

Positive Semigroups for Queueing Theory and Reliability Theory

DISSERTATION

Der Fakultät für Mathematik und Physik
der Eberhard Karls Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften

Vorgelegt von
ABDUKERIM HAJI
aus Urumqi

2007

Tag der mündlichen Qualifikation: 13. Dezember 2007
Dekan: Prof. Dr. Nils Schopohl
1. Berichterstatter: Prof. Dr. Rainer Nagel
2. Berichterstatter: Prof. Dr. Ulf Schlotterbeck

Contents

| | |
|--|----|
| Introduction | 3 |
| Chapter 1. Preliminary Results in Semigroup Theory | 7 |
| 1.1. Positive Operators | 7 |
| 1.2. Well-posedness of the abstract Cauchy problem | 11 |
| 1.3. Characteristic Equation | 12 |
| 1.4. Asymptotic Stability of Positive Semigroups | 16 |
| Chapter 2. The Dynamic $M/M^B/1$ Queueing System | 19 |
| 2.1. Introduction | 19 |
| 2.2. The Problem as an Abstract Cauchy Problem | 20 |
| 2.3. Boundary Spectrum | 22 |
| 2.4. Well-Posedness of the System | 28 |
| 2.5. Asymptotic Stability of the Solution | 38 |
| Chapter 3. The System with Primary and Secondary Failures | 41 |
| 3.1. Introduction | 41 |
| 3.2. The Problem as an Abstract Cauchy Problem | 42 |
| 3.3. Boundary Spectrum | 44 |
| 3.4. Well-posedness of the System | 51 |
| 3.5. Asymptotic Stability of the Solution | 60 |
| Chapter 4. A Parallel Maintenance System with Two Components | 63 |
| 4.1. Introduction | 63 |
| 4.2. The Problem as an Abstract Cauchy Problem | 64 |
| 4.3. Boundary Spectrum | 66 |
| 4.4. Well-posedness of the System | 78 |
| 4.5. Asymptotic Stability of the Solution | 87 |
| Table of symbols | 89 |
| Bibliography | 91 |
| Zusammenfassung in deutscher Sprache | 93 |
| Lebenslauf | 95 |

Introduction

Waiting lines, or queues, have been an everyday-life phenomenon for a long time. A modern mathematical treatment uses concepts from probability theory and aims at an understanding of the time-evolution of certain probabilities. A more detailed analysis discriminates between different types of queueing systems and uses a 3-descriptor code of the form $A/S/n$, where A stands for the distribution of arrivals of customers, S for service time distribution and server peculiarities, and n for the number of servers. We study the so-called $M/M^B/1$ model, which is described in detail below and in Chapter 2.

Another problem we are interested in is concerned with systems involving units which may fail to function, are repairable and are backed up by standby units. Again, we are interested in the evolution (in time) of failure/availability probabilities. For details we refer to Chapter 3 and 4. In both types of models we use the fact that the probabilities we are interested in are the solutions of certain systems of partial differential equations with appropriate initial/boundary conditions. This goes back to Cox [Cox55], and for the $M/M^B/1$ model in particular, to M. L. Chaundhry and J. G. C. Templeton [CT83]. In the cases studied in Chapter 3 and 4 the same observation goes back to [Gup95] and [Yeh97]. The basic idea we are following, and which goes back to G. Gupur (see [GZ98], [GLZ01] and [Gup02]), is to verify that these differential equations can be written in the form of an abstract Cauchy problem which is well-posed, so that the theory of C_0 -semigroups is applicable. In all cases which we are considering, the corresponding semigroups are even positive, which considerably facilitates the discussion of the asymptotics of solutions. We continue to give a more detailed account of the content of the respective chapters.

In Chapter 1, we first recall some basic definitions and results on Banach lattices and positive operators. We continue to outline the general framework, developed by G. Greiner [Gre87], into which all our examples fit; last we concentrate our attention to the asymptotic behaviour of positive semigroups on Banach lattices and collect the results used later.

Chapter 2 is devoted to an analysis of the $M/M^B/1$ queueing model. In this model there is a single-server which can serve B customers simultaneously. The service starts as soon as there is one customer in the queue. The arrival of the customers in the queue is at random. The arrival times of the customers as well as the service times are distributed exponentially. We first write the system as an

abstract Cauchy problem, prove well-posedness of the problem and irreducibility and positivity of the corresponding semigroup and analyze the spectrum of the generator. The main conclusion on the asymptotic behaviour of the solutions of this problem is stated in Theorem 2.5.2.

In Chapter 3, the model of a repairable system with primary as well as secondary failures is considered. The mathematical model for the system was established by Surendra M.Gupta (see [Gup95]). We rewrite the model as an abstract Cauchy problem, and prove well-posedness of the problem and positivity and irreducibility of the corresponding semigroup. Through a spectral analysis of the generator we obtain existence of a unique steady state to which all solution convergence as time tends to infinity.

In Chapter 4, we discuss a parallel maintenance system with two components. In [Yeh97], L.Yeh established the mathematical model of the system and obtained existence of a steady-state solution. In [Guo03], Guo Weihua proved the existence and uniqueness of a nonnegative solution of the system by using classical analysis methods. By using C_0 - semigroup theory, well-posedness of this problem is verified. Finally, the asymptotic behaviour of the solutions is obtained through a spectral analysis of the generator and by applying a recent result from [EFNS07].

Acknowledgements

I want to express my gratitude to Prof. Rainer Nagel without whom this work could not have been completed. I wish to thank professor Rainer Nagel for his mathematical advice, continuous encouragement, optimism and support during the preparation of my thesis.

I also wish to thank Dr. Agnes Radl for many helpful discussions and invaluable remarks. It was a great pleasure for me to work with her.

Further I want to thank all members of the Arbeitsgemeinschaft Funktionalanalyse Tübingen for the wonderful working atmosphere I enjoyed during my time in Tübingen.

I would like to thank Prof. Ulf Schlotterbeck, Prof. Abdelaziz Rhandi, Prof. Gregor Nickel, Prof. Klaus-Jochen Engel, Dr. Susanna Piazzera, Dr. Abdelhadi Es-Sahir, Dr. Tanja Eisner, Britta Dorn and Vera Keicher from whom I got many important suggestions and hints.

I want to thank Prof. Geni Gupur who arouse my interest in queueing theory.

My stay in Tübingen would not have been possible without the support from the Germany Academic Exchange Service (DAAD) to whom I want to express my gratitude as well.

I also owe sincere thanks to Xinjiang University and China Scholarship Council(CSC) for the financial support I got during my stay in Germany in 2005.

Finally, I want to express my sincere thanks to colleagues, friends and relatives in China who encouraged me to complete this work, especially to my mother and to my wife.

CHAPTER 1

Preliminary Results in Semigroup Theory

We assume that the reader is already familiar with the basic functional analysis and the theory of C_0 -semigroups on Banach spaces and refer to [EN00], [EN06], [Gol85] and [Paz83].

1.1. Positive Operators

The theory of positive operators on Banach lattices is used throughout this thesis. Therefore we recall some basic definitions and properties of Banach lattices and positive operators. These results about Banach lattices and positive operators can be found, e.g., in [Sch74], [Nag86] and [MN91].

We start by defining an order relation on vector spaces.

DEFINITION 1.1.1. A relation \geq is said to be an order relation on a nonempty set E if the following conditions are satisfied

- (i) (reflexivity) $x \leq x$ for every $x \in E$,
- (ii) (anti-symmetry) $x \leq y$ and $y \leq x$ implies $x = y$,
- (iii) (transitivity) $x \leq y$ and $y \leq z$ implies $x \leq z$.

DEFINITION 1.1.2.

- (i) A real vector space E is called an ordered vector space if there is an order relation \leq defined on E such that for $f, g \in E$

$$\begin{aligned} f \leq g &\implies f + h \leq g + h \quad \text{for all } h \in E \\ f \leq g &\implies \alpha f \leq \alpha g \quad \text{for all } \alpha \geq 0. \end{aligned}$$

- (ii) An ordered vector space E is called a vector lattice if any two elements $f, g \in E$ have the supremum (i.e. least upper bound)

$$\sup(f, g)$$

and the infimum (i.e. greatest lower bound)

$$\inf(f, g).$$

Clearly, the notation $g \geq f$ means that $f \leq g$. Moreover, $f > 0$ means that $f \geq 0$ and $f \neq 0$. If $g \leq f$, then the set

$$[g, f] := \{h \in E : g \leq h \leq f\}$$

is called an order interval. Let E be an ordered vector space. We denote by $E_+ := \{f \in E : f \geq 0\}$ the positive cone of E . If $f \in E_+$, then we say that f is positive. If E is a vector lattice, then the positive part of $f \in E$ is

$$f^+ := \sup(f, 0),$$

and the negative part of f is

$$f^- := \sup(-f, 0),$$

while the absolute value or modulus of f is

$$|f| := \sup(f, -f).$$

Note that $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

We now give the definition of lattice norm and Banach lattice

DEFINITION 1.1.3.

(i) A norm $\|\cdot\|$ on a vector lattice E is called a lattice norm if

$$|f| \leq |g| \implies \|f\| \leq \|g\|, \quad \text{for } f, g \in E.$$

(ii) A vector lattice endowed with a lattice norm is called a normed vector lattice.

(iii) A complete normed vector lattice is called a Banach lattice .

Complex Banach lattices will be used in our thesis. Therefore we now introduce the concept of a complex Banach lattice.

DEFINITION 1.1.4. Let E be a real Banach lattice, then its complexification

$$E_{\mathbb{C}} := E \times iE$$

with scalar multiplication

$$(\alpha + i\beta)(f, g) = (\alpha f - \beta g, \beta f + \alpha g) \quad \text{for } \alpha, \beta \in \mathbb{R}, (f, g) \in E_{\mathbb{C}}$$

is called a complex Banach lattice.

The space E is the real part of $E_{\mathbb{C}}$. For $f, g \in E_{\mathbb{C}}$ we write $f \geq g$ if $f, g \in E$ and if $f \geq g$ holds. The modulus of $(f, g) \in E_{\mathbb{C}}$ is

$$|(f, g)| := \sup_{0 \leq \phi < 2\pi} |(\cos \phi)f + (\sin \phi)g|.$$

We can show that the modulus indeed exists, see [Nag86, Sect.C-I 7]. Moreover, the norm on $E_{\mathbb{C}}$ is defined by

$$\|(f, g)\|_{E_{\mathbb{C}}} := \||f, g|\|_E.$$

Important classes of Banach lattices that play a significant role later are provided by AL-spaces, see [Sch74, Def.II.8.1].

DEFINITION 1.1.5. A Banach lattice E is an AL-space if

$$\|f + g\| = \|f\| + \|g\|$$

for all $f, g \in E_+$.

The space \mathbb{C} and $L_{\mathbb{C}}^1(\Omega, \mu)$ are complex Banach lattices. The underlying real vector lattices are \mathbb{R} endowed with the usual order and $L_{\mathbb{R}}^1(\Omega, \mu)$ endowed with the order

$$f \geq g \quad \text{if} \quad f(x) \geq g(x) \quad \text{for almost all } x \in \Omega.$$

Moreover, for $f \in L_{\mathbb{C}}^1(\Omega, \mu)$ the modulus is

$$|f|(x) = |f(x)|, \quad x \in \Omega.$$

In this thesis, spaces like

$$\mathbb{C}^n \times l^1(L_{\mathbb{C}}^1(\Omega, \mu))$$

occur. They are complex AL-spaces with underlying real spaces

$$\mathbb{R}^n \times l^1(L_{\mathbb{R}}^1(\Omega, \mu)).$$

Their order is given by

$$(f_i)_{i \in \mathbb{N}} \geq (g_i)_{i \in \mathbb{N}} \quad \text{if} \quad f_i \geq g_i \quad \text{for all } i \in \mathbb{N}.$$

The modulus of $(f_i)_{i \in \mathbb{N}} \in \mathbb{C}^n \times l^1(L_{\mathbb{C}}^1(\Omega, \mu))$ is

$$|(f_i)_{i \in \mathbb{N}}| = (|f_i|)_{i \in \mathbb{N}}.$$

We now turn our attention to operators and semigroups on these spaces and give the definition of positive operator and positive semigroup.

DEFINITION 1.1.6. Let E be a real Banach lattice.

(i) A linear operator T on E is called positive ($T \geq 0$ in symbols) if

$$Tf \geq 0 \quad \text{for all } f \geq 0.$$

(ii) A linear operator T on E is called strictly positive ($T \gg 0$ in symbols) if

$$Tf > 0 \quad \text{for all } f > 0.$$

(iii) A strongly continuous semigroup $(S(t))_{t \geq 0}$ on E is called positive if $S(t) \geq 0$ for all $t \geq 0$.

We can extend this definition to operators on complex vector lattices mapping the underlying real part into the real part. In this case, positivity or strict positivity means that the restriction of the operator to the real part is positive or strictly positive, respectively.

Note that for a positive operator T on a vector lattice E the inequality

$$|Tf| \leq T|f|$$

holds for all $f \in E$, see [Sch74, p.58].

The following subspaces play an important role in the theory of positive operators.

DEFINITION 1.1.7. A linear subspace F of a real or complex Banach lattice E is called an ideal in E if

$$f \in F, |g| \leq |f| \implies g \in F.$$

REMARK 1.1.8.

- (i) The ideals in \mathbb{C}^n are the subspaces

$$J_H := \{x = (x_i)_{1 \leq i \leq n} \in \mathbb{C}^n : x_i = 0 \text{ for } i \in H\},$$

where H is an arbitrary subset of $\{1, \dots, n\}$, see [Sch74, p.2].

- (ii) Let $E = L_{\mathbb{C}}^1(\Omega, \mu)$. Every closed ideal in $L_{\mathbb{C}}^1(\Omega, \mu)$ is of the form

$$I_M := \{f \in E : f(x) = 0 \text{ for almost all } x \in M\},$$

where M is a measurable subset of Ω . Conversely, every set I_M is a closed ideal in $L_{\mathbb{C}}^1(\Omega, \mu)$, see [Sch74, Example III.1.2].

The ideal E_f generated by $f \in E_+$ is the smallest ideal containing f . By [Sch74, Example II.2.1] the equality

$$E_f = \bigcup_{n \in \mathbb{N}} n[-f, f]$$

holds.

DEFINITION 1.1.9. Let $f \in E_+$. If $\overline{E_f} = E$, then f is called a quasi-interior point of E_+ .

REMARK 1.1.10. A function $f \in L_{\mathbb{C}}^1(\Omega, \mu)$ is a quasi-interior point if and only if $f(x) > 0$ for almost all $x \in \Omega$. In this case, we write $f \gg 0$.

Irreducibility of the semigroups is very useful in discussing the asymptotic behaviour. In the following we briefly recall the basic definition for positive operators and positive semigroups.

DEFINITION 1.1.11.

- (i) A positive linear operator B on E is called irreducible if there is no non-trivial closed ideal in E which invariant under B .
(ii) A positive semigroup $(S(t))_{t \geq 0}$ on E is called irreducible if there is no non-trivial closed ideal in E which invariant under $(S(t))_{t \geq 0}$.

According to [Nag86, Def.C-III 3.1], we state the following equivalent assertions to irreducibility of a semigroup on Banach lattice E .

PROPOSITION 1.1.12. *Let B be the generator of a positive semigroup $(S(t))_{t \geq 0}$. The following assertions are equivalent.*

- (i) *The semigroup $(S(t))_{t \geq 0}$ is irreducible.*
(ii) *If $f \in E$ and $f \gneq 0$, then $R(\gamma, B)f \gg 0$ for (some) all $\gamma > s(B)$.*

1.2. Well-posedness of the abstract Cauchy problem

In this section we present some definitions and tools to study problems arising in our context.

We recall the following definitions from [EN00, Def. II.6.1 (ii)].

DEFINITION 1.2.1. Let X be a Banach space and let $(B, D(B))$ be a linear operator on X , and $u_0 \in X$. The initial value problem

$$\begin{cases} \frac{du(t)}{dt} = Bu(t), & t \in [0, \infty), \\ u(0) = u_0. \end{cases} \quad (\text{ACP})$$

is called the abstract Cauchy problem associated to $(B, D(B))$ with initial value u_0 .

DEFINITION 1.2.2. A function $u(\cdot, u_0) : [0, \infty) \longrightarrow X$ is called a classical solution of (ACP) if

- (i) $u(\cdot, u_0)$ is continuously differentiable,
- (ii) $u(t, u_0) \in D(B)$ for all $t \geq 0$, and
- (iii) (ACP) holds,

According to [EN00, Def. II.6.8] we have the following definition.

DEFINITION 1.2.3. The problem (ACP) is called well-posed if

- (i) for every initial value $u_0 \in D(B)$ there exists a unique classical solution $u(\cdot, u_0)$ of (ACP),
- (ii) $D(B)$ is dense in X , and
- (iii) for every sequence $(u_n)_{n \in \mathbb{N}} \subseteq D(B)$ satisfying

$$\lim_{n \rightarrow \infty} u_n = 0$$

one has

$$\lim_{n \rightarrow \infty} u(t, u_n) = 0$$

uniformly on compact intervals $[0, t_0]$.

We now characterize the well-posedness of (ACP) as follows, see [EN00, Cor.II.6.9].

PROPOSITION 1.2.4. *For a closed operator $(B, D(B))$ on X the associated abstract Cauchy problem (ACP) is well-posed if and only if $(B, D(B))$ generates a strongly continuous semigroup on X .*

Therefore, to solve an abstract Cauchy problem means to show that the operator $(B, D(B))$ generates a strongly continuous semigroup on X . If (ACP) is

well-posed, then, by [EN00, Prop.II.6.2], the unique classical solution is given by the orbit of u_0 under the semigroup $(T(t))_{t \geq 0}$ generated by B , i.e.

$$u(t) = T(t)u_0, \quad t \geq 0.$$

Next, we are interested in generators of positive semigroups. To this purpose we give the following definition from [Nag86, p.249]

DEFINITION 1.2.5. A linear operator $(B, D(B))$ on a real Banach lattice E is called dispersive if for every $z \in D(B)$ there exists a $\chi \in E'$ such that $\|\chi\| \leq 1$, $\langle z, \chi \rangle = \|z^+\|$ and $\langle Bz, \chi \rangle \leq 0$.

Generators of positive contraction semigroups are characterised by the following theorem, see [Nag86, Thm.C-II 1.2].

THEOREM 1.2.6. (*Phillips theorem*) Let B be a densely defined operator on a real Banach lattice E . The following assertions are equivalent.

- (i) B is the generator of a positive contraction semigroup.
- (ii) B is dispersive and $\gamma - B$ is surjective for some $\gamma > 0$.

1.3. Characteristic Equation

We now consider a class of operators $(A, D(A))$ which are constructed in a particular way. We start from a closed linear operator $(A_m, D(A_m))$, called the maximal operator. Moreover, we take another Banach space ∂X the boundary space and use boundary operators $L, \Phi \in \mathcal{L}(D(A_m), \partial X)$. In the following we always assume that L is surjective.

DEFINITION 1.3.1. The operator $(A, D(A))$ is defined as

$$\begin{aligned} Ap &:= A_m p, \\ D(A) &:= \{p \in D(A_m) \mid Lp = \Phi p\}. \end{aligned}$$

Under appropriate assumptions, it is possible to characterize the spectrum $\sigma(A)$ and give an explicit representation of its resolvent. The abstract framework for this was developed by G. Greiner in [Gre87]. We sketch these results. The starting point is the operator $(A_0, D(A_0))$ which is the restriction of A_m to the kernel of L , i.e.

$$\begin{aligned} D(A_0) &:= \{p \in D(A_m) \mid Lp = 0\}, \\ A_0 p &:= A_m p. \end{aligned}$$

The domain $D(A_m)$ of the maximal operator A_m decomposes, using [Gre87, Lemma 1.2], as follows.

LEMMA 1.3.2. For $\gamma \in \rho(A_0)$ one has

$$D(A_m) = D(A_0) \oplus \ker(\gamma - A_m).$$

Since L is supposed to be surjective and $D(A_0) = \ker L$, we conclude from the above decomposition that the restriction $L|_{\ker(\gamma - A_m)}$ of L to $\ker(\gamma - A_m)$ is bijective. It follows from the closed graph theorem that the inverse of $L|_{\ker(\gamma - A_m)}$ is bounded.

DEFINITION 1.3.3. For $\gamma \in \rho(A_0)$, the operator $D_\gamma := (L|_{\ker(\gamma - A_m)})^{-1}$ is called Dirichlet operator corresponding to A_m and L .

The operators D_γ and Φ allow to characterise the spectrum $\sigma(A)$ and the point spectrum $\sigma_p(A)$ of A . Before doing so we extend the given operators to the product $X \times \partial X$ as in [KS05, Sect. 3], see also [Rad06], [HR07b].

DEFINITION 1.3.4.

- (i) $\mathcal{X} := X \times \partial X$.
- (ii) $\mathcal{A}_0 := \begin{pmatrix} A_m & 0 \\ -L & 0 \end{pmatrix}$, $D(\mathcal{A}_0) := D(A_m) \times \{0\}$.
- (iii) $\mathcal{X}_0 := X \times \{0\} = \overline{D(A_m) \times \{0\}} = \overline{D(\mathcal{A}_0)}$.
- (iv) $\mathcal{B} := \begin{pmatrix} 0 & 0 \\ \Phi & 0 \end{pmatrix}$, $D(\mathcal{B}) := D(A_m) \times \partial X$.
- (v) $\mathcal{A} := \mathcal{A}_0 + \mathcal{B} = \begin{pmatrix} A_m & 0 \\ \Phi - L & 0 \end{pmatrix}$, $D(\mathcal{A}) := D(A_m) \times \{0\}$.

REMARK 1.3.5.

- (i) Note that $\rho(\mathcal{A}_0) \supseteq \rho(A_0)$. For $\gamma \in \rho(A_0)$ the resolvent of \mathcal{A}_0 is given by

$$R(\gamma, \mathcal{A}_0) = \begin{pmatrix} R(\gamma, A_0) & D_\gamma \\ 0 & 0 \end{pmatrix}.$$

- (ii) The part $\mathcal{A}|_{\mathcal{X}_0}$ of \mathcal{A} in \mathcal{X}_0 is

$$D(\mathcal{A}|_{\mathcal{X}_0}) = D(A) \times \{0\}, \quad \mathcal{A}|_{\mathcal{X}_0} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $\mathcal{A}|_{\mathcal{X}_0}$ can be identified with the operator $(A, D(A))$.

The following shows that the spectrum of A is characterised by the spectrum of operators on the boundary space ∂X .

CHARACTERISTIC EQUATION 1.3.6. *Let $\gamma \in \rho(A_0)$. Then*

- (i)

$$\gamma \in \sigma_p(A) \iff 1 \in \sigma_p(\Phi D_\gamma).$$

- (ii) *If, in addition, there exists $\gamma_0 \in \mathbb{C}$ such that $1 \notin \sigma(\Phi D_{\gamma_0})$, then*

$$\gamma \in \sigma(A) \iff 1 \in \sigma(\Phi D_\gamma).$$

PROOF. As in [KS05, Prop. 3.3], we first show the equivalence

$$\gamma \in \sigma(\mathcal{A}) \iff 1 \in \sigma(\Phi D_\gamma). \tag{1}$$

We can decompose $\gamma - \mathcal{A}$ as

$$\gamma - \mathcal{A} = \gamma - \mathcal{A}_0 - \mathcal{B} = (\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0))(\gamma - \mathcal{A}_0). \quad (2)$$

We conclude from this that the invertibility of $\gamma - \mathcal{A}$ is equivalent to the invertibility of $\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0)$. From

$$\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0) = \begin{pmatrix} Id_X & 0 \\ -\Phi R(\gamma, A_0) & Id_{\partial X} - \Phi D_\gamma \end{pmatrix}, \quad (3)$$

one can easily see that $\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0)$ is invertible if and only if $1 \notin \sigma(\Phi D_\gamma)$. This proves (1). Since by our assumption $1 \notin \sigma(\Phi D_{\gamma_0})$, it follows that $\gamma_0 \in \rho(\mathcal{A})$. Therefore, $\rho(\mathcal{A})$ is not empty. Hence we obtain from [EN00, Prop. IV.2.17] that

$$\sigma(\mathcal{A}) = \sigma(A),$$

since A is the part of \mathcal{A} in \mathcal{X}_0 . This shows (ii).

To prove (i) observe first that \mathcal{A} and A have the same point spectrum, i.e.,

$$\sigma_p(\mathcal{A}) = \sigma_p(A).$$

Suppose now that $1 \in \sigma_p(\Phi D_\gamma)$. Then there exists $0 \neq f \in \partial X$ such that $(Id_{\partial X} - \Phi D_\gamma)f = 0$. Since $0 \neq \begin{pmatrix} D_\gamma f \\ 0 \end{pmatrix} \in D(\mathcal{A})$, we can compute

$$\begin{aligned} (\gamma - \mathcal{A}) \begin{pmatrix} D_\gamma f \\ 0 \end{pmatrix} &= \begin{pmatrix} Id_X & 0 \\ -\Phi R(\gamma, A_0) & Id_{\partial X} - \Phi D_\gamma \end{pmatrix} \begin{pmatrix} (\gamma - A_m)D_\gamma f \\ LD_\gamma f \end{pmatrix} \\ &= \begin{pmatrix} Id_X & 0 \\ -\Phi R(\gamma, A_0) & Id_{\partial X} - \Phi D_\gamma \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ (Id_{\partial X} - \Phi D_\gamma)f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

This shows that $\gamma \in \sigma_p(\mathcal{A})$.

Conversely, if we assume that $\gamma \in \sigma_p(\mathcal{A})$, then there exists $0 \neq f \in D(A_m)$ such that $(\gamma - \mathcal{A}) \begin{pmatrix} f \\ 0 \end{pmatrix} = 0$. From

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= (\gamma - \mathcal{A}) \begin{pmatrix} f \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} Id_X & 0 \\ -\Phi R(\gamma, A_0) & Id_{\partial X} - \Phi D_\gamma \end{pmatrix} \begin{pmatrix} (\gamma - A_m)f \\ Lf \end{pmatrix} \\ &= \begin{pmatrix} (\gamma - A_m)f \\ -\Phi R(\gamma, A_0)(\gamma - A_m)f + (Id_{\partial X} - \Phi D_\gamma)Lf \end{pmatrix} \end{aligned}$$

we conclude that $f \in \ker(\gamma - A_m)$ and thus

$$0 = -\Phi R(\gamma, A_0)(\gamma - A_m)f + (Id_{\partial X} - \Phi D_\gamma)Lf = (Id_{\partial X} - \Phi D_\gamma)Lf.$$

It follows from Lemma 1.3.2 that $Lf \neq 0$ and hence $1 \in \sigma_p(\Phi D_\gamma)$. \square

The boundary space ∂X will be much smaller than the state space X in most cases. So it is easier to determine the spectrum of ΦD_γ than to compute the spectrum of A directly.

For later use, we determine the resolvent of A in terms of the resolvent of A_0 , the Dirichlet operator D_γ and the boundary operator Φ .

LEMMA 1.3.7. *Suppose that there exists $\gamma_0 \in \mathbb{C}$ such that $1 \notin \sigma(\Phi D_{\gamma_0})$ and let $\gamma \in \rho(A_0) \cap \rho(A)$. Then*

$$R(\gamma, A) = R(\gamma, A_0) + D_\gamma (Id - \Phi D_\gamma)^{-1} \Phi R(\gamma, A_0).$$

PROOF. Under our assumption, we see from the Characteristic Equation 1.3.6 that $1 \notin \sigma(\Phi D_\gamma)$ and it follows from the proof that $\gamma - \mathcal{A}$ is invertible with inverse

$$(\gamma - \mathcal{A})^{-1} = (\gamma - \mathcal{A}_0)^{-1} (\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0))^{-1}.$$

Using the explicit representation (3) for $\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0)$ we obtain

$$(\mathcal{I} - \mathcal{B}R(\gamma, \mathcal{A}_0))^{-1} = \begin{pmatrix} Id_X & 0 \\ (Id_{\partial X} - \Phi D_\gamma)^{-1} \Phi R(\gamma, A_0) & (Id_{\partial X} - \Phi D_\gamma)^{-1} \end{pmatrix},$$

and hence

$$R(\gamma, \mathcal{A}) = \begin{pmatrix} R(\gamma) & D_\gamma (Id_{\partial X} - \Phi D_\gamma)^{-1} \\ 0 & 0 \end{pmatrix},$$

where $R(\gamma) = (Id_X + D_\gamma (Id_{\partial X} - \Phi D_\gamma)^{-1} \Phi) R(\gamma, A_0)$. Since A is the part of \mathcal{A} in \mathcal{X}_0 and since

$$\begin{pmatrix} R(\gamma) & 0 \\ 0 & 0 \end{pmatrix} = R(\gamma, \mathcal{A})|_{\mathcal{X}_0} = R(\gamma, \mathcal{A}|_{\mathcal{X}_0}),$$

it follows that

$$R(\gamma, A) = R(\gamma).$$

□

REMARK 1.3.8. The problems we investigate in this thesis are formulated by partial differential equations involving nontrivial boundary conditions. These problems will be rewritten as abstract Cauchy problems of the form (ACP) and we will apply semigroup theory to prove the existence and uniqueness as well as the asymptotic stability of the solutions.

All our operators will arise in the abstract form of Definition 1.3.1. Here, the maximal operator is a differential operator on its natural maximal domain while the boundary space consists of functions "on the boundary". The domain $D(A)$ of A incorporates the boundary conditions of the underlying problems.

We will determine the spectra of these operators in detail using the Characteristic Equation 1.3.6.

1.4. Asymptotic Stability of Positive Semigroups

The main subject of this thesis is to discuss the asymptotic behaviour of the solutions of the previous problems using the theory of irreducible positive semigroups. Therefore, we first collect some results on this aspect from [Nag86] and [Sch74].

Let E be a Banach lattice and $(B, D(B))$ be the generator of a positive semigroup $(S(t))_{t \geq 0}$ on E . The fixed space of the semigroup $(S(t))_{t \geq 0}$ is

$$\text{fix}(S(t))_{t \geq 0} = \bigcap_{t \geq 0} \text{fix}(S(t)) = \{z \in E : S(t)z = z \text{ for all } t \geq 0\}.$$

According to [EN00, Cor. IV.3.8 (i)] we have the equality

$$\text{fix}(S(t))_{t \geq 0} = \ker B. \quad (4)$$

To study the asymptotic behaviour of the semigroup $(S(t))_{t \geq 0}$ the following compactness property is useful.

LEMMA 1.4.1. *Let E be an AL-space and let the positive semigroup $(S(t))_{t \geq 0}$ be irreducible and bounded. If $0 \in \sigma_p(B)$, then $\{S(t) : t \geq 0\} \subseteq \mathcal{L}(E)$ is relatively compact for the weak operator topology. In particular, it is mean ergodic, i.e.*

$$\lim_{r \rightarrow \infty} \frac{1}{r} \int_0^r S(s)z \, ds$$

exists for all $z \in E$.

PROOF. From the assumption $0 \in \sigma_p(B)$ and (4) it follows that there exists $0 \neq z \in \text{fix}(S(t))_{t \geq 0}$. By the positivity of the semigroup, the inequality

$$S(t)^n |z| = S(t)^n |S(t)z| \leq S(t)^{n+1} |z| \quad (5)$$

holds for all $n \in \mathbb{N}$ and $t \geq 0$, see [Sch74, p.58]. Note that semigroup $(S(t))_{t \geq 0}$ is bounded by assumption, therefore $(S(t)^n |z|)_{n \in \mathbb{N}}$ is norm-bounded. From [Sch74, Prop.II.8.3] we know that the sequence converges to an element $z_0 \succeq 0$. In this step we use that E is an AL-space. From

$$S(t)z_0 = S(t) \lim_{n \rightarrow \infty} S(t)^n |z| = \lim_{n \rightarrow \infty} S(t)^{n+1} |z| = z_0$$

we obtain that $z_0 \in \text{fix}(S(t))_{t \geq 0}$. Thus, we can assume without loss of generality that $z \succeq 0$.

Since the semigroup is irreducible, we obtain from [Nag86, Prop.C-III 3-5(a)] that z is a quasi-interior point of E which means that

$$E_z := \bigcup_{n \geq 1} [-nz, nz]$$

is dense in E .

Take $w \in [-nz, nz]$. Then

$$-nz = -nS(t)z \leq S(t)w \leq nS(t)z = nz$$

for all $t \geq 0$. Since the order interval $[-nz, nz]$ is weakly compact, see [Sch74, p.92], the orbit $\{S(t)w : t \geq 0\}$ is relatively weakly compact. So far we have shown that the orbit of the elements w from the dense subset E_z of E are relatively weakly compact. Since the semigroup $(S(t))_{t \geq 0}$ is bounded, it follows from [EN00, Lem.V.2.7] that $\{S(t) : t \geq 0\} \subseteq \mathcal{L}(E)$ is relatively compact for the weak operator topology. By [EN00, Lem.V.2.7] we obtain that the semigroup $(S(t))_{t \geq 0}$ is mean ergodic. \square

Using the mean ergodicity of the semigroup we can decompose E into the direct sum of $\ker B$ and rgB . If the semigroup is irreducible, then $\ker B$ is one-dimensional. If in addition $\sigma(B) \cap i\mathbb{R} = \sigma_p(B) \cap i\mathbb{R} = \{0\}$, then the semigroup converges strongly to one dimensional projection onto $\ker B$. This is a consequence of the Arendt-Batty-Lyubich-Vũ Theorem 1.4.2.

THEOREM 1.4.2. *Let E be an AL-space and the positive semigroup $(S(t))_{t \geq 0}$ be irreducible, and bounded. If*

$$\sigma(B) \cap i\mathbb{R} = \sigma_p(B) \cap i\mathbb{R} = \{0\},$$

then E can be decomposed into the direct sum

$$E = E_1 \oplus E_2,$$

where $E_1 = \text{fix}(S(t))_{t \geq 0} = \ker B$ is one-dimensional and spanned by a strictly positive eigenvector $\tilde{p} \in \ker B$ of B . In addition, the restriction $(S(t)|_{E_2})_{t \geq 0}$ is strongly stable.

PROOF. Since the semigroup $(S(t))_{t \geq 0}$ is mean ergodic by Lemma 1.4.1, the space E can be decomposed into

$$E = \ker B \oplus \overline{rgB} =: E_1 \oplus E_2,$$

where $\ker B = \text{fix}(S(t))_{t \geq 0}$, E_1 and E_2 are invariant under $(S(t))_{t \geq 0}$, see [EN00, Lem.V.4.4]. There exists $\tilde{z} \in \ker B$ such that $\tilde{z} > 0$, confer the proof of Lemma 1.4.1. Moreover, by the same construction as in the proof of [EN00, Lem.V.2.20(i)], we find $z' \in E'$ such that $z' > 0$ and $B'z' = 0$. Hence we obtain that

$$\dim \ker B = 1$$

and that \tilde{z} is strictly positive, i.e. $\tilde{z} \gg 0$, see [Nag86, Prop. C-III 3.5].

We now consider the generator $(B_2, D(B_2))$ of the restricted semigroup $(S_2(t))_{t \geq 0}$ where

$$B_2v = Bv, \quad D(B_2) = D(B) \cap E_2.$$

and $S_2(t) = S(t)|_{E_2}$. Since by Lemma 1.4.1 every $z \in E$ has a relatively weakly compact orbit. $(S_2(t))_{t \geq 0}$ is totally ergodic on X_2 , i.e., $(e^{-iat}S(t))_{t \geq 0}$ is mean ergodic for all $a \in \mathbb{R}$ by [ABHN01, Prop. 4.3.12]. This implies that $\ker(B_2 - iat)$

separates $\ker(B'_2 - iat)$ for all $a \in \mathbb{R}$, see [EN00, Thm. V.4.5]. By our assumption $\ker(B_2 - iat) = \{0\}$, thus $\ker(B'_2 - iat) = \{0\}$ for all $a \in \mathbb{R}$. Hence, it follows that $\sigma_p(B'_2) \cap i\mathbb{R} = \emptyset$. Applying the Arendt-Batty-Lyubich-Vũ Theorem, see [ABHN01, Thm. 5.5.5], we obtain the strong stability of $(T_2(t))_{t \geq 0}$. \square

CHAPTER 2

The Dynamic $M/M^B/1$ Queueing System

2.1. Introduction

The $M/M^B/1$ queueing model describes a single server queue which can at most serve $B \in \mathbb{N}$ customers simultaneously. This problem has been studied in [GZ98] and [Gup02], where the authors showed the well-posedness of the $M/M^B/1$ queueing model. Here, we give a more detailed analysis of the time-dependent solution and show the existence of a unique positive steady state solution of this model. Some of our results will appear in [HR07a]

In this model, the server starts service as soon as there is at least one customer in the queue. If a customer arrives while the server is busy, then the customer joins the queue. There is assumed to be an infinite supply of customers. The customers arrive at random and their arrival obeys a Poisson process with parameter λ , the so-called *arrival rate*. The service time is exponentially distributed with parameter μ , the so-called *service rate*. The mean service rate is $\frac{1}{\mu}$.

For these parameters we assume the following.

GENERAL ASSUMPTION 2.1.1. *The parameters λ and μ fulfill*

$$0 < \lambda < \mu.$$

The ratio

$$\rho := \frac{\lambda}{\mu}$$

is called *traffic rate* or *traffic intensity*. From the above general assumption it follows that $\rho < 1$.

We need two time parameters to describe the above system. The parameter $t \in [0, \infty)$ counts the time of the evolution of the whole system, whereas $x \in [0, \infty)$ counts the elapsed service time. The service time x is reset to 0 whenever a new service starts.

$p_{0,0}(t)$ gives the probability that the queue is empty and the server is idle at time t . Moreover, $p_{n,1}(x, t)dx, n \in \mathbb{N} \cup \{0\}$, gives the probability that at time t there are n customers in the queue and the elapsed service time lies in $(x, x + dx]$, B is maximum size of service.

According to [CT83], the $M/M^B/1$ queueing model can be expressed by the equations

$$(MQ) \begin{cases} \frac{dp_{0,0}(t)}{dt} = -\lambda p_{0,0}(t) + \mu \int_0^\infty p_{0,1}(x, t) dx, \\ \frac{\partial p_{0,1}(x, t)}{\partial t} + \frac{\partial p_{0,1}(x, t)}{\partial x} = -(\lambda + \mu) p_{0,1}(x, t), \\ \frac{\partial p_{n,1}(x, t)}{\partial t} + \frac{\partial p_{n,1}(x, t)}{\partial x} = -(\lambda + \mu) p_{n,1}(x, t) + \lambda p_{n-1,1}(x, t), \quad n \geq 1. \end{cases}$$

For $x = 0$ the boundary conditions

$$(BC_{MQ}) \begin{cases} p_{0,1}(0, t) = \sum_{k=1}^B \mu \int_0^\infty p_{k,1}(x, t) dx + \lambda p_{0,0}(t), \\ p_{n,1}(0, t) = \mu \int_0^\infty p_{n+B,1}(x, t) dx, \quad n \geq 1 \end{cases}$$

are imposed and we consider the usual initial condition

$$(IC_{MQ}) \begin{cases} p_{0,0}(0) = c \in [0, 1], \\ p_n(x, 0) = f_n(x) \quad \text{for } n \geq 0, \end{cases}$$

where $f_n \in L^1[0, \infty)$. But the most important initial condition at time $t = 0$ is

$$(IC_{MQ,0}) \begin{cases} p_{0,0}(0) = 1, \\ p_{n,1}(x, 0) = 0, \quad n \geq 0, \end{cases}$$

which means that at time $t = 0$ the server as well as the queue are empty.

2.2. The Problem as an Abstract Cauchy Problem

We reformulate the underlying problem as an abstract Cauchy problem with an operator $(A_{MQ}, D(A_{MQ}))$ on the state space $X_{MQ} := \mathbb{C} \times l^1(L[0, \infty))$. For $p = (p_{0,0}, p_{0,1}(\cdot), p_{1,1}(\cdot), \dots)^t \in X_{MQ}$, the norm of p is defined as

$$\|p\| := |p_{0,0}| + \sum_{n=0}^{\infty} \|p_{n,1}(\cdot)\|_{L^1[0, \infty)}.$$

In the following, ψ denotes the linear functional

$$\psi : L^1[0, \infty) \rightarrow \mathbb{C}, \quad f \mapsto \psi(f) := \int_0^\infty f(x) dx.$$

Moreover, the operator D on $W^{1,1}[0, \infty)$ is defined as

$$Df := -\frac{d}{dx}f - (\lambda + \mu)f.$$

With these operators we build a *maximal operator* $(A_m, D(A_m))$ on X as

$$A_m^{MQ} := \begin{pmatrix} -\lambda & \mu\psi & 0 & 0 & \cdots \\ 0 & D & 0 & 0 & \cdots \\ 0 & \lambda & D & 0 & \cdots \\ 0 & 0 & \lambda & D & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

$$D(A_m^{MQ}) := \mathbb{C} \times l^1(W^{1,1}[0, \infty)).$$

As boundary space we choose

$$\partial X_{MQ} := l^1$$

and define the boundary operators as

$$L_{MQ} : D(A_m^{MQ}) \rightarrow \partial X_{MQ}, \quad \begin{pmatrix} p_{0,0} \\ p_{0,1} \\ p_{1,1} \\ \vdots \end{pmatrix} \mapsto L_{MQ} \begin{pmatrix} p_{0,0} \\ p_{0,1} \\ p_{1,1} \\ \vdots \end{pmatrix} := \begin{pmatrix} p_{0,1}(0) \\ p_{1,1}(0) \\ \vdots \end{pmatrix},$$

and the operator $\Phi_{MQ} \in \mathcal{L}(X_{MQ}, \partial X_{MQ})$ is given by operator matrix

$$\Phi_{MQ} = \begin{pmatrix} \lambda & 0 & \overbrace{\mu\psi \quad \mu\psi \quad \cdots \quad \mu\psi}^B & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu\psi & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \mu\psi & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Then, we obtain the operator $(A_{MQ}, D(A_{MQ}))$ on X_{MQ} corresponding to the underlying problem as

$$\begin{aligned} A_{MQ}p &:= A_m^{MQ}p, \\ D(A_{MQ}) &:= \{p \in D(A_m^{MQ}) \mid L_{MQ}p = \Phi_{MQ}p\}. \end{aligned}$$

With these definitions the above equations $(MQ), (BC_{MQ}), (IC_{MQ})$ can be reformulated as the abstract Cauchy problem

$$\begin{cases} \frac{dp(t)}{dt} = A_{MQ}p(t), & t \in [0, \infty), \\ p(0) = (c, f_1, f_2, \dots)^t \in X_{MQ}. \end{cases} \quad (ACP_{MQ})$$

So if A_{MQ} is the generator of a strongly continuous semigroup $(T_{MQ}(t))_{t \geq 0}$ and the initial value in (IC_{MQ}) satisfies $p(0) = (c, f_1, f_2, \dots)^t \in D(A_{MQ})$, then the unique solution of $(BC_{MQ}), (BC_{MQ})$ and (IC_{MQ}) is given by

$$\begin{aligned} p_{0,0}(t) &= (T_{MQ}(t)p(0))_1 \\ p_{n,1}(x, t) &= (T_{MQ}(t)p(0))_{n+1}(x), \quad n \geq 0. \end{aligned}$$

For this reason it suffices to study (ACP_{MQ}) .

2.3. Boundary Spectrum

In this section we use Characteristic Equation 1.3.6 to investigate the boundary spectrum $\sigma_b(A_{MQ})$ of A_{MQ} . We first characterise $\sigma(A_{MQ})$ by the spectrum of an infinite scalar matrix, i.e. an operator on the boundary space ∂X_{MQ} . To do so we apply techniques and results from [Gre87]. In particular, we need more information on the resolvent set of the operator $(A_0^{MQ}, D(A_0^{MQ}))$ defined by

$$\begin{aligned} D(A_0^{MQ}) &:= \{p \in D(A_m^{MQ}) \mid L_{MQ}p = 0\}, \\ A_0^{MQ}p &:= A_m^{MQ}p. \end{aligned}$$

The resolvent set and the resolvent of the operator A_0^{MQ} is given as the following.

LEMMA 2.3.1. *Let $S_{MQ} := \{\gamma \in \mathbb{C} \mid \Re\gamma > -\mu \text{ and } \gamma \neq -\lambda\}$. Then $S_{MQ} \subseteq \rho(A_0^{MQ})$ and for $\gamma \in S_{MQ}$ the resolvent of A_0^{MQ} is given as*

$$R(\gamma, A_0^{MQ}) = \begin{pmatrix} \frac{1}{\gamma+\lambda} & \frac{\mu}{\gamma+\lambda}\psi R(\gamma, D) & 0 & 0 & \cdots \\ 0 & R(\gamma, D) & 0 & 0 & \cdots \\ 0 & \lambda R^2(\gamma, D) & R(\gamma, D) & 0 & \cdots \\ 0 & \lambda^2 R^3(\gamma, D) & \lambda R(\gamma, D) & R(\gamma, D) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

where

$$(R(\gamma, D)f)(x) = e^{-(\gamma+\lambda+\mu)x} \int_0^x e^{(\gamma+\lambda+\mu)s} f(s) ds$$

for $f \in L^1[0, \infty)$.

PROOF. We first show that for $\gamma \in S_{MQ}$ the operator $R(\gamma, A_0^{MQ})$ is bounded. We denote by $C_c[0, \infty)$ the space of continuous functions with compact support. For $f \in C_c[0, \infty)$ we estimate

$$\begin{aligned} \|R(\gamma, D)f\|_{L^1[0, \infty)} &= \int_0^\infty |(R(\gamma, D)f)(x)| dx \\ &\leq \int_0^\infty e^{-(\Re\gamma+\lambda+\mu)x} \int_0^x e^{(\Re\gamma+\lambda+\mu)s} |f(s)| ds dx \\ &= \left[-\frac{1}{\Re\gamma + \lambda + \mu} e^{-(\Re\gamma+\lambda+\mu)x} \int_0^x e^{(\Re\gamma+\lambda+\mu)s} |f(s)| ds \right]_0^\infty \\ &\quad + \int_0^\infty \frac{1}{\Re\gamma + \lambda + \mu} e^{-(\Re\gamma+\lambda+\mu)x} e^{(\Re\gamma+\lambda+\mu)x} |f(x)| dx \\ &= \frac{1}{\Re\gamma + \lambda + \mu} \|f\|_{L^1[0, \infty)}. \end{aligned}$$

The denseness of $C_c[0, \infty)$ in $L^1[0, \infty)$ and the above estimate implies that

$$\|R(\gamma, D)\| \leq \frac{1}{\Re\gamma + \lambda + \mu}.$$

Therefore,

$$\sum_{k=0}^{\infty} \lambda^k \|R^{k+1}(\gamma, D)\| \leq \frac{1}{\Re\gamma + \lambda + \mu} \sum_{k=0}^{\infty} \left(\frac{\lambda}{\Re\gamma + \lambda + \mu} \right)^k < \infty$$

if $\Re\gamma > -\mu$. This implies that the supremum of the column sums of $R(\gamma, A_0^{MQ})$ taken in the norm are finite, and hence $R(\gamma, A_0^{MQ})$ is a bounded operator on X_{MQ} .

Clearly, the operator $R(\gamma, A_0^{MQ})$ is the inverse of $\gamma - A_0^{MQ}$. \square

The following consequence will be used for the computation of the boundary spectrum of A_{MQ}

COROLLARY 2.3.2. The resolvent set of A_0^{MQ} contains the imaginary axis, i.e.,

$$i\mathbb{R} \subseteq \rho(A_0^{MQ}).$$

The following abbreviations are used in the sequel

$$\Gamma := \gamma + \lambda + \mu$$

and

$$\Lambda := \gamma + \lambda.$$

The eigenfunctions of A_m^{MQ} are determined as follows.

LEMMA 2.3.3. For $\gamma \in \mathbb{C}$, $\Re\gamma > -\mu$, and $\gamma \neq -\lambda$ the following holds.

$$p = (p_{0,0}, p_{0,1}(\cdot), p_{0,1}(\cdot), p_{1,1}(\cdot), \dots) \in \ker(\gamma - A_m^{MQ}) \quad (6)$$

\Leftrightarrow

$$p_{0,0} = \frac{\mu c_1}{\Gamma \Lambda}, \quad (7)$$

$$p_{n,1}(x) = e^{-\Gamma x} \sum_{k=0}^n \frac{\lambda^k}{k!} x^k c_{n+1-k}, \quad n \geq 0, \quad (8)$$

$$\text{and } (c_n)_{n \geq 1} \in l^1.$$

PROOF. We first verify that each p given as in (7)-(8) is contained in $D(A_m^{MQ})$. Note that for $k \in \mathbb{N}$

$$\int_0^{\infty} e^{-\Re\Gamma x} x^k dx = \frac{k!}{(\Re\Gamma)^{k+1}}.$$

Using this we estimate the norm

$$\begin{aligned}
\|p_{n,1}\|_{L^1[0,\infty)} &= \int_0^\infty |e^{-\Gamma x} \sum_{k=0}^n \frac{\lambda^k}{k!} x^k c_{n+1-k}| dx \\
&\leq \sum_{k=0}^n \frac{\lambda^k}{k!} |c_{n+1-k}| \int_0^\infty e^{-\Re\Gamma x} x^k dx \\
&= \sum_{k=0}^n |c_{n+1-k}| \frac{\lambda^k}{(\Re\Gamma)^{k+1}}.
\end{aligned}$$

Since $\Re\gamma > -\mu$ and the series $\sum_{k=0}^\infty (\frac{\lambda}{\Re\Gamma})^k$ converges absolutely. Therefore we can estimate using the Cauchy product

$$\begin{aligned}
\sum_{n=0}^\infty \|p_{n,1}\|_{L^1[0,\infty)} &= \sum_{n=0}^\infty \sum_{k=0}^n |c_{n+1-k}| \frac{\lambda^k}{(\Re\Gamma)^{k+1}} \\
&= \frac{1}{\Re\Gamma} \left(\sum_{k=0}^\infty \left(\frac{\lambda}{\Re\Gamma}\right)^k \right) \left(\sum_{n=0}^\infty |c_{n+1}| \right) \\
&< \infty.
\end{aligned}$$

Hence, the norm $\|p\|_{D(A_m^{MQ})}$ of p is finite and $p \in D(A_m^{MQ})$. We can easily compute that each p as in (7)-(8) satisfies

$$(\gamma - A_m^{MQ})p = 0.$$

Conversely, we assume that $p \in \ker(\gamma - A_m^{MQ})$. We get a system of differential equations from $(\gamma - A_m^{MQ})p = 0$. Solving this we immediately get (7)-(8). From

$$\begin{aligned}
\sum_{n=1}^\infty |c_n| &= \sum_{n=1}^\infty |p_{n,1}(0)| \leq \sum_{n=1}^\infty |p_{n,1}|_\infty \\
&\leq \sum_{n=1}^\infty |p_{n,1}|_{W^{1,1}[0,\infty)} \leq \|p\|_{D(A_m^{MQ})} \\
&< \infty
\end{aligned}$$

we obtain that $(c_n)_{n \geq 1} \in l^1$. □

Moreover, since L_{MQ} is surjective, $L_{MQ}|_{\ker(\gamma - A_m^{MQ})} : \ker(\gamma - A_m^{MQ}) \rightarrow \partial X_{MQ}$ is invertible for any $\gamma \in \rho(A_0^{MQ})$, see Chapter 1. We denote its inverse by

$$D_\gamma^{MQ} := (L_{MQ}|_{\ker(\gamma - A_m^{MQ})})^{-1} : \partial X_{MQ} \longrightarrow \ker(\gamma - A_m^{MQ}),$$

and call it *Dirichlet operator*. We now give the explicit form of D_γ^{MQ} using the operators $\epsilon_k : \mathbb{C} \rightarrow L^1[0, \infty)$, $k \in \mathbb{N}$ defined by

$$(\epsilon_k(c))(x) := c \frac{\lambda^k}{k!} x^k e^{-(\gamma + \lambda + \mu)x}, \quad c \in \mathbb{C}, \quad x \in [0, \infty).$$

The equation $\Phi_{MQ} D_0^{MQ} c = c$ is equivalent to the following system of equations:

$$\begin{aligned} c_1 &= \left(\sum_{k=0}^B pq^k \right) c_1 + \left(\sum_{k=0}^{B-1} pq^k \right) c_2 + \left(\sum_{k=0}^{B-2} pq^k \right) c_3 + \cdots + (p + pq)c_B + pc_{B+1}, \\ c_n &= p \sum_{k=1}^{n+B} q^{n+B-k} c_k, \quad n \geq 2. \end{aligned}$$

This system is again equivalent to

$$\begin{aligned} c_1 &= \left(\sum_{k=0}^B pq^k \right) c_1 + \left(\sum_{k=0}^{B-1} pq^k \right) c_2 + \left(\sum_{k=0}^{B-2} pq^k \right) c_3 + \cdots + (p + pq)c_B + pc_{B+1}, \\ c_{B+n+1} &= \frac{c_{n+1} - qc_n}{1 - q}, \quad n \geq 2. \quad (*) \end{aligned}$$

We now define the function

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto f(x) := q^{(B+1)x} - q^{(B+1)x+1} - q^x + q.$$

Clearly, f is continuously differentiable and

$$f'(x) = (B+1)(1-q) \ln q e^{(B+1)x \ln q} - \ln q e^{x \ln q}.$$

Since the traffic intensity $\rho = \frac{\lambda}{\mu} < 1$, it follows that $q = \frac{\lambda}{\mu + \lambda} < \frac{1}{2}$ and thus $(B+1)(1-q) > 1$. Hence we can estimate

$$f'(0) = (B+1)(1-q) \ln q - \ln q < 0.$$

Therefore, there exists $x_0 > 0$ such that $f'(x) < 0$ for all $x \in (0, x_0)$, hence f is decreasing on $(0, x_0)$. But since $f(0) = 0$ and $\lim_{x \rightarrow +\infty} f(x) = q > 0$, there exists $a > 0$ such that $f(a) = 0$ or $q^{na} f(a) = 0$, respectively. Thus, we obtain that

$$q^{(B+n+1)a} = \frac{q^{(n+1)a} - qq^{na}}{1 - q}.$$

We conclude that for $c_n := q^{na}$, $n \geq 2$, the equations (*) are fulfilled. The first equation of the above system yields

$$\begin{aligned}
q^{B+1}c_1 &= \left(\sum_{k=0}^{B-1} pq^k\right)q^{2a} + \left(\sum_{k=0}^{B-2} pq^k\right)q^{3a} + \cdots + (p + pq)q^{Ba} + pq^{(B+1)a} \\
&= pq^{2a} \sum_{k=0}^{B-1} q^k + pq^{3a} \sum_{k=0}^{B-2} q^k + \cdots + (p + pq)q^{Ba} + pq^{(B+1)a} \\
&= pq^{2a} \frac{1 - q^B}{1 - q} + pq^{3a} \frac{1 - q^{B-1}}{1 - q} + \cdots + pq^{Ba} \frac{1 - q^2}{1 - q} + pq^{(B+1)a} \\
&= q^{2a}(1 - q^B) + q^{3a}(1 - q^{B-1}) + \cdots + q^{Ba}(1 - q^2) + q^{(B+1)a}(1 - q) \\
&= q^{2a}(1 + q^a + q^{2a} + \cdots + q^{(B-1)a}) \\
&\quad - q^{2a+B}(1 + q^{a-1} + q^{2(a-1)} + \cdots + q^{(B-1)(a-1)}) \\
&= q^{2a} \left[\frac{1 - q^{Ba}}{1 - q^a} - q^B \frac{1 - q^{B(a-1)}}{1 - q^{a-1}} \right] \\
&= q^{2a} \frac{(1 - q^{Ba})(1 - q^{a-1}) - (q^B - q^{Ba})(1 - q^a)}{(1 - q^a)(1 - q^{a-1})},
\end{aligned}$$

and hence

$$c_1 = q^{2a-B-1} \frac{(1 - q^{Ba})(1 - q^{a-1}) - (q^B - q^{Ba})(1 - q^a)}{(1 - q^a)(1 - q^{a-1})}.$$

Obviously, $c := (c_n)_{n \in \mathbb{N}} \in l^1$ and thus c is a fixed point of $\Phi_{MQ}D_0^{MQ}$. By the Characteristic Equation 1.3.6 we conclude that $0 \in \sigma_p(A)$. \square

Indeed, 0 is the only spectral value of A on the imaginary axis as the following lemma shows.

LEMMA 2.3.7. *Under the General Assumption 2.1.1, the spectrum $\sigma(A_{MQ})$ of A_{MQ} satisfies*

$$\sigma(A_{MQ}) \cap i\mathbb{R} = \{0\}.$$

PROOF. Let $\gamma = ai$, $a \in \mathbb{R} \setminus \{0\}$, and let $\Gamma = \gamma + \lambda + \mu$.

Recall the explicit representation of $\Phi_{MQ}D_\gamma^{MQ}$ from Remark 2.3.5. For $j \geq 2$ we estimate the j th column sum of $\Phi_{MQ}D_\gamma^{MQ}$ as

$$\sum_{i=1}^{\infty} |(\Phi_{MQ}D_\gamma^{MQ})_{ij}| \leq \frac{\mu}{|\Gamma|} \sum_{k=0}^{\infty} \left(\frac{\lambda}{|\Gamma|}\right)^k = \frac{\mu}{|\Gamma|} \frac{1}{1 - \frac{\lambda}{|\Gamma|}} = \frac{\mu}{|\Gamma| - \lambda} < 1.$$

For the first column sum we obtain

$$\sum_{i=1}^{\infty} |(\Phi_{MQ}D_\gamma^{MQ})_{i1}| \leq \frac{\mu\lambda}{(\lambda + \mu)|\Gamma|} + \frac{\mu}{|\Gamma|} \sum_{k=1}^{\infty} \left(\frac{\lambda}{|\Gamma|}\right)^k < \frac{\mu}{|\Gamma|} \sum_{k=0}^{\infty} \left(\frac{\lambda}{|\Gamma|}\right)^k = \frac{\mu}{|\Gamma| - \lambda} < 1.$$

Hence,

$$\|\Phi_{MQ}D_\gamma^{MQ}\| = \sup_{j \geq 1} \sum_{i=1}^{\infty} |(\Phi D_\gamma)_{ij}| \leq \frac{\mu}{|\Gamma| - \lambda} < 1,$$

and thus the spectral radius fulfills

$$r(\Phi_{MQ}D_\gamma^{MQ}) \leq \|Phi_{MQ}D_\gamma^{MQ}\| < 1.$$

Therefore, $1 \notin \sigma(\Phi_{MQ}D_\gamma^{MQ})$ which implies by the Characteristic Equation 1.3.6 that $\gamma \notin \sigma(A_{MQ})$, i.e.

$$\sigma(A_{MQ}) \cap i\mathbb{R} = \{0\}.$$

□

2.4. Well-Posedness of the System

In this section we prove the well-posedness of (ACP_{MQ}) . For this purpose we check that A_{MQ} fulfills conditions in the Phillips theorem, see Theorem 1.2.6.

LEMMA 2.4.1. $A_{MQ} : D(A_{MQ}) \rightarrow R(A_{MQ}) \subset X_{MQ}$ is a closed linear operator and $D(A_{MQ})$ is dense in X_{MQ} .

PROOF. We will prove the assertion in two steps.

Let us first prove that $(A_{MQ}, D(A_{MQ}))$ is closed. Suppose that

$$\begin{aligned} \lim_{n \rightarrow \infty} P_n^{MQ} &= P_0^{MQ}, \\ \lim_{n \rightarrow \infty} A_{MQ}(P_n^{MQ})^t &= (F_{MQ})^t \end{aligned}$$

for any given

$$\begin{aligned} P_n^{MQ} &= (p_{0,0}^{(n)}, p_{0,1}^{(n)}(x), p_{1,1}^{(n)}(x), p_{2,1}^{(n)}(x), \dots) \in D(A_{MQ}), \\ P_0^{MQ} &= (p_{0,0}^{(0)}, p_{0,1}^{(0)}(x), p_{1,1}^{(0)}(x), p_{2,1}^{(0)}(x), \dots) \in X_{MQ}, \end{aligned}$$

where $F_{MQ} = (f_{0,0}, f_{0,1}(x), f_{1,1}(x), f_{2,1}(x), \dots) \in X_{MQ}$.

Namely,

$$\begin{aligned} \lim_{n \rightarrow \infty} p_{0,0}^{(n)} &= p_{0,0}^{(0)}, \\ \lim_{n \rightarrow \infty} \int_0^\infty |p_{j,1}^{(n)}(x) - p_{j,1}^{(0)}(x)| dx &= 0, \quad (j = 0, 1, 2, \dots). \end{aligned}$$

It follows from this

$$\lim_{n \rightarrow \infty} \int_0^\infty p_{j,1}^{(n)}(x) dx = \int_0^\infty p_{j,1}^{(0)}(x) dx, \quad j = 0, 1, 2, \dots.$$

Furthermore,

$$\lim_{n \rightarrow \infty} A_{MQ}(P_n^{MQ})^t = \lim_{n \rightarrow \infty} \begin{pmatrix} -\lambda p_{0,0}^{(n)} + \mu \int_0^\infty p_{0,1}^{(n)}(x) dx \\ -\frac{dp_{0,1}^{(n)}(x)}{dx} - (\lambda + \mu)p_{0,1}^{(n)}(x) \\ -\frac{dp_{1,1}^{(n)}(x)}{dx} - (\lambda + \mu)p_{1,1}^{(n)}(x) + \lambda p_{0,1}^{(n)}(x) \\ -\frac{dp_{2,1}^{(n)}(x)}{dx} - (\lambda + \mu)p_{2,1}^{(n)}(x) + \lambda p_{1,1}^{(n)}(x) \\ -\frac{dp_{3,1}^{(n)}(x)}{dx} - (\lambda + \mu)p_{3,1}^{(n)}(x) + \lambda p_{2,1}^{(n)}(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} f_{0,0} \\ f_{0,1}(x) \\ f_{1,1}(x) \\ f_{2,1}(x) \\ \vdots \end{pmatrix}.$$

This is equivalent to the following system of equations:

$$\left\{ \begin{array}{l} \lim_{n \rightarrow \infty} [-\lambda p_{0,0}^{(n)} + \mu \int_0^\infty p_{0,1}^{(n)}(x) dx] = f_{0,0}, \\ \lim_{n \rightarrow \infty} [-\frac{dp_{0,1}^{(n)}(x)}{dx} - (\lambda + \mu)p_{0,1}^{(n)}(x)] = f_{0,1}(x), \\ \lim_{n \rightarrow \infty} [-\frac{dp_{1,1}^{(n)}(x)}{dx} - (\lambda + \mu)p_{1,1}^{(n)}(x) + \lambda p_{0,1}^{(n)}(x)] = f_{1,1}(x), \\ \lim_{n \rightarrow \infty} [-\frac{dp_{2,1}^{(n)}(x)}{dx} - (\lambda + \mu)p_{2,1}^{(n)}(x) + \lambda p_{1,1}^{(n)}(x)] = f_{2,1}(x), \\ \lim_{n \rightarrow \infty} [-\frac{dp_{3,1}^{(n)}(x)}{dx} - (\lambda + \mu)p_{3,1}^{(n)}(x) + \lambda p_{2,1}^{(n)}(x)] = f_{3,1}(x), \\ \vdots \end{array} \right.$$

Integrating both sides of the second equation from 0 to β , $\beta > 0$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\beta [-\frac{dp_{0,1}^{(n)}(x)}{dx} - (\lambda + \mu)p_{0,1}^{(n)}(x)] dx &= \int_0^\beta \lim_{n \rightarrow \infty} [-\frac{dp_{0,1}^{(n)}(x)}{dx} - (\lambda + \mu)p_{0,1}^{(n)}(x)] dx \\ &= \int_0^\beta f_{0,1}(x) dx. \end{aligned}$$

It yields

$$\begin{aligned} &\lim_{n \rightarrow \infty} [-p_{0,1}^{(n)}(\beta) - p_{0,1}^{(n)}(0) - (\lambda + \mu) \int_0^\beta p_{0,1}^{(n)}(x) dx] dx \\ &= -p_{0,1}^{(0)}(\beta) + p_{0,1}^{(0)}(0) - (\lambda + \mu) \int_0^\beta p_{0,1}^{(0)}(x) dx \\ &= \int_0^\beta f_{0,1}(x) dx. \end{aligned} \tag{9}$$

Similarly, integrating both sides of j^{th} equation from 0 to $\beta, \beta > 0$, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_0^\beta \left[-\frac{dp_{j,1}^{(n)}(x)}{dx} - (\lambda + \mu)p_{j,1}^{(n)}(x) + \lambda p_{j-1,1}^{(n)}(x) \right] dx \\ &= \int_0^\beta \lim_{n \rightarrow \infty} \left[-\frac{dp_{j,1}^{(n)}(x)}{dx} - (\lambda + \mu)p_{j,1}^{(n)}(x) + \lambda p_{j-1,1}^{(n)}(x) \right] dx \\ &= \int_0^\beta f_{j,1}(x) dx. \end{aligned}$$

It yields

$$\begin{aligned} & \lim_{n \rightarrow \infty} [-p_{j,1}^{(n)}(\beta) - p_{j,1}^{(n)}(0) - (\lambda + \mu) \int_0^\beta p_{j,1}^{(n)}(x) dx] dx + \lambda \int_0^\beta p_{j-1,1}^{(n)}(x) dx \\ &= -p_{j,1}^{(0)}(\beta) - p_{j,1}^{(0)}(0) - (\lambda + \mu) \int_0^\beta p_{j,1}^{(0)}(x) dx + \int_0^\beta p_{j-1,1}^{(0)}(x) dx \\ &= \int_0^\beta f_{j,1}(x) dx, j = 1, 2, 3, \dots \end{aligned} \quad (10)$$

Since $\int_0^\infty |p_{j,1}^{(0)}(x)| dx < \infty$ and $\int_0^\infty |f_j(x)| dx < \infty$ for $j = 1, 2, 3, \dots$. It follows from (9) and (10) that $p_{i,1}^{(0)}(\beta)$ is absolutely continuous for $i = 0, 1, 2, 3, \dots$ and

$$\begin{aligned} p_{0,1}'^{(0)}(\beta) &= -(\lambda + \mu)p_{0,1}^{(0)}(\beta) - f_{0,1}(\beta) \in L^1[0, \infty), \\ p_{j,1}'^{(0)}(\beta) &= -(\lambda + \mu)p_{j,1}^{(0)}(\beta) + \lambda p_{j-1,1}^{(0)}(\beta) - f_{j,1}(\beta) \in L^1[0, \infty), j = 1, 2, 3, \dots \end{aligned}$$

From the form of L_{MQ}, Φ_{MQ} and $\lim_{n \rightarrow \infty} P_n^{MQ} = P_0^{MQ}$ we show that

$$L_{MQ}(P_0^{MQ})^t = \Phi_{MQ}(P_0^{MQ})^t.$$

Therefore, $P_0^{MQ} \in D(A_{MQ})$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} p_{0,1}'^{(n)}(\beta) &= -(\lambda + \mu) \lim_{n \rightarrow \infty} p_{0,1}^{(n)}(\beta) - f_{0,1}(\beta) \\ &= p_{0,1}'^{(0)}(\beta), \\ \lim_{n \rightarrow \infty} p_{j,1}'^{(n)}(\beta) &= -(\lambda + \mu) \lim_{n \rightarrow \infty} p_{j,1}^{(n)}(\beta) + \lambda \lim_{n \rightarrow \infty} p_{j-1,1}^{(n)}(\beta) - f_{j,1}(\beta) \\ &= p_{j,1}'^{(0)}(\beta), j = 1, 2, 3, \dots \end{aligned}$$

From the above deduction we have

$$\left\{ \begin{array}{l} -\lambda p_{0,0}^{(0)} + \mu \int_0^\infty p_{0,1}^{(0)}(x) dx = f_{0,0} \\ -\frac{dp_{0,1}^{(0)}(x)}{dx} - (\lambda + \mu)p_{0,1}^{(0)}(x) = f_{0,1}(x), \\ -\frac{dp_{1,1}^{(0)}(x)}{dx} - (\lambda + \mu)p_{1,1}^{(0)}(x) + \lambda p_{0,1}^{(0)}(x) = f_{1,1}(x), \\ -\frac{dp_{2,1}^{(0)}(x)}{dx} - (\lambda + \mu)p_{2,1}^{(0)}(x) + \lambda p_{1,1}^{(0)}(x) = f_{2,1}(x), \\ -\frac{dp_{3,1}^{(0)}(x)}{dx} - (\lambda + \mu)p_{3,1}^{(0)}(x) + \lambda p_{2,1}^{(0)}(x) = f_{3,1}(x), \\ \vdots \end{array} \right.$$

This shows that $A_{MQ}(P_0^{MQ})^t = (F_{MQ})^t$, hence $(A_{MQ}, D(A_{MQ}))$ is closed operator.

We now prove that $D(A_{MQ})$ is dense in X_{MQ} .

By the definition of the norm in X_{MQ} , it is easy to see that

$$|p_{0,0}| + \sum_{n=0}^{\infty} \|p_{n,1}\|_{L[0,\infty)} < \infty$$

for any $p \in X_{MQ}$. Therefore, for any $\epsilon > 0$ there exists a positive integer N such that $\sum_{n=N}^{\infty} \|p_{n,1}\|_{L[0,\infty)} < \epsilon$. We define

$$E_{MQ} = \left\{ (p_{0,0}, p_{0,1}(x), p_{1,1}(x), \dots, p_{N,1}(x), 0, 0, \dots) \left| \begin{array}{l} p_{i,1} \in L[0, \infty), i = 0, 1, \\ 2, \dots, N, N \text{ is a finite} \\ \text{positive integer} \end{array} \right. \right\}$$

It is obvious that E_{MQ} is dense in X_{MQ} .

Let

$$G_{MQ} = \left\{ (p_{0,0}, p_{0,1}(x), p_{1,1}(x), \dots, p_{q,1}(x), 0, 0, \dots) \left| \begin{array}{l} p_{i,1} \in C_0^\infty[0, \infty), \text{ there exists} \\ \text{a number } c_i > 0, \text{ such that} \\ p_{i,1}(x) = 0 \text{ for } x \in [0, c_i], \\ i = 0, 1, 2, \dots, q. \end{array} \right. \right\}.$$

Then from [Ada75] we know that G_{MQ} is dense in E_{MQ} .

From above discussion we know that, in order to prove denseness of $D(A_{MQ})$ is dense in X_{MQ} , it is sufficient to prove that $D(A_{MQ})$ is dense in G_{MQ} .

Take any $p \in G_{MQ}$, there exists a finite number $q > 0$ and $c_i > 0$, $i = 1, 2, \dots, q$, such that

$$\begin{aligned} p(x) &= (p_{0,0}, p_{0,1}(x), p_{1,1}(x), \dots, p_{q,1}(x), 0, 0, \dots), \\ p_{i,1}(x) &= 0 \text{ for } x \in [0, c_i], i = 0, 1, 2, \dots, q. \end{aligned}$$

This leads to $p_{i,1}(x) = 0$ for $x \in [0, 2s]$, where $0 < 2s < \min\{c_0, c_1, c_2, \dots, c_q\}$.

Without loss of generality, we may assume that q is large enough so that $l = q - B$ is positive. Set

$$\begin{aligned} f^s(0) &= (p_{0,0}, f_{0,1}^s(0), f_{1,1}^s(0), f_{2,1}^s(0), \dots, f_{l,1}^s(0), \dots, f_{l+B,1}^s(0), 0, 0, \dots) \\ &= (p_{0,0}, \mu \sum_{k=1}^B \int_{2s}^{\infty} p_{k,1}(x) dx + \lambda p_{0,0}, \mu \int_{2s}^{\infty} p_{B+1,1}(x) dx, \\ &\quad \mu \int_{2s}^{\infty} p_{B+2,1}(x) dx, \dots, \mu \int_{2s}^{\infty} p_{B+l,1}(x) dx, 0, 0, \dots), \\ f^s(x) &= (p_{0,0}, f_{0,1}^s(x), f_{1,1}^s(x), \dots, f_{l,1}^s(x), \dots, f_{l+B,1}^s(x), 0, 0, \dots) \end{aligned}$$

where

$$\begin{aligned} f_{i,1}^s(x) &= \begin{cases} f_{i,1}^s(0)(1 - \frac{x}{s})^2, & x \in [0, s] \\ -c_i(x-s)^2(x-2s)^2, & x \in [s, 2s] \\ p_{i,1}(x), & x \in [2s, \infty) \end{cases} \\ c_i &= \frac{f_{i,1}^s(0) \int_0^s (1 - \frac{x}{s})^2 dx}{\int_s^{2s} (x-s)^2(x-2s)^2 dx}, \quad i = 0, 1, 2, \dots, l, \\ f_{j,1}^s(x) &= p_{j,1}(x), \quad j = l+1, l+2, \dots, l+B. \end{aligned}$$

It is easy to verify that $f^s(x) \in D(A_{MQ})$ and

$$\begin{aligned} \|p - f^s\|_B &= \sum_{n=0}^l \int_0^{\infty} |f_{n,1}^s(x) - p_{n,1}(x)| dx \\ &= \sum_{n=0}^l \int_s^{2s} |c_n|(x-s)^2(x-2s)^2 dx \\ &\quad + \sum_{n=0}^l \int_0^s |f_{n,1}^s(0)|(1 - \frac{x}{s})^2 dx \\ &= \sum_{n=0}^l |c_n| \frac{s^5}{30} + \sum_{n=0}^l |f_{n,1}^s(0)| \frac{s}{3} \rightarrow 0 \quad \text{as } s \rightarrow 0. \end{aligned}$$

This shows that $D(A_{MQ})$ is dense in G_{MQ} . Therefore $D(A_{MQ})$ is dense in X_{MQ} . \square

We will show the dispersivity of the operator A_{MQ} using the same proof as in [GLZ01].

LEMMA 2.4.2. *The operator A_{MQ} is dispersive .*

PROOF. For $p(x) = (p_{0,0}, p_{0,1}(x), p_{1,1}(x), p_{2,1}(x), \dots) \in D(A_{MQ})$, we define $q(x) = (q_{0,0}, q_{0,1}(x), q_{1,1}(x), q_{2,1}(x), \dots) \in X'_{MQ} = \mathbb{R} \times l^\infty(L^\infty[0, \infty))$ by

$$q_{0,0} = \begin{cases} 1 & \text{if } p_{0,0} > 0, \\ 0 & \text{if } p_{0,0} \leq 0. \end{cases}$$

$$q_{n,1} = \begin{cases} 1 & \text{if } p_{n,1}(x) > 0, \\ 0 & \text{if } p_{n,1}(x) \leq 0, \end{cases} \quad n = 1, 2, 3, \dots$$

Namely,

$$q_{0,0} = \begin{cases} \frac{p_{0,0}^+}{p_{0,0}} & \text{if } p_{0,0} > 0, \\ 0 & \text{if } p_{0,0} \leq 0. \end{cases}$$

$$q_{n,1} = \begin{cases} \frac{p_{n,1}^+}{p_{n,1}} & \text{if } p_{n,1}(x) > 0, \\ 0 & \text{if } p_{n,1}(x) \leq 0, \end{cases} \quad n = 1, 2, 3, \dots$$

Note that $\|q\| \leq 1$ and $\langle p, q \rangle = \|p^+\|$. We need the following equalities in order to estimate $\langle A_{MQ}p, q \rangle$ for $p \in D(A_{MQ})$:

$$\begin{aligned} \sum_{n=0}^{\infty} [p_{n,1}(0)]^+ &= [p_{0,1}(0)]^+ + \sum_{n=1}^{\infty} [p_{n,1}(0)]^+ \\ &\leq \lambda [p_{0,0}]^+ + \mu \sum_{k=1}^B \int_0^{\infty} [p_{k,1}(x)]^+ dx \\ &\quad + \mu \sum_{n=1}^{\infty} \int_0^{\infty} [p_{n+B}(x)]^+ dx \\ &= \mu \sum_{n=1}^{\infty} \int_0^{\infty} [p_{n,1}(x)]^+ dx + \lambda [p_{0,0}]^+. \end{aligned} \quad (11)$$

If we set $V_n = \{x \in [0, \infty) | p_n(x) > 0\}$ and $W_n = \{x \in [0, \infty) | p_n(x) \leq 0\}$ for $n \geq 1$, then by a short argument, from the absolute continuity of $p_n(x)$ it follows that

$$\begin{aligned} \int_0^{\infty} \frac{dp_{n,1}(x)}{dx} q_{n,1}(x) dx &= \int_{V_n} \frac{dp_{n,1}(x)}{dx} \frac{[p_{n,1}(x)]^+}{p_n(x)} dx + \int_{W_n} \frac{dp_{n,1}(x)}{dx} \frac{[p_{n,1}(x)]^+}{p_{n,1}(x)} dx \\ &= \int_{V_n} \frac{dp_{n,1}(x)}{dx} \frac{[p_{n,1}(x)]^+}{p_n(x)} dx \\ &= \int_{V_n} \frac{dp_{n,1}(x)}{dx} dx \\ &= \int_0^{\infty} \frac{d[p_{n,1}(x)]^+}{dx} dx \end{aligned}$$

$$\begin{aligned}
&= [p_{n,1}(x)]^+ \Big|_0^\infty \\
&= -[p_{n,1}(0)]^+, \quad n \geq 0.
\end{aligned} \tag{12}$$

By (11) and (12), for $p \in D(A)$, we have

$$\begin{aligned}
\langle A_{MQ}p, q \rangle &= (-\lambda p_{0,0} + \mu \int_0^\infty p_{0,1}(x) dx) q_{0,0} \\
&\quad + \int_0^\infty \left(-\frac{dp_{0,1}(x)}{dx} - (\lambda + \mu)p_{0,1}(x) \right) q_{0,1}(x) dx \\
&\quad + \sum_{n=1}^\infty \int_0^\infty \left(-\frac{dp_{n,1}(x)}{dx} - (\lambda + \mu)p_{n,1}(x) + \lambda p_{n-1,1}(x) \right) q_{n,1}(x) dx \\
&= -\lambda [p_{0,0}]^+ + \mu q_{0,0} \int_0^\infty p_{0,1}(x) dx \\
&\quad - \int_0^\infty \frac{dp_{0,1}(x)}{dx} q_{0,1}(x) dx - (\lambda + \mu) \int_0^\infty [p_{0,1}(x)]^+ dx \\
&\quad + \sum_{n=1}^\infty \left[- \int_0^\infty \frac{dp_{n,1}(x)}{dx} q_{n,1}(x) dx - (\lambda + \mu) \int_0^\infty [p_{n,1}(x)]^+ dx \right] \\
&\quad + \lambda \sum_{n=1}^\infty \int_0^\infty p_{n-1,1}(x) q_{n,1}(x) dx \\
&= -\lambda [p_{0,0}]^+ + \mu q_{0,0} \int_0^\infty p_{0,1}(x) dx \\
&\quad + [p_{0,1}]^+ - (\lambda + \mu) \int_0^\infty [p_{0,1}(x)]^+ dx \\
&\quad + \sum_{n=1}^\infty [p_{n,1}(0)]^+ - (\lambda + \mu) \int_0^\infty [p_{n,1}(x)]^+ dx \\
&\quad + \lambda \sum_{n=1}^\infty \int_0^\infty p_{n-1,1}(x) q_{n,1}(x) dx \\
&= -\lambda [p_{0,0}]^+ + \mu q_{0,0} \int_0^\infty p_{0,1}(x) dx + \sum_{n=0}^\infty [p_{n,1}(0)]^+ \\
&\quad - \sum_{n=0}^\infty (\lambda + \mu) \int_0^\infty [p_{n,1}(x)]^+ dx + \lambda \sum_{n=1}^\infty \int_0^\infty p_{n-1,1}(x) q_{n,1}(x) dx \\
&\leq -\lambda [p_{0,0}]^+ + \mu q_{0,0} \int_0^\infty p_{0,1}(x) dx \\
&\quad + \mu \sum_{n=1}^\infty \int_0^\infty [p_{n,1}(x)]^+ dx + \lambda [p_{0,0}]^+
\end{aligned}$$

$$\begin{aligned}
& - (\lambda + \mu) \sum_{n=0}^{\infty} \int_0^{\infty} [p_{n,1}(x)]^+ dx \\
& + \lambda \sum_{n=1}^{\infty} \int_0^{\infty} p_{n-1,1}(x) q_{n,1}(x) dx \\
& = \mu q_{0,0} \int_0^{\infty} p_{0,1}(x) dx - \lambda \sum_{n=0}^{\infty} \int_0^{\infty} [p_{n,1}(x)]^+ dx \\
& - \mu \int_0^{\infty} [p_{0,1}(x)]^+ dx + \lambda \sum_{n=1}^{\infty} \int_0^{\infty} p_{n-1,1}(x) q_{n,1}(x) dx \\
& \leq \mu q_{0,0} \int_0^{\infty} p_{0,1}(x) dx - \lambda \sum_{n=0}^{\infty} \int_0^{\infty} [p_{n,1}(x)]^+ dx \\
& - \mu \int_0^{\infty} [p_{0,1}(x)]^+ dx + \lambda \sum_{n=1}^{\infty} \int_0^{\infty} [p_{n-1,1}(x)]^+ dx \\
& \leq \mu q_{0,0} \int_0^{\infty} p_{0,1}(x) dx - \mu \int_0^{\infty} [p_{0,1}(x)]^+ dx \\
& = \mu(q_{0,0} - 1) \int_0^{\infty} [p_{0,1}(x)]^+ dx \leq 0. \tag{13}
\end{aligned}$$

In the inequality (13) we used

$$\begin{aligned}
\int_0^{\infty} p_{n-1,1}(x) q_{n,1}(x) dx & \leq \int_0^{\infty} [p_{n-1,1}(x)]^+ q_{n,1}(x) dx \\
& \leq \int_0^{\infty} [p_{n-1,1}(x)]^+ dx, \quad n \geq 2
\end{aligned}$$

and

$$\int_0^{\infty} p_{0,1}(x) dx \leq \int_0^{\infty} [p_{0,1}(x)]^+ dx.$$

From the inequality (13) together with Definition 1.2.5 we obtain that A_{MQ} is dispersive. \square

We also obtain the surjectivity of $\gamma - A_{MQ}$ for $\gamma > 0$.

LEMMA 2.4.3. *For $0 < \gamma \in \mathbb{R}$, we have $\gamma \in \rho(A_{MQ})$.*

PROOF. Let $0 < \gamma \in \mathbb{R}$. For $j \geq 2$, we can estimate the j^{th} column sum of ΦD_{γ} as

$$\sum_{i=1}^{\infty} (\Phi_{MQ} D_{\gamma}^{MQ})_{ij} = \frac{\mu}{\Gamma} \sum_{k=0}^{\infty} \left(\frac{\lambda}{\Gamma} \right)^k = \frac{\mu}{\Gamma} \frac{1}{1 - \frac{\lambda}{\Gamma}} = \frac{\mu}{\Gamma - \lambda} < 1$$

and the first column sum as

$$\sum_{i=1}^{\infty} (\Phi_{MQ} D_{\gamma}^{MQ})_{i1} = \frac{\mu\lambda}{(\lambda + \mu)\Gamma} + \frac{\mu}{\Gamma} \sum_{k=1}^{\infty} \left(\frac{\lambda}{\Gamma}\right)^k < \frac{\mu}{\Gamma} \sum_{k=0}^{\infty} \left(\frac{\lambda}{\Gamma}\right)^k = \frac{\mu}{\Gamma - \lambda} < 1.$$

Since the column sums are all equal from the $(B + 1)^{st}$ column on, it follows that

$$\|\Phi_{MQ} D_{\gamma}^{MQ}\| = \sup_{1 \leq j} \sum_{i=1}^{\infty} (\Phi_{MQ} D_{\gamma}^{MQ})_{ij} = \max_{1 \leq j \leq B+1} \sum_{i=1}^{\infty} (\Phi_{MQ} D_{\gamma}^{MQ})_{ij} < 1$$

and therefore

$$r(\Phi_{MQ} D_{\gamma}^{MQ}) \leq \|\Phi_{MQ} D_{\gamma}^{MQ}\| < 1.$$

Using the Characteristic Equation 1.3.6 we conclude that $\gamma \in \rho(A_{MQ})$ for $\gamma > 0$. \square

Combining Lemma 2.4.2 and Lemma 2.4.3 with Theorem 1.2.6 we get the following result.

THEOREM 2.4.4. *The operator $(A_{MQ}, D(A_{MQ}))$ is the generator of a positive strongly continuous contraction semigroup on X_{MQ} .*

From Proposition 1.2.4 and Theorem 2.4.4 we deduce the following result.

THEOREM 2.4.5. *The system $(BC_{MQ}), (BC_{MQ})$ and $(IC_{MQ,0})$ has a unique positive solution $p(x, t)$ which satisfies*

$$\|p(\cdot, t)\| = 1, \quad \forall t \in [0, \infty).$$

PROOF. We know from Proposition 1.2.4 and Theorem 2.4.4 that the associated abstract Cauchy problem (ACP_{MQ}) has a unique positive time-dependent solution $p(x, t)$ which can be expressed as

$$p(x, t) = T_{MQ}(t)p(0) = T_{MQ}(t)(1, 0, 0, 0, \dots). \quad (14)$$

Let $P(t) = p(x, t) = (p_{0,0}(t), p_{0,1}(x, t), p_{1,1}(x, t), p_{2,1}(x, t), p_{3,1}(x, t), \dots)$, then $P(t)$ satisfies the system of equations:

$$\frac{dp_{0,0}(t)}{dt} = -\lambda p_{0,0}(t) + \mu \int_0^\infty p_{0,1}(x, t) dx, \quad (15)$$

$$\frac{\partial p_{0,1}(x, t)}{\partial x} = -\frac{\partial p_{0,1}(x, t)}{\partial t} - (\lambda + \mu) p_{0,1}(x, t), \quad (16)$$

$$\frac{\partial p_{n,1}(x, t)}{\partial x} = -\frac{\partial p_{n,1}(x, t)}{\partial t} - (\lambda + \mu) p_{n,1}(x, t) + \lambda p_{n-1,1}(x, t), \quad n \geq 1, \quad (17)$$

$$p_{0,1}(0, t) = \sum_{k=1}^B \mu \int_0^\infty p_{k,1}(x, t) dx + \lambda p_{0,0}(t), \quad (18)$$

$$p_{n,1}(0, t) = \mu \int_0^\infty p_{n+B,1}(x, t) dx, \quad n \geq 1, \quad (19)$$

$$P(0) = (1, 0, 0, 0, 0, \dots). \quad (20)$$

Integrating the left sides of (16) and (17), we have

$$\begin{aligned} \int_0^\infty \frac{\partial p_{n,1}(x, t)}{\partial x} dx &= p_{n,1}(\infty, t) - p_{n,1}(0, t) \\ &= -p_{n,1}(0, t), \quad n = 0, 1, 2, 3, \dots \end{aligned} \quad (21)$$

Using (15)–(21) we compute

$$\begin{aligned} \frac{d\|P(t)\|}{dt} &= \frac{dp_{0,0}(t)}{dt} + \sum_{n=0}^\infty \int_0^\infty \frac{\partial p_{n,1}(x, t)}{\partial t} dx \\ &= -\lambda p_{0,0}(t) + \mu \int_0^\infty p_{0,1}(x, t) dx \\ &\quad - \int_0^\infty \frac{\partial p_{0,1}(x, t)}{\partial x} - (\lambda + \mu) \int_0^\infty p_{0,1}(x, t) dx \\ &\quad + \sum_{n=1}^\infty \int_0^\infty \left[-\frac{\partial p_{n,1}(x, t)}{\partial x} - (\lambda + \mu) \int_0^\infty p_{n,1}(x, t) dx \right. \\ &\quad \left. + \lambda \int_0^\infty p_{n-1,1}(x, t) dx \right] \\ &= -\lambda p_{0,0}(t) + \mu \int_0^\infty p_{0,1}(x, t) dx \\ &\quad + p_{0,1}(0, t) - (\lambda + \mu) \int_0^\infty p_{0,1}(x, t) dx \\ &\quad + \sum_{n=1}^\infty p_{n,1}(0, t) - (\lambda + \mu) \sum_{n=1}^\infty \int_0^\infty p_{n,1}(x, t) dx \end{aligned}$$

$$\begin{aligned}
& + \lambda \sum_{n=1}^{\infty} \int_0^{\infty} p_{n-1,1}(x, t) dx \\
& = -\lambda p_{0,0}(t) + \sum_{n=0}^{\infty} p_{n,1}(0, t) - \mu \sum_{n=1}^{\infty} \int_0^{\infty} p_{n,1}(x, t) dx \\
& = \sum_{n=0}^{\infty} p_{n,1}(0, t) - [\lambda p_{0,0}(t) + \mu \sum_{n=1}^{\infty} \int_0^{\infty} p_{n,1}(x, t) dx] \\
& = \sum_{n=0}^{\infty} p_{n,1}(0, t) - [\lambda p_{0,0}(t) + \mu \sum_{n=1}^B \int_0^{\infty} p_{n,1}(x, t) dx \\
& \quad + \mu \sum_{n=B+1}^{\infty} \int_0^{\infty} p_{n,1}(x, t) dx] \\
& = \sum_{n=0}^{\infty} p_{n,1}(0, t) - \sum_{n=0}^{\infty} p_{n,1}(0, t) = 0. \tag{22}
\end{aligned}$$

By (14) and (22) we obtain

$$\frac{d\|P(t)\|}{dt} = \frac{d\|T_{MQ}(t)P(0)\|}{dt} = 0.$$

Therefore,

$$\|T_{MQ}(t)P(0)\| = \|P(t)\| = \|P(0)\| = 1.$$

This shows $\|p(\cdot, t)\| = 1, \forall t \in [0, \infty)$. \square

2.5. Asymptotic Stability of the Solution

In this section we use the results on positive semigroups collected in Section 1.1 to investigate the asymptotic stability of the solution of the system.

First we show the irreducibility of the semigroup via the representation of the resolvent of A_{MQ} from Lemma 1.3.7 in terms of the resolvent of A_0^{MQ} and the operator Φ_{MQ} and D_{γ}^{MQ} .

LEMMA 2.5.1. *The semigroup $(T_{MQ}(t))_{t \geq 0}$ generated by $(A_{MQ}, D(A_{MQ}))$ is irreducible.*

PROOF. It suffices to show that there exists $\gamma > 0$ such that $0 \preceq p \in X_{MQ}$ implies $R(\gamma, A_{MQ})p \gg 0$, see Proposition 1.1.12. By Lemma 1.3.7 we have to prove that there exists $\gamma > 0$ such that $0 \preceq p \in X_{MQ}$ implies

$$R(\gamma, A_0^{MQ})p + (Id_{\partial X_{MQ}} - \Phi_{MQ}D_{\gamma}^{MQ})^{-1}\Phi_{MQ}R(\gamma, A_0^{MQ})p \gg 0.$$

Suppose that $\gamma > 0$ and $0 \preceq p \in X_{MQ}$. Then also $R(\gamma, A_0^{MQ})p \succeq 0$ and $\Phi_{MQ}R(\gamma, A_0^{MQ})p \succeq 0$. Since it follows from the proof of Lemma 2.4.3 that

$\|\Phi_{MQ}D_\gamma^{MQ}\| < 1$ for any $\gamma > 0$, the inverse of $Id_{\partial X_{MQ}} - \Phi_{MQ}D_\gamma^{MQ}$ is given by the Neumann series

$$(Id_{\partial X_{MQ}} - \Phi_{MQ}D_\gamma^{MQ})^{-1} = \sum_{n=0}^{\infty} (\Phi_{MQ}D_\gamma^{MQ})^n.$$

We know from the form of $\Phi_{MQ}D_\gamma^{MQ}$ that for every $i \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that the real number $((\Phi_{MQ}D_\gamma^{MQ})^k \Phi_{MQ}R(\gamma, A_0^{MQ})p)_i > 0$, i.e.

$$(Id_{\partial X_{MQ}} - \Phi_{MQ}D_\gamma^{MQ})^{-1} \Phi_{MQ}R(\gamma, A_0^{MQ})p \gg 0$$

and by the form of D_γ^{MQ} we have

$$D_\gamma^{MQ}(Id_{\partial X_{MQ}} - \Phi_{MQ}D_\gamma^{MQ})^{-1} \Phi_{MQ}R(\gamma, A_0^{MQ})p \gg 0.$$

This implies

$$R(\gamma, A_{MQ})p \gg 0.$$

Therefore the semigroup $(T_{MQ}(t))_{t \geq 0}$ is irreducible. \square

We now prove our main result on the asymptotic behaviour of the solution of the queueing system. Combining Lemma 2.5.1 with the results from Section 1.4 we obtain the strong convergence of the semigroup to a one-dimensional equilibrium.

THEOREM 2.5.2. *The space X_{MQ} can be decomposed into the direct sum*

$$X_{MQ} = X_{MQ}^1 \oplus X_{MQ}^2,$$

where $X_{MQ}^1 = \text{fix}(T_{MQ}(t))_{t \geq 0} = \ker A_{MQ}$ is one-dimensional and spanned by a strictly positive eigenvector $\tilde{p} \in \ker A_{MQ}$ of A_{MQ} and $(T_{MQ}(t)|_{X_{MQ}^2})_{t \geq 0}$ is strongly stable.

PROOF. By Theorem 2.4.4, Lemma 2.3.7 and Lemma 2.5.1 we know that the assumptions of Theorem 1.4.2 are fulfilled, hence the assertion follows. \square

We reformulate the above theorem as our final result.

COROLLARY 2.5.3. There exists $p' \in X'_{MQ}$, $p' \gg 0$, such that for all $p \in X_{MQ}$

$$\lim_{t \rightarrow \infty} T_{MQ}(t)p = \langle p', p \rangle \tilde{p},$$

where $\ker A_{MQ} = \langle \tilde{p} \rangle$, $\tilde{p} \gg 0$.

By Corollary 2.5.3 we obtain asymptotic stability of the solution of the $M/M^B/1$ queueing model.

COROLLARY 2.5.4. The time-dependent solution of the system (MQ) , (BC_{MQ}) and $(IC_{MQ,0})$ converges strongly to the steady-state solution as time tends to infinite, that is, $\lim_{t \rightarrow \infty} p(\cdot, t) = \alpha \tilde{p}$, where $\alpha > 0$ and \tilde{p} as in Lemma 2.5.3.

CHAPTER 3

The System with Primary and Secondary Failures

3.1. Introduction

As science and technology develop, electronic productions and networks are used everywhere. So, the stability analysis of such systems becomes more and more important.

In this section, we consider the model of a repairable system with primary as well as secondary failures. In the system there are three independent identical units. In system one of those units operates, the extra units act as warm standby. If the operating unit fails, a warm standby unit is instantaneously switched into operation. The operating unit submits three kind of failures, failures that unit itself cause as it operates, common cause failures such as fire, earthquake, flood, explosion, etc. and human error failures. There is one repairman available to repair these units. Once repaired, these units are as good as new. The repair times are arbitrarily distributed.

According to [Gup95], the model for the system with primary and secondary failures can be expressed by a system of integro-differential equations

$$(R) \left\{ \begin{array}{l} \frac{dp_0(t)}{dt} = -(\lambda + 2\alpha + \lambda_{c_0} + \lambda_{h_0})p_0(t) + \mu p_1(t) + \sum_{i=3}^5 \int_0^\infty \mu_i(x)p_i(x, t)dx, \\ \frac{dp_1(t)}{dt} = (\lambda + 2\alpha)p_0(t) - (\mu + \lambda + \alpha + \lambda_{c_1} + \lambda_{h_1})p_1(t) + \mu p_2(t), \\ \frac{dp_2(t)}{dt} = (\lambda + \alpha)p_1(t) - (\mu + \lambda + \lambda_{c_2} + \lambda_{h_2})p_2(t), \\ \frac{\partial p_i(x, t)}{\partial t} + \frac{\partial p_i(x, t)}{\partial x} = -\mu_i(x)p_i(x, t), \quad i = 3, 4, 5. \end{array} \right.$$

For $x = 0$ the boundary conditions

$$(BC_R) \left\{ \begin{array}{l} p_3(0, t) = \lambda p_2(t), \quad t > 0, \\ p_4(0, t) = \sum_{i=0}^2 \lambda_{c_i} p_i(t), \quad t > 0, \\ p_5(0, t) = \sum_{i=0}^2 \lambda_{h_i} p_i(t), \quad t > 0, \end{array} \right.$$

are prescribed, and we consider the usual initial condition

$$(IC_R) \begin{cases} p_0(0) = c \in \mathbb{C}, \\ p_i(0) = b_i, i = 1, 2; p_j(x, 0) = f_j(x), j = 3, 4, 5, \end{cases}$$

where $f_j(x) \in L^1[0, \infty)$. The most interesting initial condition

$$(IC_{R,0}) \begin{cases} p_0(0) = 1, \\ p_i(0) = 0, i = 1, 2; p_j(x, 0) = 0, j = 3, 4, 5. \end{cases}$$

Here $(x, t) \in [0, \infty) \times [0, \infty)$; $p_i(t)$ represents the probability that the system is in state i at time t , $i = 0, 1, 2$; $p_j(x, t)$ represents the probability that at time t the failed system is in state j and has an elapsed repair time of x , $j = 3, 4, 5$; λ represents failure rate of an operating unit; λ_{c_i} represents common-cause failure rates from state i to state 4, $i = 0, 1, 2$; λ_{h_i} represents human-error rates from state i to state 5, $i = 0, 1, 2$; α represents failure rate of standby unit; μ represents constant repair rate if the system is operating; $\mu_j(x)$ represents time-dependent system repair-rate when the failed system is in state j and has an elapsed repair time of x for $j = 3, 4, 5$ which satisfies $\mu_j(x) \geq 0$ ($j = 3, 4, 5$); λ_{c_i} ($i = 0, 1, 2$), λ_{h_i} ($i = 0, 1, 2$), λ, μ and α are positive constants.

We require the following for the failure rate $\mu_j(x)$.

GENERAL ASSUMPTION 3.1.1. *The function $\mu_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is measurable and bounded such that $\lim_{x \rightarrow \infty} \mu_j(x)$ exists and*

$$\mu_\infty^{(j)} := \lim_{x \rightarrow \infty} \mu_j(x) > 0, j = 3, 4, 5, \mu_\infty := \min(\mu_\infty^{(3)}, \mu_\infty^{(4)}, \mu_\infty^{(5)})$$

In [Gup95] the author established the model and studied the time-dependent availability of the system by using Laplace transform and discovered that the time-dependent availability decreases as time increases for exponential repair-time distribution. He used the steady-state solution and the time-dependent solution for calculating the system availability. But he did not discuss the well-posedness of the model and its asymptotic behavior. Investigation of the time-dependent solution of the model and its asymptotic behavior is important from the point of view of theory and applications. Therefore, in this chapter we discuss the well-posedness of the model and prove the asymptotic stability of the time-dependent solution of this system using spectral theory and semigroup methods.

3.2. The Problem as an Abstract Cauchy Problem

In this section we rewrite the underlying problem as an abstract Cauchy problem on a suitable space X_R , see [EN00, Def. II.6.1]. As the state space for our problem we choose

$$X_R := \mathbb{C}^3 \times (L^1[0, \infty))^3.$$

It is obvious that X_R is a Banach space endowed with the norm

$$\|p\| := \sum_{i=0}^2 |p_i| + \sum_{n=3}^5 \|p_n\|_{L^1[0,\infty)},$$

where $p = (p_0, p_1, p_2, p_3(x), p_4(x), p_5(x))^t \in X$.

For simplicity, let

$$\begin{aligned} a_0 &:= \lambda + 2\alpha + \lambda_{c_0} + \lambda_{h_0}, \\ a_1 &:= \mu + \lambda + \alpha + \lambda_{c_1} + \lambda_{h_1}, \\ a_2 &:= \mu + \lambda + \lambda_{c_2} + \lambda_{h_2}, \end{aligned}$$

and we denote by ψ_j the linear functionals

$$\psi_j : L^1[0, \infty) \rightarrow \mathbb{C}, \quad f \mapsto \psi_j(f) := \int_0^\infty \mu_j(x) f(x) dx, \quad j = 3, 4, 5.$$

Moreover, we define the operators D_j on $W^{1,1}[0, \infty)$ as

$$D_j f := -\frac{d}{dx} f - \mu_j f, \quad f \in W^{1,1}[0, \infty), \quad j = 3, 4, 5,$$

respectively. To define the appropriate operator $(A_R, D(A_R))$ we introduce a “maximal operator” $(A_m^R, D(A_m^R))$ on X_R given as

$$\begin{aligned} A_m^R &:= \begin{pmatrix} -a_0 & \mu & 0 & \psi_3 & \psi_4 & \psi_5 \\ \lambda + 2\alpha & -a_1 & \mu & 0 & 0 & 0 \\ 0 & \lambda + \alpha & -a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_5 \end{pmatrix}, \\ D(A_m^R) &:= \mathbb{C}^3 \times (W^{1,1}[0, \infty))^3. \end{aligned}$$

To model the boundary conditions (BC_R) we use an abstract approach as in e.g. [CENN03]. For this purpose we consider the “boundary space”

$$\partial X_R := \mathbb{C}^3,$$

and then define “boundary operators” L_R and Φ_R . As the operator L_R we take

$$L_R : D(A_m^R) \rightarrow \partial X_R, \quad \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix} \mapsto L_R \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix} := \begin{pmatrix} p_3(0) \\ p_4(0) \\ p_5(0) \end{pmatrix}$$

and the operator $\Phi_R \in \mathcal{L}(X_R, \partial X_R)$ is given by the matrix

$$\Phi_R := \begin{pmatrix} 0 & 0 & \lambda & 0 & 0 & 0 \\ \lambda_{c_0} & \lambda_{c_1} & \lambda_{c_2} & 0 & 0 & 0 \\ \lambda_{h_0} & \lambda_{h_1} & \lambda_{h_2} & 0 & 0 & 0 \end{pmatrix}.$$

The operator $(A_R, D(A_R))$ on X_R is then defined as

$$A_R p := A_m^R p, \quad D(A_R) := \{p \in D(A_m^R) \mid L_R p = \Phi_R p\}.$$

With these definitions the above equations (R) , (BC_R) and (IC_R) are equivalent to the abstract Cauchy problem

$$\begin{cases} \frac{dp(t)}{dt} = A_R p(t), & t \in [0, \infty), \\ p(0) = (c, b_1, b_2, f_1, f_2, f_3)^t \in X_R. \end{cases} \quad (ACP_R)$$

If A_R is the generator of a strongly continuous semigroup $(T_R(t))_{t \geq 0}$ and the initial value in (IC_R) satisfies $p(0) = (c, b_1, b_2, f_1, f_2, f_3)^t \in D(A_R)$, then the unique solution of (R) , (BC_R) and (IC_R) is given by

$$\begin{aligned} p_i(t) &= (T_R(t)p(0))_{i+1}, & 0 \leq i \leq 2, \\ p_n(x, t) &= (T_R(t)p(0))_{n+1}(x), & 3 \leq n \leq 5. \end{aligned}$$

For this reason it suffices to study (ACP_R) .

3.3. Boundary Spectrum

In this section we use the Characteristic Equation 1.3.6 to discuss the boundary spectrum $\sigma_b(A_R)$ of A_R . For this purpose, we first define the operator $(A_0^R, D(A_0^R))$ as

$$\begin{aligned} D(A_0^R) &:= \{p \in D(A_m^R) \mid L_R p = 0\}, \\ A_0^R p &:= A_m^R p. \end{aligned}$$

We give the representation of the resolvent of the operator A_0^R needed below to prove the irreducibility of the semigroup generated by the operator A_R .

LEMMA 3.3.1. *Let*

$$\mathbb{A} := \begin{pmatrix} -a_0 & \mu & 0 \\ \lambda + 2\alpha & -a_1 & \mu \\ 0 & \lambda + \alpha & -a_2 \end{pmatrix} \quad (23)$$

and set $S_R := \{\gamma \in \mathbb{C} \mid \Re \gamma > -\mu_\infty\} \setminus \sigma(\mathbb{A})$. Then we have

$$S_R \subseteq \rho(A_0^R).$$

Moreover, if $\gamma \in S_R$, then

$$R(\gamma, A_0^R) = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} & r_{1,4} & r_{1,5} & r_{1,6} \\ r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} & r_{2,5} & r_{2,6} \\ r_{3,1} & r_{3,2} & r_{3,3} & r_{3,4} & r_{3,5} & r_{3,6} \\ 0 & 0 & 0 & r_{4,4} & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{5,5} & 0 \\ 0 & 0 & 0 & 0 & 0 & r_{6,6} \end{pmatrix}, \quad (24)$$

where

$$\begin{aligned} r_{1,1} &= \frac{(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{1,2} &= \frac{\mu(\gamma+a_2)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{1,3} &= \frac{\mu^2}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{1,4} &= \frac{[(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)]\psi_3 R(\gamma, D_3)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{1,5} &= \frac{\mu(\gamma+a_2)\psi_4 R(\gamma, D_4)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{1,6} &= \frac{\mu^2\psi_5 R(\gamma, D_5)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{2,1} &= \frac{(\lambda+2\alpha)(\gamma+a_2)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{2,2} &= \frac{(\gamma+a_0)(\gamma+a_2)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{2,3} &= \frac{\mu(\gamma+a_0)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{2,4} &= \frac{(\lambda+2\alpha)(\gamma+a_2)\psi_3 R(\gamma, D_3)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{2,5} &= \frac{(\lambda+a_0)(\gamma+a_2)\psi_4 R(\gamma, D_4)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{2,6} &= \frac{\mu(\gamma+a_0)\psi_5 R(\gamma, D_5)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{3,1} &= \frac{(\lambda+2\alpha)(\lambda+\alpha)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{3,2} &= \frac{(\gamma+a_0)(\lambda+\alpha)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{3,3} &= \frac{(\gamma+a_0)(\gamma+a_1)-\mu(\lambda+2\alpha)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{3,4} &= \frac{(\lambda+2\alpha)(\lambda+\alpha)\psi_3 R(\gamma, D_3)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{3,5} &= \frac{(\lambda+a_0)(\lambda+\alpha)\psi_4 R(\gamma, D_4)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{3,6} &= \frac{[(\lambda+a_0)(\lambda+a_1)-\mu(\lambda+2\alpha)]\psi_5 R(\gamma, D_5)}{(\gamma+a_0)(\gamma+a_1)(\gamma+a_2)-\mu(\lambda+\alpha)(\gamma+a_0)-\mu(\lambda+2\alpha)(\gamma+a_2)}, \\ r_{4,4} &= R(\gamma, D_3), \\ r_{5,5} &= R(\gamma, D_4), \\ r_{6,6} &= R(\gamma, D_3). \end{aligned}$$

The resolvent operators of the differential operators $D_j (j = 3, 4, 5)$ is given by

$$(R(\gamma, D_j)p)(x) = e^{-\gamma x - \int_0^x \mu_j(\xi) d\xi} \int_0^x e^{\gamma s + \int_0^s \mu_j(\xi) d\xi} p(s) ds$$

for $p \in L^1[0, \infty)$.

PROOF. A combination of [Gre84, Prop. 2.1] and [Nag89, Thm. 2.4] yields that the resolvent set of A_0^R satisfies

$$\rho(A_0^R) \supseteq S_R.$$

For $\gamma \in S_R$ we can compute the resolvent of A_0^R explicitly applying the formula for the inverse of operator matrices, see [Nag89, Thm. 2.4]. This leads to the representation (24) of the resolvent of A_0^R .

Clearly, knowing the operator matrix in (24), we can directly compute that it represents the resolvent of A_0^R . \square

The following consequence is useful to compute the boundary spectrum of A_R .

COROLLARY 3.3.2. The imaginary axis belongs to the resolvent set of A_0^R , i.e.,

$$i\mathbb{R} \subseteq \rho(A_0^R).$$

The eigenvectors in $\ker(\gamma - A_m^R)$ can be computed as follows.

LEMMA 3.3.3. For $\gamma \in \mathbb{C}$, we have

$$p \in \ker(\gamma - A_m^R) \tag{25}$$

\Leftrightarrow

$$\begin{aligned} p &= (p_0, p_1, p_2, p_3(\cdot), p_4(\cdot), p_5(\cdot))^t \in D(A_m), \text{ with} \\ p_0 &= \frac{(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)} \\ &\times \sum_{j=3}^5 c_j \int_0^\infty \mu_j(x) e^{-\gamma x - \int_0^x \mu_j(\xi) d\xi} dx \end{aligned} \tag{26}$$

$$p_1 = \frac{(\lambda + 2\alpha)(\gamma + a_2) \sum_{j=3}^5 c_j \int_0^\infty \mu_j(x) e^{-\gamma x - \int_0^x \mu_j(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \tag{27}$$

$$p_2 = \frac{(\lambda + \alpha)(\lambda + 2\alpha) \sum_{j=3}^5 c_j \int_0^\infty \mu_j(x) e^{-\gamma x - \int_0^x \mu_j(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \tag{28}$$

$$p_j(x) = c_j \int_0^\infty \mu_j(x) e^{-\gamma x - \int_0^x \mu_j(\xi) d\xi} dx, j = 3, 4, 5, \tag{29}$$

where $c_3, c_4, c_5 \in \mathbb{C}$.

PROOF. If for $p \in X_R$, (26)-(29) are fulfilled, then we can easily compute that $p \in \ker(\gamma - A_m^R)$. Conversely, condition (25) gives a system of differential equations. Solving these differential equations, we see that (26)-(29) are indeed satisfied. \square

Moreover, since L_R is surjective,

$$L_R|_{\ker(\gamma - A_m^R)} : \ker(\gamma - A_m^R) \rightarrow \partial X_R$$

is invertible for each $\gamma \in \rho(A_0^R)$, see [Gre87, Lemma 1.2]. We denote its inverse by

$$D_\gamma^R := (L_R|_{\ker(\gamma - A_m^R)})^{-1} : \partial X_R \longrightarrow \ker(\gamma - A_m^R)$$

and call it ‘‘Dirichlet operator’’.

We can give the explicit form of D_γ^R as follows.

LEMMA 3.3.4. *For each $\gamma \in \rho(A_0^R)$, the operator D_γ^R has the form*

$$D_\gamma^R = \begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,3} \\ d_{2,1} & d_{2,2} & d_{2,3} \\ d_{3,1} & d_{3,2} & d_{3,3} \\ d_{4,1} & 0 & 0 \\ 0 & d_{5,2} & 0 \\ 0 & 0 & d_{6,3} \end{pmatrix},$$

where

$$\begin{aligned} d_{1,1} &= \frac{[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\ d_{1,2} &= \frac{[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\ d_{1,3} &= \frac{[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\ d_{2,1} &= \frac{(\lambda + 2\alpha)(\gamma + a_2) \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\ d_{2,2} &= \frac{(\lambda + 2\alpha)(\gamma + a_2) \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\ d_{2,3} &= \frac{(\lambda + 2\alpha)(\gamma + a_2) \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\ d_{3,1} &= \frac{(\lambda + \alpha)(\lambda + 2\alpha) \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\ d_{3,2} &= \frac{(\lambda + \alpha)(\lambda + 2\alpha) \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \end{aligned}$$

$$\begin{aligned}
d_{3,3} &= \frac{(\lambda + \alpha)(\lambda + 2\alpha) \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\
d_{4,1} &= \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx, \\
d_{5,2} &= \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx, \\
d_{6,3} &= \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx.
\end{aligned}$$

The operator $\Phi_R D_\gamma^R$ can be computed explicitly for $\gamma \in \rho(A_0^R)$.

REMARK 3.3.5. For $\gamma \in \rho(A_0^R)$ the operator $\Phi_R D_\gamma^R$ can be represented by the 3×3 -matrix

$$\Phi_R D_\gamma^R = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix},$$

where

$$\begin{aligned}
a_{1,1} &= \frac{\lambda(\lambda + \alpha)(\lambda + 2\alpha) \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\
a_{1,2} &= \frac{\lambda(\lambda + \alpha)(\lambda + 2\alpha) \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\
a_{1,3} &= \frac{\lambda(\lambda + \alpha)(\lambda + 2\alpha) \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\
a_{2,1} &= \frac{\lambda_{c_0}[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] + \lambda_{c_1}(\lambda + 2\alpha)(\gamma + a_2) + \lambda_{c_2}(\lambda + \alpha)(\lambda + 2\alpha)}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)} \\
&\quad \times \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx, \\
a_{2,2} &= \frac{\lambda_{c_0}[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] + \lambda_{c_1}(\lambda + 2\alpha)(\gamma + a_2) + \lambda_{c_2}(\lambda + \alpha)(\lambda + 2\alpha)}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)} \\
&\quad \times \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx, \\
a_{2,3} &= \frac{\lambda_{c_0}[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] + \lambda_{c_1}(\lambda + 2\alpha)(\gamma + a_2) + \lambda_{c_2}(\lambda + \alpha)(\lambda + 2\alpha)}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)} \\
&\quad \times \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx, \\
a_{3,1} &= \frac{\lambda_{h_0}[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] + \lambda_{h_1}(\lambda + 2\alpha)(\gamma + a_2) + \lambda_{h_2}(\lambda + \alpha)(\lambda + 2\alpha)}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)} \\
&\quad \times \int_0^\infty \mu_3(x) e^{-\gamma x - \int_0^x \mu_3(\xi) d\xi} dx,
\end{aligned}$$

$$\begin{aligned}
a_{3,2} &= \frac{\lambda_{h_0}[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] + \lambda_{h_1}(\lambda + 2\alpha)(\gamma + a_2) + \lambda_{h_2}(\lambda + \alpha)(\lambda + 2\alpha)}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)} \\
&\quad \times \int_0^\infty \mu_4(x) e^{-\gamma x - \int_0^x \mu_4(\xi) d\xi} dx, \\
a_{3,3} &= \frac{\lambda_{h_0}[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] + \lambda_{h_1}(\lambda + 2\alpha)(\gamma + a_2) + \lambda_{h_2}(\lambda + \alpha)(\lambda + 2\alpha)}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)} \\
&\quad \times \int_0^\infty \mu_5(x) e^{-\gamma x - \int_0^x \mu_5(\xi) d\xi} dx.
\end{aligned}$$

Using the Characteristic Equation 1.3.6 we can show that 0 is in the point spectrum of A_R .

LEMMA 3.3.6. *For the operator $(A_R, D(A_R))$ we have $0 \in \sigma_p(A_R)$.*

PROOF. By the Characteristic Equation 1.3.6 it suffices to prove that $1 \in \sigma_p(\Phi_R D_0^R)$. Since

$$\Phi_R D_\gamma^R = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix},$$

where

$$\begin{aligned}
b_{1,1} &= \frac{\lambda(\lambda + \alpha)(\lambda + 2\alpha)}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2}, \\
b_{1,2} &= \frac{\lambda(\lambda + \alpha)(\lambda + 2\alpha)}{a_0 a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2}, \\
b_{1,3} &= \frac{\lambda(\lambda + \alpha)(\lambda + 2\alpha)}{a_0 a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2}, \\
b_{2,1} &= \frac{\lambda_{c_0}[a_1 a_2 - \mu(\lambda + \alpha)] + \lambda_{c_1}(\lambda + 2\alpha)a_2 + \lambda_{c_2}(\lambda + \alpha)(\lambda + 2\alpha)}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2}, \\
b_{2,2} &= \frac{\lambda_{c_0}[a_1 a_2 - \mu(\lambda + \alpha)] + \lambda_{c_1}(\lambda + 2\alpha)a_2 + \lambda_{c_2}(\lambda + \alpha)(\lambda + 2\alpha)}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)}, \\
b_{2,3} &= \frac{\lambda_{c_0}[a_1 a_2 - \mu(\lambda + \alpha)] + \lambda_{c_1}(\lambda + 2\alpha)a_2 + \lambda_{c_2}(\lambda + \alpha)(\lambda + 2\alpha)}{a_0 a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2}, \\
b_{3,1} &= \frac{\lambda_{h_0}[a_1 a_2 - \mu(\lambda + \alpha)] + \lambda_{h_1}(\lambda + 2\alpha)a_2 + \lambda_{h_2}(\lambda + \alpha)(\lambda + 2\alpha)}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2}, \\
b_{3,2} &= \frac{\lambda_{h_0}[a_1 a_2 - \mu(\lambda + \alpha)] + \lambda_{h_1}(\lambda + 2\alpha)a_2 + \lambda_{h_2}(\lambda + \alpha)(\lambda + 2\alpha)}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2}, \\
b_{3,3} &= \frac{\lambda_{h_0}[a_1 a_2 - \mu(\lambda + \alpha)] + \lambda_{h_1}(\lambda + 2\alpha)a_2 + \lambda_{h_2}(\lambda + \alpha)(\lambda + 2\alpha)}{a_0 a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2}.
\end{aligned}$$

We can compute the j^{th} column sum ($j = 1, 2, 3$) of the 3×3 -matrix $\Phi_R D_0^R$ as follows.

$$\begin{aligned}
\sum_{i=1}^3 (\Phi_R D_0^R)_{i,j} &= b_{1,j} + b_{2,j} + b_{3,j} \\
&= \frac{\lambda(\lambda + \alpha)(\lambda + 2\alpha)}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2} \\
&+ \frac{\lambda_{c_0}[a_1 a_2 - \mu(\lambda + \alpha)] + \lambda_{c_1}(\lambda + 2\alpha)a_2 + \lambda_{c_2}(\lambda + \alpha)(\lambda + 2\alpha)}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2} \\
&+ \frac{\lambda_{h_0}[a_1 a_2 - \mu(\lambda + \alpha)] + \lambda_{h_1}(\lambda + 2\alpha)a_2 + \lambda_{h_2}(\lambda + \alpha)(\lambda + 2\alpha)}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2} \\
&= \frac{(\lambda_{c_0} + \lambda_{h_0})[a_1 a_2 - \mu(\lambda + \alpha)] + (\lambda_{c_1} + \lambda_{h_1})(\lambda + 2\alpha)a_2}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2} \\
&+ \frac{(\lambda + \lambda_{c_2} + \lambda_{c_2})(\lambda + \alpha)(\lambda + 2\alpha)}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2} \\
&= \frac{[a_0 - (\lambda + 2\alpha)][a_1 a_2 - \mu(\lambda + \alpha)] + [a_1 - (\mu + \lambda + \alpha)](\lambda + 2\alpha)a_2}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2} \\
&+ \frac{(a_2 - \mu)(\lambda + \alpha)(\lambda + 2\alpha)}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2} \\
&= \frac{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - a_1 a_2(\lambda + 2\alpha) + \mu(\lambda + \alpha)(\lambda + 2\alpha)}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2} \\
&+ \frac{a_1 a_2(\lambda + 2\alpha) - \mu(\lambda + 2\alpha)a_2 - a_2(\lambda + \alpha)(\lambda + 2\alpha)}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2} \\
&+ \frac{a_2(\lambda + \alpha)(\lambda + 2\alpha) - \mu(\lambda + \alpha)(\lambda + 2\alpha)}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2} \\
&= \frac{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2}{a_0[a_1 a_2 - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)a_2} = 1.
\end{aligned}$$

This shows that $\Phi_R D_0^R$ is column stochastic, its transpose $(\Phi_R D_0^R)^t$ is row stochastic and hence $1 \in \sigma_p((\Phi_R D_0^R)^t)$. Since $\sigma_p(\Phi_R D_0^R) = \sigma_p((\Phi_R D_0^R)^t)$, also $1 \in \sigma_p(\Phi_R D_0^R)$ holds. Therefore, by the Characteristic Equation 1.3.6 we conclude that $0 \in \sigma_p(A_R)$. \square

Indeed, 0 is even the only spectral value of A_R on the imaginary axis.

LEMMA 3.3.7. *Under the General Assumption 3.1.1, the spectrum $\sigma(A_R)$ of A_R satisfies*

$$\sigma(A_R) \cap i\mathbb{R} = \{0\}.$$

PROOF. For any $a \in \mathbb{R}, a \neq 0, C = (r_1, r_2, r_3) \in \mathbb{C}^3$, we consider the resolvent equation $(Id - \Phi_R D_{ai}^R)Q = C$. This equation is equivalent to the following system of equations:

$$\begin{cases} (1 - ah_1)q_1 - ah_2q_2 - ah_3q_3 = r_1, \\ -bh_1q_1 + (1 - bh_2)q_2 - bh_3q_3 = r_2, \\ -ch_1q_1 - ch_2q_2 + (1 - ch_3)h_3 = r_3, \end{cases} \quad (\star)$$

where

$$\begin{aligned} a &= \frac{\lambda(\lambda + \alpha)(\lambda + 2\alpha)}{(ai + a_0)[(ai + a_1)(ai + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(ai + a_2)}, \\ b &= \frac{\lambda_{c_0}[(ai + a_1)(ai + a_2) - \mu(\lambda + \alpha)] + \lambda_{c_1}(\lambda + 2\alpha)(ai + a_2) + \lambda_{c_2}(\lambda + \alpha)(\lambda + \alpha)}{(ai + a_0)[(ai + a_1)(ai + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(ai + a_2)}, \\ c &= \frac{\lambda_{h_0}[(ai + a_1)(ai + a_2) - \mu(\lambda + \alpha)] + \lambda_{h_1}(\lambda + 2\alpha)(ai + a_2) + \lambda_{h_2}(\lambda + \alpha)(\lambda + \alpha)}{(ai + a_0)[(ai + a_1)(ai + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(ai + a_2)} \end{aligned}$$

$$h_j = \int_0^\infty \mu_j(x) e^{-aix - \int_0^x \mu_j(\xi) d\xi} dx, \quad j = 3, 4, 5.$$

Since for $a \neq 0$, we have

$$\begin{aligned} \begin{vmatrix} 1 - ah_1 & -ah_2 & -ah_3 \\ -ah_1 & 1 - ah_2 & -ah_3 \\ -ch_1 & ch_2 & 1 - ch_3 \end{vmatrix} &= (1 - ah_1)(1 - bh_2)(1 - ch_3) - abch_1h_2h_3 - abch_1h_2h_3 \\ &\quad - ac(1 - bh_2)h_1h_3 - bc(1 - ah_1)h_2h_3 - ab(1 - ch_3)h_1h_2 \\ &= 1 - bh_2 - ah_1 + abh_1h_2 - ch_3 + cbh_2h_3 + ach_1h_3 \\ &\quad - abch_1h_2h_3 - ach_1h_3 - bch_2h_3 - abh_1h_2 + abch_1h_2h_3 \\ &= 1 - ah_1 - bh_2 - ch_3 \neq 0, \end{aligned}$$

i.e., the determinant of the coefficient of the equations (\star) is not equal to 0. It follows that the equation $(Id - \Phi_R D_{ai}^R)Q = C$ has exactly one solution. Therefore, $1 \notin \sigma(\Phi_R D_{ai}^R)$. This implies by the Characteristic Equation 1.3.6 that $ai \notin \sigma(A_R)$, i.e.

$$\sigma(A_R) \cap i\mathbb{R} = \{0\}.$$

□

3.4. Well-posedness of the System

The main goal in this section is to prove the well-posedness of the system. In order to prove this, we will need some lemmas.

LEMMA 3.4.1. $A_R : D(A_R) \rightarrow R(A_R) \subset X_R$ is a closed linear operator and $D(A_R)$ is dense in X_R .

PROOF. We will prove the assertion in two steps.
We first prove that A_R is closed. For any given

$$\begin{aligned} P_n &= (p_0^{(n)}, p_1^{(n)}, p_2^{(n)}, p_3^{(n)}(x), p_4^{(n)}(x), p_5^{(n)}(x)) \in D(A_R), \\ P_0 &= (p_0^{(0)}, p_1^{(0)}, p_2^{(0)}, p_3^{(0)}(x), p_4^{(0)}(x), p_5^{(0)}(x)) \in X_R. \end{aligned}$$

We suppose that

$$\begin{aligned} \lim_{n \rightarrow \infty} P_n &= P_0, \\ \lim_{n \rightarrow \infty} A_R(P_n)^t &= (F_R)^t, \end{aligned}$$

where $F_R = (f_0, f_1, f_2, f_3(x), f_4(x), f_5(x)) \in X_R$. That is,

$$\begin{aligned} \lim_{n \rightarrow \infty} p_i^{(n)} &= p_i^{(0)}, \quad (i = 0, 1, 2) \\ \lim_{n \rightarrow \infty} \int_0^\infty |p_j^{(n)}(x) - p_j^{(0)}(x)| dx &= 0, \quad (j = 3, 4, 5). \end{aligned}$$

Then we obtain from the General Assumption 3.1.1 that

$$\lim_{n \rightarrow \infty} \int_0^\infty p_j^{(n)}(x) \mu_j(x) dx = \int_0^\infty p_j^{(0)}(x) \mu_j(x), \quad j = 3, 4, 5.$$

Furthermore,

$$\lim_{n \rightarrow \infty} A_R(P_n)^t = \lim_{n \rightarrow \infty} \begin{pmatrix} -a_0 p_0^{(n)} + \mu p_1^{(n)} + \sum_{i=3}^5 \int_0^\infty \mu_i(x) p_i^{(n)}(x) dx \\ (\lambda + 2\alpha) p_0^{(n)} - a_1 p_1^{(n)} + \mu p_2^{(n)} \\ (\lambda + \alpha) p_1^{(n)} - a_2 p_2^{(n)} \\ -\frac{dp_3^{(n)}(x)}{dx} - \mu_3(x) p_3^{(n)}(x) \\ -\frac{dp_4^{(n)}(x)}{dx} - \mu_4(x) p_4^{(n)}(x) \\ -\frac{dp_5^{(n)}(x)}{dx} - \mu_5(x) p_5^{(n)}(x) \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3(x) \\ f_4(x) \\ f_5(x) \end{pmatrix}.$$

This is equivalent to the following system of equations:

$$\left\{ \begin{array}{l} \lim_{n \rightarrow \infty} [-a_0 p_0^{(n)} + \mu p_1^{(n)} + \sum_{i=3}^5 \int_0^\infty \mu_i(x) p_i^{(n)}(x) dx] = f_0, \\ \lim_{n \rightarrow \infty} [(\lambda + 2\alpha) p_0^{(n)} - a_1 p_1^{(n)} + \mu p_2^{(n)}] = f_1, \\ \lim_{n \rightarrow \infty} [(\lambda + \alpha) p_1^{(n)} - a_2 p_2^{(n)}] = f_2, \\ \lim_{n \rightarrow \infty} \left[-\frac{dp_3^{(n)}(x)}{dx} - \mu_3(x) p_3^{(n)}(x) \right] = f_3(x), \\ \lim_{n \rightarrow \infty} \left[-\frac{dp_4^{(n)}(x)}{dx} - \mu_4(x) p_4^{(n)}(x) \right] = f_4(x), \\ \lim_{n \rightarrow \infty} \left[-\frac{dp_5^{(n)}(x)}{dx} - \mu_5(x) p_5^{(n)}(x) \right] = f_5(x). \end{array} \right.$$

Integrating both sides of last three equations from 0 to $\beta > 0$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\beta \left[-\frac{dp_j^{(n)}(x)}{dx} - \mu_j(x) p_j^{(n)}(x) \right] dx &= \int_0^\beta \lim_{n \rightarrow \infty} \left[-\frac{dp_j^{(n)}(x)}{dx} - \mu_j(x) p_j^{(n)}(x) \right] dx \\ &= \int_0^\beta f_j(x) dx, \quad j = 3, 4, 5. \end{aligned}$$

This yields

$$\begin{aligned} &\lim_{n \rightarrow \infty} \left[-p_j^{(n)}(\beta) - p_j^{(n)}(0) - \int_0^\beta \mu_j(x) p_j^{(n)}(x) dx \right] \\ &= -p_j^{(0)}(\beta) - p_j^{(0)}(0) - \int_0^\beta \mu_j(x) p_j^{(0)}(x) dx \\ &= \int_0^\beta f_j(x) dx, \quad j = 3, 4, 5. \end{aligned} \tag{30}$$

We know from the boundedness of $\mu_j(x)$ that $\int_0^\infty |\mu_j(x) p_j^{(0)}(x)| dx < \infty$. Further, we have $\int_0^\infty |f_j(x)| dx < \infty$. It follows from (30) that $p_j^{(0)}(\beta)$ is absolutely continuous and

$$p_j^{(0)'}(\beta) = -\mu_j(\beta) p_j^{(0)}(\beta) - f_j(\beta) \in L^1[0, \infty).$$

Therefore, $P_0 \in D(A_R)$ and

$$\lim_{n \rightarrow \infty} p_j^{(n)'}(\beta) = \lim_{n \rightarrow \infty} [-\mu_j(\beta) p_j^{(n)}(\beta)] - f_j(\beta) = p_j^{(0)'}(\beta).$$

From the above deduction we have

$$\left\{ \begin{array}{l} -a_0 p_0^{(0)} + \mu p_1^{(0)} + \sum_{i=3}^5 \int_0^\infty \mu_i(x) p_i^{(0)}(x) dx = f_0, \\ (\lambda + 2\alpha) p_0^{(0)} - a_1 p_1^{(0)} + \mu p_2^{(0)} = f_1, \\ (\lambda + \alpha) p_1^{(0)} - a_2 p_2^{(0)} = f_2, \\ -\frac{dp_3^{(0)}(x)}{dx} - \mu_3(x) p_3^{(0)}(x) = f_3(x), \\ -\frac{dp_4^{(0)}(x)}{dx} - \mu_4(x) p_4^{(0)}(x) = f_4(x), \\ -\frac{dp_5^{(0)}(x)}{dx} - \mu_5(x) p_5^{(0)}(x) = f_5(x). \end{array} \right.$$

This shows that $A_R(P_0)^t = (F_R)^t$, hence $(A_R, D(A_R))$ is closed.

We now prove that $D(A_R)$ is dense in X_R . We define

$$E_R = \left\{ p(x) = (p_0, p_1, p_2, p_3(x), p_4(x), p_5(x)) \left| \begin{array}{l} p_i \in \mathbb{C}, \quad i = 0, 1, 2; \\ p_i(x) \in C_0^\infty[0, \infty), \quad i = 3, 4, 5 \end{array} \right. \right\}.$$

Then by [Ada75] E_R is dense in X_R . If we define

$$H_R = \left\{ p(x) = (p_0, p_1, p_2, p_3(x), p_4(x), p_5(x)) \left| \begin{array}{l} p_i(x) \in C^\infty[0, \infty) \text{ and} \\ \text{there exists a number} \\ \alpha_i \text{ such that } p_i(x) = 0, \\ \text{for } x \in [0, \alpha_i], \quad i = 3, 4, 5 \end{array} \right. \right\},$$

then H_R is dense in E_R . Therefore, in order to prove that $D(A_R)$ is dense in X_R , it suffices to prove that $D(A_R)$ is dense in H_R . Take any

$$p(x) = (p_0, p_1, p_2, p_3(x), p_4(x), p_5(x)) \in H_R,$$

then there exist numbers α_i such that $p_i(x) = 0$, for all $x \in [0, \alpha_i]$ ($i = 3, 4, 5$), i.e., $p_i(x) = 0$ for $x \in [0, s]$, here $0 < s = \min\{\alpha_3, \alpha_4, \alpha_5\}$. We introduce a function

$$\begin{aligned} \varphi^s(0) &= (\varphi_0^s, \varphi_1^s, \varphi_2^s, \varphi_3^s(0), \varphi_4^s(0), \varphi_5^s(0)) \\ &= (p_0, p_1, p_2, \lambda p_2, \sum_{i=0}^2 \lambda_{c_i} p_i, \sum_{i=0}^2 \lambda_{h_i} p_i) \\ \varphi^s(x) &= (\varphi_0^s, \varphi_1^s, \varphi_2^s, \varphi_3^s(x), \varphi_4^s(x), \varphi_5^s(x)), \end{aligned}$$

where

$$\varphi_i^s(x) = \begin{cases} \varphi_i^s(0) \left(1 - \frac{x}{s}\right)^2 & \text{if } x \in [0, s] \\ p_i(x) & \text{if } x \in [s, \infty), \end{cases} \quad i = 3, 4, 5.$$

It is easy to verify that $\varphi^s(x) \in D(A_R)$. Moreover

$$\|p - \varphi^s\| = \sum_{i=3}^5 \int_0^s |\varphi_i^s(0)| \left(1 - \frac{x}{s}\right)^2 dx = \sum_{i=3}^5 |\varphi_i^s(0)| \frac{s}{3} \rightarrow 0 \quad \text{as } s \rightarrow 0.$$

This shows that $D(A_R)$ is dense in H_R . \square

LEMMA 3.4.2. $(A_R, D(A_R))$ is a dispersive operator.

PROOF. For $p \in D(A_R)$, we may choose

$$\phi(x) = \left(\frac{[p_0]^+}{p_0}, \frac{[p_1]^+}{p_1}, \frac{[p_2]^+}{p_2}, \frac{[p_3(x)]^+}{p_3(x)}, \frac{[p_4(x)]^+}{p_4(x)}, \frac{[p_5(x)]^+}{p_5(x)} \right),$$

where

$$[p_i]^+ = \begin{cases} p_i & \text{if } p_i > 0 \\ 0 & \text{if } p_i \leq 0 \end{cases}, \quad i = 0, 1, 2; \quad [p_i(x)]^+ = \begin{cases} p_i(x) & \text{if } p_i(x) > 0 \\ 0 & \text{if } p_i(x) \leq 0 \end{cases}, \quad i = 3, 4, 5.$$

If we define $W_i = \{x \in [0, \infty) \mid p_i(x) > 0\}$ and $Q_i = \{x \in [0, \infty) \mid p_i(x) \leq 0\}$ for $i = 3, 4, 5$, then we have

$$\begin{aligned} \int_0^\infty \frac{dp_i(x)}{dx} \frac{[p_i(x)]^+}{p_i(x)} dx &= \int_{W_i} \frac{dp_i(x)}{dx} \frac{[p_i(x)]^+}{p_i(x)} dx + \int_{Q_i} \frac{dp_i(x)}{dx} \frac{[p_i(x)]^+}{p_i(x)} dx \\ &= \int_{W_i} \frac{dp_i(x)}{dx} \frac{[p_i(x)]^+}{p_i(x)} dx = \int_{W_i} \frac{dp_i(x)}{dx} dx \\ &= \int_0^\infty \frac{d[p_i(x)]^+}{dx} dx = -[p_i(0)]^+, \quad i = 3, 4, 5, \end{aligned} \quad (31)$$

$$\int_0^\infty \mu_i(x) p_i(x) dx \leq \int_0^\infty \mu_i(x) [p_i(x)]^+ dx, \quad i = 3, 4, 5. \quad (32)$$

By (31), (32) and the boundary conditions on $p \in D(A)$ we obtain that

$$\begin{aligned} \langle A_R p, \phi \rangle &= \{-a_0 p_0 + \mu p_1 + \sum_{i=3}^5 \int_0^\infty \mu_i(x) p_i(x) dx\} \frac{[p_0]^+}{p_0} \\ &\quad + \{(\lambda + 2\alpha) p_0 - a_1 p_1 + \mu p_2\} \frac{[p_1]^+}{p_1} + \{(\lambda + \alpha) p_1 - a_2 p_2\} \frac{[p_2]^+}{p_2} \\ &\quad + \sum_{i=3}^5 \int_0^\infty \left\{ -\frac{dp_i(x)}{dx} - \mu_i(x) p_i(x) \right\} \frac{[p_i(x)]^+}{p_i(x)} dx \\ &= -a_0 [p_0]^+ + \mu p_1 \frac{[p_0]^+}{p_0} + \frac{[p_0]^+}{p_0} \sum_{i=3}^5 \int_0^\infty \mu_i(x) p_i(x) dx \\ &\quad + (\lambda + 2\alpha) \frac{[p_1]^+}{p_1} p_0 - a_1 [p_1]^+ + \mu \frac{[p_1]^+}{p_1} p_1 + (\lambda + \alpha) \frac{[p_2]^+}{p_2} p_1 \end{aligned}$$

$$\begin{aligned}
& - a_2[p_2]^+ - \sum_{i=3}^5 \int_0^\infty \frac{dp_i(x)}{dx} \frac{[p_i(x)]^+}{p_i(x)} dx - \sum_{i=3}^5 \int_0^\infty \mu_i(x)[p_i(x)]^+ dx \\
& = -a_0[p_0]^+ + \mu p_1 \frac{[p_0]^+}{p_0} + \frac{[p_0]^+}{p_0} \sum_{i=3}^5 \int_0^\infty \mu_i(x)p_i(x) dx \\
& + (\lambda + 2\alpha) \frac{[p_1]^+}{p_1} p_0 - a_1[p_1]^+ + \mu \frac{[p_1]^+}{p_1} p_1 + (\lambda + \alpha) \frac{[p_2]^+}{p_2} p_1 \\
& - a_2[p_2]^+ + \sum_{i=3}^5 [p_i(0)]^+ - \sum_{i=3}^5 \int_0^\infty \mu_i(x)[p_i(x)]^+ dx \\
& = -a_0[p_0]^+ + \mu p_1 \frac{[p_0]^+}{p_0} + \frac{[p_0]^+}{p_0} \sum_{i=3}^5 \int_0^\infty \mu_i(x)p_i(x) dx \\
& + (\lambda + 2\alpha) \frac{[p_1]^+}{p_1} p_0 - a_1[p_1]^+ + \mu \frac{[p_1]^+}{p_1} p_1 + (\lambda + \alpha) \frac{[p_2]^+}{p_2} p_1 \\
& - a_2[p_2]^+ + \{[\lambda p_2]^+ + [\lambda_{c_0} p_0 + \lambda_{c_1} p_1 + \lambda_{c_2} p_2]^+\} \\
& + [\lambda_{h_0} p_0 + \lambda_{h_1} p_1 + \lambda_{h_2} p_2]^+ \} - \sum_{i=3}^5 \int_0^\infty \mu_i(x)[p_i(x)]^+ dx \\
& \leq -a_0[p_0]^+ + \mu p_1 \frac{[p_0]^+}{p_0} + \frac{[p_0]^+}{p_0} \sum_{i=3}^5 \int_0^\infty \mu_i(x)p_i(x) dx \\
& + (\lambda + 2\alpha) \frac{[p_1]^+}{p_1} p_0 - a_1[p_1]^+ + \mu \frac{[p_1]^+}{p_1} p_1 + (\lambda + \alpha) \frac{[p_2]^+}{p_2} p_1 \\
& - a_2[p_2]^+ + \{[\lambda p_2]^+ + \{\lambda_{c_0}[p_0]^+ + \lambda_{c_1}[p_1]^+ + \lambda_{c_2}[p_2]^+\} \\
& + \{\lambda_{h_0}[p_0]^+ + \lambda_{h_1}[p_1]^+ + \lambda_{h_2}[p_2]^+\} \} - \sum_{i=3}^5 \int_0^\infty \mu_i(x)[p_i(x)]^+ dx \\
& = (\lambda_{c_0} + \lambda_{h_0} - a_0)[p_0]^+ + (\lambda_{c_1} + \lambda_{h_1} - a_0)[p_1]^+ + (\lambda + \lambda_{c_2} + \lambda_{h_2} - a_0)[p_2]^+ \\
& + \mu p_1 \frac{[p_0]^+}{p_0} + \frac{[p_0]^+}{p_0} \sum_{i=3}^5 \int_0^\infty \mu_i(x)p_i(x) dx + (\lambda + 2\alpha) \frac{[p_1]^+}{p_1} p_0 \\
& + \mu \frac{[p_1]^+}{p_1} p_1 + (\lambda + \alpha) \frac{[p_2]^+}{p_2} p_1 - \sum_{i=3}^5 \int_0^\infty \mu_i(x)[p_i(x)]^+ dx \\
& = -(\lambda + 2\alpha)[p_0]^+ - (\mu + \lambda + \alpha)[p_1]^+ - \mu[p_2]^+ + \mu p_1 \frac{[p_0]^+}{p_0} \\
& + \frac{[p_0]^+}{p_0} \sum_{i=3}^5 \int_0^\infty \mu_i(x)p_i(x) dx + (\lambda + 2\alpha) \frac{[p_1]^+}{p_1} p_0 + \mu \frac{[p_1]^+}{p_1} p_2
\end{aligned}$$

$$\begin{aligned}
& + (\lambda + \alpha) \frac{[p_2]^+}{p_2} p_1 - \sum_{i=3}^5 \int_0^\infty \mu_i(x) [p_i(x)]^+ dx \\
& \leq -(\lambda + 2\alpha) [p_0]^+ - (\mu + \lambda + \alpha) [p_1]^+ - \mu [p_2]^+ + \mu p_1 \frac{[p_0]^+}{p_0} \\
& + \frac{[p_0]^+}{p_0} \sum_{i=3}^5 \int_0^\infty \mu_i(x) [p_i(x)]^+ dx + (\lambda + 2\alpha) \frac{[p_1]^+}{p_1} p_0 + \mu \frac{[p_1]^+}{p_1} p_2 \\
& + (\lambda + \alpha) \frac{[p_2]^+}{p_2} p_1 - \sum_{i=3}^5 \int_0^\infty \mu_i(x) [p_i(x)]^+ dx \\
& \leq -(\lambda + 2\alpha) [p_0]^+ - (\mu + \lambda + \alpha) [p_1]^+ - \mu [p_2]^+ + \mu [p_1]^+ \frac{[p_0]^+}{p_0} \\
& + \left(\frac{[p_0]^+}{p_0} - 1 \right) \sum_{i=3}^5 \int_0^\infty \mu_i(x) [p_i(x)]^+ dx + (\lambda + 2\alpha) \frac{[p_1]^+}{p_1} [p_0]^+ \\
& + \mu \frac{[p_1]^+}{p_1} [p_2]^+ + (\lambda + \alpha) \frac{[p_2]^+}{p_2} [p_1]^+ \\
& \leq -(\lambda + 2\alpha) [p_0]^+ - (\mu + \lambda + \alpha) [p_1]^+ - \mu [p_2]^+ + \mu [p_1]^+ \\
& + \left(\frac{[p_0]^+}{p_0} - 1 \right) \sum_{i=3}^5 \int_0^\infty \mu_i(x) [p_i(x)]^+ dx + (\lambda + 2\alpha) [p_0]^+ \\
& + \mu [p_2]^+ + (\lambda + \alpha) [p_1]^+ \\
& = \left(\frac{[p_0]^+}{p_0} - 1 \right) \sum_{i=3}^5 \int_0^\infty \mu_i(x) [p_i(x)]^+ dx \leq 0.
\end{aligned}$$

This shows from Definition 1.2.5 that $(A_R, D(A_R))$ is a dispersive operator. \square

LEMMA 3.4.3. *If $\gamma \in \mathbb{R}, \gamma > 0$, then $\gamma \in \rho(A_R)$.*

PROOF. Let $\gamma \in \mathbb{R}, \gamma > 0$, then all the entries of $\Phi_R D_\gamma^R$ are positive and we have

$$\begin{aligned}
& a_0 [(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2) - \gamma(\lambda + 2\alpha)(\gamma + a_2) \\
& = (\lambda + 2\alpha + \lambda_{c_0} + \lambda_{h_0})(\gamma + \mu + \lambda + \alpha + \lambda_{c_1} + \lambda_{h_1})(\gamma + \mu + \lambda + \lambda_{c_2} + \lambda_{h_2}) \\
& - (\lambda + 2\alpha + \lambda_{c_0} + \lambda_{h_0})\mu(\lambda + \alpha) - \mu(\lambda + 2\alpha)(\gamma + \mu + \lambda + \lambda_{c_2} + \lambda_{h_2}) \\
& - \gamma(\lambda + 2\alpha)(\gamma + \mu + \lambda + \lambda_{c_2} + \lambda_{h_2}) \\
& = [(\lambda + 2\alpha + \lambda_{c_0} + \lambda_{h_0})\gamma(\gamma + \mu + \lambda + \lambda_{c_2} + \lambda_{h_2}) \\
& - \gamma(\lambda + 2\alpha)(\gamma + \mu + \lambda + \lambda_{c_2} + \lambda_{h_2})] \\
& + [(\lambda + 2\alpha + \lambda_{c_0} + \lambda_{h_0})\mu(\gamma + \mu + \lambda + \lambda_{c_2} + \lambda_{h_2}) \\
& - \mu(\lambda + 2\alpha)(\gamma + \mu + \lambda + \lambda_{c_2} + \lambda_{h_2})]
\end{aligned}$$

$$\begin{aligned}
& + [(\lambda + 2\alpha + \lambda_{c_0} + \lambda_{h_0})(\lambda + \alpha + \lambda_{c_1} + \lambda_{h_1})(\gamma + \mu + \lambda + \lambda_{c_2} + \lambda_{h_2}) \\
& - (\lambda + 2\alpha + \lambda_{c_0} + \lambda_{h_0})\mu(\lambda + \alpha)] > 0 \\
& \implies \\
& a_0[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2) > \gamma(\lambda + 2\alpha)(\gamma + a_2). \quad (33)
\end{aligned}$$

We also have

$$\int_0^\infty \mu_j(x) e^{-\gamma s - \int_0^s \mu_j(\xi) d\xi} ds < \int_0^\infty \mu_j(x) e^{-\int_0^s \mu_j(\xi) d\xi} ds = 1. \quad (34)$$

Using (33) and (34) we can estimate the j^{th} column sum as

$$\begin{aligned}
& \sum_{i=1}^3 (\Phi_R D_\gamma^R)_{i,j} \\
& = \frac{\lambda(\lambda + \alpha)(\lambda + 2\alpha) \int_0^\infty \mu_j(x) e^{-\gamma x - \int_0^x \mu_j(\xi) d\xi} dx}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)} \\
& + \frac{\lambda_{c_0}[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] + \lambda_{c_1}(\lambda + 2\alpha)(\gamma + a_2) + \lambda_{c_2}(\lambda + \alpha)(\lambda + \alpha)}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)} \\
& \times \int_0^\infty \mu_j(x) e^{-\gamma x - \int_0^x \mu_j(\xi) d\xi} dx \\
& + \frac{\lambda_{h_0}[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] + \lambda_{h_1}(\lambda + 2\alpha)(\gamma + a_2) + \lambda_{h_2}(\lambda + \alpha)(\lambda + \alpha)}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)} \\
& \times \int_0^\infty \mu_j(x) e^{-\gamma x - \int_0^x \mu_j(\xi) d\xi} dx \\
& = \left\{ 1 - \frac{\gamma[(\lambda + 2\alpha)(\gamma + a_2) + (\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)]}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)} \right\} \\
& \times \int_0^\infty \mu_j(x) e^{-\gamma x - \int_0^x \mu_j(\xi) d\xi} dx \\
& = \left\{ 1 - \frac{\gamma[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] + \gamma(\lambda + 2\alpha)(\gamma + a_2)}{(\gamma + a_0)[(\gamma + a_1)(\gamma + a_2) - \mu(\lambda + \alpha)] - \mu(\lambda + 2\alpha)(\gamma + a_2)} \right\} \\
& \times \int_0^\infty \mu_j(x) e^{-\gamma x - \int_0^x \mu_j(\xi) d\xi} dx < 1.
\end{aligned}$$

It follows from this that $\|\Phi_R D_\gamma^R\| < 1$, and thus also

$$r(\Phi_R D_\gamma^R) \leq \|\Phi_R D_\gamma^R\| < 1.$$

Therefore, $1 \notin \sigma(\Phi_R D_\gamma^R)$. Using the Characteristic Equation 1.3.6 we conclude that $\gamma \in \rho(A_R)$ for $\gamma \in \mathbb{R}, \gamma > 0$. \square

From Lemma 3.4.1, Lemma 3.4.2 and Lemma 3.4.3 we immediately obtain the following result.

THEOREM 3.4.4. *The operator $(A_R, D(A_R))$ generates a positive contraction C_0 -semigroup $(T_R(t))_{t \geq 0}$.*

From Proposition 1.2.4 and Theorem 3.4.4 we can state our main result.

THEOREM 3.4.5. *The system $(R), (BC_R)$ and $(IC_{R,0})$ has a unique positive solution $p(x, t)$ which satisfies $\|p(\cdot, t)\| = 1, t \in [0, \infty)$.*

PROOF. From Proposition 1.2.4 and Theorem 3.4.4 we obtain that the associated abstract Cauchy problem (ACP_R) has a unique positive time-dependent solution $p(x, t)$ which can be expressed as

$$p(x, t) = T_R(t)p(0) = T_R(t)(1, 0, 0, 0, \dots). \quad (35)$$

Let $P(t) = p(x, t) = (p_0(t), p_1, p_2, p_3(x, t), p_4(x, t), p_5(x, t))$, then $P(t)$ satisfies the system of equations:

$$\frac{dp_0(t)}{dt} = -a_0p_0(t) + \mu p_1(t) + \sum_{i=3}^5 \int_0^\infty \mu_i(x)p_i(x, t)dx, \quad (36)$$

$$\frac{dp_1(t)}{dt} = (\lambda + 2\alpha)p_0(t) - a_1p_1(t) + \mu p_2(t), \quad (37)$$

$$\frac{dp_2(t)}{dt} = (\lambda + \alpha)p_1(t) - a_2p_2(t), \quad (38)$$

$$\frac{\partial p_3(x, t)}{\partial t} = -\frac{\partial p_3(x, t)}{\partial x} - \mu_3(x)p_3(x, t), \quad (39)$$

$$\frac{\partial p_4(x, t)}{\partial t} = -\frac{\partial p_4(x, t)}{\partial x} - \mu_4(x)p_4(x, t), \quad (40)$$

$$\frac{\partial p_5(x, t)}{\partial t} = -\frac{\partial p_5(x, t)}{\partial x} - \mu_5(x)p_5(x, t), \quad (41)$$

$$p_3(0, t) = \lambda p_2(t), \quad t > 0, \quad (42)$$

$$p_4(0, t) = \sum_{i=0}^2 \lambda_{c_i} p_i(t), \quad t > 0, \quad (43)$$

$$p_5(0, t) = \sum_{i=0}^2 \lambda_{h_i} p_i(t), \quad t > 0, \quad (44)$$

$$P(0) = (1, 0, 0, 0, 0, \dots). \quad (45)$$

Since

$$\int_0^\infty \frac{\partial p_j(x, t)}{\partial x} dx = p_j(\infty, t) - p_j(0, t) = -p_j(0, t), \quad j = 3, 4, 5. \quad (46)$$

Using (36)–(46) we compute

$$\begin{aligned}
\frac{d\|P(t)\|}{dt} &= \sum_{i=0}^2 \frac{dp_i(t)}{dt} + \sum_{j=3}^5 \int_0^\infty \frac{\partial p_j(x,t)}{\partial t} dx \\
&= -a_0 p_0(t) + \mu p_1(t) + \sum_{j=3}^5 \int_0^\infty \mu_j(x) p_j(x,t) dx, \\
&+ (\lambda + 2\alpha) p_0(t) - a_1 p_1(t) + \mu p_2(t) + (\lambda + \alpha) p_1(t) - a_2 p_2(t), \\
&+ \sum_{j=3}^5 \int_0^\infty \left[-\frac{\partial p_j(x,t)}{\partial x} - \mu_j(x) p_j(x,t) \right] dx \\
&= (-a_0 + \lambda + 2\alpha) p_0(t) + (\mu - a_1 + \lambda + \alpha) p_1(t) + (\mu - a_2) p_2(t) \\
&+ \sum_{j=3}^5 p_j(0,t) \\
&= -\sum_{j=3}^5 p_j(0,t) + \sum_{j=3}^5 p_j(0,t) = 0.
\end{aligned} \tag{47}$$

By (35) and (47) we obtain

$$\frac{d\|P(t)\|}{dt} = \frac{d\|T_R(t)P(0)\|}{dt} = 0.$$

Therefore,

$$\|T_R(t)P(0)\| = \|P(t)\| = \|P(0)\| = 1.$$

This shows $\|p(\cdot, t)\| = 1, \forall t \in [0, \infty)$. \square

3.5. Asymptotic Stability of the Solution

In this section we investigate the asymptotic stability of the system using the results on positive semigroup collected in Section 1.1. We express from Lemma 1.3.7 the resolvent of A_R in terms of the resolvent of A_0^R , the Dirichlet operator D_γ^R and the boundary operator Φ_R . The representation for the resolvent of A_0^R shows that it is a positive operator for $\gamma > 0$. This property is very useful in the following lemma to prove the irreducibility of the semigroup $(T_R(t))_{t \geq 0}$ generated by $(A_R, D(A_R))$. For the notation and terminology concerning positive operators we refer to the books [Sch74] and [Nag86].

LEMMA 3.5.1. *The semigroup $(T_R(t))_{t \geq 0}$ generated by $(A_R, D(A_R))$ is irreducible.*

PROOF. We know from [Nag86, Def. C-III 3.1] that the irreducibility of $(T_R(t))_{t \geq 0}$ is equivalent to the existence of $\gamma > 0$ such that $0 < p \in X$ implies

$R(\gamma, A)p \gg 0$. We now suppose that $\gamma > 0$ and $0 < p \in X_R$. Then also $R(\gamma, A_0^R)p > 0$ and $\Phi_R R(\gamma, A_0^R)p > 0$. It follows from the proof of Lemma 3.4.3 that $\|\Phi_R D_\gamma^R\| < 1$ for all $\gamma > 0$. Hence the inverse of $Id_{\partial X_R} - \Phi_R D_\gamma^R$ can be computed via the Neumann series

$$(Id_{\partial X_R} - \Phi_R D_\gamma^R)^{-1} = \sum_{n=0}^{\infty} (\Phi_R D_\gamma^R)^n.$$

We know from the form of $\Phi_R D_\gamma^R$ that for every $i \in \{1, 2, 3\}$ there exists $k \in \mathbb{N}$ such that the real number $((\Phi_R D_\gamma^R)^k \Phi_R R(\gamma, A_0^R)p)_i > 0$. Therefore,

$$(Id_{\partial X_R} - \Phi_R D_\gamma^R)^{-1} \Phi_R R(\gamma, A_0^R)p \gg 0,$$

and by the form of D_γ^R we have

$$D_\gamma^R (Id_{\partial X_R} - \Phi_R D_\gamma^R)^{-1} \Phi_R R(\gamma, A_0^R)p \gg 0.$$

This implies

$$R(\gamma, A_R)p \gg 0,$$

and hence $(T_R(t))_{t \geq 0}$ is irreducible. \square

We now prove our main result on the asymptotic behaviour. Combining Lemma 3.5.1 with the results from Section 1.4 we obtain the strong convergence of the semigroup to a one-dimensional equilibrium.

THEOREM 3.5.2. *The space X_R can be decomposed into the direct sum*

$$X_R = X_R^1 \oplus X_R^2,$$

where $X_R^1 = \text{fix}(T_R(t))_{t \geq 0} = \ker A_R$ is one-dimensional and spanned by a strictly positive eigenvector $\tilde{p} \in \ker A_R$ of A_R and the restricted semigroup $(T_R(t)|_{X_R^2})_{t \geq 0}$ is strongly stable.

PROOF. Combining Theorem 3.4.4, Lemma 3.3.6, Lemma 3.3.7, Lemma 3.5.1 with Theorem 1.4.2, we obtain the proof of the theorem. \square

We rewrite the above theorem as the following.

COROLLARY 3.5.3. There exists $p' \in X_R^1, p' \gg 0$, such that for all $p \in X_R$

$$\lim_{t \rightarrow \infty} T_R(t)p = \langle p', p \rangle \tilde{p},$$

where $\ker A_R = \langle \tilde{p} \rangle, \tilde{p} \gg 0$.

Since the semigroup gives the solutions of the original system, we obtain our final result.

COROLLARY 3.5.4. The time-dependent solution of the system $(R), (BC_R)$ and $(IC_{R,0})$ converges strongly to the steady-state solution as time tends to infinite, that is, $\lim_{t \rightarrow \infty} p(\cdot, t) = \alpha \tilde{p}$, where $\alpha > 0$ and \tilde{p} as in Corollary 3.5.3.

CHAPTER 4

A Parallel Maintenance System with Two Components

4.1. Introduction

In this section, we consider the model of a parallel maintenance system with two components. Parallel systems consisting of two repairable units are a usual phenomenon in our daily life, for example, the parallel connection of two bulbs with the same power, the parallel connection of two computers with the same power, etc. So the study of these systems is important in view of theory and practice.

The mathematical model of parallel maintenance system with two components was first put forward by L.Yeh, see [Yeh97]. He assumed that the state of a system forms a continuous-time Markov chain or a higher-dimensional Markov process after introducing some supplementary variables and derived a formula for evaluating the rate of occurrence of failures for the system. As an application of the theory, he studied the maintenance model for a two-component system. But he did not discuss the well-posedness of the model and its asymptotic behavior. In 2003, Guo Weihua proved that the model has a unique positive time-dependent solution by using classical analysis methods, see [Guo03].

According to [Yeh97], the model for the parallel maintenance system with two components can be expressed by a system of integro-differential equations

$$(PS) \left\{ \begin{array}{l} \frac{dp_0(t)}{dt} = -(\lambda_1 + \lambda_2)p_0(t) + \sum_{i=1}^2 \int_0^\infty r_i(x)p_i(t, x)dx, \\ \frac{\partial p_1(t, x)}{\partial t} + \frac{\partial p_1(t, x)}{\partial x} = -(\lambda_2 + r_1(x))p_1(t, x), \\ \frac{\partial p_2(t, x)}{\partial t} + \frac{\partial p_2(t, x)}{\partial x} = -(\lambda_1 + r_2(x))p_2(t, x), \\ \frac{\partial p_3(t, x)}{\partial t} + \frac{\partial p_3(t, x)}{\partial x} = -r_1(x)p_3(t, x) + \lambda_2 p_1(t, x), \\ \frac{\partial p_4(t, x)}{\partial t} + \frac{\partial p_4(t, x)}{\partial x} = -r_2(x)p_4(t, x) + \lambda_1 p_2(t, x). \end{array} \right.$$

For $x = 0$ the boundary conditions

$$(BC_{PS}) \begin{cases} p_1(t, 0) = \lambda_1 p_0(t) + \int_0^\infty p_4(t, x) r_2(x) dx, \\ p_2(t, 0) = \lambda_2 p_0(t) + \int_0^\infty p_3(t, x) r_1(x) dx, \\ p_i(t, 0) = 0, i = 3, 4 \end{cases}$$

are prescribed, and we consider the usual initial condition

$$(IC_{PS}) \begin{cases} p_0(0) = c \in \mathbb{C}, \\ p_j(0, x) = f_j(x), j = 1, 2, 3, 4, \end{cases}$$

where $f_j(x) \in L^1[0, \infty)$. The following initial condition is most interesting

$$(IC_{PS,0}) \begin{cases} p_0(0) = 1, \\ p_j(0, x) = 0, j = 1, 2, 3, 4. \end{cases}$$

Here $(x, t) \in [0, \infty) \times [0, \infty)$. Let $p_0(t)$ denote the probability that two units are both working at time t ; $p_1(x, t)dx$ gives the probability that unit 2 is working, unit 1 fails and the failed unit has elapsed repair time lying in $(x, x+dx]$; $p_2(x, t)dx$ describes the probability that unit 1 is working, unit 2 fails and the failed unit has elapsed repair time lying in $(x, x+dx]$; $p_3(x, t)dx$ gives the probability that both units fail, unit 1 has elapsed repair time lying in $(x, x+dx]$ and unit 2 is waiting for repair; $p_4(x, t)dx$ gives the probability that both units fail, unit 2 has elapsed repair time lying in $(x, x+dx]$ and unit 1 is waiting for repair; λ_1 represents the rate of occurrence of failures for unit 1, λ_2 represents the rate of occurrence of failures for unit 2; $r_i(x)$ ($i = 1, 2$) is the hazard function.

We require the following for the failure rate $r_i(x)$ ($i = 1, 2$).

GENERAL ASSUMPTION 4.1.1. *The functions $r_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are measurable and bounded such that $\lim_{x \rightarrow \infty} r_i(x)$ exists and*

$$r_\infty^{(i)} := \lim_{x \rightarrow \infty} r_i(x) > 0, \quad i = 1, 2, \quad r_\infty := \min(r_\infty^{(1)}, r_\infty^{(2)})$$

4.2. The Problem as an Abstract Cauchy Problem

The underlying problem is rewritten as an abstract Cauchy problem on a suitable space X_{PS} , see [EN00, Def. II.6.1]. As the state space for our problem we choose

$$X_{PS} := \mathbb{C} \times (L^1[0, \infty))^4.$$

It is obvious that X_{PS} is a Banach space endowed with the norm

$$\|p\| := |p_0| + \sum_{n=1}^4 \|p_n\|_{L^1[0, \infty)},$$

where $p = (p_0, p_1(x), p_2(x), p_3(x), p_4(x))^t \in X_{PS}$.

For simplicity, we denote by ψ_i the linear functionals

$$\psi_i : L^1[0, \infty) \rightarrow \mathbb{C}, \quad f \mapsto \psi_i(f) := \int_0^\infty r_i(x)f(x) dx, \quad i = 1, 2.$$

Moreover, we define the operators D_j on $W^{1,1}[0, \infty)$ as

$$\begin{aligned} D_1 f &:= -\frac{d}{dx}f - (\lambda_2 + r_1)f, \\ D_2 f &:= -\frac{d}{dx}f - (\lambda_1 + r_2)f, \\ D_3 f &:= -\frac{d}{dx}f - r_1f, \\ D_4 f &:= -\frac{d}{dx}f - r_2f, \quad \text{for } f \in W^{1,1}[0, \infty), \end{aligned}$$

respectively. To define the appropriate operator $(A_{PS}, D(A_{PS}))$ we introduce a “maximal operator” $(A_m^{PS}, D(A_m^{PS}))$ on X_{PS} given as

$$\begin{aligned} A_m^{PS} &:= \begin{pmatrix} -(\lambda_1 + \lambda_2) & \psi_1 & \psi_2 & 0 & 0 \\ 0 & D_1 & 0 & 0 & 0 \\ 0 & 0 & D_2 & 0 & 0 \\ 0 & \lambda_2 & 0 & D_3 & 0 \\ 0 & 0 & \lambda_1 & 0 & D_4 \end{pmatrix}, \\ D(A_m^{PS}) &:= \mathbb{C} \times (W^{1,1}[0, \infty))^4. \end{aligned}$$

To model the boundary conditions (BC_{PS}) we use an abstract approach as in e.g. [CENN03]. For this purpose we consider the “boundary space”

$$\partial X_{PS} := \mathbb{C}^4,$$

and then define “boundary operators” L_{PS} and Φ_{PS} . As the operator L_{PS} we take

$$L_{PS} : D(A_m^{PS}) \rightarrow \partial X_{PS}, \quad \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ p_4(x) \end{pmatrix} \mapsto L_{PS} \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ p_4(x) \end{pmatrix} := \begin{pmatrix} p_1(0) \\ p_2(0) \\ p_3(0) \\ p_4(0) \end{pmatrix},$$

and the operator $\Phi_{PS} \in \mathcal{L}(X_{PS}, \partial X_{PS})$ is defined by the matrix

$$\Phi_{PS} := \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & \psi_2 \\ \lambda_2 & 0 & 0 & \psi_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The operator $(A_{PS}, D(A_{PS}))$ on X_{PS} is then defined as

$$A_{PS}p := A_m^{PS}p, \quad D(A_{PS}) := \{p \in D(A_m^{PS}) \mid L_{PS}p = \Phi_{PS}p\}.$$

With these definitions the above equations (PS) , (BC_{PS}) and (IC_{PS}) are equivalent to the abstract Cauchy problem

$$\begin{cases} \frac{dp(t)}{dt} = A_{PS}p(t), & t \in [0, \infty), \\ p(0) = (c, f_1, f_2, f_3, f_4)^t \in X_{PS}. \end{cases} \quad (ACP_{PS})$$

If A_{PS} is the generator of a strongly continuous semigroup $(T_{PS}(t))_{t \geq 0}$ and the initial value in (IC_{PS}) satisfies $p(0) = (c, f_1, f_2, f_3, f_4)^t \in D(A_{PS})$, then the unique solution of (PS) , (BC_{PS}) and (IC_{PS}) is given by

$$\begin{aligned} p_0(t) &= (T_{PS}(t)p(0))_1 \\ p_n(x, t) &= (T_{PS}(t)p(0))_{n+1}(x), \quad 1 \leq n \leq 4. \end{aligned}$$

4.3. Boundary Spectrum

In this section, the boundary spectrum $\sigma_b(A_{PS})$ of A_{PS} is investigated using the Characteristic Equation 1.3.6. For this purpose, we start from the operator $(A_0^{PS}, D(A_0^{PS}))$ defined by

$$\begin{aligned} D(A_0^{PS}) &:= \{p \in D(A_m^{PS}) \mid L_{PS}p = 0\}, \\ A_0^{PS}p &:= A_m^{PS}p. \end{aligned}$$

The following lemma gives the representation of the resolvent of the operator A_0^{PS} needed below to discuss the irreducibility of the semigroup generated by the operator A_{PS} .

LEMMA 4.3.1. *For the set $S_{PS} := \{\gamma \in \mathbb{C} \mid \Re \gamma > -r_\infty\} \setminus \{-(\lambda_1 + \lambda_2)\}$ we have*

$$S_{PS} \subseteq \rho(A_0^{PS}).$$

Moreover, if $\gamma \in S_{PS}$, then

$$R(\gamma, A_0^{PS}) = \begin{pmatrix} s_{1,1} & s_{1,2} & s_{1,3} & 0 & 0 \\ 0 & s_{2,2} & 0 & 0 & 0 \\ 0 & 0 & s_{3,3} & 0 & 0 \\ 0 & s_{4,2} & 0 & s_{4,4} & 0 \\ 0 & 0 & s_{5,3} & 0 & s_{5,5} \end{pmatrix}, \quad (48)$$

where

$$\begin{aligned}
s_{1,1} &= \frac{1}{\gamma+\lambda_1+\lambda_2}, \\
s_{1,2} &= \frac{1}{\gamma+\lambda_1+\lambda_2}\psi_1 R(\gamma, D_1), \\
s_{1,3} &= \frac{1}{\gamma+\lambda_1+\lambda_2}\psi_2 R(\gamma, D_2), \\
s_{2,2} &= R(\gamma, D_1), \\
s_{3,3} &= R(\gamma, D_2), \\
s_{4,2} &= -(\gamma - \lambda_2)R(\gamma, D_3)R(\gamma, D_1) \\
s_{4,4} &= R(\gamma, D_3), \\
s_{5,3} &= -(\gamma - \lambda_1)R(\gamma, D_4)R(\gamma, D_2) \\
s_{5,5} &= R(\gamma, D_4).
\end{aligned}$$

The resolvent operators of the differential operators $D_i (j = 1, 2, 3, 4)$ is given by

$$\begin{aligned}
(R(\gamma, D_1)p)(x) &= e^{-(\gamma+\lambda_2)x - \int_0^x r_1(\xi)d\xi} \int_0^x e^{(\gamma+\lambda_2)s + \int_0^s r_1(\xi)d\xi} p(s) ds, \\
(R(\gamma, D_2)p)(x) &= e^{-(\gamma+\lambda_1)x - \int_0^x r_2(\xi)d\xi} \int_0^x e^{(\gamma+\lambda_1)s + \int_0^s r_2(\xi)d\xi} p(s) ds, \\
(R(\gamma, D_3)p)(x) &= e^{-\gamma x - \int_0^x r_1(\xi)d\xi} \int_0^x e^{\gamma s + \int_0^s r_1(\xi)d\xi} p(s) ds, \\
(R(\gamma, D_4)p)(x) &= e^{-\gamma x - \int_0^x r_2(\xi)d\xi} \int_0^x e^{\gamma s + \int_0^s r_2(\xi)d\xi} p(s) ds,
\end{aligned}$$

for $p \in L^1[0, \infty)$.

PROOF. A combination of [Gre84, Prop. 2.1] and [Nag89, Thm. 2.4] yields that the resolvent set of A_0^{PS} satisfies

$$\rho(A_0^{PS}) \supseteq S_{PS}.$$

For $\gamma \in S_{PS}$ we can compute the resolvent of A_0^{PS} explicitly applying the formula for the inverse of operator matrices, see [Nag89, Thm. 2.4]. This leads to the representation (48) of the resolvent of A_0^{PS} .

Clearly, knowing the operator matrix in (48), we can directly compute that it represents the resolvent of A_0^{PS} . \square

The following consequence will be used to compute the spectrum of A_{PS} .

COROLLARY 4.3.2. The imaginary axis belongs to the resolvent set of A_0^{PS} , i.e.,

$$i\mathbb{R} \subseteq \rho(A_0^{PS})$$

The elements in $\ker(\gamma - A_m^{PS})$ can be expressed as follows:

LEMMA 4.3.3. *For $\gamma \in \mathbb{C}$, we have*

$$p \in \ker(\gamma - A_m^{PS}) \quad (49)$$

\Leftrightarrow

$$\begin{aligned} p &= (p_0, p_1(\cdot), p_2(\cdot), p_3(\cdot), p_4(\cdot))^t \in D(A_m), \text{ with} \\ p_0 &= \frac{d_1}{\gamma + \lambda_1 + \lambda_2} \times \int_0^\infty r_1(x) e^{-(\gamma + \lambda_2)x - \int_0^x r_1(\xi) d\xi} dx \\ &+ \frac{d_2}{\gamma + \lambda_1 + \lambda_2} \times \int_0^\infty r_2(x) e^{-(\gamma + \lambda_1)x - \int_0^x r_2(\xi) d\xi} dx, \end{aligned} \quad (50)$$

$$p_1(x) = d_1 e^{-(\gamma + \lambda_2)x - \int_0^x r_1(\xi) d\xi}, \quad (51)$$

$$p_2(x) = d_2 e^{-(\gamma + \lambda_1)x - \int_0^x r_2(\xi) d\xi}, \quad (52)$$

$$p_3(x) = [d_1(1 - e^{-\lambda_2 x}) + d_3] e^{-\gamma x - \int_0^x r_1(\xi) d\xi}, \quad (53)$$

$$p_4(x) = [d_2(1 - e^{-\lambda_1 x}) + d_4] e^{-\gamma x - \int_0^x r_2(\xi) d\xi}. \quad (54)$$

PROOF. If for $p \in X_{PS}$, (50)-(54) are fulfilled, then we can easily compute that $p \in \ker(\gamma - A_m^R)$. Conversely, the condition (49) gives a system of differential equations. Solving these differential equations, we see that (50)-(54) are indeed satisfied. \square

Moreover, since L_{PS} is surjective, it follows from [Gre87, Lemma 1.2] that

$$L_{PS}|_{\ker(\gamma - A_m^{PS})} : \ker(\gamma - A_m^{PS}) \rightarrow \partial X_{PS}$$

is invertible for each $\gamma \in \rho(A_0^{PS})$. We denote its inverse by

$$D_\gamma^{PS} := (L_{PS}|_{\ker(\gamma - A_m^{PS})})^{-1} : \partial X_{PS} \longrightarrow \ker(\gamma - A_m^{PS}),$$

and call it ‘‘Dirichlet operator’’.

We can give the explicit form of D_γ^R as follows.

LEMMA 4.3.4. *For each $\gamma \in \rho(A_0^{PS})$, the operator D_γ^{PS} has the form*

$$D_\gamma^{PS} = \begin{pmatrix} d_{1,1} & d_{1,2} & 0 & 0 \\ d_{2,1} & 0 & 0 & 0 \\ 0 & d_{3,2} & 0 & 0 \\ d_{4,1} & 0 & d_{4,3} & 0 \\ 0 & d_{5,2} & 0 & d_{5,4} \end{pmatrix},$$

where

$$\begin{aligned}
d_{1,1} &= \frac{1}{\gamma + \lambda_1 + \lambda_2} \times \int_0^\infty r_1(x) e^{-(\gamma+\lambda_2)x - \int_0^x r_1(\xi) d\xi} dx, \\
d_{1,2} &= \frac{1}{\gamma + \lambda_1 + \lambda_2} \times \int_0^\infty r_2(x) e^{-(\gamma+\lambda_1)x - \int_0^x r_2(\xi) d\xi} dx, \\
d_{2,1} &= e^{-(\gamma+\lambda_2)x - \int_0^x r_1(\xi) d\xi}, \\
d_{3,2} &= e^{-(\gamma+\lambda_2)x - \int_0^x r_1(\xi) d\xi}, \\
d_{4,1} &= e^{-\gamma x - \int_0^x r_1(\xi) d\xi} (1 - e^{-\lambda_2 x}), \\
d_{4,3} &= e^{-\gamma x - \int_0^x r_1(\xi) d\xi}, \\
d_{5,2} &= e^{-\gamma x - \int_0^x r_2(\xi) d\xi} (1 - e^{-\lambda_1 x}), \\
d_{5,4} &= e^{-\gamma x - \int_0^x r_2(\xi) d\xi}.
\end{aligned}$$

The operator $\Phi_R D_\gamma^{PS}$ can be computed explicitly for $\gamma \in \rho(A_0^{PS})$.

REMARK 4.3.5. For $\gamma \in \rho(A_0^{PS})$ the operator $\Phi_{PS} D_\gamma^{PS}$ can be represented by the 4×4 -matrix

$$\Phi_{PS} D_\gamma^{PS} = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}
a_{1,1} &= \frac{\lambda_1}{\gamma + \lambda_1 + \lambda_2} \times \int_0^\infty r_1(x) e^{-(\gamma+\lambda_2)x - \int_0^x r_1(\xi) d\xi} dx, \\
a_{1,2} &= \frac{\lambda_1}{\gamma + \lambda_1 + \lambda_2} \times \int_0^\infty r_2(x) e^{-(\gamma+\lambda_1)x - \int_0^x r_2(\xi) d\xi} dx \\
&\quad + \int_0^\infty r_2(x) e^{-\gamma x - \int_0^x r_2(\xi) d\xi} (1 - e^{-\lambda_1 x}), \\
a_{2,1} &= \frac{\lambda_2}{\gamma + \lambda_1 + \lambda_2} \times \int_0^\infty r_1(x) e^{-(\gamma+\lambda_2)x - \int_0^x r_1(\xi) d\xi} dx \\
&\quad + \int_0^\infty r_1(x) e^{-\gamma x - \int_0^x r_1(\xi) d\xi} (1 - e^{-\lambda_2 x}), \\
a_{2,2} &= \frac{\lambda_2}{\gamma + \lambda_1 + \lambda_2} \times \int_0^\infty r_2(x) e^{-(\gamma+\lambda_1)x - \int_0^x r_2(\xi) d\xi} dx, \\
a_{2,3} &= \int_0^\infty r_1(x) e^{-\gamma x - \int_0^x r_1(\xi) d\xi} dx, \\
a_{1,4} &= \int_0^\infty r_2(x) e^{-\gamma x - \int_0^x r_2(\xi) d\xi} dx.
\end{aligned}$$

We now show that 0 is in the point spectrum of A_{PS} .

LEMMA 4.3.6. 0 is an eigenvalue of $(A_{PS}, D(A_{PS}))$ with geometric multiplicity one.

PROOF. Let $A_{PS}P = 0$, this is equivalent to the following system of equations.

$$-(\lambda_1 + \lambda_2)p_0 + \sum_{i=1}^2 \int_0^\infty r_i(x)p_i(x)dx = 0, \quad (55)$$

$$\frac{\partial p_1(x)}{\partial x} = -(\lambda_2 + r_1(x))p_1(x), \quad (56)$$

$$\frac{\partial p_2(x)}{\partial x} = -(\lambda_1 + r_2(x))p_2(x), \quad (57)$$

$$\frac{\partial p_3(x)}{\partial x} = -r_1(x)p_3(x) + \lambda_2 p_1(x), \quad (58)$$

$$\frac{\partial p_4(x)}{\partial x} = -r_2(x)p_4(x) + \lambda_1 p_2(x), \quad (59)$$

where

$$p_1(0) = \lambda_1 p_0 + \int_0^\infty p_4(x)r_2(x)dx, \quad (60)$$

$$p_2(0) = \lambda_2 p_0 + \int_0^\infty p_3(x)r_1(x)dx, \quad (61)$$

$$p_i(0) = 0, i = 3, 4. \quad (62)$$

Solving (55), (56), (57), (58) and (59) we obtain that

$$p_0 = \frac{a_1}{\lambda_1 + \lambda_2} \int_0^\infty r_1(x)e^{-\lambda_2 x - \int_0^x r_1(\xi) d\xi} dx + \frac{a_2}{\lambda_1 + \lambda_2} \int_0^\infty r_2(x)e^{-\lambda_1 x - \int_0^x r_2(\xi) d\xi} dx, \quad (63)$$

$$p_1(x) = a_1 e^{-\lambda_2 x - \int_0^x r_1(\xi) d\xi}, \quad (64)$$

$$p_2(x) = a_2 e^{-\lambda_1 x - \int_0^x r_2(\xi) d\xi}, \quad (65)$$

$$p_3(x) = a_3 e^{-\int_0^x r_1(\xi) d\xi} + a_1 (1 - e^{-\lambda_2 x}) e^{-\int_0^x r_1(\xi) d\xi}, \quad (66)$$

$$p_4(x) = a_3 e^{-\int_0^x r_2(\xi) d\xi} + a_2 (1 - e^{-\lambda_1 x}) e^{-\int_0^x r_2(\xi) d\xi}, \quad (67)$$

Combining (66), (67) with (60), (61) and (62) we have

$$a_1 = p_1(0) = \lambda_1 p_0 + a_2 - a_2 \int_0^\infty r_2(x)e^{-\lambda_1 x - \int_0^x r_2(\xi) d\xi} dx, \quad (68)$$

$$a_2 = p_2(0) = \lambda_2 p_0 + a_1 - a_1 \int_0^\infty r_1(x)e^{-\lambda_2 x - \int_0^x r_1(\xi) d\xi} dx, \quad (69)$$

$$a_i = p_i(0) = 0, i = 3, 4. \quad (70)$$

Using the following abbreviations,

$$\alpha := 1 - \int_0^\infty r_2(x) e^{-\lambda_1 x - \int_0^x r_2(\xi) d\xi} dx,$$

$$\beta := 1 - \int_0^\infty r_1(x) e^{-\lambda_2 x - \int_0^x r_1(\xi) d\xi} dx,$$

we obtain from (68) and (69) that

$$a_1 = \frac{\lambda_2 \alpha + \lambda_1}{1 - \alpha \beta} p_0, \quad (71)$$

$$a_2 = \frac{\lambda_1 \beta + \lambda_2}{1 - \alpha \beta} p_0. \quad (72)$$

Substituting separately (70), (71) and (72) into (64), (65), (66) and (67) we obtain that

$$p_0 = \left[\frac{\lambda_2 \alpha + \lambda_1}{(\lambda_1 + \lambda_2)(1 - \alpha \beta)} \int_0^\infty r_1(x) e^{-\lambda_2 x - \int_0^x r_1(\xi) d\xi} dx + \frac{\lambda_1 \beta + \lambda_2}{(\lambda_1 + \lambda_2)(1 - \alpha \beta)} \int_0^\infty r_2(x) e^{-\lambda_1 x - \int_0^x r_2(\xi) d\xi} dx \right] p_0, \quad (73)$$

$$p_1(x) = \frac{\lambda_2 \alpha + \lambda_1}{1 - \alpha \beta} e^{-\lambda_2 x - \int_0^x r_1(\xi) d\xi} p_0, \quad (74)$$

$$p_2(x) = \frac{\lambda_1 \beta + \lambda_2}{1 - \alpha \beta} e^{-\lambda_1 x - \int_0^x r_2(\xi) d\xi} p_0, \quad (75)$$

$$p_3(x) = \frac{\lambda_2 \alpha + \lambda_1}{1 - \alpha \beta} (1 - e^{-\lambda_2 x}) e^{-\int_0^x r_1(\xi) d\xi} p_0, \quad (76)$$

$$p_4(x) = \frac{\lambda_1 \beta + \lambda_2}{1 - \alpha \beta} (1 - e^{-\lambda_1 x}) e^{-\int_0^x r_2(\xi) d\xi} p_0. \quad (77)$$

This shows that 0 is an eigenvalue of A_{PS} . By (73), (74), (75), (76) and (77) we can easily see that the geometric multiplicity of 0 is one. \square

If X_{PS}^* denotes the dual space of X_{PS} , then

$$X_{PS}^* := \mathbb{C} \times (L^\infty[0, \infty))^4.$$

It is obvious that X_{PS}^* is a Banach space endowed with the norm

$$\|q\| := \max(|q_0|, \|q_1\|_{L^\infty[0, \infty)}, \|q_2\|_{L^\infty[0, \infty)}, \|q_3\|_{L^\infty[0, \infty)}, \|q_4\|_{L^\infty[0, \infty)}),$$

where $q = (q_0, q_1(x), q_2(x), q_3(x), q_4(x))^t \in X_{PS}^*$. Let $(A_{PS}^*, D(A_{PS}^*))$ be the adjoint operator of $(A_{PS}, D(A_{PS}))$, then $(A_{PS}^*, D(A_{PS}^*))$ can be expressed as

$$A_{PS}^*q = \begin{pmatrix} -(\lambda_1 + \lambda_2)q_0 + \lambda_1q_1(0) + \lambda_2q_2(0) \\ -\frac{dq_1(x)}{dx} + r_1(x)q_0 - (\lambda_2 + r_1(x))q_1(x) + \lambda_2q_3(x) \\ -\frac{dq_2(x)}{dx} + r_2(x)q_0 - (\lambda_1 + r_2(x))q_2(x) + \lambda_2q_4(x) \\ -\frac{dq_3(x)}{dx} + r_1(x)q_2(0) - r_1(x)q_3(x) \\ -\frac{dq_4(x)}{dx} + r_2(x)q_1(0) - r_2(x)q_4(x) \end{pmatrix},$$

$$D(A_{PS}^*) = \left\{ (q_0, q_1(x), q_2(x), q_3(x), q_4(x)) \left| \begin{array}{l} \frac{q_i(x)}{dx} \in L^\infty[0, \infty), q_i(x) \text{ is an} \\ \text{absolute continuous function,} \\ \text{and } q_i(x) \text{ is finite, } i = 1, 2, 3, 4 \end{array} \right. \right\}.$$

LEMMA 4.3.7. 0 is an eigenvalue of $(A_{PS}^*, D(A_{PS}^*))$ with geometric multiplicity one.

PROOF. Consider the equation $A_{PS}^*q = 0$. This is equivalent to the following system.

$$-(\lambda_1 + \lambda_2)q_0 + \lambda_1q_1(0) + \lambda_2q_2(0) = 0, \quad (78)$$

$$\frac{dq_1(x)}{dx} = (\lambda_2 + r_1(x))q_1(x) - \lambda_2q_3(x) - r_1(x)q_0, \quad (79)$$

$$\frac{dq_2(x)}{dx} = (\lambda_1 + r_2(x))q_2(x) - \lambda_1q_4(x) - r_2(x)q_0, \quad (80)$$

$$\frac{\partial q_3(x)}{\partial x} = r_1(x)q_3(x) - r_1(x)q_2(0), \quad (81)$$

$$\frac{\partial q_4(x)}{\partial x} = r_2(x)q_4(x) - r_2(x)q_1(0), \quad (82)$$

where

$$q_1(\infty) = q_2(\infty) = q_3(\infty) = q_4(\infty) = \omega. \quad (83)$$

Solving (79)–(82) we have

$$\begin{aligned} q_1(x) &= b_1 e^{\lambda_2 x + \int_0^x r_1(\xi) d\xi} + b_3 e^{\lambda_2 x + \int_0^x r_1(\xi) d\xi} (e^{-\lambda_2 x} - 1) \\ &\quad - \lambda_2 q_2(0) e^{\lambda_2 x + \int_0^x r_1(\xi) d\xi} \int_0^x e^{-\lambda_2 \tau + \int_0^\tau r_1(\xi) d\xi} d\tau \\ &\quad - q_2(0) e^{\lambda_2 x + \int_0^x r_1(\xi) d\xi} (e^{-\lambda_2 x} - 1) \\ &\quad - q_0 e^{\lambda_2 x + \int_0^x r_1(\xi) d\xi} \int_0^x r_1(\tau) e^{-\lambda_2 \tau + \int_0^\tau r_1(\xi) d\xi} d\tau, \quad (84) \\ q_2(x) &= b_2 e^{\lambda_1 x + \int_0^x r_2(\xi) d\xi} + b_4 e^{\lambda_1 x + \int_0^x r_2(\xi) d\xi} (e^{-\lambda_1 x} - 1) \\ &\quad - \lambda_1 q_1(0) e^{\lambda_1 x + \int_0^x r_2(\xi) d\xi} \int_0^x e^{-\lambda_1 \tau + \int_0^\tau r_2(\xi) d\xi} d\tau \end{aligned}$$

$$\begin{aligned}
& - q_1(0)e^{\lambda_1 x + \int_0^x r_2(\xi) d\xi} (e^{-\lambda_1 x} - 1) \\
& - q_0 e^{\lambda_1 x + \int_0^x r_2(\xi) d\xi} \int_0^x r_2(\tau) e^{-\lambda_1 \tau + \int_0^\tau r_2(\xi) d\xi} d\tau, \tag{85}
\end{aligned}$$

$$q_3(x) = b_3 e^{\int_0^x r_1(\xi) d\xi} + q_2(0) - q_2(0) e^{\int_0^x r_1(\xi) d\xi}, \tag{86}$$

$$q_4(x) = b_4 e^{\int_0^x r_2(\xi) d\xi} + q_1(0) - q_1(0) e^{\int_0^x r_2(\xi) d\xi}. \tag{87}$$

Multiplying $e^{-\lambda_2 x - \int_0^x r_1(\xi) d\xi}$ on both sides of (84), multiplying $e^{-\lambda_1 x - \int_0^x r_2(\xi) d\xi}$ on both sides of (85), multiplying $e^{-\int_0^x r_1(\xi) d\xi}$ on both sides of (86) and multiplying $e^{-\int_0^x r_2(\xi) d\xi}$ on both sides of (87), then using (83) we deduce

$$b_1 = \lambda_2 q_2(0) \int_0^\infty e^{\lambda_2 x + \int_0^x r_1(\xi) d\xi} dx + q_0 \int_0^\infty r_1(x) e^{-\lambda_2 x + \int_0^x r_1(\xi) d\xi} dx \tag{88}$$

$$b_2 = \lambda_1 q_1(0) \int_0^\infty e^{\lambda_1 x + \int_0^x r_2(\xi) d\xi} dx + q_0 \int_0^\infty r_2(x) e^{-\lambda_1 x + \int_0^x r_2(\xi) d\xi} dx \tag{89}$$

$$b_3 = q_2(0), \tag{90}$$

$$b_4 = q_1(0). \tag{91}$$

Substituting (88)–(89) into (84)–(87) we derive

$$\begin{aligned}
q_1(x) &= \lambda_2 q_2(0) e^{\lambda_2 x + \int_0^x r_1(\xi) d\xi} \int_x^\infty e^{\lambda_2 x + \int_0^x r_1(\xi) d\xi} dx \\
&+ q_0 e^{-\lambda_2 x + \int_0^x r_1(\xi) d\xi} \int_x^\infty r_1(x) e^{-\lambda_2 x + \int_0^x r_1(\xi) d\xi} dx, \tag{92}
\end{aligned}$$

$$\begin{aligned}
q_2(x) &= \lambda_1 q_1(0) e^{\lambda_1 x + \int_0^x r_2(\xi) d\xi} \int_x^\infty e^{\lambda_1 x + \int_0^x r_2(\xi) d\xi} dx \\
&+ q_0 e^{-\lambda_1 x + \int_0^x r_2(\xi) d\xi} \int_x^\infty r_2(x) e^{-\lambda_1 x + \int_0^x r_2(\xi) d\xi} dx, \tag{93}
\end{aligned}$$

$$q_3(x) = q_2(0), \tag{94}$$

$$q_4(x) = q_1(0). \tag{95}$$

From (92), (93), (94) and (95) it follows that

$$q_1(0) = \lambda_2 q_2(0) \int_0^\infty e^{\lambda_2 x + \int_0^x r_1(\xi) d\xi} dx + q_0 \int_0^\infty r_1(x) e^{-\lambda_2 x + \int_0^x r_1(\xi) d\xi} dx \tag{96}$$

$$q_1(0) = \lambda_1 q_1(0) \int_0^\infty e^{\lambda_1 x + \int_0^x r_2(\xi) d\xi} dx + q_0 \int_0^\infty r_2(x) e^{-\lambda_1 x + \int_0^x r_2(\xi) d\xi} dx \tag{97}$$

$$q_3(0) = q_2(0), \tag{98}$$

$$q_4(0) = q_1(0). \tag{99}$$

Solving (96)–(99) we obtain

$$q_1(0) = q_2(0) = q_3(0) = q_4(0) = q_0. \tag{100}$$

Combining (100) with (92), (93), (94) and (95) we have

$$q_1(x) = q_2(x) = q_3(x) = q_4(x) = q_0. \quad (101)$$

This shows that 0 is an eigenvalue of $(A_{PS}^*, D(A_{PS}^*))$. From (101) it follows that the geometric multiplicity of 0 is one. \square

Indeed, 0 is even the only spectral value of A_{PS} on the imaginary axis.

LEMMA 4.3.8. *Under the General Assumption 3.1.1, the spectrum $\sigma(A_{PS})$ of A_{PS} satisfies*

$$\sigma(A_{PS}) \cap i\mathbb{R} = \{0\}.$$

PROOF. Let $ai \in \sigma(A_{PS})$ for some $0 \neq a \in \mathbb{R}$ and consider the 4×4 -matrix

$$\Phi_{PS} D_{ai}^{PS} = \begin{pmatrix} b_{1,1} & b_{1,2} & 0 & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} b_{1,1} &= \frac{\lambda_1}{ai + \lambda_1 + \lambda_2} \times \int_0^\infty r_1(x) e^{-(ai+\lambda_2)x - \int_0^x r_1(\xi) d\xi} dx, \\ b_{1,2} &= \frac{\lambda_1}{ai + \lambda_1 + \lambda_2} \times \int_0^\infty r_2(x) e^{-(ai+\lambda_1)x - \int_0^x r_2(\xi) d\xi} dx \\ &\quad + \int_0^\infty r_2(x) e^{-aix - \int_0^x r_2(\xi) d\xi} (1 - e^{-\lambda_1 x}), \\ b_{2,1} &= \frac{\lambda_2}{ai + \lambda_1 + \lambda_2} \times \int_0^\infty r_1(x) e^{-(ai+\lambda_2)x - \int_0^x r_1(\xi) d\xi} dx \\ &\quad + \int_0^\infty r_1(x) e^{-aix - \int_0^x r_1(\xi) d\xi} (1 - e^{-\lambda_2 x}) dx, \\ b_{2,2} &= \frac{\lambda_2}{ai + \lambda_1 + \lambda_2} \times \int_0^\infty r_2(x) e^{-(ai+\lambda_2)x - \int_0^x r_2(\xi) d\xi} dx, \\ b_{2,3} &= \int_0^\infty r_1(x) e^{-aix - \int_0^x r_1(\xi) d\xi} dx, \\ b_{1,4} &= \int_0^\infty r_2(x) e^{-\gamma x - \int_0^x r_2(\xi) d\xi} dx. \end{aligned}$$

The General Assumption 4.1.1 implies that there exists $r \in \mathbb{R}_+$ such that $r_i(x) > 0$ for all $x \in [r, r + \frac{2\pi}{a}]$. Using the abbreviation $s_i(x) := r_i(x) e^{-\int_0^x \mu(\xi) d\xi}$, $i =$

1, 2, we compute

$$\begin{aligned}
& \left| \int_0^\infty r_i(x) e^{-\gamma x - \int_0^x r_i(\xi) d\xi} dx \right| \\
&= \left| \int_0^\infty e^{-aix} s_i(x) dx \right| \\
&\leq \left| \int_r^{r+\frac{2\pi}{a}} e^{-aix} s_i(x) dx \right| + \left| \int_0^r e^{-aix} s_i(x) dx + \int_{r+\frac{2\pi}{a}}^\infty e^{-aix} s_i(x) dx \right| \\
&\leq \left| \int_r^{r+\frac{2\pi}{a}} e^{-aix} s_i(x) dx \right| + \int_0^r s_i(x) dx + \int_{r+\frac{2\pi}{a}}^\infty s_i(x) dx.
\end{aligned}$$

The first term on the right hand side of the above inequality can be estimated as

$$\begin{aligned}
\left| \int_r^{r+\frac{2\pi}{a}} e^{-aix} s_i(x) dx \right| &= \left| \int_r^{r+\frac{\pi}{a}} e^{-aix} s_i(x) dx + \int_{r+\frac{\pi}{a}}^{r+\frac{2\pi}{a}} e^{-aix} s_i(x) dx \right| \\
&= \left| \int_r^{r+\frac{\pi}{a}} e^{-aix} s_i(x) dx + \int_r^{r+\frac{\pi}{a}} e^{-ai(x+\frac{\pi}{a})} s_i\left(x+\frac{\pi}{a}\right) dx \right| \\
&= \left| \int_r^{r+\frac{\pi}{a}} e^{-aix} s_i(x) dx - \int_r^{r+\frac{\pi}{a}} e^{-aix} s_i\left(x+\frac{\pi}{a}\right) dx \right| \\
&= \left| \int_r^{r+\frac{\pi}{a}} e^{-aix} (s_i(x) - s_i\left(x+\frac{\pi}{a}\right)) dx \right| \\
&\leq \int_r^{r+\frac{\pi}{a}} |s_i(x) - s_i\left(x+\frac{\pi}{a}\right)| dx \\
&< \int_r^{r+\frac{\pi}{a}} (s_i(x) + s_i\left(x+\frac{\pi}{a}\right)) dx \\
&= \int_r^{r+\frac{\pi}{a}} s_i(x) dx + \int_{\frac{\pi}{a}}^{\frac{2\pi}{a}} s_i(x) dx \\
&= \int_r^{r+\frac{2\pi}{a}} s_i(x) dx,
\end{aligned}$$

where we used the strict positivity of $r_i(x)$ on $[r, r + \frac{2\pi}{a}]$ in the last inequality. We thus obtain

$$\left| \int_0^\infty r_i(x) e^{-\gamma x - \int_0^x r_i(\xi) d\xi} dx \right| < \int_r^{r+\frac{2\pi}{a}} s_i(x) dx + \int_0^r s_i(x) dx + \int_{r+\frac{2\pi}{a}}^\infty s_i(x) dx$$

$$= \int_0^{\infty} s_i(x) dx = 1. \quad (102)$$

Using (102) we can estimate each column sum of absolute entries of $\Phi_{PS}D_{ai}$ as

$$\begin{aligned} \sum_{i=1}^4 |(\Phi_{PS}D_{ai}^{PS})_{i1}| &= |b_{1,1}| + |b_{2,1}| \\ &= \left| \frac{\lambda_1}{ai + \lambda_1 + \lambda_2} \times \int_0^{\infty} r_1(x) e^{-(ai+\lambda_2)x - \int_0^x r_1(\xi) d\xi} dx \right| \\ &\quad + \left| \frac{\lambda_2}{ai + \lambda_1 + \lambda_2} \times \int_0^{\infty} r_1(x) e^{-(ai+\lambda_2)x - \int_0^x r_1(\xi) d\xi} dx \right. \\ &\quad \left. + \int_0^{\infty} r_1(x) e^{-aix - \int_0^x r_1(\xi) d\xi} (1 - e^{-\lambda_2 x}) dx \right| \\ &\leq \frac{\lambda_1}{\sqrt{a^2 + (\lambda_1 + \lambda_2)^2}} \times \int_0^{\infty} |r_1(x) e^{-(ai+\lambda_2)x - \int_0^x r_1(\xi) d\xi}| dx \\ &\quad + \frac{\lambda_2}{\sqrt{a^2 + (\lambda_1 + \lambda_2)^2}} \times \int_0^{\infty} |r_1(x) e^{-(ai+\lambda_2)x - \int_0^x r_1(\xi) d\xi}| dx \\ &\quad + \int_0^{\infty} |r_1(x) e^{-aix - \int_0^x r_1(\xi) d\xi} (1 - e^{-\lambda_2 x})| dx \\ &= \frac{\lambda_1}{\sqrt{a^2 + (\lambda_1 + \lambda_2)^2}} \times \int_0^{\infty} r_1(x) e^{-\lambda_2 x - \int_0^x r_1(\xi) d\xi} dx \\ &\quad + \frac{\lambda_2}{\sqrt{a^2 + (\lambda_1 + \lambda_2)^2}} \times \int_0^{\infty} r_1(x) e^{-\lambda_2 x - \int_0^x r_1(\xi) d\xi} dx \\ &\quad + \int_0^{\infty} r_1(x) e^{-\int_0^x r_1(\xi) d\xi} (1 - e^{-\lambda_2 x}) dx \\ &< \int_0^{\infty} r_1(x) e^{-\lambda_2 x - \int_0^x r_1(\xi) d\xi} dx \\ &\quad + \int_0^{\infty} r_1(x) e^{-\int_0^x r_1(\xi) d\xi} (1 - e^{-\lambda_2 x}) dx \\ &= \int_0^{\infty} r_1(x) e^{-\int_0^x r_1(\xi) d\xi} dx = 1, \end{aligned} \quad (103)$$

$$\begin{aligned} \sum_{i=1}^4 |(\Phi_{PS}D_{ai}^{PS})_{i2}| &= |b_{1,2}| + |b_{2,2}| \\ &= \left| \frac{\lambda_1}{ai + \lambda_1 + \lambda_2} \times \int_0^{\infty} r_2(x) e^{-(ai+\lambda_1)x - \int_0^x r_2(\xi) d\xi} dx \right. \\ &\quad \left. + \int_0^{\infty} r_2(x) e^{-aix - \int_0^x r_2(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx \right| \end{aligned}$$

$$\begin{aligned}
& + \left| \frac{\lambda_2}{ai + \lambda_1 + \lambda_2} \times \int_0^\infty r_2(x) e^{-(ai+\lambda_1)x - \int_0^x r_2(\xi) d\xi} dx \right| \\
& \leq \frac{\lambda_2}{\sqrt{a^2 + (\lambda_1 + \lambda_2)^2}} \times \int_0^\infty |r_2(x) e^{-(ai+\lambda_1)x - \int_0^x r_2(\xi) d\xi}| dx \\
& + \frac{\lambda_1}{\sqrt{a^2 + (\lambda_1 + \lambda_2)^2}} \times \int_0^\infty |r_2(x) e^{-(ai+\lambda_1)x - \int_0^x r_2(\xi) d\xi}| dx \\
& + \int_0^\infty |r_2(x) e^{-aix - \int_0^x r_2(\xi) d\xi} (1 - e^{-\lambda_1 x})| dx \\
& = \frac{\lambda_2}{\sqrt{a^2 + (\lambda_1 + \lambda_2)^2}} \times \int_0^\infty r_2(x) e^{-\lambda_1 x - \int_0^x r_2(\xi) d\xi} dx \\
& + \frac{\lambda_1}{\sqrt{a^2 + (\lambda_1 + \lambda_2)^2}} \times \int_0^\infty r_2(x) e^{-\lambda_1 x - \int_0^x r_2(\xi) d\xi} dx \\
& + \int_0^\infty r_2(x) e^{-\int_0^x r_2(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx \\
& < \int_0^\infty r_2(x) e^{-\lambda_1 x - \int_0^x r_2(\xi) d\xi} dx \\
& + \int_0^\infty r_2(x) e^{-\int_0^x r_2(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx \\
& = \int_0^\infty r_2(x) e^{-\int_0^x r_2(\xi) d\xi} dx = 1, \tag{104}
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^4 |(\Phi_{PS} D_{ai}^{PS})_{i3}| & = |b_{2,3}| = \left| \int_0^\infty r_1(x) e^{-aix - \int_0^x r_1(\xi) d\xi} dx \right| \\
& < \int_0^\infty r_1(x) e^{-\int_0^x r_1(\xi) d\xi} dx = 1, \tag{105}
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^4 |(\Phi_{PS} D_{ai}^{PS})_{i4}| & = |b_{1,4}| = \left| \int_0^\infty r_2(x) e^{-aix - \int_0^x r_2(\xi) d\xi} dx \right| \\
& < \int_0^\infty r_2(x) e^{-\int_0^x r_2(\xi) d\xi} dx = 1. \tag{106}
\end{aligned}$$

Using (103), (104), (105) and (106) we derive

$$\|\Phi_{PS} D_{ai}^{PS}\| < 1,$$

thus the spectral radius fulfills

$$r(\Phi_{PS} D_{ai}^{PS}) \leq \|\Phi_{PS} D_{ai}^{PS}\| < 1.$$

By the Characteristic Equation 1.3.6 we obtain that $ai \notin \sigma(A_{PS})$ for all $a \in \mathbb{R}$, $a \neq 0$, i.e., $\sigma(A_{PS}) \cap i\mathbb{R} = \{0\}$. \square

4.4. Well-posedness of the System

In this section we prove the well-posedness of the system. In order to do this, we will need some lemmas.

LEMMA 4.4.1. $A_{PS} : D(A_{PS}) \rightarrow R(A_{PS}) \subset X_{PS}$ is a closed linear operator and $D(A_{PS})$ is dense in X_{PS} .

PROOF. We first prove that $(A_{PS}, D(A_{PS}))$ is closed operator. For given

$$\begin{aligned} P_n &= (p_0^{(n)}, p_1^{(n)}(x), p_2^{(n)}(x), p_3^{(n)}(x), p_4^{(n)}(x)) \in D(A_{PS}), \\ P_0 &= (p_0^{(0)}, p_1^{(0)}(x), p_2^{(0)}(x), p_3^{(0)}(x), p_4^{(0)}(x)) \in X_{PS}, \end{aligned}$$

we suppose that

$$\begin{aligned} \lim_{n \rightarrow \infty} P_n &= P_0, \\ \lim_{n \rightarrow \infty} A_{PS}(P_n)^t &= (F_{PS})^t, \end{aligned}$$

where $F_{PS} = (h_0, h_1(x), h_2(x), h_3(x), h_4(x)) \in X_{PS}$. That is,

$$\begin{aligned} \lim_{n \rightarrow \infty} p_0^{(n)} &= p_0^{(0)}, \\ \lim_{n \rightarrow \infty} \int_0^\infty |p_i^{(n)}(x) - p_i^{(0)}(x)| dx &= 0, \quad (i = 1, 2, 3, 4). \end{aligned}$$

Then we obtain from the General Assumption 4.1.1 that

$$\lim_{n \rightarrow \infty} \int_0^\infty p_j^{(n)}(x) r_j(x) dx = \int_0^\infty p_j^{(0)}(x) r_j(x) dx, \quad j = 1, 2.$$

Furthermore,

$$\lim_{n \rightarrow \infty} A(P_n)^t = \lim_{n \rightarrow \infty} \begin{pmatrix} -(\lambda_1 + \lambda_2)p_0^{(n)} + \sum_{j=1}^2 \int_0^\infty r_j(x) p_j^{(n)}(x) dx \\ -\frac{dp_1^{(n)}(x)}{dx} - (\lambda_2 + r_1(x))p_1^{(n)}(x) \\ -\frac{dp_2^{(n)}(x)}{dx} - (\lambda_1 + r_2(x))p_2^{(n)}(x) \\ -\frac{dp_3^{(n)}(x)}{dx} + \lambda_2 p_1^{(n)}(x) - r_1(x)p_3^{(n)}(x) \\ -\frac{dp_4^{(n)}(x)}{dx} + \lambda_1 p_2^{(n)}(x) - r_2(x)p_4^{(n)}(x) \end{pmatrix} = \begin{pmatrix} h_0 \\ h_1(x) \\ h_2(x) \\ h_3(x) \\ h_4(x) \end{pmatrix}.$$

This is equivalent to the following system of equations:

$$\left\{ \begin{array}{l} \lim_{n \rightarrow \infty} [-(\lambda_1 + \lambda_2)p_0^{(n)} + \sum_{j=1}^2 \int_0^\infty r_j(x)p_j^{(n)}(x)dx] = h_0, \\ \lim_{n \rightarrow \infty} [-\frac{dp_1^{(n)}(x)}{dx} - (\lambda_2 + r_1(x))p_1^{(n)}(x)] = h_1(x), \\ \lim_{n \rightarrow \infty} [-\frac{dp_2^{(n)}(x)}{dx} - (\lambda_1 + r_2(x))p_2^{(n)}(x)] = h_2(x), \\ \lim_{n \rightarrow \infty} [-\frac{dp_3^{(n)}(x)}{dx} + \lambda_2 p_1^{(n)}(x) - r_1(x)p_3^{(n)}(x)] = h_3(x), \\ \lim_{n \rightarrow \infty} [-\frac{dp_4^{(n)}(x)}{dx} + \lambda_1 p_2^{(n)}(x) - r_2(x)p_4^{(n)}(x)] = h_4(x). \end{array} \right.$$

Integrating both sides of the last four equations from 0 to β , we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_0^\beta [-\frac{dp_1^{(n)}(x)}{dx} - (\lambda_2 + r_1(x))p_1^{(n)}(x)] \\ &= \int_0^\beta \lim_{n \rightarrow \infty} [-\frac{dp_1^{(n)}(x)}{dx} - (\lambda_2 + r_1(x))p_1^{(n)}(x)] \\ &= \int_0^\beta h_1(x), \\ & \lim_{n \rightarrow \infty} \int_0^\beta [-\frac{dp_2^{(n)}(x)}{dx} - (\lambda_1 + r_2(x))p_2^{(n)}(x)] \\ &= \int_0^\beta \lim_{n \rightarrow \infty} [-\frac{dp_2^{(n)}(x)}{dx} - (\lambda_1 + r_2(x))p_2^{(n)}(x)] \\ &= \int_0^\beta h_2(x), \\ & \lim_{n \rightarrow \infty} \int_0^\beta [-\frac{dp_3^{(n)}(x)}{dx} + \lambda_2 p_1^{(n)}(x) - r_1(x)p_3^{(n)}(x)] \\ &= \int_0^\beta \lim_{n \rightarrow \infty} [-\frac{dp_3^{(n)}(x)}{dx} + \lambda_2 p_1^{(n)}(x) - r_1(x)p_3^{(n)}(x)] \\ &= \int_0^\beta h_3(x), \\ & \lim_{n \rightarrow \infty} \int_0^\beta [-\frac{dp_4^{(n)}(x)}{dx} + \lambda_1 p_2^{(n)}(x) - r_2(x)p_4^{(n)}(x)] \\ &= \int_0^\beta \lim_{n \rightarrow \infty} [-\frac{dp_4^{(n)}(x)}{dx} + \lambda_1 p_2^{(n)}(x) - r_2(x)p_4^{(n)}(x)] \end{aligned}$$

$$= \int_0^\beta h_4(x).$$

This yields

$$\begin{aligned} & \lim_{n \rightarrow \infty} [-p_1^{(n)}(\beta) + p_1^{(n)}(0) - \int_0^\beta (\lambda_2 + r_1(x))p_1^{(n)}(x)dx] \\ &= -p_1^{(0)}(\beta) + p_1^{(0)}(0) - \int_0^\beta (\lambda_2 + r_1(x))p_1^{(0)}(x)dx \\ &= \int_0^\beta h_1(x), \end{aligned} \tag{107}$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} [-p_2^{(n)}(\beta) + p_2^{(n)}(0) - \int_0^\beta (\lambda_1 + r_2(x))p_2^{(n)}(x)dx] \\ &= -p_2^{(0)}(\beta) + p_2^{(0)}(0) - \int_0^\beta (\lambda_1 + r_2(x))p_2^{(0)}(x)dx \\ &= \int_0^\beta h_2(x), \end{aligned} \tag{108}$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} [-p_3^{(n)}(\beta) + p_3^{(n)}(0) + \lambda_2 \int_0^\beta p_1^{(n)}(x) - \int_0^\beta r_1(x)p_3^{(n)}(x)dx] \\ &= -p_3^{(0)}(\beta) + p_3^{(0)}(0) + \lambda_2 \int_0^\beta p_1^{(0)}(x) - \int_0^\beta r_1(x)p_3^{(0)}(x)dx \\ &= \int_0^\beta h_3(x), \end{aligned} \tag{109}$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} [-p_4^{(n)}(\beta) + p_4^{(n)}(0) + \lambda_1 \int_0^\beta p_2^{(n)}(x) - \int_0^\beta r_2(x)p_4^{(n)}(x)dx] \\ &= -p_4^{(0)}(\beta) + p_4^{(0)}(0) + \lambda_1 \int_0^\beta p_2^{(0)}(x) - \int_0^\beta r_2(x)p_4^{(0)}(x)dx \\ &= \int_0^\beta h_4(x). \end{aligned} \tag{110}$$

We know from the boundedness of $r_j(x)$, $j = 1, 2$ that $\int_0^\infty |r_j(x)p_j^{(0)}(x)|dx < \infty$ and $\int_0^\infty |r_j(x)p_{j+1}^{(0)}(x)|dx < \infty$, $j = 1, 2$. Further, we have $\int_0^\infty |h_i(x)|dx < \infty$, $i = 1, 2, 3, 4$. It follows from (107), (108), (109) and (110) that $p_i^{(0)}(\beta)$ is absolutely continuous and

$$\begin{aligned} p_1^{(0)}(\beta) &= -(\lambda_2 + r_1(\beta))p_1^{(0)}(\beta) - h_1(\beta) \in L^1[0, \infty), \\ p_2^{(0)}(\beta) &= -(\lambda_1 + r_2(\beta))p_2^{(0)}(\beta) - h_2(\beta) \in L^1[0, \infty), \\ p_3^{(0)}(\beta) &= \lambda_2 p_1^{(0)}(\beta) + r_1(\beta)p_3^{(0)}(\beta) - h_3(\beta) \in L^1[0, \infty), \end{aligned}$$

$$p_4^{(0)}(\beta) = \lambda_1 p_2^{(0)}(\beta) + r_2(\beta) p_4^{(0)}(\beta) - h_4(\beta) \in L^1[0, \infty).$$

Therefore, $P_0 \in D(A_{PS})$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} p_1^{(n)}(\beta) &= \lim_{n \rightarrow \infty} [-(\lambda_2 + r_1(\beta)) p_1^{(0)}(\beta)] - h_1(\beta) = p_1^{(0)}(\beta), \\ \lim_{n \rightarrow \infty} p_2^{(n)}(\beta) &= \lim_{n \rightarrow \infty} [-(\lambda_1 + r_2(\beta)) p_2^{(0)}(\beta)] - h_2(\beta) = p_2^{(0)}(\beta), \\ \lim_{n \rightarrow \infty} p_3^{(n)}(\beta) &= \lim_{n \rightarrow \infty} [\lambda_2 p_1^{(n)}(\beta) + r_1(\beta) p_3^{(n)}(\beta)] - h_3(\beta) = p_3^{(0)}(\beta), \\ \lim_{n \rightarrow \infty} p_4^{(n)}(\beta) &= \lim_{n \rightarrow \infty} [\lambda_1 p_2^{(n)}(\beta) + r_2(\beta) p_4^{(n)}(\beta)] - h_4(\beta) = p_4^{(0)}(\beta). \end{aligned}$$

From the deduction above, we have

$$\left\{ \begin{array}{l} -(\lambda_1 + \lambda_2) p_0^{(0)} + \sum_{j=1}^2 \int_0^\infty r_j(x) p_j^{(0)}(x) dx = h_0, \\ -\frac{dp_1^{(0)}(x)}{dx} - (\lambda_2 + r_1(x)) p_1^{(0)}(x) = h_1(x), \\ -\frac{dp_2^{(0)}(x)}{dx} - (\lambda_1 + r_2(x)) p_2^{(0)}(x) = h_2(x), \\ -\frac{dp_3^{(0)}(x)}{dx} + \lambda_2 p_1^{(0)}(x) - r_1(x) p_3^{(0)}(x) = h_3(x), \\ -\frac{dp_4^{(0)}(x)}{dx} + \lambda_1 p_2^{(0)}(x) - r_2(x) p_4^{(0)}(x) = h_4(x). \end{array} \right.$$

This shows that $A_{PS}(P_0)^t = (F_{PS})^t$, hence $(A_{PS}, D(A_{PS}))$ is closed.

Now we prove that $D(A_{PS})$ is dense in X_{PS} .

We define

$$E_{PS} = \left\{ p(x) = (p_0, p_1, p_2, p_3(x), p_4(x), p_5(x)) \left| \begin{array}{l} p_0 \in \mathbb{R}, p_i(x) \in C_0^\infty[0, \infty), \\ i = 1, 2, 3, 4 \end{array} \right. \right\}.$$

Then by [Ada75] E_R is dense in X_R . If we define

$$H_{PS} = \left\{ p(x) = (p_0, p_1(x), p_2(x), p_3(x), p_4(x)) \left| \begin{array}{l} p_i(x) \in C^\infty[0, \infty) \text{ and} \\ \text{there exists a number} \\ \alpha_i \text{ such that } p_i(x) = 0, \\ \text{for } x \in [0, \alpha_i], i = 1, 2, 3, 4 \end{array} \right. \right\},$$

then H_{PS} is dense in E_{PS} . Therefore, in order to prove that $D(A_{PS})$ is dense in X_{PS} , it suffices to prove that $D(A_{PS})$ is dense in H_{PS} . Take any

$$p(x) = (p_0, p_1(x), p_2(x), p_3(x), p_4(x)) \in H_{PS},$$

then there exist numbers α_i such that $p_i(x) = 0$, for all $x \in [0, \alpha_i]$ ($i = 1, 2, 3, 4$), i.e., $p_i(x) = 0$ for $x \in [0, s]$, here $0 < s = \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. We introduce a

function

$$\begin{aligned}\tau^s(0) &= (\tau_0^s, \tau_1^s(0), \tau_2^s(0), \tau_3^s(0), \tau_4^s(0)) \\ &= (p_0, \lambda_1 p_0 + \int_0^\infty p_4(x) r_2(x) dx, \lambda_2 p_0 + \int_0^\infty p_3(x) r_1(x) dx, 0, 0) \\ \tau^s(x) &= (\tau_0^s, \tau_1^s, \tau_2^s, \tau_3^s(x), \tau_4^s(x), \tau_5^s(x)),\end{aligned}$$

where

$$\tau_i^s(x) = \begin{cases} \tau_i^s(0)(1 - \frac{x}{s})^2 & \text{if } x \in [0, s) \\ p_i(x) & \text{if } x \in [s, \infty), \end{cases} \quad i = 1, 2, 3, 4.$$

It is easy to verify that $\tau^s(x) \in D(A_R)$. Moreover

$$\|p - \tau^s\| = \sum_{i=1}^4 \int_0^s |\tau_i^s(0)| (1 - \frac{x}{s})^2 dx = \sum_{i=1}^4 |\tau_i^s(0)| \frac{s}{3} \rightarrow 0, \quad \text{as } s \rightarrow 0.$$

This shows that $D(A_{PS})$ is dense in H_{PS} , hence in X_{PS} . □

LEMMA 4.4.2. $(A_{PS}, D(A_{PS}))$ is a dispersive operator.

PROOF. For $p = (p_0, p_1(x), p_2(x), p_3(x), p_4(x)) \in D(A_{PS})$, we define

$$q = (q_0, q_1(x), q_2(x), q_3(x), q_4(x)) \in X_{PS}^*,$$

where

$$q_0 = \|P\| \operatorname{sgn}_+(p_0), q_i(x) = \|P\| \operatorname{sgn}_+(p_i(x)), i = 1, 2, 3, 4,$$

and

$$\begin{aligned}\operatorname{sgn}_+(p_0) &= \begin{cases} 1 & \text{if } p_0 > 0, \\ 0 & \text{if } p_0 \leq 0, \end{cases} \\ \operatorname{sgn}_+(p_i(x)) &= \begin{cases} 1 & \text{if } p_i(x) > 0, \\ 0 & \text{if } p_i(x) \leq 0, \end{cases} \quad i = 1, 2, 3, 4.\end{aligned}$$

If we define $L_i = \{x \in [0, \infty) | p_i(x) > 0\}$ and $M_i = \{x \in [0, \infty) | p_i(x) \leq 0\}$ for $i = 1, 2, 3, 4$, then we have

$$\begin{aligned}
\int_0^\infty \frac{dp_i(x)}{dx} \operatorname{sgn}_+(p_i(x)) dx &= \int_{L_i} \frac{dp_i(x)}{dx} \operatorname{sgn}_+(p_i(x)) dx \\
&+ \int_{M_i} \frac{dp_i(x)}{dx} \operatorname{sgn}_+(p_i(x)) dx \\
&= \int_{L_i} \frac{dp_i(x)}{dx} \operatorname{sgn}_+(p_i(x)) dx \\
&= \int_{L_i} \frac{dp_i(x)}{dx} dx \\
&= \int_0^\infty \frac{d[p_i(x)]^+}{dx} dx \\
&= -[p_i(0)]^+, \quad i = 1, 2, 3, 4,
\end{aligned} \tag{111}$$

$$\begin{aligned}
\int_0^\infty r_j(x) p_j(x) \operatorname{sgn}_+(p_i) dx &\leq \int_0^\infty r_j(x) [p_j(x)]^+ dx, \quad j = 1, 2, i = 1, 2, 3, 4, \tag{112} \\
\int_0^\infty r_k(x) p_{k+2}(x) \operatorname{sgn}_+(p_i) dx &\leq \int_0^\infty r_k(x) [p_{k+2}(x)]^+ dx, \\
&k = 1, 2, i = 1, 2, 3, 4, \tag{113}
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^4 [p_i(0)]^+ &= \sum_{i=1}^2 [p_i(0)]^+ = [\lambda_1 p_0(t) + \int_0^\infty p_4(x, t) r_2(x) dx]^+ \\
&+ [\lambda_2 p_0(t) + \int_0^\infty p_3(x, t) r_1(x) dx]^+ \\
&\leq (\lambda_1 + \lambda_2) [p_0]^+ + \int_0^\infty r_1(x) [p_3(x)]^+ dx \\
&+ \int_0^\infty r_2(x) [p_4(x)]^+ dx.
\end{aligned} \tag{114}$$

Using (111), (112), (113), (114) and the boundary conditions on $p \in D(A_{PS})$ we obtain that

$$\begin{aligned}
\langle A_{PS} p, q \rangle &= [-(\lambda_1 + \lambda_2) p_0 + \sum_{j=1}^2 \int_0^\infty r_j(x) p_j(x) dx] \|P\| \operatorname{sgn}_+(p_0) \\
&+ \int_0^\infty \left[-\frac{dp_1(x)}{dx} - (\lambda_2 + r_1(x)) p_1(x) \right] \|P\| \operatorname{sgn}_+(p_1(x)) \\
&+ \int_0^\infty \left[-\frac{dp_2(x)}{dx} - (\lambda_1 + r_2(x)) p_2(x) \right] \|P\| \operatorname{sgn}_+(p_2(x))
\end{aligned}$$

$$\begin{aligned}
& + \int_0^\infty \left[-\frac{dp_3(x)}{dx} + \lambda_2 p_1(x) - r_1(x)p_3(x) \right] \|P\| \operatorname{sgn}_+(p_3(x)) \\
& + \int_0^\infty \left[-\frac{dp_4(x)}{dx} + \lambda_1 p_2(x) - r_2(x)p_4(x) \right] \|P\| \operatorname{sgn}_+(p_4(x)) \\
& = \|P\| \left\{ [-(\lambda_1 + \lambda_2)p_0 \operatorname{sgn}_+(p_0) + \sum_{j=1}^2 \int_0^\infty r_j(x)p_j(x) \operatorname{sgn}_+(p_0) dx] \right. \\
& + \int_0^\infty \left[-\frac{dp_1(x)}{dx} \times \operatorname{sgn}_+(p_1(x)) - (\lambda_2 + r_1(x))p_1(x) \operatorname{sgn}_+(p_1(x)) \right] \\
& + \int_0^\infty \left[-\frac{dp_2(x)}{dx} \times \operatorname{sgn}_+(p_2(x)) - (\lambda_1 + r_2(x))p_2(x) \operatorname{sgn}_+(p_2(x)) \right] \\
& + \int_0^\infty \left[-\frac{dp_3(x)}{dx} \times \operatorname{sgn}_+(p_3(x)) + \lambda_2 p_1(x) \operatorname{sgn}_+(p_3(x)) \right. \\
& \left. - r_1(x)p_3(x) \operatorname{sgn}_+(p_3(x)) \right] \\
& + \int_0^\infty \left[-\frac{dp_4(x)}{dx} \times \operatorname{sgn}_+(p_4(x)) + \lambda_1 p_2(x) \operatorname{sgn}_+(p_4(x)) \right. \\
& \left. - r_2(x)p_4(x) \operatorname{sgn}_+(p_4(x)) \right] \Big\} \\
& \leq \|P\| \left\{ [-(\lambda_1 + \lambda_2)[p_0]^+ + \sum_{j=1}^2 \int_0^\infty r_j(x)[p_j(x)]^+ dx] \right. \\
& + \int_0^\infty [-(\lambda_2 + r_1(x))[p_1(x)]^+] dx + \int_0^\infty [-(\lambda_1 + r_2(x))[p_2(x)]^+] dx \\
& + \lambda_2 \int_0^\infty [p_1(x)]^+ dx - \int_0^\infty r_1(x)[p_3(x)]^+ dx + \lambda_1 \int_0^\infty [p_2(x)]^+ dx \\
& \left. - \int_0^\infty r_2(x)[p_4(x)]^+ dx + \sum_{i=1}^4 [p_i(0)]^+ \right\} \\
& = \|P\| \left\{ [-(\lambda_1 + \lambda_2)[p_0]^+ - \int_0^\infty r_1(x)[p_3(x)]^+ dx \right. \\
& \left. - \int_0^\infty r_2(x)[p_4(x)]^+ dx + \sum_{i=1}^4 [p_i(0)]^+ \right\} \leq 0.
\end{aligned}$$

By Definition 1.2.5 we obtain that $(A_{PS}, D(A_{PS}))$ is a dispersive operator. \square

LEMMA 4.4.3. *If $\gamma \in \mathbb{R}, \gamma > 0$, then $\gamma \in \rho(A_{PS})$.*

PROOF. Let $\gamma \in \mathbb{R}, \gamma > 0$, then all the entries of $\Phi_{PS} D_\gamma^{PS}$ are positive and we can estimate each column sum as

$$a_{1,1} + a_{2,1} = \frac{\lambda_1}{\gamma + \lambda_1 + \lambda_2} \times \int_0^\infty r_1(x) e^{-(\gamma + \lambda_2)x - \int_0^x r_1(\xi) d\xi} dx$$

$$\begin{aligned}
& + \frac{\lambda_2}{\gamma + \lambda_1 + \lambda_2} \times \int_0^\infty r_1(x) e^{-(\gamma+\lambda_2)x - \int_0^x r_1(\xi) d\xi} dx \\
& + \int_0^\infty r_1(x) e^{-\gamma x - \int_0^x r_1(\xi) d\xi} (1 - e^{-\lambda_2 x}) \\
& = \frac{\lambda_1 + \lambda_2}{\gamma + \lambda_1 + \lambda_2} \times \int_0^\infty r_1(x) e^{-(\gamma+\lambda_2)x - \int_0^x r_1(\xi) d\xi} dx \\
& + \int_0^\infty r_1(x) e^{-\gamma x - \int_0^x r_1(\xi) d\xi} (1 - e^{-\lambda_2 x}) \\
& < \int_0^\infty r_1(x) e^{-(\gamma+\lambda_2)x - \int_0^x r_1(\xi) d\xi} dx \\
& + \int_0^\infty r_1(x) e^{-\gamma x - \int_0^x r_1(\xi) d\xi} (1 - e^{-\lambda_2 x}) \\
& = \int_0^\infty r_1(x) e^{-\gamma x - \int_0^x r_1(\xi) d\xi} dx < 1, \\
a_{1,2} + a_{2,2} & = \frac{\lambda_1}{\gamma + \lambda_1 + \lambda_2} \times \int_0^\infty r_2(x) e^{-(\gamma+\lambda_1)x - \int_0^x r_2(\xi) d\xi} dx \\
& + \int_0^\infty r_2(x) e^{-\gamma x - \int_0^x r_2(\xi) d\xi} (1 - e^{-\lambda_1 x}) \\
& + \frac{\lambda_2}{\gamma + \lambda_1 + \lambda_2} \times \int_0^\infty r_2(x) e^{-(\gamma+\lambda_2)x - \int_0^x r_2(\xi) d\xi} dx \\
& = \frac{\lambda_1 + \lambda_2}{\gamma + \lambda_1 + \lambda_2} \times \int_0^\infty r_2(x) e^{-(\gamma+\lambda_1)x - \int_0^x r_2(\xi) d\xi} dx \\
& + \int_0^\infty r_2(x) e^{-\gamma x - \int_0^x r_2(\xi) d\xi} (1 - e^{-\lambda_1 x}) \\
& < \int_0^\infty r_2(x) e^{-(\gamma+\lambda_1)x - \int_0^x r_2(\xi) d\xi} dx \\
& + \int_0^\infty r_2(x) e^{-\gamma x - \int_0^x r_2(\xi) d\xi} (1 - e^{-\lambda_1 x}) \\
& = \int_0^\infty r_2(x) e^{-\gamma x - \int_0^x r_2(\xi) d\xi} dx < 1, \\
a_{2,3} & = \int_0^\infty r_1(x) e^{-\gamma x - \int_0^x r_1(\xi) d\xi} dx < \int_0^\infty r_1(x) e^{-\int_0^x r_1(\xi) d\xi} dx = 1, \\
a_{1,4} & = \int_0^\infty r_2(x) e^{-\gamma x - \int_0^x r_2(\xi) d\xi} dx < \int_0^\infty r_2(x) e^{-\int_0^x r_2(\xi) d\xi} dx = 1.
\end{aligned}$$

It follows from this that $\|\Phi_{PS} D_\gamma^{PS}\| < 1$, and thus also

$$r(\Phi_{PS} D_\gamma^{PS}) \leq \|\Phi_{PS} D_\gamma^{PS}\| < 1.$$

Therefore, $1 \notin \sigma(\Phi_{PS}D_\gamma^{PS})$. Using the Characteristic Equation 1.3.6 we conclude that $\gamma \in \rho(A_{PS})$ for $\gamma \in \mathbb{R}, \gamma > 0$. \square

Combining Lemma 4.4.1, Lemma 4.4.2, Lemma 4.4.3 with Theorem 1.2.6 we immediately obtain the following result.

THEOREM 4.4.4. *The operator $(A_{PS}, D(A_{PS}))$ generates a positive contraction C_0 -semigroup $(T_{PS}(t))_{t \geq 0}$.*

Using Proposition 1.2.4 and Theorem 4.4.4 we can state our main result.

THEOREM 4.4.5. *The system $(PS), (BC_{PS})$ and $(IC_{PS,0})$ has a unique positive solution $p(t, x)$ which satisfies $\|p(t, \cdot)\| = 1, t \in [0, \infty)$.*

PROOF. From Proposition 1.2.4 and Theorem 4.4.4 we obtain that the associated abstract Cauchy problem (ACP_{PS}) has a unique positive time-dependent solution $p(t, x)$, which can be expressed as

$$p(t, x) = T_{PS}(t)p(0) = T_{PS}(t)(1, 0, 0, 0, \dots). \quad (115)$$

Let $P(t) = p(t, x) = (p_0(t), p_1(t, x), p_2(t, x), p_3(t, x), p_4(t, x))$, then $P(t)$ satisfies the system of equations:

$$\frac{dp_0(t)}{dt} = -(\lambda_1 + \lambda_2)p_0(t) + \sum_{i=1}^2 \int_0^\infty r_i(x)p_i(x, t)dx, \quad (116)$$

$$\frac{\partial p_1(t, x)}{\partial t} = -\frac{\partial p_1(t, x)}{\partial x} - (\lambda_2 + r_1(x))p_1(t, x), \quad (117)$$

$$\frac{\partial p_2(t, x)}{\partial t} = -\frac{\partial p_2(t, x)}{\partial x} - (\lambda_1 + r_2(x))p_2(t, x), \quad (118)$$

$$\frac{\partial p_3(t, x)}{\partial t} = -\frac{\partial p_3(t, x)}{\partial x} - r_1(x)p_3(t, x) + \lambda_2 p_1(t, x), \quad (119)$$

$$\frac{\partial p_4(t, x)}{\partial t} = -\frac{\partial p_4(t, x)}{\partial x} - r_2(x)p_4(t, x) + \lambda_1 p_2(t, x), \quad (120)$$

$$p_1(t, 0) = \lambda_1 p_0(t) + \int_0^\infty p_4(t, x)r_2(x)dx, \quad (121)$$

$$p_2(t, 0) = \lambda_2 p_0(t) + \int_0^\infty p_3(t, x)r_1(x)dx, \quad (122)$$

$$p_i(t, 0) = 0, i = 3, 4, \quad (123)$$

$$P(0) = (1, 0, 0, 0, 0, \dots). \quad (124)$$

Since

$$\int_0^\infty \frac{\partial p_i(t, x)}{\partial x} dx = p_i(t, \infty) - p_i(t, 0) = -p_i(t, 0), i = 1, 2, 3, 4. \quad (125)$$

Using (116)–(125) we compute

$$\begin{aligned}
\frac{d\|P(t)\|}{dt} &= \frac{dp_0(t)}{dt} + \sum_{i=1}^4 \int_0^\infty \frac{\partial p_i(t, x)}{\partial t} dx \\
&= -(\lambda_1 + \lambda_2)p_0(t) + \sum_{i=1}^2 \int_0^\infty r_i(x)p_i(t, x)dx, \\
&\quad + \int_0^\infty \left[-\frac{\partial p_1(t, x)}{\partial x} - (\lambda_2 + r_1(x))p_1(t, x) \right], \\
&\quad + \int_0^\infty \left[-\frac{\partial p_2(t, x)}{\partial x} - (\lambda_1 + r_2(x))p_2(t, x) \right], \\
&\quad + \int_0^\infty \left[-\frac{\partial p_3(t, x)}{\partial x} - r_1(x)p_3(t, x) + \lambda_2 p_1(t, x) \right], \\
&\quad + \int_0^\infty \left[-\frac{\partial p_4(t, x)}{\partial x} - r_2(x)p_4(t, x) + \lambda_1 p_2(t, x) \right], \\
&= -\lambda_1 p_0(t) - \lambda_2 p_0(t) - \int_0^\infty r_1(x)p_3(t, x)dx \\
&\quad - \int_0^\infty r_2(x)p_4(t, x)dx + \sum_{i=1}^4 p_i(0, t) \\
&= -\sum_{i=1}^4 p_i(0, t) + \sum_{i=1}^4 p_i(0, t) = 0. \tag{126}
\end{aligned}$$

By (115) and (126) we obtain

$$\frac{d\|P(t)\|}{dt} = \frac{d\|T_{PS}(t)P(0)\|}{dt} = 0.$$

Therefore,

$$\|T_{PS}(t)P(0)\| = \|P(t)\| = \|P(0)\| = 1.$$

This shows $\|p(\cdot, t)\| = 1, \forall t \in [0, \infty)$. \square

4.5. Asymptotic Stability of the Solution

While the semigroup $(T_{PS}(t))_{t \geq 0}$ generated by $(A_{PS}, D(A_{PS}))$ is not irreducible, we know that its fixed space is one-dimensional with a strictly positive eigenvector and no other imaginary eigenvalues of A except 0 (see Lemma 4.3.6 and Lemma 4.3.8). This means that the semigroup $(T(t))$ is relatively weakly compact and we can apply [EFNS07, Thm. 2.5] to obtain "almost weak convergence".

THEOREM 4.5.1. *For all $p \in X_{PS}$ there exist $p' \in X'_{PS}, p' \gg 0$ and a set $M \subset \mathbb{R}_+$ with density 1 such that*

$$T_{PS}(t)p \xrightarrow{\sigma} \langle p', p \rangle \tilde{p} \text{ as } t \in M, t \rightarrow \infty,$$

where $\ker A_{PS} = \langle \tilde{p} \rangle, \tilde{p} \gg 0$.

Since the semigroup gives the solutions of the original system, we obtain the asymptotic behaviour of this system.

COROLLARY 4.5.2. The time-dependent solution of the system $(PS), (BC_{PS})$ and $(IC_{PS,0})$ converges almost weakly to the steady-state solution as time tends to infinity.

Table of symbols

| | |
|---------------------------------|--|
| $D(A)$ | domain of A |
| $\text{fix}(S(t))_{t \geq 0}$ | fixed space of the semigroup $(S(t))_{t \geq 0}$ |
| $\ker T$ | kernel of T |
| $\mathcal{L}(X)$ | space of bounded linear operators on X |
| $L^1(\Omega, \mu)$ | space of complex valued integrable functions on Ω with respect to μ |
| $L_{\mathbb{C}}^1(\Omega, \mu)$ | space of complex valued integrable functions on Ω with respect to μ |
| $L_{\mathbb{R}}^1(\Omega, \mu)$ | space of real valued integrable functions on Ω with respect to μ |
| $\Re z$ | real part of Z |
| $r(T)$ | spectral radius of T |
| $rg(T)$ | range of T |
| $\rho(A)$ | resolvent set of A |
| $R(\gamma, A)$ | resolvent of A in γ |
| $s(A)$ | spectral bound of A |
| $\sigma(A)$ | spectrum of A |
| $\sigma_b(A)$ | boundary spectrum of A |
| $\sigma_p(A)$ | point spectrum of A |
| $\sigma_r(A)$ | residual spectrum of A |

Bibliography

- [ABHN01] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics, vol. 96, Birkhäuser Verlag, 2001.
- [Ada75] R.A. Adams, *Sobolev Spaces*, Academic Press, 1975.
- [CENN03] V. Casarino, K.-J. Engel, R. Nagel, and G. Nickel, *A semigroup approach to boundary feedback systems*, Integr. Equ. Oper. Theory **47** (2003), 289–306.
- [Cox55] D.R. Cox, *The analysis of non-markovian stochastic processes by the inclusion of supplementary variables*, Proc. Cambridge Philos. Soc. **51** (1955), 433–441.
- [CT83] M.L. Chaudhry and J.G.C. Templeton, *A first Course in Bulk Queues*, John Wiley & Sons, New York, 1983.
- [EFNS07] T. Eisner, B. Farkas, R. Nagel, and A. Serény, *Weakly and almost weakly stable C_0 -semigroups*, Int. J. Dyn. Syst. Diff. Eq. **1** (2007), 44–57.
- [EN00] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, 2000.
- [EN06] ———, *A Short Course on Operator Semigroups*, Springer-Verlag, 2006.
- [GLZ01] G. Gupur, Xue-zhi Li, and Guang-tian Zhu, *Functional Analysis Method in Queueing Theory*, Research Information Ltd., Hertfordshire, 2001.
- [Gol85] J.A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Mathematical Monographs, Oxford University Press, 1985.
- [Gre84] G. Greiner, *A typical Perron-Frobenius theorem with applications to an age-dependent population equation*, Infinite-dimensional systems (Retzhof, 1983), Lecture Notes in Math., vol. 1076, Springer, 1984, pp. 86–100.
- [Gre87] G. Greiner, *Perturbing the boundary conditions of a generator*, Houston J. Math. **13** (1987), 213–229.
- [Guo03] Wei-hua Guo, *The existence and uniqueness of the solution for a parallel maintenance system with two components*, J.Xinyang Nor.UNi.(in Chinese) **16** (2003), 270–272.
- [Gup95] S.M Gupta, *Stochastic analysis of systems with primary and secondary failures*, Microelectron. Reliab. **35** (1995), 65–71.
- [Gup02] G. Gupur, *Resolvent set of the $M/M^B/1$ operator*, Computers Math. App. **44** (2002), 67–82.
- [GZ98] G. Gupur and H.-X. Zhang, *Existence and uniqueness of positive solution of $M/M^B/1$ model*, Advances in 98 (1998), 280–285.
- [HR07a] A. Haji and A. Radl, *Asymptotic stability of the solution of the $M/M^B/1$ queueing model*, Computers Math. App. **53** (2007), 1411–1420.
- [HR07b] ———, *A semigroup approach to queueing systems*, Semigroup Forum **75** (2007), 610–624.
- [KS05] M. Kramar and E. Sikolya, *Spectral properties and asymptotic periodicity of flows in networks*, Math. Z. **249** (2005), 139–162.
- [MN91] P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, 1991.

- [Nag86] R. Nagel (ed.), *One-parameter Semigroups of Positive Operators*, Lecture Notes in Mathematics, vol. 1184, Springer-Verlag, 1986.
- [Nag89] ———, *Towards a matrix theory for unbounded operator matrices*, Math. Z. **201** (1989), 57–68.
- [Paz83] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, 1983.
- [Rad06] A. Radl, *Semigroups applied to transport and queueing processes*, Dissertation, Tübingen, 2006.
- [Sch74] H.H. Schaefer, *Banach Lattices and Positive Operators*, Die Grundlehren der mathematischen Wissenschaften, Band 215, Springer-Verlag, 1974.
- [Yeh97] L. Yeh, *The rate of occurrence of failures*, J. Appl. Prob. **34** (1997), 234–247.

Zusammenfassung in deutscher Sprache

Wir diskutieren in dieser Arbeit ein Warteschlangen-Modell, und zwar, in der üblichen Notation, das $M/M^B/1$ Modell, sowie zwei Zuverlässigkeitsmodelle, welche jeweils durch abstrakte Cauchyprobleme beschrieben werden. Die in allen drei Fällen gemeinsame Vorgehensweise ist so, dass die Wohlgestelltheit des jeweiligen Cauchyproblems gezeigt wird. Danach wird über eine Spektralanalyse des Generators die Asymptotik der Lösungen bestimmt. Wir erhalten jeweils "steady-state solutions", gegen die die Lösungen für $t \rightarrow \infty$ konvergieren. In den ersten beiden Fällen ist die Halbgruppe irreduzibel und die Konvergenz ist in der Norm. Im letzten Fall ist Irreduzibilität nicht gegeben. Es gelingt aber durch Anwendung eines neuen Resultats aus [EFNS07] der Nachweis des fast-schwachen Konvergenz.

Lebenslauf

| | |
|-----------------|---|
| 14.07.1963 | geboren in Kashkar, Xinjiang Uigur Autonome Region, V. R. China |
| 1970–1975 | Besuch der Grundschule in Kashkar |
| 1975–1980 | Besuch des Gymnasiums in Kashkar |
| 06.07.1980 | Abitur |
| 1980–1985 | Studium der Mathematik (Vordiplom) an der Universität Xinjiang in Urumqi, Xinjiang Uigur Autonome Region, V. R. China |
| 1985–1988 | Studium der Mathematik (Diplom) an der Universität Xinjiang |
| 1988–1991 | Mathematikassistent an der Universität Xinjiang |
| 1991–1996 | Mathematikdozent an der Universität Xinjiang |
| 1996– | Mathematikprofessor an der Universität Xinjiang |
| 2004–2005 | einjähriger Forschungsaufenthalt an der Eberhard-Karls Universität Tübingen (CSC Stipendium) |
| 2007 | dreimonatiger Studien- und Forschungsaufenthalt an der Eberhard-Karls Universität Tübingen (DAAD Stipendium) |
| seit 16.11.2005 | Doktorand an der Fakultät für Mathematik und Physik der Eberhard-Karls Universität Tübingen |

Meine akademischen Lehrer waren

in Mathematik:

A. Wakeli, T. Awut, Chen De Huang, Ge Qing, Yang Nian Xi, M. Muhamhaydar, R. Talip, K. Kadir, M. Israyil, Qagidir, A. Zunun, O. Abdurahman, A. Abdulla, O. Haxim (in Urumqi)
R. Nagel, U. Schlotterbeck, G. Nickel (Tübingen)

in Informatik:

N. Mamat, A. Hajahmat, T. Ibrayim (Urumqi)
B. Eberhardt (Tübingen)