The Dot-Depth Hierarchy
v.
Iterated Block Products of DA

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Abstract

Like the sequence of the classes of the dot-depth hierarchy the sequence of classes given by the n-fold iterated block product of DA has the class of starfree regular languages as its limit. It is shown that this DA-block-product hierarchy grows more slowly than the dot-depth hierarchy: in fact already Σ_2^L of the dot-depth hierarchy contains properness witnesses for all levels of the DA-block-product hierarchy.

1 Introduction

The dot-depth hierarchy is a way to classify the complexity of starfree regular languages: the lower a starfree language sits in the dot-depth hierarchy the less complex it is supposed to be. But there exist alternative ways to classify the starfree languages which are only partially comparable with the dot-depth hierarchy, for example the until/since depth from temporal logic [TW04].

Another classification of the starfree languages is considered here: the hierarchy given by the n-fold iterated block product of DA. DA is the set of monoids corresponding as syntactic monoids to the languages in Δ_2^L of the the dot-depth hierarchy, a very robust class with many characterizations [TT02]. The block product $\Box$ is also coming from the algebraic side and is the two-sided version of the wreath product on finite monoids, resp. on classes of monoids, see [RT89, ST02, TW04]. In this paper, DA and block products of DA will be identified with their corresponding language classes.

It is easy to see that the iterated block product $\mathrm{DA}^{n\Box}$ of DA, defined strongly bracketed as

$$\mathrm{DA}^{n\Box} := \mathrm{DA} \,\Box\, (\ldots(\mathrm{DA} \,\Box\, \mathrm{DA})),$$

is a subset of Δ_{n+1}^L of the dot-depth hierarchy, so the two hierarchies are in one direction comparable. It is also known that Δ_{n+1}^L contains languages from $\mathrm{DA}^{n\Box}$ which are not in the full level DD_n^L of the dot-depth hierarchy – this fact can be interpreted in the way that some parts of the DA-block-product hierarchy are growing as fast as the dot-depth hierarchy. The main result of this note is that other parts of the DA-block-product hierarchy are growing slowly compared with the dot-depth hierarchy: it is shown that already Σ_2^L contains for every $n \geq 1$ witnesses of the properness of the inclusion $\mathrm{DA}^{n\Box} \subset \mathrm{DA}^{(n+1)\Box}$. A graphical summary of the results is sketched in Figure 2.

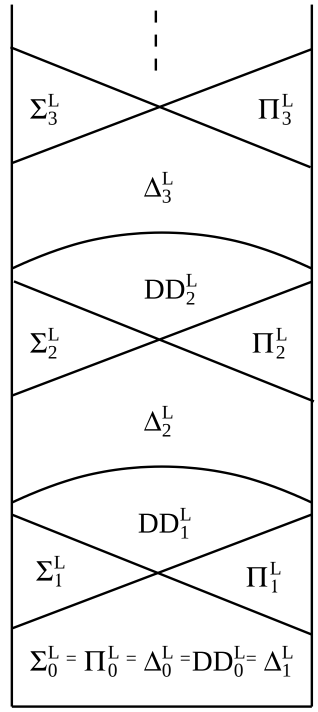

Figure 1: The dot-depth hierarchy

2 Preliminairies

The dot-depth of a starfree regular language counts the minimal nesting depth of concatenations (= "dot products") one needs to represent the language by a starfree regular expression. There are two versions of the dot-depth hierarchy: the classical one by Cohen & Brzozowski [CH71] and the variant by Straubing and Thérien [St81, The81]. They only differ slightly, see [St94], i.e. the level $n+1$ of one contains the level n of the other. We consider in this paper only the second version, and we will use a logical characterization of its levels [Tho82, PP86]. The dot-depth hierarchy consists for every $n \geq 0$ of the classes $\Sigma_n, \Pi_n, \mathrm{DD}_n$, and Δ_n, each of which is formally a mapping from the sets of finite alphabets to a set of regular languages over this alphabet. The class Σ_n is, according to a characterization of Thomas [Tho82] and Perrin & Pin [PP86], the set of languages definable with a Σ_n alternation prefix in first-order logic on words with the signature $[<]$ plus a unary predicate for each letter of the respective alphabet, see [St94, PW97]. Π_n is the set of complements of languages in Σ_n, DD_n^L (usually called L_n) is the Boolean closure of Σ_n, and Δ_n is defined as $\Sigma_n \cap \Pi_n$. It hold the proper inclusions as depicted in Figure 1, see for example [St94, PW97].

The *syntactical monoid* M_L of a language L over alphabet A consists of the equivalence classes $[u]$ for $u \in A^*$ defined by the the equivalence relation

$$[u] = [v] \iff \forall w, z \in A^* : wuz \in L \iff wvz \in L. \tag{1}$$

The monoid operation can be defined by $[u][v] := [uv]$, especially it holds for all words u, v, w, z from Σ^*:

$$\text{if } [u] = [v] \text{ then } [wuz] = [wvz]. \tag{2}$$

A language is regular iff its syntactical monoid is finite, and it is starfree iff moreover there exists a number ω such for all $x \in A^*$ it holds

$$[x^\omega] = [x^\omega x^n] \text{ for every } n \geq 0. \tag{3}$$

The class of monoids DA, which naming letters stand for the algebraic notions "D-classes" and "aperiodic", is the algebraic pendant of the language class Δ_2^L from the dot-depth hierarchy, in the sense that a language A is in Δ_2^L if and only if its syntactical monoid M_A is in DA, see for example [PW97, TT02]. By this correspondence, and because this paper tries to stay on the language side only, DA will stand for Δ_2^L from now on. The following characterization of DA, which is very close to the algebraic definiton of DA, see [TT02], will be used extensively.

Lemma 1 (DA) *A language L over alphabet Σ belongs to* DA *iff for all words $x, y, z \in \Sigma^*$ it holds in M_L:*

$$[(xyz)^\omega y (xyz)^\omega] = [(xyz)^\omega]. \tag{4}$$

For the definition of the block product we also stay on the language side (besides a little dip into the syntactic monoid), see [TW04].

Definition 1 (block product) *The* block product $K \Box J$ *of a language J over alphabet Σ and a language $K \in$ DA over alphabet $M_J \times \Sigma \times M_J$ (where M_J is the syntactic monoid of J) is the language over alphabet Σ consisting of all words $x = x_1 \cdots x_n$ in Σ^* such that the following word $\tau(x)$ is in K:*

$$\tau(x) := ([\varepsilon], x_1, [x_2 \cdots x_n]) \;\; ([x_1], x_2, [x_3 \cdots x_n]) \;\cdots\; ([x_1 \cdots x_{n-1}], x_n, [\varepsilon]). \tag{5}$$

The block product $\mathcal{K} \Box \mathcal{J}$ of two classes of languages $\mathcal{K}$ and $\mathcal{J}$ is the set of block products $K \Box J$ such that $K \in \mathcal{K}$ and $J \in \mathcal{J}$

The block product is in general not associative, see for example [ST02]. Therefore, we have two extrem cases (and many in between) concerning the bracketing: The *strongly iterated block product of n languages $K_n, \ldots, K_1$* (we prefer them to be numbered from the right) is defined as

$$K_n \Box (K_{n-1} \Box (\ldots (K_2 \Box K_1) \ldots))$$

while the *n-fold weakly iterated block product* is defined as

$$((\ldots (K_n \Box K_{n-1}) \ldots) \Box K_2) \Box K_1.$$

Let $\mathrm{DA}^{n\Box}$ be the set of all n-fold strongly iterated block products of DA languages. It holds that every weakly iterated block product of DA languages is in $\mathrm{DA}^{n\Box}$, see for example [ST02], likewise every other bracketing of an n-fold block product of DA languages results in a language contained in $\mathrm{DA}^{n\Box}$. This justifies that we speak of $\mathrm{DA}^{n\Box}$ as the n-fold iterated block product of DA, without mentioning the strong bracketing.

The class DA and every block product expression built on it, like $\mathrm{DA}^{n\Box}$, is a *variety of languages*, i.e. it is closed under Boolean operations, under left and right quotients and under inverse homomorphic images, see [Pin86, ST02].

We state the following facts about the relation of $\mathrm{DA}^{n\Box}$ and the dot-depth hierarchy. They can be derived from results in the literature, the proofs below are only sketched.

Theorem 1 *Let $n \geq 1$.*
(a) $\mathrm{DA}^{n\Box} \subseteq \Delta^L_{n+1}$,
(b) $\mathrm{DA}^{n\Box}$ *contains languages in* $\Delta^L_{n+1} - \mathrm{DD}^L_n$,
(c) $\bigcup_{n\geq 1} \mathrm{DA}^{n\Box}$ *equals the class of starfree languages.*

Proof. (a) For $n = 1$ this holds by definition. For the induction consider a language L in $\mathrm{DA}^{(n+1)\Box}$, i.e. $L = L_1 \Box L_0$ with $L_1 \in \mathrm{DA}$ and $L_0 \in \mathrm{DA}^{n\Box}$. In order to get a Σ_{n+2} expression for L take the Σ_2 expression for L_0 and plug the Π_{n+1} expression for L_1, which exists by induction hypothesis, into it. The two $\forall$ levels collapse and in total it is a Σ_{n+2} expression. In order to get a Π_{n+2} expression for L plug the Σ_{n+1} formula for L_1 into the Π_2 expression for L_0. This shows $L \in \Sigma^L_{n+2} \cap \Pi^L_{n+2} = \Delta^L_{n+2}$.
(b) Consider for $n \geq 2$ the following language D_n on alphabet $\{0, 1, \ldots, 2n-2\}$, see [BL+04]: $D_2 = 0^*1\{0, 1, 2\}^*$, and for $n \geq 3$ D_n consists of the words w such that the occurences of the letters $2n-3$ and $2n-2$ in w are considered as markers, and w is in D_n iff the marker after the first factor between two such markers which is in D_{n-1} is $2n-1$. D_n is not only in Δ_n, as it is argued in [BL+04], but even in $\mathrm{DA}^{(n-1)\Box}$. And moreover (thanks to Klaus Wagner, Würzburg, for this hint), D_n can be shown to be not in DD^L_n by the result of [Tr02, BL+04] that $\mathrm{Leaf}^P(D_n) = \Delta^p_n$, together with the oracle result separating the levels of PH and the relativizable result that PH collapses if BH collpases.
(c) Part (a) above verifies that each $\mathrm{DA}^{n\Box}$, and therefore the limit of this sequence, consists of starfree languages only. On the other hand every starfree language L is covered by some $\mathrm{DA}^{n\Box}$: let ϕ be a first order formula for L, which exists by the classical result starfree = first-order definable of McNaughton & Papert [MP71]. Then the quantifier depth (n.b.: not the quantifier alternation depth) of ϕ is such an n: each nested quantifier can be simulated by a $\mathrm{DA} \Box \ldots$ operation (actually, by a $\mathrm{DD}^L_1 \Box \ldots$ operation). **q.e.d.**

Note that by the results of Theorem 1 it still could be the case that for example $\mathrm{DA}^{n\Box} = \Delta^L_{n+1}$ for all $n \geq 1$, or that $\mathrm{DA}^{n\Box}$ is a class in between Δ^L_n and Δ^L_{n+1}, or that a similar close relation to the dot-depth hierarchy holds. In the following section it is shown that this is not the case.

3 Σ^L_2 is not contained in an iterated block product of DA

The following languages L_n, for $n \geq 2$, over alphabet $\Sigma_n := \{1, \ldots, n\}$ are from Σ^L_2 and will be shown to be witnesses for the properness of the inclusion $\mathrm{DA}^{(n-1)\Box} \subset \mathrm{DA}^{n\Box}$.

$$L_2 = \{1, 2\}^* 11 \{1, 2\}^*, \qquad (6)$$

$$L_{n+1} := \Sigma_{n+1}^* L_n L_n \Sigma_{n+1}^*. \tag{7}$$

where L_n is considered as a language over the larger alphabet Σ_{n+1}. For example,

$$L_3 = \{1,2,3\}^* 11 \{1,2\}^* 11 \{1,2,3\}^*$$

(because $\{1,2,3\}^*\{1,2\}^* = \{1,2,3\}^*$ etc.), and

$$L_4 = \{1,2,3,4\}^* 11 \{1,2\}^* 11 \{1,2,3\}^* 11 \{1,2\}^* 11 \{1,2,3,4\}^*.$$

(With some fantasy the reader can see overlapping waves in these languages.) These examples show that L_n can also described as $L_n = \Sigma_n^* M_n \Sigma_n^*$ where M_n is defined via the following recursion:

$$M_2 = 11, \tag{8}$$

$$M_n = M_{n-1} \Sigma_{n-1}^* M_{n-1}. \tag{9}$$

Theorem 2 (Main) *For every $n \geq 2$ it holds: The language L_n is an element of $\Sigma_2^L \cap \mathrm{DA}^{n\Box}$ but not of $\mathrm{DA}^{(n-1)\Box}$.*

This theorem is the conjunction of the following Lemma 2, Corollary 1, and Lemma 6, which will be proven now, using more sub-lemmata.

A *marked product of sub-alphabets* over an alphabet A is a regular expression

$$A_0 a_1 A_1 \dots a_n A_n$$

with $n \geq 0$, $a_0, \dots, a_n$ "markers" = letters from A, and $A_0, \dots, A_n$ sub-alphabets, i.e. subsets of A. Example: $\{0,1,2\}^* 2 0^* 2 \{0,1,2\}^*$ expressing "there exists two 2's with no 1's between them". It is easy to see that a language described by a marked product of sub-alphabets is in Σ_2^L, and in fact, by the results of Arfi [Ar87], Σ_2^L equals the set of all finite unions of them.

Lemma 2 *For every $n \geq 2$ it holds: The language L_n is an element of Σ_2^L.*

Proof. Every L_n (for $n \geq 2$) is by the representation $\Sigma_n^* M_n \Sigma_n^*$ a marked product of sub-alphabets: $M_2 = 1\emptyset^*1$ is a marked product of sub-alphabets with two outmost markers 1, and $M_{n+1} = M_n \Sigma_n^* M_n$ keeps its two outmost markers 1. **q.e.d.**

Lemma 3 *For every $n \geq 1$ it holds: Any language described by a marked product of sub-alphabets with at most $2^n - 1$ markers is in $\mathrm{DA}^{n\Box}$.*

Proof. Induction start $n = 1$. A marked product $A_0 a_1 A_1$ is in Σ_2^L, see above. On the other hand, $A_0 a_1 A_1$ can be expressed by the following Π_2 expression "there exists a position carrying letter a_1, and all positions carry letters from $A_0 \cup A_1 \cup \{a_1\}$, and it never occurs that a position has a letter from $A_1 - (A_0 \cup \{a_1\})$ and larger position has a letter from $A_0 - (A_1 \cup \{a_1\})$, and between every two positions with a letter from $A_0 - (A_1 \cup \{a_1\})$ and a letter from $A_1 - (A_0 \cup \{a_1\})$ there is a position in between carrying letter a_1". This shows that $A_0 a_1 A_1$ is in $\Sigma_2^L \cap \Pi_2^L = \Delta_2^L$.

Induction step for $n \geq 2$. Given a marked product $L = A_0 a_1 A_1 \dots a_m A_m$ over alphabet A with $m \leq 2^n - 1$, let a_k be the marker in the middle of the expression, i.e. $k = m/2$ if m is odd and $k =$

$(m+1)/2$ if m is even. Then $L = L_0 a_k L_1$ with $L_0 = A_0 a_1 A_1 \dots a_{k-1} A_{k-1}$ and $L_1 = A_k \dots a_m A_m$, and both L_0 and L_1 are marked products of sub-alphabets with not more than $2^{n-1} - 1$ markers. Therefore, the induction hypothesis applies to L_0 and L_1, i.e. both L_0 and L_1 are in $\mathrm{DA}^{(n-1)\Box}$. Let $P := L_0 \times L_1$ be their product language which is by the variety closure properties still an element of $\mathrm{DA}^{(n-1)\Box}$. Let Q be the Σ_1^L language consisting of the union of the languages $B^*(p, a_k, q)B^*$ on the alphabet $B = M_P \times A \times M_P$ such that p stands for acceptance of L_0 and q for acceptance of L_1. The language $Q \Box P$ is by this representation from $\mathrm{DA}^{n\Box}$ and equals L. **q.e.d.**

Because L_n has 2^{n-1} markers (the 1's) we have the following corollary.

Corollary 1 *For every $n \geq 2$ it holds: L_n is in $\mathrm{DA}^{n\Box}$.*

It remains to prove that L_n is not in $\mathrm{DA}^{(n-1)\Box}$. Assume that L_n equals a language K from $\mathrm{DA}^{(n-1)\Box}$, i.e.

$$K := K_{n-1} \Box (\dots (K_2 \Box K_1).) \tag{10}$$

where each K_i is in DA. We will specify two words u_n, v_n such that $u_n \notin L_n$ and $v_n \in L_n$ but u_n and v_n are indistinguishable by K, i.e. $u_n \in K \iff v_n \in K$.

Define u_n and v_n for $2 \leq n$ by induction:

$$u_2 = (21)^\omega \tag{11}$$

$$v_2 = (21)^\omega 1 (21)^\omega \tag{12}$$

where ω is the constant from Lemma 1 for K_1. For $n \geq 3$ define the abbreviation w_n, and u_n, v_n the following way:

$$w_n = u_{n-1} n u_{n-1} v_{n-1} \tag{13}$$

$$u_n := \underbrace{w_n^\omega}_{\mathrm{I}} \underbrace{w_n^\omega}_{\mathrm{II}} \underbrace{w_n^\omega}_{\mathrm{III}} \underbrace{w_n^\omega}_{\mathrm{IV}} \tag{14}$$

$$v_n := \underbrace{w_n^\omega}_{\mathrm{I}} \underbrace{w_n^\omega}_{\mathrm{II}} \underbrace{v_{n-1}}_{\mathrm{IIa}} \underbrace{w_n^\omega}_{\mathrm{III}} \underbrace{w_n^\omega}_{\mathrm{IV}} \tag{15}$$

where ω is the constant from Lemma 1 for K_{n-1} (no indexing of ω necessary, it will be clear from context which one is meant).

We show that $u_n \notin L_n$ and $v_n \in L_n$ via the following stronger invariant.

Lemma 4 *Consider a word $g = g_1 \cdots g_m$ where each g_i is either u_n or v_n. The factors of g which are elements of M_n are the following: exactly one such factor within each of the g_i for which $g_i = v_n$.*

Proof. For $n = 2$ the lemma can be checked easily. Let $n \geq 3$ und consider a word g from $\{u_n, v_n\}^*$. Because M_n does not use the letter n, a potential factor of g which is in M_n can only be found in the parts $u_{n-1}v_{n-1}u_{n-1}$ and $u_{n-1}v_{n-1}v_{n-1}u_{n-1}$, the latter occuring within the v_n's of g. The parts $u_{n-1}v_{n-1}u_{n-1}$ contain by induction hypothesis only one factor which is from M_{n-1}. By $M_n = M_{n-1}\Sigma_{n-1}M_{n-1}$ we need two factors from M_{n-1} for a word in M_n. Therefore, these parts $u_{n-1}v_{n-1}u_{n-1}$ do not contain a factor from M_n, what proves one part of Lemma 4 for this n. The parts $u_{n-1}v_{n-1}v_{n-1}u_{n-1}$ contain by induction hypothesis exactly 2 factors of a word from M_{n-1}. Therefore these two factors together with the word in between build a factor belonging to $M_n = M_{n-1}\Sigma_{n-1}M_{n-1}$, and this is the only such factor. The parts $u_{n-1}v_{n-1}v_{n-1}u_{n-1}$ are the parts corresponding to the the occurrences of v_n in g. Therefore, Lemma 4 holds also for this n. **q.e.d.**

Corollary 2 *For every $n \geq 2$ it holds: $u_n \notin L_n$, $v_n \in L_n$.*

Proof. From Lemma 4 it follows that for $g = g_1 = u_n$ there is no occurrence of a factor from M_n, therefore u_n is not contained in $L_n = \Sigma^* M_n \Sigma^*$, while for $g = g_1 = v_n$ is there an (actually, exactly one) occurrence of a factor from M_n, therefore v_n is contained in $L_n = \Sigma^* M_n \Sigma^*$. **q.e.d.**

We will proof by induction the following crucial invariant.

Lemma 5 *For $n \geq 2$ it holds in the syntactic monoid of $K = K_{n-1} \,\Box\, (\ldots(K_2 \,\Box\, K_1)\ldots)$ the following:*

$$[v_n] = [u_n] = [u_n u_n] = [v_n v_n] = [u_n v_n] = [v_n u_n]. \tag{16}$$

Proof. Induction start: In case $n = 2$ the block product $K = K_1$ is a single DA language. In order to verify the first of the equations in 16 note that $[v_2] = [(21)^\omega 1 (21)^\omega] = [(21)^\omega (21)^\omega] = [u_2 u_2]$ by equation 4 in Lemma 1 setting $x := 2$, $y := 1$ $z := \varepsilon$. Moreover, $[u_2] = [(21)^\omega] = [(21)^\omega (21)^\omega] = [u_2 u_2]$ by equation 3. The other equations follow immediately from these two by equation 2.

Induction step for $n \geq 3$: Define $J := K_{n-2} \,\Box\, (\ldots(K_2 \,\Box\, K_1).)$, this way $K = K_{n-1} \,\Box\, J$. We go to the definition of the block product $K_{n-1} \,\Box\, J$, and will analyze the words $\tau(z u_n z')$ and $\tau(z v_n z')$, see equation 5 in Definition 1. z and z' are two arbitrary words from Σ_n, we need them later in order to show that from $[\tau(z u_n z')] = [\tau(z v_n z')]$ in the syntactic monoid of K_{n-1} it follows $[u_n] = [v_n]$ in the syntactic monoid of $K_{n-1} \,\Box\, J$. Note that $\tau(z u_n z')$ and $\tau(z v_n z')$ are words on alphabet $M_J \times \Sigma \times M_J$ which have the same length as $z u_n z'$ and $z v_n z'$, respectively, so we can keep the partition of the positions of u_n and v_n into the parts I to IV, as in equations 14 and 15, plus two parts 0 and V for the positions of z and z', respectively. We will show that there exist words p_0, p, x, y, s, s_0 over alphabet $M_J \times \Sigma \times M_J$ such that $\tau(u_n)$ and $\tau(v_n)$ can be written the following way:

$$\tau(z u_n z') = \tau(\underbrace{z}_{0}\underbrace{w_n^\omega}_{\text{I}}\underbrace{w_n^\omega}_{\text{II}}\underbrace{w_n^\omega}_{\text{III}}\underbrace{w_n^\omega}_{\text{IV}}\underbrace{z'}_{\text{V}}) = \underbrace{p_0}_{0}\underbrace{p}_{\text{I}}\underbrace{(xy)^\omega}_{\text{II}}\underbrace{(xy)^\omega}_{\text{III}}\underbrace{s}_{\text{IV}}\underbrace{s_0}_{\text{V}} \tag{17}$$

$$\tau(z v_n z') = \tau(\underbrace{z}_{0}\underbrace{w_n^\omega}_{\text{I}}\underbrace{w_n^\omega}_{\text{II}}\underbrace{v_n}_{\text{IIa}}\underbrace{w_n^\omega}_{\text{III}}\underbrace{w_n^\omega}_{\text{IV}}\underbrace{z'}_{\text{V}}) = \underbrace{p_0}_{0}\underbrace{p}_{\text{I}}\underbrace{(xy)^\omega}_{\text{II}}\underbrace{y}_{\text{IIa}}\underbrace{(xy)^\omega}_{\text{III}}\underbrace{s}_{\text{IV}}\underbrace{s_0}_{\text{V}} \tag{18}$$

To verify the above three equations 17 and 18 we have to show the following:

(a) $\tau(zu_nz')$ and $\tau(zv_nz')$ coincide on parts 0, I, II, III, IV and V.

(b) There exists a word h $(= xy)$ such that the two restrictions of $\tau(zu_nz')$ to parts II and III are of the form h^ω

(c) This periodic pattern h from (b) has a suffix y which equals $\tau(zv_nz')$ restricted to part IIa.

ad (a): We show that the words $\tau(zu_nz')$ and $\tau(zu_nz')$ coincide on parts 0, I, II, III, IV, and V: Let i be a position in part 0, I, or II of the words $zu_nz' = b_1 \dots b_m$ and $zv_nz' = b'_1 \dots b'_{m'}$. The two triples $([b_1 \dots b_{i-1}], b_i, [b_{i+1} \dots b_m])$ at position i of $\tau(zu_nz')$ and $([b'_1 \dots b'_{i-1}], b'_i, [b'_{i+1} \dots b_{m'}])$ at position i of $\tau(zv_nz')$ will of course coincide on their left and middle component because zu_nz' and zv_nz' are identical up to that position. But moreover they also coincide on the right component of the triple: The two words $b_{i+1} \dots b_m$ and $b'_{i+1} \dots b'_{m'}$ only differ by the extra factor v_{n-1} in $b'_{i+1} \dots b'_{n'}$ from part IIa. But this v_{n-1} is immediately left to a u_{n-1} (u_{n-1} is a prefix of part III), and by induction hypothesis we have $[v_{n-1}u_{n-1}] = [u_{n-1}]$ in the syntactic monoid of J. Therefore, by equation 2, $[b_{i+1} \dots b_m] = [b'_{i+1} \dots b'_{m'}]$, i.e. the third components of the two tripels are also equal. By symmetrical arguments and $[v_{n-1}v_{n-1}] = [v_{n-1}]$ by induction hypothesis we have that $\tau(zu_nz')$ and $\tau(zv_nz')$ also coincide on parts III, IV, and V.

ad (b): Let i be a position in the j-th factor w_n $(1 \leq j \leq \omega)$ of part II of zu_nz'. Then the triple of $\tau(zu_nz')$ at that position i has the form

$$([zw_n^\omega w_n^{j-1} f], a, [gw_n^{\omega-j} w_n^\omega z'])$$

where f and g are the prefix and the suffix of the factor w_n left and right of that position i, respectively, i.e. $fag = w_n$. Note that by equation 3 it holds $[zw_n^\omega w_n^{j-1}] = [zw_n^\omega]$ in the syntactic monoid of J, so we can by equation 2 rewrite the left component as $[zw_n^\omega f]$. Likewise (now via adding w_n^{j-1} instead of dropping it) the right component can be rewritten as $[gw_n^{\omega-1} w_n^\omega z']$. This way we have at the position i in the j-th factor w_n of part II of $\tau(zu_nz')$ the triple

$$([zw_n^\omega f], a, [gw_n^{\omega-1} w_n^\omega z']).$$

But this is exactly the same triple as the triple at the i-th position of the first factor w_n in part II of $\tau(zu_nz')$. By setting h to be the suffix of length $|w_n|$ of part II of $\tau(z'u_nz)$ we get the desired property (b) for part II. By symmetrical arguments (b) also holds for part III.

ad (c): Consider a position i in part IIa, i.e. $v_n = b_1 \cdots b_{i-1} b_i b_{i+1} \cdots b_m$. The triple at the i-th position in part IIa of $\tau(zv_nz')$ will be

$$([zw_n^\omega w_n^{\omega-1} u_{n-1} n \underline{u_{n-1} v_{n-1}} b_1 \cdots b_{i-1}], b_i, [b_{i+1} \cdots b_m \underline{u_{n-1}} n u_{n-1} v_{n-1} w_n^{\omega-1} w_n^\omega z']).$$

By induction hypothesis it holds $[u_{n-1}v_{n-1}] = [v_{n-1}]$ in the syntactic monoid of J, therefore the first component the factor $u_{n-1}v_{n-1}$ left of b_1 can be rewritten by u_{n-1}, and likewise in the third component the factor u_{n-1} right of b_m can be rewritten by $v_{n-1}u_{n-1}$, as this is indicated by the underlinings in the triples above and below. This way the above triple equals

$$([zw_n^\omega w_n^{\omega-1} u_{n-1} n \underline{u_{n-1}} b_1 \cdots b_{i-1}], b_i, [b_{i+1} \cdots b_m \underline{v_{n-1} u_{n-1}} n u_{n-1} v_{n-1} w_n^{\omega-1} w_n^\omega z']).$$

But this is exactly the triple which one gets by looking at the i-th position in the suffix v_{n-1} of part II of the word $\tau(zv_nz')$.

We have shown (a), (b), and (c), i.e. $\tau(zu_nz')$ and $\tau(zv_nz')$ can be written in the form of equations 17 and 18. This gives the following equation 19 in the syntactic monoid of K_{n-1}:

$$[\tau(zu_nz')] = [\underbrace{p_0}_{0}\underbrace{p}_{\mathrm{I}}\underbrace{(xy)^\omega}_{\mathrm{II}}\underbrace{(xy)^\omega}_{\mathrm{III}}\underbrace{s}_{\mathrm{IV}}\underbrace{s_0}_{\mathrm{V}}] = [\underbrace{p_0}_{0}\underbrace{p}_{\mathrm{I}}\underbrace{(xy)^\omega}_{\mathrm{II}}\underbrace{y}_{\mathrm{IIa}}\underbrace{(xy)^\omega}_{\mathrm{III}}\underbrace{s}_{\mathrm{IV}}\underbrace{s_0}_{\mathrm{V}}] = [\tau(zv_nz')] \tag{19}$$

The middle equation symbol above holds by the following equality in the syntactic monoid of K_{n-1} which is a case of equation 4 (no renaming of the variables x, y necessary, $z := \varepsilon$):

$$[\underbrace{(xy)^\omega}_{\mathrm{II}}\underbrace{(xy)^\omega}_{\mathrm{III}}] = [\underbrace{(xy)^\omega}_{\mathrm{II}}\underbrace{y}_{\mathrm{IIa}}\underbrace{(xy)^\omega}_{\mathrm{III}}] \tag{20}$$

We have shown $[\tau(zu_nz')] = [\tau(zv_nz')]$ in the syntactic monoid of K_{n-1} for all words $z, z' \in \Sigma_n^*$. From this it follows $\tau(zu_nz') \in K_{n-1} \iff \tau(zv_nz') \in K_{n-1}$ for all $z, z' \in \Sigma_n^*$. This means, by the definition of block product: $zu_nz' \in K_{n-1} \Box J \iff zv_nz' \in K_{n-1} \Box J$ for all $z, z' \in \Sigma_n^*$. By the definition of the elements of the syntactic monoid we have the equality

$$[u_n] = [v_n] \tag{21}$$

in the syntactic monoid of $K_{n-1} \Box J$.

This shows that the first equation in Lemma 5 holds. Now we show the second equation $[u_nu_n] = [u_n]$. Let z, z' be again some words from Σ_n^*. Let τ again be the function in equation 5 in the definition of block product. It holds for $\tau(zu_nu_nz')$ the following:

$$\tau(zu_nu_nz') = \tau(\underbrace{z}_{0}\underbrace{w_n^\omega}_{\mathrm{I}}\underbrace{w_n^{3\omega}}_{\mathrm{IIb}}\underbrace{w_n^{3\omega}}_{\mathrm{IIIb}}\underbrace{w_n^\omega}_{\mathrm{IV}}\underbrace{z'}_{\mathrm{V}}) = \underbrace{p_0}_{0}\underbrace{p}_{\mathrm{I}}\underbrace{(xy)^{3\omega}}_{\mathrm{IIb}}\underbrace{(xy)^{3\omega}}_{\mathrm{IIIb}}\underbrace{s}_{\mathrm{IVa}}\underbrace{s_0}_{\mathrm{V}} \tag{22}$$

The first equality is the definition of u_n, the second equality holds by the same argumentation like for claim (a) above. In the syntactic monoid of K_{n-1} it holds by equation 3 $[(xy)^{3\omega}] = [(xy)^\omega]$. Therefore, and by equations 22 and 17 together with equation 2, it holds in the syntactic monoid of K_{n-1}:

$$[\tau(zu_nu_nz')] = [\underbrace{p_0}_{0}\underbrace{p}_{\mathrm{I}}\underbrace{(xy)^{3\omega}}_{\mathrm{IIb}}\underbrace{(xy)^{3\omega}}_{\mathrm{IIIb}}\underbrace{s}_{\mathrm{IVa}}\underbrace{s_0}_{\mathrm{V}}] = [\underbrace{p_0}_{0}\underbrace{p}_{\mathrm{I}}\underbrace{(xy)^\omega}_{\mathrm{II}}\underbrace{(xy)^\omega}_{\mathrm{III}}\underbrace{s}_{\mathrm{IVa}}\underbrace{s_0}_{\mathrm{V}}] = [\tau(zu_nz')] \tag{23}$$

From $[\tau(zu_nu_nz')] = [\tau(zu_nz')]$ in the syntactic monoid of K_{n-1} for all $z, z, \in \Sigma_n^*$ we can like above conclude that in the syntactic monoid of $K_{n-1} \Box J$ it holds:

$$[u_nu_n] = [u_n] \tag{24}$$

We have shown $[u_n] = [v_n]$ and $[u_n] = [u_nu_n]$ in the syntactic monoid of $K_{n-1} \Box J$. The other equations follow immediately from these two by equation 2. **q.e.d.**

Lemma 6 *For every $n \geq 2$ it holds: L_n is not an element of* $\mathrm{DA}^{(n-1)\Box}$.

Proof. Let $n \geq 2$ and consider L_n as a language over alphabet Σ_n. Assume that L_n is in $\mathrm{DA}^{(n-1)\Box}$. Then there exist $n-1$ languages $K_{n-1}, \ldots, K_1$ all of them from DA such that for $K = K_{n-1} \Box (\ldots (K_2 \Box K_1) \ldots)$ it holds $L_n = K$. By Corollary 2, $u_n \in L_n$ and $v_n \notin L_n$. But on the other hand, by Lemma 4, it holds $[u_n] = [v_n]$ in the syntactic monoid of K, from which it follows $u_n \in K \iff v_n \in K$, i.e., u_n and v_n are indistinguishable in K. Therefore, L_n cannot be equal to K. It follows that L_n cannot be from $\mathrm{DA}^{(n-1)\Box}$. **q.e.d.**

From Theorems 1 and 2 we can conclude:

Corollary 3 *Let $n \geq 1$ and $k \geq 2$. If $n < k$ then each of the four classes Σ_k^L, Π_k^L, DD_k^L, and Δ_{k+1}^L contains $\mathrm{DA}^{n\Box}$ properly. If $n \geq k$ then each of these four classes is incomparable with $\mathrm{DA}^{n\Box}$.*

Figure 2 gives a visual summary of the results in Theorems 1 and 2, and Corollary 3.

4 Open Questions and Acknowledgements

A problem left open is whether the weakly and the strongly bracketed n-fold iterated block product of DA coincide. Another interesting question is whether the class DA $\Box$ DA or at least $(\mathrm{DA} \Box \mathrm{DA}) \cap \Sigma_2^L$ is decidable. By the results of Arfi [Ar87] the latter question can be reduced to the decidability of the following computational problem: Given a marked product $A_0 a_1 A_1 \ldots a_n A_n$ of sub-alphabets, does it belong to DA $\Box$ DA?

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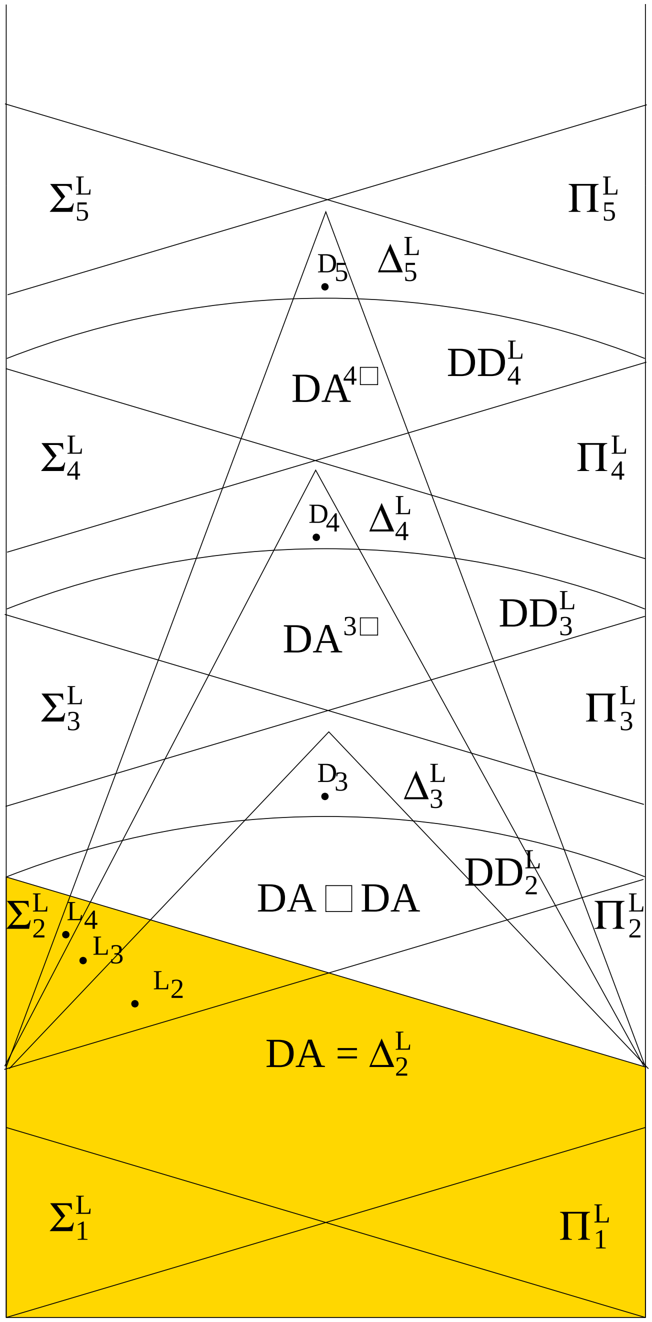

Figure 2: Σ_2^L v. iterated block products of DA

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