

Glueball phenomenology
within a nonlocal approach

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Abstract

In this thesis we describe the properties of glueball phenomenology within a nonlocal covariant constituent approach.

The search for glueballs, their theoretical description and the mixing with quarkonia mesons is an active and unsolved issue of hadronic QCD. Different models and assignments have been proposed, but up to now no certain statement about their existence can be done.

After introducing the theoretical framework in which we will work in, the attention will be focused on the problem of the scalar glueball, which lattice QCD predicts to be the lightest gluonic state with a mass between 1.4-1.8 GeV. In the same mass region one encounters many scalar resonances; mixing between the bare glueball and quarkonia states is therefore likely. In a covariant constituent approach one cannot define rigorously a mixing matrix connecting the bare to physical fields. However, we propose a definition which satisfies the correct requirements and which can be compared to other phenomenological studies.

The two-photon decay of isoscalar-scalar states is believed to be crucial to pin down the flavor content of the resonances between 1 and 2 GeV. We discuss and calculate the two-photon decay rates of the mixed states glueball-quarkonia, getting results which are consistent with the current experimental upper limits.

We also analyze the strong decays of the scalar resonances by using a phenomenological Lagrangian inspired by chiral perturbation theory. The decay of a scalar into two pseudoscalar mesons is another decisive point to test a mixing scenario.

When introducing the model a careful description of the so-called compositeness condition is presented; it is a useful tool to describe the properties of a bound state. A connection with the non-relativistic limit is established and worked out by analyzing the analogous mechanism occurring for positronium states.

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Chapter 1

Introduction

1.1 Constituent quarks

1.1.1 The need for quarks

The quarks were first introduced to put order in the chaotic realm of hadron spectroscopy. In this way a clear and schematic interpretation of hadrons as bound objects made out of quarks has been achieved: the bosonic hadrons, called mesons, are quark-antiquark states, while the fermionic hadrons, called baryons, are three quark states.

In order to reproduce the electric charge of the proton and the neutron two types of quarks (flavors) with a fractionary charge $2/3$ and $-1/3$, called respectively u and d , were proposed. In this way the proton has a flavor content " uud " and the neutron " udd ". These quarks have a mass of the order of $M_p/3$ (M_p is the proton mass) and are referred to as "constituent quarks". The static properties of the nucleon can be fairly good described within these simple assumptions.

In the same scheme one can interpret the pion triplet π^+ , π^0 , π^- as $u\bar{d}$, $\sqrt{1/2}(u\bar{u} - d\bar{d})$, $d\bar{u}$. Successively the quark-flavor s , slightly heavier than u and d , with the charge $-1/3$, has been postulated to describe properly the known ground states of baryons and mesons.

The occurrence of bound states means that an attraction among quarks takes place. Its origin cannot be the electromagnetic force (too weak to do it; furthermore, the state $\pi^+ = u\bar{d}$ is made out of two quarks with positive charge). There must be another interaction type, which we call "strong" interaction.

For a proper and exhaustive description of these ideas we refer to [1, 2] and Refs. therein.

1.1.2 Color's primary

The spin-statistic's problem of the spin $3/2$ state Δ^{++} is renowned; its charge $+2$ is compatible with a flavor wave function " uuu ". A Δ^{++} with $S_z = 3/2$ is expressed by

$$|\Delta^{++}, S_z = 3/2\rangle = |space\rangle |flavor-spin\rangle = |l = l' = 0\rangle |u \uparrow, u \uparrow, u \uparrow\rangle, \quad (1.1)$$

where l and l' are the relative spatial angular momenta, set to zero because we consider a ground state hadron. This state is overall symmetric, in disagreement with the Pauli principle. The existence of another degree of freedom has been originally invoked to avoid this paradox: each quark comes in 3 colors, R, G and B (red, green, blue); the color wave

function must be antisymmetric, leaving only one possibility:

$$|color\rangle = \sqrt{\frac{1}{6}}(RGB + BRG + GBR - RBG - GRB - BGR) \quad (1.2)$$

In this way the state

$$|\Delta^{++}, S_z = 3/2\rangle = |space\rangle |flavor-spin\rangle |color\rangle \quad (1.3)$$

is, as required, antisymmetric.

Each baryon has the same antisymmetric color wave function, thus explaining why the color does not generate a proliferation of states. The introduced color configuration corresponds to a "white" object, i.e. is invariant under a $SU(3)$ rotation in the color space. Going even further, it has been postulated that each bound state of quarks is "white", also invariant under rotation in the color space. For a meson (quark-antiquark state) a white configuration corresponds to

$$|color\rangle = \sqrt{\frac{1}{3}}(\bar{R}R + \bar{G}G + \bar{B}B). \quad (1.4)$$

The quarks are colored, but the bound states are white. Confinement can be introduced here: only white states occur in nature.

As a direct consequence of this statement no quarks can be seen isolated. Such strong statement has proven to hold true up to now, although the full mathematical verification has still to come. Confinement is a property of the strong interaction among quarks.

Summarizing, each quark carries two quantum numbers, the flavor (u,d,s) and the color (R,G,B). The three flavors have different electric charge and mass; however, due to their similar mass, the use of group theory is very useful for the classification of hadrons. One speaks of an approximate $SU_f(3)$ flavor symmetry, valid at first order for strongly interacting quarks but clearly not conserved by the electromagnetic interaction.

On the other hand the color symmetry $SU_c(3)$ is assumed to be exact; a red u-quark and a blue u-quark have exactly the same mass and the same charge. Due to confinement only colorless states appear: the color can be thought of as the "charge" of the strong interaction among quarks. The potential between R and \bar{R} , for instance, is attractive and increases with the distance, in such a way that the quarks cannot be "liberated". Computer simulations confirmed such a behavior.

In the language of Quantum Field Theory [3] a "potential" corresponds to an exchanged particle. The best and most famous example of a QFT is QED, written under the requirement of $U(1)$ local gauge invariance and describing the electrons interacting through photon exchange. A new quantum field theory for the description of the strongly interacting quarks, called QCD, has been built in analogy to QED: the *local* color gauge invariance under $SU_c(3)$ transformations leads to the introduction of 8 bosonic massless mediators, called gluons: the name by itself corresponds to the resulting strong (confining) interaction among quarks.

The color, originally introduced to avoid a spin-statistical collapse, has been risen up to the basic charge of a non-abelian, renormalizable local gauge theory.

The non-abelian nature of $SU(3)$ implies a direct interaction among gluons, which are themselves colored objects, coming in 8 different color combinations. The gluons are not "white", ergo we cannot see a free gluon, pretty much as we can't free quarks.

1.2 Glueball hypothesis

The fact that in QCD the gluons "shine in their own light" is the major difference with QED and is thought of being responsible for the confining process discussed above. Another striking consequence can be conceived ([4, 5] and Refs. therein): the gluons could generate "gluonic mesons", i.e. bound states of gluons called glueballs. In fact, as strongly interacting particles, the gluons, as well as the quarks, could form "white" bound states generating a full glueball spectrum.

In analogy to quarkonia (ordinary) mesons one can describe the glueball as a bound state of two "constituent" massive gluons; this sentence may seem in disagreement with what was stated above about the massless nature of the gluons. We will soon come back to this point, which represents an important step in the construction of a phenomenological effective model.

The glueballs have been being searched for about 30 years, but, up to now, no certain statement about their existence can be done. The major difficulty to test such a hypothesis is that a glueball would have the same quantum numbers as an ordinary $\bar{q}q$ quarkonia meson, without saying "Look at me, I'm a glueball". This makes the identification more difficult. Furthermore, a glueball can mix with quarkonia mesons with the same quantum numbers, thus further complicating the issue.

Other possibilities can occur: one could also have "hybrids", i.e. states like $\bar{q}qg$ (constituent quark-antiquark pair and a constituent gluon). The interesting aspect is that one can construct states with exotic quantum numbers, which are not allowed for a quark-antiquark pair. Also, in this case the search for such states is in progress, and their existence has either been proved or disproved, depending on the theoretical interpretation.

For review papers about exotics (no $\bar{q}q$ or qqq states) we refer to [5, 6, 7, 8] and Refs. therein.

In this thesis we will mostly concentrate on the glueballs, with special attention on the scalar glueball, whose eventual existence is currently under hot debate.

In the last decade progress has been done thanks to better experimental results, new theoretical interpretations and more precise computer Lattice simulations. Lattice glueballs in the pure Yang-Mills sector of QCD exist, and show a complete spectrum where the ground state is the $J^{PC} = 0^{++}$ scalar glueball with a mass between $1.4 - 1.8 \text{ GeV}$ [9, 10, 11]. In the same mass region one experimentally identifies [12] ten scalar states: the isoscalars $f_0(1370)$, $f_0(1500)$, $f_0(1710)$, the isodoublets $K^*(1430)$ and the isotriplet $a_0(1450)$. The experimental identification of these resonances and the study of their decays has been improved in the last years, even if a further improvement would be very useful. In fact, the nature of these states, especially concerning the three isoscalars f_0 , is still not known. Are they the ground state scalar quarkonia mesons or excited states? Is there the glueball among them? Does one have mixing among the glueball and the quarkonia states?

It is then clear that a solution of the scalar glueball puzzle involves a solution of the scalar quarkonia problem. The states listed above are not the only scalar states one has below 2 GeV . Other states, below 1 GeV , are found: $\sigma = f_0(400 - 1200)$, $f_0(980)$, $a_0(980)$, are well established [12]. Another broad state called $k(700)$ ([12, 13] and Refs. therein) has been proposed, but its existence is not universally accepted and omitted from the summary table of [12]. Many questions arise: which are the scalar $\bar{q}q$ ground states? The nonet below 1 GeV or between $1 - 2 \text{ GeV}$? If this last possibility is true, how can one interpret the scalar states below 1 GeV ?

There are by far too many scalar states to interpret all of them as quark-antiquark scalar

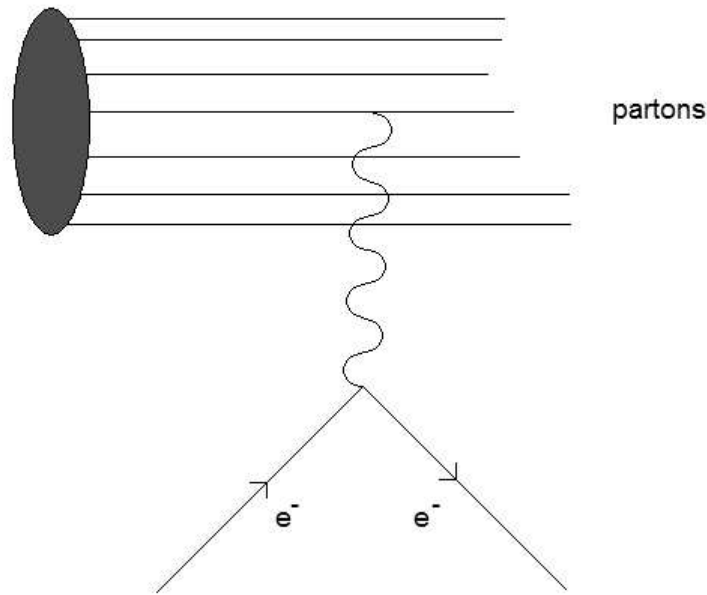


Figure 1.1: Electron-proton scattering.

objects; there is indeed "room" for the scalar glueball, and not only.

Before discussing this issue further, we first describe the other part of QCD , the high energy domain.

1.3 Partons

Let us look at the proton from another point of view. A very energetic electron hits a proton; this process can be explained in terms of partons [1, 2, 14, 15, 16].

The proton is made of many elementary objects, called partons, which interact weakly among each other. We have to choose a reference frame to study the system; the physics is invariant, but the choice of a convenient frame simplifies the work. In this case a system where the proton is moving fast is very useful: in fact, we can neglect the transversal momenta of the partons constituting the proton and consider only the longitudinal (i.e. in the direction of motion) components. Some partons are charged, some neutral; the incoming electron scatters elastically with one charged parton, which in first approximation is free and does not interact with the others (see Fig. 1.1). Under this simple assumption one can satisfactorily describe the dynamics of the process.

A careful analysis shows that the charged partons are almost massless and carry fractional charges ($2/3, 1/3, -1/3, -2/3$), therefore we recognize the quark flavors u, d , and s and their antiparticles. At the same time almost half of the proton momentum is carried by massless neutral partons, which we interpret as gluons.

According to this view the proton is made up of three "high energy" quarks uud , called valence quarks, which determine its quantum numbers, plus quark-antiquark pairs and gluons "flying around".

The fact that the partons can be considered as "almost free" shows the asymptotic free nature of the theory: at high energies the quarks and the gluons interact weakly, rendering a perturbative approach applicable. This is valid only for high energy electron-proton collisions.

One can note a different description of the proton: before we spoke of 3 constituent effective quarks " uud " with a large effective mass ($\sim M_p/3$), now we speak of many nearly massless partonic quarks and massless partonic gluons.

Both descriptions are indeed valid and describe properly some characteristics of the proton, depending on the energy or resolution scale. We realize that a unified view is necessary to understand properly the proton and more generally the QCD states.

The quarks and gluons of the partonic description are the high energy quarks and gluons; at the same time, when one considers the proton in its rest frame, it can be described as a bound state of three "quasi-particle", which are the constituent quarks. The connection can be heuristically established as following:

Constituent u = Valence u dressed with $\bar{u}u$, $\bar{d}d$, $\bar{s}s$ and gluon clouds.

The valence u is nearly massless, while the constituent u is massive because of the dressing phenomenon. The valence u gets heavier because it is surrounded with quark-antiquark pairs and gluons. The valence and the constituent quarks have the same quantum numbers, but a very different mass (from ~ 5 MeV for the valence to ~ 300 MeV for the constituent). The same follows for the other flavors.

Ergo:

Proton: Valence (almost massless) uud plus $\bar{u}u$, $\bar{d}d$, $\bar{s}s$ and gluons = Constituents (massive) uud .

The fact the valence quark gets massive through dressing changes the symmetry of the system; a system with massless quarks is "more symmetric" than one with massive quarks (whose masses also differ): this is essential in the context of spontaneous chiral symmetry breaking, which we will discuss in the sixth chapter.

The proton is a rather complicated object; the fact that the constituent picture works is of great use and a non-trivial fact. Of course, the non-interacting picture where the partons move freely is valid at high energy in a suitable reference frame. In the rest frame of the proton the quarks are bound and confined in the proton. We chose the proton for this discussion for historical reasons, but all these considerations are automatically valid, with the due changes, for every QCD bound state.

The original naive quark model was non-relativistic; the constituent quarks were interacting by a potential increasing with the distance. The origin of the potential is the gluon exchange: these gluons are often called "background" gluons, and are responsible for the string tension and for confinement, ergo for QCD bound states.

Later also Quantum Field Theory effective approaches were developed to describe the proton as a bound state of three constituent quarks described by fermionic fields; these models are relativistic, and the approximation of considering the proton as built out of three "elementary" constituents (i.e. neglecting the fact that we are actually dealing with three quasi-particles) has turned out to be good. If it had not been like that, we would have had serious troubles in describing the static properties of the QCD bound states.

Similarly, phenomenological QFT models for the description of the pions and other mesons were used to describe their decays.

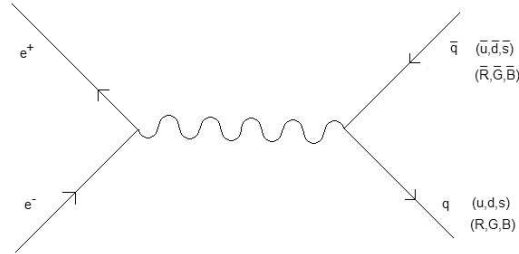


Figure 1.2: Electron-positron annihilation originating a quark-antiquark couple.

In this work we attempt to describe the ground state scalar mesons as a bound state of an (effective) quark-antiquark pair in a scalar configuration. We have already said that we are dealing with "bad-behaving objects", the uncontrollable children of QCD. There is indeed still a lot of work to do to derive the scalar quarkonia properties starting directly from QCD. Here we will use a relativistic QFT phenomenological model to achieve a fair description of these elusive objects.

But the glueball will jump in as well. It will be described as a bound state of two constituent gluons; the arguments exposed before about the constituent and the valence quarks are directly applicable also for the constituent gluons and the valence ones.

1.4 Jets

In the fast-moving system half of the proton momentum was carried by the gluons, thus already proving their existence, whatever means to "exist" for objects we can't detect directly, but such a discussion oversteps into philosophy. There are even more stringent proves that the gluons are really there. In order to discuss this, which represents, also from an historical point of view, the confirmation of QCD, we consider the electron-positron high energy scattering [1, 2, 14, 15, 16].

In the Fig. 1.2 the following situation is depicted: the electron and the positron generate a virtual photon, which subsequently radiates a quark-antiquark pair. The two quarks fly away in the opposite direction; however, we will not see a quark in our detectors because of the confinement discussed before. A hadronization process takes place, whose easiest visualization reads: a quark-antiquark couple is created in the vacuum, the original emitted quark forms a meson with the antiquark from the vacuum, similarly for the original emitted antiquark with the quark of the vacuum pair. Instead of two quarks, two mesons (white QCD states) are emitted (Fig. 1.3.a).

In a high energy process many quark-antiquark couples are produced in vacuum due to energy conservation, and the system is considerably more complicated than the simple two-meson event described above. However, the principle is the same: in the end only white objects fly out. One measures two hadron-jets (made of many baryons and mesons, see Fig. 1.3.b) flying in the opposite direction. The hadrons are revealed in the experimental

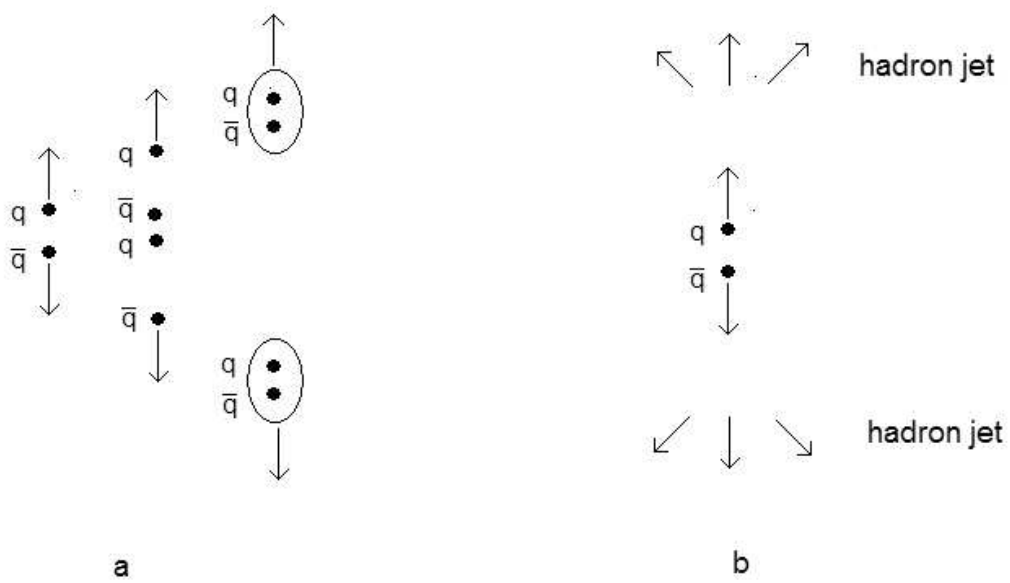


Figure 1.3: Quark-antiquark hadronization. In (a) two mesons fly away, in (b), at high energy, two hadronic jets are generated.

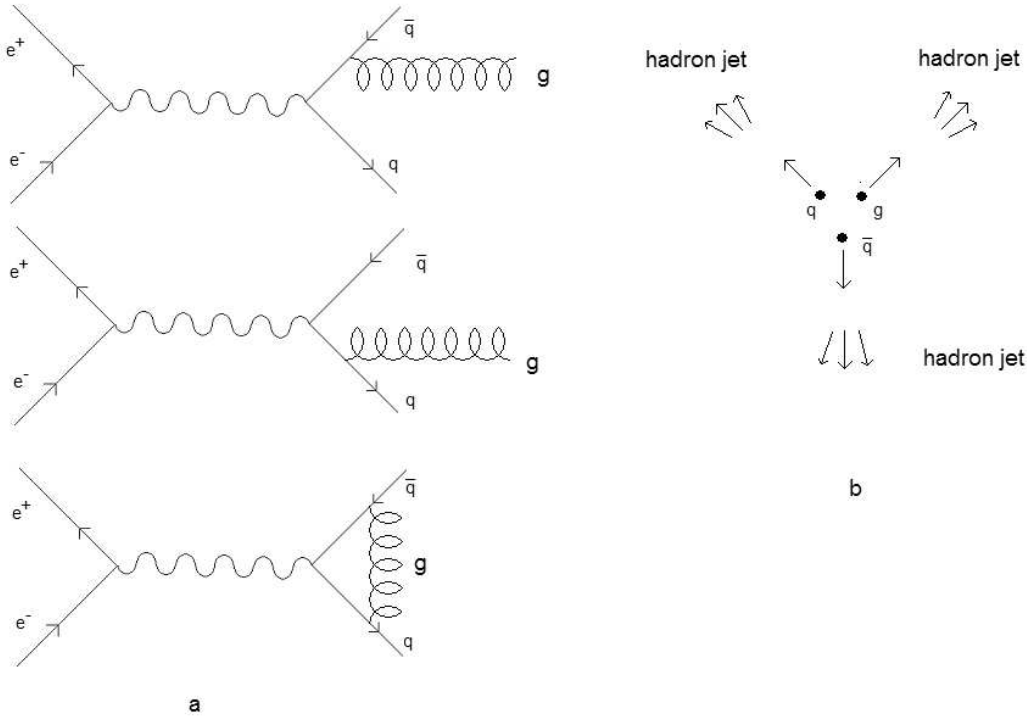


Figure 1.4: In (a) next-to-leading order diagrams are depicted; in (b) a three-jet event (where the third jet originates from the radiated gluon) is drawn.

detectors, built isotropic around the scattering centre.

The two-jet events also proved the existence of the color degrees of freedom; in the calculation of the probability of Fig. 1.2 we realize that the $q\bar{q}$ pair can come in three different color combinations $\bar{R}R$, $\bar{G}G$ and $\bar{B}B$. Thus, the total cross section, with respect to $\mu^+\mu^-$ production, is enhanced by a factor $N_c = 3$. This enhancement has been seen, thus confirming the correctness of the "color" hypothesis.

Another possibility can take place [17]: from one of the two outgoing quarks a gluon can be radiated (see the first and the second diagram of Fig. 1.4.a, where the next to leading order diagrams are depicted); the subsequent hadronization generates three jets instead of two (Fig. 1.4.b). The detection of these 3 jets event at PETRA from the TASSO collaboration [18] constituted the direct prove of the gluon existence.

The description of the high energy processes from a theoretical point of view is achieved at a great level of accuracy by using QCD: at high energy, the weakly interacting quarks and gluons allow a perturbative approach. The probability for a gluon emission, for example, is calculated within this framework. Then it is simply assumed that the hadronization probability is one, which means the radiated particles transform in white objects.

On the other hand, when one works at low energies (soft gluon "background" exchange for the QCD bound state formation) a direct application of perturbative QCD is not possible; the interaction becomes very strong, and alternative methods must be used, for instance like:

- Non-relativistic constituent quark models: the constituent quarks interact through a confining potential ([5, 19] and Refs. therein).

- Relativistic constituent quark models: one works at a QFT level with basic constituent quark fields. A typical example is the Nambu Jona-Lasino (NJL) model (original papers in [20], review papers [21, 22, 23]) working well in the pseudoscalar mesonic sector, and extended to finite density and temperature.

- Dyson-Schwinger equations: a non-perturbative method for the calculation of the quark and gluon propagators. Then, a Bethe-Salpeter analysis for the bound states is applied ([24, 25] and Refs. therein)

- Chiral Perturbation theory: in this case the basic degrees of freedom are the physical states, like mesons and baryons. The pions, being the lightest hadrons, play a decisive role. This approach is in the limit of zero energy equivalent to QCD ([26, 27, 28] and Refs. therein).

As mentioned before, we will use a phenomenological relativistic constituent QFT model. We will employ a nonlocal Lagrangian for the description of the hadron properties [29].

1.5 Back to glueballs

1.5.1 A possible assignment

As we have seen, the existence of gluons is clearly established by high energy experiments.

At the same time, the self-interacting nature of gluons and the resulting strong attraction among them led to the prediction of gluonic bound states.

As for quarks, we distinguish among the "high energy massless gluons" and the constituent massive gluons, whose mass range is between 0.6-1.2 *GeV*, even more massive than the constituent quarks.

This effective high mass, also confirmed in Lattice simulations [30], has its origin from the dressing of the valence gluon with virtual gluons and virtual quark-antiquark pairs around it. In the computer simulations one actually considered only the gluon dressing, i.e. only the gluonic part of the theory has been studied, the so-called Yang-Mills QCD sector. Already in this sub-theory glueballs exist. The effect of dynamical quarks is under study, but it seems not to alter drastically the original Y-M results.

The glueball is interpreted as a state of two constituent gluons; its scalar (lightest) mass is of the order of two times the constituent gluon mass, around 1.5 *GeV*.

The experimental identification of glueballs on one side, and their theoretical comprehension on the other side, would constitute a big progress in the understanding of the hadronic world. The glueballs are in a sense a missing link, predicted by many approaches and authors, but not yet found or ruled out.

In this thesis we analyze the scalar glueball and its probable mixing with quarkonia mesons. The Lattice scalar glueball mass leads to focus the attention on the scalar states between 1-2 *GeV*. Out of a quark and an antiquark one can build nine scalar quarkonia states:

- $\bar{u}d, \bar{d}u, \sqrt{1/2}(\bar{u}u - \bar{d}d) : I = 1$ states (pion flavor content)
- $\bar{u}s, \bar{s}u, \bar{d}s, \bar{s}d : I = 1/2$ states (kaon flavor content)
- $\sqrt{1/2}(\bar{u}u + \bar{d}d), \bar{s}s : I = 0$ states (eta and eta-prime flavor content).

The experimental identification of 10 scalar states between 1 – 2 *GeV* (see section 1.2), in particular three isoscalars instead of two ($f_0(1370)$, $f_0(1500)$ and $f_0(1710)$), can be explained if the glueball degree of freedom shows up. One has nine scalar quark-antiquark states plus a scalar glueball intruding among them.

The QCD quark-gluon interaction is flavor-blind; a gluon is flavor "democratic", i.e. interacts with a *u* exactly as with a *d* and with a *s*. In the basic quark-gluon vertex the color

is interchanged, but the flavor remains untouched.

In this sense the glueball (2-gluon bound states) is a flavor blind object. If we introduce mixing, and we allow that the two gluons transform into a $\bar{q}q$ pair, we realize that the probability of creating a $\bar{u}u$, or a $\bar{d}d$, or a $\bar{s}s$ is the same (up to u - d - s mass differences, negligible between u and d , approximate between $n = u, d$ and s). This means that a glueball behaves "like" $\sqrt{1/3}(\bar{u}u + \bar{d}d + \bar{s}s)$ (flavor singlet, also showing the isoscalar nature of the glueball), and should also decay correspondingly; after removal of phase space, one predicts a decay branching ratios into the pseudoscalar mesons $\pi\pi, \bar{K}K, \eta\eta, \eta\eta'$ like $3 : 4 : 1 : 0$, simply corresponding to the charge multiplicities of the pseudoscalar couples; the decay into $\eta\eta'$ is forbidden because a flavor singlet cannot decay into a singlet an octet.

None of the three existing isoscalar resonances shows such a decay pattern. This indeed can be understood if one considers that the glueball can transform into a quarkonia meson. The fact that these three states lie in the same mass region supports this idea; in fact, mixing among states is enhanced if the states have a similar mass.

Under this assumption each isoscalar resonance is a mixture of quarkonia and glueball; such a mixing would also lead to a distortion of the flavor blind decay discussed above. In fact, no state would be a pure gluonic state.

Summarizing, the physical resonances can be interpreted as following:

- $a_0(1450) = \bar{u}d, \bar{d}u, \sqrt{1/2}(\bar{u}u - \bar{d}d) : I = 1.$
- $K^*(1430) = \bar{u}s, \bar{s}u, \bar{d}s, \bar{s}d : I = 1/2.$
- $f_0(1370), f_0(1500), f_0(1710) : \text{mixture of } \bar{n}n = \sqrt{1/2}(\bar{u}u + \bar{d}d), \bar{s}s \text{ and glueball: } I = 0.$

A precise experimental knowledge of the strong decays is necessary to test the mixing and to deduce the amount of bare states $\bar{n}n, gg$ and $\bar{s}s$ in the flavor wave functions of the physical resonances.

These ideas were originally introduced in [31] almost ten years ago. At that time the state $f_0(1710)$ was still not established; there was indeed a resonance, whose angular momentum was still not known ($J = 0$ and $J = 2$ were the two options). Further experimental analysis confirmed the scalar $J = 0$ nature of the state. In their work the state $f_0(1370)$ has a dominant $\bar{n}n$ content, $f_0(1710)$ a dominant $\bar{s}s$ one, while the state with the biggest "glue" amount is $f_0(1500)$. The strong decays were analyzed under the mixing dominance hypothesis, i.e. the direct decay of two gluons into two pseudoscalar mesons is neglected.

In order to get these results the starting point where pure $\bar{n}n, gg$ and $\bar{s}s$ states with the bare level ordering $M_{\bar{n}n} < M_{gg} < M_{\bar{s}s}$. In this way the intermediate state $f_0(1500)$ gets naturally the biggest glueball strength. This state has been clearly established by Crystal Barrel at LEAR [32] in proton-antiproton annihilation, and is also seen in central pp collisions [33] and J/Ψ decays [34]. The main interest in the $f_0(1500)$ as a possible candidate with at least partial glueball content rests on several phenomenological and theoretical observations [35]. The $f_0(1500)$ is produced in gluon rich production mechanisms, whereas no signal is seen in $\gamma\gamma$ collisions [36].

In the same scheme a Lattice analysis has been performed [37]; however, due to their high bare glueball mass ($M_{gg} \sim 1.7 \text{ GeV}$) a reversed bare level ordering $M_{\bar{n}n} < M_{\bar{s}s} < M_{gg}$ has been considered. The consequence is that the state with the biggest glueball admixture is $f_0(1710)$. The two different approaches share indeed the same underlying idea; the present experimental results indicate a large $\bar{s}s$ amount in $f_0(1710)$, as confirmed by the small ratio [12]

$$(f_0(1710) \rightarrow \pi\pi) / (f_0(1710) \rightarrow \bar{K}K) = 0.2 \pm 0.024 \pm 0.036.$$

This fact favors the first interpretation. However, Sexton et al. [38] (see also [39]) made a Lattice study, where they showed that such a small $\pi\pi/\overline{K}K$ ratio is also possible for a bare glueball. The results have not been reproduced by other authors, the errors are still big, but they would mean that the coupling constant for the 2-pion decay is much suppressed as compared to the $\overline{K}K$ or $\eta\eta$ ones, and in turn would imply a strong violation of flavor symmetry. Considering the uncertainty of this result and its dramatic consequences, we rather tend to accept the first interpretation, where flavor symmetry still plays a crucial role (as in all the other nonets) and where a small $\pi\pi/\overline{K}K$ ratio implies a large $\overline{s}s$ content in the correspondent physical decaying state.

1.5.2 Widths

The width of a scalar state $\overline{n}n$ is predicted to be large ($\sim 500 \text{ MeV}$) by various models [31, 40]. This is in agreement with $f_0(1370)$, which is broad [12] ($\Gamma = 300 - 500 \text{ MeV}$) and, although the experimental uncertainties are large, is compatible with a dominant $\overline{n}n$ state [12, 41].

On the contrary, the resonance $f_0(1500)$ is relatively narrow ($\Gamma = 109 \text{ MeV}$). This is also in agreement with a large inert "glueball" component, a "non-decaying part" in its wave function, rendering it narrow. The resonance decays dominantly through its $\overline{n}n$ and $\overline{s}s$ components.

The state $f_0(1710)$ has a width of 140 MeV , compatible with $\overline{s}s$, as the ratios confirm [12, 41].

One can then go further and consider also a direct glueball decay into pseudoscalar mesons; in this case the relative quarkonia-glueball phases play a decisive role in the decays. In the work of Close and Kirk [42] a strongly decaying glueball is introduced, changing considerably the discussion presented above.

The arguments presented up to here are mostly qualitative.

1.5.3 Scalar states below 1 GeV

We forgot the scalar states below 1 GeV $\sigma = f_0(400 - 1200)$, $f_0(980)$, $a_0(980)$, $k(700)$. What about them? If the ground quarkonia scalar states are between $1 - 2 \text{ GeV}$ as discussed above, what are they?

Jaffe [43] proposed an intriguing explanation: they are not $\overline{q}q$ states, but four quark states. In Jaffe's hypothesis they are a bound state of a diquark qq with an antidiquark $\overline{q}\overline{q}$. A strong attraction can occur [8], binding these objects, which are, of course, white. These states would also be "exotics", ergo not ordinary quark-antiquark ones.

Another possible interpretation is as mesonic molecules [44, 45]. The σ is then a pion-pion bound states, while $f_0(980)$, $a_0(980)$ are kaonic molecules.

The disentanglement of such states is still not complete. Only when both scalar "nonets" below and above 1 GeV are simultaneously understood we will be able to claim to have solved the scalar puzzle.

In any case, if Jaffe's 4-quark states or pseudoscalar molecules, we would not deal with basic quark-antiquark objects, consolidating the quark-antiquark scalar ground state assignment between 1 and 2 GeV .

1.5.4 Other interpretations

In the following we summarize some of the proposed interpretations, further composed of sub-interpretations, for the scalar states in a schematic way [13, 46, 47]. The first one, *A*, corresponds to the description above.

A :

- Scalars between 1-2 *GeV* : ground state ($1\ ^3P_0$) $\bar{q}-q$ + the glueball and consequent mixing with the isoscalar quarkonia states
- Scalars below 1 *GeV* : four quark states:
 - Jaffe's four quark states: $\bar{q}\bar{q}-qq$; the building blocks are diquarks.
 - Meson molecules: $\bar{q}q-\bar{q}q$; the building blocks are pseudoscalar mesons.

It is very difficult to distinguish among the last two options for the scalar states below 1 *GeV*. The physical resonances could also be a mixture of the two proposed solutions.

However, this scenario, which we will follow in this thesis, clearly divides the states below 1 *GeV* from the ones above, which are quark-antiquark states mixed with the glueball.

The mixing of the scalar states below and above 1 *GeV* has also been studied [48].

Note that both [31] and [37] interpretations discussed above are contained in scenario *A*; in this sense "they are not so different", if compared to the other scenarios (see below).

B :

- Scalars below 1 *GeV* : ground state $\bar{q}-q$ ($1\ ^3P_0$) ($n = 1$, where n is the radial quantum number)

According to the Nambu relation the scalar-isoscalar state $\bar{n}n = \sqrt{1/2}(\bar{u}u + \bar{d}d)$ has a mass $M_\sigma \simeq 2m^* \sim 600\ MeV$, where one can recognize the $\sigma = f_0(400 - 1200)$.

However, the Nambu relation is extracted from the Nambu Jona-Lasinio model [20, 21, 22, 23], which is solid at very low energy, but whose validity for scalar states may be questionable. As cited above, the scalar states are $L = 1$, and we expect their mass above 1 *GeV*, in the energy region of the other *P* states.

- Scalars between 1-2 *GeV* : first excited ($n = 2, 2\ ^3P_0$) $\bar{q}-q$ states, with possible mixing with the glueball.

Scenario *B*, as well as *A*, still separates the states below and above 1 *GeV*. Other possibilities, which do not consider such a separation, have been proposed.

C :

- $f_0(980)$, $a_0(980)$, $f_0(1500)$, $K^*(1430)$: ground $\bar{q}-q$ states ($1\ ^3P_0$).

The resonance $f_0(1370)$ is not considered as a real state.

The $\sigma = f_0(400 - 1200)$ is then interpreted as a light glueball, which lives to shortly to mix. Such an interpretation is in agreement with some theoretical works within *QCD* sum rules [49], where a light glueball is predicted. Other considerations about this hypothesis are explained in [51].

In this scenario [7, 47, 50] only one scalar nonet is recognized, and the glueball does not mix significantly with the quarkonia mesons. The existence of a light glueball is not in agreement with nowadays Lattice results: an improvement in computer simulations, taking into account dynamical quarks, is highly desired to distinguish among the scenarios.

In this scenario the two isoscalar states $f_0(980)$ and $f_0(1500)$ are like [47]:

$$f_0(980) \sim \sqrt{1/6}(\bar{n}n + 2\bar{s}s)$$

$$f_0(1500) \sim \sqrt{1/3}(\bar{n}n - \bar{s}s),$$

i.e. as η_0 and η_8 , but with a crucial sign difference. Such strong mixing between the strange and the non-strange sectors occur through instantons, in a similar way as in the pseudoscalar nonet (but with an opposite sign).

The state $f_0(1370)$ is considered as the "tail" of the σ , i.e. itself a high mass manifestation of the scalar glueball; $a_0(1450)$ and $f_0(1710)$ are excited states.

A similar approach, which allocates most of the glueball strength in the $f_0(1370)$, is considered in Ref. [52].

Other possibilities have been considered. We do not aim to report all of them, but just to give an idea of the current state. Celenza et al. [53] proposed, for instance:

D :

$f_0(980), a_0(980), f_0(1370), K^*(1430) : \bar{q}-q$ ground states ($1\ ^3P_0$).

The model used to get such a classification is a modified NJL model with confinement, where the confinement shifts up the mass of the $\bar{n}n$ from $2m^* \sim 600\ MeV$ up to $\sim 1\ GeV$, therefore $f_0(980)$ is interpreted as $\bar{n}n$. $f_0(1370)$ is then a $\bar{s}s$ state, $f_0(1500)$ an excited state.

As one can note, the number and the intrinsic differences of the possible interpretations are a clear sign of the difficulty of the subject, and of the not-yet-solved underlying problem.

As stated above, I will follow the interpretation A , which to my humble opinion is the most solid. In this work we wish to study some consequences of A , showing that it can be the right one within a phenomenological context.

1.6 Thesis content

In this section we present the content of the following chapters.

1.6.1 Second chapter:

In the second chapter, working with scalar fields, we first introduce some basic concepts of Quantum Field Theory, like the canonical (second) quantization, following from the basic requirement of microcausality and like the principles of a perturbative expansion.

Then follows a brief discussion concerning renormalizability, and how we can interpret non-renormalizable theories, which can provide a useful description of some physical systems.

The use of a nonlocal non-renormalizable theory for the study of a bound state of two scalar particles is discussed. All the definitions and properties presented here for the scalar case are valid throughout the whole work.

At the end of the chapter the non-relativistic limit and the connection of the non-locality with the Bethe-Salpeter analysis is established.

1.6.2 Third chapter:

After an introduction of the Dirac field, the bound state formation of a fermion and an antifermion is studied.

In particular, pseudoscalar and scalar states and their spin decomposition is developed.

A non-local Lagrangian for their description is introduced.

The method is the same as the one of the second chapter, here extended to Dirac fields.

1.6.3 Fourth chapter:

In the fourth chapter we study the two-photon decay of a bound state.

After having recalled how a decay takes place and which are the peculiarities of the electromagnetic field, the 2γ decay of pseudoscalar and scalar bound states is discussed.

The issue of the electromagnetic $U(1)$ gauge invariance together with the non-locality of our approach is treated.

The results for the two-photon decay of the pion and for the scalar and pseudoscalar positronium are reported. The positronium study with a non-local Lagrangian is an application of the non-relativistic limit of our approach.

The results for the scalar quark-antiquark states are reported in the sixth chapter, after that the glueball and its mixing are introduced. However, the basic theoretical analysis of the two-photon decay for a scalar quarkonium state is discussed here.

1.6.4 Fifth chapter:

"Mixing" is a typical physical phenomenon, which is introduced by a classical example. This chapter is simple and pedagogical, but contains some important notions which will be discussed in the following one.

The mixing between fields in a Klein-Gordon theory is presented here, together with the consequent decay of mixed states.

1.6.5 Sixth chapter:

After a short description of QCD, the mixing between the bare glueball and quarkonia scalar-isoscalar mesons is performed within a phenomenological quantum field theoretical approach, taking into account the momentum dependence of the quantities under study.

The problem of the definition of a mixing matrix within such an approach is discussed; a mixing matrix, which satisfies the correct requirements, is proposed. Its results are compared to those of other phenomenological works.

The two-photon decays of the resulting mixed states (identified with $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$) are calculated.

1.6.6 Seventh chapter:

In the last chapter we study the two-pseudoscalar strong decays of the scalar states between 1-2 GeV . We use two different Lagrangian forms: the first simply takes into account $SU(3)$ flavor relations and the second is inspired by chiral perturbation theory.

A χ^2 -analysis is applied in order to get the best description of the experimental data.

Chapter 2

Scalar field

2.1 Classical scalar field

2.1.1 Introduction

We start from the study of scalar fields, because it embodies many features of Quantum Field Theory, but is free from spin and vectorial complications; this is not only a theoretical construct, because one encounters scalar fields pretty often in nature, as for example the scalar mesons, which play a central role in this thesis. Furthermore another scalar particle, the Higgs, is of fundamental importance in the theoretical construction of the Standard Model, even if its experimental discovery has still to come.

For what concerns the first four sections of the present chapter we refer to the following standard QFT books [3, 54, 55, 56, 57, 58, 59, 60]. The intent of these sections is a summary of the typical quantum field theory themes, presented in a succinct form; the discussion centers on those arguments which to me are crucial in QFT and which typically represent a hard conceptual step in the understanding of QFT. The sequence of the themes is a personal way to introduce the subject. The inclusion of this summary also follows from the desire of a (as much as possible) self-contained work.

We consider a single scalar real field $\varphi = \varphi(x)$, where $x \equiv x^\mu = (t, \mathbf{x})$, with the following Lagrangian

$$\mathcal{L}(\varphi(x), \partial_\mu \varphi(x)) = \frac{1}{2} (\partial_\mu \varphi(x)) (\partial^\mu \varphi(x)) - V(\varphi(x)), \quad (2.1)$$

where V is a function of φ .

In this way the Lagrangian is invariant under transformations of the Poincaré group, being a basic postulate of QFT; in fact, if we apply the shift

$$x \rightarrow x' = \Lambda x + a \quad (2.2)$$

(where Λ is a 4x4 Lorentz matrix and a a generic four-vector) considering that

$$\varphi(x) \rightarrow \varphi'(x') = \varphi(x), \quad (2.3)$$

we have, as desired

$$\mathcal{L}(\varphi(x), \partial_\mu \varphi(x)) = \mathcal{L}(\varphi'(x'), \partial'_\mu \varphi'(x')). \quad (2.4)$$

The equation of motion for the field φ reads:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \frac{\partial \mathcal{L}}{\partial \varphi} \quad (2.5)$$

The Hamiltonian density and the momentum density are given by the equations:

$$\mathcal{H} = \mathcal{P}^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} (\partial_0 \varphi) - \mathcal{L}; \quad (2.6)$$

$$\mathcal{P}^i = -\frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} (\partial_i \varphi). \quad (2.7)$$

The corresponding integrals of these quantities over all space give us the four-momentum of the field

$$P^\mu = \int d^3x \mathcal{P}^\mu, \quad (2.8)$$

for which one has, as a consequence of the translational symmetry :

$$\frac{dP^\mu}{dt} = 0. \quad (2.9)$$

2.1.2 Klein-Gordon case

We concentrate now on a sub-case of the previous one, that we get for $V(\varphi) = \frac{1}{2}m^2\varphi^2$, where m is a positive constant; the equation of motion, obtained from (2.5), is the famous Klein-Gordon one:

$$(\square + m^2) \varphi(x) = 0. \quad (2.10)$$

2.1.2.1 Plane waves

Let us consider a plane wave function of the form

$$\varphi(x) = N e^{-ikx}, \quad (2.11)$$

where the four-vector $k = (k^0, \mathbf{k})$ is introduced, with $k^0 = E$ and \mathbf{k} , as we will show, *being connected* with the energy and the impulse of the wave. If we plug (2.11) in (2.10) we find

$$(k^0)^2 = \mathbf{k}^2 + m^2; \quad (2.12)$$

this equation reminds us of the Einstein mass-equation, but here we are not dealing with a single particle, but with a wave; secondly, the function given in (2.11) is not real.

2.1.2.2 Solution in a box

Through plane waves we can construct a general *real* solution for $\varphi(x)$; a very useful class of solutions is found by confining our system to a box of side L ; to do this we consider periodic boundary conditions of the kind

$$\varphi(t, x, y, z) = \varphi(t, x + L, y, z) = \varphi(t, x, y + L, z) = \varphi(t, x, y, z + L). \quad (2.13)$$

This is of course an *artificial* condition, but we have to imagine that we deal in a limiting process with a box which is enlarged to infinity. The plane waves we consider are those fulfilling the previous equation, which gives:

$$\mathbf{k} = 2\pi \frac{\mathbf{n}}{L}, \quad (2.14)$$

where \mathbf{n} is a three-vector of whole numbers ($n_i = 0, \pm 1, \pm 2, \dots; i = 1, 2, 3$).

We can easily see that the set of plane waves $\left\{ \varphi_{\mathbf{k}} = \frac{1}{\sqrt{V}} e^{i\mathbf{k}\mathbf{x}}, \mathbf{k} = 2\pi \frac{\mathbf{n}}{L} \right\}$ is orthonormal and complete:

$$\int_V d^3x \varphi_{\mathbf{k}'}^*(\mathbf{x}) \varphi_{\mathbf{k}}(\mathbf{x}) = \delta_{\mathbf{k}, \mathbf{k}'}; \quad (2.15)$$

$$\frac{1}{V} \sum_{\mathbf{k}=2\pi \frac{\mathbf{n}}{L}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} = \delta^3(\mathbf{x}-\mathbf{x}'). \quad (2.16)$$

A generic real function $\varphi(t, \mathbf{x})$ fulfilling (2.13) can be written as a superposition of the plane waves:

$$\varphi(t, \mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}=2\pi \frac{\mathbf{n}}{L}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a(t, \mathbf{k}) e^{i\mathbf{k}\mathbf{x}} + a^*(t, \mathbf{k}) e^{-i\mathbf{k}\mathbf{x}} \right). \quad (2.17)$$

The term $\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}$, with $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$, has been introduced for future simplicity; the coefficients $a(t, \mathbf{k})$ depend on time, and inserting the last equation in (2.10) we find:

$$\frac{d^2 a(t, \mathbf{k})}{dt^2} + (\mathbf{k}^2 + m^2) a(t, \mathbf{k}) = 0, \quad (2.18)$$

that we can easily solve

$$a(t, \mathbf{k}) = a_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} \quad (2.19)$$

and find the final expression:

$$\varphi(t, \mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}=2\pi \frac{\mathbf{n}}{L}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}} + a_{\mathbf{k}}^* e^{i\mathbf{k}\mathbf{x}} \right)_{k^0=\omega_{\mathbf{k}}}. \quad (2.20)$$

In some cases it is easier to use this form, and then at the very end apply the limit $L \rightarrow \infty$.

2.1.2.3 Solution in the whole space

Sending V to infinity is not so trivial, since the basis $\left\{ \frac{1}{\sqrt{V}} e^{i\mathbf{k}\mathbf{x}} \right\}$ goes to zero and \mathbf{k} becomes a continuous variable; we better consider the set $\left\{ \varphi_{\mathbf{k}} = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\mathbf{x}} \right\}$ for which we have the generalization to the whole space of the relations (2.15) and (2.16):

$$\int d^3x \varphi_{\mathbf{k}'}^*(\mathbf{x}) \varphi_{\mathbf{k}}(\mathbf{x}) = \delta(\mathbf{k} - \mathbf{k}'); \quad (2.21)$$

$$\int d^3k \varphi_{\mathbf{k}}^*(\mathbf{x}') \varphi_{\mathbf{k}}(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}'). \quad (2.22)$$

It is important to note that the single function $\varphi_{\mathbf{k}}(\mathbf{x})$ is not normalized; we have then, replacing $\sqrt{V} \rightarrow (2\pi)^{3/2}$ and $\sum \rightarrow \int d^3k$ the following solution:

$$\varphi(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(\mathbf{k})}} \left(a(\mathbf{k}) e^{-i\mathbf{k}\mathbf{x}} + a^*(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}} \right)_{k^0=\omega(\mathbf{k})}. \quad (2.23)$$

2.1.2.4 Energy and Momentum

We stated that $k = (k^0, \mathbf{k})$ is connected with the energy and momentum of the field, and we would like to have a closer look at it. We consider therefore the following solution (in a box), corresponding to the mode \mathbf{k} only:

$$\varphi_{\mathbf{k}}(t, \mathbf{x}) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{-ikx} + a_{\mathbf{k}}^* e^{ikx} \right)_{k^0 = \omega_{\mathbf{k}}} \quad (2.24)$$

The Hamiltonian density in the Klein-Gordon case looks like:

$$\mathcal{H} = \frac{1}{2} \left((\partial_0 \varphi)^2 + (\nabla \varphi)^2 + m^2 \varphi^2 \right), \quad (2.25)$$

and if we insert $\varphi_{\mathbf{k}}(t, \mathbf{x})$ and calculate the Hamiltonian, i.e. the energy of the wave, we find (after some simple steps):

$$H = \int_V d^3x \left(\frac{k^0}{V} |a_{\mathbf{k}}|^2 \right) = k^0 |a_{\mathbf{k}}|^2; \quad (2.26)$$

similarly for the three-momentum:

$$P^i = k^i |a_{\mathbf{k}}|^2. \quad (2.27)$$

We can simply interpret our wave as made of $|a_{\mathbf{k}}|^2 = N_{\mathbf{k}}$ bulks of energy, each of them with energy $k^0 = \sqrt{\mathbf{k}^2 + m^2}$ and with momentum \mathbf{k} ; every bulk fulfils the Einstein mass-energy equation, and therefore we can say we are dealing with $N_{\mathbf{k}}$ "particles" with mass m .

For a general solution like in (2.17) we find

$$H = \sum_{\mathbf{k}=2\pi\frac{\mathbf{n}}{L}} k^0 |a_{\mathbf{k}}|^2, \quad (2.28)$$

and in general

$$P^\mu = \sum_{\mathbf{k}=2\pi\frac{\mathbf{n}}{L}} k^\mu |a_{\mathbf{k}}|^2. \quad (2.29)$$

It is also clear how to interpret at this stage the coefficients $a_{\mathbf{k}}$: $|a_{\mathbf{k}}|^2$ represents the number of particles $N_{\mathbf{k}}$ with impulse \mathbf{k} and energy k^0 . The total number of particles is

$$N = \sum_{\mathbf{k}=2\pi\frac{\mathbf{n}}{L}} |a_{\mathbf{k}}|^2. \quad (2.30)$$

The generalization to the whole space is straightforward.

2.2 Quantization

2.2.1 From functions to operators

We now want to describe the corresponding quantum field theory; the function $\varphi(x)$ becomes an operator:

$$\varphi(x) \rightarrow \hat{\varphi}(x), \quad (2.31)$$

and in general

$$A(\varphi(x), \partial_\mu \varphi(x)) \rightarrow \widehat{A}(\widehat{\varphi}(x), \partial_\mu \widehat{\varphi}(x)), \quad (2.32)$$

where A is a generic functional of the field and its derivatives.

The operators act on a vectorial space, called space of the physical states \mathcal{F} , which we assume to be an Hilbert space (i.e. a vectorial space, which is at the same time complete and in which a scalar product is defined). A generic vector in \mathcal{F} is indicated as $|s\rangle$, that does not depend on the space-time (Heisenberg picture). All the physical information of a system is contained in $|s\rangle$. In order to give a correct probabilistic interpretation of the theory we only consider normalized states with $\langle s | s \rangle = 1$. The analogous of the classic quantity A is now the mean value of the operator:

$$A_s \leftrightarrow \langle s | \widehat{A} | s \rangle. \quad (2.33)$$

In order to assure a correspondence with the classical case, we assume that the equation of motion (2.5) is valid also for the quantized field $\widehat{\varphi}(x)$:

$$(\square + m^2) \widehat{\varphi}(x) = 0. \quad (2.34)$$

In order to have $\langle s | \widehat{\varphi}(x) | s \rangle$ real, we demand that the operator $\widehat{\varphi}(x)$ is Hermitian (this is a condition which doesn't hold in general, but that we impose now to discuss the analogous of the real scalar field in the quantized case):

$$\widehat{\varphi}(x) = \widehat{\varphi}^\dagger(x). \quad (2.35)$$

Because of the last two equations, and reminding us of (2.20), we can write down $\widehat{\varphi}(x)$ as:

$$\begin{aligned} \widehat{\varphi}(x) &= \frac{1}{\sqrt{V}} \sum_{\mathbf{k}=2\pi\frac{\mathbf{n}}{L}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{-ikx} + a_{\mathbf{k}}^\dagger e^{ikx} \right)_{k^0=\omega_{\mathbf{k}}} \\ (L \rightarrow \infty) &= \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a(\mathbf{k}) e^{-ikx} + a(\mathbf{k})^\dagger e^{ikx} \right)_{k^0=\omega_{\mathbf{k}}} \end{aligned} \quad (2.36)$$

where $a_{\mathbf{k}}$ are now operators, and no longer complex numbers.

Now, the solution (2.36) doesn't tell us which properties the operators $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ have. We must impose something more in order to get information about them, and this comes in the next subsection; according to the classical formulas, we expect that $a_{\mathbf{k}}$ is related to the description of a particle with momentum \mathbf{k} .

2.2.2 Microcausality

In order that the principles of special relativity are satisfied, we have to require that

$$\begin{aligned} [\widehat{\varphi}(x), \widehat{\varphi}(y)] &= 0 \\ \text{if } (x - y)^2 &< 0; \end{aligned} \quad (2.37)$$

in fact, for space-like distances there cannot be interference between the two fields. The knowledge of the field in x cannot influence the one in y , if their separation is space-like. This is our starting point (following [59]), based on physical considerations, namely that no physical interaction can travel faster than the light, also in a quantum context.

From the previous equation it follows that

$$[\widehat{\varphi}(t, \mathbf{x}), \widehat{\varphi}(t, \mathbf{y})] = 0 \quad (2.38)$$

(note that the commutator is zero for $\mathbf{x} \neq \mathbf{y}$ in virtue of (2.37), but it is zero also for equal space points because it becomes trivially the commutator of an operator with itself).

Using now the expression (2.36) in the whole space and making the Fourier transform, we find:

$$e^{i\omega t} a^\dagger(\mathbf{k}) + e^{-i\omega t} a(-\mathbf{k}) = \sqrt{2\omega} \int \frac{d^3x}{(2\pi)^{3/2}} e^{i\mathbf{k}\mathbf{x}} \widehat{\varphi}(t, \mathbf{x}) \quad (2.39)$$

from which, for every $\mathbf{k}_1, \mathbf{k}_2$ and t we have (plugging into (2.38)):

$$\begin{aligned} & e^{i(\omega_1+\omega_2)t} [a^\dagger(\mathbf{k}_1), a^\dagger(\mathbf{k}_2)] + e^{-i(\omega_1+\omega_2)t} [a(-\mathbf{k}_1), a(-\mathbf{k}_2)] \\ & - e^{i(\omega_1-\omega_2)t} [a^\dagger(\mathbf{k}_1), a(-\mathbf{k}_2)] + e^{-i(\omega_1-\omega_2)t} [a(-\mathbf{k}_1), a^\dagger(\mathbf{k}_2)] \\ = & \sqrt{2\omega_1} \sqrt{2\omega_2} \int \frac{d^3x}{(2\pi)^{3/2}} \frac{d^3y}{(2\pi)^{3/2}} e^{i\mathbf{k}_1\mathbf{x}} e^{i\mathbf{k}_2\mathbf{y}} [\widehat{\varphi}(t, \mathbf{x}), \widehat{\varphi}(t, \mathbf{y})] = 0, \end{aligned} \quad (2.40)$$

and so we find $\forall \mathbf{k}_1, \mathbf{k}_2$:

$$[a^\dagger(\mathbf{k}_1), a^\dagger(\mathbf{k}_2)] = [a(\mathbf{k}_1), a(\mathbf{k}_2)] = 0. \quad (2.41)$$

Note that the same cannot in general be said for the commutator $[a(\mathbf{k}_1), a^\dagger(\mathbf{k}_2)]$; in order to evaluate this quantity, which plays a central role in the theory, we first note that for $\mathbf{x} \neq \mathbf{y}$:

$$\left[\widehat{\varphi}(t, \mathbf{x}), \frac{\partial}{\partial t} \widehat{\varphi}(t, \mathbf{y}) \right] = 0 \quad (2.42)$$

(here we cannot say anything about the commutator at equal space points).

The Fourier transform of $\frac{\partial}{\partial t} \widehat{\varphi}(t, \mathbf{x})$

$$e^{i\omega t} a^\dagger(\mathbf{k}) - e^{-i\omega t} a(-\mathbf{k}) = -i \sqrt{\frac{2}{\omega}} \int \frac{d^3x}{(2\pi)^{3/2}} e^{i\mathbf{k}\mathbf{x}} \frac{\partial}{\partial t} \widehat{\varphi}(t, \mathbf{x}), \quad (2.43)$$

together with (2.39), leads to:

$$\begin{aligned} & [a(\mathbf{k}_1), a^\dagger(\mathbf{k}_2)] \\ = & e^{i(\omega_1-\omega_2)t} \int \frac{d^3x}{(2\pi)^{3/2}} \frac{d^3y}{(2\pi)^{3/2}} e^{-i\mathbf{k}_1\mathbf{x}+i\mathbf{k}_2\mathbf{y}} \cdot \\ & \left(i \sqrt{\frac{\omega_2}{\omega_1}} \left[\frac{\partial}{\partial t} \widehat{\varphi}(t, \mathbf{x}), \widehat{\varphi}(t, \mathbf{y}) \right] - i \sqrt{\frac{\omega_1}{\omega_2}} \left[\widehat{\varphi}(t, \mathbf{x}), \frac{\partial}{\partial t} \widehat{\varphi}(t, \mathbf{y}) \right] \right) \end{aligned}$$

The crucial point is what happens for equal space points, which is, taken equal time for the fields, the only case in which the commutator (2.42) can be non-zero. The meaning of a nonzero commutator is the impossibility to know simultaneously the value of a field $\widehat{\varphi}(t, \mathbf{x})$ at the point \mathbf{x} and its "velocity" $\frac{\partial}{\partial t} \widehat{\varphi}(t, \mathbf{x})$. This is of course the new thing in comparison to

the classical case, where the solution is a function, for which we directly know also the value of the time-derivative.

The simplest assumption we can do, according to the constraints, is:

$$\left[\widehat{\varphi}(t, \mathbf{x}), \frac{\partial}{\partial t} \widehat{\varphi}(t, \mathbf{y}) \right] = iZ \delta^3(\mathbf{x} - \mathbf{y}), \quad (2.44)$$

where Z is a positive constant; it then follows

$$\left[a(\mathbf{k}_1), a^\dagger(\mathbf{k}_2) \right] = Z \delta^3(\mathbf{k}_1 - \mathbf{k}_2). \quad (2.45)$$

Z can be set to 1 through a rescaling of the field; indeed, the physical results depend on it, and one can determine its value, which is nothing else but the Plank constant \hbar . However, from a mathematical point of view, it is convenient to set it to one in order not to clutter up too much the equations.

The meaning of (2.44) is that we cannot know at the same time the value of a field at \mathbf{x} , and "its speed", so its derivative with respect to time. If one knows the value of the field, one can't know how it varies in the time; all typical quantum-mechanical considerations are applicable here.

If Z were 0, we could know at the same time the value of the field and its time-derivative, but in this case we would be back to the classical theory, where the use of operators and Hilbert space instead of functions would be an unnecessary complication.

Concluding this section, we write down the commutation relation for the introduced a -operators:

$$\begin{aligned} [a(\mathbf{k}_1), a(\mathbf{k}_2)] &= [a^\dagger(\mathbf{k}_1), a^\dagger(\mathbf{k}_2)] = 0, \\ [a(\mathbf{k}_1), a^\dagger(\mathbf{k}_2)] &= \delta^3(\mathbf{k}_1 - \mathbf{k}_2). \end{aligned} \quad (2.46)$$

These relations are exactly the commutation relations that one meets in QM studying harmonic oscillators; we deal with an infinite set of quantum oscillators, one for each \mathbf{k} .

Now, knowing the commutation relations we can in principle calculate the value of each operator starting from a classical functional F .

Summarizing, in this approach to QFT we first go from functions to operators imposing a link with the classical world: $\varphi_s(x) \leftrightarrow \langle s | \widehat{\varphi}(x) | s \rangle$. Secondly, we imposed microcausality, which constrained a lot the features of a and a^\dagger , and in the end we assumed (2.44), which seems the most natural assumption if one wants to respect causality (note that (2.44) is nothing else but the canonical quantization prescription $[\widehat{\varphi}(t, \mathbf{x}), \widehat{\pi}(t, \mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})$, where $\pi = \partial L / \partial(\partial_0 \varphi)$ is the conjugate momentum; most textbooks start from here).

What is presented here is of course not a derivation, but arguments of plausibility in order to get the canonical commutation relations starting from the basic idea of microcausality, necessary to have agreement with special relativity. The last point to discuss now is the following: we started from the commutation relation (2.37) to end up with the canonical commutation relation (2.42) written for equal times; one may ask what happens if we calculate now (2.37) for generic x and y , and not only in the equal time case.

We can easily do it by substituting the expression (2.36) for the fields:

$$[\widehat{\varphi}(x), \widehat{\varphi}(y)] =$$

$$\int \int \frac{d^3 k}{(2\pi)^3 \sqrt{2\omega(\mathbf{k})}} \frac{d^3 k'}{(2\pi)^3 \sqrt{2\omega(\mathbf{k}')}} \left[\left(a(\mathbf{k}) e^{-ikx} + a(\mathbf{k})^\dagger e^{ikx} \right), \left(a(\mathbf{k}') e^{-iky} + a(\mathbf{k}')^\dagger e^{iky} \right) \right], \quad (2.47)$$

and then making use of (2.46) one gets:

$$[\widehat{\varphi}(x), \widehat{\varphi}(y)] = \int \frac{d^3 k}{(2\pi)^3 2\omega(\mathbf{k})} \left(e^{-ik(x-y)} - e^{ik(x-y)} \right) = i\Delta(x-y). \quad (2.48)$$

We have introduced the function "capital delta" because the commutator of the 2 fields is a c-number. We can immediately see that $\Delta(x-y)$ is real and odd; in particular, for $x^0 = y^0 = t$, we get

$$i\Delta(x-y) = [\widehat{\varphi}(t, \mathbf{x}), \widehat{\varphi}(t, \mathbf{y})] = \int \frac{d^3 k}{(2\pi)^3 2\omega(\mathbf{k})} \sin(\mathbf{k}\mathbf{x}) = 0, \quad (2.49)$$

getting again (2.38), as expected. The function $\Delta(x-y)$ is also a Lorentz invariant: this means that $\Delta(\Lambda(x-y)) = \Delta(x-y)$, where Λ is a Lorentz-transformation (the term $d^3 k/2\omega(\mathbf{k})$ is a Lorentz invariant measure). If $(x-y)$ is space like, one can always perform a transformation such that for the rotated variables it holds $x^0 = y^0$, then showing that $\Delta(x-y) = 0$ for space-like separation, confirming that all the theoretical construction makes perfectly sense.

2.2.3 Constructing states

Clearly also the four momentum becomes an operator:

$$P^\mu \rightarrow \widehat{P}^\mu. \quad (2.50)$$

Classically this object is a constant of motion, therefore we expect that the corresponding operators do not depend on the time variable; in fact, calculating it from the classical definition (2.8) and using (2.46) we find (in the box):

$$\widehat{P}^\mu = \sum_{\mathbf{k}=2\pi\frac{\mathbf{n}}{L}} \frac{k^\mu}{2} \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}} a_{\mathbf{k}}^\dagger \right) = \sum_{\mathbf{k}=2\pi\frac{\mathbf{n}}{L}} k^\mu \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2} \right). \quad (2.51)$$

Note that the series is divergent because of the constant 1/2; but only the differences between four-momenta with respect to the ground states make physical sense, therefore we can consider

$$\widehat{P}^\mu = \sum_{\mathbf{k}=2\pi\frac{\mathbf{n}}{L}} k^\mu \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \right). \quad (2.52)$$

We can easily prove now that

$$[\widehat{P}^\mu, a_{\mathbf{k}}^\dagger] = k^\mu a_{\mathbf{k}}^\dagger, \quad (2.53)$$

$$[\widehat{P}^\mu, a_{\mathbf{k}}] = -k^\mu a_{\mathbf{k}}. \quad (2.54)$$

More generally, the so-called Heisenberg equation holds:

$$\partial_\mu \widehat{\varphi}(x) = i \left[\widehat{P}^\mu, \widehat{\varphi}(x) \right]. \quad (2.55)$$

The classical theory admits an energy $H \geq 0$, or, taking into account that the energy is defined up to a constant, we can say that the classical energy is limited from below: $H \geq c$. We assume that this still holds true when considering quantized fields. The corresponding state, whose energy is the minimal value c , is called the vacuum and is indicated with $|0\rangle$:

$$H|0\rangle = c|0\rangle. \quad (2.56)$$

From (2.54), we have:

$$[\hat{H}, a_{\mathbf{k}}]|0\rangle = -k^0 a_{\mathbf{k}}|0\rangle, \quad (2.57)$$

from which

$$\hat{H}a_{\mathbf{k}}|0\rangle = (c - k^0)a_{\mathbf{k}}|0\rangle; \quad (2.58)$$

it means that the state $a_{\mathbf{k}}|0\rangle$ has an energy smaller than the one of the vacuum, contrary to the hypothesis. Furthermore if we take the states $(a_{\mathbf{k}})^n|0\rangle$ we can construct states with lower and lower energy. A theory without a minimum of the energy would also be unstable, because a state could fall down into lower and lower energy states, as a well without a bottom. So the only possibility is:

$$a_{\mathbf{k}}|0\rangle = 0. \quad (2.59)$$

Setting for simplicity $c = 0$, we have from (2.53):

$$\hat{P}^\mu a_{\mathbf{k}}^\dagger|0\rangle = k^\mu a_{\mathbf{k}}^\dagger|0\rangle \quad (2.60)$$

with a straightforward physical interpretation: the state $a_{\mathbf{k}}^\dagger|0\rangle$ represents a quantum of four-momentum k^μ , i.e. a particle, fulfilling the Einstein relation $k^0 = \sqrt{\mathbf{k}^2 + m^2}$.

In general we can construct states as $a_{\mathbf{k}_1}^\dagger a_{\mathbf{k}_2}^\dagger|0\rangle$, whose four momentum is simply $k_1^\mu + k_2^\mu$; furthermore, from the commutation relations (2.46) it follows that Bose statistics holds:

$$a_{\mathbf{k}_1}^\dagger a_{\mathbf{k}_2}^\dagger|0\rangle = a_{\mathbf{k}_2}^\dagger a_{\mathbf{k}_1}^\dagger|0\rangle \quad (2.61)$$

Remembering the derivation of the previous section, we can see that the boson nature of the particles comes directly from microcausality; it was exactly from this assumption that the commutator for 2 $a_{\mathbf{k}}^\dagger$ vanishes.

The last thing to do is to build up normalized states; again, making use of the basic commutation relations (2.46) it is not difficult to show that the general state

$$|s\rangle = |\dots, n_{\mathbf{k}_1}, \dots, n_{\mathbf{k}_2}, \dots\rangle = \dots \frac{(a_{\mathbf{k}_1}^\dagger)^{n_{\mathbf{k}_1}}}{\sqrt{n_{\mathbf{k}_1}!}} \dots \frac{(a_{\mathbf{k}_2}^\dagger)^{n_{\mathbf{k}_2}}}{\sqrt{n_{\mathbf{k}_2}!}} \dots |0\rangle \quad (2.62)$$

is normalized and represents a state with $n_{\mathbf{k}_1}$ particles with impulse \mathbf{k}_1 , $n_{\mathbf{k}_2}$ particles with impulse \mathbf{k}_2 and so on. The set of all these states for $n_{\mathbf{k}} = 0, 1, 2, \dots$ for each \mathbf{k} form a complete basis in the space of physical states \mathcal{F} .

2.2.4 Coherent waves

We introduced the quantization by going from functions to operators and stating that the analogous of the classical function is now the mean value $\langle s|\hat{\varphi}(x)|s\rangle$. If now we consider $|s\rangle = a_{\mathbf{k}}^\dagger|0\rangle$, using the fundamental commutation relations, we get:

$$\langle s|\hat{\varphi}(x)|s\rangle = 0; \quad (2.63)$$

this result might seem disappointing, because we would expect as a result a plane wave....but it is not like that: looking closer, $|s\rangle = a_{\mathbf{k}}^\dagger |0\rangle$ represents one particle, one "quantum" of energy, while classically one has waves, which are made out of many quanta! But it is not enough: if we consider a state like (2.62) we would still get $\langle s | \hat{\varphi}(x) | s \rangle = 0$. In order to construct a state which, for large energies, coincides with our idea of a classical field, we consider

$$|s\rangle = e^{-\frac{1}{2}|a|^2} \sum_{n=0}^{\infty} \frac{a^n}{\sqrt{n!}} \left(a_{\mathbf{k}}^\dagger\right)^n |0\rangle, \quad (2.64)$$

where a is a complex number.

Note that what we are doing is simply to consider $a \cdot a_{\mathbf{k}}^\dagger$ instead of $a_{\mathbf{k}}^\dagger$ only, and summing over the possible states formed from $a_{\mathbf{k}}^\dagger$: it is not difficult to prove that [54] (using $Z = \hbar \neq 1$):

- $\langle s | s \rangle = 1$ (normalization of the state)
- $|s\rangle$ is an eigenstates of a $a_{\mathbf{k}}$:

$$a_{\mathbf{k}} |s\rangle = a |s\rangle \quad (2.65)$$

- the mean value of the number of quanta is:

$$N = \langle s | \hat{N} | s \rangle = |a|^2; \quad (2.66)$$

$|a|^2$ represent the number of quanta; it can be compared to the classical one (see section 1.1.4)

- the fluctuation for \hat{N} is:

$$\Delta N = \sqrt{\langle s | \hat{N}^2 | s \rangle - \bar{N}^2} = |a|; \quad (2.67)$$

we therefore see that

$$\Delta N/N = 1/|a| \quad (2.68)$$

goes to zero for large $|a|$

- the mean value for the field is:

$$\langle s | \hat{\varphi}(x) | s \rangle = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega}} \left(a e^{-ikx} + a^* e^{ikx} \right)_{k^0=\omega_{\mathbf{k}}}, \quad (2.69)$$

which is exactly the classical expression (see (2.24))!

- the fluctuation of the field is:

$$\Delta \varphi(x) = \sqrt{\langle s | \hat{\varphi}^2 | s \rangle - (\langle s | \hat{\varphi} | s \rangle)^2} = \sqrt{\frac{\hbar \omega}{2V}}. \quad (2.70)$$

We see that, if $\hbar \omega \ll |a|$, the fluctuation of the field is negligible, and we are back to the classical case where \hbar is zero.

The generalization to many momenta is now straightforward.

2.2.5 Transformations

2.2.5.1 Classical survey

Classically we transform functions like

$$\varphi(x) \rightarrow \varphi'(x) \quad (2.71)$$

If $\varphi'(x)$ is still a solution of the equation of motion we have a symmetry of the equation, and therefore of the underlying Lagrangian.

For example, because of the structure of the Lagrangian, if we make the shift

$$x \rightarrow x' = \Lambda x + a \quad (2.72)$$

(where Λ is a 4×4 Lorentz matrix and a a generic four-vector) and we consider

$$\varphi(x) \rightarrow \varphi'(x') = \varphi(x) \quad (2.73)$$

then

$$\mathcal{L}(\varphi(x), \partial_\mu \varphi(x)) = \mathcal{L}(\varphi'(x'), \partial'_\mu \varphi'(x')). \quad (2.74)$$

It means that, if $\varphi(x)$ is a solution in x -space, (solution of the equation of motion written for x) $\varphi'(x')$ is the corresponding solution in x' -space. This means that, if I am in the x -system, and I see the solution $\varphi(x)$, then my friend who is in the x' -system observes contemporarily the solution $\varphi'(x')$. If then, as in this case, both are solutions of the same underlying equation¹ (but with different boundary conditions) I can say that the theory is invariant under the considered transformation.

Making a more mathematical consideration, this kind of symmetry corresponds to a way of finding other solutions of the basic equation. In fact, in our case, if $\varphi(x)$ is a solution also $\varphi'(x) = \varphi(\Lambda^{-1}(x - a))$ is a solution (this is exactly what our friend in the x' -space sees, if he would call the variable x , something absolutely allowed!). Note that we can analyze covariance remaining in x : it is enough to check whether $\varphi(\Lambda^{-1}(x - a))$ is a solution of the equation of motion².

Very often one doesn't deal with a scalar field only, but with a generic vectorial structure φ_i where $i = 1, 2, \dots, N$; in this case it is possible that if we apply to a vectorial field a space-time transformation like (2.72), the transformation $\varphi_i(x) \rightarrow \varphi'_i(x') = \varphi_i(x)$, is not a symmetry of \mathcal{L} ; it doesn't mean that the theory is not covariant (not yet)! We have to search for a matrix B

$$\varphi'_i(x') = B_{ij} \varphi_j(x) \quad (2.75)$$

such that

$$\mathcal{L}(\varphi_i(x)) = \mathcal{L}(\varphi'_i(x')) \quad (2.76)$$

then we can say that the theory is still covariant and that our friend in x' will observe a field $\varphi'_i(x')$; mathematically it means that the fields $\varphi'_i(x) = B_{ij} \varphi_j(\Lambda^{-1}(x - a))$ are still a solution of the equations of motion (for each $i = 1, 2, \dots, N$).

Intuitively this is related to the vectorial nature of $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_N)$; a "rotation" in space can also rotate the vector φ .

¹Note that both are solutions because the action is the same: being $d^4x = d^4x'$ and having the same Lagrangian, it is clear that $dS = dS'$.

²One would tend to generalize to $x \rightarrow f(x)$ and $\varphi(x) \rightarrow \varphi'(x) = \varphi(f^{-1}(x))$, but in this case $d^4x \neq d^4x'$; we can therefore discuss the symmetry of the equation, but not the correspondence to other observers, because in this case one has to use general relativity.

2.2.5.2 Quantum case

In the quantum case a transformation means to consider a unitary operator U acting on the elements of the Hilbert space \mathcal{F} :

$$|s\rangle \rightarrow |s'\rangle = U |s\rangle ; \quad (2.77)$$

the unitarity guaranties that the norm of $|s'\rangle$ is still one (if, of course, $|s\rangle$ is itself normalized to 1).

For example, if we transform the mean field value we have:

$$\langle s | \widehat{\varphi}(x) | s \rangle \rightarrow \langle s' | \widehat{\varphi}(x) | s' \rangle = \langle s | U^\dagger \widehat{\varphi}(x) U | s \rangle . \quad (2.78)$$

We can immediately note that, instead of transforming the state, we could transform the operator as $\widehat{\varphi}(x) \rightarrow U^\dagger \widehat{\varphi}(x) U$; in general we have two possibilities to establish a transformation:

- We rotate the states $|s\rangle \subset F$ like $|s\rangle \rightarrow |s'\rangle = U |s\rangle$ and we leave the operators untouched.
- We rotate the operators \widehat{A} like $\widehat{A} \rightarrow U^\dagger \widehat{A} U$ and we do not modify the states.

Translation In order to establish a connection with the classical case, and to work out an example, we consider the (four-dimensional) translation:

$$x \rightarrow x' = x + a \quad (2.79)$$

Classically we have, for small a :

$$\varphi(x) \rightarrow \varphi'(x) = \varphi(x - a) \simeq \varphi(x) - a^\mu \partial_\mu \varphi(x); \quad (2.80)$$

we expect to have a similar transformation for the mean value $\langle s | \widehat{\varphi}(x) | s \rangle$:

$$\langle s | \widehat{\varphi}(x) | s \rangle \rightarrow \langle s | \widehat{\varphi}(x) | s \rangle - a^\mu \langle s | \partial_\mu \widehat{\varphi}(x) | s \rangle , \quad (2.81)$$

and so for the field operator:

$$\widehat{\varphi}(x) \rightarrow \widehat{\varphi}(x) - a^\mu \partial_\mu \widehat{\varphi}(x). \quad (2.82)$$

We look for an operator U for which this happens; in analogy with Quantum Mechanics [61, 62] we take:

$$U = e^{ia^\mu \widehat{P}_\mu}. \quad (2.83)$$

We have then, taking small a and using the Heisenberg equation (2.55):

$$\widehat{\varphi}(x) \rightarrow U^\dagger \widehat{\varphi}(x) U \simeq \left(1 - ia^\mu \widehat{P}_\mu\right) \widehat{\varphi}(x) \left(1 + ia^\mu \widehat{P}_\mu\right) \quad (2.84)$$

$$\simeq \widehat{\varphi}(x) - ia^\mu \left[\widehat{P}_\mu, \widehat{\varphi}(x) \right] = \widehat{\varphi}(x) - a^\mu \partial_\mu \widehat{\varphi}(x). \quad (2.85)$$

In this way we have found the quantum field version of the translation operator; note that, for a generic (not small) a we have:

$$e^{-ia^\mu \widehat{P}_\mu} \widehat{\varphi}(x) e^{ia^\mu \widehat{P}_\mu} = \widehat{\varphi}(x - a). \quad (2.86)$$

Lorentz Transformation We now consider the transformation of the Poincaré group, which consists of Lorentz transformations plus translations:

$$x \rightarrow x' = \Lambda x + a \quad (2.87)$$

We now want to discuss the form of the corresponding quantum operator $U(\Lambda, a)$ which acts on the state as $|s\rangle \rightarrow |s'\rangle = U(\Lambda, a)|s\rangle$; instead of giving its explicit form, it's enough to know how U acts on the $a^\dagger(\mathbf{k})$. In fact, we have seen that a generic state can be than expressed as in (2.62). What we need to know is

$$U(\Lambda, a) a^\dagger(\mathbf{k}) U^\dagger(\Lambda, a); \quad (2.88)$$

as one can note, this is not really the transformation of an operator, that is in general $\widehat{F} \rightarrow U^\dagger \widehat{F} U$; but, if we consider for example the state $|s\rangle = a^\dagger(\mathbf{k})|0\rangle$, we have:

$$|s\rangle \rightarrow |s'\rangle = U(\Lambda, a)|s\rangle = U(\Lambda, a) a^\dagger(\mathbf{k})|0\rangle \quad (2.89)$$

$$= U(\Lambda, a) a^\dagger(\mathbf{k}) U^\dagger(\Lambda, a) U(\Lambda, a)|0\rangle \quad (2.90)$$

$$= U(\Lambda, a) a^\dagger(\mathbf{k}) U^\dagger(\Lambda, a)|0\rangle = |s'\rangle, \quad (2.91)$$

where we have used that $U(\Lambda, a)|0\rangle = |0\rangle$, so that the vacuum is invariant under translation and Lorentz transformation; this is one of the basic postulates on the nature of the vacuum!

This is why we are interested in (2.88); we can easily generalize to a generic state. For example, with two particles we have:

$$U(\Lambda, a) a^\dagger(\mathbf{k}_1) a^\dagger(\mathbf{k}_2) U^\dagger(\Lambda, a) \quad (2.92)$$

$$= \left(U(\Lambda, a) a^\dagger(\mathbf{k}_1) U^\dagger(\Lambda, a) \right) \left(U(\Lambda, a) a^\dagger(\mathbf{k}_2) U^\dagger(\Lambda, a) \right), \quad (2.93)$$

having inserted the unity $1 = U^\dagger(\Lambda, a) U(\Lambda, a)$ in the middle; it is clear again that we need to know how to evaluate the block of (2.88).

Now, in order to find it, we must consider the transformation of the field operator; classically we have

$$\varphi(x) \rightarrow \varphi'(x) = \varphi(\Lambda^{-1}(x - a)), \quad (2.94)$$

therefore in QFT

$$\widehat{\varphi}(x) \rightarrow \widehat{\varphi}'(x) \quad (2.95)$$

$$= U^\dagger(\Lambda, a) \widehat{\varphi}(x) U(\Lambda, a) = \widehat{\varphi}(\Lambda^{-1}(x - a)) \quad (2.96)$$

(now we have the correct operator transformation $U^\dagger \widehat{F} U$). Taking the last equation and (2.36), we have:

$$\begin{aligned} & \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} (U^\dagger(\Lambda, a) a(\mathbf{k}) U(\Lambda, a) e^{-ikx} \\ & + U^\dagger(\Lambda, a) a^\dagger(\mathbf{k}) U(\Lambda, a) e^{ikx})_{k^0=\omega_{\mathbf{k}}} = \\ & \int \frac{d^3 k'}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega'_{\mathbf{k}}}} \left(a(\mathbf{k}') e^{-ik'\Lambda^{-1}(x-a)} + a^\dagger(\mathbf{k}') e^{ik'\Lambda^{-1}(x-a)} \right)_{k^0=\omega'_{\mathbf{k}}}. \end{aligned} \quad (2.97)$$

In the second integral we can perform a transformation $k' = \Lambda^{-1}k$ and taking into account that Λ^{-1} is also a proper Lorentz transformation one has $(\Lambda^{-1}k) \cdot (\Lambda^{-1}x) = kx$, therefore we find:

$$\begin{aligned} & \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} (U^\dagger(\Lambda, a) a(\mathbf{k}) U(\Lambda, a) e^{-ikx} \\ & + U^\dagger(\Lambda, a) a^\dagger(\mathbf{k}) U(\Lambda, a) e^{ikx})_{k^0=\omega_{\mathbf{k}}} = \\ & \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a(\Lambda^{-1}\mathbf{k}) e^{-ik(x-a)} + a(\Lambda^{-1}\mathbf{k})^\dagger e^{ik(x-a)} \right)_{k^0=\omega_{\mathbf{k}}}, \end{aligned} \quad (2.98)$$

and by comparison of the two members of the last equation we find

$$U^\dagger(\Lambda, a) a^\dagger(\mathbf{k}) U(\Lambda, a) = a^\dagger(\Lambda^{-1}\mathbf{k}) e^{-ika}. \quad (2.99)$$

We want to evaluate (2.88), therefore we consider that $U(\Lambda, a) = U^{-1}(\Lambda^{-1}, -a)$:

$$\begin{aligned} U(\Lambda, a) a^\dagger(\mathbf{k}) U^\dagger(\Lambda, a) &= \\ U^\dagger(\Lambda^{-1}, -a) a^\dagger(\mathbf{k}) U(\Lambda^{-1}, -a) &= a^\dagger(\Lambda\mathbf{k}) e^{ika} \end{aligned} \quad (2.100)$$

where the last passage has been performed using (2.99) with $\Lambda \rightarrow \Lambda^{-1}$ and $a \rightarrow -a$; our final result is:

$$U(\Lambda, a) a^\dagger(\mathbf{k}) U^\dagger(\Lambda, a) = a^\dagger(\Lambda\mathbf{k}) e^{ika} \quad (2.101)$$

which allows us to transform a generic state under an operation of the Poincaré group.

Note that the translation is a particular case of this one, when $\Lambda = 1$ (to verify this use (2.83) plus the equations (2.53) and (2.54)).

2.3 Propagator

2.3.1 Classical point of view

As done up to now, we start from the classical case; we introduce in the Lagrangian a source term:

$$\mathcal{L}(\varphi(x), \partial_\mu \varphi(x)) = \frac{1}{2} (\partial_\mu \varphi(x)) (\partial^\mu \varphi(x)) - \frac{1}{2} m^2 \varphi^2(x) - J(x) \varphi(x) \quad (2.102)$$

where $J(x)$ is a given function. Of course, in nature the function J is connected with other fields, interacting with φ ; considering J as a given function is an approximation, indeed a very useful one to study some characteristics of the problem. If J were simply a given function we would immediately violate translational invariance; it is somehow analogous to the introduction of a potential term in quantum mechanics, which immediately breaks translational invariance, but as known this potential term is nothing else but an approximation of interactions with other particles.

After having done these preliminary considerations, we consider the equation of motion (see equation (2.5)):

$$(\square + m^2) \varphi(x) = -J(x); \quad (2.103)$$

we can write the solution like

$$\varphi(x) = \varphi_0(x) + \int d^4y G(x-y) J(y) \quad (2.104)$$

where $\varphi_0(x)$ is a solution of the free K-G equation, and $G(x - y)$ is given from

$$(\square_x + m^2) G(x - y) = -\delta^4(x - y), \quad (2.105)$$

therefore being a solution of the interacting K-G equation in presence of a delta-source at the point y , which is a parameter, while x is the dynamical variable of the equation. We can see that, apart from the point $x = y$, $G(x - y)$ is also a solution of the free K-G eq.

Writing $G(x - y)$ like

$$G(x - y) = \int \frac{d^4q}{(2\pi)^4} (-iG(q^2)) e^{-iq(x-y)} \quad (2.106)$$

(we introduced the $-i$ for future convenience) and plugging in (2.105) one gets

$$G(q) = \frac{i}{q^2 - m^2}. \quad (2.107)$$

Therefore:

$$G(x - y) = \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m^2} e^{-iq(x-y)} \quad (2.108)$$

Everything seems fine, but there is indeed a problem; performing the integration $\int_{-\infty}^{\infty} dq^0$ we realize that there are 2 singularities on the integration path, for $q^0 = \pm\sqrt{\mathbf{q}^2 + m^2}$; a prescription how to go around these singularities must be given. There are four ways how to go around them, and it is not clear which choice should be performed; only QFT can "decide". We give now some heuristic arguments lurking in QFT. If we have propagation of quanta, we can think that a particle is created in y from the delta-potential; we have 2 possibilities (see Fig. 2.1) :

- $x^0 > y^0$: the particle is created in y and propagates in the future to x ; it must be a positive energy state, so the picked up mode should be like $\exp[-iE(x^0 - y^0)]$ with $E = \sqrt{\mathbf{q}^2 + m^2}$ (think of the operator $i\partial/\partial x^0$ in QM acting on this mode, than you get a positive energy E). This means that we have to pick up the pole for $q^0 = \sqrt{\mathbf{q}^2 + m^2}$ and close the integral in the lower half-plane. In fact, on the imaginary axis of the lower half plane the integrand converges, thus allowing the application of the residue theorem [58, 63, 64].

- $x^0 < y^0$: the particle is created in y and propagates in the past to x . In this case negative modes should be the solution, in such a way that a negative energy state propagating backward in time is perceived as a positive energy state propagating forward in time. We therefore expect that in this case modes like $\exp[iE(x^0 - y^0)]$ should be picked up (in this case $i\partial/\partial x^0$ picks up a negative energy state $-E$). Note that if there were other quantum numbers (like the charge), they would appear reversed. We then would speak of an antiparticle propagation; in the case under present study particles and antiparticles coincide). In this case we close "up", in the upper half plane, where an application of the residue theorem is possible.

The other two possibilities (negative energy forward in time and positive energy backward) are ruled out because the theory doesn't have to contain negative energy states (condition (2.59)).

If we perform the integration following the Feynman path and closing the path "down" and "up" respectively, as described above, we get:

$$G_F(x - y) = \int_{C_F} \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m^2} e^{-iq(x-y)}$$

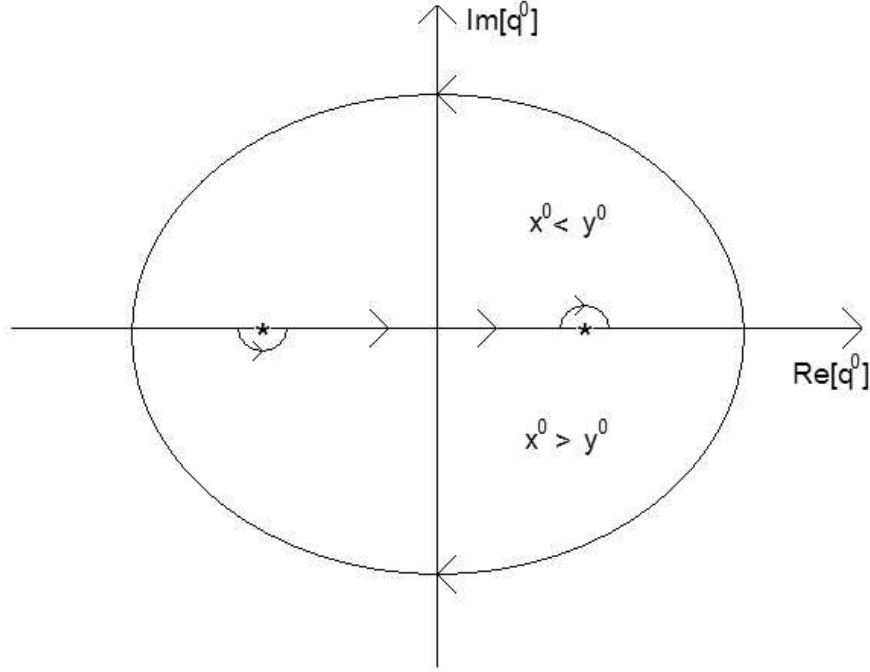


Figure 2.1: Integration contour.

$$\begin{aligned}
 &= \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 - m^2 + i\varepsilon} e^{-iq(x-y)} = \\
 &-i \int \frac{d^3 q}{(2\pi)^3 2E} \left(\vartheta(x^0 - y^0) e^{-iq(x-y)} + \vartheta(y^0 - x^0) e^{iq(x-y)} \right)_{q^0 = E = \sqrt{\mathbf{q}^2 + m^2}} \quad (2.109)
 \end{aligned}$$

As we can see, following this path (mathematically analogous to the ε prescription, see second integral) positive energies are picked up for $x^0 > y^0$ and negative for $x^0 < y^0$, according to our interpretation.

This propagator becomes very important in QFT, being the basis of perturbative calculations.

The given interpretation is the Feynman's one. It is not unique. The "stuff" with the propagation backward in time may appear weird. Consider again the case $x^0 < y^0$. We note that we can interpret x as the parameter and y as dynamical variable. G is in fact a solution of the following equation as well:

$$(\square_y + m^2) G(y - x) = -\delta^4(y - x). \quad (2.110)$$

We can then think that the particle is created in x and propagates to y (which is now in the future....). Under this point of view a positive energy solution is $\exp[-iq^0(y^0 - x^0)] = \exp[iq^0(x^0 - y^0)]$, exactly as found with the Feynman interpretation.

Now, what happens if, instead of the term $J(x)\varphi(x)$ I have a function of the field $u(\varphi(x))$? The equation of motion looks like

$$(\square + m^2) \varphi(x) = -\frac{\partial u(\varphi(x))}{\partial \varphi(x)}. \quad (2.111)$$

We can still write a generic solution in the form

$$\varphi(x) = \varphi_0(x) - \int d^4y G(x-y) \frac{\partial u(\varphi(y))}{\partial \varphi(y)}, \quad (2.112)$$

but the situation is much more complicated: in fact, $\partial u(\varphi(y))/\partial \varphi(y)$ is a function of φ , so it means that the field φ is outside and inside the integral. We have an integral equation of the second kind of the Fredholm type in 4 dimensions [63].

There is not a general solution of this problem; one way is to "hope" that the free solution is a "good starting point" and that one actually has $\varphi(x) \approx \varphi_0(x)$, then treating the integral term as a small perturbation, writing an expansion series.

There is no guaranty that it works: if you have a better guess, guess!

Such a possibility depends clearly also on the "strength" of the interaction; one may hope that, if it is small, than it might be valid.

These considerations already show the difficulty of the problem when interacting fields are taken into account.

2.3.2 QFT point of view

In *QFT* one considers the following object

$$\langle 0 | T [\widehat{\varphi}(x) \widehat{\varphi}(y)] | 0 \rangle \quad (2.113)$$

where T is the time ordering operator:

- $T [\widehat{\varphi}(x) \widehat{\varphi}(y)] = \widehat{\varphi}(x) \widehat{\varphi}(y)$ if $x^0 > y^0$;
- $T [\widehat{\varphi}(x) \widehat{\varphi}(y)] = \widehat{\varphi}(y) \widehat{\varphi}(x)$ if $x^0 < y^0$.

Eq (2.113) is an example of a Green function, i.e. a vacuum mean value of time-ordered field operators; the physical meaning of T is "*always create before annihilate*" : the field whose time component comes later is on the left. For example, if $x^0 > y^0$ I have $\varphi(x)\varphi(y) | 0 \rangle$, which means that first I create a particle at the time y^0 in \mathbf{y} and then I annihilate at the time $x^0 > y^0$ in \mathbf{x} . This choice of T assures that we are considering the propagation of positive-energy objects and it is the *QFT* counterpart of the discussion in the previous subsection. When we evaluate (2.113) we realize that this quantity is the field propagator as calculated in the previous subsection:

$$\begin{aligned} \langle 0 | T [\widehat{\varphi}(x) \widehat{\varphi}(y)] | 0 \rangle &= iG(x-y) = \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\varepsilon} e^{-iq(x-y)} \\ &= \int \frac{d^3q}{(2\pi)^3 2E} \left(\vartheta(x^0 - y^0) e^{-iq(x-y)} - \vartheta(y^0 - x^0) e^{iq(x-y)} \right)_{q^0 = E = \sqrt{\mathbf{q}^2 + m^2}} \end{aligned} \quad (2.114)$$

In fact, if $x^0 > y^0$, we are considering the amplitude for a particle created in y and propagating to x and vice-versa if $x^0 < y^0$, exactly as explained in detail before.

In general in an interacting *QFT* one encounters not only two but n-point Green functions of the form

$$G(x_1, x_2, \dots, x_n) = \langle 0 | T [\widehat{\varphi}(x_1) \widehat{\varphi}(x_2) \dots \widehat{\varphi}(x_n)] | 0 \rangle. \quad (2.115)$$

2.4 Interacting fields

2.4.1 Introduction

We now consider a Lagrangian with an interaction term $u(\varphi)$

$$\mathcal{L} = \frac{1}{2} ((\partial_\mu \varphi)^2 - m^2 \varphi^2) - u(\varphi); \quad (2.116)$$

as a typical example we consider [3, 56, 57]

$$u(\varphi) = \frac{\lambda}{4!} \varphi^4. \quad (2.117)$$

We mentioned the problem in section 2.3.1 studying the propagator. The issue is very complicated, and no general solution to the problem is known, even in this simple scalar version with a quartic interaction term. One way out is to hope that a perturbative expansion works, so that one can start from the free solutions and then, order by order in the parameter λ , calculate the corrections due to the interaction. Such a procedure is normally successful for scattering or decay of particles, but not for the formation of bound states, which is a typical nonperturbative phenomenon.

As in the free case we start in the Heisenberg picture with the field equation of motion (2.5) for the quantized field:

$$(\square + m^2) \widehat{\varphi}(x) + \frac{\partial u(\widehat{\varphi})}{\partial \widehat{\varphi}} = 0, \quad (2.118)$$

which is now much more complicated. No general solutions are known in this case, not even for the relatively simple φ^4 interaction theory.

We then introduce the equal time quantization rules as in section 2.2

$$[\widehat{\varphi}(t, \mathbf{x}), \widehat{\varphi}(t, \mathbf{y})] = 0, \quad \left[\widehat{\varphi}(t, \mathbf{x}), \frac{\partial \widehat{\varphi}(t, \mathbf{y})}{\partial t} \right] = i \delta^3(\mathbf{x} - \mathbf{y}) \quad (2.119)$$

It is not so difficult to prove that (try to do it), as a consequence of the commutation relations, the Heisenberg equation holds:

$$\partial_\mu \widehat{\varphi}(x) = i \left[\widehat{P}_\mu, \widehat{\varphi}(x) \right] \quad (2.120)$$

where the \widehat{P}_μ are given from (2.8).

The Heisenberg equation can be generalized to a generic functional of the field and its derivatives $\widehat{A} = A(\widehat{\varphi}, \partial_\nu \widehat{\varphi})$ with

$$\partial_\mu \widehat{A} = i \left[\widehat{P}_\mu, \widehat{A} \right]. \quad (2.121)$$

The problem now is that we can't solve it. In order to "save" all the work done in the free case we will consider the interaction picture, where the field equations of motion are the same as in the free case.

Let us first write $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ and $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ where $\mathcal{H}_1 = -\mathcal{L}_1 = u(\widehat{\varphi})$; an operator in the Schrodinger and then in the interaction picture looks like:

- $\widehat{A}^S = e^{-iHt} \widehat{A} e^{iHt}$,

- $\widehat{A}^I = e^{iH_0^S t} \widehat{A}^S e^{-iH_0^S t}$.

From the first definition it follows that

$$\frac{\partial \widehat{A}^S}{\partial t} = 0. \quad (2.122)$$

Furthermore, from the second definition, considering that $\widehat{P}_k = \widehat{P}_{0,k} = \widehat{P}_{0,k}^S = \widehat{P}_{0,k}^I$ ($k = 1, 2, 3$ and all of them are time independent), we have

$$\partial_\mu \widehat{A}^I(x) = i \left[\widehat{P}_{0,\mu}^I, \widehat{A}^I(x) \right], \quad (2.123)$$

i.e. the fields in the interaction picture fulfill the *free* equation of motion; this means that one has

$$(\square + m^2) \widehat{\varphi}^I(x) = 0. \quad (2.124)$$

From the definitions of the operators in the interaction picture follows that the canonical commutation relation holds:

$$\left[\widehat{\varphi}^I(t, \mathbf{x}), \frac{\partial \widehat{\varphi}^I(t, \mathbf{y})}{\partial t} \right] = i\delta(\mathbf{x} - \mathbf{y}). \quad (2.125)$$

The last two relations show that now I can treat $\widehat{\varphi}^I(x)$ exactly as in the free case I did with $\widehat{\varphi}(x)$; one has the same expansion in terms of free waves, together with the same commutation relation. But the question is: where is now the interaction? Where has it gone? If we change the fields, we have also to change the states; we have in our case

$$|s(t)\rangle^I = e^{-iH_0^S t} |s(t)\rangle^S \quad (2.126)$$

from which one gets

$$i \frac{d |s(t)\rangle^I}{dt} = H_1^I |s(t)\rangle^I \quad (2.127)$$

meaning that the interaction term affects now the time dependence of the states. We consider the time evolution operator $U(t, t_0)$ such that

$$|s(t)\rangle^I = U(t, t_0) |s(t_0)\rangle^I; \quad (2.128)$$

we then consider the limit $t \rightarrow \infty$ and $t_0 \rightarrow -\infty$ and define the so called S matrix like

$$S = U(\infty, -\infty). \quad (2.129)$$

If we suppose that the interaction is switched off in the very past and in the very future, we have an asymptotic initial state $|s_{in}\rangle^I = |s_{in}\rangle^S = |s_{in}\rangle^H$ and an asymptotic final state $|s_{fin}\rangle^I = |s_{fin}\rangle^S = |s_{fin}\rangle^H$.

The amplitude for the transition $|s_{in}\rangle \rightarrow |s_{fin}\rangle$ is then given by the matrix element $\langle s_{fin} | S | s_{in} \rangle$; one can show [3, 54, 58, 60] that the explicit form for S is the following:

$$S = T e^{-i \int d^4x \mathcal{H}_1^I(x)}, \quad (2.130)$$

where T is the time ordering operator as described previously; it appears naturally by evaluating this quantity. This expression allows an expansion in terms of the (assumed) small parameter λ , that appears in the interaction term:

$$S = \sum_{n=0}^{\infty} S^{(n)} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \dots d^4x_n T[\mathcal{H}_1^I(x_1) \dots \mathcal{H}_1^I(x_n)] \quad (2.131)$$

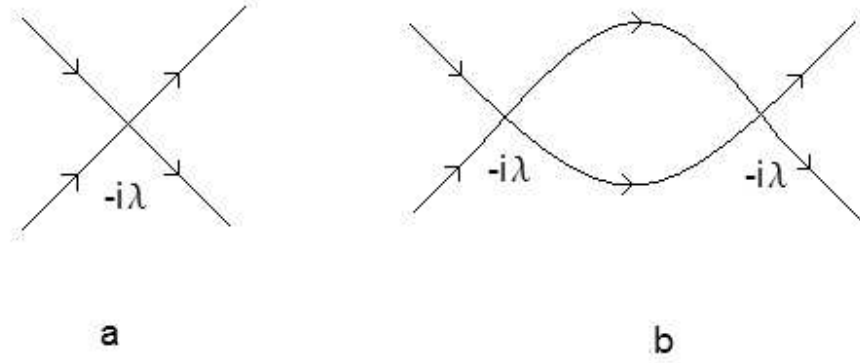


Figure 2.2: First and second order diagrams for two-body scattering.

For example, analyzing the n -th order $S^{(n)}$ one can reduce, through the use of the Wick theorem, the calculation to the evaluation of two-point correlation functions, i.e. the propagator:

$$\langle 0 | T [\varphi^I(x) \varphi^I(y)] | 0 \rangle = G(x - y); \quad (2.132)$$

this is exactly the quantity evaluated previously; one can then appreciate the power of the interaction picture, otherwise we could not evaluate this quantity. In the following we will omit the index I , but it is understood that the operators are expressed in this way.

Moreover, it is well known that one can find a one to one mathematical correspondence of the different orders of the S matrix with a graphical representation called Feynman diagrams; to each piece of a diagram one can associate a mathematical quantity (Feynman rules) and then write down the amplitude (at a given order) just looking at the diagrams corresponding to that order.

2.4.2 Lowest order

The diagram at the lowest order is depicted in Fig. 2.2.a. It represents the scattering of two particles. We have four external legs, because our Lagrangian is quartic in the field φ .

The initial and the final states are

$$|in\rangle = a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger |0\rangle, \quad |fin\rangle = a_{\mathbf{p}_3}^\dagger a_{\mathbf{p}_4}^\dagger |0\rangle. \quad (2.133)$$

The S matrix at the first order looks like

$$S^{(1)} = -i\lambda \int d^4x T[\varphi^4(x)]. \quad (2.134)$$

Making use of (2.17) we find

$$\langle fin | S^{(1)} | in \rangle = \frac{(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)}{(\sqrt{V})^4 \sqrt{2\omega_{\mathbf{p}_1} 2\omega_{\mathbf{p}_2} 2\omega_{\mathbf{p}_3} 2\omega_{\mathbf{p}_4}}} (-iM_1) \quad (2.135)$$

where the invariant amplitude $-iM_1$ is given by

$$-iM_1 = -i\lambda. \quad (2.136)$$

It is the contribution of the vertex, and it is simply the coupling constant. The combinatorial factor $4!$ has been "eaten" by the different combination arising from the full expression of the time product T . The fact that $M \simeq M_1 = \lambda > 0$ means that the interaction is repulsive, as one can show following the arguments of [3].

Note that the other possibility with $\lambda < 0$, for which one has attraction between two particles, would generate an interaction "potential" $u(\varphi) = \frac{\lambda}{4!}\varphi^4$ without a minimum; the theory would then be unstable [3]. As we will see, such a case will be very useful in the framework of an effective theory.

If we then consider the graph at the second order Fig. 2.2.b we have

$$-iM_2 = -i\frac{1}{2}\lambda^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q+p/2)^2 - m^2} \frac{1}{(q-p/2)^2 - m^2}. \quad (2.137)$$

where $p = p_1 + p_2$. We have a λ for each vertex, plus the two propagators of the internal lines and an extra-integration over q , because this momentum is not fixed. The factor $1/2$ in front is a symmetry factor arising when one properly considers the contractions. We come back later to these diagrams.

2.4.3 Mass shift

We expect that some properties of the theory are modified by the interaction; for instance, we expect that the mass changes. In fact, the mass is given from the pole of the two point correlation function, whose first and second order diagrams are depicted in Fig. 2.3. These diagrams modify the propagator. The "new" mass is the pole of the modified propagator. The mass shift is a property that one cannot see at a finite order. We can formally calculate the full modified propagator by considering the most general proper diagram (which cannot be cut into two distinct pieces), whose amplitude is denoted as $-i\Sigma$, and by performing the sum up to order ∞ as in the figure 2.4, getting a geometrical series, whose result is:

$$G(q^2) = \frac{i}{q^2 - m^2 - \Sigma(p^2)}. \quad (2.138)$$

The "new" mass is a pole of this function; we call it m_P with the subscript P as "physical" because this is the physical mass which is measured in an experiment; in fact we cannot switch off the interaction to measure the bare one.

The equation for m_P is then

$$m_P^2 - m^2 - \Sigma(m_P^2) = 0; \quad (2.139)$$

we can express m_P as a function of m through a series in λ . At first order in λ we can approximate $-i\Sigma$ with the term in Fig. 2.3.a, which gives rise to a momentum independent correction:

$$m_P^2 = m^2 + \lambda \int \frac{d^4q_E}{(2\pi)^4} \frac{1}{q_E^2 + m^2}, \quad (2.140)$$

where in the second integral we have performed a Wick rotation. We immediately realize that something goes wrong, because the integral is divergent. But we can do some physical

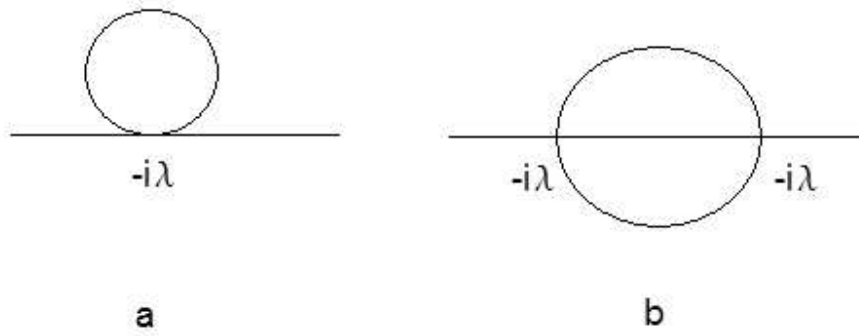


Figure 2.3: Self interaction at first and second order.

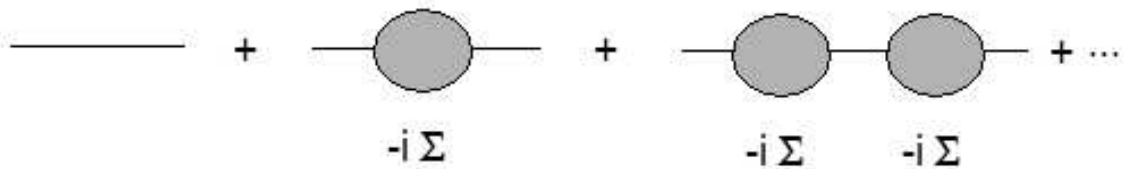


Figure 2.4: Schematic sum of all proper diagrams.

consideration [60] : the divergence occurs because very high momenta are allowed to circulate in the loop. Very high momenta mean very short distances and high energies, even above the Planck scale, where we do not expect such a quantum field theory in flat Minkowski space to be valid. There must be "something" that doesn't allow such big momenta to propagate, something we do not know. Let us introduce a cut-off Λ in the integral in order to make it finite:

$$\int \frac{d^4 q_E}{(2\pi)^4} \rightarrow \int_{\Lambda} \frac{d^4 q_E}{(2\pi)^4} \quad (2.141)$$

we can think of an hard cut-off, but also of anything that at the energy scale of the (unknown) Λ cuts the contribution of the high momenta in some way. The precise choice of this "cutting" takes the name of regularization, but we are here not too much concerned with it.

In this way m_P is finite, and, although Λ is large, we suppose that λ is small enough to allow a perturbative expansion. To this order we have a quadratic dependence on Λ , thus schematically $m_P = m + C\lambda\Lambda^2$ where C is some number depending on the evaluation of the integral and on the details of the adopted regularization scheme [60].

The second order correction comes from the contribution depicted in Fig. 2.3.b, which is now momentum dependent.

More formally, expanding $\Sigma(p^2)$ like

$$\Sigma(p^2) = \Sigma(m_P^2) + (p^2 - m_P^2)\Sigma'(m_P^2) + \tilde{\Sigma}(p^2) \quad (2.142)$$

from which we have

$$G(q^2) = \frac{iZ_2}{q^2 - m_P^2 - Z_2\tilde{\Sigma}(p^2)} \quad (2.143)$$

where $Z_2 = (1 - \Sigma'(m_P^2))^{-1}$ is a number which can be reabsorbed in the field, as we have already done in section 2.2. One may still be puzzled by the presence of the term $Z_2\tilde{\Sigma}(p^2)$ in the denominator; at second order we can write $\Sigma_2(p^2) = \lambda^2\Pi_2(p^2)$, therefore

$$Z_2\tilde{\Sigma}(p^2) = (\lambda^2/(1 - \lambda^2\Pi_2'(m_P^2)))\tilde{\Pi}_2(p^2) \simeq \lambda^2\tilde{\Pi}_2(p^2) = \tilde{\Sigma}_2(p^2). \quad (2.144)$$

We then note that, because of the derivatives, $\tilde{\Sigma}_2(p^2)$ is Λ independent, or better, dependent like $1/\Lambda + 1/\Lambda^2 + \dots$, that can be omitted because Λ is very large.

The final result is that the propagator is for $q^2 \simeq m_P^2$

$$G(q^2) = \frac{1}{q^2 - m_P^2}(1 + \text{corrections}) \quad (2.145)$$

where the corrections do not depend on Λ .

If we express all the physical results as function of m_P , what we should do, there is no presence of Λ coming from the mass shift, because it is reabsorbed in the physical mass. The point is whether one can do it at all orders and for all possible Green functions, i.e. for all physical processes. This is not possible in general, but only in the case of so-called renormalizable theories. The quartic interaction term we are considering belongs to this kind.

2.4.4 Scattering

Let us now consider the scattering of two particles. At lowest order the amplitude is simply $-iM = -i\lambda$. At second order one has the loop diagram of Fig. 2.2.b, which again contains an ultraviolet infinity, or better, depends logarithmically on the cutoff Λ .

$$-iM_2 = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q+p/2)^2 - m^2} \frac{1}{(q-p/2)^2 - m^2} \quad (2.146)$$

We are not interested now in the evaluation of the loop integral, but we can express it in terms of the kinematic variables $s = (k_1 + k_2)^2$, $t = (k_1 - k_3)^2$ and $u = (k_1 - k_4)^2$. One gets [60]

$$-iM = -i\lambda + iC\lambda^2 \left[\log\left(\frac{\Lambda^2}{s}\right) + \log\left(\frac{\Lambda^2}{t}\right) + \log\left(\frac{\Lambda^2}{u}\right) \right] = -i\lambda + iC\lambda^2 L \quad (2.147)$$

where C is again some finite number.

Now, the bare coupling is not measurable, exactly as the bare mass m . We should express the result in terms of measurable quantities. Let us introduce a physical coupling λ_P defining it as the cross section for the two body scattering at some given values for (s_0, t_0, u_0) . At second order we have

$$-i\lambda_P = -i\lambda + iC\lambda^2 L_0 \quad (2.148)$$

If now I want to evaluate M in terms of λ_P instead of λ I can simply invert the last equation getting $-i\lambda = -i\lambda_P - iC\lambda_P^2 L_0$ (this is valid at the second order). We find therefore:

$$-iM = -i\lambda_P + iC\lambda_P^2 (L - L_0) \quad (2.149)$$

where the quantity $(L - L_0)$ is Λ independent:

$$(L - L_0) = \left[\log\left(\frac{s_0}{s}\right) + \log\left(\frac{t_0}{t}\right) + \log\left(\frac{u_0}{u}\right) \right]. \quad (2.150)$$

Again, we reabsorb the cutoff dependence of the amplitudes in the physical coupling λ_P . Measuring it, I then can calculate the amplitude at any other given scattering angle and energy.

For pure convenience one often sets s_0, t_0 and u_0 to μ^2 , and when I calculate a physical quantity I get an explicit dependence on it. One may then ask: what is the meaning of it? I got rid of Λ to get another extra-parameter μ , which was not present in the Lagrangian.

The point is that the choice of μ is absolutely arbitrary, and I could chose any. I can consider λ_P as a function of the parameter μ :

$$\lambda_P(\mu) = \lambda - 3C\lambda^2 \log\left(\frac{\Lambda^2}{\mu^2}\right). \quad (2.151)$$

The choice of μ depends on the energy range I'm working in, and it is astute to take it within this range; in this way in the amplitude

$$-iM = -i\lambda_P(\mu) + iC\lambda_P^2(\mu) \left[\log\left(\frac{\mu^2}{s}\right) + \log\left(\frac{\mu^2}{t}\right) + \log\left(\frac{\mu^2}{u}\right) \right], \quad (2.152)$$

there are objects like $\log(\mu^2/s)$, which are small for s not much different from μ^2 . If we were so stupid to take μ very large, the second order correction would increase, and be comparable to the first term, if not bigger.

Finally, we impose that the amplitude M doesn't depend on μ : $dM/d\mu = 0$, from which we find the equation (always at the second order in the coupling constant):

$$\mu \frac{d\lambda_P(\mu)}{d\mu} = 6C\lambda_P^2(\mu). \quad (2.153)$$

This is the subject of the so called renormalization group theory [3, 16, 60], which studies the μ dependence of the coupling constant. Here we are just describing some preliminary and simple features of it, showing that it comes naturally from the definition of the physical coupling constant (also called the renormalized one) in order to get rid of the cutoff dependence.

In the end the physical quantities depend only on m_P and λ_P (for a given μ) and we do not have an explicit appearance of the cutoff.

So, there is a cutoff, but the physical processes, expressed in terms of the physical mass and coupling constant, DO NOT depend on it.

This is the main point of "*renormalization*".

One now can ask: "Fine, you do not have Λ in the amplitudes because you have redefined the mass and the coupling, i.e. the two and the four point correlation functions. But what happens if one considers the 6 point Green function, i.e. the scattering of three particles? Are you sure that no other quadratic or logarithmic dependence on Λ would appear again, but without any other "physical" quantity to redefine? (The two you had are already taken)"

Well, this is exactly the difference between renormalizable and unrenormalizable theories. If, after having introduced m_P and λ_P (and rescaled the field), I do not have other direct dependencies on Λ , than the theory is said to be renormalizable. If, on the contrary, other "infinities" appear, the theory is not, and a direct dependence on the cutoff is unavoidable. This may seem puzzling because we do not know how the cutoff comes into the game. The Lagrangian itself doesn't tell us how one should cut the high momenta, and an infinite number of choices is possible, then raising doubts on the predictive power of such a theory.

Non-renormalizable theories are indeed very useful if one can interpret physically the cutoff, which then is a number on which the results depend explicitly.

In the end, the so called "power counting scheme" [16, 55, 60] tells us how to recognize if a theory is renormalizable or not. One has to check the dimensions of the coupling constant, and if it has zero or positive dimensions then the theory is renormalizable; on the other hand if it is negative it is not. In our case λ is dimensionless, therefore the theory is renormalizable.

2.5 Phenomenology

2.5.1 Effective theory: an example

We can look at the problem from another point of view; we have seen that our φ^4 theory is renormalizable, but we could interpret it as a low-energy effective theory of some other field theory. In this case we do not need to renormalize the φ^4 theory. We consider the cutoff as a parameter of the theory, whose physical meaning is connected to some other field theory. We illustrate it by a simple example [60]. Let us consider the following Lagrangian with two

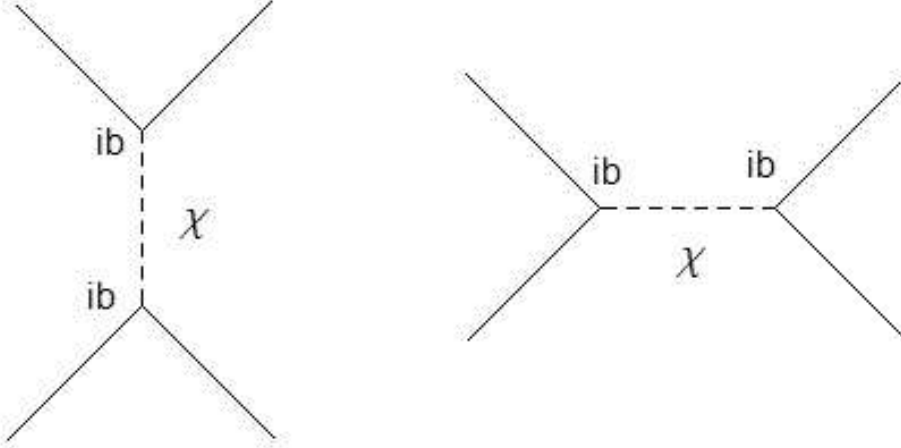


Figure 2.5: Two-body scattering by a boson exchange.

fields, χ and φ .

$$\mathcal{L} = \frac{1}{2} ((\partial_\mu \varphi)^2 - m^2 \varphi^2) + \frac{1}{2} ((\partial_\mu \chi)^2 - m_\chi^2 \chi^2) + b \chi \varphi^2. \quad (2.154)$$

The interaction term is of third order, the coupling b has the dimension of an energy, therefore the theory is renormalizable. Now, imagine that we are interested in the evaluation of the scattering of 2 " φ ". At lowest order we have the diagrams of Fig. 2.5.

Let us suppose that the mass of the particle χ is large, larger than the exchanged momentum. We can therefore approximate

$$\frac{i}{p^2 - m_\chi^2} \simeq -\frac{i}{m_\chi^2}. \quad (2.155)$$

We realize that we would obtain the same results if we would consider the φ^4 Lagrangian with

$$i\lambda = -i \frac{b^2}{m_\chi^2} 12 \quad (2.156)$$

where 12 is just a combinatorial factor (see later). A quartic scalar theory with

$$\lambda = -\frac{b^2}{m_\chi^2} 12 \quad (2.157)$$

is a "low energy effective theory" of the previous one, which is valid for small momenta of the particles participating in the scattering. We realize that in this case it would not make

sense to consider a bubble as we have done in the previous section. In fact, for energies larger than m_χ^2 one cannot ignore the full form of the propagator of the particle χ . The cutoff in this case is of the order of m_χ , and means that new physics appears at this energy scale.

Furthermore, the coupling λ turns out to be negative, which on one hand means that the χ exchange generates an attractive interaction among 2φ , and on the other hand that a fundamental theory with $\lambda < 0$ is impossible, because, as noted previously, it is unstable.

The main message is that the same φ^4 theory can be considered as a renormalizable theory, in the sense that the results do not depend on the cutoff (although there is one, but we don't know its value) or can be thought of as an effective theory, as in the last example, where we do not consider any redefinition of the mass and coupling. If a theory is not renormalizable the only possible use is to consider it as an effective one.

This simple example shows us that also a theory with a cut-off can be useful, being the approximation, a low-energy limit, of some other field theory.

2.5.2 Cutoff function

In the evaluation of the bubble in section 2.4.4 we introduced some not specified regularization on the integrals. Let us consider again the "bubble" where we introduce a function $\tilde{\Phi}^2(q)$ which decreases for high (Euclidean) momenta in order to get a finite integral:

$$\frac{1}{2}(-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q+p/2)^2 - m^2} \frac{1}{(q-p/2)^2 - m^2} \tilde{\Phi}^2(q). \quad (2.158)$$

The meaning of $\tilde{\Phi}^2$ will become clear later on. One can choose a step function like $\tilde{\Phi}(q_E) = \theta(\Lambda^2 - q_E^2)$, or a gaussian $\exp(-q_E^2/\Lambda^2)$, etc. We do not employ a renormalization, but we consider the result as dependent on the parameter Λ , exactly as it depends on m and λ . One then has to fix Λ by comparison with experimental values.

Many questions can arise. Nobody tells us which function one should use to cut the integrals. We have an infinite number of choices, and this is why such an approach was first regarded with care, because it depends on the cutoff function. As said before, we can do it, if we can justify the physical meaning of this function. Our aim is to show that such a cutoff function can be useful to describe a bound state formed by 2 particles φ . If the particles compose a bound object, the momenta larger than a certain value, let's say Λ , do not contribute, because the momentum space wave function goes to zero for large momenta. First, we can have a bound state only if the interaction is attractive; to this end we consider the Lagrangian

$$\mathcal{L} = \frac{1}{2} ((\partial_\mu \varphi)^2 - m^2 \varphi^2) + \frac{K}{2} \varphi^4 \quad (2.159)$$

where $K > 0$ (K instead of λ for the coupling in order to distinguish this approach from the previous one) and so we have opposite sign with respect to the previous case, thus inducing an attraction between the bosons. Here we do not introduce a factor 4! but only a factor 2 for future simplicity. We will not be interested in the two body scattering, but on the formation of a bound state.

The Lagrangian (2.159) is incomplete, because it comes without the cutoff function. If we calculate the bubble contribution, the cut-off function doesn't appear and the "bubble" diverges.

We can render the Lagrangian complete and finite by introducing the cutoff function in it, modifying the interaction term by a nonlocal extension:

$$\mathcal{L}_1 = \frac{K}{2}\varphi^4 \rightarrow \frac{K}{2} \left(\int d^4y \varphi(x+y/2)\Phi(y)\varphi(x-y/2) \right)^2 = \frac{K}{2}J^2 \quad (2.160)$$

where

$$J = \int d^4y \varphi(x+y/2)\Phi(y)\varphi(x-y/2), \quad (2.161)$$

thus giving a total Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 = \frac{1}{2} ((\partial_\mu\varphi)^2 - m^2\varphi^2) + \frac{K}{2}J^2. \quad (2.162)$$

In this case, by making the calculation of the bubble, we get the result of (2.158), where $\tilde{\Phi}(q)$ is the (four dimensional) Fourier transform of $\Phi(y)$. The Feynman rule connected to the vertex is modified by the nonlocal extension. If we consider 2-body scattering we get

$$\frac{4!}{2}iK \rightarrow \frac{4!}{2}iK\tilde{\Phi}\left(\frac{p_1-p_2}{2}\right)\tilde{\Phi}\left(\frac{p_3-p_4}{2}\right), \quad (2.163)$$

i.e. a function of the relative four-momenta.

Note that now we have $+iK$ while before we had $-i\lambda$. We also understand why in the bubble one gets the $\tilde{\Phi}^2(q)$, just as a consequence of the Feynman rules. The scattering amplitude comes "together with the vertex functions in momentum space, i.e. with the amplitudes that the particles have such a relative momentum". It is then tempting to interpret $\tilde{\Phi}(q)$ as the wave function in momentum space, but care should be used. $\tilde{\Phi}(q)$ depends on the four-momentum and not only on the three-dimensional one, it is not clear how to define a classical wave function in such a context. We will come back to this point.

Five comments are still in order:

- This is just one of the possible ways to introduce a nonlocal extension for the regularization of the loops. Others can also be introduced [65, 66].
- In the literature one often considers a local Lagrangian form, and then introduces later the cutoff. The cutoff function is often the step one (in the four or the three momentum [23]) like in some applications of the *NJL* model. Although the first Lagrangian is local, one is actually dealing with a nonlocal approach, even if this is not explicit.
- If we are in the rest frame of the two body scattering, we have $E_1 = E_2$ and $E_3 = E_4$, and therefore in the amplitudes only the relative three-momentum appears, but in general also the energy component (i.e. the temporal one in space-time) appears.
- One may worry about the relativistic invariance of the Lagrangian (2.162). To ensure covariance one has to take a vertex function depending on y^2 , whose Fourier transform is $\tilde{\Phi}(q^2)$. In the non-relativistic limit it is also useful to consider non-covariant vertex functions.
- This kind of approach can be used as an effective tool to study bound states [29, 67, 68, 69, 70]. The "underlying theories" are then QED and QCD.

2.6 Mass of the bound state

The determination of bound states in the framework of a *QFT* is very complicated. One has to solve the so-called Bethe-Salpeter equation [24, 25, 71, 72]. In the non-relativistic limit

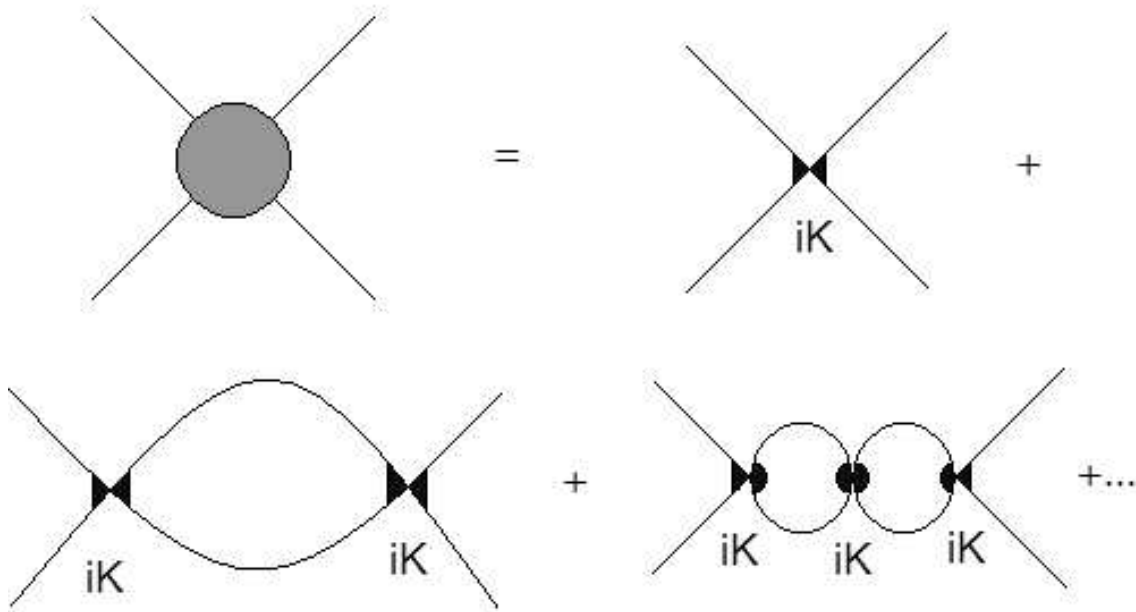


Figure 2.6: Two-body scattering; the shadowed part represents the vertex function.

there are some valid approximations, which allow to relate the Bethe-Salpeter equation to the classical Schroedinger one with a classical potential. Far from the non-relativistic limit no general solutions for the Bethe-Salpeter equation are known; one has to rely on some extra-assumptions, which are often not *a priori* justifiable: in *QCD* this kind of approach is a very hard task, and only relatively recently there were some attempts in this direction [24, 25].

In our low energy Lagrangian (2.162) the B-S equation looks very simple; it is a nice toy-model to understand some features of the problem.

Our goal is to relate the mass of the bound state (if it exists) to the parameters m , K and Λ . We just give some heuristic arguments to understand the problem, thus not discussing the full derivation of the B-S equation, for which we refer to the specialized works cited above.

In two-body scattering we sum the diagrams up to order infinity as in the figure 2.6.

As mentioned before, in order to study bound state properties one has to sum an infinite set of diagrams, and we do it considering only the bubbles. This is also a simplification: the consideration of other terms would complicate a lot the present discussion.

We now have to be careful with the symmetry factors; this is perhaps the only disadvantage of working with a quartic scalar theory, because one has always to take care of these pesky numbers. When considering the scattering in the most general case all possible combi-

nations must be taken into account. The formation of a bound state (if it exists) can occur among the first and second particle, or the first and the third, or the first and the fourth. Here we want to consider the formation of a bound state out of the two incoming particles, which have momenta p_1 and p_2 . The bound state momentum will be then given in this channel as $p = p_1 + p_2$. This means that we pick up only a channel among the 3 possible combinations. This is achieved by considering the normal Feynman rule plus

$$\frac{4!}{2}K \rightarrow \frac{4!}{2} \frac{K}{3} = 4K \quad (2.164)$$

because we take one channel out of three for each vertex. The first order is than

$$-iM_{1,channel1} = 2\tilde{\Phi}\left(\frac{p_1 - p_2}{2}\right)(iK)\tilde{\Phi}\left(\frac{p_3 - p_4}{2}\right)2 \quad (2.165)$$

where we factorized a factor 2 in the beginning and a factor 2 in the end. Such factors occur at every order because the initial and the final states are made out of two identical bosons. Now, let us go to the next order; we find

$$-iM_{2,channel1} = 2\tilde{\Phi}\left(\frac{p_1 - p_2}{2}\right) [(iK)\Sigma(p^2)(iK)] \tilde{\Phi}\left(\frac{p_3 - p_4}{2}\right)2 \quad (2.166)$$

where $\Sigma(p^2)$ is the bubble contribution:

$$\Sigma(p^2) = 2i \int \frac{d^4q}{(2\pi)^4} \frac{i}{(q + p/2)^2 - m^2} \frac{i}{(q - p/2)^2 - m^2} \tilde{\Phi}^2(q) \quad (2.167)$$

The extra factor 2 is, again, connected with the loop of two bosons.

Yes, I know, these numbers like 2, 4! and so on are pretty tedious and bothering, but if one wants to get the correct results has to take them into account.

We define the matrix T relative to the scattering in the chosen channel in the following way

$$-iM_{channel1} = -iM_{1,channel1} + -iM_{2,channel1} + \dots = \quad (2.168)$$

$$2\tilde{\Phi}\left(\frac{p_1 - p_2}{2}\right) (-iT(p^2)) \tilde{\Phi}\left(\frac{p_3 - p_4}{2}\right)2 \quad (2.169)$$

where

$$T(p^2) = -K + K\Sigma(p^2)K + \dots = -(1 - K\Sigma(p^2)^{-1})K \quad (2.170)$$

$$= -(K^{-1} - \Sigma(p^2))^{-1}. \quad (2.171)$$

In this case K and Σ are scalars, but we prefer to express them in a form which is valid when they become matrices. The occurrence of a bound state is then visible as a pole in the T matrix. In fact, it means that in this channel the two particles combine to form a bound state and the amplitude for $p^2 \simeq M^2$, where M is the bound state mass, will be proportional to $1/(p^2 - M^2)$, being of course dominant for $p^2 \rightarrow M^2$. The pole equation of the T matrix is [23, 73]:

$$K^{-1} - \Sigma(p^2 = M^2) = 0. \quad (2.172)$$

This is a case with a very simple mass equation, due to the fact that T depends on p^2 only. Furthermore we are paid back of all the efforts to get the correct symmetry factors. The mass equation looks very clean!

Such a simple example already shows some typical features of this kind of approach. In order to calculate $\Sigma(p^2)$, also called mass operator, one has to perform a Wick-rotation, finding (in the rest frame of the bound object $p = (\sqrt{p^2}, \mathbf{0})$):

$$\Sigma(p^2) = 2 \int \frac{d^4 q_E}{(2\pi)^4} \frac{\tilde{\Phi}^2(q_E^2)}{(q_E^2 + m^2 - p^2/4)^2 + (q^{(4)})^2 \cdot p^2}. \quad (2.173)$$

This quantity is positive, therefore we realize that the occurrence of a bound state is possible only if $K > 0$, as we correctly anticipated. Furthermore we note that it is not difficult to solve such an expression numerically. As one expects there is a pole in the integration if $M > 2m$. A bound state can occur only for $M < 2m$, otherwise there is no binding energy. The bound object cannot decay into its subcomponents. This would indeed be the case if $M > 2m$, with the appearance of imaginary part in the mass operator function, which corresponds to the on-shell production of both φ particles.

Note that $\Sigma(p^2)$, although evaluated in the rest frame, is valid in every reference frame because of the covariance of the theory, as long we employ a covariant vertex function $\Phi(y) = \Phi(y^2)$. If we employ a non-covariant vertex function this is no longer true. In this case the expression (2.173) is valid only in the rest frame with $p^2 = (p^{(0)})^2$. All the formulas remain unchanged, but they are strictly valid only in the rest frame for the bound state.

From figure 2.7, we realize that, the larger the coupling, the stronger is then the attraction and the binding energy, with a smaller mass for the bound state.

Now, if $(p_1 + p_2)^2 = p^2 \simeq M^2$ the dominant contribution to the two-body scattering is given from the bound state! Expanding $\Sigma(p^2)$ around M^2 , and using the mass-pole equation, we have

$$\begin{aligned} T(p^2) &= \frac{1}{\Sigma'(p^2 = M^2)(p^2 - M^2) + \Sigma^{reg}(p^2)} \xrightarrow{p^2 \simeq M^2} \\ &= \frac{1}{\Sigma'(p^2 = M^2)(p^2 - M^2)} = \frac{g^2}{p^2 - M^2}. \end{aligned} \quad (2.174)$$

The constant g

$$g = (\Sigma'(p^2 = M^2))^{-1/2} \quad (2.175)$$

is the coupling constant of the bound state with the two composing φ particles. In fact, just looking at the diagram in Fig. 2.8 we realize that we have

$$2\tilde{\Phi}\left(\frac{p_1 - p_2}{2}\right) \left(\frac{g^2}{p^2 - M^2}\right) \tilde{\Phi}\left(\frac{p_3 - p_4}{2}\right) \quad (2.176)$$

i.e. the propagation of a field of mass M with coupling constant g to 2 φ . More precisely, we find such a contribution from the interaction Lagrangian [22, 23, 74]

$$\mathcal{L} = gBJ \quad (2.177)$$

where B describes a field with mass M and J is the non-local current

$$J = \int d^4 y \varphi(x + y/2) \Phi(y) \varphi(x - y/2). \quad (2.178)$$

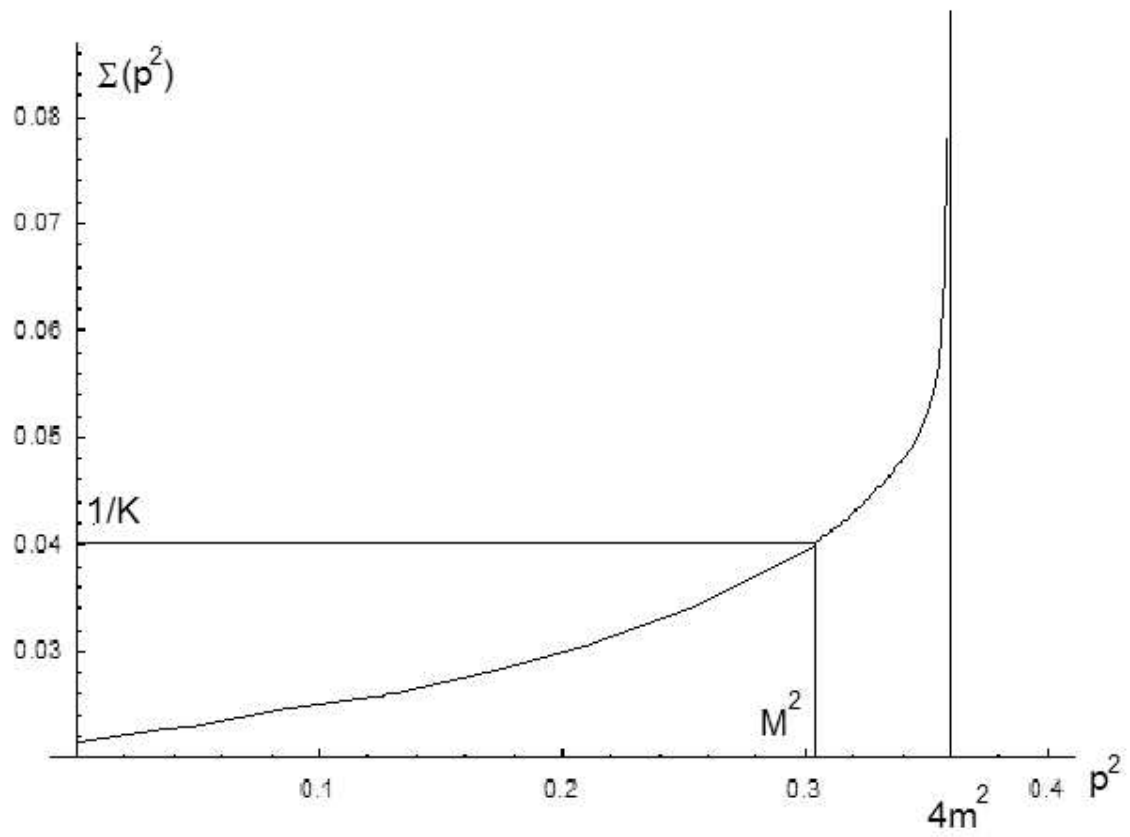


Figure 2.7: Mass operator function.

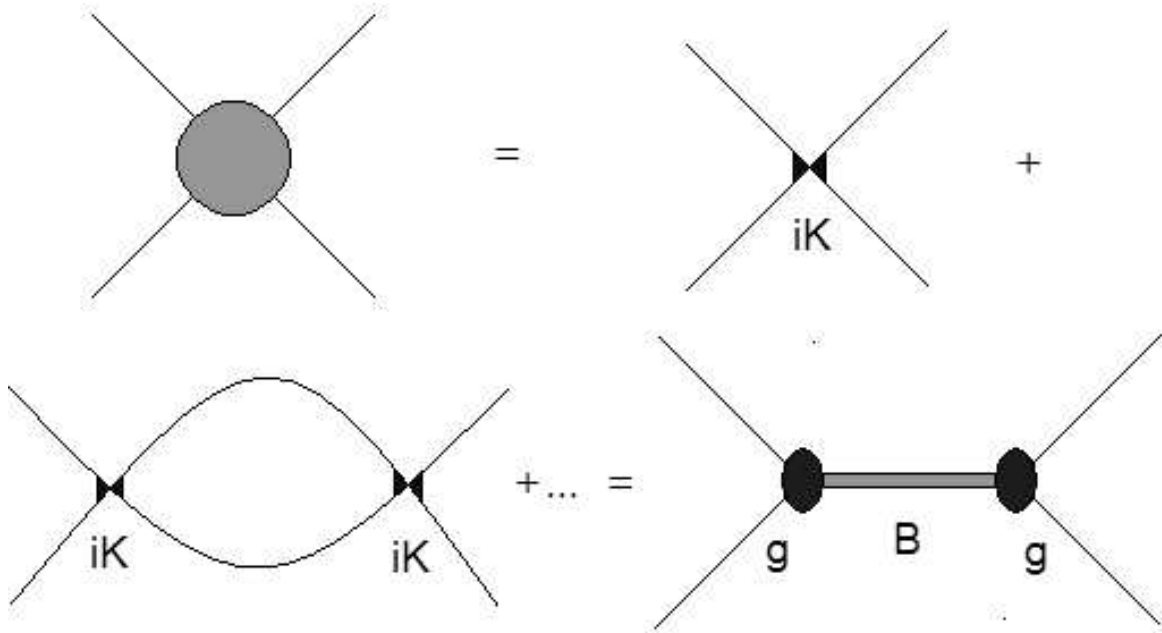


Figure 2.8: Two-body scattering with the bound state formation.

In the following section we analyze a more formal way to introduce the bound field B by matching two theories. We will find again the condition (2.175), which we will derive in two different ways. Now, we just note that we can write it as

$$\begin{aligned}
 g &= (\Sigma'(p^2 = M^2))^{-1/2} = \\
 &= \lim_{p^2 \rightarrow M^2} \sqrt{(p^2 - M^2)T(p^2)} = \left(\frac{\partial(T^{-1})}{\partial p^2} \right)_{p^2=M^2}^{-1/2}. \quad (2.179)
 \end{aligned}$$

The last two expressions are suited to be generalized in case of more fields and mixing among them (see chapter 6).

2.7 Bound field B

2.7.1 Matching between two Lagrangians

As we have seen, we are describing a bound state with mass M formed by two particles φ . We can now ask if we can introduce a field $B(x)$ which describes the dynamics of this bound

object. To this end let us consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2} ((\partial_\mu \varphi)^2 - m^2 \varphi^2) + gBJ - \frac{1}{2} \alpha B^2 \quad (2.180)$$

where α and g are two constants that we have to determine in order to have an equivalent description of the Lagrangian $\mathcal{L} = \frac{1}{2} ((\partial_\mu \varphi)^2 - m^2 \varphi^2) + K \cdot J^2/2$. We expect to find g as in the previous section. The first thing one should note is that the field B doesn't have a dynamical part. We can exploit the Euler-Lagrange equation of motion of B in order to relate it to the field φ :

$$B(x) = \frac{g}{\alpha} J(x) \quad (2.181)$$

We can then see that B really represents two fields φ . Plugging the expression back into the Lagrangian, we find

$$\mathcal{L} = \frac{1}{2} ((\partial_\mu \varphi)^2 - m^2 \varphi^2) + \frac{1}{2} \frac{g^2}{\alpha} J^2 \quad (2.182)$$

from which I can immediately deduce $K = g^2/\alpha$. But this is not all; we expect that the field B has the propagator $1/(p^2 - M^2)$ (for $p^2 \simeq M^2$, where M is the bound state mass). This seems "strange" because in (2.180) there is no dynamical part for the field B . At lowest order the propagator for B is simply a constant $-i/\alpha$, but then, because of the interaction with φ , we have a series of bubbles that modify the propagator. We find:

$$\frac{i}{-\alpha} \rightarrow \text{bubbles} \frac{i}{-\alpha + g^2 \Sigma(p^2)} \quad (2.183)$$

$$= \frac{i}{-\alpha + g^2 \Sigma(M^2) + g^2 \Sigma'(M^2)(p^2 - M^2) + g^2 \tilde{\Sigma}(p^2)} \quad (2.184)$$

$$\rightarrow \text{p}^2 \rightarrow M^2 \frac{i}{g^2 \Sigma'(M^2)(p^2 - M^2)} \quad (2.185)$$

Note that $-\alpha + g^2 \Sigma(M^2) = 0$. If we want to have a normalized propagator for the B field we must take

$$g = \frac{1}{\sqrt{\Sigma'(M^2)}} \quad (2.186)$$

$$\alpha = \Sigma(M^2)/\Sigma'(M^2) \quad (2.187)$$

The first of the last two relations is also known as compositeness condition [29, 67, 68, 70, 74]. It was derived at the beginning of the sixties in a different context [75]. Here we know by construction that B represents a bound state, because we have related (2.180) to the Bethe-Salpeter equation for the bound state of (2.162). In (2.180) we have introduced B as an auxiliary field, without a dynamical part. We then imposed that the propagator for B (taking into account the bubbles) is the free one with residue 1, thus imposing a kind of normalization.

Many phenomenological Lagrangians [29, 67, 68, 69, 70, 76] start directly from the form $\mathcal{L} = gBJ$ (bound state-composing particles interaction term). In the beginning it may be puzzling that a field interacts with its constituents, but it should be clear in the derivation here presented, because this last Lagrangian is completely related to (2.162), being nothing else than a rewritten form of it, indeed a very useful one, because it allows the calculation of

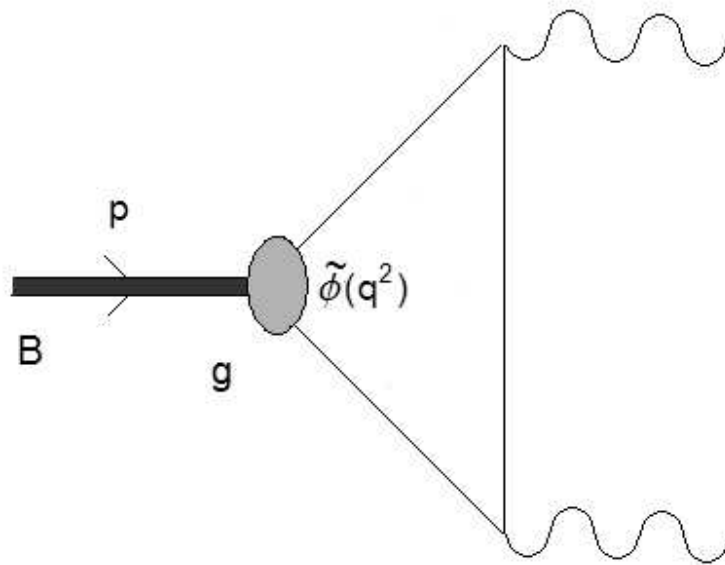


Figure 2.9: Two-photon decay of the bound state.

the decay of the bound state. If the field φ interacts with photons, we would have the decay $B \rightarrow 2\gamma$, represented by the diagram in Fig. 2.9.

We don't want to discuss the details of it (see chapter 4 for a detailed treatment), but it is clear that $\mathcal{L} = gBJ$ is particularly useful for such a calculation. The coupling constant g enters directly in the decay rate.

The Feynman rule of this interaction term is simple, and reads $ig\tilde{\Phi}(q)$, as one can easily prove explicitly. The Fourier transform of the vertex function naturally appears in the evaluation of physical amplitudes.

Finally, we would like to stress that the introduction of auxiliary fields can be generalized and formalized in the path integral formulation of QFT [3, 29, 60]. Here we have just presented the underlying basic ideas in a somewhat more intuitive way.

2.7.2 Further equivalence

As a further text of the equivalence of (2.162) with (2.180) we consider two-body scattering at the lowest order considering all possible combinations. In the first case we get

$$-iM_1 = \frac{4!}{2} iK \tilde{\Phi}\left(\frac{p_1 - p_2}{2}\right) \tilde{\Phi}\left(\frac{p_3 - p_4}{2}\right). \quad (2.188)$$

Considering then (2.180) we note that we have indeed three possibilities to generate two-body scattering, corresponding to the 3 channels for the bound state formation. At lowest order the bound state propagator reduces to g^2/α . We then have (see Fig. 2.10)

$$\begin{aligned} -iM_1 &= 4 \cdot 3i \cdot \frac{g^2}{\alpha} \tilde{\Phi}\left(\frac{p_1 - p_2}{2}\right) \tilde{\Phi}\left(\frac{p_3 - p_4}{2}\right) = \\ &= 4 \cdot 3i \cdot \frac{1}{\Sigma(M^2)} \tilde{\Phi}\left(\frac{p_1 - p_2}{2}\right) \tilde{\Phi}\left(\frac{p_3 - p_4}{2}\right) \\ &= 4 \cdot 3i \cdot K \cdot \tilde{\Phi}\left(\frac{p_1 - p_2}{2}\right) \tilde{\Phi}\left(\frac{p_3 - p_4}{2}\right) \end{aligned} \quad (2.189)$$

where in the last passage we have made use of the pole equation $K = 1/\Sigma(p^2 = M^2)$ (see eq. (2.172)). This indeed proves that we correctly choose a channel to analyze the bound state properties. The equivalence can of course be proved order by order.

2.7.3 Compositeness condition

As mentioned before, the relation (2.186) was found in the sixties in a different way [75]; in order to sketch that idea, let us write the following Lagrangian

$$\mathcal{L} = \frac{1}{2} ((\partial_\mu B_0)^2 - M_0^2 B_0^2) + g_0 B_0 J + \mathcal{L}_{free,\varphi} \quad (2.190)$$

where $\mathcal{L}_{free,\varphi}$ is the free Lagrangian for the field φ . The bubbles of fields φ are finite because of the cutoff function $\Phi(y)$ in J . So, considered as effective, no renormalization is applied to this field. On the contrary, we consider the field B_0 as the bare one, with g_0 and M_0 the bare coupling constant and mass. It is a bit strange that we consider just one field as "bare" and not the other, but the reason will be clear soon. As we can note, apart from this difference, this Lagrangian is very similar to (2.154), where we were describing an interacting theory of two elementary particles. On the other hand, (2.190) is also similar to (2.180),

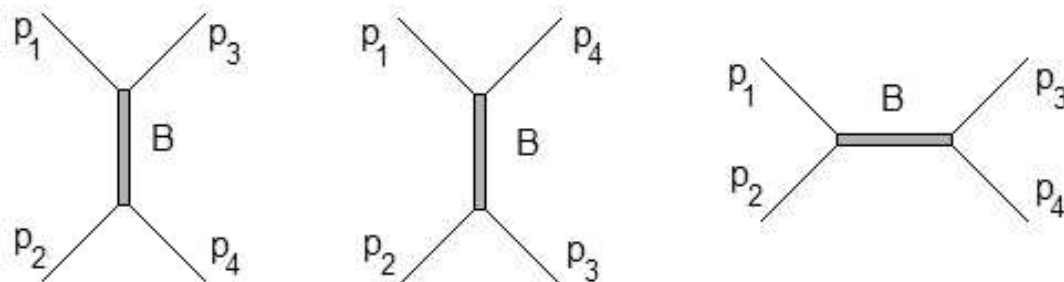


Figure 2.10: Two-body scattering in the three combinatorial possibilities.

where, by construction, we were describing a bound object. Here we would like to describe, as in the previous section, a bound state field B . At this stage, just looking at (2.190), we can't understand if B (the dressed field of the bare B_0) describes a bound state of 2φ or an elementary object interacting with φ .

Let us now consider the "dressing" (Fig. 2.11) of the propagator of B_0 , similar to the one seen in section (1.4.3):

$$\frac{i}{p^2 - M_0^2} \rightarrow \frac{i}{p^2 - M_0^2 + g_0^2 \Sigma(p^2)} \quad (2.191)$$

where $-i\Sigma(p^2)$ is the bubble contribution. The new mass is:

$$M^2 - M_0^2 + g_0^2 \Sigma(M^2) = 0 \quad (2.192)$$

and therefore we find

$$\frac{iZ_2}{p^2 - M^2 + Z_2 g_0^2 \widetilde{\Sigma}(p^2)}. \quad (2.193)$$

As proposed back in the sixties, in order to study a bound state we postulate

$$Z_2 = \frac{1}{1 + g_0^2 \Sigma'(M^2)} = 0. \quad (2.194)$$

The condition $Z_2 = 0$ means that $g_0 \rightarrow \infty$, and, as we will see, is equivalent to the analysis of the previous subsection. First, note that because of this condition the denominator of the propagator is well defined; the term $Z_2 g_0^2$ is then

$$Z_2 g_0^2 = \frac{g_0^2}{1 + g_0^2 \Sigma'(M^2)} \rightarrow_{g_0 \rightarrow \infty} \frac{1}{\Sigma'(M^2)}. \quad (2.195)$$

The dressed field and coupling are $B = B_0/\sqrt{Z_2}$ and $g = g_0\sqrt{Z_2}$, so the propagator for B in the vicinity of M^2 is $1/(p^2 - M^2)$. If now, we write the Lagrangian in terms of the dressed fields we have

$$\mathcal{L} = -\frac{1}{2} Z_2 M_0^2 B^2 + g B J + \mathcal{L}_{0,\varphi} \quad (2.196)$$

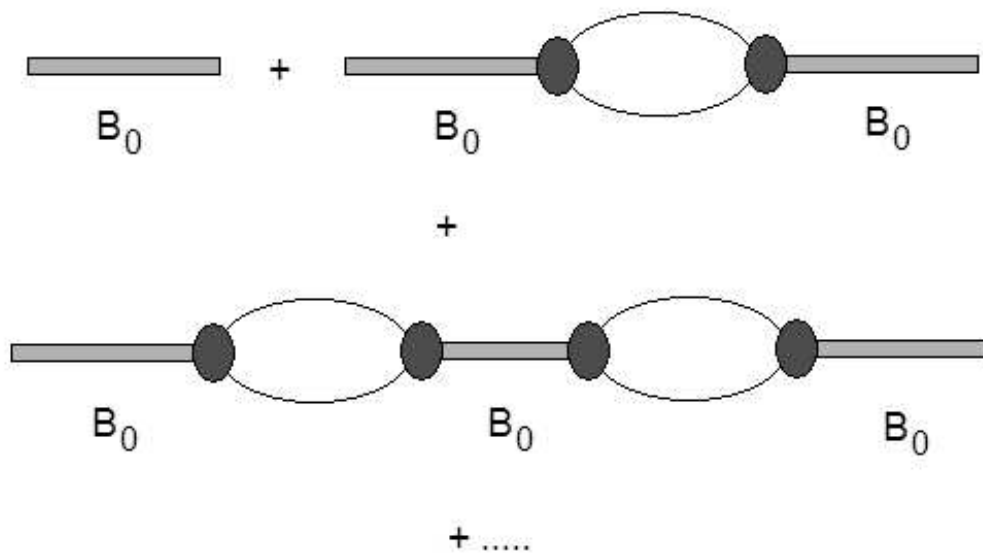


Figure 2.11: Self interaction of the bare field.

where we note that the condition $Z_2 = 0$ is responsible for the zero dynamical part of the dressed field B , exactly as we had before. Furthermore:

$$Z_2 M_0^2 = \frac{M^2 + g_0^2 \Sigma(M^2)}{1 + g_0^2 \Sigma'(M^2)} \xrightarrow{g_0 \rightarrow \infty} = \frac{\Sigma(M^2)}{\Sigma'(M^2)} \quad (2.197)$$

exactly as the parameter α evaluated in the previous section.

For g we find as well the same result:

$$g^2 = g_0^2 Z_2 = \frac{g_0^2}{1 + g_0^2 \Sigma'(M^2)} \xrightarrow{g_0 \rightarrow \infty} = \frac{1}{\Sigma'(M^2)} \quad (2.198)$$

and so precisely the compositness condition $g = 1/\sqrt{\Sigma'(M^2)}$. All these results are a consequence of $g_0 \rightarrow \infty$, i.e. of the condition $Z_2 = 0$. The final dressed Lagrangian we find is

$$\mathcal{L} = -\frac{1}{2}\alpha B^2 + gBJ + \mathcal{L}_{0,\varphi} \quad (2.199)$$

identical to (2.180).

2.8 Vertex function in the non relativistic limit

2.8.1 Up to now...

Before going on, let us take a breath and summarize the main concepts related to the bound states and to the interaction Lagrangian $\mathcal{L} = gBJ$:

- Feynman rule: it reads $ig\tilde{\Phi}(q)$ where $q = (p_1 - p_2)/2$.
- Coupling constant: it is given from the compositness condition $g = (\Sigma'(M^2))^{-1/2}$ where $\Sigma(p^2) = 2i \int \frac{d^4q}{(2\pi)^4} G(p/2+q)G(p/2-q)\tilde{\Phi}^2(q)$ with $G(p)$ being the propagator of the φ particle.
- The propagator of the field B is obtained by summing all the "bubbles" and is given from $i(p^2 - M^2 - \tilde{\Sigma}(p^2)/\Sigma'(M^2))^{-1} \rightarrow i(p^2 - M^2)^{-1}$ for $p^2 \rightarrow M^2$.

If the bound state field is off-shell one has also to take into account the modification of the propagator.

2.8.2 Comparison

We want now to look closer to the meaning of $\Phi(y)$ by a comparison with the non-relativistic limit. In this case the bound state in the rest frame is described by [3, 59]

$$|s\rangle = \frac{1}{\sqrt{V}} \int \frac{d^3k}{(2\pi)^{3/2}} A(\mathbf{k}) a^\dagger(\mathbf{k}) a^\dagger(-\mathbf{k}) |0\rangle, \quad (2.200)$$

where $A(\mathbf{k})$ is the wave function in momentum space of the bound system of two φ . $A(\mathbf{k})$ is the amplitude of finding a relative momentum $2\mathbf{k}$ between the two particles. In our approach we have introduced the current J

$$J(x) = \int d^4y \varphi(x+y/2)\Phi(y)\varphi(x-y/2); \quad (2.201)$$

it is then natural to consider the state $J(x)|0\rangle$, where J creates two particles " φ " correlated by the vertex function $\Phi(y)$. We then perform a Fourier transform of this state to get the corresponding state depending on the four-momentum p of the bound state:

$$|j(p)\rangle = \int d^4x J(x)|0\rangle e^{-ipx} \quad (2.202)$$

and $|j(p = (M, \mathbf{0}))\rangle$ represents the state in the rest frame. After a bit of algebra we get explicitly

$$|j(p = (M, \mathbf{0}))\rangle = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{2\sqrt{\mathbf{k}^2 + m^2}} \tilde{\Phi}(0, \mathbf{k}) a^\dagger(\mathbf{k}) a^\dagger(-\mathbf{k}) |0\rangle. \quad (2.203)$$

Apart from the factor $1/(\sqrt{\mathbf{k}^2 + m^2})$ (which is a constant in the NR limit) we find that $|j(p = (M, \mathbf{0}))\rangle$ has the same form as $|s\rangle$, with $\tilde{\Phi}(0, \mathbf{k})$ instead of $A(\mathbf{k})$. One then is tempted to interpret $\tilde{\Phi}(0, \mathbf{k})$ as the wave function in momentum space of the bound state; however, as said before, care is required with such an identification. We clearly realize that $\tilde{\Phi}$ and A are connected, but the argument we presented still cannot constitute justification for such an identification, and now we list some remarks in this sense:

- $\Phi(y)$ depends on the four-vector $y = (y^0, \mathbf{y})$ (and $\tilde{\Phi}(q)$ on $q = (q^0, \mathbf{q})$), i.e. on a temporal variable which doesn't have a counterpart in the NR limit.
- We just considered the states $|s\rangle$ and $|j(p = (M, \mathbf{0}))\rangle$ showing the similarities, but we didn't give a precise connection between these quantities.
- $|s\rangle$ has an energy

$$E = \int d^3k (2\sqrt{\mathbf{k}^2 + m^2}) |A(\mathbf{k})|^2 > 2m \quad (2.204)$$

because of this the decay $|s\rangle \rightarrow 2\varphi$ is possible but actually unwanted; in the phenomenological QFT approach the decay $B \rightarrow 2\varphi$ is kinematically forbidden, because the condition $E = M < 2m$ is satisfied.

Nevertheless, states like $|s\rangle$ have been useful to calculate many properties of the bound states like positronium or muonium $\mu^+\mu^-$ [3, 59]; we have then to find a physical process which allows a direct comparison of A with $\tilde{\Phi}$. One possibility is to calculate a decay of the bound state. Previously we have mentioned the possible 2-photon decay of the bound state; in order to simplify our life, let us consider something analogous with a scalar particle χ instead of the photon, in order not to have vectorial complications (and gauge considerations) typical of electromagnetic properties.

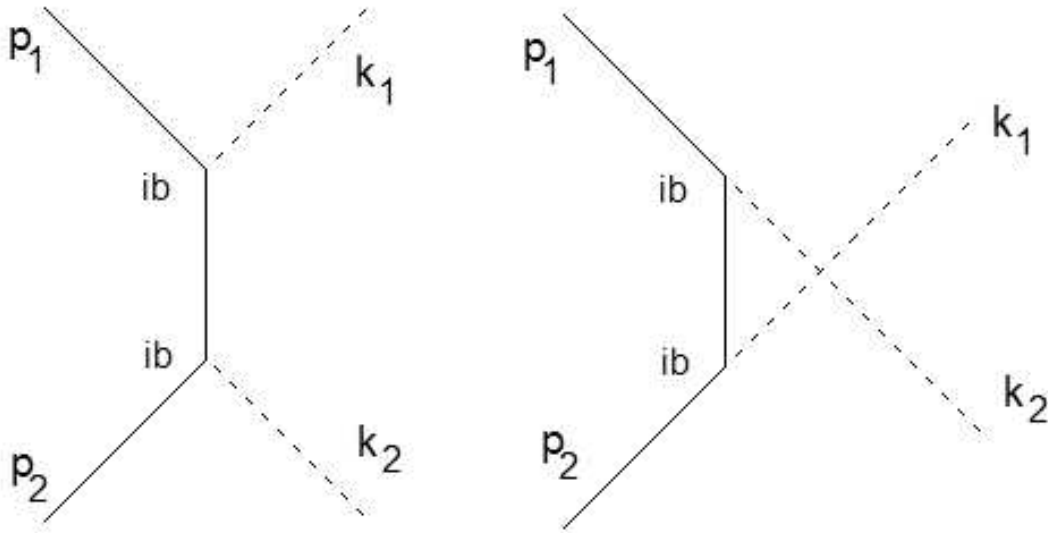
We want to evaluate the decay into two χ , and we do it in two different ways, before using $|s\rangle$ and then using the phenomenological field theory approach.

2.8.3 $|s\rangle \rightarrow 2\chi$

Let us first consider the Lagrangian

$$\mathcal{L} = \frac{1}{2} ((\partial_\mu \varphi)^2 - m^2 \varphi^2) + \frac{1}{2} ((\partial_\mu \chi)^2 - m_\chi^2 \chi^2) + b\chi\varphi^2 \quad (2.205)$$

where b is the coupling $\chi - \varphi^2$ and where we assume that the coupling constant b is small and that a perturbative treatment is indeed allowed. The reaction $2\chi \rightarrow 2\varphi$ is possible, and

Figure 2.12: Two-boson χ production

the corresponding amplitude is 2.12:

$$-iM = \frac{(ib)^2 i}{(p_1 - k_1) - m^2} + \frac{(ib)^2 i}{(p_1 - k_2) - m^2} \quad (2.206)$$

Now, in this Lagrangian we didn't consider any bound state generation of 2φ ; in order to calculate the decay $|s\rangle \rightarrow 2\chi$ we simply consider as initial state $|s\rangle$ instead of $|p_1, p_2\rangle$ (see the discussion about the positronium in [59]).

The amplitude for this process is proportional to

$$\frac{1}{\sqrt{V}} \int \frac{d^3q}{(2\pi)^{3/2}} A(\mathbf{q}) \left(\frac{(ib)^2 i}{(p_1 - k_1) - m^2} + \frac{(ib)^2 i}{(p_1 - k_2) - m^2} \right) \delta(p_1 + p_2 - k_1 - k_2) \quad (2.207)$$

where

$$\begin{aligned} p_1 &= (\sqrt{\mathbf{q}^2 + m^2}, \mathbf{q}), \\ p_2 &= (\sqrt{\mathbf{q}^2 + m^2}, -\mathbf{q}), \\ k_1 &= (\sqrt{\mathbf{k}^2 + m_\chi^2}, \mathbf{k}), \\ k_2 &= (\sqrt{\mathbf{k}^2 + m_\chi^2}, -\mathbf{k}). \end{aligned}$$

Because of momentum conservation we have $\sqrt{\mathbf{q}^2 + m^2} = \sqrt{\mathbf{k}^2 + m_\chi^2}$ and then

$$(p_1 - k_1) - m^2 = -((\mathbf{p} - \mathbf{q})^2 + m^2), \quad (2.208)$$

$$(p_1 - k_2) - m^2 = -((\mathbf{p} + \mathbf{q})^2 + m^2). \quad (2.209)$$

Considering that, because of the integration over \mathbf{q} , both terms give rise to the same contribution, we then can rewrite the amplitude as proportional to

$$\frac{1}{\sqrt{V}} \int \frac{d^3q}{(2\pi)^{3/2}} A(\mathbf{q}) \frac{2(ib)^2 i}{(\mathbf{p} - \mathbf{q})^2 + m^2}. \quad (2.210)$$

We speak only of proportionality because this is what we care in the comparison between A and $\tilde{\Phi}$.

Note that this is the typical way for calculating the decay of a bound state in the non-relativistic limit, making a direct use of the wave function in momentum space.

2.8.4 $B \rightarrow 2\chi$

We now want to calculate the same process within the bound state nonlocal field theory with the vertex function $\Phi(y)$. The interaction Lagrangian interesting to us for the decay into 2χ (Fig. 1.13) is

$$\mathcal{L}_{int} = gB \cdot J + b\chi\varphi^2 \quad (2.211)$$

The Feynman diagram connected to the decay is represented by a loop. This is the main difference with respect to the previous case. In that case the two incoming φ particles were on shell, and we just made a convolution of the initial state by introducing the wave function in momentum space. In this case we have a loop where the two φ particles are also off shell, and the vertex function enters in the Feynman rule for the vertex B - 2φ . The amplitude (Fig. 2.13) reads then (taking already into account the exchange diagram)

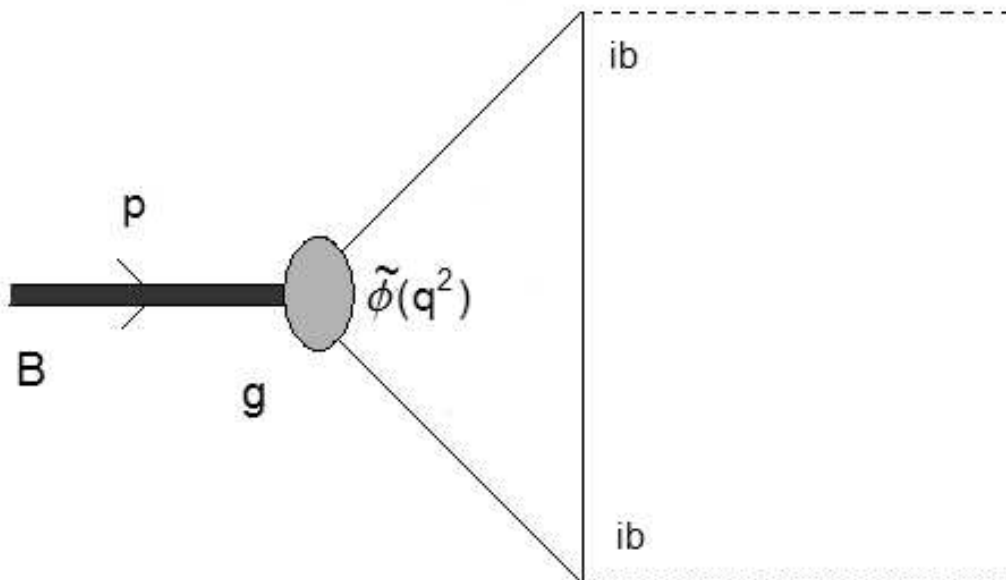
$$-iM = i(2g) \int \frac{d^4q}{(2\pi)^4} \frac{i}{(p/2 + q)^2 - m^2} \cdot \frac{i}{(p/2 - q)^2 - m^2} \frac{i}{(p/2 + q - k_1)^2 - m^2} \tilde{\Phi}(q). \quad (2.212)$$

This expression is valid for a generic vertex function and also far from the non-relativistic limit; what we want to do here is to establish a connection with the NR case, therefore we evaluate the integral over q^0 in this limit; by closing "down" the residue which gives the largest contribution is the one for $q^0 = -M/2 + \sqrt{\mathbf{q}^2 + m^2} > 0$ (note that, in virtue of the non-relativistic limit, q^0 is positive but is a very small number). In fact, from the second propagator we get the term

$$\frac{1}{M(M - 2\sqrt{\mathbf{q}^2 + m^2})} \quad (2.213)$$

which is large, because in the non-relativistic limit $M \sim 2m$ and \mathbf{q}^2 is small as compared to m . The other two poles give rise to smaller contribution. The third propagator reduces to

$$\frac{1}{q^0 - (\mathbf{p} - \mathbf{q})^2 - m^2} \sim \frac{-1}{(\mathbf{p} - \mathbf{q})^2 + m^2} \quad (2.214)$$

Figure 2.13: Decay of the bound state into two bosons χ .

being possible because q^0 is small as compared to m . Therefore, putting all these pieces together, we get in the NR limit the amplitude

$$2ig \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{q}^2 + m^2}} \frac{1}{M(M - 2\sqrt{\mathbf{q}^2 + m^2})} \frac{1}{(\mathbf{p} - \mathbf{q})^2 + m^2} \tilde{\Phi}(q = (-M/2 + \sqrt{\mathbf{q}^2 + m^2}, \mathbf{q})). \quad (2.215)$$

In order to compare directly to the previous case we realize that $\tilde{\Phi}$ should not have a q^0 dependence (which in any case is small, being $q^0 \sim 0$). This is possible if we assume that

$$\tilde{\Phi}(y) = \delta(y^0) \cdot \Phi(y) \quad (2.216)$$

from which $\tilde{\Phi}$ depends on the three-momentum only. This is in the spirit of Eddington's words: "a proton yesterday and an electron today do not make an atom". With this choice for the vertex function we impose that the two particles exist contemporary. Such an approximation for the vertex function goes under the name of instantaneous approximation [71, 72], because one neglects the time dependence of the vertex function.

Comparing finally the two expressions for the amplitudes for the decay into 2χ we find

$$\tilde{\Phi}(\mathbf{q}) = c\sqrt{\mathbf{q}^2 + m^2}(M - 2\sqrt{\mathbf{q}^2 + m^2})A(\mathbf{q}) \quad (2.217)$$

where c is some proportionality constant we are not much concerned about because each method is separately normalized.

Note that the equivalence of the two results is pretty remarkable, considering that in the first case we had 2 particles on shell, while in the second a loop diagram where no internal particle is on-shell.

The equivalence occurs through an extensive use of the non-relativistic condition $M \sim 2m$, which we apply at every step of the calculation. If this condition is not satisfied such a direct connection between $\tilde{\Phi}$ and A cannot be found, but in this case the concept of a "wave function" is not anymore valid itself. The vertex function, also in a covariant form, may be considered as a kind of generalization of the wave function for the description of the finite size of the bound state.

Our result (2.217) shows also that what we naively did in the beginning of this subsection was not so wrong, in the sense that the vertex function in momentum space is proportional to the wave function, and the extra-factor in front of it comes from the residual loop evaluation. At the end of this chapter we will re-obtain this result in a more formal way, by making use of the Bethe-Salpeter equation.

Our result is then similar to the assumption of [77, 78] in the positronium case. We will actually exploit these considerations in the evaluation of the two-photon decay of the positronium, because it is a good test of the bound state approach, for which good experimental results are known and for which a non-relativistic limit is completely justified. In the light quark-antiquark case, where one has a similar object, but with the much more complicated light quark dynamics of QCD , such non-relativistic limit is not anymore valid. In this case one can make an Ansatz for the vertex function to model the bound nature of a meson. In some models of QCD [23] one uses as a vertex function a step function $\theta(p^2 - \Lambda^2)$ which cuts in a crude way the high momenta. Other smoother forms are then considered [67, 69], like gaussian or dipole forms.

Relatively recently there are approaches to solve directly the non-perturbative QCD dynamics by making use of the so called Dyson-Schwinger equations [24, 25]. In this case the propagators are not anymore the free ones, and the vertex bound states-elementary fields is calculated directly from QCD . A certain number of approximations must be used in this case, in order to solve the rather complicated coupled equations. In this way one may hope to calculate the "vertex function" directly from the basic underlying theory. In the next subsection we point to some general mathematical considerations when evaluating the mass operator also in the presence of non-free propagators.

2.9 Parabola

Up to now we have used for the field φ the free propagator form $i/(q^2 - m^2)$. It may happen that such an approximation is not valid; for example, quarks and gluons in QCD have propagators which, at low energies, are very different from the free ones that one could deduce by looking at the QCD Lagrangian (see the related discussions in chapters 3 and 5). Sometimes one has to take into account explicitly the running behavior of the mass $m = m(p^2)$, i.e. of a non-free propagator [24, 79].

In this subsection we don't want to analyze such a phenomenon, but rather describe a general feature connected with the portion of complex plane which comes in the game by evaluating the mass operator with a general form of the propagators.

First, consider a generic propagator $G = G(q^2)$ and then replace $q^2 \rightarrow z$, where z is a complex variable. We then have $G(z)$, which we assume to be an analytic function in the complex plane, apart from some singularities. One may ask: q^2 is a real number, positive or negative, why should one be interested in the extension to the complex plane? The answer is: because of the Wick rotation. By performing it, we introduce complex arguments in the propagator.

In general the "bubble" is (eq. (2.167))

$$\Sigma(p^2) = 2i \int \frac{d^4q}{(2\pi)^4} G((q + p/2)^2) G((q - p/2)^2) \tilde{\Phi}^2(q). \quad (2.218)$$

In the rest frame we have $p = (M, \mathbf{0})$, therefore, Wick-rotating like $q^{(0)} = iq^{(4)}$, we get:

$$(q + p/2)^2 = -q_E^2 + \frac{M^2}{4} + iq^{(4)}M \quad (2.219)$$

where $q_E^2 = \mathbf{q}^2 + (q^{(4)})^2$. We can see that the propagator G is evaluated at a complex number $z = -q_E^2 + M^2/4 + iq^{(4)}M$, where the imaginary part comes from the Wick rotation. Similarly the second propagator is evaluated at z^* . We therefore have the following integral:

$$\Sigma(p^2) = 2 \int \frac{d^4q_E}{(2\pi)^4} G(z) G(z^*) \tilde{\Phi}^2(q_E); \quad (2.220)$$

first, note that the integral is real; we have $G(z^*) = G^*(z)$ (because the original propagator is real, therefore expandable with real coefficients). We have

$$\Sigma(p^2) = 2 \int \frac{d^4q_E}{(2\pi)^4} |G(z)|^2 \tilde{\Phi}^2(q_E). \quad (2.221)$$

We can now check which part of the complex plane is "hit" in the integration (Fig. 2.14) :

- $q^{(4)} = 0 \rightarrow z = -q_E^2 + M^2/4$ where the "maximum" is for $M^2/4$, which is positive and real, thus in the Minkowsky part of the plane.

- I then vary $q^{(4)}$, moving on the imaginary axis. For the real part being zero, I have that $-M/2 \leq q^{(4)} \leq M/2$, therefore the intersection with the imaginary axis is for $\pm M^2/2$.

In general I have

- $Re[z] = -q_E^2 + M^2/4$;
- $Im[z] = q^{(4)}M$, with $-\sqrt{q_E^2} \leq q^{(4)} \leq \sqrt{q_E^2}$.

The extrema are given from the following parabola equation:

$$Re[z] = \frac{M^2}{4} - \frac{(Im[z])^2}{M^2} \quad (2.222)$$

and all the points of the complex plane hit in the integration are inside this parabola. This result is general (also in the most general Bethe-Salpeter approach). If we display the poles of the analytic function $G(z)$ in the complex plane we can immediately see whether the poles are encountered in the integration or not.

If no poles are inside it, everything is fine, otherwise care is needed because infinities may appear. We then see that, for a free propagator form, everything is fine if $M < 2m$, because the pole of the propagator occurs for $z = m^2$, which is outside the parabola if $2m > M$ (Fig. 2.14).

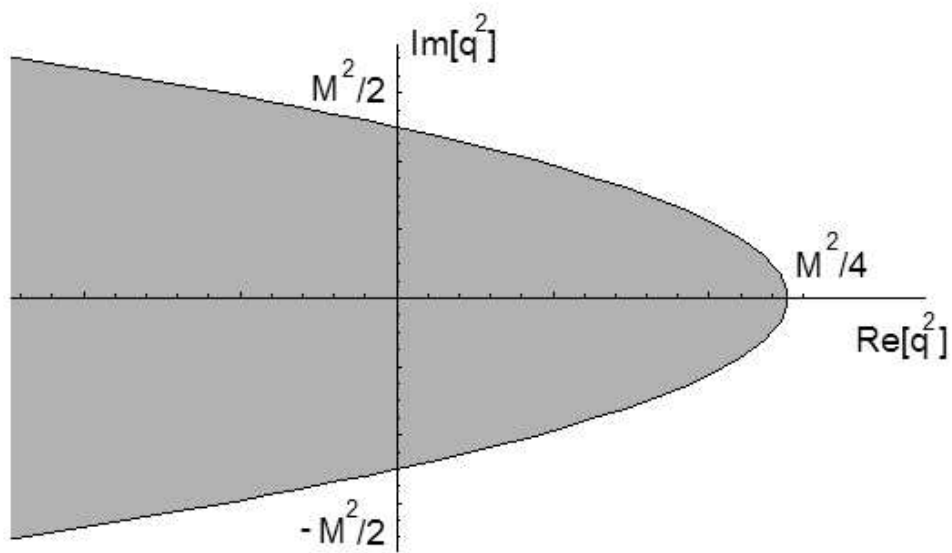


Figure 2.14: Parabola: the shadowed region is encountered in the integration.

2.10 Our approach and the B-S equation

2.10.1 Bethe-Salpeter equation

We mentioned the Bethe-Salpeter equation in section 2.6. Our pole-equation for the determination of the bound state mass is actually a solution of the Bethe-Salpeter equation for our particular case.

We now want to briefly describe this important and general approach for the study of a bound state in a quantum field theory and re-derive our previous equations as a particular case of the general one. We consider a two-body scattering for the particles φ , which interact in some way. Let us call $K(q, k; p)$ the irreducible two-body interaction vertex where

$$q = (p_1 - p_2)/2, \quad (2.223)$$

$$k = (p_3 - p_4)/2 \quad (2.224)$$

$$p = p_1 + p_2 = p_3 + p_4. \quad (2.225)$$

The full calculation of this object is in general not possible, because one has an infinity of Feynman diagrams to calculate, but one will make assumption, keeping (hopefully) the leading terms.

For example, if we consider a one-boson exchange theory described by the Lagrangian (2.154, section 2.5.1), the kernel will be at lowest order

$$K(q, k; p) = b^2 \left(\frac{i}{(k - q)^2 - m_\chi^2} \right) \quad (2.226)$$

i.e. the boson propagator (we are for simplicity not considering the annihilation term).

The full kernel, also in such a simple field theory, is indeed very complicated. One has to sum an infinity of diagrams, which is practically impossible. Therefore only some terms are included (as in the previous equation), "hoping" that they are dominating in the bound state formation process. This operation goes under the name of "truncation", because one cuts, truncates, the full sum keeping only the first term(s).

In Fig. 2.15 some first terms are depicted. In the evaluation of K the external lines do not play any role. The first term correspond to eq. (2.226).

In our case from the non-local Lagrangian (2.162) we have, again at lowest order, a separable kernel with

$$K(q, k; p) = 2K\tilde{\Phi}(q)\tilde{\Phi}(k). \quad (2.227)$$

One then has in general an expansion as in Fig. 2.8.

What follows is not a derivation of the B-S equation (for which we refer to [24, 25, 71, 72]), but some pictorial representation of it. Once the kernel is specified, one considers the two-body scattering summing to order infinity the kernel interaction, as in Fig 2.16.a, where $T = T(q, k; p)$ is the total (amputated) scattering amplitude. The adjective "amputated" means that the external lines are not considered. At lowest order $T = K$, but the bound state formation, being an intrinsic non-perturbative phenomenon of a field theory, needs an expansion as in the figure. This expansion can be easily expressed in Fig. 2.16.b, where the T appears on the left and on the right of the equation; the full mathematical expression reads

$$T(q, k; p) = K(q, k; p) + \int d^4k' T(q, k'; p) G(p/2 + k') G(p/2 - k') K(k', k; p) \quad (2.228)$$

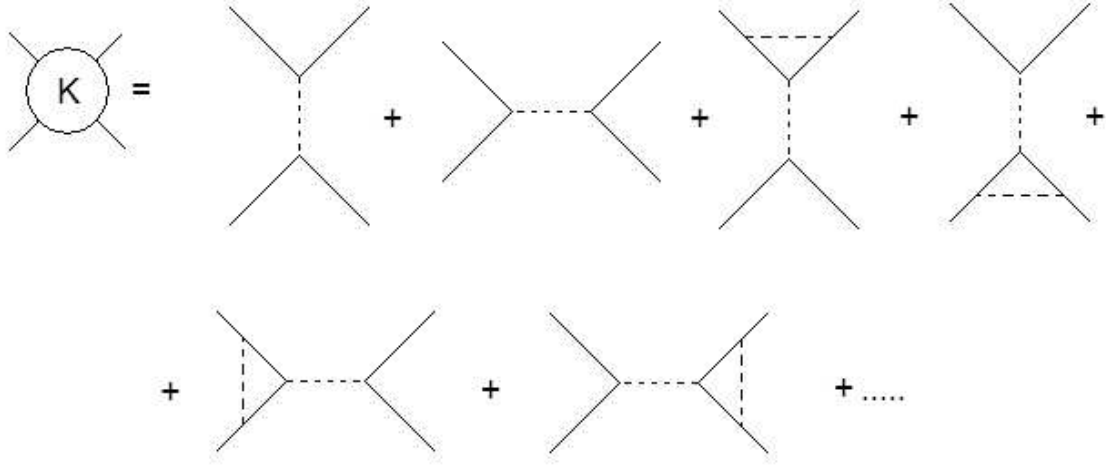


Figure 2.15: Two-body kernel for a boson exchange theory.

If T has a pole for p^2 we have a bound state, pretty much as described in section 2.6, but now in this generalized context. Even more generally, when one deals with many fields, T becomes a matrix. The poles of T , corresponding to the formation of bound states, are then found from the equation $\text{Det}[T^{-1}] = 0$.

But this is not the end of the story; if the total momentum $p = p_1 + p_2$ is such that a bound state for $p^2 = M^2$ appears, then we have a pole of T and we can depict the situation as in Fig. 2.16.c, where the propagator of the bound state appears; with $\Gamma(q; p)$ one denotes the general bound state-constituents vertex term. In order to find it we "compare" Fig. 2.16.b with 2.16.c, getting figure 2.16.d, whose mathematical counterpart is

$$\Gamma(q; p) \frac{i}{p^2 - M^2} \Gamma(k; p) = K(q, k; p) + \Gamma(q; p) \frac{i}{p^2 - M^2} i \int \frac{d^4 k'}{(2\pi)^4} K(k', k; p) G(p/2 + k') G(p/2 - k') \Gamma(k'; p) \quad (2.229)$$

If we exploit again the condition $p^2 \rightarrow M^2$ we realize that the first term on the rhs of the last equation doesn't contribute; we then get the so called Bethe-Salpeter equation for the determination of the vertex function Γ with

$$\Gamma(k; p) = i \int \frac{d^4 k'}{(2\pi)^4} K(k', k; p) G(p/2 + k') G(k/2 - k') \Gamma(k'; p) \quad (2.230)$$

(see Fig. 2.16.e). It is an integral equation of the Fredholm type, which is very difficult to solve. From it one can find the mass of the bound state, for the corresponding value of $p^2 = M^2$ at which the Bethe-Salpeter equation is fulfilled.

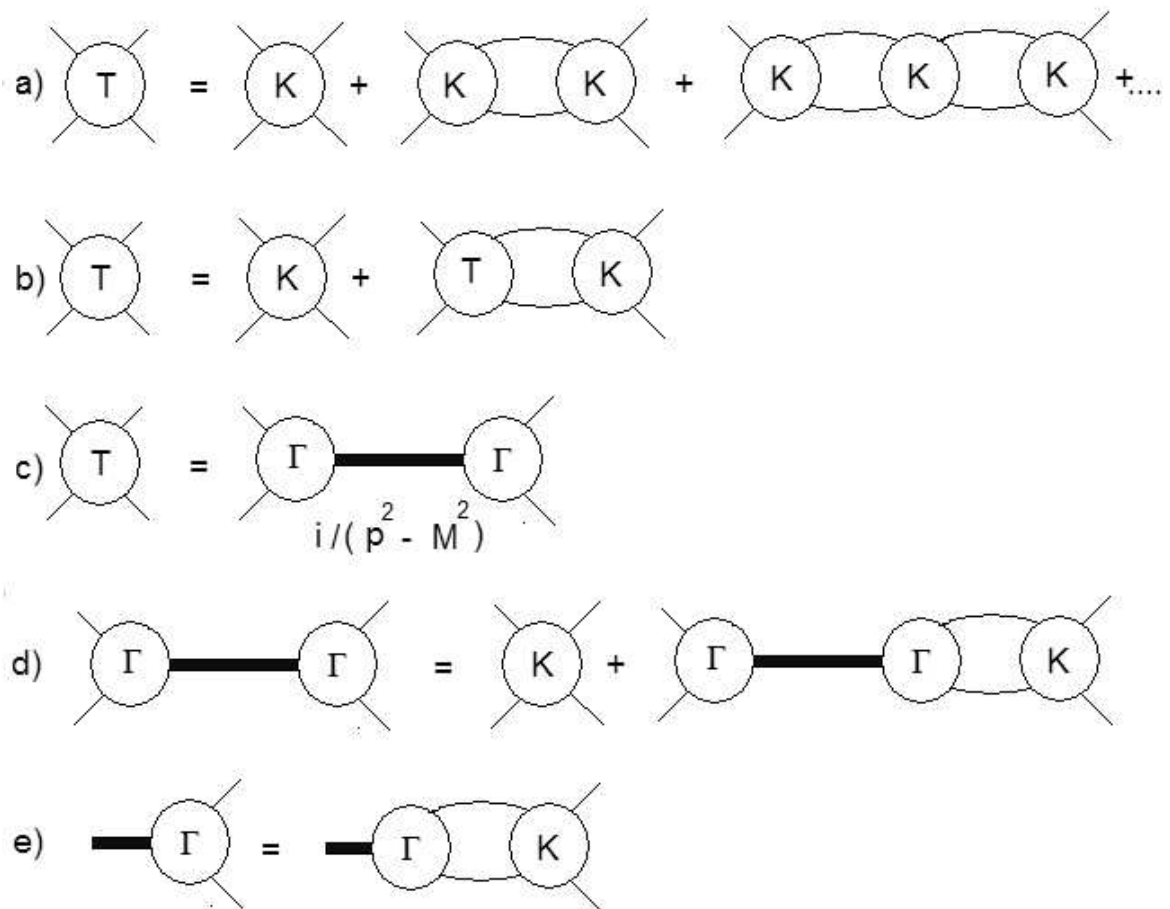


Figure 2.16: Bethe-Salpeter pictorial representation.

In the end, note that the discussion about the parabola of the previous subsection is still valid, also in the most general Bethe-Salpeter approach.

2.10.2 Separable kernel

If we plug our separable interaction kernel $K(k, q; p) = 2K\tilde{\Phi}(q)\tilde{\Phi}(k)$ inside the Bethe-Salpeter equation we find

$$\Gamma(k; p) = c(p^2)2K\tilde{\Phi}(k) \quad (2.231)$$

where

$$c(p^2) = i \int \frac{d^4q}{(2\pi)^4} \tilde{\Phi}(q)G(p/2 + q)G(k/2 - q)\Gamma(q; p). \quad (2.232)$$

Plugging in this last equation the previous expression $\Gamma(q; p) = c(p^2)K\tilde{\Phi}(q)$ we then find

$$1 = K(2i) \int \frac{d^4q}{(2\pi)^4} \tilde{\Phi}^2(q)G(p/2 + q)G(p/2 - q), \quad (2.233)$$

i.e. nothing else than our already famous pole equation

$$\frac{1}{K} = \Sigma(p^2). \quad (2.234)$$

Then $c(p^2)$ can be found from the normalization condition for the Bethe-Salpeter equations; we didn't discuss it here, and we refer to the above cited works. Its physical meaning is the normalization of the bound state propagator. In our case it reads

$$c(p^2 = M^2) = \frac{1}{\sqrt{\Sigma'(M^2)}} \frac{1}{K} = \frac{g}{K} \quad (2.235)$$

from which we have

$$\Gamma(k; p) = g\tilde{\Phi}(k) \quad (2.236)$$

which is exactly the vertex term in momentum space for the Lagrangian (2.180), together with the compositeness condition for the coupling constant.

2.10.3 Non-relativistic limit and the vertex function

A lot of effort has been done to find the non relativistic limit of the Bethe-Salpeter equation, in order to get the "usual" Schroedinger equation for a bound state [55, 71, 72]. Here we do not intend to discuss this issue in detail, but we just discuss how the three-dimensional wave function is usually introduced. The Bethe-Salpeter amplitude reads

$$A(y/2, -y/2; p) = \langle 0 | T[\varphi(y/2)\varphi(-y/2)] | B(p) \rangle \quad (2.237)$$

where $|B(p)\rangle$ is the bound state with four-momentum p . In momentum space one has the Fourier transform

$$A(q; p) = \int d^4x e^{iqx} A(y/2, -y/2; p). \quad (2.238)$$

It is possible to show [71, 72] that this quantity is

$$A(q; p) = G(p/2 + q)\Gamma(k; p)G(p/2 - q) = \frac{\Gamma(q; p)}{((p/2 + q)^2 - m^2)((p/2 - q)^2 - m^2)}, \quad (2.239)$$

i.e. the bound state-components vertex multiplied by the field propagators. The last passage is valid in presence of free propagators. A pictorial representation is indeed possible: the amplitude $A(q; p)$ connects the bound-state to the constituents, i.e. the vertex $\Gamma(q; p)$ plus the propagators of the 2 constituents (see the first piece of Fig. 2.16.e).

It is possible to show that in the non-relativistic limit the wave function in momentum space is given from [71]

$$A(\mathbf{q}; p) = ic \int_{-\infty}^{+\infty} \frac{dq^0}{2\pi} A(q; p). \quad (2.240)$$

where c is a normalization factor. In this way contact among Bethe-Salpeter and Quantum Mechanics is found. See, for example, [71, 72] for a detailed description of this point. The reason for this limit can be also easily understood: if in (2.237) we plug in the non-relativistic form for the bound state

$$|B(p = (M, \mathbf{0}))\rangle = |s\rangle = \frac{1}{\sqrt{V}} \int \frac{d^3k}{(2\pi)^{3/2}} A(\mathbf{k}) a^\dagger(\mathbf{k}) a^\dagger(-\mathbf{k}) |0\rangle \quad (2.241)$$

we get

$$A(q; p = (M, \mathbf{0})) = \delta(q^0) \frac{1}{\sqrt{V}(2E)} A(\mathbf{q}) \quad (2.242)$$

which, upon integration over q^0 gives, apart from some constant, exactly (2.240).

In our separable case we then have the following relation between the wave function and the vertex:

$$A(\mathbf{q}; p) = ic \int_{-\infty}^{+\infty} \frac{dq^0}{2\pi} \frac{g\tilde{\Phi}(q)}{((p/2 + q)^2 - m^2)((p/2 - q)^2 - m^2)}. \quad (2.243)$$

If now we suppose an instantaneous approximation with $\tilde{\Phi}(q) = \tilde{\Phi}(\mathbf{q})$, we can evaluate the residues of the integration over q^0 :

$$A(\mathbf{q}; p) = c \cdot g \cdot \tilde{\Phi}(\mathbf{q}) \frac{1}{2M\sqrt{\mathbf{q}^2 + m^2}} \cdot \left(\frac{1}{(M - 2\sqrt{\mathbf{q}^2 + m^2})} + \frac{1}{(M + 2\sqrt{\mathbf{q}^2 + m^2})} \right), \quad (2.244)$$

where the first term in the parenthesis comes from the pole for $q^{(0)} = -M/2 + \sqrt{\mathbf{k}^2 + m^2}$ and the second from $q^{(0)} = M/2 + \sqrt{\mathbf{k}^2 + m^2}$. The first pole is clearly dominant because of the non-relativistic condition $M \simeq 2m$; keeping only these term we find again equation (2.217):

$$\tilde{\Phi}(\mathbf{q}) = (const) * 2M\sqrt{\mathbf{q}^2 + m^2} (M - 2\sqrt{\mathbf{q}^2 + m^2}) A(\mathbf{q}; p). \quad (2.245)$$

It is gratifying to find the same results in two different ways, the first by studying a decay of the bound states in an intuitive way and the second from the more formal Bethe-Salpeter equation.

If we also keep the second term we have

$$\tilde{\Phi}(\mathbf{q}) = (const) * \sqrt{\mathbf{q}^2 + m^2} (M^2 - 4(\mathbf{q}^2 + m^2)) A(\mathbf{q}; p). \quad (2.246)$$

Note that we don't care about the value of the constant in front of these expressions. In fact, every physical prediction is invariant under the redefinition

$$\tilde{\Phi}(q) \rightarrow \xi \tilde{\Phi}(q) \tag{2.247}$$

where ξ is some constant. In fact, such a redefinition implies also

$$g \rightarrow \frac{g}{\xi} \tag{2.248}$$

thus leaving the Bethe-Salpeter vertex function $g\tilde{\Phi}(q)$ unchanged.

Chapter 3

Bound states of fermions

3.1 Fermion field

3.1.1 Dirac equation

The Dirac equation is the relativistic version of the Schroedinger one, and it is built under the following requirements [58]:

(a) to have a correct continuity equation, the Dirac equation must contain only a first order time derivative;

(b) to have covariance it must also have first order space derivatives;

(c) linearity and homogeneity;

(d) it should describe the spin, therefore the wave function should be a column-vector;

(e) compatibility with the Klein-Gordon equation, i.e. validity of the Einstein relation $E^2 = \mathbf{p}^2 + m^2$;

(f) the probability density must be positive definite and of the form $\rho = j^0 = \Psi^\dagger(x)\Psi(x)$; there must exist a corresponding conserved current $\partial_\mu j^\mu = 0$, so that $\int d^3x \cdot j^0 = 1$ can hold at every time, necessary for a probabilistic interpretation.

The points (a-d) are fulfilled by the equation

$$(i\gamma^\mu \partial_\mu - m) \Psi(x) = 0 \quad (3.1)$$

where γ^μ are matrices and the wave function $\Psi(x)$ is a column-vector of the same dimension as the matrices.

The conditions defining the γ -matrices are found using the last 2 points (e) and (f):

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot 1_4 \quad (3.2)$$

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0 \quad (3.3)$$

The condition (3.2) is obtained imposing (e); in fact multiplying (3.1) by $(i\gamma^\nu \partial_\nu + m)$ from the left we get

$$(i\gamma^\nu \partial_\nu + m) (i\gamma^\mu \partial_\mu - m) \Psi(x) = 0 \quad (3.4)$$

and making use of (3.2) we find exactly the K-G equation. From (3.2) one can also prove that the dimension of the γ -matrices is even and at least four [3, 55, 58].

The condition (3.3) on the gammas is found imposing (f); the current is

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi \quad (3.5)$$

where $\bar{\Psi} = \Psi^\dagger(x)\gamma^0$.

There is an infinite number of possible choice for the γ -matrices; here we will use the Dirac representation which allows one to "see" better the non-relativistic limit and the spin properties. The explicit form is

$$\gamma^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \quad (3.6)$$

where 1_2 is the 2×2 identity matrix and the σ^k are the Pauli matrices.

There are other matrices with whom one often works:

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3; \quad (3.7)$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \quad (3.8)$$

It is important to note that the matrices $1_4, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5$ form a basis of 16 matrices for the space of 4×4 complex matrices.

The references listed in section 2.1.1 explain in detail the Dirac equation.

3.1.2 Plane wave solutions

We search for solutions in the plane wave form

$$\Psi(x) = u(\mathbf{p})e^{-ipx} \quad (3.9)$$

where $u(p)$ is a column-matrix and $p \cdot x = g_{\mu\nu}p^\mu x^\nu = Et - \mathbf{p}\mathbf{x}$. Inserting in (3.1) we find

$$(\gamma^\mu p_\mu - m)u(\mathbf{p}) = 0 \quad (3.10)$$

Let us first consider the case $\mathbf{p} = 0$; by using the Dirac representation we have

$$\begin{pmatrix} E - m & 0 & 0 & 0 \\ 0 & E - m & 0 & 0 \\ 0 & 0 & E + m & 0 \\ 0 & 0 & 0 & E + m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = 0. \quad (3.11)$$

We now define the spinors $\chi^{(s)}$ where $s = \pm 1/2$ as

$$\chi^{(1/2)} = \chi^{(+)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \chi^{(-1/2)} = \chi^{(-)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.12)$$

Then it is clear that we have 4 possible solutions, 2 with $E = m$ and $u(\mathbf{0})$ being a linear combination of

$$u^{(\pm)}(\mathbf{0}) = \begin{pmatrix} \chi^{(\pm)} \\ 0 \end{pmatrix} \quad (3.13)$$

and 2 with $E = -m$ and $u(\mathbf{0})$ being a linear combination of

$$u_*^{(\pm)}(0) = \begin{pmatrix} 0 \\ \chi^{(\pm)} \end{pmatrix} \quad (3.14)$$

(with 0 we denote the zero column matrix of dimension 2).

We understand the need for dimension 4; 2 for the sign of the energy and 2 for the spin structure; in fact, the spin operator $\mathbf{S} = \frac{1}{2}\boldsymbol{\Sigma}$ (see section 3.2) has in the Dirac representation the form

$$S^k = \frac{1}{2}\Sigma^k = \frac{1}{2} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \quad (3.15)$$

and the basis of solutions is made of spin eigenstates along the z -axis.

We have so:

- $E = m$, spin up and spin down;
- $E = -m$, spin up and spin down.

If $\mathbf{p} \neq 0$ the matrix equation reads

$$Eu(\mathbf{p}) = \begin{pmatrix} m & \sigma_i p_i \\ \sigma_i p_i & -m \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = E \begin{pmatrix} u_A \\ u_B \end{pmatrix} \quad (3.16)$$

where u_A and u_B are spinors and with m we denote $m \cdot 1_2$.

We have the two spinorial equations

$$(\sigma_i p_i)u_B = (E - m)u_A \quad (3.17)$$

$$(\sigma_i p_i)u_A = (E + m)u_B \quad (3.18)$$

From the last equation we get

$$u_B = \frac{\sigma_i p_i}{E + m}u_A, \quad (3.19)$$

and substituting it in the first we find

$$E^2 = \mathbf{p}^2 + m^2. \quad (3.20)$$

The basis for the solutions with $E > 0$ is:

$$u^{(s)}(\mathbf{p}) = N \begin{pmatrix} \chi^{(s)} \\ \frac{\sigma_i p_i}{E+m}\chi^{(s)} \end{pmatrix}; \quad (3.21)$$

in this case the solutions tend to the ones analyzed for zero momentum in the limit $\mathbf{p} \rightarrow 0$. N is a normalization constant.

Similarly for $E < 0$ we have

$$u_*^{(s)}(\mathbf{p}) = N \begin{pmatrix} \frac{\sigma_i p_i}{E-m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix} = N \begin{pmatrix} \frac{-\sigma_i p_i}{|E|+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix}. \quad (3.22)$$

We have again 4 independent solutions, two with positive energy and two with negative energy. It is important to note that the 2 solutions with the same energy are not spin-eigenstates; this is the case only when $\mathbf{p} \rightarrow 0$.

The problem of the energy is not trivial; it seems that the energy does not admit a minimum, creating than difficulties in the interpretation of the theory. Actually, in the classical case there is not trace of negative energy; in the classical version we have only $E = \mathbf{p}^2/2m$ and not the one with the opposite sign.

At a relativistic level we have a complete symmetry of the two solutions. The non-existence of a minimum for the energy would generate a catastrophe because an electron with $E > 0$ could collapse into lower energy states radiating off an infinite quantity of energy.

Dirac himself tried to find a way out of this paradox in the famous interpretation of the Dirac sea: all the negative energy states are supposed to be filled, and in virtue of the Pauli principle no positive state can decay into negative energy states. On the other hand it is possible to excite such a non-trivial "ground state" pulling out an electron from the sea and leaving a hole in it, which can be interpreted as a positron, a positive energy state with the same mass m as the electron but inverted quantum properties.

The problem of the Dirac theory is connected with the idea of a one-particle theory: exactly as the Schroedinger equation, the Dirac one describes just one particle, and not more. This doesn't seem to be possible in nature; in quantum field theory the number of particles is not fixed anymore and the electron-field Ψ itself is quantized. The vacuum becomes a non-trivial object where virtual particles can create and annihilate in it. As done in the quantization of the scalar field (see chapter 2), we will impose the existence of a ground state, i.e. a state for which the energy is minimized. But there is a subtle point: this vacuum doesn't mean that only the two solutions with positive energy survive, but that antiparticles appear.

What classically is a negative energy solution will be reinterpreted as a positive energy solution with inverted quantum numbers, i.e. an antiparticle.

3.1.3 Lagrangian form

In the second chapter we started from the Lagrangian formulation of the scalar field; in this case the Lagrangian, whose equation of motion is the Dirac one, is:

$$\mathcal{L} = \bar{\Psi}(x) (i\gamma^\mu \partial_\mu - m) \Psi(x). \quad (3.23)$$

There is still a point we should take care of; the condition $\partial_\mu j^\mu = 0$ is, of course, conserved also at the Lagrangian level, being the Noether current corresponding to the transformation $\Psi \rightarrow e^{i\alpha} \Psi$, but nothing is said about the value of the conserved charge $Q = \int d^3x \cdot j^0$. The condition $Q = 1$ is an extra-condition necessary for a "wave function interpretation" of the Dirac equation, i.e. the probability at every time of finding the fermion somewhere in the space must be one. Such a condition is necessary in a "one particle theory" such as the Dirac equation. The energy density is

$$\mathcal{H} = \Psi^\dagger (-i\alpha \cdot \nabla + \beta m) \Psi \quad (3.24)$$

where $\alpha^k = \gamma^0 \gamma^k$ and $\beta = \gamma^0$. On shell (i.e. when the equations of motion are satisfied) we get $\mathcal{H} = \Psi^\dagger i \frac{\partial}{\partial t} \Psi$, where we find the famous operatorial association $E \longleftrightarrow i\partial/\partial t$. The appearance of negative energy is also clear at this level.

3.1.4 Feynman interpretation

In QFT the negative energy states are interpreted in the Feynman way, which we briefly discussed in the second chapter (2.3.1). If a negative energy particle is propagating back in

time, what we will see is a positive energy state with some inverted characteristics. Let us consider the plane wave solution

$$\Psi(x) = u_*^{(s)}(\mathbf{p})e^{-ipx}; \quad (3.25)$$

it represents a state with $p^0 = E < 0$, spin s , momentum \mathbf{p} . If now the propagation is backward in time, we would experience a positive energy state with spin $-s$ and impulse $-\mathbf{p}$. If then we consider the shift $\mathbf{p} \rightarrow -\mathbf{p}$ and $s \rightarrow -s$ we have the state

$$\Psi(x) = u_*^{(-s)}(-\mathbf{p})e^{ip'x} \quad (3.26)$$

where now $p' = (|p^0| > 0, \mathbf{p})$; this is still a negative energy state ($i\partial/\partial t$ projects $-|p^0| < 0$) with spin $-s$ and impulse $-\mathbf{p}$, but if we reinterpret it, it represents a positive energy state with energy $|p^0| > 0$, spin s , momentum \mathbf{p} .

We introduce therefore the following spinors $v^{(s)}(\mathbf{p})$ as

$$v^{(s)}(\mathbf{p}) = N \begin{pmatrix} \frac{\sigma_i p_i}{E+m} \varepsilon \chi^{(s)} \\ \varepsilon \chi^{(s)} \end{pmatrix} \quad (3.27)$$

where

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad (3.28)$$

ε changes the direction of the spin, in accord with the given interpretation. We have so

$$v^{(+)}(\mathbf{p}) = -u_*^{(-)}(-\mathbf{p}), \quad (3.29)$$

$$v^{(-)}(\mathbf{p}) = u_*^{(+)}(-\mathbf{p}). \quad (3.30)$$

The presence of the minus in the first of the last two equations seems not important here; it seems just an arbitrary phase; it has indeed its meaning, connected with the charge conjugation and with the spin structure of an antiparticle.

Now one may ask what does it mean to speak of a negative energy state propagating backward in time; it sounds a bit weird. This is because we try to describe some aspects which are not fully included in this theory, but this interpretation helps the understanding: there are in fact only two ways to get positive energy states, one is from the natural $E > 0$ states, the second by this reinterpretation of the negative energy states. In both cases we are left with positive energy states propagating forward in time, i.e. with physical meaningful quantities.

3.1.5 Normalization and general solution

The spinors $u^{(s)}(\mathbf{p})$ are normalized imposing that

$$\bar{u}^{(s)}(\mathbf{p})u^{(r)}(\mathbf{p}) = \delta_{sr} \quad (3.31)$$

from which one gets:

$$N = \sqrt{\frac{E+m}{2m}}. \quad (3.32)$$

As a consequence, for the spinors $v^{(s)}(\mathbf{p})$ one gets

$$\bar{v}^{(s)}(\mathbf{p})v^{(r)}(\mathbf{p}) = -\delta_{sr}; \quad (3.33)$$

this minus sign has a tremendous influence with respect to microcausality, where anticommutation (and not commutation) relations will be necessary.

We now take a positive energy solution like

$$\Psi_p^{(s)}(x) = C u^{(s)}(\mathbf{p}) e^{-ipx} \quad (3.34)$$

and we normalize it in such a way that (in the box):

$$\int d^3x \Psi_p^{(s)\dagger}(x) \Psi_{p'}^{(r)}(x) = \delta_{sr} \delta_{\mathbf{p}\mathbf{p}'} \quad (3.35)$$

finding for C

$$C = \frac{1}{\sqrt{V}} \sqrt{\frac{m}{E}}; \quad (3.36)$$

the same is valid for the negative energy solutions.

We write then a general solution of the Dirac equation as

$$\Psi(x) = \sum_{s=\pm 1/2} \sum_{\mathbf{p}} \frac{1}{\sqrt{V}} \sqrt{\frac{m}{E}} \left(b^{(s)}(\mathbf{p}) u^{(s)}(\mathbf{p}) e^{-ipx} + d^{(s)*}(\mathbf{p}) v^{(s)}(\mathbf{p}) e^{ipx} \right) \quad (3.37)$$

or in the continuum limit

$$\Psi(x) = \sum_{s=\pm 1/2} \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}} \left(b^{(s)}(\mathbf{p}) u^{(s)}(\mathbf{p}) e^{-ipx} + d^{(s)*}(\mathbf{p}) v^{(s)}(\mathbf{p}) e^{ipx} \right) \quad (3.38)$$

where we have integrated over all the momenta and we have positive and negative energy decomposition; $b^{(s)}(\mathbf{p})$ and $d^{(s)*}(\mathbf{p})$ are the coefficients of the plane wave expansion. They are generic complex numbers, apart from the normalization constraint $Q = 1$.

3.2 Transformations

We now discuss some transformations and symmetries of the Dirac equation. We do it briefly for the Lorentz ones (deeply discussed in many books) and a bit more deeply for parity and charge transformations, which play an important role in the description of bound states such as mesons.

3.2.1 Lorentz transformation

The Dirac equation was built respecting covariance (point (b) of section 3.1), as the Einstein equation $E^2 = \mathbf{p}^2 + m^2$ confirms.

Under a Lorentz transformation $x' = \Lambda x$ it can be proved [3, 55, 58] that the wave function transforms like

$$\Psi'(x') = e^{-\frac{i}{4} \omega^{\mu\nu} \sigma_{\mu\nu}} \Psi(x) \quad (3.39)$$

where $\Lambda_\nu^\mu = g_\nu^\mu + \omega_\nu^\mu$. Boosts and rotations are included in this transformation. Ψ is a vectorial, better a spinorial field, whose components are rotated under a space-time rotation. In particular, the spin is given by

$$i \frac{[\gamma^i, \gamma^j]}{2} = \frac{\sigma^{ij}}{2} = \varepsilon^{ijk} S^k = \varepsilon^{ijk} \frac{\Sigma^k}{2}; \quad (3.40)$$

it can be easily proved that the S^k satisfy the usual commutation relations

$$[S^k, S^i] = i\varepsilon^{kij} S^j \quad (3.41)$$

and their explicit form in the Dirac representation is exactly the one given in section 3.2. The spin is an angular momentum intrinsic in the nature of the fermionic particle and comes directly in the game when one analyzes covariance.

3.2.2 Parity

We now consider the space inversion:

$$x = (t, \mathbf{x}) \rightarrow (t, -\mathbf{x}) = x' \quad (3.42)$$

corresponding to the improper Lorentz transformation

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (3.43)$$

Be $\Psi(x)$ a solution in the x frame:

$$(i\gamma^\mu \partial_\mu - m) \Psi(x) = (i\gamma^0 \partial_0 + i\gamma^k \partial_k - m) \Psi(x) = 0 \quad (3.44)$$

We search a new function $\Psi'(x')$ that is a solution of the Dirac equation in the x' frame and that is connected to $\Psi(x)$ by $\Psi'(x') = S_p \Psi(x)$, where S_p is a linear unitary 4x4 matrix independent on space-time.

For $\Psi'(x')$ we have

$$(i\gamma^\mu \partial'_\mu - m) \Psi'(x') = (i\gamma^0 \partial_0 - i\gamma^k \partial_k - m) S_p \Psi(x) = 0 \quad (3.45)$$

Multiplying from the left by S_p^{-1} we get

$$\left(i (S_p^{-1} \gamma^0 S_p) \partial_0 - i (S_p^{-1} \gamma^k S_p) \partial_k - m \right) \Psi(x) = 0 \quad (3.46)$$

which is satisfied if

$$(S_p \gamma^0 S_p^{-1}) = \gamma^0, \quad (3.47)$$

$$(S_p \gamma^k S_p^{-1}) = -\gamma^k. \quad (3.48)$$

A solution is then

$$S_p = \eta_p \gamma^0 \quad (3.49)$$

where η_p is just a phase we can set to one.

Remembering the discussion of Chapter 2, section 2.2.5, we realize that we have a symmetry of the equation. If $\Psi(x)$ is a solution of the Dirac equation, then $\Psi'(t, \mathbf{x}) = \gamma^0 \Psi(t, -\mathbf{x})$ is a solution as well. This is actually the wave function that our colleague in the x' frame experiences.

From the form of γ^0 in the Dirac representation we can see "in nuce" the opposite intrinsic parity of a particle and an antiparticle; in fact there is an extra minus down right.

This fact is particularly important for the determination of the parity of a meson, which is a bound state of two fermions, a quark and an antiquark.

3.2.3 C-Transformation

Up to now we dealt with a one-particle theory with positive and negative energy solutions; as anticipated, the theory is in this form not complete. QFT shows that the negative energy states are reinterpreted as antiparticles, according to the discussed Feynman-interpretation.

We can in any case discuss the C-transformation, i.e. the interchange of particles with antiparticles, noting that it has to transform positive energy states into negative energy states. So, an electron with spin up and momentum \mathbf{p} will be transformed into a negative energy state with spin down and impulse $-\mathbf{p}$, which can be thought as flying back in time, being then a positron with spin up and impulse \mathbf{p} . This can be understood consistently only in the framework of QFT: here it's not completely clear what one means by saying "going backwards in time", and it must be accepted as an heuristic approach to the problem.

Nevertheless we can do it: in order to change the sign of the energy and the momentum it is enough to consider the conjugate of the wave function:

$$\Psi \rightarrow \Psi^* \quad (3.50)$$

We have to find a new solution of the Dirac equation which contains Ψ^* ; more precisely we search a solution like

$$S_c \bar{\Psi}^t = S_c (\gamma^0)^t \Psi^* \quad (3.51)$$

where $(\gamma^0)^t$ is introduced for convenience.

In order to do it let us start taking the conjugate of the Dirac equation:

$$(-i(\gamma^\mu)^* \partial_\mu - m) \Psi^*(x) = 0; \quad (3.52)$$

then, by inserting $(\gamma^0)^t (\gamma^0)^t = 1$ we get

$$(-i(\gamma^\mu)^* (\gamma^0)^t \partial_\mu - m (\gamma^0)^t) \bar{\Psi}^t = 0. \quad (3.53)$$

Now, consider that

$$(\gamma^\mu)^* (\gamma^0)^t = (\gamma^0 (\gamma^\mu)^\dagger)^t = (\gamma^0 \gamma^0 \gamma^\mu \gamma^0)^t = (\gamma^0)^t (\gamma^\mu)^t \quad (3.54)$$

we find the following equation

$$(i(-\gamma^\mu)^t \partial_\mu - m) \bar{\Psi}^t = 0. \quad (3.55)$$

We have to find S_c such that

$$(i\gamma^\mu \partial_\mu - m) S_c \bar{\Psi}^t = 0; \quad (3.56)$$

multiplying from the left with S_c^{-1} the last equation is fulfilled if

$$S_c^{-1} \gamma^\mu S_c = (-\gamma^\mu)^t; \quad (3.57)$$

this defines S_c .

Let us consider the Dirac representation; in this case we can find S_c like

$$S_c = \eta_C i \gamma^2 \gamma^0 = -\eta_C \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}; \quad (3.58)$$

where η_C is just a phase we fix to -1 (in order to get rid of the extra-minus in front of the matrix).

If we take the two positive energy solutions

$$\Psi^{(s)} = u^{(s)}(\mathbf{p})e^{-ipx} \quad (3.59)$$

and we consider the transformation

$$\Psi \rightarrow \Psi_c = S_c \bar{\Psi}^t, \quad (3.60)$$

taking into account that

$$\Psi^{(s)*} = u^{(s)*}(\mathbf{p})e^{ipx} = N \left(\begin{array}{c} \chi^{(s)} \\ \frac{(\sigma_1 p_1 - \sigma_2 p_2 + \sigma_3 p_3)}{E+m} \chi^{(s)} \end{array} \right) e^{ipx} \quad (3.61)$$

(σ_2 changes sign through complex conjugation) we have

$$\begin{aligned} \Psi_c^{(s)} &= S_c \gamma^0 \Psi^{(s)*} = \\ &= \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} N \left(\begin{array}{c} \chi^{(s)} \\ \frac{(\sigma_1 p_1 - \sigma_2 p_2 + \sigma_3 p_3)}{E+m} \chi^{(s)} \end{array} \right) e^{ipx} = v^{(s)}(\mathbf{p})e^{ipx}. \end{aligned} \quad (3.62)$$

(Note that ε commutes with σ_2 but anticommutes with σ_1 and σ_3).

We get exactly the solution written through the spinors $v^{(s)}$, as we correctly argued in section 2.4. This procedure also shows that the spinors $v^{(s)}(\mathbf{p})$ are the correct ones for the description of antiparticles.

3.3 Second Quantization

We have now to go from the Dirac equation theory to quantum field theory, which describe many particles. The function $\Psi(x)$ becomes a field:

$$\Psi(x) \rightarrow \hat{\Psi}(x); \quad (3.63)$$

instead of the function $\Psi(x)$ we have an operator $\hat{\Psi}(x)$; all the physical information about the system is contained in a vector $|s\rangle$, belonging to the vectorial space F of the physical states, which by assumption is an Hilbert-space. We follow step by step the arguments of the second chapter (see 2.2.1), apart from the requirement of Hermiticity of the field operator, which here is not imposed because the quantum mechanical field is not real.

For the quantized field $\hat{\Psi}(x)$, instead of the complex number $b^{(s)}(\mathbf{p})$ and $d^{(s)*}(\mathbf{p})$, we have now two operators $b^{(s)}(\mathbf{p})$ and $d^{(s)\dagger}(\mathbf{p})$. The field operator $\hat{\Psi}(x)$ looks like:

$$\hat{\Psi}(x) = \sum_{s=\pm 1/2} \int \frac{d^3 p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}} \left(b^{(s)}(\mathbf{p}) u^{(s)}(\mathbf{p}) e^{-ipx} + d^{(s)\dagger}(\mathbf{p}) v^{(s)}(\mathbf{p}) e^{ipx} \right). \quad (3.64)$$

Calling P_μ the four-momentum operator we assume the validity of the Heisenberg equation:

$$\partial_\mu \hat{\Psi}(x) = i \left[P_\mu, \hat{\Psi}(x) \right], \quad (3.65)$$

from which it follows that

$$\left[P_\mu, b^{(s)}(\mathbf{p}) \right] = -p_\mu b^{(s)}(\mathbf{p})$$

$$\begin{aligned}
[P_\mu, d^{(s)\dagger}(\mathbf{p})] &= p_\mu d^{(s)\dagger}(\mathbf{p}) \\
[P_\mu, b^{(s)\dagger}(\mathbf{p})] &= p_\mu b^{(s)\dagger}(\mathbf{p}) \\
[P_\mu, d^{(s)}(\mathbf{p})] &= -p_\mu d^{(s)}(\mathbf{p}).
\end{aligned} \tag{3.66}$$

We assume that the theory admits a vacuum $|0\rangle$, (see the discussion of chapter 2, section 2.2.3 and in the beginning of the present chapter) for which we fix the four-momentum to be zero:

$$P_\mu |0\rangle = 0; \tag{3.67}$$

the vacuum is *by definition* the state with minimal energy, which we set to zero. In order this being possible it is clear from the relations (3.66) that we need

$$b^{(s)}(\mathbf{p}) |0\rangle = d^{(s)}(\mathbf{p}) |0\rangle = 0, \tag{3.68}$$

otherwise we would not have a minimum for the energy. In this way we get rid of the negative energy states in a similar way as in the second chapter. Note that in that case there was no negative energy at a classic field level, while here also at the Dirac equation level (first quantization) there was. In any case the elimination of these states is based on the fundamental assumption of the existence of a minimum of the energy, i.e. on the existence of the vacuum.

We are left with four possibilities: 2 from $b^{(s)\dagger}(\mathbf{p})$ and 2 from $d^{(s)\dagger}(\mathbf{p})$; the state $b^{(s)\dagger}(\mathbf{p}) |0\rangle$ represents an electron with spin s and four-momentum p ; analogously $d^{(s)\dagger}(\mathbf{p}) |0\rangle$ represents a positron with spin s and four-momentum p .

At this point we have to introduce the commutation rules; we know from experiments that two fermions cannot occupy the same state, according to the well known Pauli principle. Therefore we postulate:

$$\{b^{(s)}(\mathbf{p}_1), b^{(r)\dagger}(\mathbf{p}_2)\} = \delta_{rs}\delta(\mathbf{p}_1 - \mathbf{p}_2); \tag{3.69}$$

$$\{d^{(s)}(\mathbf{p}_1), d^{(r)\dagger}(\mathbf{p}_2)\} = \delta_{rs}\delta(\mathbf{p}_1 - \mathbf{p}_2), \tag{3.70}$$

and zero for all the others anticommutators.

If we consider a state with two electrons we have:

$$|s\rangle = b^{(s)\dagger}(\mathbf{p}_1)b^{(r)\dagger}(\mathbf{p}_2) |0\rangle = -b^{(r)\dagger}(\mathbf{p}_2)b^{(s)\dagger}(\mathbf{p}_1) |0\rangle; \tag{3.71}$$

this happens in accord with the Fermi-Dirac statistic; furthermore, if $\mathbf{p}_1 = \mathbf{p}_2$ and $s = r$ then we have $|s\rangle = 0$, in accord with the Pauli principle.

The introduced anticommutation rules are also consistent with microcausality; if we have an observable $A(t, \mathbf{x})$ we expect that

$$[A(t, \mathbf{x}), A(t, \mathbf{y})] = 0 \tag{3.72}$$

if $\mathbf{x} \neq \mathbf{y}$; first note that $\Psi(x)$ is itself not an observable (it is not Hermitian), but with $\Psi(x)$ we can construct observable quantities such as $\Psi^\dagger(x)\Psi(x)$, $\bar{\Psi}(x)\Psi(x)$, these quantities satisfy

microcausality, justifying the choice of the anticommutation relations; this can be proved noting that we have

$$\{\Psi(t, \mathbf{x}), \bar{\Psi}(t, \mathbf{y})\} = \gamma^0 \delta^3(\mathbf{x} - \mathbf{y}),$$

$$\{\Psi(t, \mathbf{x}), \Psi(t, \mathbf{y})\} = \{\bar{\Psi}(t, \mathbf{x}), \bar{\Psi}(t, \mathbf{y})\} = 0 \quad (3.73)$$

and using the identity

$$[AB, CD] = A\{B, C\}D - AC\{B, D\} - C\{A, D\}B + \{C, A\}DB. \quad (3.74)$$

In virtue of (3.73) this quantity vanishes for space like separations, in accord with microcausality. The relations (3.73) are the key of this phenomenon.

If one introduced commutation relation instead of anticommutation one would find a series of inconsistencies [3]. Similarly, if one tried to impose anticommutation relations in the scalar case, a violation of microcausality would occur. This is "in nuce" the spin-statistic connection, asserting that bosons satisfy the Bose-Einstein statistic, and fermions the Fermi-Dirac statistics.

In the end, let us note that in (3.64) the creation and annihilation operators come together with the corresponding spinors. At a diagrammatic level it means to attach at each external line a spinor, more precisely a $u^{(r)}(\mathbf{p})$ for an incoming fermion, $\bar{u}^{(r)}(\mathbf{p})$ for an outgoing one, $\bar{v}^{(r)}(\mathbf{p})$ for an incoming antifermion and $v^{(r)}(\mathbf{p})$ for an outgoing one (see references in the end of section 2.1.1).

3.3.1 Parity

We now want to discuss parity and charge transformation in the context of QFT. $\Psi(x)$ is now an operator (we omit the "hat" for simplicity); if we consider a parity transformation

$$x = (t, \mathbf{x}) \rightarrow (t, -\mathbf{x}) = x', \quad (3.75)$$

there must exist in QFT an operator U_P with whom the states transform:

$$|s\rangle \rightarrow U_P |s\rangle; \quad (3.76)$$

equivalently we can transform an operator as (see 2.2.5)

$$F \rightarrow U_P^{-1} F U_P. \quad (3.77)$$

Let us now take into account the field operator $\Psi(x)$; at the level of the Dirac equation, we have seen (3.2.2) how the wave function transforms under parity transformations:

$$\Psi'(t, \mathbf{x}) = \gamma^0 \Psi(t, -\mathbf{x}); \quad (3.78)$$

we then expect to have

$$U_P^{-1} \Psi(x) U_P = \gamma^0 \Psi(t, -\mathbf{x}). \quad (3.79)$$

Considering the field expansion (3.64) we have:

$$U_P^{-1} \Psi(x) U_P =$$

$$\begin{aligned}
& \sum_{s=\pm 1/2} \int \frac{d^3 p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}} \left(U_P^{-1} b^{(s)}(\mathbf{p}) U_P u^{(s)}(\mathbf{p}) e^{-ipx} + U_P^{-1} d^{(s)\dagger}(\mathbf{p}) U_P v^{(s)}(\mathbf{p}) e^{ipx} \right) \\
& \qquad \qquad \qquad = \gamma^0 \Psi(t, -\mathbf{x}) = \\
& \sum_{s=\pm 1/2} \int \frac{d^3 p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}} \left(b^{(s)}(\mathbf{p}) \gamma^0 u^{(s)}(\mathbf{p}) e^{-iEt - i\mathbf{p}\mathbf{x}} + d^{(s)*}(\mathbf{p}) \gamma^0 v^{(s)}(\mathbf{p}) e^{iEt + i\mathbf{p}\mathbf{x}} \right). \quad (3.80)
\end{aligned}$$

Changing the sign in the three-momentum integration of the last integral and taking into account that

$$\gamma^0 u^{(s)}(-\mathbf{p}) = u^{(s)}(\mathbf{p}) \quad (3.81)$$

$$\gamma^0 v^{(s)}(-\mathbf{p}) = -v^{(s)}(\mathbf{p}) \quad (3.82)$$

(this is easily seen in the Dirac representation, but it is true in any case) we then find by comparison the transformation for the creation and annihilation operators:

$$U_P b^{(s)\dagger}(\mathbf{p}) U_P^{-1} = b^{(s)\dagger}(-\mathbf{p}), \quad (3.83)$$

$$U_P d^{(s)\dagger}(\mathbf{p}) U_P^{-1} = -d^{(s)\dagger}(-\mathbf{p}). \quad (3.84)$$

As we can see there is again a minus sign when compared to the positron creation operator; this is due to the already mentioned opposite parity of a fermion and antifermion. This minus sign plays a crucial role in the study of the mesons, which are bound states of a fermion and an antifermion: the parity of a meson state is reversed because of this minus sign. Considering that the vacuum is invariant under parity transformations, we have for an electron state:

$$\begin{aligned}
|e^-(\mathbf{p}, s)\rangle &= b^{(s)\dagger}(\mathbf{p}) |0\rangle \rightarrow U_P |e^-(\mathbf{p}, s)\rangle = U_P b^{(s)\dagger}(\mathbf{p}) |0\rangle = \\
&= U_P b^{(s)\dagger}(\mathbf{p}) U_P U_P^{-1} |0\rangle = b^{(s)\dagger}(-\mathbf{p}) |0\rangle = |e^-(-\mathbf{p}, s)\rangle; \quad (3.85)
\end{aligned}$$

for the positron state we have following analogous transformations:

$$|e^+(\mathbf{p}, s)\rangle \rightarrow -|e^+(-\mathbf{p}, s)\rangle. \quad (3.86)$$

The origin of this minus is contained in the nature of γ^0 and in the spinorial structure of the $\Psi(x)$.

3.3.2 Charge Transformation

We want to do the same for the charge transformation; of course we expect that the corresponding operator U_C transforms the electron creators to the positron ones and vice-versa; in formulas:

$$U_C b^{(s)\dagger}(\mathbf{p}) U_C^{-1} = d^{(s)\dagger}(\mathbf{p}), \quad (3.87)$$

$$U_C d^{(s)\dagger}(\mathbf{p}) U_C^{-1} = b^{(s)\dagger}(\mathbf{p}). \quad (3.88)$$

This is indeed the case if U_C is given by

$$U_C^{-1}\Psi(x)U_C = S_c\bar{\Psi}^T. \quad (3.89)$$

This justifies *a posteriori* what we did previously only in the framework of the Dirac equation.

The last equations can be proved taking into account that:

$$S_c\bar{u}^{(s)T}(\mathbf{p}) = v^{(s)}(\mathbf{p}) \quad (3.90)$$

$$S_c\bar{v}^{(s)T}(\mathbf{p}) = u^{(s)}(\mathbf{p}). \quad (3.91)$$

The total effect of the C -transformation is to invert particles to antiparticles; this one also is important in the meson study.

Ergo, we have for the transformation of an electron and a positron:

$$|e^-(\mathbf{p}, s)\rangle \rightarrow |e^+(\mathbf{p}, s)\rangle; \quad (3.92)$$

$$|e^+(\mathbf{p}, s)\rangle \rightarrow |e^-(\mathbf{p}, s)\rangle. \quad (3.93)$$

3.4 Propagator

3.4.1 Basic formulas

We now turn to the fermion propagator, in particular discussing some properties of the quark propagator.

As discussed in the second chapter (2.3), the propagator in QFT is given by the two point Green function

$$S_{\alpha,\beta}(x-y) = -i\langle 0|T[\Psi_\alpha(x)\bar{\Psi}_\beta(y)]|0\rangle = \int \frac{d^4q}{(2\pi)^4} (-iS_{\alpha,\beta}(q)) e^{-iq(x-y)}, \quad (3.94)$$

where T refers to the time ordered product, this time defined for fermion operators as

$$T[\Psi_\alpha(x)\bar{\Psi}_\beta(y)] = \begin{cases} \Psi_\alpha(x)\bar{\Psi}_\beta(y) & \text{if } x^0 > y^0 \\ -\bar{\Psi}_\beta(y)\Psi_\alpha(x) & \text{if } y^0 > x^0 \end{cases}. \quad (3.95)$$

The extra-minus results from the anticommutation properties of a fermion field. Using the expression (3.64) one can easily calculate the propagator, finding

$$S(q) = \frac{i}{\not{q} - m} = i\frac{\not{q} + m}{q^2 - m^2} \quad (3.96)$$

This is of course the free propagator, when no interaction is considered. Including interactions causes a modification of the propagator, as discussed in the second chapter for the scalar case (section 2.4.3). A general form for the fermionic propagator is then [24]

$$S(q) = i\frac{Z(q^2)}{\not{q} - m(q^2)} \quad (3.97)$$

where the physical mass is then given by $m_P^2 = m(q^2 = m_P^2)$. The calculation of the running functions $Z(q^2)$ and $m(q^2)$ within QFT is a very difficult task [24, 25, 79]. We now give some examples from the two most famous quantum theories, namely QED and QCD .

3.4.2 QED case

We consider the electron. The physical mass of this particle is known and precisely measured, and we call it m_e . The *QED* Lagrangian [3, 54, 55, 58] describes the interaction of electrons and photons, and is written out as:

$$\mathcal{L}_{QED} = \bar{\Psi}(x) (i\gamma^\mu \partial_\mu - m) \Psi(x) + e\bar{\Psi}(x)\gamma^\mu\Psi(x)A_\mu(x) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (3.98)$$

where $\Psi(x)$ is the fermionic electron field, m its (bare) mass, A_μ the photon field with e the (bare) coupling constant. $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field tensor responsible for the description of free photons. The fact that the photon itself carries an index μ means that it has a spin, more precisely spin $S = 1$. It is not my purpose now to discuss the rich structure of the *QED* Lagrangian, but only describe some features of the electron propagator.

After summing over the self-energy Feynman diagrams, one can show [54] that the following form for the electron propagator is valid:

$$S_e(q) = \frac{i}{\not{q} - m_e} (1 + \text{corrections}). \quad (3.99)$$

This formula is valid for small q^2 , because in this regime the QED coupling is small. As discussed in the first chapter, in the framework of a renormalizable QFT the physical coupling constant becomes a function of μ , i.e. the energy scale at which we are performing an experiment. In QED one then has $e_P = e_P(\mu^2)$ where the suffix P again means "physical". With "corrections" we intend in fact a series in e_P . This propagator, being valid for small momenta, has indeed a pole for m_e , i.e. for the physical electron mass, as it should be.

If one would consider *QED* at very high momenta, where e_P becomes very large, the previous expression would not be longer valid. In fact, the series with whom one expresses the corrections would not be perturbative anymore. The coupling constant grows for increasing μ^2 , and it is still an open question to understand what is going on in the non-perturbative *QED* sector. *QED* is a renormalizable theory, as the φ^4 theory we described in the previous chapter. However, we don't know what happens at large momenta. A cut-off enters in the game, on which the results depend on. This dependence, although much suppressed for our energy scale, may change the results in the high energy sector. In this sense QED can also be a low energy effective theory of a larger framework we still have to discover, with the cut-off depending on that theory.

3.4.3 QCD case

The QCD Lagrangian describes quarks and gluons (see the introduction, chapter 1) and has a similar form to the QED one. The gluon-quark interaction term looks like [3], restricting to the light quark flavor u

$$\mathcal{L}_{QCD,quark-gluon} = g\bar{u}(x)\gamma^\mu \frac{\lambda^a}{2} u(x) G_\mu^a(x) \quad (3.100)$$

where $u(x)$ is the u-quark fermion field, g the bare coupling and $G_\mu^a(x)$ the gluon field. The extra-index a refers to another quantum number, the color, which is exchanged in the interaction. λ^a is a $SU(3)$ Gell-Mann matrix. We refer to the sixth chapter (and Refs. there) for a more precise discussion of the Lagrangian; here we are interested in some aspects of

the quark propagator. The running renormalized physical coupling $g_P(\mu^2)$ shows an opposite behavior to the QED case of $e_P(\mu^2)$: it is big for small μ and small for large ones. In case of an high-energy process involving a u-quark, a very useful form for the u-quark propagator has been used:

$$S_u(q) = \frac{i}{\not{q} - m_u}(1 + \text{corrections}) \quad (3.101)$$

where $m_u \simeq 5 \text{ MeV}$. The corrections can here be calculated within *QCD* and a perturbative expansion is possible because the coupling is small for high quark momenta.

A very important point should now be stressed. $S_u(q)$ is valid for large $|q^2|$, i.e. $S_u(q)$ does NOT have a pole for $q^2 = m_u^2$. The parameter m_u is not the physical mass of the quark, but a useful parameter [12, 80]. No free quark has been seen up to now.

The previous high energy approximation for the quark propagator is not valid in the low-energy regime. This is the range set by the scale of *QCD* bound states, i.e. hadrons. At this level another form for the quark propagator has been used, giving rise to a successful descriptions of the bound state phenomenology for hadron masses below 1 GeV :

$$S_u(q) = \frac{i}{\not{q} - m_u^*} \quad (3.102)$$

where m_u^* is of the order of $300 - 500 \text{ MeV}$ (depending on the model; see [21, 22, 23, 24, 29, 67, 81]), i.e. much larger than the previous u-quark mass. No "corrections" are indicated here because in this QCD regime no expansion in the coupling is possible. This form is a phenomenological one, where the quark mass m_u^* has been fitted to reproduce some experimental quantities, as, for example, the pion mass. The non-perturbative QCD regime causes a dramatic change in the quark propagator: a very large "mass" comes out at low energy. A note of care: the last form for $S_u(q)$ is valid for $|q^2| < m_u^{*2}$, and again no pole is present. The parameter m_u^* is NOT the physical quark mass.

Here we are discussing at a propagator level the phenomenon of "dressing" discussed in the Introduction: the "high energy" light quarks get heavy because they "attract" quark-antiquark pairs from the quantum sea around them.

To extract the quark propagator directly from QCD has been proven to be a difficult task. A full description of this quantity should show the low energy and the high energy limits discussed above. Recently many progresses have been made for the evaluation of the quark propagator, specially from Lattice QCD [82] and from Dyson-Schwinger equation studies [24, 25, 79, 83]. The running mass function $m_u(q^2)$ has been calculated for Euclidean momenta, i.e. for $q^2 < 0$. The correct limits

$$\lim_{q^2 \rightarrow -\infty} m_u(q^2) = m_u \simeq 5 \text{ MeV} \quad (3.103)$$

$$\lim_{q^2 \rightarrow 0^-} m_u(q^2) = m_u^* \simeq 300 - 500 \text{ MeV} \quad (3.104)$$

are found. The typical schematic behavior of the running mass function has been displayed in Fig. 3.1, where one can see in the Euclidean sector the connection of the nonperturbative to the perturbative regime.

The calculation of this quantity in the Minkowski region ($q^2 > 0$) is even more difficult. First attempts in this direction have been done within the DSE method, trying to extrapolate this quantity to the other part of the plane [79, 83]. In order to calculate hadron properties one needs to know the quark propagator inside the parabola described in section 2.9, which

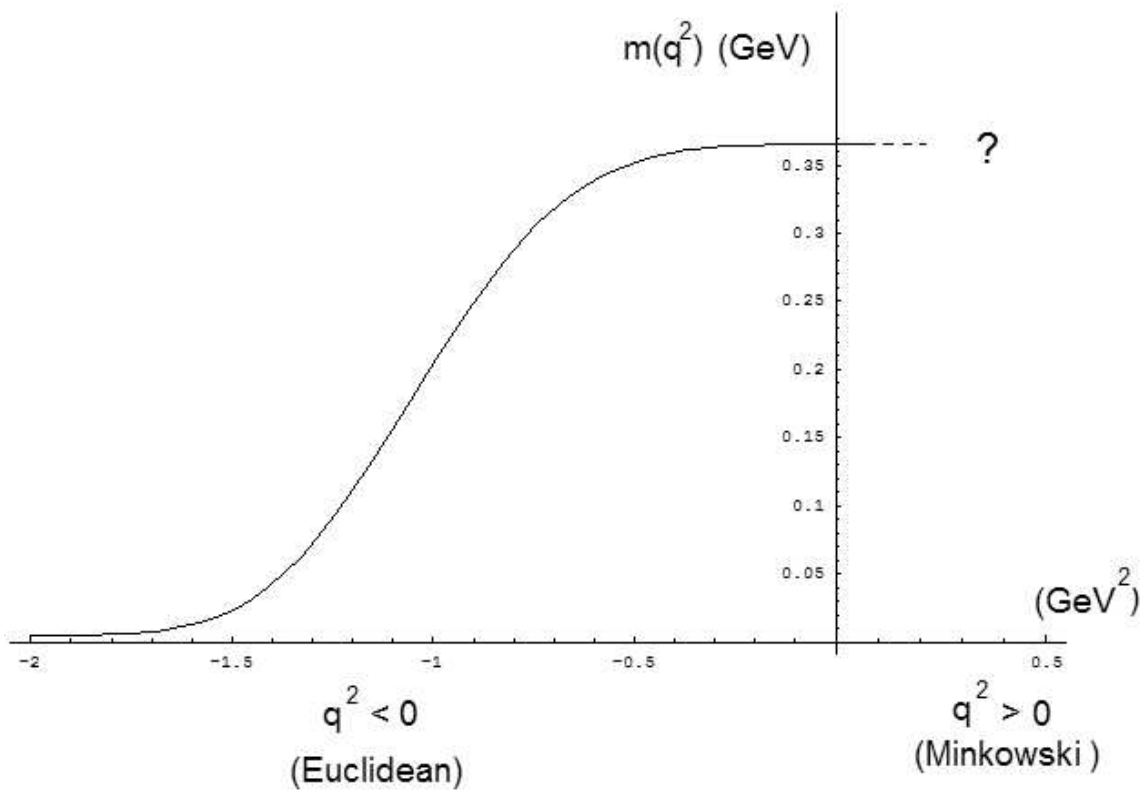


Figure 3.1: Running mass function.

has a consistent part in the Minkowski region. The precise form of the mass function in the Minkowski region is still unknown; there are different speculations and hypotheses [24, 79, 83], but the issue is far from being solved. This is why we have to put a question mark on the right side of figure 3.1.

Similar behaviors for the quark propagators are found also for the other two light quark species, d and s . The d quark propagator is very similar to the u one, so that usually one employs the simplification $m_n = m_u = m_d$ and similarly at low energies $m_n^* = m_u^* = m_d^*$; in reality, the d -quark is slightly heavier (few MeV), but this difference can safely be neglected, especially in low energy studies. On the contrary, this cannot be done for the s flavor, for which one has $m_s \approx 100 MeV$ in the high energy sector and $m_s^* = 500 - 700 MeV$ in the low energy one.

The problem of the (non)existence of poles of the quark propagator is still an open issue. Even if a pole is present, no physical quark should exist. Studies about this problem are nowadays in progress.

In phenomenological QCD studies some forms for the quark propagator without poles have been used. A typical example is the following [84, 85]:

$$S_u(q) = \frac{i}{\not{q} - m_u^*} (1 - \exp[\beta(q^2 - m^2)]) \quad (3.105)$$

where the extra-term removes the pole for $q^2 = m_u^{*2}$. The propagator is now an entire function, and the singularity is shifted for $q^2 \rightarrow \infty$. Such a propagator can be valid only at low energy, because the large energy limit is not fulfilled. It should be stressed that studies with this kind of propagators cannot give precise results, but only show some patterns and features of the difficult underlying theory. Such propagator forms should be used until better QCD-derived propagators are available.

Unfortunately the difficulty in performing the calculations grows with the complexity of the propagator. Already this form implies some care; the QED Ward identity, for example, is not anymore valid.

We conclude this section noting that the non-perturbative features of QCD really constitute a hard work, but they are necessary for a deep and complete understanding of the physical world.

3.5 Pseudoscalar case

3.5.1 Spin decomposition

In the study of bound states we often have to deal with bilinears, i.e. with operators of the form $\bar{\Psi}(x)\Gamma\Psi(x)$, where $\Gamma = 1_4, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5$ and where $\Psi(x)$ is a fermionic field (typically a quark or a lepton). The objects $\bar{\Psi}(x)\Gamma\Psi(x)$ are covariant, and transform in a determined way under parity and charge conjugation. We will consider closer the pseudoscalar and the scalar case, which are obtained with γ^5 and 1_4 respectively. We will also write a Lagrangian for the description of such bound states, in analogy with the arguments of section 2.5.

The object $\bar{\Psi}(x)i\gamma^5\Psi(x)$ is pseudoscalar, as one can see by considering its transformation properties under Lorentz and parity transformations. In this section we want to study these characteristics in a somewhat different way, introducing a pseudoscalar current and evaluating the spin-decomposition of it.

In chapter 2 we considered the scalar field φ and the nonlocal current $J(x) = \int d^4y \varphi(x + y/2) \varphi(x - y/2) \Phi(y)$. We now extend these consideration to the pseudoscalar quantity

$$J_P(x) = \int d^4y \bar{\Psi}(x + y/2) i\gamma^5 \Psi(x - y/2) \Phi_P(y). \quad (3.106)$$

We have seen that the state $\int d^4x e^{-ipx} J(x) |0\rangle$ contains the bound state structure of two scalar particles and that the vertex function $\Phi(y)$ is directly connected to the wave function of the composite object. We now analyze the structure

$$\int d^4x e^{-ipx} J_P(x) |0\rangle \quad (3.107)$$

where we have the extra complication of the spinorial structure of the field operators. Expressing the fields as in (3.64) we get for $J_P(x) |0\rangle$

$$\begin{aligned} & J_P(x) |0\rangle \\ = & \sum_{s_1, s_2 = \pm 1/2} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \sqrt{\frac{m}{E_1}} \sqrt{\frac{m}{E_2}} e^{i(p_1 + p_2)x} \left(\bar{u}^{(s_1)}(\mathbf{p}_1) i\gamma^5 v^{(s_2)}(\mathbf{p}_2) \right) \\ & \cdot b^{\dagger(s_1)}(\mathbf{p}_1) d^{\dagger(s_2)}(\mathbf{p}_2) |0\rangle \int d^4y e^{i\frac{(p_1 - p_2)y}{2}} \Phi_P(y). \end{aligned} \quad (3.108)$$

Then, we consider $\int d^4x e^{-ipx} J_P(x) |0\rangle$ in the frame $p = p_1 + p_2 = (M, \mathbf{0})$ where with M we indicate the total energy of the system, which corresponds to the mass of the bound state. Introducing the Fourier transform of $\Phi(y)$ we get a state proportional to

$$\begin{aligned} & \int d^3q \frac{m}{E} \left(\sum_{s_1, s_2 = \pm 1/2} \bar{u}^{(s_1)}(\mathbf{q}) i\gamma^5 v^{(s_2)}(-\mathbf{q}) \right) \\ & \cdot b^{\dagger(s_1)}(\mathbf{q}) d^{\dagger(s_2)}(-\mathbf{q}) |0\rangle \tilde{\Phi}_P(q = (0, \mathbf{q})). \end{aligned} \quad (3.109)$$

The term $b^{\dagger(s_1)}(\mathbf{q}) d^{\dagger(s_2)}(-\mathbf{q}) |0\rangle$ clearly shows the particle-antiparticle content of the state, to which the modulating vertex function is attached. The spin structure is contained in the term $(\bar{u}^{(s_1)}(\mathbf{q}) i\gamma^5 v^{(s_2)}(-\mathbf{q}))$, which we can express like (after a bit of algebra):

$$\left(\bar{u}^{(s_1)}(\mathbf{q}) i\gamma^5 v^{(s_2)}(-\mathbf{q}) \right) = i \frac{E + m}{2m} \left(1 + \frac{\mathbf{q}^2}{(E + m)^2} \right) \varepsilon^{s_1, s_2} \quad (3.110)$$

where ε^{s_1, s_2} is the antisymmetric Ricci-tensor of dimension 2. Note that the presence of the matrix ε for the spinor $v^{(s_2)}(-\mathbf{q})$ is fundamental to get this result (see eqs. (3.27) and (3.28)). We already discussed it in sections 2.3 and 2.4. If one would not take it into account, we would obtain wrong results and the wrong non-relativistic limit. The Ricci tensor ε^{s_1, s_2} is fundamental for the description of a spin zero state. In fact we get

$$\begin{aligned} |s\rangle = & C \int d^3q \frac{m}{E} \frac{E + m}{2m} \left(1 + \frac{\mathbf{q}^2}{(E + m)^2} \right) \tilde{\Phi}(q) \\ & \left(b^{\dagger(+1/2)}(\mathbf{q}) d^{\dagger(-1/2)}(-\mathbf{q}) - b^{\dagger(-1/2)}(\mathbf{q}) d^{\dagger(+1/2)}(-\mathbf{q}) \right) |0\rangle \end{aligned} \quad (3.111)$$

where C is a normalization constant.

The spin structure is then

$$|spin\rangle = \left(b^{\dagger(+1/2)}(\mathbf{q})d^{\dagger(-1/2)}(-\mathbf{q}) - b^{\dagger(-1/2)}(\mathbf{q})d^{\dagger(+1/2)}(-\mathbf{q}) \right) |0\rangle \quad (3.112)$$

which clearly shows a spin 0 state like $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$. If we apply the parity operator U_P and the charge conjugation U_C on the state $|s\rangle$ we find

$$U_P |s\rangle = (-1)^{L+1} |s\rangle \quad (3.113)$$

$$U_C |s\rangle = (-1)^L |s\rangle \quad (3.114)$$

where L is the relative angular momentum coming from the transformation of $\tilde{\Phi}(q = (0, \mathbf{q}))$

$$\tilde{\Phi}(q = (0, -\mathbf{q})) = (-1)^L \tilde{\Phi}(q = (0, \mathbf{q})). \quad (3.115)$$

If we assume a \mathbf{q}^2 dependence of it we simply have $L = 0$, and we are therefore describing an object of spin zero, angular momentum zero, negative parity and positive charge conjugation. All these properties can be summarized in

$$J^{PC} = 0^{-+} \quad (3.116)$$

where $J = S + L$ is the total angular momentum.

In the study of the parapositronium (electron-positron bound state with $L = S = 0$, see for example [59]) one indeed considers a state like

$$|parapositronium\rangle =$$

$$\frac{1}{\sqrt{V}} \int \frac{d^3q}{(2\pi)^{3/2}} A(\mathbf{q}) \frac{1}{\sqrt{2}} \left(b^{\dagger(+1/2)}(\mathbf{q})d^{\dagger(-1/2)}(-\mathbf{q}) - b^{\dagger(-1/2)}(\mathbf{q})d^{\dagger(+1/2)}(-\mathbf{q}) \right) |0\rangle \quad (3.117)$$

where the same structure is present. Just a note of care about the direct connection of the wave function in momentum space and the vertex function must be done. According to the related discussion of chapter 2, section 2.8 we expect to have

$$\tilde{\Phi}_P(\mathbf{q}) = c(2\sqrt{\mathbf{q}^2 + m^2} - M)A(\mathbf{q}), \quad (3.118)$$

where the vertex function depends on \mathbf{q} only and c is a proportionality constant. Such an expression is valid in the non-relativistic limit, a condition that is fulfilled in the parapositronium case. Indeed, we will analyze this case in the next chapter, where we will see that the scalar result can be also applied to the present spinorial case.

Far from the non-relativistic limit the vertex function describes the finite dimensions of the bound state, which are modelled by a covariant Ansatz.

3.5.2 Lagrangian approach

We have seen that the current $J_P(x)$ is the correct one for the description of a pseudoscalar object. Following the discussion of the previous chapter we consider the Lagrangian

$$\mathcal{L} = \bar{\Psi}(x) (i\gamma^\mu \partial_\mu - m) \Psi(x) + \frac{K_P}{2} J_P^2(x) \quad (3.119)$$

with the introduced current

$$J_P(x) = \int d^4y \bar{\Psi}(x+y/2) i\gamma^5 \Psi(x-y/2) \Phi_P(y). \quad (3.120)$$

Scattering in the fermion-antifermion channel is at first order

$$\begin{aligned} -iM_{1,channel1} &= \tilde{\Phi}\left(\frac{p_1 - p_2}{2}\right) \left(\bar{v}^{(r_2)}(\mathbf{p}_2) i\gamma^5 u^{(r_1)}(\mathbf{p}_1) \right) (iK_P) \cdot \\ &\quad \cdot \left(\bar{u}^{(r_3)}(\mathbf{p}_3) i\gamma^5 v^{(r_4)}(\mathbf{p}_4) \right) \tilde{\Phi}\left(\frac{p_3 - p_4}{2}\right). \end{aligned} \quad (3.121)$$

The second order, always in this channel, is

$$\begin{aligned} &-iM_{2,channel1} \\ &= \tilde{\Phi}\left(\frac{p_1 - p_2}{2}\right) \left(\bar{v}^{(r_2)}(\mathbf{p}_2) i\gamma^5 u^{(r_1)}(\mathbf{p}_1) \right) (iK_P (-i\Sigma_P(p^2)) iK_P) \cdot \\ &\quad \cdot \left(\bar{u}^{(r_3)}(\mathbf{p}_3) i\gamma^5 v^{(r_4)}(\mathbf{p}_4) \right) \tilde{\Phi}\left(\frac{p_3 - p_4}{2}\right) \end{aligned} \quad (3.122)$$

where $\Sigma_P(p^2)$ is the "bubble" contribution which reads

$$\Sigma_P(p^2) = -i \int \frac{d^4q}{(2\pi)^4} Tr[S(q+p/2) i\gamma^5 S(q-p/2) i\gamma^5] \tilde{\Phi}_P^2(q). \quad (3.123)$$

$S(q)$ is the quark propagator, the trace has been taken together with an extra-minus in front due to the presence of a fermion loop.

Summing up all the bubbles up to order infinity we get (see Fig. 3.2)

$$\begin{aligned} -iM_{channel1} &= -iM_{1,channel1} - iM_{2,channel1} - \dots = \\ &= \tilde{\Phi}\left(\frac{p_1 - p_2}{2}\right) \left(\bar{v}^{(r_2)}(\mathbf{p}_2) i\gamma^5 u^{(r_1)}(\mathbf{p}_1) \right) (-iT(p^2)) \\ &\quad \cdot \left(\bar{u}^{(r_3)}(\mathbf{p}_3) i\gamma^5 v^{(r_4)}(\mathbf{p}_4) \right) \tilde{\Phi}\left(\frac{p_3 - p_4}{2}\right) \end{aligned} \quad (3.124)$$

with

$$\begin{aligned} T(p^2) &= -K_P + K_P \Sigma_P(p^2) K_P + \dots \\ &= -(1 - K_P \Sigma_P(p^2))^{-1} K_P = -(K_P^{-1} - \Sigma_P(p^2))^{-1}. \end{aligned} \quad (3.125)$$

The mass M of the bound state is then given by the pole of the T matrix

$$K_P = \frac{1}{\Sigma_P(p^2 = M^2)} \quad (3.126)$$

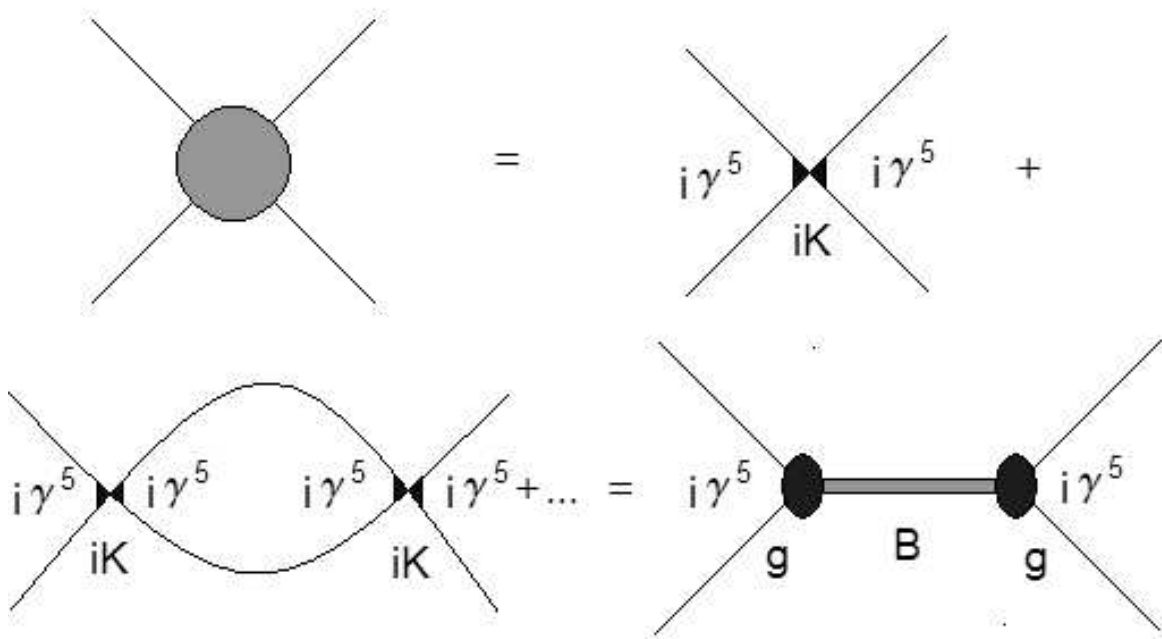


Figure 3.2: Two-body scattering and bound state formation.

Apart from the differences due the spinorial nature of the fields, the meaning of these formulas and the relative explanation is the same as in section 2.6.

By performing a Taylor expansion for $p^2 = M^2$, we can express T like

$$T = \frac{g_P^2}{p^2 - M^2} \quad (3.127)$$

with

$$g_P = \frac{1}{\sqrt{\Sigma'_P(p^2 = M^2)}} \quad (3.128)$$

being the bound state-constituents coupling constant.

3.5.3 Bound state field

As in the second chapter, section 2.7, we can introduce a composite field $P(x)$ and the corresponding Lagrangian

$$\mathcal{L} = \bar{\Psi}(x) (i\gamma^\mu \partial_\mu - m) \Psi(x) + g_P P(x) J_P(x) - \frac{1}{2} \alpha^2 P(x)^2 \quad (3.129)$$

where equivalence is achieved by imposing

$$g_P = \frac{1}{\sqrt{\Sigma'_P(p^2 = M^2)}}, \quad (3.130)$$

$$\alpha = \frac{\Sigma_P(p^2 = M^2)}{\Sigma'_P(p^2 = M^2)}. \quad (3.131)$$

In this case the propagator of the composite pseudoscalar field P , after having summed over all the bubbles, is exactly $i/(p^2 - M^2)$.

In order to calculate the decay properties of such a pseudoscalar field one considers then the interaction Lagrangian [29, 67]

$$\mathcal{L}_{int} = g_P P(x) J_P(x) \quad (3.132)$$

together with the compositeness condition $g_P = (\Sigma'_P(p^2 = M^2))^{-1/2}$. The basic Feynman diagram described by this interaction Lagrangian is shown in Fig. 3.3. We have the same ingredients as in the scalar case, plus the extra matrix $i\gamma^5$ due to the spinorial nature of the bound states.

3.5.4 Quarks

In nature we encounter pseudoscalar objects pretty often. In QED, the theory of electrons and photons, a typical example is the already mentioned parapositronium, a bound state of an electron and a positron. In QCD, as anticipated in the first chapter and in section 3.4.2, the quarks carry two other quantum numbers, the flavors u, d and s and the colors R, G, B . The fermionic quark field carries two extra-indices $q = q_{i,c}$ where $i = u, d, s$ and $c = R, G, B$.

When we write down the current we have to take into account these two extra-indices. As discussed in 1.2 and we will discuss this in more detail in the sixth chapter, each quark-antiquark mesonic state has the same color wave function $\sqrt{\frac{1}{3}}(\bar{R}R + \bar{G}G + \bar{B}B)$, which corresponds to a "white" object. On the other hand, one has all the possible flavor decompositions.

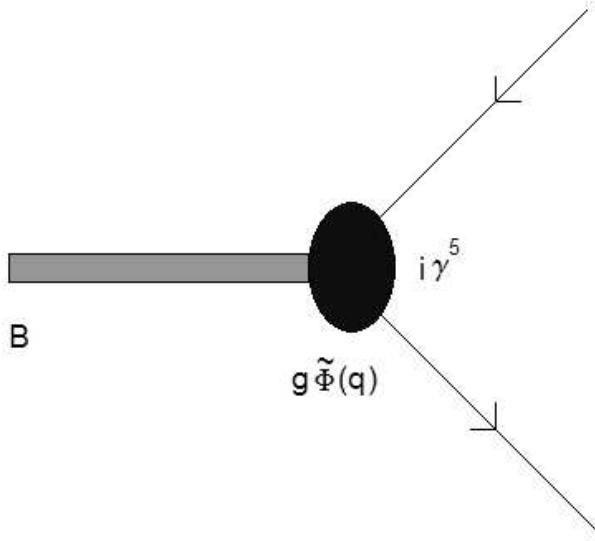


Figure 3.3: Vertex for the interaction of the bound state field with its components.

A generic pseudoscalar current is then written as

$$J_P(x) = \int d^4y \bar{q}_{i,c}(x+y/2) (T_f)_{i,j} i\gamma^5 q_{j,c}(x-y/2) \Phi_P(y) \quad (3.133)$$

where a summation over c is understood (leading exactly to the desired color wave function) and T_f is a flavour matrix. In this way we can describe the pseudoscalar meson nonet, where the pion plays a crucial role. In order to describe, for instance, the neutral pion $\pi^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d)$ we consider the matrix

$$T_{\pi^0} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.134)$$

leading to

$$J_{\pi^0}(x) = \frac{1}{\sqrt{2}} \int d^4y [\bar{u}_c(x+y/2) i\gamma^5 u_c(x-y/2) - \bar{d}_c(x+y/2) i\gamma^5 d_c(x-y/2)] \Phi_{\pi^0}(y) \quad (3.135)$$

where $\Phi_{\pi^0}(y)$ is the pion vertex function.

There are indeed nine independent possibilities to combine 3 quarks and 3 antiquarks, corresponding to the $8 + 1$ Gell-Mann λ^a matrices plus the identity (see again chapter 6).

The positronium is a non-relativistic object, while the pion is highly relativistic. Indeed the current is very similar (apart from the extra flavor and color decomposition for the

pion), but we expect a different form for the vertex function: a non-relativistic one for the positronium, a covariant one for the pion.

3.5.5 Evaluation of the mass operator

3.5.5.1 Free propagator

In this subsection we discuss the calculation of the mass operator, i.e. the "bubble" diagram given by

$$\Sigma_P(p^2) = -i \int \frac{d^4 q}{(2\pi)^4} \text{Tr}[S(q+p/2)i\gamma^5 S(q-p/2)i\gamma^5] \tilde{\Phi}_P^2(q). \quad (3.136)$$

This discussion is not necessary for the understanding of the underlying physical ideas, but it gives a possible way to evaluate the mass operator. First, we consider the free propagator

$$S(q) = \frac{i}{\not{q} - m} = i \frac{\not{q} + m}{q^2 - m^2}. \quad (3.137)$$

Then, considering that

$$\text{Tr}[(\not{p}_1 + m)i\gamma^5(\not{p}_2 + m)i\gamma^5] = 4(p_1 \cdot p_2 - m^2), \quad (3.138)$$

and performing a Wick-rotation ($q^0 = iq^4$) in the rest frame for the bound state $p = (\sqrt{p^2}, \mathbf{0})$, we get

$$\Sigma_P(p^2) = \int \frac{d^4 q_E}{(2\pi)^4} \frac{4(q_E^2 + p^2/4 + m^2)}{(q_E^2 - p^2/4 + m^2)^2 + (q^{(4)})^2 p^2} \tilde{\Phi}_P^2(q_E). \quad (3.139)$$

If covariance is preserved, i.e. $\tilde{\Phi}_P(q_E) = \tilde{\Phi}_P(q_E^2)$, this formula is valid in every covariant reference frame. If not, it is valid in the rest frame only. Then, going to spherical coordinates with $s^2 = \mathbf{q}^2$ we find

$$\Sigma_P(p^2) = \frac{1}{\pi^3} \int_0^\infty s^2 ds \int_{-\infty}^\infty dq^{(4)} \frac{s^2 + (q^{(4)})^2 + p^2/4 + m^2}{(s^2 + (q^{(4)})^2 - p^2/4 + m^2)^2 + (q^{(4)})^2 p^2} \tilde{\Phi}_P^2(q_E). \quad (3.140)$$

Finally, one can solve (3.140) numerically. It consists of a two-dimensional integration which can be handled, for example, by using Mathematica. Different choices for the vertex function do not constitute a numerical problem within this approach.

One can note from (3.140) that for $p^2 > 4m^2$ the previous integral diverges, because a pole is encountered in the integration. The form of the mass operator for the same parameter set analyzed in the scalar sector ($m = 0.3 \text{ GeV}$, $\Lambda = 1 \text{ GeV}$ and Gaussian form for the vertex function) is shown in Fig. 3.4. A similar qualitative behavior when compared to the scalar case is found. The mass operator goes to infinity for $p^2 \rightarrow 4m^2$, i.e. at the constituent production threshold.

3.5.5.2 Quark case

What changes if we do the same calculation for π^0 ? In order to calculate it we have to specify the quark propagator. It is a matrix in the Dirac, flavor and color space with

$$S_{cd} = \delta_{cd} \begin{pmatrix} S_u & 0 & 0 \\ 0 & S_d & 0 \\ 0 & 0 & S_s \end{pmatrix} \quad (3.141)$$

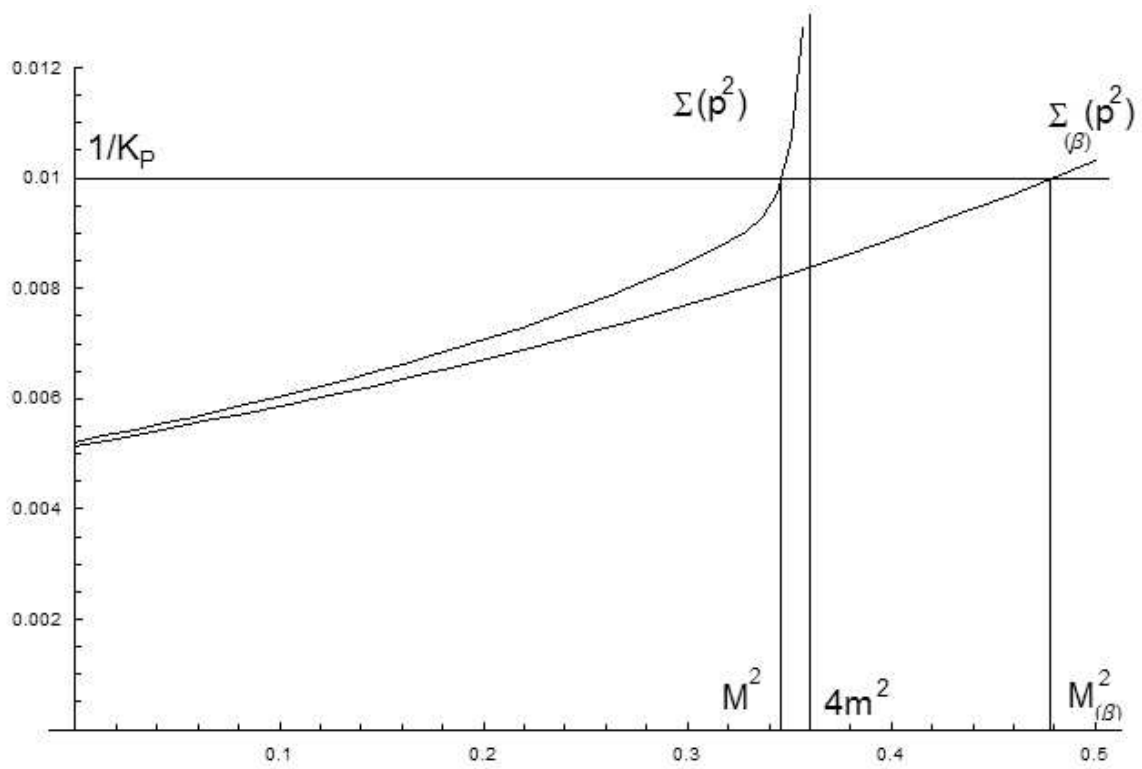


Figure 3.4: Comparison of the two mass operators in dependence on p^2 for the free and entire propagators.

where each S_i is a Dirac 4×4 matrix. The color part gives the identity matrix δ_{cd} (a consequence of $SU_c(3)$ color symmetry, see chapter 6), the flavor part is diagonal, but not the identity (we would have this if the propagators for the different flavors were identical, and this would mean also $SU_f(3)$ flavor symmetry).

When calculating the bubble, we find an extra factor $N_c = 3$ (number of colors) and the extra flavor decomposition. We then get

$$\begin{aligned} \Sigma_{\pi^0}(p^2) = & \\ & -i\frac{N_c}{2} \int \frac{d^4q}{(2\pi)^4} Tr[S_u(q+p/2)i\gamma^5 S_u(q-p/2)i\gamma^5] \tilde{\Phi}_{\pi^0}^2(q) \\ & -i\frac{N_c}{2} \int \frac{d^4q}{(2\pi)^4} Tr[S_d(q+p/2)i\gamma^5 S_d(q-p/2)i\gamma^5] \tilde{\Phi}_{\pi^0}^2(q) \end{aligned} \quad (3.142)$$

where one can see the u and the d bubble contributions. Under the hypothesis $S_n = S_u = S_d$ (well justified, as discussed before) we have

$$\Sigma_{\pi^0}(p^2) = -iN_c \int \frac{d^4q}{(2\pi)^4} Tr[S_n(q+p/2)i\gamma^5 S_n(q-p/2)i\gamma^5] \tilde{\Phi}_{\pi^0}^2(q) \quad (3.143)$$

i.e. formally the same expression as for the case of positronium, where the only difference is the extra-color factor N_c and, of course, the different vertex function.

3.5.5.3 Entire propagator

Let us consider now the calculation of the bubble when the propagator is described by an entire function [84, 85]

$$S(q) = \frac{i}{\not{q} - m} (1 - \exp[\beta(q^2 - m^2)]) . \quad (3.144)$$

After performing the Wick rotation one gets

$$\begin{aligned} \Sigma_{P,\beta}(p^2) = & \\ & \frac{1}{\pi^3} \int_0^\infty s^2 ds \int_{-\infty}^\infty dq^{(4)} \frac{s^2 + (q^{(4)})^2 + p^2/4 + m^2}{(s^2 + (q^{(4)})^2 - p^2/4 + m^2)^2 + (q^{(4)})^2 p^2} \\ & \cdot \left(1 - 2 \cos[\beta q^{(4)} \sqrt{p^2}] e^{-\beta(q_E^2 - p^2/4 + m^2)} + e^{-2\beta(q_E^2 - p^2/4 + m^2)} \right) \tilde{\Phi}_P^2(q_E) . \end{aligned} \quad (3.145)$$

The term in the parenthesis has a zero exactly where the denominator has one, thus there is no pole also for $p^2 > 4m^2$. The suffix β has been introduced to distinguish this mass operator from the one obtained with the free propagator form. One can extend the calculation for larger bound state masses, because no decay into a fermion and an antifermion is possible. This example directly shows the utility of propagators containing no poles; no "imaginary part" is found in the calculation of the mass operator or similar Feynman diagrams.

The situation and the comparison with the previous mass operator is displayed in Fig. 3.4. In the limit $\beta \rightarrow \infty$ the mass operator $\Sigma_{P,\beta}(p^2)$ (3.145) tends to $\Sigma_P(p^2)$ (eq. (3.139)). If β is large the two forms are similar for small p^2 . However, when p^2 increases the difference

between the two mass operator functions becomes larger. $\Sigma_{P,\beta}(p^2)$ does not have any pole; for K_P displayed in the figure there is a crucial bound state mass difference. The bound state mass from (3.139) is obviously $M^2 < 4m^2$, while M_β^2 from (3.145) is larger than $4m^2$. In this last case the bound state cannot decay into the two subcomponents, it does not matter how energetic is the bound state. This is a schematic way to describe confinement.

An extension to the quark case is now straightforward.

3.6 Scalar case

3.6.1 Spin decomposition

We now study the scalar case $\bar{\Psi}(x)\Psi(x)$ where we have the identity matrix between the fields. This object is scalar, i.e. $J = 0$ and with positive parity. We will show that it corresponds to a $S = L = 1$ state, which combine to give a scalar.

The current looks like

$$J_S(x) = \int d^4y \bar{\Psi}(x+y/2)\Psi(x-y/2)\Phi_S(y); \quad (3.146)$$

as before we analyze $\int d^4x e^{-ipx} J_S(x) |0\rangle$ in the rest frame where $p = (M, \mathbf{0})$. What follows is worked out as in the pseudoscalar case, apart from the replacement $i\gamma^5 \rightarrow 1_4$. We therefore find a state proportional to

$$\int d^3q \frac{m}{E} \left(\sum_{s_1, s_2 = \pm 1/2} \bar{u}^{(s_1)}(\mathbf{q}) v^{(s_2)}(-\mathbf{q}) \right) \cdot b^{\dagger(s_1)}(\mathbf{q}) d^{\dagger(s_2)}(-\mathbf{q}) |0\rangle \tilde{\Phi}_S(q = (0, \mathbf{q})) \quad (3.147)$$

which, despite its appearance, is more complicated than the previous one, because we cannot so easily decompose the spin part. We have indeed

$$\left(\bar{u}^{(s_1)}(\mathbf{q}) v^{(s_2)}(-\mathbf{q}) \right) = -2 \left(\chi^{\dagger(s_1)} \frac{\sigma_i q_i}{E+m} \varepsilon \chi^{(s_2)} \right) \quad (3.148)$$

with three different pieces from $\sigma_i q_i = \sigma_1 q_1 + \sigma_2 q_2 + \sigma_3 q_3$. Let us for example consider $\sigma_3 q_3$ from which we get

$$-2\chi^{\dagger(s_1)} \frac{\sigma_3 q_3}{E+m} \varepsilon \chi^{(s_2)} = \frac{-2q_3}{E+m} \sigma_1^{s_1, s_2} \quad (3.149)$$

which gives rise to the spin structure

$$\left(b^{\dagger(+1/2)}(\mathbf{q}) d^{\dagger(-1/2)}(-\mathbf{q}) + b^{\dagger(-1/2)}(\mathbf{q}) d^{\dagger(1/2)}(-\mathbf{q}) \right) |0\rangle. \quad (3.150)$$

This is clearly $|S = 1, S_z = 0\rangle$ belonging to the triplet, as announced before. Similarly the other two pieces coming from $\sigma_1 q_1 + \sigma_2 q_2$ belong to the spin triplet; from $\sigma_1 q_1$ one gets the structure $|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle$ and from $\sigma_2 q_2$ the analogous one $|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle$.

Note that the presence of the momentum q_i in $\sigma_i q_i$ changes the angular momentum of by a unit factor. If $\tilde{\Phi}_S$ is symmetric in \mathbf{q}^2 it means that we have $L = 1$, as anticipated. The

apparently "innocuous" identity matrix hides a rich spin-angular structure. We write the final state like

$$|s\rangle = C \int d^3q \frac{m}{E} \left(\sum_{s_1, s_2 = \pm 1/2} \chi^{\dagger(s_1)} \frac{\sigma_i \varepsilon}{E+m} \chi^{(s_2)} \right) \tilde{\Phi}_i(\mathbf{q}) b^{\dagger(s_1)}(\mathbf{q}) d^{\dagger(s_2)}(-\mathbf{q}) |0\rangle \quad (3.151)$$

where $\tilde{\Phi}_i(\mathbf{q}) = q_i \tilde{\Phi}_S(q = (0, \mathbf{q}))$.

Applying parity and charge conjugation we find

$$U_P |s\rangle = (-1)^{L+1} |s\rangle \quad (3.152)$$

$$U_C |s\rangle = (-1)^{L+1} |s\rangle \quad (3.153)$$

where

$$\Phi_i(-\mathbf{q}) = (-1)^L \Phi_i(\mathbf{q}). \quad (3.154)$$

If $\tilde{\Phi}_S$ depends on \mathbf{q}^2 , what we assume, one has $L = S = 1$ and $J = 0$ because the total state is a scalar. Thus we have

$$J^{PC} = 0^{++}. \quad (3.155)$$

In nature we have the excited p-wave positronium with these quantum numbers, but more important and mysterious are the scalar mesons, i.e. quark-antiquark states in the scalar configuration.

We expect also nine of them (considering only the ground state) as we already discussed in the introduction. Indeed there are more, and it is not easy to disentangle their nature. One can build up 0^{++} objects also out of four quarks or out of two gluons. Many theories give also conflictive results and interpretation of the observed physical states. The understanding of their properties and characteristics is one of the still unsolved puzzles of modern high energy physics.

Before going on, we summarize the characteristics of a fermion-antifermion object with L and S quantum numbers. The results presented here show that

$$P = (-1)^{L+1}, \quad (3.156)$$

$$C = (-1)^{L+S}. \quad (3.157)$$

The extra-minus of $L + 1$ comes from the intrinsic odd nature of an antifermion (see sections 3.1 and 3.2).

3.6.2 Lagrangian form

We follow step by step what we did before by simply replacing $J_P \rightarrow J_S$

$$\mathcal{L} = \bar{\Psi}(x) (i\gamma^\mu \partial_\mu - m) \Psi(x) + \frac{K_S}{2} J_S^2(x). \quad (3.158)$$

The bound state mass is as usual $K_S^{-1} = \Sigma_S(p^2 = M^2)$ with the "bubble"

$$\Sigma_S(p^2) = -i \int \frac{d^4q}{(2\pi)^4} Tr[S(q+p/2)S(q-p/2)] \tilde{\Phi}_S^2(q). \quad (3.159)$$

The interaction Lagrangian with the composite field $S(x)$ reads

$$\mathcal{L}_{int} = g_S S(x) J_S(x) \quad (3.160)$$

together with the "compositeness condition"

$$g_S = \frac{1}{\sqrt{\Sigma'_S(p^2 = M^2)}}. \quad (3.161)$$

The evaluation of the mass operator can be worked out in the same way as for the pseudoscalar case. The only difference comes from the trace. In this case we have

$$Tr[(\not{p}_1 + m)(\not{p}_2 + m)] = 4(p_1 \cdot p_2 + m^2) \quad (3.162)$$

Therefore, in the Euclidean integration simply take $(q_E^2 + p^2/4 - m^2)$ instead of $(q_E^2 + p^2/4 + m^2)$. There is just a sign difference; the rest does not change.

We finally note that if we have an equal strength in the pseudoscalar and scalar channel ($K_P = K_S$) the scalar bound state will be more massive than the corresponding pseudoscalar one. In fact, the sign difference lowers the mass operator contribution in the scalar case.

Chapter 4

Two-photon decay

4.1 Introduction

The study of decays of resonances plays an important role in elaborating their inner structure. Quantum Field Theory naturally explains the decays, which can be calculated from the corresponding Feynman diagrams.

In the study of mesons (quark-antiquark bound states) one has to deal with a big number of resonances and the analysis of the decay pattern is fundamental to understand the nature of the bound state and to determine its quantum numbers.

An instable meson decays strongly into lighter mesons according to some conservation rules. The vector meson ρ , for example, whose quantum numbers are $J^{PC} = 1^{--}$ ($L = 0, S = 1$) decays dominantly into two pseudoscalar pions.

Electromagnetic decays are also very important because they are sensitive to electrically charged components, in our case to the quark flavor content of a meson. Each quark flavor has its own fractional charge ($e_u = 2/3, e_d = e_s = -1/3$). A typical example is the "famous" decay of the neutral pion $\pi^0 \rightarrow 2\gamma$. The fact that a neutral object decays into photons reveals that it must consist of charged subcomponents.

The last class of decays is driven by the weak interactions, where the quarks couple to the massive bosons W^\pm and Z^0 .

In this chapter we will concentrate on the two-photon decay of a bound state. Before doing this we discuss a very simple example, the decay of a scalar into two scalars.

4.2 A simple but instructive exercise

We consider two scalar fields χ and φ whose interaction is given by the Lagrangian

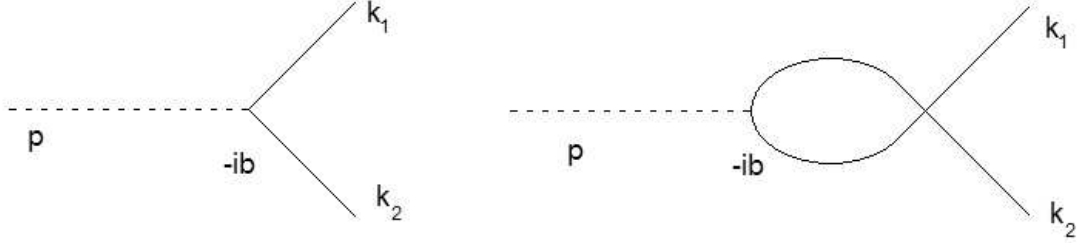
$$\mathcal{L}_{int} = b\chi\varphi^2, \quad (4.1)$$

which we already introduced in section 2.5.1 (2.154).

In the following we suppose $m_\chi > 2m_\varphi$, so that a decay of χ into two φ is kinematically allowed. The amplitude for $\chi \rightarrow 2\varphi$ at first order is the sum of two diagrams (Fig. 4.1) with

$$-iM = i2b, \quad (4.2)$$

which in this simple case is just a constant. The factor two occurs because the two diagrams give the same contribution ib .

Figure 4.1: First order decay diagrams for the χ -particle.

The initial and the final states are given by

$$|in\rangle = b_{\mathbf{p}}^{\dagger} |0\rangle, \quad (4.3)$$

$$|fin\rangle = a_{\mathbf{k}_1}^{\dagger} a_{\mathbf{k}_2}^{\dagger} |0\rangle, \quad (4.4)$$

and the corresponding matrix element by

$$\langle fin | S^{(1)} | in \rangle = \frac{1}{V^{3/2}} \frac{1}{\sqrt{2\omega_{\mathbf{k}_1} 2\omega_{\mathbf{k}_2} 2\omega_{\mathbf{p}}}} (2\pi)^4 \delta^4(p - k_1 - k_2) (-iM_1) \quad (4.5)$$

where $-iM_1$ is the invariant amplitude given above. This expression is also valid for a general amplitude $-iM$. The kinematical factors are in fact present for a generic decay into two scalar particles. V is the volume of the "box" in which we express the fields.

What we want to find is the lifetime of χ , i.e. probability for the process $\chi \rightarrow 2\varphi$. First, we take the square root of the amplitude, which is a probability:

$$\left| \langle fin | S^{(1)} | in \rangle \right|^2 = \frac{1}{V^3} \frac{1}{2\omega_{\mathbf{k}_1} 2\omega_{\mathbf{k}_2} 2\omega_{\mathbf{p}}} (2\pi)^8 (\delta^4(p - k_1 - k_2))^2 |-iM|^2 \quad (4.6)$$

but we immediately encounter some problems. How to handle the square of the Dirac function? Why there is a volume dependence? Such problems appear when one tries to evaluate physical quantities (cross sections or decay rates), and can be rigorously solved by using wave packets. In the following we will use the so called Fermi-trick to get around this difficulty.

In the second chapter, section 2.4.2, when we introduced the S matrix, we assumed that the interaction does not act for $t \rightarrow \pm\infty$. In these limits the free plane waves are then justified. This is easy to understand if we think of a scattering among two particles, where, if they are very distant, the interaction can be neglected. But the decay is an intrinsic property of a particle. It does not matter where and at which time the particle is, it has always some probability to decay. We then consider the following situation: the particle χ is created at $t = 0$, and we ask which is the probability to find 2φ at the time $t > 0$.

We can then manipulate the square of the delta using the Fermi trick [59, 58]:

$$\begin{aligned}
& (2\pi)^8 (\delta^4(p - k_1 - k_2))^2 \\
= & (2\pi)^4 \delta^4(p - k_1 - k_2) \int d^4x e^{ix(p-k_1-k_2)} = (2\pi)^4 \delta^4(p - k_1 - k_2) \int d^4x \\
\rightarrow & (2\pi)^4 \delta^4(p - k_1 - k_2) \int_V d^3x \int_0^t dt = (2\pi)^4 \delta^4(p - k_1 - k_2) V t \tag{4.7}
\end{aligned}$$

where the integral $\int d^4x$ has been substituted by $\int_V d^3x \int_0^t dt$. In fact, we confined our system inside a box of volume V and we considered the system in the time interval $t=0$ and t (in the two-body scattering case one takes a large time interval T , here t is supposed to be small; see below).

Furthermore, we have to take into account the density of final states. The quantity $|\langle fin | S^{(1)} | in \rangle|^2$ represents the probability for $|in\rangle \rightarrow |fin\rangle$, where $|fin\rangle$ has definite momenta k_1 and k_2 . We have to consider generic momenta in the final state. The probability for the decay $\chi \rightarrow 2\varphi$ when the two particles φ have momenta between $(\mathbf{k}_1, \mathbf{k}_1 + d^3k_1)$ and $(\mathbf{k}_2, \mathbf{k}_2 + d^3k_2)$ is given by

$$|\langle fin | S^{(1)} | in \rangle|^2 V \frac{d^3k_1}{(2\pi)^3} V \frac{d^3k_2}{(2\pi)^3}. \tag{4.8}$$

In fact, the factor $V d^3k_1 / (2\pi)^3$ represents the number of states with three-momentum between $(\mathbf{k}_1, \mathbf{k}_1 + d^3k_1)$. In a box the three-momenta are quantized with $\mathbf{k} = 2\pi\mathbf{n}/L$, and the volume of a single state is given by $(2\pi/L)^3 = (2\pi)^3/V$. The "number of states" included in d^3k_1 is then given by

$$d^3k_1 / (\text{volume of a single state}) = V d^3k_1 / (2\pi)^3. \tag{4.9}$$

Making use of (4.7), Eq. (4.8) becomes:

$$\begin{aligned}
& |\langle fin | S^{(1)} | in \rangle|^2 V \frac{d^3k_1}{(2\pi)^3} V \frac{d^3k_2}{(2\pi)^3} \\
= & \frac{1}{2\omega_{\mathbf{k}_1} 2\omega_{\mathbf{k}_2} 2\omega_{\mathbf{p}}} (2\pi)^4 \delta^4(p - k_1 - k_2) \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} |-iM|^2 t. \tag{4.10}
\end{aligned}$$

Hence the result does not depend on the normalization volume V . The last thing to do is to integrate over all possible final momenta. We then define the decay width Γ as

$$\Gamma = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{|-iM|^2}{2\omega_{\mathbf{k}_1} 2\omega_{\mathbf{k}_2} 2\omega_{\mathbf{p}}} (2\pi)^4 \delta^4(p - k_1 - k_2). \tag{4.11}$$

The probability that at the instant t we find 2φ is simply Γt , or, reversing the problem, the probability to find the particle χ at the instant t is given by

$$p(t) = 1 - \Gamma t. \tag{4.12}$$

This expression is of perturbative origin. This means that it is valid only if the "corrections" are small, i.e. if $t \ll \Gamma^{-1}$. In fact, without such a constraint, after enough time we would obtain the meaningless result of a negative probability. Nevertheless we can find the probability of finding $p(t)$ at a generic t from the recursive use of the previous one:

$$p(t + dt) = p(t) (1 - \Gamma dt) \tag{4.13}$$

which gives

$$p(t) = e^{-\Gamma t}. \quad (4.14)$$

The lifetime is defined as $\tau = \Gamma^{-1}$. Again, all these considerations do not depend on the particular decay process, but are general. We used the square of the Dirac delta function together with some physical considerations to get a meaningful result.

The last thing to do is to evaluate Γ :

$$\Gamma = \frac{1}{2(2\pi)^2} \int d^3k_1 d^3k_2 \frac{|-iM(\mathbf{k}_1, \mathbf{k}_2)|^2}{2\omega_{\mathbf{k}_1} 2\omega_{\mathbf{k}_2} 2\omega_{\mathbf{p}}} \delta^4(p - k_1 - k_2). \quad (4.15)$$

In the rest frame of the decaying particle with $p = (m_\chi, \mathbf{0})$ we have

$$\delta^4(p - k_1 - k_2) = \delta^3(\mathbf{k}_1 + \mathbf{k}_2) \delta(m_\chi - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2}). \quad (4.16)$$

Therefore, solving the integral over d^3k_2 (by use of the Dirac delta function) we are left with

$$\Gamma = \frac{1}{2(2\pi)^2} \int d^3k_1 \frac{|-iM(\mathbf{k}_1, -\mathbf{k}_1)|^2}{(2\omega_{\mathbf{k}_1})^2 2m_\chi} \delta(m_\chi - 2\omega_{\mathbf{k}_1}). \quad (4.17)$$

We can write

$$\delta(m_\chi - 2\omega_{\mathbf{k}_1}) = \frac{4m_\chi}{k_f} \delta(|\mathbf{k}_1| - k_f), \quad (4.18)$$

where

$$k_f = \sqrt{\frac{m_\chi^2}{4} - m^2} \quad (4.19)$$

is the modulus of the momentum of the outgoing particles. Note that this last expression makes sense only if $m_\chi > 2m$, otherwise the decay is not allowed. Going to spherical coordinates with $d^3k_1 = d\Omega |\mathbf{k}_1|^2 d|\mathbf{k}_1|$ and solving the integral over $d|\mathbf{k}_1|$ we get the final result:

$$\begin{aligned} \Gamma &= s_f \frac{k_f}{32\pi^2 m_\chi^2} \int d\Omega |-iM(\mathbf{k}_1, -\mathbf{k}_1, |\mathbf{k}_1| = k_f)|^2 \\ &= s_f \frac{k_f}{32\pi^2 m_\chi^2} \int d\Omega |-iM|^2 \end{aligned} \quad (4.20)$$

Question: "What is that s_f in front ?" Answer: " s_f is a symmetry factor. In the case of a decay into two identical bosons it is $1/2$."

When $|-iM|^2$ does not depend on the angles, we have

$$\Gamma = s_f \frac{k_f}{8\pi m_\chi^2} |-iM|^2. \quad (4.21)$$

This is the case when we consider $|-iM|^2 = |-iM_1|^2 = 4b^2$. Ergo, the final result (at first order) for this decay (including $s_f = 1/2$) is:

$$\Gamma_{\chi \rightarrow 2\varphi} = b^2 \frac{k_f}{4\pi m_\chi^2}. \quad (4.22)$$

The final expression is astonishingly simple. Many useful results have been derived at lowest order in a similar fashion. We will also use this result in chapter 7, when we will evaluate the strong decays of scalar mesons.

In the following we would like to study the two-photon decay. The complication is connected to the vectorial nature of the photon field and to gauge invariance. In order to study it properly we first do a brief digression concerning the photon field.

4.3 Digression: the photon

4.3.1 Photon propagator

The electromagnetic Lagrangian is expressed by the field A_μ and the corresponding classical Lagrangian [54, 57]

$$\mathcal{L}_{em} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (4.23)$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The electric and the magnetic fields are given from

$$E^i = F^{0i}, \quad (4.24)$$

$$B^i = \varepsilon^{ijk}F^{jk}. \quad (4.25)$$

As well known, a redefinition of the potential A^μ like $A^\mu(x) \rightarrow A^\mu(x) + \partial^\mu\eta(x)$ does not alter the Lagrangian and leaves the physical fields E^i and B^i unchanged. This occurrence is called gauge invariance.

Let us now consider a "source" in order to study the propagator of the photon, in a similar way to what was done in chapter 2, section 2.3, in the scalar case. We examine the Green function problem from a "classical" point of view. We then write down the Lagrangian

$$\mathcal{L}_{em} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J_\mu A^\mu \quad (4.26)$$

whose equation of motion is

$$\square A^\mu - \partial^\mu(\partial_\rho A^\rho) = J^\mu. \quad (4.27)$$

Note that (4.26) is still gauge invariant, provided that $\partial_\mu J^\mu = 0$. Now, in the second chapter (2.3) we simply had $J(x)$ and we studied the Green function considering a delta function as a source. We want now to do the same, but we have to take care of the extra-index μ , which is a tensorial covariant index, therefore the corresponding "identity" in this case is $g^{\mu\nu}$. We then consider

$$J(x) \rightarrow J^\mu(x) = g^{\mu\nu}\delta(x-y) \quad (4.28)$$

where ν and y should be considered as parameters.

We then introduce the two point Green function $G^{\mu\nu}(x-y)$ as solution of

$$\square G^{\mu\nu}(x-y) - \partial^\mu(\partial_\rho G^{\rho\nu}(x-y)) = g^{\mu\nu}\delta(x-y). \quad (4.29)$$

In complete analogy to the scalar case we are then led to consider $G^{\mu\nu}(x-y)$ as the electromagnetic quantum propagator (the photon propagator) in the position space, while in momentum space one has $G^{\mu\rho}(q)$ defined as the Fourier transform of $G^{\mu\nu}(x-y)$:

$$G^{\mu\nu}(x-y) = \int \frac{d^4q}{(2\pi)^4} iG^{\mu\nu}(q)e^{-iq(x-y)}. \quad (4.30)$$

Everything seems fine, apart from a "small detail": $G^{\mu\nu}(q)$ does NOT exist. In order to see it we first write $G^{\mu\nu}(q)$ in the general covariant form

$$G^{\mu\nu}(q) = B(q^2)g^{\mu\nu} + C(q^2)q^\mu q^\nu \quad (4.31)$$

and then plug (4.30) into (4.29). No solution for the functions $B(q^2)$ and $C(q^2)$ can be found. Well, the mathematicians say that the equation (4.29) does not have an inverse.

In classical electrodynamics it was always a clever idea to fix a gauge. The gauge freedom was exploited to simplify our life. Why not to do the same? The famous Lorentz gauge $\partial_\mu A^\mu = 0$ considerably simplifies the equation of motion, which takes the simple form $\square A^\mu = J^\mu$. The equation for the propagator can now easily be found with

$$G^{\mu\nu}(q) = g^{\mu\nu} \frac{-i}{q^2}. \quad (4.32)$$

If we fix a gauge then the propagator does exist! Note that the photon has zero mass. The previous form for the propagator is usually employed in the calculations. The simplified equation of motion in the Lorenz gauge $\square A^\mu = J^\mu$ does not follow directly from the Lagrangian (4.26). The gauge condition $\partial_\mu A^\mu = 0$ has been imposed as an extra constraint. We would like to derive the propagator directly from a suitable Lagrangian. How to do it? To this end we consider

$$\mathcal{L}_{em,\alpha} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J_\mu A^\mu - \frac{1}{2\alpha}(\partial_\rho A^\rho)^2 \quad (4.33)$$

where the last term $\frac{1}{2\alpha}(\partial_\rho A^\rho)^2$ is defined as the gauge fixing term. This term breaks explicitly gauge invariance. The equation of motion now reads

$$\square A^\mu - (1 - \frac{1}{\alpha})\partial^\mu(\partial_\rho A^\rho) = J^\mu. \quad (4.34)$$

If, for example, $\alpha = 1$ we find the equation of motion of the Lorenz gauge with $\square A^\mu = J^\mu$. But we should note that, although we find this equation of motion directly from the Lagrangian (4.33) with $\alpha = 1$, the corresponding gauge condition $\partial_\mu A^\mu = 0$ doesn't follow from it. We will soon come back to this point.

The photon propagator as calculated from (4.33) now exists with

$$G^{\mu\nu}(q) = -i\frac{1}{q^2} \left(g^{\mu\nu} + (\alpha - 1)\frac{q^\mu q^\nu}{q^2} \right) \quad (4.35)$$

as one can see plugging (4.30) into (4.34).

4.3.2 External photons

Up to now we only spoke about the photon propagator, i.e. about internal photon line in a Feynman diagram. What happens if we have an incoming or outgoing photon? If we tried to quantize (4.23) we would have problems. In fact, $\pi^0 = \partial\mathcal{L}_{em}/\partial(dA_0/dt) = 0$, and the canonical commutation relations could not be fulfilled. Let us then consider the "free" Lagrangian without source but with the gauge fixing term with $\alpha = 1$:

$$\mathcal{L}_{em,\alpha=1} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(\partial_\rho A^\rho)^2. \quad (4.36)$$

Now, the equation of motion of this Lagrangian reads $\square A^\mu = 0$. In principle one could chose any α , but $\alpha = 1$ simplifies considerably the issue because the eq. of motion is the simplest.

There is a very important point to stress. The eq. of motion for (4.23) is $\square A^\mu - \partial^\mu(\partial_\rho A^\rho) = 0$. If we fix the Lorenz gauge $\partial_\rho A^\rho = 0$ we have indeed two equations, which must be simultaneously fulfilled:

$$\square A^\mu = 0, \quad (4.37)$$

$$\partial_\rho A^\rho = 0. \quad (4.38)$$

Now, from the eq. of motion of $\mathcal{L}_{em,\alpha=1}$ only one equation $\square A^\mu = 0$ is fulfilled, but NOT $\partial_\rho A^\rho = 0$. This is why (4.23) together with the Lorenz gauge $\partial_\rho A^\rho = 0$ does NOT coincide with $\mathcal{L}_{em,\alpha=1}$ (not even in a classical sense). This point is crucial.

We go on quantizing $\mathcal{L}_{em,\alpha=1}$ by imposing the canonical commutation relations

$$[A^\mu(t, \mathbf{x}), \pi^\nu(t, \mathbf{y})] = -g^{\mu\nu} \delta(\mathbf{x} - \mathbf{y}). \quad (4.39)$$

The eq. of motion $\square A^\mu = 0$ allows us to write

$$A_\mu(x) = \sum_{\alpha=0}^3 \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(\mathbf{k})}} \cdot \left(a^\alpha(\mathbf{k}) \varepsilon_\mu^{(\alpha)}(\mathbf{k}) e^{-ikx} + a^{\dagger\alpha}(\mathbf{k}) \varepsilon_\mu^{*(\alpha)}(\mathbf{k}) e^{ikx} \right)_{k^0=\omega(\mathbf{k})=|\mathbf{k}|}, \quad (4.40)$$

where the creator and annihilator $a^{\dagger\alpha}(\mathbf{k})$ and $a^\alpha(\mathbf{k})$ have been introduced as in the scalar case. We have still to specify the four four-vectors $\varepsilon_\mu^{(\alpha)}(\mathbf{k})$, necessary to express the vectorial nature of the field. We have to define a basis for them; we impose the following normalization

$$\varepsilon^{(\alpha)}(\mathbf{k}) \cdot \varepsilon^{(\beta)}(\mathbf{k}) = g^{\alpha\beta}. \quad (4.41)$$

Furthermore, we conventionally impose that the $\varepsilon^{(\alpha)}(\mathbf{k})$ for $\alpha = 1, 2, 3$ have a zero temporal component

$$\varepsilon_0^{(\alpha)}(\mathbf{k}) = 0 \quad (4.42)$$

and we also chose $\alpha = 1, 2$ to be orthogonal to the photon three-momentum

$$\varepsilon_i^{(\alpha)}(\mathbf{k}) \cdot k_i = 0. \quad (4.43)$$

With these choices $\varepsilon^{(0)}(\mathbf{k})$ is simply

$$\varepsilon^{(0)}(\mathbf{k}) = (1, 0, 0, 0). \quad (4.44)$$

From the canonical commutation relations one then has

$$[a^\mu(\mathbf{k}_1), a^{\dagger\nu}(\mathbf{k}_2)] = -g^{\mu\nu} \delta(\mathbf{k}_1 - \mathbf{k}_2). \quad (4.45)$$

One immediately realize that something goes wrong. If $\mu = \nu = 0$ we have

$$[a^0(\mathbf{k}_1), a^{\dagger 0}(\mathbf{k}_2)] = -\delta(\mathbf{k}_1 - \mathbf{k}_2). \quad (4.46)$$

This means that the state $|s\rangle = a^{\dagger 0}(\mathbf{k}) |0\rangle$ has a negative norm with $\langle s | s \rangle = -1$. This is a very "strange object". It cannot describe a physical object. How to get rid of it? Well, Gupta

and Bleuler [54, 56] had a very smart idea. As we stated above, the Lagrangian $\mathcal{L}_{em,\alpha=1}$ does not imply $\partial_\rho A^\rho = 0$, but we know that this condition is necessary in order to have an equation of motion like $\square A^\mu = 0$ starting from (4.23). Gupta and Bleuler then said that among all the states $|s\rangle$ the physical ones $|s_{phys}\rangle$ are such that

$$\partial_\mu A^{\mu(+)} |s_{phys}\rangle = 0 \quad (4.47)$$

where $A^{\mu(+)}$ refers to the positive energy part of the field expansion (4.40), i.e. the first part with the destruction operators $a^\alpha(\mathbf{k})$. This means that

$$\langle s_{phys} | \partial_\mu A^\mu | s_{phys} \rangle = 0, \quad (4.48)$$

i.e. the missing condition $\partial_\mu A^\mu = 0$ is reimposed on the level of the physical matrix elements. In this way one can show that the negative norm states do not belong to the physical ones. We refer to [54, 59, 56] for the careful study of this problem. The main point is that in virtue of the Gupta-Bleuler condition (4.47) the physical states are formed with $a^{\dagger 1,2}(\mathbf{k})$ (see [59] for a precise description of this issue). We can decompose $A_\mu(x)$ (4.40) into two pieces

$$\begin{aligned} A_\mu(x) &= A_\mu^{(phys)} + A_\mu^{(unphys)} = \\ &= \sum_{\alpha=1,2} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} \left(a^\alpha(\mathbf{k}) \varepsilon_\mu^{(\alpha)}(\mathbf{k}) e^{-ikx} + a^{\dagger\alpha}(\mathbf{k}) \varepsilon_\mu^{*(\alpha)}(\mathbf{k}) e^{ikx} \right)_{k^0=\omega(\mathbf{k})=|\mathbf{k}|} \\ &+ \sum_{\alpha=0,3} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} \left(a^\alpha(\mathbf{k}) \varepsilon_\mu^{(\alpha)}(\mathbf{k}) e^{-ikx} + a^{\dagger\alpha}(\mathbf{k}) \varepsilon_\mu^{*(\alpha)}(\mathbf{k}) e^{ikx} \right)_{k^0=\omega(\mathbf{k})=|\mathbf{k}|}. \end{aligned} \quad (4.49)$$

Only $A_\mu^{(phys)}$ is relevant for on-shell physical photons. But this does not mean that we can get rid of the second piece! For example, if we calculate the photon propagator we get, as expected

$$\langle 0 | T[A^\mu(x)A^\nu(y)] | 0 \rangle = \int \frac{d^4q}{(2\pi)^4} \left(\frac{-ig^{\mu\nu}}{q^2} \right) e^{-iq(x-y)}. \quad (4.50)$$

We need the full decomposition of the fields $A^\mu(x)$, and only the piece $A_\mu^{(phys)}$ would not be enough (it is indeed possible to do, but one would find a non-covariant propagator).

In the end let us note that $A_\mu^{(phys)}$ correspond to the temporal gauge because we have $A_0^{(phys)} = \nabla \mathbf{A}^{(phys)} = 0$, but only the full expression is covariant, while $A_\mu^{(phys)}$ alone is not.

4.3.3 QED, gauge invariance and Ward identity

If in (4.26) one replaces the static source with the dynamical electron source $J^\mu = -e\bar{\Psi}\gamma^\mu\Psi$ and introduces the free Dirac electron part, then one gets the QED Lagrangian [3, 54, 60, 58]:

$$\mathcal{L}_{QED} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi - e\bar{\Psi}\gamma^\mu\Psi A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (4.51)$$

This Lagrangian is still invariant under gauge transformations

$$A^\mu(x) \rightarrow A^\mu(x) + \partial^\mu\eta(x), \quad (4.52)$$

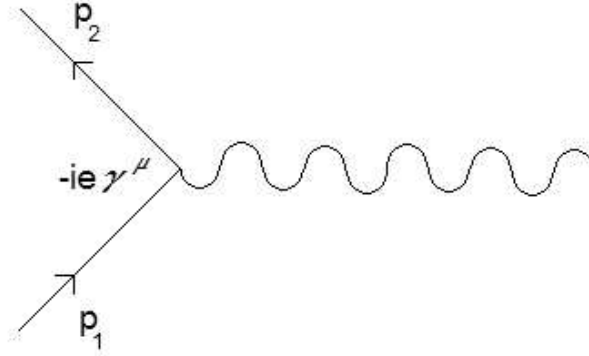


Figure 4.2: Fermion-photon vertex.

provided that simultaneously one makes the change

$$\Psi \rightarrow^{-ie\eta(x)} \Psi. \quad (4.53)$$

This is the form in which the QED Lagrangian is presented, as we have also done in Chapter 2. If we want to describe the interaction of a generic fermion with photons simply replace the electron charge $-e$ with the charge of the fermion (for example, if one has a quark u replace $-e \rightarrow \frac{2}{3}e$).

We said that care should be taken to quantize the photon, and that a gauge fixing term is needed to get a meaningful propagator. One has to do it also for the QED Lagrangian by introducing

$$\mathcal{L}_{QED,\alpha} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi - e\bar{\Psi}\gamma^\mu\Psi A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2\alpha}(\partial_\rho A^\rho)^2. \quad (4.54)$$

This Lagrangian is no longer gauge invariant. The QED Feynman rules can be easily read from this expression [3, 54, 58]. The main question we ask here is if the amplitude of a physical process depends on the parameter α . We realize that this would be catastrophic! The photon propagator depends on α (see (4.35)), and no physical prediction would then be possible! One can show that in each physical amplitude with internal photons the term $(\alpha - 1)q^\mu q^\nu / q^2$ of (4.35) does not contribute. If we replace a photon line with $q^\mu q^\nu$, where q is the momentum of the photon, the amplitude goes to zero. This means that only the first term $-ig^{\mu\nu}/q^2$ of the general quark propagator (4.35) contributes to the physical amplitude.

An example: in a scattering with an incoming and an outgoing electron (Fig. 4.2) the corresponding current is $J^\mu = \bar{u}^{(r)}(\mathbf{p}_1)\gamma^\mu u^{(r)}(\mathbf{p}_2)$; it is not difficult to prove, using the expression for the spinor given section 3.1.2, that

$$\bar{u}^{(r)}(\mathbf{p}_1)\gamma^\mu u^{(r)}(\mathbf{p}_2)(p_2 - p_1)_\mu = 0. \quad (4.55)$$

In this simple case the reason for it is the conservation of the electron current $\partial_\mu J^\mu = \partial_\mu \bar{\Psi}\gamma^\mu\Psi = 0$, which in momentum space reads $q_\mu J^\mu(q) = 0$

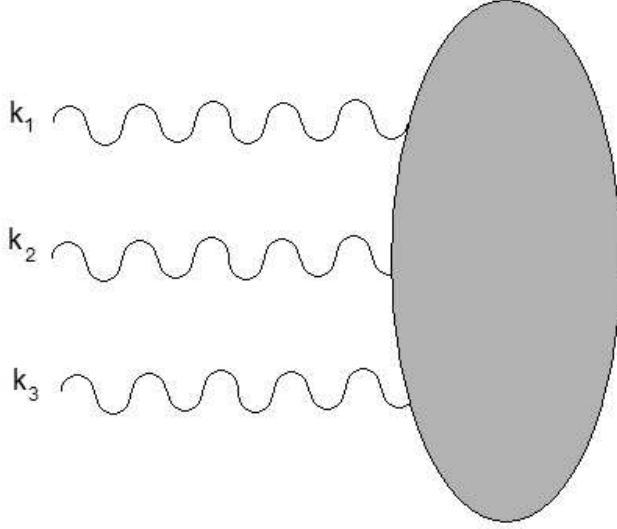


Figure 4.3: 3 external incoming photons.

But this is valid in general. Although the Lagrangian $\mathcal{L}_{QED,\alpha}$ given in (4.54) depends on the arbitrary parameter α and is not gauge invariant, one has in QED still this gauge invariance, in the sense that the results do not depend on the gauge fixing term, i.e. do not depend on α . But this is not only valid for QED, but for each gauge invariant theory involving photons.

Last point: if we have a process with external photons (let us say n incoming photons; in Fig. 4.3 the case $n = 3$ is schematically depicted) we can write the amplitude as

$$M^{\alpha_1, \alpha_2, \dots, \alpha_n} = \varepsilon_{\mu_1}^{(\alpha_1)}(\mathbf{k}_1) \varepsilon_{\mu_2}^{(\alpha_2)}(\mathbf{k}_2) \dots \varepsilon_{\mu_n}^{(\alpha_n)}(\mathbf{k}_n) M^{\mu_1, \mu_2, \dots, \mu_n}; \quad (4.56)$$

the previous discussion means that, if we substitute an external photon term $\varepsilon_{\mu_a}^{(\alpha_a)}(\mathbf{k}_a)$ (where $a = 1, 2, \dots, n$) with $(k_a)_\mu$ we get zero:

$$(k_a)_{\mu_a} M^{\mu_1, \mu_2, \dots, \mu_n} = 0. \quad (4.57)$$

This equation, also known as Ward Identity [3, 54, 56], constrains the possible forms of a decay amplitude when external photons are present, and is of great practical use. For example, if the photon polarizations (the α_a) are not known, one has to average over initial configurations and sum over final ones. The following identity is then very useful:

$$\sum_{a=1}^n \sum_{\alpha_a=1,2} \left| \varepsilon_{\mu_1}^{(\alpha_1)}(\mathbf{k}_1) \varepsilon_{\mu_2}^{(\alpha_2)}(\mathbf{k}_2) \dots \varepsilon_{\mu_n}^{(\alpha_n)}(\mathbf{k}_n) M^{\mu_1, \mu_2, \dots, \mu_n} \right|^2 = -M^{\mu_1, \mu_2, \dots, \mu_n} M_{\mu_1, \mu_2, \dots, \mu_n}^*, \quad (4.58)$$

where summation over μ_a is understood. The reason for this comes from the identity

$$\sum_{\alpha_a=1,2} \varepsilon_{\mu}^{(\alpha_a)}(\mathbf{k}_a) \varepsilon_{\nu}^{(\alpha_a)}(\mathbf{k}_a) = -g_{\mu\nu} - A_{\mu\nu} \quad (4.59)$$

where the term $A_{\mu\nu}$ turns out to be proportional to $(k_a)_{\mu}$ and $(k_a)_{\nu}$, therefore not contributing in virtue of (4.57) [54].

4.4 Decay of a pseudoscalar particle into 2 photons

4.4.1 Point-like pseudoscalar field

Let us consider a pseudoscalar field $P(x)$ and its direct coupling to two photons. We have to construct a Lagrangian which is parity and gauge invariant. A^{μ} is a four-vector, which under parity transforms like

$$A^0(x) \rightarrow A^0(t, -\mathbf{x}), \quad A^i(x) \rightarrow -A^i(t, -\mathbf{x}), \quad (4.60)$$

in accord with the classical parity transformations of the electric and magnetic fields:

$$E^i(x) \rightarrow -E^i(t, -\mathbf{x}), \quad B^i(x) \rightarrow B^i(t, -\mathbf{x}). \quad (4.61)$$

Note that the QED Lagrangian is invariant under parity transformation, in agreement with the experiment. The only way to couple a pseudoscalar fields to two photons is then with another pseudoscalar object like

$$\mathcal{L}_{P\gamma\gamma,int} = e^2 g_{P\gamma\gamma} P(x) F^{\mu\nu}(x) \tilde{F}_{\mu\nu}(x) \quad (4.62)$$

where the factor e^2 has been introduced for future simplicity, and where

$$\tilde{F}_{\mu\nu}(x) = \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}(x). \quad (4.63)$$

The term $F^{\mu\nu}(x) \tilde{F}_{\mu\nu}(x)$ is indeed known from classical electrodynamic as being $\mathbf{E} \cdot \mathbf{B}$, which is clearly a pseudoscalar object (see transformations above). Furthermore, the Lagrangian is also gauge invariant.

Our goal is the calculation of the decay rate $P \rightarrow 2\gamma$, which is depicted in Fig. 4.4. We have two external photons with momenta k_1 and k_2 , therefore the total squared amplitude reads (summing over final states):

$$|-iM|^2 = \sum_{\alpha_1, \alpha_2=1,2} |M^{\mu\nu}(k_1, k_2) \varepsilon_{\mu}^{\alpha_1}(\mathbf{k}_1) \varepsilon_{\nu}^{\alpha_2}(\mathbf{k}_2)|^2 = |M^{\mu\nu}(k_1, k_2) M_{\mu\nu}(k_1, k_2)|. \quad (4.64)$$

(Note that one should take $(\varepsilon_{\mu}^{\alpha_1}(\mathbf{k}_1))^*$, but our $\varepsilon_{\mu}^{\alpha}(\mathbf{k})$ are real; the same for the element $M^{\mu\nu}$) where the last passage of the previous expression is a direct application of (4.58). Gauge invariance and the pseudoscalar two-photon final state imply the following form for $M^{\mu\nu}$:

$$M^{\mu\nu}(k_1, k_2) = B \varepsilon^{\mu\nu\rho\sigma} k_{1,\rho} k_{2,\sigma} \quad (4.65)$$

where B is a scalar. In fact one has

$$k_{1,\mu} M^{\mu\nu}(k_1, k_2) = k_{2,\nu} M^{\mu\nu}(k_1, k_2) = 0 \quad (4.66)$$

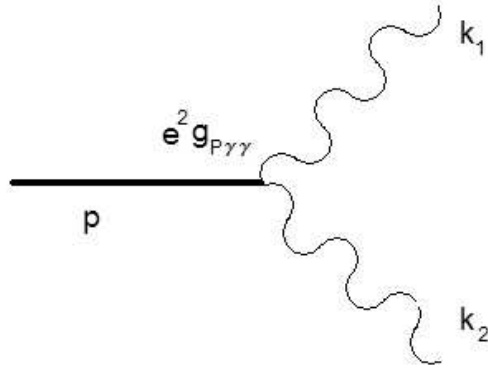


Figure 4.4: Decay of P(S) into two photons.

that is eq. (4.57); furthermore, if we exchange $k_1 \longleftrightarrow k_2$ we get a minus sign. The expression (4.65) is valid for each amplitude which has two photons in the final state in a pseudoscalar configuration. In fact, it has been written down by only exploiting general principles. We then have for the general amplitude

$$|-iM|^2 = |B| 8\omega^4 \quad (4.67)$$

where $\omega = M/2$, where M is the mass of the pseudoscalar field P .

The scalar quantity B depends of course on the particular interaction Lagrangian; in our case, as one can easily prove developing the Feynman rules of (4.62), a very simple expression for B at lowest order (Fig.4.4) is found with

$$B = 2e^2 g_{P\gamma\gamma}, \quad (4.68)$$

where the factor 2 comes from the exchange diagram. With equation (4.20) we can write the final decay rate into two photons as

$$\Gamma_{P \rightarrow 2\gamma} = \frac{\pi}{4} \alpha^2 M^3 (2g_{P\gamma\gamma})^2 \quad (4.69)$$

where the fine structure constant $\alpha = e^2/4\pi \simeq 1/137$ has been introduced. In this case $g_{P\gamma\gamma}$ is an unknown parameter of the Lagrangian (4.62), where the pseudoscalar field $P(x)$ is treated as an elementary particle. What we want to do in the following is to give a microscopic interpretation of $g_{P\gamma\gamma}$ when $P(x)$ describes a bound state.

4.4.2 Two-photon decay of a pseudoscalar bound state

In the third chapter, section 3.5, we discussed the introduction of a pseudoscalar bound state made out of two fermions (typically light quarks or electrons). The corresponding Lagrangians

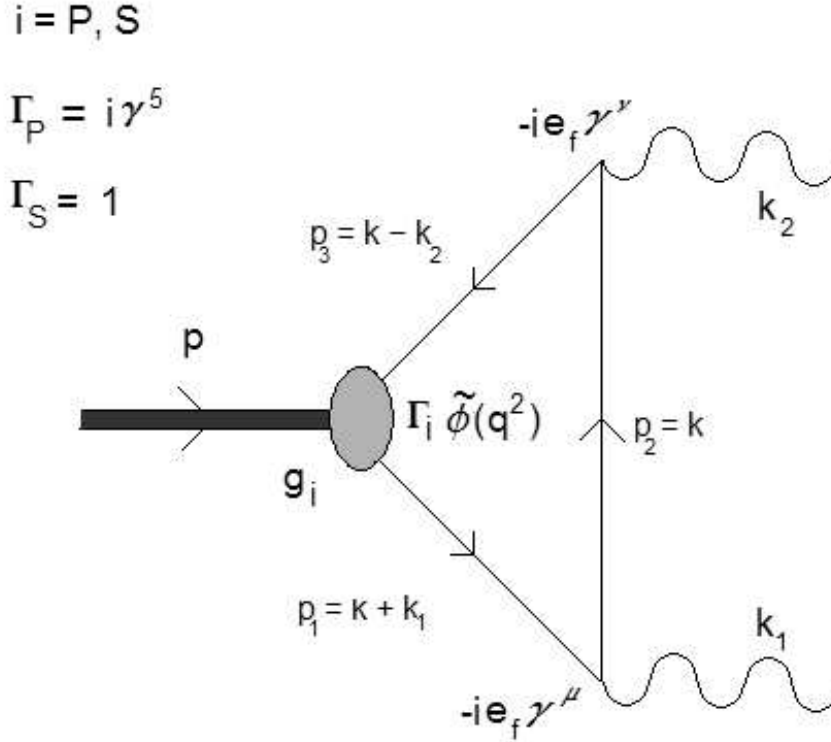


Figure 4.5: Two-photon decay by the triangle diagram; scalar and pseudoscalar cases are displayed.

are given in sections 3.5.2 and 3.5.3. Let us now introduce the photon into the pseudoscalar bound state Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi + e_f \bar{\Psi}\gamma^\mu \Psi A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{K_P}{2}J_P^2(x) \quad (4.70)$$

where $J_P(x)$ is the pseudoscalar current

$$J_P(x) = \int d^4y \bar{\Psi}(x+y/2)i\gamma^5\Psi(x-y/2)\Phi_P(y). \quad (4.71)$$

Similarly one can consider the auxiliary pseudoscalar bound state $P(x)$ with the Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi + g_P P(x)J_P(x) + e_f \bar{\Psi}\gamma^\mu \Psi A_\mu - \frac{1}{2} \frac{\Sigma_P(M^2)}{\Sigma'_P(M^2)} P^2 \quad (4.72)$$

where g_P is given from the compositeness condition $g_P = (\Sigma'_P(M^2))^{-1/2}$ (see chapters 2 and 3).

We realize that the Feynman diagram of Fig. 4.5 is possible; it describes the decay of the pseudoscalar object $P(x)$ into two photons by a fermion loop. What we have to do is to calculate the amplitude for this triangle diagram. But before doing it, we should ask a fundamental question: are the equivalent Lagrangians (4.70) and (4.72) gauge invariant?

4.4.3 Restoration of gauge invariance

The answer to the previous question is: NO! And the reason for it is the nonlocal current $J_P(x)$. A local transformation $\Psi(x) \rightarrow e^{ie_f \eta(x)} \Psi(x)$ does not leave $J_P(x)$ unchanged. This would be the case only if $\Phi(y) \rightarrow \delta^4(y)$, i.e. in the local limit for the current (but in this case the mass operator would diverge and we would not be able to describe the bound nature of the state).

If gauge invariance is not conserved, it means that in general we cannot make use of the useful relations of section 4.3. For example, we cannot say "a priori" that the triangle diagram amplitude can be written out as $M_{triangle}^{\mu\nu}(k_1, k_2) = B \varepsilon^{\mu\nu\rho\sigma} k_{1,\rho} k_{2,\sigma}$. Indeed, in the pseudoscalar case we are lucky. The triangle diagram as calculated from (4.72) has this gauge invariant form, even if this Lagrangian is not gauge invariant; we will see this in the next subsection. At the same time we realize that we need a general method to recover gauge invariance when a nonlocal current is introduced. In other cases we may not be so lucky.

As shown in [86], a restoration of gauge invariant is achieved by making the following shift in the current $J_P(x)$:

$$\Psi(x - y/2) \rightarrow \left(\exp[ie_f \int_{x, L_1}^{x-y/2} dz^\mu A_\mu(z)] \right) \Psi(x - y/2) \quad (4.73)$$

$$\bar{\Psi}(x - y/2) \rightarrow \bar{\Psi}(x - y/2) \left(\exp[-ie_f \int_{x, L_2}^{x+y/2} dz^\mu A_\mu(z)] \right) \quad (4.74)$$

where $L_{1(2)}$ is a generic path connecting x to $x - (+)y/2$, to be evaluated for the integral. It is actually a one dimensional integral. Let us parametrize L_1 with $z = z(\tau)$ such that $z(0) = x$ and $z(1) = x - y/2$; it means that we have

$$\int_{x, L_1}^{x-y/2} dz^\mu A_\mu(z) = \int_0^1 d\tau \frac{dz^\mu(\tau)}{d\tau} A_\mu(z(\tau)) \quad (4.75)$$

One can show that the results do not depend on the choice of the paths [67, 68].

The "new" current looks like

$$J_{P,restored}(x) =$$

$$\int d^4y \bar{\Psi}(x + y/2) e^{-ie_f \int_{x, L_2}^{x+y/2} dz^\mu A_\mu(z)} i \gamma^5 e^{ie_f \int_{x, L_1}^{x-y/2} dz^\mu A_\mu(z)} \Psi(x - y/2) \Phi_P(y) \quad (4.76)$$

ant preserves local gauge invariance, due to the introduced exponents. If now we replace $J_P(x)$ by $J_{P,restored}(x)$ in the Lagrangians (4.70) and (4.70) we get gauge invariant expressions. The interaction Lagrangian of the bound state with its components is then

$$\mathcal{L}_{restored} = g_P P(x) J_{P,restored}(x) \quad (4.77)$$

which is $U(1)$ local gauge invariant.

This is fine, but not the end of the story. The introduction of $J_{P,restored}(x)$ modifies the interaction. As a gift, we get direct photon-bubble interaction terms. For example, expanding

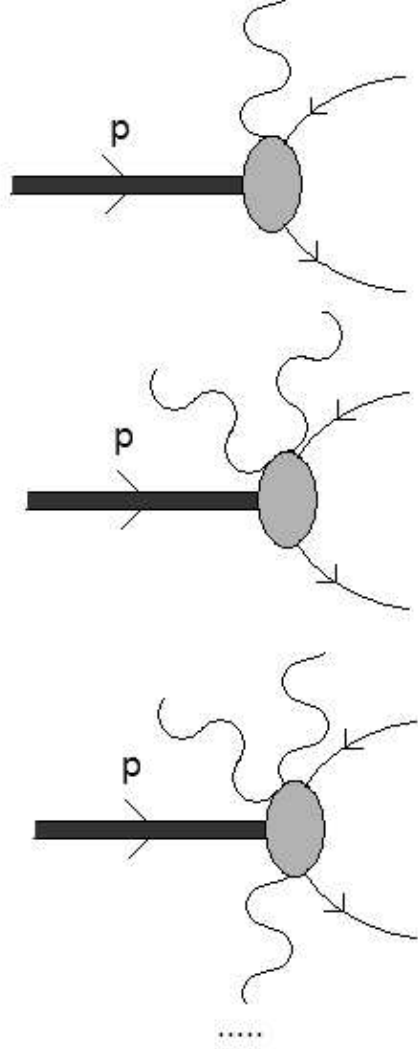


Figure 4.6: Photon-bubble interaction diagrams.

the exponents in $g_P P(x) J_{P,restored}(x)$ at first order in e we find the following forms:

$$\begin{aligned}
 & g_P P(x) J_{P,restored}(x) \\
 = & g_P P(x) J_P(x) \\
 & + \int d^4 y \bar{\Psi}(x+y/2) i\gamma^5 \left(-ie_f \int_{x,L1}^{x-y/2} dz^\mu A_\mu(z) \right) \Psi(x-y/2) \Phi_P(y) \\
 & + \int d^4 y \bar{\Psi}(x+y/2) \left(-ie_f \int_{x,L2}^{x+y/2} dz^\mu A_\mu(z) \right) i\gamma^5 \Psi(x-y/2) \Phi_P(y) + \dots \quad (4.78)
 \end{aligned}$$

i.e. we find a direct photon-bubble interaction. At the n -th order we have a n -photon-bubble interaction term (Fig. 4.6). This procedure to gauge nonlocal Lagrangians is valid in the case of a generic nonlocal current.

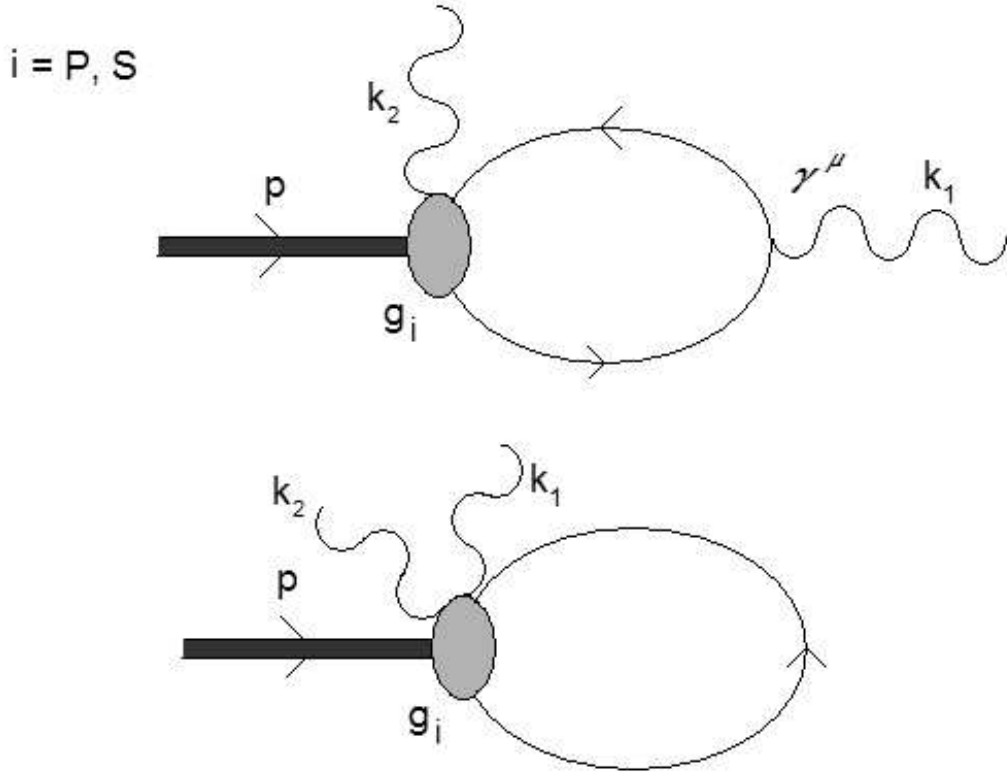


Figure 4.7: Bubble and tadpole diagrams.

In the two-photon decay, where we stop at second order (in addition to the triangle diagram) we also find other two contributing diagrams, depicted in Fig. 4.7. The calculation of these diagrams, known as bubble and tadpole diagrams, is in general not so easy. Their inclusion is indeed formally necessary for a complete and meaningful treatment of the two-photon decay of the bound state, but fortunately the contributions of these extra-diagrams are generally small and numerically negligible [67]. In the pseudoscalar case (Fig. 4.7 for $i = P$) one is indeed lucky, because their contribution is zero and the triangle diagram is by itself gauge invariant. One then has

$$M_{physical, P \rightarrow 2\gamma}^{(2)\mu\nu} = M_{triangle}^{\mu\nu} + M_{bubble}^{\mu\nu} + M_{tadpole}^{\mu\nu} = M_{triangle}^{\mu\nu} \quad (4.79)$$

because $M_{bubble} = M_{tadpole} = 0$. In fact, they are respectively proportional to $Tr[i\gamma^5(\not{p}_1 + m)\gamma^\mu(\not{p}_2 + m)]$ and $Tr[i\gamma^5(\not{p}_1 + m)]$, both vanishing. If one would like to calculate the electromagnetic form factor, would also have non-vanishing bubble-photon vertex contributions, even if they are numerically suppressed (see for example [67]). Similarly, in the scalar case the bubble and the tadpole diagrams for the two-photon decay give rise to non-zero contributions.

A physical amplitude calculated from the gauge-restored Lagrangian is gauge invariant, because such is the underlying Lagrangian. A generic physical amplitude at the n -th order will have the form

$$M_{physical}^{(n)} = M_{no-photon-bubble}^{(n)} + M_{1-photon-bubble}^{(n)} + \dots + M_{n-photon-bubble}^{(n)}. \quad (4.80)$$

Even if the total amplitude $M_{physical}^{(n)}$ is gauge invariant, it often happens that each single term of the sum is not.

4.4.4 Triangle diagram

Let us now concentrate on the triangle diagram (Fig. 4.5 with $i = P$ and $\Gamma_P = i\gamma^5$). We have, denoting k_1 and k_2 the four-momenta of the outgoing photons:

$$-iM^{\mu\nu}(k_1, k_2) = -ie_f^2 2g_P \int \frac{d^4k}{(2\pi)^4} Tr[i\gamma^5 S(p_1)\gamma^\mu S(p_2)\gamma^\nu S(p_3)]\tilde{\Phi}_P(q), \quad (4.81)$$

where $p_1 = k + k_1$, $p_2 = k$, $p_3 = k - k_2$, $q = k + (k_1 - k_2)/2$ and where $S(k)$ is the free fermion propagator.

The evaluation of the trace gives

$$Tr[i\gamma^5 (\not{k} + \not{k}_1 + m)\gamma^\mu (\not{k} + m)\gamma^\nu (\not{k} - \not{k}_2 + m)] = 4m\varepsilon^{\mu\nu\rho\sigma} k_{1,\rho} k_{2,\sigma}, \quad (4.82)$$

thus finding the gauge invariant structure analyzed in section 4.3.1 of this chapter, and in accord with the discussion of the previous subsection. We then get:

$$|-iM|^2 = \left(e_f^2 2g_P 4m(-i) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)(p_3^2 - m^2)} \tilde{\Phi}_P(q) \right)^2 8\omega^4 \quad (4.83)$$

with $\omega = M/2$. The decay rate is

$$\Gamma_{P \rightarrow 2\gamma} = \frac{\pi}{4} \alpha^2 M^3 (2g_P I)^2, \quad (4.84)$$

where I is the integral

$$I = -i4m \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)(p_3^2 - m^2)} \tilde{\Phi}_P(q). \quad (4.85)$$

The parameter $g_{P\gamma\gamma}$ introduced in (4.62) receives in this way a microscopic interpretation, being connected to the fermion loop; more precisely, by comparing (4.69) with (4.84) we have

$$g_{P\gamma\gamma} = g_P I = g_P 4m(-i) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)(p_3^2 - m^2)} \tilde{\Phi}_P(q). \quad (4.86)$$

The Lagrangian (4.62) describes the same decay, but at a different level; the loop is parametrized by a constant and a local interaction. Imagine to take a magnifying glass and to observe carefully this local vertex; you would then note that it is not a point, but a loop, where a fermion and an antifermion annihilate to generate two photons. This is contained in the Lagrangian (4.72). We are "going down" looking closer at the same physical process. Furthermore, when the photons are virtual and not real, a loop evaluation allows to take into account microscopically the variation of the effective local coupling on the virtual four-momenta.

These kinds of connection often happen in physics. To describe the process from a more basic point of view means a more difficult calculation to do. With the Lagrangian (4.62) there was no loop to calculate, and no fermion at all. On the other hand, the Lagrangian

(4.72) requires more work for the calculation of the coupling constant g_P and of the integral I . Going at a even more elementary level would mean to describe the fermionic bound state by the underlying theory (QCD with quark and gluons for mesons and QED for positronium or QED bound states in general), thus "calculating" the vertex function from the underlying theory.

4.4.5 $\pi^0 \rightarrow 2\gamma$

Let us now look in more detail at the neutral pion decay into two photons. The only differences are that the pion is made of two different quark flavors with the composition $\sqrt{\frac{1}{2}}(\bar{u}u - \bar{d}d)$ and that the quarks carry color. The pion current, as already discussed in the third chapter (see 3.5.4) is written as

$$J_{\pi^0}(x) = \int d^4y \bar{q}^a(x+y/2) i\gamma^5 T_{\pi^0} q^a(x-y/2) \Phi_{\pi^0}(y) \quad (4.87)$$

where

$$q^a = \begin{pmatrix} u^a \\ d^a \\ s^a \end{pmatrix}, \quad T_{\pi^0} = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.88)$$

and $a = R, G, B$ runs over the three color configurations. The bubble reads

$$\Sigma_{\pi^0}(p^2) = -iN_c \int \frac{d^4q}{(2\pi)^4} Tr[S_{(n)}(q+p/2) i\gamma^5 S_{(n)}(q-p/2) i\gamma^5] \tilde{\Phi}_{\pi^0}^2(q) \quad (4.89)$$

where N_c is the number of colors and $S_{(n)}$ is the light quark propagator (equal for the two flavors u and d , as usually supposed) in the low energy limit:

$$S_{(n)}(q) = \frac{i}{\not{q} - m_n^*} \quad (4.90)$$

with $m_n^* = m_u^* = m_d^*$. No trace of the matrix T_{π^0} is left in $\Sigma_{\pi^0}(p^2)$ because $Tr[T_{\pi^0}^2] = 1$ (and because the u and d propagators are equal).

One may ask: "Why didn't you introduce a factor $1/\sqrt{N_c}$ to get rid of the color factor in $\Sigma_{\pi^0}(p^2)$? " Well, it is just a convention. If you want you can do it".

The composite pion-quark interaction Lagrangian looks like

$$\mathcal{L}_{\pi^0-\bar{n}n} = g_{\pi^0-\bar{n}n} \pi^0 J_{\pi^0}(x) \quad (4.91)$$

where as usual

$$g_{\pi^0-\bar{n}n} = \left(\frac{\partial \Sigma_{\pi^0}(p^2)}{\partial p^2} \right)_{p^2=M_{\pi^0}^2}^{-1/2}. \quad (4.92)$$

Note that with the adopted convention $g_{\pi^0-\bar{n}n} \propto 1/\sqrt{N_c}$. This is in agreement with the large N_c theory (see [87] and Refs. therein).

The formula for the decay into two photons, taking into account the fractional charge for the quarks ($e_u/e = 2/3$, $e_d/e = -1/3$), the number of colors and the flavour structure, reads

$$\Gamma_{\pi^0 \rightarrow 2\gamma} = \frac{\pi}{4} \alpha^2 M_{\pi^0}^3 \left(2g_{\pi^0-\bar{n}n} N_c \left(\frac{4}{9} - \frac{1}{9} \right) I^{(n)} \right)^2 \quad (4.93)$$

where

$$I^{(n)} = 4m_n^* \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p_1^2 - m_n^{*2})(p_2^2 - m_n^{*2})(p_3^2 - m_n^{*2})} \tilde{\Phi}_{\pi^0}(q). \quad (4.94)$$

The physical quantity $\Gamma_{\pi^0 \rightarrow 2\gamma}$ is then proportional to N_c . This is convention independent (it is a physical result); if we had introduced a factor $1/\sqrt{N_c}$ in J_{π^0} we would have $g_{\pi^0 - \bar{n}n} \propto N_c^0$, but the amplitude would still be proportional to $N_c^{1/2}$.

In the work of [67] a covariant pion vertex function

$$\tilde{\Phi}_{\pi^0}(q) = \exp\left[\frac{q^2}{\Lambda^2}\right] \quad (4.95)$$

is used, with $\Lambda = 1 \text{ GeV}$ and $m_n^* = 0.235 \text{ GeV}$. With this relatively simple theoretical set-up many properties of the pion are correctly described, such as the two-photon decay, the weak pion decay constant, the pion radius and the electromagnetic form factor. The decay rate, for example, is $\Gamma_{\pi^0 \rightarrow 2\gamma} = 7.15 \text{ eV}$, in good agreement with experiment.

The vertex function $\tilde{\Phi}_{\pi^0}(q)$ is an Ansatz of the model, and describes the bound nature of the quarks in the pion. The choice of the gaussian is the simplest, but in principle any choice is *a priori* valid, as long as the vertex function falls off sufficiently fast for Euclidean momenta (a Wick-rotation is always understood when evaluating the integrals) in order to avoid ultraviolet divergences. The use of a different vertex function does not alter the results drastically, if the momentum cut-off has a similar value [69].

4.5 The parapositronium

4.5.1 Why to study it within this approach?

The method for the study of bound states we have introduced has been mostly used for the QCD bound states, i. e. the hadrons, both mesons and baryons, light and heavy [67, 68, 69, 70, 76]. The vertex function describes the bound nature of the state. In the second chapter, considering the scalar bound state in the non-relativistic limit, we have shown how to connect the vertex function to the wave function. This result has been achieved in two ways, first by a "comparison" for an hypothetical two-body decay into two scalars, and then from the Bethe-Salpeter approach.

The parapositronium is a pseudoscalar non-relativistic QED bound state, for which a non-relativistic wave function is known. Also the decay into two photons has been measured. We have the possibility to calculate the decay rate by the triangle loop and then to compare it with the experimental results and with other employed methods.

This may help us to understand closer what the vertex function really describes, and its limitations. In this sense, see also the works [77, 78]. At the same time it shows that the method is general, and can be applied to QCD bound states and to QED ones as well.

4.5.2 Usual approach

The usual way to introduce the (para)positronium bound state is (see 3.5.1 and [59]):

$$\begin{aligned} & |parapositronium\rangle \\ &= \int \frac{d^3q}{(2\pi)^{3/2}} \frac{A(\mathbf{q})}{\sqrt{V}} \frac{1}{\sqrt{2}} \left(b^{\dagger(+1/2)}(\mathbf{q}) d^{\dagger(-1/2)}(-\mathbf{q}) - b^{\dagger(-1/2)}(\mathbf{q}) d^{\dagger(+1/2)}(-\mathbf{q}) \right) |0\rangle \end{aligned} \quad (4.96)$$

where $A(\mathbf{q})$ describes the scalar bound state wave function in momentum space and is given by

$$A(\mathbf{q}) = N \left(1 + \frac{4}{\alpha^2 m_e^2} \mathbf{q}^2 \right)^{-2}. \quad (4.97)$$

N is a normalization constant, $\alpha = 1/137$ and $m_e = 0.511 \text{ MeV}$. $A(\mathbf{q})$ is the Fourier transform of the spatial wave function

$$\Psi(\mathbf{x}) = \frac{1}{\sqrt{\pi}} \left(\frac{\alpha m_e}{2} \right)^{3/2} \exp[-|\mathbf{x}| \frac{\alpha m_e}{2}], \quad (4.98)$$

which can be calculated from the Schrodinger equation with a Coulomb potential.

The mass of the parapositronium is given as

$$M = 2m_e - |E_{bind}| = 2m_e - \frac{\alpha^2 m_e}{4} \quad (4.99)$$

where $|E_{bind}| = \alpha^2 m_e / 4$ is the binding energy of the state.

For a generic level with principal quantum number n the bound state mass is

$$M_n = 2m_e - |E_{bind,n}| = 2m_e - \frac{\alpha^2 m_e}{4 \cdot n^2} \quad (4.100)$$

Here we do not consider hyperfine and spin splitting.

The decay into two photons at lowest order is evaluated in a similar way as the one described one in section 2.8.3; it is a QED two-photon production, where one considers (4.96) as a initial state. It reads (see [3, 59, 55] and the original works in [88])

$$\Gamma_{p-ps \rightarrow 2\gamma} = \frac{4\pi\alpha^2}{m_e^2} |\Psi(\mathbf{x} = \mathbf{0})|^2 = \frac{\alpha^5 m_e}{2}. \quad (4.101)$$

Some drawbacks of the approach are known [77]: in this calculation the electron and the positron are on-shell, while one should have a loop; in fact, the two composing particles of a bound state cannot be simultaneously on-shell because of four-momentum conservation. Furthermore, the energy in the rest frame of the state (4.96) is larger than $2m_e$, thus meaning that the this state could decay into a positron and an electron, while we know that this is not allowed, being $M = 2m_e - |E_{bind}| < 2m_e$.

The non-relativistic nature of the parapositronium justifies these approximations, but it would be nice to have a method which is valid simultaneously for relativistic bound states such as mesons and for non relativistic ones like positronium. This is also the aim of this part of the thesis.

4.5.3 Comparison

At the end of the first chapter we introduced the Bethe-Salpeter equation and the Bethe-Salpeter amplitude, and its relation to the wave function in momentum space.

In our fermion case described by the Lagrangian

$$\mathcal{L} = g_P P(x) J_P(x) = g_P P(x) \int d^4 y \bar{\Psi}(x + y/2) i\gamma^5 \Psi(x - y/2) \Phi_P(y) \quad (4.102)$$

the Bethe-Salpeter vertex is $\Gamma(q; p) = \Gamma(q) = g_P i\gamma^5 \tilde{\Phi}_P(q)$.

4.5.3.1 B-S amplitude: first method

The Bethe Salpeter amplitude is now a matrix ([24, 71] and section 2.10):

$$\begin{aligned} A(q; p) &= \frac{i}{\not{q} + \not{p}/2 - m} \Gamma(q; p) \frac{i}{\not{q} - \not{p}/2 - m} = \\ &= -ig_P \tilde{\Phi}_P(q) \frac{(\not{q} + \not{p}/2 + m) \gamma^5 (\not{q} - \not{p}/2 - m)}{((q + p/2)^2 - m^2) ((q - p/2)^2 - m^2)}. \end{aligned} \quad (4.103)$$

We work in the rest frame of the positronium $p = (M, \mathbf{0})$ (with M the positronium mass) and we suppose that $\tilde{\Phi}_P(q) = \tilde{\Phi}_P(\mathbf{q})$ (instantaneous approximation: see [71] and section 2.8).

Upon integrating over q^0 , taking only the contribution from the dominant pole in $-M/2 + \sqrt{\mathbf{q}^2 + m^2}$ we find

$$\begin{aligned} \tilde{A}(\mathbf{q}, p_{rest}) &= \int_{-\infty}^{+\infty} \frac{dq^0}{2\pi} A(q; p_{rest}) = \\ g_P \tilde{\Phi}_P(\mathbf{q}) &\frac{[(\not{q} + \not{p}/2 + m) \gamma^5 (\not{q} - \not{p}/2 - m)]_{q^0 = -M/2 + \sqrt{\mathbf{q}^2 + m^2}}}{2M \sqrt{\mathbf{q}^2 + m^2} (2\sqrt{\mathbf{q}^2 + m^2} - M)}. \end{aligned} \quad (4.104)$$

We now evaluate the spin-part, finding

$$\tilde{A}(\mathbf{q}, p_{rest}) = \tilde{\Phi}_P(\mathbf{q}) \frac{i\gamma^5 (M\sqrt{\mathbf{q}^2 + m^2} - Mm\gamma^0)}{2M \sqrt{\mathbf{q}^2 + m^2} (2\sqrt{\mathbf{q}^2 + m^2} - M)}. \quad (4.105)$$

4.5.3.2 B-S amplitude: second method

We now evaluate the same quantity starting from the basic definition in position space

$$\Psi_{\alpha\beta}(y/2, -y/2; p) = \langle 0 | T[\Psi_\alpha(y/2)\Psi_\beta(-y/2)] | B \rangle \quad (4.106)$$

In order to calculate it we have to specify $|B\rangle$; we take the non-relativistic form discussed in the previous subsection, which we however write as follows

$$|B\rangle = N \int d^3q A(\mathbf{q}) \left(\sum_{s_1, s_2 = \pm 1/2} \bar{u}^{(s_1)}(\mathbf{q}) i\gamma^5 v^{(s_2)}(-\mathbf{q}) \right) b^{\dagger(s_1)}(\mathbf{q}) d^{\dagger(s_2)}(-\mathbf{q}) |0\rangle \quad (4.107)$$

According to the description of section 3.5 the spin structure has the correct behavior. $A(\mathbf{q})$ is the momentum space wave function of eq. (4.97). For small momenta \mathbf{q} this form is completely equivalent to (4.96).

In general we could write

$$|B\rangle = N \int d^3q F(\mathbf{q}) \left(\sum_{s_1, s_2 = \pm 1/2} \bar{u}^{(s_1)}(\mathbf{q}) i\gamma^5 v^{(s_2)}(-\mathbf{q}) \right) b^{\dagger(s_1)}(\mathbf{q}) d^{\dagger(s_2)}(-\mathbf{q}) |0\rangle \quad (4.108)$$

where the function $F(\mathbf{q})$ must be the wave function for small momenta:

$$F(\mathbf{q}) = A(\mathbf{q})(1 + \text{relativistic corrections}). \quad (4.109)$$

In this work we are not interested in the relativistic corrections.

We now can evaluate the Fourier transform of $\Psi_{\alpha\beta}(y/2, -y/2; p)$, i.e. the B-S amplitude in momentum space

$$\begin{aligned} A_{\alpha\beta}(q; p) &= \int d^4y \Psi_{\alpha\beta}(y/2, -y/2; p) e^{iqy} = \\ &= \text{const} \cdot \delta(q^0) \frac{m}{E} \frac{[(\not{q}_1 + m) i \gamma^5 (\not{q}_2 - m)]_{\alpha\beta}}{4m^2} A(\mathbf{q}) \end{aligned} \quad (4.110)$$

where $q_1 = (E = \sqrt{\mathbf{q}^2 + m^2}, \mathbf{q})$ and $q_2 = (E, -\mathbf{q})$.

The integration upon q^0 is trivial in this case. After rewriting conveniently the matrix part we find

$$\tilde{A}(\mathbf{q}, p_{rest}) = \text{const} \cdot A(\mathbf{q}) \frac{2\sqrt{\mathbf{q}^2 + m^2}}{4mME} \left[i \gamma^5 \left(M \sqrt{\mathbf{q}^2 + m^2} - M m \gamma^0 \right) \right]. \quad (4.111)$$

4.5.3.3 Connection of both methods and discussion

We now compare (4.105) with (4.111); we can find a connection between the wave function and the vertex function:

$$\tilde{\Phi}_P(\mathbf{q}) = (\text{const}) \left(2\sqrt{\mathbf{q}^2 + m^2} - M \right) A(\mathbf{q}). \quad (4.112)$$

As already discussed in the end of section 2.10, the constant doesn't play any role. We can safely set it to 1.

It should be stressed that this study doesn't represent a precision test of QED (for which many good and advanced works are available: see [89] and Refs. therein), but an attempt to understand the meaning of the nonlocality of eq. (4.102). When we study a weakly bound state such as positronium within this kind of a nonlocal approach some nonrelativistic approximations are valid; in this way we can better understand how the vertex function is connected to the wave function. The fundamental result in this sense is that the vertex function is proportional to the wave function.

In the presented nonlocal approach we cannot calculate the vertex function, or the corresponding wave function. The vertex function is an Ansatz, a starting point.

However, when the nonrelativistic limit is not valid, and we don't know how to calculate the wave function, we can still introduce a vertex function as an effective phenomenological way to describe the finite size of the state under study.

The positronium case is interesting because we know the wave function, and through the arguments exposed before we deduce the vertex function, and we can calculate, as in the following subsection, a decay process; this allows us to test the approach in a case where we have a clear idea of the finite size of the state (a study along this line is performed in [90]).

On the contrary, in the study of mesons and baryons, especially in the low energy sector, we cannot calculate the wave function. We have some information about the dimensions of the objects, but no precise calculation in the nonperturbative QCD regime is possible. A vertex function introduced in this context is a possible way to model the bound state properties, until "better ideas" and explicit QCD solutions are found. If this is doable, is still unknown!

4.5.4 Two-photon decay

We can calculate the two photon decay using the formulas developed in section 4.4.4, namely:

$$\Gamma_{P \rightarrow 2\gamma} = \frac{\pi}{4} \alpha^2 M^3 (2g_P I)^2 \quad (4.113)$$

$$I = 4m \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)(p_3^2 - m^2)} \tilde{\Phi}_P(q). \quad (4.114)$$

We analyze separately g_P and the integral I .

4.5.4.1 Coupling constant g_P

The mass operator expressed as a function of $v = p^2$ is

$$\Sigma(v) = -4i \int \frac{d^4 q}{(2\pi)^4} \frac{-q^2 + v + m^2}{((q + p/2)^2 - m^2)((q - p/2)^2 - m^2)} \left(\tilde{\Phi}_P(\mathbf{q}) \right)^2. \quad (4.115)$$

By doing the integration over q^0 keeping the leading pole in $-\sqrt{v}/2 + \sqrt{\mathbf{q}^2 + m^2}$ we get

$$\Sigma(v) = \int \frac{d^3 q}{(2\pi)^3} 4 \frac{\left(\tilde{\Phi}_P(\mathbf{q}) \right)^2}{2 \left(2\sqrt{\mathbf{q}^2 + m^2} - \sqrt{v} \right)}. \quad (4.116)$$

Carrying on with the derivative with respect to v

$$\frac{d\Sigma(v)}{dv} = \int \frac{d^3 q}{(2\pi)^3} \frac{\left(\tilde{\Phi}_P(\mathbf{q}) \right)^2}{\sqrt{v} \left(2\sqrt{\mathbf{q}^2 + m^2} - \sqrt{v} \right)^2}. \quad (4.117)$$

We now plug in the expression $\tilde{\Phi}_P(\mathbf{q}) = \left(2\sqrt{\mathbf{q}^2 + m^2} - M \right) A(\mathbf{q})$ found in the previous subsection, and we evaluate the derivative at the point $v = M^2$:

$$\left(\frac{d\Sigma(v)}{dv} \right)_{v=M^2} = \frac{1}{M} \int \frac{d^3 q}{(2\pi)^3} A(\mathbf{q})^2. \quad (4.118)$$

This term is the normalization term for the positronium wave function. The coupling constant g_P deduced from the compositeness condition reads

$$g_P = \sqrt{\frac{M}{\int \frac{d^3 q}{(2\pi)^3} A(\mathbf{q})^2}}. \quad (4.119)$$

In the nonrelativistic limit the compositeness condition acquires a very simple form; it is nothing else but the normalization factor!

4.5.4.2 Triangle contribution

Keeping the dominant pole, the triangle contribution contained in I reads

$$\begin{aligned} I &= \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{\Phi}_P(\mathbf{q})}{2M^2 \sqrt{\mathbf{q}^2 + m^2} \left(2\sqrt{\mathbf{q}^2 + m^2} - M\right) \left(\sqrt{\mathbf{q}^2 + m^2} - q^{(3)}\right)} \\ &= \int \frac{d^3q}{(2\pi)^3} \frac{A(\mathbf{q})}{2M^2 \sqrt{\mathbf{q}^2 + m^2} \left(\sqrt{\mathbf{q}^2 + m^2} - q^{(3)}\right)} \end{aligned} \quad (4.120)$$

$$\simeq \frac{1}{2M^2 m^2} \int \frac{d^3q}{(2\pi)^3} A(\mathbf{q}). \quad (4.121)$$

We immediately notice that I is proportional to the spatial wave function at the origin, i. e. the amplitude of finding both the electron and the positron at the centre of mass of the system. The annihilation into photons occurs mostly under these conditions.

We can now calculate the decay rate $\Gamma_{P \rightarrow 2\gamma}$:

$$\begin{aligned} \Gamma_{P \rightarrow 2\gamma} &= \frac{\pi}{4} \alpha^2 M^3 (2g_P I)^2 = \frac{4\pi\alpha^2}{m^2} \left[\frac{\int \frac{d^3q}{(2\pi)^3} A(\mathbf{q})}{\sqrt{\int \frac{d^3q}{(2\pi)^3} A(\mathbf{q})^2}} \right]^2 = \\ &= \frac{4\pi\alpha^2}{m^2} |\Psi(0)|^2 = \frac{\alpha^5 m}{2}. \end{aligned} \quad (4.122)$$

We also understand why we never cared about the proper normalization factors of the momentum space wave function and of the vertex function: simply because the method is by itself normalized.

In this limit we clearly get the wave function with its own normalization, coming from the evaluation of the bound state-subcomponents coupling constant g_P .

The nonrelativistic limit is analytically recovered, as desired.

4.5.4.3 Results

We now list the results for the two-photon decay:

- Non-relativistic first order result:

$$\Gamma_{P \rightarrow 2\gamma} = \frac{\alpha^5 m}{2} = 5.287 \cdot 10^{-12} \text{ MeV} \quad (4.123)$$

- Full three-dimensional result, where the triangle contribution is expressed as in (4.120), without the approximation leading to the nonrelativistic result:

$$(\Gamma_{P \rightarrow 2\gamma})_{3 \text{ dim}} = 5.227 \cdot 10^{-12} \text{ MeV} \quad (4.124)$$

This result is completely equivalent to the one of [91].

- Full calculation from (4.85) (for g_P as well) applying a Wick-rotation:

$$(\Gamma_{P \rightarrow 2\gamma})_{wick} = 5.201 \cdot 10^{-12} \text{ MeV} \quad (4.125)$$

As one can note, the results are very similar. The full Wick rotation, where all the poles are taken into account, gives a very similar result to the three-dimensional one, thus justifying the assumption of a dominant pole in the integration.

The experimental result reads [92]

$$(\Gamma_{P \rightarrow 2\gamma})_{\text{exp}} = 7990.9 \pm 1.7 \mu s^{-1}; \quad (4.126)$$

which when converted to MeV (reminding that $\mu s^{-1} = 6.58 \cdot 10^{-16} MeV$):

$$(\Gamma_{P \rightarrow 2\gamma})_{\text{exp}} = (5.2580 \pm 0.0019) \cdot 10^{-12} MeV, \quad (4.127)$$

is very close to the predicted values.

In order to do better, one has then to take into account the relativistic and QED next-to-leading order corrections, but this work goes beyond the scope of the present analysis, which is centered on the interpretation of the introduced nonlocal vertex function.

In the already cited work of [91] and in [93] this discussion is actually done for the two-photon decay of heavy-quarkonia states; however, apart from different color factors, one can treat the positronium states and the heavy quarkonia ones in the same way. The reason for this is essentially contained in the non-relativistic nature of both types of bound states.

This fact again shows the useful analogy between QED and QCD bound states [90], even if the light quark bound states are indeed much more difficult to handle.

4.5.4.4 Radial excitation with $n = 2$

Let us briefly consider the analogous $n = 2$ state 2^1S_0 (with the same quantum numbers $J^{PC} = 0^{-+}$). The bound state mass and the wave function read

$$M_2 = 2m_e - |E_{\text{bind},2}| = 2m_e - \frac{\alpha^2 m_e}{4 \cdot 4}, \quad (4.128)$$

$$\Psi_2(\mathbf{x}) = \frac{1}{2\sqrt{8}\sqrt{\pi}} \left(\frac{\alpha m_e}{2}\right)^{3/2} \left(2 - \left(\frac{2}{\alpha m_e}\right)|\mathbf{x}|\right) \exp[-|\mathbf{x}| \frac{\alpha m_e}{2 \cdot 2}]. \quad (4.129)$$

The nonrelativistic first order result is

$$(\Gamma_{2P \rightarrow 2\gamma}) = \frac{4\pi\alpha^2}{m^2} |\Psi_2(0)|^2 = \frac{\alpha^5 m}{2 \cdot 8} = 6.609 \cdot 10^{-13} MeV. \quad (4.130)$$

The wave function in momentum space is

$$A_2(\mathbf{q}) = \text{const} \cdot \frac{\cos[2 \arctan[2 \left(\frac{2}{\alpha m_e}\right) |\mathbf{q}|]]}{\left(1 + 4 \left(\frac{2}{\alpha m_e}\right)^2 |\mathbf{q}|^2\right)^2}. \quad (4.131)$$

The rest is then exactly as before. The vertex function is

$$\tilde{\Phi}_{2P}(\mathbf{q}) = \left(2\sqrt{\mathbf{q}^2 + m^2} - M_2\right) A_2(\mathbf{q}). \quad (4.132)$$

The result for the full Wick rotation is $6.501 \cdot 10^{-13} MeV$, while the three-dimensional one is $6.534 \cdot 10^{-13} MeV$. The same considerations done previously are still valid.

4.6 Scalar particle decay into 2 photons

4.6.1 Point-like scalar field

We now consider the decay of a scalar point-like object into two photons. In order to guarantee local gauge invariance and parity conservation we write down the following interaction term

$$\mathcal{L}_{S\gamma\gamma,int} = e^2 g_{S\gamma\gamma} S(x) F^{\mu\nu}(x) F_{\mu\nu}(x). \quad (4.133)$$

The discussion is similar to the pseudoscalar one; the gauge invariant amplitude in a scalar configuration reads

$$M^{\mu\nu}(k_1, k_2) = (k_1^\nu k_2^\mu - (k_1 \cdot k_2)) B, \quad (4.134)$$

which is general. With the Lagrangian $\mathcal{L}_{S\gamma\gamma,int}$ one has $B = 2e^2 g_{S\gamma\gamma}$ leading to the total decay width

$$\Gamma_{S \rightarrow 2\gamma} = \frac{\pi}{4} \alpha^2 M^3 (2g_{S\gamma\gamma})^2 \quad (4.135)$$

where M is now the mass of the scalar particle.

4.6.2 Two-photon decay of a scalar bound state

In the scalar case we have the following Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi + e_f \bar{\Psi}\gamma^\mu \Psi A_\mu - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{K_S}{2} J_S^2(x) \quad (4.136)$$

where $J_S(x)$ is the scalar current

$$J_S(x) = \int d^4 y \bar{\Psi}(x+y/2)\Psi(x-y/2)\Phi_S(y). \quad (4.137)$$

Similarly one can consider the auxiliary scalar bound state $S(x)$ with the Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi + g_S S(x) J_S(x) + e_f \bar{\Psi}\gamma^\mu \Psi A_\mu - \frac{1}{2} \frac{\Sigma_S(M^2)}{\Sigma'_S(M^2)} S^2 \quad (4.138)$$

where g_S is given from the compositeness condition $g_S = (\Sigma'_S(M^2))^{-1/2}$ (see chapters 1 and 2).

We have again a triangle diagram to evaluate, where instead of the matrix $i\gamma^5$ we have the identity. This fact complicates the evaluation.

4.6.3 Triangle diagram and gauge invariance

Calling k_1 and k_2 the four-momentum of the outgoing photons the amplitude of the triangle diagram (Fig. 4.5 with $i = S$) reads

$$-iM_{triangle}^{\mu\nu}(k_1, k_2) = -ie_f^2 2g_S \int \frac{d^4 k}{(2\pi)^4} Tr[S(p_1)\gamma^\mu S(p_2)\gamma^\nu S(p_3)]\tilde{\Phi}_S(q) \quad (4.139)$$

where $p_1 = k + k_1$, $p_2 = k$, $p_3 = k - k_2$, $q = k + (k_1 - k_2)/2$ and where $S(k)$ is the free fermion propagator.

The evaluation of the trace gives

$$\begin{aligned} & \text{Tr}[(\not{k} + \not{k}_1 + m) \gamma^\mu (\not{k} + m) \gamma^\nu (\not{k} - \not{k}_2 + m)] \\ &= 4m((k_1^\nu k_2^\mu - (k_1 \cdot k_2)g^{\mu\nu})) + 4m(4k^\mu k^\nu - g^{\mu\nu}(k^2 - m^2)) \end{aligned} \quad (4.140)$$

where the first piece is gauge invariant and the second is not. This is the "bad" surprise! We are not so lucky as in the pseudoscalar case. The triangle diagram by itself it is not gauge invariant. One then has to consider the restoration of gauge invariance with

$$\begin{aligned} & J_{S,restored}(x) \\ &= \int d^4y \bar{\Psi}(x+y/2) e^{-ie_f \int_{x,L_2}^{x+y/2} dz^\mu A_\mu(z)} e^{ie_f \int_{x,L_1}^{x-y/2} dz^\mu A_\mu(z)} \Psi(x-y/2) \Phi_S(y) \end{aligned} \quad (4.141)$$

leading to a gauge invariant scalar-components interaction Lagrangian

$$\mathcal{L}_{restored} = g_S S J_{S,restored}. \quad (4.142)$$

As discussed previously we have now also the direct photon-bubble coupling, therefore the total amplitude will be

$$M_{total}^{\mu\nu} = M_{triangle}^{\mu\nu} + M_{bubble}^{\mu\nu} + M_{tadpole}^{\mu\nu} \quad (4.143)$$

which is now gauge invariant, since it arises from a gauge invariant interaction.

4.6.4 Where is the problem?

The point is how to manipulate the gauge breaking piece of $M_{triangle}^{\mu\nu}$

$$I^{\mu\nu}(k_1, k_2) = \int \frac{d^4k}{(2\pi)^4} (4k^\mu k^\nu - g^{\mu\nu}(k^2 - m^2)) g(k, k_1, k_2) \quad (4.144)$$

where

$$g(k, k_1, k_2) = \frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)(p_3^2 - m^2)} \tilde{\Phi}_S(q) \quad (4.145)$$

with $p_1 = k + k_1$, $p_2 = k$, $p_3 = k - k_2$, $q = k + (k_1 - k_2)/2$.

We now suppose to have a covariant vertex function $\tilde{\Phi}_S(q) = \tilde{\Phi}_S(q^2)$. We then exploit covariance to express $I^{\mu\nu}$ in the most general form taking into account that it is symmetric:

$$I^{\mu\nu} = (k_1^\nu k_2^\mu - (k_1 \cdot k_2)g^{\mu\nu}) B + k_1^\mu k_2^\nu B + g^{\mu\nu} \eta + (k_1^\mu k_1^\nu + k_2^\mu k_2^\nu) \alpha \quad (4.146)$$

where the term η explicitly breaks gauge invariance. We can simply find B and η by multiplying the previous equation by $k_1^\mu k_2^\nu$ and by $g^{\mu\nu}$:

$$(k_1 \cdot k_2) \eta = \int \frac{d^4k}{(2\pi)^4} (4(k \cdot k_1)(k \cdot k_2) - (k_1 \cdot k_2)(k^2 - m^2)) g(k, k_1, k_2), \quad (4.147)$$

$$-2(k_1 \cdot k_2) B + 4\eta = \int \frac{d^4k}{(2\pi)^4} 4m^2 g(k, k_1, k_2). \quad (4.148)$$

We can solve these equations finding

$$B = - \int \frac{d^4k}{(2\pi)^4} \left[\frac{2k^2}{(k_1 \cdot k_2)} - \frac{8(k \cdot k_1)(k \cdot k_2)}{(k_1 \cdot k_2)^2} \right] g(k, k_1, k_2) \quad (4.149)$$

$$\eta = \int \frac{d^4k}{(2\pi)^4} \left[\frac{4(k \cdot k_1)(k \cdot k_2)}{(k_1 \cdot k_2)} - (k^2 - m^2) \right] g(k, k_1, k_2). \quad (4.150)$$

4.6.5 A mistake not to do:

What would have happened if we had not introduced η ? Let us consider the NJL model. The NJL Lagrangian is local, and at first sight gauge invariance is satisfied. BUT often a cut-off is then introduced, and this cut-off breaks gauge invariance. In fact, it is analogous to a vertex function $\theta(-p^2 - \Lambda^2)$ in our approach, which is *different* from the identity in the momentum space and *different* from a Dirac delta function in x space. In the pion case this fact does not have bad consequences because, as we have seen, also in the non-local case one has a gauge invariant amplitude. The bubble and the tadpole diagrams are zero. In the scalar case to neglect η , and then to consider a gauge invariant amplitude without introducing the restoration of gauge invariance as indicated in section 4.6.3 and 4.6.4, means to have the wrong expressions:

$$B_{wrong} = - \int \frac{d^4k}{(2\pi)^4} \frac{4m^2}{2(k_1 \cdot k_2)} g(k, k_1, k_2) = - \frac{4m^2}{M^2} \int \frac{d^4k}{(2\pi)^4} g(k, k_1, k_2). \quad (4.151)$$

This *false* result would lead to the total amplitude

$$|-iM_{wrong}|^2 = (e_f^2 2g_S I_{wrong})^2 8\omega^4 \quad (4.152)$$

with

$$I_{wrong} = -i4m \left(1 - \frac{4m^2}{M^2}\right) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)(p_3^2 - m^2)} \tilde{\Phi}_S(q) \quad (4.153)$$

surely simple to solve but unfortunately not correct. Some authors [94, 95] used this form for the calculation. This exercise tells us that we should really be careful for what concerns gauge invariance.

4.6.6 Back to $M_{triangle}^{\mu\nu}$

As we have seen we can express $M_{triangle}^{\mu\nu}$ by a gauge invariant piece $\left(M_{triangle}^{\mu\nu}\right)_{g,i}$ and a gauge breaking one $\left(M_{triangle}^{\mu\nu}\right)_{g,b}$:

$$M_{triangle}^{\mu\nu} = \left(M_{triangle}^{\mu\nu}\right)_{g,i} + \left(M_{triangle}^{\mu\nu}\right)_{g,b}. \quad (4.154)$$

The gauge invariant piece reads

$$\left(M_{triangle}^{\mu\nu}\right)_{g,i} = (k_1^\nu k_2^\mu - (k_1 \cdot k_2)g^{\mu\nu}) 2g_S e_f^2 I, \quad (4.155)$$

with $I = I_1 + I_2$ with

$$I_1 = -i4m \int \frac{d^4k}{(2\pi)^4} g(k, k_1, k_2), \quad (4.156)$$

$$I_2 = 4mB = 4mi \int \frac{d^4k}{(2\pi)^4} \left[\frac{2k^2}{(k_1 \cdot k_2)} - \frac{8(k \cdot k_1)(k \cdot k_2)}{(k_1 \cdot k_2)^2} \right] g(k, k_1, k_2), \quad (4.157)$$

and the gauge breaking one

$$\left(M_{triangle}^{\mu\nu}\right)_{g,b} = 2g_S e_f^2 g^{\mu\nu} 4m \cdot \eta. \quad (4.158)$$

It is possible to do a similar job also for the tadpole and the bubble diagrams (Fig. 4.7 for $i = S$) (see [67]); the total amplitude is then

$$\begin{aligned} M_{tot}^{\mu\nu} &= \left(M_{triangle}^{\mu\nu}\right)_{g.i.} + \left(M_{triangle}^{\mu\nu}\right)_{g.b.} + \left(M_{bubble}^{\mu\nu}\right)_{g.i.} + \\ &\quad + \left(M_{bubble}^{\mu\nu}\right)_{g.b.} + \left(M_{tadpole}^{\mu\nu}\right)_{g.i.} + \left(M_{tadpole}^{\mu\nu}\right)_{g.b.} \\ &= \left(M_{triangle}^{\mu\nu}\right)_{g.i.} + \left(M_{bubble}^{\mu\nu}\right)_{g.i.} + \left(M_{tadpole}^{\mu\nu}\right)_{g.i.}, \end{aligned} \quad (4.159)$$

where the gauge breaking terms cancel. Furthermore, a numerical study reveals that $\left(M_{tadpole}^{\mu\nu}\right)_{g.i.}$ and $\left(M_{tadpole}^{\mu\nu}\right)_{g.b.}$ are generally suppressed in comparison to $\left(M_{triangle}^{\mu\nu}\right)_{g.i.}$.

Neglecting the bubble and the tadpole contributions we have the following final decay rate

$$\Gamma_{S \rightarrow 2\gamma} = \frac{\pi}{4} \alpha^2 M^3 (2g_S I)^2 \quad (4.160)$$

where $I = I_1 + I_2$ which we here rewrite explicitly:

$$I_1 = -i4m \int \frac{d^4 k}{(2\pi)^4} \frac{\tilde{\Phi}_S(q)}{(p_1^2 - m^2)(p_2^2 - m^2)(p_3^2 - m^2)} \quad (4.161)$$

$$I_2 = 4mB =$$

$$\begin{aligned} &i4m \int \frac{d^4 k}{(2\pi)^4} \left[\frac{2k^2}{(k_1 \cdot k_2)} - \frac{8(k \cdot k_1)(k \cdot k_2)}{(k_1 \cdot k_2)^2} \right] \\ &\quad \cdot \frac{\tilde{\Phi}_S(q)}{(p_1^2 - m^2)(p_2^2 - m^2)(p_3^2 - m^2)} \end{aligned} \quad (4.162)$$

where $p_1 = k + k_1$, $p_2 = k$, $p_3 = k - k_2$, $q = k + (k_1 - k_2)/2$.

4.6.7 Formal derivation of the gauge invariant part

We can find the gauge invariant part of the triangle diagram in a more formal way, which can be applied to any diagram. Here we just sketch briefly the procedure. In the expression for $M_{triangle}^{\mu\nu}$

$$-iM_{triangle}^{\mu\nu}(k_1, k_2) = -ie_f^2 2g_S \int \frac{d^4 k}{(2\pi)^4} Tr[S(p_1)\gamma^\mu S(p_2)\gamma^\nu S(p_3)]\tilde{\Phi}_S(q) \quad (4.163)$$

do the following replacements [67]:

$$\gamma^\mu \rightarrow \gamma_{\perp, k_1}^\mu = \gamma^\mu - k_1^\mu \frac{k_1}{k_1^2}, \quad (4.164)$$

$$\gamma^\nu \rightarrow \gamma_{\perp, k_2}^\nu = \gamma^\nu - k_2^\nu \frac{k_2}{k_2^2}. \quad (4.165)$$

One gets

$$\left(-iM_{triangle}^{\mu\nu}(k_1, k_2)\right)_{g.i.} = -ie_f^2 2g_S \int \frac{d^4 k}{(2\pi)^4} Tr[S(p_1)\gamma_{\perp, k_1}^\mu S(p_2)\gamma_{\perp, k_2}^\nu S(p_3)]\tilde{\Phi}_S(q) \quad (4.166)$$

which by construction is gauge invariant.

A bit of Algebra shows that this object is exactly the same as the one of eq. (4.155).

This procedure can easily be generalized to a generic Feynman diagram containing photons.

4.7 The scalar positronium

We conclude this chapter by studying the scalar positronium.

4.7.1 Non-relativistic expressions

We consider a $n = 2$ state with $L = 1$. The mass is $M_{n=2}$, as the already analyzed excited positronium. In spherical coordinates the wave function is

$$\Psi_{n=2,l=1,m} = \varphi_{n=2,l=1}(r)Y_{l=1,m}(\theta, \varphi) \quad (4.167)$$

where [62]

$$\varphi_{n=2,l=1}(r) = \frac{1}{2\sqrt{24}\sqrt{\pi}} \left(\frac{\alpha m}{2}\right)^{5/2} r \cdot \exp\left[-\frac{r \cdot \alpha m}{4}\right]. \quad (4.168)$$

The decay into two photons in the non-relativistic limit reads

$$\Gamma_{S \rightarrow 2\gamma} = \frac{9\alpha^2}{m^4} \left| \frac{d}{dr} (\varphi_{21}(r))_{r=0} \right|^2 = \frac{3\alpha^7 m}{256} = 6.61 \cdot 10^{-18} \text{ MeV}. \quad (4.169)$$

4.7.2 Our approach

We first have to determine the vertex function. This case is a bit more complicated than the previous ones. As discussed in section 3.6 the $L = 1$ structure comes out from the spinor decomposition of the scalar current.

First, let us consider the Fourier transform of the vertex function, which we express like

$$A_{21m}(\mathbf{q}) = \varphi_{21}(|\mathbf{q}|)Y_{1m}(\theta_q, \varphi_q). \quad (4.170)$$

where

$$\varphi_{21}(|\mathbf{q}|) = N \frac{|\mathbf{q}|}{\left(1 + \frac{16}{\alpha^2 m^2} |\mathbf{q}|^2\right)^3}. \quad (4.171)$$

In section 3.6 we have seen that the extra $|\mathbf{q}|$ comes from the spinorial evaluation, therefore the non-relativistic bound state $|B\rangle$ in the scalar case can be expressed in the following way:

$$|B\rangle = N \int d^3q \frac{\varphi_{21}(|\mathbf{q}|)}{|\mathbf{q}|} \left(\sum_{s_1, s_2 = \pm 1/2} \bar{u}^{(s_1)}(\mathbf{q}) v^{(s_2)}(-\mathbf{q}) \right) b^{\dagger(s_1)}(\mathbf{q}) d^{\dagger(s_2)}(-\mathbf{q}) |0\rangle. \quad (4.172)$$

Repeating the same steps already done in the parapositronium study we find:

$$\Phi(|\mathbf{q}|) = \left(2\sqrt{|\mathbf{q}|^2 + m^2} - M_{n=2}\right) \frac{\varphi_{21}(|\mathbf{q}|)}{|\mathbf{q}|} = \frac{\left(2\sqrt{|\mathbf{q}|^2 + m^2} - M_{n=2}\right)}{\left(1 + \frac{16}{\alpha^2 m^2} |\mathbf{q}|^2\right)^3}. \quad (4.173)$$

The residual factor has the same meaning as discussed before.

We can now calculate the two-photon decay by making the Wick rotation and by using the formulas (4.160), (4.161), (4.162). Well, if you do it, you obtain a result for the two-photon decay of the order $1.65 \cdot 10^{-18} \text{ MeV}$, which is quite different from the non-relativistic limit! The reason for this discrepancy can be traced to the expression (4.162), which has been calculated under the hypothesis of relativistic covariance, which in this case is broken by the use of an "instantaneous" vertex function.

However, we can make a "trick" to perform the calculation. We know that the dominant pole is for $q^0 = -M/2 + \sqrt{|\mathbf{q}|^2 + m^2} \simeq 0$. Therefore, in the integral (4.139) it is allowed to substitute $\Phi(|\mathbf{q}|)$ by $\Phi(-q^2)$. This last quantity is by no means a covariant expression for the wave function of the scalar positronium, but just a mathematical trick. The q^0 dependence, in fact, does not seriously modify the results. Now it is allowed to do all the manipulations shown above. The result for the two-photon decay rate is $6.37 \cdot 10^{-18} \text{ MeV}$, in good agreement with the non-relativistic limit. In [91] the decay rate of $6.55 \cdot 10^{-18} \text{ MeV}$ is found.

It may seem a bit of overkill to apply a covariant analysis to the scalar positronium. But this is a consistency check that the formulas we developed are correct, and can be applied to the calculation of the relativistic scalar mesons.

We could of course improve the result for the scalar positronium by including the relativistic corrections and higher order QED processes. But the goal is, as explained before, the study of a nonlocality, and not a precision test for QED.

Chapter 5

Mixing

5.1 Ellipses

5.1.1 Independent oscillators

We start from a mechanics example. Let us consider a point-like classical particle, whose motion in the plane (x, y) is described by the Lagrangian

$$L = \frac{1}{2} \left(\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right) - \frac{1}{2} \omega_1^2 x^2 - \frac{1}{2} \omega_2^2 y^2. \quad (5.1)$$

The potential $V(x, y) = \frac{1}{2} \omega_1^2 x^2 + \frac{1}{2} \omega_2^2 y^2$ is an ellipsoid; the equations of motion are two independent harmonic oscillators

$$\frac{d^2 x}{dt^2} + \omega_1^2 x = 0 \quad (5.2)$$

$$\frac{d^2 y}{dt^2} + \omega_2^2 y = 0 \quad (5.3)$$

and the general solutions for $x(t)$ and $y(t)$ are

$$x(t) = a e^{-i\omega_1 t} + a^* e^{i\omega_1 t} \quad (5.4)$$

$$y(t) = b e^{-i\omega_2 t} + b^* e^{i\omega_2 t}, \quad (5.5)$$

where we took into account that the two coordinates are real. The complex numbers a and b are determined from the four initial conditions, i.e. the two initial positions and velocities.

Note the similarity of this simple system with the QFT approach for two free fields! We also have written the solution for $x(t)$ and $y(t)$ in a similar form.

5.1.2 Mixing term

Let us now consider a x - y mixing term in the Lagrangian:

$$L \rightarrow L' = \frac{1}{2} \left(\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right) - \frac{1}{2} \omega_1^2 x^2 - \frac{1}{2} \omega_2^2 y^2 - \varepsilon xy. \quad (5.6)$$

The equations of motion are not anymore independent. In fact we have

$$\frac{d^2x}{dt^2} + \omega_1^2 x + \varepsilon y = 0 \quad (5.7)$$

$$\frac{d^2y}{dt^2} + \omega_2^2 y + \varepsilon x = 0 \quad (5.8)$$

It seems that the problem has become more complicated. But this is actually not true. We are just looking at it from the "wrong" point of view. The Lagrangian L is written in a "clever" form because the axes x and y coincide with the axes of the ellipsoid. Then its form is easy as the equations of motion.

Which form has the "new" potential $V'(x, y) = \frac{1}{2}\omega_1^2 x^2 + \frac{1}{2}\omega_2^2 y^2 + \varepsilon xy$? It is still an ellipsoid. But x and y are not anymore its axes. In order to recover the independent equations of motion we have to make a rotation in the x - y plane.

5.1.3 Matrix approach

We study the problem from a matrix point of view. Let us introduce the vector

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \quad (5.9)$$

and then we write L' like

$$L' = \frac{1}{2} \frac{dv^t}{dt} \cdot \frac{dv}{dt} - \frac{1}{2} v^t \begin{pmatrix} \omega_1^2 & \varepsilon \\ \varepsilon & \omega_2^2 \end{pmatrix} v \quad (5.10)$$

$$= \frac{1}{2} \frac{dv^t}{dt} \cdot \frac{dv}{dt} + \frac{1}{2} v^t \Omega v \quad (5.11)$$

where Ω is the symmetric matrix

$$\Omega = \begin{pmatrix} \omega_1^2 & \varepsilon \\ \varepsilon & \omega_2^2 \end{pmatrix}. \quad (5.12)$$

We search the eigenvalues of this matrix from

$$\det[\lambda \cdot 1_2 - \Omega] = (\lambda - \omega_1^2)(\lambda - \omega_2^2) - \varepsilon^2 = 0 \quad (5.13)$$

with solutions $\lambda_1 = \omega_1'^2$ and $\lambda_2 = \omega_2'^2$. The eigenvectors are then given by

$$\Omega v_i = (\omega_i')^2 v_i \quad (5.14)$$

with the normalization conditions

$$v_i \cdot v_j = \delta_{ij} \quad (5.15)$$

where $i = 1, 2$. We now consider the transformation matrix $B \in SO(2)$

$$B = \begin{pmatrix} v_1^t \\ v_2^t \end{pmatrix} \quad (5.16)$$

which has the eigenvectors in its rows. From the previous definitions it follows that

$$B\Omega B^t = \Omega' = \begin{pmatrix} \omega_1'^2 & 0 \\ 0 & \omega_2'^2 \end{pmatrix}. \quad (5.17)$$

All these relations are easily extendable to $i = 1, \dots, N$ but in the two-dimensional case an explicit solution is not too complicated with:

$$\omega_{1,2}'^2 = \frac{1}{2} \left[\omega_1^2 + \omega_2^2 \pm \sqrt{(\omega_1^2 + \omega_2^2)^2 - 4(\omega_1^2 \omega_2^2 - \varepsilon^2)} \right] \quad (5.18)$$

$$v_1 = \frac{1}{\sqrt{1 + ((\omega_1'^2 - \omega_2^2)/\varepsilon)^2}} \begin{pmatrix} (\omega_1'^2 - \omega_2^2)/\varepsilon \\ 1 \end{pmatrix}, \quad (5.19)$$

$$v_2 = \frac{1}{\sqrt{1 + ((\omega_2'^2 - \omega_2^2)/\varepsilon)^2}} \begin{pmatrix} (\omega_2'^2 - \omega_2^2)/\varepsilon \\ 1 \end{pmatrix}. \quad (5.20)$$

We then define

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = v' = Bv = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (5.21)$$

where we have explicitly written B in term of an angle θ . This is possible because B is an orthogonal $SO(2)$ matrix.

The Lagrangian in terms of v' looks like

$$L' = \frac{1}{2} \left(\left(\frac{dx'}{dt} \right)^2 + \left(\frac{dy'}{dt} \right)^2 \right) - \frac{1}{2} \omega_1'^2 x'^2 - \frac{1}{2} \omega_2'^2 y'^2 \quad (5.22)$$

which again consists of two independent oscillators, and is easily solvable as before. If we want the solutions in terms of (x, y) we rotate back, but this is generally not necessary.

The problem as seen from (x', y') is again symmetric. The "mixing term" disappears. It was just a sign of a not-optimal choice of axes. The relation between the original coordinates (x, y) and the new ones (x', y') is achieved by the transformation matrix B .

5.1.4 Small mixing limit

But what can we say about the new frequencies ω_1' and ω_2' ? Let us briefly analyze the small mixing limit, corresponding to

$$\varepsilon^2 \ll (\omega_2^2 - \omega_1^2) \quad (5.23)$$

where we suppose $\omega_2 > \omega_1$. We have:

$$\begin{aligned} \omega_{1,2}'^2 &= \frac{1}{2} \left[\omega_2^2 + \omega_1^2 \pm \sqrt{(\omega_2^2 + \omega_1^2)^2 - 4(\omega_1^2 \omega_2^2 - \varepsilon^2)} \right] \\ &= \frac{1}{2} \left[\omega_2^2 + \omega_1^2 \pm \sqrt{(\omega_2^2 - \omega_1^2)^2 + 4\varepsilon^2} \right] \\ &\simeq \frac{1}{2} (\omega_2^2 + \omega_1^2) \pm \frac{1}{2} \left[(\omega_2^2 - \omega_1^2) + \frac{2\varepsilon^2}{(\omega_2^2 - \omega_1^2)} \right], \end{aligned} \quad (5.24)$$

from which follows:

$$\omega_1'^2 = \omega_1^2 - \frac{\varepsilon^2}{(\omega_2^2 - \omega_1^2)}, \quad (5.25)$$

$$\omega_2'^2 = \omega_2^2 + \frac{\varepsilon^2}{(\omega_2^2 - \omega_1^2)}. \quad (5.26)$$

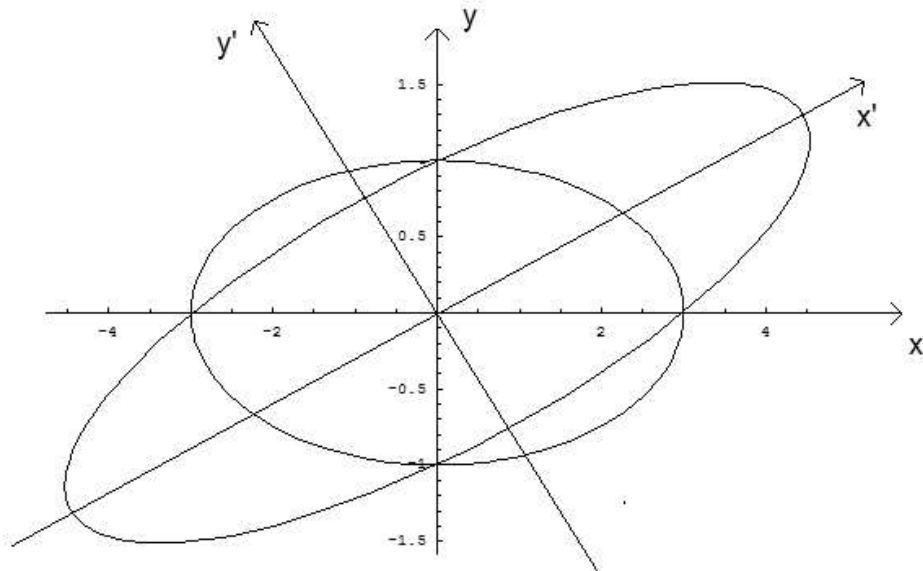


Figure 5.1: Comparison of ellipses; the inclined ellipsis corresponds to x-y mixing.

We have then $\omega_1'^2 < \omega_1^2$ and $\omega_2'^2 > \omega_2^2$, showing a typical mixing phenomenon, the "repulsion" of the frequencies: the lower frequency decreases, the higher one increases.

Let us observe the problem from a geometrical point of view. We consider the intersection with the plane $z = 1/2$ with the potential without mixing term $V(x, y) = \frac{1}{2}\omega_1^2 x^2 + \frac{1}{2}\omega_2^2 y^2$. It is an ellipsis with equation

$$\omega_1^2 x^2 + \omega_2^2 y^2 = 1, \quad (5.27)$$

whose axis are ω_1^{-1} and ω_2^{-1} .

If we consider the intersection with $V'(x, y) = \frac{1}{2}\omega_1^2 x^2 + \frac{1}{2}\omega_2^2 y^2 + \varepsilon xy$, we get a thinner and longer ellipsis, where the new axis are $(\omega_1')^{-1}$ and $(\omega_2')^{-1}$; in fact, the long axis is longer than the previous one: $(\omega_1')^{-1} > \omega_1^{-1}$. Similarly: $(\omega_2')^{-1} < \omega_2^{-1}$. This second ellipsis is then also rotated with respect to the previous one, and this shows again the need of a coordinate change. This situation is depicted in Fig. 5.1.

5.2 Field mixing

5.2.1 Rotation of fields

We now study the analogous problem for two fields; we call them G and S (for now elementary). The Lagrangian, together with the mixing term, looks like

$$\mathcal{L} = \frac{1}{2}(\partial_\mu G)^2 - \frac{1}{2}M_G^2 G^2 + \frac{1}{2}(\partial_\mu S)^2 - \frac{1}{2}M_S^2 S^2 - fGS. \quad (5.28)$$

If for the mixing parameter we have $f = 0$ we simply have two free fields with masses M_G and M_S . If $f \neq 0$ mixing is introduced, the two fields are not anymore orthogonal. But we are still dealing with a "free case" problem. As in the mechanical case, we have to perform a rotation, this time a field rotation, in order to recover a free Lagrangian. Let us introduce

$$\varphi = \begin{pmatrix} G \\ S \end{pmatrix}, \quad \Omega = \begin{pmatrix} M_G^2 & f \\ f & M_S^2 \end{pmatrix} \quad (5.29)$$

and rewrite

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi^t)(\partial^\mu \varphi) - \frac{1}{2}\varphi^t \Omega \varphi. \quad (5.30)$$

We transform Ω with the $SO(2)$ transformation matrix B as

$$B\Omega B^t = \Omega' = \begin{pmatrix} M_{G'}^2 & 0 \\ 0 & M_{S'}^2 \end{pmatrix} \quad (5.31)$$

(see the first section of this chapter). The rotated fields are

$$\begin{pmatrix} G' \\ S' \end{pmatrix} = \varphi' = B\varphi = B \begin{pmatrix} G \\ S \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} G \\ S \end{pmatrix} \quad (5.32)$$

which shows the composition of the "new" G' and S' fields as functions of the "bare ones" G and S . In terms of G' and S' the Lagrangian looks like

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \varphi'^t)(\partial^\mu \varphi') - \frac{1}{2}\varphi'^t \Omega' \varphi' = \\ &= \frac{1}{2}(\partial_\mu G')^2 - \frac{1}{2}M_{G'}^2 G'^2 + \frac{1}{2}(\partial_\mu S')^2 - \frac{1}{2}M_{S'}^2 S'^2 \end{aligned} \quad (5.33)$$

with the "new" rotated masses $M_{G'}$ and $M_{S'}$, found from the matrix equation

$$\text{Det}[\lambda \cdot 1 - \Omega] = (\lambda - M_G^2)(\lambda - M_S^2) - f^2 = 0. \quad (5.34)$$

No trace of mixing is present in (5.33): it is simply a free Lagrangian for G' and S' . The rotation brought us to the system where the physical states are defined. It is important to note that, if $f \neq 0$ the bare masses M_G and M_S do not have a direct physical meaning. The corresponding states G and S , which are not orthogonal, do not describe the asymptotic physical states. Their meaning is fully justified only in the limit $f \rightarrow 0$. The masses one measures in a hypothetical experiment are then $M_{G'}$ and $M_{S'}$ corresponding to the rotated physical fields G' and S' .

Nevertheless, it is often useful to start the discussion from some bare fields G and S , and then analyze the mixing and the corresponding new masses. In fact, they may correspond to



Figure 5.2: Corrections to the G-propagator.

more intuitive objects and it may be convenient to write the starting Lagrangian in terms of them. Furthermore, even if it is impossible to switch off the mixing in nature, this is conceptually and numerically doable. The direct connection (5.32) between the bare unphysical fields and the rotated physical ones also helps for an intuitive understanding of the mixing features. We can think that the field G' , for instance, is a quantum composition of G and S where $\cos\theta$ and $\sin\theta$ are the corresponding amplitudes.

In the small mixing limit the new masses are

$$M_{G'}^2 = M_G^2 - \frac{f^2}{M_S^2 - M_G^2} \quad (5.35)$$

$$M_{S'}^2 = M_S^2 + \frac{f^2}{M_S^2 - M_G^2} \quad (5.36)$$

where we have supposed that $M_S > M_G$. We again have the phenomenon of level repulsion of the eigenvalues with $M_{G'}^2 < M_G^2$ and $M_{S'}^2 > M_S^2$.

5.2.2 A funny exercise: mixing and renormalization

Let us again consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu G)^2 - \frac{1}{2}M_G^2 G^2 + \frac{1}{2}(\partial_\mu S)^2 - \frac{1}{2}M_S^2 S^2 - fGS; \quad (5.37)$$

what happens if we consider the mixing term fGS as a perturbation and we consider the mass renormalization as in section 2.4? M_G and M_S are then the "bare" unphysical masses, and we have to search the poles of the propagator for the fields G and S when the "mixing interaction" is included.

For the propagator of the field G we then have the summation (see Fig. 5.2)

$$\begin{aligned} \frac{i}{p^2 - M_G^2} &\rightarrow \frac{i}{p^2 - M_G^2} \left(1 + \frac{i}{p^2 - M_G^2} (-f^2) \frac{i}{p^2 - M_S^2} + \dots \right) = \\ &= \frac{i}{p^2 - M_G^2} \sum_{n=0}^{\infty} \left(\frac{f^2}{(p^2 - M_G^2)(p^2 - M_S^2)} \right)^n \\ &= i \frac{(p^2 - M_S^2)}{(p^2 - M_G^2)(p^2 - M_S^2) - f^2}. \end{aligned} \quad (5.38)$$

The "new" poles of this expression are exactly the solution of the determinant equation shown before. If we then choose as solution of the pole the mass $M_{G'}$, we have for $p^2 \simeq M_{G'}^2$, the propagator

$$\frac{i}{p^2 - M_{G'}^2} \quad (5.39)$$

i.e. the propagator of the field G' . This simple exercise shows indeed something important: summing up to infinite order we find, as expected, the correct physical masses. But this is not possible at any finite order of perturbation theory. The field rotation $\varphi \rightarrow \varphi'$ is in general a non-trivial field change. It allows us to find the physical states and to define properly the physical fields. This transformation is non-perturbative, and it constitutes a nice and simple example to prove that not always a perturbative approach is the most useful.

5.3 Three fields

We now extend the mixing to the case of three fields. This case is of physical interest to us. We consider three scalar fields: N , G , S . The first, N , represents the light quark-antiquark state $\bar{n}n = \sqrt{1/2}(\bar{u}u + \bar{d}d)$, the second, G , the bound state of two gluons, and the third, S , a composition of the strange quark pair $\bar{s}s$. At this level we will consider the three fields N , G , and S as elementary and point-like.

The quark-antiquark scalar mesons have a mass between 1-2 GeV (see introduction and Refs. therein). Lattice QCD predicts a similar mass for the glueball, and the respective states can mix. We will then analyze in more detail in the next chapter how this mixing comes out from the basic quark and gluon degrees of freedom, but here we want to illustrate it from the easier Klein-Gordon point of view. We then write down the Lagrangian in the most general mixing configuration as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu N)^2 - \frac{1}{2}M_G^2 N^2 + \frac{1}{2}(\partial_\mu G)^2 - \frac{1}{2}M_G^2 G^2 + \frac{1}{2}(\partial_\mu S)^2 - \frac{1}{2}M_S^2 S^2 \\ &\quad - fGS - \sqrt{2}frGN - \varepsilon NS \end{aligned} \quad (5.40)$$

where we introduced the three mixing parameters f , r , and ε . Introducing φ and Ω like

$$\varphi = \begin{pmatrix} N \\ G \\ S \end{pmatrix}; \quad \Omega = \begin{pmatrix} M_N^2 & \sqrt{2}fr & \varepsilon \\ \sqrt{2}fr & M_G^2 & f \\ \varepsilon & f & M_S^2 \end{pmatrix}. \quad (5.41)$$

we have again

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi^t)(\partial^\mu \varphi) - \frac{1}{2}\varphi^t \Omega \varphi. \quad (5.42)$$

We diagonalize Ω with the $SO(3)$ transformation matrix B as

$$B\Omega B^t = \Omega' = \begin{pmatrix} M_{N'}^2 & 0 & 0 \\ 0 & M_{G'}^2 & 0 \\ 0 & 0 & M_{S'}^2 \end{pmatrix} \quad (5.43)$$

where the physical masses appear. The rotated field are

$$\begin{pmatrix} N' \\ G' \\ S' \end{pmatrix} = \varphi' = B\varphi = B \begin{pmatrix} N \\ G \\ S \end{pmatrix} \quad (5.44)$$

In the mass region between 1-2 GeV three scalar-isoscalar states $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$ are seen. As discussed in the introduction, one naively expects that only two of such states can be part of a quark-antiquark nonet (see also next chapter). A possible explanation to the puzzle is to say that the glueball comes into the game. But no state among these three resonance has the right strong decay pattern one would expect for a pure glueball state. A way out is then to admit that the bare glueball (gluon-gluon state in a minimal configuration) mixes with the quark-antiquark configuration (terms fGS and $\sqrt{2}frGN$ in our toy Lagrangian) generating the three observed resonances. We then interpret the three orthogonal state N' , G' and S' of our simple Lagrangian as the physical resonances $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$.

The B matrix thus tells us what the amount of $\bar{n}n$, gluons and $\bar{s}s$ in each physical resonance is.

If we start from the bare level configuration $M_N < M_G < M_S$, which seems now the favored case (see [8]), we will have the following trend:

- $M_{N'} < M_N$
- $M_{S'} > M_S$,

i.e. the outer states are repelled, as in the two-state mixing case. For the intermediate there is not a general behavior, but it depends on the special mixing configuration one adopts.

With this assignment, when mixing is small, it typically follows that the state $G' \equiv f_0(1500)$ has the largest gluonic content.

We just note that this is not the only possible solution of the problem. Other scalar states have been observed, such as $f_0(980)$ and the enigmatic broad $\sigma \equiv f_0(400 - 1200)$. It is still an open issue to disentangle the nature of these states; if one follows our previous assignment (see scenario A in section 1.5.4 and Refs. therein), then these states are not $\bar{q}q$ states, but something different, as four-quark states or meson molecules. But other mixing schemes are considered, where also these states are $q\bar{q}$ and may have a glueball decomposition. Actually I think that all possible compositions of glueball and scalar states have already been proposed (see 1.5.4).

In this work we will mostly analyze the "standard" and original scheme, where the glueball mixes with the scalar states above 1 GeV .

In the next chapter we will compare the Klein-Gordon approach with a more fundamental non-local one, where one deals with the basic quark and gluon degrees of freedom.

5.4 Two-photon decay of the mixed states

Let us consider the zero-mixing limit and analyze in this case the two-photon decay. According to what we have seen in the previous chapter and considering the quark-antiquark nature

of the states N and S we write

$$\begin{aligned}\mathcal{L}_{nomix,2\gamma} &= \frac{1}{2}(\partial_\mu N)^2 - \frac{1}{2}M_G^2 N^2 \frac{1}{2}(\partial_\mu G)^2 - \frac{1}{2}M_G^2 G^2 + \frac{1}{2}(\partial_\mu S)^2 - \frac{1}{2}M_S^2 S^2 \\ &+ g_{scalar} \left(\frac{4}{9} + \frac{1}{9} \right) N (F^{\mu\nu} F_{\mu\nu}) \\ &+ g_{scalar} \left(\frac{1}{9} \right) S (F^{\mu\nu} F_{\mu\nu})\end{aligned}\quad (5.45)$$

where:

- G doesn't couple directly to the photons; in fact, G describes two (or more) gluons, which do not interact with photons.

- g_{scalar} is a constant which describes the loop diagrams for the scalar quark-antiquark configuration. The fact that we take the same constant g_{scalar} for both the N and the S decay means that we are assuming that the s-quark loop is equal to the u-d quark one (this is valid in the $SU_f(3)$ flavor symmetry limit).

- The factors $\left(\frac{4}{9} + \frac{1}{9}\right)$ for N and $\frac{1}{9}$ for S take explicitly into account the charge of the quark content. This is assumed to be the "dominant" difference among the two states.

We then have following two-photon decay rates for the three fields N , G and S :

$$\Gamma_{N \rightarrow 2\gamma} = \frac{\pi}{4} \alpha^2 M_N^3 \left(2 \left(\frac{4}{9} + \frac{1}{9} \right) g_{scalar} \right)^2 \quad (5.46)$$

$$\Gamma_{G \rightarrow 2\gamma} = 0 \quad (5.47)$$

$$\Gamma_{S \rightarrow 2\gamma} = \frac{\pi}{4} \alpha^2 M_S^3 \left(2 \left(\frac{1}{9} \right) g_{scalar} \right)^2. \quad (5.48)$$

- The coupling g_{scalar} is not only valid for the two considered states N and S but also for the isospin 1 state $a_0^0(1450) \equiv \sqrt{1/2}(\bar{u}u - \bar{d}d)$; the only difference is the opposite flavour phase; the decay rate for this state is then

$$\Gamma_{a_0^0 \rightarrow 2\gamma} = \frac{\pi}{4} \alpha^2 M_{a_0}^3 \left(2 \left(\frac{4}{9} - \frac{1}{9} \right) g_{scalar} \right)^2 \quad (5.49)$$

where the flavour factor $\left(\frac{4}{9} - \frac{1}{9}\right)$ is the same as for the pion.

We understand why it is useful to start with the bare fields N , G , and S in this case: one can write in a simple and physical meaningful way the two-photon interaction terms. We then introduce the mixing; the goal is to derive the two-photon decay of the rotated physical fields N' , G' and S' . To this end let us introduce the column vector F

$$F = \begin{pmatrix} g_{scalar} \left(\frac{5}{9} \right) (F^{\mu\nu} F_{\mu\nu}) \\ 0 \\ g_{scalar} \left(\frac{1}{9} \right) (F^{\mu\nu} F_{\mu\nu}) \end{pmatrix} \quad (5.50)$$

from which we can write the full Lagrangian in the following matrix form (see 5.43)

$$\mathcal{L}_{2\gamma} = \frac{1}{2}(\partial_\mu \varphi^t)(\partial^\mu \varphi) - \frac{1}{2}\varphi^t \Omega \varphi + \varphi^t F. \quad (5.51)$$

Introducing then $\varphi' = B\varphi$ (Eqs. (5.43) and (5.44)) we get

$$\mathcal{L}_{2\gamma} = \frac{1}{2}(\partial_\mu \varphi'^t)(\partial^\mu \varphi') - \frac{1}{2}\varphi'^t \Omega' \varphi' + \varphi'^t B F. \quad (5.52)$$

where the first part is the free one for the rotated field φ' . The two-photon interaction term reads explicitly

$$\begin{aligned}
\mathcal{L}_{2\gamma,int} &= \varphi'^t B F = \\
&= g_{scalar} N' (F^{\mu\nu} F_{\mu\nu}) \left(B^{N'N} \frac{5}{9} + B^{N'S} \frac{1}{9} \right) \\
&\quad + g_{scalar} G' (F^{\mu\nu} F_{\mu\nu}) \left(B^{G'N} \frac{5}{9} + B^{G'S} \frac{1}{9} \right) \\
&\quad + g_{scalar} S' (F^{\mu\nu} F_{\mu\nu}) \left(B^{S'N} \frac{5}{9} + B^{S'S} \frac{1}{9} \right), \tag{5.53}
\end{aligned}$$

where we denoted the elements of the mixing matrix B as $B^{i,a}$ with $i = N', G', S'$ and $a = N, G, S$. The term $B^{N'G}$, for example, tell us the amount of gluons in the physical state N' . The two-photon decay takes place only through the N and the S components, because the gluons, being chargeless particles, cannot decay directly into two photons.

The final formulas for the decay of the mixed states are then:

$$\begin{aligned}
\Gamma_{N' \rightarrow 2\gamma} &= \frac{\pi}{4} \alpha^2 M_{N'}^3 \left(2 \left(B^{N'N} \frac{5}{9} + B^{N'S} \frac{1}{9} \right) g_{scalar} \right)^2, \\
\Gamma_{G' \rightarrow 2\gamma} &= \frac{\pi}{4} \alpha^2 M_{G'}^3 \left(2 \left(B^{G'N} \frac{5}{9} + B^{G'S} \frac{1}{9} \right) g_{scalar} \right)^2, \\
\Gamma_{S' \rightarrow 2\gamma} &= \frac{\pi}{4} \alpha^2 M_{S'}^3 \left(2 \left(B^{S'N} \frac{5}{9} + B^{S'S} \frac{1}{9} \right) g_{scalar} \right)^2. \tag{5.54}
\end{aligned}$$

Note that these decays are sensible to the mixing decompositions, first because of the different two-photon interactions of the bare fields (which in turn depend on the different charge of the subcomponents), secondly because a constructive or destructive interference can change the decay rate. This decay, together with the strong decay channels, which are more difficult to develop because of the possible direct decay of the glueball state, should in principle allow to derive the mixing parameters and tell us if the three-mixing scheme hypothesis is valid and sufficient to describe the results of the resonances $N' \equiv f_0(1370)$, $G' \equiv f_0(1500)$ and $S' \equiv f_0(1710)$.

Unfortunately the two-photon decay rates of these three resonances are not well known. Only the decay rate of the $N' \equiv f_0(1370)$ has already been seen. It is however not clear if this two-photon signal comes from this resonance or from the high-mass tail of $f_0(400 - 1200)$, or from both, and in this last case in which amounts.

This is one of the reasons why the existence of the glueball and its eventual mixing is still an open issue under intensive debate.

We will also study in the last chapter the strong decays of mixed states. The formalism described in this chapter can be applied with minor changes to the decay of a scalar into two pseudoscalars.

5.5 Coupling with external currents

Let us now consider the following interaction, where the three fields N , G and S couple to three external currents J_n , J_g and J_s :

$$\mathcal{L}_{int} = g_N N J_n + g_G G J_g + g_S S J_s = \varphi'^t \hat{g} J \tag{5.55}$$

where

$$\widehat{g} = \begin{pmatrix} g_N & 0 & 0 \\ 0 & g_G & 0 \\ 0 & 0 & g_S \end{pmatrix} \quad (5.56)$$

and

$$J = \begin{pmatrix} J_n \\ J_g \\ J_s \end{pmatrix}. \quad (5.57)$$

After diagonalization, we have in terms of the rotated fields φ'

$$\varphi'^t B \widehat{g} J = \varphi'^t \widehat{g}' J \quad (5.58)$$

where

$$B \widehat{g} = \widehat{g}' = \begin{pmatrix} g_{N'}^{nn} & g_{N'}^{gg} & g_{N'}^{ss} \\ g_{G'}^{nn} & g_{G'}^{gg} & g_{G'}^{ss} \\ g_{S'}^{nn} & g_{S'}^{gg} & g_{S'}^{ss} \end{pmatrix}. \quad (5.59)$$

\widehat{g}' describes the coupling constants of the rotated fields to the three currents J_n , J_g and J_s . Explicitly, for G' one has

$$L_{G'} = g_{G'}^{nn} G' J_n + g_{G'}^{gg} G' J_g + g_{G'}^{ss} G' J_s \quad (5.60)$$

and similarly for N' and S' . Because of the mixing, the rotated field G' couples also with J_n and J_s , while the original bare field G couples only to J_g . The physical meaning will be better explained in the next chapter; the current J_g will represent a two-gluon current, and J_n and J_s two $\bar{q}q$ currents corresponding to $\sqrt{1/2}(\bar{u}u + \bar{d}d)$ and $\bar{s}s$ respectively. The "bare" G couples only to gluons, while the rotated G' couples also to quarks. It means that the corresponding state is a mixture of quarks and gluons.

Let us note that the elements of the Klein-Gordon mixing matrix B are related to the coupling constants as follows

$$B^{ia} = g_i^a / g_a \quad (5.61)$$

with $i = N', G', S'$ and $a = N, G, S$.

We will use such a relation to *define* the mixing matrix when the mixing will be described in terms of gluons and quarks. We will have to extend it properly, but it will be the correct extension when one is working within a Bethe-Salpeter approach.

Chapter 6

The glueball and its mixing with \bar{q} - q components

6.1 QCD

6.1.1 The Lagrangian

In this work we often spoke about quarks and gluons. We gave an historical presentation in the introduction, we discussed briefly the peculiar characteristics of the quark mass at low and high energy (chapter 3), we described how to take into account the flavor and the color when writing down a non-local quark current (chapter 3), and we finally analyzed the two-photon decay of the neutral pion and of a scalar quark-antiquark state (chapter 4). Now we briefly describe the main properties of the underlying theory [3, 56, 60, 55, 96, 97, 98, 99].

There are 6 different types of quarks (flavors): u,d,s,c,b,t. Here we will restrict to the three light ones, i.e. u, d and s; the quarks interact among each other by gluon exchange, in an elementary quark-quark-gluon vertex, similar to the lepton-lepton-photon one of QED. In order to see how the interaction comes out let us postulate that the quarks carry a color-charge, which is analogous to the electric one, but which can take three different colors $c = R, G, B$ (red, green, blue). This means that the quark field carries two indices, one for the flavor and one for the color:

$$q(x) = q_{i,c}(x) \quad (6.1)$$

with $i = u, d, s$ and $c = R, G, B$. $q(x)$ is a 36-vector column. The corresponding free fermionic Lagrangian is

$$\mathcal{L}_{quark,free} = \bar{q}_{i,c}(x) (i\gamma^\mu \partial_\mu - m_i) q_{i,c}(x) \quad (6.2)$$

where the summation over i and c is understood; m_i are the high-energy or current "quark masses", where we used the double quotes to stress that the word "mass" should be taken with care (see section 3.4.2). This Lagrangian is invariant under global $SU_c(3)$ color transformations

$$q_{i,c}(x) \rightarrow \left(e^{i\theta_a \frac{1}{2}\lambda_a} \right)_{cd} q_{i,d}(x) \quad (6.3)$$

where the 8 traceless matrices λ_a are the generator of the $SU(3)$ group and θ_a are 8 arbitrary real constants. The commutation relations for λ_a are

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2} \right] = if_{abc} \frac{\lambda_c}{2} \quad (6.4)$$

where the non-zero totally antisymmetric constants f_{abc} are

$$f_{123} = 1, \quad f_{458} = f_{678} = \sqrt{3}/2$$

$$f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{346} = 1/2. \quad (6.5)$$

We have seen that the QED Lagrangian is invariant under local $U(1)$ transformations; the basic postulate of QCD is invariance under *local* $SU_c(3)$ transformations, which are found from the global ones by considering the eight parameters $\theta_a(x)$ as generic functions of the space-time variable x

$$q_{i,c}(x) \rightarrow \left(e^{i\theta_a(x)\frac{1}{2}\lambda_a} \right)_{cd} q_{i,d}(x). \quad (6.6)$$

The free quark Lagrangian does not satisfy such a requirement; according to the Yang-Mills procedure we achieve $SU_c(3)$ local gauge invariance by considering

$$\mathcal{L}_{QCD} = \bar{q}_{i,c}(x) (i\gamma^\mu (D_\mu)_{cd} - m_i) q_{i,d}(x) - \frac{1}{4} G_{\mu\nu}^a G^{a,\mu\nu} \quad (6.7)$$

where:

- D_μ , given by

$$(D_\mu)_{cd} = \delta_{cd}\partial_\mu + ig\frac{1}{2}(\lambda_a)_{c,d}G_\mu^a \quad (6.8)$$

is the covariant derivative; the 8 vectorial boson fields G_μ^a describe the gluons. We assume that, when transforming the quark fields like (6.6), the gluons at the same time change like

$$G_\mu^a \rightarrow G_\mu^a - \frac{1}{g}\partial_\mu\theta_a - f_{abc}\theta_b G_\mu^c. \quad (6.9)$$

In this way the first piece of the Lagrangian is $SU_c(3)$ local gauge invariant.

- The introduction of the covariant derivative gives us the gluon-quark interaction term

$$L_{gluon-quarks} = g\bar{q}_{i,c}(x)\frac{1}{2}(\lambda_a)_{c,d}\gamma^\mu q_{i,d}(x)G_\mu^a \quad (6.10)$$

which gives origin to elementary quark-quark-gluon vertex; g is the corresponding coupling constant. Note that at each vertex also the λ_a color matrices act.

- The last piece of the QCD Lagrangian describes the dynamical part of the gluons with

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - gf_{abc}G_\mu^b G_\nu^c. \quad (6.11)$$

This is the non-abelian generalization of the $F^{\mu\nu}$ field tensor of electrodynamics. The extra-piece $gf_{abc}G_\mu^b G_\nu^c$ is necessary in order to guarantee local gauge invariance of $G_{\mu\nu}^a G^{a,\mu\nu}$.

This is the crucial difference from QED. This extra-term generates three and four-gluon interaction terms in the gluonic part of the QCD Lagrangian. It is believed that this difference forms the basis of the peculiar QCD phenomenology, that is:

→ confinement: only color-neutral objects appear in nature; these are quarks and gluon combination which do not transform under color transformations; a typical example is the mesonic quark-antiquark configuration

$$\bar{q}_{i,c}(x)q_{j,c}(x) \quad (6.12)$$

corresponding to the color wave function $\bar{R}R + \bar{G}G + \bar{B}B$. The current $\bar{q}_{i,c}(x)q_{j,c}(x)$ is invariant under local $SU_c(3)$ transformations (6.6), so we may say that it is "locally white". In phenomenological studies the weaker constraint of "global color invariance" corresponding to the meson current

$$\bar{q}_{i,c}(x_1)q_{j,c}(x_2) \quad (6.13)$$

is often required; this object is invariant under global $SU_c(3)$ (6.3).

Another example is the baryonic three quark configuration

$$\varepsilon^{abc}q_{i,a}q_{j,b}q_{k,c} \quad (6.14)$$

where ε^{abc} is the totally antisymmetric Ricci tensor. As above, local color gauge invariance is satisfied if the fields are considered at the same space-time point x , otherwise the weaker global color symmetry is imposed.

This antisymmetric color decomposition is the one discussed in the introduction, originally proposed to overcome the statistical Δ^{++} problem.

→ asymptotic freedom: the quarks are almost free at small distances, but interact non-perturbative at large ones. We have seen that a theory has to undergo a renormalization process, and that the coupling constant becomes momentum dependent; in QCD at lowest order one gets

$$\alpha_{strong}(\mu^2) = \frac{g^2(\mu^2)}{4\pi} = \frac{4\pi}{(11 - \frac{2}{3}N_f) \ln \left[\frac{\mu^2}{\Lambda_{QCD}^2} \right]} \quad (6.15)$$

where N_f is the number of flavors, 3 in our case. This function, valid for $\mu^2 \gg \Lambda_{QCD}^2$, decreases for increasing $\mu^2 = -q^2$ and q is the gluon four-momentum. It means that if we try to pull a quark out of a hadron the return force will grow for increasing distance. This makes the operation impossible; we cannot set the quark free. What it may happens in such an operation is the appearance of other hadrons (at the cost of additional quark-antiquark pair creation), but in any case quark-confined structures (section 1.4). On the other hand, for large μ^2 the interaction decreases, realizing asymptotic freedom.

We can summarize this discussion by writing:

- $\mu^2 \gg \Lambda_{QCD}^2$: a perturbative approach is applicable
- $\mu^2 \leq \Lambda_{QCD}^2$: non-perturbative regime and confinement; the QCD confined states "are here", the "hic sunt leones" of the theory.

6.1.2 Symmetries

We now list the symmetries of QCD; the first one is of course the local $SU_c(3)$, which we studied in the previous subsection. We actually constructed the QCD Lagrangian by imposing this local symmetry. This in turn led us to the introduction of 8 gluon fields, which, in virtue of the non-abelian nature of the theory, interact among each other. But there are other symmetries of QCD :

6.1.2.1 $U_V(1)$:

We have invariance under the transformation

$$q_{i,c} \rightarrow e^{i\alpha} q_{i,c} \quad (6.16)$$

whose corresponding charge is the baryon number

$$B = \frac{1}{3} \int d^3x q_{i,c}^\dagger q_{i,c}. \quad (6.17)$$

6.1.2.2 $SU_f(3)$:

Let us now consider the global $SU_f(3)$ flavour transformation

$$q_{i,c} \rightarrow \left(e^{i\theta_k \frac{1}{2} \lambda_k} \right)_{ij} q_{j,c}. \quad (6.18)$$

The gluon-quark interaction term is invariant under this transformation; in fact, the gluons interact with each flavor in the same way. There is "flavor-democracy" in this interaction term; only the color is interchanged, but not the flavor. The mass term of the QCD Lagrangian is invariant only if the three masses are equal. Let us split the discussion into the perturbative and the non-perturbative QCD domains:

→ *PQCD* : in the high energy region the momenta are much larger than the light quark masses $m_u = 2 - 5 \text{ MeV}$, $m_d = 4 - 7 \text{ MeV}$ and $m_s = 70 - 130 \text{ MeV}$. This means that we can consider all the light quarks as massless, thus having an approximate $SU_f(3)$ symmetry.

→ *NPQCD* : in the low energy domain one has, as discussed in section 3.4, a non-perturbative modification of the light quark propagators with the generation of high effective quark masses $m_u^* \sim m_d^* = 200 - 500 \text{ MeV}$, $m_s^* = 500 - 700 \text{ MeV}$. These three masses are not equal but of the same order; also in this limit we can speak of an approximate realization of $SU_f(3)$ flavor symmetry.

It is important to distinguish conceptually the flavor and the color $SU(3)$ transformations; the color one is exact, with the color being the "charge" of the gluonic interaction, while the flavor one is an approximate symmetry valid for the light quark sector in the limit of equal flavor masses (more generally in the limit of equal flavor propagators).

6.1.2.3 $SU_{A,f}(3)$:

The axial flavor transformation is similar to the previous one with an extra γ^5 in the exponential:

$$q_{i,c} \rightarrow \left(e^{i\gamma^5 \theta_k \frac{1}{2} \lambda_k} \right)_{ij} q_{j,c}. \quad (6.19)$$

Again, the QCD interaction term is invariant because γ^5 commutes with γ^μ . What about the mass term? Well, this is invariant only if the three quark masses are zero. This means that:

→ *PQCD* : the quark masses are negligible, therefore in this energy domain $SU_{A,f}(3)$ is a good (approximate) symmetry of the theory, but....

→ *NPQCD* : large effective quark masses are, due to non non-perturbative effects, developed! In this low-energy region $SU_{A,f}(3)$ is NOT realized in terms of a degenerate spectrum; a look at the mesonic spectrum confirms this; this transformation, by the γ^5 , changes the parity of a mesonic state, thus transforming a pseudoscalar state into a scalar one. A symmetry under this transformation would mean, for instance, an equal spectrum for the pseudoscalar $J^{PC} = 0^{-+}$ and the scalar $J^{PC} = 0^{++}$, which is clearly not realized because the pseudoscalar resonances are much lighter than the scalar ones.

This phenomenon is also known under the name of "spontaneous breaking of the $SU_{A,f}(3)$ symmetry group". Although the QCD Lagrangian is (approximately) invariant under

$SU_{A,f}(3)$ transformations, the physical QCD ground state $|0\rangle_{QCD}$ is not. It follows that the charges do not annihilate the vacuum

$$Q_k^A |0\rangle_{QCD} \neq 0, \quad (6.20)$$

where the Q_a^A are explicitly

$$Q_k^A = \int d^3x \left(\bar{q}_{i,c} \gamma^5 \left(\frac{\lambda_k}{2} \right)_{ij} q_{j,c} \right) \quad (6.21)$$

where $k = 1, \dots, 8$.

The Goldstone theorem then says that for each conserved charge which does not annihilate the vacuum, there is a massless boson: in fact, if one considers the eight states

$$|\pi_k\rangle \propto Q_k^A |0\rangle_{QCD} \neq 0 \quad (6.22)$$

we have

$$H_{QCD} Q_k^A |0\rangle_{QCD} \simeq Q_k^A H_{QCD} |0\rangle_{QCD} = 0, \quad (6.23)$$

thus meaning that we expect eight "almost" massless pseudoscalar bosons $|\pi_k\rangle$: "pseudoscalar", because such are the charges Q_k^A and "almost massless" because the symmetry of the theory is only approximate; this is why we have $[H_{QCD}, Q_k^A] \simeq 0$ and not exactly equal to zero (in turn related to the small but non-zero perturbative current quark masses).

6.1.2.4 $U_A(1)$

The QCD interaction term is also invariant under the axial $U_A(1)$ transformation

$$q_{i,c} \rightarrow e^{i\alpha\gamma^5} q_{i,c}. \quad (6.24)$$

The mass term, again, is invariant under this operation only if the quark masses are zero. One is then lead to think that there is a ninth Goldstone boson, but this is not the case. It is possible to prove [3, 56, 60] that also in the limit of zero quark masses one has at the quantized level

$$\partial_\mu (\bar{q}_{i,c} \gamma^\mu \gamma^5 q_{i,c}) = \frac{g^2 N_f}{8\pi^2} \left(\frac{1}{8} \varepsilon_{\mu\nu\rho\sigma} G^{a,\mu\nu} G^{a,\rho\sigma} \right) \neq 0. \quad (6.25)$$

This is a so-called anomaly, i.e. a symmetry of the classical Lagrangian which does not survive after quantization. The axial current is related to the pseudoscalar gluon field configuration, which is not zero. This means that the charge

$$Q_0^A = \int d^3x \left(\bar{q}_{i,c} \gamma^5 \left(\frac{\lambda_0}{2} \right)_{ij} q_{j,c} \right), \quad (6.26)$$

where $\lambda_0 = \sqrt{2/3}1_3$, doesn't annihilate the vacuum. The corresponding bound state is a pseudoscalar "flavor singlet", i.e. $\sqrt{1/3}(\bar{u}u + \bar{d}d + \bar{s}s)$; we recognize the observed state η' , which is approximately a flavor singlet and is heavier than the other 8 pseudoscalar states, thus confirming the non-Goldstone boson nature of the ninth state.

6.2 Mesons

6.2.1 Flavor basis

Let us now look closer at the flavor combinations for a quark-antiquark bound state. We have seen that $SU_f(3)$ is an approximate symmetry of the theory. One then constructs the states using the generators of the symmetry group, that is considering the nine objects $\bar{q}_{i,c}(\lambda^k)_{i,j} \Gamma q_{j,c}$ where $k = 0, 1, \dots, 9$ with $\lambda^0 = \sqrt{2/3}1_3$, Γ is a Dirac matrices responsible for the spin-quantum numbers, thus unimportant for the flavor decomposition; with $\Gamma = i\gamma^5$, for instance, one has pseudoscalar objects. For $k = 1, \dots, 8$ the current $\bar{q}_{i,c}(\lambda^k)_{i,j} \Gamma q_{j,c}$ changes under $SU_f(3)$ rotations, while for $k = 0$ one has the flavour singlet, which is invariant. One finds the octet states:

$$\begin{aligned} & \bar{q}_{i,c}(\lambda^1)_{i,j} \Gamma q_{j,c} \pm \bar{q}_{i,c}(\lambda^2)_{i,j} \Gamma q_{j,c}, \quad \bar{q}_{i,c}(\lambda^3)_{i,j} \Gamma q_{j,c}, \\ & \bar{q}_{i,c}(\lambda^4)_{i,j} \Gamma q_{j,c} \pm \bar{q}_{i,c}(\lambda^5)_{i,j} \Gamma q_{j,c}, \quad \bar{q}_{i,c}(\lambda^6)_{i,j} \Gamma q_{j,c} \pm \bar{q}_{i,c}(\lambda^7)_{i,j} \Gamma q_{j,c} \\ & \bar{q}_{i,c}(\lambda^8)_{i,j} \Gamma q_{j,c}, \end{aligned}$$

which written in terms of flavors are

$$\bar{u}d, \bar{d}u, \sqrt{\frac{1}{2}}(\bar{u}u - \bar{d}d), \bar{u}s, \bar{s}u, \bar{d}s, \bar{s}d, \sqrt{\frac{1}{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s). \quad (6.27)$$

One then has the flavor singlet state,

$$\bar{q}_{i,c}(\lambda^0)_{i,j} \Gamma q_{j,c}, \quad (6.28)$$

which is invariant under $SU_f(3)$ rotation and in terms of flavor reads

$$\sqrt{\frac{1}{3}}(\bar{u}u + \bar{d}d + \bar{s}s). \quad (6.29)$$

The three states $\bar{u}d, \bar{d}u, \sqrt{1/2}(\bar{u}u - \bar{d}d)$ arise from $a = 1, 2, 3$ and correspond to the subgroup $SU_f(2)$ relative to the quark flavors u and d ; this subgroup is called "isospin" and the three states are then isospin vectors. In the pseudoscalar case they are the renowned pions π^-, π^+, π^0 .

In the limit of an exact symmetry under $SU_f(3)$ transformations the octet states are degenerated. But we know that $SU_f(3)$ symmetry is only approximate, therefore we expect deviations from this trend. In order to analyze it we consider in the following a quartic Lagrangian.

6.2.2 Example: $SU_V(3)$ quartic Lagrangian

First, note that we are restricting to quark-antiquark states, but a physical meson state may be more complicated in terms of Fock states:

$$|s\rangle = c_1 |\bar{q}q\rangle + c_2 |\bar{q}q\bar{q}q\rangle + \dots \quad (6.30)$$

We assume that the first contribution is dominant. This may be true in some cases, but, as we have seen in the introduction, not in general. In the previous chapters we analyzed the

appearance of bound states by making use of a quartic Lagrangian. Why not to do the same now? We have then seen that the symmetry $SU_V(3)$ is still approximately valid also at low energy, i.e. in the bound state energy domain. Referring to the channel Γ ($\Gamma = i\gamma^5$ for the pseudoscalar nonet, $\Gamma = \gamma^\mu$ for the vectorial one, $\Gamma = 1$ for the scalar, ...) we write down the interaction term

$$\mathcal{L} = \frac{G_\Gamma}{4} \sum_{k=0}^8 \left(\bar{q} \Gamma \lambda^k q \right)^2 \quad (6.31)$$

which is $SU_V(3)$ invariant. Now, in order to analyze the structure, we consider this local version, but, in the spirit of the previous chapters, it is our intention to extend it to the non-local case, in order to take into account the bound state nature of the mesons.

6.2.2.1 π^+

In the local version some regularization scheme must be applied in order to regularize the integrals. Apart from this point, one can proceed as in the third chapter.

In the pseudoscalar channel one has for π^+ ($G_\Gamma = G_P$)

$$G_P^{-1} = \Sigma_{\pi^+}(p^2 = M_{\pi^+}^2) \quad (6.32)$$

where

$$\Sigma_{\pi^+}(p^2) = -iN_c \int_\Lambda \frac{d^4q}{(2\pi)^4} \text{Tr} [i\gamma^5 S_u(q+p/2) i\gamma^5 S_d(q+p/2)]. \quad (6.33)$$

\int_Λ means that some regularization of the integral is applied, we do not care now precisely which one. We then can introduce the composite field π^+ together with the Lagrangian

$$\mathcal{L} = g_{\pi^+} \pi^+ (\bar{u}(x) i\gamma^5 d(x)) \quad (6.34)$$

and

$$g_{\pi^+} = \left(\frac{\partial \Sigma_{\pi^+}(M_{\pi^+}^2)}{\partial p^2} \right)^{-1/2}. \quad (6.35)$$

The same procedure can be applied for all the states $\bar{u}d, \bar{d}u, \bar{u}s, \bar{s}u, \bar{d}s, \bar{s}d$ without problems.

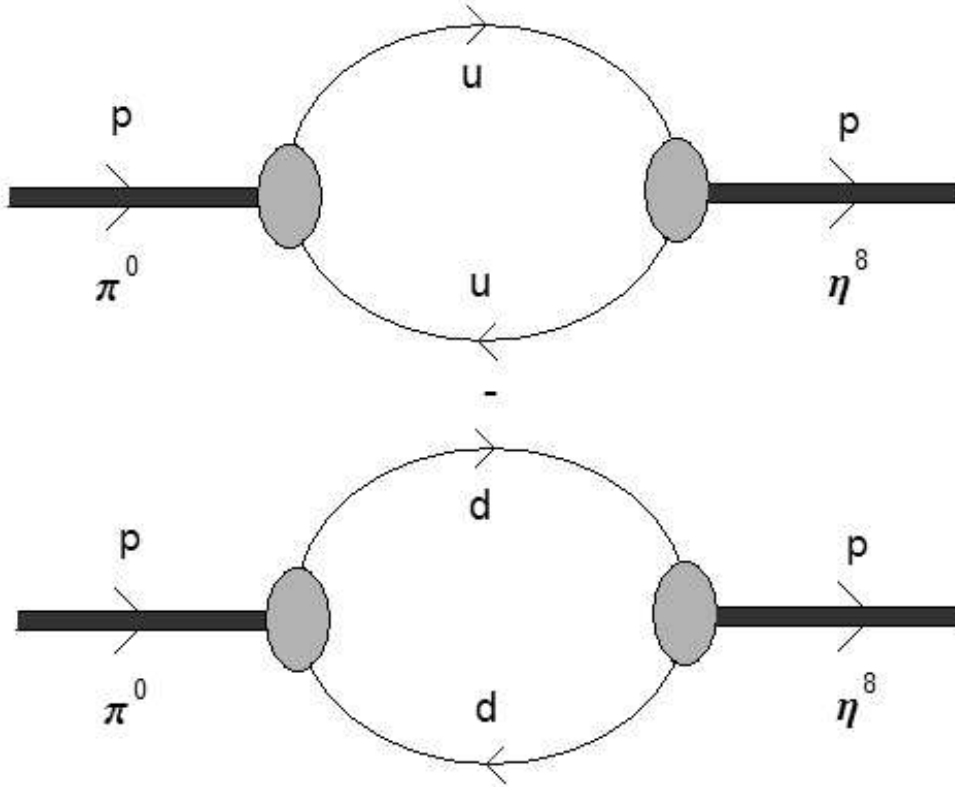
Care is needed for the equal-flavor states.

6.2.2.2 Equal-flavor case

We now restrict to the equal flavor case, which corresponds to $k = 0, 3, 8$ in the summation in the Lagrangian term. If we introduce the bound state fields π^0, η_8 and η_0 we are then lead to

$$\begin{aligned} \mathcal{L}_{equal-flavor} &= g_{\pi^0} \pi^0 \sqrt{\frac{1}{2}} (\bar{u} i\gamma^5 u - \bar{d} i\gamma^5 d) \\ &+ g_{\eta_8} \sqrt{\frac{1}{6}} (\bar{u} i\gamma^5 u + \bar{d} i\gamma^5 d - 2\bar{s} i\gamma^5 s) \\ &+ g_{\eta_0} \sqrt{\frac{1}{3}} (\bar{u} i\gamma^5 u + \bar{d} i\gamma^5 d + \bar{s} i\gamma^5 s). \end{aligned} \quad (6.36)$$

Let us ask the following question: are the three introduced fields, which correspond to two octet and one singlet states, really orthogonal? In order to find the coupling constants g_{π^0} ,

Figure 6.1: Diagrams for π^0 - η^8 mixing.

g_{η_8} and g_{η_0} we have to sum up all the quark-antiquark bubbles of the corresponding states. We can proceed as in the second and the third chapters (and for the other mesons as π^+) only if transitions like $\pi^0 \longleftrightarrow \eta^8$ vanish, i.e. if the three states are orthogonal. The corresponding diagrams for this mixing are depicted in Fig. 6.1.

Such diagrams mean that the state π^0 and η^8 can in principle mix, and that the real physical states are in the general case a mixture of these. The mixing amplitude is proportional to

$$\int_{\Lambda} \frac{d^4 q}{(2\pi)^4} \{Tr[i\gamma^5 S_u(q + p/2) i\gamma^5 S_u(q + p/2)] - Tr[i\gamma^5 S_d(q + p/2) i\gamma^5 S_d(q + p/2)]\}, \quad (6.37)$$

which is equal to zero if $S_u(q) = S_d(q)$. The states π^0 and η^8 (and similarly π^0 and η^0) are then orthogonal in the limit of equal u and d propagators. This is indeed a good approximation, because, as already discussed, the masses of these two flavors are almost equal. In the following we will work under the assumption $S_u(q) = S_d(q) = S_n(q)$ where $n = \{u, d\}$. The same cannot be said for the η^8 and η^0 states. In this case the mixing term for the transition $\eta_0 \longleftrightarrow \eta_8$ is proportional to

$$\int_{\Lambda} \frac{d^4 q}{(2\pi)^4} \{Tr[i\gamma^5 S_n(q + p/2) i\gamma^5 S_n(q + p/2)]\}$$

$$-Tr[i\gamma^5 S_s(q + p/2)i\gamma^5 S_s(q + p/2)] \neq 0 \quad (6.38)$$

because $S_n(q) \neq S_s(q)$. Although the s and the n masses are of the same order, the difference cannot be neglected.

6.2.2.3 Orthogonal states

How are then the physical states determined? Let us note that we can write the quartic Lagrangian as

$$\frac{G_\Gamma}{4} \sum_{k=0,3,8} (\bar{q}\Gamma\lambda^k q)^2 = \frac{G_\Gamma}{2} [(\bar{u}\Gamma u)^2 + (\bar{d}\Gamma d)^2 + (\bar{s}\Gamma s)^2] \quad (6.39)$$

where we restricted to $k = 0, 3, 8$, i.e. to the equal-flavor combinations. We could then define the bound states fields U , D , and S and consider

$$\mathcal{L}_{equal-flavor} = g_U U(\bar{u}\Gamma u) + g_D D(\bar{d}\Gamma d) + g_S S(\bar{s}\Gamma s). \quad (6.40)$$

This would not be wrong. These states are automatically orthogonal, we can find the masses and the coupling constants in the usual way.

Why then normally one does not introduce such states? They seem to be the best choice. But at the same time we know that it is very useful to group the three pions together. The isospin is conserved in the strong interaction, and the pions form a isospin triplet; the very similar masses of π^0 and π^\pm confirm this symmetry. Furthermore, the isospin triplet is found in each octet to be practically degenerate.

In terms of U and D the π^0 state is $\sqrt{1/2}(U - D)$; the orthogonal state is $\sqrt{1/2}(U + D)$. We then realize that another decomposition is possible:

$$\begin{aligned} \frac{1}{2} \sum_{k=0,3,8} \left(\bar{q}\Gamma \frac{\lambda^k}{2} q \right)^2 &= (\bar{u}\Gamma u)^2 + (\bar{d}\Gamma d)^2 + (\bar{s}\Gamma s)^2 = \\ &= \left(\sqrt{\frac{1}{2}}(\bar{u}\Gamma u - \bar{d}\Gamma d) \right)^2 + \left(\sqrt{\frac{1}{2}}(\bar{u}\Gamma u + \bar{d}\Gamma d) \right)^2 + (\bar{s}\Gamma s)^2. \end{aligned} \quad (6.41)$$

Note that in the case $u = d = n$ these two last decompositions are perfectly equivalent. We then write down the Lagrangian correspondent to the last decomposition:

$$\mathcal{L}_{equal-flavor} = g_{\pi^0} \pi^0 \sqrt{\frac{1}{2}}(\bar{u}\Gamma u - \bar{d}\Gamma d) + g_N N \sqrt{\frac{1}{2}}(\bar{u}\Gamma u + \bar{d}\Gamma d) + g_S S(\bar{s}\Gamma s). \quad (6.42)$$

(We write π^0 , but in a generic nonet we refer to the corresponding $\sqrt{1/2}(\bar{u}u - \bar{d}d)$ state). Note that the three states π^0 , N and S are also orthogonal (because we supposed $S_u = S_d = S_n$). In this way we split the iso-vector $\sqrt{1/2}(\bar{u}u - \bar{d}d)$ (vector in the $SU_f(2)$ sub-group relative to u, d space) to the isoscalars $\sqrt{1/2}(\bar{u}u + \bar{d}d)$ and $\bar{s}s$, the last two being invariant under the $SU_f(2)$ isospin sub-group transformations

$$q_{i,c} \rightarrow \left(e^{i\theta_k \frac{1}{2}\lambda_k} \right)_{ij} q_{j,c} \quad (6.43)$$

with $k = 1, 2, 3$.

Then, the basis $\sqrt{1/2}(\bar{u}u - \bar{d}d)$, $\sqrt{1/2}(\bar{u}u + \bar{d}d)$ and $\bar{s}s$ we will work with corresponds to $SU_{u,d}(2) \times U_s(1)$

Note that in the case $u = d = n$ it is only a matter of utility if we use $\bar{u}u, \bar{d}d$ (i.e. at the composite level U and D) or $\sqrt{1/2}(\bar{u}u - \bar{d}d)$ and $\sqrt{1/2}(\bar{u}u + \bar{d}d)$ (corresponding to π^0 or N). This last choice is preferred because of the mentioned isospin considerations, i.e. because the neutral π^0 and the π^\pm are degenerated.

6.2.2.4 η and η' : a fist look at the mixing problem

The Lagrangian (6.31) admits, as we have seen, as physical states π^0 , N and S . So, out of only $SU_f(3)$ symmetry in a quartic Lagrangian one gets these physical states, which, together with the other $\bar{u}d, \bar{d}u, \bar{u}s, \bar{s}u, \bar{d}s, \bar{s}d$ form a nonet.

But we ask what happens if we consider an extra-mixing like

$$G_{mix}(\bar{s}\Gamma s) \left(\sqrt{\frac{1}{2}}(\bar{u}\Gamma u + \bar{d}\Gamma d) \right). \quad (6.44)$$

In this case π^0 is still a physical state, but N and S are not, because they mix through this term. This is indeed what happens in the pseudoscalar sector because of the $U_A(1)$ anomaly (see [21] and later). On the other hand in the vectorial, pseudovector and tensor mesons the physical states are very close to N and S , i.e. such a mixing N - S is negligible. The scalar case is controversial, and we will discuss it separately later. The glueball intrusion in the nonet can increase this mixing.

In the pseudoscalar case ($\Gamma = i\gamma^5$) the two physical states called η and η' are actually "closer" to the octet $\sqrt{1/6}(\bar{u}u + \bar{d}d - 2\bar{s}s)$ and to the singlet $\sqrt{1/3}(\bar{u}u + \bar{d}d + \bar{s}s)$ decompositions; the reason for this is a mixing term like 6.44 (see [21, 22]).

We will see later how to treat the general mixing case, but we will do it in the framework of the glueball mixing. For the moment we content ourselves to have understood the problem we have to face and how to decompose properly the flavor basis.

The difficulty is how to define a mixing scheme for the bound states; in fact, in this approach, the fields N and S , as η and η' , are not elementary. They describe bound objects, made out of the more elementary quarks.

In the end, we add a note concerning notation: in the pseudoscalar case the fields N and S are pseudoscalar, in the scalar case, where $\Gamma = 1_4$, they are scalar. In the vectorial case, they are vectorial, and so on.

6.3 The *NJL* model

6.3.1 The Lagrangian and its symmetries

In this context we are compelled to describe briefly one of the most famous QCD models, the Nambu Jona-Lasinio one [20, 21, 22, 23, 73]. In the previous subsection we considered a fourth-order Lagrangian invariant under $SU_f(3)$ transformations (6.31), but not invariant under $SU_{A,f}(3)$. As we have seen in the first section, this last symmetry, valid at high energy, is spontaneously broken at low ones: the vacuum does not respect the symmetries of the interaction Lagrangian.

Considering the scalar and the pseudoscalar channels only, the Nambu Jona-Lasinio Lagrangian reads:

$$\mathcal{L}_{NJL} = \bar{q}(i\gamma^\mu \partial_\mu - \hat{m})q + \frac{G}{4} \sum_{k=0}^8 \left[\left(\bar{q}i\gamma^5 \lambda^k q \right)^2 + \left(\bar{q}\lambda^k q \right)^2 \right] \quad (6.45)$$

where $\hat{m} = \text{Diag}\{m_u, m_d, m_s\}$ is the current quark mass matrix. In the limit of zero quark masses the NJL-Lagrangian is $SU_f(3) \times SU_{A,f}(3)$ invariant. Note that a single term of the interaction Lagrangian is not $SU_{A,f}(3)$ invariant, but the sum satisfies this requirement.

Let us then consider the equation of motion for the flavor u :

$$\begin{aligned} & (i\gamma^\mu \partial_\mu - m_u)u - \frac{\partial \mathcal{L}_{NJL,int}}{\partial \bar{u}} \\ &= (i\gamma^\mu \partial_\mu - m_u)u + G(\bar{u}u)u + G(\bar{u}i\gamma^5 u)u \dots = 0 \end{aligned} \quad (6.46)$$

If we consider the mean value of the operators on the right for the vacuum state we find

$$(i\gamma^\mu \partial_\mu - m_u)u - G \langle 0 | \bar{u}u | 0 \rangle u = 0 \quad (6.47)$$

(all the other terms like $\langle 0 | \bar{u}i\gamma^5 u | 0 \rangle = 0$ vanish because of symmetry constraints; in this case, for instance, because the vacuum has even parity). The "new" effective mass is then

$$m_u^* = m_u - G \langle 0 | \bar{u}u | 0 \rangle \quad (6.48)$$

where $\langle \bar{u}u \rangle = \langle 0 | \bar{u}u | 0 \rangle$ is the scalar quark condensate of flavor u . If its value is non-vanishing and large, we go from a small mass m_u of the Lagrangian to the larger effective mass $m_u^* > m_u$, thus realizing a spontaneous breaking of the $SU_{A,f}(3)$. Completely similar relations are valid for the other flavors. The quantity $\langle 0 | \bar{u}u | 0 \rangle$ is the order parameter of the spontaneous chiral symmetry breaking.

Quantities like $\langle 0 | \bar{u}u | 0 \rangle$, as each loop-diagram, diverge in the context of the NJL model, therefore a regularization must be introduced. Many schemes have been used, such as a three-dimensional cut-off (useful for a finite density extension), a four-dimensional one, Pauli-Villars scheme and so on [22].

The introduction of a cut-off means that the theory is not longer local, therefore care in conservating of symmetries is needed (see the related discussion in section 3.6).

The NJL Lagrangian (6.45) is also $U_A(1)$ invariant in the limit of zero quark masses. But we know that this symmetry is not conserved at the QCD quantum level. The NJL model can then be improved by considering an extra-term, known as the t'Hooft interaction

$$\mathcal{L}_{t'Hooft} = -K_{t'Hooft} \left[\det_{i,j} [\bar{q}_i(1 + \gamma^5)q_j] + \det_{i,j} [\bar{q}_i(1 - \gamma^5)q_j] \right] \quad (6.49)$$

which generates 6-point interaction, is $SU_f(3) \times SU_{A,f}(3)$ invariant but breaks the $U_A(1)$ symmetry. The origin of this term is connected to the QCD ground states and instantons. It is fundamental for a proper description of the η - η' states. Such a term, in fact, generates a $N - S$ mixing as in equation (6.44); this mixing is automatically present in the pseudoscalar as well as in the scalar sector. For a better description of these issues we refer to NJL review papers [21, 22, 23].

6.3.2 Meson states within NJL and the scalar problem

The NJL model incorporates the important feature of the $SU_{A,f}(3)$ spontaneous symmetry breaking, even if in a somewhat "strange" way: the quark-condensate is calculated through a quark loop, similar to the one seen in section 2.4.3 within the φ^4 theory; in that case we were going from the bare unphysical mass to a physical one, in a renormalization process, here from a current mass to the larger effective low-energy one. Furthermore, the NJL theory is not renormalizable: $[\text{coupling constant}] = [\text{energy}]^{-2}$. We should then interpret this approach as a phenomenological schematic way for the description of spontaneous symmetry breaking; the cut-off Λ one has to introduce then "cuts" the high-energy momenta, which become inert and inactive, in accord with the basic QCD idea of asymptotic freedom.

In this way many low energy theorems can be consistently derived; also the pseudoscalar nonet, together with the η - η' mixing [21, 22], is correctly described. Attempts to describe light baryons starting from the NJL model have also been performed [23, 100, 101].

The Lagrangian also contains the scalar channel: what about these resonances? They can be calculated within the model, and the typical result for the $\sigma = N = \sqrt{1/2}(\bar{u}u + \bar{d}d)$ is a mass of $\sim 2m_n^* \simeq 600 \text{ MeV}$. The first "temptation" is to interpret it as the resonance $\sigma = f_0(400 - 1200)$ [21], but the situation is not so simple. It is not clear if this resonance has a quark-antiquark nature, or a multi-quark one (as a two-pion bound state or a four-quark state). Furthermore, experimentally one finds the scalar states between 1-2 GeV ; the isovector states $\bar{u}d, du, \sqrt{1/2}(\bar{u}u - \bar{d}d)$ are assigned to the $a_0(1450)$, the strange states $\bar{u}s, \bar{s}u, \bar{u}d, \bar{d}u$ to the resonance $K^*(1450)$. The NJL scalar masses are typically smaller than these ones. Some authors interpreted the states predicted by the NJL model with the two low-mass scalar states $f_0(980)$ ($I = 0$) and $a_0(980)$ ($I = 1$) states, but, as mentioned in the previous chapter and discussed in the introduction, there are serious theoretical arguments to believe in a multi-quark nature of these two resonances.

We conclude this section by saying that the NJL model works very well at an energy scale well below 1 GeV and for the pseudoscalar nonet, but, if we want to describes states between 1-2 GeV , where probably the quark-antiquark scalar states lie, we may be at the border (if not outside) of its applicability [13].

However, within the NJL model studies about the scalar mesons and the glueball have been performed [94, 102, 103], where the glueball degree of freedom is introduced by a dilaton [102, 103] or by a scalar field constraint via the QCD trace anomaly [94].

In the present approach we resort directly to the elementary gluon fields in a non-local interaction Lagrangian.

6.4 Non-local approach

6.4.1 Scalar-isoscalar term

How to describe then the scalar states above 1 GeV ? We can try to apply our non-local approach developed in the previous chapters. First, we limit to the study of the isoscalar-scalar states $N = \sqrt{1/2}(\bar{u}u + \bar{d}d)$ and $S = \bar{s}s$. These two states are those interesting to us because of the glueball mixing. To this end we consider the non-local currents [74, 104]

$$J_n(x) = \frac{1}{\sqrt{2}} \int d^4y (\bar{n}(x+y/2)n(x-y/2)) \Phi(y)$$

$$\text{with } n(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix} \quad (6.50)$$

and

$$J_s(x) = \int d^4y (\bar{s}(x+y/2)s(x-y/2)) \Phi(y). \quad (6.51)$$

together with isoscalar-scalar Lagrangian interaction term

$$\mathcal{L}_q = \frac{K_N}{2} (J_n^2(x) + J_s^2(x)). \quad (6.52)$$

\mathcal{L}_q is a non-local generalization of the scalar-isoscalar piece of the NJL Lagrangian [69]. But this is true from a mathematical point of view. In the NJL model one relates the quark propagators to the quark condensates and tries to describe in a unified way pseudoscalar and scalar mesons. This is probably too much to ask for. In our approach we do not implement directly the chiral structure, but we concentrate on the scalar states above 1 GeV, specially on the isoscalar ones N and S : we do it by writing the non-local quark-antiquark currents for these states, taking into account their scalar nature and through $\Phi(y)$ their bound-state characteristics. We didn't write down the dynamical part of the Lagrangian; we do it later, when discussing the propagator.

Note that we employ the same vertex function for both states, which is a simplification of the problem, but justified because the dimensions of the two mesons is of the same order.

Another point of the discussion is the following: a summation over color is understood. These objects are white, as they should be according to confinement. The non-local currents J_n and J_s are, of course, globally white but not anymore locally. A local color $SU_c(3)$ does not leave them unchanged. It should be stressed that we are working with a phenomenological approach, so we simply require that the global color is white. To work with local $SU_c(3)$ invariance means to work with QCD directly. One may say: "the NJL model, being local, deals with local $SU_c(3)$ invariant currents"

There are 2 answers:

1: the NJL model is local up to the introduction of a cut-off, which mathematically corresponds to a θ function in momentum space.

2: the NJL Lagrangian as a whole is of course not locally gauge invariant because of the free part.

In general, a phenomenological description of a QCD bound state has to require the global color invariance, but not the local. This last point leads to gluon fields, i.e. to exact QCD.

6.4.2 Generalization and its limitations

Even if we will mostly work with the isoscalar states, a generalization to the whole nonet is (almost) straightforward. The $a_0^0 = \sqrt{1/2}(\bar{u}u - \bar{d}d)$ current is for example

$$J_{a_0^0}(x) = \frac{1}{\sqrt{2}} \int d^4y (\bar{u}(x+y/2)u(x-y/2) - \bar{d}(x+y/2)d(x-y/2)) \Phi(y) \quad (6.53)$$

and similarly for the other isovector states. It corresponds to a Lagrangian piece $\frac{K_N}{2} J_{a_0^0}^2(x)$.

For the kaon-like state $\bar{u}s$ we write

$$J_{\bar{u}s} = \int d^4y \bar{u}(x+y/2)s(x-y/2)\Phi(y). \quad (6.54)$$

One may criticize it, because the masses of u and s are different, i.e. the quark-propagators are not equal. The center of gravity is not at x anymore. A possible way out is to consider

$$J_{\bar{u}s} = \int d^4y_1 d^4y_2 \delta \left(x - \frac{m_u^* y_1 + m_s^* y_2}{m_u^* + m_s^*} \right) \bar{u}(y_1) s(y_2) \Phi(y_1 - y_2). \quad (6.55)$$

The Dirac delta function assures that the meson sits in the center of gravity of the two-body system. Such a form for nonlocal currents is discussed in [68] and Refs. therein.

In general we write down

$$\mathcal{L}_{nonet} = \frac{K_N}{4} \sum_{k=0}^8 J_k^2 \quad (6.56)$$

with

$$J_k = \int d^4y_1 d^4y_2 \delta \left(x - \frac{m_i^* y_1 + m_j^* y_2}{m_i^* + m_j^*} \right) \bar{q}_{i,c}(y_1) (\lambda_k)_{ij} q_{j,c} \Phi(y_1 - y_2). \quad (6.57)$$

The quark masses are parameters; as explained before we do not fix them through the quark condensates. As usual, we assume $m_n^* = m_u^* = m_d^* < m_s^*$. It is a simple exercise to recover the currents J_n and J_s when the masses are equal.

We immediately realize that there is a problem; if we use the typical low-energy masses, we find that the resonance masses ($\sim 1.5 \text{ GeV}$) are larger than two times the effective quark masses, thus leading to poles in the bubble integrations. We discuss this problem in the following subsection.

The Lagrangian is $SU_f(3)$ invariant, because we used the same vertex function for all the states. If we expand \mathcal{L}_{nonet} we find for the isoscalar piece exactly \mathcal{L}_q of eq. (6.52), as desired.

We are not considering the $U_A(1)$ anomaly; these and other modifications would split the Lagrangian into the more general form

$$\mathcal{L}_{nonet} = \sum_{k=0}^8 \frac{K_k}{4} J_k^2 \quad (6.58)$$

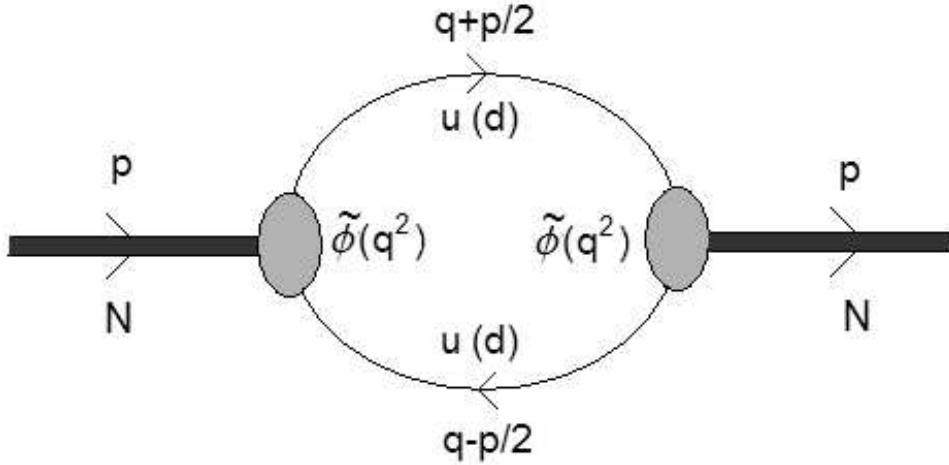
where each channel has its own strength; another possible complication is to consider for each channel a different vertex functions. We just mention it, but do not analyze these cases because they lead to a large number of parameters, which are already not few (later for this issue).

A problem connected to the scalar nonet is that, if we consider $a_0(1450)$ and $K^*(1450)$ as members of it (see section 1.5), one has $M_{K^*(1430)} = 1.412 \pm 0.006 \text{ GeV} < M_{a_0(1450)} = 1.474 \pm 0.019 \text{ GeV}$.

How can one understand it if $m_s^* > m_{n=u,d}^*$? This is possible if the constant K_{a_0} is smaller (a smaller strength leads to a larger mass: see chapter 2) than for $K_{K(1430)}$, but not if they are equal. Even a small difference may cause relatively large mass differences. This shows that $SU_f(3)$ invariance alone is not enough for the full and precise description of the scalar nonet. In any case at first order it is a good starting point, especially for our scalar-isoscalar sector, and, as we will see, for the glueball mixing.

6.4.3 Scalar-isoscalar masses

The scalar meson masses M_N and M_S are deduced from the poles of the T-matrix of the scalar-isoscalar Lagrangian (6.52) and are given by the solutions of the Bethe-Salpeter equations:


 Figure 6.2: Self energy diagram for N .

$$K_N - \frac{1}{\Sigma_N(M_N^2)} = K_N - \frac{1}{\Sigma_S(M_S^2)} = 0. \quad (6.59)$$

The mass operator $\Sigma_{N(S)}(p^2)$, where p is the meson momentum, is deduced from the quark loop diagram of Fig. 6.2 and given by

$$\Sigma_{N(S)}(p^2) = -iN_c \int \frac{d^4q}{(2\pi)^4} \text{Tr} [S_{n(s)}(q+p/2)S_{n(s)}(q-p/2)] \tilde{\Phi}^2(q^2), \quad (6.60)$$

where $N_c = 3$ is the number of colors and $S_{n(s)}$ is the quark propagator. A Wick rotation is then applied in order to calculate $\Sigma_{N(S)}(p^2)$. As evident from (6.59), M_N and M_S are not independent. Once the vertex function $\tilde{\Phi}(q)$ is chosen and using the bare nonstrange meson mass M_N as an additional input, the coupling constant K_N and the bare strange meson mass M_S are fixed.

By introducing the auxiliary meson fields $N(x)$ and $S(x)$ (N for the $\sqrt{1/2}(\bar{u}u + \bar{d}d)$ and S for the $\bar{s}s$ meson) previous procedure can be summarized by the Lagrangian :

$$\mathcal{L}'_q = g_N N(x) J_n(x) + g_S S(x) J_s(x) \quad (6.61)$$

with the condition that

$$g_{N(S)} = \left(\frac{\partial \Sigma_{N(S)}(p^2)}{\partial p^2} \right)_{p^2=M_{N(S)}^2}^{-1/2}, \quad (6.62)$$

where the last relation is the compositeness condition, discussed throughout all the work. The compositeness condition requires that the renormalization constant of the meson fields $N(x)$ and $S(x)$ is set to zero, hence physical meson states are exclusively described by the dressing with the constituent degrees of freedom.

We will use a gaussian form for the vertex function with

$$\tilde{\Phi}(q_E^2) = \exp[-q_E^2/\Lambda^2], \quad (6.63)$$

where q_E is the Euclidean momentum. This particular choice for the vertex function preserves covariance and was used in previous studies of light and heavy hadron system [67, 68, 69, 70]. Any covariant choice for $\tilde{\Phi}$ is appropriate as long as it falls off sufficiently fast to render the resulting Feynman diagrams ultraviolet finite. The size parameter Λ will be varied within reasonable range, checking the dependence of the results on it.

6.4.4 Quark propagator

We now turn to the discussion of the quark propagator. The general form (see [24] and chapter 3) is

$$S(p) = i \frac{Z(p^2)}{(\not{p} - m(p^2))}. \quad (6.64)$$

Considering the low energy limit as discussed in the second chapter we have $Z(p^2) \sim \text{constant}$ and $m(p^2) \sim m^*$, where m^* is the effective or constituent quark mass. These masses are typically in the range of 0.25 to 0.45 GeV for the $u(d)$ quarks and 0.5 to 0.7 GeV for the s flavour. The low energy limit for the quark propagator is too naive for our purposes, since it leads to poles in the mass operator for meson masses of about 1.5 GeV. In the following we consider two possible ways to avoid these infinities.

6.4.4.1 Free propagators with a large mass

The mass function for $n = u, d$ flavor can be decomposed as $m_n(p^2) = m_n^* + \sigma_n(p^2)$ and the contribution of σ_n is assumed to be replaced by a large average value. To avoid poles, that is the unphysical decay of the quark-antiquark meson into two quarks, the averaged mass function is chosen as $\langle m_n(p^2) \rangle = m_n^* + \langle \sigma_n(p^2) \rangle = \mu_n \geq 0.86 \text{ GeV}$ (here for the u - and d -flavour), where $2 \cdot \mu_n$ is a lower limit set by the mass of the $f_0(1710)$ resonance. We then have the free form

$$S_n(p) = \frac{i}{(\not{p} - \mu_n)}. \quad (6.65)$$

We also introduce an analogous parameter μ_s for the s flavour, which for $\mu_s \neq \mu_n$ contains flavor symmetry violation. The basic assumption is that the effective mass for bound states in the energy region between 1 – 2 GeV is larger than for the pseudoscalar nonet, thus not generating poles. Above parametrization of the quark propagator is the simplest choice; although it neglects the p^2 dependence of the quark mass, the approximation by a free quark propagator with a large effective mass allows to test the approach and leads to considerable simplifications concerning the technical evaluation. A similar quark propagator was also used in Ref. [94], where the decay of scalar mesons into two photons was analyzed.

The total Lagrangian, including the free part, is then in this case

$$\mathcal{L} = \bar{q}(i\not{\partial} - \hat{\mu})q + \frac{K_N}{2} (J_n^2(x) + J_s^2(x)) \quad (6.66)$$

where $\hat{\mu} = \text{Diag}[\mu_n, \mu_n, \mu_s]$.

The full \mathcal{L}' Lagrangian with the auxiliary fields N and S but completely equivalent to \mathcal{L} is then

$$\begin{aligned} \mathcal{L}' = & -\frac{1}{2} \frac{\Sigma_N(M_N^2)}{\Sigma'_N(M_N^2)} N^2 - \frac{1}{2} \frac{\Sigma_S(M_S^2)}{\Sigma'_S(M_S^2)} S^2 + g_N N(x) J_n(x) \\ & + g_S S(x) J_s(x) + \bar{q}(i\hat{\not{\partial}} - \hat{\mu})q \end{aligned} \quad (6.67)$$

where $g_{N(S)}$ is given from the compositeness condition (6.62) (see the full discussion in chapters 1 and 2).

6.4.4.2 Entire propagator

We also consider the quark propagator introduced in chapter 2, which is described by an entire function:

$$S_i(p) = \frac{i}{\not{p} - m_i^*} (1 - \exp(\beta(p^2 - m_i^{*2}))). \quad (6.68)$$

This parametrization has been used both in the study of meson and baryon properties [84, 85] and serves as one possible way to model confinement; the factor multiplying the free quark propagator removes the pole and on-shell $\bar{q}q$ creation is avoided. The effective quark masses m_i^* (with $i = n, s$) are taken from [81] with $m_n^* = 0.462 \text{ GeV}$, $m_s^* = 0.657 \text{ GeV}$. In Ref. [81] the effective quark masses are calculated within a Dyson-Schwinger approach and, as functions of the Euclidean momentum, display an almost constant behavior up to high values. The parameter β is constrained from below to generate a behavior like $i/(\not{p} - m_i^*)$ for small (and for euclidean) momenta; β is also constrained from above, requiring that the propagator does not diverge for momenta up to $\sim 2 \text{ GeV}$.

In the end, we want to write down the full Lagrangian, including the dynamical part. A free Dirac Lagrangian cannot generate the propagator (6.68). We can do it by writing

$$\mathcal{L} = \bar{q}(S^{-1}(i\hat{\not{\partial}}))q + \frac{K_N}{2} (J_n^2(x) + J_s^2(x)) \quad (6.69)$$

where

$$S(i\hat{\not{\partial}}) = \delta^{ab} \delta^{ij} \frac{(\hat{\not{\partial}} - m_i^*)}{1 - \exp[\beta(\square - m_i^{*2})]}. \quad (6.70)$$

Of course, this is just a *formal* way to introduce the entire propagator in the Lagrangian, but the physical reasons behind it were listed in the third chapter and at the beginning of the present chapter. The expression for \mathcal{L}' is straightforward.

We also understand why we postponed the discussion of the dynamical part, and we started with the interaction Lagrangian only. In fact, the dynamical part is connected with the propagator, which, at low energy, reflects the non-trivial structure of QCD.

6.5 The glueball

6.5.1 Introduction

We have seen that the QCD physical states are "white". Can we construct "white" gluon bound states? Of course, we can. The following object, for example

$$G_{\mu\nu}^a(x) G^{a,\mu\nu}(x) \quad (6.71)$$

is invariant under local gauge color transformation

$$G_\mu^a \rightarrow G_\mu^a - \frac{1}{g} \partial_\mu \theta_a - f_{abc} \theta_b G_\mu^c. \quad (6.72)$$

As discussed previously, from a phenomenological point of view we limit to "global white objects"

$$G_{\mu\nu}^a(x_1) G^{a,\mu\nu}(x_2) \quad (6.73)$$

which are invariant under the global color gluon transformation

$$G_\mu^a \rightarrow G_\mu^a - f_{abc} \theta_b G_\mu^c. \quad (6.74)$$

where the θ_a are constant. Note that the non-abelian nature of QCD implies that also in the global case the gluon fields transform, in contrast to the QED case. The current $G_{\mu\nu}^a(x_1) G^{a,\mu\nu}(x_2)$ contains two, three and four gluon states.

The fundamental question is the following: is a only-gluons bound state possible in QCD?

The existence of the current is in fact not enough; also in QED we can construct analogous currents, like $F^{\mu\nu} F_{\mu\nu}$. But the photon-photon interaction, although different from zero (they do not interact directly but through electron loops [55]) is not strong enough to generate stable bound states. This is why you have never heard a state called photonball or photonium.

In QCD the situation is different: the gluons undergo third and fourth order self interaction. This fact is thought to be at the basis of asymptotic freedom and confinement, but another fundamental question is the possible existence of gluonic bound states. The interaction may be strong enough to generate them.

In this case we would have a whole spectrum. The current $G_{\mu\nu}^a(x_1) G^{a,\mu\nu}(x_2)$ is scalar, thus it corresponds to a $J^{PC} = 0^{++}$ glueball. One would have then the pseudoscalar configuration as

$$G_{\mu\nu}^a(x_1) G_{\rho\sigma}^a(x_2) \varepsilon^{\mu\nu\rho\sigma} \quad (6.75)$$

and so on and so forth.

In this work we will concentrate on the scalar glueball. We immediately realize that there is a complication: two gluons can transform into a quark and an antiquark, thus meaning that in QCD a scalar glueball configuration can transform into a scalar meson (parity, C-parity and J are conserved at the QCD level). This makes the situation more difficult, because it is possible that the physical states are a mixture of gluons and quark-antiquarks. How to analyze it?

If the scalar $\bar{q}q$ states and the "gg" glueball have a similar mass, the mixing may also be large (see the general discussion about mixing in the previous chapter and in the first one).

If in nature there is mixing, we cannot switch it off to study the bare states. This operation, while being practically unapplicable, is conceptually possible. Lattice QCD can study the Yang-Mills part of the QCD Lagrangian, i.e. the gluon part only

$$\mathcal{L}_{Y-M} = -\frac{1}{4} G_{\mu\nu}^a G^{a,\mu\nu} \quad (6.76)$$

and study the bound states of this sub-theory. Lattice QCD predicts pure gluonic bound state, where the lightest one is the scalar $J^{PC} = 0^{++}$ with a mass of the order of 1.5 GeV! (See introduction and Refs therein).

So, Lattice QCD answer to the question "Do we have gluonic bound states" is loud and clear "Yes, we do"! The mass 1.5 GeV is furthermore in the same region where the scalar

mesons which are pion-like and kaon-like, $a_0(1450)$ and $K^*(1430)$, lie. One then has three and not only two scalar-isoscalar states $f_0(1350)$, $f_0(1500)$ and $f_0(1710)$; only two are expected to complete the nonet (see the introduction). The fact that we have three states brings us to believe that the glueball intrudes among them, and that mixing occurs among them.

In these arguments we DO NOT consider the other scalar resonances $f_0(400 - 1200)$, $f_0(980)$ and $a_0(980)$, which we interpret as multiquarks bound state. We therefore believe that the ground state scalar meson nonet (decuplet with the scalar glueball) lies completely above 1 GeV , where the other $L = S = 1$ p-wave states are observed (the tensor mesons $J = 2^{++}$ and the pseudovector ones $J = 1^{++}$). In fact, as we have seen in the second chapter, the scalar configuration corresponds to a p-wave state; it is then natural to believe that the scalar $\bar{q}q$ resonances lie close in mass to the correspondingly $J = 1, 2$ multiplets.

This point of view, as stressed in the first chapter, is not unique; other interpretations are possible, and as mentioned before all possible combinations and permutations of octet, nonet, glueballs and so on have already been proposed.

In any case, if Lattice QCD is right, we do have glueballs. A crucial question is then: "what happens if one introduces dynamical quarks?" It seems that the glueball mass tends to be a bit smaller, but it does not change drastically.

In the following we will trust the Lattice prediction concerning the existence of the scalar glueball in the mass region around 1.5 GeV and we will study the mixing with the scalar mesons in the same mass region.

6.5.2 Gluonic quartic non-local interaction

The goal of this subsection is to write down a quartic non-local Lagrangian for the description of the scalar glueball. As noted above, the current $G_{\mu\nu}^a(x)G^{a,\mu\nu}(x)$ contains 2, 3 and 4 gluons; we split the tensor $G_{\mu\nu}^a$ into the one-gluon and the two-gluon terms

$$G_{\mu\nu}^a = \bar{G}_{\mu\nu}^a - gf_{abc}G_\mu^b G_\nu^c \quad (6.77)$$

where

$$\bar{G}_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a \quad (6.78)$$

is the abelian part of the tensor. The product is then

$$G_{\mu\nu}^a(x_1)G^{a,\mu\nu}(x_2) = \bar{G}_{\mu\nu}^a(x_1)\bar{G}^{a,\mu\nu}(x_2) + \text{"3gluons"} + \text{"4gluons"}. \quad (6.79)$$

In the following we will consider only the two-gluon term $\bar{G}_{\mu\nu}^a(x_1)\bar{G}^{a,\mu\nu}(x_2)$, under the assumption that it is dominant. This means that we describe a glueball as made out of constituent 2 gluons only; as we will discuss later, these two gluons are not the high-energy massless objects of the perturbative QCD, but two low-energy effective constituent gluons with a high non-perturbative effective mass.

Let us then consider the following interaction Lagrangian [74, 104]

$$\mathcal{L}_g = \frac{K_G}{2} J_g^2, \quad (6.80)$$

where

$$J_g = \int d^4 y \bar{G}_{\mu\nu}^a(x + y/2)\bar{G}^{a,\mu\nu}(x - y/2)\Phi_G(y) \quad (6.81)$$

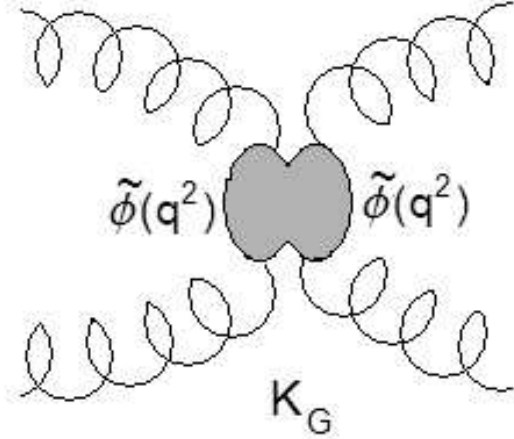


Figure 6.3: Effective quartic gluon interaction.

is the non-local glueball current where we used the "truncated" tensor $\overline{G}_{\mu\nu}^a(x_1)\overline{G}^{a,\mu\nu}(x_2)$ only. This Lagrangian contains a non-local separable four-gluon vertex. $\Phi_G(y)$ is the glueball vertex-function. Again, this quartic interaction (see Fig. 6.3) should not be confused with the more fundamental four-point gluon interaction of the QCD Lagrangian. It should be stressed that the gluons introduced in this section are not the "background" gluons responsible for confinement [107, 108], but two effective constituent gluons forming the glueball. For simplicity we will use the same vertex function as in the previous case, setting the glueball size equal to the one of the quarkonia states.

If instead of the truncated current we would use the "full" one

$$\begin{aligned} J_{g,full} &= \int d^4y G_{\mu\nu}^a(x+y/2)G^{a,\mu\nu}(x-y/2)\Phi_G(y) \\ &= J_g + \text{"3gluons"} + \text{"4gluons"} \end{aligned} \quad (6.82)$$

then we would also have 5,6,7,8 gluon-point Feynman diagrams. The use of the truncation is therefore necessary in order to be able to solve the corresponding mathematical problem. The current $J_{g,full}$ is invariant under global color transformations, but the truncated current J_g not. So, we should do the calculations with J_g but always keep in mind that this is an approximation of the full global "white" current $J_{g,full}$.

6.5.3 Glueball mass operator and composite field

The coupling constant K_G again is linked to the bare glueball mass by the pole-equation:

$$K_G - \frac{1}{\Sigma_G(M_G^2)} = 0, \quad (6.83)$$

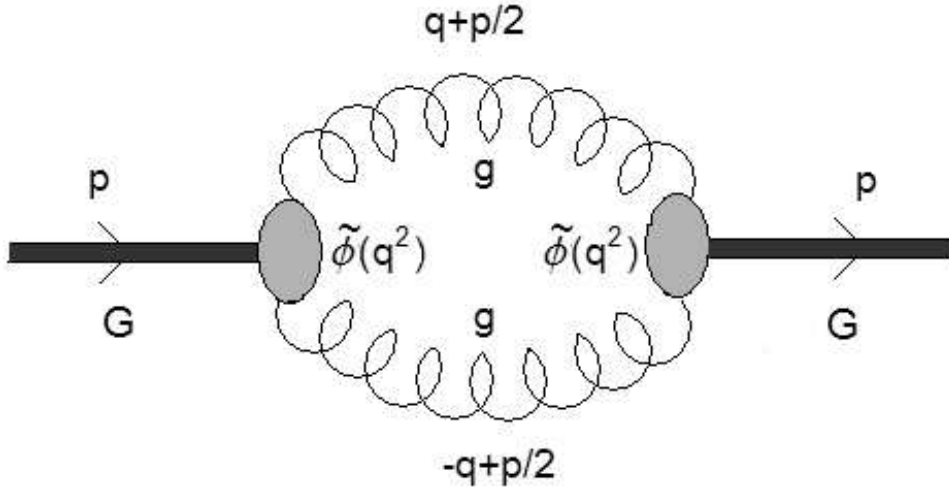


Figure 6.4: Glueball mass operator.

where the mass operator $\Sigma_G(p^2)$, as indicated in Fig. 6.4, is given as:

$$\Sigma_G(p^2) = i2(N_c^2 - 1) \int \frac{d^4q}{(2\pi)^4} [8(q_1 \cdot q_2)^2 + 4q_1^2 q_2^2] D(q_1^2) D(q_2^2) \tilde{\Phi}^2(q^2). \quad (6.84)$$

where $q_1 = q + p/2$ and $q_2 = -q + p/2$. $D(q^2)$ is the scalar part of the gluon propagator, which in the Landau gauge is [109]

$$D_{\mu\nu}^{ab} = \delta^{ab} (g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) D(q^2). \quad (6.85)$$

In order to get $\Sigma_G(p^2)$ one has to take care of the derivative in the interaction, which generates the term in the square brackets in the integrand. As before, the formalism can alternatively be defined by a Lagrangian containing the scalar glueball field $G(x)$ with:

$$\mathcal{L}'_g = g_G G(x) J_g(x), \quad (6.86)$$

where the coupling g_G is deduced from the compositeness condition

$$g_G = \left(\frac{\partial \Sigma_G(p^2)}{\partial p^2} \right)_{p^2=M_G^2}^{-1/2}. \quad (6.87)$$

The last two defining equations are equal to the ones used in [110].

Note that a coupling with the "full" gluonic current $J_{g,full}$ would lead to vertices coupling the glueball to 3 and 4 gluons, and to extra mass-operator diagrams with 3 and 4 internal gluon lines. As explained before, these extra pieces are supposed to be negligible.

6.5.4 Gluon propagator

For the gluon propagator we choose the free one [109, 111]

$$D(q^2) = \frac{i}{q^2 - m_g^2}, \quad (6.88)$$

where the effective mass m_g should be large, with $2 \cdot m_g$ larger than the bare glueball mass deduced from lattice simulations between 1.4-1.8 GeV. [9, 37, 10, 11]. The generation of a constituent gluon mass is analogous to the effective quark mass, and one can relate its value to the gluon condensate [112]. Typical values for the effective gluon mass are in the range of 0.6-1.2 GeV. Lattice simulations give a value of 0.6-0.7 GeV [30]; in the QCD formulation of [113] a higher value of about 1.2 GeV is deduced. A similar value is then found in [114] as a off-shell gluon mass in the so called maximal abelian gauge. The gauge (in)dependence of the effective gluon mass is still an open issue [111, 115]. For our purposes it is sufficient to take a constituent gluon mass within the range 0.6-1.2 GeV; we chose an intermediate value of 0.9 GeV, which is in accord with the effective constituent gluon mass found in the study of gluon-dynamics in [116].

This propagator is the easiest choice we can do, but also meaningful. Note that, the extra graphs with 3 and 4 internal gluon lines are naturally suppressed with such a choice, because of the large denominator attached to each internal gluonic line.

Also here we wish to write down the full Lagrangian, including the "free" term. As we have seen, we choose the Landau gauge for the gluon propagator. The corresponding constituent gluonic Lagrangian is then

$$\mathcal{L} = -\frac{1}{4} \overline{G}_{\mu\nu}^a \overline{G}^{a,\mu\nu} - \frac{1}{2} m_g^2 G_\mu^a G^{a,\mu} - \frac{1}{2\alpha} (\partial_\mu G^{a,\mu}) (\partial^\mu G_\mu^a) + \frac{K_G}{2} J_g^2 \quad (6.89)$$

taking the limit $\alpha \rightarrow 0$ (which corresponds to the Landau gauge). Note that this Lagrangian is clearly not locally gauge invariant. The same argument we presented elsewhere is valid: to work with local color gauge invariance means to work with QCD and not with a QCD-model. The presented Lagrangian is meant to be a low-energy effective gluonic Lagrangian with an effective gluon-mass for the description of the scalar glueball bound state. For a generic α one has the following gluon propagator

$$D_{\mu\nu}^{ab} = \delta^{ab} \left(g_{\mu\nu} + q_\mu q_\nu \frac{\alpha - 1}{q^2 - \alpha m_g^2} \right) \frac{i}{q^2 - m_g^2} \quad (6.90)$$

and for $\alpha \rightarrow 0$ we get the desired result [111].

The gluons play other roles in the present considerations: one has the so-called "background gluons", which are responsible for the string tension, i.e. for the attraction among the quark and the antiquark; we have the constituent gluons forming the glueball, also interacting by the exchange of background gluons.

In the quarkonia sector we used a quartic interaction Lagrangian; this means that the "background" gluons, responsible for the string tension, are described by a constant propagator (and taken into account in the coupling constant K_N). The gluons, exchanged among the quark-antiquark couple, are "soft"; the gluon propagator of eq (6.88) becomes a constant for small momenta, thus being in accord with that assumption. The exchange of nonperturbative gluons is also responsible for the appearance of a vertex function, i.e. for the finite size of

the quarkonia mesons, even if this happens in a non-trivial way; in this model calculation the vertex function is parametrized by (6.63).

These arguments are also valid for the soft background gluons exchanged by the two constituent gluons forming the glueball (constant K_G). On the other hand, when evaluating the glueball mass-operator (Eq. (6.84) and Fig. 6.4), the two constituent gluons (i.e. valence gluons dressed with sea-gluons) can have also large time-like momenta, because the glueball mass ($\sim 1.5 GeV$) is relatively large. In this case the use of a constant for the gluon propagator is not adequate. This is why we use the form of Eq (6.88), with a large effective mass: this choice is at the same time valid to justify the quartic interactions employed above and for the constituent gluons building up the glueball.

6.6 Mixing term

Two (or more) gluons forming a scalar bound state can transform into a quark-antiquark also bound in a scalar bound state, because the quantum numbers J^{PC} are conserved quantities in QCD. A gluon couples to all the quarks with the same strength, because as we discussed, the QCD gluon-quark interaction term is $SU_f(3)$ invariant. This means that two gluons couple to $\bar{u}u$, $\bar{d}d$ and $\bar{s}s$ with the same strength, leading to the schematic interaction

$$gg (\bar{u}u + \bar{d}d + \bar{s}s). \quad (6.91)$$

We say that the glueball is "flavor blind", and that it couples to the flavor singlet combination.

In our previous analysis of the $\bar{q}q$ states we have shown that the best basis is $\sqrt{1/2}(\bar{u}u - \bar{d}d)$, $\bar{n}n = \sqrt{1/2}(\bar{u}u + \bar{d}d)$ and $\bar{s}s$, therefore the glueball-quarkonia mixing term has the schematic form

$$gg (\sqrt{2}\bar{n}n + \bar{s}s). \quad (6.92)$$

The glueball does not mix with the isovector $\sqrt{1/2}(\bar{u}u - \bar{d}d)$. In fact, an interaction like

$$gg (\bar{u}u - \bar{d}d) \quad (6.93)$$

would imply an opposite strength in the channels $\bar{u}u$ and $\bar{d}d$, in complete contrast with the underlying QCD theory.

In terms of the introduced non-local currents J_g , J_n and J_s we then write down the following mixing interaction

$$\mathcal{L}_{mix} = K_{mix} J_g (\sqrt{2} J_n + J_s), \quad (6.94)$$

which generates a two-gluon to two-quark scattering, both in a scalar configuration. In terms of bound states, it generates a mixing of glueball with quarkonia states. Again, it should be stressed that Eq. (6.94) does not describe the fundamental quark-gluon vertex, but an effective coupling of the scalar glueball current with the quarkonia ones, suitable for the description of the mixing among these two bound states.

We do not introduce a direct N - S mixing, under the hypothesis that it is a higher perturbation [31, 40, 42]. In any case a generalization to this case would be straightforward. Indeed, some authors ([7, 47]) considered a large flavor $\bar{n}n$ - $\bar{s}s$ mixing; its presence can change non trivially the mixing and the decay properties. The previous authors introduce a large

flavour mixing to accommodate in the same nonet the states $f_0(1500)$ and $f_0(980)$, which are both interpreted as quark-antiquark scalar-isoscalar states with large mixing, i.e. respectively like $\sqrt{1/3}(\sqrt{2}\bar{n}n - \bar{s}s)$ and $\sqrt{1/6}(\sqrt{2}\bar{n}n + 2\bar{s}s)$. Note that opposite phases with respect to the pseudoscalar states η^0 and η^8 . As explained in the introduction and in section 6.5.1, we do not interpret $f_0(980)$ as a quark-antiquark state, and we group the scalar meson plus glueball between 1.3-1.7 GeV. However, the hypothesis of a relatively large flavour mixing is interesting. We will come back to it in chapter 7. In the present chapter we do not consider a flavour mixing, in order to compare with phenomenological works, where such a possibility was not studied.

6.7 Rotated fields

6.7.1 T matrix and the physical masses

The full interaction Lagrangian resulting from the sum of the quark, gluonic and mixing terms reads

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_q + \mathcal{L}_g + \mathcal{L}_{mix} \\ &= \frac{K_N}{2} (J_n^2 + J_s^2) + \frac{K_G}{2} J_g^2 + K_{SG} J_g (\sqrt{2} J_n + J_s).\end{aligned}\quad (6.95)$$

In order to have a compact expression we omitted the free parts.

The correspondent T matrix has to take into account that we have now three channels: $\bar{n}n$, gg and $\bar{s}s$. We then have a 3×3 matrix, which is not diagonal; in fact, mixing among the different channels, like $\bar{n}n \leftrightarrow gg$, is possible. Its form is an extension of the forms studied in the second *and* in the third chapters:

$$T = -(1 - K \cdot \Sigma)^{-1} K = -(K^{-1} - \Sigma)^{-1}, \quad (6.96)$$

with the non-diagonal coupling matrix

$$K = \begin{pmatrix} K_N & \sqrt{2}K_{SG} & 0 \\ \sqrt{2}K_{SG} & K_G & K_{SG} \\ 0 & K_{SG} & K_S = K_N \end{pmatrix}, \quad (6.97)$$

and the diagonal mass operator

$$\Sigma(p^2) = \begin{pmatrix} \Sigma_N(p^2) & 0 & 0 \\ 0 & \Sigma_G(p^2) & 0 \\ 0 & 0 & \Sigma_S(p^2) \end{pmatrix}. \quad (6.98)$$

The masses of the mixed states are obtained from the zeros of the determinant in the denominator of the T matrix with $Det[1 - K\Sigma] = 0$.

In the limiting case of $K_{SG} = 0$ the determinant reduces to

$$(1 - K_N \Sigma_N(p^2))(1 - K_S \Sigma_S(p^2))(1 - K_G \Sigma_G(p^2)) = 0, \quad (6.99)$$

which results in the defining equations of the bare masses without mixing (6.59) and (6.83)).

For $K_{SG} \neq 0$ we have mixing, where, starting from the unrotated masses (M_N, M_G, M_S) , we end up with the mixed states (N', G', S') of masses $(M_{N'}, M_{G'}, M_{S'})$, which we interpret

as the physical resonances $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$. Quantitative predictions in the three-state mixing schemes strongly depend on the assumed level ordering of the bare states before mixing. In previous works [31, 37] essentially two schemes were considered: $M_N < M_G < M_S$ and $M_N < M_S < M_G$. In phenomenological studies latter level ordering seems to be excluded, when analyzing the hadronic two-body decay modes of the f_0 states [40, 42]. In the current work we will refer to the first of these two possibilities, where the bare glueball mass is centered between the quarkonia states before mixing. We will see that this case is indeed favored in our work.

We also compare our pole equation $\text{Det}[1 - K\Sigma] = 0$ to other approaches.

But first we discuss further consequences of the covariant non-local approach, such as the meson-constituent coupling constants, which are crucial in the calculation of the decays of the mixed states.

6.7.2 Coupling constants

As a result of the mixing the physical rotated fields N' , G' , S' couple to the $\bar{n}n$, gg and $\bar{s}s$ configurations. The leading contribution to the T -matrix (6.96) in the limit $p^2 \simeq M_i^2$ with $i = N', G', S'$ is given by the pole at $p^2 = M_i^2$ with

$$T^{ab} = \frac{g_i^a g_i^b}{p^2 - M_i^2}, \quad (6.100)$$

where $a, b = \bar{n}n$, gg and $\bar{s}s$ refer to the constituent components. A similar expression for the T matrix is also given in [21, 22], where the η - η' mixing was analyzed in the context of the NJL model. The constant $g_{G'}^{\bar{n}n}$, for example, represents the coupling of the mixed state G' to the quark-antiquark configuration of flavor u and d . The modulus of the nine coupling constants is then given by

$$|g_i^a| = \lim_{p^2 \rightarrow M_i^2} \sqrt{(p^2 - M_i^2) T^{a,a}} = \left(\frac{\partial(T^{a,a})^{-1}}{\partial p^2} \right)_{p^2=M_i^2}^{-1/2} \quad (6.101)$$

which we solve numerically. In the next section, where we consider the mixing of two fields, we also give an explicit expression for this quantity. The expression (6.101) is the generalization of the compositeness condition (eqs. (6.62) and (6.87)).

In the limit $K_{SG} \rightarrow 0$ the T -matrix is diagonal and the coupling constants $g_{G'}^a$, for instance, become

$$g_{G'}^{\bar{n}n} = g_{G'}^{\bar{s}s} = 0, \quad (6.102)$$

$$|g_{G'}^{gg}| = g_G^{gg} = \left(\frac{\partial \Sigma_G(p^2)}{\partial p^2} \right)_{p^2=M_{G'}^2=M_G^2}^{-1/2} \quad (6.103)$$

where the last equation is the glueball compositeness condition (6.87). In this case there is no mixing and the glueball couples only to gluons.

We still have to discuss the sign of the coupling constants. By convention we chose $g_{N'}^{\bar{n}n}$, $g_{G'}^{gg}$ and $g_{S'}^{\bar{s}s}$ as positive numbers (to have the correct zero-mixing limit). The sign is determined from the off-diagonal elements of the T -matrix:

$$g_i^a = \text{sign}(\alpha_i^a) \left(\frac{\partial(T^{a,a})^{-1}}{\partial p^2} \right)_{p^2=M_i^2}^{-1/2} \quad (6.104)$$

where

$$\alpha_i^a = \lim_{p^2 \rightarrow M_i^2} (p^2 - M_i^2) T^{i,a} = \left(\frac{\partial(T^{i,a})^{-1}}{\partial p^2} \right)_{p^2=M_i^2}^{-1}. \quad (6.105)$$

Having obtained these coupling constants we again can write an effective interaction Lagrangian for the G' field (see Fig. 6.5)

$$\mathcal{L}'_{G'} = g_{G'}^{\bar{n}n} G' J_n + g_{G'}^{gg} G' J_g + g_{G'}^{\bar{s}s} G' J_s, \quad (6.106)$$

where the coupling between the mixed states and the constituent configurations are made explicit. This Lagrangian allows to calculate the decay of G' as we will see explicitly for the two-photon decay; the coupling constants directly enter in the decay rate of the state. One then has completely analogous expressions for the fields N' and S' :

$$\mathcal{L}'_{N'} = g_{N'}^{\bar{n}n} N' J_n + g_{N'}^{gg} N' J_g + g_{N'}^{\bar{s}s} N' J_s, \quad (6.107)$$

$$\mathcal{L}'_{S'} = g_{S'}^{\bar{n}n} S' J_n + g_{S'}^{gg} S' J_g + g_{S'}^{\bar{s}s} S' J_s. \quad (6.108)$$

The strength of the coupling is directly connected to the mixing strength, the explicit mixing matrix is discussed in the following.

6.7.3 Mixing matrix M

When dealing with elementary scalar particles, and not with composite ones, mixing between the bare meson fields (N, G, S) can be expressed by the Lagrangian introduced in the section 5.3, which we rewrite here as

$$\begin{aligned} \mathcal{L}_{K-G} = & \frac{1}{2}(\partial_\mu N)^2 - \frac{1}{2}M_N^2 N^2 + \frac{1}{2}(\partial_\mu G)^2 - \frac{1}{2}M_G^2 G^2 + \frac{1}{2}(\partial_\mu S)^2 - \frac{1}{2}M_S^2 S^2 \\ & + fGS + \sqrt{2}frGN, \end{aligned} \quad (6.109)$$

where f and r are mixing parameters ($r \neq 1$ takes into account breaking of the flavour blindness hypothesis). In order to allow a comparison we do not consider a direct mixing between N and S , id est $\varepsilon = 0$. In this simple case one has to diagonalize

$$T_{K-G} = \begin{pmatrix} -M_N^2 & \sqrt{2}fr & 0 \\ \sqrt{2}fr & -M_G^2 & f \\ 0 & f & -M_S^2 \end{pmatrix} \quad (6.110)$$

where the mixing matrix $B \subset SO(3)$ is obtained from the condition

$$B \cdot T_{K-G} \cdot B^t = D_{K-G} = \text{Diag}[-M_{N'}^2, -M_{G'}^2, -M_{S'}^2], \quad (6.111)$$

where the primed Klein-Gordon masses refer to the mixed states (see the full discussion in the previous chapter). The matrix $B \subset SO(3)$ ($i = N', G', S'$ and $a = N, G, S$) connects the unmixed and the mixed states by

$$\begin{pmatrix} |N'\rangle \\ |G'\rangle \\ |S'\rangle \end{pmatrix} = B \begin{pmatrix} |N\rangle \\ |G\rangle \\ |S\rangle \end{pmatrix}. \quad (6.112)$$

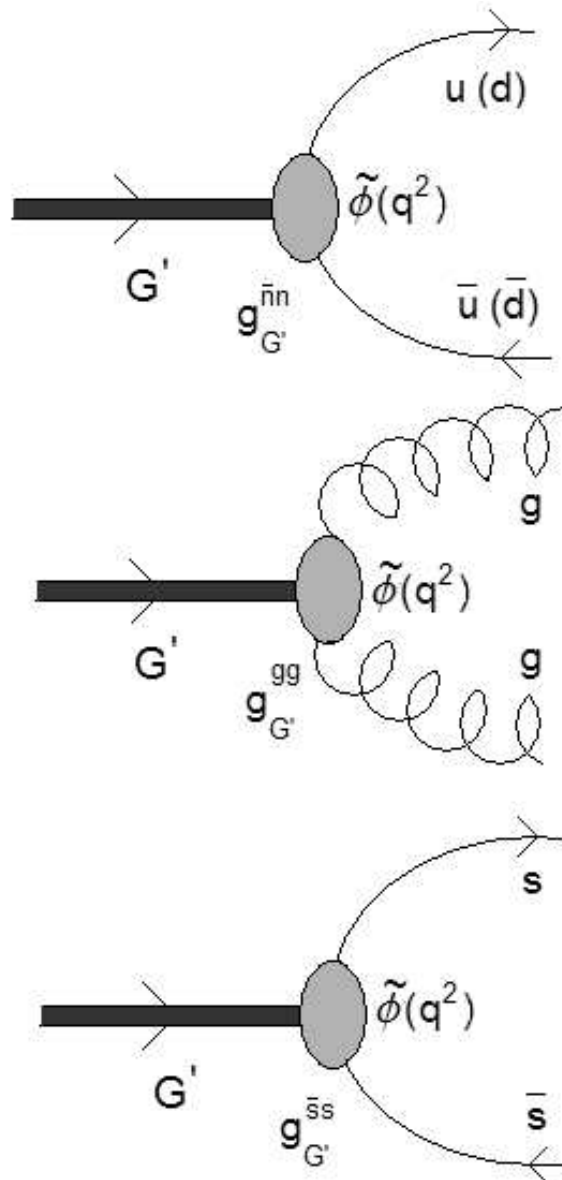


Figure 6.5: G' couples both to quarks and gluons, representing a mixed state.

In the following we want to determine an analogous matrix in our Bethe-Salpeter approach. We consider $|G'\rangle$ which can be written as a superposition of the bare states:

$$|G'\rangle = M^{G',N} |N\rangle + M^{G',G} |G\rangle + M^{G',S} |S\rangle \quad (6.113)$$

where $M^{G',N}$, for example, is the admixture of $|N\rangle = \sqrt{1/2} |\bar{u}u + \bar{d}d\rangle$ to the mixed state $|G'\rangle$. Unlike in the Klein-Gordon case, care should be taken with such an expression, since here the bare states $|N\rangle$, $|G\rangle$ and $|S\rangle$ are not well defined. They correspond to the Bethe-Salpeter solutions in the case of zero mixing, but, when including mixing, they are not normalized vectors of the Hilbert space anymore [22]. To obtain a corresponding expression as in the Klein-Gordon case, we exploit the knowledge of the coupling constants; as in the K-G case with external currents (Section 4.5), we have seen that the elements of the mixing matrix are $B^{ia} = g_i^a/g_a$.

Therefore, the coupling constant $g_{G'}^{\bar{n}n}$ (evaluated in the previous subsection) is related to g_N (the coupling constant of the bare state N to $\bar{n}n$ in the case of no mixing) by the relation

$$g_{G'}^{\bar{n}n} = M^{G',N} g_N(p^2 = M_{G'}^2), \quad (6.114)$$

where $g_N(p^2 = M_{G'}^2)$ is determined from (6.62) evaluated at the physical mass with $p^2 = M_{G'}^2$. Eq. (6.114) states that the $\bar{n}n$ to G' coupling equals the $\bar{n}n$ admixture in G' (which is the matrix element $M^{G',N}$ we want to determine) times the N to $\bar{n}n$ coupling evaluated on mass-shell of G' . It is an extension of the discussion of section 5.5 to the bound state theory, taking into account the momentum dependence of the quantities.

Using (6.62) and (6.104) we obtain for $M^{G',N}$:

$$M^{G',N} = \frac{g_{G'}^{\bar{n}n}}{g_N(p^2 = M_{G'}^2)} = \text{sign}(\alpha_{G'}^{\bar{n}n}) \left[\left(\frac{\partial(T^{\bar{n}n, \bar{n}n})^{-1}}{\partial p^2} \right) \left(\frac{\partial \Sigma_N(p^2)}{\partial p^2} \right) \right]_{p^2=M_{G'}^2}^{-1/2}. \quad (6.115)$$

Generalizing this result for a generic component of the mixing matrix M we have

$$M^{i,a} = \frac{g_i^a}{g_a(p^2 = M_i^2)} = \text{sign}(\alpha_i^a) \left[\left(\frac{\partial(T^{a,a})^{-1}}{\partial p^2} \right) \left(\frac{\partial \Sigma_a(p^2)}{\partial p^2} \right) \right]_{p^2=M_i^2}^{-1/2}. \quad (6.116)$$

with $i = N', G', S'$ and $a = N \equiv \bar{n}n, G \equiv gg, S \equiv \bar{s}s$. Last expression, while not being based on a rigorous derivation, can be regarded as a *definition* of the mixing matrix in a Bethe-Salpeter approach. We explicitly show in Appendix A for the reduced problem of two mixed fields that this definition of the mixing matrix is the analogous one the Klein-Gordon case.

Furthermore, the components of the mixed state $|i\rangle$ are correctly normalized with

$$1 = (M^{iN})^2 + (M^{iG})^2 + (M^{iS})^2 \quad (6.117)$$

as verified both numerically and analytically (Appendix A). This is a further confirmation of the consistency of the definition (6.116). While the rows of M are properly normalized, this does not hold for the respective columns of M because of the p^2 dependence of the T matrix and of the mass operators. The rows are evaluated at different on-shell values of p^2 values with $p^2 = M_i^2$. This implies that the matrix M is not orthogonal, but here we will demonstrate that deviations from orthogonality are small for both choices of the quark propagator.

In the end, let us note that this mixing approach is different from the one used in [21]; there one diagonalizes the Dyson matrix $D = K^{-1}TK^{-1}$, here we present another approach, which to my knowledge was not discussed elsewhere.

6.7.4 Comparison with other mixing schemes

We again consider the Klein-Gordon case and its mass equation $Det[p^2 + T_{K-G}] = 0$ which reads:

$$(p^2 - M_N^2)(p^2 - M_G^2)(p^2 - M_S^2) - f^2 ((p^2 - M_N^2) + 2r^2(p^2 - M_S^2)) = 0. \quad (6.118)$$

In our approach the mass equation $Det[T^{-1}] = 0$ (see Eq. (6.96)) is written out as:

$$\begin{aligned} & (1 - K_N \Sigma_N(p^2))(1 - K_G \Sigma_G(p^2))(1 - K_S \Sigma_S(p^2)) - \\ & - K_{SG}^2 \{ \Sigma_G(p^2) \Sigma_S(p^2) (1 - K_N \Sigma_N(p^2)) \\ & + 2 \Sigma_G(p^2) \Sigma_S(p^2) (1 - K_S \Sigma_S(p^2)) \} = 0. \end{aligned} \quad (6.119)$$

In order to compare the last expression with (6.118) we introduce the functions $\eta_a(p^2)$ as

$$\eta_a(p^2) = \frac{(1 - K_a \Sigma_a(p^2))}{(p^2 - M_a^2)} \quad (6.120)$$

for each $a = N, G, S$. The $\eta_a(p^2)$ do not contain poles for $p^2 = M_a^2$ since $(1 - K_a \Sigma_a(p^2 = M_a^2)) = 0$ (see (6.83) and (6.59)). When substituting (6.120) in (6.119) we get:

$$\begin{aligned} & (p^2 - M_N^2)(p^2 - M_G^2)(p^2 - M_S^2) - K_{SG}^2 \frac{\Sigma_G(p^2) \Sigma_S(p^2)}{\eta_G(p^2) \eta_S(p^2)} \\ & \cdot \{ (p^2 - M_N^2) + 2 \frac{\Sigma_N(p^2) \eta_S(p^2)}{\eta_N(p^2) \Sigma_S(p^2)} (p^2 - M_S^2) \} = 0 \end{aligned} \quad (6.121)$$

Comparing (6.121) with (6.118) we deduce that the mixing parameters f and r of the Klein-Gordon approach become p^2 -dependent functions in the case of composite scalar fields. In particular we make the following identification

$$\begin{aligned} f^2 & \rightarrow f^2(p^2) = K_{SG}^2 \frac{\Sigma_G(p^2) \Sigma_S(p^2)}{\eta_G(p^2) \eta_S(p^2)}, \\ r^2 & \rightarrow r^2(p^2) = \frac{\Sigma_N(p^2) \eta_S(p^2)}{\eta_N(p^2) \Sigma_S(p^2)}. \end{aligned} \quad (6.122)$$

It should be noted that the composite approach generates a value of $r \neq 1$ which reflects the deviation from the flavor blindness hypothesis. Although we set up the quark-gluon interaction assuming this hypothesis, on the composite hadronic level we obtain a breaking of this symmetry due to the flavor dependence of the quark propagators. As we will see in our numerical evaluation (see section 6.8 and Figs.6.6 and 6.7) the functions $f(p^2)$ and $r(p^2)$ vary slowly in the momentum range of interest, thus justifying *a posteriori* the Klein-Gordon approach.

Replacing in (6.110) the constants f and r by the running functions $f(p^2)$ and $r(p^2)$ we have

$$T_{K-G} \rightarrow T_{K-G}(p^2) = \begin{pmatrix} -M_N^2 & \sqrt{2}f(p^2)r(p^2) & 0 \\ \sqrt{2}f(p^2)r(p^2) & -M_G^2 & f(p^2) \\ 0 & f(p^2) & -M_S^2 \end{pmatrix}, \quad (6.123)$$

where the Klein-Gordon mass equation $Det[p^2 + T_{K-G}(p^2)] = 0$ coincides with the pole equation $Det[T^{-1}] = 0$ by construction. When diagonalizing $T_{K-G}(p^2)$ we obtain a momentum-dependent transition matrix $B(p^2) \subset SO(3)$ (for each p^2), which cannot be directly interpreted as the mixing matrix. To extract the composition of a mixed state one has to consider its on-shell mass value. For example, the mixing coefficients for the diagonal state $|G'\rangle$ are obtained from the second row of $B(p^2 = M_{G'}^2)$ at the corresponding on-shell value. We then can define a mixing matrix M' (the prime serves to distinguish it from M defined in the previous subsection) where we have $M'^{G',a} = B^{G',a}(p^2 = M_{G'}^2)$ with $(a = N, G, S)$; in general we get:

$$M'^{i,a} = B^{i,a}(p^2 = M_i^2). \quad (6.124)$$

Above procedure arises from the analogy to the Klein-Gordon case and indicates that M' is a natural generalization of B , provided that we evaluate the rotated states at their corresponding on-shell mass value. Again, due to the p^2 -dependence and the on-shell evaluation the mixing matrix M' is not orthogonal, where in the limit of $f(p^2) = const$ and $r(p^2) = const$ one has $M' = B$ as desired. The numerical results of M (6.116) and M' (6.124) are very similar, as we will show in section 6.8. The analytical argument leading to $M \sim M'$ are presented in Appendix A. Because of the weak p^2 dependence of the mixing functions $f(p^2)$ and $r(p^2)$, the mixing matrix M (and $M' \sim M$) will be almost orthogonal, as shown in section 6.8.

In the three-state mixing scheme many phenomenological approaches [31, 40, 42] refer to an interaction matrix, which is diagonalized linearly with respect to the masses:

$$T_{QM} = \begin{pmatrix} -M_N & \sqrt{2}f^*r^* & 0 \\ \sqrt{2}f^*r^* & -M_G & f^* \\ 0 & f^* & -M_S \end{pmatrix}. \quad (6.125)$$

Here we introduce the parameters f^* and r^* to distinguish them from those in (6.118). The resulting equation for the masses of the mixed states is then:

$$\begin{aligned} & (\sqrt{p^2} - M_N)(\sqrt{p^2} - M_G)(\sqrt{p^2} - M_S) - \\ & - f^{*2} \left((\sqrt{p^2} - M_N) + 2r^{*2}(\sqrt{p^2} - M_S) \right) = 0. \end{aligned} \quad (6.126)$$

To relate our approach to the linear mass case with parameters f^* and r^* we start from (6.121) and decompose $(p^2 - M_i^2) = (\sqrt{p^2} - M_i)(\sqrt{p^2} + M_i)$. Then we have

$$\begin{aligned} & (\sqrt{p^2} - M_N)(\sqrt{p^2} - M_G)(\sqrt{p^2} - M_S) - \frac{f^2(p^2)}{(\sqrt{p^2} + M_S)(\sqrt{p^2} + M_G)} \\ & \cdot \left((\sqrt{p^2} - M_N) + 2r^2(p^2) \frac{(\sqrt{p^2} + M_S)}{(\sqrt{p^2} + M_G)} (\sqrt{p^2} - M_S) \right) = 0. \end{aligned} \quad (6.127)$$

Comparing (6.127) to (6.126) we find the "running" behavior for f^* and r^* with

$$\begin{aligned} f^{*2} &\rightarrow f^{*2}(p^2) = \frac{f^2(p^2)}{(\sqrt{p^2} + M_S)(\sqrt{p^2} + M_G)} \\ r^{*2} &\rightarrow r^{*2}(p^2) = r^2(p^2) \frac{(\sqrt{p^2} + M_S)}{(\sqrt{p^2} + M_G)}, \end{aligned} \quad (6.128)$$

where extra-factors occur when relating the quadratic to the linear mass case. The consistent limit of our approach is actually the Klein-Gordon case, but Eq. (6.128) yields a direct comparison to the linear mass case, where many phenomenological approaches are working in [40, 42].

6.8 Numerical results and discussion

6.8.1 Mixing matrix and coupling constants

First we proceed to constrain the parameters of the model. The mass scale of the non-strange quarkonia in the scalar meson nonet is set by the physical state $a_0(1450)$. We therefore assume that the mass of the unmixed N meson state is close to this value. Using $M_N = 1.377$ GeV as an input, which is the value deduced in the phenomenological analysis of Ref. [42], the mass of the bare S meson state and the elementary quark coupling constant K_N are fixed from Eq.(6.59) once the cut-off Λ is specified. The bare scalar glueball mass is predicted from lattice calculations in the range of $1611 \pm 30 \pm 160$ MeV [9]. The analysis of [42] prefers a value of 1.443 GeV, which we use as an additional input. The bare glueball mass fixes in turn the gluonic coupling K_G of Eq. (6.83).

We now consider separately the phenomenological consequences of the two proposed choices for the quark propagators.

6.8.1.1 Case 1, free quark propagator

Mass spectrum: As discussed in Section 6.4.4 we consider large effective quark masses in order to avoid poles in the integration. For the quark mass we choose the threshold value $\mu_n = 0.86$ GeV to prevent unphysical on-shell production of a quark-antiquark pair.

For the vertex function we choose a cut-off of $\Lambda = 1.5$ GeV comparable to the values chosen in the analysis of [67] for the light meson sector. We will subsequently vary Λ within reasonable range, in order to check the dependence of the results on the specific value (see Appendix B). Actually, we find a remarkable stability of the results under changes of Λ as discussed further on. The mixing coefficient K_{SG} and the effective strange quark mass μ_s in the quark propagator are fixed by the experimental masses $M_{G'} = 1507 \pm 5$ MeV and $M_{S'} = 1713 \pm 7$ MeV. We then can compare K_{SG} with phenomenological approaches and with estimates from the lattice following the expressions of (6.122) and (6.128). Here we do not use the mass of the resonance $N' \equiv f_0(1370)$ as a further constraint, since it is rather broad and ill-determined. For the optimal choice of $\mu_s = 0.989$ GeV and $K_{SG} = 0.55$ GeV⁻¹ we obtain for the masses of the physical N' and the bare S states:

$$M_S = 1.696 \text{ GeV}, \quad M_{N'} = 1.297 \text{ GeV}. \quad (6.129)$$

The obtained quark "mass difference" $\mu_s - \mu_n = 129$ MeV is close to the upper limit of the current quark mass difference of about 130 MeV. With these quark mass values the bare

mass level scheme $M_N < M_G < M_S$ comes out naturally. A reversal of the bare scheme with $M_N < M_S < M_G$ would require a rather small mass difference $\mu_s - \mu_n$ in conflict with phenomenology. In fact, if we assume that the average $\langle \sigma_n(p^2) \rangle$ (see 6.4.4) is the same for n and s , thus one should find also for the scalar meson above 1 GeV the typical s - n mass difference.

Mixing matrix: The mixing matrix M linking the rotated and the unrotated states is:

$$\begin{pmatrix} |N'\rangle \\ |G'\rangle \\ |S'\rangle \end{pmatrix} = \begin{pmatrix} 0.80 & 0.59 & 0.10 \\ -0.60 & 0.76 & 0.26 \\ 0.08 & -0.27 & 0.96 \end{pmatrix} \begin{pmatrix} |N\rangle \\ |G\rangle \\ |S\rangle \end{pmatrix} \quad (6.130)$$

showing a similar pattern as in the phenomenological work of [42]: the center state $|G'\rangle$, identified with the $f_0(1500)$, has a dominant glueball component and the quark components $|N\rangle$ and $|S\rangle$ which are out of phase. Latter effect causes destructive interference for the $K\bar{K}$ decay mode consistent with experimental observation [31]. On the other hand $|N'\rangle$ and $|S'\rangle$ show a tendency to be dominated by the $\bar{n}n$ and $\bar{s}s$ constituent components as also deduced in [40, 42].

As already indicated, the mixing matrix of (6.130) is not orthogonal. The deviation from orthogonality is displayed by $M \cdot M^t$ (which is just the identity in the Klein-Gordon limit):

$$M \cdot M^t = \begin{pmatrix} 1 & -0.0028 & -0.0019 \\ -0.0028 & 1 & -0.0074 \\ -0.0019 & -0.0074 & 1 \end{pmatrix}, \quad (6.131)$$

where the off-diagonal elements turn out to be very small. This in turn implies that the Klein-Gordon limit is fully appropriate to set up the mixing scheme of the scalar meson states. Furthermore, the unities on the diagonal are in accord with (6.117).

In Section 6.7.4 we have introduced the mixing matrix M' . The numerical evaluation yields

$$M' = \begin{pmatrix} 0.79 & 0.60 & 0.09 \\ -0.61 & 0.75 & 0.24 \\ 0.08 & -0.26 & 0.96 \end{pmatrix}, \quad (6.132)$$

where the result is nearly identical to the one for M . The reasons for this coincidence are given in Appendix A.

The pattern observed in the mixing matrix only weakly depends on the particular choice of Λ . Similarly, a change in the value of μ_n up to 1.1 GeV alters the mixing pattern only within 5 %. The weak dependence of the results on Λ and μ_n is explicitly indicated in Appendix B.

Coupling constants: In the quantum mechanical approach the amplitude for the decay of the mixed state $|i\rangle$ (with $i = N', G', S'$) into two pions is fed by the $|N\rangle$ component in the strong coupling limit [31]. Hence the 2π decay amplitude is proportional to $M^{i,N}$ defined in Eq. (6.116). In this limit we obtain for the ratio of 2π decay amplitudes $|A_{N'\rightarrow 2\pi}/A_{G'\rightarrow 2\pi}| \propto |M^{N',N}/M^{G',N}| = 0.80/0.60 = 1.33$ (in the preferred solution of [42] one has for this ratio 1.13). In our full approach the decay amplitude is related to the respective coupling constant, hence we obtain for the ratio $|A_{N'\rightarrow 2\pi}/A_{G'\rightarrow 2\pi}| \propto |g_{N'}^{\bar{n}n}/g_{G'}^{\bar{n}n}| = 1.64$, which shows a

clear enhancement. This might explain why the state $N' \equiv f_0(1370)$ is considerably broader than the $G' \equiv f_0(1500)$. The relatively strong deviations of mixing coefficients and coupling constants is a non-negligible effect, which can be traced to the fact that the coupling constants $g_a(p^2) = (\partial\Sigma_a(p^2)/\partial p^2)^{-1/2}$ with $a = N, G, S$ are momentum dependent. Again, the relation $|g_{N'}^{\bar{n}n}/g_{G'}^{\bar{n}n}| > |M^{N',N}/M^{G',N}|$ is stable when changing the parameters Λ and μ_n .

The complete set of resulting coupling constants is summarized as

$$\left(\begin{array}{l} g_{N'}^{\bar{n}n} : g_{G'}^{\bar{n}n} : g_{S'}^{\bar{n}n} = 7.35 : -4.48 : 0.40 \\ g_{N'}^{\bar{s}s} : g_{G'}^{\bar{s}s} : g_{S'}^{\bar{s}s} = 1.20 : 2.69 : 8.10 \\ g_{N'}^{gg} : g_{G'}^{gg} : g_{S'}^{gg} = 0.91 : 0.95 : -0.23 \end{array} \right). \quad (6.133)$$

These results indicated here are of course model dependent, but they point to an interesting aspect in the comparison with other studies: the ratio of coupling constants, relevant for the strong two-body decay modes, can vary rather sensibly from the ratios of mixing matrix elements. This deviation is a consequence of the bound-state nature of the scalar mesons in a covariant framework.

It would of course be interesting to calculate the decays into two pseudoscalar mesons by loop diagrams directly. But this would require a consistent knowledge of the quark propagators and a careful study of the pseudoscalar meson-quark vertex functions to treat both the scalar mesons above 1 GeV and the light pseudoscalar mesons in a unified way. In the next chapter we will analyze the strong decays at a composite level, i.e. without evaluating the correspondent triangle diagrams, but parametrizing it (see discussion in section 3.4).

Running mixing functions: As already explained in the section devoted to the comparison with other mixing schemes, here we obtain a momentum dependent mixing strength. We developed two formulations, $f(p^2)$ of Eq. (6.122) and $f^*(p^2)$ of Eq. (6.128), in order to compare to the Klein-Gordon case and the quantum mechanical linear mass limit. Our results for p^2 -dependence of the mixing strength is summarized in Fig. 6.6. The running function $f^*(p^2)$ varies in the range between 60 and 64 MeV for the p^2 values of interest. Our result should be compared to the value of $85 \pm 10 MeV$ obtained in [42]. Lattice calculations in the quenched approximation obtain $f^* = 43 \pm 31 MeV$ [37] with a large uncertainty but of similar magnitude. Other quantum mechanical studies use fitted mixing parameters of $f^* = 77 MeV$ [37], $f^* = 64 \pm 13 MeV$ [37] and $f^* = 80 MeV$ [40]. Since $f^*(p^2)$ does not vary drastically in the region considered, this weak dependence explains why the mixing matrix M (and analogously M') is "almost" orthogonal.

Another characteristics of the non-local covariant approach is the dynamical generation of flavour blindness breaking with $r > 1$. The values of the matched "running" functions $r(p^2)$ and $r^*(p^2)$ (Fig. 6.6) are by 10-20 percent larger than unity. The lattice result of $r^* = 1.20 \pm 0.07$ [37] is in rather good agreement with our evaluation.

The characteristics of the mixing parameters are again rather stable when changing the parameters.

6.8.1.2 Case 2, entire function

In the following we summarize our results for the quark propagator of Eq. (6.68), described by an entire function modelling confinement.

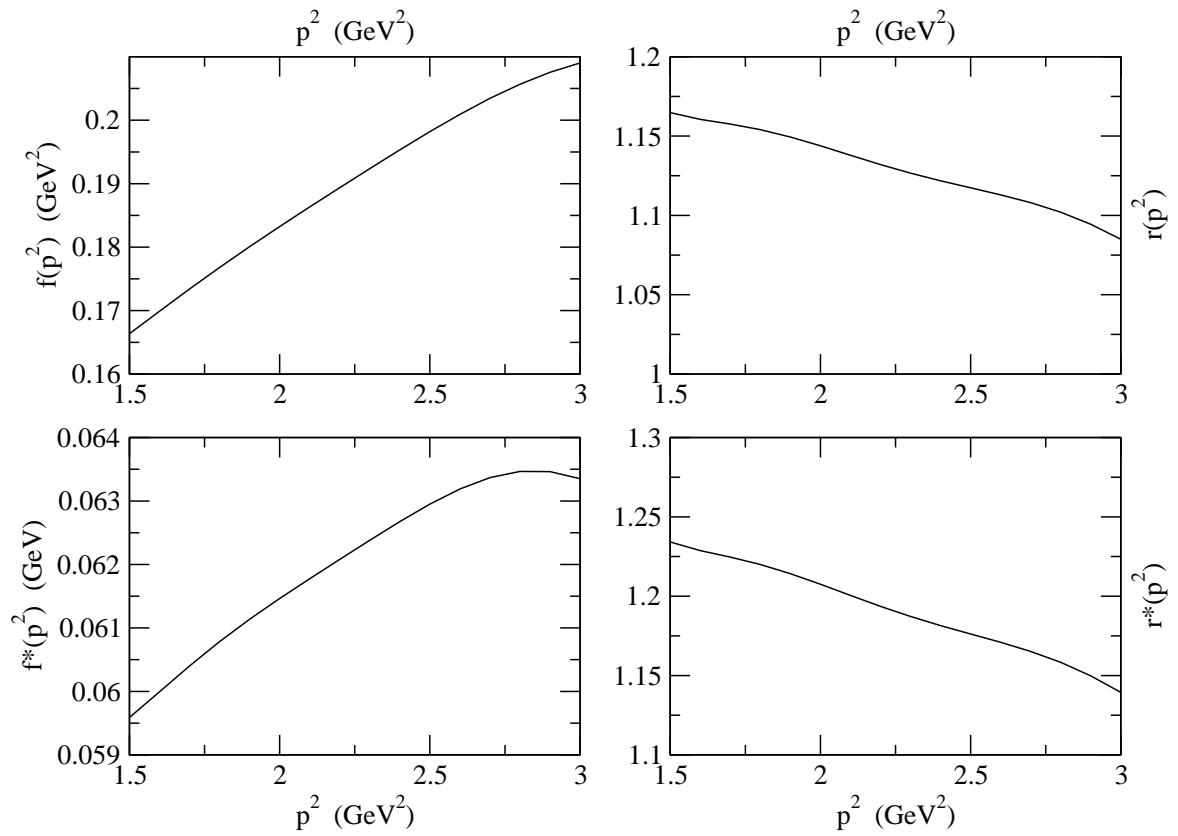


Figure 6.6: p^2 dependence of the running mixing functions for the free quark propagator.

Mass spectrum: We fix the parameters in the same fashion as for the previous case (that is to $M_N = 1.377 \text{ GeV}$, $M_G = 1.443 \text{ GeV}$, $M_{G'} = 1.507 \text{ GeV}$ and $M_{S'} = 1.713 \text{ GeV}$), but this time we have the parameter β of the quark propagator of Eq. (6.68) together with the mixing strength K_{SG} . For an identical cut-off with $\Lambda = 1.5 \text{ GeV}$ we get:

$$M_S = 1.698 \text{ GeV}, M_{N'} = 1.297 \text{ GeV} \quad (6.134)$$

obtained for the fit values of $\beta = 2.44 \text{ GeV}^{-2}$ and $K_{SG} = 0.284 \text{ GeV}^{-1}$. The two results of Eq. (6.134) are essentially identical to the previous case of Eq. (6.129). Also in this case the reversed bare level ordering with $M_N < M_S < M_G$ is disfavored, requiring a decrease of β of the order of 10^{-2} . This is in contrast to the requirement that the propagator behaves as a free one in the limit of small Minkowski or Euclidean momenta.

Mixing matrix: The mixing matrix M linking the rotated and the unrotated states is practically unchanged when compared to case 1:

$$\begin{pmatrix} |N'\rangle \\ |G'\rangle \\ |S'\rangle \end{pmatrix} = \begin{pmatrix} 0.80 & 0.59 & 0.10 \\ -0.59 & 0.76 & 0.25 \\ 0.07 & -0.27 & 0.96 \end{pmatrix} \begin{pmatrix} |N\rangle \\ |G\rangle \\ |S\rangle \end{pmatrix}, \quad (6.135)$$

where $M \cdot M^t$ is very close to the identity matrix.

Coupling constants: The results concerning the admixture and coupling constant ratios is analogous to the previous case. For the ratio of mixing amplitudes we get $|M^{N',N}/M^{G',N}| = 0.79/0.59 = 1.36$, while for the coupling constants we have $|g_{N'}^{\bar{n}n}/g_{G'}^{\bar{n}n}| = 1.65$, again almost unchanged with respect to case 1. The complete table reads:

$$\begin{pmatrix} g_{N'}^{\bar{n}n} : g_{G'}^{\bar{n}n} : g_{S'}^{\bar{n}n} = 3.82 : -2.32 : 0.22 \\ g_{N'}^{\bar{s}s} : g_{G'}^{\bar{s}s} : g_{S'}^{\bar{s}s} = 0.63 : 1.37 : 4.11 \\ g_{N'}^{gg} : g_{G'}^{gg} : g_{S'}^{gg} = 0.91 : 0.95 : -0.23 \end{pmatrix}. \quad (6.136)$$

Running mixing functions Results for $f(p^2)$, $f^*(p^2)$, $r(p^2)$ and $r^*(p^2)$ are summarized in Fig. 6.7, with the same quantitative behavior as in Fig. 6.6.

We conclude this section by noting that the quark propagator, modelling confinement, gives rise to very similar results as the free propagator with a large effective quark mass. Again, changes in the cutoff Λ do not alter the qualitative features of the results. This result is rather encouraging, since the model predictions considered here, seemingly do not depend on the particular choice of the quark propagator.

6.8.2 Two-photon decay

6.8.2.1 Analytic expression

In this work we also analyze the two-photon decay rates of the scalar isoscalar f_0 mesons considered. The two-photon decay width constitutes a crucial test to analyze the charge content of the scalar mesons [42, 117, 118]. The glueball component does not couple directly to the two-photon state and leads to a suppression of the decay width when present in the mixed state.

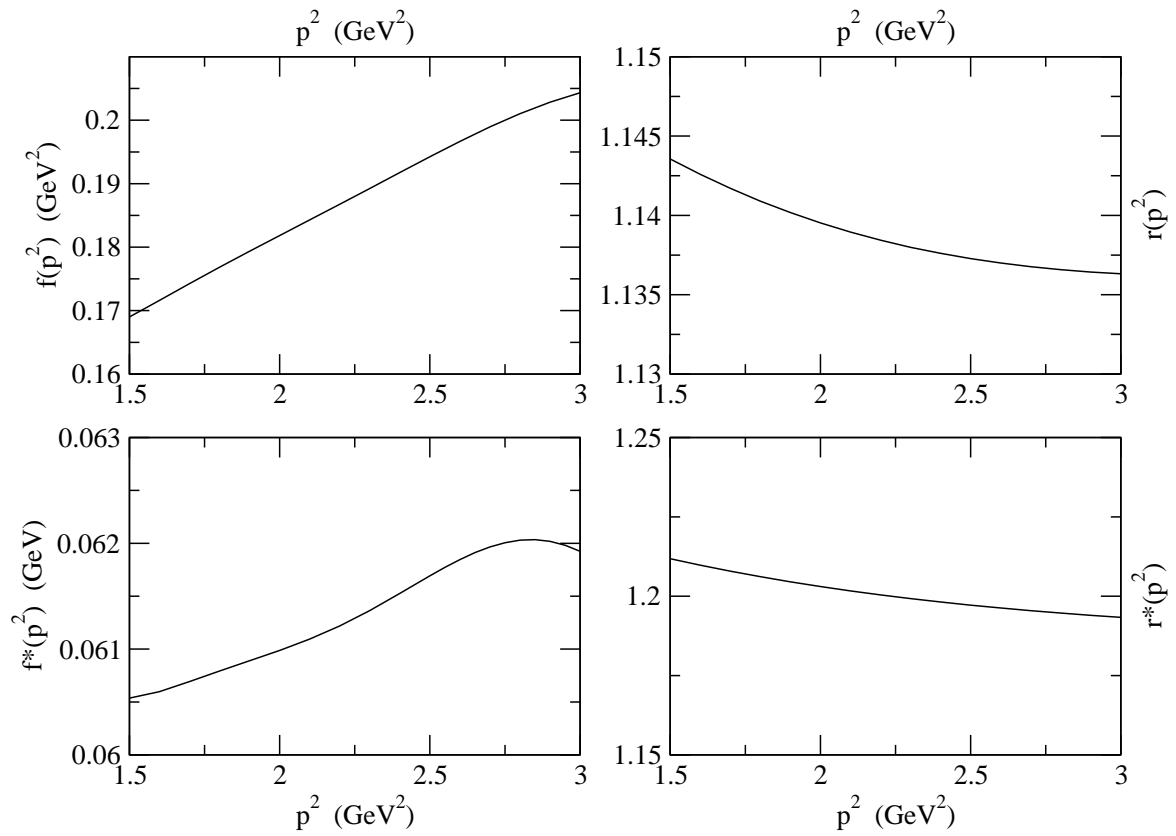


Figure 6.7: p^2 dependence of the running mixing parameters for the entire propagator.

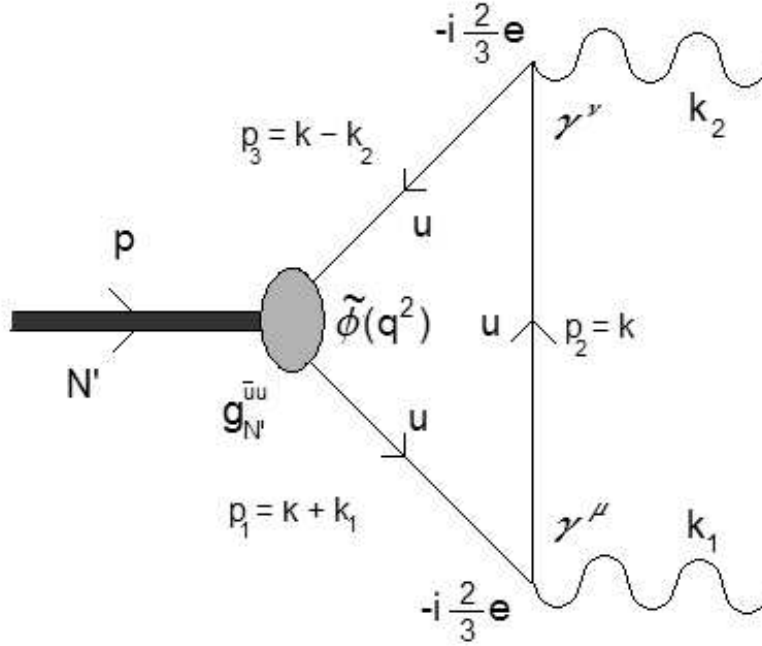


Figure 6.8: Mixed state N' decaying into two photons through a u -loop.

For the decay of the $N' \equiv f_0(1370)$ meson by its $\bar{u}u$ constituents into two photons we have to consider the triangle diagram of Fig. 6.8. Analogous diagrams occur for the d and s flavors in the loop. We consider in this case only the free propagator, so that we do not have a modification of the QED Ward-identities (this would be necessary with the entire propagator).

The calculation of the amplitude was explained in section 4.6, here we have just to extend with respect to the flavor decomposition. The analytic expression for the dominant gauge invariant part $M_{triangle,\perp}^{\mu\nu}$

$$M_{triangle,\perp}^{\mu\nu} = (k_1^\nu k_2^\mu - (k_1 \cdot k_2) g^{\mu\nu}) \cdot I, \quad (6.137)$$

where $I = I_1 + I_2$ is given by

$$I_1 = -i \cdot 4\mu_n \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(p_1^2 - \mu_n^2)(p_2^2 - \mu_n^2)(p_3^2 - \mu_n^2)} \tilde{\Phi}(q^2), \quad (6.138)$$

$$I_2 = i \cdot 4\mu_n \int \frac{d^4 k}{(2\pi)^4} \frac{\left(\frac{2k^2}{(k_1 \cdot k_2)} - 8 \frac{(k \cdot k_1)(k \cdot k_2)}{(k_1 \cdot k_2)^2} \right)}{(p_1^2 - \mu_n^2)(p_2^2 - \mu_n^2)(p_3^2 - \mu_n^2)} \tilde{\Phi}(q^2). \quad (6.139)$$

The term I_1 also occurs in the neutral pion decay into two photons, whereas the additional term I_2 turns out to have opposite sign to I_1 and tends to lower the decay rate.

With above results we finally obtain for the two-photon decay width of the mixed scalar meson state like N' :

$$\Gamma_{N' \rightarrow 2\gamma} = \frac{\pi}{4} \alpha^2 M_{N'}^3 \left[2 \left(\frac{g_{N'}^{\bar{n}n}}{\sqrt{2}} \right) N_c \left(\frac{4}{9} + \frac{1}{9} \right) (I) + 2g_{N'}^{\bar{s}s} N_c \left(\frac{1}{9} \right) (I^{(s)}) \right]^2 \quad (6.140)$$

where $N_c = 3$ is the number of colors and $I^{(s)}$ is obtained from I by replacing μ_n with μ_s in the quark propagator. Only the coupling constants $g_{N'}^{\bar{n}n}$ and $g_{N'}^{\bar{s}s}$ of the mixed field N' contribute to the decay rate, while $g_{N'}^{gg}$ does not because gluons do not couple directly to photons. The extra-factor 2 in front of the coupling constants comes from the exchange diagram. Analogous expressions follow for the decay rates of the other two resonances S' and G' .

6.8.2.2 Two-photon decay widths

We first consider the decay widths of the bare states N and S when mixing is neglected. In this case the coupling constants $N\text{-}\bar{n}n$ and $S\text{-}\bar{s}s$ are given by (6.62). We obtain

$$\Gamma_{N \rightarrow 2\gamma} = 0.821 \text{ keV}, \quad \Gamma_{S \rightarrow 2\gamma} = 0.08 \text{ keV}. \quad (6.141)$$

As explained previously, the amplitude of the triangle diagram which gives the dominant contribution to the decay is proportional to $(I_1 + I_2)$. The extra term I_2 , absent in the neutral pion case, has the opposite sign to I_1 and the ratio $|I_2/I_1|$ grows with increasing mass of the resonance. In the decay of the bare state N the term I_2 lowers the decay rate by a factor of 5.22.

For the two-photon decays of the physical states, where the coupling constants of the mixed states are used, we get:

$$\Gamma_{N' \rightarrow 2\gamma} = 0.453 \text{ keV}, \quad \Gamma_{G' \rightarrow 2\gamma} = 0.273 \text{ KeV}, \quad \Gamma_{S' \rightarrow 2\gamma} = 0.125 \text{ keV}, \quad (6.142)$$

resulting in the ratios of

$$\Gamma_{N' \rightarrow 2\gamma} / \Gamma_{S' \rightarrow 2\gamma} = 3.62, \quad \Gamma_{G' \rightarrow 2\gamma} / \Gamma_{S' \rightarrow 2\gamma} = 2.19. \quad (6.143)$$

which are similar to the ones obtained in [42] with $\Gamma_{N' \rightarrow 2\gamma} / \Gamma_{S' \rightarrow 2\gamma} = 3.56$ and $\Gamma_{G' \rightarrow 2\gamma} / \Gamma_{S' \rightarrow 2\gamma} = 2.36$. Current experimental upper limits for $G' \equiv f_0(1500)$ and for $S' \equiv f_0(1710)$ are [119]:

$$\Gamma_{f_0(1500) \rightarrow 2\gamma} \frac{\Gamma_{f_0(1500) \rightarrow \pi\pi}}{\Gamma_{f_0(1500)tot}} < 0.46 \text{ keV}, \quad (6.144)$$

$$\Gamma_{f_0(1710) \rightarrow 2\gamma} \frac{\Gamma_{f_0(1710) \rightarrow K\bar{K}}}{\Gamma_{f_0(1710)tot}} < 0.110 \text{ keV}. \quad (6.145)$$

Multiplying our theoretical results by the experimental ratios [119] $\frac{\Gamma_{f_0(1500) \rightarrow \pi\pi}}{\Gamma_{f_0(1500)tot}} = 0.454 \pm 0.104$ and $\frac{\Gamma_{f_0(1710) \rightarrow K\bar{K}}}{\Gamma_{f_0(1710)tot}} = 0.38_{-0.19}^{+0.09}$ we find:

$$(\Gamma_{f_0(1500) \rightarrow 2\gamma})_{theory} \frac{\Gamma_{f_0(1500) \rightarrow \pi\pi}}{\Gamma_{f_0(1500)tot}} = 0.124 \pm 0.028 \text{ keV} \quad (6.146)$$

$$(\Gamma_{f_0(1710) \rightarrow 2\gamma})_{theory} \frac{\Gamma_{f_0(1710) \rightarrow K\bar{K}}}{\Gamma_{f_0(1710)tot}} = 0.0475_{-0.0237}^{+0.0112} \text{ keV}, \quad (6.147)$$

in accord with the experimental upper limits.

The experimental two-photon decay width of the scalar resonance $f_0(1370)$ has been seen; originally in [120] two values were indicated, i.e. $3.8 \pm 1.5 \text{ keV}$ and $5.4 \pm 2.3 \text{ keV}$. However, it is not clear if the two-photon signal comes from the $f_0(1370)$ or from the high mass end of the broad $f_0(400 - 1200)$. The PDG currently [119, 12] seems to favor this last possibility, but it states in a footnote that this data could also be valid for the $f_0(1370)$. We therefore interpret the two experimental values as an upper limit for the two-photon decay width of the $f_0(1370)$. The result for $\Gamma_{N' \equiv f_0(1370) \rightarrow 2\gamma}$ is an order of magnitude smaller than these upper limits. A precise experimental determination of the two-photon decay values of these scalar states would clearly help in understanding their structure.

For what concerns the cut-off dependence of the decay rates, we note that an increase in the cut-off leads to a weak increase of the decay widths as indicated in Appendix B, but the ratios remain stable.

6.8.2.3 Final discussion

In this chapter we utilized a covariant constituent approach to analyze glueball-quarkonia mixing in the scalar meson sector above 1 GeV . We used simple forms for the quark and gluon propagators in order to avoid unphysical threshold production of quarks and gluons. Although quark and gluon propagators are directly accessible in lattice simulations in the Euclidean region, an extrapolation to the Minkowski region, also needed here, is not straightforward. We therefore considered relatively simple choices of the propagators, which allowed us to point out some features of the mixing of covariant bound states. We tried to work out similarities and differences with phenomenological approaches, in particular with respect to the analysis of [42]. Although in a covariant approach the mixing matrix is in general not orthogonal, in the present case only small deviations from orthogonality are obtained. This in turn leads to a mixing pattern rather similar to that of Ref. [42]. The mixing matrix M has been introduced by knowledge of the coupling constants of the mixed and unmixed states taking into account their p^2 dependence. The resulting matrix M is analogous to the Klein-Gordon case as shown both numerically and in part analytically.

Many properties we analyzed, such as the appearance of the "running" mixing parameters $f(p^2)$ and $r(p^2)$, are rather independent on the choice of the particular quark propagator. The numerical results for $f(p^2)$ and $r(p^2)$ are in qualitative accord with the lattice evaluations. We generate a dynamical breaking of the glueball flavor blindness corresponding to r slightly bigger than unity, which is directly connected to the isospin symmetry violation at the level of the quark propagators. Again, all these considerations do not depend on the choice of the two proposed quark propagator forms and on the employed parameter sets. Another interesting result of our approach is that the bare level ordering $M_N < M_G < M_S$ is naturally favored for both propagator choices.

As a further application we also evaluated the two-photon decay rates in the context of the mixing model. The predicted results are in accord with the present experimental upper limits. The respective ratios are also in agreement with the phenomenological estimate of [42].

Chapter 7

Strong decays

7.1 Introduction

In this chapter we study the strong decays of the scalar mesons between 1-2 GeV into two pseudoscalar ones. The analysis of strong decays, together with the two-photon one, is necessary to test the glueball mixing hypothesis and to pin down the composition of their wave functions.

As discussed in section 1.5 in the introduction, the resonances $f_0(1500)$ and $f_0(1710)$ are relatively narrow, with $\Gamma_{f_0(1500)} = 109 \pm 5 MeV$ and $\Gamma_{f_0(1710)} = 140 \pm 10 MeV$ [12]. The experimental data on the decay of these two states improved in the last years; specially for what concerns $f_0(1500)$, [12] reports the ratios in the form $\Gamma_{f_0(1500) \rightarrow p1p2} / \Gamma_{f_0(1500)}$ where "p1p2" represents a pseudoscalar meson pair, like pion-pion.

For the broad state $f_0(1370)$ the uncertainties are still large. No average or fit is presented in [12]; the results from different experiments are often contradictory. However, the results from WA102 (see [41]) indicate a large $\bar{n}n$ component in its wave function with

$$\Gamma_{f_0(1370) \rightarrow \bar{K}K} / \Gamma_{f_0(1370) \rightarrow \pi\pi} = 0.46 \pm 0.19;$$

the results from Crystal Barrel (summarized in [32] and subsequently analyzed in [105]) confirm such a trend (see also [46] for a recent review). The main problem connected with this resonance is its large width (200-500 MeV) and its partial overlap with the broad low-lying σ .

In this last chapter we aim to describe the two-pseudoscalar decays within phenomenological approaches. To perform this analysis we make a step back with respect to what was done in the previous chapter. Within the nonlocal approach described throughout all the work the decay of $G' \equiv f_0(1500)$ into two pions would be "expressed" through a triangle diagram, similar to the one of the two-photon decay (see Fig. 7.1).

As mentioned in the previous chapter, the problem connected with such a calculation is the simultaneous and consistent treatment of the scalar states (plus glueball) above 1 GeV and the pseudoscalar states below 1 GeV . Thereby the choice of the propagators plays a crucial role. The free propagators with a large effective mass can be justified in a phenomenological treatment of scalar states above 1 GeV , but do not work for pseudoscalar mesons, where a lower quark mass is needed. In such a calculation one would really need a p^2 momentum dependent running quark mass function $m = m(p^2)$. One could make suitable "Ansatz" for $m(p^2)$, but this would increase the number of parameters and raise questions about the

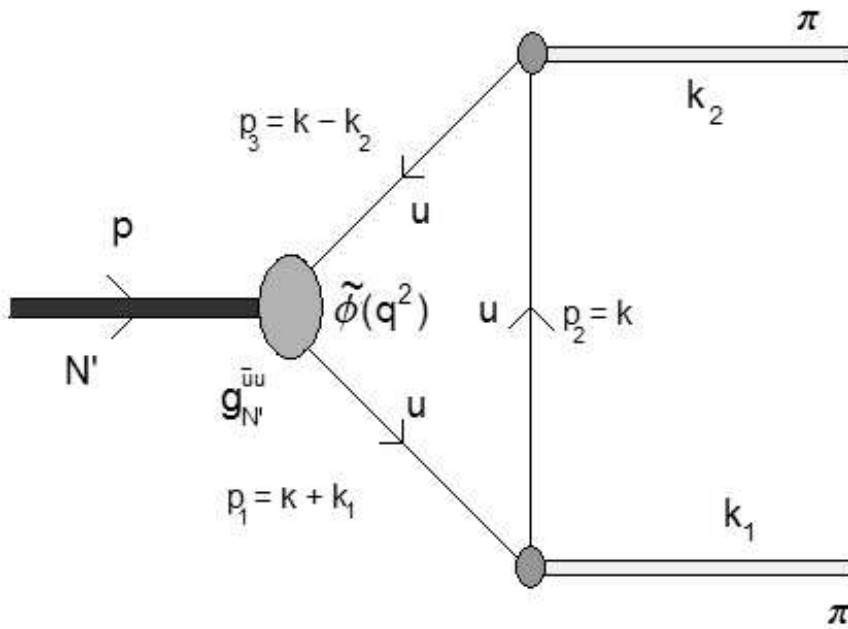


Figure 7.1: Decay of N' into two pions through a quark-loop.

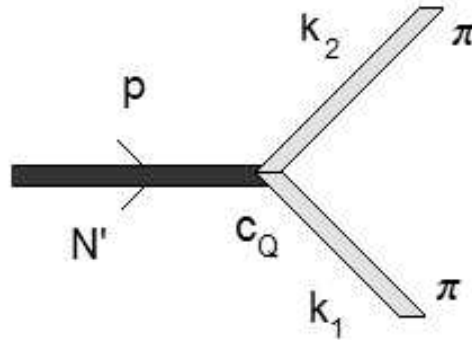


Figure 7.2: Transition of N' into two pions with a point-like vertex.

reliability of the approach. A consistent treatment could be achieved, for instance, by using Dyson-Schwinger equations by calculating the quark propagator and then, through the Bethe-Salpeter equations, calculate the pseudoscalar and scalar quarkonia masses. The problem is that the scalar masses found in a DSE approach are typically too small, below 1 GeV [81] (in [106], with a similar approach, the scalars are not found). A part from this not yet solved difficulty, one should also consistently calculate the glueball mass, and then the glueball mixing. Such desirable "links" are still outside of reach. I actually think it will be possible to do it in the future.

After these preliminary remarks we concentrate on the evaluation of the two-pseudoscalar decay within two phenomenological Lagrangians where the only degrees of freedom are mesons. The decay into two pions, for instance, will be described by the point-like diagrams of Fig. 7.2.

In the vertex of Fig. 7.2 we parametrize the triangle diagram, and we take into account the different flavor contributions. This job is similar to what was done in the fifth chapter.

In a sense, we do the opposite of what done in section 4.4.4; instead of looking at the process with a magnifying glass, we put the magnifying glass away. In this way we simplify our job and we can describe all the scalar nonet plus glueball in a compact way, even if we renounce to a microscopic interpretation of the decays.

The first of the two employed Lagrangians is inspired by flavour symmetry.; although rather simplified, it constitutes a very useful exercise for discussions and comparisons with other works.

The second Lagrangian is taken in the context of chiral perturbation theory [26, 27, 28]. Such an approach is rigorously defined for small momenta and masses; an application to physical states between $1\text{-}2 \text{ GeV}$ cannot *a priori* be justified; loops and higher order corrections may be large. However, if intended as a phenomenological approach, can be a

very useful tool to describe the scalar nonet and its mixing with the scalar glueball.

7.2 Flavour Lagrangian

7.2.1 Scalar and pseudoscalar nonets

Let us define the matrix

$$\begin{aligned}
\frac{\widehat{S}}{\sqrt{2}} &= \frac{1}{\sqrt{2}} \sum_{a=0}^8 S_a \lambda_a \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} S_3 + \frac{1}{\sqrt{3}} S_0 + \frac{1}{\sqrt{6}} S_8 & \frac{1}{\sqrt{2}} (S_1 - iS_2) & \frac{1}{\sqrt{2}} (S_4 - iS_5) \\ \frac{1}{\sqrt{2}} (S_1 + iS_2) & -\frac{1}{\sqrt{2}} S_3 + \frac{1}{\sqrt{3}} S_0 + \frac{1}{\sqrt{6}} S_8 & \frac{1}{\sqrt{2}} (S_6 - iS_7) \\ \frac{1}{\sqrt{2}} (S_4 + iS_5) & \frac{1}{\sqrt{2}} (S_6 + iS_7) & \frac{1}{\sqrt{3}} S_0 - \frac{2}{\sqrt{6}} S_8 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} a_0^+ + \frac{1}{\sqrt{3}} S_0 + \frac{1}{\sqrt{6}} S_8 & a_0^+ & K^{*+} \\ a_0^- & -\frac{1}{\sqrt{2}} a_0^+ + \frac{1}{\sqrt{3}} S_0 + \frac{1}{\sqrt{6}} S_8 & K^{*0} \\ K^{*-} & \overline{K}^{*0} & \frac{1}{\sqrt{3}} S_0 - \frac{2}{\sqrt{6}} S_8 \end{pmatrix}, \quad (7.1)
\end{aligned}$$

where the λ_a are the Gell-Mann matrices in flavor space. The fields K^* and a_0 refer to the physical states $K^*(1430)$ and $a_0(1450)$. The isoscalar S_0 and S_8 are related to $N \equiv \bar{\pi}n$ and $S \equiv \bar{s}s$ through

$$\begin{pmatrix} S_0 \\ S_8 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} N \\ S \end{pmatrix}. \quad (7.2)$$

Similarly we define the pseudoscalar nonet field like

$$\begin{aligned}
\frac{\widehat{P}}{\sqrt{2}} &= \frac{1}{\sqrt{2}} \sum_{a=0}^8 P_a \lambda_a = \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{3}} \eta_0 + \frac{1}{\sqrt{6}} \eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{3}} \eta_0 + \frac{1}{\sqrt{6}} \eta_8 & K^0 \\ K^- & \overline{K}^0 & \frac{1}{\sqrt{3}} \eta_0 - \frac{2}{\sqrt{6}} \eta_8 \end{pmatrix}, \quad (7.3)
\end{aligned}$$

where one has pions and kaons; the physical states η and η' are linked to η_0 and η_8 through

$$\begin{pmatrix} \eta_0 \\ \eta_8 \end{pmatrix} = \begin{pmatrix} \cos \theta_{ps} & -\sin \theta_{ps} \\ \sin \theta_{ps} & \cos \theta_{ps} \end{pmatrix} \begin{pmatrix} \eta' \\ \eta \end{pmatrix} \quad (7.4)$$

where θ_{ps} is the pseudoscalar mixing angle (-10° from the quadratic Gell-Mann-Okubo formula for the pseudoscalar nonet [35, 12], $\sim -20^\circ$ from studies based on chiral perturbation theory [121]; see also section 7.4).

7.2.2 Lagrangian for the decay into pseudoscalars: inert glueball

The Lagrangian consists of two parts: the first one is quadratic and describes the scalar glueball-quarkonia mixing:

$$\begin{aligned} \mathcal{L}_{quadratic} = & \frac{1}{2} ((\partial_\mu N)^2 - M_G^2 N^2) + \frac{1}{2} ((\partial_\mu G)^2 - M_G^2 G^2) + \frac{1}{2} ((\partial_\mu S)^2 - M_S^2 S^2) \\ & - fGS - \sqrt{2}frGN - \varepsilon NS \\ & + \text{free part for } K^* \text{ and } a^0 + \text{free part for } \pi, K, \eta, \eta' \end{aligned} \quad (7.5)$$

The masses of pseudoscalar and scalar states are taken from [12] and are (in MeV): $M_\pi = 134.976$, $M_K = 493.677$, $M_\eta = 547.75$, $M_{\eta'} = 957.78$, $M_{a_0} = 1.474$, $M_{K^*} = 1.412$.

The masses of N , G , and S together with the respective mixing are object of a separated study (see the next subsection).

The second part of the Lagrangian has to connect the scalar mesons with the pseudoscalar ones, describing the decay of a scalar resonance into two pseudoscalars respecting flavour symmetry. The interaction form fulfilling the desired requirements reads:

$$\mathcal{L}_{decay} = c_Q Tr[\widehat{S}\widehat{P}\widehat{P}]. \quad (7.6)$$

The full Lagrangian is then

$$\mathcal{L}_{flavor} = \mathcal{L}_{quadratic} + \mathcal{L}_{decay}. \quad (7.7)$$

An important point should immediately be noticed: the bare glueball G does NOT decay. It is not included in \widehat{S} . This is in accord with [31], where the glueball was "inert". The scalar quarkonia dominate the decays; the direct decay of a scalar glueball into two-pseudoscalar mesons is supposed to be negligible when compared to the quarkonia ones. If such an hypothesis is true is still not known.

7.2.3 Expressions for the decay rates

As discussed in the fifth chapter one has to diagonalize the matrix

$$\Omega = \begin{pmatrix} M_N^2 & \sqrt{2}fr & \varepsilon \\ \sqrt{2}fr & M_G^2 & f \\ \varepsilon & f & M_S^2 \end{pmatrix} \quad (7.8)$$

from which one finds the masses and the compositions of the physical isoscalar states (see section 5.3); the physical fields $|i\rangle$ with $i = N' \equiv f_0(1370)$, $G' \equiv f_0(1500)$ and $S' \equiv f_0(1710)$ are given by

$$|i\rangle = B^{iN} |N\rangle + B^{iG} |G\rangle + B^{iS} |S\rangle \quad (7.9)$$

where B is the transformation matrix

$$B\Omega B^t = \Omega' = \begin{pmatrix} M_{N'}^2 & 0 & 0 \\ 0 & M_{G'}^2 & 0 \\ 0 & 0 & M_{S'}^2 \end{pmatrix}. \quad (7.10)$$

In the following we introduce the notation $p_{sp_1p_2}$, where "s" denotes a scalar state and "p₁" and "p₂" two pseudoscalar ones, with:

$$p_{sp_1p_2} = \frac{1}{2} \sqrt{\frac{M_s^4 + (M_{p_1}^2 - M_{p_2}^2)^2 - 2(M_{p_1}^2 + M_{p_2}^2)M_s^2}{M_s^2}}. \quad (7.11)$$

The final expressions for the decays can be calculated following the discussion of section 4.2.

7.2.3.1 Decays of the mixed isoscalar states

The decays of $|i\rangle$ into $\pi\pi$, $\bar{K}K$, $\eta\eta$ and $\eta\eta'$, as calculated from the interaction Lagrangian, are given by:

$$\Gamma_{i \rightarrow \pi\pi} = 3 \frac{\frac{1}{2} \sqrt{M_i^2 - 4M_\pi^2}}{16\pi M_i^2} [4c_Q B^{iN}]^2, \quad (7.12)$$

$$\Gamma_{i \rightarrow \bar{K}K} = 4 \frac{\frac{1}{2} \sqrt{M_i^2 - 4M_K^2}}{16\pi M_i^2} [2c_Q (B^{iN} + \sqrt{2}B^{iS})]^2, \quad (7.13)$$

$$\Gamma_{i \rightarrow \eta\eta} = \frac{\frac{1}{2} \sqrt{M_i^2 - 4M_\eta^2}}{16\pi M_i^2} [2c_Q \Lambda_{1\eta\eta}]^2 \quad (7.14)$$

and

$$\begin{aligned} \Lambda_{1\eta\eta} &= (B^{iN} + \sqrt{2}B^{iS}) - \frac{1}{\sqrt{3}} (B^{iN} - \sqrt{2}B^{iS}) \cdot \cos(2\theta_{ps}) \\ &\quad - \frac{2}{3} (\sqrt{2}B^{iN} - 2B^{iS}) \cdot \sin(2\theta_{ps}), \end{aligned} \quad (7.15)$$

$$\Gamma_{i \rightarrow \eta\eta'} = \frac{p_{i\eta\eta'}}{8\pi M_i^2} [c_Q \Lambda_{1\eta\eta'}]^2 \quad (7.16)$$

and

$$\Lambda_{1\eta\eta'} = \frac{2}{3} \left[2 (\sqrt{2}B^{iN} - 2B^{iS}) \cos(2\theta_{ps}) + (-B^{iN} + \sqrt{2}B^{iS}) \sin(2\theta_{ps}) \right]. \quad (7.17)$$

Note that the matrix elements B^{iG} do not appear in the decays because we suppose that the decay of the bare glueball is suppressed.

7.2.3.2 $a_0(1450)$ decays

The state isovector $a_0(1450)$ decays into $\bar{K}K$, $\pi\eta$ and $\pi\eta'$ with the following partial decay widths:

$$\Gamma_{a_0 \rightarrow \bar{K}K} = 2 \frac{\frac{1}{2} \sqrt{M_{a_0}^2 - 4M_K^2}}{8\pi M_{a_0}^2} [2c_Q]^2, \quad (7.18)$$

$$\Gamma_{a_0 \rightarrow \pi\eta} = \frac{p_{a_0\pi\eta}}{8\pi M_{a_0}^2} \left[\frac{4}{\sqrt{3}} c_Q (\cos(\theta_{ps}) - \sqrt{2} \sin(\theta_{ps})) \right]^2, \quad (7.19)$$

$$\Gamma_{a_0 \rightarrow \pi\eta'} = \frac{p_{a_0\pi\eta'}}{8\pi M_{a_0}^2} \left[\frac{4}{\sqrt{3}} c_Q (\sqrt{2} \cos(\theta_{ps}) + \sin(\theta_{ps})) \right]^2. \quad (7.20)$$

7.2.3.3 $K^*(1430)$ decays

The state $K^*(1450)$ decays into the pseudoscalar channels πK and $K\eta$:

$$\Gamma_{K^* \rightarrow \pi K} = 3 \frac{p_{K^* \pi K}}{8\pi M_{K^*}^2} [2c_Q]^2, \quad (7.21)$$

$$\Gamma_{K^* \rightarrow K\eta} = \frac{p_{K^* K\eta}}{8\pi M_{K^*}^2} \left[c_Q \frac{2}{\sqrt{3}} \left(\cos(\theta_{ps}) + 2\sqrt{2} \sin(\theta_{ps}) \right) \right]^2. \quad (7.22)$$

7.2.3.4 Two-photon decays

The calculation of the two-photon decay expressions has been carried out in section 5.4 and summarized by equations (5.54).

7.2.4 χ^2 analysis

We consider in this section a χ^2 study to find the best parameters for the description of the decays. In the following we will work with $r = 1$ (small deviations from unity may be possible, but they are negligible [31, 42, 74]). We have six free parameters: the bare masses M_N , M_G and M_S , two mixing parameters f and ε and one decay strength c_Q . Before proceeding with the χ^2 analysis we have to do some considerations:

- We will consider a direct N - S mixing, that is flavor mixing, through ε ; this is not done in [31, 42, 37]. A large N - S mixing in the scalar sector is the starting point of [47, 50] (see 1.5.4), even if the authors of these works make a different assignment. The origin of this mixing is connected with instanton effects, as in the pseudoscalar channel, but with opposite sign (see also [122]). In our case it means to have $\varepsilon > 0$.

- We use $\theta_{ps} = -10^\circ$, as in [35].

- We do not include $\Gamma_{f_0(1500) \rightarrow \eta\eta'}$ in the fit, because the $\eta\eta'$ production is very close to threshold; modifications due to the finite width of the state are supposed to be large.

- We take the experimental values and relative errors from the quoted averages of [12]; we only take "bold" results, i.e. accepted by [12].

In particular, as an experimental input, we take:

→ For the mixed states we choose the assignments $N' \equiv f_0(1370)$, $G' \equiv f_0(1500)$ and $S' \equiv f_0(1710)$ with the corresponding mass values of $M_{N' \equiv f_0(1370)} = 1.35 \pm 0.15 \text{ GeV}$, $M_{G' \equiv f_0(1500)} = 1.507 \pm 0.005 \text{ GeV}$, $M_{S' \equiv f_0(1710)} = 1.714 \pm 0.005 \text{ GeV}$.

→ The accepted partial decay widths entering in the fit are:

$$\Gamma_{G' \rightarrow \pi\pi} = 0.0380 \pm 0.0050 \text{ GeV};$$

$$\Gamma_{G' \rightarrow \bar{K}K} = 0.0094 \pm 0.0017 \text{ GeV};$$

$$\Gamma_{G' \rightarrow \eta\eta} = 0.0056 \pm 0.0014 \text{ GeV}.$$

$$\Gamma_{S' \rightarrow \pi\pi} / \Gamma_{S' \rightarrow \bar{K}K} = 0.20 \pm 0.06;$$

$$\Gamma_{S' \rightarrow \eta\eta} / \Gamma_{S' \rightarrow \bar{K}K} = 0.48 \pm 0.15.$$

- The state $f_0(1710)$ has only been observed in the decays to two pseudoscalar mesons. The decay into the final state 4π , which can be fed by higher meson resonances, is suppressed [41]. We will impose the extra-condition that

$$(\Gamma_{S'})_{2\text{pseudoscars}} = (\Gamma_{S'})_{tot} \quad (7.23)$$

Such a condition is necessary to obtain meaningful total widths; the partial decay widths of $f_0(1500)$ and the ratios for $f_0(1710)$, are not sufficient; without the condition on the full width there could be minima of χ^2 for which $(\Gamma_{S'})_{tot}$ is larger than 1 GeV , a clearly unacceptable solution, because we know that it is at most 140 MeV .

We will then use:

$$(\Gamma_{S'})_{2pseudoscalars} = \Gamma_{S' \rightarrow \pi\pi} + \Gamma_{S' \rightarrow \bar{K}K} + \Gamma_{S' \rightarrow \eta\eta} + \Gamma_{S' \rightarrow \eta\eta'} = 0.140 \pm 0.01 \text{ GeV}.$$

• Although the ratios $\Gamma_{a_0 \rightarrow \bar{K}K} / \Gamma_{a_0 \rightarrow \pi\eta}$ and $\Gamma_{a_0 \rightarrow \pi\eta'} / \Gamma_{a_0 \rightarrow \pi\eta}$ are given in the PDG compilation, we do not include them in the fit; the reason is simple: these ratios are fixed and do not depend on any of the 6 parameters.

After these consideration, we consider:

$$\begin{aligned} \chi^2 &= \chi^2[M_N, M_G, M_S, f, \varepsilon, c_Q] \\ &= \left(\frac{M_{N'} - 1.35}{0.15} \right)^2 + \left(\frac{M_{G'} - 1.507}{0.005} \right)^2 + \left(\frac{M_{S'} - 1.714}{0.005} \right)^2 \\ &\quad + \left(\frac{1}{0.0050} (\Gamma_{G' \rightarrow \pi\pi} - 0.0380) \right)^2 + \left(\frac{1}{0.0017} (\Gamma_{G' \rightarrow \bar{K}K} - 0.0094) \right)^2 \\ &\quad + \left(\frac{1}{0.0014} (\Gamma_{G' \rightarrow \eta\eta} - 0.0056) \right)^2 + \left(\frac{1}{0.06} \left(\frac{\Gamma_{S' \rightarrow \pi\pi}}{\Gamma_{S' \rightarrow \bar{K}K}} - 0.20 \right) \right)^2 \\ &\quad + \left(\frac{1}{0.15} \left(\frac{\Gamma_{S' \rightarrow \eta\eta}}{\Gamma_{S' \rightarrow \bar{K}K}} - 0.48 \right) \right)^2 \\ &\quad + \left(\frac{1}{0.01} (\Gamma_{S' \rightarrow \pi\pi} + \Gamma_{S' \rightarrow \bar{K}K} + \Gamma_{S' \rightarrow \eta\eta} + \Gamma_{S' \rightarrow \eta\eta'} - 0.14) \right)^2 \end{aligned} \quad (7.24)$$

A minimum is obtained for the following parameters:

$$M_N = 1.377, M_G = 1.481, M_S = 1.687, f = 0.120, \varepsilon = 0.294, c_Q = 0.010; \chi^2 = 8.914.$$

The parameter f is in qualitative agreement with other works (see section 6.8 and Refs. there and [74]). The parameter ε was not considered in the sixth chapter; however, the obtained value points to a substantial influence of it on the results.

In the following we list in a table the quantities of the fit, comparing them to the experimental values and the corresponding χ_i^2 .

Quantity	Exp	Theory	χ_i^2
$M_{N'}$	$1350 \pm 150 \text{ MeV}$	1314 MeV	0.056
$M_{G'}$	$1507 \pm 5 \text{ MeV}$	1507 MeV	~ 0
$M_{S'}$	$1714 \pm 5 \text{ MeV}$	1714 MeV	~ 0
$\Gamma_{G' \rightarrow \pi\pi}$	$38.0 \pm 5.0 \text{ MeV}$	38.6 MeV	0.007
$\Gamma_{G' \rightarrow \bar{K}K}$	$9.4 \pm 1.7 \text{ MeV}$	10.7 MeV	0.58
$\Gamma_{G' \rightarrow \eta\eta}$	$5.6 \pm 1.4 \text{ MeV}$	2.46 MeV	5.45
$\Gamma_{S' \rightarrow \pi\pi} / \Gamma_{S' \rightarrow \bar{K}K}$	0.20 ± 0.06	0.194	0.077
$\Gamma_{S' \rightarrow \eta\eta} / \Gamma_{S' \rightarrow \bar{K}K}$	0.48 ± 0.15	0.228	2.802
$(\Gamma_{S'})_{2pseudoscalars}$	$140 \pm 10 \text{ MeV}$	140.07 MeV	~ 0
χ_{tot}^2	-	-	8.91

Note that $\chi_{tot}^2 / N = 0.99 < 1$; the fit is indeed acceptable.

With the parameters of the fit we can calculate many other quantities, which can be compared to other models and to various experiments (although there is no accepted average in [12]).

7.2.5 Consequences of the fit

- Mixing matrix

The mixing matrix linking the rotated physical fields to the bare ones reads:

$$\begin{pmatrix} N' \\ G' \\ S' \end{pmatrix} = \begin{pmatrix} 0.88 & 0.39 & 0.27 \\ -0.40 & 0.91 & -0.01 \\ -0.26 & -0.09 & 0.96 \end{pmatrix} \begin{pmatrix} N \\ G \\ S \end{pmatrix}. \quad (7.26)$$

The trend discussed in [31, 40, 42] again is found; N' is mostly $\bar{n}n$, G' mainly the glueball and S' is dominantly $\bar{s}s$. However, we do not have a phase difference between the $\bar{n}n$ and $\bar{s}s$ components in G' . The $\bar{s}s$ amount is practically absent, and considering that the glueball doesn't decay, the decay modes of this state are driven by its $\bar{n}n$ component. This is in accord with [35, 40].

- $f_0(1370)$ resonance:

We compare our results to the particular results from WA102 collaboration [41]:

Quantity	Exp (WA102)	Theory
$\Gamma_{N' \rightarrow \bar{K}K} / \Gamma_{N' \rightarrow \pi\pi}$	0.46 ± 0.19	0.47
$\Gamma_{N' \rightarrow \eta\eta} / \Gamma_{N' \rightarrow \pi\pi}$	0.16 ± 0.07	0.10
$(\Gamma_{N'})_{2pseudoscalars}$	"small"	318.68 MeV

(7.27)

The ratios are in good agreement with the results; however, the full width is not. WA102 reports $\Gamma_{N' \rightarrow 4\pi} / \Gamma_{N' \rightarrow \pi\pi} = 34.0_{-9}^{+22}$, which points to a dominant 4π (arising from intermediate 2σ and 2ρ decay modes) decay width. The decays into $\pi\pi$, KK and $\eta\eta$ are then of the order of few MeV [42]. Ref. [53] mentions the possibility that the large measured 4π comes from a four-quark component of this state. However, the error on $\Gamma_{N' \rightarrow 4\pi} / \Gamma_{N' \rightarrow \pi\pi}$ is still large; a better experimental results would be desirable.

- $f_0(1710)$ resonance:

Quantity	Exp (WA102)	Theory
$\Gamma_{S' \rightarrow \bar{K}K} / \Gamma_{S' \rightarrow \pi\pi}$	5.0 ± 0.7	5.0
$\Gamma_{S' \rightarrow \eta\eta} / \Gamma_{S' \rightarrow \pi\pi}$	2.4 ± 0.6	1.2
$\Gamma_{S' \rightarrow \eta\eta'} / \Gamma_{S' \rightarrow \pi\pi}$	< 0.18	3.12
$(\Gamma_{S'})_{2pseudoscalars}$	"large"	140.0 MeV

(7.28)

The first two ratios are acceptable and were previously included in the fit. We also used the extra constraint of $(\Gamma_{S'})_{2pseudoscalars} = (\Gamma_{S'})_{tot}$, which is in agreement with WA102, used to motivate such a choice.

On the other hand, the ratio $\Gamma_{S' \rightarrow \eta\eta} / \Gamma_{S' \rightarrow \pi\pi}$ is in complete disagreement with the upper limit. If this result of WA102 should be confirmed by future experiments, the described mixing scenario should be discarded.

- $a_0(1450)$ resonance:

We compare the ratios for a_0 and we give the full decay width in two pseudoscalars:

Quantity	Exp	Theory
$\Gamma_{a_0 \rightarrow \bar{K}K} / \Gamma_{a_0 \rightarrow \pi\eta}$	0.88 ± 0.23	0.86
$\Gamma_{a_0 \rightarrow \pi\eta'} / \Gamma_{a_0 \rightarrow \pi\eta}$	0.35 ± 0.16	0.64
$(\Gamma_{a_0})_{2pseudoscalars}$?	173 MeV

(7.29)

The absolute experimental decay width into two pseudoscalars is not known; the theoretical one is however acceptable.

- $K^*(1430)$ resonance:

Quantity	Exp	Theory
$\Gamma_{K^* \rightarrow \pi K}$	dominant	108 MeV
$\Gamma_{K^* \rightarrow K\eta} / \Gamma_{K^* \rightarrow \pi K}$?	0.021

(7.30)

The measured full width of this state is 294 ± 23 MeV. If the decay $\Gamma_{K^* \rightarrow \pi K}$ is effectively dominant, then we underestimate it by a factor 3. At this point we should note that a simultaneous description of K^* and a_0 has always been a problem. Theoretically one generally has [31, 123]:

$$(\Gamma_{a_0})_{2pseudoscalars} > (\Gamma_{K^*})_{2pseudoscalars} \quad (7.31)$$

seemingly in disaccord with the current experimental states.

- Two-photon decay ratios:

From the two-photon decay expressions derived in section 5.4, Eq. (5.54), using the results of the mixing matrix, we get the following ratios:

$$\Gamma_{N' \rightarrow 2\gamma} : \Gamma_{G' \rightarrow 2\gamma} : \Gamma_{S' \rightarrow 2\gamma} = 1 : 0.279 : 0.012. \quad (7.32)$$

The ratio $\Gamma_{G' \rightarrow 2\gamma} / \Gamma_{N' \rightarrow 2\gamma}$ is therefore of the same magnitude (even if smaller) of the result found in the previous chapter. The somewhat smaller ratio is compatible with the considerations presented in [?, 35].

However, the present results show a very small $\Gamma_{S' \rightarrow 2\gamma} / \Gamma_{N' \rightarrow 2\gamma}$ decay ratio, different from the previous results. The reason for this discrepancy is the opposite phase of the $\bar{\pi}n$ and $\bar{s}s$ components, which sensibly lowers the decay for the S' state. Only better experimental results can help us to pin down the flavor wave function of the physical resonance. With the present approach we cannot determine the full two-photon decay widths, because we do not know the constant g_{scalar} of equations (5.54). The ratios clearly do not depend on this unknown parameter.

7.3 Strong decays of the glueball

7.3.1 Lagrangian

Based on the qualitative arguments of [31, 40], we first assumed that the direct glueball decay is suppressed. Now we want to take explicitly into account the possibility that the glueball decays by itself; we have to include G in the decay Lagrangian. We can do it by considering that the glueball is a flavor singlet; we write down the following quantity:

$$\frac{\widehat{S}}{\sqrt{2}} \rightarrow \frac{\widehat{S}^\bullet}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(\sum_{a=0}^8 S_a \lambda_a + r_G G \lambda_0 \right) =$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} a_0^0 + \frac{1}{\sqrt{3}} S_0 + \frac{1}{\sqrt{6}} S_8 + \frac{r_G}{\sqrt{3}} G & a_0^+ & K^{*+} \\ a_0^- & -\frac{1}{\sqrt{2}} a_0^0 + \frac{1}{\sqrt{3}} S_0 + \frac{1}{\sqrt{6}} S_8 + \frac{r_G}{\sqrt{3}} G & K^{*0} \\ K^{*-} & \overline{K}^{*0} & \frac{1}{\sqrt{3}} S_0 - \frac{2}{\sqrt{6}} S_8 + \frac{r_G}{\sqrt{3}} G \end{pmatrix}.$$

The decay Lagrangian is obtained from the previous one when substituting \widehat{S} by \widehat{S}^\bullet :

$$\mathcal{L}_{decay}^\bullet = c_Q \text{Tr}[\widehat{S}^\bullet \widehat{P} \widehat{P}]. \quad (7.33)$$

The parameter r_G takes into account the strength of the glueball decay relative to the one of quarkonium; if $r_G = 0$ we are back to the previous case, where a non-decaying glueball was considered. If $r_G \ll 1$ the quarkonia components dominate the decay [31, 40]; if $r_G = 1$ the glueball decays with equal strength as quarkonium, if $r_G > 1$ the glueball dominates the decay. This last option has indeed been studied in [42], where the glueball strength is larger than the corresponding one of the quarkonia component.

In the end, we define c_G as

$$c_G = r_G \cdot c_Q, \quad (7.34)$$

which is the absolute glueball decay strength.

7.3.2 Expressions for the decays

The quadratic Lagrangian and the glueball-mixing expressed in eq. (7.5) does not change when including a direct glueball decay; however, we have to write down the decay expressions for the isoscalar states $|i\rangle$; the decay rates depend on the glueball amount B^{iG} as well.

Performing the calculation we find (see section 4.3):

$$\Gamma_{i \rightarrow \pi\pi} = 3 \frac{\frac{1}{2} \sqrt{M_i^2 - 4M_\pi^2}}{16\pi M_i^2} \left[4c_Q B^{iN} + 4\sqrt{\frac{2}{3}} B^{iG} c_G \right]^2, \quad (7.35)$$

$$\Gamma_{i \rightarrow \overline{K}K} = 4 \frac{\frac{1}{2} \sqrt{M_i^2 - 4M_K^2}}{16\pi M_i^2} \cdot \left[2c_Q \left(B^{iN} + \sqrt{2} B^{iS} \right) + 2B^{iG} c_G \left(\frac{\sqrt{2} + 1}{\sqrt{3}} \right) \right]^2, \quad (7.36)$$

$$\Gamma_{i \rightarrow \eta\eta} = \frac{\frac{1}{2} \sqrt{M_i^2 - 4M_\eta^2}}{16\pi M_i^2} \left[2c_Q \Lambda_{1\eta\eta} + 4\sqrt{\frac{2}{3}} B^{iG} c_G \right]^2, \quad (7.37)$$

$$\Gamma_{i \rightarrow \eta\eta'} = \frac{p_{i\eta\eta'}}{8\pi M_i^2} [c_Q \Lambda_{1\eta\eta'}]^2. \quad (7.38)$$

Note that $\Gamma_{i \rightarrow \eta\eta'}$ does not change; in fact, a flavor singlet like the glueball cannot decay into a singlet and an octet. The decays of a_0 and K^* are, of course, unchanged with respect to the previous case (they do not mix with the glueball).

7.3.3 χ^2 analysis

In the following we again perform a χ^2 analysis; the set-up is the same as in the previous section, where we simply have the additional parameter c_G (or r_G).

We then have

$$\chi^2 = \chi^2[M_N, M_G, M_S, f, \varepsilon, c_Q, c_G], \quad (7.39)$$

which is the same expression for χ^2 as in equation (7.24). Considering that the previous χ^2 analysis was good (with $\chi^2/N < 1$) we do not expect strong variations from this case.

In fact in the fit we get:

$$M_N = 1.414, M_G = 1.479, M_S = 1.691, f = 0.107, \varepsilon = 0.264, c_Q = 0.010, c_G = 0.00070; \\ \chi^2 = 8.864.$$

The ratio $r_G = c_G/c_Q = 0.067 \ll 1$ points to a "weak" decaying glueball. We expect results similar to the inert glueball case with a correspondent analogous discussion.

The χ^2 table looks like:

Quantity	Exp	Theory	χ_i^2
$M_{N'}$	$1350 \pm 150 \text{ MeV}$	1355 MeV	0.001
$M_{G'}$	$1507 \pm 5 \text{ MeV}$	1507 MeV	0.003
$M_{S'}$	$1714 \pm 5 \text{ MeV}$	1714 MeV	0.011
$\Gamma_{G' \rightarrow \pi\pi}$	$38.0 \pm 4.6 \text{ MeV}$	38.6 MeV	0.01
$\Gamma_{G' \rightarrow \bar{K}K}$	$9.4 \pm 1.7 \text{ MeV}$	10.4 MeV	0.40
$\Gamma_{G' \rightarrow \eta\eta}$	$5.6 \pm 1.4 \text{ MeV}$	2.46 MeV	5.63
$\Gamma_{S' \rightarrow \pi\pi} / \Gamma_{S' \rightarrow \bar{K}K}$	0.20 ± 0.06	0.206	0.01
$\Gamma_{S' \rightarrow \eta\eta} / \Gamma_{S' \rightarrow \bar{K}K}$	0.48 ± 0.15	0.228	2.80
$(\Gamma_{S'})_{2pseudoscars}$	$140 \pm 10 \text{ MeV}$	140.07 MeV	0.0001
χ_{tot}^2	-	-	8.864

The mixing matrix is:

$$\begin{pmatrix} N' \\ G' \\ S' \end{pmatrix} = \begin{pmatrix} 0.85 & 0.45 & 0.26 \\ -0.45 & 0.88 & -0.04 \\ -0.26 & -0.08 & 0.96 \end{pmatrix} \begin{pmatrix} N \\ G \\ S \end{pmatrix}. \quad (7.41)$$

$f_0(1370)$:

Quantity	Exp (WA102)	Theory
$\Gamma_{N' \rightarrow \bar{K}K} / \Gamma_{N' \rightarrow \pi\pi}$	0.46 ± 0.19	0.50
$\Gamma_{N' \rightarrow \eta\eta} / \Gamma_{N' \rightarrow \pi\pi}$	0.16 ± 0.07	0.11
$(\Gamma_{N'})_{2pseudoscalars}$	"small"	317.91 MeV

$f_0(1710)$:

Quantity	Exp (WA102)	Theory
$\Gamma_{S' \rightarrow \bar{K}K} / \Gamma_{S' \rightarrow \pi\pi}$	5.0 ± 0.7	4.85
$\Gamma_{S' \rightarrow \eta\eta} / \Gamma_{S' \rightarrow \pi\pi}$	2.4 ± 0.6	1.11
$\Gamma_{S' \rightarrow \eta\eta'} / \Gamma_{S' \rightarrow \pi\pi}$	< 0.18	3.01
$(\Gamma_{S'})_{2pseudoscalars}$	"large"	139.9 MeV

$a_0(1450)$:

Quantity	Exp	Theory	
$\Gamma_{a_0 \rightarrow \bar{K}K} / \Gamma_{a_0 \rightarrow \pi\eta}$	0.88 ± 0.23	0.86	(7.44)
$\Gamma_{a_0 \rightarrow \pi\eta'} / \Gamma_{a_0 \rightarrow \pi\eta}$	0.35 ± 0.16	0.64	
$(\Gamma_{a_0})_{2pseudoscalars}$?	174 MeV	

$K^*(1430)$:

Quantity	Exp	Theory	
$\Gamma_{K^* \rightarrow \pi K}$	dominant	109 MeV	(7.45)
$\Gamma_{K^* \rightarrow K\eta} / \Gamma_{K^* \rightarrow \pi K}$?	0.021	

For the two-photon decay we get:

$$\Gamma_{N' \rightarrow 2\gamma} : \Gamma_{G' \rightarrow 2\gamma} : \Gamma_{S' \rightarrow 2\gamma} = 1 : 0.354 : 0.011. \quad (7.46)$$

The discussion is analogous to the previous case. The critical points are $(\Gamma_{N'})_{2pseudoscalars}$ and $\Gamma_{S' \rightarrow \eta\eta} / \Gamma_{S' \rightarrow \pi\pi}$, which are in complete disagreement with the WA102 experiment. If the experimental results will be confirmed, one has to search for other possible solutions.

7.4 Chiral approach

7.4.1 Lagrangian

In a next step we intend to describe the decay of the scalar nonet into two pseudoscalar mesons within a chiral approach [28, 124, 125, 126, 127]. We first do not consider the possible direct glueball decay, thus concentrating on the quarkonia meson fields. The problem of scalar resonances and their possible decays was first analyzed in [125], to which we refer for a careful analysis of this issue. Here we intend to apply the approach developed in [125] to the scalar states within 1-2 GeV [127]. Although, as already stressed in the introduction of this chapter, the higher order loop corrections are expected to be large in this mass region, we intend to perform a tree-level evaluation of the scalar meson decays. We do not expect precise results, but a theoretical guide to discuss the decays. We also do not calculate the loop corrections, which in the light of the present experimental status, it would overcomplicate the issue, especially since new parameters enter at the one-loop level.

The effective Lagrangian to calculate the scalar decays at first order (no loops), involving the scalar and the pseudoscalar matrices \hat{S} and \hat{P} (see section 7.2.1) and in the large N_c limit, reads:

$$\begin{aligned} \mathcal{L}_{eff} = & \frac{c_Q^d}{\sqrt{2}F^2} Tr \left[\hat{S} \left(\partial_\mu \hat{P} \right) \left(\partial^\mu \hat{P} \right) \right] \\ & - \frac{c_Q^m}{\sqrt{2}F^2} \frac{B}{2} Tr \left[2\hat{S}\hat{P}\hat{M}\hat{P} + \hat{S}\hat{P}\hat{P}\hat{M} + \hat{S}\hat{M}\hat{P}\hat{P} \right], \end{aligned} \quad (7.47)$$

where:

- c_Q^d and c_Q^m are the two decay constants. In the flavor case we had just one parameter, here we get two. This in turn reflects the richer structure of the chiral Lagrangian: deviations from flavor symmetry can be described within this approach.

- The current mass matrix \widehat{M} given by

$$\widehat{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d = m_u & 0 \\ 0 & 0 & m_s \end{pmatrix} \quad (7.48)$$

- The parameter B links the current quark masses to the physical meson masses M_π and M_K [28]:

$$M_\pi^2 = 2Bm_u, \quad (7.49)$$

$$M_K^2 = B(m_u + m_s). \quad (7.50)$$

Note that the quantities Bm_u and Bm_s are determined from:

$$Bm_u = \frac{M_\pi^2}{2}, \quad (7.51)$$

$$Bm_s = M_K^2 - \frac{M_\pi^2}{2}. \quad (7.52)$$

Exploiting the last two relations we will be able to express the decay rates in terms of the known masses M_π and M_K , instead of B , m_u and m_s . This is possible (although on tree-level we are left with two parameters) because B always appears in combination either with m_u or with m_s .

- The parameter F ; at lowest order it is the pion decay constant. We use:

$$F = 92.4 \text{ MeV}. \quad (7.53)$$

At higher order one has a different F attached to each pseudoscalar mesons; here, performing a tree-level calculation, we always employ the same F .

- Within chiral perturbation theory one can calculate also the mass of the octet pseudoscalar state η_8 , which reads:

$$M_{\eta_8} = \frac{1}{3} (4M_K^2 - M_\pi^2). \quad (7.54)$$

But in nature we have mixing among the singlet η_0 and the octet η_8 . We can describe it at the field composite level by the mixing Lagrangian (see chapter 5):

$$L = \frac{1}{2} (\partial_\mu \eta_0)^2 - \frac{1}{2} M_{\eta_0}^2 \eta_0^2 + \frac{1}{2} (\partial_\mu \eta_8)^2 - \frac{1}{2} M_{\eta_8}^2 \eta_8^2 + z_{mix} \eta_0 \eta_8. \quad (7.55)$$

The parameters M_{η_0} and z_{mix} are unknown, but can be fixed by imposing that the physical rotated masses coincide with the experimental masses form $M_\eta = 0.54775 \text{ GeV}$ and $M_{\eta'} = 0.95778 \text{ GeV}$. One then finds $M_{\eta_0} = 0.94786$ and $z_{mix} = -0.106316$. As a consequence, following the discussion of chapter 5, we can deduce the corresponding mixing angle, finding $\theta_{ps} = -9.95^\circ$. In the following we will use -10° , as in the previous case.

This result, as explained in [121], is valid only at first order in chiral perturbation theory, while when one goes to higher orders a doubling of the pseudoscalar mixing angle is found. However, we will limit our study to a first order evaluation, and we will consistently employ the corresponding value of -10° .

7.4.2 Decay expressions

From (7.47) we can calculate the decay rates as in sections 4.2 and 7.2; one just has to take care of the derivatives. Every decay amplitude consists of two pieces, coming from the two parts of the effective Lagrangian: the first one proportional to c_Q^d and the second to c_Q^m .

The decay rates for $|i\rangle$ read:

$$\Gamma_{i \rightarrow \pi\pi} = 3 \frac{\frac{1}{2} \sqrt{M_i^2 - 4M_\pi^2}}{16\pi M_i^2} \left[\frac{4B^{iN}}{\sqrt{2}F^2} \left(\frac{M_i^2 - 2M_\pi^2}{2} c_Q^d + M_\pi^2 c_Q^m \right) \right]^2, \quad (7.56)$$

$$\Gamma_{i \rightarrow \bar{K}K} = 4 \frac{\frac{1}{2} \sqrt{M_i^2 - 4M_K^2}}{16\pi M_i^2} \left[\frac{2}{\sqrt{2}F^2} \left(B^{iN} + \sqrt{2}B^{iS} \right) \left(\frac{M_i^2 - 2M_K^2}{2} c_Q^d + M_K^2 c_Q^m \right) \right]^2. \quad (7.57)$$

For the $\eta\eta$ decay we have:

$$\Gamma_{i \rightarrow \eta\eta} = \frac{\frac{1}{2} \sqrt{M_i^2 - 4M_\eta^2}}{16\pi M_i^2} \left[-\frac{2c_Q^d \Lambda_{1\eta\eta}}{\sqrt{2}F^2} \frac{M_i^2 - 2M_\eta^2}{2} + \frac{2c_Q^m \Lambda_{2\eta\eta}}{\sqrt{2}F^2} \right]^2 \quad (7.58)$$

where $\Lambda_{1\eta\eta}$ is, as in section 7.2,

$$\begin{aligned} \Lambda_{1\eta\eta} &= \left(B^{iN} + \sqrt{2}B^{iS} \right) - \frac{1}{\sqrt{3}} \left(B^{iN} - \sqrt{2}B^{iS} \right) \cdot \cos(2\theta_{ps}) \\ &\quad - \frac{2}{3} \left(\sqrt{2}B^{iN} - 2B^{iS} \right) \cdot \sin(2\theta_{ps}), \end{aligned} \quad (7.59)$$

while $\Lambda_{2\eta\eta}$ is different and depends on M_π and M_K :

$$\begin{aligned} \Lambda_{2\eta\eta} &= -2 \left[\frac{M_\pi^2}{2} B^{iN} + \sqrt{2} \left(M_K^2 - \frac{M_\pi^2}{2} \right) \right] \\ &\quad + \frac{2}{3} \left[\frac{M_\pi^2}{2} B^{iN} - \sqrt{2} \left(M_K^2 - \frac{M_\pi^2}{2} \right) \right] \cdot \cos(2\theta_{ps}) \\ &\quad + \frac{4}{3} \left[\sqrt{2} \frac{M_\pi^2}{2} B^{iN} - 2 \left(M_K^2 - \frac{M_\pi^2}{2} \right) \right] \cdot \sin(2\theta_{ps}) \end{aligned} \quad (7.60)$$

For the $\eta\eta'$ decay we have:

$$\Gamma_{i \rightarrow \eta\eta'} = \frac{p_{i\eta\eta'}}{8\pi M_i^2} \left[-\frac{c_Q^d \Lambda_{1\eta\eta'}}{\sqrt{2}F^2} \frac{M_i^2 - M_\eta^2 - M_{\eta'}^2}{2} + \frac{c_Q^m \Lambda_{2\eta\eta'}}{\sqrt{2}F^2} \right]^2 \quad (7.61)$$

where

$$\Lambda_{1\eta\eta'} = \frac{2}{3} \left[2 \left(\sqrt{2}B^{iN} - 2B^{iS} \right) \cos(2\theta_{ps}) + \left(-B^{iN} + \sqrt{2}B^{iS} \right) \sin(2\theta_{ps}) \right]. \quad (7.62)$$

and

$$\begin{aligned} \Lambda_{2\eta\eta'} &= -\frac{8}{3} \left[\sqrt{2} \frac{M_\pi^2}{2} B^{iN} - 2 \left(M_K^2 - \frac{M_\pi^2}{2} \right) \right] \cdot \cos(2\theta_{ps}) \\ &\quad - \frac{4}{3} \left[-\frac{M_\pi^2}{2} B^{iN} + \sqrt{2} \left(M_K^2 - \frac{M_\pi^2}{2} \right) \right] \cdot \sin(2\theta_{ps}) \end{aligned} \quad (7.63)$$

Note that, again, the matrix elements B^{iG} do not appear in the decays because in the present treatment we suppose that direct glueball decays are suppressed.

The decay rates for $a_0(1450)$ are:

$$\Gamma_{a_0 \rightarrow \bar{K}K} = 2 \frac{\frac{1}{2} \sqrt{M_{a_0}^2 - 4M_K^2}}{8\pi M_{a_0}^2} \left[\frac{2}{\sqrt{2}F^2} \left(\frac{M_{a_0}^2 - 2M_K^2}{2} c_Q^d + M_K^2 c_Q^m \right) \right]^2, \quad (7.64)$$

$$\begin{aligned} \Gamma_{a_0 \rightarrow \pi\eta} &= \frac{p_{a_0\pi\eta}}{8\pi M_{a_0}^2} \left\{ \frac{4}{\sqrt{3}} \frac{c_Q^d}{\sqrt{2}F^2} \left(\cos(\theta_{ps}) - \sqrt{2} \sin(\theta_{ps}) \right) \frac{M_{a_0}^2 - M_\pi^2 - M_\eta^2}{2} \right. \\ &\quad \left. + \frac{4}{\sqrt{3}} \frac{c_Q^m}{\sqrt{2}F^2} \left(\cos(\theta_{ps}) - \sqrt{2} \sin(\theta_{ps}) \right) M_\pi^2 \right\}^2 \end{aligned} \quad (7.65)$$

$$\begin{aligned} \Gamma_{a_0 \rightarrow \pi\eta'} &= \frac{p_{a_0\pi\eta'}}{8\pi M_{a_0}^2} \left\{ \frac{4}{\sqrt{3}} \frac{c_Q^d}{\sqrt{2}F^2} \left(\sqrt{2} \cos(\theta_{ps}) + \sin(\theta_{ps}) \right) \frac{M_{a_0}^2 - M_\pi^2 - M_{\eta'}^2}{2} \right. \\ &\quad \left. + \frac{4}{\sqrt{3}} \frac{c_Q^d}{\sqrt{2}F^2} \left(\sqrt{2} \cos(\theta_{ps}) + \sin(\theta_{ps}) \right) M_\pi^2 \right\}^2 \end{aligned} \quad (7.66)$$

The decay rates for $K^*(1430)$ are:

$$\Gamma_{K^* \rightarrow \pi K} = 3 \frac{p_{K^*\pi K}}{8\pi M_{K^*}^2} \left[\frac{2c_Q^d}{\sqrt{2}F^2} \frac{M_{aK^*}^2 - M_\pi^2 - M_K^2}{2} + \frac{2c_Q^m}{\sqrt{2}F^2} \frac{M_\pi^2 + M_K^2}{2} \right]^2, \quad (7.67)$$

$$\begin{aligned} \Gamma_{K^* \rightarrow K\eta} &= \frac{p_{K^*K\eta}}{8\pi M_{K^*}^2} \left\{ \frac{c_Q^d}{\sqrt{2}F^2} \frac{2}{\sqrt{3}} \left(\cos(\theta_{ps}) + 2\sqrt{2} \sin(\theta_{ps}) \right) \frac{M_{K^*}^2 - M_K^2 - M_\eta^2}{2} \right. \\ &\quad \left. + \frac{2c_Q^m}{\sqrt{6}F^2} \left(\frac{\cos(\theta_{ps})}{2} \left(5 \left(M_K^2 - \frac{M_\pi^2}{2} \right) - \frac{M_\pi^2}{2} \right) + 2\sqrt{2} M_K^2 \sin(\theta_{ps}) \right) \right\}^2 \end{aligned} \quad (7.68)$$

7.4.3 χ^2 analysis

7.4.3.1 Step 1, flavor symmetry breaking

The χ^2 -fit depends now on 6 variables: the bare masses M_N , M_G , M_S , the mixing parameters f , ε , and the decay strengths c_Q^d , c_Q^m . Here we include the experimental values of $\Gamma_{a_0 \rightarrow \bar{K}K} / \Gamma_{a_0 \rightarrow \pi\eta}$ and $\Gamma_{a_0 \rightarrow \pi\eta'} / \Gamma_{a_0 \rightarrow \pi\eta}$. Note that these two ratios only depend on the ratio c_Q^m / c_Q^d , and not on all the others.

This fact suggests the following strategy: we first find the best ratio c_Q^m / c_Q^d to describe the two experimental values related to a_0 . We consider:

$$\begin{aligned} \chi_1^2 &= \chi_1^2[y = c_Q^m / c_Q^d] \\ &= \left[\frac{1}{0.16} \left(\frac{\Gamma_{a_0 \rightarrow \pi\eta'}}{\Gamma_{a_0 \rightarrow \pi\eta}} - 0.35 \right) \right]^2 + \left[\frac{1}{0.23} \left(\frac{\Gamma_{a_0 \rightarrow \bar{K}K}}{\Gamma_{a_0 \rightarrow \pi\eta}} - 0.88 \right) \right]^2. \end{aligned} \quad (7.69)$$

Minimizing χ_1^2 with respect to $y = c_Q^m / c_Q^d$, using $\theta_{ps} = -10^\circ$, we find

$$y = c_Q^m / c_Q^d = 0.41353. \quad (7.70)$$

With this value we have:

$$\frac{\Gamma_{a_0 \rightarrow \pi\eta'}}{\Gamma_{a_0 \rightarrow \pi\eta}} = 0.29; \quad (7.71)$$

$$\frac{\Gamma_{a_0 \rightarrow KK}}{\Gamma_{a_0 \rightarrow \pi\eta}} = 0.88. \quad (7.72)$$

compared to the experimental values of respectively 0.35 ± 0.16 and 0.88 ± 0.23 . This result for y , however, sensibly depends on the pseudoscalar mixing angle θ_{ps} . In the following we will still use $\theta_{ps} = -10^\circ$.

If one uses $\theta_{ps} = -21.8^\circ$ [121], would obtain $y = 1.19$ together with the decay ratios

$$\frac{\Gamma_{a_0 \rightarrow \pi\eta'}}{\Gamma_{a_0 \rightarrow \pi\eta}} = 0.13; \quad (7.73)$$

$$\frac{\Gamma_{a_0 \rightarrow KK}}{\Gamma_{a_0 \rightarrow \pi\eta}} = 0.88. \quad (7.74)$$

7.4.3.2 Step 2

We now consider (7.24), which in this case depends on the full set $M_N, M_G, M_S, f, \varepsilon, c_Q^d, c_Q^m$.

However, following the previous subsection, we can write $c_Q^m = 0.41353 \cdot c_Q^d$. We therefore have one parameter less. We perform the usual χ^2 analysis, following the prescriptions and the discussions of section 7.2. We find a minimum for:

$$M_N = 1.442, M_G = 1.485, M_S = 1.696, f = 0.080, \varepsilon = 0.223, c_Q^d = 0.00817; \chi^2 = 10.57.$$

Although the χ^2 is a slightly worse than in the flavor case, we should not forget that we improved the ratios for a_0 and that this approach can in principle be improved by going to higher orders.

The full table is:

Quantity	Exp	Theory	χ_i^2
$M_{N'}$	$1350 \pm 150 \text{ MeV}$	1400 MeV	0.1
$M_{G'}$	$1507 \pm 5 \text{ MeV}$	1507 MeV	~ 0
$M_{S'}$	$1714 \pm 5 \text{ MeV}$	1714 MeV	~ 0
$\Gamma_{G' \rightarrow \pi\pi}$	$38.0 \pm 4.6 \text{ MeV}$	38.21 MeV	~ 0
$\Gamma_{G' \rightarrow \bar{K}K}$	$9.4 \pm 1.7 \text{ MeV}$	10.4 MeV	0.35
$\Gamma_{G' \rightarrow \eta\eta}$	$5.6 \pm 1.4 \text{ MeV}$	1.83 MeV	7.9
$\Gamma_{S' \rightarrow \pi\pi} / \Gamma_{S' \rightarrow \bar{K}K}$	0.20 ± 0.06	0.205	0.01
$\Gamma_{S' \rightarrow \eta\eta} / \Gamma_{S' \rightarrow \bar{K}K}$	0.48 ± 0.15	0.257	2.21
$(\Gamma_{S'})_{2pseudoscalars}$	$140 \pm 10 \text{ MeV}$	139.94 MeV	~ 0
χ_{tot}^2	-	-	10.56

The mixing matrix is:

$$\begin{pmatrix} N' \\ G' \\ S' \end{pmatrix} = \begin{pmatrix} 0.85 & 0.46 & 0.25 \\ -0.46 & 0.88 & -0.05 \\ -0.24 & -0.07 & 0.97 \end{pmatrix} \begin{pmatrix} N \\ G \\ S \end{pmatrix}. \quad (7.76)$$

$f_0(1370)$:

Quantity	Exp (WA102)	Theory	
$\Gamma_{N' \rightarrow \bar{K}K} / \Gamma_{N' \rightarrow \pi\pi}$	0.46 ± 0.19	0.36	(7.77)
$\Gamma_{N' \rightarrow \eta\eta} / \Gamma_{N' \rightarrow \pi\pi}$	0.16 ± 0.07	0.06	
$(\Gamma_{N'})_{2pseudoscalars}$	"small"	147 MeV	

$f_0(1710)$:

Quantity	Exp (WA102)	Theory	
$\Gamma_{S' \rightarrow \bar{K}K} / \Gamma_{S' \rightarrow \pi\pi}$	5.0 ± 0.7	4.86	(7.78)
$\Gamma_{S' \rightarrow \eta\eta} / \Gamma_{S' \rightarrow \pi\pi}$	2.4 ± 0.6	1.24	
$\Gamma_{S' \rightarrow \eta\eta'} / \Gamma_{S' \rightarrow \pi\pi}$	< 0.18	1.69	
$(\Gamma_{S'})_{2pseudoscalars}$	"large"	139.9 MeV	

$a_0(1450)$:

Quantity	Exp	Theory	
$\Gamma_{a_0 \rightarrow \bar{K}K} / \Gamma_{a_0 \rightarrow \pi\eta}$	0.88 ± 0.23	0.88	(7.79)
$\Gamma_{a_0 \rightarrow \pi\eta'} / \Gamma_{a_0 \rightarrow \pi\eta}$	0.35 ± 0.16	0.29	
$(\Gamma_{a_0})_{2pseudoscalars}$?	80.38 MeV	

$K^*(1430)$:

Quantity	Exp	Theory	
$\Gamma_{K^* \rightarrow \pi K}$	dominant	56.65 MeV	(7.80)
$\Gamma_{K^* \rightarrow K\eta} / \Gamma_{K^* \rightarrow \pi K}$?	0.058	

- Two-photon decay ratios

For the two-photon decay we get:

$$\Gamma_{N' \rightarrow 2\gamma} : \Gamma_{G' \rightarrow 2\gamma} : \Gamma_{S' \rightarrow 2\gamma} = 1 : 0.340 : 0.005. \quad (7.81)$$

7.4.4 Comments

The pattern emerging from this fit is qualitative similar to the flavor case, but we also have some differences.

$(\Gamma_{N'})_{2pseudoscalars}$ is smaller (by a factor 3), and this improves the situation, even if is still to large when compared to WA103. $\Gamma_{S' \rightarrow \eta\eta'} / \Gamma_{S' \rightarrow \pi\pi}$ is now not as large as before, even if the tendency is the same. The ratios for a_0 are of course improved, but the full width $(\Gamma_{a_0})_{2pseudoscalars}$ is decreased by a factor of 2. As a consequence also $\Gamma_{K^* \rightarrow \pi K}$ is small, in disagreement with the experimental result [12].

The crucial points are still $(\Gamma_{N'})_{2pseudoscalars}$ and $\Gamma_{S' \rightarrow \eta\eta'} / \Gamma_{S' \rightarrow \pi\pi}$; the acceptance of the model with these parameters depends on future experiments, which will measure these quantities.

For what concerns the two-photon decay we find the same trend of the flavor case. The destructive interference of the u,d and s components of the resonance $f_0(1710)$ lower the corresponding two-photon decay rate for this state.

7.5 Glueball in the chiral approach

7.5.1 Lagrangian

We now include the direct glueball decay in the chiral formalism. The glueball is considered as a flavor singlet mesonic field G , whose decays will be described by two constants c_G^d and c_G^m :

$$\begin{aligned} \mathcal{L}_{eff+glueball} = & \frac{c_Q^d}{\sqrt{2}F^2} Tr \left[\widehat{S} \left(\partial_\mu \widehat{P} \right) \left(\partial^\mu \widehat{P} \right) \right] \\ & - \frac{c_Q^m}{\sqrt{2}F^2} \frac{B}{2} Tr \left[2\widehat{S}\widehat{P}\widehat{M}\widehat{P} + \widehat{S}\widehat{P}\widehat{P}\widehat{M} + \widehat{S}\widehat{M}\widehat{P}\widehat{P} \right] \\ & + \frac{c_G^d}{\sqrt{2}F^2} Tr \left[G\lambda_0 \left(\partial_\mu \widehat{P} \right) \left(\partial^\mu \widehat{P} \right) \right] \\ & - \frac{c_G^m}{\sqrt{2}F^2} \frac{B}{2} Tr \left[G\lambda_0 \left(2\widehat{P}\widehat{M}\widehat{P} + \widehat{P}\widehat{P}\widehat{M} + \widehat{M}\widehat{P}\widehat{P} \right) \right] \end{aligned} \quad (7.82)$$

We have two parameters more, the two constants c_G^d and c_G^m ; when they are zero, we are back to the previous case. The chiral approach cannot tell us anything about their value.

7.5.2 Expressions for the decays

Instead of writing down explicitly the expressions, which in this case are pretty lengthy, we write them in a symbolic form. Let us consider a generic matrix element deduced from \mathcal{L}_{eff} (see equation (7.47)), which we denote like $M_{ip_1p_2}$, where "i" refers to a scalar-isoscalar state and "p₁" and "p₂" refer to two pseudoscalar mesons; $M_{ip_1p_2}$ depends on B^{iN} , B^{iS} , c_Q^d and c_Q^m ; explicitly:

$$M_{ip_1p_2} = M_{ip_1p_2}[B^{iN}, B^{iS}, c_Q^d, c_Q^m]. \quad (7.83)$$

The modified matrix element, originating from (7.82), is given by:

$$M_{ip_1p_2}^{glueball} = M_{ip_1p_2}[B^{iN}, B^{iS}, c_Q^d, c_Q^m] + M_{ip_1p_2}[\sqrt{\frac{2}{3}}B^{iG}, \sqrt{\frac{1}{3}}B^{iG}, c_G^d, c_G^m]. \quad (7.84)$$

In this way we can calculate explicitly the decay rates. For instance, the decay of $|i\rangle$ into two pions reads:

$$\begin{aligned} \Gamma_{i \rightarrow \pi\pi} = & 3 \frac{\frac{1}{2}\sqrt{M_i^2 - 4M_\pi^2}}{16\pi M_i^2} \left\{ \frac{4B^{iN}}{\sqrt{2}F^2} \left(\frac{M_i^2 - 2M_\pi^2}{2} c_Q^d + M_\pi^2 c_Q^m \right) \right. \\ & \left. + \frac{4}{\sqrt{2}F^2} \sqrt{\frac{2}{3}} \left(\frac{M_i^2 - 2M_\pi^2}{2} c_G^d + M_\pi^2 c_G^m \right) \right\}^2 \end{aligned} \quad (7.85)$$

The full expressions for the decay rates are reported in Appendix C.

7.5.3 χ^2 analysis

We still use the result of the flavor symmetry breaking fit (step 1) of the previous subsection, thus having $y = c_Q^m/c_Q^d = 0.41353$.

We now perform a χ^2 -fit with the following parameters: $M_N, M_G, M_S, f, \varepsilon, c_Q^d, c_G^d, c_G^m$. A minimum is obtained for the following values of the parameters:

$$M_N = 1.403, M_G = 1.467, M_S = 1.92, f = 0.13, \varepsilon = 0.25, c_Q^d = 0.0082, c_G^d = 0.00072, \\ c_G^m = -0.0023; \chi^2 = 10.56.$$

The full table is:

Quantity	Exp	Theory	χ_i^2
$M_{N'}$	$1350 \pm 150 \text{ MeV}$	1331 MeV	0.01
$M_{G'}$	$1507 \pm 5 \text{ MeV}$	1507 MeV	~ 0
$M_{S'}$	$1714 \pm 5 \text{ MeV}$	1714 MeV	~ 0
$\Gamma_{G' \rightarrow \pi\pi}$	$38.0 \pm 4.6 \text{ MeV}$	38.05 MeV	~ 0
$\Gamma_{G' \rightarrow \bar{K}K}$	$9.4 \pm 1.7 \text{ MeV}$	10.64 MeV	0.56
$\Gamma_{G' \rightarrow \eta\eta}$	$5.6 \pm 1.4 \text{ MeV}$	1.86 MeV	7.75
$\Gamma_{S' \rightarrow \pi\pi} / \Gamma_{S' \rightarrow \bar{K}K}$	0.20 ± 0.06	0.197	0.001
$\Gamma_{S' \rightarrow \eta\eta} / \Gamma_{S' \rightarrow \bar{K}K}$	0.48 ± 0.15	0.34	2.22
$(\Gamma_{S'})_{2\text{pseudoscalars}}$	$140 \pm 10 \text{ MeV}$	139.94 MeV	~ 0
χ_{tot}^2	-	-	10.56

The mixing matrix is:

$$\begin{pmatrix} N' \\ G' \\ S' \end{pmatrix} = \begin{pmatrix} 0.82 & 0.51 & 0.26 \\ -0.52 & 0.85 & -0.03 \\ -0.23 & -0.11 & 0.97 \end{pmatrix} \begin{pmatrix} N \\ G \\ S \end{pmatrix}. \quad (7.87)$$

$f_0(1370)$:

Quantity	Exp (WA102)	Theory
$\Gamma_{N' \rightarrow \bar{K}K} / \Gamma_{N' \rightarrow \pi\pi}$	0.46 ± 0.19	0.31
$\Gamma_{N' \rightarrow \eta\eta} / \Gamma_{N' \rightarrow \pi\pi}$	0.16 ± 0.07	0.05
$(\Gamma_{N'})_{2\text{pseudoscalars}}$	"small"	122.3 MeV

$f_0(1710)$:

Quantity	Exp (WA102)	Theory
$\Gamma_{S' \rightarrow \bar{K}K} / \Gamma_{S' \rightarrow \pi\pi}$	5.0 ± 0.7	5.06
$\Gamma_{S' \rightarrow \eta\eta} / \Gamma_{S' \rightarrow \pi\pi}$	2.4 ± 0.6	1.30
$\Gamma_{S' \rightarrow \eta\eta'} / \Gamma_{S' \rightarrow \pi\pi}$	< 0.18	1.75
$(\Gamma_{S'})_{2\text{pseudoscalars}}$	"large"	140.1 MeV

$a_0(1450)$:

Quantity	Exp	Theory
$\Gamma_{a_0 \rightarrow \bar{K}K} / \Gamma_{a_0 \rightarrow \pi\eta}$	0.88 ± 0.23	0.88
$\Gamma_{a_0 \rightarrow \pi\eta'} / \Gamma_{a_0 \rightarrow \pi\eta}$	0.35 ± 0.16	0.29
$(\Gamma_{a_0})_{2\text{pseudoscalars}}$?	80.31 MeV

$K^*(1430)$:

Quantity	Exp	Theory
$\Gamma_{K^* \rightarrow \pi K}$	dominant	56.60 MeV
$\Gamma_{K^* \rightarrow K\eta} / \Gamma_{K^* \rightarrow \pi K}$?	0.058

For the two-photon decay ratios we find:

$$\Gamma_{N' \rightarrow 2\gamma} : \Gamma_{G' \rightarrow 2\gamma} : \Gamma_{S' \rightarrow 2\gamma} = 1 : 0.528 : 0.004. \quad (7.92)$$

7.5.4 Comments

The pattern didn't change drastically when including the glueball. Note the stability of the mixing matrix. In the four analyzed cases it is practically unchanged, and similar to the analysis of the previous chapter.

There is a converging of results for the mixing matrix, even if different methods are used, even if there are differences in the parameter choice. In the sixth chapter we had no flavor mixing; in the present one, by including it, we still find similar results. The difference between the present mixing matrix and the one of chapter 6 is due to the flavor mixing, introduced in the present analysis.

Of course, there are indeed some differences, as the two-photon decay ratios show; but such differences strongly depend on the precise values of the mixing matrix. Improvement is needed to extract a precise and reliable mixing matrix.

The chiral approach is suitable for future improvements, as soon as better experimental and lattice results will be available. It can be a successful tool to explain intuitively the dynamics of the elusive scalar mesons.

Chapter 8

Conclusion

The main subject of this thesis has been the investigation of scalar glueball-quarkonia mixing, the corresponding two-photon and two-pseudoscalar meson decays related to the scalar meson resonances between 1 and 2 GeV .

In the first chapter we listed the present status of the glueball search, discussing various possible assignments. The search for glueballs or for mixed states with a glueball component in their wave function is an active subject in hadronic physics, both on experimental and theoretical sides. In fact, the existence of the glueball would confirm on the low-energy side a basic peculiarities of QCD: the strong interaction among gluons.

Lattice QCD indeed predicts the existence of glueballs, whose lightest state is scalar and with a mass between 1.4-1.8 GeV [10]. Mixing with nearby quarkonia mesons is likely. Among all the possible interpretations of the scalar mesons below 2 GeV we described in detail the original assignment of [31], which according to my personal opinion is still the most solid. We therefore used the phenomenological assignment of [31] as a starting point of our theoretical analysis.

We studied the scalar glueball and its mixing with quarkonia mesons within a covariant nonlocal quantum field theoretical approach, which has been developed in the second and in the third chapters. In a first step, we introduced the model for scalar fields; in this simple theoretical framework, after a brief introduction of quantum field theory, we studied the main features of bound state formation, the non-relativistic limit and the connection of our approach to a Bethe-Salpeter analysis. The so-called compositeness condition, used throughout the whole thesis, has been introduced for scalar fields and carefully analyzed in this context.

We subsequently extended the study to bound states of fermions; in particular, we concentrated on pseudoscalar and scalar bound states of two fermionic fields by using nonlocal currents.

The study of decays is a crucial point to compare theoretical predictions to experimental results; the decay of mesonic resonances gives us information about their inner structure. In particular, the two-photon decay is sensitive to the quark-flavor content of the state.

The two-photon transition of scalar and pseudoscalar bound states in the context of the nonlocal model has been developed in the fourth chapter. The issue of local $U(1)$ gauge invariance has been carefully discussed. In this context we also analyzed the two-photon decay of positronium states, which allows to test the non-relativistic limit of our approach.

We then turned to mixing, first introduced with a classical mechanics example, and then

carried out in the framework of a Klein-Gordon theory. Mixing of fields is a physical phenomenon, which one often encounters in modern physics. We concentrated on the basic peculiarities of mixing, such as physical and bare fields and the mixing matrix, and we already introduced the decay into two photons of mixed state.

The analysis of mixing of bound states in the covariant nonlocal model has been carried out in the sixth chapter; the bare glueball and the bare quarkonia states, respectively described by nonlocal currents of two constituent gluons and quarks, are allowed to mix. The physical fields are then a mixture of the bare ones; however, in contrast to the Klein-Gordon theory, an orthogonal mixing matrix connecting the physical to bare fields cannot be rigorously defined. We proposed a possible definition of a mixing matrix, which satisfies the correct requirements when compared to the Klein-Gordon and to the Quantum Mechanics limits.

In the mixing configuration of chapter six we do not take into account the flavor mixing between $|\bar{n}n\rangle$ and $|\bar{s}s\rangle$; the mixing is solely driven by a flavor-blind glueball.

Our results of chapter six infer a mixing matrix in agreement with the results of [31, 42, 40] (see section 6.8 and Refs. [74, 104])

$$\begin{pmatrix} |f_0(1370)\rangle = |N'\rangle \\ |f_0(1500)\rangle = |G'\rangle \\ |f_0(1710)\rangle = |S'\rangle \end{pmatrix} = \begin{pmatrix} 0.80 & 0.59 & 0.10 \\ -0.60 & 0.76 & 0.26 \\ 0.08 & -0.27 & 0.96 \end{pmatrix} \begin{pmatrix} |\bar{n}n\rangle = |N\rangle \\ |gg\rangle = |G\rangle \\ |\bar{s}s\rangle = |S\rangle \end{pmatrix}.$$

Although the mixing matrix is not orthogonal, only small deviations from the orthogonal limit are found: $M \cdot M^t \simeq 1$.

The resonance $f_0(1370)$ is dominantly composed by the light flavors u and d, $f_0(1500)$ contains the largest gluonic amount and $f_0(1710)$ is mostly made up of strange quarks.

The results for the two-photon decays of mixed states are below the current experimental upper limits [12] with

$$\Gamma_{N' \rightarrow 2\gamma} = 0.453 \text{ keV}, \quad \Gamma_{G' \rightarrow 2\gamma} = 0.273 \text{ KeV}, \quad \Gamma_{S' \rightarrow 2\gamma} = 0.125 \text{ keV}.$$

An experimental improvement is desired to test the presented solution. However, the ratios $\Gamma_{N' \rightarrow 2\gamma}/\Gamma_{S' \rightarrow 2\gamma} = 3.62$, $\Gamma_{G' \rightarrow 2\gamma}/\Gamma_{S' \rightarrow 2\gamma} = 2.19$ are in agreement with the phenomenological study of [42].

Furthermore, we compared the Klein-Gordon results to our approach; the two mixing parameters f and r introduced in section 6.7.3 become functions of the four-momentum p of the bound states, and are reported in Figs 6.6 and 6.7. Even if we started from a flavor blind mixing configuration (corresponding to $r = 1$ in the Klein-Gordon approach) we find small deviations of flavor blindness in accord with Lattice QCD.

The results of the mixing matrix and of the two-photon decay rates for variation of the parameters (such as the cut-off Λ) change only within a few percent, as discussed in Appendix B.

In the seventh chapter we studied the strong decays of the scalar mesons between 1-2 GeV with a phenomenological Lagrangian based on flavor symmetry and inspired by chiral-perturbation-theory.

In this chapter we also included a possible flavor mixing between $|\bar{n}n\rangle$ and $|\bar{s}s\rangle$, and we performed a χ^2 -analysis to obtain the best description of data listed in [12]. We considered separately the cases of an inert and of a decaying glueball. However, when including the direct

glueball decays, the results point to a suppressed strength in comparison to the quarkonia decay strength.

The mixing matrix (from section 7.5.3) found in the seventh chapter shows the following trend (see also [127]):

$$\begin{pmatrix} |f_0(1370)\rangle = |N'\rangle \\ |f_0(1500)\rangle = |G'\rangle \\ |f_0(1710)\rangle = |S'\rangle \end{pmatrix} = \begin{pmatrix} 0.82 & 0.51 & 0.26 \\ -0.52 & 0.85 & -0.03 \\ -0.23 & -0.11 & 0.97 \end{pmatrix} \begin{pmatrix} |\bar{n}n\rangle = |N\rangle \\ |gg\rangle = |G\rangle \\ |\bar{s}s\rangle = |S\rangle \end{pmatrix}.$$

Some differences with respect to the previous case appear, mostly caused by the flavor mixing, but the general trend is confirmed.

The consequence of the mixing scenario presented in the last chapter points to some results, such as a large ratio $\Gamma_{S' \rightarrow \eta\eta'} / \Gamma_{S' \rightarrow \pi\pi}$ (> 1 in the four cases of chapter seven), which now is not in agreement with the experimental result of WA1202 collaboration (< 0.18). Further experimental results concerning the strong decays are also desirable to accept or reject the proposed mixing scenario.

A peculiar difference encountered in chapter seven is the enhanced ratio $\Gamma_{N' \rightarrow 2\gamma} / \Gamma_{S' \rightarrow 2\gamma}$, which stems from a small $\Gamma_{S' \rightarrow 2\gamma}$; the reason for it is a destructive interference of the $\bar{n}n$ and $\bar{s}s$ components.

The glueball still constitutes a missing link in the hadronic world; at the present state of our knowledge we still cannot make any final statement on the real nature of glueballs; further studies are needed to find or to rule out its existence. In this work we presented and discussed, within phenomenological QFT methods, possible solutions of the glueball puzzle, which are in agreement with the present averages of [12].

Further work, both theoretically and experimentally, is needed to clarify this central problem of hadronic physics.

Appendix A

The two-field reduced problem

In Section 6.7.3 we introduced the mixing matrix M (see eq. (6.116)) and then in Section 6.7.4 alternatively M' (see eq. (6.124)). For both mixing matrices, as indicated in Section 5.8, we find similar results. To demonstrate the close connection between M and M' we consider the reduced problem for the mixing of two fields, G and S , thus leaving out the field N from the discussion. In this case we have the rotated fields G' and S' only, and for G' we have $|G'\rangle = M^{G'G} |G\rangle + M^{G'S} |S\rangle$. K and Σ are now 2x2 matrices with:

$$K = \begin{pmatrix} K_G & K_{SG} \\ K_{SG} & K_S \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_G & 0 \\ 0 & \Sigma_S \end{pmatrix}. \quad (\text{A.1})$$

Then we write the full expression for the matrix T^{-1} :

$$T^{-1} = -(K^{-1} - \Sigma) = \begin{pmatrix} -\frac{K_S}{\text{Det}[K]} + \Sigma_G & \frac{K_{SG}}{\text{Det}[K]} \\ \frac{K_{SG}}{\text{Det}[K]} & -\frac{K_G}{\text{Det}[K]} + \Sigma_S \end{pmatrix}, \quad (\text{A.2})$$

from which we get

$$T = \frac{1}{\text{Det}[T^{-1}]} \begin{pmatrix} -\frac{K_G}{\text{Det}[K]} + \Sigma_S & -\frac{K_{SG}}{\text{Det}[K]} \\ -\frac{K_{SG}}{\text{Det}[K]} & -\frac{K_S}{\text{Det}[K]} + \Sigma_G \end{pmatrix}. \quad (\text{A.3})$$

The coupling constant $g_{G'}^{gg}$ (6.104) is then explicitly given as

$$\begin{aligned} g_{G'}^{gg} &= \lim_{p^2 \rightarrow M_{G'}^2} \sqrt{(p^2 - M_{G'}^2) T^{gg,gg}} = \lim_{p^2 \rightarrow M_{G'}^2} \sqrt{\frac{(p^2 - M_{G'}^2)}{\text{Det}[T^{-1}]} \left(-\frac{K_G}{\text{Det}[K]} + \Sigma_S \right)} = \\ &= \left(\sqrt{\left(\frac{\partial \text{Det}[T^{-1}]}{\partial p^2} \right)^{-1} \left(-\frac{K_G}{\text{Det}[K]} + \Sigma_S \right)} \right)_{p^2=M_{G'}^2}. \end{aligned} \quad (\text{A.4})$$

The expressions for the other coupling constants follow from (6.104). Similarly, the explicit expression for $M^{G'G}$ is

$$\begin{aligned} M^{G'G} &= \sqrt{g_{G'}^{gg} \left(\frac{\partial \Sigma_G}{\partial p^2} \right)_{p^2=M_{G'}^2}} = \\ &= \left(\sqrt{\left(\frac{\partial \text{Det}[T^{-1}]}{\partial p^2} \right)^{-1} \left(-\frac{K_G}{\text{Det}[K]} + \Sigma_S \right) \left(\frac{\partial \Sigma_G}{\partial p^2} \right)} \right)_{p^2=M_{G'}^2}. \end{aligned} \quad (\text{A.5})$$

One has then similar relations for the other elements.

First we show that the components of the state $|G'\rangle = M^{G'G}|G\rangle + M^{G'S}|S\rangle$ are correctly normalized. Using (A.5) and (A.3) we have

$$\begin{aligned} 1 &= (M^{G'G})^2 + (M^{G'S})^2 = \\ &= \left[\left(\frac{\partial \text{Det}[T^{-1}]}{\partial p^2} \right)^{-1} \left(\left(-\frac{K_G}{\text{Det}[K]} + \Sigma_S \right) \frac{\partial \Sigma_G}{\partial p^2} + \left(-\frac{K_S}{\text{Det}[K]} + \Sigma_G \right) \frac{\partial \Sigma_S}{\partial p^2} \right) \right]_{p^2=M_{G'}^2} \end{aligned} \quad (\text{A.6})$$

We come now to the equivalence of M and M' . Let us consider the ratio $\xi = |M^{G'G}/M^{G'S}|$, which can be calculated explicitly from the basic definition (6.116) and from (A.3) :

$$\xi = \left| \frac{M^{G'G}}{M^{G'S}} \right| = \left(\sqrt{\frac{-\frac{K_G}{\text{Det}[K]} + \Sigma_S}{-\frac{K_S}{\text{Det}[K]} + \Sigma_G} \left(\frac{\partial \Sigma_G / \partial p^2}{\partial \Sigma_S / \partial p^2} \right)} \right)_{p^2=M_{G'}^2} = \quad (\text{A.7})$$

$$= \left(\left(\frac{-K_G + \text{Det}[K] \cdot \Sigma_S}{K_{SG}} \right) \sqrt{\frac{\partial \Sigma_G / \partial p^2}{\partial \Sigma_S / \partial p^2}} \right)_{p^2=M_{G'}^2}, \quad (\text{A.8})$$

where the last term has been obtained making use of the equation $\text{Det}[T^{-1}] = 0$, which holds for $p^2 = M_{G'}^2$.

When we consider the analogous ratio calculated from the elements of M' (6.116) (in this case we just have the running parameter $f(p^2)$ from equation (6.122) and not $r(p^2)$) we find:

$$\xi' = \left| \frac{M'^{G'G}}{M'^{G'S}} \right| = \frac{M_S^2 - M_{G'}^2}{K_{SG}} \sqrt{\left(\frac{\eta_G(p^2)}{\Sigma_G(p^2)} \frac{\eta_S(p^2)}{\Sigma_S(p^2)} \right)_{p^2=M_{G'}^2}}. \quad (\text{A.9})$$

We want to prove that $\xi \simeq \xi'$ (i.e. $M \simeq M'$) in the weak mixing limit (K_{SG} small) and in the $f(p^2) = \text{const}$ limit, for which $M' = M_{K-G}$.

A small mixing strength K_{SG} implies that we can neglect the term proportional to K_{SG}^2 in the determinant of equation (A.8). Introducing the quantity $\eta_S(p^2)$ of Eq. (6.120) we find:

$$\xi \simeq \frac{K_G(M_S^2 - M_{G'}^2)\eta_S(p^2)}{K_{SG}} \sqrt{\left(\frac{\partial \Sigma_G(p^2) / \partial p^2}{\partial \Sigma_S(p^2) / \partial p^2} \right)_{p^2=M_{G'}^2}}. \quad (\text{A.10})$$

For $p^2 \simeq M_G^2$ the following approximation is valid:

$$\frac{\eta_G(p^2)}{\Sigma_G(p^2)} \simeq K_G^2 \Sigma'_G(p^2). \quad (\text{A.11})$$

In fact, when K_{SG} is small, $M_{G'}^2$ is close to M_G^2 . This allows us to write

$$K_G \sqrt{\Sigma'_G(p^2 = M_{G'}^2)} \simeq \sqrt{\left(\frac{\eta_G(p^2)}{\Sigma_G(p^2)} \right)_{p^2=M_{G'}^2}}. \quad (\text{A.12})$$

Let us now consider the second limit, for which $f(p^2) = \text{const}$ in the region of interest, i.e. between $M_{G'}^2$ and M_S^2 , in the 2 field case, and between $M_{N'}^2$ and M_S^2 , in the 3 field case. The condition $f(p^2) = \text{const}$ is satisfied if $\Sigma_a(p^2)/\eta_a(p^2) = c_a$, where c_a is a constant for

$a = G, S$. In this case one has $M' = M_{K-G}$, which is orthogonal. We are then considering the orthogonal limit for M' . The condition $\Sigma_a(p^2)/\eta_a(p^2) = c_a$ for the case $a = S$ implies the following form for $\Sigma_S(p^2)$:

$$\Sigma_S(p^2) = \frac{c_S}{(p^2 - M_S^2) + c_S K_S}. \quad (\text{A.13})$$

This is of course an approximate form for $\Sigma_S(p^2)$ valid in the limit of a constant c_S . Note that the condition $\Sigma_S(p^2 = M_S^2) - 1/K_S = 0$ is fulfilled. With this form for $\Sigma_S(p^2)$ we have

$$\frac{\eta_S(p^2)}{\sqrt{\partial \Sigma_S(p^2)/\partial p^2}} = \sqrt{\frac{\eta_S(p^2)}{\Sigma_S(p^2)}} = \frac{1}{\sqrt{c_S}} \quad (\text{A.14})$$

which is valid in the interval where (A.13) is valid.

Plugging the approximations (A.12) and (A.14) in (A.8) we have indeed

$$\xi = \xi' = \frac{M_S^2 - M_{G'}^2}{K_{SG}} \sqrt{\left(\frac{\eta_G(p^2)}{\Sigma_G(p^2)} \frac{\eta_S(p^2)}{\Sigma_S(p^2)} \right)_{p^2=M_{G'}^2}} = \frac{M_S^2 - M_{G'}^2}{K_{SG} \sqrt{c_S c_G}}, \quad (\text{A.15})$$

thus having similar results for M and $M' = M_{K-G}$ for the used approximations. As shown in section 5.8, in the three field mixing case, similar matrices are found. This is then true for all the parameters studied in our work and for both propagator choices.

Note that the two conditions discussed in this appendix are satisfied: K_{SG} is small (corresponding to a small difference $M_{S'}^2 - M_S^2$) and the function $f(p^2)$ is almost constant in the momentum interval between $M_{N'}^2$ and $M_{S'}^2$ (see Fig. 6.6 and Fig. 6.7).

Appendix B

Results for parameter variation

To explicitly indicate the dependence on the cut-off value of the vertex function here we summarize our results for $\Lambda = 2 \text{ GeV}$ with the fit values $\mu_s = 0.985 \text{ GeV}$, $K_{SG} = 0.122 \text{ GeV}^{-1}$. The masses of physical $N' \equiv f_0(1370)$ and bare S state are $M_{N'} = 1.287 \text{ GeV}$ and $M_S = 1.696 \text{ GeV}$.

Mixing matrix:

$$M = \begin{pmatrix} 0.79 & 0.61 & 0.11 \\ -0.61 & 0.74 & 0.27 \\ 0.08 & -0.28 & 0.95 \end{pmatrix}. \quad (\text{B.1})$$

Set of coupling constants:

$$\begin{pmatrix} g_{N'}^{\bar{n}n} : g_{G'}^{\bar{n}n} : g_{S'}^{\bar{n}n} = 5.56 : -3.65 : 0.35 \\ g_{N'}^{\bar{s}s} : g_{G'}^{\bar{s}s} : g_{S'}^{\bar{s}s} = 0.94 : 2.04 : 6.24 \\ g_{N'}^{gg} : g_{G'}^{gg} : g_{S'}^{gg} = 0.65 : 0.69 : -0.20 \end{pmatrix}. \quad (\text{B.2})$$

Two-photon decay widths:

$$\begin{aligned} \Gamma_{N \rightarrow 2\gamma} &= 0.897 \text{ keV} & \Gamma_{S \rightarrow 2\gamma} &= 0.095 \text{ keV} \\ \Gamma_{N' \rightarrow 2\gamma} &= 0.465 \text{ keV} & \Gamma_{G' \rightarrow 2\gamma} &= 0.326 \text{ keV} & \Gamma_{S' \rightarrow 2\gamma} &= 0.152 \text{ keV} , \\ \Gamma_{N' \rightarrow 2\gamma} / \Gamma_{S' \rightarrow 2\gamma} &= 3.05 & \Gamma_{G' \rightarrow 2\gamma} / \Gamma_{S' \rightarrow 2\gamma} &= 2.14 . \end{aligned}$$

For an even further increase value of the cut-off width $\Lambda = 2.5 \text{ GeV}$ ($\mu_s = 0.982 \text{ GeV}$; $K_{SG} = 0.041 \text{ GeV}^{-1}$) we get $M_{N'} = 1.287 \text{ GeV}$ and $M_S = 1.695 \text{ GeV}$.

Mixing matrix:

$$M = \begin{pmatrix} 0.78 & 0.61 & 0.11 \\ -0.62 & 0.74 & 0.27 \\ 0.08 & -0.29 & 0.95 \end{pmatrix},$$

Set of coupling constants:

$$\begin{pmatrix} g_{N'}^{\bar{n}n} : g_{G'}^{\bar{n}n} : g_{S'}^{\bar{n}n} = 4.64 : -3.18 : 0.32 \\ g_{N'}^{\bar{s}s} : g_{G'}^{\bar{s}s} : g_{S'}^{\bar{s}s} = 0.79 : 1.70 : 5.27 \\ g_{N'}^{gg} : g_{G'}^{gg} : g_{S'}^{gg} = 0.51 : 0.55 : -0.17 \end{pmatrix}.$$

Two-photon decay widths:

$$\begin{aligned}\Gamma_{N \rightarrow 2\gamma} &= 0.911 \text{ keV} , \Gamma_{S \rightarrow 2\gamma} = 0.101 \text{ keV}, \\ \Gamma_{N' \rightarrow 2\gamma} &= 0.455 \text{ keV} \quad \Gamma_{G' \rightarrow 2\gamma} = 0.347 \text{ keV} \quad \Gamma_{S' \rightarrow 2\gamma} = 0.165 \text{ keV} , \\ \Gamma_{N' \rightarrow 2\gamma} / \Gamma_{S' \rightarrow 2\gamma} &= 2.99 \quad \Gamma_{G' \rightarrow 2\gamma} / \Gamma_{S' \rightarrow 2\gamma} = 2.28.\end{aligned}$$

As a last point we consider the increase of the effective quark mass parameter μ_n from 0.86 to 1.1 *GeV* in order to check its influence on the results. For $\Lambda = 1.5 \text{ GeV}$ we get $\mu_s = 1.2 \text{ GeV}$ and $K_{SG} = 1.00 \text{ GeV}^{-1}$. The masses are $M_{N'} = 1.307 \text{ GeV}$ and $M_S = 1.695 \text{ GeV}$. The mixing matrix is

$$M = \begin{pmatrix} 0.81 & 0.57 & 0.09 \\ -0.57 & 0.78 & 0.25 \\ 0.07 & -0.27 & 0.96 \end{pmatrix},$$

where again no decisive variation from the previous cases is seen.

Appendix C

Expressions for the decays

We now list the results for the decay of the isoscalar state $|i\rangle$, when the glueball is allowed to decay in the framework of the chiral approach (see section 7.5).

$$\begin{aligned}\Gamma_{i \rightarrow \pi\pi} &= 3 \frac{\frac{1}{2}\sqrt{M_i^2 - 4M_\pi^2}}{16\pi M_i^2} \left\{ \frac{4B^{iN}}{\sqrt{2}F^2} \left(\frac{M_i^2 - 2M_\pi^2}{2} c_Q^d + M_\pi^2 c_Q^m \right) \right. \\ &\quad \left. + \frac{4B^{iG}}{\sqrt{3}F^2} \left(\frac{M_i^2 - 2M_\pi^2}{2} c_G^d + M_\pi^2 c_G^m \right) \right\}.\end{aligned}\quad (\text{C.1})$$

$$\begin{aligned}\Gamma_{i \rightarrow \bar{K}K} &= 4 \frac{\frac{1}{2}\sqrt{M_i^2 - 4M_K^2}}{16\pi M_i^2} \left\{ \frac{2}{\sqrt{2}F^2} (B^{iN} + \sqrt{2}B^{iS}) \left(\frac{M_i^2 - 2M_K^2}{2} c_Q^d + M_K^2 c_Q^m \right) \right. \\ &\quad \left. + \frac{4B^{iG}}{\sqrt{3}F^2} \left(\frac{M_i^2 - 2M_K^2}{2} c_G^d + M_K^2 c_G^m \right) \right\}^2.\end{aligned}\quad (\text{C.2})$$

For the $\eta\eta$ decay we have:

$$\begin{aligned}\Gamma_{i \rightarrow \eta\eta} &= \frac{\frac{1}{2}\sqrt{M_i^2 - 4M_\eta^2}}{16\pi M_i^2} \left\{ -\frac{2c_Q^d \Lambda_{1\eta\eta}}{\sqrt{2}F^2} \frac{M_i^2 - 2M_\eta^2}{2} + \frac{2c_Q^m \Lambda_{2\eta\eta}}{\sqrt{2}F^2} \right\} \\ &\quad + B^{iG} \left(-\frac{2c_G^d \Pi_{1\eta\eta}}{\sqrt{2}F^2} \frac{M_i^2 - 2M_\eta^2}{2} + \frac{2c_G^m \Pi_{2\eta\eta}}{\sqrt{2}F^2} \right) \Big\}^2\end{aligned}\quad (\text{C.3})$$

where $\Lambda_{1\eta\eta}$ is as in section 7.2

$$\begin{aligned}\Lambda_{1\eta\eta}[B^{iN}, B^{iS}] &= (B^{iN} + \sqrt{2}B^{iS}) - \frac{1}{\sqrt{3}} (B^{iN} - \sqrt{2}B^{iS}) \cdot \cos(2\theta_{ps}) \\ &\quad - \frac{2}{3} (\sqrt{2}B^{iN} - 2B^{iS}) \cdot \sin(2\theta_{ps}),\end{aligned}\quad (\text{C.4})$$

while $\Lambda_{2\eta\eta}$ is different and depends on M_π and M_K :

$$\begin{aligned}\Lambda_{2\eta\eta}[B^{iN}, B^{iS}] &= -2 \left[\frac{M_\pi^2}{2} B^{iN} + \sqrt{2} \left(M_K^2 - \frac{M_\pi^2}{2} \right) \right] \\ &+ \frac{2}{3} \left[\frac{M_\pi^2}{2} B^{iN} - \sqrt{2} \left(M_K^2 - \frac{M_\pi^2}{2} \right) \right] \cdot \cos(2\theta_{ps}) \\ &+ \frac{4}{3} \left[\sqrt{2} \frac{M_\pi^2}{2} B^{iN} - 2 \left(M_K^2 - \frac{M_\pi^2}{2} \right) \right] \cdot \sin(2\theta_{ps});\end{aligned}\quad (\text{C.5})$$

$\Pi_{1\eta\eta}$ and $\Pi_{2\eta\eta}$ are given by:

$$\Pi_{1\eta\eta} = \Lambda_{1\eta\eta} \left[\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}} \right], \quad (\text{C.6})$$

$$\Pi_{2\eta\eta} = \Lambda_{2\eta\eta} \left[\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}} \right]. \quad (\text{C.7})$$

For the $\eta\eta'$ decay we have:

$$\begin{aligned}\Gamma_{i \rightarrow \eta\eta'} &= \frac{p_{i\eta\eta'}}{8\pi M_i^2} \left\{ \left(-\frac{c_Q^d \Lambda_{1\eta\eta'}}{\sqrt{2}F^2} \frac{M_i^2 - M_\eta^2 - M_{\eta'}^2}{2} + \frac{c_Q^m \Lambda_{2\eta\eta'}}{\sqrt{2}F^2} \right) \right. \\ &\left. + B^{iG} \left(-\frac{c_G^d \Pi_{1\eta\eta'}}{\sqrt{2}F^2} \frac{M_i^2 - M_\eta^2 - M_{\eta'}^2}{2} + \frac{c_G^m \Pi_{2\eta\eta'}}{\sqrt{2}F^2} \right) \right\}^2\end{aligned}\quad (\text{C.8})$$

where

$$\Lambda_{1\eta\eta'}[B^{iN}, B^{iS}] = \frac{2}{3} \left[2 \left(\sqrt{2} B^{iN} - 2 B^{iS} \right) \cos(2\theta_{ps}) + \left(-B^{iN} + \sqrt{2} B^{iS} \right) \sin(2\theta_{ps}) \right]. \quad (\text{C.9})$$

$$\begin{aligned}\Lambda_{2\eta\eta'}[B^{iN}, B^{iS}] &= -\frac{8}{3} \left[\sqrt{2} \frac{M_\pi^2}{2} B^{iN} - 2 \left(M_K^2 - \frac{M_\pi^2}{2} \right) \right] \cdot \cos(2\theta_{ps}) \\ &- \frac{4}{3} \left[-\frac{M_\pi^2}{2} B^{iN} + \sqrt{2} \left(M_K^2 - \frac{M_\pi^2}{2} \right) \right] \cdot \sin(2\theta_{ps}),\end{aligned}\quad (\text{C.10})$$

and where:

$$\Pi_{1\eta\eta'} = \Lambda_{1\eta\eta'} \left[\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}} \right], \quad (\text{C.11})$$

$$\Pi_{2\eta\eta'} = \Lambda_{2\eta\eta'} \left[\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}} \right]. \quad (\text{C.12})$$

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