$\mathcal{C}^2$ rectifiability and Q valued functions

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Zusammenfassung in deutscher Sprache

In dieser Arbeit werden integrale n Varifaltigkeiten in $\mathbb{R}^{n+m}$ betrachtet, welche eine Bedingung an die verallgemeinerte mittlere Krümmung in L^p-Räumen erfüllen. Genauer wird der Zusammenhang von Größen, welche den klassischen Tilt- und Height-Excess umfassen und verallgemeinern, untersucht, insbesondere im Hinblick auf die Frage möglicher $\mathcal{C}^2$-Rektifizierbarkeit solcher Varifaltigkeiten. Das Hauptresultat besagt, daß die Abweichung der integralen Varifaltigkeit von einer eventuell mehrwertigen Ebene (Height-Excess) durch die Abweichung der approximativen Tangentialräume der integralen Varifaltigkeit von besagter Ebene (Tilt-Excess) und die mittlere Krümmung kontrolliert werden kann.

Introduction

This work is concerned with $\mathcal{C}^2$ rectifiability of integral n varifolds in $\mathbb{R}^{n+m}$, $m, n \in \mathbb{N}$ which are of locally bounded first variation. More precisely, given assumptions on the mean curvature, the relationship between $\mathcal{C}^2$ rectifiability and decay of height or tilt quantities is examined.

First, some definitions will be recalled. Suppose throughout the introduction that m, n are as above and U is a nonempty, open subset of $\mathbb{R}^{n+m}$. Using [Sim83, Theorem 11.8] as a definition, μ is a rectifiable [an integral] n varifold in U if and only if μ is a Radon measure on U and for μ almost all $x \in U$ there exists an approximate tangent plane $T_x\mu \in G(n+m,n)$ with multiplicity $\theta^n(\mu,x)$ of μ at x [and $\theta^n(\mu,x) \in \mathbb{N}$], $G(n+m,n)$ denoting the set of n dimensional, unoriented planes in $\mathbb{R}^{n+m}$. The distributional first variation of mass of μ equals

$$(\delta\mu)(\eta) = \int \operatorname{div}_\mu \eta \,\mathrm{d}\mu \quad \text{whenever } \eta \in C^1_{\mathrm{c}}(U, \mathbb{R}^{n+m})$$

where $\operatorname{div}_\mu \eta(x)$ is the trace of $D\eta(x)$ with respect to $T_x\mu$. $\|\delta\mu\|$ denotes the total variation measure associated to $\delta\mu$ and μ is said to be of locally bounded first variation if and only if $\|\delta\mu\|$ is a Radon measure. The tilt-excess and the height-excess of μ are defined by

$$\begin{aligned}\operatorname{tiltex}_\mu(x,\varrho,T) &:= \varrho^{-n} \textstyle\int_{B_\varrho(x)} |T_\xi\mu - T|^2 \,\mathrm{d}\mu(\xi),\\ \operatorname{heightex}_\mu(x,\varrho,T) &:= \varrho^{-n-2} \textstyle\int_{B_\varrho(x)} \operatorname{dist}(\xi - x, T)^2 \,\mathrm{d}\mu(\xi)\end{aligned}$$

whenever $x \in \mathbb{R}^{n+m}$, $0 < \varrho < \infty$, $B_\varrho(x) \subset U$, $T \in G(n+m,n)$; here $S \in G(n+m,n)$ is identified with the orthogonal projection of $\mathbb{R}^{n+m}$ onto S and $|\cdot|$ denotes the norm induced by the usual inner product on $\operatorname{Hom}(\mathbb{R}^{n+m}, \mathbb{R}^{n+m})$. From the above definition of a rectifiable n varifold μ one obtains that μ almost all of U is covered by a countable collection of n dimensional submanifolds of $\mathbb{R}^{n+m}$ of class $\mathcal{C}^1$. This concept is extended to higher orders of differentiability by adapting a definition of Anzellotti and Serapioni in [AS94] as follows: A rectifiable n varifold μ in U is called countably rectifiable of class $\mathcal{C}^{k,\alpha}$ [$\mathcal{C}^k$], $k \in \mathbb{N}$, $0 < \alpha \leq 1$, if and only if there exists a countable collection of n dimensional submanifolds of $\mathbb{R}^{n+m}$ of class $\mathcal{C}^{k,\alpha}$ [$\mathcal{C}^k$] covering μ almost all of U. Throughout the introduction this will be abbreviated to $\mathcal{C}^{k,\alpha}$ [$\mathcal{C}^k$] rectifiability. Note that $\mathcal{C}^{k,1}$ rectifiability and $\mathcal{C}^{k+1}$ rectifiability agree by [Fed69, 3.1.15].

Decays of tilt-excess or height-excess have been successfully used in [All72, Bra78, Sch04a, Sch04b]. The link to $\mathcal{C}^2$ rectifiability is provided in [Sch04b], see below. In order to explain some of these results, a mean curvature condition is introduced. An integral n varifold in U is said to satisfy (H_p), $1 \le p \le \infty$, if and only if either $p > 1$ and for some $\vec{\mathbf{H}}_\mu \in L^p_{\mathrm{loc}}(\mu, \mathbb{R}^{n+m})$, called the generalised mean curvature of μ,

$$(\delta\mu)(\eta) = -\textstyle\int \vec{\mathbf{H}}_\mu \bullet \eta \,\mathrm{d}\mu \quad \text{whenever } \eta \in C^1_{\mathrm{c}}(U, \mathbb{R}^{n+m}) \tag{H_p}$$

or $p = 1$ and

$$\mu \text{ is of locally bounded first variation;} \tag{H_1}$$

here $\bullet$ denotes the usual inner product on $\mathbb{R}^{n+m}$. Brakke has shown in [Bra78, 5.7] that

$$\mathrm{tiltex}_\mu(x, \varrho, T_x\mu) = o_x(\varrho), \; \mathrm{heightex}_\mu(x, \varrho, T_x\mu) = o_x(\varrho) \quad \text{as } \varrho \downarrow 0$$

for μ almost every $x \in U$ provided μ satisfies (H_1) and

$$\mathrm{tiltex}_\mu(x, \varrho, T_x\mu) = o_x(\varrho^{2-\varepsilon}), \; \mathrm{heightex}_\mu(x, \varrho, T_x) = o_x(\varrho^{2-\varepsilon}) \quad \text{as } \varrho \downarrow 0$$

for every $\varepsilon > 0$ for μ almost every $x \in U$ provided μ satisfies (H_2). In case of codimension 1 and $p > n$ Schätzle has proved the following result yielding optimal decay rates.

Theorem 5.1 in [Sch04a]. *If $m = 1$, $p > n$, $p \ge 2$, and μ is an integral n varifold in U satisfying (H_p), then*

$$\mathrm{tiltex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2), \; \mathrm{heightex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2) \quad \textit{as } \varrho \downarrow 0$$

for μ almost all $x \in U$.

The importance of the improvement from $2 - \varepsilon$ to 2 stems mainly from the fact that the quadratic decay of tilt-excess can be used to compute the mean curvature vector $\vec{\mathbf{H}}_\mu$ in terms of the local geometry of μ which had already been noted in [Sch01, Lemma 6.3]. In [Sch04b] Schätzle provides the above mentioned link to $\mathcal{C}^2$ rectifiability as follows:

Theorem 3.1 in [Sch04b]. *If μ is an integral n varifold in U satisfying (H_2) then the following two statements are equivalent:*

(1) μ is $\mathcal{C}^2$ rectifiable.

(2) For μ almost every $x \in U$ there holds

$$\mathrm{tiltex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2), \; \mathrm{heightex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2) \quad \textit{as } \varrho \downarrow 0.$$

The quadratic decay of $\mathrm{heightex}_\mu$ implies $\mathcal{C}^2$ rectifiability without the condition (H_2) as may be seen from the proof in [Sch04b]. However, (1) would not imply (2) if μ were merely required to satisfy (H_p) for some p with $1 \le p < \frac{2n}{n+2}$, an example will be provided in C.5. On the other hand, it is evident from the Caccioppoli type inequality relating tiltex_μ to $\mathrm{heightex}_\mu$ and mean curvature, see e.g. [Bra78, 5.5], that quadratic decay of $\mathrm{heightex}_\mu$ implies quadratic decay for tiltex_μ under the condition (H_2). This leads to the following question:

Problem. Does quadratic decay of tiltex_μ imply quadratic decay of $\mathrm{heightex}_\mu$ under the condition (H_2)?

More generally, suppose that μ is an integral n varifold in U satisfying (H_p), $1 \leq p \leq \infty$, and $0 < \alpha \leq 1$, $1 \leq q < \infty$. Does

$$\limsup_{r \downarrow 0} r^{-\alpha - n/q} \big(\textstyle\int_{B_r(x)} |T_\xi \mu - T_x \mu|^q \,\mathrm{d}\mu(\xi) \big)^{1/q} < \infty$$

for μ almost all $x \in U$ imply

$$\limsup_{r \downarrow 0} r^{-1-\alpha - n/q} \big(\textstyle\int_{B_r(x)} \operatorname{dist}(\xi - x, T_x \mu)^q \,\mathrm{d}\mu(\xi) \big)^{1/q} < \infty$$

for μ almost all $x \in U$?

The answer to the second question will be shown in 2.8–2.10 to be in the affirmative if and only if either $p \geq n$ or $p < n$ and $\alpha q \leq \frac{np}{n-p}$, yielding in particular a positive answer to the first question. The main task is to prove the following theorem which in fact provides a quantitative estimate together with the usual embedding in L^q spaces.

Theorem 2.8. *Suppose $Q \in \mathbb{N}$, $0 < \alpha \leq 1$, $1 \leq p \leq n$, and μ is an integral n varifold in U satisfying (H_p).*

Then the following two statements hold:

(1) If $p < n$, $1 \leq q_1 < n$, $1 \leq q_2 \leq \min\{\frac{nq_1}{n-q_1}, \frac{1}{\alpha} \cdot \frac{np}{n-p}\}$, then for μ almost all $a \in U$ with $\theta^n(\mu, a) = Q$ there holds

$$\begin{aligned} &\limsup_{r \downarrow 0} r^{-\alpha-1-n/q_2} \| \operatorname{dist}(\cdot - a, T_a\mu) \|_{L^{q_2}(\mu \llcorner B_r(a))} \\ &\leq \Gamma_{(1)} \limsup_{r \downarrow 0} r^{-\alpha - n/q_1} \| T_\mu - T_a \mu \|_{L^{q_1}(\mu \llcorner B_r(a))} \end{aligned}$$

where $\Gamma_{(1)}$ is a positive, finite number depending only on m, n, Q, q_1, and q_2.

(2) If $p = n$, $n < q \leq \infty$, then for μ almost all $a \in U$ with $\theta^n(\mu, a) = Q$ there holds

$$\begin{aligned} &\limsup_{r \downarrow 0} r^{-\alpha-1} \| \operatorname{dist}(\cdot - a, T_a\mu) \|_{L^\infty(\mu \llcorner B_r(a))} \\ &\leq \Gamma_{(2)} \limsup_{r \downarrow 0} r^{-\alpha - n/q} \| T_\mu - T_a \mu \|_{L^{q}(\mu \llcorner B_r(a))} \end{aligned}$$

where $\Gamma_{(2)}$ is a positive, finite number depending only on m, n, Q, and q.

Here T_μ denotes the function mapping x to $T_x\mu$ whenever the latter exists. The connection to higher order rectifiability is provided by the following simple adaption of [Sch04b, Appendix A].

Lemma 3.1. *Suppose $0 < \alpha \leq 1$, μ is a rectifiable n varifold in U, and A denotes the set of all $x \in U$ such that $T_x\mu$ exists and*

$$\limsup_{\varrho \downarrow 0} \varrho^{-n-1-\alpha} \textstyle\int_{B_\varrho(x)} \operatorname{dist}(\xi - x, T_x\mu) \,\mathrm{d}\mu(\xi) < \infty.$$

Then $\mu \llcorner A$ is $\mathcal{C}^{1,\alpha}$ rectifiable.

The analog of Theorem 2.8 in the case of weakly differentiable functions can be proved simply by using the Sobolev Poincaré inequality in conjunction with an iteration procedure. In the present case, however, the curvature condition is needed to exclude a behaviour like the one shown by the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{i=0}^{\infty} (2^{-i}) \chi_{[2^{-i-1}, 2^{-i}[}(x) \quad \text{whenever } x \in \mathbb{R}$$

at 0; in fact an example of this behaviour occurring on a set of positive $\mathcal{L}^1$ measure is provided by $f^{1/2} \circ g$ where g is the distance function from a compact set C such that $\mathcal{L}^1(C) > 0$ and for some $0 < \lambda < 1$

$$\liminf_{r \downarrow 0} r^{-3/2} \mathcal{L}^1([x + \lambda r, x + r[\sim C) > 0 \quad \text{whenever } x \in C.$$

Therefore the strategy to prove Theorem 2.8 is to provide a special Sobolev Poincaré type inequality for integral varifolds involving curvature, see 2.4. In the construction weakly differentiable functions are replaced by Lipschitzian Q valued functions, a Q valued function being a function with values in $Q_Q(\mathbb{R}^m) \cong (\mathbb{R}^m)^Q / \sim$ where $\sim$ is induced by the action of the group of permutations of $\{1, \ldots, Q\}$ on $(\mathbb{R}^m)^Q$.

Roughly speaking, the construction performed in a ball $B_r(a) \subset U$ proceeds as follows. Firstly, a graphical part G of μ in $B_r(a)$ is singled out. The complement of G can be controlled in mass by the curvature, whereas its geometry cannot be controlled in a suitable way as may be seen from the example in C.2 used to demonstrate the sharpness of the curvature condition. On the graphical part G the varifold μ might not quite correspond to the graph of a Q valued function but still have "holes" or "missing layers". Nevertheless, it will be shown that μ behaves just enough like a Q valued function to make it possible to reduce the problem to this case. Finally, for Q valued functions Almgren's bi Lipschitzian equivalence of $Q_Q(\mathbb{R}^m)$ to a subset of $\mathbb{R}^{mP}$ for some $P \in \mathbb{N}$ which is a Lipschitz retract of the whole space directly yields a Poincaré inequality. More details about the technical difficulties occurring in the construction and how they are solved will be given at the beginning of Section 1.

To conclude the introduction, it will be indicated why tilt quantities with exponent different from 2 may become relevant. The above mentioned decay rates for tilt-excess (or height-excess) shown by Brakke in case the integral varifold μ satisfies (H_1) imply that μ is $\mathcal{C}^{1,1/2}$ rectifiable but for every $1/2 < \alpha \leq 1$ there is no example known to the author of such a μ which is not $\mathcal{C}^{1,\alpha}$ rectifiable. In contrast, for any $1/2 + \frac{1}{2(n-1)} < \alpha \leq 1$, $n > 1$, there exists an example, see C.4, showing that tilt-excess and height-excess do not decay with power 2α, i.e. the power corresponding to $\mathcal{C}^{1,\alpha}$ rectifiability via Theorem 2.8 and Lemma 3.1. The 1 tilt does behave better in this respect. In fact, it will be shown that decay of the 1 tilt implies $\mathcal{C}^2$ rectifiability and locality of mean curvature:

Lemma 3.2. *Suppose μ is an integral n varifold in U satisfying (H_1) and A denotes the set of all $x \in U$ such that $T_x\mu$ exists and*

$$\limsup_{\varrho \downarrow 0} \varrho^{-1-n} \textstyle\int_{B_\varrho(x)} |T_\xi \mu - T_x \mu| \, d\mu(\xi) < \infty.$$

Then $\mu \llcorner A$ is $\mathcal{C}^2$ rectifiable and for every n dimensional submanifold M of $\mathbb{R}^{n+m}$ of class $\mathcal{C}^2$ there holds

$$\vec{\mathbf{H}}_\mu(x) = \vec{\mathbf{H}}_M(x) \quad \textit{for } \mu \textit{ almost every } x \in A \cap M$$

where $\vec{\mathbf{H}}_M$ denotes the mean curvature of M and $-\vec{\mathbf{H}}_\mu$ corresponds to the absolutely continuous part of $\delta\mu$ with respect to μ.

The first part of the lemma is a direct consequence of Theorem 2.8 and Lemma 3.1 whereas the second part is an adaption of [Sch01, Lemma 6.3] with the help of the differentiation theorem provided in B.1.

The work is organised as follows. In Section 1 the approximation of μ by a Q valued function is constructed. In Section 2 the approximation is used to prove the Sobolev Poincaré type inequality 2.6 and Theorem 2.8. Section 3 provides Lemmas 3.1 and 3.2. The Appendices A and B provide more basic properties of rectifiable varifolds which are needed to prove the various results contained in the body of the text in a precise fashion which then allows the example given in Appendix C to demonstrate the sharpness of these results. Finally, Appendix D collects for the convenience of the reader the results needed from Almgren's Big Regularity Paper [Alm00].

The notation follows [Sim83]. Additionally to the symbols already defined, $\operatorname{im} f$ and $\operatorname{dmn} f$ denote the image and the domain of a function f respectively, $T^\perp$ is the orthogonal complement of T for $T \in G(n+m, n)$, γ_n denotes the best constant in the Isoperimetric Inequality as defined in A.3, and $f(\phi)$ denotes the ordinary push forward of a measure ϕ by a function f, i.e. $f(\phi)(A) := \phi(f^{-1}(A))$ whenever $A \subset Y$, if ϕ is a measure on X and $f : X \to Y$. Definitions are denoted by '=' or, if clarity makes it desirable, by ':='. To simplify verification, in case a statement asserts the existence of a constant, small (ε) or large (Γ), depending on certain parameters this number will be referred to by using the number of the statement as index and what is supposed to replace the parameters in the order of their appearance given in brackets, for example $\varepsilon_{A.10}(m, n, 1 - \delta_3/2)$.

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1 Approximation of integral varifolds

In this section an approximation procedure for integral n varifolds μ in $\mathbb{R}^{n+m}$ by Q valued functions is carried out. Similar constructions occur in [Alm00, Chapter 3] and [Bra78, Chapter 5]. Basically, a part of μ which is suitably close to a Q valued plane is approximated "above" a subset Y of $\mathbb{R}^n$ by a Lipschitzian Q valued function. The sets where this approximation fails are estimated in terms of μ and $\mathcal{L}^n$ measure.

In order to obtain an approximation useful for proving the main lemma 2.4 for the Sobolev Poincaré type inequalities 2.6 and 2.8 in the next section, the following three problems had to be solved.

Firstly, in the above mentioned estimate one can only allow for tilt and mean curvature terms and not for a height term as it is present in [Bra78, 5.4]. This is done using a new version of Brakke's multilayer monotonicity which allows for variable offsets, see 1.8.

Secondly, the seemingly most natural way to estimate the height of μ above the complement of Y, namely measure times maximal height h, would not produce sharp enough an estimate. In order to circumvent this difficulty, a "graphical part" G of μ defined mainly in terms of mean curvature is used which is larger than the part where μ equals the "graph" of the Q valued function. Points in G still satisfy a one sided Lipschitz condition with respect to points above Y, see 1.10 and 1.14 (4). Using this fact in conjunction with a covering argument, the actual error in estimating the q height in a ball $\bar{B}_t(\zeta)$ where $\mathcal{L}^n(\bar{B}_t(\zeta) \cap Y)$ and $\mathcal{L}^n(\bar{B}_t(\zeta) \sim Y)$ are comparable, can be estimated by $\mathcal{L}^n(\bar{B}_t(\zeta) \sim Y)^{1/q} \cdot t$ instead of $\mathcal{L}^n(\bar{B}_t(\zeta) \sim Y)^{1/q} \cdot h$; the replacement of h by t being the decisive improvement which allows to estimate the q^* height ($q^* = \frac{nq}{n-q}$, $1 \leq q < n$) instead of the q height in 2.4.

Thirdly, to obtain a sharp result with respect to the assumptions on the mean curvature, all curvature conditions are phrased in terms of isoperimetric ratios. Therefore, it seems to be impossible to derive lower bounds for the density and monotonicity results for the density ratios by integration from the monotonicity formula, see e.g. [Sim83, (17.3)], as in [Sim83, Theorems 17.6, 17.7]. Instead, lower bounds are obtained via slicing from the Isoperimetric Inequality of Michael and Simon, see Appendix A, and it is shown that nonintegral bounds for density ratios are preserved provided the varifold is additionally close to a Q valued plane, see 1.4. Both results appear to be generally useful in deriving sharp estimates involving mean curvature.

1.1. If $m, n \in \mathbb{N}$, $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$, $T \in G(n+m, n)$, and μ is a stationary, integral n varifold in $B_r(a)$ with $T_x\mu = T$ for μ almost all $x \in B_r(a)$, then $T^\perp(\operatorname{spt}\mu)$ is discrete and closed in $T^\perp(B_r(a))$ and for every $x \in \operatorname{spt}\mu$

$$y \in B_r(a),\ y - x \in T \quad \text{implies} \quad \theta^n(\mu, y) = \theta^n(\mu, x) \in \mathbb{N};$$

hence with $S_x = \{y \in B_r(a) : y - x \in T\}$

$$\mu \llcorner S_x = \theta^n(\mu, x)\mathcal{H}^n \llcorner S_x \quad \text{whenever } x \in B_r(a).$$

A similar assertion may be found in [Alm00, 3.6] and is used in [Bra78, 5.3 (16)].

1.2 Lemma. *Suppose $0 < M < \infty$, $M \notin \mathbb{N}$, $0 < \lambda_1 < \lambda_2 < 1$, $m, n \in \mathbb{N}$, $T \in G(n+m, n)$, F is the family of all stationary, integral n varifolds in $B_1^{n+m}(0)$ such that*

$$T_x\mu = T \quad \text{for } \mu \text{ almost all } x \in B_1^{n+m}(0), \quad \mu(B_1^{n+m}(0)) \leq M\omega_n,$$

and N is the supremum of all numbers

$$(\omega_n r^n)^{-1}\mu(\bar{B}_r^{n+m}(0))$$

corresponding to all $\mu \in F$ and $\lambda_1 \leq r \leq \lambda_2$.

Then for some $\mu \in F$ and some $\lambda_1 \leq r \leq \lambda_2$

$$N = (\omega_n r^n)^{-1}\mu(\bar{B}_r^{n+m}(0)) < M.$$

Proof. The proof uses the structure of the elements of F described in 1.1. Since

$$(\omega_n r^n)^{-1} \mu(\bar{B}_r^{n+m}(0))$$

depends continuously on $(\mu, r) \in F \times [\lambda_1, \lambda_2]$, the first part of the conclusion is a consequence of the fact that F is compact with respect to the weak topology (cf. [All72, 6.4]). To prove the second part, one notes

$$(r^2 - \varrho^2)^{n/2} < (1 - \varrho^2)^{n/2} r^n \quad \text{whenever } 0 < \varrho \leq r < 1$$

and computes

$$\begin{aligned}
\mu(\bar{B}_r^{n+m}(0)) &= \sum_{x \in \bar{B}_r^{n+m}(0) \cap T^\perp(\operatorname{spt} \mu)} \theta^n(\mu, x) \omega_n (r^2 - |T^\perp(x)|^2)^{n/2} \\
&\leq \Big(\sum_{x \in \bar{B}_r^{n+m}(0) \cap T^\perp(\operatorname{spt} \mu)} \theta^n(\mu, x) \omega_n (1 - |T^\perp(x)|^2)^{n/2} \Big) r^n \\
&\leq \mu(B_1^{n+m}(0)) r^n \leq M \omega_n r^n.
\end{aligned}$$

If $\operatorname{spt} \mu \not\subset T$, then the first inequality in the computation is strict. Otherwise, the last inequality is strict because $M \notin \mathbb{N}$. □

1.3 Remark. Any $\mu \in F$ satisfies

$$(\omega_n r^n)^{-1} \mu(\bar{B}_r^{n+m}(0)) \to \omega_n^{-1} \mu(B_1^{n+m}(0)) \quad \text{as } r \uparrow 1,$$

and this number may equal M. Therefore the conclusion $N < M$ would fail if $\lambda_2 = 1$. However, the supremum in the definition of N can be extended over all $0 < r \leq \lambda_2 r$ with $N < M$ still being valid as will be shown in 1.4.

1.4 Lemma (Quasi monotonicity)**.** *Suppose* $0 < M < \infty$*,* $M \notin \mathbb{N}$*,* $0 < \lambda < 1$*, and* $m, n \in \mathbb{N}$*.*

Then there exists a positive, finite number ε *with the following property.*

If $a \in \mathbb{R}^{n+m}$*,* $0 < r < \infty$*,* μ *is an integral* n *varifold in* $B_r(a)$ *with locally bounded first variation,*

$$\mu(B_r(a)) \leq M \omega_n r^n,$$

and whenever $0 < \varrho < r$

$$\begin{gathered}
\|\delta \mu\|(\bar{B}_\varrho(a)) \leq \varepsilon\, \mu(\bar{B}_\varrho(a))^{1-1/n}, \\
\textstyle\int_{\bar{B}_\varrho(a)} |T_x \mu - T| \, \mathrm{d}\mu(x) \leq \varepsilon\, \mu(\bar{B}_\varrho(a)) \quad \text{for some } T \in G(n+m, n),
\end{gathered}$$

(here $0^0 := 1$*), then*

$$\mu(\bar{B}_\varrho(a)) \leq M \omega_n \varrho^n \quad \text{whenever } 0 < \varrho \leq \lambda r.$$

Proof. Using induction, one verifies that it is enough to prove the statement with $\lambda^2 r \leq \varrho \leq \lambda r$ replacing $0 < \varrho \leq \lambda r$ in the last line which is readily accomplished by a contradiction argument using 1.2 and Allard's compactness theorem for integral varifolds [All72, 6.4]. □

1.5 Remark. Clearly,

$$(\omega_n \varrho^n)^{-1} \mu(\bar{B}_\varrho(a)) \le M \lambda^{-n} \quad \text{whenever } 0 < \varrho < r.$$

1.6. Suppose $-\infty < a < b < \infty$, $I = [a, b]$, $f : I \to \mathbb{R}$ is nondecreasing and continuous from the left, $g : I \to \mathbb{R}$ is continuous, and $f(a) \ge g(a)$, $f(b) < g(b)$.

Then there exists ξ with $a \le \xi < b$ such that

$$f(\xi) = g(\xi), \quad \text{and} \quad f(t) \ge g(t) \quad \text{whenever } \xi \ge t \in I;$$

in fact one may take $\xi = \inf(\{t \in I : f(t) < g(t)\})$.

1.7 Lemma (Multilayer monotonicity)**.** *Suppose $m, n, Q \in \mathbb{N}$, $0 < \delta \le 1$, and $0 \le s < 1$.*

Then there exists a positive, finite number ε with the following property.

If $X \subset \mathbb{R}^{n+m}$, $T \in G(n+m, n)$, $0 < r < \infty$,

$$|T(y-x)| \le s|y-x| \quad \textit{whenever } x, y \in X,$$

μ is an integral n varifold in $\bigcup_{x \in X} B_r(x)$ with locally bounded first variation,

$$\textstyle\sum_{x \in X} \theta^n_*(\mu, x) \ge Q - 1 + \delta,$$

and whenever $0 < \varrho < r$, $x \in X \cap \operatorname{spt} \mu$

$$\|\delta\mu\|(\bar{B}_\varrho(x)) \le \varepsilon\, \mu(\bar{B}_\varrho(x))^{1-1/n}, \quad \textstyle\int_{\bar{B}_\varrho(x)} |T_\xi \mu - T| \, \mathrm{d}\mu(\xi) \le \varepsilon\, \mu(\bar{B}_\varrho(x)),$$

then

$$\mu\big(\textstyle\bigcup_{x \in X} B_\varrho(x)\big) \ge (Q - \delta)\omega_n \varrho^n \quad \textit{whenever } 0 < \varrho \le r.$$

Proof. If the lemma were false for some $m, n, Q \in \mathbb{N}$, $0 < \delta < 1/2$, and $0 < s < 1$, there would exist a sequence ε_i with $\varepsilon_i \downarrow 0$ as $i \to \infty$ and sequences X_i, T_i, r_i, and μ_i showing that ε_i does not satisfy the conclusion of the lemma.

Clearly, one could assume for some $T \in G(n+m, n)$

$$T_i = T \quad \text{for } i \in \mathbb{N},$$

$X_i \subset \operatorname{spt} \mu_i$ for $i \in \mathbb{N}$, and in view of A.10 also

$$\# X_i \le Q \quad \text{for } i \in \mathbb{N}.$$

One would observe that 1.6 could be used to deduce the existence of a sequence $0 < \varrho_i < r_i$ such that

$$\mu_i\big(\textstyle\bigcup_{x \in X_i} B_{\varrho_i}(x)\big) \le (Q - \delta)\omega_n (\varrho_i)^n,$$
$$\mu_i\big(\textstyle\bigcup_{x \in X_i} B_\varrho(x)\big) \ge (Q - 1 + \delta/2)\omega_n \varrho^n \quad \text{whenever } 0 < \varrho \le \varrho_i.$$

There would hold for $x \in X_i$, $i \in \mathbb{N}$

$$\|\delta\mu_i\|(B_{\varrho_i}(x)) \le \varepsilon_i (Q\omega_n)^{1-1/n} (\varrho_i)^{n-1},$$
$$\textstyle\int_{B_{\varrho_i}(x)} |T_\xi \mu_i - T| \, \mathrm{d}\mu_i(\xi) \le \varepsilon_i Q \omega_n (\varrho_i)^n.$$

Rescaling, one would infer the existence of sequences of integral n varifolds ν_i in $\mathbb{R}^{n+m}$, $X_i \subset \operatorname{spt} \nu_i$, and ε_i with $\varepsilon_i \downarrow 0$ as $i \to \infty$ such that for some $T \in G(n+m,n)$, $0 < M < \infty$, $Q \in \mathbb{N}$, $0 < \delta < 1/2$, and $0 < s < 1$

$$\begin{gathered} \# X_i \le Q, \quad s^{-1}|T(y-x)| \le |y-x| \quad \text{for } x,y \in X_i, \\ \|\delta \nu_i\|(B_1(x)) \le \varepsilon_i M, \quad \textstyle\int_{B_1(x)} |T_\xi \nu_i - T| \,\mathrm{d}\nu_i(\xi) \le \varepsilon_i M \quad \text{for } x \in X_i, \\ \nu_i\big(\textstyle\bigcup_{x \in X_i} B_1(x)\big) \le (Q-\delta)\omega_n, \\ \nu_i\big(\textstyle\bigcup_{x \in X_i} B_\varrho(x)\big) \ge (Q-1+\delta/2)\omega_n \varrho^n \quad \text{whenever } 0 < \varrho \le 1. \end{gathered}$$

The proof will be concluded by showing that objects with the properties described in the preceding paragraph do not exist. If they existed, one could assume first

$$X_i \subset \bar{B}^{n+m}_M(0) \quad \text{for } i \in \mathbb{N}$$

by moving pieces of ν_i by translations (here ν is a piece of ν_i if and only if $\nu = \nu_i \llcorner Z$ for some connected component Z of $\bigcup_{x \in X_i} B_1(x))$ and then, since $X_i \neq \emptyset$ for $i \in \mathbb{N}$, passing to a subsequence,

$$X_i \to X \quad \text{in Hausdorff distance as } i \to \infty, \quad \# X \le Q$$

for some nonempty, closed subset X of $\bar{B}^{n+m}_M(0)$ (cf. [Fed69, 2.10.21]). Noting that given $0 < \varrho_1 < \varrho_2 < 1$

$$\textstyle\bigcup_{x \in X} B_{\varrho_1}(x) \subset \bigcup_{x \in X_i} B_{\varrho_2}(x), \quad \bigcup_{x \in X} B_{\varrho_2}(x) \supset \bigcup_{x \in X_i} B_{\varrho_1}(x)$$

for large i, one could assume, possibly passing to another subsequence (cf. [All72, 6.4]), that for some stationary, integral n varifold ν in

$$U := \textstyle\bigcup_{x \in X} B_1(x)$$

satisfying

$$T_x \nu = T \quad \text{for } \nu \text{ almost all } x \in U$$

there would hold

$$\textstyle\int \varphi \,\mathrm{d}\nu_i \to \int \varphi \,\mathrm{d}\nu \quad \text{as } i \to \infty \text{ for } \varphi \in C^0_{\mathrm{c}}(\mathbb{R}^{n+m}) \text{ with } \operatorname{spt} \varphi \subset U.$$

The inclusions previously noted, would show

$$\begin{gathered} \nu(U) \le (Q-\delta)\omega_n, \\ \nu\big(\textstyle\bigcup_{x \in X} B_\varrho(x)\big) \ge (Q-1+\delta/2)\omega_n \varrho^n \quad \text{for } 0 < \varrho \le 1. \end{gathered}$$

Since for $y, z \in X$

$$\begin{gathered} s^{-1}|T(y-x)| \le |y-x|, \\ \{x \in \mathbb{R}^{n+m} : y-x \in T\} \cap \{x \in \mathbb{R}^{n+m} : z-x \in T\} = \emptyset \quad \text{if } y \neq z, \end{gathered}$$

these inequalities would imply by 1.1

$$\begin{aligned} Q-1+\delta/2 &\le \liminf_{\varrho \downarrow 0} \nu\big(\textstyle\bigcup_{x \in X} B_\varrho(x)\big)/(\omega_n \varrho^n) \\ &= \textstyle\sum_{x \in X} \theta^n(\nu,x) \le \nu(U)/\omega_n \le Q-\delta; \end{aligned}$$

a contradiction to $\sum_{x \in X} \theta^n(\nu,x) \in \mathbb{N}$. □

1.8 Lemma (Multilayer monotonicity with variable offset)**.** *Suppose $m, n, Q \in \mathbb{N}$, $0 \leq M < \infty$, $\delta > 0$, and $0 \leq s < 1$.*

Then there exists a positive, finite number ε with the following property.

If $X \subset \mathbb{R}^{n+m}$, $T \in G(n+m, n)$, $0 \leq d < \infty$, $0 < r < \infty$, $0 < t < \infty$, $f : X \to \mathbb{R}^{n+m}$,

$$|T(y-x)| \leq s|y-x|, \quad |T(f(y)-f(x))| \leq s|f(y)-f(x)|,$$
$$f(x) - x \in \bar{B}^{n+m}_d(0) \cap T, \quad d \leq Mt, \quad d+t \leq r$$

for $x, y \in X$, μ is an integral n varifold in $\bigcup_{x \in X} B_r(x)$ with locally bounded first variation,

$$\textstyle\sum_{x \in X} \theta^n_*(\mu, x) \geq Q - 1 + \delta, \quad \mu(B_r(x)) \leq M\omega_n r^n \quad \text{for } x \in X \cap \operatorname{spt}\mu,$$

and whenever $0 < \varrho < r$, $x \in X \cap \operatorname{spt}\mu$

$$\|\delta\mu\|(\bar{B}_\varrho(x)) \leq \varepsilon\, \mu(\bar{B}_\varrho(x))^{1-1/n}, \quad \textstyle\int_{\bar{B}_\varrho(x)} |T_\xi\mu - T| \,\mathrm{d}\mu(\xi) \leq \varepsilon\, \mu(\bar{B}_\varrho(x)),$$

then

$$\mu\big(\textstyle\bigcup_{x \in X} \{y \in B_t(f(x)) : |T(y-x)| > s|y-x|\}\big) \geq (Q-\delta)\omega_n t^n.$$

Proof. The proof skips some arguments already presented in 1.7.

If the lemma were false for some $m, n, Q \in \mathbb{N}$, $0 \leq M < \infty$, $0 < \delta < 1$, and $0 < s < 1$, there would exist a sequence ε_i with $\varepsilon_i \downarrow 0$ as $i \to \infty$ and sequences X_i, T_i, d_i, r_i, t_i, f_i, and μ_i showing that ε_i does not satisfy the conclusion of the lemma.

In view of 1.4, 1.5 one could assume $d_i + t_i = r_i$ for $i \in \mathbb{N}$ by replacing M by $2M$. Using isometries and homotheties one could also assume for some $T \in G(n+m, n)$

$$T_i = T, \quad r_i = 1$$

for $i \in \mathbb{N}$. Finally, one could assume as in 1.7, possibly replacing M by a larger number,

$$X_i \subset \operatorname{spt}\mu_i, \quad \# X_i \leq Q, \quad X_i \subset \bar{B}^{n+m}_M(0)$$

for $i \in \mathbb{N}$.

Therefore passing to a subsequence (cf. [Fed69, 2.10.21]), there would exist a nonempty, closed subset X of $\bar{B}^{n+m}_M(0)$, $0 \leq d < \infty$, $0 \leq t < \infty$, and a nonempty, closed subset f of $\mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$ such that $\# X \leq Q$,

$$d_i \to d \text{ and } t_i \to t \text{ as } i \to \infty,$$
$$X_i \to X \text{ and } f_i \to f \text{ in Hausdorff distance as } i \to \infty.$$

There would hold

$$s^{-1}|T(y-x)| \leq |y-x| \quad \text{for } x, y \in X, \quad d \leq Mt, \quad d+t = 1, \quad t > 0.$$

Moreover, since

$$\begin{aligned}(1-s^2)^{1/2}|y_i - x_i| &\leq \big|T^\perp(y_i - x_i)\big| \\ &= \big|T^\perp(f_i(y_i) - f_i(x_i))\big| \leq |f_i(y_i) - f_i(x_i)|\end{aligned}$$

for $x_i, y_i \in X_i$, and $i \in \mathbb{N}$, f were a function and one could readily verify $\operatorname{dmn} f = X$, and

$$f(x) - x \in \bar{B}_d^{n+m}(0) \cap T \quad \text{for } x \in X,$$
$$s^{-1}|T(f(y) - f(x))| \leq |f(y) - f(x)| \quad \text{for } x, y \in X.$$

Possibly passing to another subsequence, one could construct (cf. [All72, 6.4]) a stationary, integral n varifold μ in $U := \bigcup_{x \in X} B_1(x)$ with

$$T_x\mu = T \quad \text{for } \mu \text{ almost all } x \in U$$

such that

$$\textstyle\int \varphi \, \mathrm{d}\mu_i \to \int \varphi \, \mathrm{d}\mu \quad \text{as } i \to \infty \text{ for } \varphi \in C_c^0(\mathbb{R}^{n+m}) \text{ with } \operatorname{spt} \varphi \subset U.$$

According to 1.7 one would estimate for large i

$$\mu_i\big(\textstyle\bigcup_{x \in X_i} B_\varrho(x)\big) \geq (Q - \delta)\omega_n \varrho^n \quad \text{whenever } 0 < \varrho \leq 1,$$

hence

$$\mu\big(\textstyle\bigcup_{x \in X} B_\varrho(x)\big) \geq (Q - \delta)\omega_n \varrho^n \quad \text{whenever } 0 < \varrho \leq 1.$$

Therefore, passing to the limit $\varrho \downarrow 0$, one would infer the lower bound (noting 1.1)

$$\textstyle\sum_{x \in X} \theta^n(\mu, x) \geq Q - \delta.$$

For $y, z \in \mathbb{R}^{n+m}$, $0 < \varrho < \infty$ define $V(y, z, \varrho)$ to be the set of all $x \in B_\varrho(z)$ such that

$$s^{-1}|T(y - x)| > |y - x|,$$

and note that every compact subset K of $\bigcup_{x \in X} V(x, f(x), t)$ would satisfy

$$K \subset \textstyle\bigcup_{x \in X_i} V(x, f_i(x), t_i) \quad \text{for large } i;$$

hence

$$\mu\big(\textstyle\bigcup_{x \in X} V(x, f(x), t)\big) \leq \liminf\limits_{i \to \infty} \mu_i\big(\textstyle\bigcup_{x \in X_i} V(x, f_i(x), t_i)\big) \leq (Q - \delta)\omega_n t^n.$$

On the other hand 1.1 would imply in conjunction with the fact

$$\{x \in \mathbb{R}^{n+m} : x - y \in T\} \cap \{x \in \mathbb{R}^{n+m} : x - z \in T\} = \emptyset$$

for $y, z \in X$ with $y \neq z$ and the lower bound previously derived

$$\mu\big(\textstyle\bigcup_{x \in X} V(x, f(x), t)\big) \geq \big(\textstyle\sum_{x \in X} \theta^n(\mu, x)\big)\omega_n t^n \geq (Q - \delta)\omega_n t^n,$$

hence $\sum_{x \in X} \theta^n(\mu, x) = Q - \delta$ which is incompatible with $Q - \delta \notin \mathbb{N}$. $\square$

1.9 Lemma. *Suppose $m, n \in \mathbb{N}$, $0 < \delta < 1$, $0 \le s < 1$, and $0 \le M < \infty$.*

Then there exists a positive, finite number ε with the following property.

If $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$, $T \in G(n+m, n)$, $0 \le d < \infty$, $0 < t < \infty$, $\zeta \in \mathbb{R}^{n+m}$,

$$\max\{d, r\} \le Mt, \quad \zeta \in \bar{B}_d^{n+m}(0) \cap T, \quad d + t \le r,$$

μ is an integral n varifold in $B_r(a)$ with locally bounded first variation, $a \in \operatorname{spt} \mu$,

$$\|\delta\mu\|(B_r(a)) \le \varepsilon\, \mu(B_r(a))^{1-1/n}, \quad \mu(B_r(a)) \le M\omega_n r^n,$$
$$\textstyle\int_{B_r(a)} |T_\xi \mu - T| \,\mathrm{d}\mu(\xi) \le \varepsilon\, \mu(B_r(a))$$

and for $0 < \varrho < r$

$$\|\delta\mu\|(\bar{B}_\varrho(a)) \le (2\gamma_n)^{-1} \mu(\bar{B}_\varrho(a))^{1-1/n}$$

(see A.3), then

$$\mu(\{x \in B_t(a+\zeta) : |T(x-a)| > s|x-a|\}) \ge (1-\delta)\omega_n t^n.$$

Proof. A contradiction argument using A.8, 1.1, and [All72, 6.4] yields the result. □

1.10 Lemma. *Suppose $m, n, Q \in \mathbb{N}$, $0 < \delta_1 \le 1$, $0 < \delta_2 \le 1$, $0 \le s < 1$, $0 \le s_0 < 1$, $0 \le M < \infty$, and $0 < \lambda < 1$ is uniquely defined by the requirement*

$$(1-\lambda^2)^{n/2} = (1-\delta_2) + \Big(\frac{(s_0)^2}{1-(s_0)^2} \Big)^{n/2} \lambda^n.$$

Then there exists a positive, finite number ε with the following property.

If $X \subset \mathbb{R}^{n+m}$, $T \in G(n+m, n)$, $0 \le d < \infty$, $0 < r < \infty$, $0 < t < \infty$, $\zeta \in \mathbb{R}^{n+m}$,

$$\# T(X) = 1, \quad \zeta \in \bar{B}_d^{n+m}(0) \cap T, \quad d \le Mt, \quad d + t \le r,$$

μ is an integral n varifold in $\bigcup_{x \in X} B_r(x)$ with locally bounded first variation, $\theta^n(\mu, x) \in \mathbb{N}$ for $x \in X$,

$$\textstyle\sum_{x \in X} \theta^n(\mu, x) = Q, \quad \mu(B_r(x)) \le M\omega_n r^n \quad \text{for } x \in X,$$

and whenever $0 < \varrho < r$, $x \in X$

$$\|\delta\mu\|(\bar{B}_\varrho(x)) \le \varepsilon\, \mu(\bar{B}_\varrho(x))^{1-1/n}, \quad \textstyle\int_{\bar{B}_\varrho(x)} |T_\xi \mu - T| \,\mathrm{d}\mu(\xi) \le \varepsilon\, \mu(\bar{B}_\varrho(x))$$

satisfying

$$\mu\big(\textstyle\bigcup_{x \in X} \{y \in B_t(x+\zeta) : |T(y-x)| > s_0|y-x|\}\big) \le (Q+1-\delta_2)\omega_n t^n,$$

then the following two statements hold:

(1) If $0 < \tau \le \lambda t$, then

$$\mu\big(\textstyle\bigcup_{x \in X} \bar{B}_\tau(x)\big) \le (Q+\delta_1)\omega_n \tau^n.$$

(2) If $y \in \operatorname{spt} \mu$ *with* $\operatorname{dist}(y, X) \le \lambda t/2$ *and*

$$\|\delta \mu\|(\bar{B}_\varrho(y)) \le (2\gamma_n)^{-1} \mu(\bar{B}_\varrho(y))^{1-1/n} \quad \textit{for } 0 < \varrho < \delta_1 \operatorname{dist}(y, X),$$

then for some $x \in X$

$$|T(y-x)| \ge s|y-x|.$$

Proof of (1). One may first assume $\max\{\delta_1, \delta_2\} \le 1/2$ and then $\lambda^2 \le \tau/t \le \lambda$ by iteration of the result observing that the remaining assertion implies inductively

$$\mu\big(\textstyle\bigcup_{x\in X} \bar{B}_{\lambda^{-i}\tau}(x)\big) \le (Q+\delta_1)\omega_n(\lambda^{-i}\tau)^n$$

whenever $i \in \mathbb{N}$, $\lambda^{-i}\tau \le \lambda t$. Moreover, in view of 1.4, 1.5, only the case $r = d+t$ needs to be considered.

The remaining assertion will be proved by contradiction. If it were false for some $m, n, Q \in \mathbb{N}$, $0 < \delta_1 \le 1/2$, $0 < \delta_2 \le 1/2$, $0 < s_0 < 1$, and $0 \le M < \infty$, there would exist a sequence ε_i with $\varepsilon_i \downarrow 0$ as $i \to \infty$ and sequences X_i, T_i, d_i, r_i, t_i ζ_i, μ_i, and τ_i with $i \in \mathbb{N}$ showing that ε_i does not satisfy the assertion.

The argument follows the pattern of 1.8. First, one could assume for some $T \in G(n+m, n)$

$$T_i = T, \quad r_i = 1$$

for $i \in \mathbb{N}$ and then noting $\# X_i \le Q$ that $X_i \subset \bar{B}^{n+m}_M(0)$ and hence, possibly passing to a subsequence, the existence of real numbers d, t, τ, of a nonempty, closed subset X of $\bar{B}^{n+m}_M(0)$, of $\zeta, \in \mathbb{R}^{n+m}$, and of a stationary, integral n varifold μ in $U := \bigcup_{x\in X} B_1(x)$ such that $\# X \le Q$, and, as $i \to \infty$,

$$d_i \to d, \quad t_i \to t, \quad \tau_i \to \tau, \quad \zeta_i \to \zeta,$$
$$X_i \to X \quad \text{in Hausdorff distance,}$$
$$\textstyle\int \varphi \, \mathrm{d}\mu_i \to \int \varphi \, \mathrm{d}\mu \quad \text{for } \varphi \in C^0_{\mathrm{c}}(\mathbb{R}^{n+m}) \text{ with } \operatorname{spt} \varphi \subset U,$$

and additionally

$$T_x\mu = T \quad \text{for } \mu \text{ almost all } x \in U.$$

Clearly,

$$d \le Mt, \quad d+t = 1, \quad t > 0, \quad \lambda^2 \le \tau/t \le \lambda,$$
$$\# T(X) = 1, \quad \zeta \in \bar{B}^{n+m}_d(0) \cap T,$$

and one would readily verify

$$\mu\big(\textstyle\bigcup_{x\in X}\{y \in B_t(x+\zeta) : |T(y-x)| > s_0|y-x|\}\big) \le (Q+1-\delta_2)\omega_n t^n,$$
$$\mu\big(\textstyle\bigcup_{x\in X} \bar{B}_\tau(x)\big) \ge (Q+\delta_1)\omega_n \tau^n.$$

Moreover, 1.7 would imply with $S_x := \{z \in \mathbb{R}^{n+m} : T^\perp(z-x) = 0\}$ for $x \in \mathbb{R}^{n+m}$

$$\mu\big(\textstyle\bigcup_{x\in X} B_\varrho(x)\big) \ge (Q-\delta_1)\omega_n\varrho^n \quad \text{for } 0 < \varrho \le 1,$$
$$\textstyle\sum_{x\in X} \theta^n(\mu, x) \ge Q,$$
$$\textstyle\sum_{x\in X} \theta^n(\mu, x)\big(\mathcal{H}^n \llcorner S_x\big)(A) \le \mu(A) \quad \text{for } A \subset U.$$

Therefore if $x \in X$, $y \in \operatorname{spt} \mu$, $T^\perp(y) \notin T^\perp(X)$, $0 < |T^\perp(y-x)| = h < t$, then one would find

$$\begin{aligned}
&\{z \in S_y : |T(z-x)| \leq s_0|z-x|\} = S_y \cap \bar{B}_{(s_0^{-2}-1)^{-1/2}h}(x + T^\perp(y-x)),\\
&\big((1-(h/t)^2)^{n/2} - (s_0^{-2}-1)^{-n/2}(h/t)^n\big)\omega_n t^n\\
&= \big(\mathcal{H}^n \llcorner S_y\big)(B_t(x+\zeta)) - \big(\mathcal{H}^n \llcorner S_y\big)(\{z \in \mathbb{R}^{n+m} : |T(z-x)| \leq s_0|z-x|\})\\
&\leq \big(\mathcal{H}^n \llcorner S_y\big)(\{z \in B_t(x+\zeta) : |T(z-x)| > s_0|z-x|\})\\
&\leq (1-\delta_2)\omega_n t^n,
\end{aligned}$$

hence $h \geq \lambda t$, in particular, since $\lambda t \geq \tau$ and $\# T(X) = 1$,

$$(\operatorname{spt} \mu) \cap \textstyle\bigcup_{x \in X} B_\tau(x) = \bigcup_{x \in X} S_x \cap B_\tau(x),$$
$$\mu\big(\textstyle\bigcup_{x \in X} \bar{B}_\tau(x)\big) = Q\omega_n \tau^n$$

contradicting the previously derived lower bound because $\tau > 0$. □

Proof of (2). On may first assume $\max\{\delta_1, \delta_2\} \leq 1/2$, then

$$\lambda^2/2 \leq \operatorname{dist}(y, X)/t \leq \lambda/2$$

by part (1), and $1 \leq r/t \leq M+1$ by 1.4, 1.5.

The remaining assertion will be proved by contradiction. If it were false for some $m, n, Q \in \mathbb{N}$, $0 < \delta_1 \leq 1/2$, $0 < \delta_2 \leq 1/2$, $0 \leq s_0 < 1$, $0 \leq s < 1$, and $0 \leq M < \infty$, there would exist a sequence ε_i with $\varepsilon_i \downarrow 0$ as $i \to \infty$ and sequences X_i, T_i, d_i, r_i, t_i ζ_i, μ_i, and y_i with $i \in \mathbb{N}$ showing that ε_i does not satisfy the assertion.

The argument follows the pattern of part (1). First, one could assume for some $T \in G(n+m, n)$

$$T_i = T, \quad r_i = 1$$

for $i \in \mathbb{N}$ and then noting $\# X_i \leq Q$ that $X_i \subset \bar{B}^{n+m}_M(0)$ and hence, possibly passing to a subsequence, the existence of real numbers d, t, of a nonempty, closed subset X of $\bar{B}^{n+m}_M(0)$, of $\zeta \in \mathbb{R}^{n+m}$, and of a stationary, integral n varifold μ in $U := \bigcup_{x \in X} B_1(x)$ such that $\# X \leq Q$, and, as $i \to \infty$,

$$d_i \to d, \quad t_i \to t, \quad \zeta_i \to \zeta,$$
$$X_i \to X \quad \text{in Hausdorff distance},$$
$$\textstyle\int \varphi \, \mathrm{d}\mu_i \to \int \varphi \, \mathrm{d}\mu \quad \text{for } \varphi \in C^0_{\mathrm{c}}(\mathbb{R}^{n+m}) \text{ with } \operatorname{spt} \varphi \subset U,$$

and additionally

$$T_x \mu = T \quad \text{for } \mu \text{ almost all } x \in U.$$

Clearly,

$$d \leq Mt, \quad d + t \leq 1, \quad 0 < t \leq 1, \quad \# T(X) = 1, \quad \zeta \in \bar{B}^{n+m}_d(0) \cap T,$$

and one would readily verify

$$\mu\big(\textstyle\bigcup_{x \in X} \{y \in B_t(x+\zeta) : |T(y-x)| > s_0|y-x|\}\big) \leq (Q + 1 - \delta_2)\omega_n t^n.$$

It would hold $y \in \operatorname{spt}\mu$ by A.8 and

$$0 < \operatorname{dist}(y, X)/t \leq \lambda/2,$$
$$|T(y-x)| \leq s|y-x| \quad \text{for } x \in X, \quad T^\perp(y) \notin T^\perp(X),$$

hence there would exist $x \in X$ with $|y-x| \leq \lambda t/2$ implying $0 < |T^\perp(y-x)| < t$. Finally, one would obtain as in the last paragraph of the proof of part (1) that

$$\lambda t \leq \left|T^\perp(y-x)\right|$$

which is incompatible with

$$\left|T^\perp(y-x)\right| \leq |y-x| \leq \lambda t/2$$

because $\lambda t > 0$. □

1.11 Remark. Combining A.15 and 1.10 (2), one obtains the following proposition about tangent planes:

Suppose m, n, p, U, and μ are as in A.1 with $p < n$ and μ integral. Then for every $0 < s < 1$

$$\lim_{r \downarrow 0} \mu\big(\{y \in U : s^{-1} \operatorname{dist}(y-x, (T_x\mu)^\perp) < |y-x| < r\}\big)/r^{n^2/(n-p)} = 0$$

for μ almost every $x \in U$.

The exponent $n^2/(n-p)$ cannot be replaced by any larger number as C.2 (4) shows.

1.12 Definition. Suppose $m, n, Q \in \mathbb{N}$, and $T \in G(n+m, n)$.

Then P is called a Q *valued plane parallel to* T if and only if for some $S \in Q_Q(T^\perp)$ (see D.1)

$$P = \big(\theta^0(\|S\|, \cdot) \circ T^\perp\big)\mathcal{H}^n.$$

S is uniquely determined by P. For any two Q valued planes P_1 and P_2 parallel to T associated to $S_1, S_2 \in Q_Q(T^\perp)$ one defines (see D.1)

$$\mathcal{G}(P_1, P_2) := \mathcal{G}(S_1, S_2).$$

In case $S = \sum_{i=1}^{Q} [\![z_i]\!]$ for some $z_1, \ldots, z_Q \in T^\perp$, then

$$\|S\| = \sum_{i=1}^{Q} \delta_{z_i}, \quad P = \sum_{i=1}^{Q} \mathcal{H}^n \llcorner \{x \in \mathbb{R}^{n+m} : T^\perp(x) = z_i\}$$

where δ_x denotes the Dirac measure at the point x.

1.13. In studying approximations of integral varifolds the following notation will be convenient. Suppose $m, n \in \mathbb{N}$, and $T \in G(n+m, n)$. Then there exist orthogonal projections $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $\sigma : \mathbb{R}^{n+m} \to \mathbb{R}^m$ such that $T = \operatorname{im} \pi^*$ and $\pi \circ \sigma^* = 0$, hence

$$T = \pi^* \circ \pi, \quad T^\perp = \sigma^* \circ \sigma, \quad \mathbb{1}_{\mathbb{R}^{n+m}} = \pi^* \circ \pi + \sigma^* \circ \sigma.$$

Whenever $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$, $0 < h \leq \infty$ the *closed cylinder* $C(T, a, r, h)$ is defined by

$$\begin{aligned} C(T, a, r, h) &= \{x \in \mathbb{R}^{n+m} : |T(x-a)| \leq r \text{ and } |T^\perp(x-a)| \leq h\} \\ &= \{x \in \mathbb{R}^{n+m} : |\pi(x-a)| \leq r \text{ and } |\sigma(x-a)| \leq h\}. \end{aligned}$$

This definition extends Allard's definition in [All72, 8.10] where $h = \infty$.

1.14 Lemma (Approximation by Q valued functions)**.** *Suppose $m, n, Q \in \mathbb{N}$, $0 < L < \infty$, $1 \leq M < \infty$, and $0 < \delta_i \leq 1$ for $i \in \{1, 2, 3, 4\}$.*

Then there exists a positive, finite number ε with the following property.

If a, r, h, T, π, and σ are as in 1.13, $h > 2\delta_4 r$,

$$U = \{x \in \mathbb{R}^{n+m} : \operatorname{dist}(x, C(T, a, r, h)) < 2r\},$$

μ is an integral n varifold in U with locally bounded first variation,

$$\begin{gathered} (Q - 1 + \delta_1)\omega_n r^n \leq \mu(C(T, a, r, h)) \leq (Q + 1 - \delta_2)\omega_n r^n, \\ \mu(C(T, a, r, h + \delta_4 r) \sim C(T, a, r, h - 2\delta_4 r)) \leq (1 - \delta_3)\omega_n r^n, \\ \mu(U) \leq M\omega_n r^n, \end{gathered}$$

$0 < \varepsilon_1 \leq \varepsilon$, B denotes the set of all $x \in C(T, a, r, h)$ with $\theta^{*n}(\mu, x) > 0$ such that

$$\begin{array}{lll} \textit{either} & \|\delta\mu\|(\bar{B}_\varrho(x)) > \varepsilon_1\, \mu(\bar{B}_\varrho(x))^{1-1/n} & \textit{for some } 0 < \varrho < 2r, \\ \textit{or} & \int_{\bar{B}_\varrho(x)} |T_\xi\mu - T| \,\mathrm{d}\mu(\xi) > \varepsilon_1\, \mu(\bar{B}_\varrho(x)) & \textit{for some } 0 < \varrho < 2r, \end{array}$$

and G denotes the set of all $x \in C(T, a, r, h) \cap \operatorname{spt}\mu$ such that

$$\begin{gathered} \|\delta\mu\|(B_{2r}(x)) \leq \varepsilon\, \mu(B_{2r}(x))^{1-1/n}, \\ \int_{B_{2r}(x)} |T_\xi\mu - T| \,\mathrm{d}\mu(\xi) \leq \varepsilon\, \mu(B_{2r}(x)), \\ \|\delta\mu\|(\bar{B}_\varrho(x)) \leq (2\gamma_n)^{-1}\mu(\bar{B}_\varrho(x))^{1-1/n} \quad \textit{for } 0 < \varrho < 2r, \end{gathered}$$

then there exist an $\mathcal{L}^n$ measurable subset Y of $\mathbb{R}^n$ and a function $f : Y \to Q_Q(\mathbb{R}^m)$ with the following seven properties:

(1) $Y \subset \bar{B}_r(\pi(a))$ and f is Lipschitzian with $\operatorname{Lip} f \leq L$.

(2) Defining $A = C(T, a, r, h) \sim B$ and $A(y) = \{x \in A : \pi(x) = y\}$ for $y \in \mathbb{R}^n$, the sets A and B are Borel sets and there holds (see D.1)

$$\begin{gathered} \sigma(A \cap \operatorname{spt}\mu) \subset \bar{B}_{h-\delta_4 r}(\sigma(a)), \quad \operatorname{spt} f(y) \subset \sigma(A(y)), \\ \|f(y)\| = \sigma\big(\theta^n(\mu, \cdot)\mathcal{H}^0 \llcorner A(y)\big) \end{gathered}$$

whenever $y \in Y$.

(3) Defining the sets

$$C = \bar{B}_r(\pi(a)) \sim (Y \sim \pi(B)), \quad D = C(T, a, r, h) \cap \pi^{-1}(C),$$

there holds

$$\mathcal{L}^n(C) + \mu(D) \leq \Gamma_{(3)}\, \mu(B).$$

with $\Gamma_{(3)} = \max\{3 + 2Q + (12Q + 6)5^n, 4(Q + 2)/\delta_1\}$.

(4) *If* $x_1 \in G$, *then*

$$|\sigma(x_1 - a)| \leq h - \delta_4 r$$

and for $y \in Y \cap \bar{B}_{\lambda_{(4)}}(\pi(x_1))$ *there exists* $x_2 \in A(y)$ *with* $\theta^n(\mu, x_2) \in \mathbb{N}$ *and*

$$\left|T^\perp(x_2 - x_1)\right| \leq L\,|T(x_2 - x_1)|,$$

where $0 < \lambda_{(4)} < 1$ *depends only on* n, δ_2, *and* δ_4. *Moreover,* $G \supset A \cap \operatorname{spt} \mu$ *and (see D.1, D.4)*

$$(\pi \bowtie \sigma)\big(G \cap \pi^{-1}(Y)\big) = \operatorname{graph}_Q f.$$

(5) $\overline{Y} \sim Y$ *has measure* 0 *with respect to* $\mathcal{L}^n$ *and* $\pi(\mu \llcorner G)$.

(6) *If* $\mathcal{L}^n(\bar{B}_r(\pi(a)) \sim Y) \leq \frac{1}{2}\omega_n(\lambda_{(4)} r/6)^n$, $1 \leq q < \infty$, $S \in Q_Q(\mathbb{R}^m)$, $P = (\theta^0(\|S\|, \cdot) \circ \sigma)\mathcal{H}^n$ *is the* Q *valued plane associated to* S *via* σ, *and* $g : Y \to \mathbb{R}$ *is defined by* $g(y) = \mathcal{G}(f(y), S)$ *for* $y \in Y$, *then*

$$\begin{aligned}&\|\operatorname{dist}(\cdot, \operatorname{spt} P)\|_{L^q(\mu \llcorner G)}\\ &\qquad \leq (12)^{n+1} Q\big(\|g\|_{L^q(\mathcal{L}^n \llcorner Y)} + \Gamma_{(6)} \mathcal{L}^n(\bar{B}_r(\pi(a)) \sim Y)^{1/q + 1/n}\big),\end{aligned}$$

where $\Gamma_{(6)}$ *is a positive, finite number depending only on* q, *and* n, *and*

$$\begin{aligned}&\sup\{\operatorname{dist}(x, \operatorname{spt} P) : x \in G\}\\ &\qquad \leq \|g\|_{L^\infty(\mathcal{L}^n \llcorner Y)} + 2\big(\mathcal{L}^n(\bar{B}_r(\pi(a)) \sim Y)/\omega_n\big)^{1/n}.\end{aligned}$$

(7) *For* $\mathcal{L}^n$ *almost all* $y \in Y$ *the following is true:*

  (a) f *is approximately strongly affinely approximable at* y.

  (b) *Whenever* $x \in G$ *with* $\pi(x) = y$

$$(\pi \bowtie \sigma)(T_x\mu) = \operatorname{Tan}\big(\operatorname{graph}_Q \operatorname{ap} Af(y), (y, \sigma(x))\big)$$

  where $\operatorname{Tan}(S, a)$ *denotes the classical tangent cone of* S *at* a *in the sense of [Fed69, 3.1.21].*

  (c) $\|T_x\mu - T\| \leq \|\operatorname{ap} Af(y)\|$ *for* $x \in G$ *with* $\pi(x) = y$.

  (d) $\|\operatorname{ap} Af(y)\|^2 \leq Q(1 + (\operatorname{Lip} f)^2) \max\{\|T_x\mu - T\|^2 : x \in \pi^{-1}(\{y\}) \cap G\}$.

Choice of constants. One can assume $3L \leq \delta_4$.

Choose $0 < s_0 < 1$ close to 1 such that $2(s_0^{-2} - 1)^{1/2} \leq \delta_4$, define $\lambda = \lambda_{1.10}(n, \delta_2, s_0)/4$, choose $s_0 \leq s < 1$ close to 1 satisfying

$$(s^{-2} - 1)^{1/2} \leq \lambda/4, \quad Q^{1/2}(s^{-2} - 1)^{1/2} \leq L,$$

and define $\varepsilon > 0$ so small that

$$\begin{gathered}\varepsilon \leq (2\gamma_n)^{-1}, \quad Q - 1 + \delta_1/2 \leq (1 - n\varepsilon^2)(Q - 1 + \delta_1),\\ Q - 1/2 \leq (1 - n\varepsilon^2)(Q - 1/4), \quad 1 - n\varepsilon^2 \geq 1/2,\end{gathered}$$

and not larger than the minimum of the following seven numbers

$$\begin{gathered}\varepsilon_{A.10}(m,n,1-\delta_3/2), \quad \varepsilon_{1.8}(m,n,1,M,\delta_3/2,s), \\ \varepsilon_{1.8}(m,n,Q+1,M,\delta_2/2,s), \quad \varepsilon_{1.8}(m,n,Q,M,1/4,s), \\ \varepsilon_{1.9}(m,n,\min\{\delta_2/3,\delta_3/2\},s,\max\{M,2\}), \quad \varepsilon_{1.8}(m,n,Q,M,\delta_2/3,s), \\ \varepsilon_{1.10}(m,n,Q,1,\delta_2,s,s_0,M).\end{gathered}$$

Clearly, ε_1 satisfies the same inequalities as ε and one can assume $a=0$, and $r=1$. $\square$

Proof of (1) *and* (2). Since $\theta^{*n}(\mu,\cdot)$ is a Borel function, one may verify that A and B are Borel sets (cp. [Fed69, 2.9.14]).

First, the following *basic properties of* A are proved: For $x \in A \cap \operatorname{spt}\mu$

$$\begin{gathered}\theta^n_*(\mu,x) \geq \delta_3/2, \\ \{\xi \in \pi^{-1}(\bar{B}^n_1(0)) : |T(\xi-x)| > s|\xi-x|\} \subset \sigma^{-1}(\bar{B}_{\min\{\lambda/2,\delta_4\}}(\sigma(x))), \\ \sigma(A \cap \operatorname{spt}\mu) \subset \bar{B}^m_{h-\delta_4}(0).\end{gathered}$$

The first is implied by A.10. The second is a consequence of the fact that for $\xi \in \pi^{-1}(\bar{B}^n_1(0))$ with $|T(\xi-x)| > s|\xi-x|$

$$|\sigma(\xi)-\sigma(x)| < (s^{-2}-1)^{1/2}|\pi(\xi)-\pi(x)| \leq 2(s^{-2}-1)^{1/2} \leq \min\{\lambda/2,\delta_4\}.$$

To prove the third, note that 1.8 applied with

Q, δ, X, d, r, t, and f replaced by
1, $\delta_3/2$, $\{x\}$, 1, 2, 1, and $T^\perp|\{x\}$

yields

$$\mu\big(\pi^{-1}(\bar{B}^n_1(0)) \cap \sigma^{-1}(\bar{B}_{\delta_4}(\sigma(x)))\big) \geq (1-\delta_3/2)\omega_n,$$

so that $h-\delta_4 < |\sigma(x)| \leq h$ would be incompatible with

$$\mu(C(T,0,1,h+\delta_4) \sim C(T,0,1,h-2\delta_4)) \leq (1-\delta_3)\omega_n.$$

Next, it will be shown *if* $X \subset A \cap \operatorname{spt}\mu$, $\theta^n(\mu,x) \in \mathbb{N}_0$ *for* $x \in X$,

$$s^{-1}|T(x_2-x_1)| \leq |x_2-x_1| \quad \textit{whenever } x_1,x_2 \in X,$$

then $\sum_{x\in X}\theta^n(\mu,x) \leq Q$. Using the basic properties of A to verify

$$\begin{aligned}\{\xi \in B_1(T^\perp(x)) : |T(\xi-x)| > s|\xi-x|\} &\subset \pi^{-1}(\bar{B}^n_1(0)) \cap \sigma^{-1}(\bar{B}_{\delta_4}(\sigma(x))) \\ &\subset C(T,0,1,h)\end{aligned}$$

there holds

$$\begin{aligned}\mu\big(\textstyle\bigcup_{x\in X}\{\xi \in B_1(T^\perp(x)) : |T(\xi-x)| > s|\xi-x|\}\big) &\leq \mu(C(T,0,1,h)) \\ &\leq (Q+1-\delta_2)\omega_n\end{aligned}$$

and 1.8 applied with

$$Q,\ \delta,\ d,\ r,\ t, \text{ and } f \text{ replaced by}$$
$$Q+1,\ \delta_2/2,\ 1,\ 2,\ 1, \text{ and } T^\perp | X$$

yields

$$\textstyle\sum_{x \in X} \theta^n(\mu, x) < Q + \delta_2/2 < Q + 1,$$

hence $\sum_{x \in X} \theta^n(\mu, x) \leq Q$. In particular, $\sum_{x \in A(y)} \theta^n(\mu, x) \leq Q$ whenever $y \in \bar{B}_1^n(0)$ and $\theta^n(\mu, x) \in \mathbb{N}_0$ for each $x \in A(y)$.

Let Y be the set of all $y \in \bar{B}_1^n(0)$ such that

$$\textstyle\sum_{x \in A(y)} \theta^n(\mu, x) = Q \quad \text{and} \quad \theta^n(\mu, x) \in \mathbb{N}_0 \text{ for } x \in A(y),$$

Z be the set of all $z \in \bar{B}_1^n(0)$ such that

$$\textstyle\sum_{x \in A(z)} \theta^n(\mu, x) \leq Q - 1 \quad \text{and} \quad \theta^n(\mu, x) \in \mathbb{N}_0 \text{ for } x \in A(z),$$

and $N = \bar{B}_1^n(0) \sim (Y \cup Z)$. Clearly, $Y \cap Z = \emptyset$. Note by the concluding remark of the preceding paragraph $\mathcal{L}^n(N) = 0$ because $\theta^n(\mu, x) \in \mathbb{N}_0$ for $\mathcal{H}^n$ almost all $x \in U$. Since $\theta^n(\mu, \cdot)$ is a Borel function whose domain is a Borel set and A is a Borel set, Y and Z are $\mathcal{L}^n$ measurable by [Fed69, 3.2.22 (3)]. Let $f : Y \to Q_Q(\mathbb{R}^m)$ be defined by

$$f(y) = \sigma_\# \big(\textstyle\sum_{x \in A(y)} \theta^n(\mu, x) [\![x]\!] \big) \quad \text{whenever } y \in Y.$$

One infers from the assertion of the preceding paragraph and D.12

$$\mathcal{G}(f(y_2), f(y_1)) \leq Q^{1/2} (s^{-2} - 1)^{1/2} |y_2 - y_1| \quad \text{for } y_1, y_2 \in Y.$$

(1) and (2) are now evident. □

Proof of (3). For the estimate some preparations are needed. Let ν denote the Radon measure defined by the requirement

$$\nu(X) = \textstyle\int_X J^\mu T \, \mathrm{d}\mu \quad \text{for every Borel subset } X \text{ of } U$$

where J^μ denotes the Jacobian with respect μ. Note

$$|T_x \mu - T| \leq \varepsilon \quad \text{for } \mu \text{ almost all } x \in A,$$

hence $1 - J^\mu T(x) \leq 1 - (J^\mu T)(x)^2 \leq n \varepsilon^2$. Therefore

$$(1 - n\varepsilon^2)\, \mu \llcorner A \leq \nu \llcorner A.$$

This implies the *coarea estimate*

$$\begin{aligned} &(1 - n\varepsilon^2)\, \mu\big(C(T, 0, 1, h) \cap \pi^{-1}(W) \big) \\ &\leq \mu\big(B \cap \pi^{-1}(W) \big) + Q \mathcal{L}^n(Y \cap W) + (Q-1) \mathcal{L}^n(Z \cap W) \end{aligned}$$

for every subset W of $\mathbb{R}^n$; in fact the estimate holds for every Borel set by [Fed69, 3.2.22 (3)] and $\pi(\mu \llcorner B)$ is a Radon measure by [Fed69, 2.2.17]. Also note that in view of the choice of $\Gamma_{(3)}$ one can assume

$$\mu(B) \leq (\delta_1/4) \omega_n,$$

which implies $\mathcal{L}^n(Y) > 0$ because it follows from the coarea estimate applied with $W = \bar{B}^n_1(0)$

$$\begin{aligned}(Q - 1 + \delta_1/2)\omega_n &\leq (1 - n\varepsilon^2)\mu(C(T, 0, 1, h)) \\ &\leq \mu(B) + Q\mathcal{L}^n(Y) + (Q - 1)\mathcal{L}^n(Z) \\ &\leq (\delta_1/4)\omega_n + (Q - 1 + \delta_1/4)\omega_n + \mathcal{L}^n(Y) - (\delta_1/4)\mathcal{L}^n(Z),\end{aligned}$$

hence $\mathcal{L}^n(Z) \leq (4/\delta_1)\mathcal{L}^n(Y)$.

In order to derive an upper bound for the $\mathcal{L}^n$ measure of Z, the following assertion will be proved. *If $z \in Z$ with $\theta^n(\mathcal{L}^n \llcorner \mathbb{R}^n \sim Z, a) = 0$, then there exist $\zeta \in \mathbb{R}^n$ and $0 < t < \infty$ with*

$$z \in \bar{B}_t(\zeta) \subset \bar{B}^n_1(0), \quad \mathcal{L}^n(\bar{B}_{5t}(\zeta)) \leq 6 \cdot 5^n \, \mu\big(B \cap \pi^{-1}(\bar{B}_t(\zeta))\big).$$

Since $\mathcal{L}^n(Y) > 0$, some element $\bar{B}_t(\zeta)$ of the family of balls

$$\{\bar{B}_\theta((1 - \theta)z) : 0 < \theta \leq 1\}$$

will satisfy

$$z \in \bar{B}_t(\zeta) \subset \bar{B}^n_1(0), \quad 0 < \mathcal{L}^n(Y \cap \bar{B}_t(\zeta)) \leq \tfrac{1}{2}\mathcal{L}^n(Z \cap \bar{B}_t(\zeta)).$$

Hence there exists $y \in Y \cap B_t(\zeta)$. Noting for $\xi \in A(y)$ with $\theta^n(\mu, \xi) > 0$, and $\eta \in \mathbb{R}^{n+m}$ with $|\eta_{-\pi^*(\zeta - y), 1}(\xi) - \eta| < t$,[1]

$$\begin{gathered} t \leq 1, \quad \pi(\xi) = y, \\ |\pi(\eta) - \zeta| = |\pi(\xi + \pi^*(\zeta - y) - \eta)| \leq |\eta_{-\pi^*(\zeta - y), 1}(\xi) - \eta| < t, \\ B_t(\eta_{-\pi^*(\zeta - y), 1}(\xi)) \subset \pi^{-1}(\bar{B}_t(\zeta)), \end{gathered}$$

and, recalling the basic properties of A,

$$\{\kappa \in B_t(\eta_{-\pi^*(\zeta - y), 1}(\xi)) : |T(\kappa - \xi)| > s|\kappa - \xi|\} \subset C(T, 0, 1, h) \cap \pi^{-1}(\bar{B}_t(\zeta)),$$

one can apply 1.8 with

$$\begin{gathered} \delta,\ X,\ d,\ r, \text{ and } f \text{ replaced by} \\ 1/4,\ \{\xi \in A(y) : \theta^n(\mu, \xi) > 0\},\ t,\ 2, \text{ and} \\ \eta_{-\pi^*(\zeta - y), 1} | \{\xi \in A(y) : \theta^n(\mu, \xi) > 0\} \end{gathered}$$

to obtain

$$(Q - 1/4)\omega_n t^n \leq \mu\big(C(T, 0, 1, h) \cap \pi^{-1}(\bar{B}_t(\zeta))\big).$$

The coarea estimate with $W = \bar{B}_t(\zeta)$ now implies

$$\begin{aligned} &(Q - 1/2)\omega_n t^n \\ &\leq \mu\big(B \cap \pi^{-1}(\bar{B}_t(\zeta))\big) + Q\mathcal{L}^n(Y \cap \bar{B}_t(\zeta)) + (Q - 1)\mathcal{L}^n(Z \cap \bar{B}_t(\zeta)) \\ &= \mu\big(B \cap \pi^{-1}(\bar{B}_t(\zeta))\big) + (Q - 1/2)\omega_n t^n \\ &\quad + \tfrac{1}{2}\mathcal{L}^n(Y \cap \bar{B}_t(\zeta)) - \tfrac{1}{2}\mathcal{L}^n(Z \cap \bar{B}_t(\zeta)), \end{aligned}$$

[1] Recall from [Sim83] that the functions $\eta_{a,r} : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ are given by $\eta_{a,r}(x) = r^{-1}(x - a)$ for $a, x \in \mathbb{R}^{n+m}$, $0 < r < \infty$.

hence

$$\tfrac{2}{3}\mathcal{L}^n(\bar{B}_t(\zeta)) \le \mathcal{L}^n(Z \cap \bar{B}_t(\zeta)) \le 4\,\mu\big(B \cap \pi^{-1}(\bar{B}_t(\zeta))\big)$$

and the assertion follows.

$\mathcal{L}^n$ almost all $z \in Z$ satisfy the assumptions of the last assertion (cf. [Fed69, 2.9.11]) and Vitali's covering theorem (cf. [Fed69, 2.8.5]) implies

$$\mathcal{L}^n(Z) \le 6 \cdot 5^n\, \mu(B).$$

Clearly,

$$\mathcal{L}^n(\pi(B)) \le \mathcal{H}^n(B) \le \mu(B).$$

Since $C \sim N \subset Z \cup \pi(B)$, it follows

$$\mathcal{L}^n(C) \le (1 + 6 \cdot 5^n)\, \mu(B).$$

Finally, applying the coarea estimate with $W = C$ yields

$$(1 - n\varepsilon^2)\, \mu(D) \le \mu(B) + Q\mathcal{L}^n(C) \le (1 + Q + 6Q \cdot 5^n)\, \mu(B). \qquad \square$$

Proof of (4). Assuming now that x_1 and y satisfy the conditions of (4), it will be shown that one can take $\lambda_{(4)} = \lambda$. Verifying

$$\{\xi \in \pi^{-1}(\bar{B}^n_1(0)) : |T(\xi - x_1)| > s|\xi - x_1|\} \subset \sigma^{-1}(\bar{B}_{\min\{\lambda/2, \delta_4\}}(\sigma(x_1))),$$

defining $\delta_5 = \min\{\delta_2/3, \delta_3/2\}$ and applying 1.9 with

δ, M, a, r, d, t, and ζ replaced by
δ_5, $\max\{M, 2\}$, x_1, 2, 1, 1, and $-T(x_1)$

yields

$$\mu\big(\pi^{-1}(\bar{B}^n_1(0)) \cap \sigma^{-1}(\bar{B}_{\min\{\lambda/2,\delta_4\}}(\sigma(x_1)))\big) \ge (1 - \delta_5)\omega_n$$

so that $h - \delta_4 < |\sigma(x_1)| \le h$ would be incompatible with

$$\mu\big(C(T, 0, 1, h + \delta_4) \sim C(T, 0, 1, h - 2\delta_4)\big) \le (1 - \delta_3)\omega_n$$

and the first part of (4) follows.

To prove the second part, one defines $X = \{\xi \in A(y) : \theta^n(\mu, \xi) \in \mathbb{N}\}$ and first observes that 1.8 applied with

δ, d, r, t, and f replaced by,
$\delta_2/3$, 1, 2, 1, and $\eta_{\pi^*(y),1}|X$

yields

$$\mu\big(\textstyle\bigcup_{x \in X} \{\xi \in B_1(x - \pi^*(y)) : |T(\xi - x)| > s|\xi - x|\}\big) \ge \big(Q - \tfrac{\delta_2}{3}\big)\omega_n.$$

On the other hand

$$\mu(C(T, 0, 1, h)) \le (Q + 1 - \delta_2)\omega_n.$$

Therefore, using the basic properties of A, for some $x \in X$

$$C(T, 0, 1, h) \cap \sigma^{-1}(\bar{B}_{\lambda/2}(\sigma(x_1))) \cap \sigma^{-1}(\bar{B}_{\lambda/2}(\sigma(x))) \neq \emptyset,$$

hence $|\sigma(x_1 - x)| \leq \lambda$ and

$$\operatorname{dist}(x_1, X) \leq |\pi(x_1 - x)| + |\sigma(x_1 - x)| \leq 2\lambda = \lambda_{1.10}(n, \delta_2, s_0)/2 \leq 1.$$

Finally, the point $x_2 \in X$ may be constructed by applying 1.10 (2) with

$$\begin{gathered} \delta_1,\ \lambda,\ d,\ r,\ t,\ \zeta, \text{ and } y \text{ replaced by} \\ 1,\ \lambda_{1.10}(n, \delta_2, s_0),\ 1,\ 2,\ 1,\ -\pi^*(y), \text{ and } x_1 \end{gathered}$$

noting

$$\{\xi \in B_1(x - \pi^*(y)) : |T(\xi - x)| > s_0 |\xi - x|\} \subset C(T, 0, 1, h)$$

for $x \in X$.

The postscript follows readily from the second part and $\varepsilon_1 \leq \varepsilon \leq (2\gamma_n)^{-1}$. □

Proof of (5). Recalling $(\mu \llcorner A)/2 \leq \nu \llcorner A$ and $\mathcal{L}^n(N) = 0$, it is enough to prove

$$\overline{Y} \subset N \cup Y, \quad \pi^{-1}(\overline{Y}) \cap G \subset A \cap \operatorname{spt} \mu$$

in view of the coarea formula [Fed69, 3.2.22 (3)].

Suppose for this purpose $y \in \overline{Y}$. Since f is Lipschitzian, there exists a unique $S \in Q_Q(\mathbb{R}^m)$ such that

$$(y, S) \in \overline{\operatorname{graph} f}.$$

Let $R = \pi^{-1}(\{y\}) \cap \sigma^{-1}(\operatorname{spt} S)$. Since $A \cap \operatorname{spt} \mu$ is closed (cp. [Fed69, 2.9.14]),

$$R \subset A \cap \operatorname{spt} \mu$$

and (4) implies $G \cap \pi^{-1}(\{y\}) \subset R$, the second inclusion follows.

Choose a sequence $y_i \in Y$ with $y_i \to y$ as $i \to \infty$ and abbreviate $X_i = \{\xi \in A(y_i) : \theta^n(\mu, \xi) \in \mathbb{N}\}$ for $i \in \mathbb{N}$. 1.8 applied with

$$\begin{gathered} \delta,\ X,\ d,\ r, \text{ and } f \text{ replaced by} \\ 1/4,\ X_i,\ 0,\ 2, \text{ and } \mathbb{1}_{X_i} \end{gathered}$$

yields for $i \in \mathbb{N}$

$$\mu\big(\textstyle\bigcup_{x \in X_i} \bar{B}_t(x)\big) \geq (Q - 1/4)\omega_n t^n \quad \text{whenever } 0 < t < 2.$$

Since $f(y_i) \to S$ in Hausdorff distance as $i \to \infty$ the same estimate holds with X_i replaced by R and

$$Q - 1/4 \leq \limsup_{t \downarrow 0} \frac{\mu\big(\bigcup_{x \in R} \bar{B}_t(x)\big)}{\omega_n t^n} \leq \sum_{x \in R} \theta^{*n}(\mu, x)$$

implies $y \notin Z$, hence the first inclusion. □

Proof of (6). Let $\psi := \mu \llcorner G$. Using $(\pi(\psi)) \llcorner Y \leq 2(\pi(\nu \llcorner G)) \llcorner Y \leq 2Q\mathscr{L}^n \llcorner Y$,

$$\{x \in G \cap \pi^{-1}(Y) : \operatorname{dist}(x, \operatorname{spt} P) > \gamma\} \subset G \cap \pi^{-1}(\{y \in Y : g(y) > \gamma\})$$

for $0 < \gamma < \infty$, one infers

$$\|\operatorname{dist}(\cdot, \operatorname{spt} P)\|_{L^q(\mu \llcorner G \cap \pi^{-1}(Y))} \leq 2Q\|g\|_{L^q(\mathscr{L}^n \llcorner Y)}.$$

Hence only $\|\operatorname{dist}(\cdot, \operatorname{spt} P)\|_{L^q(\mu \llcorner G \sim \pi^{-1}(Y))}$ needs to be estimated in the first part of (6).

Whenever $z \in \bar{B}_1^n(0) \sim \overline{Y}$ there exist $\zeta \in \mathbb{R}^n$ and $0 < t \leq \lambda/6$ such that

$$z \in \bar{B}_t(\zeta) \subset \bar{B}_1^n(0), \quad \mathscr{L}^n(\bar{B}_t(\zeta) \cap Y) = \mathscr{L}^n(\bar{B}_t(\zeta) \sim Y)$$

as may be verified by consideration of the family of closed balls

$$\{\bar{B}_\theta((1-\theta)z) : 0 < \theta \leq \lambda\}.$$

Therefore [Fed69, 2.8.5] yields a countable set I and $\zeta_i \in \mathbb{R}^n$, $0 < t_i \leq \lambda/6$ and $y_i \in Y \cap \bar{B}_{t_i}(\zeta_i)$ for each $i \in I$ such that

$$\begin{gathered}\bar{B}_{t_i}(\zeta_i) \subset \bar{B}_1^n(0), \quad \mathscr{L}^n(\bar{B}_{t_i}(\zeta_i) \cap Y) = \mathscr{L}^n(\bar{B}_{t_i}(\zeta_i) \sim Y),\\ \bar{B}_{t_i}(\zeta_i) \cap \bar{B}_{t_j}(\zeta_j) = \emptyset \quad \text{whenever } i, j \in I \text{ with } i \neq j,\\ \bar{B}_1^n(0) \sim \overline{Y} \subset \textstyle\bigcup_{i \in I} E_i \subset \bar{B}_1^n(0)\end{gathered}$$

where $E_i = \bar{B}_{5t_i}(\zeta_i) \cap \bar{B}_1^n(0)$ for $i \in I$. Let

$$h_i := \mathcal{G}(f(y_i), S), \quad X_i := \{\xi \in A(y_i) : \theta^n(\mu, \xi) \in \mathbb{N}\}$$

for $i \in I$, $J := \{i \in I : h_i \geq 18t_i\}$, and $K := I \sim J$.

In view of (5) there holds

$$\|d\|_{L^q(\mu \llcorner G \sim \pi^{-1}(Y))} \leq \|d\|_{L^q(\psi \llcorner \pi^{-1}(\bigcup_{j \in J} E_j))} + \|d\|_{L^q(\psi \llcorner \pi^{-1}(\bigcup_{i \in K} E_i))}$$

for every ψ measurable function $d : \mathbb{R}^{n+m} \to [0, \infty[$. In order to estimate the terms on the right hand side for $d = \operatorname{dist}(\cdot, \operatorname{spt} P)$, two observations will be useful. *If $i \in I$, $x_1 \in G \cap \pi^{-1}(E_i)$, then*

$$\operatorname{dist}(x_1, \operatorname{spt} P) \leq 6t_i + h_i;$$

in fact $|\pi(x_1) - y_i| \leq 6t_i \leq \lambda$ and (4) yields a point $x_2 \in X_i$ and

$$\left|T^\perp(x_2 - x_1)\right| \leq L\,|T(x_2 - x_1)| = L\,|\pi(x_1) - y_i| \leq 6t_i,$$

implying

$$\operatorname{dist}(x_1, \operatorname{spt} P) \leq \left|T^\perp(x_2 - x_1)\right| + \operatorname{dist}(x_2, \operatorname{spt} P) \leq 6t_i + h_i.$$

Moreover,

$$|x_2 - x_1| \leq |T(x_2 - x_1)| + \left|T^\perp(x_2 - x_1)\right| \leq 12t_i, \quad x_1 \in \bar{B}_{12t_i}(x_2),$$

hence

$$G \cap \pi^{-1}(E_i) \subset \textstyle\bigcup_{x \in X_i} \bar{B}_{12t_i}(x)$$

and 1.10 (1) applied with

$$\begin{gathered}\delta_1,\ s,\ \lambda,\ X,\ d,\ r,\ t,\ \zeta, \text{ and } \tau \text{ replaced by}\\ 1,\ 0,\ \lambda_{1.10}(n,\delta_2,s_0),\ X_i,\ 1,\ 2,\ 1,\ -\pi^*(y_i), \text{ and } 12t_i\end{gathered}$$

yields

$$\psi\big(\pi^{-1}(E_i)\big) \leq (Q+1)\omega_n(12t_i)^n \quad \textit{whenever } i \in I.$$

Now, the first term will be estimated. Note, if $j \in J$, then

$$\begin{aligned} \operatorname{dist}(x, \operatorname{spt} P) \leq \tfrac{4}{3} h_j \quad & \text{whenever } x \in G \cap \pi^{-1}(E_j),\\ \tfrac{4}{3}h_j \leq 2\mathcal{G}(f(y),S) \quad & \text{whenever } y \in Y \cap \bar{B}_{t_j}(\zeta_j),\end{aligned}$$

because

$$\mathcal{G}(f(y),S) \geq \mathcal{G}(f(y_j),S) - L|y-y_j| \geq h_j - 2Lt_j \geq \tfrac{2}{3}h_j.$$

Using this fact and the preceding observations, one estimates with $J(\gamma) := \{j \in J : \frac{4}{3}h_j > \gamma\}$ for $0 < \gamma < \infty$

$$\begin{aligned} \psi\big(\pi^{-1}({\textstyle\bigcup_{j\in J}} E_j) \cap \{x \in \mathbb{R}^{n+m} : \operatorname{dist}(x,\operatorname{spt} P) > \gamma\}\big) &\leq {\textstyle\sum_{j\in J(\gamma)}} \psi\big(\pi^{-1}(E_j)\big)\\ \leq {\textstyle\sum_{j\in J(\gamma)}}(Q+1)\omega_n(12t_j)^n &\leq (Q+1)(12)^n\mathcal{L}^n\big({\textstyle\bigcup_{j\in J(\gamma)}}\bar{B}_{t_j}(\zeta_j)\big)\\ &\leq 2(Q+1)(12)^n\mathcal{L}^n\big({\textstyle\bigcup_{j\in J(\gamma)}}\bar{B}_{t_j}(\zeta_j)\cap Y\big)\\ &\leq 2(Q+1)(12)^n\,\mathcal{L}^n(\{y \in Y : \mathcal{G}(f(y),S) > \gamma/2\}),\end{aligned}$$

hence

$$\|\operatorname{dist}(\cdot,\operatorname{spt} P)\|_{L^q(\psi \llcorner \pi^{-1}(\bigcup_{j\in J} E_j))} \leq (2(Q+1)(12)^n)2\,\|g\|_{L^q(\mathcal{L}^n \llcorner Y)}.$$

To estimate the second term, one notes, if $i \in K$, $x \in G \cap \pi^{-1}(E_i)$, then

$$\operatorname{dist}(x,\operatorname{spt} P) < 24t_i.$$

Therefore one estimates with $K(\gamma) := \{i \in K : 24t_i > \gamma\}$ for $0 < \gamma < \infty$ and $u : \mathbb{R}^n \to \mathbb{R}$ defined by $u = \sum_{i\in I} 2t_i\,\chi_{\bar{B}_{t_i}(\zeta_i)}$

$$\begin{aligned} \psi\big(\pi^{-1}({\textstyle\bigcup_{i\in K}} E_i) \cap \{x \in \mathbb{R}^{n+m} : \operatorname{dist}(x,\operatorname{spt} P) > \gamma\}\big) &\leq {\textstyle\sum_{i\in K(\gamma)}}\psi\big(\pi^{-1}(E_i)\big)\\ \leq {\textstyle\sum_{i\in K(\gamma)}}(Q+1)\omega_n(12t_i)^n &\leq (Q+1)(12)^n\mathcal{L}^n\big({\textstyle\bigcup_{i\in K(\gamma)}}\bar{B}_{t_i}(\zeta_i)\big)\\ &\leq (Q+1)(12)^n\mathcal{L}^n\big(\{y\in\mathbb{R}^n : u(y) > \gamma/(12)\}\big),\end{aligned}$$

hence

$$\|\operatorname{dist}(\cdot,\operatorname{spt} P)\|_{L^q(\psi \llcorner \pi^{-1}(\bigcup_{i\in K} E_i))} \leq (Q+1)(12)^{n+1}\|u\|_{L^q(\mathcal{L}^n)}.$$

Combining these two estimates and

$$\begin{aligned} \mathcal{L}^n\big({\textstyle\bigcup_{i\in I}}\bar{B}_{t_i}(\zeta_i)\big) &\leq 2\mathcal{L}^n(\bar{B}^n_1(0)\sim Y),\\ \textstyle\int |u|^q\,\mathrm{d}\mathcal{L}^n = \sum_{i\in I}(2t_i)^q\omega_n(t_i)^n &\leq 2^q\omega_n^{-q/n}\big({\textstyle\sum_{i\in I}}\mathcal{L}^n(\bar{B}_{t_i}(\zeta_i))\big)^{1+q/n}\\ &\leq 2^{q+1+q/n}\omega_n^{-q/n}\big(\mathcal{L}^n(\bar{B}^n_1(0)\sim Y)\big)^{1+q/n},\end{aligned}$$

one obtains the first part of the conclusion of (6).

To prove the second part, suppose $x_1 \in G$. Since

$$\bar{B}_\theta((1-\theta)\pi(x_1)) \subset \bar{B}^n_1(0), \quad \mathcal{L}^n(\bar{B}_\theta((1-\theta)\pi(x_1)) \cap Y) > 0$$

for $(\mathcal{L}^n(\bar{B}^n_1(0) \sim Y)/\omega_n)^{1/n} < \theta < 1$, there exists for any $\delta > 0$ a $y \in Y$ with

$$\begin{gathered} \mathcal{G}(f(y), S) \leq \|g\|_{L^\infty(\mathcal{L}^n \llcorner Y)}, \\ |\pi(x_1) - y| \leq 2\big(\mathcal{L}^n(\bar{B}^n_1(0) \sim Y)/\omega_n\big)^{1/n} + \delta, \end{gathered}$$

in particular $|\pi(x_1) - y| \leq \lambda$ for small δ. Therefore (4) may be applied to construct a point $x_2 \in A(y)$ with $\theta^n(\mu, x_2) \in \mathbb{N}$ and

$$\big|T^\perp(x_2 - x_1)\big| \leq L\,|T(x_2 - x_1)| \leq |\pi(x_1) - y|.$$

Finally,

$$\begin{aligned} \operatorname{dist}(x_1, \operatorname{spt} P) &\leq \operatorname{dist}(x_2, \operatorname{spt} P) + \big|T^\perp(x_2 - x_1)\big| \\ &\leq \mathcal{G}(f(y), S) + 2\big(\mathcal{L}^n(\bar{B}^n_1(0) \sim Y)/\omega_n\big)^{1/n} + \delta \end{aligned}$$

and δ can be chosen arbitrarily small. □

Proof of (7). Combine (1), (4), D.2, D.11, and estimates for orthogonal projections, see e.g. [All72, 8.9 (5)]. □

1.15 Remark. The idea to prove (4) was taken from [Alm00, 3.8 (4)].

2 A Sobolev Poincaré type inequality for integral varifolds

In this section the two main theorems, 2.6 and 2.8, are proved, the first being a Sobolev Poincaré type inequality at some fixed scale r but involving of necessity mean curvature, the second considering the limit r tends to 0. For this purpose the distance of an integral n varifold from a Q valued plane is introduced. One cannot use ordinary planes in 2.6 (without additional assumptions) as may be seen from the fact that any Q valued plane is stationary with vanishing tilt. In 2.8–2.10 an answer to the Problem posed in the introduction is provided.

2.1 Definition. Suppose $m, n, Q \in \mathbb{N}$, $1 \leq q \leq \infty$, $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$, $0 < h \leq \infty$, $T \in G(n+m, n)$, P is a Q valued plane parallel to T (see 1.12), μ is an integral n varifold in an open superset of $C(T, a, r, h)$, A is the $\mathcal{H}^n$ measurable set of all $x \in T \cap \bar{B}_r(T(a))$ such that for some $R(x), S(x) \in Q_Q(\mathbb{R}^{n+m})$

$$\begin{aligned} \|R(x)\| &= \theta^n(P \llcorner C(T, a, r, h), \cdot)\mathcal{H}^0 \llcorner T^{-1}(\{x\}), \\ \|S(x)\| &= \theta^n(\mu \llcorner C(T, a, r, h), \cdot)\mathcal{H}^0 \llcorner T^{-1}(\{x\}) \end{aligned}$$

and $g : A \to \mathbb{R}$ is the $\mathcal{H}^n$ measurable function defined by $g(x) = \mathcal{G}(R(x), S(x))$ for $x \in A$.[2]

[2] The asserted measurabilities may be shown by use of the coarea formula (cf. [Fed69, 3.2.22 (3)]).

Then the *q height of μ with respect to P in* $C(T, a, r, h)$, denoted by

$$H_q(\mu, a, r, h, P),$$

is defined to be the sum of

$$r^{-1-n/q} \| \operatorname{dist}(\cdot, \operatorname{spt} P) \|_{L^q(\mu \llcorner C(T,a,r,h))}$$

and the infimum of the numbers

$$r^{-1-n/q} \|g\|_{L^q(\mathcal{H}^n \llcorner Y)} + r^{-1-n/q} \mathcal{H}^n(T \cap \bar{B}_r(T(a)) \sim Y)^{1/q+1/n}$$

corresponding to all $\mathcal{H}^n$ measurable subsets Y of A. The *q tilt of μ with respect to T in* $C(T, a, r, h)$ is defined by

$$T_q(\mu, a, r, h, T) = r^{-n/q} \|T_\mu - T\|_{L^q(\mu \llcorner C(T,a,r,h))}.$$

Moreover,

$$H_q(\mu, a, r, h, Q, T)$$

is defined to be the infimum of all numbers $H_q(\mu, a, r, h, P)$ corresponding to all Q valued planes P parallel to T.

2.2 Remark. $T_q(\mu, a, r, h, T)$ generalises $\operatorname{tiltex}_\mu$ in an obvious way.

$H_q(\mu, a, r, h, P)$ measures the distance of μ in $C(T, a, r, h)$ from the Q valued plane P. To obtain a reasonable definition of distance, neither the first nor the second summand would be sufficient. The first summand is 0 if $\mu = P \llcorner B$ for some $\mathcal{H}^n$ measurable set B. The second summand is 0 if $\mu = P + \mathcal{H}^n \llcorner B$ for some $\mathcal{H}^n$ measurable subset B of $C(T, a, r, h)$ with $\mathcal{H}^n(B) < \infty$ and $\mathcal{H}^n(T(B)) = 0$. From a more technical point of view, the second summand is added because it is useful in the iteration procedure occurring in 2.8 where the distance of Q valued planes corresponding to different radii r has to be estimated.

2.3 Remark. One readily checks that $H_q(\mu, a, r, h, P) = 0$ implies

$$\mu \llcorner C(T, a, r, h) = P \llcorner C(T, a, r, h)$$

and $H_q(\mu, a, r, h, Q, T) = 0$, $h < \infty$ implies $H_q(\mu, a, r, h, P) = 0$ for some Q valued plane P parallel to T.

More generally, the infima occurring in the definition of $H_q(\mu, a, r, h, P)$ and $H_q(\mu, a, r, h, Q, T)$ are attained. However, this latter fact will neither be used nor proved in this work.

2.4 Lemma. *Suppose $m, n, Q \in \mathbb{N}$, $1 \le M < \infty$, and $0 < \delta \le 1$.*

Then there exists a positive, finite number ε with the following property.

If $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$, $0 < h \le \infty$, $T \in G(n+m, n)$, $\delta r < h$, μ is an integral n varifold in an open superset of $C(T, a, 3r, h + 2r)$ with locally bounded first variation satisfying

$$(Q - 1 + \delta)\omega_n r^n \le \mu(C(T, a, r, h)) \le (Q + 1 - \delta)\omega_n r^n,$$
$$\mu(C(T, a, r, h + \delta r) \sim C(T, a, r, h - \delta r)) \le (1 - \delta)\omega_n r^n,$$
$$\mu(C(T, a, 3r, h + 2r)) \le M \omega_n r^n,$$
$$\|\delta\mu\|(C(T, a, 3r, h + 2r)) \le \varepsilon r^{n-1}, \quad T_1(\mu, a, 3r, h + 2r, T) \le \varepsilon,$$

G is the set of all $x \in C(T,a,r,h) \cap \operatorname{spt}\mu$ such that

$$\|\delta\mu\|(\bar{B}_\varrho(x)) \leq (2\gamma_n)^{-1}\,\mu(\bar{B}_\varrho(x))^{1-1/n} \quad \textit{whenever } 0 < \varrho < 2r,$$

and A is the set defined as G with ε replacing $(2\gamma_n)^{-1}$, then the following two statements hold:

(1) If $1 \leq q < n$, $q^* = \frac{nq}{n-q}$, then

$$\begin{aligned} &H_{q^*}(\mu \llcorner G, a, r, h, Q, T) \\ &\qquad \leq \Gamma_{(1)}\big(T_q(\mu, a, 3r, h+2r, T) + (r^{-n}\mu(C(T,a,r,h) \sim A))^{1/q}\big) \end{aligned}$$

where $\Gamma_{(1)}$ is a positive, finite number depending only on m, n, Q, M, δ, and q.

(2) If $n < q \leq \infty$, then

$$\begin{aligned} &H_\infty(\mu \llcorner G, a, r, h, Q, T) \\ &\qquad \leq \Gamma_{(2)}\big(T_q(\mu, a, 3r, h+2r, T) + (r^{-n}\mu(C(T,a,r,h) \sim A))^{1/q}\big). \end{aligned}$$

where $\Gamma_{(2)}$ is a positive, finite number depending only on m, n, Q, M, δ, and q.

Proof. Let

$$\begin{gathered} \Gamma_0 := \Gamma_{D.15}(m,Q), \quad \Gamma_1 := \Gamma_{1.14(3)}(Q,n,\delta/2), \quad L := 1, \\ \varepsilon_0 := \varepsilon_{1.14}(m,n,Q,1,M,\delta/2,\delta/2,\delta/2,\delta/2), \\ \varepsilon_1 := \varepsilon_0, \quad \lambda := \lambda_{1.14(4)}(n,\delta/2,\delta/2) \end{gathered}$$

and choose $0 < \varepsilon \leq \varepsilon_0$ such that

$$\begin{gathered} \varepsilon \leq \varepsilon_0(n\gamma_n)^{1-n}, \quad 3^n\varepsilon \leq \varepsilon_0(n\gamma_n)^{-n}, \\ \Gamma_1 N(n+m)3^n\varepsilon \leq \tfrac{1}{2}\omega_1(\lambda/6) \quad \text{if } n = 1, \\ \Gamma_1 N(n+m)\big(3^n\varepsilon + \varepsilon^{n/(n-1)}\big) \leq \tfrac{1}{2}\omega_n(\lambda/6)^n \quad \text{if } n > 1; \end{gathered}$$

recall that $N(n+m)$ denotes the best constant in Besicovitch's covering theorem in $\mathbb{R}^{n+m}$, see [Sim83, Lemma 4.6].

Assume $a = 0$ and $r = 1$. Choose orthogonal projections $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $\sigma : \mathbb{R}^{n+m} \to \mathbb{R}^m$ with $\pi \circ \sigma^* = 0$ and $\operatorname{im} \pi^* = T$. Applying 1.14, one obtains sets Y, B and a Lipschitzian function $f : Y \to Q_Q(\mathbb{R}^m)$ with the properties listed there. Using 1.14 (1) (2) and D.15 and noting the existence of a retraction of $\mathbb{R}^m$ to $\bar{B}^m_h(0)$ with Lipschitz constant 1 (cf. [Fed69, 4.1.16]), one constructs an extension $g : \bar{B}^n_1(0) \to Q_Q(\mathbb{R}^m)$ of f with $\operatorname{Lip} g \leq \Gamma_0$ and $\operatorname{spt} g(x) \subset \bar{B}^m_h(0)$ for $x \in \bar{B}^n_1(0)$.

Next, it will be verified that the set G agrees with the set G defined in 1.14; in fact for $x \in G$ using A.8 yields

$$\begin{gathered} \|\delta\mu\|(B_2(x)) \leq \|\delta\mu\|(C(T,0,3,h+2)) \leq \varepsilon \leq \varepsilon_0\,\mu(B_2(x))^{1-1/n}, \\ \textstyle\int_{B_2(x)} |T_\xi\mu - T|\,\mathrm{d}\mu(\xi) \leq \int_{C(T,0,3,h+2)} |T_\xi\mu - T|\,\mathrm{d}\mu(\xi) \leq 3^n\varepsilon \leq \varepsilon_0\,\mu(B_2(x)). \end{gathered}$$

In order to be able to apply 1.14 (6), it will be shown

$$\mathcal{L}^n(\bar{B}^n_1(0) \sim Y) \leq \tfrac{1}{2}\omega_n(\lambda/6)^n.$$

Let B_1 be the set of all $x \in B$ such that

$$\|\delta\mu\|(\bar{B}_\varrho(x)) > \varepsilon_0\, \mu(\bar{B}_\varrho(x))^{1-1/n} \quad \text{for some } 0 < \varrho < 2,$$

and let B_2 be the set of all $x \in B$ such that

$$\textstyle\int_{\bar{B}_\varrho(x)} |T_\xi\mu - T| \,\mathrm{d}\mu(\xi) > \varepsilon_0\, \mu(\bar{B}_\varrho(x)) \quad \text{for some } 0 < \varrho < 2.$$

Clearly, Besicovitch's covering theorem implies

$$\mu(B_2) \leq N(n+m)3^n T_1(\mu, 0, 3, h+2, T) \leq N(n+m)3^n\varepsilon.$$

Moreover, $B_1 = \emptyset$ if $n = 1$, and Besicovitch's covering theorem implies in case $n > 1$

$$\mu(B_1) \leq N(n+m)\, \|\delta\mu\|(C(T,0,3,h+2))^{n/(n-1)} \leq N(n+m)\varepsilon^{n/(n-1)}.$$

Therefore the desired estimate is implied by 1.14 (3) and the choice of ε.

To prove part (1), let $1 \leq q < n$, $q^* = \frac{nq}{n-q}$, define

$$\Gamma_2 = 1 + (12)^{n+1} Q \max\{1, \Gamma_{1.14(6)}(q^*, n)\},$$
$$\Gamma_3 = 2\Gamma_{D.14(1)}(m,n,Q,q), \quad \Gamma_4 = N(n+m)^{1/q}\varepsilon^{-1}3^{n/q},$$
$$\Gamma_5 = 2^{1/2} Q m^{1/2}, \quad \Gamma_6 = \Gamma_0 m^{1/2} Q^{1/2},$$

choose $S \in Q_Q(\mathbb{R}^m)$ such that (see D.13)

$$h_{q^*}(g,S) \leq \Gamma_3\, t_q(g), \quad \operatorname{spt} S \subset \bar{B}^m_h(0)$$

with the help of D.14 (1) noting again [Fed69, 4.1.16] and denote by $P := (\theta^0(\|S\|,\cdot) \circ \sigma)\mathcal{H}^n$ the Q valued plane associated to S via σ. The estimate for $H_{q^*}(\mu \llcorner G, 0, 1, h, P)$ is obtained by combining the following six inequalities:

$$H_{q^*}(\mu \llcorner G, 0, 1, h, P) \leq \Gamma_2 \big(h_{q^*}(g,S) + \mathcal{L}^n(\bar{B}^n_1(0) \sim Y)^{1/q} \big), \tag{I}$$
$$h_{q^*}(g,S) \leq \Gamma_3\, t_q(g), \tag{II}$$
$$\mathcal{L}^n(\bar{B}^n_1(0) \sim Y)^{1/q} \leq (\Gamma_1)^{1/q}\, \mu(B)^{1/q}, \tag{III}$$
$$\mu(B \cap A)^{1/q} \leq \Gamma_4\, T_q(\mu, 0, 3, h+2, T), \tag{IV}$$
$$t_q(g|Y) \leq \Gamma_5\, T_q(\mu, 0, 1, h, T), \tag{V}$$
$$t_q(g|\bar{B}^n_1(0) \sim Y) \leq \Gamma_6\, \mathcal{L}^n(\bar{B}^n_1(0) \sim Y)^{1/q}. \tag{VI}$$

(I) is implied by 1.14 (2) (4) (6) and $\operatorname{spt} S \subset \bar{B}^m_h(0)$, (II) is implied by the choice of S, (III) is implied by 1.14 (3), (VI) is elementary (cf. D.2). To prove (IV), note that for every $x \in B \cap A$ there exists $0 < \varrho < 2$ such that

$$\varepsilon_0\, \mu(\bar{B}_\varrho(x)) < \textstyle\int_{\bar{B}_\varrho(x)} |T_\xi\mu - T| \,\mathrm{d}\mu(\xi),$$

hence by Hölder's inequality

$$(\varepsilon_0)^q\, \mu(\bar{B}_\varrho(x)) < \textstyle\int_{\bar{B}_\varrho(x)} |T_\xi\mu - T|^q \,\mathrm{d}\mu(\xi)$$

and Besicovitch's covering theorem implies (IV). Observing that

$$\{y \in Y : |\operatorname{ap} Ag(y)| > \gamma\} \sim \pi\big(\{\xi \in G \cap \pi^{-1}(Y) : |T_\xi\mu - T| > \gamma/\Gamma_5\}\big)$$

has $\mathcal{L}^n$ measure 0 by 1.14 (7d) and D.2, inequality (V) is a consequence of

$$\begin{aligned} &\mathcal{L}^n(\{y \in Y : |\operatorname{ap} Ag(y)| > \gamma\}) \\ &\leq \mathcal{H}^n(\{\xi \in G \cap \pi^{-1}(Y) : |T_\xi\mu - T| > \gamma/\Gamma_5\}) \\ &\leq \mu(\{\xi \in G \cap \pi^{-1}(Y) : |T_\xi\mu - T| > \gamma/\Gamma_5\}). \end{aligned}$$

The proof of part (2) exactly parallels the proof of part (1) with ∞, q, and D.14 (2) replacing q^*, q, and D.14 (1). ☐

2.5 Remark. Part (2) can be sharpened using Lorentz spaces to

$$\begin{aligned} &H_\infty(\mu \llcorner G, a, r, h, Q, T) \\ &\leq \Gamma\big(T_{n,1}(\mu, a, 3r, h+2r, T) + (r^{-n}\mu(C(T,a,r,h) \sim A))^{1/n}\big) \end{aligned}$$

with a positive, finite number Γ depending only on m, n, Q, M, and δ. Here $T_{n,1}$ is the obvious generalisation of T_q to Lorenz spaces.

A similar improvement is possible for part (1) using Peetre's theorem.

2.6 Theorem. *Suppose $m, n, Q \in \mathbb{N}$, $1 \leq M < \infty$, $0 < \delta \leq 1$, $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$, $T \in G(n+m, n)$, $1 \leq p \leq n$, μ is an integral n varifold in an open superset of $C(T, a, 3r, 3r)$ satisfying (H_p) and*

$$\begin{gathered} \psi = \|\delta\mu\| \quad \text{if } p = 1, \quad \psi = |\vec{\mathbf{H}}_\mu|^p\mu \quad \text{if } p > 1, \\ (Q-1+\delta)\omega_n r^n \leq \mu(C(T,a,r,r)) \leq (Q+1-\delta)\omega_n r^n, \\ \mu(C(T,a,r,(1+\delta)r) \sim C(T,a,r,(1-\delta)r)) \leq (1-\delta)\omega_n r^n, \\ \mu(C(T,a,3r,3r)) \leq M\omega_n r^n. \end{gathered}$$

Then the following two statements hold:

(1) *If $p < n$, $1 \leq q < n$, then*

$$\begin{aligned} &H_{\frac{nq}{n-q}}(\mu, a, r, r, Q, T) \\ &\qquad \leq \Gamma_{(1)}\big(T_q(\mu, a, 3r, 3r, T) + (r^{p-n}\psi(C(T,a,3r,3r)))^{\frac{n-q}{q(n-p)}}\big) \end{aligned}$$

where $\Gamma_{(1)}$ is a positive, finite number depending only on m, n, Q, M, δ, p, and q.

(2) *If $p = n$ and $\psi(C(T,a,3r,3r)) \leq \varepsilon_{(2)}$ where $\varepsilon_{(2)}$ is a positive, finite number depending only on m, n, Q, M, and δ, then*

  (a) *$H_{\frac{nq}{n-q}}(\mu, a, r, r, Q, T) \leq \Gamma_{(2a)}\, T_q(\mu, a, 3r, 3r, T)$ whenever $1 \leq q < n$,*

  (b) *$H_\infty(\mu, a, r, r, Q, T) \leq \Gamma_{(2b)}\, T_q(\mu, a, 3r, 3r, T)$ whenever $n < q \leq \infty$*

where $\Gamma_{(2a)}$, $\Gamma_{(2b)}$ are positive, finite numbers depending only on m, n, Q, M, δ, and q.

Proof. To prove part (1), assume $a = 0$, $r = 1$, define $q^* = \frac{nq}{n-q}$, and suppose that ε, A, and G are as in 2.4. Observing

$$\begin{aligned}
H_{q^*}(\mu, 0, 1, 1, Q, T) - H_{q^*}(\mu \llcorner G, 0, 1, 1, Q, T) &\leq 2\mu(C(T, 0, 1, 1) \sim G)^{1/q^*} \\
&\quad + \mathcal{H}^n(T(\{\xi \in C(T, 0, 1, 1) : \theta^{*n}(\mu, \xi) > 0\} \sim G))^{1/q} \\
&\leq (2 + \omega_n^{1/n})\mu(C(T, 0, 1, 1) \sim G)^{1/q^*}, \\
\mu(C(T, 0, 1, 1) \sim G) &\leq N(n + m)(2\gamma_n)^{\frac{np}{n-p}} \psi(C(T, 0, 3, 3))^{\frac{n}{n-p}}, \\
\mu(C(T, 0, 1, 1) \sim A) &\leq N(n + m)\varepsilon^{-\frac{np}{n-p}} \psi(C(T, 0, 3, 3))^{\frac{n}{n-p}}, \\
\|\delta\mu\|(C(T, 0, 3, 3)) &\leq \mu(C(T, 0, 3, 3))^{1-1/p}\psi(C(T, 0, 3, 3))^{1/p} \\
&\leq (M\omega_n)^{1-1/p}\psi(C(T, 0, 3, 3))^{1/p}, \\
T_1(\mu, 0, 3, 3, T) &\leq 3^{-n+n/q}(M\omega_n)^{1-1/q}T_q(\mu, 0, 3, 3, T), \\
H_{q^*}(\mu, 0, 1, 1, Q, T) &\leq \mu(C(T, 0, 1, 1))^{1/q^*} + \omega_n^{1/q} \leq M^{1/q^*}\omega_n^{1/q^*} + \omega_n^{1/q},
\end{aligned}$$

a suitable number $\Gamma_{(1)}$ is readily constructed using 2.4 (1).

Part (2) is proved similarly using 2.4 (2). $\square$

2.7 Remark. In case μ additionally satisfies

$$\mu(\{x \in C(T, a, r, r) : \theta^n(\mu, x) = Q\}) \geq \delta\omega_n r^n,$$

there exists $z \in T^\perp$ such that for $P := Q\mathcal{H}^m \llcorner \{x \in \mathbb{R}^{n+m} : T^\perp(x) = z\}$

$$H_{\frac{nq}{n-q}}(\mu, a, r, r, P) \leq \Gamma\big(T_q(\mu, a, 3r, 3r, T) + (r^{p-n}\psi(C(T, a, 3r, 3r)))^{\frac{n-q}{q(n-p)}}\big)$$

provided $p < n$, $1 \leq q < n$ where Γ is a positive, finite number depending only on m, n, Q, M, δ, p, and q.

In fact from 1.14 (2) (3) and the coarea formula [Fed69, 3.2.22 (3)] one obtains for the set Y_0 of all $y \in T \cap B_r(T(a))$ such that for some $x_0 \in C(T, a, r, r)$ with $T(x_0) = y$

$$\theta^n(\mu, x_0) = Q, \qquad \theta^n(\mu, x) = 0 \quad \text{for } x \in T^{-1}(\{y\}) \cap C(T, a, r, r) \sim \{x_0\}$$

the estimate

$$\mathcal{L}^1(Y_0) \geq (2\delta/3)\omega_n r^n$$

provided the right hand side of the inequality in question is suitably small (depending only on m, n, Q, M, δ, p, and q), hence for any Q valued plane P' parallel to T such that

$$(2H_{\frac{nq}{n-q}}(\mu, a, r, r, P'))^q \leq (\delta/3)\omega_n$$

there holds

$$((\delta/3)\omega_n)^{1/q-1/n}\frac{\operatorname{diam} T^\perp(\operatorname{spt} P')}{2r} \leq 2H_{\frac{nq}{n-q}}(\mu, a, r, r, P')$$

and suitable z and Γ are readily constructed.

A similar remark holds for the second part.

2.8 Theorem. *Suppose $m, n, Q \in \mathbb{N}$, $0 < \alpha \leq 1$, $1 \leq p \leq n$, U is an open subset of $\mathbb{R}^{n+m}$, and μ is an integral n varifold in U satisfying (H_p).*

Then the following two statements hold:

(1) *If $p < n$, $1 \leq q_1 < n$, $1 \leq q_2 \leq \min\{\frac{nq_1}{n-q_1}, \frac{1}{\alpha} \cdot \frac{np}{n-p}\}$, then for μ almost all $a \in U$ with $\theta^n(\mu, a) = Q$ there holds*

$$\begin{aligned}&\limsup_{r \downarrow 0} r^{-\alpha-1-n/q_2} \| \operatorname{dist}(\cdot - a, T_a\mu)\|_{L^{q_2}(\mu \llcorner B_r(a))} \\ &\leq \Gamma_{(1)} \limsup_{r \downarrow 0} r^{-\alpha-n/q_1} \|T_\mu - T_a\mu\|_{L^{q_1}(\mu \llcorner B_r(a))}\end{aligned}$$

where $\Gamma_{(1)}$ is a positive, finite number depending only on m, n, Q, q_1, and q_2.

(2) *If $p = n$, $n < q \leq \infty$, then for μ almost all $a \in U$ with $\theta^n(\mu, a) = Q$ there holds*

$$\begin{aligned}&\limsup_{r \downarrow 0} r^{-\alpha-1} \| \operatorname{dist}(\cdot - a, T_a\mu)\|_{L^{\infty}(\mu \llcorner B_r(a))} \\ &\leq \Gamma_{(2)} \limsup_{r \downarrow 0} r^{-\alpha-n/q} \|T_\mu - T_a\mu\|_{L^{q}(\mu \llcorner B_r(a))}\end{aligned}$$

where $\Gamma_{(2)}$ is a positive, finite number depending only on m, n, Q, and q.

Proof. For $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$ such that $B_{7r}(a) \subset U$ denote by $G_r(a)$ the set of all $x \in \bar{B}_{5r}(a) \cap \operatorname{spt} \mu$ satisfying

$$\|\delta\mu\|(\bar{B}_\varrho(x)) \leq (2\gamma_n)^{-1} \mu(\bar{B}_\varrho(x))^{1-1/n} \quad \text{whenever } 0 < \varrho < 2r.$$

To prove (1), one may assume first that $q_2 \geq \frac{n}{n-1}$ possibly replacing q_2 by a larger number since $\min\{\frac{nq_1}{n-q_1}, \frac{1}{\alpha} \cdot \frac{np}{n-p}\} \geq \frac{n}{n-1}$, and thus also that $q_2 = \frac{nq_1}{n-q_1}$ possibly replacing q_1 by a smaller number. Define $M = 6^n Q$, $\delta = 1/2$, $q = q_1$, $q^* = q_2$,

$$\varepsilon = \min\{\varepsilon_{2.4}(m, n, Q, M, \delta), (2\gamma_n)^{-1}\}, \quad \Gamma = \Gamma_{2.4(1)}(m, n, Q, M, \delta, q).$$

Denote by C_i for $i \in \mathbb{N}$ the set of all $x \in \operatorname{spt} \mu$ such that $B_{1/i}(x) \subset U$ and

$$\|\delta\mu\|(\bar{B}_\varrho(x)) \leq \varepsilon\, \mu(\bar{B}_\varrho(x))^{1-1/n} \quad \text{whenever } 0 < \varrho < 1/i.$$

The conclusion will be shown for $a \in \operatorname{dmn} T_\mu$ such that

$$\begin{gathered}\theta^n(\mu, a) = Q, \quad \theta^{n-1}(\|\delta\mu\|, a) = 0, \\ \lim_{r \downarrow 0} r^{-n^2/(n-p)} \mu(\bar{B}_r(x) \sim C_i) = 0 \quad \text{for some } i \in \mathbb{N}.\end{gathered}$$

Note that according to [Fed69, 2.9.5] and A.12 (1) (4) (5) with s replaced by n this is true for μ almost all $a \in U$ with $\theta^n(\mu, a) = Q$, fix such a, i, and abbreviate $T := T_a\mu$.

For a there holds

$$\begin{gathered}\lim_{r \downarrow 0} \frac{\mu(C(T, a, r, r))}{\omega_n r^n} = Q, \\ \lim_{r \downarrow 0} \frac{\mu(C(T, a, r, 3r/2) \sim C(T, a, r, r/2))}{\omega_n r^n} = 0\end{gathered}$$

and one can assume for some $0 < \gamma < \infty$

$$\limsup_{r \downarrow 0} r^{-\alpha} T_q(\mu, a, r, r, T) < \gamma.$$

Choose $0 < s < \min\{(2i)^{-1}, \operatorname{dist}(a, \mathbb{R}^{n+m} \sim U)/7\}$ so small that for $0 < \varrho < s$

$$\begin{gathered}
(Q - 1/2)\omega_n \varrho^n \leq \mu(C(T, a, \varrho, \varrho)) \leq (Q + 1/2)\omega_n \varrho^n, \\
\mu(C(T, a, \varrho, 3\varrho/2) \sim C(T, a, \varrho, \varrho/2)) \leq (1/2)\omega_n \varrho^n, \\
\mu(C(T, a, 3\varrho, 3\varrho)) \leq \mu(\bar{B}_{5\varrho}(a)) \leq \omega_n 6^n Q \varrho^n, \\
\|\delta\mu\|(C(T, a, 3\varrho, 3\varrho)) \leq \varepsilon \varrho^{n-1}, \quad T_1(\mu, a, 3\varrho, 3\varrho, T) \leq \varepsilon, \\
T_q(\mu, a, 3\varrho, 3\varrho, T) + (\varrho^{-n} \mu(C(T, a, \varrho, \varrho) \sim C_i))^{1/q} \leq 4\gamma \varrho^\alpha;
\end{gathered}$$

in particular 2.4 (1) can be applied to any such ϱ with (r, h) replaced by (ϱ, ϱ). For each $0 < \varrho < s$ use 2.3 to choose Q valued planes P_ϱ parallel to T such that

$$H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, P_\varrho) \leq 2H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, Q, T),$$

denote by A_ϱ the $\mathcal{H}^n$ measurable sets of all $x \in T \cap \bar{B}_\varrho(T(a))$ such that for some $R_\varrho(x), S_\varrho(x) \in Q_Q(\mathbb{R}^{n+m})$

$$\begin{aligned}
\|R_\varrho(x)\| &= \theta^n(P_\varrho \llcorner C(T, a, \varrho, \varrho), \cdot)\, \mathcal{H}^0 \llcorner T^{-1}(\{x\}), \\
\|S_\varrho(x)\| &= \theta^n(\mu \llcorner G_\varrho(a) \cap C(T, a, \varrho, \varrho), \cdot)\, \mathcal{H}^0 \llcorner T^{-1}(\{x\}),
\end{aligned}$$

and by $g_\varrho : A_\varrho \to \mathbb{R}$ the $\mathcal{H}^n$ measurable functions defined by

$$g_\varrho(x) = \mathcal{G}(R_\varrho(x), S_\varrho(x)) \quad \text{for } x \in A_\varrho.$$

By 2.3 there exist $\mathcal{H}^n$ measurable subset Y_ϱ of A_ϱ such that

$$\begin{aligned}
2H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, P_\varrho) &\geq \varrho^{-n/q} \| \operatorname{dist}(\cdot, \operatorname{spt} P_\varrho)\|_{L^{q^*}(\mu \llcorner G_\varrho(a) \cap C(T, a, \varrho, \varrho))} \\
&+ \varrho^{-n/q} \|g_\varrho\|_{L^{q^*}(\mathcal{H}^n \llcorner Y_\varrho)} + \varrho^{-n/q} \mathcal{H}^n(T \cap \bar{B}_\varrho(T(a)) \sim Y_\varrho)^{1/q}.
\end{aligned}$$

Possibly replacing s by a smaller number, one may assume for $0 < \varrho < s$ that

$$(2H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, P_\varrho))^q \leq 2^{-n-2}\omega_n$$

by 2.4 (1) and also that

$$\mu(C(T, a, \varrho, \varrho) \sim C_i) \leq 2^{-n-2} \omega_n \varrho^n.$$

Noting $C_i \cap C(T, a, \varrho/2, \varrho) \subset G_\varrho(a) \cap G_{\varrho/2}(a)$, one obtains directly from the additional assumptions on s that

$$\begin{aligned}
&\mathcal{H}^n(T \cap \bar{B}_\varrho(T(a)) \sim Y_\varrho) \leq 2^{-n-2} \omega_n \varrho^n, \\
&\mathcal{H}^n(T \cap \bar{B}_{\varrho/2}(T(a)) \sim Y_{\varrho/2})\} \leq 2^{-n-2} \omega_n \varrho^n, \\
&\mathcal{H}^n(\{x \in Y_{\varrho/2} \cap Y_\varrho : S_\varrho(x) \neq S_{\varrho/2}(x)\}) \\
&\leq \mathcal{H}^n\big(T(\{x \in C(T, a, \varrho/2, \varrho) : \theta^{*n}(\mu, x) \geq 1\} \sim C_i)\big) \\
&\leq \mu(C(T, a, \varrho, \varrho) \sim C_i) \leq 2^{-n-2} \omega_n \varrho^n,
\end{aligned}$$

hence for $B_\varrho := Y_\varrho \cap Y_{\varrho/2} \cap \{x : S_\varrho(x) = S_{\varrho/2}(x)\}$

$$\mathcal{H}^n(B_\varrho) \geq \tfrac{1}{4}\omega_n(\varrho/2)^n \quad \text{for } 0 < \varrho < s,$$

in particular

$$\operatorname{dmn} R_\varrho = A_\varrho \supset Y_\varrho \supset B_\varrho \neq \emptyset, \quad \mathcal{G}(P_\varrho, Q\mathcal{H}^n \llcorner T) \leq Q^{1/2}\varrho.$$

By integration over the set B_ϱ one obtains

$$\begin{aligned}
&(\tfrac{1}{4}\omega_n(\varrho/2)^n)^{1/q-1/n}\mathcal{G}(P_\varrho, P_{\varrho/2}) \\
&\leq \|g_\varrho\|_{L^{q^*}(\mathcal{H}^n \llcorner Y_\varrho)} + \|g_{\varrho/2}\|_{L^{q^*}(\mathcal{H}^n \llcorner Y_{\varrho/2})} \\
&\leq \varrho^{n/q}4\big(H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, Q, T) + H_{q^*}(\mu \llcorner G_{\varrho/2}(a), \varrho/2, \varrho/2, Q, T)\big)
\end{aligned}$$

for $0 < \varrho < s$. Therefore 2.4 (1) implies

$$\mathcal{G}(P_\varrho, P_{\varrho/2}) \leq \Gamma_1\gamma\varrho^{1+\alpha}$$

where $\Gamma_1 = 2^{3+n/q+2/q-2/n}\omega_n^{1/n-1/q}\Gamma$, hence

$$\mathcal{G}(Q\mathcal{H}^n \llcorner T, P_\varrho) \leq \textstyle\sum_{i=0}^\infty \mathcal{G}(P_{2^{-i}\varrho}, P_{2^{-i-1}\varrho}) \leq 2\Gamma_1\gamma\varrho^{1+\alpha}$$

because $\mathcal{G}(P_\varrho, Q\mathcal{H}^n \llcorner T) \to 0$ as $\varrho \downarrow 0$. From the definition of the q^* height of μ in $C(T, a, \varrho, \varrho)$ one obtains

$$\begin{aligned}
&H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, Q\mathcal{H}^n \llcorner T) - H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, P_\varrho) \\
&\leq \varrho^{-n/q}\big(\mu(C(T, a, \varrho, \varrho))^{1/q^*} + \mathcal{H}^n(Y_\varrho)^{1/q^*}\big)\mathcal{G}(Q\mathcal{H}^n \llcorner T, P_\varrho) \leq \Gamma_2\gamma\varrho^\alpha
\end{aligned}$$

for $0 < \varrho < s$ where $\Gamma_2 = \omega_n^{1/q^*}2(Q+1)^{1/q^*}2\Gamma_1$, hence

$$\begin{aligned}
&\limsup_{\varrho \downarrow 0} \varrho^{-\alpha} H_{q^*}(\mu \llcorner G_\varrho(a), a, \varrho, \varrho, Q\mathcal{H}^n \llcorner T) \\
&\qquad\qquad \leq (8\Gamma + \Gamma_2) \limsup_{\varrho \downarrow 0} \varrho^{-\alpha} T_q(\mu, a, \varrho, \varrho, T)
\end{aligned}$$

by 2.4 (1). Combining this with the fact that

$$\lim_{\varrho \downarrow 0} \varrho^{-\alpha-1-n/q^*} \|\operatorname{dist}(\cdot - a, T_a\mu)\|_{L^{q^*}(\mu \llcorner B_\varrho(a) \sim G_\varrho(a))} = 0,$$

the conclusion follows.

(2) may be proved by a similar argument using 2.4 (2) and A.9 instead of 2.4 (1) and A.12 (1). $\square$

2.9 Remark. As in 2.5, in (2) the L^q norm can be replaced by $L^{n,1}$, in particular $n = q = 1$ is admissible. The latter fact can be derived without the use of Lorentz spaces, of course.

2.10 Remark. If $1 \leq p < n$, $1 \leq q_1 \leq q_2 < \infty$, $\frac{1}{\alpha} \cdot \frac{np}{n-p} < q_2$, then the conclusion of (1) fails for some μ; in fact one can assume $q_1 = q_2$ possibly enlarging q_1 and then take $\alpha_2 = \alpha$ and α_1 slightly larger than α_2 in C.2. Clearly, also in (2) the assumption $p = n$ cannot be weakened.

3 About the significance of the 1 tilt

The purpose of this section is to show the significance of the 1 tilt introduced in the preceding section for $\mathcal{C}^2$ rectifiability and locality of mean curvature of integral varifolds. Firstly, decay of the 1 tilt implies by 2.8 decay of the 1 height – the leading integral quantity for $\mathcal{C}^2$ rectifiability. Secondly, concerning locality of mean curvature, using the differentiation theorem B.1 an adaption of [Sch01, Lemma 6.3] (or [Sch04a, Prop. 6.1] or [Sch04b, Theorem 4.1]) yields the desired result.

3.1 Lemma. *Suppose $m, n \in \mathbb{N}$, $0 < \alpha \leq 1$, U is an open subset of $\mathbb{R}^{n+m}$, μ is a rectifiable n varifold in U, and A denotes the set of all $x \in U$ such that $T_x\mu$ exists and*

$$\limsup_{\varrho \downarrow 0} \varrho^{-n-1-\alpha} \int_{B_\varrho(x)} \operatorname{dist}(\xi - x, T_x\mu) \, d\mu(\xi) < \infty.$$

Then $\mu \llcorner A$ is countably rectifiable of class $\mathcal{C}^{1,\alpha}$.

Proof. The proof mainly requires an extension of the results in [Sch04b, Appendix A] to the case $\alpha < 1$ which may be accomplished using [Ste70, VI.2.2.2, VI.2.3.1–3]. □

3.2 Lemma. *Suppose $m, n \in \mathbb{N}$, U is an open subset of $\mathbb{R}^{n+m}$, μ is an integral n varifold in U of locally bounded first variation, and A denotes the set of all $x \in U$ such that $T_x\mu$ exists and*

$$\limsup_{\varrho \downarrow 0} \varrho^{-1-n} \int_{B_\varrho(x)} |T_\xi\mu - T_x\mu| \, d\mu(\xi) < \infty.$$

Then $\mu \llcorner A$ is countably rectifiable of class $\mathcal{C}^2$ and for every n dimensional submanifold M of $\mathbb{R}^{n+m}$ of class $\mathcal{C}^2$ there holds

$$\vec{\mathbf{H}}_\mu(x) = \vec{\mathbf{H}}_M(x) \quad \text{for } \mu \text{ almost every } x \in A \cap M$$

where $\vec{\mathbf{H}}_M$ denotes the mean curvature of M and $-\vec{\mathbf{H}}_\mu$ corresponds to the absolutely continuous part of $\delta\mu$ with respect to μ.

Proof. The first part follows from 2.8 and 3.1.

Suppose now M is a n dimensional submanifold of $\mathbb{R}^{n+m}$ of class $\mathcal{C}^2$. Since the conclusion is local, one may assume the existence of an orthogonal frame adapted to M in U, i.e. of $\tau_i : U \to \mathbb{R}^{n+m}$, $\nu_j : U \to \mathbb{R}^{n+m}$, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$ of class $\mathcal{C}^1$ such that $\tau_1(x), \ldots, \tau_n(x), \nu_1(x), \ldots, \nu_m(x)$ form an orthonormal base of $\mathbb{R}^{n+m}$ whenever $x \in U$ and such that $\tau_1(x), \ldots, \tau_n(x)$ form an orthonormal base of T_xM whenever $x \in U \cap M$. Define $S : U \to \operatorname{Hom}(\mathbb{R}^{n+m}, \mathbb{R}^{n+m})$, $H : U \to \mathbb{R}^{n+m}$ by

$$S(x)(v) = \sum_{i=1}^{n} (\tau_i(x) \bullet v)\tau_i(x) \quad \text{for } x \in U,\ v \in \mathbb{R}^{n+m},$$

$$H(x) = -\sum_{i=1}^{n} \sum_{j=1}^{m} \partial_{\tau_i(x)}\nu_j(x) \bullet \tau_i(x)\nu_j(x) \quad \text{for } x \in U;$$

here $\partial_v f(x)$ denotes the directional derivative of f at x with direction v. Clearly, $T_x\mu = S(x)$ for μ almost every $x \in A \cap M$. Hence, since S is of class $\mathcal{C}^1$,

$$\limsup_{\varrho \downarrow 0} \varrho^{-n-1} \textstyle\int_{\bar{B}_\varrho(x)} |T_\xi \mu - S(\xi)| \,\mathrm{d}\mu(\xi) < \infty \quad \text{for } \mu \text{ almost every } x \in A \cap M.$$

By B.1 applied with $\nu = |T_\mu - S| \mu \llcorner (A \cap M)$, this implies

$$\lim_{\varrho \downarrow 0} \varrho^{-n-1} \textstyle\int_{\bar{B}_\varrho(x)} |T_\xi \mu - S(\xi)| \,\mathrm{d}\mu(\xi) = 0 \quad \text{for } \mu \text{ almost every } x \in A \cap M.$$

The conclusion will be shown for a point $x \in A \cap M$ which additionally satisfies

$$T_x\mu = S(x), \quad \theta^{*n}(\|\delta\mu\|, x) < \infty, \quad \vec{\mathbf{H}}_\mu(x) \in (T_x\mu)^\perp,$$
$$\lim_{\varrho \downarrow 0} \varrho^{-n} \textstyle\int \eta(\varrho^{-1}(y-x)) \,\mathrm{d}\delta\mu(y) \to -\theta^n(\mu, x) \big(\textstyle\int \eta \,\mathrm{d}\mathcal{H}^n \llcorner T_x\mu \big) \bullet \vec{\mathbf{H}}_\mu(x)$$

as $\varrho \downarrow 0$ whenever $\eta \in C^0_{\mathrm{c}}(\mathbb{R}^{n+m}, \mathbb{R}^{n+m})$, as μ almost all do according to [Fed69, 2.9.5, 2.9.9, 2.9.10] and [Bra78, Theorem 5.8].

Let $v \in (T_x\mu)^\perp$, $\varphi \in C^1_c(\mathbb{R}^{n+m})$ such that $\int \varphi \,\mathrm{d}\mathcal{H}^n \llcorner T_x\mu \neq 0$, and define $\eta_\varrho : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$

$$\eta_\varrho(y) = \varphi(\varrho^{-1}(y-x))(v - S_0(y)(v)) \quad \text{for } y \in \mathbb{R}^{n+m},\ 0 < \varrho < \infty$$

where $S_0 : \mathbb{R}^{n+m} \to \mathrm{Hom}(\mathbb{R}^{n+m}, \mathbb{R}^{n+m})$ is a function of class $\mathcal{C}^1$ agreeing with S in an open neighbourhood V of x. One computes, noting $S(x)(v) = 0$,

$$\begin{aligned}
\lim_{\varrho \downarrow 0} \varrho^{-n} (\delta\mu)(\eta_\varrho) &= \lim_{\varrho \downarrow 0} \textstyle\int \varphi(\varrho^{-1}(y-x)) v \,\mathrm{d}\delta\mu(y) \\
&= -\theta^n(\mu, x) \big(\textstyle\int \varphi \,\mathrm{d}\mathcal{H}^n \llcorner T_x\mu \big) v \bullet \vec{\mathbf{H}}_\mu(x), \\
(\delta\mu)(\eta_\varrho) &= \textstyle\int T_\mu \bullet D\eta_\varrho \,\mathrm{d}\mu \quad \text{for } 0 < \varrho < \infty, \\
\lim_{\varrho \downarrow 0} \varrho^{-n} &\textstyle\int (T_\mu - S_0) \bullet D\eta_\varrho \,\mathrm{d}\mu = 0,
\end{aligned}$$

as well as

$$\begin{aligned}
S(x) \bullet D\eta_\varrho(x) &= \sum_{i=1}^n \sum_{j=1}^m \partial_{\tau_i(x)} (\nu_j \bullet \eta_\varrho \nu_j)(x) \bullet \tau_i(x) \\
= \sum_{i=1}^n \sum_{j=1}^m &(\nu_j \bullet \eta_\varrho)(x) \partial_{\tau_i(x)} \nu_j(x) \bullet \tau_i(x) = -H(x) \bullet \eta_\varrho(x)
\end{aligned}$$

whenever $x \in V$, $0 < \varrho < \infty$,

$$\varrho^{-n} \textstyle\int S_0 \bullet D\eta_\varrho \,\mathrm{d}\mu = -\varrho^{-n} \textstyle\int H \bullet \eta_\varrho \,\mathrm{d}\mu \to -\theta^n(\mu, x) \big(\textstyle\int \varphi \,\mathrm{d}\mathcal{H}^n \llcorner T_x\mu \big) v \bullet \vec{\mathbf{H}}_M(x)$$

as $\varrho \downarrow 0$. Recalling $\vec{\mathbf{H}}_M(x), \vec{\mathbf{H}}_\mu(x) \in (T_x\mu)^\perp = (T_xM)^\perp$, one finally obtains

$$\vec{\mathbf{H}}_M(x) = \vec{\mathbf{H}}_\mu(x)$$

as claimed. □

3.3 Remark. The method of [Sch01, Lemma 6.3] (or [Sch04a, Prop. 6.1] or [Sch04b, Theorem 4.1]) suffices to prove the locality of mean curvature in case the 1 tilt is replaced by the q tilt for some $1 < q < \infty$. The differentiation theorem B.1 is used to extend this method to the case $q = 1$.

3.4 Remark. It is not known to the author if $\mu(U \sim A) > 0$ for some μ.

A The Isoperimetric Inequality and its applications

In this appendix the connections between smallness assumptions on the mean curvature and lower bounds for density ratios are investigated by use of the Isoperimetric Inequality of Michael and Simon, see e.g. [Sim83, Theorem 18.6]. Despite the fact that it is well known how to derive such lower bounds by use of an isoperimetric inequality (see [Fed69, 5.1.6] for currents and [All72, 8.3] for varifolds), this method appears to be only rarely used in literature in the present case. Since the sharpness of the results in this work depends crucially on a seemingly slight weakening (see A.8) of the assumptions used in [All72, 8.3], the author felt obliged to briefly document the deduction of the lower density bound, see A.6–A.8. The main theorem concerning the size of various sets where the mean curvature is large is then derived in A.12.

A.1. In this appendix the following situation will be studied:

$m, n \in \mathbb{N}$, $1 \leq p \leq n$, U is an open subset of $\mathbb{R}^{n+m}$, μ is a rectifiable n varifold in U of locally bounded first variation, $\theta^n(\mu, x) \geq 1$ for μ almost all $x \in U$, and, in case $p > 1$,

$$(\delta\mu)(\eta) = -\textstyle\int \vec{\mathbf{H}}_\mu \bullet \eta \, \mathrm{d}\mu \quad \text{whenever } \eta \in C^1_{\mathrm{c}}(U, \mathbb{R}^{n+m})$$

for some $\vec{\mathbf{H}}_\mu \in L^p_{\mathrm{loc}}(\mu, \mathbb{R}^{n+m})$.

In doing so, the following abbreviation will be used:

$$\psi = \|\delta\mu\| \quad \text{if } p = 1, \quad \psi = |\vec{\mathbf{H}}_\mu|^p \mu \quad \text{else.}$$

A.2 Theorem. *Suppose $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, μ is a rectifiable n varifold in $\mathbb{R}^{n+m}$ such that $\mu(\mathbb{R}^{n+m}) < \infty$ and $\|\delta\mu\|(\mathbb{R}^{n+m}) < \infty$.*

Then for some positive, finite number γ depending only on n

$$\mu\big(\big\{x \in \mathbb{R}^{n+m} : \theta^n(\mu, x) \geq 1\big\}\big) \leq \gamma \, \mu(\mathbb{R}^{n+m})^{1/n} \|\delta\mu\|(\mathbb{R}^{n+m}).$$

Proof. This follows from [All72, Theorem 7.1] with a constant γ depending on $n + m$ (what would be sufficient for the purpose of this work). A slight modification of [Sim83, Lemma 18.7, Theorem 18.6] yields the stated result. □

A.3 Definition. For $n \in \mathbb{N}$ let γ_n denote the best constant γ in A.2.

A.4 Remark. Taking $m = 0$, $\mu = \mathcal{L}^n \llcorner \bar{B}^n_1(0)$ yields

$$\gamma_n \geq \omega_n^{-1/n}/n.$$

Does equality hold?

A.5 Definition. Suppose $k, m, n \in \mathbb{N}$, $k \leq n$, U is an open subset of $\mathbb{R}^{n+m}$, and $f : U \to \mathbb{R}^k$ is a Lipschitzian function.

Then for $x \in \mathbb{R}^k$ the Radon measure $\langle \mu, f, x \rangle$ is defined by

$$\langle \mu, f, x \rangle (\varphi) := \lim_{\varrho \downarrow 0} \omega_k^{-1} \varrho^{-k} \textstyle\int_{f^{-1}(\bar{B}_\varrho(x))} (J^\mu f) \varphi \, \mathrm{d}\mu \quad \text{for } \varphi \in C^0_{\mathrm{c}}(U)$$

whenever this limit exists for all $\varphi \in C^0_{\mathrm{c}}(U)$; here J^μ denotes the Jacobian with respect to μ.

A.6 Lemma. *Suppose $k, m, n \in \mathbb{N}$, $k \leq n$, U is an open subset of $\mathbb{R}^{n+m}$, μ is a rectifiable n varifold in U with $\mu(U) < \infty$, and $f : U \to \mathbb{R}^k$ is a Lipschitzian function.*

Then the following three statements hold:

(1) *For $\mathcal{L}^k$ almost all $x \in \mathbb{R}^k$ the slice $\langle \mu, f, x \rangle$ exists and satisfies*

$$\langle \mu, f, x \rangle = \theta^n(\mu, \cdot)\mathcal{H}^{n-k} \llcorner f^{-1}(\{x\}).$$

(2) *Whenever $k = 1$, $-\infty < a < b < \infty$*

$$\textstyle\int_a^b \langle \mu, f, t \rangle (U) \, \mathrm{d}\mathcal{L}^1 t \leq (\operatorname{Lip} f) \, \mu(\{x \in U : a < f(x) < b\}).$$

(3) *If $k = 1$ and μ is of locally bounded first variation, then there holds for $\mathcal{L}^1$ almost all $t \in \mathbb{R}$*

$$\delta(\mu \llcorner A_t)(\eta) = ((\delta\mu) \llcorner A_t)(\eta) + \textstyle\int \eta \bullet \dfrac{\nabla^\mu f}{|\nabla^\mu f|} \, \mathrm{d} \langle \mu, f, t \rangle$$

whenever $\eta \in C^1_{\mathrm{c}}(U, \mathbb{R}^{n+m})$ where $A_t = \{x \in U : f(x) < t\}$ and ∇^μ denotes the gradient with respect to μ.

Proof. Similar results valid for general varifolds and smooth f which are readily modified to treat the present case can be found in [All72, 4.10]. □

A.7 Lemma. *Suppose m, n, p, U, and μ are as in A.1, $p = 1$, $a \in \operatorname{spt} \mu$, $I =]0, \operatorname{dist}(a, \mathbb{R}^{n+m} \sim U)[$, and $u : I \to \mathbb{R}$, $v : I \to \mathbb{R}$ are defined by*

$$u(t) = \mu(\bar{B}_t(a)), \quad v(t) = \|\delta\mu\|(\bar{B}_t(a))$$

whenever $t \in I$.

Then for $\mathcal{L}^1$ almost all $t \in I$

$$\gamma_n^{-1} \leq u(t)^{1/n-1} v(t) + u(t)^{1/n-1} u'(t).$$

Proof. Let $0 < t < \infty$, $0 < h < \infty$ with $\bar{B}_{t+h}(a) \subset U$, and $d : U \to \mathbb{R}$ defined by $d(x) := |x - a|$ whenever $x \in U$. Since

$$\textstyle\int_t^{t+h} \langle \mu, d, r \rangle (U) \, \mathrm{d}\mathcal{L}^1 r \leq \mu(B_{t+h}(a) \sim \bar{B}_t(a)),$$

there exists $t < \varrho < t + h$ with

$$\langle \mu, d, \varrho \rangle (U) \leq h^{-1} \mu(B_{t+h}(a) \sim \bar{B}_t(a)),$$
$$\|\delta(\mu \llcorner \bar{B}_\varrho(a))\| \leq \|\delta\mu\| \llcorner \bar{B}_\varrho(a) + \langle \mu, d, \varrho \rangle$$

by A.6, hence

$$u(t) \leq (\mu \llcorner \bar{B}_\varrho(a))(U) \leq u(t+h), \quad (\|\delta\mu\| \llcorner \bar{B}_\varrho(a))(U) \leq v(t+h).$$

Therefore A.2 implies

$$\gamma_n^{-1} \leq u(t)^{1/n-1} v(t+h) + u(t)^{1/n-1} (u(t+h) - u(t))/h$$

and the conclusion follows by taking the limit $h \downarrow 0$, see [Fed69, 2.9.19]. □

A.8 Remark. A basic consequence of A.7 is the following: *If* $0 < r < \infty$, $\bar{B}_r(a) \subset U$,

$$\|\delta\mu\|(\bar{B}_\varrho(a)) \le (2\gamma_n)^{-1}\mu(\bar{B}_\varrho(a))^{1-1/n} \quad \textit{whenever } 0 < \varrho < r,$$

then

$$\mu(\bar{B}_\varrho(a)) \ge (2n\gamma_n)^{-n}\varrho^n \quad \textit{whenever } 0 < \varrho < r;$$

in fact one observes

$$(2\gamma_n)^{-1} \le u(\varrho)^{1/n-1}u'(\varrho), \quad (2n\gamma_n)^{-1} \le (u^{1/n})'(\varrho)$$

for $\mathcal{L}^1$ almost all $0 < \varrho < r$ and integration with the help of [Fed69, 2.9.19] yields the asserted inequality. Moreover, *if* $0 < r < \infty$, $a_i \in \mathbb{R}^{n+m}$ *for* $i \in \mathbb{N}$, μ_i *is a sequence of rectifiable* n *varifolds with locally bounded first variation in* $\mathbb{R}^{n+m}$ *and* $\theta^n(\mu_i, x) \ge 1$ *for* μ_i *almost all* $x \in \mathbb{R}^{n+m}$,

$$\|\delta\mu_i\|(\bar{B}_\varrho(a_i)) \le (2\gamma_n)^{-1}\mu_i(\bar{B}_\varrho(a_i))^{1-1/n} \quad \textit{whenever } 0 < \varrho < r,\ i \in \mathbb{N},$$

$a_i \to a$ *as* $i \to \infty$ *for some* $a \in \mathbb{R}^{n+m}$, $\int f \,\mathrm{d}\mu_i \to \int f \,\mathrm{d}\mu$ *for* $f \in C^0_c(\mathbb{R}^{n+m})$ *as* $i \to \infty$ *for some Radon measure* μ, *then* $a \in \operatorname{spt}\mu$.

The first consequence is also derived in [All72, 8.3] from a condition implying

$$\|\delta\mu\|(V) \le (2\gamma_n)^{-1}\mu(V)^{1-1/n}$$

for any open subset V of $B_r(a)$ (here $0^0 = 1$) what is, of course, stronger than the assumption above in case $n > 1$.

A.9 Remark. Clearly, the conditions of the preceding remark can be deduced by means of Hölder's inequality from similar conditions involving the measure ψ defined in A.1. In particular, if $p = n$ and $\psi(\{a\}) < (2\gamma_n)^{-n}$, then

$$\|\delta\mu\|(\bar{B}_\varrho(x)) \le (2\gamma_n)^{-1}\mu(\bar{B}_\varrho(x))^{1-1/n} \quad \text{whenever } 0 < \varrho < r,\ x \in \operatorname{spt}\mu \cap \bar{B}_r(a)$$

is satisfied for all sufficiently small positive radii r (here $0^0 = 1$).

A.10 Lemma. *Suppose* $m, n \in \mathbb{N}$, *and* $\delta > 0$.

Then there exists a positive number ε *with the following property.*

If $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$, m, n, p, U, *and* μ *are related as in A.1 with* $U = B_r(a)$, $p = 1$, $a \in \operatorname{spt}\mu$, *and*

$$\|\delta\mu\|(\bar{B}_\varrho(a)) \le (2\gamma_n)^{-1}\mu(\bar{B}_\varrho(a))^{1-1/n} \quad \textit{for } 0 < \varrho < r,$$
$$\|\delta\mu\|(B_r(a)) \le \varepsilon\,\mu(B_r(a))^{1-1/n},$$

then

$$\mu(B_r(a)) \ge (1-\delta)\omega_n r^n.$$

Proof. If the lemma were false for some $m, n \in \mathbb{N}$, and $\delta > 0$, there would exist a sequence ε_i with $\varepsilon_i \downarrow 0$ as $i \to \infty$ and sequences a_i, r_i, and μ_i showing that $\varepsilon = \varepsilon_i$ does not satisfy the conclusion of the lemma.

One could assume $a_i = 0$, $r_i = 1$ for $i \in \mathbb{N}$ and it would hold

$$\|\delta\mu_i\|(B^{n+m}_1(0)) \le \tfrac{1}{i}\mu(B^{n+m}_1(0))^{1-1/n} \le ((1-\delta)\omega_n)^{1-1/n}/i$$

for $i \in \mathbb{N}$. By A.8 one could construct (as the limit of a subsequence of μ_i) a stationary, rectifiable n varifold μ in $B_1^{n+m}(0)$ with

$$\theta^n(\mu, x) \geq 1 \quad \text{whenever } x \in \operatorname{spt} \mu, \quad 0 \in \operatorname{spt} \mu, \quad \mu(B_1^{n+m}(0)) \leq (1 - \delta)\omega_n,$$

see [Sim83, Corollary 17.8, Theorem 42.7]. But the monotonicity formula for μ would imply, see [Sim83, (17.5)],

$$1 \leq \theta^n(\mu, 0) \leq \frac{\mu(B_1^{n+m}(0))}{\omega_n} \leq 1 - \delta,$$

a contradiction. $\square$

A.11 Remark. A.9 and A.10 imply the following proposition.

If m, n, p, U, μ and ψ are as in A.1, $p = n$, then

$$\theta_*^n(\mu, a) \geq 1 \quad \textit{whenever } a \in \operatorname{spt} \mu \textit{ and } \psi(\{a\}) = 0.$$

Clearly, the condition $\psi(\{a\}) = 0$ is redundant in case $\|\delta\mu\|$ is absolutely continuous with respect to μ (i.e. $\delta\mu$ has no singular part with respect to μ).

A.12 Theorem. *Suppose m, n, p, U, μ, and ψ are as in A.1, $p < n$, $0 \leq s < \infty$, $0 < \varepsilon \leq (2\gamma_n)^{-p/(n-p)}$, $4\gamma_n n < \Gamma < \infty$,*

$$A = \big\{x \in U : \theta^{*n-p}(\psi, x) < (\varepsilon/\Gamma)^{n-p}/\omega_{n-p}\big\},$$

denote by B_i for $i \in \mathbb{N}$ the set of all $x \in U$ such that either $\bar{B}_{1/i}(x) \not\subset U$ or

$$\psi(\bar{B}_\varrho(x)) > \varepsilon^{n-p}\, \mu(\bar{B}_\varrho(x))^{1-p/n} \quad \textit{for some } 0 < \varrho < 1/i,$$

and denote by X_i for $i \in \mathbb{N}$ the set of all $a \in U$ such hat

$$\lim_{r \downarrow 0} \mu\big(B_i \cap \bar{B}_r(a)\big)/r^{sn/(n-p)} = 0.$$

Then the following five conclusions hold:

(1) The sets X_i are Borel sets and

$$\mathcal{H}^s\big(A \sim \textstyle\bigcup_{i \in \mathbb{N}} X_i\big) = 0.$$

(2) For $\mathcal{H}^s$ almost all $a \in A \cap \bigcap_{i \in \mathbb{N}} B_i$

$$\lim_{r \downarrow 0} \mu(\bar{B}_r(a))/r^{sn/(n-p)} = 0.$$

(3) If $a \in (\operatorname{spt} \mu) \sim B_i$, then $\bar{B}_{1/i}(a) \subset U$ and

$$(2n\gamma_n)^{-n} \varrho^n \leq \mu(\bar{B}_\varrho(a)) \quad \textit{for } 0 < \varrho < 1/i.$$

(4) If $\|\delta\mu\|$ is absolutely continuous with respect to μ, then

$$\mathcal{H}^{n-p}(U \sim A) = 0.$$

(5) If $p = 1$, then

$$\mathcal{H}^{n-1}(X \sim A) \le (\Gamma/\varepsilon)^{n-1} \omega_{n-1} \|\delta\mu\|(X \sim A) \quad \textit{for } X \subset U.$$

Proof. Clearly, $B_{i+1} \subset B_i$, $X_i \subset X_{i+1}$ for $i \in \mathbb{N}$. (1) implies (2) because the sets $\{x \in B_i : \bar{B}_{1/i}(x) \subset U\}$ are open (cp. [Fed69, 2.9.14]). (3) is a consequence of A.8, A.9. (4) and (5) follow from [Fed69, 2.10.6, 2.10.19 (3)]. Therefore only (1) remains to be proved.

X_i are Borel sets.

Define for $i \in \mathbb{N}$ the set A_i of all $x \in U$ such that $B_{1/i}(x) \subset U$ and

$$\psi(\bar{B}_\varrho(x)) \le (\varepsilon/\Gamma)^{n-p} \varrho^{n-p} \quad \text{whenever } 0 < \varrho < 1/i.$$

The sets A_i are closed (cp. [Fed69, 2.9.14]) and satisfy $A \subset \bigcup_{i \in \mathbb{N}} A_i$. Let C denote the set of all $x \in \operatorname{spt} \mu$ such that

$$\limsup_{\varrho \downarrow 0} \frac{\psi(\bar{B}_\varrho(x))}{\mu(\bar{B}_\varrho(x))^{1-p/n}} < \varepsilon^{n-p}$$

and note $\mu(U \sim C) = 0$ by [Fed69, 2.9.5]. By [Fed69, 2.10.6, 2.10.19 (4)] it is enough to prove $a \in X_{2i}$ for a point $a \in A_i$ with $\theta^s(\psi \llcorner U \sim A_i, a) = 0$.

For this purpose the following assertion will be proved. *For each $x \in B_{2i} \cap B_{1/(2i)}(a) \cap C$ there exists $0 < \varrho < \infty$ with*

$$\bar{B}_\varrho(x) \subset B_{2|x-a|}(a) \sim A_i, \quad \mu(\bar{B}_\varrho(x)) < \varepsilon^{-n} \psi(\bar{B}_\varrho(x))^{n/(n-p)}.$$

Choose $y \in A_i$ with $|y - x| = \operatorname{dist}(x, A_i)$ and let J be the set of all $0 < \varrho < 1/(2i)$ with

$$\mu(\bar{B}_\varrho(x)) < \varepsilon^{-n} \psi(\bar{B}_\varrho(x))^{n/(n-p)}.$$

Then $J \ne \emptyset$, because $x \in B_{2i}$, $\bar{B}_{1/(2i)}(x) \subset B_{1/i}(a) \subset U$, and, since $x \in C$, $\inf J > 0$. Therefore $t := \inf J$ satisfies

$$\begin{gathered} 0 < t < 1/(2i), \quad \mu(\bar{B}_t(x)) \le \varepsilon^{-n} \psi(\bar{B}_t(x))^{n/(n-p)}, \\ \mu(\bar{B}_\varrho(x)) \ge \varepsilon^{-n} \psi(\bar{B}_\varrho(x))^{n/(n-p)} \quad \text{for } 0 < \varrho < t. \end{gathered}$$

Noting

$$\begin{gathered} |y - x| = \operatorname{dist}(x, A_i) \le |x - a| \le 1/(2i), \quad t + |y - x| < 1/i, \\ \bar{B}_t(x) \subset \bar{B}_{t+|y-x|}(y) \subset B_{1/i}(y) \subset U, \end{gathered}$$

one estimates

$$\begin{aligned} \psi(\bar{B}_t(x))^{n/(n-p)} &\le \psi(\bar{B}_{t+|y-x|}(y))^{n/(n-p)} \\ &\le (\varepsilon/\Gamma)^n (t + |y - x|)^n < \varepsilon^n 2^{-n} (1 + |y - x|/t)^n (2n\gamma_n)^{-n} t^n \end{aligned}$$

and, using the inequalities derived from the definition of t and A.8, A.9,

$$\mu(\bar{B}_t(x)) \le \varepsilon^{-n} \psi(\bar{B}_t(x))^{n/(n-p)} < 2^{-n} (1 + |y - x|/t)^n \mu(\bar{B}_t(x)),$$

hence

$$(1 + |y - x|/t)^n > 2^n, \quad |y - x| > t$$

and the assertion follows by taking $\varrho \in J$ slightly larger than t.

Let $0 < r < 1/(2i)$. Then the preceding assertion in conjunction with Besicovitch's covering theorem implies the existence of countable, pairwise disjoint collections of closed balls $F_1, \ldots, F_{N(n+m)}$ satisfying

$$B_{2i} \cap \bar{B}_r(a) \cap C \subset \bigcup_{j=1}^{N(n+m)} \bigcup_{S \in F_j} S \subset B_{2r}(a) \sim A_i,$$
$$\mu(S) < \varepsilon^{-n} \psi(S)^{n/(n-p)} \quad \text{for } S \in \bigcup_{j=1}^{N(n+m)} F_j;$$

recall that $N(n+m)$ denotes the best constant in Besicovitch's covering theorem in $\mathbb{R}^{n+m}$, see [Sim83, Lemma 4.6]. Hence

$$\begin{aligned} &\mu(B_{2i} \cap \bar{B}_r(a)) = \mu(B_{2i} \cap \bar{B}_r(a) \cap C) \\ &\le \textstyle\sum_{j=1}^{N(n+m)} \sum_{S \in F_j} \mu(S) \le \varepsilon^{-n} \sum_{j=1}^{N(n+m)} \sum_{S \in F_j} \psi(S)^{n/(n-p)} \\ &\le \varepsilon^{-n} \textstyle\sum_{j=1}^{N(n+m)} \big(\sum_{S \in F_j} \psi(S) \big)^{n/(n-p)} \le \varepsilon^{-n} N(n+m) \psi(B_{2r}(a) \sim A_i)^{n/(n-p)} \end{aligned}$$

and (1) follows by taking the limit $r \downarrow 0$. □

A.13 Remark. It can happen that $\mathcal{H}^n \big(A \cap (\operatorname{spt} \mu) \cap \bigcap_{i \in \mathbb{N}} B_i \big) > 0$. An example is given in C.2. In fact taking $\mu \llcorner \mathbb{R}^{n+1} \sim T$ one sees from (3) and C.2 (4) that $T \subset \bigcap_{i \in \mathbb{N}} B_i$.

A.14 Remark. A.12 implies in particular that $\mathcal{H}^n$ almost all $x \in U$ satisfy

$$\text{either} \quad \theta^n_*(\mu, x) \ge (2n\gamma_n)^{-n}/\omega_n \quad \text{or} \quad \theta^{n^2/(n-p)}(\mu, x) = 0$$

and, in case $\|\delta\mu\|$ is absolutely continuous with respect to μ, that $\mathcal{H}^{n-p}$ almost all $x \in U$ satisfy

$$\text{either} \quad \theta^n_*(\mu, x) \ge (2n\gamma_n)^{-n}/\omega_n \quad \text{or} \quad \theta^n(\mu, x) = 0.$$

Moreover, the exponent $n^2/(n-p)$ cannot be replaced by any larger number as may be seen by taking $\mu \llcorner \mathbb{R}^{n+1} \sim T$ with μ as in C.2. Hence, the same holds for the exponent $sn/(n-p)$ in (2) if $s = n$.

A.15 Corollary. *Suppose m, n, p, U, and μ are as in A.1, $p < n$, $\varepsilon > 0$, $0 < \lambda < \infty$, and for $a \in U$ let B_a denote the set of all $x \in U$ such that $\bar{B}_{\lambda|x-a|}(x) \subset U$ and*

$$\|\delta\mu\|(\bar{B}_\varrho(x)) > \varepsilon\, \mu(\bar{B}_\varrho(x))^{1-1/n} \quad \textit{for some } 0 < \varrho < \lambda|x - a|.$$

Then $\mathcal{H}^n$ almost all $a \in U$ satisfy

$$\lim_{r \downarrow 0} \mu(B_a \cap \bar{B}_r(a))/r^{n^2/(n-p)} = 0.$$

Proof. Take $s = n$ in A.12 (1) and note A.12 (5) as well as the fact that for each $i \in \mathbb{N}$ and $a \in A$

$$B_a \cap \bar{B}_r(a) \subset B_i \quad \text{for small } r$$

if ε in the definition of the sets B_i is replaced by $\min\{\varepsilon, (2\gamma_n)^{-1}\}^{p/(n-p)}$. □

A.16 Remark. $n^2/(n-p)$ cannot be replaced by any larger number because otherwise the proposition derived in 1.11 would hold for that number.

B A differentiation theorem

Here, a theorem concerning the differentiation of functions or measures with respect to rectifiable varifolds satisfying a lower density bound and curvature conditions is proved. It is for use in 3.2 but might be of independent interest.

B.1 Theorem. *Suppose m, n, p, U, and μ are as in A.1, ν measures U with $\nu(U \sim \operatorname{spt} \mu) = 0$, A is μ measurable with $\nu(A) = 0$, and $1 \leq q < \infty$. In case $p < n$ additionally suppose for some $1 \leq r \leq \infty$ and some nonnegative function $f \in L^r_{\mathrm{loc}}(\mu)$ that*

$$\nu = f\mu \quad \textit{and} \quad q \leq 1 + (1 - 1/r)\frac{p}{n-p}.$$

Then for $\mathcal{H}^n$ almost all $a \in A$

$$\limsup_{s \downarrow 0} \nu(\bar{B}_s(a))/s^{nq} \quad \textit{equals either } 0 \textit{ or } \infty.$$

Proof. For $i \in \mathbb{N}$ let B_i denote the set of all $x \in U$ such that either $\bar{B}_{1/i}(x) \not\subset U$ or

$$\|\delta\mu\|(\bar{B}_\varrho(x)) > (2\gamma_n)^{-1}\mu(\bar{B}_\varrho(x))^{1-1/n} \quad \text{for some } 0 < \varrho < 1/i.$$

First, the case $A \subset \{x \in U : \theta^{*n}(\mu, x) > 0\}$ will be treated. In this case A is measurable and σ finite with respect to $\mathcal{H}^n$ by [Fed69, 2.10.19 (1) (3)]. Hence one may assume A to be compact. Define

$$A_i = \{a \in A : \nu(\bar{B}_s(a))/ \leq i\, s^{nq} \text{ for } 0 < s < 1/i\}$$

whenever $i \in \mathbb{N}$, $1/i < \operatorname{dist}(A, \mathbb{R}^{n+m} \sim U)$. The sets A_i are compact (cp. [Fed69, 2.9.14]) and their union equals

$$\big\{a \in A : \limsup_{s \downarrow 0} \nu(\bar{B}_s(a))/s^{nq} < \infty\big\}.$$

It therefore suffices to show for each $i \in \mathbb{N}$ with $1/i < \operatorname{dist}(A, \mathbb{R}^{n+m} \sim U)$

$$\lim_{s \downarrow 0} \nu(\bar{B}_s(a))/s^{nq} = 0 \quad \text{for } \mathcal{H}^n \text{ almost all } a \in A_i.$$

In fact, this equality will be proved for all $a \in A_i$ satisfying

$$\|\delta\mu\|(\{a\}) = 0, \quad \theta^n(\mu \llcorner U \sim A_i, a) = 0, \quad \theta^n(f^r\mu, a) = 0 \quad \text{if } r < \infty,$$
$$\limsup_{s \downarrow 0} \mu(B_j \cap \bar{B}_s(a))/s^{n^2/(n-p)} = 0 \quad \text{for some } j \in \mathbb{N},\ j \geq 2i, \text{ if } p < n$$

as $\mathcal{H}^n$ almost all $a \in A_i$ do according to [Fed69, 2.10.19 (3) (4)] and A.12.

In case $p = n$ one chooses $j \in \mathbb{N}$, $j \geq 2i$, using A.9 such that

$$B_j \cap \bar{B}_{1/j}(a) = \emptyset.$$

Let $0 < s < 1/j$. For $x \in \bar{B}_s(a) \cap (\operatorname{spt} \mu) \sim (B_j \cup A_i)$ there exists $y \in A_i$ with $|x - y| = \operatorname{dist}(x, A_i)$, hence

$$\begin{gathered} t := |x-y| \leq |x-a| \leq s < 1/j \leq 1/(2i), \\ \bar{B}_{|x-y|/2}(x) \subset \bar{B}_{3|x-y|/2}(y) \cap \bar{B}_{2s}(a) \sim A_i, \\ \nu(\bar{B}_{t/2}(x)) \leq \nu(\bar{B}_{3t/2}(y)) \leq i3^{nq}(t/2)^{nq} \leq c\,\mu(\bar{B}_{t/2}(x))^q \end{gathered}$$

where $c = i3^{nq}(2\gamma_n n)^{nq}$. Therefore one infers from Besicovitch's covering theorem the existence of countable, pairwise disjoint collections $F_1, \ldots, F_{N(n+m)}$ of closed balls such that

$$\begin{gathered} \bar{B}_s(a) \cap (\operatorname{spt}\mu) \sim (B_j \cup A_i) \subset \textstyle\bigcup_{k=1}^{N(n+m)} \bigcup_{S \in F_k} S \subset \bar{B}_{2s}(a) \sim A_i, \\ \nu(S) \leq c\,\mu(S)^q \quad \text{whenever } S \in \textstyle\bigcup_{k=1}^{N(n+m)} F_k, \end{gathered}$$

hence

$$\begin{gathered} \nu(\bar{B}_s(a) \sim B_j) = \nu(\bar{B}_s(a) \cap (\operatorname{spt}\mu) \sim (B_j \cup A_i)) \leq cN(n+m)\,\mu(\bar{B}_{2s}(a) \sim A_i)^q, \\ \lim_{s \downarrow 0} \nu(\bar{B}_s(a) \sim B_j)/s^{nq} = 0. \end{gathered}$$

To conclude the proof of the first case, one observes

$$\begin{gathered} \nu(B_j \cap \bar{B}_s(a)) = 0 \quad \text{if } p = n, \\ \nu(B_j \cap \bar{B}_s(a)) \leq \mu(B_j \cap \bar{B}_s(a))^{1-1/r} \|f\|_{L^r(\mu \llcorner \bar{B}_s(a))} \quad \text{if } p < n \end{gathered}$$

implying

$$\lim_{s \downarrow 0} \nu(B_j \cap \bar{B}_s(a))/s^{nq} = 0$$

because $(1-1/r)\frac{n}{n-p} + 1/r \geq q$ in case $p < n$.

It remains to treat the case $A \subset \{x \in U : \theta^n(\mu, x) = 0\}$. Using A.9 and A.14 one obtains

$$\begin{gathered} A \cap \operatorname{spt}\mu \text{ is countable} \quad \text{if } p = n, \\ \theta^{n^2/(n-p)}(\mu, a) = 0 \quad \text{for } \mathcal{H}^n \text{ almost all } a \in A \quad \text{if } p < n \end{gathered}$$

and the claim follows by using Hölder's inequality as in the preceding paragraph noting by [Fed69, 2.10.19 (4)]

$$\theta^n(f^r\mu, a) = 0 \quad \text{for } \mathcal{H}^n \text{ almost all } a \in A \quad \text{if } r < \infty. \qquad \square$$

B.2 Remark. This theorem generalises [Fed69, 2.9.17] and [CZ61, Theorem 10 (ii)]. The case treated by Federer roughly corresponds to the case $p = n$, $q = 1$ with μ satisfying a doubling condition. The case treated by Calderón and Zygmund corresponds to $p = n$, $m = 0$, $\mu = \mathcal{L}^{n+m}$ and ν absolutely continuous with respect to μ. The method of proof is based on Federer's proof and A.12 is used because of the absence of a doubling condition.

One can easily deduce from B.1 a Rademacher type theorem for first order differentiability of functions in L^q spaces with respect to μ similar to [Zie89, Theorem 3.8.1] (see also [CZ61, Theorem 5]). In the present case such a theorem would involve bounds on the exponent of the L^q spaces as may be seen from C.2 (5).

B.3 Remark. If $q = 1$, the condition $\nu(U \sim \operatorname{spt}\mu) = 0$ cannot be omitted as may be seen from [Fed69, 2.9.18 (2)].

B.4 Remark. If $p < n$ the condition $q \leq 1 + (1-1/r)p/(n-p)$ cannot be omitted as can be shown using C.2. In fact given μ and T as in C.2 a counterexample is provided by $\nu := \mu \llcorner \mathbb{R}^{n+1} \sim T$ in case $r = \infty$ and if $1 < r < \infty$ applying

C.2 (5) with $s = nq$ and $\alpha_1 q_1 = \alpha_2 q_2$ slightly larger than $\frac{np}{n-p}$ yields a function f such that $\nu := f\mu$ does not satisfy the conclusion of B.1. Finally, if $r = 1$ the condition is also violated for a slightly larger r, hence reducing this case to the previous one.

B.5 Remark. Note that the preceding two remarks remain valid if $\mathcal{H}^n$ is replaced by μ in the conclusion of B.1.

C An example concerning tilt and height decays of integral varifolds

This appendix provides the example which was used throughout the text to demonstrate the sharpness of various results obtained.

C.1 Definition. Suppose $x \in \mathbb{R}^{n+m}$ and $0 < \varrho < \infty$.

Then $Q_\varrho(x) := \{y \in \mathbb{R}^{n+m} : |y_i - x_i| \leq \varrho \text{ for } i = 1, \ldots, n+m\}$. To avoid ambiguity, $Q_\varrho^{n+m}(0)$ will be written instead of $Q_\varrho(0)$.

C.2 Example. Suppose $n \in \mathbb{N}$, $1 \leq p < n$, $0 < \alpha_i \leq 1$, $1 \leq q_i < \infty$ for $i \in \{1, 2\}$, such that

$$\alpha_2 q_2 \leq \alpha_1 q_1, \quad \frac{1}{p} > 1 + \frac{\alpha_2 q_2}{\alpha_1 q_1} \Big(\frac{1}{n} + \frac{1}{\alpha_2 q_2} - 1 \Big)$$

In case $\alpha_1 q_1 = \alpha_2 q_2$ the last condition reads $\alpha_2 q_2 > \frac{np}{n-p}$.

Then there exists a rectifiable n varifold μ in $\mathbb{R}^{n+1}$, $T \in G(n+1, n)$ and $0 < \Gamma < \infty$ with the following properties:

(1) $T \subset \operatorname{spt} \mu$ and $(\operatorname{spt} \mu) \sim T$ is an n dimensional manifold of class $\mathcal{C}^\infty$.

(2) $\theta^n(\mu, x) = 1$ for $x \in \operatorname{spt} \mu$ and $T_x \mu = T$ for $x \in T$.

(3) For some $\vec{\mathbf{H}}_\mu \in L^p_{\mathrm{loc}}(\mu, \mathbb{R}^{n+1})$ there holds $(\delta\mu)(\eta) = -\int \vec{\mathbf{H}}_\mu \bullet \eta \, \mathrm{d}\mu$ whenever $\eta \in C^1_{\mathrm{c}}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$.

(4) Whenever $x \in T$ and $0 < \varrho \leq 1$

$$\Gamma^{-1} \varrho^{\alpha_2 q_2} \leq \varrho^{-n} \mu(\{\xi \in \bar{B}_\varrho(x) : \operatorname{dist}(\xi - x, T) \geq \varrho / \Gamma\}),$$
$$\varrho^{-n} \mu(\bar{B}_\varrho(x) \sim T) \leq \Gamma \varrho^{\alpha_2 q_2},$$
$$\varrho^{-1-n/q_2} \big(\textstyle\int_{\bar{B}_\varrho(x)} \operatorname{dist}(\xi - x, T_x\mu)^{q_2} \, \mathrm{d}\mu(\xi) \big)^{1/q_2} \approx \varrho^{\alpha_2},$$
$$\varrho^{-n/q_1} \big(\textstyle\int_{\bar{B}_\varrho(x)} |T_\xi \mu - T_x \mu|^{q_1} \, \mathrm{d}\mu(\xi) \big)^{1/q_1} \approx \varrho^{\alpha_1},$$

here $a \approx b$ means that $a \leq \Gamma_1 b$ and $b \leq \Gamma_1 a$ for some positive, finite number Γ_1 depending only on n, and α_i, q_i for $i \in \{1, 2\}$.

(5) Whenever $1 < r < \infty$, $n + (1 - 1/r)\alpha_2 q_2 < s < \infty$ there exists a nonnegative function $f \in L^r_{\mathrm{loc}}(\mu)$ such that $f(x) = 0$ for $x \in T$, and

$$\varrho^s \approx \textstyle\int_{\bar{B}_\varrho(x)} f \, \mathrm{d}\mu \quad \text{whenever } x \in T, \ 0 < \varrho \leq 1,$$

here $a \approx b$ means $a \leq \Gamma_2 b$ and $b \leq \Gamma_2 a$ for some positive, finite number Γ_2 depending only on n and s.

Construction of example. Let $a := \alpha_2 q_2/n + 1$, $b := (\alpha_1 q_1 - \alpha_2 q_2)/a + 1 \geq 1$. Define for $i \in \mathbb{N}_0$

$$W_i := \big\{ Q_{2^{-i-2}}(x) : 2^{i+1} x \in \mathbb{Z}^n \big\}.$$

Clearly, $\bigcup_{Q \in W_i} \overline{Q} = \mathbb{R}^n$ and W_i is pairwise disjoint. Let

$$F_i := \big\{]2^{-i-1}, 2^{-i}[\times W : W \in W_i \big\} \quad \text{for } i \in \mathbb{N}_0, \quad F := \textstyle\bigcup_{i \in \mathbb{N}_0} F_i.$$

Clearly, $\bigcup_{S \in F} \overline{S} =]0,1] \times \mathbb{R}^n$ and F is pairwise disjoint. Let $T := \{0\} \times \mathbb{R}^n$.

Next, it will be indicated how to construct for every $0 < \sigma \leq \varrho < \infty$ a compact n dimensional submanifold M of $\mathbb{R}^{n+1}$ of class $\mathcal{C}^\infty$ such that

$$\begin{gathered} M \subset Q_\varrho^{n+1}(0), \quad (\Gamma_0)^{-1} \varrho^n \leq \mathcal{H}^n(M) \leq \Gamma_0 \varrho^n, \quad |\vec{\mathbf{H}}_M| \leq \Gamma_0 \sigma^{-1}, \\ \mathcal{H}^n(\{ x \in M : |T_x M - T| \geq 1 \}) \geq (\Gamma_0)^{-1} \sigma \varrho^{n-1}, \\ \mathcal{H}^n(\{ x \in M : \vec{\mathbf{H}}_M(x) \neq 0 \text{ or } T_x M \neq T \}) \leq \Gamma_0 \sigma \varrho^{n-1} \end{gathered}$$

where Γ_0 is a positive, finite number depending only on n. To construct M, one may assume $\varrho = 1$. Choose a concave function $f : [-1/2, 1/2] \to [0,1]$ and $0 < \Gamma_1 < \infty$ such that

$$\begin{gathered} f(-1/2) = \sigma/4 = f(1/2), \\ f(s) = \sigma/2 \quad \text{whenever } s \in [-1/2 + \sigma/4, 1/2 - \sigma/4] \end{gathered}$$

and such that

$$N := \{ (s,t) \in [-1/2, 1/2] \times \mathbb{R} : |t| = f(s) \} \cup (\{-1/2, 1/2\} \times [-\sigma/4, \sigma/4])$$

is a 1 dimensional submanifold of class $\mathcal{C}^\infty$ with $|\vec{\mathbf{H}}_N| \leq \Gamma_1 \sigma^{-1}$. Noting

$$\mathcal{H}^1(\operatorname{graph} f | [-1/2, -1/2 + \sigma/4] \cap [1/2 - \sigma/4, 1/2]) \leq \sigma,$$

one can take

$$M := \{ (y,z) \in \mathbb{R} \times \mathbb{R}^n : (|z|, y) \in N \}.$$

For each $i \in \mathbb{N}_0$ and $Q \in F_i$ choose a n dimensional submanifold M_Q of the type just constructed corresponding to $\varrho_i := 2^{-ia-2}$, $\sigma_i := 2^{-iba-2}$ contained in Q and let M be the union those submanifolds. Take $\mu := \mathcal{H}^n \llcorner (T \cup M)$. (1) is now evident.

To prove the estimates, fix $x \in T$ and define for $i, j \in \mathbb{N}_0$

$$b_{i,j} := \#\big\{ Q \in F_j : Q \cap Q_{2^{-i}}(x) \neq \emptyset \big\}, \quad c_{i,j} := \#\big\{ Q \in F_j : Q \subset Q_{2^{-i}}(x) \big\}.$$

Clearly, $b_{i,j} = c_{i,j} = 0$ if $j < i$. If $j \geq i$, one estimates

$$b_{i,j} \leq \big(2^{j-i+2} + 1\big)^n \leq \big(5 \cdot 2^{j-i}\big)^n, \quad c_{i,j} \geq \big(2^{j-i+2} - 1\big)^n \geq \big(3 \cdot 2^{j-i}\big)^n.$$

One calculates

$$\mu(Q_{2^{-i}}(x)\sim T) \leq \sum_{j=0}^{\infty} b_{i,j}\Gamma_0(\varrho_j)^n \leq (5/4)^n\Gamma_0(2^{-i})^{an}(1-2^{n(1-a)})^{-1},$$

$$n - ba(1-p) + (1-n)a = -\alpha_1 q_1 + p(\alpha_1 q_1 - \alpha_2 q_2 + \alpha_2 q_2/n + 1) < 0,$$

$$\textstyle\int_{Q_{2^{-i}}(x)\sim T} |\vec{\mathbf{H}}_M|^p \,\mathrm{d}\mu \leq \sum_{j=0}^{\infty} b_{i,j}(\Gamma_0)^{p+1}(\sigma_j)^{1-p}(\varrho_j)^{n-1}$$

$$\leq 5^n(\Gamma_0)^{n+1}(2^{-i})^{ba(1-p)+n-1}(1-2^{n-ba(1-p)+(1-n)a})^{-1} < \infty,$$

$$\textstyle\int_{Q_{2^{-i}}(x)} \operatorname{dist}(\xi - x, T)^{q_2}\,\mathrm{d}\mu(\xi) \leq 2^{-iq_2}\mu(Q_{2^{-i}}(x)\sim T),$$

$$\textstyle\int_{Q_{2^{-i}}(x)} |T_\xi\mu - T|^{q_1}\,\mathrm{d}\mu(\xi) \leq (2n)^{q_1}\sum_{j=0}^{\infty} b_{i,j}\Gamma_0\sigma_j(\varrho_j)^{n-1}$$

$$\leq (2n)^{q_1}(5/4)^n\Gamma_0(2^{-i})^{ba+a(n-1)}(1-2^{n-ba-a(n-1)})^{-1},$$

$$\textstyle 2^{(i+1)q_2}\int_{Q_{2^{-i}}(x)} \operatorname{dist}(\xi - x, T)^{q_2}\,\mathrm{d}\mu(\xi)$$

$$\geq \mu(\{\xi \in Q_{2^{-i}}(x) : \operatorname{dist}(\xi - x, T) \geq 2^{-i-1}\}) \geq (\Gamma_0)^{-1}(\varrho_i)^n = (4^n\Gamma_0)^{-1}2^{-ian},$$

$$\textstyle\int_{Q_{2^{-i}}(x)} |T_\xi\mu - T|^{q_1}\mu(\xi) \geq (\Gamma_0)^{-1}\sigma_i(\varrho_i)^{n-1} = (4^n\Gamma_0)^{-1}(2^{-i})^{ab+a(n-1)}.$$

Therefore (3) and (4) are proved and the first estimate of (4) implies (2).

To prove (5), define f by $f(y) := 2^{(na-s)i}$ if $y \in \bigcup_{S\in F_i} S$ for some $i \in \mathbb{N}_0$ and $f(y) = 0$ else. Then for $i \in \mathbb{N}_0$

$$\textstyle\int_{Q_{2^{-i}}(x)} |f|\,\mathrm{d}\mu \leq \sum_{j=0}^{\infty} b_{i,j}2^{(na-s)j}\Gamma_0(\varrho_j)^n \leq (5/4)^n\Gamma_0(2^{-i})^s(1-2^{n-s})^{-1},$$

$$\textstyle\int_{Q_{2^{-i}}(x)} |f|^r\,\mathrm{d}\mu \leq \sum_{j=0}^{\infty} b_{i,j}2^{(na-s)rj}\Gamma_0(\varrho_j)^n$$

$$\leq (5/4)^n\Gamma_0(2^{-i})^{(s-na)r+na}(1-2^{n+(na-s)r-na})^{-1} < \infty$$

because

$$n + (na-s)r - an = \alpha_2 q_2(r-1) + r(n-s) < 0.$$

The estimate from below is similar to the one from above. □

C.3 Remark. The integral n varifold μ constructed depends only on n and the products $\alpha_i q_i$ for $i \in \{1,2\}$. Moreover, the assumption $\alpha_i \leq 1$ for $i \in \{1,2\}$ could be replaced by $\alpha_i < \infty$ for $i \in \{1,2\}$.

C.4 Remark. Taking $p = 1$, $\alpha_1 = \alpha_2$, and $q_1 = q_2 = 2$ in the last two estimates of (4) shows that for every $n \in \mathbb{N}$, $n > 1$, $1/2 + (2(n-1))^{-1} < \alpha \leq 1$, there exists an integral n varifold μ of $\mathbb{R}^{n+1}$ of locally bounded first variation such that for some A with $\mu(A) > 0$

$$\lim_{\varrho\downarrow 0} \varrho^{-2\alpha}\operatorname{heightex}_\mu(x,\varrho,T_x\mu) = \infty, \quad \lim_{\varrho\downarrow 0} \varrho^{-2\alpha}\operatorname{tiltex}_\mu(x,\varrho,T_x\mu) = \infty$$

for $x \in A$. In [Bra78, 5.7] Brakke showed in arbitrary codimension that the above limits equal 0 almost everywhere with respect to μ if $\alpha = 1/2$.

C.5 Remark. Similarly to the preceding remark, taking $\alpha_1 = \alpha_2 = 1$, $q_1 = q_2 = q$ and noting (1), one obtains for every $p^* = \frac{np}{n-p} < q < \infty$ an integral n varifold μ satisfying (H_p) which is countably rectifiable of class $\mathcal{C}^2$ such that for some A with $\mu(A) > 0$

$$\lim_{\varrho \downarrow 0} \varrho^{-2-n/q} \big(\textstyle\int_{B_\varrho(x)} \operatorname{dist}(\xi - x, T_x\mu)^q \, d\mu(\xi) \big)^{1/q} = \infty,$$
$$\lim_{\varrho \downarrow 0} \varrho^{-1-n/q} \big(\textstyle\int_{B_\varrho(x)} |T_\xi\mu - T_x\mu|^q \, d\mu(\xi) \big)^{1/q} = \infty$$

for $x \in A$. In particular, if $p < \frac{2n}{n+2}$ then countable rectifiability of class $\mathcal{C}^2$ does not imply quadratic decay of neither $\operatorname{tiltex}_\mu$ nor $\operatorname{heightex}_\mu$. If $p = 2$, countable rectifiability of class $\mathcal{C}^2$ is equivalent to quadratic decay of both quantities, see [Sch04b, Theorem 3.1].

D Elementary properties of Q valued functions

The purpose of this appendix is to collect for the convenience of the reader Almgren's definitions and results concerning Q valued functions (cf. [Alm00]) needed to prove the Poincaré inequality for integral varifolds. There is also included an elementary but useful decomposition of a Lipschitzian Q valued function into a countable collection of ordinary Lipschitzian functions. This decomposition directly entails the rectifiability of the Q valued graph which had been proved by Almgren using the compactness theorem for integral currents and also a simple proof of the Rademacher theorem for Q valued functions. Another proof of the Rademacher theorem avoiding Almgren's bi Lipschitzian embedding of $Q_Q(\mathbb{R}^{n+m})$ into a Euclidean space based on continuous selection results can be found in [Gob06].

D.1 (cf. [Alm00, 1.1 (1)])**.** Suppose $Q \in \mathbb{N}$ and V is a finite dimensional Euclidean vector space.

$Q_Q(V)$ is defined to be the set of all 0 dimensional integral currents R such that $R = \sum_{i=1}^Q [\![x_i]\!]$ for some $x_1, \ldots, x_Q \in V$. A metric $\mathcal{G}$ on $Q_Q(V)$ is defined such that

$$\mathcal{G}\big(\textstyle\sum_{i=1}^Q [\![x_i]\!], \sum_{i=1}^Q [\![y_i]\!]\big) = \min \Big\{ \big(\sum_{i=1}^Q |x_i - y_{\pi(i)}|^2 \big)^{1/2} : \pi \in \mathcal{S}(Q) \Big\}$$

whenever $x_1, \ldots, x_Q, y_1, \ldots, y_Q \in V$ where $\mathcal{S}(Q)$ denotes the set of permutations of $\{1, \ldots, Q\}$. The function mapping $(x, T) \in V \times Q_Q(V)$ to $\theta^0(\|T\|, x)$ is upper semicontinuous. Note that in case $T = \sum_{i=1}^Q [\![x_i]\!]$ for some $x_1, \ldots, x_Q \in V$

$$\theta^0(\|T\|, x) = \#\{i : x_i = x\} \quad \text{whenever } x \in V.$$

Whenever $f : X \to Q_Q(V)$ the *Q valued image of f* and *Q valued graph of f* are defined by

$$\operatorname{im}_Q f = \{v \in V : v \in \operatorname{spt} f(x) \text{ for some } x \in X\},$$
$$\operatorname{graph}_Q f = \{(x, v) \in X \times V : v \in \operatorname{spt} f(x)\}.$$

In case X is a topological space and f is continuous, the function mapping $(x, v) \in X \times V$ to $\theta^0(\|f(x)\|, v)$ is upper semicontinuous. In particular,

$$\operatorname{graph}_Q f = \{(x, v) \in X \times V : \theta^0(\|f(x)\|, v) \geq 1\}$$

is closed in $X \times V$.

D.2 (cf. [Alm00, 1.1 (9)–(11)])**.** Suppose $m, n, Q \in \mathbb{N}$.

A function $f : \mathbb{R}^n \to Q_Q(\mathbb{R}^m)$ is called *affine* if and only if there exist affine functions $f_i : \mathbb{R}^n \to \mathbb{R}^m$, $i = 1, \ldots, Q$ such that

$$f(x) = \textstyle\sum_{i=1}^{Q} [\![f_i(x)]\!] \quad \text{whenever } x \in \mathbb{R}^n.$$

$f_1, \ldots, f_Q$ are uniquely determined up to order. One defines semi norms such that

$$|f| = \big(\textstyle\sum_{i=1}^{Q} |Df_i(a)|^2 \big)^{1/2}, \quad \|f\| = \limsup_{x \to a} \mathcal{G}(f(x), f(a))/|x - a|$$

whenever $a \in \mathbb{R}^n$. Among their basic properties are the following inequalities:

$$\max\{\|Df_i(a)\| : i = 1, \ldots, Q\} \leq \|f\| \leq Q^{1/2} \max\{\|Df_i(a)\| : i = 1, \ldots, Q\},$$
$$|f| \leq m^{1/2} Q^{1/2} \|f\|, \quad \operatorname{Lip} f = \|f\| \leq |f|.$$

Let $a \in A \subset \mathbb{R}^n$, $f : A \to Q_Q(\mathbb{R}^m)$. f is called *affinely approximable at a* if and only if A contains a neighbourhood of a and there exists an affine function $g : \mathbb{R}^n \to Q_Q(\mathbb{R}^m)$ such that

$$\lim_{x \to a} \mathcal{G}(f(x), g(x))/|x - a| = 0.$$

f is called *approximately affinely approximable at a* if and only if there exists an affine function $g : \mathbb{R}^n \to Q_Q(\mathbb{R}^m)$ such that (see [Fed69, 2.9.12, 3.1.2])

$$\operatorname{ap} \lim_{x \to a} \mathcal{G}(f(x), g(x))/|x - a| = 0.$$

The function g is unique in both cases and denoted by $Af(a)$ and $\operatorname{ap} Af(a)$ respectively. f is called *strongly affinely approximable at a* if and only if $Af(a)$ has the following property: If $Af(a)(x) = \sum_{i=1}^{Q} [\![g_i(x)]\!]$ whenever $x \in \mathbb{R}^n$ for some affine functions $g_i : \mathbb{R}^n \to \mathbb{R}^m$ and $g_i(a) = g_j(a)$ for some i and j, then $Dg_i(a) = Dg_j(a)$. Similarly, one defines *approximately strongly affinely approximable at a*.

If f is affinely approximable [approximately affinely approximable] at a, then

$$\|Af(a)\| \leq \operatorname{Lip} f \quad [\|\operatorname{ap} Af(a)\| \leq \operatorname{Lip} f].$$

D.3. Recall from [Fed69, 3.3.1] that for $m, n, Q \in \mathbb{N}$, $a \in \mathbb{R}^{n+m}$, $0 \leq r < \infty$, $V \in G(n + m, m)$, $0 < s < 1$, $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ an orthogonal projection

$$X(a, r, V, s) := \{x \in \mathbb{R}^{n+m} : s^{-1} \operatorname{dist}(x - a, V) < |x - a| < r\},$$
$$X(a, r, \ker p, s) = \{x \in \mathbb{R}^{n+m} : s^{-1} |p(x - a)| < |x - a| < r\}$$

where $\ker p$ denotes the kernel of p.

D.4 Definition (cf. [Alm00, T.1 (23)])**.** Whenever $f : X \to Y$, $g : X \to Z$ the *join* $f \bowtie g : X \to Y \times Z$ is defined by

$$(f \bowtie g)(x) = (f(x), g(x)) \quad \text{whenever } x \in X.$$

D.5 Theorem. *Suppose $m, n, Q \in \mathbb{N}$, B is a Borel subset of $\mathbb{R}^n$, and $f : B \to Q_Q(\mathbb{R}^m)$ is a Lipschitzian function.*

Then $\operatorname{graph}_Q f$ *is a Borel set and there exists a sequence $f_1, f_2, f_3, \ldots$ of Borel subsets of* $\operatorname{graph}_Q f$ *such that each f_i is a Borel function with* $\operatorname{Lip} f_i \leq \operatorname{Lip} f$ *and*

$$\#\{i : (x, y) \in f_i\} = \theta^0(\|f(x)\|, y) \quad \textit{whenever } (x, y) \in B \times \mathbb{R}^m.$$

In particular, $\operatorname{graph}_Q f$ *and* $\operatorname{im}_Q f$ *are countably n rectifiable (in the sense of [Fed69, 3.2.14 (2)]).*

Proof. Assume $\operatorname{Lip} f > 0$ and let $E = \operatorname{graph}_Q f$, $s = (1 + (\operatorname{Lip} f)^2)^{-1/2}$, and $p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $q : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ the projections.

If $a \in E$, $0 < 2r \leq \operatorname{dist}(q(a), (\operatorname{spt} f(p(a))) \sim \{q(a)\})$, $z \in E \cap B_r(a)$, then

$$\begin{gathered} q(a) \in \operatorname{spt} f(p(a)), \quad q(z) \in \operatorname{spt} f(p(z)), \\ |q(z) - q(a)| \leq |z - a| < r, \quad |q(z) - q(a)| = \operatorname{dist}(q(z), \operatorname{spt} f(p(a))), \\ |q(z) - q(a)| \leq \mathcal{G}(f(p(z)), f(p(a))) \leq (\operatorname{Lip} f)|p(z) - p(a)|, \\ |z - a| \leq s^{-1}|p(z) - p(a)|, \quad z \notin X(a, r, \ker p, s). \end{gathered}$$

Therefore E is the union of

$$E_i := \{a \in E : E \cap X(a, 1/i, \ker p, s) = \emptyset\}$$

corresponding to $i \in \mathbb{N}$.

It is now elementary to verify (cp. [Fed69, 3.3.5]) that each subset of E_i with diameter less that $1/i$ is a Lipschitzian function with Lipschitz constant at most $(s^{-2} - 1)^{1/2} = \operatorname{Lip} f$. Noting the fact that if $g = \operatorname{graph} g \subset \mathbb{R}^n \times \mathbb{R}^m$ is a Lipschitzian function so is $\overline{g} = \overline{\operatorname{graph} g}$ and $\operatorname{Lip} \overline{g} = \operatorname{Lip} g$, one uses D.1 to construct $f_1, f_2, f_3, \ldots$ with the required properties. Since $f_i \subset \operatorname{im}(\mathbb{1}_{\operatorname{dmn} f_i} \bowtie f_i)$ for each i and $\operatorname{im}_Q f = q(\operatorname{graph}_Q f)$, the postscript follows. ☐

D.6 Remark. Concerning the assertion of the theorem, recall the following relation between Borel functions and functions which are Borel sets from [Fed69, 2.2.10, 2.2.14].

Suppose A is a Borel subset of $\mathbb{R}^n$, $p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the projection and $f \subset \mathbb{R}^n \times \mathbb{R}^m$ with $p(f) = A$. Then f is a Borel function if and only if f is a Borel set and $p|f$ is univalent. Moreover, if B is a Borel subset of $\mathbb{R}^n \times \mathbb{R}^{n+m}$ with $p|B$ univalent, then $p(B)$ is a Borel set. Hence the assumption on A is not needed in the second part of the equivalence.

D.7 Remark. If the last condition on the sequence f_i is dropped, one can require the f_i to form a Borel partition of $\operatorname{graph}_Q f$.

D.8 Remark. Almgren proved in [Alm00, 1.5 (8)] that $\operatorname{graph}_Q f$ is countably $(\mathcal{H}^n, n)$ rectifiable in the sense of [Fed69, 3.2.14 (3)] using the compactness theorem for integral currents [Fed69, 4.2.17 (2)].

D.9 Remark. In [Gob06, Section 5] an example with $n = 2$, $m = 2$ and B the unit sphere is given such that no continuous function $g : B \to \mathbb{R}^m$ satisfies $g(x) \in \operatorname{spt} f(x)$ whenever $x \in B$. Hence, in general the domain of the functions f_i will not equal B.

D.10 Theorem. *Suppose $m, n, Q \in \mathbb{N}$, $A \subset B \subset \mathbb{R}^n$, B is open, $Q \in \mathbb{N}$, $f : B \to Q_Q(\mathbb{R}^m)$, and*

$$\limsup_{x \to a} \mathcal{G}(f(x), f(a))/|x - a| < \infty \quad \textit{whenever } a \in A.$$

Then f is strongly affinely approximable at $\mathcal{L}^n$ almost all points of A.

Proof. As in [Fed69, 3.1.5, 3.1.9] one proves that $A \subset C$, where C is the union of a countable family of closed sets such that f restricted to each member is Lipschitzian and that it is enough to show that f is approximately strongly affinely approximable at $\mathcal{L}^n$ almost all points of C.

Using D.5, one now constructs a countable Borel covering $f_1, f_2, f_3, \dots$ of $\operatorname{graph}_Q f|C$ consisting of Lipschitzian Borel functions such that

$$\#\{i : (x, y) \in f_i\} = \theta^0(\|f(x)\|, y) \quad \text{whenever } (x, y) \in C \times \mathbb{R}^m.$$

Define $I(x) = \{i \in \mathbb{N} : x \in \operatorname{dmn} f_i\}$ for $x \in C$ and note $\# I(x) = Q$ and

$$f(y) = \sum_{i \in I(x)} [\![f_i(y)]\!] \quad \text{whenever } y \in \bigcap_{i \in I(x)} \operatorname{dmn} f_i.$$

According to [Fed69, 2.9.11, 3.1.2, 3.1.7] $\mathcal{L}^n$ almost all $x \in C$ satisfy:

$$\begin{gathered} i \in I(x) \quad \text{implies} \quad f_i \text{ is approximately differentiable at } x, \\ i, j \in I(x),\ f_i(x) = f_j(x) \quad \text{implies} \quad \operatorname{ap} D f_i(x) = \operatorname{ap} D f_j(x). \end{gathered}$$

At such a point x there holds $\theta^n\big(\mathcal{L}^n \llcorner \mathbb{R}^n \sim \bigcap_{i \in I(x)} \operatorname{dmn} f_i, x\big) = 0$, and f is therefore approximately strongly affinely approximable with

$$\operatorname{ap} A f(x)(v) = \sum_{i \in I(x)} [\![f_i(x) + \langle v, \operatorname{ap} D f_i(x) \rangle]\!] \quad \text{for } v \in \mathbb{R}^n. \qquad \square$$

D.11 Remark. Similarly, one proves the following proposition:

If A is $\mathcal{L}^n$ measurable, $f : A \to Q_Q(\mathbb{R}^m)$ is Lipschitzian, I is countable, and to each $i \in I$ there corresponds a function $f_i \subset \operatorname{graph}_Q f$ with $\mathcal{L}^n$ measurable domain and $\operatorname{Lip} f_i \leq \operatorname{Lip} f$ *such that*

$$\#\{i : (x, y) \in f_i\} = \theta^0(\|f(x)\|, y) \quad \textit{whenever } (x, y) \in A \times \mathbb{R}^m,$$

then f is approximately strongly affinely approximable with

$$\operatorname{ap} A f(a)(v) = \textstyle\sum_{i \in I(a)} [\![f_i(x) + \langle v, \operatorname{ap} D f_i(x) \rangle]\!] \quad \textit{whenever } v \in \mathbb{R}^n$$

at $\mathcal{L}^n$ almost all $a \in A$ where $I(a) = \{i \in I : a \in \operatorname{dmn} f_i\}$.

The existence of such functions f_i is a consequence of D.5 applied to $\overline{\operatorname{graph} f}$ replacing f.

D.12. If $0 < d < \infty$, $m \in \mathbb{N}$, $S, T \in Q_Q(\mathbb{R}^m)$, and for each subset X of $\operatorname{spt} S$

$$\textstyle\sum_{x \in X} \theta^0(\|S\|, x) + \sum_{y \in Y} \theta^0(\|T\|, y) \leq Q$$

where $Y = (\operatorname{spt} S) \sim \bigcup_{x \in X} B_d(x)$, then

$$\mathcal{G}(S, T) < Q^{1/2} d;$$

in fact if $S = \sum_{i=1}^{Q} [\![x_i]\!]$, $T = \sum_{i=1}^{Q} [\![y_i]\!]$ for some $x_1, \ldots, x_Q, y_1, \ldots, y_Q \in \mathbb{R}^m$ one may verify the existence of a permutation σ of $\{1, \ldots, Q\}$ such that $|x_i - y_{\sigma(i)}| < d$ for $i \in \{1, \ldots, Q\}$ by Hall's theorem on perfect matches (cf. [LP86, Theorem 1.1.3]).

D.13 Definition. Suppose $m, n, Q \in \mathbb{N}$, $S \in Q_Q(\mathbb{R}^m)$, $1 \leq q \leq \infty$, A is $\mathcal{L}^n$ measurable, and $f : A \to Q_Q(\mathbb{R}^m)$ is an $\mathcal{L}^n \llcorner A$ measurable function.

Then the *q height of f with respect to S* is defined to be the $L^q(\mathcal{L}^n \llcorner A)$ (semi) norm of the function mapping $x \in A$ to $\mathcal{G}(f(x), S)$, denoted by $h_q(f, S)$, and, if f is additionally Lipschitzian, then the *q tilt of f* is defined to be the $L^q(\mathcal{L}^n \llcorner A)$ (semi) norm of the function mapping $x \in A$ to $|\operatorname{ap} A f(x)|$, denoted by $t_q(f)$. Moreover, the *q height of f* is defined to be the infimum of the numbers $h_q(f, S)$ corresponding to all $S \in Q_Q(\mathbb{R}^m)$ and denoted by $h_q(f)$.

D.14 Theorem. *Suppose $m, n, Q \in \mathbb{N}$, $f : \bar{B}_1^n(0) \to Q_Q(\mathbb{R}^m)$, and* $\operatorname{Lip} f < \infty$.

Then the following two statements hold:

(1) If $1 \leq q < n$, $q^* = \frac{qn}{n-q}$, then there exists a positive, finite number $\Gamma_{(1)}$ depending only on m, n, Q, and q such that

$$h_{q^*}(f) \leq \Gamma_{(1)}\, t_q(f).$$

(2) If $q < n \leq \infty$, then there exists a positive, finite number $\Gamma_{(2)}$ depending only on m, n, Q, and q such that

$$h_\infty(f) \leq \Gamma_{(2)}\, t_q(f).$$

Proof. Using Almgren's functions $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$ [Alm00, 1.2 (3), 1.3 (1), 1.4 (3) (5)], the assertion is readily deduced from classical embedding results. □

D.15 Theorem ([Alm00, 1.3 (2)])**.** *Suppose $m, n, Q \in \mathbb{N}$, $A \subset \mathbb{R}^n$, and $f : A \to Q_Q(\mathbb{R}^m)$.*

Then there exists $g : \mathbb{R}^n \to Q_Q(\mathbb{R}^m)$ such that

$$g|A = f, \quad \operatorname{Lip} g \leq \Gamma \operatorname{Lip} f$$

where Γ is a positive, finite number depending only on m and Q.

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