{[\to q, a_1], [b_1 \to q], [q \to p]\big\}$. Since $q$ did not become a germ in this construction, neither one of the ESSs constituting an external branching on this proxy can be chosen. If instead of $c_1 \to p$ the sequent $q \to p$ had been chosen, a third iteration step with the only germ $q$ would have to be performed, which would have lead to a dependent choice between the sequents $\to q, a_1$ and $b_1 \to q$. Hence, there are $1 + 2$ different connexion sets to be developed in $\mathfrak{E}_{\widetilde{T}}$ from the germ $w$:

$$\mathfrak{X}_{\widetilde{T},1} = \big\{[c_1 \rightarrow p], [p \rightarrow w], [w \rightarrow r], [r \rightarrow b_2, c_2]\big\}$$
$$\mathfrak{X}_{\widetilde{T},2} = \big\{[\rightarrow q, a_1], [q \rightarrow p], [p \rightarrow w], [w \rightarrow r], [r \rightarrow b_2, c_2]\big\}$$
$$\mathfrak{X}_{\widetilde{T},3} = \big\{[b_1 \rightarrow q], [q \rightarrow p], [p \rightarrow w], [w \rightarrow r], [r \rightarrow b_2, c_2]\big\}$$

These examples shall suffice to demonstrate that the explosion calculus is far from being suited as a method akin to proof search. The employment of (Prx) in the explosion procedure disperses contexts with considerable effort by means of the introduction of proxy variables. A decision procedure has to reverse this dispersal for each possible connexion set, which is contained in the explosion set. Although relatively efficient ways of doing this could be described, such as the retaining of common shared sets of ESSs, any such procedure is far more involved than a proof search in RK. The price of having generated an extensive and entirely structural meaning of a sequent is to have to put considerable additional effort into generating and checking its connexion sets.

# Part III

# Logical Tomography

# Chapter 10

# Hypergraphs and Bipartite Graphs

In this part, we will develop the notions that are required to relate ESSs and explosion sets to suitable set-theoretic entities. The most immediate approach is to interpret an elementary structural sequent $\gamma \rightarrow \delta$ as a tuple $(\widehat{\gamma}, \widehat{\delta})$ for suitable interpretations $\widehat{\gamma}$ and $\widehat{\delta}$. Then, an entire explosion set $\mathfrak{E} = \{\gamma_i \rightarrow \delta_i\}_i$ can be related to a set $\widehat{\mathfrak{E}} = \big\{(\widehat{\gamma_i}, \widehat{\delta_i})\big\}_i$ of such tuples. The set $\widehat{\mathfrak{E}}$ is the desired relational interpretation of the explosion set. For a suitable carrier set $\mathfrak{V}_{\widehat{\mathfrak{E}}} = \{\widehat{\gamma_i}, \widehat{\delta_i}\}_i$, which encompasses all the interpretations $\widehat{\gamma_i}$ and $\widehat{\delta_i}$ of the antecedents and succedents of the explosion set, the whole explosion set is interpreted by the graph $\langle \mathfrak{V}_{\widehat{\mathfrak{E}}}, \widehat{\mathfrak{E}} \rangle$. For an ESS $\gamma \rightarrow \delta$, both $\gamma$ and $\delta$ are sequences of proper occurrence variables and proxy variables, and the most immediate interpretation would give $\mathfrak{V}_{\widehat{\mathfrak{E}}} \subseteq (\mathcal{O} \cup \mathcal{P})^* = \mathcal{V}^*$.

However, elements of $\mathcal{V}^*$ are ordered tuples, in which each occurrence of a variable is related to a specific position within the tuple. The exact position of individual occurrence variables (and formulae in general) within a sequent can be changed without any prerequisite by means of the exchange rules, however. The explosion calculus in particular makes an unrecorded use of exchange rules in its meta rules. Therefore, the ESSs, which are obtained as leaves of an explosion derivation, do not in any way represent applications of exchange rules. Of course, this was never intended in the first place. Gentzen had envisioned the structure of sequents as less tight than that of formulae in the sense that it is manipulable by means of the structural rules of LK in a manner that allows certain changes of the formal structure while retaining the logical content. The exchange rules of LK are thus understood as providing a means for compensating the order of formulae inherent as a notational artifact of linear notation. Since one of our goals is to give a non-syntactic relational interpretation of sequents, it is certainly an imperative to remove the intrinsic element of syntactic representations: the imposition of a linear order upon its components. It is generally known that the combination of exchange rules and contraction rules in LK allow antecedent and succedent of a sequent to be treated as ordered representations of sets of formulae: exchange rules allow to obtain any permutation of the order, in which the formulae are given, while the contraction rules (in top-down perspective of the rules) can be used to remove duplicities of formulae.

Our use of occurrence instances of sequents had the purpose of explicitly accounting of duplicities by effectively removing them. It is for this reason that the contraction rules do not play any role in RK. However, we have explicitly maintained the exchange rules up to this point. In moving from sequents as syntactical entities to their relational interpretation, we will cast off this last remnant of syntactical representation. By means of repeated applications of the exchange rules, every possible rearrangement of the antecedent and succedent of a sequent can be obtained. Hence, ESSs $\gamma \to \delta$ and $\gamma' \to \delta'$, for which there is a derivation consisting only of applications of (XA) and (XS)

$$\frac{\gamma' \to \delta'}{\gamma \to \delta}$$

should be modelled by the same relational entity. For our interpretation this would suggest to let $\widehat{\gamma} \subseteq \mathcal{O} \cup \mathcal{P}$ and $\widehat{\delta} \subseteq \mathcal{O} \cup \mathcal{P}$, which would lead to $\mathfrak{V}_{\widehat{\mathfrak{E}}} \subseteq \mathbb{P}(\mathcal{O} \cup \mathcal{P}) = \mathbb{P}(\mathcal{V})$, where $\mathbb{P}$ is the powerset operator. The interpretation of an explosion set $\mathfrak{E}$ would then be the graph $\langle \mathfrak{V}_{\widehat{\mathfrak{E}}}, \widehat{\mathfrak{E}} \rangle$. Although this would be a perfectly reasonable representation as far as the interpretation of a sequent goes, it has the drawback of using sets of variables as fundamental entities. This takes the notion of a formula, which has already been reduced to a somewhat featureless level by only considering variables, i.e. atomic formulae, as constituents of ESSs, to yet another level of abstraction. The representation of the ESSs as pairs of sets of occurrence variables is not in question. However, sets of variables should not be considered as the carrier of these tuples, but rather the individual variables themselves. Briefly, the interpretation of explosion sets, which will be introduced in detail in the next chapter, is as follows. For an explosion set $\mathfrak{E} = \{\gamma_i \to \delta_i\}_i$, we will define $\widehat{\mathfrak{E}} = \big\{(\widehat{\gamma_i}, \widehat{\delta_i})\big\}_i = \big\{(\{\!\!\{\gamma_i\}\!\!\}, \{\!\!\{\delta_i\}\!\!\})\big\}_i$ and $\mathcal{V}_{\widehat{\mathfrak{E}}} = \mathcal{V} \cap \bigcup_i (\{\!\!\{\gamma_i\}\!\!\} \cup \{\!\!\{\delta_i\}\!\!\})$, and we let $\langle \mathcal{V}_{\widehat{\mathfrak{E}}}, \widehat{\mathfrak{E}} \rangle$ be the hypergraph that serves as *relational interpretation* of $\mathfrak{E}$.[1] That

[1]The terminology is intended to reflect Hertz' tenet that reasoning is primarily concerned with relations between abstract elements.

is, instead of using graphs on subsets of $\mathbb{P}(\mathcal{V})$, we will use hypergraphs on subsets of $\mathcal{V}$ as models for explosion sets.

In the remainder of this chapter, the required graph-theoretical notions required for the treatment of hypergraphs will be introduced. While certain notions can be generalised from standard graphs, the additional structure of hyperarcs will require the introduction of new concepts.

## 10.1 Directed Hypergraphs and Hyperarcs

In this context, we will use the term *standard graph* to refer to the usual understanding of the term "graph". Although undirected hypergraphs will not be used in the interpretation of explosion sets, we shall introduce them very briefly in order to present the graph-theoretical context of directed hypergraphs and also to slowly set the mood for the peculiarities that distinguish hypergraphs from standard graphs.

The notion of a standard graph or directed standard graph on a set of vertices $V$ can be extended to accomodate the possibility of an edge or arc connecting more than two vertices simply by replacing the binary relation on $V$ by a binary relation on $\mathbb{P}(V)$. This is very straightforward, especially considering that we only have to concern ourselves with finite sets.

**Definition 10.1** *A* hypergraph *is a pair $\langle V, \mathfrak{H} \rangle$, where $V$ is a finite set of* vertices *(or* nodes*) and $\mathfrak{H} \subseteq \mathbb{P}(V)$ is a finite set of* hyperedges*.*

Since $\mathfrak{H} \subseteq \mathbb{P}(V)$, $\mathfrak{H}$ is a finite family of subsets of $V$. Therefore, a hypergraph can be visualized by a Venn diagram. If we write $\mathbb{P}_n(M)$ for the set of all subsets of $M$ of cardinality $n$, then in the special case of $\mathfrak{H} \subseteq \mathbb{P}_2(V)$ the hypergraph is just a standard graph.[2] Even in the general case, some notions from standard graphs, suitably modified, can be used to characterise hypergraphs.

**Definition 10.2** *In a hypergraph $\langle V, \mathfrak{H} \rangle$, a vertex $v \in V$ is* incident *with the hyperedge $\mathfrak{e} \in \mathfrak{H}$, if $v \in \mathfrak{e}$. Two vertices $v, w$ with $v \neq w$ are* adjacent *(or* neighbouring*), if there is a hyperedge $\mathfrak{e} \in \mathfrak{H}$ such that $v \in \mathfrak{e}$ and $w \in \mathfrak{e}$. For a vertex $v$, the* neighbourhood *of $v$, denoted by $v^\circ$, is the set of all neighbouring vertices. Hyperedges $\mathfrak{e}, \mathfrak{f} \in \mathfrak{H}$ with $\mathfrak{e} \neq \mathfrak{f}$ are $n$-*adjacent*, if $|\mathfrak{e} \cap \mathfrak{f}| = n > 0$. They are* adjacent*, if they are $n$-adjacent for some $n > 0$. A hyperedge $\mathfrak{e}$ is said to be* subsumed *by the hyperedge $\mathfrak{f}$, if $\mathfrak{e} \subseteq \mathfrak{f}$.*

[2]The set of edges of a standard graph is occasionally given in just this manner instead of as a binary relation, i.e. instead of a set of pairs.

Adjacency of vertices is a more general notion for hypergraphs, since the number of neighbouring vertices of some vertex $v$ is not necessarily equal to the number of hyperedges, to which $v$ is incident. Moreover, adjacency of hyperedges is no longer a primitive notion, since the number of vertices mediating such an adjacency is variable. For example, in the hypergraph $\langle V, \mathfrak{H}\rangle$ with $V = \{p, q, r, s, t\}$ and $\mathfrak{H} = \big\{\{p, q, r\}, \{r, s\}, \{p, r, t\}\big\}$, hyperedges $\{p, q, r\}$ and $\{p, r, t\}$ are 2-adjacent, since both contain the vertices $p$ and $r$, whereas hyperedges $\{p, q, r\}$ and $\{r, s\}$ as well as hyperedges $\{r, s\}$ and $\{p, r, t\}$ are 1-adjacent.

Since undirected hypergraphs are not relevant for the suggested interpretation of elementary structural sequents, we shall not, apart from a few references, consider them further. Instead, we swiftly proceed to the definitions of directed hypergraphs and hyperarcs.

**Definition 10.3** *A* directed hypergraph *is a pair $\mathcal{H} = \langle V, \mathfrak{H}\rangle$, where $V$ is a finite set of* vertices *or* vertices *and $\mathfrak{H} \subseteq \mathbb{P}(V) \times \mathbb{P}(V)$ is a finite set of* directed hyperedges *or* hyperarcs. *For hyperarc $\mathfrak{e} = (S, T)$, $S$ is the set of* initial vertices, *denoted by ${}^\bullet\mathfrak{e}$, and $T$ is the set of* terminal *vertices, denoted by $\mathfrak{e}^\bullet$, and $S \cup T$ is the set of* supporting *vertices, denoted by ${}^\bullet\mathfrak{e}^\bullet$; the* indegree $\deg^-(\mathfrak{e})$ *is the number of its initial vertices, i.e.* $\deg^-(\mathfrak{e}) = |{}^\bullet\mathfrak{e}|$, *and the* outdegree $\deg^+(\mathfrak{e})$ *is the number of its terminal vertices, i.e.* $\deg^+(\mathfrak{e}) = |\mathfrak{e}^\bullet|$. *For a vertex $v$, the* outdegree $\deg^+(v)$ *is the number of edges, for which $v \in {}^\bullet\mathfrak{e}$, and the* indegree $\deg^-(v)$ *is the number of edges, for which $v \in \mathfrak{e}^\bullet$. Vertex $v$ is* initial *in $\mathcal{H}$, if* $\deg^-(v) = 0$*; it is* terminal *in $\mathcal{H}$, if* $\deg^+(v) = 0$*; it is* external *in $\mathcal{H}$, if it is initial or terminal; it is* internal *in $\mathcal{H}$, if it is neither initial nor terminal.*

For example, in the directed hypergraph $\mathcal{H} = \langle V, \mathfrak{H}\rangle$ with $V = \{p, u, v, w, x, y\}$ and $\mathfrak{H} = \big\{\big(\{u, v\}, \{w\}\big), \big(\{w\}, \{p\}\big), \big(\{p\}, \{x\}\big), \big(\{p\}, \{y\}\big)\big\}$, for vertex $p$, it is $\deg^-(p) = 1$ and $\deg^+(p) = 2$, and for the hyperarc $\mathfrak{e} = \big(\{u, v\}, \{w\}\big)$, it is $\deg^-(\mathfrak{e}) = 2$ and $\deg^+(\mathfrak{e}) = 1$. Vertices $u$ and $v$ are initial in $\mathcal{H}$, vertices $x$ and $y$ are terminal in $\mathcal{H}$, and vertices $p$ and $w$ are internal in $\mathcal{H}$. For directed standard graphs, in- and outdegrees of edges are always 1, so the notion has been specifically introduced for directed hypergraphs. See figure 10.1 for the rendering of a sizeable directed hypergraph.[3]

In view of the above definition, plain hypergraphs are sometimes called *undirected* hypergraphs. Of course, any undirected hypergraph can easily be turned into a directed hypergraph by giving each of its hyperedges an orientation.

[3] We will use *graph representations* for displaying hypergraphs. This notions will be formally defined shortly.

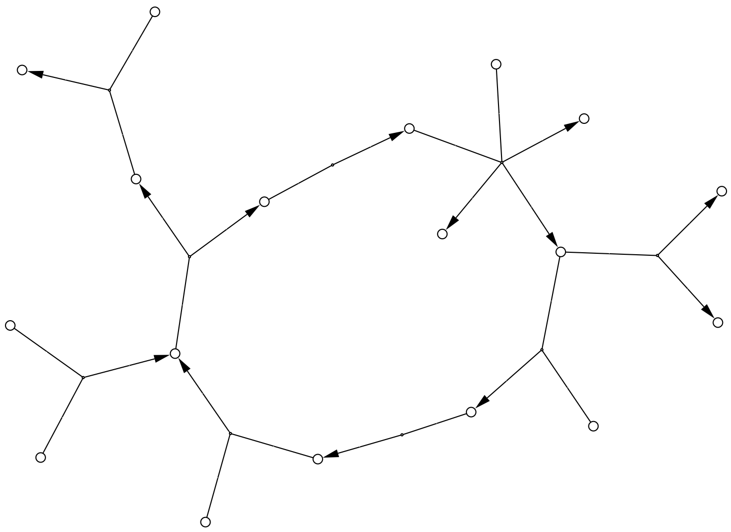


Figure 10.1: A directed hypergraph

**Definition 10.4** *A* strict orientation *of the hypergraph* $\langle V, \mathfrak{H} \rangle$ *is a directed hypergraph* $\langle V, \vec{\mathfrak{H}} \rangle$*, such that for every hyperedge* $\mathfrak{e} \in \mathfrak{H}$*, the set* $\vec{\mathfrak{H}}$ *contains exactly one hyperarc* $\vec{\mathfrak{e}} \in \mathbb{P}(V) \times \mathbb{P}(V)$*, where* $^\bullet\vec{\mathfrak{e}} \cap \vec{\mathfrak{e}}^\bullet = \emptyset$ *and* $^\bullet\vec{\mathfrak{e}} \cup \vec{\mathfrak{e}}^\bullet = \mathfrak{e}$.

Every directed edge $\vec{\mathfrak{e}}$ of a strict orientation is thus an ordered 2-partition of the underlying edge $\mathfrak{e}$ in the undirected hypergraph. Hence, every hyperedge $\mathfrak{e}$ has $2^{|\mathfrak{e}|}$ possible orientations, and the hypergraph $\langle V, \mathfrak{H} \rangle$ has $\prod_{\mathfrak{e} \in \mathfrak{H}} 2^{|\mathfrak{e}|}$ different orientations.[4] Conversely, every directed hypergraph $\langle V, \vec{\mathfrak{H}} \rangle$ has an *underlying undirected hypergraph* $\langle V, \mathfrak{H} \rangle$, which is obtained by replacing every hyperarc $(S, T)$ by the hyperedge $S \cup T$. Due to the loss of extra structure, it is in general only $|\mathfrak{H}| \leq |\vec{\mathfrak{H}}|$. For example, both hyperarcs $\big(\{p, q\}, \{r\}\big)$ and $\big(\{p\}, \{q, r\}\big)$ collapse onto the single hyperedge $\{p, q, r\}$. Also note that it is possible that $S \cap T \neq \emptyset$. For example, the hyperarc $\big(\{a, b\}, \{b, c\}\big)$ collapses onto the hyperedge $\{a, b, c\}$.

Some of the notions introduced in definition 10.2 have to be modified in order to accomodate directed hypergraphs.

**Definition 10.5** *In a directed hypergraph* $\langle V, \mathfrak{H} \rangle$*, a vertex* $v \in V$ *is* incident *with the hyperarc* $\mathfrak{e} \in \mathfrak{H}$*, if* $v \in {}^\bullet\mathfrak{e}$ *or* $v \in \mathfrak{e}^\bullet$*. Vertex* $v$ immediately preceeds *vertex* $w$*, denoted by* $v \prec w$*, if there is a hyperarc* $\mathfrak{e} \in \mathfrak{H}$ *such that* $v \in {}^\bullet\mathfrak{e}$ *and*

[4]The condition $^\bullet\vec{\mathfrak{e}} \cap \vec{\mathfrak{e}}^\bullet = \emptyset$ is not necessary, which makes the definition "strict".

*$w \in \mathfrak{e}^\bullet$; in this case, $w$ is said to* immediately succeed *$v$, denoted by $w \succ v$. For a vertex $w$, the* predecessor neighbourhood *of $w$, denoted by $w^{\succ}$, is the set of all immediately preceeding vertices; the* successor neighbourhood *of $v$, denoted by $v^{\prec}$, is the set of all immediately succeeding vertices.*

Due to the fact that a directed hyperedge is an ordered pair, the notion of incidence is no longer fine enough. In its stead, we have the two symmetric binary relations $\prec$ and $\succ$ on $V$, which are defined by the hyperarcs in $\mathfrak{H}$. A hyperarc $\mathfrak{e}$ with $\deg^-(\mathfrak{e}) = m$ and $\deg^+(\mathfrak{e}) = n$ contributes $m \cdot n$ pairs to both relations. Of course, general adjacency on hypergraphs can be defined as the relation $\prec \cup \succ$. For the hyperarc $\mathfrak{e} = \big(\{u, v\}, \{w\}\big)$ from the preceding example, it is $u \prec w$ and $v \prec w$, and, thus, $w^{\succ} = \{u, v\}$.

As for vertices, adjacency is too general a notion for directed hypergraphs. In its stead, we must introduce several new notions.

**Definition 10.6** *In a directed hypergraph $\langle V, \mathfrak{H}\rangle$, a hyperarc $\mathfrak{e} \in \mathfrak{H}$ $k$-*preceeds *hyperarc $\mathfrak{f} \in \mathfrak{H}$, denoted by $\mathfrak{e} \prec^k \mathfrak{f}$, if $|\mathfrak{e}^\bullet \cap {}^\bullet\mathfrak{f}| = k > 0$; hyperarc $\mathfrak{e}$* preceeds *$\mathfrak{f}$, denoted by $\mathfrak{e} \prec \mathfrak{f}$, if $\mathfrak{e}$ $k$-preceeds $\mathfrak{f}$ for some $k > 0$. A hyperarc $\mathfrak{f}$ $k$-*succeeds *hyperarc $\mathfrak{e}$, denoted by $\mathfrak{f} \succ^k \mathfrak{e}$, if $\mathfrak{e}$ $k$-succeeds $\mathfrak{f}$. A hyperarc $\mathfrak{f}$* succeeds *hyperarc $\mathfrak{e}$, denoted by $\mathfrak{f} \succ \mathfrak{e}$, if $\mathfrak{e}$ preceeds $\mathfrak{f}$. Hyperarcs $\mathfrak{e}$ and $\mathfrak{f}$ with $\mathfrak{e} \neq \mathfrak{f}$ are* $k$-seceding*, denoted by $\mathfrak{e} \curlywedge^k \mathfrak{f}$, if $|{}^\bullet\mathfrak{e} \cap {}^\bullet\mathfrak{f}| = k > 0$; they are* seceding*, denoted by $\mathfrak{e} \curlywedge \mathfrak{f}$, if they are $k$-seceding for some $k > 0$. A vertex $v$ with $\deg^+(v) \geq 2$ is called a* secession vertex*. Hyperarcs $\mathfrak{e}$ and $\mathfrak{f}$ with $\mathfrak{e} \neq \mathfrak{f}$ are* $k$-conceding*, denoted by $\mathfrak{e} \curlyvee^k \mathfrak{f}$, if $|\mathfrak{e}^\bullet \cap \mathfrak{f}^\bullet| = k > 0$; they are* conceding*, denoted by $\mathfrak{e} \curlyvee \mathfrak{f}$, if they are $k$-conceding for some $k > 0$. A vertex $v$ with $\deg^-(v) \geq 2$ is called a* concession vertex*. The hyperarc $\mathfrak{e}$ is* subsumed *by the hyperarc $\mathfrak{f}$, if ${}^\bullet\mathfrak{e} \subseteq {}^\bullet\mathfrak{f}$ and $\mathfrak{e}^\bullet \subseteq \mathfrak{f}^\bullet$.*

The notion of adjacency for hyperarcs breaks up into four notions, of which precession and succession (which are symmetric to one another) are the relations that reflect the directedness of hyperarcs. Two seceding hyperarcs share at least one common inital vertex, whereas two conceding hyperarcs share at least one common terminal vertex. In the examples of figure 10.2, the secession vertices and the concession vertex are rendered in grey. Note that two hyperarcs can be both seceding and conceding. For example, consider the hyperarcs $\mathfrak{e} = \big(\{p, q\}, \{r, s, u\}\big)$ and $\mathfrak{f} = \big(\{q, t, u\}, \{r, s, p\}\big)$, for which ${}^\bullet\mathfrak{e} \cap {}^\bullet\mathfrak{f} = \{q\}$ and $\mathfrak{e}^\bullet \cap \mathfrak{f}^\bullet = \{r, s\}$. In this very involved case, it is even $\mathfrak{e} \prec \mathfrak{f}$ and $\mathfrak{f} \prec \mathfrak{e}$, because $\mathfrak{e}^\bullet \cap {}^\bullet\mathfrak{f} = \{u\}$ and $\mathfrak{f}^\bullet \cap {}^\bullet\mathfrak{e} = \{p\}$.[5] As it is obvious from the

[5]Fortunately, as we will see, such mixed cases will not appear in the interpretations of explosion sets. Instead, for any two hyperarcs $\mathfrak{e}$ and $\mathfrak{f}$, we will always have at most one of $\mathfrak{e} \prec \mathfrak{f}$, $\mathfrak{f} \prec \mathfrak{e}$, $\mathfrak{e} \curlywedge \mathfrak{f}$ and $\mathfrak{e} \curlyvee \mathfrak{f}$.

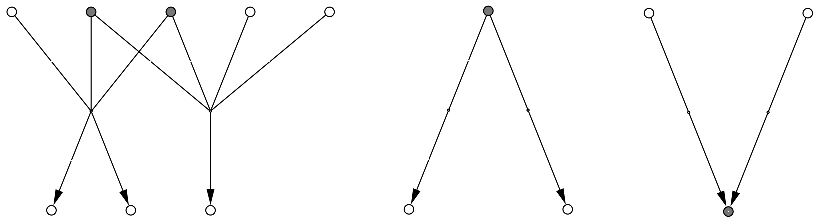


Figure 10.2: 2-seceding, 1-seceding and 1-conceding hyperarcs

definition, both $\curlywedge$ and $\curlyvee$ are symmetric relations. They are not transitive, however. For let $\mathfrak{e} = \big(\{p\},\{u\}\big)$, $\mathfrak{f} = \big(\{p,q\},\{v\}\big)$ and $\mathfrak{g} = \big(\{q\},\{w\}\big)$; then it is both $\mathfrak{e} \curlywedge \mathfrak{f}$, because $\mathfrak{e}$ and $\mathfrak{f}$ share the initial node $p$, and $\mathfrak{f} \curlywedge \mathfrak{g}$, because $\mathfrak{f}$ and $\mathfrak{f}$ share the initial node $q$, but $\mathfrak{e} \not\curlywedge \mathfrak{g}$, since ${}^\bullet\mathfrak{e} \cap {}^\bullet\mathfrak{g} = \emptyset$. It follows from the definitions that an internal vertex $v$ of a hypergraph is incident with two hyperarcs $\mathfrak{e}, \mathfrak{f}$ such that $\mathfrak{e} \prec \mathfrak{f}$, whereas for an external vertex there are no such hyperarcs. External vertices can be coincident with two or more hyperarcs, however. For example, a vertex, which is initial to a hypergraph, can be incident with two seceding hyperarcs.

As it is the case with standard graphs, the interesting aspect of hypergraphs is certainly their manner of connectedness. It is therefore useful to be able to abstract from the actual supporting vertices. For this purpose we introduce the generalisation of the notion of graph isomorphism.

**Definition 10.7** *Directed hypergraphs $\langle V, \mathfrak{H}\rangle$ and $\langle W, \mathfrak{I}\rangle$ are* isomorphic*, if there is a bijection $\sigma : V \to W$, such that $\mathfrak{e} \in \mathfrak{H}$ if and only if $\sigma(\mathfrak{e}) \in \mathfrak{I}$.*

In the definition, $\sigma(\mathfrak{e})$ is used in the manner $\sigma(\mathfrak{e}) = \sigma\big((S,T)\big) = (\sigma(S), \sigma(T))$, where $\sigma(M) = \{\sigma(v) | v \in M\}$. If two directed hypergraphs are supported by sets of vertices $V, W \subseteq U$, then they are isomorphic if there is a permutation $\sigma : U \to U$ such that $\mathfrak{e} \in \mathfrak{H}$ if and only if $\sigma(\mathfrak{e}) \in \mathfrak{I}$.

Another notion, which is useful in the study of standard graphs is that of a rooted graph. This is easily generalised in the following definition.

**Definition 10.8** *A* rooted *directed hypergraph is a tuple $\mathcal{H} = \langle V, \mathfrak{H}, w\rangle$, such that $\langle V, \mathfrak{H}\rangle$ is a directed hypergraph and $w$ is a distinguished element of $V$, called the* root *of $\mathcal{H}$.*

All notions, which have been introduced so far as well as those that will be introduced in the following sections can be trivially extended over rooted directed hypergraphs. For example, two rooted directed hypergraphs are isomorphic, if the directed hypergraphs are isomorphic, and the root of the first is mapped onto the root of the second. Most of the time, we shall omit the designation "rooted" and just speak of directed hypergraphs, even though a root is provided.

We shall sometimes want to add labels to vertices. This is straightforward, and again, there is no difference to be made between directed and undirected hypergraphs.

**Definition 10.9** *A tuple* $\langle V, \mathfrak{H}, \lambda\rangle$ *is called a* vertex-labelled directed hypergraph *on* $L$*, if* $\langle V, \mathfrak{H}\rangle$ *is a directed hypergraph and* $\lambda : V \to L$ *is a (partial) labelling function, which maps vertices of* $V$ *to elements of* $L$*.*

The labelling function can be a partial function, as it is not required that every vertex is assigned to an element of $L$. In particular, certain vertices can be emphasised by having elements of $L$ assigned to them. For a rooted directed hypergraph $\langle V, \mathfrak{H}, w\rangle$, its vertex-labelled variant is given by the tuple $\langle V, \mathfrak{H}, w, \lambda\rangle$.

## Graph Representations of Hypergraphs

The visualisation of directed hypergraphs themselves is still uncharted territory. Although the notion proves to be very useful for modellings in various emerging practical fields, especially in bioinformatics and in cheminformatics, directed hypergraphs have not been investigated in great depth in traditional graph theory. As a consequence, there has not been a particular demand for visualising hypergraphs. Fortunately, there is a simple solution to this problem. Instead of visualising the hypergraph itself, we consider a standard bipartite graph, which suitably represents each hyperarc by several standard arcs.

**Definition 10.10** *The* graph representation *of directed hypergraph* $\langle V, \mathfrak{H}\rangle$ *is the directed bipartite graph* $\langle V, \mathfrak{H}, {}^{\bullet}E \uplus E^{\bullet}\rangle$*, where* ${}^{\bullet}E$ *and* $E^{\bullet}$ *are defined as follows:*

$$ {}^{\bullet}E = \{(p, \mathfrak{e}) \mid p \in V \text{ and } \mathfrak{e} \in \mathfrak{H}, \text{ such that } p \in {}^{\bullet}\mathfrak{e}\} $$
$$ E^{\bullet} = \{(\mathfrak{e}, p) \mid \mathfrak{e} \in \mathfrak{H} \text{ and } p \in V, \text{ such that } p \in \mathfrak{e}^{\bullet}\} $$

The definition is extended over rooted directed hypergraphs in the obvious manner. For a vertex-labelled directed hypergraph $\langle V, \mathfrak{H}, \lambda\rangle$, we have to consider the vertex-labelled directed bipartite graph $\langle V, \mathfrak{H}, {}^{\bullet}E \uplus E^{\bullet}, \lambda'\rangle$, where $\lambda'$ is the trivial extension of the labelling function $\lambda$ to the domain $V \uplus \mathfrak{H}$, i.e. $\lambda'|_V = \lambda$ and $\lambda'|_{\mathfrak{H}}$ is everywhere undefined.

For every hyperarc $\mathfrak{e}$ contained in $\mathfrak{H}$, the relation ${}^{\bullet}E$ contains a pair $(p, \mathfrak{e})$ for every initial vertex $p$ of $\mathfrak{e}$. Likewise, $E^{\bullet}$ relates hyperarcs to their terminal vertices. A hyperarc $\mathfrak{e}$ with $\deg^{-}(\mathfrak{e}) = m$ and $\deg^{+}(\mathfrak{e}) = n$, therefore, induces $m + n$ arcs in the corresponding bipartite graph. For example, the hyperarc $\mathfrak{e} = \big(\{p, q, r\}, \{s, t\}\big)$ produces the standard arcs $(p, \mathfrak{e}), (q, \mathfrak{e}), (r, \mathfrak{e}), (\mathfrak{e}, s), (\mathfrak{e}, t)$ for a suitable bipartite graph.

Renderings of bipartite graphs will be used here for the purpose of visualising hypergraphs. In those renderings, elements of $\mathfrak{H}$ will be represented by tiny circles, elements of $V$ will be represented by large cirlces, and a root will be represented by a large shaded circle. Note that arcs from the set $E^{\bullet}$ will be rendered as arrows, whereas arcs from the set ${}^{\bullet}E$ will be rendered by mere lines, although elements of ${}^{\bullet}E$ are ordered pairs, just as elements of $E^{\bullet}$ are. The reason for this is that hyperarcs $\mathfrak{e}$ with large number of initial vertices would have a large amount of incoming arrowheads on the vertex representing $\mathfrak{e}$. For example, consider the central vertices in the leftmost example of figure 10.2. It is obvious that arrowheads would make the bipartite graphs extremely unsightly. Moreover, this convention shall support the intuition that the rendering of hypergraphs by bipartite graphs is only an auxiliary construction solely for the purpose of visualising hypergraphs. This visualisation should not distract from the fact that the chosen realm of interpretation for explosion sets is that of directed hypergraphs. Note that, from the visualisation, possible orderings of the initial and terminal vertices could mistakenly be assumed, for instance by reading them clockwise relative to the hyperarc vertex or from the top downward. Such orderings are mere artifacts of the chosen visualisation, however.

## The Taxonomy of Hyperarcs

It will prove very useful to define notions for distinguishing hyperarcs on the basis of the size of their sets of inital and terminal vertices. The size of these sets gives a hyperarc a specific shape, when it is visualized. For instance, if for a hyperarc $\mathfrak{e}$, it is both $\deg^{-}(\mathfrak{e}) = 1$ and $\deg^{+}(\mathfrak{e}) = 1$, then the hyperarc is shaped like a standard arc in a standard graph. If $\deg^{+}(\mathfrak{e}) > 1$, then $\mathfrak{e}$ breaks up and fans out in the direction of its terminal vertices. We will consider a hyperarc's terminal vertices as lying in its *forward direction*, and its initial vertices as lying in its *backward direction*.

**Definition 10.11** *A hyperarc $\mathfrak{e}$ is called a* backward branching hyperarc (B-arc)*, if* $\deg^{+}(\mathfrak{e}) \leq 1$*; it is called a* forward branching hyperarc (F-arc)*, if* $\deg^{-}(\mathfrak{e}) \leq 1$*; it is called a* standard hyperarc (S-arc)*, if both* $\deg^{-}(\mathfrak{e}) = 1$ *and* $\deg^{+}(\mathfrak{e}) = 1$.

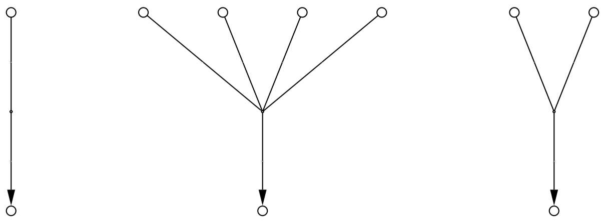


Figure 10.3: General, proper and elementary fusion arcs

A hyperarc is classified as B-arc, if its set of terminal vertices, the forward end, is a singleton set or the empty set. Intuitively, the backward end of the B-arc can still be interesting, but the forward end is not. Respectively, the interesting end of an F-arc is the set of terminal vertices, whereas its set of initial vertices is a singleton set or the empty set. Note that a standard hyperarc $\mathfrak{e}$ with $\deg^-(\mathfrak{e}) = \deg^+(\mathfrak{e}) = 1$ is both a B-arc and a F-arc. For hyperarcs lacking initial vertices or terminal vertices, the interesting end is the one, which is not devoid of any node. A hyperarc $\mathfrak{f}$ with $\deg^-(\mathfrak{f}) = 0$ and $\deg^+(\mathfrak{f}) = 1$ is also both a B-arc and a F-arc, as is a hyperarc $\mathfrak{g}$ with $\deg^-(\mathfrak{g}) = 1$ and $\deg^+(\mathfrak{g}) = 0$ and even the pathological case of a hyperarc $\mathfrak{h}$ with $\deg^-(\mathfrak{h}) = 0$ and $\deg^+(\mathfrak{h}) = 0$. Those hyperarcs, which are neither B-arcs nor F-arcs, are, in a sense, too interesting, because they cross-relate more than one initial vertex with more than one terminal vertex. However, such a hyperarc could be cut into a B-arc and a F-arc by introducing a new vertex.

Some additional notions will enable us to speak of specific kinds of B-arcs and F-arcs, examples for which are given in figures 10.3 and 10.4.

**Definition 10.12** *A hyperarc* $\mathfrak{e}$ *is called*

- *a* fusion arc*, if it is a B-arc with* $\deg^-(\mathfrak{e}) \geq 1$*;*
- *a* proper fusion arc*, if it is a B-arc with* $\deg^-(\mathfrak{e}) \geq 2$*;*
- *an* elementary fusion arc*, if it is a B-arc with* $\deg^-(\mathfrak{e}) = 2$*.*
- *a* fission arc*, if it is a F-arc with* $\deg^+(\mathfrak{e}) \geq 1$*;*
- *a* proper fission arc*, if it is a F-arc with* $\deg^+(\mathfrak{e}) \geq 2$*;*
- *an* elementary fission arc*, if it is a F-arc with* $\deg^+(\mathfrak{e}) = 2$*.*

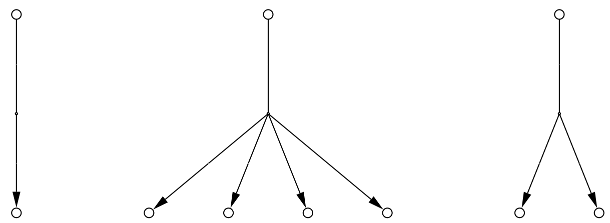


Figure 10.4: General, proper and elementary fission arcs

These notions express the relation of the indegree and the outdegree of a hyperarc. Since for a B-arc $\mathfrak{e}$ it is $\deg^+(\mathfrak{e}) \leq 1$, a fusion arc has the property $\deg^-(\mathfrak{e}) \geq 1 \geq \deg^+(\mathfrak{e})$. For a proper fusion arc the set of initial vertices must not be a singleton, i.e. $\deg^-(\mathfrak{e}) > 1 \geq \deg^+(\mathfrak{e})$, and for an elementary fusion arc the set of initial vertices contains exactly two vertices, i.e. it is specifically $\deg^-(\mathfrak{e}) = 2 \geq 1 \geq \deg^+(\mathfrak{e})$. Intuitively, assuming that attention is a quantifiable resource, when moving the focus of attention from the inital vertices of the proper fusion arc to the terminal node, the attention can be relaxed or fused, since there are less terminal vertices to consider than initial vertices. For F-arcs and fission arcs, the dual relationships apply. This reflects the intuition that, when moving the focus of attention from the initial node to the terminal vertices, the attention has to be split up to account for all the vertices which become relevant. Trivial arcs, which have exactly one initial node and exactly one terminal node, are both (improper) fusion arcs and fission arcs. The notions of proper fusion arc and fission arc rule out this pathological case.

## 10.2 Partial and Total Traversals

It will be necessary to talk about certain parts of hypergraphs. In this section we will generalise many notions from standard graphs to hypergraphs and introduce a very important new one, that of a flow. The various notions of subgraphs are generalised from standard graphs. They can be applied to both directed and undirected hypergraphs.

**Definition 10.13** *A (directed) hypergraph $\mathcal{H} = \langle V, \mathfrak{H} \rangle$ is called a* subgraph *of the (directed) hypergraph $\mathcal{H}' = \langle V', \mathfrak{H}' \rangle$, written $\mathcal{H} \subseteq \mathcal{H}'$, if $V \subseteq V'$ and $\mathfrak{H} \subseteq \mathfrak{H}'$. It is called a* vertex-induced subgraph*, if $\mathfrak{H} = \mathfrak{H}' \cap (\mathbb{P}(V) \times \mathbb{P}(V))$. It is called an* edge-induced subgraph*, if there is no $V'' \subset V$ such that $\langle V'', \mathfrak{H} \rangle$ is a graph.*

Suitable generalisation of traversals, such as walks, paths and cycles, are also useful for directed hypergraphs. For traversals within standard graphs, directed or undirected, there is always a single vertex to be reached when following any edge from any given vertex. Hence, all possible traversals of a standard graph are sequential. Due to the generally non-standard shape of hyperarcs, we can in general only obtain partial sequential traversals. For example, a movement from $p$ along a hyperarc $\big(\{p\},\{q,r\}\big)$ requires a choice one of the vertices $q$ and $r$ as target of the traversal. However, partiality is only necessary, if the sequentiality of traversals within hypergraphs is to be maintained, that is, if the movements along some hyperarc should always lead from a single vertex to another single vertex. Let us begin with the generalisation of the traversal notions from standard graph theory, in which sequentiality is maintaied, and which can, therefore, only lead to partial concepts.[6]

**Definition 10.14** *Let $\mathcal{H} = \langle V, \mathfrak{H}\rangle$ be a directed hypergraph. A* directed partial walk *of length $n$ in $\mathcal{H}$ is an alternating sequence of vertices and hyperarcs $\mathfrak{w} = (v_0, \mathfrak{e}_1, v_1, \ldots, v_{n-1}, \mathfrak{e}_n, v_n)$ with $v_i \in V$ for all $0 \leq i \leq n$ and $\mathfrak{e}_i \in \mathfrak{H}$ for all $1 \leq i \leq n$, such that, for all $1 \leq i \leq n$, it is $v_{i-1} \in {}^{\bullet}\mathfrak{e}_i$ and $v_i \in \mathfrak{e}_i^{\bullet}$. It is called a* directed partial path *of length $n$, if for all $1 \leq i < j \leq n$ it is $\mathfrak{e}_i \neq \mathfrak{e}_j$. If it is further $v_0 = v_n$, then $\mathfrak{w}$ is called a* directed partial circuit *of length $n$. Moreover, if $n \geq 2$ and $v_i \neq v_j$ for all $0 \leq i < j < n$, then $\mathfrak{w}$ is called a* directed partial cycle.

Instead of recording a walk in full detail, i.e. including vertices and hyperarcs, it is sufficient to write down the hyperarcs: $\mathfrak{w} = (\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n)$. Note that this practise can lead to ambiguities if, for some $1 < i \leq n$ and $k \geq 2$, hyperarc $\mathfrak{e}_{i-1}$ $k$-preceeds $\mathfrak{e}_i$. As we will see later on, this cannot be the case with interpretations of explosion sets, and for this reason the abbreviation is safe to use within this context. The defined notions are all partial in the following sense: For every hyperarc $\mathfrak{e}_i$, only two vertices are considered, namely some $v_{i-1} \in {}^{\bullet}\mathfrak{e}_i$ and some $v_i \in \mathfrak{e}_i^{\bullet}$. A partial walk $\mathfrak{w}$ at vertex $v_{i-1}$ continues over some hyperarc $\mathfrak{e}_i$ with $v_{i-1} \in {}^{\bullet}\mathfrak{e}_i$, and immediately afterwards another vertex $v_i$ is chosen from $\mathfrak{e}_i^{\bullet}$ as continuation. The remaining vertices in ${}^{\bullet}\mathfrak{e}_i \setminus \{v_{i-1}\}$ and $\mathfrak{e}_i^{\bullet} \setminus \{v_i\}$ are not taken into consideration. Note that the standard notions of walk, path, circuit and cycle are obtained, if the walk contains only standard edges, i.e. if $\deg^-(\mathfrak{e}_i) = \deg^+(\mathfrak{e}_i) = 1$ for $1 \leq i \leq n$. In figure 10.5, the directed partial cycle, which is contained in the directed hypergraph of figure 10.1, is emphasised.

[6] We shall not call these notions hyperwalks, hyperpaths etc. but retain the terminology of standard graphs and rather explicitly call them *partial* walks, *partial* paths and so on.

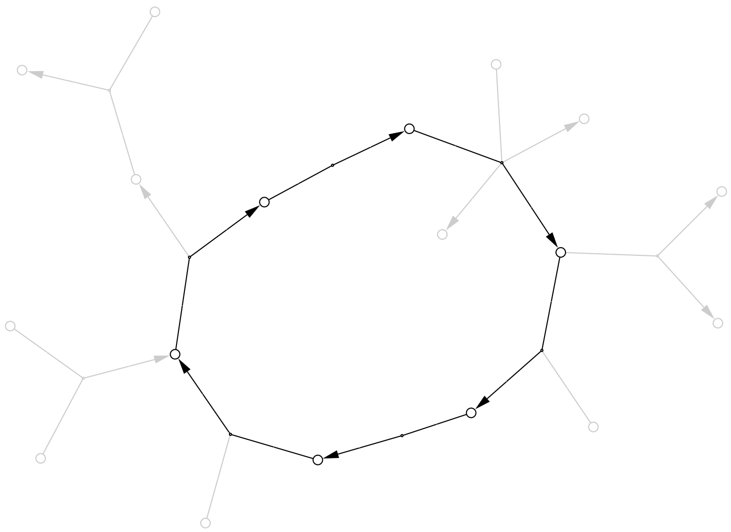


Figure 10.5: A directed hypergraph containing a partial cycle

It is apparent that the notion of a partial walk and its derivatives is no more than an attempt of retaining familiar notions from standard graphs. They do not extend to the additional expressiveness of hypergraphs in an adequate manner. Let us review the fundamental problems that are involved. Partial paths in hypergraphs, as generalisations of paths in standard graphs, are alternating sequences of vertices and nodes. In traversing a partial path, for each hyperedge taken, one of its terminal vertices is chosen as starting point for the following step, and the previous starting point can be released. It is evident that a sequential account of a traversal, which fans out to more than two terminal vertices and funnels in from more than one initial vertex, is impossible. This would already fail in the very simple case that a hyperarc has a single initial node and two terminal nodes. There are even more problematic instances of hyperarcs, which have to be accounted for. Consider, for example, the hyperarc $\big(\{v, w\}, \emptyset\big)$, which is reached in a traversal via the vertex $v$. Should a traversal have to end at this point or should it also account for the possibility of reaching this dead end via $w$? We shall opt for the latter requirement for our notion of total traversal. Moreover, it should certainly be possible to connect other hyperarcs via the vertex $w$. In summary, we want to develop a *spreading* traversal of hyperarcs, which is capable of reversing from an initial vertex to other initial vertices of the same hyperarc and vice versa. At the same time, just as a partial path can begin and end at any node, it should be possible to develop the traversal for only a selection of the initial and terminal vertices of any hyperarc. Finally, to retain at least a local sequentiality, a single vertex should be initial node to at most one hyperarc and terminal node to at most one hyperarc and, at the same time, any two hyperarcs must not have more than one vertex in common.

We will first provide a top-down characterisation of the desired traversal, which we call *spread*. This means that rather than describing, how a spread is successively developed, we will characterise, what kind of directed hypergraph already constitutes a spread. A procedure for generating spreads will be given afterwards. It is necessary to introduce a few auxiliary notions in order to work up to the desired notion. Firstly, a notion of *weak* connectedness has to be introduced, which is a notion of connectedness that ignores the directions of hyperarcs.

**Definition 10.15** *A directed hypergraph* $\mathcal{H} = \langle V, \mathfrak{H} \rangle$ *is* connected*, if, for every pair of vertices* $u$ *and* $w$*, there is some* $n$ *and an alternating sequence of vertices and hyperedges* $\mathfrak{w} = (v_0, \mathfrak{e}_1, v_1, \ldots, v_{n-1}, \mathfrak{e}_n, v_n)$ *with* $v_i \in V$ *for all* $0 \leq i \leq n$ *and* $\mathfrak{e}_i \in \mathfrak{H}$ *for all* $1 \leq i \leq n$*, such that* $u = v_0$ *and* $w = v_n$*, and, for all* $1 \leq i \leq n$*, it is* $v_{i-1} \in {}^\bullet\mathfrak{e}_i^\bullet$ *and* $v_i \in {}^\bullet\mathfrak{e}_i^\bullet$*, and, for all* $1 \leq i < j \leq n$*, it is* $\mathfrak{e}_i \neq \mathfrak{e}_j$*. A* maximally connected component *of* $\mathcal{H}$ *is a connected subgraph* $\langle V', \mathfrak{H}' \rangle$ *such that, for all vertices* $v \in V \setminus V'$ *there is no* $w \in V'$ *such that there is a hyperarc* $\mathfrak{e}$ *with* $\{v, w\} \subseteq {}^\bullet\mathfrak{e}^\bullet$*. A vertex* $v \in V$ *is a* cut vertex*, if the vertex induced subgraph on* $V \setminus \{v\}$ *has more maximally connected components than the original graph.*

By refering to the support of hyperarcs, their direction is ignored. Thereby connections between vertices can also be traced via two initial nodes or two terminal nodes of the same hyperarc. This notion of connectedness, therefore, refers to paths in underlying undirected hypergraphs.

In a second step, we have to now narrow our attention to particular kinds of connected directed hypergraphs, whose structure can accomodate spreads.

**Definition 10.16** *A connected directed hypergraph* $\langle V, \mathfrak{H} \rangle$ *is called a* directed cut hypergraph*, if it meets the following conditions:*

*1) every internal vertex is a cut vertex;*

*2) for every initial vertex* $v \in V$*, it is* $\deg^+(v) = 1$*;*

*3) for every terminal vertex* $v \in V$*, it is* $\deg^-(v) = 1$*;*

*4) for all hyperarcs* $\mathfrak{e}, \mathfrak{f}$*, it is* $|{}^\bullet\mathfrak{e}^\bullet \cap {}^\bullet\mathfrak{f}^\bullet| \leq 1$.

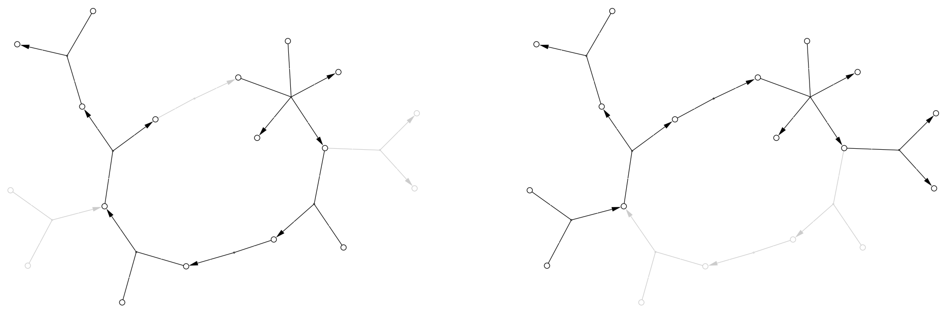


Figure 10.6: Two maximal strands of a directed hypergraph

A directed cut hypergraph must be connected, which means that its hyperarcs and vertices contained a single component. Moreover, removing any internal vertex, i.e. any vertex, which is both initial and terminal vertex, disconnects the directed cut hypergraph. Every internal vertex is, therefore, indispensible for the connectedness of the graph. For this reason, a directed cut hypergraph cannot contain a partial cylce. The next two conditions demand that every initial node and every terminal node of a directed cut hypergraph connects to at most a single hyperarc, which means that no weak connection of two hyperarcs can be traced over external nodes. The inequalities are required to accomodate the trivial directed cut hypergraph consisting only of a single vertex and no hyperarcs. The last condition prevents that two hyperarcs share more than one vertex. Note that this is not a consequence of the other conditions. A counterexample is a directed hypergraph containing hyperarcs $\big(\{p\},\{q,r\}\big)$ and $\big(\{q,r\},\{s\}\big)$. Removing $q$ results in the removal of both hyperarcs, leaving the unconnected vertices $p, r, s$. Removing $r$ instead would also leave the hypergraph disconnected. Hence both $q$ and $r$ are cut vertices.

On the basis of this suitably well-behaved class of graphs, the following definition finally introduces the desired notion.

**Definition 10.17** *A directed cut hypergraph $\mathcal{H} = \langle V, \mathfrak{H} \rangle$ is called a* strand*, if, for every internal $v \in V$, it is $\deg^-(v) = \deg^+(v) = 1$. It is called a* maximal strand *in hypergraph $\mathcal{H}'$, if $\mathcal{H} \subseteq \mathcal{H}'$ and there is no $\mathcal{H}''$ with $\mathcal{H} \subset \mathcal{H}'' \subseteq \mathcal{H}'$ that is a strand.*

In a general directed cut hypergraph, the indegree and outdegree of any internal node $v$ is not limited beyond its defining condition that it must neither be $\deg^-(v) = 0$ nor $\deg^+(v) = 0$. Hence, any internal node of a directed cut hypergraph can be a secession vertex, from which two or more hyperarcs depart, or a concession vertex, into which two or more hyperarcs lead. This is prohibited in a strand, as any internal vertex must not only be incident with exactly two hyperarcs, but it must be so as initial vertex in one and as terminal vertex in the other. Hence, a hyperarc $\mathfrak{e}$ of a strand can, in fact, connect to as many as $\deg^-(\mathfrak{e}) + \deg^+(\mathfrak{e})$ different hyperarcs. Whereas there is always just a single vertex to keep track of after each step in a partial traversal of a hypergraph, a total traversal of a hypergraph requires significantly more effort and bookkeeping, as becomes evident in the following lemma.

**Lemma 10.18** *Let $\mathcal{H} = \langle V, \mathfrak{H} \rangle$ be a directed hypergraph. A subgraph $\mathcal{H}'$ is a maximal strand in $\mathcal{H}$, if and only if it can be constructed by the following procedure.*

- *Initialisation: For some $w \in V$, let*

$$V_0 = \{w\},$$
$$\mathfrak{H}_0 = \emptyset,$$
$$I_0 = \{w\},$$
$$T_0 = \{w\}.$$

- *Iteration: If possible, select one of the following options:*

  *1) For some $v \in T_i$ and some $\mathfrak{e} \in \mathfrak{H} \setminus \mathfrak{H}_i$ such that $v \in {}^\bullet\mathfrak{e}$ and $V_i \cap {}^\bullet\mathfrak{e}^\bullet = \{v\}$, let*

$$V_{i+1} = V_i \cup {}^\bullet\mathfrak{e}^\bullet,$$
$$\mathfrak{H}_{i+1} = \mathfrak{H}_i \cup \{\mathfrak{e}\},$$
$$I_{i+1} = I_i \cup ({}^\bullet\mathfrak{e} \setminus \{v\}),$$
$$T_{i+1} = (T_i \setminus \{v\}) \cup \mathfrak{e}^\bullet.$$

  *2) For some $v \in I_i$ and some $\mathfrak{e} \in \mathfrak{H} \setminus \mathfrak{H}_i$ such that $v \in \mathfrak{e}^\bullet$ and $V_i \cap {}^\bullet\mathfrak{e}^\bullet = \{v\}$, let*

$$V_{i+1} = V_i \cup {}^\bullet\mathfrak{e}^\bullet,$$
$$\mathfrak{H}_{i+1} = \mathfrak{H}_i \cup \{\mathfrak{e}\},$$
$$I_{i+1} = (I_i \setminus \{v\}) \cup {}^\bullet\mathfrak{e},$$
$$T_{i+1} = T_i \cup (\mathfrak{e}^\bullet \setminus \{v\}).$$

*If $k$ be the natural number, for which none of the options is possible, then let $\mathcal{H}' = \langle V_k, \mathfrak{H}_k \rangle$.*

PROOF: For "$\Leftarrow$", we observe that the procedure generates a family of subgraphs $\langle V_i, \mathfrak{H}_i \rangle_{0 \leq i \leq k}$ of $\mathcal{H}$, in which each $\langle V_i, \mathfrak{H}_i \rangle \subset \langle V_{i+1}, \mathfrak{H}_{i+1} \rangle$ for $0 \leq i < k$. The sets $I_i$ and $T_i$ contain all the initial vertices and all the terminal vertices of the respective subgraph. A vertex $v \in I_i$ can be used to extend the subgraph by adding a hyperedge, which has $v$ as a terminal vertex; similarly, a vertex $v \in T_i$ can be used to extend the subgraph by adding a hyperedge, which has $v$ as an initial vertex. In the initialisation step an arbitrary starting vertex is chosen, which is both initial and terminal vertex for puspose of the development of the subgraph in both forward and backward directions. The iteration step is repeated as long as there are fresh vertices and hyperarcs that do not connect to the present graph in a second vertex. In each step of the iteration one of the alternatives 1) and 2) has to be selected. The difference is that in case 1), the chosen vertex is used to add a hyperarc, of which it is an initial vertex, whereas in case 2) it is a terminal vertex of the chosen hyperarc. The considered hyperarc $\mathfrak{e}$ is added to $\mathfrak{H}_i$, and its support ${}^\bullet\mathfrak{e}^\bullet$ is added to $V_i$, thereby extending the subgraph. The terminal vertices of $\mathfrak{e}$ are added to $T_i$, as any hyperarc, which has one of those vertices as initial nodes, succeeds $\mathfrak{e}$ and can be selected in a following iteration step; in a similar manner, the initial vertices of $\mathfrak{e}$ are added to $I_i$, thereby allowing all hyperarcs, which precede $\mathfrak{e}$, to be added at a later stage. In any case, the vertex $v$ has to be removed from that set, either $I_i$ or $T_i$, from which it was selected at the outset of the iteration step, as it has in this very step fulfilled its role as terminal vertex in case 1) or as initial vertex in case 2). Since $v$ thereby fulfils its role of internal node connecting the new hyperarc $\mathfrak{e}$, it must also be exempt from being added to the set of terminal nodes or, respectively, initial nodes. Thereby, each vertex $v$ is used at most once as initial vertex in some hyperarc and at most once as terminal vertex in some other hyperarc, which guarantees $\deg^-(v) \leq 1$ and $\deg^+(v) \leq 1$. The construction maintains $I_i$ as the set of initial vertices of $\langle V_i, \mathfrak{H}_i \rangle$ and $T_i$ as the set of its terminal vertices. Hence, every vertex $v \in V_i \setminus (I_i \cup T_i)$ must be an internal vertex of $\langle V_i, \mathfrak{H}_i \rangle$. Neither present in $I_i$ and $T_i$, it must be initial node of some hyperarc and at the same time terminal node of another, which renders $\deg^-(v) \geq 1$ and $\deg^+(v) \geq 1$. With the added condition above, this yields $\deg^-(v) = \deg^+(v) = 1$ for all internal nodes $v$. The condition that $V_i \cap {}^\bullet\mathfrak{e}^\bullet = \{v\}$ further ensures that a selected hyperarc does not accidentally connect to the hitherto constructed subgraph in any other node, and therefore guarantees that $v$ is a cut vertex. This already implies that $|{}^\bullet\mathfrak{e}^\bullet \cap {}^\bullet\mathfrak{f}^\bullet| \leq 1$ for all $\mathfrak{f} \in \mathfrak{H}_i$. Hence, for $0 < i \leq k$ the procedure traces a strand in $\mathcal{H}$.

If $\mathcal{H}' = \langle V_k, \mathfrak{H}_k \rangle$ is not the entirety of $\mathcal{H}$ and, hence, trivially maximal, and if it is not a maximally connected component of $\mathcal{H}$, then there must be some hyperarc $\mathfrak{e}$ of $\mathcal{H}$, which is not included in $\mathfrak{H}_k$, but for which ${}^\bullet\mathfrak{e}^\bullet \cap V_k \neq \emptyset$.

We consider the various possibilities of how $\mathfrak{e}$ and $\mathcal{H}'$ can be connected.

- If ${}^\bullet\mathfrak{e} \cap T_k \neq \emptyset$, then another step 1) of the procedure would include $\mathfrak{e}$. As the procedure has terminated, this case it not possible.
- If $\mathfrak{e}^\bullet \cap T_k \neq \emptyset$, then two hyperarcs of $\mathcal{H}$ concede on the same vertex $v \in \mathfrak{e}^\bullet \cap T_k$. Hence, it is $\deg^-(v) > 1$. Consequently, including $\mathfrak{e}$ would violate definition 10.16 3).
- If $\mathfrak{e}^\bullet \cap I_k \neq \emptyset$, then another step 2) of the procedure would include $\mathfrak{e}$. As the procedure has terminated, this case it not possible.
- If ${}^\bullet\mathfrak{e} \cap I_k \neq \emptyset$, then two hyperarcs of $\mathcal{H}$ secede from the same vertex $v \in {}^\bullet\mathfrak{e} \cap I_k$. Hence, it is $\deg^+(v) > 1$. Consequently, including $\mathfrak{e}$ would violate definition 10.16 2).
- Otherwise, it must be ${}^\bullet\mathfrak{e}^\bullet \cap (V_k \setminus (I_k \cup T_k)) \neq \emptyset$. Let $v$ be a vertex of that intersection. As $v$ is an internal vertex of the strand $\mathcal{H}'$, it must be $\deg^-(v) = \deg^+(v) = 1$ with respect to it. But in $\mathcal{H}$ the hyperarc $\mathfrak{e}$ also has $v$ as either initial or terminal node, which renders either $\deg^-(v) > 1$ or $\deg^+(v) > 1$ with respect to $\mathcal{H}$. Consequently, including $\mathfrak{e}$ would violate definition 10.17.

Therefore, $\mathcal{H}'$ is a maximal strand in $\mathcal{H}$.

For "$\Rightarrow$", the given strand $\mathcal{H}'$ is traced. We initialise the procedure with an arbitrary vertex $w$ of $\mathcal{H}'$. Each iteration step adds some hyperarc of $\mathcal{H}'$. Since $\mathcal{H}'$ is a strand, each internal vertex $v$ is incident with two hyperarcs, in one it is initial and in the other one it is terminal. In the first case, the corresponding hyperarc can be added to $\mathfrak{H}_i$ by option 2), in the second case it can be added by option 1). In either case, the vertices of the new hyperarc extend the appropriate sets, its initial vertices are added to $I_i$, its terminal vertices to $T_i$ and all of them to $V_i$. The vertex $v$ is removed appropriately from the new sets, as it has been used up in the trace. It is safe to do this, because two hyperarcs of $\mathcal{H}'$ can share at most a single vertex, and, hence, the addition of a hyperarc can never connect back to a vertex that has already been visited. In any case, the new iteration step has extended the trace to another hyperarc of $\mathcal{H}'$ and its respective vertices. The procedure continues until all the hyperarcs of $\mathcal{H}'$ have been added. In this case, $I_k$ contains the initial vertices of $\mathcal{H}'$, and $T_k$ contains its terminal vertices. As $\mathcal{H}'$ is a maximal strand in $\mathcal{H}$, there are no hyperedges in $\mathcal{H}$ that could be added to $\mathcal{H}'$ without destroying some property of strands. As the iteration step can only add hyperarcs that maintain the necessary properties, it must terminate as soon as all of $\mathcal{H}'$ has been traced. □

Compared to the notion of a path in standard graphs and its generalisation to a partial path in hypergraphs, the notion of a strand, which corresponds to a total traversal of a hypergraph, is somewhat involved. This is, of course, due to the fact that hyperarcs have a much richer structure than standard arcs. This extra structure requires that the state of a traversal is not only a single vertex but a set of vertices, each of which can connect to yet another hyperarc, which adds any number of new vertices to the state, while only removing a single one.

We will see in the following chapter that maximal strands are the hypergraphical correspondence to germinated connexion sets. As an entire explosion set generally contains several germinated connexion sets, which have non-empty intersections, explosion sets will not correspond to maximal strands, but only to the more general directed cut hypergraphs.

# Chapter 11

# Explosion Sets as Hypergraphs

In this chapter, we will consider the specifics of the hypergraph interpretation of explosion sets and their connexion sets. It will be shown that connexion sets are represented by maximal strands and explosion sets to general directed cut hypergraphs. Particular consideration will be given to the question of how different RK-derivations of the same sequent can be obtained from such an interpretation. We will further demonstrate how to construct a directed cut hypergraph directly from a given sequent, which is, of course, simply a different if instructive perspective on the explosion procedure.

## 11.1 Interpreting Explosion Sets

The elements of the interpretation of explosion sets, which shall be employed, have already been briefly indicated in the previous chapter. At this point, we will provide a more detailed demonstration and give a number of examples.

The key to the interpretation is to relate a sequence of atomic formulae $\gamma$ to a set $\widehat{\gamma} \stackrel{def}{=} \{\!|\gamma|\!\}$. Recall the the set $\mathcal{V}$ of occurrence variables is the union of the set of proper occurrence variables $\mathcal{O}$, which are the variables replacing propositional variables, and the set of proxy variables $\mathcal{P}$, which are introduced by applications of (Prx) in the explosion procedure. As all the elementary structural sequents of an explosion set contain only occurrence variables, all of the formulae of $\gamma$ are occurrence variables and, therefore, distinct from one another. As a consequence, none of these variables are identified in $\widehat{\gamma}$, and we have $\mathbf{len}(\gamma) = |\widehat{\gamma}|$. This is an important property, since the identification of formulae within a set-theoretic framework would result in the loss of important structure. The sequent $\mathcal{E} = [\gamma \rightarrow \delta]$ will be interpreted by the tuple $\widehat{\mathcal{E}} = (\widehat{\gamma}, \widehat{\delta})$. All of the tuples corresponding to the ESSs of an explosion set will then make up a binary relation on sets of those occurrence variables, which do occur in the explosion set. We can then define the interpretation of an explosion set as follows.

**Definition 11.1** *For an explosion set* $\langle \mathfrak{E} = \{\gamma_i \to \delta_i\}_i, w\rangle$*, the* relational interpretation *thereof is the rooted directed hypergraph* $\langle \mathcal{V}_{\widehat{\mathfrak{E}}}, \widehat{\mathfrak{E}}, w\rangle$*, where*

$$\begin{aligned} \widehat{\mathfrak{E}} &\stackrel{def}{=} \{(\widehat{\gamma}_i, \widehat{\delta}_i)\}_i, \\ \mathcal{V}_{\widehat{\mathfrak{E}}} &\stackrel{def}{=} \mathcal{V} \cap \textstyle\bigcup_i (\widehat{\gamma}_i \cup \widehat{\delta}_i). \end{aligned}$$

*The* relational interpretation *of the occurrence instance* $\widetilde{\mathcal{S}}$ *of a sequent is the relational interpretation of the explosion set* $\langle \mathfrak{E}_{\widetilde{\mathcal{S}}}, w_{\widetilde{\mathcal{S}}}\rangle$*.*

**Proposition 11.2** *Let* $\widetilde{\mathcal{S}}$ *and* $\widetilde{\mathcal{S}}'$ *be two different occurrence instances of the sequent* $\mathcal{S}$*. Then the relational interpretations of* $\widetilde{\mathcal{S}}$ *and* $\widetilde{\mathcal{S}}'$ *are isomorphic.*

PROOF: Let $\langle \mathfrak{E}_{\widetilde{\mathcal{S}}}, w_{\widetilde{\mathcal{S}}}\rangle$ and $\langle \mathfrak{E}_{\widetilde{\mathcal{S}}'}, w_{\widetilde{\mathcal{S}}'}\rangle$ be the explosion sets of $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}}'$. As each occurrence of a propositional variable $a$ in $\mathcal{S}$ is mapped to a unique occurrence variable $a_i$ in $\widetilde{\mathcal{S}}$ and $a_j$ in $\widetilde{\mathcal{S}}'$, there is a permutation $\tau : \mathcal{O} \to \mathcal{O}$ such that $\widetilde{\mathcal{S}}\tau = \widetilde{\mathcal{S}}'$. Then $\mathfrak{E}_{\widetilde{\mathcal{S}}}\tau = \mathfrak{E}_{\widetilde{\mathcal{S}}\tau}$ and $\mathfrak{E}_{\widetilde{\mathcal{S}}'}$ are explosion sets of the same occurrence instance. According to proposition 6.12, there is a renaming $\sigma : \mathcal{P} \to \mathcal{P}$ such that $\mathfrak{E}_{\widetilde{\mathcal{S}}}\tau\sigma = \mathfrak{E}_{\widetilde{\mathcal{S}}'}$. Of course, it is especially $\sigma(\tau(w_{\widetilde{\mathcal{S}}})) = w_{\widetilde{\mathcal{S}}'}$. As $\mathcal{V} = \mathcal{O} \uplus \mathcal{P}$, the composition $\sigma \circ \tau$ is a permutation of $\mathcal{V}$. Hence, the relational interpretations of $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}}'$ are isomorphic. □

The fact that the explosion sets of different occurrence instances have isomorphic relational interpretations allows the following definition.

**Definition 11.3** *Let* $\widetilde{\mathcal{S}}$ *be an occurrence instance of a sequent* $\mathcal{S}$ *with restoration function* $\rho$*, and let* $\langle V, \mathfrak{H}, w\rangle$ *be the relational interpretation of* $\widetilde{\mathcal{S}}$*. Then the* relational interpretation *of* $\mathcal{S}$ *is the labelled rooted directed hypergraph* $\langle V, \mathfrak{H}, w, \rho\rangle$*.*

Recall that the restoration function $\rho : \mathcal{O} \to \mathcal{A}$ maps an occurrence instance of a sequent back to the sequent itself, that is $\widetilde{\mathcal{S}}\rho = \mathcal{S}$. As, for a relational interpretation, it must be $V \subseteq \mathcal{V}$, the function $\rho$, trivially extended over $\mathcal{V} = \mathcal{O} \uplus \mathcal{P}$, assigns a propositional variable $a$ as label to those vertices, which are proper occurrence variables representing occurrences of $a$, and no labels to proxy variables.

Of course, it is also possible to use other labelling functions, such as the identity function $id_{\mathcal{O}}$ on the set of occurrence variables, trivially extended to $\mathcal{V}$, if the proper occurrence variables should be displayed as labels, or the

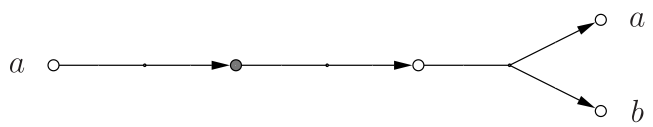



Figure 11.1: The relational interpretation of $a \to a \vee b$

identity function $id_{\mathcal{P}}$ on the set of proxy variables, again trivially extended to $\mathcal{V}$, if the vertices should be labelled only with proxy variables. Another labelling function that will become very relevant shortly is the inverse of the occurrence mapping $\pi$ for an occurrence instance $\widetilde{\mathcal{S}}$.

As an example, consider the sequent $\mathcal{S} = [a \to a \vee b]$ and its occurrence instance $\widetilde{\mathcal{S}} = [a_1 \to a_2 \vee b_1]$. The explosion procedure yields the following explosion set:

$$\mathfrak{E} = \big\{[a_1 \to w], [w \to p], [p \to a_2, b_1]\big\}$$

The binary relation on $\mathbb{P}(\mathcal{V})$ corresponding to $\mathfrak{E}$ is:

$$\widehat{\mathfrak{E}} = \big\{\big(\{a_1\}, \{w\}\big), \big(\{w\}, \{p\}\big), \big(\{p\}, \{a_2, b_1\}\big)\big\}$$

The relational interpretation of $\mathcal{S}$ is the labelled hypergraph $\langle \mathcal{V}_{\widehat{\mathfrak{E}}}, \widehat{\mathfrak{E}}, w, \rho\rangle$, where $\mathcal{V}_{\widehat{\mathfrak{E}}} = \{a_1, w, p, a_2, b_1\}$ and $\rho = \{a_1 \mapsto a, a_2 \mapsto a, b_1 \mapsto b\}$. See figure 11.1 for the representation of this hypergraph, and recall the convention that the watershed proxy is emphasised by a grey vertex.

The example shows how the commutativity of positive occurrences of the disjunction is directly reflected in the interpretation. The positive occurrence of the disjunction $a_2 \vee b_1$ in the succedent is represented by the ESS $p \to a_2, b_1$. Both this ESS and the ESS $p \to b_1, a_2$, which would be obtained by an explosion of $a_1 \to b_2 \vee a_1$, have the same interpretation $(\{p\}, \{a_2, b_1\})$.[1] This property is due to the fact that the succedent $b_1, a_2$ is a permutation of $a_2, b_1$, which are both interpreted by the same set $\{a_2, b_1\}$. In the same manner, our interpretation results in the inherent commutativity of negative occurrences of the conjunction.

Consider the sequent $\mathcal{T} = [a, b \to a \,\&\, b]$ as an example for a positive occurrence of the conjunction. The explosion set of $\widetilde{\mathcal{T}}$ is:

[1]As always, this is understood up to renaming of proxy variables. If necessary, a permutation of proxy variables guarentees that the same proxy variable $p$ is used for the detachment of $a_2 \vee b_1$ and $b_1 \vee a_2$.

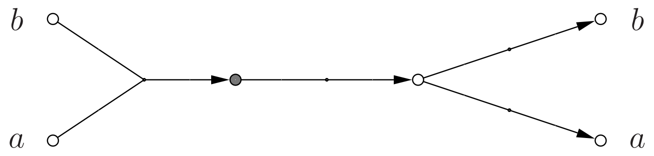



Figure 11.2: The relational interpretation of $a, b \to a \,\&\, b$

$$\mathfrak{F} = \{[a_1, b_1 \to v], [v \to q], [q \to a_2], [q \to b_2]\}$$

The following relation is the relational interpretation of $\mathfrak{F}$:

$$\widehat{\mathfrak{F}} = \{(\{a_1, b_1\}, \{v\}), (\{v\}, \{q\}), (\{q\}, \{a_2\}), (\{q\}, \{b_2\})\}$$

The relational interpretation of the explosion set $\mathfrak{F}$ is the graph $\langle \mathcal{V}_{\widehat{\mathfrak{F}}}, \widehat{\mathfrak{F}}, v, \zeta \rangle$, where $\mathcal{V}_{\widehat{\mathfrak{F}}} = \{a_1, b_1, w, p, a_2, b_1\}$ and $\zeta = \{a_1 \mapsto a, a_2 \mapsto a, b_1 \mapsto b, b_2 \mapsto b\}$. Figure 11.2 depicts this hypergraph. The conjunction is represented in $\mathfrak{F}$ by the two ESSs $q \to a_2$ and $q \to b_2$. In $\widehat{\mathfrak{F}}$, these ESSs are interpreted by the pairs $(\{p\}, \{a_2\})$ and $(\{p\}, \{b_2\})$. Since $\widehat{\mathfrak{F}}$ is also a set, the order, in which these pairs occur, is of no consequence. Hence, the occurrence instance $a_1, b_1 \to b_2 \,\&\, a_2$, in which the order of $a_2$ and $b_2$ is reversed, has the same relational interpretation as $\widetilde{\mathcal{T}}$.

A similar argument gives us commutativity of negative occurrences of the disjunction. Hence, in addition to the desired properties, the proposed interpretation of explosion sets gives identical representations for sequents that differ only in the order of the immediate subformulae of disjunctive and conjunctive formulae.

## 11.2 Connexion Sets and Strands

Having established that the relational interpretation of an explosion set of an explosion set corresponds to a logical tomograph, we will now consider particular subsets of the explosion set of a sequent, namely germinated connexion sets of chapter 9, and investigate their relational interpretations. The connexion sets of chapter 8 differ from those only the manner, in which the latter are to be developed, as determined by an already given derivation, and hence are merely a special case.

Recall that, given an explosion set $\langle \mathfrak{E}, w \rangle$, elementary structural sequents, which are connected via proxy variables, can be consecutively selected from

$\mathfrak{E}$, starting at the watershed proxy $w$. This procedure yields a subset $\mathfrak{X} \subseteq \mathfrak{E}$, in which any two ESSs share at most a single proxy variable. Such a subset of $\mathfrak{E}$ is called a connexion set. The procedure for constructing connexion sets resembles that for generating a total traversal of a hypergraph.

**Proposition 11.4** *Let $\langle \mathfrak{E}, w\rangle$ be an explosion set and $\mathfrak{X}$ be a germinated connexion set thereof. Then the relational interpretation $\langle \mathcal{V}_{\widehat{\mathfrak{X}}}, \widehat{\mathfrak{X}}, w\rangle$ is a maximal strand in the relational interpretation $\langle \mathcal{V}_{\widehat{\mathfrak{E}}}, \widehat{\mathfrak{E}}, w\rangle$.*

PROOF: Consider the construction of a germinated connexion set of definition 9.4. For the initialisation of the procedure, $\mathfrak{X}_0 = \big\{[\delta \to w], [w \to \gamma]\big\}$, the relational interpretation $\mathcal{X}_0 = \langle \mathcal{V}_{\widehat{\mathfrak{X}}_0}, \widehat{\mathfrak{X}}_0, w\rangle$ is obviously a strand. The set of germs is initialised to $G_0 = \{\!\!\{\gamma\}\!\!\} \cup \{\!\!\{\delta\}\!\!\}$, all of which, as branching proxies or proper occurrence variables, are external vertices in $\mathfrak{X}_0$. For the iteration step, we assume that $\mathcal{X}_i$ is a strand. For one of the external vertices $p$ of $G_i$, some ESS $\mathcal{E}$ is selected, in which $p$ occurs as base proxy. In fact, $p$ is an internal vertex of $\mathcal{X}_{i+1}$, as a base proxy occurs in the antecedent of an ESS, if its branching proxy occurs in the succedent of another ESS, and vice versa. We obtain the relational interpretation $\mathcal{X}_{i+1} = \langle \mathcal{V}_{\widehat{\mathfrak{X}}_i} \cup {}^{\bullet}\widehat{\mathcal{E}}^{\bullet}, \widehat{\mathfrak{X}}_i \cup \{\widehat{\mathcal{E}}\}, w\rangle$. Although the construction only demands $G_i \cap \{\!\!\{\mathcal{E}\}\!\!\} \neq \emptyset$, the intersection is always a singleton, as an ESSs from an explosion set at most shares a single proxy variable with any other ESS, and each proper occurrence variable only has a single occurrence in one ESS. Because of this, it is even $\mathcal{V}_{\widehat{\mathfrak{X}}_i} \cap {}^{\bullet}\widehat{\mathcal{E}}^{\bullet} = \{p\}$. This establishes properties 1) and 4) of definition 10.16 for the new hypergraph. For 2) and 3), we observe that the external vertices of $\mathcal{X}_{i+1}$ are given by the set $G_{i+1} = (G_i \cup {}^{\bullet}\widehat{\mathcal{E}}^{\bullet}) \setminus \{p\}$, all of which are branching proxies or proper occurrence variables. As these properties held for all vertices in $G_i$ and, independently, hold for all vertices in ${}^{\bullet}\widehat{\mathcal{E}}^{\bullet}$, they also hold for all external vertices of $\mathcal{X}_{i+1}$. As $p$ is no longer external to $\mathcal{X}_{i+1}$, it is now $\deg^-(p) = \deg^+(p) = 1$ Hence, $\mathcal{X}_{i+1}$ is a strand. Finally, the termination condition of the construction of a germinated connexion set is $G_k \cap \mathcal{P} = \emptyset$, i.e. $G_k$ contains only proper occurrence variables, each of which occurs exactly in a single ESS already included in the germinated connexion set. Consequently, no other ESS can be added and, correspondingly, no other hyperarc can be added to $\langle \mathcal{V}_{\widehat{\mathfrak{X}}}, \widehat{\mathfrak{X}}, w\rangle$. Hence, it is maximal in $\langle \mathcal{V}_{\widehat{\mathfrak{E}}}, \widehat{\mathfrak{E}}, w\rangle$ . □

**Corollary 11.5** *Let $\langle \mathfrak{E}, w\rangle$ be an explosion set and $\mathfrak{X}$ be a germinated connexion set thereof. Then all of the internal vertices of its relational interpretation $\langle \mathcal{V}_{\widehat{\mathfrak{X}}}, \widehat{\mathfrak{X}}, w\rangle$ are elements of $\mathcal{P}$, and all of its external vertices are elements of $\mathcal{O}$.*

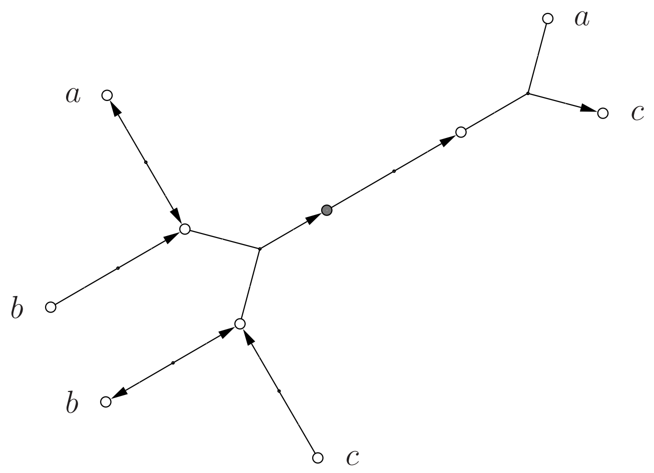



Figure 11.3: The relational interpretation of $a \supset b, b \supset c \rightarrow a \supset c$

PROOF: The termination condition of the procedure that generates a germinated connexion set is $G_i \cap \mathcal{P} = \emptyset$. As $G_i \subseteq \mathcal{V} = \mathcal{O} \uplus \mathcal{P}$, it must be $G_i \subseteq \mathcal{O}$. Consequently, all the external vertices in $\langle \mathcal{V}_{\widehat{\mathfrak{X}}}, \widehat{\mathfrak{X}}, w \rangle$, which are all elements of $G_i$, must be elements of $\mathcal{O}$. At the same time, all the elements that are ever removed from the set of germs during the construction of $\mathfrak{X}$ are proxy variables, i.e. elements of $\mathcal{P}$. As we saw above, those make up all of the internal vertices of $\langle \mathcal{V}_{\widehat{\mathfrak{X}}}, \widehat{\mathfrak{X}}, w \rangle$. □

This corollary has a bearing on the relational interpretation of a sequent $\mathcal{S}$, which, according to definition 11.3, is a labelled rooted directed hypergraph $\langle V, \mathfrak{H}, w, \rho \rangle$, where $\rho$ is the restoration function $\mathcal{O} \rightarrow \mathcal{A}$. For, if, in every maximal strand in that hypergraph, all the external vertices are elements of $\mathcal{O}$ and all the internal vertices are elements ogf $\mathcal{P}$, then $\rho$ labels all of the external vertices of that hypergraph (and only those), because every vertex of it belongs to at least one maximal strand.

Just as an explosion set encompasses a family of connexion sets, the relational interpretation thereof encompasses a family of maximal strands. As an example, let us consider the sequent $\mathcal{S} = [a \supset b, b \supset c \rightarrow a \supset c]$ and its occurrence instance $\widetilde{\mathcal{S}} = [a_1 \supset b_1, b_2 \supset c_1 \rightarrow a_2 \supset c_2]$. Its explosion set is:

$$\mathfrak{E} = \left\{ \begin{array}{l} [\rightarrow p, a_1], [b_1 \rightarrow p], [\rightarrow q, b_2], [c_1 \rightarrow q], \\ [p, q \rightarrow w], [w \rightarrow p], [p \rightarrow a_2, c_2] \end{array} \right\}$$

The relational interpretation of $\mathfrak{E}$ is presented in figure 11.3. It is easy to

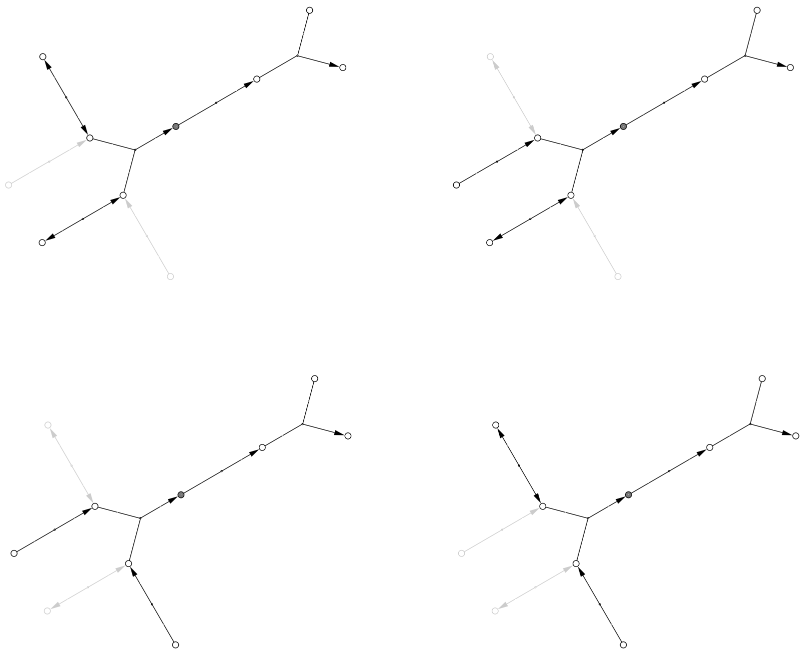


Figure 11.4: The four maximal strands of $a \supset b, b \supset c \rightarrow a \supset c$

see that four different connexion sets can be germinated from the watershed proxy $w$. They are:

$$\begin{aligned}
\mathfrak{X}_{\widetilde{\mathcal{S}},1} &= \{[\rightarrow p, a_1], [\rightarrow q, b_2], [p, q \rightarrow w], [w \rightarrow p], [p \rightarrow a_2, c_2]\} \\
\mathfrak{X}_{\widetilde{\mathcal{S}},2} &= \{[b_1 \rightarrow p], [\rightarrow q, b_2], [p, q \rightarrow w], [w \rightarrow p], [p \rightarrow a_2, c_2]\} \\
\mathfrak{X}_{\widetilde{\mathcal{S}},3} &= \{[b_1 \rightarrow p], [c_1 \rightarrow q], [p, q \rightarrow w], [w \rightarrow p], [p \rightarrow a_2, c_2]\} \\
\mathfrak{X}_{\widetilde{\mathcal{S}},4} &= \{[\rightarrow p, a_1], [c_1 \rightarrow q], [p, q \rightarrow w], [w \rightarrow p], [p \rightarrow a_2, c_2]\}
\end{aligned}$$

Figure 11.4 represents the four maximal strands, which correspond to these connexion sets. The two choices, which are involved in the construction of the germinated connexion sets giving rise to the four maximal strands, are clearly visible as two pairs of conceding hyperarcs.

Proposition 11.4 has already established the most general correspondence between germinated connexion sets and maximal strands. The case of connexion sets, which correspond to particular sequents occurring in a bottom-up derivation of some sequent, is merely a special case.

**Corollary 11.6** *Let $\widetilde{\Xi}$ be the occurrence instance of a RK-derivation of $\widetilde{\mathcal{S}}$, and let $\mathfrak{S} = (\widetilde{\mathcal{S}}_0, \ldots, \widetilde{\mathcal{S}}_k)$ be a branch of $\widetilde{\Xi}$ with $\widetilde{\mathcal{S}}_0 = \widetilde{\mathcal{S}}$. Let $\langle \mathfrak{E}, w \rangle$ be* *an explosion set of* $\widetilde{\mathcal{S}}$ *with ocurrence mapping* $\pi$. *Let* $\mathfrak{X}_{\mathfrak{S},i}$ *be the connexion set corresponding to* $\widetilde{\mathcal{S}}_i$. *Then the family of relational interpretations* $\{\langle \mathcal{V}_{\widehat{\mathfrak{X}}_{\mathfrak{S},i}}, \widehat{\mathfrak{X}}_{\mathfrak{S},i}, w\rangle\}_{0\leq i\leq k}$ *is a family of strands in the relational interpretation* $\langle \mathcal{V}_{\widehat{\mathfrak{E}}}, \widehat{\mathfrak{E}}, w\rangle$, *such that, for* $0 \leq i < k$, *it is* $\mathcal{V}_{\widehat{\mathfrak{X}}_{\mathfrak{S},i}} \subseteq \mathcal{V}_{\widehat{\mathfrak{X}}_{\mathfrak{S},i+1}}$ *and* $\widehat{\mathfrak{X}}_{\mathfrak{S},i} \subseteq \widehat{\mathfrak{X}}_{\mathfrak{S},i+1}$.

PROOF: We construct a germinating connexion set by following the branch $\mathfrak{S}$ upward. Sequent $\widetilde{\mathcal{S}}_0$ corresponds to the initialisation of the construction. For each step, it has to checked, whether $\widetilde{\mathcal{S}}_i$ has main formula $C_i$ in $\widetilde{\Xi}$. If it does, then we select an ESS $\mathcal{E}$, which has $\pi(C_i)$ as base proxy and $\pi(D_i)$ as side proxy for a side formula $D_i$ in $\widetilde{\mathcal{S}}_{i+1}$, and add it to $\mathfrak{X}_{\mathfrak{S},i}$. Recall that, for each sequent in $\mathfrak{S}$, proposition 8.10 guarantees that such a selection results in a connexion set. Otherwise, we let $\mathfrak{X}_{\mathfrak{S},i+1} = \mathfrak{X}_{\mathfrak{S},i}$.[2] In any case, it is $\mathfrak{X}_{\mathfrak{S},i} \subseteq \mathfrak{X}_{\mathfrak{S},i+1}$, and, thereby, also $\widehat{\mathfrak{X}}_{\mathfrak{S},i} \subseteq \widehat{\mathfrak{X}}_{\mathfrak{S},i+1}$, which further implies $\mathcal{V}_{\widehat{\mathfrak{X}}_{\mathfrak{S},i}} \subseteq \mathcal{V}_{\widehat{\mathfrak{X}}_{\mathfrak{S},i+1}}$. As a consequence of the preceding proposition, all of the relational interpretations of this construction are strands. □

To every branch of a derivation $\widetilde{\Xi}$ corresponds a family of strands ordered by inclusion. The topmost sequent in that branch corresponds to the largest strand of that family.

The following corollary concerning the relational interpretation corresponding to individual sequents occurring in RK-derivations, is a trivial consequence of the preceding corollary.

**Corollary 11.7** *Let* $\widetilde{\Xi}$ *be the occurrence instance of a RK-derivation of* $\widetilde{\mathcal{S}}$, *and let* $\widetilde{\mathcal{T}}$ *be a sequent occurring in* $\widetilde{\Xi}$. *Let* $\langle \mathfrak{E}, w\rangle$ *be an explosion set of* $\widetilde{\mathcal{S}}$ *with occurrence mapping* $\pi$. *Let* $\mathfrak{X}_{\widetilde{\mathcal{T}}}$ *be the connexion set corresponding to* $\widetilde{\mathcal{T}}$. *Then* $\langle \mathcal{V}_{\widehat{\mathfrak{X}}_{\widetilde{\mathcal{T}}}}, \widehat{\mathfrak{X}}_{\widetilde{\mathcal{T}}}, w\rangle$ *is a strand in* $\langle \mathcal{V}_{\widehat{\mathfrak{E}}}, \widehat{\mathfrak{E}}, w\rangle$.

PROOF: Simply consider that relational interpretation of the construction above, which corresponds to $\widetilde{\mathcal{T}}$ in the development of the family of strands corresponding to the branch in $\widetilde{\Xi}$ that contains $\widetilde{\mathcal{T}}$. □

These results are indeed important, as they show that the relational interpretation of the explosion set of a sequent $\widetilde{\mathcal{S}}$ already contains all the relational interpretations of all the sequents, which can occur in any RK-derivation thereof. Note, however, that occurrence instances have to be considered, because the construction of connexion sets has been formulated with the help of occurrence mappings $\pi$. These mappings are only functions, if all formulae, which occur in a sequent, are distinct. After the constructions have been

[2]Consequently, a sequent, which is the premiss of an exchange rule, and its conclusion are interpreted by the same strand.

performed, it is no longer necessary to use occurrence instances. Hence, we can revert to standard formulae, sequents and derivations by applying the restoration function $\rho$ to the involved entities.

Bearing this in mind, we consider the following derivation $\Xi$ of the sequent $a \supset b, b \supset c \to a \supset c$, of which the relational interpretation has already been given in figure 11.3.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \to a, b, c}{\to a, b, a \supset c}\,(\supset\mathrm{S}_m)}{\to a, a \supset c, b}\,(\mathrm{XS})}{\to a \supset c, a, b}\,(\mathrm{XS}) \quad \dfrac{\dfrac{a, c \to a, c}{c \to a, a \supset c}\,(\supset\mathrm{S}_m)}{c \to a \supset c, a}\,(\mathrm{XS})}{b \supset c \to a \supset c, a}\,(\supset\mathrm{A}_m) \quad \dfrac{\dfrac{\dfrac{\dfrac{a, b \to c, b \quad c, a, b \to c}{b \supset c, a, b \to c}\,(\supset\mathrm{A}_m)}{a, b \supset c, b \to c}\,(\mathrm{XA})}{a, b, b \supset c \to c}\,(\mathrm{XA})}{b, b \supset c \to a \supset c}\,(\supset\mathrm{S}_m)}{a \supset b, b \supset c \to a \supset c}\,(\supset\mathrm{A}_m)$$

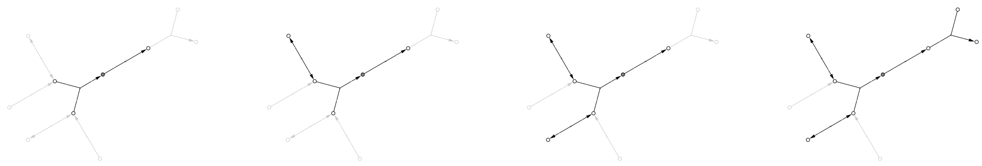
For the moment we are not interested that each of the branches of $\Xi$ could be closed off by an application of $(\mathrm{ax}^*)$. Instead, we want to consider how each of its four branches gives rise to a family of strands. Instead of $\Xi$, we consider the connexion tree of the occurrence instance $a_1 \supset b_1, b_2 \supset c_1 \to a_2 \supset c_2$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\{a_2, r \to c_2\}}{\{\,\}}\,(\supset\mathrm{S}_m)}{\{\,\}}\,(\mathrm{XS})}{\{\to q, b_2\}}\,(\mathrm{XS}) \quad \dfrac{\dfrac{\{a_2, r \to c_2\}}{\{\,\}}\,(\supset\mathrm{S}_m)}{\{c_1 \to q\}}\,(\mathrm{XS})}{\{\to p, a_1\}}\,(\supset\mathrm{A}_m) \quad \dfrac{\dfrac{\dfrac{\dfrac{\{\to q, b_2\} \quad \{c_1 \to q\}}{\{\,\}}\,(\supset\mathrm{A}_m)}{\{\,\}}\,(\mathrm{XA})}{\{a_2, r \to c_2\}}\,(\mathrm{XA})}{\{b_1 \to p\}}\,(\supset\mathrm{S}_m)}{\{p, q \to w\} \cup \{w \to r\}}\,(\supset\mathrm{A}_m)$$

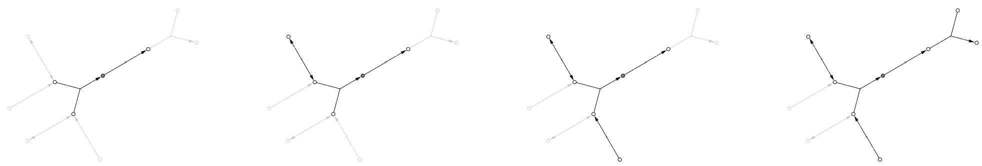
In exploring the four branches of this derivation, the relational interpretations corresponding to its intermediate sequents are obtained by adding the interpretations of the ESSs as they are encountered. Hence, from the four branches of $\widetilde{\Xi}$, families of strands can be generated. They are depicted in figure 11.5 with the family of strands corresponding to the respective branch developed from left to right. In each case, the initial hypergraph consists of the two hyperarcs connected to the root $w$. Similar explorations can be generated for any RK-derivation of the sequent $a \supset b, b \supset c \to a \supset c$.

Just as each connexion set, which can be constructed along the branch of any derivation of some sequent, is always contained in the explosion set of that sequent, so is the maximal strand corresponding to the connexion set always contained in the relational interpretation of the explosion set. All the maximal strands share the common root $w$. It is in this sense that the directed hypergraph, which is the relational interpretation of an explosion set, contains every branch of every possible derivation of its generating sequent. Hence, the relational interpretation of an explosion set contains all the information that is already contained in the explosion set itself.

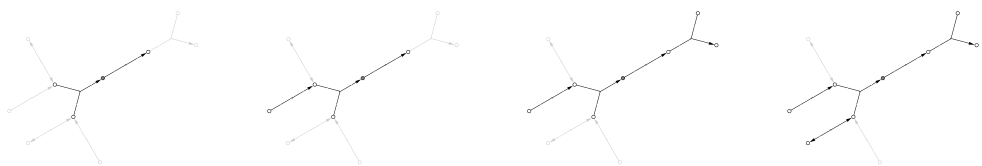
Leftmost branch:



Second to left branch:



Second to right branch:



Rightmost branch:

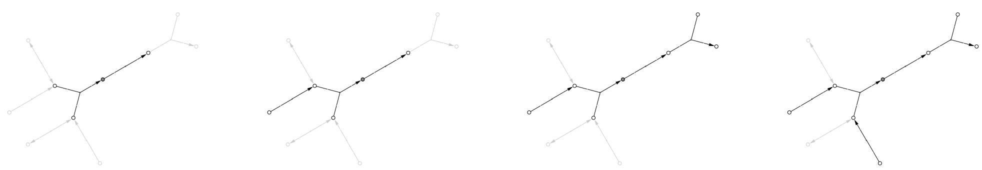


Figure 11.5: Exploring the four branches of $\Xi$

# Chapter 12

# Decision Procedures on Hypergraphs

Since the relational interpretation of a sequent is obtained from the explosion set of that sequent, the ideas on decision procedures for explosion sets of chapter 9 should carry over to hypergraphs in a straightforward manner. Hence, we will introduce bridging actions on strands as a corespondence of cut-actions on germinated connexion sets and examine the particularities of this correspondence. Bridging actions on strands will generate new hyperarcs that relate initial nodes of the strand with its terminal nodes, and the most simple refutation procedure compares the labels of the initial vertices and the terminal vertices of these hyperarcs.

However, there is a more elegant approach to the decision problem for relational interpretations of explosion sets that does not resort to a comparison of labels. This approach instead focusses on the identification of partial cycles. As the hypergraphs that are obtained as relational interpretations are cut hypergraphs, strands cannot contain such cycles. For this reason, co-identity arcs have to be systematically added to a given hypergraph. A co-identity arc is a hyperarc that connects the terminal node and the initial node *of matching labels* within such a hypergraph. The co-identity arcs that have to be added to a given relational interpretation will be considered as a new sort of hyperarcs, which is always explicitly considered in addition to an already given relational interpretation of a sequent.

## 12.1 Bridging Actions on Strands

The notion of cut-actions on connexion sets was crucial for the suggested refutation and decision procedures described in chapter 9. Recall that cut-actions are derivations using only proxy cut rule and exchange rules, which derive from a given connexion set $\mathfrak{X}_{\mathfrak{S}}$ a sequent $\widetilde{\mathcal{U}}$ that is related to the leaf $\mathcal{U}$ of the branch $\mathfrak{S}$ of a given RK-derivation of a sequent $\mathcal{S}$ by means of the restoration function $\rho$, i.e. $\rho(\widetilde{\mathcal{U}}) = \mathcal{U}$. It was established in proposition 9.6 that every cut-action on a gerimated connexion set generates such a leaf. The key to the refutation procedure was the observation made in the corollary of that lemma, which states that, if the restoration of that leaf is not an instance of the RK-axiom (ax*), then $\mathcal{S}$ is not provable in RK. Now, according to proposition 11.4, the relational interpretation of a germinated connexion set is a maximal strand. Hence, the correspondence to the refutation condition mentioned above is easy to give, at least intuitively.

A cut-action on a connexion set, i.e. a set of ESSs, successively applies proxy cuts, thereby yielding new sequents. As in the case of connexion sets, the contextual formulae are always atomic, the rule (Prx), as used in cut-actions, always has this form:

$$\frac{\gamma_1 \to \delta_1, p \qquad p, \gamma_2 \to \delta_2}{\gamma_1, \gamma_2 \to \delta_1, \delta_2}\ \text{(Prx)}$$

Hence, a proxy cut collects the atomic formulae in the antecedents and succedents of its two premises, apart from the proxy variable that is the cut formula, into a single antecedent and a single succedent of the conclusion. In general, the conclusion is not an ESS, as it may contain multiple atoms in both antecedent and succedent.

A strand, the relational interpretation of a connexion set, is a cut graph. That is, every internal vertex of the strand has the property that removing it, and thereby the hyperarcs it is incident with, renders the remaining hypergraph disconnected. Now, recall that the internal vertices correspond to the proxy variables in the ESSs in a connexion set. The first proxy cut of a cut-action produces from two ESSs, which share a proxy variable, a new atomic sequent. Those two ESSs play no further role in the cut-action, as they have already been used. Correspondingly, the first step of what shall correspond to a cut-action on a strand should remove a cut vertex and the hyperarcs it is incident with. The newly disconnected hypergraph has new terminal vertices. They are those vertices that used to be the initial nodes of that hyperarc, which had the cut vertex as its terminal node. The hypergraph also has new initial vertices, namely those vertices that used to be the terminal nodes of the other hyperarc, which had the cut vertex as its initial node. In order to reconnect the hypergraph resulting from the removal of the cut vertex and the two hypergraphs, a new hyperarc has to be added that restores the new external vertices to internal vertices, i.e. that has the new terminal vertices as its initial vertices and the new initial vertices as its terminal vertices. The new hyperarc obviously corresponds to the conclusion of the proxy cut of two ESSs. This process can be iterated by removing other cut vertices in this manner. Formally, the procedure that was sketched out above can be defined formally by means of the following notion.

**Definition 12.1** *Let* $\langle V, \mathfrak{H} \rangle$ *be a directed hypergraph. Let* $v$ *be a cut vertex thereof, and let* $\mathfrak{e}$ *be the preceding hyperarc with* $\mathfrak{e}^{\bullet} = \{v\}$ *and* $\mathfrak{f}$ *be the succeeding hyperarc with* ${}^{\bullet}\mathfrak{f} = \{v\}$*. A* bridged cut *on* $v$ *is the hypergraph* $\langle V', \mathfrak{H}' \rangle$ *such that* $V' = V \setminus \{v\}$ *and* $\mathfrak{H}' = (\mathfrak{H} \setminus \{\mathfrak{e}, \mathfrak{f}\}) \cup \{({}^{\bullet}\mathfrak{e}, \mathfrak{f}^{\bullet})\}$*. The hyperarc* $({}^{\bullet}\mathfrak{e}, \mathfrak{f}^{\bullet})$ *is called the* bridge *of the bridged cut.*

While a cut simply removes the vertex $v$ and yields the vertex induced subgraph on $V \setminus \{v\}$, a bridged cut further adds a hyperarc that connects the former initial vertices of $\mathfrak{e}$ to the former terminal vertices of $\mathfrak{f}$.

Bridged cuts can be used to bridge cut vertices in arbitrary directed hypergraphs. However, for the purpose of iterating bridged cuts, it is useful to consider hypergraphs that are strands. In this case bridged cuts can be iterated along the entire strand as follows.

**Definition 12.2** *Let* $\langle V, \mathfrak{H} \rangle$ *be a strand. The* bridging action *on the strand is any sequence of bridged cuts developed as follows:*

- *Initialisation: Let* $\langle V_0, \mathfrak{H}_0 \rangle$ *be the original strand.*
- *Iteration: If* $\langle V_i, \mathfrak{H}_i \rangle$ *has an internal vertex* $v$*, then let* $\langle V_{i+1}, \mathfrak{H}_{i+1} \rangle$ *be the bridged cut on* $v$*; otherwise terminate the procedure.*

*The final* $\langle V_k, \mathfrak{H}_k \rangle$*, which has no internal vertex left, is called the* result *of the bridging action.*

As it is not specified, which cut vertex should be chosen at any stage, the procedure is non-deterministic. This is not a difficulty, however, as we shall see promptly.

For example, consider figure 12.1, which shows three possible bridging actions for a strand.[1] The strand itself, consisting of four hyperarcs, is depicted in black, the successive bridged cuts are depicted in progressively lighter shades of grey. Every bridge replaces the vertex and the two hyperarcs immediately above it in the corresponding part of the illustration. In the bottommost example, the first two bridged cuts are independent of one

[1]The strand consists mostly of S-arcs. Of course, this is not a typical case, but it serves the clarity of the illustration.

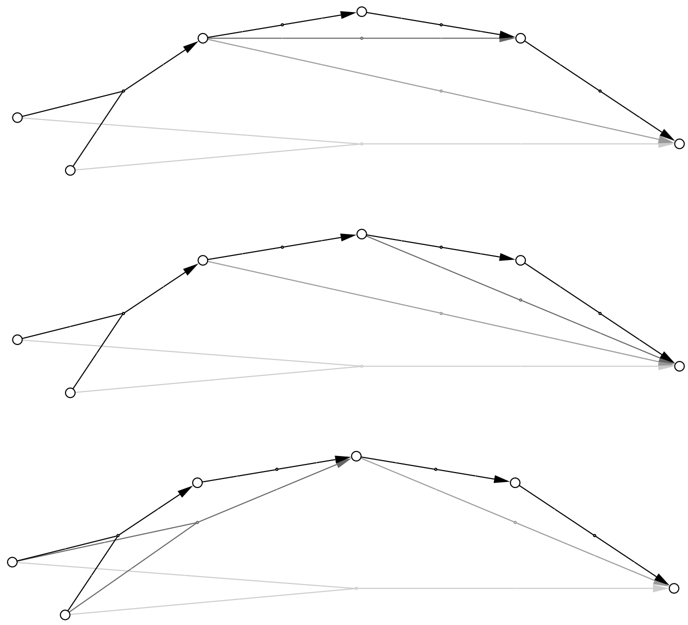


Figure 12.1: Different bridging actions on the same strand

another and could occur in any order. In each case there are exactly three bridged cuts, the result of the bridging actions is in all cases the same hyperarc, and its vertices are the external vertices of the original strand. The following lemma generalises these important properties.

**Lemma 12.3** *Let $\mathcal{H} = \langle V, \mathfrak{H} \rangle$ be a strand, and let $\langle V_k, \mathfrak{H}_k \rangle$ and $\langle V_l, \mathfrak{H}_l \rangle$ be results of two bridging actions on it. Then the following holds:*

*1) $k = l$;*

*2) $\mathfrak{H}_k = \mathfrak{H}_l$ are singletons containing a hyperarc, which has the initial vertices of $\mathcal{H}$ as its initial vertices and the terminal vertices of $\mathcal{H}$ as its terminal vertices;*

*3) $V_k = V_l$ both contain the external vertices of $\mathcal{H}$.*

Proof: Let $\mathcal{H}$ have $m$ internal vertices. A strand is a connected directed cut hypergraph having certain properties. One property is that every external vertex is incident with exactly one hyperarc. Furthermore, for every internal vertex $v$, it is $\deg^-(v) = \deg^+(v) = 1$. Hence, no pairs of hyperarcs of $\mathcal{H}$ can be seceding or conceding. Consequently, two hyperarcs can at most be succeeding on a single internal vertex. Hence, $\mathcal{H}$ must contain $m+1$ hyperarcs. Now, in one bridging action each bridging cut will remove one of the $m$ internal vertices, say $v_i$, and its preceding and succeeding hyperarcs, say $\mathfrak{e}_i$ and $\mathfrak{f}_i$. Each bridge $({}^\bullet\mathfrak{e}_i, \mathfrak{f}_i^\bullet)$, however, leaves the status of the initial vertices of $\mathfrak{e}_i$ and of the terminal vertices of $\mathfrak{f}_i$ within the remaining hypergraph unchanged: vertices that were internal before the bridging cut remain internal, and those that were external before remain external. Regardless of the order in which the internal vertices are cut and bridged in the bridging action, there will always be exactly $m$ bridging cuts, i.e. $k = l = m$. Furthermore, each of them removing two hyperarcs and adding a new one. Consequently, after $m$ bridging cuts, the originally $m+1$ hyperarcs result in a single hyperarc in $\mathfrak{H}_m$. Moreover, since all of the internal vertices of $\mathcal{H}$ have been removed, only its external vertices remain in $V_m$. Finally, the single hyperarc then connects all of the initial vertices of the strand to all of its terminal ones. □

Since the result of a bridging action is a hypergraph consisting of a single hyperarc and vertices that are incident with it, the hyperarc itself can also be called the result of the bridging action.

A cut-action on a connexion set results in a sequent that is the leaf of a RK-derivation, and the test whether that leaf is an instance of (ax$^*$) or not is the basis for the refutation procedure in chapter 9. The result of a bridging action is a single hyperarc. If the strand is a maximal strand in a relational interpretation of an explosion set, then the vertices of this resulting hyperarc are all elements of $\mathcal{O}$. If the restoration function $\rho$ is used as labelling, then those vertices have atomic formulae as their labels. This allows us to formulate the result sketched out in the beginning of this section.

**Proposition 12.4** *Let $\mathcal{H} = \langle V, \mathfrak{H}, w, \rho\rangle$ be the relational interpretation of a sequent $\mathcal{S}$. Let $\mathfrak{e}$ be the hyperarc, which is the result of the bridging action on some maximal strand of $\mathcal{H}$. If $\rho({}^\bullet\mathfrak{e}) \cap \rho(\mathfrak{e}^\bullet) = \emptyset$, then $\mathcal{S}$ does not have a RK-proof.*

Proof: If $\mathcal{H}$ is the relational interpretation of a sequent $\mathcal{S}$, then $\langle V, \mathfrak{H}, w\rangle$ is the relational interpretation of the corresponding occurrence instance $\widetilde{\mathcal{S}}$ and, consequently, of an explosion set $\langle \mathfrak{E}, w\rangle$. Let $\mathfrak{X} \subseteq \mathfrak{E}$ be some germinated connexion set thereof. Let further $\rho(\widetilde{\mathcal{U}})$ be the restoration of the end sequent of the cut-action $\mathcal{C}(\mathfrak{X})$. According to corollary 9.7, if $\rho(\widetilde{\mathcal{U}})$ is not an instance of (ax$^*$), then $\mathcal{S}$ does not have a RK-proof. Now, according to proposition 11.4, the relational interpretation of $\mathfrak{X}$ is a maximal strand in $\mathcal{H}$. Then the result of the bridging action, $\mathfrak{e}$, is the relational interpretation of $\widetilde{\mathcal{U}}$, because each step in the cut-action can be matched by a bridged cut removing the vertex corresponding to the proxy variable that is cut and the hyperarcs corresponding to the premises of the cut. Hence, the initial vertices of $\mathfrak{e}$ correspond to the proper occurrence variables in the antecedent of $\widetilde{\mathcal{U}}$ and, correspondingly, its terminal vertices correspond to the proper occurrence variables in the succedent of $\widetilde{\mathcal{U}}$. If $\rho(\widetilde{\mathcal{U}})$ is not an instance of (ax*), then there is no propositional atom, which occurs both in its antecedent and in its succedent. The corresponding condition in terms of the result of the bridging action, the hyperarc $\mathfrak{e}$, is that none of its initial and terminal vertices bear the same label, i.e. $\rho(^\bullet\mathfrak{e}) \cap \rho(\mathfrak{e}^\bullet) = \emptyset$. □

Following the remarks on a refutation procedure for explosion sets and, in particular, defition 9.8, a refutation procedure for relational interpretations of sequents can be given as follows:

**Definition 12.5** *Let $\langle V, \mathfrak{H}, w, \rho\rangle$ be the relational interpretation of a sequent $\mathcal{S}$, and let $N$ be the set of internal vertices thereof. The* refutation procedure *for $\mathcal{S}$ is given by the following algorithm.*

- *Initialisation: Let*

$$V_0 = \{\!\{\gamma\}\!\} \cup \{w\} \cup \{\!\{\delta\}\!\},$$
$$\mathfrak{H}_0 = \{(\widehat{\gamma}, \{w\}), (\{w\}, \widehat{\delta})\},$$
$$E_0 = \{\!\{\gamma\}\!\} \cup \{\!\{\delta\}\!\},$$
$$I_0 = \{\!\{\gamma\}\!\} \setminus N,$$
$$T_0 = \{\!\{\delta\}\!\} \setminus N.$$

- *Iteration: While $E_i \cap N \neq \emptyset$, select a hyperarc $\mathfrak{e} \in \mathfrak{H} \setminus \mathfrak{H}_i$ such that $v \in E_i \cap {}^\bullet\mathfrak{e}^\bullet$ and let*

$$V_{i+1} = V_i \cup {}^\bullet\mathfrak{e}^\bullet,$$
$$\mathfrak{H}_{i+1} = \mathfrak{H}_i \cup \{\mathfrak{e}\},$$
$$E_{i+1} = (E_i \cup {}^\bullet\mathfrak{e}^\bullet) \setminus \{v\},$$
$$I_{i+1} = I_i \cup ({}^\bullet\mathfrak{e} \setminus N),$$
$$T_{i+1} = T_i \cup (\mathfrak{e}^\bullet \setminus N).$$

*Let $k$ be the number such that $E_k \cap N = \emptyset$. If $\rho(I_k) \cap \rho(T_k) = \emptyset$, then reject $\mathcal{S}$ on the basis of the counterexample $\langle V_k, \mathfrak{H}_k\rangle$, otherwise restart with the initialisation step.*

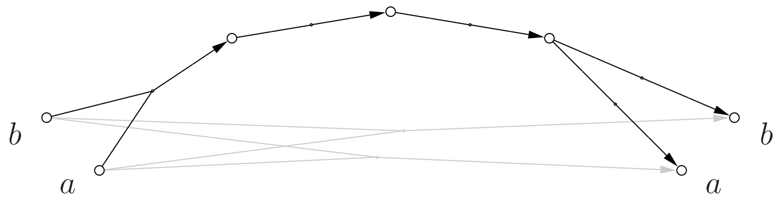



Figure 12.2: The results of the bridging actions on $a \,\&\, b \rightarrow a \,\&\, b$

The procedure generates a family of strands $\langle V_i, \mathfrak{H}_i \rangle_{1 \leq i \leq k}$, the last of which contains as external vertices only external vertices of the relational interpretation of $\mathcal{S}$.[2] Set $E_i$ contains all of the external vertices of the corresponding strand, including those that are internal to the relational interpretation of $\mathcal{S}$. Sets $I_i$ and $T_i$ collect all of the initial and, respectively, terminal nodes of the current strand that are also external to the relational interpretation of $\mathcal{S}$. Consequently, $I_k$ and $T_k$ are the initial and terminal vertices of the maximal strand therein. But then, the result of any bridging action on $\langle V_k, \mathfrak{H}_k \rangle$ is the hyperarc $(I_k, T_k)$. According to the preceding proposition 12.4, $\mathcal{S}$ does not have a RK-proof, if $\rho({}^\bullet\mathfrak{e}) \cap \rho(\mathfrak{e}^\bullet) = \emptyset$ for the result $\mathfrak{e}$ of the bridging action. In this case, this corresponds to the condition $\rho(I_k) \cap \rho(T_k) = \emptyset$ of the procedure. If the condition is not met, this merely means that the attempted refutation has failed for the generated maximal strand, whereas another maximal strand of the relational interpretation of $\mathcal{S}$ might still refute the provability of $\mathcal{S}$.

See figure 12.2 for an example of the sequent $a\&b \rightarrow a\&b$, whose relational interpretation has two maximal strands. For both of these strands, the result of the bridging action is also displayed in light grey. One of these hyperarcs has initial vertices with labels $a$ and $b$ and a terminal vertex with label $a$, the other one has the same initial vertices and a terminal vertex with label $b$. For both of these hyperarcs, the condition of the refutation procedure fails. Consequently, the refutation procedure on the relational interpretation will fail to refute the provability of $a \,\&\, b \rightarrow a \,\&\, b$ in RK.

As we remarked in section 9.3, a decision procedure can be obtained by first generating all possible maximal strands of a given relational interpretation of a sequent $\mathcal{S}$ and then checking the crucial property for each of those strands, rejecting the provability of $\mathcal{S}$ as soon as one of the maximal strands has that property and confirming its provability if none of them has that

[2]Compare this procedure to that of lemma 10.18, but bear in mind that sets $I_i$ and $T_i$ serve different purposes in the two procedures.

property. In the example of figure 12.2, such a decision procedure would obviously confirm the provability of $a \mathbin{\&} b \to a \mathbin{\&} b$, because, as we have just seen, both strands fail the refutation procedure.

These procedures were designed to closely mimic the corresponding procedures for explosion sets. The crucial property in view of refutability and decidability of a sequent is the relation of the labels of the initial and terminal vertices of the maximal strands in its relational interpretation. If there is a single maximal strand, in which the intersection of the set of labels of the initial vertices and the set of labels of the terminal vertices is empty, then the sequent is not provable. Based on these observations, a more elegant approach can be presented in the following section.

## 12.2 Partial Cycles and Co-Identity Arcs

Instead of unraveling the maximal strands of a hypergraph and comparing the labels of its external vertices, it would be preferable to refute or decide the provability of a sequent based on some property of the structure of the hypergraph. However, the labels of the external vertices contain essential information about their *identity* in view of the sequent, from which the hypergraph was obtained. This identification is given by means of the labelling function $\rho$, which is originally used to restore the disambiguation of propositional variables into occurrence instances. Recall that the unique representation of occurrences of porpositional atoms was of crucial importance for an adequate treatment of explosion sets. This forced disambiguation was carried over to relational interpretations of explosion sets by the manner in which this interpretation is performed. In this sense, the restoration function $\rho$ undoes the excessive disambiguation. Taken as a labelling function for a relational interpretation, it assigns labels to its external vertices. As this function is not injective, it can be seen as marking certain vertices as identical. What is required is a means by which vertices can be identified without having to resort to labels.

The most immediate (and at the same time the most extreme) possibility of identification is to merge vertices bearing the same label into a single vertex. This would require a modification of the property that is crucial for refutability and decidability, which consisted in tracing a maximal strand and checking its initial and terminal vertices for identical labels. The problem is that in the process of merging vertices bearing the same label, both initial and terminal vertices can be merged into a single vertex, which, being both initial vertex of some hyperarc and terminal vertex of another hyperarc, is no longer an external vertex of the new hypergraph. As the termination condition of a run of the refutation procedure depends on the fact that eventually no further internal vertices can be found, the procedure cannot terminate as it should in that case. This apparent problem is, however, the key to the sought after structural criterion. For the procedure cannot terminate at this point, because it has traced a part of the new hypergraph that is no longer a strand but a partial cycle! In the case that the traced maximal strand does not contain any labels shared by both initial and terminal vertices in the original hypergraph, it terminates as before.

Of course, it cannot be expected that a procedure designed for a particular type of hypergraph, specifically a directed cut hypergraph, can be employed on hypergraphs that might no longer have one of the most important properties. If any initial and terminal vertices are merged, a hypergraph loses the cut property, because removing the merged vertex from the resulting graph will not disconnect it in general.[3] More specifically, merging initial vertices with terminal vertices of the same label creates partial cycles, because a maximal strand that contains a partial path from the initial vertex of that label to the corresponding terminal vertex has its first and final vertex merged. Hence, the hypergraph resulting from this merger contains a partial cycle where the original hypergraph contained a maximal strand.[4] Consequently, a decision procedure for thus merged relational interpretations of sequents could be formulated as follows: If the given hypergraph contains any maximal strand that is not contained in a cycle, then the sequent, of which the hypergraph is the relational interpretation, is not provable.

The problem of the suggested solution by merging vertices lies in the fact that the hypergraph obtained as relational interpretation of a sequent has to be modified in a significant manner. While the operation itself is trivial, through merging of vertices the explicit information about different occurrences of the same propositional variables is lost. As a merged vertex is in general no longer an external vertex, i.e. either initial or terminal, the different polarities of its various occurrences are lost. Of course, the information is still implicitly present in the hyperarcs the merged vertex is incident with. However, this raises another, even more pressing problem. As soon as the labels are removed from merged vertices, the ones that induce partial cycles can no longer be distinguished from formerly internal vertices. The

[3]In the case that one or more of the the hyperarcs incident with the merged vertex is a proper fusion arc or a proper fission arc, other initial or terminal vertices might get disconnected by the removal of the merged vertex.

[4]The partial cycle still covers hyperarcs that make up a maximal strand; in fact, removing any one of the hyperarcs from those comprising the partial cycle yields a maximal strand. Before, a maximal strand was obtained simply by tracing hyperedges until the external vertices of the embedding hypergraph were reached.

idea of explicitly representing every aspect of the meaning of a sequent would thereby be undone. What is required is an approach that provides the information about the identity of vertices in a manner that leaves hypergraph obtained as relational interpretation intact. The aim must be to obtain a hypergraph that contains a partial cycle wherever the relational interpretation of some sequent contains a maximal strand that has initial and terminal vertices bearing the same label, but without merging vertices.

## Co-Identity Arcs

Fortunately, the solution is extremely simple. Instead of merging vertices that have the same label, additional hyperarcs are systematically added from terminal vertices to initial vertices of the same label. For example, in order to express the identity of an initial vertex $v$ of some label $a$ and a terminal vertex $w$ of label $a$, the S-arc $(\{w\}, \{v\})$ is added. If there are $m$ initial vertices of some label and $n$ terminal vertices of the same label, $m \cdot n$ of these S-arcs have to be added to the hypergraph for that label alone. The original hyperarcs that represent the structural meaning of the sequent and those that are added by this method are considered to be of different sorts; the former are called *meaning arcs*, and the latter are called *co-identity arcs*.[5]

**Definition 12.6** *Let $\mathcal{H} = \langle V, \mathfrak{H}, w, \rho \rangle$ be the relational interpretation of a sequent $\mathcal{S}$. The directed rooted hypergraph $\mathcal{H}' = \langle V, \mathfrak{H}, \mathfrak{I}, w \rangle$ with*

$$\mathfrak{I} = \big\{ (\{w\}, \{v\}) \mid w \text{ is terminal and } v \text{ is initial in } \mathcal{H} \text{ and } \rho(v) = \rho(w) \big\}$$

*is called the* relational interpretation *of $\mathcal{S}$ with* structural identity information. *For a $\mathcal{H}'$ thus given, the graph $\langle V, \mathfrak{H}, w \rangle$ is called the* occurrence trunk.

The occurrence trunk is what remains of the relational interpretation of a sequent $\mathcal{S}$ with structural identity information, after the co-identity arcs have been removed. Essentially, this is the same graph as the relational interpretation of an occurrence instance $\widetilde{\mathcal{S}}$ of $\mathcal{S}$ (see definition 11.1).

The hypergraph, which is obtained by removing the restoration function $\rho$ and instead adding co-identity arcs, does not retain any remnant of logical

[5]The standard axiom (Ax) of LK is occasionally called (Id) for "identity", as a sequent $A \to A$ can be considered to express the identity of "assumption", i.e. antecedent, and "assertion", i.e. succedent. S-arcs relate terminal vertices of some label, i.e. vertices that were obtained by interpreting the succedent position of ESSs, with initial vertices of that label, i.e. vertices that were obtained from antecedents of ESSs. Hence, the S-arcs connecting these terminal vertices to the corresponding initial ones express the notion of co-identity.

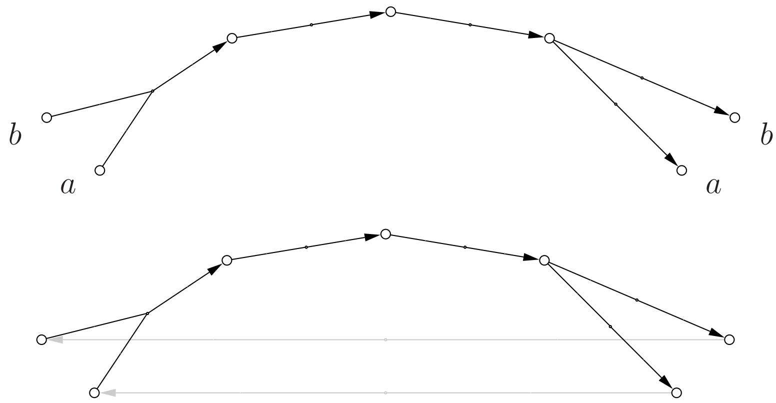



Figure 12.3: Replacing labels by co-identity arcs

language. The restoration function $\rho$, which relates vertices to logical atoms, is no longer present. Of course, since $\mathcal{H}$ was constructed from an explosion set, its vertices are still elements of $\mathcal{O} \cup \mathcal{P}$, so even though the function $\rho$ is given up, a vertex in $V$ could still be checked for memembership in $\mathcal{O}$ or $\mathcal{P}$. However, the nature of the vertices is utterly immaterial at this stage, and, hence, any hypergraph isomorphic to the construction contains all the relevant information about the sequent $\mathcal{S}$. Any vertex that is incident with some hyperarc in $\mathfrak{I}$ is a vertex that was external in $\mathcal{H}$, i.e. a vertex that represents the occurrence of a propositional variable in $\mathcal{S}$. Any other vertex was internal in $\mathcal{H}$ and is either the root $w$ or represets the occurrence of a complex formula in $\mathcal{S}$.

For example, in figure 12.3 the top hypergraph depicts the relational interpretations of the sequent $a \,\&\, b \rightarrow a \,\&\, b$, the bottom one has the labels replaced by structural identity information. The part of the bottom hypergraph that is depicted in black is the occurrence trunk. According to the preceding lemma, the hypergraph is also the relational interpretations with structural identity information for sequents $a \,\&\, c \rightarrow a \,\&\, c$ and $c \,\&\, d \rightarrow c \,\&\, d$ and so forth. This has an interesting consequence.

**Lemma 12.7** *Let $\mathcal{S}$ be a sequent, let $\tau : \mathcal{A} \rightarrow \mathcal{A}$ be a permutation of propositional variables and let $\mathcal{T}$ be a sequent with $\mathcal{T} = \mathcal{S}\tau$. Then the relational interpretation with structural identity information of $\mathcal{S}$ and that of $\mathcal{T}$ are isomorphic.*

Proof: The permutation $\tau$ simultaneously substitutes all occurrences of propositional variables. Hence, for all pairs of occurrences of propositional variables in $\mathcal{S}$ it holds that, they are the occurrences of the same variable if and only if the corresponding occurrences in $\mathcal{T}$ are also occurrences of the same variable. If $\langle V, \mathfrak{H}, w, \rho \rangle$ is the relational interpretation of $\mathcal{S}$ and $\langle V', \mathfrak{H}', w', \rho' \rangle$ is the relational interpretation of $\mathcal{T}$, then the hypergraphs are isomorphic and $\rho' = \tau \circ \rho$. In the construction of the co-identity arcs of $\mathfrak{I}$, the condition $\rho(v) = \rho(w)$ and the condition $\tau(\rho(v)) = \tau(\rho(w))$ coincide for all $v, w$, because $\tau$ is a bijection. Consequently, $\langle V, \mathfrak{H}, \mathfrak{I}, w \rangle$ and $\langle V', \mathfrak{H}', \mathfrak{I}, w' \rangle$ are isomorphic. □

This result confirms that the identity of the propositional variables themselves is lost. For, if $\rho(v) = \rho(w) = a$ for initial $v$ and terminal $w$ in $\mathcal{H}$, then $\mathfrak{I}$ contains the co-identity arc $(\{w\}, \{v\})$, but this hyperarc is not related to $a$ in any way.

With the desired purely structural notion in place, we can turn to the question of a suitable refutation procedure.

**Proposition 12.8** *Let $\mathcal{H} = \langle V, \mathfrak{H}, \mathfrak{I}, w \rangle$ be the relational interpretation of a sequent $\mathcal{S}$ with structural identity information. Consider any maximal strand $\mathcal{H}'$ of the occurrence trunk of $\mathcal{H}$, and let $I$ and $T$ be the sets of initial and terminal vertices of $\mathcal{H}'$. If there is no co-identity arc $\mathfrak{i} \in \mathfrak{I}$ with ${}^\bullet\mathfrak{i} \cap T \neq \emptyset$ and $\mathfrak{i}^\bullet \cap I \neq \emptyset$, then $\mathcal{S}$ does not have a RK-proof.*

Proof: We can obtain hyperarc $\mathfrak{e}$ as the result of the bridging action of $\mathcal{H}'$. Observe that ${}^\bullet\mathfrak{e} = I$ and $\mathfrak{e}^\bullet = T$. According to proposition 12.4, provability of $\mathcal{S}$ is refuted, if $\rho(I) \cap \rho(T) = \emptyset$, i.e. if there are no vertices $v \in I$ and $w \in T$ with $\rho(v) = \rho(w)$. If that is the case, however, then, according to the definition, there is no co-identity arc $\mathfrak{i}$ connecting any of the vertices in $T$ to any of those in $I$. □

The co-identity arcs are only referred to in order to check a property similar to the one previously given by labels. However, the desired structural criterion is easily derived from the proposition. In the following case, a partial cycle in the entire hypergraph, i.e. including the co-identity arcs, is sought.

**Corollary 12.9** *Let $\mathcal{H} = \langle V, \mathfrak{H}, \mathfrak{I}, w \rangle$ be the relational interpretation of a sequent $\mathcal{S}$ with structural identity information. Consider any maximal strand $\langle V', \mathfrak{H}', w \rangle$ in the occurrence trunk of $\mathcal{H}$. If $\langle V', \mathfrak{H}', \mathfrak{I}, w \rangle$ does not contain a partial cycle, then $\mathcal{S}$ does not have a RK-proof.*

Proof: According to the preceding proposition, provability of $\mathcal{S}$ is refuted, if, for some maximal strand $\langle V', \mathfrak{H}', w \rangle$, no co-identity arc $\mathfrak{i} \in \mathfrak{I}$ can be found that connects one of its terminal nodes to one of its initial nodes. If such an $\mathfrak{i}$ did exist, then it would connect one of the terminal vertices of the maximal strand to one of its initial vertices. But then, the graph consisting of the maximal strand and $\mathfrak{i}$ would contain a partial cycle. Consequently, provability of $\mathcal{S}$ is refuted, if there is a strand maximal in the occurrence trunk of $\mathcal{H}$, which cannot be extended into a hypergraph containing a partial cycle by the addition of hyperarcs in $\mathfrak{I}$. $\square$

This result immediately suggests the structural decision procedure for relational interpretations $\mathcal{H}$ with structural identity information of a sequent $\mathcal{S}$. Every maximal strand in the occurrence trunk of $\mathcal{H}$, which is extended by the structural identity information, contains a partial cycle, if and only if the provability of $\mathcal{S}$ cannot be refuted, i.e. if and only if $\mathcal{S}$ is provable. It follows from corollary 9.10 that the relational interpretation of a sequent $\mathcal{S}$ has at most $2^{\mathbf{C}_{\mathcal{S}}^{+}+\mathbf{D}_{\mathcal{S}}^{-}+\mathbf{I}_{\mathcal{S}}^{-}}$ different maximal strands, which have to be considered by such a procedure.

# Chapter 13

# The Growth Procedure

We will show that it is not necessary to generate the relational representation of an explosion set via the explosion procedure. Instead, a the directed cut hypergraph of a sequent can be "grown" directly. For this purpose, we will demonstrate properties of the relational interpretation of an explosion set and use this as a basis for a growth procedure.

Recall the taxonomy of hyperarcs that was introduced in chapter 10. It can be used to characterise elements of the relational interpretation of a sequent by drawing parallels to chapter 7. We begin by considering the relational interpretation of the base structure of a sequent.

**Proposition 13.1** *Let* $\widetilde{\mathcal{S}}$ *be an occurrence instance of a sequent, and let* $\langle V, \mathfrak{H}, w\rangle$ *be its relational interpretation. Then* $\mathfrak{H}$ *contains a B-arc* $\mathfrak{e}$ *with* $\mathfrak{e}^\bullet = \{w\}$ *and a F-arc* $\mathfrak{f}$ *with* ${}^\bullet\mathfrak{f} = \{w\}$*.*

Proof: According to proposition 7.2, the explosion set of $\widetilde{\mathcal{S}}$ contains ESSs $\gamma \to w$ and $w \to \delta$, where $w$ is the watershed proxy. The relational interpretation of $\gamma \to w$ is the hyperarc $(\widehat{\gamma}, \widehat{w}) = (\widehat{\gamma}, \{w\})$, which is a B-arc with the single terminal vertex $w$. The relational interpretation of $w \to \delta$ is the hyperarc $(\widehat{w}, \widehat{\delta}) = (\{w\}, \widehat{\delta})$, a F-arc with the single initial vertex $w$. $\square$

Of course, the same proposition holds for relational interpretations that include some labelling function, such as the relational interpretation $\langle V, \mathfrak{H}, w, \rho\rangle$ of a sequent $\mathcal{S}$. The labelling will, in general, be of no consequence for the hyperarcs that are the relational interpretation of the base structure of a sequent.

The structure of complex formulae can be interpreted by one or two hyperarcs, depending on how their structure is given by one or two ESSs. This is summarised in the following proposition and its corollaries, which are counterparts to proposition 7.4 and its corollaries 7.5, 7.6 and 7.7. In this case, the occurrence mapping $\pi$, which is always available for an occurrence instance $\widetilde{\mathcal{S}}$, is required to keep track of the correspondences.

**Proposition 13.2** *Let $\widetilde{\mathcal{S}}$ be an occurrence instance of a sequent with occurrence mapping $\pi$, and let $\langle V, \mathfrak{H}, w\rangle$ be its relational interpretation. Then the following holds:*

1) *For every negative occurrence of a subformula $A \& B$ in $\widetilde{\mathcal{S}}$, $\mathfrak{H}$ contains a unique elementary fusion arc $(\{x, y\}, \{p\})$, where $x = \pi(A)$, $y = \pi(B)$ and $p = \pi(A \& B)$.*

2) *For every positive occurrence of a subformula $A \& B$ in $\widetilde{\mathcal{S}}$, $\mathfrak{H}$ contains two unique S-arcs $(\{p\}, \{x\})$ and $(\{p\}, \{y\})$, where $x = \pi(A)$, $y = \pi(B)$ and $p = \pi(A \& B)$.*

Proof: For 1), recall from proposition 7.4 that the explosion set of $\widetilde{\mathcal{S}}$ contains a unique ESSs $\pi(A), \pi(B) \to \pi(A \& B)$. Its relational interpretation is just the hyperarc $\big(\{\pi(A), \pi(B)\}, \{\pi(A \& B)\}\big)$. Case 2) is analogous. $\square$

The observations made for explosion sets of occurrence instances of sequents still hold for their respective relational interpretations. The same holds for negative and positive occurrences of disjunctive, implicative and negated formulae.

**Corollary 13.3** *Let $\widetilde{\mathcal{S}}$ and $\langle V, \mathfrak{H}, w\rangle$ be as before. Then the following holds:*

1) *For every negative occurrence of a subformula $A \vee B$ in $\widetilde{\mathcal{S}}$, $\mathfrak{H}$ contains two unique S-arcs $(\{x\}, \{p\})$ and $(\{y\}, \{p\})$, where $x = \pi(A)$, $y = \pi(B)$ and $p = \pi(A \vee B)$.*

2) *For every positive occurrence of a subformula $A \vee B$ in $\widetilde{\mathcal{S}}$, $\mathfrak{H}$ contains a unique elementary fission arc $(\{p\}, \{x, y\})$, where $x = \pi(A)$, $y = \pi(B)$ and $p = \pi(A \vee B)$.*

**Corollary 13.4** *Let $\widetilde{\mathcal{S}}$ and $\langle V, \mathfrak{H}, w\rangle$ be as before. Then the following holds:*

1) *For every negative occurrence of a subformula $A \supset B$ in $\widetilde{\mathcal{S}}$, $\mathfrak{H}$ contains a unique F-arc $(\emptyset, \{p, x\})$ and a unique S-arc $(\{y\}, \{p\})$, where $x = \pi(A)$, $y = \pi(B)$ and $p = \pi(A \supset B)$.*

2) *For every positive occurrence of a subformula $A \supset B$ in $\widetilde{\mathcal{S}}$, $\mathfrak{H}$ contains a unique elementary fusion arc $(\{x, p\}, \{y\})$, where $x = \pi(A)$, $y = \pi(B)$ and $p = \pi(A \supset B)$.*

**Corollary 13.5** *Let $\widetilde{\mathcal{S}}$ and $\langle V, \mathfrak{H}, w\rangle$ be as before. Then the following holds:*

1) *For every negative occurrence of a subformula $\neg A$ in $\widetilde{\mathcal{S}}$, $\mathfrak{H}$ contains a unique F-arc $(\emptyset, \{p, x\})$, where $x = \pi(A)$ and $p = \pi(\neg A)$.*

2) *For every positive occurrence of a subformula $\neg A$ in $\widetilde{\mathcal{S}}$, $\mathfrak{H}$ contains a unique B-arc $(\{x, p\}, \emptyset)$, where $x = \pi(A)$ and $p = \pi(\neg A)$.*

All of the above correspondences taken together not only confirm that properties of explosion sets carry over to their relational interpretations. The interpretation is straightforward enough to render those facts quite obvious. However, combined with in interesting labelling function, these properties give rise to interesting insights.

Consider the relational interpretation $\langle V, \mathfrak{H}, w, \rho\rangle$ of a sequent $\mathcal{S}$, which can be restored from its occurrence instance $\widetilde{\mathcal{S}}$ by means of the restoration function $\rho$. First of all, it is obvious that the same observations that were made in the propositions and the corrolaries hold in the presence of any labelling. Now, instead of taking the restoration function $\rho : \mathcal{O} \to \mathcal{A}$, a more general labelling function can be constructed from the occurrence mapping $\pi$ of an occurrence instance $\widetilde{\mathcal{S}}$ of $\mathcal{S}$ and the restoration function $\rho$, namely the function $\rho \circ \pi^{-1}$. Recall that $\pi^{-1} : \mathcal{V}_{\Pi_{\widetilde{\mathcal{S}}}} \to \mathcal{F}_{\widetilde{\mathcal{S}}}$ is the function that assigns to each occurrence variable a formula according to some explosion derivation $\Pi_{\widetilde{\mathcal{S}}}$. The inductive extension of $\rho$ maps all formulae over $\mathcal{O}$, i.e. occurrence instances of formulae, to all formulae over $\mathcal{A}$, i.e. the proper logical formulae. Clearly, $\mathcal{F}_{\widetilde{\mathcal{S}}}$ is a subset of the set of formulae over $\mathcal{O}$, which makes the composition feasible. As $\mathcal{V}_{\widehat{\mathfrak{E}}} = \mathcal{V}_{\Pi_{\widetilde{\mathcal{S}}}}$ for some explosion derivation $\Pi_{\widetilde{\mathcal{S}}}$ that produces the explosion set $\mathfrak{E}$ in the first place, the labelling function $\rho \circ \pi^{-1}$ is a function $\mathcal{V}_{\widehat{\mathfrak{E}}} \to \mathcal{F}_{\mathcal{S}}$, i.e. a function that assigns to each vertex of the relational interpretation of a sequent $\mathcal{S}$ some occurrence of a formula. Using this labelling not only labels the external vertices of the relational interpretation to propositional variables, as it is the case with the labelling function $\rho$. Instead, all of the vertices bear as labels the original formulae of the sequent $\mathcal{S}$ that correspond to them by abstracting from occurrence instances. Note that restricting $(\rho \circ \pi^{-1})$ to external vertices $\mathcal{O}$ yields $\rho$.

**Proposition 13.6** *Let $\langle V, \mathfrak{H}, w\rangle$ be the relational interpretation of the sequent $\widetilde{\mathcal{S}}$, which is an occurrence instance of $\mathcal{S}$ with restoration function $\rho$, and let $\pi$ be the occurrence mapping of $\widetilde{\mathcal{S}}$. Let $\lambda = \rho \circ \pi^{-1}$ be a labelling function of the vertices. Then the following properties holds:*

1) *In the hyperarc $(S, \{w\}) \in \mathfrak{H}$, for every $x \in S$, $\lambda(x)$ is a formula of the antecedent of $\mathcal{S}$.*

*In the hyperarc* $(\{w\}, T) \in \mathfrak{H}$*, for every* $x \in T$*,* $\lambda(x)$ *is a formula of the succedent of* $\mathcal{S}$*.*

2) *In any other hyperarc* $(S, T) \in \mathfrak{H}$*, there is one vertex* $p \in S \cup T$ *such that, for all vertices* $x \in S \cup T$ *with* $x \neq p$*,* $\lambda(x)$ *is an immediate subformula of* $\lambda(p)$*.*

Proof: For 1), recall from proposition 13.1 that $w$ is a vertex that is incident with exactly two hyperarcs, both of which result from ESSs that are the base structure of the sequent: $\gamma \to w$ and $w \to \delta$. Hence, $S = \widehat{\gamma}$ and $T = \widehat{\delta}$. Then, for each $x \in S$, $\pi^{-1}(x)$ is some formula of the antecedent of $\widetilde{\mathcal{S}}$, which is restored to a formula of the antecedent of $\mathcal{S}$ by $\rho$. Analogously, for each $x \in T$, $\pi^{-1}(x)$ is some formula of the succedent of $\widetilde{\mathcal{S}}$, which is restored to a formula of the antecedent of $\mathcal{S}$ by $\rho$. For 2), it must be understood that, for every other hyperarc $(S, T) \in \mathfrak{H}$ and every $x \in S \cup T$, $\pi^{-1}(x)$ is some occurrence instance of a formula in $\widetilde{\mathcal{S}}$. Now, $(S \cup T)\pi^{-1}$ must contain a single formula $C$ of maximal complexity (see below). Let $p \in S \cup T$ be the vertex with $p = \pi(C)$. As $C$ is some occurrence of a complex formula in $\widetilde{\mathcal{S}}$, according to proposition 13.2 and its corollaries, there must be a unique hyperarc in $\mathfrak{H}$ such that $p = \pi(C)$ is a vertex thereof and, for all other vertices $x$ thereof, $x = \pi(A)$, where $A$ is an immediate subformula of $C$. If $(S \cup T)\pi^{-1}$ were to contain two or more formulae of equal complexity, it would be a contradiction to proposition 13.2 or one of its corollaries. □

The combination of the preceding results suggests a procedure for the direct, incremental construction of the relational interpretation of a sequent.[1] The following procedure initialises with the hypergraph made up from the B-arc and the F-arc that are, according to 1) of the preceding proposition, the structural representation of the sequent's antecedent and succedent. In each step of the procedure, one or two new hyperarcs will be attached to an external vertex of the existant hypergraph (which is an internal vertex of the resulting hypergraph), depending on the formula label of the vertex and whether it is an initial vertex or a terminal vertex. These additions are done in such a manner that the hyperarcs that are obtained in one step are exactly those that would be obtained as relational interpretations of the ESS or ESSs corresponding to that formula, i.e. the labels of the new vertices must be the subformulae of the label of the selected vertex. Note that it is not necessary to resort to occurrence instances of formulae, because the different occurrences of formulae are distinguished by the vertices that bear them as their labels.

[1]It is this procedure or the notion of the hypergraphs, which are derived by it, that is picked up in the title for this dissertation.

**Definition 13.7** *Let $\mathcal{S} = [A_0, \ldots, A_m \to B_0, \ldots, B_n]$ be a standard sequent. The* logical tomograph *of sequent $\mathcal{S}$ is a labelled rooted directed hypergraph $\mathcal{H}_\mathcal{S} = \langle V, \mathfrak{H}, w, \lambda \rangle$, which is generated by the following* growing procedure.

- *Initialisation: Let $w, x_1, \ldots, x_m, y_1, \ldots, y_n \in \mathcal{V}$ be pairwise distinct vertices such that, for all $1 \leq i \leq m$, let $x_i \in \mathcal{O}$, if $A_i$ is an atomic formula, or $x_i \in \mathcal{P}$ otherwise, and, for all $1 \leq i \leq n$, let $y_i \in \mathcal{O}$, if $B_i$ is an atomic formula, or $y_i \in \mathcal{P}$ otherwise. Let*

$$V_0 = \{w, x_1, \ldots, x_m, y_1, \ldots, y_n\},$$
$$\mathfrak{H}_0 = \big\{\big(\{x_1, \ldots, x_m\}, \{w\}\big), \big(\{w\}, \{y_1, \ldots, y_n\}\big)\big\},$$
$$\lambda_0 = \{x_i \mapsto A_i \mid 1 \leq i \leq m\} \cup \{y_i \mapsto B_i \mid 1 \leq i \leq n\}.$$

- *Iteration: For some external vertex $p$ of $\langle V_i, \mathfrak{H}_i, w, \lambda_i \rangle$ with $\lambda_i(p) = A \star B$ for $\star \in \{\&, \vee, \supset\}$ or $\lambda_i(p) = \neg A$ do the following:*

  *(a) If $\lambda_i(p) = A \,\&\, B$, let $x, y$ be new vertices such that $x \in \mathcal{O}$, if $A$ is an atomic formula, or $x \in \mathcal{P}$ otherwise, or $y \in \mathcal{O}$, if $B$ is an atomic formula, and $y \in \mathcal{P}$ otherwise. Let*

  $$V_{i+1} = V_i \cup \{x, y\},$$
  $$\lambda_{i+1} = \lambda_i \cup \{x \mapsto A\} \cup \{y \mapsto B\}.$$

  *If $p$ is an initial vertex, then let*

  $$\mathfrak{H}_{i+1} = \mathfrak{H}_i \cup \big\{\big(\{x, y\}, \{p\}\big)\big\};$$

  *if $p$ is a terminal vertex, then let*

  $$\mathfrak{H}_{i+1} = \mathfrak{H}_i \cup \big\{\big(\{p\}, \{x\}\big), \big(\{p\}, \{y\}\big)\big\};$$

  *(b) If $\lambda_i(p) = A \vee B$, let $x, y$ be new vertices such that $x \in \mathcal{O}$, if $A$ is an atomic formula, or $x \in \mathcal{P}$ otherwise, or $y \in \mathcal{O}$, if $B$ is an atomic formula, and $y \in \mathcal{P}$ otherwise. Let*

  $$V_{i+1} = V_i \cup \{x, y\},$$
  $$\lambda_{i+1} = \lambda_i \cup \{x \mapsto A\} \cup \{y \mapsto B\}.$$

  *If $p$ is an initial vertex, then let*

  $$\mathfrak{H}_{i+1} = \mathfrak{H}_i \cup \big\{\big(\{x\}, \{p\}\big), \big(\{y\}, \{p\}\big)\big\};$$

  *if $p$ is a terminal vertex, then let*

  $$\mathfrak{H}_{i+1} = \mathfrak{H}_i \cup \big\{\big(\{p\}, \{x, y\}\big)\big\};$$

  *(c) If $\lambda_i(p) = A \supset B$, let $x, y$ be new vertices such that $x \in \mathcal{O}$, if $A$ is an atomic formula, or $x \in \mathcal{P}$ otherwise, or $y \in \mathcal{O}$, if $B$ is an atomic formula, and $y \in \mathcal{P}$ otherwise. Let*

$$V_{i+1} = V_i \cup \{x, y\},$$
$$\lambda_{i+1} = \lambda_i \cup \{x \mapsto A\} \cup \{y \mapsto B\}.$$

*If p is an initial vertex, then let*

$$\mathfrak{H}_{i+1} = \mathfrak{H}_i \cup \big\{ \big(\emptyset, \{p, x\}\big), \big(\{y\}, \{p\}\big) \big\};$$

*if p is a terminal vertex, then let*

$$\mathfrak{H}_{i+1} = \mathfrak{H}_i \cup \big\{ \big(\{x, p\}, \{y\}\big) \big\}.$$

*(s) If $\lambda_i(p) = \neg A$, let q be new vertices such that $x \in \mathcal{O}$, if A is an atomic formula, or $x \in \mathcal{P}$ otherwise. Let*

$$V_{i+1} = V_i \cup \{x\},$$
$$\lambda_{i+1} = \lambda_i \cup \{x \mapsto A\}.$$

*If p is an initial vertex, then let*

$$\mathfrak{H}_{i+1} = \mathfrak{H}_i \cup \big\{ \big(\emptyset, \{p, x\}\big) \big\};$$

*if p is a terminal vertex, then let*

$$\mathfrak{H}_{i+1} = \mathfrak{H}_i \cup \big\{ \big(\{x, p\}, \emptyset\big) \big\}.$$

*As each iteration step turns an external vertex, which has a complex formula as a label, into an internal one and introduces one or two new external nodes, which have the immediate subformula or subformulae as label or labels, the procedure terminates after k steps. Let $V = V_k$, $\mathfrak{H} = \mathfrak{H}_k$ and $\lambda = \lambda_k$.*

Observe that each step of the iteration adds one or two hyperedges, which only share a single vertex with the previous hypergraph. All of the vertices, which eventually become internal vertices in the logical tomograph, are elements of $\mathcal{P}$. Obviously, the labelling of vertices is necessary to control the procedure. However, when it has terminated, it is not required that $\lambda_k$ is used as labelling function. Using instead the restriction $\lambda = \lambda_k|_{\mathcal{O}}$, as we shall do in the examples, removes the labels from the internal vertices of $\mathcal{H}_{\mathcal{S}}$.

Figure 13.1 shows the growing of the logical tomograph of the sequent $a \,\&\, b, (a \supset c) \,\&\, (b \supset d) \rightarrow c \,\&\, d$. Note that in each step only the labels of the respective external vertices are displayed. Step ① is the initialisation of the procedure, which introduces two hyperarcs, one of which has initial nodes, say $p$ and $q$, which are labelled by $a \,\&\, b$ and $(a \supset c) \,\&\, (b \supset d)$, the other of which has a terminal node, say $r$, which is labelled by $c \& d$. Since none of the labels is an atomic formula, all of the vertices are taken from $\mathcal{P}$. Their shared internal node is the root $w$ of all of the following directed hypergraphs. For step ②, out of the three external nodes, which are to be developed, vertex $q$ with the label $(a \supset c) \,\&\, (b \supset d)$ is selected. Two new vertices, say $s$ and $t$ with labels $a \supset c$ and $b \supset d$, are introduced as new initial vertices and

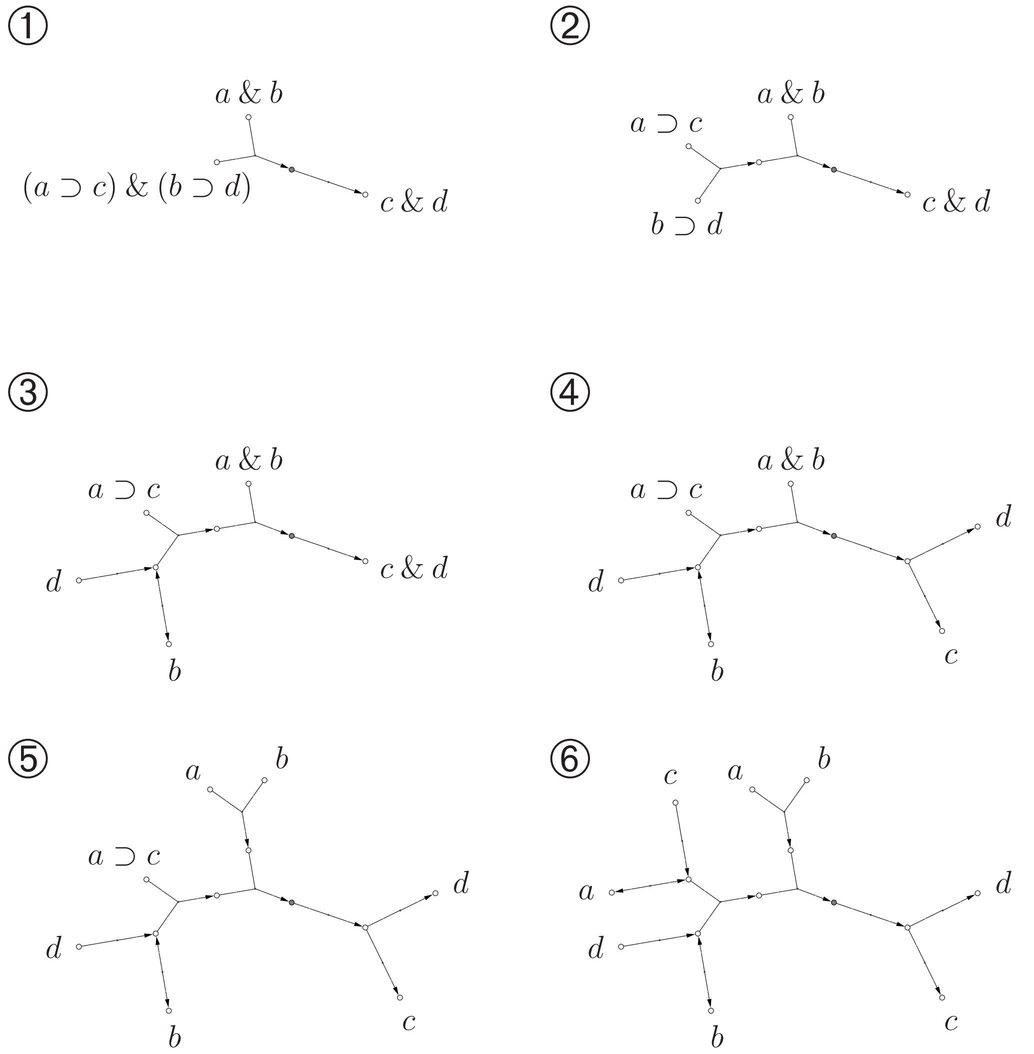



Figure 13.1: Growing the tomograph for $a \,\&\, b, (a \supset c) \,\&\, (b \supset d) \rightarrow c \,\&\, d$

connected to that node by means of the elementary fusion arc $\big(\{s,t\},\{q\}\big)$. This results in a directed hypergraph having four external vertices. In step ③, the initial vertex $t$ is developed. Due to its label $b \supset d$, two vertices have to be introduced, and since the subformulae of the implication are atomic, these vertices have to be elements of $\mathcal{O}$, say $b_1$ and $d_1$. Hyperarcs $\big(\emptyset,\{t,b_1\}\big)$ and $\big(\{d_1\},\{t\}\big)$, which concede on the vertex $t$, are added. Neither of $b_1$ and $d_1$ have complex formulae as labels, so these vertices will remain external vertices to the very end of the construction. Step ④ visits terminal vertex $r$, which is labelled by $c \,\&\, d$. As the subformulae are atomic, two vertices $c_1$ and $d_2$, which are elements of $\mathcal{O}$, are introduced and labelled by $c$ and $d$, and hyperarcs $\big(\{r\},\{c_1\}\big)$ and $\big(\{r\},\{d_2\}\big)$, which secede from $r$, are added to the directed hypergraph. For step ⑤, we consider the initial vertex $p$, which has the label $a \,\&\, b$. Again, the subformulae are atomic and demand the introduction of two elements of $\mathcal{O}$, say $a_1$ and $b_2$, which are labelled accordingly. The fusion hyperarc $\big(\{a_1,b_2\},\{p\}\big)$ is added. Eventually, step ⑥ treats the initial vertex $s$, which has the label $a \supset c$. Vertices $a_2$ and $c_2$, taken from $\mathcal{O}$, are introduced and labelled accordingly, and hyperarcs $\big(\emptyset,\{s,a_2\}\big)$ and $\big(\{c_2\},\{s\}\big)$, which concede at $s$, are added. As all vertices bear atomic formulae as labels, the procedure terminates.

The following observation is an obvious, but very important property of hypergraphs, which are obtained by this procedure.

**Proposition 13.8** *Let $\mathcal{S}$ be any sequent. Then its logical tomograph $\mathcal{H}_{\mathcal{S}}$ is a directed cut hypergraph.*

Proof: We show that each step of the construction results in a directed hypergraph, which satisfies properties 1) to 4) of definition 10.16. The initialisation step introduces two hyperarcs $\mathfrak{e} = \big(\{x_1,\ldots,x_m\},\{w\}\big)$ and $\mathfrak{f} = \big(\{w\},\{y_1,\ldots,y_n\}\big)\}$. The root $w$ is an internal node, which is a cut vertex, satisfying 1). It is also ${}^\bullet\mathfrak{e}^\bullet \cap {}^\bullet\mathfrak{f}^\bullet = \{w\}$, which establishes 4). Moreover, each of the $x_i$ is an initial vertex of $\langle V_0, \mathfrak{H}_0\rangle$ and only incident with $\mathfrak{e}$, and each of the $y_i$ is a terminal vertex of $\langle V_0, \mathfrak{H}_0\rangle$ and only incident with $\mathfrak{f}$, satisfying 2) and 3). For the iteration step, we observe that in each case either one or two new vertices $x$ and, if applicable, $y$ and one or two new hyperarcs are added to the existing hypergraph on one of its external vertices $p$ and no other vertex of the existing graph. Thereby property 4) is maintaned. An inspection of the cases reveals that in any case, the vertex $p$ becomes an internal vertex. Removing $p$ from the new hypergraph disconnects $x$ and, if applicable, $y$, and, thus, $p$ is a cut vertex; This establishes property 1) for the iteration step. Finally, the new vertices are external and are only incident with one of the new hyperarcs, which maintains 2) and 3). Therefore, a logical tomograph is a directed cut hypergraph. □

It will be useful to relate formula occurrences of the sequent $\mathcal{S}$ to particular elements of its logical tomograph. In particular, positive and negative formula occurrences can be distinguished by the type of vertices that have them as labels.

**Lemma 13.9** *Let $\mathcal{S}$ be a sequent and $\mathcal{H}_\mathcal{S}$ be its logical tomograph as above. If a complex formula $C$ has a positive (negative) occurrence in the sequent $\mathcal{S}$, then there is a vertex $p$ of $\mathcal{H}_\mathcal{S}$ with $\lambda(p) = C$, such that, at some stage $l$ in the procedure, $p$ is terminal (initial) in $\langle V_l, \mathfrak{H}_l \rangle$, and, at stage $l+1$ in the procedure, $p$ is internal in $\langle V_{l+1}, \mathfrak{H}_{l+1} \rangle$.*

PROOF: A straightforward induction is hampered by the fact that the procedure can develop the tomograph at different vertices. Every formula occurrence $C$ in $\mathcal{S} = [A_1, \ldots, A_m \to B_1, \ldots, B_n]$ is the subformula of either some formula occurrence $A_i$ or some formula occurrence $B_j$. We further consider the sequence $(C_0, \ldots C_k)$ of formulae, where each $C_{i+1}$ is an immediate subformula occurrence of $C_i$ and $C_k = C$. In the construction of $\mathcal{H}_\mathcal{S}$, each $C_i$, which is complex, is addressed eventually at stage $l_i$ and one or two vertices are introduced, (one of) which is labelled with $C_{i+1}$. We will now inspect the sequence of stages $(l_0, l_1, \ldots, l_k)$.

According to definition A.2, all formulae $A_i$ in the antecedent of $\mathcal{S}$ have negative occurrences, and all formulae $B_i$ in the succedent of $\mathcal{S}$ have positive occurrences. If $C_0 = A_j$ for some $1 \leq j \leq m$, then $C_0$ is a negative occurrence. Per definition, $\mathfrak{H}_0$ contains the hyperarcs $\big(\{p_1, \ldots, p_m\}, \{w\}\big)$ and $\big(\{w\}, \{q_1, \ldots, q_n\}\big)$, and, hence, it is $\lambda_{l_0}(p_j) = C_0$ for an initial node $p_j$ of $\langle V_{l_0}, \mathfrak{H}_{l_0} \rangle$. An investigation of those cases, in which an initial node is treated in the construction, reveals that in each instance a hyperarc is added, which has $p_j$ as terminal node, which leaves $p_j$ an internal node of the hypergraph at stage $l_0 + 1$. In the case that $C_0 = B_j$ for some $1 \leq j \leq n$, then it is $\lambda_{l_0}(q_j) = C_0$ for a terminal node $q_j$, which becomes an internal node at stage $l_0 + 1$, and this status is not changed until stage $l_1$.

At stage $l_i$, we have to consider either a positive occurrence of $C_i$, which can be assumed to be the label of a terminal vertex $p$ or a negative occurrence of $C_i$, which can be assumed to be the label of an initial vertex $p$. From the eight cases that have to be distinguished, we treat only two, as the remaining cases can be treated correspondingly.

- If $C_i = A \vee B$ is a positive occurrence in $\mathcal{S}$ and $p$ is a terminal vertex, the procedure adds the hyperarc $\big(\{p\}, \{x, y\}\big)$, where both $x$ and $y$ are new terminal vertices in the new hypergraph with $\lambda_{l_i+1}(x) = A$ and $\lambda_{l_i+1}(y) = B$. According to definition A.1, both $A$ and $B$ have positive occurrences in $\mathcal{S}$. If $C_{i+1} = A$, then it has a positive occurrence in $\mathcal{S}$

and $x$ is a terminal vertex of $\langle V_{l_i+1}, \mathfrak{H}_{l_i+1}\rangle$; if $C_{i+1} = B$, then it has a positive occurrence in $\mathcal{S}$ and $y$ is an terminal vertex of $\langle V_{l_i+1}, \mathfrak{H}_{l_i+1}\rangle$.

- If $C_i = A \supset B$ is a negative occurrence in $\mathcal{S}$ and $p$ is an initial vertex, the procedure adds hyperarcs $\big(\emptyset, \{p, x\}\big)$ and $\big(\{y\}, \{p\}\big)$, where both $x$ and $y$ are new vertices. Hence, $x$ is terminal and $y$ is initial to the new hypergraph with $\lambda_{l_i+1}(x) = A$ and $\lambda_{l_i+1}(y) = B$. According to definition A.1, $A$ has a positive occurrence in $\mathcal{S}$ and $B$ has a negative occurrence in $\mathcal{S}$. If $C_{i+1} = A$, then it has a positive occurrence in $\mathcal{S}$ and $x$ is a terminal vertex of $\langle V_{l_i+1}, \mathfrak{H}_{l_i+1}\rangle$; if $C_{i+1} = B$, then it has a negative occurrence in $\mathcal{S}$ and $y$ is an initial vertex of $\langle V_{l_i+1}, \mathfrak{H}_{l_i+1}\rangle$.

In any case, after the addition of these respective hyperarc or hyperarcs, $p$ is an internal node of the new hypergraph at stage $l_i + 1$ and remains so until stage $l_{i+1}$. □

Note that a logical tomograph is not a strand in general. In three of the eight cases in the iteration of the procedure, two new hyperarcs are added to an external vertex $p$ of the hypergraph, which leaves either $\deg^-(p) > 1$ or $\deg^+(p) > 1$.

The following result establishes that the development of a logical tomograph $\mathcal{H}_\mathcal{S}$ sufficiently distinguishes the different formula occurrences of the sequent $\mathcal{S}$.

**Lemma 13.10** *Let $\mathcal{S}$ be a sequent and $\widetilde{\mathcal{S}}$ an occurrence instance thereof. Then the logical tomograph $\mathcal{H}_\mathcal{S}$ of $\mathcal{S}$ and the logical tomograph $\mathcal{H}_{\widetilde{\mathcal{S}}}$ of $\widetilde{\mathcal{S}}$ are isomorphic.*

Proof: Sequent $\mathcal{S}$ and its occurrence instance $\widetilde{\mathcal{S}}$ only differ in their atomic formulae, as $\widetilde{\mathcal{S}}\rho = \mathcal{S}$ for the restoration function $\rho : \mathcal{O} \to \mathcal{A}$. Formula occurrences are retained in $\mathcal{H}_\mathcal{S}$ and $\mathcal{H}_{\widetilde{\mathcal{S}}}$ only by the labelling functions $\lambda$ and $\lambda'$ (for which it is $\rho \circ \lambda' = \lambda$). We retrace the generation of the two tomographs interleavedly, first relating their roots and the vertices representing the formula occurrences of antecedent and succedent by a bijective mapping $\sigma$ satisfying $\lambda(p) = \rho(\lambda'(\sigma(p)))$, and then by extending this mapping to all vertices, which are added during the procedure. □

The lemma establishes that the procedure develops a hypergraph in such a manner that all occurrences of subformulae of a sequent $\mathcal{S}$ are uniquely represented by vertices in $\mathcal{H}_\mathcal{S}$. We are now set up to formulate the main result of this section, which establishes that the growth procedure does exactly what it is supposed to do.

**Proposition 13.11** *Let $\mathcal{R}$ be the relational interpretation of a sequent $\mathcal{S}$, and let $\mathcal{H}_\mathcal{S}$ be the logical tomograph of $\mathcal{S}$. Then $\mathcal{R}$ and $\mathcal{H}_\mathcal{S}$ are isomorphic.*

PROOF: According to lemma 13.10, the logical tomographs $\mathcal{H}_\mathcal{S}$ and $\mathcal{H}_{\widetilde{\mathcal{S}}}$ are isomorphic, and thus we will use the latter. We show that, for every hyperarc that is added in the construction of $\mathcal{H}_{\widetilde{\mathcal{S}}}$, there is a corresponding hyperarc in $\mathcal{R}$, and that thereby all of the hyperarcs of $\mathcal{R}$ are tapped, and define a bijection $\sigma$ between the vertices of the two hypergraphs. Let $\widetilde{\mathcal{S}}$ have occurrence mapping $\pi$ and let $\mathcal{H}_{\widetilde{\mathcal{S}}} = \langle V, \mathfrak{H}, \dot{w}, \lambda\rangle$. (The vertices of $\mathcal{H}_{\widetilde{\mathcal{S}}}$ are marked with a dot to facilitate readability.)

Let $\mathcal{S}$ be the sequent $\Gamma \rightarrow \Delta$. Consider the relational interpretation of the base structure of a sequent, $(\widehat{\gamma}, \{w\})$ and $(\{w\}, \widehat{\delta})$, both hyperarcs in $\mathcal{R}$. At the same time, $\mathcal{H}_{\widetilde{\mathcal{S}}}$ is initialised with hyperarcs $(\{\dot{p_1}, \ldots, \dot{p_m}\}, \{\dot{w}\})$ and $(\{\dot{w}\}, \{\dot{q_1}, \ldots, \dot{q_n}\})$, where $m = \mathbf{len}(\Gamma) = \mathbf{len}(\gamma)$ and $n = \mathbf{len}(\Delta) = \mathbf{len}(\delta)$, and so we can already initialise $\sigma = \{\dot{w} \mapsto w\} \cup \{\dot{x} \mapsto \pi(C) \mid \lambda(\dot{x}) = C\}$; the root of $\mathcal{H}_\mathcal{S}$ is mapped to the watershed proxy, and every other vertex $\dot{x}$ is mapped to that occurrence variable, which is the image of the label of $\dot{x}$ under the occurrence mapping.

Whenever a vertex $\dot{x}$ of $\mathcal{H}_{\widetilde{\mathcal{S}}}$ is considered, new vertices and hyperarcs might be added, depending on the formula $C = \lambda(\dot{x})$ occurring as its label and whether $\dot{x}$ is an initial or a terminal node. For this formula we also obtain an occurrence variable $\pi(\lambda(\dot{x}))$, for which the bijection $\sigma$ has already been defined to yield $\sigma(\dot{x}) = \pi(C)$. Now, if $C$ is an atomic formula $a$, then $\dot{x}$ need not be considered any further, because $\pi(a) = a \in \mathcal{O}$. For neither does the growth procedure connect any hyperarc to a vertex $\dot{x}$ with label $a$, as new hyperarcs are only added to vertices bearing some complex formulae as a label, nor can the relational interpretation $\mathcal{R}$ contain more than a single hyperarc connected to a vertex in $\mathcal{O}$, as it resulted from a proper occurrence variable, each of which only occur in a single ESS of the explosion set. Otherwise, we have to distinguish eight cases. We will exemplarily consider two of them.

- If $C = A \mathbin{\&} B$ is a positive occurrence in $\mathcal{S}$, then $\dot{p}$ is a terminal node at that stage of the construction. Hence, hyperarcs $(\{\dot{p}\}, \{\dot{x}\})$ and $(\{\dot{p}\}, \{\dot{y}\})$ are added to the hypergraph for new vertices $\dot{x}$ and $\dot{y}$ with $\lambda(\dot{x}) = A$ and $\lambda(\dot{y}) = B$. According to proposition 13.2, item 2), $\mathcal{R}$ contains two hyperarcs $(\{p\}, \{x\})$ and $(\{p\}, \{y\})$ with $p = \pi(A \mathbin{\&} B)$, $x = \pi(A)$ and $y = \pi(B)$. We extend $\sigma$ to $\sigma \cup \{\dot{x} \mapsto \pi(A), \dot{y} \mapsto \pi(B)\}$, and as $A$ and $B$ have not yet been considered, $\sigma$ is still an injection.

- If $C = \neg A$ is a negative occurrence in $\mathcal{S}$, then $\dot{p}$ is a initial node at that stage of the construction. Hence, the hyperarc $(\emptyset, \{\dot{p}, \dot{x}\})$ is added to the hypergraph for a new vertex $\dot{x}$ with $\lambda(\dot{x}) = A$. According to lemma 13.5, item 1), $\mathcal{R}$ contains a hyperarc $(\emptyset, \{p, x\})$ with $p = \pi(\neg A)$ and $\pi(A)$. Then, $\sigma$ can be extended to $\sigma \cup \{\dot{x} \mapsto \pi(A)\}$. As $A$ has not yet been considered, $\sigma$ is still an injection.

For the remaining cases, proposition 13.2 and corollaries 13.3, 13.4 and 13.5 have to be considered.

This establishes that an injection $\sigma$ between vertices can be constructed. It is easy to see that $\sigma$ must also be a surjection. Every hyperarc in $\mathcal{R}$ is obtained from an ESS of the explosion set of $\widetilde{\mathcal{S}}$ has a base proxy $p$, for which it is $C = \pi^{-1}(p)$ for some complex formula occurrence of $\widetilde{\mathcal{S}}$. But then $\mathcal{H}_{\widetilde{\mathcal{S}}}$ contains some vertex $\dot{p}$ with $\lambda(\dot{p}) = C$, which is certainly addressed during the construction, which renders $\sigma$ a surjection. □

In view of this result, we can use the notion of relational interpretation of an explosion set and that of logical tomograph interchangeably. Hence, for a given sequent, these two notions merely specify the manner, in which the corresponding directed hypergraph has been obtained. This is a brief summary of the two alternative methods for obtaining a rooted cut hypergraph for a given sequent $\mathcal{S}$:

1) Generate a relational interpretation of the explosion set of $\mathcal{S}$ by executing the following three steps:
   (a) Let $\widetilde{\mathcal{S}}$ be an occurrence instance of $\mathcal{S}$ with restoration function $\rho$.
   (b) Obtain the explosion set $\langle \mathfrak{E}, w \rangle$ by an explosion derivation of $\widetilde{\mathcal{S}}$.
   (c) Obtain the relational interpretation $\langle \mathcal{V}_{\widehat{\mathfrak{E}}}, \widehat{\mathfrak{E}}, w, \rho \rangle$.
2) Directly generate the logical tomograph $\mathcal{H}_{\mathcal{S}}$ by the growing procedure that was described in this chapter.

Method 1) follows the rather involved path by which the different ideas have been successively presented. Method 2) is the much more concise and immediate method. Already starting out with a core hypergraph that has formulae as labels, the vertices serve as occurrence markers, relating each formula occurrence a unique vertex. The growth procedure adds exactly those vertices and hyperedges to a vertex of the existant hypergraph that correspond to the ESSs that would be obtained in the explosion procedure from the formula that is the label of that vertex. In this sense, the growth procedure is a more immediate method for obtaining the relational interpretation of a sequent.

Of course, it is possible to abandon the labelling function altogether in favour of structural identity information provided by co-identity arcs in the manner introduced in the last chapter. These arcs would have to be derived from the external vertices of the logical tomograph, i.e. from the function $\lambda = \lambda_k|_{\mathcal{O}}$, as follows.

**Definition 13.12** *Let $\mathcal{H}_{\mathcal{S}} = \langle V, \mathfrak{H}, w, \lambda \rangle$ be the logical tomograph of a sequent $\mathcal{S}$. The directed rooted hypergraph $\mathcal{H}' = \langle V, \mathfrak{H}, \mathfrak{I}, w \rangle$ with*

$$\mathfrak{I} = \big\{ (\{w\}, \{v\}) \mid w \textit{ is terminal and } v \textit{ is initial in } \mathcal{H}_{\mathcal{S}} \textit{ and } \lambda(v) = \lambda(w) \big\}$$

*is called the* logical tomograph *of $\mathcal{S}$ with* structural identity information*. For a $\mathcal{H}'$ thus given, the graph $\langle V, \mathfrak{H}, w \rangle$ is called the* occurrence trunk.

This definition is exactly the same as definition 12.6, and, consequently, the following holds.

**Corollary 13.13** *Let $\mathcal{R}$ be the relational interpretation of sequent $\mathcal{S}$ with structural identity information, and let $\mathcal{H}_{\mathcal{S}}$ be the logical tomograph of $\mathcal{S}$ with structural identity information. Then $\mathcal{R}$ and $\mathcal{H}_{\mathcal{S}}$ are isomorphic.*

PROOF: This is a trivial consequence of proposition 13.11 □

The refutation procedures and the decision procedures that were outlined in the previous chapter are applicable to logical tomographs with structural identity information.

# Chapter 14

# Discussion

It was the purpose of this investigation to demonstrate how formal reasoning can be traced back to a purely structural reasoning and, moreover, to find a method for presenting the basic constituents of such a structural reasoning in a more intuitive and accessible manner than by elementary structural sequents. It has always been the exercise of structural proof theory from the time of its inception by Gerhard Gentzen in [Gen35], to exhibit how individual elements of logical language relate to particular structural aspects of the derivations that serve as justifications for these elements. Usually these structural aspects are represented by derivation trees, which are furnished with syntactic entities, such as formulae or, for logistic calculi, sequents.

Rather than agreeing to Gentzen's work as point of origin of structural proof theory and thereby endorsing these traditional presentations of structural aspects, a significant part of this investigation consisted in demonstrating that the inspiration for the structural aspects, which Gentzen employed for his purposes, can be traced back to the works of Paul Hertz, and to use those as a source of inspiration. Our approach towards exposing these structural aspects was therefore guided by taking some of Hertz' somewhat curious positions seriously, the most prominent of which can be summarised as follows:

- Reasoning should be concerned with the manipulation of relations, such as relations of causes and effects or relations of assumptions and assertions. Individual components of these relations, which are called *sentences*, are the elementary logical constituents.

- The realm of relations, about which we reason, is not interesting in its potential for an infinity of arrangements, but in the particular arrangement of even a finite number of them.

- Reasoning is not about language. By emphasising logical syntax, the actual matter, namely the aforementioned structural relations, is lost out of sight.

By focussing on these three positions, it became almost immediately apparent that the elements of Hertz structural logic, which he calls *sentences*, can be retrieved by undoing in a particular manner Gentzen-style sequents by means of a procedure that explodes them into its elementary structural constituents by means of a modified calculus that employs only the cut rule restricted to atomic cut formulae and local logical rules, i.e. logical rules, in which the context is a single propositional atom. We have demonstrated how the collection of the thereby generated elementary structural sequents, the explosion set, can be considered to represent the meaning of the original sequent. Furthermore, it was demonstrated that the provability of a Gentzen-style sequent can be decided by investigating its explosion set.

It became apparent that the elementary structural sequents obtained by this procedure are ideally represented by graph-theoretical entities. Due to the somewhat involved form of sequents and Hertz' sentences of higher degree, we had to resort to directed hypergraphs. As it turned out, it is in particular rooted directed cut hypergraphs, which can serve as the relational interpretations of the structural constituents of logical sequents. We have shown how every sequent containing occurrences of complex formulae can be either dissected by means of the explosion calculus and then transformed into a the relational interpretation thereof or, equivalently, directly decomposed into a logical tomograph. In any case, the result of the procedure is a rooted directed cut hypergraph, the hyperarcs of which are made up of the elementary structural relations. Finally, the question of provability was shown to be decidable by considering partial cycles in the resulting hypergraphs that are enhanced by structural identity information.

In summary, we have shown that propositional logic can be presented entirely structurally in the spirit of Paul Hertz without losing any expressiveness. The overemphasis of primarily formal entities and their derivations, which obstructs the view of the structural foundations of logic, might be loosened or at least supplemented by the elementary structural and graph-theoretical perspective that we have presented.

## Perspectives

Several areas for further investigations, which have been deliberately avoided, were already mentioned in the introduction. For one, it remains to be seen, whether concise hypergraphical representations of sequents of the predicate calculus can be given. It seems promising to consider Hertz' macro sentences for this purpose, as it is conceivable that the issue of scope can be captured in that way. Moreover, in connection with macro sentences, it might be a fruitful endeavour to experiment with inference rules, which are based on syllogisms other than the modus barbara, the only syllogism that Hertz himself ever considered. In a similar manner, it should be explored, whether the non-local effect of certain intuitionistic rules can be accomodated in our approach in an elegant manner. Rather than a brute force interpretation, in which a new explosion set has to be generated for each critical rule application, it is conceivable to instead introduce a selection function, which assigns to each state of an intuitionistic derivation a particular subset of a single explosion set. It is unclear at present, however, whether such an approach would result in an intuitive representation.

An entirely different field of investigation is opened up by the following observation. There is a close correspondence between directed hypergraphs and directed bipartite graphs, which was exploited for the purpose of displaying hypergraphs. At the same time, directed bipartite graphs are employed for depicting the structure of discrete distributed system as perti nets. It might be well worth studying the dynamics of hypergraphs in a manner related to the dynamics of petri nets. This is particularly relevant in view of Paul Hertz' allusions that reasoning is indeed closely related to descriptive dynamics. In addition to finding and investigating a suitable dynamics for logical tomographs, the modification of such a dynamics under the addition of shortcutting or intermediating hyperarcs could be a rewarding area of study.

# Appendix A

# On Formula Occurrences

**Definition A.1** *An occurrence of a subformula of a formula $C$ is either a* positive occurrence *or a* negative occurrence. *The two notions are mutually recursively defined as follows:*

- *$C$ is a positive occurrence in $C$.*
- *If $A \,\&\, B$ is a positive occurrence in $C$, then both $A$ and $B$ are positive occurrences in $C$. If $A \,\&\, B$ is a negative occurrence in $C$, then both $A$ and $B$ are negative occurrences in $C$.*
- *If $A \vee B$ is a positive occurrence in $C$, then both $A$ and $B$ are positive occurrences in $C$. If $A \vee B$ is a negative occurrence in $C$, then both $A$ and $B$ are negative occurrences in $C$.*
- *If $A \supset B$ is a positive occurrence in $C$, then $A$ is a negative occurrence in $C$ and $B$ is a positive occurrence in $C$. If $A \supset B$ is a negative occurrence in $C$, then $A$ is a positive occurrence in $C$ and $B$ is a negative occurrence in $C$.*
- *If $\neg A$ is a positive occurrence in $C$, then $A$ is a negative occurrence in $C$. If $\neg A$ is a negative occurrence in $C$, then $A$ is a positive occurrence in $C$.*

**Definition A.2** *An occurrence of a subformula $F$ of a sequent $\mathcal{S}$ is* positive, *if it is a positive occurrence of a formula, which occurs in the succedent of $\mathcal{S}$, or if it is a negative occurrence of a formula $C$, which occurs in the succedent of $\mathcal{S}$. An occurrence of a subformula $F$ of a sequent $\mathcal{S}$ is* negative, *if it is a negative occurrence of a formula, which occurs in the succedent of $\mathcal{S}$, or if it is a positive occurrence of a formula $C$, which occurs in the succedent of $\mathcal{S}$.*

**Definition A.3** *For a formula $C$, the functions $\mathbf{C}^+_C$ and $\mathbf{C}^-_C$, counting the number of positive/negative occurrences of conjunctions occuring in $C$, are defined by mutual recursion on the auxiliary functions $\mathbf{C}^\oplus_C$ and $\mathbf{C}^\ominus_C$ as follows:*

| | | |
|---|---|---|
| $C = a \in \mathcal{A}$ | $\mathbf{C}^+_a = 0$<br>$\mathbf{C}^\oplus_a = 0$ | $\mathbf{C}^-_a = 0$<br>$\mathbf{C}^\ominus_a = 0$ |
| $C = A \,\&\, B$ | $\mathbf{C}^+_{A \,\&\, B} = \mathbf{C}^+_A + \mathbf{C}^+_B$<br>$\mathbf{C}^\oplus_{A \,\&\, B} = \mathbf{C}^\oplus_A + \mathbf{C}^\oplus_B$ | $\mathbf{C}^-_{A \,\&\, B} = \mathbf{C}^-_A + \mathbf{C}^-_B$<br>$\mathbf{C}^\ominus_{A \,\&\, B} = \mathbf{C}^\ominus_A + \mathbf{C}^\ominus_B$ |
| $C = A \vee B$ | $\mathbf{C}^+_{A \vee B} = \mathbf{C}^+_A + \mathbf{C}^+_B + 1$<br>$\mathbf{C}^\oplus_{A \vee B} = \mathbf{C}^\oplus_A + \mathbf{C}^\oplus_B$ | $\mathbf{C}^-_{A \vee B} = \mathbf{C}^-_A + \mathbf{C}^-_B$<br>$\mathbf{C}^\ominus_{A \vee B} = \mathbf{C}^\oplus_A + \mathbf{C}^\ominus_B + 1$ |
| $C = A \supset B$ | $\mathbf{C}^+_{A \supset B} = \mathbf{C}^\oplus_A + \mathbf{C}^+_B$<br>$\mathbf{C}^\oplus_{A \supset B} = \mathbf{C}^+_A + \mathbf{C}^\oplus_B$ | $\mathbf{C}^-_{A \supset B} = \mathbf{C}^\ominus_A + \mathbf{C}^-_B$<br>$\mathbf{C}^\ominus_{A \supset B} = \mathbf{C}^-_A + \mathbf{C}^\ominus_B$ |
| $C = \neg A$ | $\mathbf{C}^+_{\neg A} = \mathbf{C}^\oplus_A$<br>$\mathbf{C}^\oplus_{\neg A} = \mathbf{C}^+_A$ | $\mathbf{C}^-_{\neg A} = \mathbf{C}^\ominus_A$<br>$\mathbf{C}^\ominus_{\neg A} = \mathbf{C}^-_A$ |

**Definition A.4** *For a sequent $\mathcal{S}$: $C_1, \ldots, C_m \rightarrow D_1, \ldots, D_n$, the functions $\mathbf{C}^+_\mathcal{S}$ and $\mathbf{C}^-_\mathcal{S}$, counting the number of positive/negative occurrences of conjunctions occuring in $\mathcal{S}$, are given as follows:*

$$\mathbf{C}^+_\mathcal{S} = \sum_{i=1}^{m} \mathbf{C}^-_{C_i} + \sum_{i=1}^{n} \mathbf{C}^+_{D_i}$$

$$\mathbf{C}^-_\mathcal{S} = \sum_{i=1}^{m} \mathbf{C}^+_{C_i} + \sum_{i=1}^{n} \mathbf{C}^-_{D_i}$$

**Definition A.5** *For a formula $C$, the functions $\mathbf{D}^+_C$ and $\mathbf{D}^-_C$, counting the number of positive/negative occurrences of disjunctions occuring in $C$, are* *defined by mutual recursion on the auxiliary functions* $\mathbf{D}_C^{\oplus}$ *and* $\mathbf{D}_C^{\ominus}$ *as follows:*

| | | |
|---|---|---|
| $C = a \in \mathcal{A}$ | $\mathbf{D}_a^{+} = 0$<br>$\mathbf{D}_a^{\oplus} = 0$ | $\mathbf{D}_a^{-} = 0$<br>$\mathbf{D}_a^{\ominus} = 0$ |
| $C = A \,\&\, B$ | $\mathbf{D}_{A \& B}^{+} = \mathbf{D}_A^{+} + \mathbf{D}_B^{+}$<br>$\mathbf{D}_{A \& B}^{\oplus} = \mathbf{D}_A^{\oplus} + \mathbf{D}_B^{\oplus}$ | $\mathbf{D}_{A \& B}^{-} = \mathbf{D}_A^{-} + \mathbf{D}_B^{-}$<br>$\mathbf{D}_{A \& B}^{\ominus} = \mathbf{D}_A^{\ominus} + \mathbf{D}_B^{\ominus}$ |
| $C = A \vee B$ | $\mathbf{D}_{A \vee B}^{+} = \mathbf{D}_A^{+} + \mathbf{D}_B^{+} + 1$<br>$\mathbf{D}_{A \vee B}^{\oplus} = \mathbf{D}_A^{\oplus} + \mathbf{D}_B^{\oplus}$ | $\mathbf{D}_{A \vee B}^{-} = \mathbf{D}_A^{-} + \mathbf{D}_B^{-}$<br>$\mathbf{D}_{A \vee B}^{\ominus} = \mathbf{D}_A^{\oplus} + \mathbf{D}_B^{\ominus} + 1$ |
| $C = A \supset B$ | $\mathbf{D}_{A \supset B}^{+} = \mathbf{D}_A^{\oplus} + \mathbf{D}_B^{+}$<br>$\mathbf{D}_{A \supset B}^{\oplus} = \mathbf{D}_A^{+} + \mathbf{D}_B^{\oplus}$ | $\mathbf{D}_{A \supset B}^{-} = \mathbf{D}_A^{\ominus} + \mathbf{D}_B^{-}$<br>$\mathbf{D}_{A \supset B}^{\ominus} = \mathbf{D}_A^{-} + \mathbf{D}_B^{\ominus}$ |
| $C = \neg A$ | $\mathbf{D}_{\neg A}^{+} = \mathbf{D}_A^{\oplus}$<br>$\mathbf{D}_{\neg A}^{\oplus} = \mathbf{D}_A^{+}$ | $\mathbf{D}_{\neg A}^{-} = \mathbf{D}_A^{\ominus}$<br>$\mathbf{D}_{\neg A}^{\ominus} = \mathbf{D}_A^{-}$ |

**Definition A.6** *For a sequent* $\mathcal{S}$*:* $C_1, \ldots, C_m \rightarrow D_1, \ldots, D_n$*, the functions* $\mathbf{D}_{\mathcal{S}}^{+}$ *and* $\mathbf{D}_{\mathcal{S}}^{-}$*, counting the number of positive/negative occurrences of disjunctions occuring in* $\mathcal{S}$*, are given as follows:*

$$\mathbf{D}_{\mathcal{S}}^{+} = \sum_{i=1}^{m} \mathbf{D}_{C_i}^{-} + \sum_{i=1}^{n} \mathbf{D}_{D_i}^{+}$$

$$\mathbf{D}_{\mathcal{S}}^{-} = \sum_{i=1}^{m} \mathbf{D}_{C_i}^{+} + \sum_{i=1}^{n} \mathbf{D}_{D_i}^{-}$$

**Definition A.7** *For a formula* $C$*, the functions* $\mathbf{I}_C^{+}$ *and* $\mathbf{I}_C^{-}$*, counting the number of positive/negative occurrences of implications occuring in* $C$*, are* *defined by mutual recursion on the auxiliary functions* $\mathbf{I}^{\oplus}_{C}$ *and* $\mathbf{I}^{\ominus}_{C}$ *as follows:*

| | | |
|---|---|---|
| $C = a \in \mathcal{A}$ | $\mathbf{I}^{+}_{a} = 0$<br>$\mathbf{I}^{\oplus}_{a} = 0$ | $\mathbf{I}^{-}_{a} = 0$<br>$\mathbf{I}^{\ominus}_{a} = 0$ |
| $C = A \,\&\, B$ | $\mathbf{I}^{+}_{A \,\&\, B} = \mathbf{I}^{+}_{A} + \mathbf{I}^{+}_{B}$<br>$\mathbf{I}^{\oplus}_{A \,\&\, B} = \mathbf{I}^{\oplus}_{A} + \mathbf{I}^{\oplus}_{B}$ | $\mathbf{I}^{-}_{A \,\&\, B} = \mathbf{I}^{-}_{A} + \mathbf{I}^{-}_{B}$<br>$\mathbf{I}^{\ominus}_{A \,\&\, B} = \mathbf{I}^{\ominus}_{A} + \mathbf{I}^{\ominus}_{B}$ |
| $C = A \vee B$ | $\mathbf{I}^{+}_{A \vee B} = \mathbf{I}^{+}_{A} + \mathbf{I}^{+}_{B}$<br>$\mathbf{I}^{\oplus}_{A \vee B} = \mathbf{I}^{\oplus}_{A} + \mathbf{I}^{\oplus}_{B}$ | $\mathbf{I}^{-}_{A \vee B} = \mathbf{I}^{-}_{A} + \mathbf{I}^{-}_{B}$<br>$\mathbf{I}^{\ominus}_{A \vee B} = \mathbf{I}^{\oplus}_{A} + \mathbf{I}^{\ominus}_{B}$ |
| $C = A \supset B$ | $\mathbf{I}^{+}_{A \supset B} = \mathbf{I}^{\oplus}_{A} + \mathbf{I}^{+}_{B} + 1$<br>$\mathbf{I}^{\oplus}_{A \supset B} = \mathbf{I}^{+}_{A} + \mathbf{I}^{\oplus}_{B}$ | $\mathbf{I}^{-}_{A \supset B} = \mathbf{I}^{\ominus}_{A} + \mathbf{I}^{-}_{B}$<br>$\mathbf{I}^{\ominus}_{A \supset B} = \mathbf{I}^{-}_{A} + \mathbf{I}^{\ominus}_{B} + 1$ |
| $C = \neg A$ | $\mathbf{I}^{+}_{\neg A} = \mathbf{I}^{\oplus}_{A}$<br>$\mathbf{I}^{\oplus}_{\neg A} = \mathbf{I}^{+}_{A}$ | $\mathbf{I}^{-}_{\neg A} = \mathbf{I}^{\ominus}_{A}$<br>$\mathbf{I}^{\ominus}_{\neg A} = \mathbf{I}^{-}_{A}$ |

**Definition A.8** *For a sequent* $\mathcal{S}$*:* $C_1, \ldots, C_m \to D_1, \ldots, D_n$*, the functions* $\mathbf{I}^{+}_{\mathcal{S}}$ *and* $\mathbf{I}^{-}_{\mathcal{S}}$*, counting the number of positive/negative occurrences of conjunctions occuring in* $\mathcal{S}$*, are given as follows:*

$$\mathbf{I}^{+}_{\mathcal{S}} = \sum_{i=1}^{m} \mathbf{I}^{-}_{C_i} + \sum_{i=1}^{n} \mathbf{I}^{+}_{D_i}$$

$$\mathbf{I}^{-}_{\mathcal{S}} = \sum_{i=1}^{m} \mathbf{I}^{+}_{C_i} + \sum_{i=1}^{n} \mathbf{I}^{-}_{D_i}$$

**Definition A.9** *For a formula* $C$*, the functions* $\mathbf{N}^{+}_{C}$ *and* $\mathbf{N}^{-}_{C}$*, counting the number of positive/negative occurrences of implications occuring in* $C$*, are* *defined by mutual recursion on the auxiliary functions* $\mathbf{N}_C^{\oplus}$ *and* $\mathbf{N}_C^{\ominus}$ *as follows:*

| | | |
|---|---|---|
| $C = a \in \mathcal{A}$ | $\mathbf{N}_a^{+} = 0$<br>$\mathbf{N}_a^{\oplus} = 0$ | $\mathbf{N}_a^{-} = 0$<br>$\mathbf{N}_a^{\ominus} = 0$ |
| $C = A \,\&\, B$ | $\mathbf{N}_{A \,\&\, B}^{+} = \mathbf{N}_A^{+} + \mathbf{N}_B^{+}$<br>$\mathbf{N}_{A \,\&\, B}^{\oplus} = \mathbf{N}_A^{\oplus} + \mathbf{N}_B^{\oplus}$ | $\mathbf{N}_{A \,\&\, B}^{-} = \mathbf{N}_A^{-} + \mathbf{N}_B^{-}$<br>$\mathbf{N}_{A \,\&\, B}^{\ominus} = \mathbf{N}_A^{\ominus} + \mathbf{N}_B^{\ominus}$ |
| $C = A \vee B$ | $\mathbf{N}_{A \vee B}^{+} = \mathbf{N}_A^{+} + \mathbf{N}_B^{+}$<br>$\mathbf{N}_{A \vee B}^{\oplus} = \mathbf{N}_A^{\oplus} + \mathbf{N}_B^{\oplus}$ | $\mathbf{N}_{A \vee B}^{-} = \mathbf{N}_A^{-} + \mathbf{N}_B^{-}$<br>$\mathbf{N}_{A \vee B}^{\ominus} = \mathbf{N}_A^{\oplus} + \mathbf{N}_B^{\ominus}$ |
| $C = A \supset B$ | $\mathbf{N}_{A \supset B}^{+} = \mathbf{N}_A^{\oplus} + \mathbf{N}_B^{+} + 1$<br>$\mathbf{N}_{A \supset B}^{\oplus} = \mathbf{N}_A^{+} + \mathbf{N}_B^{\oplus}$ | $\mathbf{N}_{A \supset B}^{-} = \mathbf{N}_A^{\ominus} + \mathbf{N}_B^{-}$<br>$\mathbf{N}_{A \supset B}^{\ominus} = \mathbf{N}_A^{-} + \mathbf{N}_B^{\ominus} + 1$ |
| $C = \neg A$ | $\mathbf{N}_{\neg A}^{+} = \mathbf{N}_A^{\oplus}$<br>$\mathbf{N}_{\neg A}^{\oplus} = \mathbf{N}_A^{+}$ | $\mathbf{N}_{\neg A}^{-} = \mathbf{N}_A^{\ominus}$<br>$\mathbf{N}_{\neg A}^{\ominus} = \mathbf{N}_A^{-}$ |

**Definition A.10** *For a sequent* $\mathcal{S}$*:* $C_1, \ldots, C_m \to D_1, \ldots, D_n$*, the functions* $\mathbf{N}_{\mathcal{S}}^{+}$*,* $\mathbf{N}_{\mathcal{S}}^{-}$ *and* $\mathbf{N}_{\mathcal{S}}$*, counting the number of positive/negative/total occurrences of negations occuring in* $\mathcal{S}$*, are given as follows:*

$$\mathbf{N}_{\mathcal{S}}^{+} = \sum_{i=1}^{m} \mathbf{N}_{C_i}^{-} + \sum_{i=1}^{n} \mathbf{N}_{D_i}^{+}$$

$$\mathbf{N}_{\mathcal{S}}^{-} = \sum_{i=1}^{m} \mathbf{N}_{C_i}^{+} + \sum_{i=1}^{n} \mathbf{N}_{D_i}^{-}$$

$$\mathbf{N}_{\mathcal{S}} = \mathbf{N}_{\mathcal{S}}^{+} + \mathbf{N}_{\mathcal{S}}^{-}$$

# Bibliography


[ADH98] Armen S. Asratian, Tristan M.J. Denley, and Roland Häggkvist. *Bipartite Graphs and their Applications.* Cambridge University Press, 1998.

[Bau90] Bernd Baumgarten. *Petri-Netze. Grundlagen und Anwendungen.* BI Wissenschatsverlag, 1990.

[Ber76] Claude Berge. *Graphs and Hypergraphs.* North-Holland, 1976.

[Boo84] George Boolos. Don't eliminate cut. *Journal of Philosophical Logic*, 13:373–378, 1984.

[DSH93] Kosta Došen and Peter Schroeder-Heister, editors. *Substructural logics.* Oxford University Press, 1993.

[Gen33] Gerhard Gentzen. Über die Existenz unabhängiger Axiomensysteme zu unendlichen Satzsystemen. *Mathematische Annalen*, 107:329–350, 1933.

[Gen36a] Gerhard Gentzen. Die Widerspruchsfreiheit der reinen Zahlentheorie. *Mathematische Annalen*, 112:493–565, 1936.

[Gen36b] Gerhard Gentzen. Die Widerspruchsfreiheit der Stufenlogik. *Mathematisches Zentralblatt*, 41:357–366, 1936.

[Gen38] Gerhard Gentzen. Die gegenwärtige Lage in der mathematischen Grundlagenforschung. *Deutsche Mathematik*, 3:255–268, 1938.

[Gen43] Gerhard Gentzen. Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie. *Mathematische Annalen*, 119:140–161, 1943.

[Gen35] Gerhard Gentzen. Untersuchungen über das logische Schließen. I-II. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1934/35.

[Gir87] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[Gir01] Jean-Yves Girard. Locus solum: From the rules of logic to the logic of rules. *Mathematical Structures in Computer Science*, 11(3):301–506, 2001.

[Her04] Paul Hertz. *Untersuchungen über unstetige Bewegungen des Elektrons*. PhD thesis, Georg-August-Universität Göttingen, 1904.

[Her22a] Paul Hertz. Statistische Mechanik. *Ergebnisse der exakten Naturwissenschaften*, 1, 1922.

[Her22b] Paul Hertz. Über Axiomensysteme für beliebige Satzsysteme. I. Teil. Sätze ersten Grades. *Mathematische Annalen*, 87:246–269, 1922.

[Her23] Paul Hertz. Über Axiomensysteme für beliebige Satzsysteme. II. Teil. Sätze höheren Grades. *Mathematische Annalen*, 89:76–100, 1923.

[Her28] Paul Hertz. Reichen die üblichen syllogistischen Regeln für das Schließen in der positiven Logik elementarer Sätze aus? *Annalen der Philosophie und philosophischen Kritik*, 7:272–277, 1928.

[Her29] Paul Hertz. Über Axiomensysteme für beliebige Satzsysteme. *Mathematische Annalen*, 101:457–514, 1929.

[Her30] Paul Hertz. Über den Kausalbegriff im Makroskopischen, besonders in der klassischen Physik. *Erkenntnis*, 1(1):211–227, 1930.

[Her31] Paul Hertz. Vom Wesen des Logischen, insbesondere der Bedeutung des modus barbara. *Erkenntnis*, 2(1):369–392, 1931.

[Her37a] Paul Hertz. Regelmäßigkeit, Kausalität und Zeitrichtung. *Erkenntnis*, 6(5/6), 1937.

[Her37b] Paul Hertz. Sprache und Logik. *Erkenntnis*, 7(1):309–324, 1937.

[Hil23] David Hilbert. Die logischen Grundlagen der Mathematik. *Mathematische Annalen*, 88:151–165, 1923.

[Kos92] Arnold Koslow. *A Structuralist Theory of Logic*. Cambridge University Press, 1992.

[KSH06] Reinhard Kahle and Peter Schroeder-Heister, editors. *Proof-Theoretic Semantics*. Special issue of *Synthese*. 2006.

[Mos94] Yiannis Moschovakis. Sense and denotation as algorithm and value. *Lecture Notes in Logic*, 2:210–249, 1994.

[Mos06] Yiannis Moschovakis. A logical calculus of meaning and synonymy. *Linguistics and Philosophy*, 29:27–89, 2006.

[SH02] Peter Schroeder-Heister. Resolution and the origins of structural reasoning: Early proof-theoretic ideas of Hertz and Gentzen. *The Bulletin of Symbolic Logic*, 8:246–265, 2002.

[TS96] A.S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 1996.