Functional Partial Differential Equations and Evolution Semigroups

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Zusammenfassung in Deutscher Sprache

Thema dieser Dissertation ist die Wohlgestelltheit und Asymptotik von nichtautonomen funktionalen Partielle-Differentialgleichungen der Form

$$\frac{d}{dt}u(t) = B(t)u(t) + \Phi(t, u_t), \quad t \geq 0, \tag{DPDG}$$

und

$$\frac{d}{dt}Fu_t = B(t)Fu_t + \Phi(t, u_t), \quad t \geq 0. \tag{NPDG}$$

Wesentliches Hilfsmittel zur Diskussion sind Evolutionshalbgruppen, die von der Evolutionsfamilie auf einer Halbgeraden erzeugt werden (siehe [**14**], [**23**, Chap. VI.9], [**72**], [**55**], [**56**]), sowie die Theorie der Randstörung eines Generators (siehe [**29**]).

In Kapitel 1 werden die grundlegenden Konzepte über Evolutionhalbguppen und Evolutionsfamilien auf einer Halbgeraden behandelt. Hier werden alle Hilfsmittel bereitgestellt, die wir später benötigen, einschliesslich der Ergebnisse über die exponentielle Dichotomie allgemeiner Evolutiongleichungen.

In Kapitel 2 betrachten wir zuerst die Gleichung (DPDG) mit nichtautonomer Vergangenheit der Form

$$\begin{cases} \frac{\partial}{\partial t}u(t,0) &= Bu(t,0) + \Phi u(t,.), \qquad t \geq 0, \\ \frac{\partial}{\partial t}u(t,s) &= \frac{\partial}{\partial s}u(t,s) + A(s)u(t,s), \qquad t \geq 0 \geq s. \end{cases}$$

Wir konstruieren eine stark stetige Halbgruppe, die diese Gleichung löst. Dann benutzen wir die Charakterisierung der hyperbolischen Halbgruppe (siehe [**58**, Theorem 2.6.2]), um die Robustheit der exponentiellen Dichotomie der Lösungen zu erhalten. Am Ende des Kapitels studieren wir mit dem Methoden und Ergebnissen aus Kapitel 1 die Robustheit der exponentiellen Dichotomie der Lösungen der allgemeinen nichtautonomen (DPDG). Wir bekommen ähnliche Ergebnisse für Delayoperatoren, die nur auf einem endlichem Intervall wirken.

In Kapitel 3 schlagen wir eine Halbgruppenbehandlung zu autonomer (NPDG) der Form

$$\begin{cases} \frac{\partial}{\partial t}Fu_t &= BFu_t + \Phi u_t \quad \text{falls } t \geq 0, \\ u_0(t) &= \varphi(t) \quad \text{falls } t \leq 0, \end{cases}$$

vor.

Durch Anwendung die Theorie der Störungen des Operator auf dem Rand können wir eine Lösungshalbgruppe für die obige Gleichung unter Bedingungen an den Differenzoperator F konstruieren und die Wohlgestelltheit der Gleichung zeigen.

In Kapitel 4 benutzen wir die ähnlichen Ideen wie in Kapitel 2, um die Wohlgestelltheit und die Robustheit der exponentiellen Stabilität der Lösungen der (NPDG) mit nichtautonomen Vergangenheit der Form

$$\begin{cases} \frac{\partial}{\partial t} F(u(t,\cdot)) & = BFu(t,\cdot) + \Phi u(t,\cdot), \qquad t \geq 0, \\ \frac{\partial}{\partial t} u(t,s) & = \frac{\partial}{\partial s} u(t,s) + A(s)u(t,s), \qquad t \geq 0 \geq s, \end{cases}$$

zu bekommen. Schliesslich, in Kapitel 5, erweitern wir unsere Methoden und Ergebnisse aus Kapitel 1, um allgemeine nichtautonome (NPDG) zu betrachten. Wir erhalten ähnliche Resultate für Delayoperatoren und Differenzoperatoren, die auf endlichem Intervall wirken.

Contents

Introduction

Functional partial differential equations (FPDE's) arise from various applications. We refer to Hale [**34, 35**], Wu [**81**], Wu and Xia [**82**] for numerous examples and applications of FPDE's. In general, non-autonomous FPDE's can be written in an abstract form as

$$\frac{d}{dt}u(t) \quad = \quad B(t)u(t) + \Phi(t, u_t), \quad t \geq 0, \tag{0.1}$$

for FPDE's of *retarded* type, and as

$$\frac{d}{dt}Fu_t \quad = \quad B(t)Fu_t + \Phi(t, u_t), \quad t \geq 0, \tag{0.2}$$

for FPDE's of *neutral* type.
Here, the linear (unbounded) operators $B(t)$ "generate" a strongly continuous evolution family $(U(t,s))_{t\geq s\geq 0}$ on a Banach space X, Φ and F, called the *delay operator* and the *difference operator*, respectively, are X-valued mappings defined on $C([-r,0],X)$ (or on $C_0(\mathbb{R}_-,X)$). Finally, the history function u_t, $t \geq 0$, is defined by $u_t(\theta) := u(t+\theta)$ for $\theta \in [-r,0]$ (or $\theta \in \mathbb{R}_-$). Precise definitions and assumptions will be given later in this thesis.

For the equation (0.1), there are many systematic treatments dealing with the well-posedness and stability of the solutions (see, e.g., Fitzgibbon [**24, 25**], Pazy [**63**], Ruess [**32, 67, 68**] and Wu [**81**] and references therein). However, regarding more general asymptotic behavior of the solutions to (0.1), e.g., exponential dichotomy, there still are many open problems. Recently, the technique of "evolution semigroup" has been applied to the problem of existence and robustness of exponential dichotomy with great success. One of advantages of using this method is that it allows one to relate the exponential dichotomy of the solutions of (0.1) to properties of the spectra of the corresponding evolution semigroup generated by the equation (0.1) defined on $\mathbb{R}$. We refer to Aulbach and Nguyen Van Minh [**2**] and Gühring, Räbiger and Schnaubelt [**31**] for results on the robustness of exponential dichotomy of the solutions to (0.1) in this case. However, for the equation (0.1) defined only on the half-line $\mathbb{R}_+$ the problem of finding conditions for the robustness of exponential dichotomy seems to be open. The main difficulty is that, in case the equation (0.1) is defined only on the half-line $\mathbb{R}_+$, one may not immediately have the corresponding right translation evolution semigroup as in the case of the whole line. One needs to take into account a suitable boundary condition at zero to define the corresponding right translation evolution semigroup.

Therefore, as a first attempt to fill this gap, we consider a special (but interesting) case of the equation (0.1), i.e., the case of FPDE's with *non-autonomous past* which have been proposed by Brendle and Nagel [**10**]. These equations are an appropriate model for many biological and physical systems (see Fragnelli [**28**]) and have the form

$$\begin{cases} \frac{\partial}{\partial t}u(t,0) &= Bu(t,0) + \Phi u(t,.), \qquad t \geq 0, \\ \frac{\partial}{\partial t}u(t,s) &= \frac{\partial}{\partial s}u(t,s) + A(s)u(t,s), \qquad t \geq 0 \geq s. \end{cases} \tag{0.3}$$

Here, the function $u(\cdot,\cdot)$ takes values in a Banach space X, B is a linear operator on X, and Φ, called the *delay operator*, is a linear operator from a space of X-valued functions on $\mathbb{R}_-$ into X. Finally, $A(s)$ are (unbounded) operators on X for which the non-autonomous Cauchy problem

$$\begin{cases} \frac{dx(t)}{dt} &= -A(t)x(t), \; t \leq s \leq 0, \\ x(s) &= x_s \in X, \end{cases} \tag{0.4}$$

is well-posed with exponential bound. In particular, there exists an exponentially bounded backward evolution family $\mathcal{U} = (U(t,s))_{t \leq s \leq 0}$ solving (0.4), i.e., the solutions of (0.4) are given by $x(t) = U(t,s)x(s)$ for $t \leq s \leq 0$.

We then apply the theory of evolution semigroups as developed by Chicone and Latushkin [**14**], Schnaubelt [**23**, Chap. VI.9], [**72**] and others (see [**55, 56**]) to study the well-posedness and the asymptotic behavior of the solutions of the equation (0.3). Our approach may be summarized as follows.

We first define an abstract differential (maximal) operator G on the space $C_0(\mathbb{R}_-, X)$ (see Definition 1.23). We then perturb this operator at the boundary using the delay operator Φ (and the operator B) to obtain a restriction $G_{B,\Phi}$ of G. For this restriction we compute explicitly its resolvent using its representation by an integral equation. We then show the Hille-Yosida estimates to obtain a generator of a semigroup $(T_{B,\Phi}(t))_{t \geq 0}$ which solves (0.3) in a mild sense (see [**10**, Sections 1 and 2]).

At this point we would like to note that the idea of perturbing a (maximal) operator at the boundary to obtain a generator was introduced by Greiner [**29**]. It has been then used, e.g., by Engel [**22**], Casarino *et al.* [**12, 13**] to obtain generators of semigroups solving many important equations arising in applications. The idea of representing an operator using an integral equation has been used, e.g., by Nguyen Van Minh [**53**], Nguyen Van Minh, Räbiger and Schnaubelt [**55**] to study exponential dichotomy of evolution equations (see also [**40, 54, 56**]). In our thesis we combine these two ideas to solve the equation (0.3) and to study the asymptotic behavior of its solutions. We then use this approach to investigate the more general equation

$$\begin{cases} \frac{\partial}{\partial t}F(u(t,\cdot)) &= BFu(t,\cdot) + \Phi u(t,\cdot), \qquad t \geq 0, \\ \frac{\partial}{\partial t}u(t,s) &= \frac{\partial}{\partial s}u(t,s) + A(s)u(t,s), \qquad t \geq 0 \geq s, \end{cases} \tag{0.5}$$

which can be called a *neutral partial differential equation with non-autonomous past.*

Our thesis is divided into five chapters. In Chapter 1, we briefly recall some basic concepts of evolution families and evolution semigroups on a half-line. All the auxiliary results for the later use are presented here. We also include in this chapter the results on the asymptotic behavior of the solutions to general evolution equations on a half-line with special emphasis on exponential dichotomy.

In Chapter 2, we first deal with the equation (0.3). Using the evolution semigroup approach and the ideas described above, we are able to construct a strongly continuous semigroup solving (0.3). We then use the characterization of hyperbolic semigroups (see Neerven [**58**, Theorem 2.6.2]) to study the robustness of the exponential dichotomy of the solutions to (0.3). It turns out that the robustness of the exponential dichotomy can be obtained in an elegant manner using the evolution semigroup approach. At the end of Chapter 2, we use the methods and the results from Chapter 1 to study the robustness of exponential dichotomy of the solutions to non-autonomous partial functional differential equations (0.1). We obtain similar results for delay operators acting on a finite interval $[-r, 0]$.

In Chapter 3, we propose a semigroup approach to linear autonomous neutral partial functional differential equations of the form

$$\begin{cases} \frac{\partial}{\partial t} F u_t &= BFu_t + \Phi u_t \quad \text{for } t \geq 0, \\ u_0(t) &= \varphi(t) \quad \text{for } t \leq 0. \end{cases} \tag{0.6}$$

It is interesting that the picture of well-posedness of the equation (0.6) can be clearly seen through Tübingen glasses. That is, using the idea of perturbing a (maximal) operator at the boundary we can construct a strongly continuous semigroup solving (0.6) and obtain the well-posedness of the equation (0.6) under appropriate conditions on the difference operator F. In this way we extend and improve results by Hale [**34**], Wu [**81**, Chap 2.3] and Datko [**18**].

In Chapter 4, we use the same approach and ideas as in Chapter 2 to obtain the well-posedness and the robustness of exponential stability of partial neutral functional differential equations with non-autonomous past (0.5). Finally, in Chapter 5, we extend the methods and results from Chapter 1 to study non-autonomous partial neutral functional differential equations (0.2) and obtain the similar results for equations with delay and difference operators acting on a finite interval $[-r, 0]$.

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CHAPTER 1

Evolution Families and Evolution Semigroups on a Half-line

In this chapter we introduce some basic concepts of evolution families and evolution semigroups on a half-line. We also present the auxiliary results for the next chapters as well as some results on the asymptotic behavior of the solutions to evolution equations, with special emphasis on exponential dichotomy for evolution equations on the half-line.

1. Motivation and preliminaries

It is known that, in order to deal with evolution families on the whole line, the technique of "evolution semigroup" has been applied with great success (see [**14**], [**53**], [**56**], [**23**, Chap. VI.9] and references therein). It turns out that the asymptotic behavior of evolution families on the whole line, especially their exponential dichotomies, can be characterized by spectral properties of the corresponding evolution semigroups. We refer to Schnaubelt [**72**] for an excellent review on this approach. However, for evolution families on the half-line $\mathbb{R}_+$, the situation becomes more complicated. One of the main difficulties in dealing with evolution families on $\mathbb{R}_+$ is that we do not immediately have the corresponding right translation evolution semigroups as in the case of evolution families on the whole line. The main point is that we have to include a suitable boundary condition to define the corresponding right translation evolution semigroup. Therefore, it is necessary to develop an analogous theory for the evolution families and evolution semigroups on the half-line. We begin with the definition of an evolution family $\mathcal{U}$ on $\mathbb{R}_+$.

Definition 1.1. A family of bounded linear operators $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ on a Banach space X is a *(strongly continuous, exponentially bounded) evolution family on the half-line* if

(i) $U(t,t) = Id$ and $U(t,r)U(r,s) = U(t,s)$ for $t \geq r \geq s \geq 0$,
(ii) the map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$,
(iii) there are constants $N \geq 1$ and $\alpha \in \mathbb{R}$ such that $\|U(t,s)\| \leq Ne^{\alpha(t-s)}$ for $t \geq s \geq 0$.

Then the constant

$$\omega(\mathcal{U}) := \inf\{\alpha \in \mathbb{R} : \text{ there is } N \geq 1 \text{ such that } \|U(t,s)\| \leq Ne^{\alpha(t-s)}, \quad t \geq s \geq 0\}$$

is called the *growth bound* of $\mathcal{U}$.

This notion of evolution families arises naturally from the theory of evolution equations which are well-posed (see, e.g., [**62**, Chap. 5], [**60**], [**71**]). In fact, in the terminology of [**62**, Chap. 5] and [**60**], an evolution family arises from the following well-posed evolution equation

$$\begin{cases} \frac{du(t)}{dt} &= A(t)u(t), \ t \geq s \geq 0, \\ u(s) &= x_s \in X, \end{cases} \tag{1.1}$$

where $A(t)$ are (in general unbounded) linear operators for $t \geq 0$. We refer to Nagel and Nickel [**57**] for a detailed discussion of well-posedness for non-autonomous abstract Cauchy problems on $\mathbb{R}$.

For a complex Banach space X we will consider the function space (endowed with the sup-norm)

$$C_0(\mathbb{R}_+, X) := \{v : \mathbb{R}_+ \to X : v \text{ is continuous and } \lim_{t\to\infty} v(t) = 0\} =: C_0$$

In order to overcome difficulties in dealing with evolution families on the half-line as stated above, one idea proposed by Brendle and Nagel [**10**] is to extend $(U(t,s))_{t\geq s\geq 0}$ to an evolution family $(\tilde{U}(t,s))_{t\geq s}$ on $\mathbb{R}$. This extension allows us to use the known techniques for evolution families on $\mathbb{R}$ from [**14**], [**53**], [**56**], or [**23**, Chap. VI.9]. This extension will be

$$\tilde{U}(t,s) := \begin{cases} U(t,s) & \text{for } t \geq s \geq 0, \\ U(t,0) & \text{for } t \geq 0 \geq s, \\ U(0,0) = Id & \text{for } 0 \geq t \geq s \end{cases}$$

yielding the corresponding right translation evolution semigroup.

Definition 1.2. On $C_0(\mathbb{R}, X)$, the *right translation evolution semigroup* $(\tilde{T}(t))_{t\geq 0}$ corresponding to $(\tilde{U}(t,s))_{t\geq s}$ is given by

$$(\tilde{T}(t)\tilde{f})(s) := \tilde{U}(s,s-t)\tilde{f}(s-t) = \begin{cases} U(s,s-t)\tilde{f}(s-t) & \text{for } s \geq s-t \geq 0, \\ U(s,0)\tilde{f}(s-t) & \text{for } s \geq 0 \geq s-t, \\ \tilde{f}(s-t) & \text{for } 0 \geq s \geq s-t, \end{cases}$$

for all $\tilde{f} \in C_0(\mathbb{R}, X), s \in \mathbb{R}, t \geq 0$.

We can see that this semigroup is strongly continuous on $C_0(\mathbb{R}, X)$ (see [**23**, Lemma VI.9.10]) and denote its generator by $(\tilde{G}, D(\tilde{G}))$. We then have the following properties of this operator.

Lemma 1.3. *For $\tilde{u}$, $\tilde{f}$ in $C_0(\mathbb{R}, X)$ and $\lambda \in \mathbb{C}$ the following assertions hold.*

(i) *$\tilde{u} \in D(\tilde{G})$ and $(\lambda - \tilde{G})\tilde{u} = \tilde{f}$ if and only if $\tilde{u}$ and $\tilde{f}$ satisfy the integral equation*

$$\tilde{u}(t) = e^{-\lambda(t-s)}\tilde{U}(t,s)\tilde{u}(s) + \int_s^t e^{-\lambda(t-\xi)}\tilde{U}(t,\xi)\tilde{f}(\xi)d\xi \qquad \text{for all } t \geq s. \tag{1.2}$$

(ii) *The operator $(\tilde{G}, D(\tilde{G}))$ is a* local operator *in the sense that for $\tilde{u} \in D(\tilde{G})$ and $\tilde{u}(s) = 0$ for all $a < s < b$ we have that $[\tilde{G}\tilde{u}](s) = 0$ for all $a < s < b$.*

Proof. (i) Let $\tilde{u} \in D(\tilde{G})$ and $\tilde{G}\tilde{u} = \lambda\tilde{u} - f$. We first observe that the evolution semigroup, corresponding to the evolution family $(e^{-\lambda(t-s)}\tilde{U}(t,s))_{t\geq s}$, is $(e^{-\lambda t}\tilde{T}(t))_{t\geq 0}$ with the generator $-\lambda+\tilde{G}$. Therefore, by basic semigroup theory (see, e.g., [**23**, Lemma II.1.3]), we have that

$$e^{-\lambda t}\tilde{T}(t)\tilde{u} - \tilde{u} = \int_0^t e^{-\lambda\xi}\tilde{T}(\xi)(-\lambda + \tilde{G})\tilde{u} = -\int_0^t e^{-\lambda\xi}\tilde{T}(\xi)\tilde{f}d\xi,$$

i.e.,

$$\tilde{u} = e^{-\lambda t}\tilde{T}(t)\tilde{u} + \int_0^t e^{-\lambda\xi}\tilde{T}(\xi)\tilde{f}d\xi \quad \text{for } t \geq 0.$$

By the definition of $\tilde{T}(t)$ we obtain that $\tilde{u}$ and $\tilde{f}$ satisfy the equation (1.2).

Conversely, if $u,\ f \in C_0(\mathbb{R}, X)$ satisfy the equation (1.2), then by reversing the above argument we obtain

$$e^{-\lambda t}\tilde{T}(t)\tilde{u} - \tilde{u} = \int_0^t e^{-\lambda\xi}\tilde{T}(\xi)f(\xi)d\xi, \quad t \geq 0.$$

In particular, this implies that $u \in D(\tilde{G})$ and $\tilde{G}\tilde{u} = \lambda\tilde{u} - f$.

(ii) By (i) we have that

$$\tilde{u}(t) = \tilde{U}(t,s)\tilde{u}(s) - \int_s^t \tilde{U}(t,\xi)[\tilde{G}\tilde{u}](\xi)d\xi \qquad \text{for } \tilde{u} \in D(\tilde{G}) \text{ and } t \geq s.$$

Therefore, if $\tilde{u}(s) = 0$ for all $a < s < b$, then

$$\frac{1}{s-t}\int_t^s \tilde{U}(t,\xi)[\tilde{G}\tilde{u}](\xi)d\xi = 0 \quad \text{for all } a < t < s < b.$$

By the strong continuity of $\tilde{U}(t,s)$ we obtain, for $s \downarrow t$, that

$$[\tilde{G}\tilde{u}](t) = \tilde{U}(t,t)[\tilde{G}\tilde{u}](t) = 0 \quad \text{for all } t \in (a,b).$$

□

The locality of $\tilde{G}$ allows us to define an operator G on $C_0 := C_0(\mathbb{R}_+, X)$ in the following way (see Brendle and Nagel [**10**, Definition 2.7]).

Definition 1.4. Take

$$D(G) := \{\tilde{f}|_{\mathbb{R}_+} : \tilde{f} \in D(\tilde{G})\}$$

and define

$$[Gf](t) := [\tilde{G}\tilde{f}](t) \text{ for } t \geq 0 \text{ and } f = \tilde{f}|_{\mathbb{R}_+}.$$

Analogously to Lemma 1.3, we now have the following description of G.

Lemma 1.5. *Let $u, f \in C_0 = C_0(\mathbb{R}_+, X)$ and $\lambda \in \mathbb{C}$. Then $u \in D(G)$ and $(\lambda - G)u = f$ if and only if u and f satisfy*

$$u(t) = e^{-\lambda(t-s)}U(t,s)u(s) + \int_s^t e^{-\lambda(t-\xi)}U(t,\xi)f(\xi)d\xi \qquad \text{for } t \geq s \geq 0. \tag{1.3}$$

Proof. If $u,\ f \in C_0$ satisfy the equation (1.3), then we extend $u,\ f$ to $\mathbb{R}$ by

$$\tilde{u}(t) := \begin{cases} u(t) & \text{for } t \geq 0 \\ e^{-\lambda t}g(t) & \text{for } t < 0, \end{cases}$$
$$\tilde{f}(t) := \begin{cases} f(t) & \text{for } t \geq 0 \\ e^{-\lambda t}g'(t) & \text{for } t < 0. \end{cases}$$

Here, $g : \mathbb{R}_- \to X$ is continuously differentiable with compact support such that $g(0) = u(0),\ g'(0) = f(0)$. We then have that $\tilde{u}$ and $\tilde{f}$ belong to $C_0(\mathbb{R}, X)$. A straightforward computation yields that $\tilde{u}$ and $\tilde{f}$ satisfy the equation (1.2). Therefore, by Lemma 1.3, we obtain $(\lambda - \tilde{G})\tilde{u} = \tilde{f}$. By the definition of G we have that $(\lambda - G)u = f$.

Conversely, if $u \in D(G)$ and $(\lambda - G)u = f$, then, by the definition of G, there exist $\tilde{u}, \tilde{f} \in C_0(\mathbb{R}, X)$ such that $\tilde{u}|_{\mathbb{R}_+} = u$, $\tilde{f}|_{\mathbb{R}_+} = f$ and $(\lambda - \tilde{G})\tilde{u} = \tilde{f}$. By Lemma 1.3, $\tilde{u}$ and $\tilde{f}$ satisfy the equation (1.2). Restricting this equation to $\mathbb{R}_+$, we have that u and f satisfy (1.3). □

The operator G becomes a generator only if we restrict it to a smaller domain, e.g., $D := \{u \in D(G) : [Gu](0) = 0\}$ (see [**55**, Lemma 1.1]). However, for later applications we consider a more general case and make the following assumption.

ASSUMPTION 1.6. Let $(B, D(B))$ be the generator of a strongly continuous semigroup $(e^{tB})_{t\geq 0}$ on the Banach space X satisfying $\|e^{tB}\| \leq Me^{\omega_2 t}$ for some constants $M \geq 1$ and $\omega_2 \in \mathbb{R}$.

DEFINITION 1.7. On the space $C_0 = C_0(\mathbb{R}_+, X)$ we define a right translation evolution semigroup $(T^r_{B,0}(t))_{t\geq 0}$ by

$$[T^r_{B,0}(t)f](s) = \begin{cases} U(s, s-t)f(s-t) & \text{for } s-t \geq 0 \\ U(s,0)e^{(t-s)B}f(0) & \text{for } s-t \leq 0 \end{cases} \quad \text{for all } f \in E.$$

One can easily verify that $(T^r_{B,0}(t))_{t\geq 0}$ is strongly continuous. We denote its generator by $G^r_{B,0}$.

We then have the following properties of $G^r_{B,0}$ and $(T^r_{B,0}(t))_{t\geq 0}$.

PROPOSITION 1.8. *The following assertions hold.*

(i) *The generator of* $(T^r_{B,0}(t))_{t\geq 0}$ *is given by*

$$\begin{aligned} D(G^r_{B,0}) &:= \{f \in D(G) : f(0) \in D(B) \text{ and } (G(f))(0) = Bf(0)\}, \\ G^r_{B,0}f &:= Gf \text{ for } f \in D(G^r_{B,0}). \end{aligned}$$

(ii) *The set* $\{\lambda \in \mathbb{C} : Re\lambda > \omega(\mathcal{U}) \text{ and } \lambda \in \rho(B)\}$ *is contained in* $\rho(G^r_{B,0})$. *Moreover, for* λ *in this set, the resolvent is given by*

$$[R(\lambda, G^r_{B,0})f](t) = e^{-\lambda t}U(t,0)R(\lambda, B)f(0) + \int_0^t e^{-\lambda(t-\xi)}U(t,\xi)f(\xi)d\xi \quad \text{for } f \in C_0,\ t \geq 0.$$

(iii) *The semigroup* $(T^r_{B,0}(t))_{t\geq 0}$ *satisfies*

$$\|T^r_{B,0}(t)\| \leq Ke^{\omega t},\ t \geq 0,$$

with $K := MN$ *and* $\omega := \max\{\omega_1, \omega_2\}$ *for the constants* M, N, ω_1 *and* ω_2 *appearing in Definition 1.1 and Assumption 1.6.*

PROOF. (i) This can be found in [**10**, Proposition 2.8].

(ii) Observe that for $f \in C_0$, $\lambda \in \rho(B)$ and $Re\lambda > \omega(\mathcal{U})$ the function

$$u(t) := e^{\lambda t}U(t,0)R(\lambda, B)f(0) + \int_0^t e^{\lambda(t-\xi)}U(t,\xi)f(\xi)d\xi, \qquad t \geq 0,$$

belongs to C_0 and is the unique solution of the equation (1.3) with the initial condition $u(0) = R(\lambda, B)f(0)$. This condition is equivalent to $(\lambda - B)u(0) = f(0) = [(\lambda - G)u](0)$ or $[Gu](0) = Bu(0)$. This means that $u \in D(G_{B,0})$ and $u = R(\lambda, G_{B,0})f$.

(iii) This follows immediately from the definition of $(T^r_{B,0}(t))_{t\geq 0}$. □

Beside the function space $C_0 := C_0(\mathbb{R}_+, X)$, we also consider its following closed subspace

$$C_{00} \;:=\; \{v \in C_0 : v(0) = 0\}.$$

Furthermore, for a closed subspace Z of X, we define the subspace C_Z of C_0 by

$$C_Z := \{f \in C_0 : f(0) \in Z\}. \tag{1.4}$$

In order to characterize exponential dichotomies of evolution families we need the following special cases of the evolution semigroup $(T^r_{B,0}(t))_{t\geq 0}$.

DEFINITION 1.9. The evolution semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on C_0 is given by

$$\begin{aligned} T(t) \;&:=\; T^r_{0,0}(t) \quad \text{for } t \geq 0, \; i.e., \\ [T(t)f](s) \;&:=\; \begin{cases} U(s, s-t)f(s-t) & \text{for } 0 \leq t \leq s, \\ U(s,0)f(0) & \text{for } 0 \leq s \leq t. \end{cases} \end{aligned} \tag{1.5}$$

It can be seen easily that this evolution semigroup leaves invariant C_{00}. Hence, we can define the semigroup $(T_0(t))_{t\geq 0}$ as the restriction of $\mathcal{T}$ to the space C_{00}. We denote the generators of $(T_0(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ by G_0 and G_X, respectively.

In the next section we will need the following operators.

DEFINITION 1.10.
(a) On the space C_0 we define the operator I_X by

$$I_X := -G \quad \text{on the domain } D(I_X) := D(G). \tag{1.6}$$

(b) For a closed subspace Z of X we define the operator I_Z by

$$D(I_Z) := D(I_X) \cap C_Z \;\text{ and } I_Z u := I_X u \text{ for } u \in D(I_Z). \tag{1.7}$$

The following properties is an immediate corollary of Lemma 1.5 and Proposition 1.8.

LEMMA 1.11.
(a) *Let $u, f \in C_{00}$. Then $u \in D(G_0)$ and $G_0 u = -f$ if and only if*

$$u(t) = \int_0^t U(t,\xi)f(\xi)d\xi \quad \textit{for all } t \geq 0. \tag{1.8}$$

(b) *Let $u \in C_0$ and $f \in C_{00}$. Then $u \in D(G_X)$ and $G_X u = -f$ if and only if*

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)d\xi \quad \textit{for all } t \geq s \geq 0. \tag{1.9}$$

(c) *The operator G_0 is injective and the part of $-I_Z$ in C_{00}, i.e., $D(G_0) := \{u \in D(I_Z) \cap C_{00} : I_Z u \in C_{00}\}$ and $G_0 u = -I_Z u$ for $u \in D(G_0)$.*
(d) *The operator $(I_X, D(I_X))$ is an extension of $(-G_X, D(G_X))$ and*

$$\ker I_X = \ker G_X = \{u \in C_0 : u(t) := U(t,0)u(0), \; t \geq 0\}.$$

PROOF. Since $(T_0(t))_{t\geq 0}$ is the restriction of $(T(t))_{t\geq 0}$ to C_{00}, we have that G_0 is the restriction of G_X to C_{00}. By definition of $(T(t))_{t\geq 0}$ (see (1.5)), we obtain that $G_X = G^r_{0,0}$. Therefore, the assertions (a) and (b) follow from Proposition 1.8 (a) and Lemma 1.5 by taking $\lambda = 0$ and $B = 0$.

The assertions (c) and (d) follows from (a), (b) and the definitions of the operators I_Z and I_X, respectively. □

We refer the readers to [**55**, Lemma 1.1, Remarks 1.2, 1.4] for another proof of the above lemma using directly the integral equation (1.9).

2. Exponential dichotomy

In this section we will characterize the exponential dichotomy of evolution families using the operators I_X and I_Z defined in (1.6), (1.7), respectively. Let us outline the history of the problem. We consider the linear differential equation

$$\frac{dx}{dt} = A(t)x, \quad t \geq 0, \ x \in X,$$

where $A(t)$ are (in general) unbounded linear operators on a Banach space X. One of the central research interests regarding the asymptotic behavior of the solutions to the above equation is to find conditions for the solutions to be *stable* or to have *exponential dichotomy*. In the case that $A(\cdot)$ is a continuous matrix function, Perron [**64**] first observed a relation between the asymptotic behavior of the solutions and the properties of the differential operator $\frac{d}{dt} - A(t)$ as an operator on the space $BC(\mathbb{R}_+, \mathbb{R}^n)$ of $\mathbb{R}^n$-valued bounded continuous functions on the half-line $\mathbb{R}_+$. These results served as a starting point for numerous work on the qualitative theory of solutions to differential equations. We refer the reader to the books by Massera and Schäffer [**51**] and Daleckii and Krein [**16**] for a characterization of the exponential dichotomy of solutions to the above equation in terms of the surjectiveness of the differential operator $\frac{d}{dt} - A(t)$ in the case of infinite dimensions with bounded $A(t)$. Levitan and Zhikov [**49**, Chap. 10] extended this to the case of unbounded $A(t)$ for equations defined on $\mathbb{R}$. For equations defined on the half-line with unbounded $A(t)$, in order to characterize the exponential dichotomy, apart from the surjectiveness of the differential operator $\frac{d}{dt} - A(t)$, one needs additional conditions, namely a complement of the stable subspaces (see [**16**], [**51**], [**55**]).

Recently there has been an increasing interest in the unbounded case (see, e.g., [**8**], [**9**], [**23**], [**40**], [**44**], [**54**], [**58**], [**73**]). In particular, we mention the recent paper [**55**] in which a new characterization of exponential dichotomy was given in Hilbert spaces using only conditions on $\frac{d}{dt} - A(t)$ and $A(t) - \frac{d}{dt}$ (more precisely, its closure). These conditions are closely related to the evolution semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ associated to an evolution family $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ on the half-line defined as in (1.5).

In this section we shall use the concept of *exponentially dichotomous operators* (see [**5, 9**]) and introduce *quasi-exponentially dichotomous operators* to characterize the exponential dichotomy of evolution family $\mathcal{U}$. Our main results are contained in Theorems 1.15, 1.16 and 1.18 extending known results for finite dimensional spaces (see [**5, 9**]). Before doing so, we now make precise the notion of exponential dichotomy.

Definition 1.12. An evolution family $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ on the Banach space X is said to have an *exponential dichotomy* on $[0,\infty)$ if there exist bounded linear projections $P(t)$, $t \geq 0$, on X and positive constants N, ν such that

(a) $U(t,s)P(s) = P(t)U(t,s)$, $t \geq s \geq 0$,
(b) Take $Q(t) := Id - P(t)$ and denote the restriction $U(t,s)_| : Q(s)X \to Q(t)X$, $t \geq s \geq 0$, by $U_Q(t,s)$. Then $U_Q(t,s)$ is an isomorphism (and we denote its inverse by $U_Q(s,t) : \ker P(t) \to \ker P(s)$ for $t \geq s \geq 0$),
(c) $\|U(t,s)x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in P(s)X$, $t \geq s \geq 0$,

(d) $\|U_Q(s,t)x\| \le Ne^{-\nu(t-s)}\|x\|$ for $x \in Q(t)X,\ t \ge s \ge 0$.

The bounded linear projections $(P(t))_{t\ge 0}$ are called *dichotomy projections*, while the positive constants $N,\ \nu$ are called the *dichotomy constant* and *dichotomy exponent*, respectively.

The following lemma, whose proof can be found in [**55**, Lemma 4.2] and in [**23**, Lemma VI.9.17], supplies some properties of evolution families having an exponential dichotomy.

LEMMA 1.13. *Let $\mathcal{U} = (U(t,s))_{t\ge s\ge 0}$ be an evolution family having an exponential dichotomy on $[0,+\infty)$ with the corresponding dichotomy projections $(P(t))_{t\ge 0}$ and constants $N > 0,\ \nu > 0$. Then the following assertions hold.*

(a) $M := \sup_{t\ge 0} \|P(t)\| < \infty$.
(b) $[0,t] \ni s \mapsto U_Q(s,t) \in \mathcal{L}(Q(t)X, X)$ *is strongly continuous for* $t \ge 0$.
(c) $t \mapsto P(t)$ *is strongly continuous.*
(d) $U_Q(t,s)x = U_Q(t,r)U_Q(r,s)x$ *for* $x \in Q(s)X$ *and* $t,r,s \ge 0$.
(e) $\|U(t,s)P(s)\| \le MNe^{-\nu(t-s)}$ *for* $t \ge s \ge 0$.
(f) $\|U_Q(s,t)Q(t)\| \le MNe^{-\nu(t-s)}$ *for* $t \ge s \ge 0$.

In the paper [**55**, Theorems 4.3, 4.5], the authors have proven the following characterization of exponential dichotomy.

THEOREM 1.14. *Assume that $\mathcal{U} = (U(t,s))_{t\ge s\ge 0}$ is an evolution family on the Banach space X and Z is a closed linear subspace of X. Let the subspaces $X_0(t_0)$ of X be defined as*

$$X_0(t_0) := \{x \in X : \lim_{t\to\infty} U(t,t_0)x = 0\} \quad \text{for } t_0 \ge 0. \tag{1.10}$$

Then the following assertions are equivalent.

(i) $\mathcal{U}$ *has an exponential dichotomy with the corresponding dichotomy projections* $(P(t))_{t\ge 0}$ *satisfying* $\ker P(0) = Z$.
(ii) $I_Z : D(I_Z) \subseteq C_Z \to C_0$ *is invertible.*
(iii) I_X *is surjective and the space $X_0(0)$ defined by (1.10) is complemented with its complement being Z.*

In this section we shall characterize the exponential dichotomy of evolution families by other properties of the operators I_X and I_Z. These operators are called *exponentially dichotomous* and *quasi-exponentially dichotomous* operators, respectively. The former has been introduced in [**5**] and [**9**]. For sake of completeness, we recall its definition. To do this, we use the basic facts on strongly continuous semigroups (see [**23, 62**] and references therein). Besides ordinary semigroups defined on the positive half-line, henceforth also called *right semigroups*, we also consider semigroups defined on the negative semiaxis. These are called *left semigroups*.

Let $(V(t))_{t\in J}$ be a strongly continuous right or left semigroup. Here, J is the half-line $[0,\infty)$ or $(-\infty,0]$ according to $(V(t))_{t\in J}$ being a right or left semigroup, respectively. It is known that there exist constants M and ω such that

$$\|V(t)\| \le Me^{\omega|t|},\ t \in J.$$

If the above inequality is satisfied for a given real number ω and some positive constant M, we say that $(V(t))_{t\in J}$ is of *exponential type* ω. Semigroups of negative exponential type are called *exponentially decaying*.

We introduce next the concepts of *exponentially dichotomous* and *quasi-exponentially dichotomous* operators. Let $(S, D(S))$ be a linear operator on a Banach space Y. Let $\mathcal{P} : Y \to Y$ be a bounded projection of Y commuting with S, i.e., $\mathcal{P}D(S) \subset D(S)$ and $\mathcal{P}Sy = S\mathcal{P}y$ for all $y \in D(S)$. Put $Y_- := \mathrm{Im}\mathcal{P}$ and $Y_+ := \ker \mathcal{P}$. Then

$$Y = Y_- \oplus Y_+ \tag{1.11}$$

and this decomposition reduces S. By this we mean that

$$D(S) = [D(S) \cap Y_-] \oplus [D(S) \cap Y_+], \tag{1.12}$$

S maps $D(S) \cap Y_-$ into Y_- and maps $D(S) \cap Y_+$ into Y_+. With respect to the decomposition (1.11) and (1.12), the operator S has the matrix representation

$$S = \begin{pmatrix} S_- & 0 \\ 0 & S_+ \end{pmatrix}. \tag{1.13}$$

Here $S_- := S\,|_{Y_-}\ (Y_- \to Y_-)$ is the restriction of S to Y_-, $S_+ := S\,|_{Y_+}\ (Y_+ \to Y_+)$ is the restriction of S to Y_+. The domain $D(S_-)$ of S_- is $D(S) \cap Y_-$, the domain $D(S_+)$ of S_+ is $D(S) \cap Y_+$. Thus, (1.12) can be written as $D(S) = D(S_-) \oplus D(S_+)$.

The operator S is said to be *exponentially dichotomous* if the operators S_- and S_+ in (1.13) are generators of exponentially decaying strongly continuous left and right semigroups, respectively. In that case, the projection $\mathcal{P}$ is called the separating projection for S. We say that S is of *exponential type* $\omega(< 0)$ if this is true for the semigroups generated by S_- and S_+.

The operator S is said to be *quasi-exponentially dichotomous* of exponential type $\omega < 0$ if, in (1.13), the operator S_+ is the generator of an exponentially decaying strongly continuous right semigroup of exponential type $\omega(< 0)$ and the operator S_- is a *left Hille-Yosida* operator of type ω, that is an operator satisfying the following conditions.

There exists a positive constant M such that the set $\{\lambda \in \mathbb{C} : Re\lambda < -\omega\}$ is contained in the resolvent set $\rho(S_-)$ and the resolvent $R(\lambda; S_-)$ satisfies

$$\|R(\lambda; S_-)^n\| \leq \frac{M}{(-Re\lambda - \omega)^n} \quad \textit{for } Re\lambda < -\omega \textit{ and all } n \in \mathbb{N}. \tag{1.14}$$

We refer the readers to [**23**, Chap. II.3] and [**59**, Chap. 3] for more information about Hille-Yosida operators and their role, e.g., in the theory of adjoint semigroups. Note that, if a left Hille-Yosida operator is densely defined, then it is the generator of a strongly continuous left semigroup. Therefore, a quasi-exponentially dichotomous operator is exponentially dichotomous if and only if the corresponding operator S_- is densely defined.

We now come to our first main result. It characterizes the exponential dichotomy of an evolution family in terms of the exponential dichotomy of the operator I_X.

THEOREM 1.15. *Let $\mathcal{U} = (U(t,s))_{t \geq s \geq 0}$ be an evolution family on the Banach space X. Then the following assertions are equivalent.*

(i) *$\mathcal{U}$ has an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t \geq 0}$ satisfying* $\ker P(0) = X$.
(ii) *I_X is exponentially dichotomous.*

PROOF. We first note that condition (i) does not imply that the evolution family $\mathcal{U}$ has a trivial exponential dichotomy. We refer the readers to [**55**, Example 4.6] for an example of an evolution family which has a nontrivial exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geq 0}$ satisfying $\ker P(0) = X$.

(ii)⇒(i): Assume that I_X is an exponentially dichotomous operator of exponential type $\omega(< 0)$. By a remark in Section 1 of [**5**], the strip $\{\lambda \in \mathbb{C} : |\mathrm{Re}\lambda| < -\omega\}$ is contained in $\rho(I_X)$. In particular, I_X is invertible. By Theorem 1.14, $\mathcal{U}$ has an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geq 0}$ satisfying $\ker P(0) = X$.

(i)⇒(ii): We prove this in several steps.

1) Let the condition (i) be satisfied. We define $\mathcal{P} : C_0 \to C_0$ by $(\mathcal{P}f)(t) = P(t)f(t)$ for $f \in C_0$. Then $\mathcal{P}$ is a projection on C_0. We now prove that

$$\mathcal{P}I_X = I_X\mathcal{P}\mid_{D(I_X)}. \tag{1.15}$$

From the definition of I_X and the equality $P(t)U(t,s) = U(t,s)P(s)$ we have

$$P(t)u(t) = U(t,s)P(s)u(s) + \int_s^t U(t,\xi)P(\xi)[I_X(u)](\xi)d\xi \quad \text{for } u \in D(I_X).$$

Again by definition of I_X, we obtain that $P(\cdot)u(\cdot) \in D(I_X)$ and

$$I_X(P(\cdot)u(\cdot)) = P(\cdot)[I_X(u)](\cdot).$$

This yields (1.15).
Note that (1.15) implies that $\mathcal{P}(D(I_X)) \subseteq D(I_X)$, whence,

$$D(I_X) = [D(I_X) \cap \mathrm{Im}\mathcal{P}] \oplus [D(I_X) \cap \mathrm{Ker}\mathcal{P}].$$

2) Here we contruct a strongly continuous, exponentially decaying left semigroup $(S_0(t))_{t\leq 0}$ on $\mathrm{Im}\mathcal{P}$.
Note that

$$\mathrm{Im}\mathcal{P} = \{f \in C_0 : f(s) \in \mathrm{Im}P(s),\ s \geq 0\},$$

$\mathrm{Im}\mathcal{P} \subseteq C_{00}$ (because $\ker P(0) = X$), and $\mathrm{Im}\mathcal{P}$ is invariant under the semigroup $(T_0(t))_{t\geq 0}$. Hence, we can define the operators $(S_0(t))_{t\leq 0}$ on $\mathrm{Im}\mathcal{P}$ by

$$S_0(t)f := T_0(-t)f \text{ for } f \in \mathrm{Im}\mathcal{P} \text{ and } t \leq 0. \tag{1.16}$$

We recall from Section 1, Definition 1.9, that the semigroup $(T_0(t))_{t\geq 0}$ on C_{00} is given by

$$[T_0(t)f](s) = \begin{cases} U(s,s-t)f(s-t) & \text{for } s \geq t \geq 0 \\ 0 & \text{for } 0 \leq s \leq t \end{cases} \quad \text{for all } f \in C_{00}. \tag{1.17}$$

From the exponential dichotomy of $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ and the definition of $(T_0(t))_{t\geq 0}$ one can easily see that $(S_0(t))_{t\leq 0}$ is a strongly continuous, exponentially decaying left semigroup satisfying

$$\|S_0(t)\| \leq Ne^{\nu t},\ t \leq 0,$$

where the positive constants N and ν are defined by the exponential dichotomy of $\mathcal{U}$.

3) We show next that the generator of $(S_0(t))_{t\leq 0}$ is $I_X\mid_{\mathrm{Im}\mathcal{P}} =: I_X^-$.
Since $(T_0(t))_{t\geq 0}$ leaves $\mathrm{Im}\mathcal{P}$ invariant, we may denote the restriction of $(T_0(t))_{t\geq 0}$ on $\mathrm{Im}\mathcal{P}$ by $(\hat{T}_0(t))_{t\geq 0}$ and the generator of $(\hat{T}_0(t))_{t\geq 0}$ by $\hat{G}_0$.

Denote by K the generator of $(S_0(t))_{t\le 0}$. By the equality (1.16) and Lemma 1.11, we obtain that

$$\begin{aligned} Kf &= \lim_{t\uparrow 0}\frac{S_0(t)f-f}{t} = \lim_{t\downarrow 0} -\frac{\hat{T}_0(t)f-f}{t} \\ &= -\hat{G}_0 f = I_X^- f \quad \text{for } f\in D(K) \text{ or } f\in D(I_X^-), \end{aligned}$$

hence, $K = I_X^-$ (here we use the fact that $D(I_X^-)\subseteq \mathrm{Im}\mathcal{P}\subseteq C_{00}$).

4) We now contruct a strongly continuous, exponentially decaying right semigroup $(S(t))_{t\ge 0}$ on $\ker\mathcal{P}$. Note that

$$\ker\mathcal{P} = \{f\in C_0 : f(t)\in\ker P(t),\ t\ge 0\}.$$

We define the right semigroup $(S(t))_{t\ge 0}$ on $\ker\mathcal{P}$ by

$$(S(t)f)(s) = U_Q(s,s+t)f(s+t) \quad \text{for } t,s\ge 0.$$

By the definition of $U_Q(s,t)$, we have that $(S(t)f)(s) = U_Q(s,s+t)f(s+t)\in\ker P(s)$ for $f\in\ker\mathcal{P}$ and $t,s\ge 0$. Therefore, $S(t):\ker\mathcal{P}\to\ker\mathcal{P}$. By the exponential dichotomy of $(U(t,s))_{t\ge s\ge 0}$ and Lemma 1.13, we can easily see that $(S(t))_{t\ge 0}$ is a strongly continuous, exponentially decaying right semigroup satisfying

$$\|S(t)\|\le Ne^{-\nu t},\ t\ge 0.$$

5) We conclude the proof by showing that the generator of $(S(t))_{t\ge 0}$ is $I_X\,|_{\ker\mathcal{P}} =: I_X^+$. Denote by L the generator of $(S(t))_{t\ge 0}$. We shall prove the following:

Let $u,f\in\ker\mathcal{P}$. Then $u\in D(L)$ and $Lu=f$ if and only if (1.9) holds.

In fact, let $Lu=f$. The general theory of linear semigroups (see, e.g., [**23**, Lemma II.1.3]) yields

$$S(\tau)u-u = \int_0^\tau S(\xi)Lud\xi = \int_0^\tau S(\xi)fd\xi \text{ for } \tau\ge 0.$$

Thus

$$S(\tau)u = u + \int_0^\tau S(\xi)fd\xi,$$

hence

$$U_Q(s,s+\tau)u(s+\tau) = u(s) + \int_0^\tau U_Q(s,s+\xi)f(s+\xi)d\xi \text{ for } s,t\ge 0.$$

Applying $U_Q(s+\tau,s)$ on both sides and noting that $u,f\in\ker\mathcal{P}$ we have

$$u(s+\tau) = U(s+\tau,s)u(s) + \int_0^\tau U(s+\tau,s+\xi)f(s+\xi)d\xi.$$

Putting $t := s+\tau$ we obtain the equation (1.9).

Conversely, if $u,f\in\ker\mathcal{P}$ satisfy the equation (1.9), then by reversing the above argument we obtain

$$S(t)u-u = \int_0^t S(\xi)fd\xi \text{ for } t\ge 0.$$

In particular, this implies $u\in D(L)$ and $Lu=f$.

By the definition of I_X^+ we have that $L = I_X^+$. □

Our next result characterizes the exponential dichotomy of an evolution family by the quasi-exponential dichotomy of the operator I_Z (see [**41**]).

THEOREM 1.16. *Let* $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ *be an evolution family on the Banach space* X *and let* $Z \neq \{0\}$ *be a closed subspace of* X. *Then the following assertions are equivalent.*

(i) $\mathcal{U}$ *has an exponential dichotomy with the corresponding dichotomy projections* $(P(t))_{t\geq 0}$ *satisfying* $\ker P(0) = Z$.
(ii) I_Z *is quasi-exponentially dichotomous.*

PROOF. (ii)⇒(i): Assume that I_Z is a quasi-exponentially dichotomous operator of exponential type $\omega < 0$. Then $I_Z = I_Z^- \oplus I_Z^+$. Here, I_Z^- is a left Hille-Yosida operator. Furthermore, since I_Z^+ is the generator of an exponentially decaying right semigroup of exponential type $\omega < 0$, we have that $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \omega\} \subset \rho(I_Z^+)$. Thus, $\{\lambda \in \mathbb{C} : |\mathrm{Re}\lambda| < -\omega\} \subset \rho(I_Z)$. In particular, I_Z is invertible. By Theorem 1.14, $\mathcal{U}$ has an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geq 0}$ satisfying $\ker P(0) = Z$.

(i)⇒(ii): We prove this in several steps.

1) Let the condition (i) be satisfied. We define $\mathcal{P} : C_0 \to C_0$ by $(\mathcal{P}f)(t) = P(t)f(t)$ for $f \in C_0$. Then $\mathcal{P}$ is a projection on C_0. We now prove that

$$\mathcal{P} I_Z = I_Z \mathcal{P} \mid_{D(I_Z)} . \tag{1.18}$$

From the definition of I_Z and the equality $P(t)U(t,s) = U(t,s)P(s)$, we have

$$P(t)u(t) = U(t,s)P(s)u(s) + \int_s^t U(t,\xi)P(\xi)[I_Z(u)](\xi)d\xi \text{ for } u \in D(I_Z),$$

and $P(0)u(0) = 0 \in Z$ since $u(0) \in Z$.
Again by definition of I_Z, we obtain that $P(\cdot)u(\cdot) \in D(I_Z)$ and

$$I_Z(P(\cdot)u(\cdot)) = P(\cdot)[I_Z(u)](\cdot).$$

This yields (1.18).
Note that (1.18) includes $\mathcal{P}(D(I_Z)) \subseteq D(I_Z)$, whence,

$$D(I_Z) = [D(I_Z) \cap \mathrm{Im}\mathcal{P}] \oplus [D(I_Z) \cap \mathrm{Ker}\mathcal{P}].$$

2) Here we prove that $I_Z^- := I_Z \mid_{\mathrm{Im}\mathcal{P}}: \mathrm{Im}\mathcal{P} \to \mathrm{Im}\mathcal{P}$ is a left Hille-Yosida operator.
Note that

$$\mathrm{Im}\mathcal{P} = \{f \in C_0 : f(s) \in \mathrm{Im}P(s),\ s \geq 0\},$$

and

$$D(I_Z^-) = D(I_Z) \cap \mathrm{Im}\mathcal{P} \subseteq \mathrm{Im}\mathcal{P} \cap C_{00} \text{ since } \mathrm{Im}P(0) \cap Z = \{0\}. \tag{1.19}$$

It is easily seen that $(T_0(t))_{t\geq 0}$ leaves $\mathrm{Im}\mathcal{P} \cap C_{00}$ invariant. Hence, we may denote the restriction of $(T_0(t))_{t\geq 0}$ to $\mathrm{Im}\mathcal{P} \cap C_{00}$ by $(\tilde{T}_0(t))_{t\geq 0}$ and the generator of $(\tilde{T}_0(t))_{t\geq 0}$ by $\tilde{G}_0$. By Lemma 1.11, we have that $(-\tilde{G}_0, D(-\tilde{G}_0)) \subseteq (I_Z^-, D(I_Z^-))$. By the exponential dichotomy of $\mathcal{U}$, the semigroup $(\tilde{T}_0(t))_{t\geq 0}$ is exponentially decaying of exponential type $-\nu < 0$. Hence, by the Hille-Yosida Theorem, the inclusion $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > -\nu\} \subset \rho(\tilde{G}_0)$ holds. We will prove that the set $\mathcal{K} := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < \nu\}$ is contained in $\rho(I_Z^-)$.

In fact, for $\lambda \in \mathcal{K}$, if $f \in \mathrm{Im}\mathcal{P} \cap C_{00}$ and $(\lambda - I_Z^-)f = 0$, then $I_Z^- f = \lambda f \in \mathrm{Im}\mathcal{P} \cap C_{00}$. This implies that $\tilde{G}_0 f = -\lambda f$. This observation, together with the fact that $-\lambda \in \rho(\tilde{G}_0)$, yields that $f = 0$. Hence, $\lambda - I_Z^-$ is injective. To prove the surjectivity of $\lambda - I_Z^-$, we observe that for $f \in \mathrm{Im}\mathcal{P}$ and $\lambda \in \mathcal{K}$ the function

$$u(t) := \int_0^t e^{\lambda(t-\xi)} U(t,\xi) f(\xi) d\xi \tag{1.20}$$

belongs to $\mathrm{Im}\mathcal{P} \cap C_{00}$ and satisfies the equation

$$u(t) = \int_0^t U(t,\xi)(\lambda u(\xi) + f(\xi)) d\xi, \; t \geq 0.$$

This yields $I_Z^- u = \lambda u + f$. Thus, $\lambda - I_Z^-$ is surjective. Therefore, we obtain $\lambda \in \rho(I_Z^-)$.

We now show the resolvent estimate (1.14) for the resolvent $R(\lambda; I_Z^-)$. Indeed, by the formula (1.20), the resolvent $R(\lambda; I_Z^-)$ is given by

$$[R(\lambda; I_Z^-) f](t) = -\int_0^t e^{\lambda(t-\xi)} U(t,\xi) f(\xi) d\xi \quad \text{for } \mathrm{Re}\lambda < \nu \text{ and } f \in \mathrm{Im}\mathcal{P}. \tag{1.21}$$

By this expression, we obtain that

$$\begin{aligned} \frac{d}{d\lambda}[R(\lambda; I_Z^-) f](t) &= -\frac{d}{d\lambda} \int_0^t e^{\lambda(t-\xi)} U(t,\xi) f(\xi) d\xi \\ &= -\int_0^t (t-\xi) e^{\lambda(t-\xi)} U(t,\xi) f(\xi) d\xi \\ &\qquad \text{for} \quad \mathrm{Re}\lambda < \nu \text{ and } f \in \mathrm{Im}\mathcal{P}. \end{aligned}$$

Proceeding by induction, we obtain

$$\frac{d^n}{d\lambda^n}[R(\lambda; I_Z^-) f](t) = -\int_0^t (t-\xi)^n e^{\lambda(t-\xi)} U(t,\xi) f(\xi) d\xi \quad \text{for } n \in \mathbb{N}. \tag{1.22}$$

On the other hand, from the resolvent identity

$$R(\lambda; I_Z^-) - R(\mu; I_Z^-) = (\mu - \lambda) R(\lambda; I_Z^-) R(\mu; I_Z^-)$$

it follows that for every $\lambda \in \rho(I_Z^-)$, the mapping $\lambda \to R(\lambda; I_Z^-)$ is holomorphic and

$$\frac{d}{d\lambda} R(\lambda; I_Z^-) = -R(\lambda; I_Z^-)^2.$$

Proceeding again by induction, we find

$$\frac{d^n}{d\lambda^n} R(\lambda; I_Z^-) = (-1)^n n! R(\lambda; I_Z^-)^{n+1}. \tag{1.23}$$

Comparing (1.22) and (1.23) yields

$$[R(\lambda; I_Z^-)^n f](t) = (-1)^n \frac{1}{(n-1)!} \int_0^t (t-\xi)^{n-1} e^{\lambda(t-\xi)} U(t,\xi) f(\xi) d\xi.$$

Hence,

$$\begin{aligned} \|[R(\lambda; I_Z^-)^n f](t)\| &\leq \frac{N}{(n-1)!} \int_0^t (t-\xi)^{n-1} e^{(\mathrm{Re}\lambda - \nu)(t-\xi)} \|f\| d\xi \\ &= \frac{N}{(n-1)!} \int_0^t \eta^{n-1} e^{(\mathrm{Re}\lambda - \nu)\eta} \|f\| d\eta \\ &\qquad \text{(by changing variable } \eta := t - \xi) \end{aligned}$$

$$\leq \frac{N}{(n-1)!}\int_0^\infty \eta^{n-1}e^{(\mathrm{Re}\lambda-\nu)\eta}\|f\|d\eta$$
$$= \frac{N}{(\nu-\mathrm{Re}\lambda)^n}\|f\| \quad \text{for all } t\geq 0,\ \mathrm{Re}\lambda<\nu,\ \text{and } n\in\mathbb{N}.$$

Therefore,

$$\|R(\lambda;I_Z^-)^n\| \leq \frac{N}{(\nu-\mathrm{Re}\lambda)^n} \quad \text{for } \mathrm{Re}\lambda<\nu \text{ and } n\in\mathbb{N}.$$

3) By the same way as in proof of Theorem 1.15 (parts **4)**, **5)**) (with the condition $Z\neq\{0\}$) we can prove that the operator $I_Z^+ := I_Z\mid_{\ker\mathcal{P}}\colon \ker\mathcal{P}\to\ker\mathcal{P}$ is the generator of an exponentially decaying right semigroup defined on $\ker\mathcal{P}$ by

$$(S(t)f)(s) = U_Q(s,s+t)f(s+t),\ t,s\geq 0.$$

□

3. Perturbations

In this section we study the robustness of the exponential dichotomy of evolution families under small perturbations. More precisely, let H be a strongly continuous and uniformly bounded function from $\mathbb{R}_+$ into the space $\mathcal{L}(X)$. Then it is known (see [**23**, Theorem VI.9.19]) that there exists a unique evolution family $(U_H(t,s))_{t\geq s\geq 0}$ satisfying the variation of constants formula

$$U_H(t,s)x = U(t,s)x + \int_s^t U(t,\xi)H(\xi)U_H(\xi,s)x d\xi, \quad t\geq s\geq 0, x\in X. \tag{1.24}$$

We will prove that, if $(U(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy and the norm $\|H(\cdot)\| := \sup_{t\geq 0}\|H(t)\|$ is sufficiently small, then $(U_H(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy as well. We note that, if we consider $(U(t,s))_{t\geq s\geq 0}$ to be "generated" by a concrete well-posed non-autonomous Cauchy problem, e.g., as in (1.1), then we have that $(U_H(t,s))_{t\geq s\geq 0}$ is "generated" by the perturbed problem of (1.1), i.e., by

$$\begin{cases} \frac{du(t)}{dt} &= (A(t)+H(t))u(t),\ t\geq s\geq 0,\\ u(s) &= x_s\in X. \end{cases} \tag{1.25}$$

Therefore, our result reveals that the exponential dichotomy of the solutions to the problem (1.1) is robust under small perturbations by bounded operators $H(t)$. We also note that, if we consider the evolution family $(U(t,s))_{t\geq s}$ on $\mathbb{R}$, then the result is well-known (see [**23**, Theorem VI.9.24] and references therein).

THEOREM 1.17. *Let the evolution family $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ have an exponential dichotomy and let H be a strongly continuous and uniformly bounded function from $\mathbb{R}_+$ into the space $\mathcal{L}(X)$. Then, if the norm $\|H(\cdot)\| := \sup_{t\geq 0}\|H(t)\|$ is sufficiently small, the evolution family $\mathcal{U}_H = (U_H(t,s))_{t\geq s\geq 0}$ defined as in (1.24) has an exponential dichotomy as well.*

PROOF. Since $\mathcal{U}$ has an exponential dichotomy, by Theorem 1.14, we have that for $f\in C_0$ the equation (1.9) has at least a solution $u(\cdot)$ given by (see [**55**, (4.1)])

$$u(t) = \int_0^t U(t,\xi)f(\xi)d\xi - \int_t^\infty U_Q(t,\xi)Q(\xi)f(\xi)d\xi, \quad t\geq 0. \tag{1.26}$$

We now define the Green's function

$$G(t,\tau) := \begin{cases} P(t)U(t,\tau) & \text{for } t \ge \tau \ge 0, \\ -U_Q(t,\tau)Q(\tau) & \text{for } 0 \le t \le \tau. \end{cases}$$

Then the formula (1.26) can be rewritten as

$$u(t) = \int_0^\infty G(t,\xi)f(\xi)d\xi, \quad t \ge 0. \tag{1.27}$$

We thus obtain the general form of the solutions belonging to C_0 of the equation (1.9) by adding to the solution already obtained an arbitrary term of the form $U(t,0)y$ for some arbitrary element y of the subspace $X_0(0)$ defined in (1.10).

Therefore, all the solutions belonging to C_0 of equation (1.9) are represented by the formula

$$u(t) = U(t,0)y + \int_0^\infty G(t,\xi)f(\xi)d\xi, \ t \ge 0, \tag{1.28}$$

where $y = P(0)u(0)$ is an arbitrary element of the subspace $X_0(0)$.

In the equation (1.24), we now put $x(t) := U_H(t,0)x$. Then, this can be rewritten as

$$x(t) = U(t,0)x + \int_0^t U(t,\xi)f(\xi)d\xi, \quad t \ge 0, \tag{1.29}$$

corresponding to the function $f(t) = H(t)x(t)$. The formula (1.28) permits us to represent the solution belonging to C_0 in the form

$$x(t) = U(t,0)y + \int_0^\infty G(t,\xi)H(\xi)x(\xi)d\xi, \quad t \ge 0, \tag{1.30}$$

for some $y \in X_0(0)$.

We now prove that, if $\|H(\cdot)\|$ is small enough, then we can represent this solution in a nicer form. More precisely, we can write

$$x(t) = \Gamma(t)y, \quad y \in X_0(0), \tag{1.31}$$

for some bounded operator-valued function $\Gamma(t)$ on $\mathbb{R}_+$. To that purpose, we consider the operator $S : C_0 \to C_0$ defined as

$$[Su](t) := \int_0^\infty G(t,\xi)H(\xi)u(\xi)d\xi, \ t \ge 0.$$

Using the exponential dichotomy of $\mathcal{U}$, we estimate the norm of S in $\mathcal{L}(C_0)$ by

$$\begin{aligned} \|Su\| &\le N\|H(\cdot)\| \sup_{t\ge 0} \int_0^\infty e^{-\nu|t-\xi|}d\xi\|u\| \\ &\le \frac{2}{\nu}N\|H(\cdot)\|\|u\| \quad \text{for } u \in C_0. \end{aligned}$$

Hence, $\|S\| \le \frac{2}{\nu}N\|H(\cdot)\|$. We note that the equation (1.30) can be written as

$$(I-S)x(\cdot) = U(\cdot,0)y \quad \text{for } y \in X_0(0). \tag{1.32}$$

Therefore, if $\|H(\cdot)\| < \frac{\nu}{2N}$, then $\|S\| < 1$ and hence the operator $I - S$ is invertible. Thus, the representation (1.31) follows with $\Gamma(\cdot)y := (I-S)^{-1}(U(\cdot,0)y)$ for $y \in X_0(0)$.

Furthermore, by applying Neumann's series for $(I-S)^{-1}$, we obtain

$$\|\Gamma(t)\| \leq \frac{N}{1-\frac{2N\|H(\cdot)\|}{\nu}} \quad \text{for all } t \geq 0. \tag{1.33}$$

We now define the *stable subspaces* of X corresponding to the perturbed evolution family $\mathcal{U}_H$ by

$$\tilde{X}_0(t_0) := \{x \in X : \lim_{t\to\infty} U_H(t,t_0)x = 0\} \quad \text{for } t_0 \geq 0. \tag{1.34}$$

We prove next that $\tilde{X}_0(0)$ is closed and complemented. To that purpose, let $x \in \tilde{X}_0(0)$ be arbitrary. Then the function $U_H(\cdot,0)x$ is a solution belonging to C_0 of the equation (1.29). Therefore, by (1.30) and (1.31) we have

$$x = U_H(0,0)x = y + \int_0^\infty G(0,\xi)H(\xi)\Gamma(\xi)y\,d\xi = (I - Q(0)RP(0))y,$$

where $y \in X_0(0)$, and

$$R = \int_0^\infty Q(0)U_Q(0,\xi)Q(\xi)H(\xi)\Gamma(\xi)P(0)\,d\xi$$

is a bounded operator. By Lemma 1.13 and the estimate (1.33) we have that

$$\|R\| \leq \frac{M^2N^2\|H(\cdot)\|}{\nu - 2N\|H(\cdot)\|}. \tag{1.35}$$

Therefore, the bounded operator $I - Q(0)RP(0)$ maps the subspace $X_0(0)$ onto $\tilde{X}_0(0)$. This operator has the bounded inverse

$$(I - Q(0)RP(0))^{-1} = I + Q(0)RP(0),$$

and hence the subspace $\tilde{X}_0(0)$ is closed. The operator

$$\begin{aligned}\tilde{P}(0) &= (I - Q(0)RP(0))P(0)(I - Q(0)RP(0))^{-1}\\ &= (I - Q(0)RP(0))P(0)(I + Q(0)RP(0)) = P(0) - Q(0)RP(0)\end{aligned}$$

is a projection whose range coincides with $\tilde{X}_0(0)$. The complementary projection has the form

$$\tilde{Q}(0) = I - \tilde{P}(0) = Q(0) + Q(0)RP(0) = Q(0)(I + RP(0)),$$

showing that $\tilde{X}_1(0) = X_1(0)$.

We estimate next the trajectories $U_H(t,0)x$ with the initial value x belonging to the subspaces $\tilde{X}_0(0)$ and $\tilde{X}_1(0)$, respectively.

We first consider the trajectories $U_H(t,0)x$ with $x \in \tilde{X}_0(0)$. Putting $x(t) := U_H(t,0)x$, by (1.30) we have that, for $t \geq s \geq 0$,

$$\begin{aligned}x(t) &= U(t,0)y + \int_0^\infty G(t,\xi)H(\xi)x(\xi)\,d\xi \quad \text{for some } y \in X_0(0)\\ &= P(t)U(t,0)y + \int_0^\infty G(t,\xi)H(\xi)x(\xi)\,d\xi\\ &= P(t)U(t,s)\left(U(s,0)y + \int_0^\infty G(s,\xi)H(\xi)x(\xi)\,d\xi\right)\\ &\quad - \int_0^\infty P(t)U(t,s)G(s,\xi)H(\xi)x(\xi)\,d\xi + \int_0^\infty G(t,\xi)H(\xi)x(\xi)\,d\xi\end{aligned}$$

$$\begin{aligned} &= P(t)U(t,s)x(s) - \int_0^s P(t)U(t,s)P(s)U(s,\xi)H(\xi)x(\xi)d\xi \\ &\quad + \int_s^\infty P(t)U(t,s)U_Q(s,\xi)Q(\xi)H(\xi)x(\xi)d\xi + \int_0^\infty G(t,\xi)H(\xi)x(\xi)d\xi \\ &= P(t)U(t,s)x(s) + \int_s^\infty G(t,\xi)H(\xi)x(\xi)d\xi. \end{aligned}$$

Therefore, by the exponential dichotomy of $\mathcal{U}$ we obtain that

$$\|x(t)\| \le Ne^{-\nu(t-s)}\|x(s)\| + N\|H(\cdot)\| \int_s^\infty e^{-\nu|t-\xi|}\|x(\xi)\|d\xi \quad \text{for } t \ge s \ge 0.$$

Applying now Gronwall's Lemma from [**16**, Lemma III.2.2] we have that there exist positive constants N_1 and μ such that

$$\|U_H(t,0)x\| \le N_1 e^{-\mu(t-s)}\|U_H(s,0)x\| \quad \text{for } x \in \tilde{X}_0(0) \text{ and } t \ge s \ge 0. \tag{1.36}$$

Similarly, if we consider the trajectories $U_H(t,t_0)x$ with $x \in \tilde{X}_0(t_0)$ and $t_0 \ge 0$, then we obtain that there exist positive constants N_1 and μ such that

$$\|U_H(t,t_0)x\| \le N_1 e^{-\mu(t-s)}\|U_H(s,t_0)x\| \quad \text{for } x \in \tilde{X}_0(t_0) \text{ and } t \ge s \ge t_0 \ge 0. \tag{1.37}$$

We estimate next the trajectories $U_H(t,0)x$ with $x \in \tilde{X}_1(0) = X_1(0)$. Putting $x(t) := U_H(t,0)x$, by (1.24) we have that, for $0 \le t \le s$,

$$\begin{aligned} x(t) &= U(t,0)x + \int_0^t U(t,\xi)H(\xi)x(\xi)d\xi \\ &= Q(t)U(t,0)x + \int_0^t U(t,\xi)H(\xi)x(\xi)d\xi \\ &= Q(t)U_Q(t,s)Q(s)\left(U(s,0)x + \int_0^s U(s,\xi)H(\xi)x(\xi)d\xi\right) \\ &\quad -Q(t)U_Q(t,s)Q(s)\int_0^s U(s,\xi)H(\xi)x(\xi)d\xi + \int_0^t U(t,\xi)H(\xi)x(\xi)d\xi \\ &= Q(t)U_Q(t,s)Q(s)x(s) - \int_0^s U_Q(t,s)Q(s)U(s,\xi)H(\xi)x(\xi)d\xi \\ &\quad + \int_0^t (P(t)+Q(t))U(t,\xi)H(\xi)x(\xi)d\xi \\ &= Q(t)U_Q(t,s)Q(s)x(s) - \int_0^t Q(t)U(t,\xi)H(\xi)x(\xi)d\xi \\ &\quad - \int_t^s U_Q(t,\xi)Q(\xi)H(\xi)x(\xi)d\xi + \int_0^t (P(t)+Q(t))U(t,\xi)H(\xi)x(\xi)d\xi \\ &= U_Q(t,s)Q(s)x(s) + \int_0^t P(t)U(t,\xi)H(\xi)x(\xi)d\xi - \int_t^s U_Q(t,\xi)Q(\xi)H(\xi)x(\xi)d\xi. \end{aligned}$$

Here we use the fact that

$$U_Q(t,s)Q(s)U(s,\xi) = \begin{cases} U_Q(t,\xi)Q(\xi) & \text{for } 0 \le t \le \xi \le s, \\ Q(t)U(t,\xi) & \text{for } 0 \le \xi \le t \le s. \end{cases}$$

Therefore, by the exponential dichotomy of $\mathcal{U}$, we obtain that

$$\|x(t)\| \le Ne^{-\nu(s-t)}\|x(s)\| + N\|H(\cdot)\| \int_0^s e^{-\nu|t-\xi|}\|x(\xi)\|d\xi \quad \text{for } 0 \le t \le s.$$

By this inequality, it follows from [**16**, Corollary III.2.3] that there exist positive constants N_1 and μ such that

$$\|U_H(t,0)x\| \le N_1 e^{-\mu(s-t)}\|U_H(s,0)x\| \quad \text{for } x \in \tilde{X}_1(0) \text{ and } s \ge t \ge 0. \tag{1.38}$$

Finally, we put $\tilde{X}_1(t) := U_H(t,0)\tilde{X}_1(0)$ for $t \ge 0$. Then it is straightforward to see that $X = \tilde{X}_0(t) \oplus \tilde{X}_1(t)$. Let now $\tilde{P}(t)$ be the projection from X onto $\tilde{X}_0(t)$ with kernel $\tilde{X}_1(t)$. Then, by (1.37), (1.38), and by the definitions of $\tilde{X}_0(t)$ and $\tilde{X}_1(t)$, we have that the evolution family $\mathcal{U}_H = (U_H(t,s))_{t\ge s\ge 0}$ has an exponential dichotomy with dichotomy projections $(\tilde{P}(t))_{t\ge 0}$. □

The following perturbation theorem is of different nature. That is, we consider the same evolution family $(U(t,s))_{t\ge s\ge 0}$ on X but with different subspaces Z. More precisely, for closed subspaces Z_1 and Z_2 of X, we shall investigate the relation between the invertible operators I_{Z_1} and I_{Z_2} associated to the evolution family having exponential dichotomies with the corresponding dichotomy projections $(P_1(t))_{t\ge 0}$ and $(P_2(t))_{t\ge 0}$ satisfying $\ker P_1(0) = Z_1$ and $\ker P_2(0) = Z_2$, respectively. The following theorem can be considered as an extension of the results known for finite dimensional spaces (see [**9**, Theorem 1.3]).

THEOREM 1.18. *Let Z_1 and Z_2 be closed subspaces of X. Let I_{Z_1} and I_{Z_2} be operators defined as in (1.7) corresponding to Z_1 and Z_2, respectively. Assume that I_{Z_1} and I_{Z_2} are invertible. Let $(P_1(t))_{t\ge 0}$ and $(P_2(t))_{t\ge 0}$ be the corresponding dichotomy projections defined by the exponential dichotomies of the evolution family $\mathcal{U} = (U(t,s))_{t\ge s\ge 0}$ and satisfy* $\ker P_1(0) = Z_1$ *and* $\ker P_2(0) = Z_2$*, respectively.*
Then $X = Z_2 \oplus ImP_1(0)$ and

$$I_{Z_2}^{-1} = I_{Z_1}^{-1} - KMI_{Z_1}^{-1}. \tag{1.39}$$

Here, the operator $K : X \to C_0$ is given by

$$[Kx](t) := U(t,0)P_2(0)x,\ x \in X,\ t \ge 0, \tag{1.40}$$

and $M : C_0 \to X$ is given by $Mf := f(0)$.

PROOF. From the proof of [**55**, Theorem 4.5, Corollary 3.3], we have that

$$\mathrm{Im}P_1(0) = \mathrm{Im}P_2(0) = X_0(0) := \{x \in X : \lim_{t\to\infty} U(t,0)x = 0\}.$$

Hence, $X = Z_2 \oplus \mathrm{Im}P_2(0) = Z_2 \oplus \mathrm{Im}P_1(0)$. We define the operator B as

$$\begin{aligned} B : X &\to C_0 \\ [Bx](t) &:= U(t,0)P_1(0)x \text{ for all } x \in X. \end{aligned}$$

By the exponential dichotomy of $\mathcal{U}$, the operator B in an element of $\mathcal{L}(X, C_0)$. By Lemma 1.11 (d), we obtain

$$\ker I_X = \mathrm{Im}B. \tag{1.41}$$

The equality $\mathrm{Im}P_1(0) = \mathrm{Im}P_2(0)$ yields $P_2(0) = P_1(0)P_2(0)$. This implies that $K = BP_2(0)$. Note that $\mathrm{Im}B = \ker I_X$, and $MB = P_1(0)$. In addition, $BP_1(0) = B$. Hence,

$$BMB = B. \tag{1.42}$$

Let now $g \in C_0$ be arbitrary. Since I_X extends both I_{Z_1} and I_{Z_2}, we have that

$$I_X(I_{Z_2}^{-1} - I_{Z_1}^{-1})g = I_{Z_2}I_{Z_2}^{-1}g - I_{Z_1}I_{Z_1}^{-1}g = 0.$$

By (1.41), this implies that there exists $x \in X$ such that

$$(I_{Z_2}^{-1} - I_{Z_1}^{-1})g = Bx. \tag{1.43}$$

By (1.42), this leads to

$$(I_{Z_2}^{-1} - I_{Z_1}^{-1})g = BMBx = BM(I_{Z_2}^{-1} - I_{Z_1}^{-1})g. \tag{1.44}$$

Note that (1.43) and $MB = P_1(0)$ imply that

$$M(I_{Z_2}^{-1} - I_{Z_1}^{-1})g = MBx = P_1(0)x \in \mathrm{Im}P_1(0).$$

This, together with the fact that $\mathrm{Im}P_1(0) = \mathrm{Im}P_2(0)$, yields

$$M(I_{Z_2}^{-1} - I_{Z_1}^{-1})g = P_2(0)M(I_{Z_2}^{-1} - I_{Z_1}^{-1})g. \tag{1.45}$$

Since $I_{Z_2}^{-1}g \in D(I_{Z_2}) \subseteq C_{Z_2}$, we have that $MI_{Z_2}^{-1}g \in Z_2$.
However, $Z_2 = \ker P_2(0)$, thus, $P_2(0)MI_{Z_2}^{-1}g = 0$. Hence, (1.45) implies that

$$M(I_{Z_2}^{-1} - I_{Z_1}^{-1})g = -P_2(0)MI_{Z_1}^{-1}g. \tag{1.46}$$

It follows from (1.46) and (1.44) that

$$(I_{Z_2}^{-1} - I_{Z_1}^{-1})g = -BP_2(0)MI_{Z_1}^{-1}g. \tag{1.47}$$

Since $K = BP_2(0)$, this leads to

$$I_{Z_2}^{-1}g = I_{Z_1}^{-1}g - KMI_{Z_1}^{-1}g. \tag{1.48}$$

However, $g \in C_0$ is arbitrary, thus, (1.39) holds. □

4. An example

We illustrate our results by the following example.

EXAMPLE 1.19. We consider the problem

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \sum_{k,l=1}^{n} D_k a_{kl}(t,x) D_l u(t,x) + \delta u(t,x) + b(t,x)u(t,x) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \geq s \geq 0,\ x \in \Omega \\ \sum_{k,l=1}^{n} n_k(x) a_{kl}(t,x) D_l u(t,x) = 0,\ t \geq s \geq 0,\ x \in \partial\Omega \\ u(s,x) = f(x), \quad x \in \Omega. \end{cases} \tag{1.49}$$

Here $D_k := \frac{\partial}{\partial x_k}$ and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ oriented by outer unit normal vectors $n(x)$. The coefficients $a_{k,l}(t,x) \in C_b^{\mu}(\mathbb{R}_+, L^\infty(\Omega))$, $\mu > \frac{1}{2}$, are supposed to be real, symmetric, and uniformly elliptic in the sense that

$$\sum_{k,l=1}^{n} a_{kl}(t,x) v_k v_l \geq \eta |v|^2 \quad \text{for a.e. } x \in \Omega \text{ and some constant } \eta > 0,$$

while the coefficient $b(t,x)$ belongs to $C_b(\mathbb{R}_+, L^\infty(\Omega))$. Finally, the constant δ is defined by

$$\delta := -\frac{1}{2}\eta\lambda,$$

where $\lambda < 0$ denotes the largest eigenvalue of Neumann Laplacian Δ_N on Ω. We now chose the Hilbert space $X = L^2(\Omega)$ and define the operators $C(t)$ via the standard scalar product in X as

$$(C(t)f, g) = -\sum_{k,l=1}^{n} \int_\Omega a_{kl} D_k f(x)(t,x)\overline{D_l g(t,x)}dx$$

with $D(C(t)) = \{f \in W^{2,2}(\Omega) : \sum_{k,l}^n n_k(x)a_{kl}(t,x)D_l f(x) = 0,\ x \in \partial\Omega\}$. We then write the problem (1.49) as an abstract Cauchy problem

$$\begin{cases} \frac{d}{dt}u(t,\cdot) & = A(t)u(t,\cdot) + H(t)u(t,\cdot),\ t \geq s \geq 0 \\ u(s,\cdot) & = f \in X, \end{cases}$$

where $A(t) := C(t)+\delta$ and $H(t) : X \to X$ defined by $(H(t)f)(x) := b(t,x)f(x)$ for $f \in X$.

By Schnaubelt [**74**, Chap. 2, Theorem 2.8, Example 2.3], we have that the the operators $A(t)$ generate an evolution family having an exponential dichotomy with the dichotomy exponent ν and dichotomy constant N provided that the Hölder constants of $a_{k,l}$ is sufficiently small. By Theorem 1.17 we now obtain that, if

$$\sup_{t\geq 0} \|b(t,\cdot)\|_{L^\infty(\Omega)} < \frac{\nu}{2N},$$

then the evolution family solving the problem (1.49) also has an exponential dichotomy.

5. Evolution semigroups and evolution families on $\mathbb{R}_-$

For later applications we briefly consider *backward evolution families* defined on $\mathbb{R}_-$ as follows.

DEFINITION 1.20. A family of bounded linear operators $\mathcal{U} = (U(t,s))_{t\leq s\leq 0}$ on a Banach space X is called a *(strongly continuous, exponentially bounded) backward evolution family* on $\mathbb{R}_-$ if

(i) $U(t,t) = Id$ and $U(t,r)U(r,s) = U(t,s)$ for $t \leq r \leq s \leq 0$,
(ii) the map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$,
(iii) there are constants $N \geq 1$ and $\omega_1 \in \mathbb{R}$ such that $\|U(t,s)\| \leq Ne^{\omega_1(s-t)}$ for $t \leq s \leq 0$.

The constant

$$\omega(\mathcal{U}) := \inf\{\alpha \in \mathbb{R} : \exists N \geq 1 \text{ such that } \|U(t,s)\| \leq Ne^{\alpha(s-t)} \quad \text{for all } t \leq s \leq 0\}$$

is called the *growth bound* of $\mathcal{U}$.

This notion of backward evolution families arises when we consider well-posed evolution equations on the negative half-line $\mathbb{R}_-$

$$\begin{cases} \frac{du(t)}{dt} & = -A(t)u(t),\ t \leq s \leq 0, \\ u(s) & = x_s \in X. \end{cases} \tag{1.50}$$

More precisely, we will say that the Cauchy problem (1.50) is well-posed with exponential bound if there exists an exponentially bounded backward evolution family $\mathcal{U} = (U(t,s))_{t\leq s\leq 0}$ solving (1.50), i.e., the solutions of (1.50) are given by $x(t) = U(t,s)x(s)$ for $t \leq s \leq 0$. Clearly, for backward evolution families on $\mathbb{R}_-$, we have the same results as in the case of "forward" evolution families on $\mathbb{R}_+$. For later use, we summarize the construction of the corresponding left translation evolution semigroups and some auxiliary results.

First, the evolution family $(U(t,s))_{t\le s\le 0}$ is extended to a backward evolution family on $\mathbb{R}$ by setting

$$\tilde{U}(t,s) := \begin{cases} U(t,s) & \text{for } t \le s \le 0, \\ U(t,0) & \text{for } t \le 0 \le s, \\ U(0,0) = Id & \text{for } 0 \le t \le s. \end{cases}$$

DEFINITION 1.21. On $\tilde{E} := C_0(\mathbb{R}, X)$, we define the *left translation evolution semigroup* $(\tilde{T}(t))_{t\ge 0}$ corresponding to $(\tilde{U}(t,s))_{t\le s}$ by

$$(\tilde{T}(t)\tilde{f})(s) := \tilde{U}(s,s+t)\tilde{f}(s+t) = \begin{cases} U(s,s+t)\tilde{f}(s+t) & \text{for } s \le s+t \le 0, \\ U(s,0)\tilde{f}(s+t) & \text{for } s \le 0 \le s+t, \\ \tilde{f}(s+t) & \text{for } 0 \le s \le s+t, \end{cases}$$

for all $\tilde{f} \in \tilde{E}, s \in \mathbb{R}, t \ge 0$.

We also denote its generator by $(\tilde{G}, D(\tilde{G}))$.

As in Lemma 1.3, we then have following properties of the operator $(\tilde{G}, D(\tilde{G}))$.

LEMMA 1.22. *For $\tilde{u}$, $\tilde{f}$ in $\tilde{E}$ and $\lambda \in \mathbb{C}$ the following assertions hold.*

(i) $\tilde{u} \in D(\tilde{G})$ *and* $(\lambda - \tilde{G})\tilde{u} = \tilde{f}$ *if and only if* $\tilde{u}$ *and* $\tilde{f}$ *satisfy the integral equation*

$$\tilde{u}(t) = e^{\lambda(t-s)}\tilde{U}(t,s)\tilde{u}(s) + \int_t^s e^{\lambda(t-\xi)}\tilde{U}(t,\xi)\tilde{f}(\xi)d\xi \qquad \text{for all } t \le s. \tag{1.51}$$

(ii) *The operator* $(\tilde{G}, D(\tilde{G}))$ *is a* local operator *in the sense that for* $\tilde{u} \in D(\tilde{G})$ *and* $\tilde{u}(s) = 0$ *for all* $a < s < b$ *we have that* $[\tilde{G}\tilde{u}](s) = 0$ *for all* $a < s < b$.

The locality of $\tilde{G}$ allows us to define an operator G on $E := C_0(\mathbb{R}_-, X)$ as follows.

DEFINITION 1.23. Take

$$D(G) := \{\tilde{f}|_{\mathbb{R}_-} : \tilde{f} \in D(\tilde{G})\}$$

and define

$$[Gf](t) := [\tilde{G}\tilde{f}](t) \quad \text{for } t \le 0 \text{ and } f = \tilde{f}|_{\mathbb{R}_-}.$$

Analogously to Lemma 1.22, we now have the following description of G.

LEMMA 1.24. *Let* $u, f \in E = C_0(\mathbb{R}_-, X)$, *and* $\lambda \in \mathbb{C}$. *Then* $u \in D(G)$ *and* $(\lambda - G)u = f$ *if and only if* u *and* f *satisfy*

$$u(t) = e^{\lambda(t-s)}U(t,s)u(s) + \int_t^s e^{\lambda(t-\xi)}U(t,\xi)f(\xi)d\xi \qquad \text{for } t \le s \le 0. \tag{1.52}$$

Similarly to Definition 1.7, for an operator B as in Assumption 1.6 we construct the following left translation evolution semigroup which will play an important role in our investigation of delay partial differential equations with non-autonomous past (see Chapter 2).

DEFINITION 1.25. On the space $E = C_0(\mathbb{R}_-, X)$ we define a left translation evolution semigroup $(T_{B,0}(t))_{t\geq 0}$ by

$$[T_{B,0}(t)f](s) = \begin{cases} U(s, s+t)f(s+t) & \text{for } s+t \leq 0 \\ U(s,0)e^{(t+s)B}f(0) & \text{for } s+t \geq 0 \end{cases} \quad \text{for all } f \in E.$$

We denote its generator by $G_{B,0}$.

As in Proposition 1.8, we then have the following properties of $G_{B,0}$ and $(T_{B,0}(t))_{t\geq 0}$.

PROPOSITION 1.26. *The following assertions hold.*

(i) *The generator of $(T_{B,0}(t))_{t\geq 0}$ is given by*

$$\begin{aligned} D(G_{B,0}) &:= \{f \in D(G) : f(0) \in D(B) \text{ and } (G(f))(0) = Bf(0)\}, \\ G_{B,0}f &:= Gf \text{ for } f \in D(G_{B,0}). \end{aligned}$$

(ii) *The set $\{\lambda \in \mathbb{C} : Re\lambda > \omega(\mathcal{U}) \text{ and } \lambda \in \rho(B)\}$ is contained in $\rho(G_{B,0})$. Moreover, for λ in this set, the resolvent is given by*

$$[R(\lambda, G_{B,0})f](t) = e^{\lambda t}U(t,0)R(\lambda, B)f(0) + \int_t^0 e^{\lambda(t-\xi)}U(t,\xi)f(\xi)d\xi \quad \text{for } f \in E,\ t \leq 0.$$

(iii) *The semigroup $(T_{B,0}(t))_{t\geq 0}$ satisfies*

$$\|T_{B,0}(t)\| \leq Ke^{\omega t}, \quad t \geq 0,$$

with $K := MN$ and $\omega := \max\{\omega_1, \omega_2\}$ for the constants M, N, ω_1 and ω_2 appearing in Definition 1.20 and Assumption 1.6.

REMARK 1.27. The above notions and results will be used in Chapters 2 and 4 to study delay and neutral partial differential equations with non-autonomous past.

CHAPTER 2

Delay Partial Differential Equations

This chapter is devoted to the investigation of delay partial differential equations (DPDE's) with non-autonomous past (see the equations (2.2) and (2.3) below) and non-autonomous delay partial differential equations (see the equation (2.25) below). We note that DPDE's with non-autonomous past are special cases of non-autonomous DPDE's. We refer to Nickel and Rhandi [**61**] for information about the relation between DPDE's with non-autonomous past and non-autonomous DPDE's. In Section 1, we briefly explain why the study of partial functional differential equation plays an important role in many biological, chemical and physical systems. In Section 2, using the theory of evolution semigroups introduced in the previous chapter, we are able to obtain results on the well-posedness for linear DPDE's with non-autonomous past as well as on robustness of exponential stability and dichotomy of the solutions. In Section 3, we consider some concrete examples to illustrate our results. Finally, in Section 4, we extend our methods and results from Chapter 1 to study non-autonomous DPDE's with the delay operators acting on a finite interval $[-r, 0]$.

1. Motivation

Many biological, chemical and physical processes can be mathematically modeled via functional partial differential equations. In the following we will quote several examples to illustrate this fact.

Let us start with an example from population ecology. It is known that the central aspects of a model for the growth of a population in a spatially heterogeneous environment are rules for local growth or kinetics, and schemes for distribution of individuals or communication among local environments. The most familiar model incorporating these features are reaction diffusion equations. If we treat a single-species population moving along a line, then, using the linearization as in [**15**, Chap. 2] we are led to the following equation near the equilibrium states (see Wu [**81**, p. 5] and references therein for more details)

$$\frac{\partial u(x,t)}{\partial t} = a\frac{\partial^2 u(x,t)}{\partial x^2} + bu(t,x) + \int_{-T}^{0} [d\eta(s,t)]u(x,t+s).$$

Here, $u(x,t)$ is the density of the diffusing species, and the function of bounded variation $\eta(t,s)$ is the rates of exchange within the diffusing species.

The next example arises from control theory. As an example we consider the system of mechanisms of a computer using the information from the transducers to generate the appropriate control signals (see [**3, 79**]). Then, the controlled signals may be delayed in time due to the possible presence of time delays in actuation and in information transmission and processing. A simplified mathematical description of the overall control system may be given by (see Wang [**79**])

$$\frac{\partial u(x,t)}{\partial t} = a\frac{\partial^2 u(x,t)}{\partial x^2} + c[f(u(x,t-r)) - u(x,t)]$$

defined on a one-dimensional spatial domain $0 < x < 1$, where $u(x,t-r)$ is the time-delayed temperature distribution and f represents a distributed temperature source function depending on $u(x,t-r)$. In practice, it is of interest to determine the stability of its equilibrium states with respect to various perturbations using the principle of linearized stability and perturbation theory (see [**3, 4, 38, 76, 79**] and references therein).

Our last example comes from structured population models. Gyllenberg and Heijmans [**33**] (see also [**52, 80**]) considered a model for the dynamics of a population of single cells which can be distinguished from each other according to their size and the particular cell cycle phase they are in. For the concrete model in which the cell cycle consists of two distint phases they derived the following equation

$$\begin{aligned}\frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}[g(x)n(t,x)] &= -\mu(x)n(t,x) - b(x)n(t,x)\\ &\quad + \frac{2p(y^{-1}(x))b(y^{-1}(x))}{y'(y^{-1}(x))}n(t-\tau, y^{-1}(x)).\end{aligned}$$

Here t denote time and x the cell size. The unknown n is the size distribution of cells in the first phase. The functions g, μ, and b are the rate at which cells of size x grow, die, and transit to the second phase, respectively. The constant $\tau > 0$ is the constant duration of the second phase, $y(x)$ is the size of a new-born cell whose mother entered the second phase (exactly τ time units before) with size x, and $p(x)$ is the fraction of cells who survive the second phase given that they entered it with size x.

We refer to [**3, 4, 38, 52, 79, 80, 81**] for detailed discussions on various examples of biological, chemical and physical models which may be described via functional partial differential equations. With appropriate choices of phase spaces and partial differential operators the above equations may be formulated (linearized near the equilibrium states) in an abstract form as

$$\begin{cases} \frac{d}{dt}u(t) &= Bu(t) + \Phi u_t \quad \text{for } t \geq 0,\\ u_0 &= \varphi. \end{cases} \tag{2.1}$$

Here, the function $x(\cdot)$ takes values in a Banach space X, B is some linear (partial differential) operator on X, the *history function* $x_t : \mathbb{R}_- \to X$ is defined by

$$x_t(s) := x(s+t), \quad s \leq 0.$$

Finally the delay operator Φ is a linear operator from a space of X-valued functions on $\mathbb{R}_-$ into X. From [**23**, Corollary VI.6.3] we know that, if we choose a relevant space of initial data, e.g., $E := C_0(\mathbb{R}_-, X)$, then there exists a semigroup $(T_{B,\Phi}(t))_{t\geq 0}$ solving the equation (2.1) on E. If we now consider the function $u : \mathbb{R}_+ \times \mathbb{R}_- \to X$ defined as

$$u(t,s) = [T_{B,\Phi}(t)f](s),$$

then we obtain the equality

$$\frac{\partial}{\partial t}u(t,s) = \frac{\partial}{\partial s}u(t,s),$$

which is known as the balance law between the velocity of the evolution process in the past and in the future (see [**20**, p. 39-40]). However, in many applications this balance law may not be true. An idea introduced by Brendle and Nagel [**10**] to control the unbalance is to suppose that the value of the history function is modified according to an evolution law. Consequently, this modification leads to the following system of equations which are known as *delay partial differential equations with non-automous past* (see [**10**, Eqns. (1) and (2)])

$$\frac{\partial}{\partial t}u(t,0) = Bu(t,0) + \Phi u(t,.), \qquad t \geq 0, \tag{2.2}$$

$$\frac{\partial}{\partial t}u(t,s) = \frac{\partial}{\partial s}u(t,s) + A(s)u(t,s), \qquad t \geq 0 \geq s. \tag{2.3}$$

Here, the function $u(\cdot,\cdot)$ takes values in a Banach space X, the (partial differential) operator B and the delay operator Φ are the operators as before. Finally, $A(s)$ are (unbounded) operators on X for which the non-autonomous Cauchy problem

$$\begin{cases} \frac{dx(t)}{dt} &= -A(t)x(t), \ t \leq s \leq 0, \\ x(s) &= x_s \in X, \end{cases} \tag{2.4}$$

is well-posed with exponential bound. As discussed in Chapter 1, Section 4, there exists an exponentially bounded backward evolution family $\mathcal{U} = (U(t,s))_{t \leq s \leq 0}$ solving (2.4), i.e., the solutions of (2.4) are given by

$$x(t) = U(t,s)x(s) \quad \text{for } t \leq s \leq 0.$$

These two equations describe a system with delay (the equation (2.2)) acting on a non-autonomous past (the equation (2.3)). They have been solved using semigroup methods in the space $C_0(\mathbb{R}_-, X)$ in [**10**] or in the space $L^p(\mathbb{R}_-, X)$ in [**26**]. We may note that the model of "non-autonomous past" seems to be particularly well suited for biological systems with delays and diffusion. We refer to Fragnelli [**28**] for concrete examples. In this chapter, we will study the asymptotic behavior of the equations (2.2) and (2.3). More precisely, via evolution semigroup method we will study the robustness of exponential stability and dichotomy of the solutions under small perturbations of the delay operator.

2. Linear delay partial differential equations with non-autonomous past

In this section, we use the theory of evolution semigroups as recalled in Chapter 1, Section 1, where we defined an abstract differential operator G on $C_0(\mathbb{R}_-, X)$ (see Definition 1.23). We now use the delay operator Φ (and the operator B) to define a certain restriction $G_{B,\Phi}$ of G. For this restriction we then compute explicitly its resolvent and show the Hille-Yosida estimates. In this way, we obtain a semigroup $(T_{B,\Phi}(t))_{t\geq 0}$ which solves (2.2) and (2.3) in a mild sense (see [**10**, Sections 1 and 2]). The advantage of our method, using a direct description of the resolvent, is that it yields explicit stability estimates. In particular, we can show that the exponential stability and exponential dichotomy of this semigroup, hence of the solutions of (2.2) and (2.3), is robust under small perturbations of the delay operator Φ.

2.1. Evolution semigroups with delay. In this subsection we consider a bounded linear operator $\Phi : E := C_0(\mathbb{R}_-, X) \to X$, called *delay operator*, and the linear operator B generating a strongly continuous semigroup $(e^{tB})_{t\geq 0}$ on the Banach space X as in Assumption 1.6. We use them to define the following restriction of the operator G from Definition 1.23.

DEFINITION 2.1. The operator $(G_{B,\Phi}, D(G_{B,\Phi}))$ on $E = C_0(\mathbb{R}_-, X)$ is given by

$$\begin{aligned} D(G_{B,\Phi}) &:= \{f \in D(G) : f(0) \in D(B) \text{ and } (Gf)(0) = Bf(0) + \Phi f\}, \\ G_{B,\Phi} f &:= Gf \text{ for } f \in D(G_{B,\Phi}). \end{aligned}$$

We recall that in [**10**] the authors, using extrapolation methods from [**66**], proved that the operator $G_{B,\Phi}$ generates a strongly continuous semigroup $(T_{B,\Phi}(t))_{t\geq 0}$. In this subsection we compute the resolvent of $G_{B,\Phi}$ and show that it satisfies the conditions of the Hille-Yosida theorem. This approach allows us to obtain information on the robustness of the system under small perturbations of the delay operator Φ.

THEOREM 2.2. *Let $e_\lambda : X \to E$ be the function defined by $[e_\lambda x](t) := e^{\lambda t} U(t,0)x$ for $t \leq 0$, $x \in X$ and $Re\lambda > \omega(\mathcal{U})$, where $\omega(\mathcal{U})$ is the growth bound of $\mathcal{U}$ defined as in Definition 1.20. Let the constants K and ω be defined as in Proposition 1.26. Then the following assertions hold.*

(i) *The set $\{\lambda \in \mathbb{C} : Re\lambda > K\|\Phi\| + \omega\} \subset \rho(G_{B,\Phi})$, and for $Re\lambda > K\|\Phi\| + \omega$ the resolvent of $G_{B,\Phi}$ satisfies*

$$R(\lambda, G_{B,\Phi})f = e_\lambda R(\lambda, B)\Phi R(\lambda, G_{B,\Phi})f + R(\lambda, G_{B,0})f, \quad f \in E. \tag{2.5}$$

(ii)

$$\|R(\lambda, G_{B,\Phi})\| \leq \frac{K}{(Re\lambda - K\|\Phi\| - \omega)} \qquad \text{for } Re\lambda > K\|\Phi\| + \omega.$$

(iii) *For $Re\lambda > K^2\|\Phi\| + \omega$ we have*

$$\|R(\lambda, G_{B,\Phi})^n\| \leq \frac{K}{(Re\lambda - K^2\|\Phi\| - \omega)^n} \qquad \text{for all } n \in \mathbb{N}. \tag{2.6}$$

(iv̈) *The operator $G_{B,\Phi}$ is densely defined.*

PROOF. (i) Note that, for $\lambda > K\|\Phi\| + \omega$, the equation

$$u(t) = e^{\lambda t} U(t,0) R(\lambda, B)(f(0) + \Phi u) + \int_t^0 e^{\lambda(t-\xi)} U(t,\xi) f(\xi) d\xi \qquad \text{for } t \leq 0 \tag{2.7}$$

is equivalent to

$$u = e_\lambda R(\lambda, B)\Phi u + R(\lambda, G_{B,0})f. \tag{2.8}$$

If for each $f \in E$ and $Re\lambda > K\|\Phi\| + \omega$ this equation has a unique solution $u \in E$, then $u(0) = R(\lambda, B)(f(0) + \Phi u)$. This is equivalent to

$$(\lambda - B)u(0) = [(\lambda - G)u](0) + \Phi u \quad \text{or} \quad [Gu](0) = Bu(0) + \Phi u.$$

Hence, by Lemma 1.24, $u \in D(G_{B,\Phi})$ and $u = R(\lambda, G_{B,\Phi})f$. Therefore, to prove (i) we have to verify that, for each $f \in E$ and $Re\lambda > K\|\Phi\| + \omega$, the equation (2.8) has a unique solution $u \in E$. Let $M_\lambda : E \to E$ be the linear operator defined as $M_\lambda := e_\lambda R(\lambda, B)\Phi$.

Since λ satisfies $Re\lambda > K\|\Phi\| + \omega$, we have that M_λ is bounded with

$$\|M_\lambda\| \leq \frac{K\|\Phi\|}{Re\lambda - \omega} < 1.$$

Therefore, the operator $I - M_\lambda$ is invertible, and the equation (2.8) has a unique solution $u = (I - M_\lambda)^{-1} R(\lambda, G_{B,0})f$. Thus,

$$R(\lambda, G_{B,\Phi})f = M_\lambda R(\lambda, G_{B,\Phi})f + R(\lambda, G_{B,0})f,$$

and (2.5) follows.

(ii) By the Neumann series $(I-M_\lambda)^{-1}=\sum_{n=0}^{\infty} M_\lambda^n$ we have that, for $Re\lambda > K\|\Phi\|+\omega$,

$$\begin{aligned}\|R(\lambda,G_{B,\Phi})\| &= \|\sum_{n=0}^{\infty} M_\lambda^n R(\lambda,G_{B,0})\| \\ &\le \frac{K}{(Re\lambda-\omega)}\sum_{n=0}^{\infty}\|M_\lambda^n\| \\ &\le \frac{K}{(Re\lambda-\omega)}\sum_{n=0}^{\infty}\left(\frac{K\|\Phi\|}{Re\lambda-\omega}\right)^n \\ &= \frac{K}{(Re\lambda-K\|\Phi\|-\omega)}.\end{aligned}$$

(iii) We shall prove this by induction. By (2.5) we obtain that

$$\begin{aligned}R(\lambda,G_{B,\Phi})^n &= e_\lambda R(\lambda,B)\Phi R(\lambda,G_{B,\Phi})^n + R(\lambda,G_{B,0})R(\lambda,G_{B,\Phi})^{n-1} \\ &= e_\lambda R(\lambda,B)\Phi R(\lambda,G_{B,\Phi})^n + R(\lambda,G_{B,0})e_\lambda R(\lambda,B)\Phi R(\lambda,G_{B,\Phi})^{n-1}+ \\ &\qquad +R(\lambda,G_{B,0})^2R(\lambda,G_{B,\Phi})^{n-2} \\ &= \cdots \\ &= e_\lambda R(\lambda,B)\Phi R(\lambda,G_{B,\Phi})^n + R(\lambda,G_{B,0})e_\lambda R(\lambda,B)\Phi R(\lambda,G_{B,\Phi})^{n-1}+ \\ &\quad +R(\lambda,G_{B,0})^2 e_\lambda R(\lambda,B)\Phi R(\lambda,G_{B,\Phi})^{n-2}+\cdots+R(\lambda,G_{B,0})^n. \qquad (2.9)\end{aligned}$$

Clearly, (2.6) holds for $n=1$. If it holds for $n-1$, we prove it for n.
In fact, for $Re\lambda > K^2\|\Phi\|+\omega$, we obtain, by (2.9) and induction hypothesis, that

$$\begin{aligned}\|R(\lambda,G_{B,\Phi})^n\| \le\ & \frac{K^2\|\Phi\|}{Re\lambda-\omega}\|R(\lambda,G_{B,\Phi})^n\|+ \\ &+\frac{K^3\|\Phi\|}{(Re\lambda-\omega)^2(Re\lambda-\omega-K^2\|\Phi\|)^{n-1}}+ \\ &+\frac{K^3\|\Phi\|}{(Re\lambda-\omega)^3(Re\lambda-\omega-K^2\|\Phi\|)^{n-2}}+ \\ &+\cdots+\frac{K^3\|\Phi\|}{(Re\lambda-\omega)^n(Re\lambda-\omega-K^2\|\Phi\|)}+\frac{K}{(Re\lambda-\omega)^n}.\end{aligned}$$

Putting $a:=Re\lambda-\omega$; $b:=Re\lambda-\omega-K^2\|\Phi\|$, this yields

$$\begin{aligned}\frac{b}{a}\|R(\lambda,G_{B,\Phi})^n\| &\le K\left[\frac{K^2\|\Phi\|}{a^2b}\left(\frac{1}{b^{n-2}}+\frac{1}{ab^{n-3}}+\cdots+\frac{1}{a^{n-2}}\right)+\frac{1}{a^n}\right] \\ &= K\left[\frac{K^2\|\Phi\|}{a^2b}\left(\frac{\frac{1}{a^{n-1}}-\frac{1}{b^{n-1}}}{\frac{1}{a}-\frac{1}{b}}\right)+\frac{1}{a^n}\right] \\ &= \frac{K}{ab^{n-1}} \text{ (note that } a-b=K^2\|\Phi\|).\end{aligned}$$

Hence,

$$\|R(\lambda,G_{B,\Phi})^n\| \le \frac{K}{b^n}=\frac{K}{(Re\lambda-\omega-K^2\|\Phi\|)^n}.$$

(iv) For $\lambda > K\|\Phi\|+\omega$, by (2.5) we have that

$$D(G_{B,\Phi})=R(\lambda,G_{B,\Phi})E=(I-e_\lambda R(\lambda,B)\Phi)^{-1}R(\lambda,G_{B,0})E=(I-e_\lambda R(\lambda,B)\Phi)^{-1}D(G_{B,0}).$$

Since $D(G_{B,0})$ is dense in E, and $I-e_\lambda R(\lambda,B)\Phi$ is an isomorphism we obtain that $D(G_{B,\Phi})$ is dense in E. □

The Hille-Yosida Theorem now yields the following results.

COROLLARY 2.3. *The operator $G_{B,\Phi}$ is the generator of a strongly continuous semigroup $(T_{B,\Phi}(t))_{t\geq 0}$ satisfying*

$$\|T_{B,\Phi}(t)\| \leq Ke^{(K^2\|\Phi\|+\omega)t}, \quad t \geq 0,$$

where the constants K and ω are defined as in Proposition 1.26.

COROLLARY 2.4. *If the backward evolution family $\mathcal{U}$ and the semigroup $(e^{tB})_{t\geq 0}$ are exponentially stable and $\|\Phi\|$ is small enough, then the semigroup $(T_{B,\Phi}(t))_{t\geq 0}$ is also exponentially stable.*

PROOF. The assumption that $\mathcal{U}$ and $(e^{tB})_{t\geq 0}$ are exponentially stable means that $\omega = \max\{\omega_1, \omega_2\} < 0$. Therefore, if $\|\Phi\| < -\frac{\omega}{K^2}$, then the semigroup $(T_{B,\Phi}(t))_{t\geq 0}$ is also exponentially stable. □

2.2. Spectra and hyperbolicity of evolution semigroups. In this subsection we first compute the spectra of the evolution semigroup $(T_{B,0}(t))_{t\geq 0}$ on $E = C_0(\mathbb{R}_-, X)$ and its generator. This will be used to prove the robustness of the hyperbolicity of the semigroup $(T_{B,\Phi}(t))_{t\geq 0}$ under small perturbations by the delay operator Φ. We first compare $(T_{B,0}(t))_{t\geq 0}$ to its restriction to the subspace $C_{00} := \{f \in E : f(0) = 0\}$. This restriction has already been studied in Chapter 1 for evolution families on $\mathbb{R}_+$ (see Chapter 1, Definition 1.9 and Lemma 1.11).

LEMMA 2.5. *Let the semigroup $(T_{B,0}(t))_{t\geq 0}$ on $E = C_0(\mathbb{R}_-, X)$ be defined as in Definition 1.25 wiht the gnerator $G_{B,0}$. Denote by $(T_0(t))_{t\geq 0}$ the restriction of $(T_{B,0}(t))_{t\geq 0}$ to the subspace C_{00} and let G_0 be its generator. Then the following assertions hold.*

(i)
$$\sigma(T_{B,0}(t)) \subseteq \sigma(T_0(t)) \cup \sigma(e^{tB}) \quad \textit{for} \quad t \geq 0. \tag{2.10}$$

(ii)
$$\sigma(G_{B,0}) \cup \sigma(B) = \sigma(G_0) \cup \sigma(B). \tag{2.11}$$

PROOF. (i) Endow $X \oplus C_{00}$ with the 1-norm

$$\|(x, f)\| := \|f\| + \|x\| \text{ for } (x, f) \in X \oplus C_{00}.$$

For a fixed continuous real valued function φ with compact support satisfying $\varphi(0) = 1$, we consider the linear operator

$$\begin{array}{rcl} \mathcal{J} : E = C_0(\mathbb{R}_-, X) & \to & X \oplus C_{00}, \\ f & \mapsto & (f(0), f - \varphi(\cdot)f(0)). \end{array}$$

Then $\mathcal{J}$ is an isomorphism and its inverse is given by

$$\begin{array}{rcl} \mathcal{J}^{-1} : X \oplus C_{00} & \to & E, \\ (x, f) & \mapsto & f + \varphi(\cdot)z. \end{array}$$

Therefore, by similarity, the operators

$$\hat{T}(t) := \mathcal{J}T_{B,0}(t)\mathcal{J}^{-1} = \begin{pmatrix} e^{tB} & 0 \\ (T_{B,0}(t) - e^{tB})\varphi(\cdot) & T_0(t) \end{pmatrix}, \ t \geq 0,$$

form a semigroup satisfying $\sigma(\hat{T}(t)) = \sigma(T_{B,0}(t))$. Let now $\lambda \in \rho(T_0(t)) \cap \rho(e^{tB})$. Then the operator

$$\begin{pmatrix} \lambda - e^{tB} & 0 \\ (T_{B,0}(t) - e^{tB})\varphi(\cdot) & \lambda - T_0(t) \end{pmatrix}$$

is invertible with inverse

$$\begin{pmatrix} (\lambda - e^{tB})^{-1} & 0 \\ -(\lambda - T_0(t))^{-1}[(T_{B,0}(t) - e^{tB})\varphi(.)](\lambda - e^{tB})^{-1} & (\lambda - T_0(t))^{-1} \end{pmatrix}.$$

Hence $\lambda \in \rho(\hat{T}(t)) = \rho(T_{B,0}(t))$. This means that $\rho(T_0(t)) \cap \rho(e^{tB}) \subseteq \rho(T_{B,0}(t))$. Thus, (i) follows.

(ii) By Proposition 1.26, we have $\rho(G_0) \cap \rho(B) \subseteq \rho(G_{B,0})$. Hence,

$$\sigma(G_{B,0}) \subseteq \sigma(G_0) \cup \sigma(B). \tag{2.12}$$

It remains to prove that

$$\sigma(G_0) \subseteq \sigma(G_{B,0}) \cup \sigma(B). \tag{2.13}$$

In fact, if $\lambda - G_{B,0}$ is injective, then so is $\lambda - G_0$ because G_0 is the restriction of $G_{B,0}$ to C_{00}.

Let now $\lambda \in \rho(B)$ and $\lambda - G_{B,0}$ be surjective. We will verify that $\lambda - G_0$ is also surjective. Indeed, let $f \in C_{00}$ be arbitrary. Then, by the surjectivity of $\lambda - G_{B,0}$, there exists a function $u \in D(G_{B,0})$ such that $(\lambda - G_{B,0})u = f$. By definition of $G_{B,0}$ we have that $0 = f(0) = \lambda u(0) - [G_{B,0}u](0) = (\lambda - B)u(0)$. Therefore, $u(0) = 0$ and $u \in C_{00}$. Hence, $(\lambda - G_0)u = (\lambda - G_{B,0})u = f$. Thus, $\lambda - G_0$ is surjective. This yields

$$\rho(G_{B,0}) \cap \rho(B) \subseteq \rho(G_0),$$

and inclusion (2.13) follows. □

In [**55**, Corollary 2.4] it has been proved that a Spectral Mapping Theorem holds for the semigroup $(T_0(t))_{t\geq 0}$. More precisely, we have

$$\sigma(G_0) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq \omega(\mathcal{U})\}$$

and

$$\sigma(T_0(t))\backslash\{0\} = e^{t\sigma(G_0)}, \quad t > 0. \tag{2.14}$$

By this and Lemma 2.5 we obtain the following.

Theorem 2.6. *Let the operators G_0 and $T_{B,0}(t)$ be defined as in Lemma 2.5. Then the spectral equality*

$$[\sigma(T_{B,0}(t)) \cup \sigma(e^{tB})]\backslash\{0\} = [e^{t\sigma(G_0)} \cup \sigma(e^{tB})]\backslash\{0\}, \quad t \geq 0, \tag{2.15}$$

holds.

Proof. By Lemma 2.5 and (2.14) we have that

$$\begin{aligned} [\sigma(T_{B,0}(t)) \cup \sigma(e^{tB})]\backslash\{0\} &\overset{\text{by (2.10)}}{\subseteq} [\sigma(T_0(t)) \cup \sigma(e^{tB})]\backslash\{0\} \\ &\overset{\text{by (2.14)}}{=} [e^{t\sigma(G_0)} \cup \sigma(e^{tB})]\backslash\{0\} \\ &= [e^{t\sigma(G_0)} \cup e^{t\sigma(B)} \cup \sigma(e^{tB})]\backslash\{0\} \\ &= [e^{t(\sigma(G_0)\cup\sigma(B))} \cup \sigma(e^{tB})]\backslash\{0\} \\ &\overset{\text{by (2.11)}}{=} [e^{t(\sigma(G_{B,0})\cup\sigma(B))} \cup \sigma(e^{tB})]\backslash\{0\} \end{aligned}$$

$$\begin{aligned} &= [e^{t\sigma(G_{B,0})} \cup e^{t\sigma(B)} \cup \sigma(e^{tB})]\backslash\{0\} \\ &\subseteq [\sigma(T_{B,0}(t)) \cup \sigma(e^{tB})]\backslash\{0\}. \end{aligned}$$

Thus, (2.15) follows. □

Therefore, using the spectral characterization of hyperbolic semigroups (see [**23**, Theorem V.1.15]), the above theorem allows the following consequence.

COROLLARY 2.7. *If the operator $(B, D(B))$ generates a hyperbolic semigroup $(e^{tB})_{t\geq 0}$ and if the backward evolution family $\mathcal{U} = (U(t,s))_{t\leq s\leq 0}$ is exponentially stable, then the semigroup $(T_{B,0}(t))_{t\geq 0}$ is hyperbolic.*

PROOF. The assumption that $\mathcal{U}$ is exponentially stable means that $\omega(\mathcal{U}) < 0$, hence $s(G_0) < 0$ by (2.14). Therefore, $\sigma(G_0) \cap i\mathbb{R} = \emptyset$. By the hyperbolicity of $(e^{tB})_{t\geq 0}$ we have

$$(e^{t\sigma(G_0)} \cup \sigma(e^{tB})) \cap e^{i\mathbb{R}} = \emptyset.$$

The hyperbolicity of $(T_{B,0}(t))_{t\geq 0}$ now follows from (2.15) and [**23**, Theorem V.1.15]. □

The main purpose of this section is to prove the robustness of hyperbolicity of the semigroup $(T_{B,\Phi}(t))_{t\geq 0}$ under small perturbations of the delay operator Φ. To do so we need the following characterization of hyperbolic semigroups (see [**58**, Theorem 2.6.2]).

THEOREM 2.8. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with the generator A. Then the following assertions are equivalent.*

(i) *$(T(t))_{t\geq 0}$ is hyperbolic.*
(ii) *$i\mathbb{R} \subset \rho(A)$ and*

$$(C,1)\text{-}\sum_{k\in\mathbb{Z}} R(i\omega + ik, A)x := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} R(i\omega + ik, A)x$$

converges for all $\omega \in \mathbb{R}$ and $x \in X$.

We note that the above theorem is taken from [**58**, Theorem 2.6.2], while its proof is essentially due to G. Greiner and M. Schwarz [**30**, Theorem 1.1 and Corollary 1.2]. A continuous version of the above theorem is proved by M. Kaashoek and S. Verduyn Lunel in [**44**, Theorem 4.1].

In order to apply this theorem we have to compute the resolvent $R(\lambda, G_{B,\Phi})$ starting from the resolvent $R(\lambda, G_{B,0})$. This can be done as follows.

LEMMA 2.9. *Let the backward evolution $\mathcal{U}$ be exponentially stable and the operator $(B, D(B))$ be the generator of a hyperbolic semigroup $(e^{tB})_{t\geq 0}$. Then for sufficiently small $\|\Phi\|$ there exist an open strip Σ containing the imaginary axis and a function H_λ which is analytic and uniformly bounded on Σ such that*

$$R(\lambda, G_{B,\Phi}) = H_\lambda R(\lambda, G_{B,0}) \quad \text{for } \lambda \in \Sigma. \tag{2.16}$$

PROOF. By [**44**, Theorem 4.1] and the hyperbolicity of $(e^{tB})_{t\geq 0}$, we obtain that there exist constants $P_1, \nu > 0$ such that

$$\|R(\lambda, B)\| \leq P_1 \quad \text{for all} \quad |Re\lambda| < \nu.$$

By the exponential stability of $\mathcal{U}$, there exist constants $\omega_1 > 0$ and K_1 such that

$$\|U(t,s)\| < K_1 e^{-\omega_1(s-t)} \quad \text{for all} \quad t \leq s \leq 0. \tag{2.17}$$

Let now ω be a real number such that $0 < \omega < \min\{\omega_1, \nu\}$. We then put

$$\Sigma := \{\lambda \in \mathbb{C} : |Re\lambda| < \omega\}$$

and

$$P := \sup_{\lambda \in \Sigma} \|R(\lambda, B)\|. \tag{2.18}$$

As in the proof of Theorem 2.2, we first verify that for each $f \in E$ and $\lambda \in \Sigma$ the equation (2.8) has a unique solution $u \in E$.

Let $M_\lambda : E \to E$ be the linear operator defined as $M_\lambda := e_\lambda R(\lambda, B)\Phi$ with e_λ as in Theorem 2.2. For $\lambda \in \Sigma$, this operator is bounded and satisfies

$$\|M_\lambda\| \leq K_1 P \|\Phi\| < 1 \quad \text{if, in addition,} \quad \|\Phi\| < \frac{1}{K_1 P}.$$

Therefore, the operator $I - M_\lambda$ is invertible, and the equation (2.8) has a unique solution $u = (I - M_\lambda)^{-1} R(\lambda, G_{B,0}) f$. Putting $H_\lambda := (I - M_\lambda)^{-1}$ we obtain

$$R(\lambda, G_{B,\Phi}) = H_\lambda R(\lambda, G_{B,0}).$$

Since,

$$H_\lambda = (I - M_\lambda)^{-1} = \sum_{n=0}^{\infty} M_\lambda^n, \tag{2.19}$$

it follows that

$$\begin{aligned} \|H_\lambda\| &\leq \sum_{n=0}^{\infty} \|M_\lambda\|^n \\ &\leq \sum_{n=0}^{\infty} (K_1 P \|\Phi\|)^n \\ &= \frac{1}{1 - K_1 P \|\Phi\|} \quad \text{for all } \lambda \in \Sigma \text{ and } \|\Phi\| < \frac{1}{K_1 P}. \end{aligned}$$

Since $\|M_\lambda^n\| \leq (K_1 P \|\Phi\|)^n$ for all $\lambda \in \Sigma$ and the series $\sum_{n=0}^{\infty} (K_1 P \|\Phi\|)^n$ converges for $\|\Phi\| < \frac{1}{K_1 P}$, we obtain that, if $\|\Phi\| < \frac{1}{K_1 P}$, then the Neumann series (2.19) converges uniformly for all $\lambda \in \Sigma$. This fact, together with the analyticity of M_λ, yields the analyticity of H_λ. □

We now come to our main result of this subsection and obtain conditions for exponential dichotomy of solutions of the equations (2.2) and (2.3).

THEOREM 2.10. *Let the backward evolution $\mathcal{U}$ be exponentially stable and the operator $(B, D(B))$ be the generator of a hyperbolic semigroup $(e^{tB})_{t \geq 0}$. Then, for sufficiently small $\|\Phi\|$, the semigroup $(T_{B,\Phi}(t))_{t \geq 0}$ is hyperbolic.*

PROOF. By Corollary 2.7, the evolution semigroup $(T_{B,0}(t))_{t\geq 0}$ is hyperbolic. We first prove that, for sufficiently small $\|\Phi\|$, the sum $\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} R(i\omega+ik, G_{B,\Phi})$ is bounded in $\mathcal{L}(E)$. In fact, by Lemma 2.9, we have

$$
\begin{aligned}
& \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}[R(i\omega+ik,G_{B,\Phi})f](s) = \\
= \; & \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}[(1+M_{i\omega+ik}+M_{i\omega+ik}^2+\cdots)R(i\omega+ik,G_{B,0})f](s) = \\
= \; & \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}[R(i\omega+ik,G_{B,0})f](s) + \\
+ \; & \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{(i\omega+ik)s}U(s,0)R(i\omega+ik,B)\Phi R(i\omega+ik,G_{B,0})f+\cdots \qquad (2.20)
\end{aligned}
$$

for $s\in\mathbb{R}_-$.

Note that the semigroup $(T_{B,0}(t))_{t\geq 0}$ is hyperbolic, hence $e^{-2\pi i\omega}\in\rho(T_{B,0}(2\pi))$ for all $\omega\in\mathbb{R}$. Using the formula (see [**23**, Lemma II.1.9])

$$R(\lambda,G_{B,0})(1-e^{-\lambda t}T_{B,0}(t)) = \int_0^t e^{-\lambda s}T_{B,0}(s)ds \quad \text{for } \lambda\in\rho(G_{B,0}),$$

we obtain

$$R(i\omega+ik,G_{B,0}) = \int_0^{2\pi} e^{-(i\omega+ik)t}T_{B,0}(t)(1-e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}dt.$$

The first term of (2.20) can now be computed as

$$
\begin{aligned}
& \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}R(i\omega+ik,G_{B,0})f = \\
= \; & \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\int_0^{2\pi} e^{-(i\omega+ik)t}T_{B,0}(t)(1-e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}fdt = \\
= \; & \int_0^{2\pi}\left[\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}e^{-ikt}\right]e^{-i\omega t}T_{B,0}(t)(1-e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}fdt \\
= \; & \int_0^{2\pi}\sigma_N(t)e^{-i\omega t}T_{B,0}(t)(1-e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}fdt.
\end{aligned}
$$

Here, $\sigma_N(t) := \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}e^{-ikt}$. Since

$$\sigma_N(t) = \frac{1-\cos(Nt)}{N(1-\cos t)} \geq 0 \text{ and } \int_0^{2\pi}\sigma_N(t)dt = 2\pi \qquad (2.21)$$

(see [**30**, Theorem 1.1]), the norm of the first term in (2.20) can be estimated by

$$\|\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}R(i\omega+ik,G_{B,0})f\| \leq C_1\|f\| \qquad (2.22)$$

with $C_1 := 2\pi\sup_{0\leq w\leq 1}\{\|(1-e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}\|\}\sup_{0\leq t\leq 2\pi}\{\|T_{B,0}(t)\|\}$.

We now compute the second term of (2.20). For $s \in \mathbb{R}_-$, we have

$$
\begin{aligned}
&\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} M_{i\omega+ik}R(i\omega+ik, G_{B,0})f(s) = \\
= \;& \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{(i\omega+ik)s}U(s,0)R(i\omega+ik,B)\Phi R(i\omega+ik,G_{B,0})f \\
= \;& \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{(i\omega+ik)s}U(s,0)\int_0^{2\pi} e^{-(i\omega+ik)\tau}e^{\tau B}(1-e^{2\pi B})^{-1}d\tau \\
&\qquad \Phi\int_0^{2\pi} e^{-(i\omega+ik)t}T_{B,0}(t)(1-e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}f dt \\
= \;& \int_0^{2\pi}\int_0^{2\pi}\left[\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{-ik(t+\tau-s)}\right] e^{-i\omega(t+\tau-s)}U(s,0)e^{\tau B}(1-e^{2\pi B})^{-1} \\
&\qquad \Phi T_{B,0}(t)(1-e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}f d\tau dt \\
= \;& \int_0^{2\pi}\int_0^{2\pi} \sigma_N(t+\tau-s)e^{-i\omega(t+\tau-s)}U(s,0)e^{\tau B}(1-e^{2\pi B})^{-1} \\
&\qquad \Phi T_{B,0}(t)(1-e^{-2\pi i\omega}T_{B,0}(2\pi))^{-1}f d\tau dt.
\end{aligned}
$$

Therefore, using (2.17) and (2.21), the norm of the second term of (2.20) can be estimated by

$$C_1K_1C_2\|\Phi\|\|f\| \quad \text{with } C_2 := 2\pi\|(1-e^{2\pi B})^{-1}\| \sup_{0\le t\le 2\pi}\{\|e^{tB}\|\} \tag{2.23}$$

and K_1, C_1 as in (2.17), (2.22), respectively.

By induction, the norm of the n^{th} term of (2.20) is estimated by

$$C_1(K_1C_2\|\Phi\|)^n\|f\|.$$

Moreover, the series $\sum_{n=0}^{\infty} C_1(K_1C_2\|\Phi\|)^n$ converges if $\|\Phi\| < \frac{1}{K_1C_2}$. Hence, for these $\|\Phi\|$ the sum $\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} R(i\omega+ik, G_{B,\Phi})$ is bounded in $\mathcal{L}(E)$.

We now prove the convergence of $(C,1)$-$\sum_{k\in\mathbb{Z}} R(i\omega+ik,G_{B,\Phi})f$ for $\omega\in\mathbb{R}$ and $f \in E$. This can be done by using the idea from [**30**, Theorem 1.1]. By [**69**, III.4.5], it is sufficient to show convergence on a dense subset. From $i\mathbb{R}\subset\rho(G_{B,\Phi})$ and the spectral mapping theorem for the residual spectrum (see [**23**, Theorem IV.3.7]) we obtain that $e^{-2\pi i\omega}$ does not belong to the residual spectrum $R\sigma(T_{B,\Phi}(2\pi))$. This implies that $(1 - e^{-2\pi i\omega}T_{B,\Phi}(2\pi))E$ is a dense subset of E. Let $f := (1-e^{-2\pi i\omega}T_{B,\Phi}(2\pi))g$. Then

$$\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} R(i\omega+ik,G_{B,\Phi})(1-e^{-2\pi i\omega}T_{B,\Phi}(2\pi))g = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\int_0^{2\pi} e^{-(i\omega+ik)s}T_{B,\Phi}(s)g ds. \tag{2.24}$$

Now $e^{-i\omega\cdot}\, T_{B,\Phi}(\cdot)g$ is a continuous function with Fourier coefficients

$$Q_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-(i\omega+ik)s}T_{B,\Phi}(s)g ds.$$

Therefore, by Fejer's Theorem [**48**, Theorem I.3.1], the sum in (2.24) converges as $N\to\infty$. The assertion of the theorem now follows from Theorem 2.8. □

3. Examples

In this section we investigate some concrete examples. In the first example we shall determine the "sufficient smallness" of $\|\Phi\|$ ensuring the robustness of exponential stability.

EXAMPLE 2.11. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. The Dirichlet Laplacian generates an analytic semigroup $(e^{t\Delta})_{t\geq 0}$ on $X := L^2(\Omega)$. We then take operators $A(s)$ as

$$A(s) := a(s)\Delta,$$

where the function $a(\cdot) \in L^1_{loc}(\mathbb{R}_-)$ satisfies $a(\cdot) \geq \gamma > 0$ for some constant γ. These operators generate a backward evolution family $(U(r,s))_{r\leq s\leq 0}$ given by

$$U(r,s) = e^{(\int_r^s a(\tau)d\tau)\Delta} \qquad \text{for } r \leq s \leq 0.$$

We then have

$$\|U(r,s)\| = e^{(\int_r^s a(\tau)d\tau)\lambda_0} \leq e^{\gamma\lambda_0(s-t)} \qquad \text{for } r \leq s \leq 0,$$

where $\lambda_0 < 0$ denotes the largest eigenvalue of Δ. Therefore, we can choose in Definition 1.20 the constants $N = 1$ and $\omega_1 = \gamma\lambda_0 < 0$. We now define the delay operator Φ by

$$\Phi f := \int_{-\infty}^0 \varphi(s)f(s)ds \text{ for } f \in E,$$

where $\varphi(\cdot) \in L^1(\mathbb{R})$. We then have

$$\|\Phi\| \leq \|\varphi(\cdot)\|_{L^1}.$$

Let now B generate a semigroup $(e^{tB})_{t\geq 0}$ satisfying $\|e^{tB}\| \leq Me^{\omega_2 t}$ with $\omega_2 < 0$. From the definition of $(T_{B,0}(t))_{t\geq 0}$ we obtain

$$\|T_{B,0}(t)\| \leq Me^{\max\{\gamma\lambda_0,\ \omega_2\}t}, \qquad t \geq 0.$$

Hence, in Corollar 2.3 we can choose $K = M$. Therefore, if

$$\|\varphi(\cdot)\|_{L^1} < -\frac{\max\{\gamma\lambda_0, \omega_2\}}{M^2},$$

then the semigroup $(T_{B,\Phi}(t))_{t\geq 0}$ is exponentially stable.

The next example gives an explicit estimate for the size of perturbations of Φ under which the exponential dichotomy persists.

EXAMPLE 2.12. We consider again Example 4.8 with the same backward evolution family $U(r,s) := e^{(\int_r^s a(\tau)d\tau)\Delta}$ and the same delay operator $\Phi f := \int_{-\infty}^0 \varphi(s)f(s)ds$. However, let now B generate a hyperbolic semigroup $(e^{tB})_{t\geq 0}$ satisfying $\|R(\lambda, B)\| \leq P_1$ for $|Re\lambda| < \omega_2$ (for instance, we can take B to be a sectorial operator satisfing $\sigma(B)\cap i\mathbb{R} = \emptyset$ as in [**50**, Example 2.1.4] or [**73**, Example 4.2]). Take $0 < \omega < \min\{-\gamma\lambda_0, \omega_2\}$ and put

$$\Sigma := \{\lambda \in \mathbb{C} : |Re\lambda| < \omega\}$$

and

$$P := \max\{\sup_{\lambda\in\Sigma}\{\|R(\lambda, B)\|\}, 2\pi\|(1-e^{2\pi B})^{-1}\| \sup_{0\leq t\leq 2\pi}\{\|e^{tB}\|\}\}.$$

We obtain that the semigroup $(T_{B,\Phi}(t))_{t\geq 0}$ is hyperbolic if

$$\|\varphi(\cdot)\|_{L^1} < \frac{1}{P}.$$

4. Non-autonomous delay partial differential equations

We now consider the general non-autonomous delay partial differential equations with delay operators acting on a finite inteval $[-r,0]$. More precisely, we study a semi-linear non-autonomous delay partial differential equations of the form

$$(DPDE) \quad \begin{cases} \frac{\partial}{\partial t}u_t &= B(t)u(t) + \Phi(t,u_t) \text{ for } t \geq a, \\ u_a &= \phi \in C := C([-r,0],X). \end{cases} \tag{2.25}$$

We assume that $B(t)$ are (unbounded) linear operators such that the corresponding Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} &= B(t)u(t),\ t \geq s \geq 0, \\ u(s) &= x_s \in X, \end{cases} \tag{2.26}$$

is well-posed with exponential bound. This means that there exists an exponentially bounded evolution family $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ solving (2.26), i.e., the solutions of (2.26) are given by $x(t) = U(t,s)x(s)$ for $t \geq s \geq 0$. We will use the term "*the operators $B(t)$ generate the evolution family $(U(t,s))_{t\geq s\geq 0}$*" to indicate the above well-posedness.

We refer to [**24, 25, 32, 63, 67, 68, 78, 81**] for concrete examples and special cases of the above equation, e.g., the case when $B(t) := B$ is independent of t. The fact that the operators $B(t)$ generate the evolution family $(U(t,s))_{t\geq s\geq 0}$ allows us to solve the equation (2.25) in a mild manner. Roughly speaking, we will prove that the mild solution of DPDE (2.25) exists uniquely on the interval $[a-r,b]$ of the real line provided that the delay operator $\Phi(t,\phi)$ is Lipschitz continuous with respect to $\phi \in C$ uniformly for all $t \in [a,b]$. Moreover, our estimates allow us to obtain the robustness of exponential stability and of exponential dichotomy of the system under small perturbations of the delay operator Φ. We begin with the proposition about existence and uniqueness of the mild solution to the equation (2.25). The proof of this proposition can be done using the same idea as in [**63**], [**78**] and [**81**, Theorem 2.1.1] (see also [**24, 25, 32**]). However we will present this proof for sake of completeness.

PROPOSITION 2.13. *Let the operators $B(t)$ generate the evolution family $(U(t,s))_{t\geq s\geq 0}$. Suppose that the delay operator $\Phi : [a,b]\times C \to X$ is continuous and satisfies a Lipschitz condition*

$$\|\Phi(t,\phi) - \Phi(t,\psi)\| \leq L\|\phi - \psi\| \text{ for all } t \in [a,b],\ \phi,\psi \in C := C([-r,0],X), \tag{2.27}$$

where L is a positive constant. Then, for given $\phi \in C$ there exists a unique continuous function $u : [a-r,b] \to X$ which solves the initial value problem

$$\begin{cases} u(t) &= U(t,a)\phi(0) + \int_a^t U(t,s)\Phi(s,u_s)ds, \quad a \leq t \leq b, \\ u_a &= \phi. \end{cases} \tag{2.28}$$

Moreover, the solutions depend continuously on the initial conditions.

PROOF. For a given $\phi \in C$, we define a mapping

$$\mathcal{H}_\phi : C([a-r,b],X) \to C([a-r,b],X)$$

by

$$(\mathcal{H}_\phi u)(t) = \begin{cases} U(t,a)\phi(0) + \int_a^t U(t,s)\Phi(s,u_s)ds & \text{for } a \le t \le b \\ \phi(t-a) & \text{for } a-r \le t \le a. \end{cases}$$

Hence, denoting by $\|\cdot\|_\infty$ the norm in $C([a-r,b],X)$, we obtain

$$\|(\mathcal{H}_\phi u)(t) - (\mathcal{H}_\phi v)(t)\| \le PL(t-a)\|u-v\|_\infty \text{ for } a \le t \le b \tag{2.29}$$

and $(\mathcal{H}_\phi u)(t) - (\mathcal{H}_\phi v)(t) = 0$ for $a-r \le t \le a$, where $P := \sup_{a\le s\le t\le b}\|U(t,s)\|$. By induction we obtain

$$\|(\mathcal{H}_\phi^n u)(t) - (\mathcal{H}_\phi^n v)(t)\| \le \frac{(PL(t-a))^n}{n!}\|u-v\|_\infty \text{ for } a \le t \le b \tag{2.30}$$

and $(\mathcal{H}_\phi^n u)(t) - (\mathcal{H}_\phi^n v)(t) = 0$ for $a-r \le t \le a$. Hence,

$$\|\mathcal{H}_\phi^n u - \mathcal{H}_\phi^n v\|_\infty \le \frac{(PL(b-a))^n}{n!}\|u-v\|_\infty.$$

For n large enough we have that $\frac{(PL(b-a))^n}{n!} < 1$. Therefore, by the contraction principle, $\mathcal{H}_\phi$ has a unique fixed point u in $C([a-r,b],X)$. By the definition of $\mathcal{H}_\phi$, we have that u is a solution of the problem (2.28).

The uniqueness of u and the continuous dependence of u on the initial data can be proved as follows. Let v be the solution of the equation (2.25) on $[a-r,b]$ with the initial value ψ. Then

$$u(t) - v(t) = (\mathcal{H}_\phi u)(t) - (\mathcal{H}_\psi v)(t)$$

holds for $a-r \le t \le b$. By definition of $\mathcal{H}_\phi$, we have that

$$\begin{aligned} \|(\mathcal{H}_\phi u)(t) - (\mathcal{H}_\psi v)(t)\| &\le Ke^{\alpha(t-a)}\|\phi-\psi\| + \\ &\quad + \int_a^t KLe^{\alpha(t-s)}\|u_s - v_s\|ds \quad \text{for } a \le t \le b, \text{ and} \\ \|(\mathcal{H}_\phi u)(t) - (\mathcal{H}_\psi v)(t)\| &\le K\|\phi-\psi\| \quad \text{for } a-r \le t \le a, \end{aligned} \tag{2.31}$$

with the constants K and α appearing in Definition 1.1.

Therefore,

$$\begin{aligned} \|u(t)-v(t)\| &\le Ke^{(\alpha(t-a)}\|\phi-\psi\| + \\ &\quad + \int_a^t KLe^{\alpha(t-s)}\|u_s-v_s\|ds \text{ for } a \le t \le b, \text{ and} \\ \|u(t)-v(t)\| &\le K\|\phi-\psi\| \text{ for } a-r \le t \le a. \end{aligned} \tag{2.32}$$

Hence, if $\alpha \ge 0$, then

$$\|u_t - v_t\| \le Ke^{\alpha(t-a)}\|\phi-\psi\| + \int_a^t KLe^{\alpha(t-s)}\|u_s-v_s\|ds \text{ for all } a \le t \le b, \tag{2.33}$$

and if $\alpha < 0$, then

$$\|u_t - v_t\| \le Ke^{-\alpha r}e^{\alpha(t-a)}\|\phi-\psi\| + \int_a^t KLe^{-\alpha r}e^{\alpha(t-s)}\|u_s-v_s\|ds \text{ for all } a \le t \le b. \tag{2.34}$$

Now, the inequalities (2.33), (2.34) and Gronwall's inequality imply that

$$\|u_t - v_t\| \leq \begin{cases} Ke^{(\alpha+KL)(t-a)}\|\phi - \psi\| & \text{if } \alpha \geq 0 \\ Ke^{-\alpha r}e^{(\alpha+KLe^{-\alpha r})(t-a)}\|\phi - \psi\| & \text{if } \alpha < 0. \end{cases} \tag{2.35}$$

Therefore, the uniqueness and the continuous dependence on the initial data of u follow. □

REMARK 2.14. Assume that the delay operator $\Phi : [0, \infty) \times C \to X$ is continuous and satisfies the inequality (2.27) uniformly for all $t \geq 0$ and $\phi, \psi \in C$. Then the unique solution $u(t)$ of (2.25) is defined on $[-r, \infty)$. If, in addition $\Phi(t, 0) = 0$ for $t \geq 0$, then $u = 0$ is a solution of (2.25) with the initial condition $\phi = 0$. We say that the solution u with the initial condition $u_0 = \phi$ is *exponentially stable* if there exist positive constants K and ω such that $\|u_t\| \leq Ke^{-t\omega}\|\phi\|$ for all $t \geq 0$. Then the inequality (2.35) yields a sufficient condition for the solution u to be exponentially stable. That is, if the operators $B(t)$ generate an exponentially stable evolution family (i.e., $\alpha < 0$) and the Lipschitz constant L is small enough (i.e., $L < -\frac{\alpha}{Ke^{-\alpha r}}$), then the solution u is exponentially stable.

We note that, if we impose appropriate conditions on $B(\cdot)$, Φ and initial data ϕ, then we can deduce that a mild solution defined as in (2.28) is differentiable and satisfies the equation (2.25) (see [**81**, Chap. 2]).

We now consider the robustness of exponential dichotomy of the system under small perturbations. To that purpose, we restrict ourself to the case of linear delay operators. That is, Φ is now a strongly continuous and uniformly bounded function from $\mathbb{R}_+$ into the space $\mathcal{L}(C([-r, 0], X), X)$, and hence we will write $\Phi(t)(u_t)$ instead of $\Phi(t, u_t)$. More precisely, we consider the equation

$$\begin{cases} \frac{\partial}{\partial t}u_t & = B(t)u(t) + \Phi(t)(u_t) \text{ for } t \geq s \geq 0 \\ u_s & = \phi \in C := C([-r, 0], X), \end{cases} \tag{2.36}$$

or in the mild form

$$\begin{cases} u(t) & = U(t, a)\phi(0) + \int_a^t U(t, s)\Phi(s)(u_s)ds, \quad a \leq t \leq b \\ u_a & = \phi, \end{cases} \tag{2.37}$$

where, as above, the evolution family $(U(t, s))_{t\geq s\geq 0}$ is generated by $B(t)$.

Clearly, Φ satisfies the inequality (2.27) uniformly for all $t \geq 0$ and $\phi, \psi \in C$ (with Lipschitz constant $L = \sup_{t\geq 0}\|\Phi(t)\|$). Therefore, by Remark 2.14, we obtain that, for $\phi \in C([-r, 0], X)$, the equation (2.37) has a unique solution $u(\cdot)$. This solution is defined on $[-r, \infty)$ and depends continuously on the initial data. This fact allows us to define a strongly continuous evolution family $\mathcal{V} = (V(t, s))_{t\geq s\geq 0}$ on the Banach space $C([-r, 0], X)$ as

$$V(t, s)\phi := u_t(\cdot, \phi), \tag{2.38}$$

where the function $u_t(\cdot, \phi)$ is the solution of the equation (2.37) satisfying $u_s(\cdot, \phi) = \phi$.

Therefore, by (2.37), we have the following relation between the evolution family $(V(t,s))_{t\geq s\geq 0}$ and the evolution family $(U(t,s))_{t\geq s\geq 0}$ generated by $B(\cdot)$:

$$V(t,s)\phi(\theta) = \begin{cases} U(t+\theta,s)\phi(0)+ & \\ \qquad + \int_s^{t+\theta} U(t+\theta,\xi)\Phi(\xi)(V(\xi,s)\phi)d\xi & \text{for } t \geq t+\theta \geq s \\ \phi(t+\theta-s) & \text{for } t-r \leq t+\theta \leq s. \end{cases} \tag{2.39}$$

We now come to our last result of this chapter about the robustness of exponential dichotomy of the solutions to the equation (2.37).

THEOREM 2.15. *Assume that the operators $B(t)$ generate the evolution family $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ having an exponential dichotomy. Let Φ be a strongly continuous and uniformly bounded function from $\mathbb{R}_+$ into the space $\mathcal{L}(C([-r,0],X),X)$. Then, if the norm $\|\Phi(\cdot)\| := \sup_{t\geq 0}\|\Phi(t)\|$ is sufficiently small, the evolution family $\mathcal{V} = (V(t,s))_{t\geq s\geq 0}$ defined as in (2.38) has an exponential dichotomy as well.*

PROOF. The proof is a minor modification of the proof of Theorem 1.17 in Chapter 1. We just have to use the variation of constants formula (2.39) instead of the formula (1.24). □

CHAPTER 3

A Semigroup Approach to Linear Neutral Partial Differential Equations

1. Motivation

In the beginning of the twentieth century (see [**11, 70**]), neutral differential equations were considered to be a special type of *differential-difference equations*. Examples of such equations are

$$u''(t) - u'(t-1) + u(t) = 0,$$
$$u'(t) - u(t-1) - u(t-\sqrt{2}) = 0,$$
$$u'(t) - 2u(t) + u'(t-1) - 2u(t-1) = 0,$$

(see [**6, 7, 11, 37, 70**]), or in general form of differential order n and difference order m:

$$F\big[t, u(t), u(t-r_1), ..., u(t-r_m), u'(t), u'(t-r_1), ..., u'(t-r_m), ... \\ ..., u^{(n)}(t), u^{(n)}(t-r_1), ..., u^{(n)}(t-r_m)\big] = 0$$

for some function F of $(m+1)(n+1)$ variables.

In order to understand the origins of names such as "delay", "neutral", etc., let us consider the general form of linear differential-difference equations of differential order 1 and difference order 1, i.e.,

$$a_0 u'(t) + a_1 u'(t-\omega) + b_0 u(t) + b_1 u(t-\omega) = f(t) \quad \text{for fixed } \omega > 0. \tag{3.1}$$

If $a_0 = a_1 = 0$, then this equation is simply called a *difference equation*. It does not contain any differential terms.

If $a_0 \neq 0,\ a_1 = 0$, then this equation is called a *differential-difference equation of retarded type* or simply a *delay differential equation*, because it describes a system depending on its states in the past.

If $a_0 = 0,\ a_1 \neq 0$, then this equation is called a *differential-difference equation of advanced type* or simply an *advanced differential equation*, because it describes a system depending on its states in the future.

Finally, if $a_0 \neq 0,\ a_1 \neq 0$, then this type of differential-difference equations is of mixed type with "retarded" and "advanced" arguments. Therefore, in this case, the equation is called a *differential-difference equation of neutral type* or simply a *neutral differential equation*. We refer to Bellman and Cooke [**6**, Chap. 2] for the full history of the problem.

Recently, Wu and Xia [**82**] have considered a network of transmission lines and have shown that the corresponding system of hyperbolic equations is equivalent to a system of neutral differential-difference equations. Such an equation can be formulated as

$$\frac{\partial}{\partial t} F u_t = a \frac{\partial^2}{\partial x^2} F u_t + \Phi u_t \tag{3.2}$$

which is called a *neutral partial differential equation* (NPDE). Here, the function u belongs to $C([-r,0],X)$ for some $r \geq 0$ and a Banach space X of functions on the unit circle

S^1, e.g., $X = H^1(S^1)$ or $X = C(S^1)$, the *history function* u_t is defined by $u_t(\theta) := u(t+\theta)$ for $\theta \in [-r, 0]$ and $t \geq 0$. Finally, F and Φ, called the *difference operator* and *delay operator*, respectively, are bounded linear operators from $C([-r,0],X)$ into X. A systematic treatment for the above equation is due to Hale [**34, 35**], in which he has given the basic existence and uniqueness results as well as properties of the solution operator.

In this chapter we propose a semigroup approach to linear NPDE's. It turns out that the well-posedness of linear NPDE's and the robustness of exponential stability of their solutions can be obtained in an elegant manner via semigroup methods. To this purpose we formulate linear NPDE's in an abstract form as

$$(NPDE) \quad \begin{cases} \frac{\partial}{\partial t} F u_t &= BFu_t + \Phi u_t \quad \text{for } t \geq 0, \\ u_0(t) &= \varphi(t) \quad \text{for } t \leq 0. \end{cases} \tag{3.3}$$

In the examples, B is some linear partial differential operator, while the operators F and Φ are called *difference operator* and *delay operator*, respectively. We refer to Hale [**34, 35**], Wu [**81**, Chap 2.3], Wu and Xia [**82**], Adimy and Ezzinbi [**1**] for concrete examples. In order to treat these equations in an abstract manner, we choose a Banach space X and consider the solution $u(\cdot)$ as a function from $\mathbb{R}$ to X. Then, the corresponding *history function* is defined as

$$u_t(s) := u(t+s) \text{ for all } t \geq 0,\ s \leq 0.$$

Moreover, B is a linear operator on X (representing a concrete partial differential operator), while F and Φ are linear operators from an X-valued function space , e.g., $C_0(\mathbb{R}_-, X)$ into X. More precisely, we make the following assumption.

ASSUMPTION 3.1. On the Banach spaces X and $E := C_0(\mathbb{R}_-, X)$ we consider the following operators.

(i) Let $(B, D(B))$ be the generator of a strongly continuous semigroup $(e^{tB})_{t\geq 0}$ on X satisfying $\|e^{tB}\| \leq Me^{\omega_1 t}$ for some constants $M \geq 1$ and $\omega_1 \in \mathbb{R}$.
(ii) Let the *difference operator* $F : E \to X$ and the *delay operator* $\Phi : E \to X$ be bounded and linear.

Under these assumptions we will solve the equation $(NPDE)$ by constructing an appropriate strongly continuous semigroup on the space E. This semigroup will be obtained by proving that a certain operator (see Definition 3.4) satisfies the Hille-Yosida conditions as long as we can write the difference operator as $F = \delta_0 - \Psi$ with Ψ being "small" (see (3.8)). If the delay and difference operators only act on a finite interval $[-r, 0]$, it can be shown that the smallness of Ψ can be replaced by the condition "having no mass in 0" (see Definition 3.8).

In the case of ordinary neutral functional differential equations on finite dimensional spaces X, we refer the readers to Hale and Verduyn Lunel [**36**, Chap. 9], Engel [**21**], Kappel and Zhang [**45, 46**] for results about well-posedness and asymptotic behavior of the solutions as well as the use of the condition "having no mass in 0" (or, "nonatomic at zero", see Remark 3.9). In the case of infinite dimensional spaces X, such a condition appeared in Schwarz [**71**] (see also Datko [**18**]), where the generator property has been shown under dissipativity conditions for ordinary neutral functional differential equations. Hale [**34, 35**] and Wu [**81**, Chap 2.3] assumed B to generate an analytic semigroup and also obtained a semigroup solving $(NPDE)$ in a mild sense if Ψ is nonatomic at zero.

2. Neutral semigroups with infinite delay

Under Assumption 3.1, we consider the operator $(G_m, D(G_m))$ on $E := C_0(\mathbb{R}_-, X)$, defined by

$$\begin{aligned} D(G_m) &:= \{f \in E \cap C^1(\mathbb{R}_-, X) : f' \in E\}, \\ G_m f &:= f' \quad \text{for } f \in D(G_m). \end{aligned} \tag{3.4}$$

We are now looking at various restrictions of this (maximal) operator yielding generators of strongly continuous semigroups. We start with a simple case.

DEFINITION 3.2. On the space $E = C_0(\mathbb{R}_-, X)$ we define a the operators $T_{B,0}(t)$ by

$$[T_{B,0}(t)f](s) = \begin{cases} f(s+t), & s+t \leq 0, \\ e^{(t+s)B} f(0), & s+t \geq 0, \end{cases}$$

for $f \in E$ and $t \geq 0$.

Moreover, we define the operator $(G_{B,0}, D(G_{B,0}))$ by

$$\begin{aligned} D(G_{B,0}) &:= \{f \in D(G_m) : f(0) \in D(B), f' \in E, \text{ and } f'(0) = Bf(0)\}, \\ G_{B,0} f &:= f' \quad \text{for } f \in D(G_{B,0}). \end{aligned}$$

For the reader's convenience, we recall the following properties of $G_{B,0}$ and $(T_{B,0}(t))_{t\geq 0}$ from Proposition 1.26.

PROPOSITION 3.3. *The following assertions hold.*

(i) $(T_{B,0}(t))_{t\geq 0}$ *is a strongly continuous semigroup on the space* E *with the generator* $(G_{B,0}, D(G_{B,0}))$.

(ii) *The set* $\{\lambda \in \mathbb{C} : Re\lambda > 0 \text{ and } \lambda \in \rho(B)\}$ *is contained in* $\rho(G_{B,0})$. *Moreover, for* λ *in this set, the resolvent is given by*

$$[R(\lambda, G_{B,0})f](t) = e^{\lambda t} R(\lambda, B) f(0) + \int_t^0 e^{\lambda(t-\xi)} f(\xi) d\xi \quad \text{for } f \in E,\ t \leq 0. \tag{3.5}$$

(iii) *The semigroup* $(T_{B,0}(t))_{t\geq 0}$ *satisfies*

$$\|T_{B,0}(t)\| \leq M e^{\omega_2 t},\ t \geq 0, \tag{3.6}$$

with $\omega_2 := \max\{0, \omega_1\}$ *for the constants* M *and* ω_1 *appearing in Assumption 3.1.*

We now take the delay operator Φ and the difference operator F (see Assumption 3.1) to define a different restriction of the operator G_m.

DEFINITION 3.4. The operator $G_{B,F,\Phi}$ is defined by

$$\begin{aligned} G_{B,F,\Phi} f &:= f' \quad \text{on the domain} \\ D(G_{B,F,\Phi}) &:= \{f \in D(G_m) : Ff \in D(B) \text{ and } F(f') = BFf + \Phi f\}. \end{aligned} \tag{3.7}$$

Our aim is to find conditions on F such that the operator $G_{B,F,\Phi}$ becomes the generator of a strongly continuous semigroup. To do so, we write F in the form

$$Ff := f(0) - \Psi f, \qquad f \in E, \tag{3.8}$$

for some bounded linear operator $\Psi : E \to X$. The domain of $G_{B,F,\Phi}$ can then be rewritten as

$$D(G_{B,F,\Phi}) = \{f \in D(G_m) : f(0) - \Psi f \in D(B) \text{ and } f'(0) = B(f(0) - \Psi f) + \Phi f + \Psi f'\}.$$

It is now our main result that, if the operator Ψ is "small", then $G_{B,F,\Phi}$ is densely defined and satisfies the Hille-Yosida estimates, hence generates a strongly continuous semigroup.

To that purpose and for each $\lambda \in \mathbb{C}$ satisfying $\mathrm{Re}\lambda > 0$, we define the operator $e_\lambda : X \to E$ by

$$[e_\lambda x](t) := e^{\lambda t} x \quad \text{for } t \le 0,\ x \in X. \tag{3.9}$$

THEOREM 3.5. *Assume that the difference operator F is of the form (3.8) such that Ψ satisfies the condition $\|\Psi\| < 1$. Then the following assertions hold.*

(i) *$\lambda \in \rho(G_{B,F,\Phi})$ for each $\lambda > \omega_2 + \frac{M\|\Phi\|}{1-\|\Psi\|}$ (with the constants ω_2 and M as in Proposition 3.3). For such λ the resolvent of $G_{B,F,\Phi}$ has the form*

$$\begin{aligned} R(\lambda, G_{B,F,\Phi})f &= e_\lambda[\Psi R(\lambda, G_{B,F,\Phi}) + R(\lambda, B)(\Phi R(\lambda, G_{B,F,\Phi}) - \Psi)]f + \\ &\quad + R(\lambda, G_{B,0})f \quad \text{for } f \in E. \end{aligned} \tag{3.10}$$

(ii) *For $L := \frac{M+M\|\Psi\|}{1-\|\Psi\|}$ and $\lambda_0 := \omega_2 + \frac{M\|\Phi\|}{1-\|\Psi\|}$ we have*

$$\|R(\lambda, G_{B,F,\Phi})\| \le \frac{L}{(\lambda - \lambda_0)} \qquad \text{for } \lambda > \lambda_0.$$

(iii) *For $\lambda > \omega_0 := \max\{2\lambda_0, \omega_2 + L\|\Phi\|\}$ and $P := 3e[(M+L)\|\Psi\| + 2M + 1]$ we have that*

$$\|R(\lambda, G_{B,F,\Phi})^n\| \le \frac{P}{(\lambda - \omega_0)^n} \qquad \text{for all } n \in \mathbb{N}.$$

(iv̈) *The operator $G_{B,F,\Phi}$ is densely defined.*

PROOF. (i) We first observe that for $u, f \in E$ and $\lambda \in \mathbb{C}$, we have that $u \in D(G_m)$ and $\lambda u - G_m u = f$ if and only if u, f satisfy the equation

$$u(t) = e^{\lambda(t-s)} u(s) + \int_t^s e^{\lambda(t-\xi)} f(\xi) d\xi \quad \text{for } t \le s \le 0. \tag{3.11}$$

Note that, for $\lambda > \omega_2$ and by (3.5) the equation

$$u(t) = e^{\lambda t}[\Psi u + R(\lambda, B)(f(0) + \Phi u - \Psi f)] + \int_t^0 e^{\lambda(t-\xi)} f(\xi) d\xi \text{ for } t \le 0 \tag{3.12}$$

is equivalent to

$$u = e_\lambda(\Psi u + R(\lambda, B)\Phi u) - e_\lambda R(\lambda, B)\Psi f + R(\lambda, G_{B,0})f. \tag{3.13}$$

If, for each $f \in E$ and $\lambda > \lambda_0$, this equation has a unique solution $u \in E$, then $u(0) = \Psi u + R(\lambda, B)(f(0) + \Phi u - \Psi f)$. This is equivalent to

$$(\lambda - B)(u(0) - \Psi u) = [(\lambda - G_m)u](0) + \Phi u - \Psi(\lambda - G_m)u$$

or

$$u'(0) = B(u(0) - \Psi u) + \Phi u + \Psi u'.$$

Hence, by the above observation and the definition of $G_{B,F,\Phi}$, we have that $u \in D(G_{B,F,\Phi})$ and $u = R(\lambda, G_{B,\Phi,F})f$. Therefore, to prove (i) we have to verify that for $\lambda > \lambda_0$ and each $f \in E$ the equation (3.12) has a unique solution $u \in E$.

Let $M_\lambda : E \to E$ be the linear operator defined as

$$M_\lambda := e_\lambda(\Psi + R(\lambda, B)\Phi). \tag{3.14}$$

Since $\lambda > \omega_2 + \frac{M\|\Phi\|}{1-\|\Psi\|}$, we have that M_λ is bounded and satisfies

$$\|M_\lambda\| \leq \|\Psi\| + \frac{M\|\Phi\|}{\lambda - \omega_2} < 1.$$

Therefore, the operator $I - M_\lambda$ is invertible, and the equation (3.12) has a unique solution $u = (I - M_\lambda)^{-1}(R(\lambda, G_{B,0})f - e_\lambda R(\lambda, B)\Psi f)$. Thus,

$$R(\lambda, G_{B,F,\Phi})f = M_\lambda R(\lambda, G_{B,F,\Phi})f - e_\lambda R(\lambda, B)\Psi f + R(\lambda, G_{B,0})f \tag{3.15}$$

and the identity (3.10) follows.

(ii) By the Neumann series $(I - M_\lambda)^{-1} = \sum_{n=0}^{\infty} M_\lambda^n$ we have that

$$\begin{aligned}
\||R(\lambda, G_{B,F,\Phi})\| &= \|\sum_{n=0}^{\infty} M_\lambda^n(R(\lambda, G_{B,0}) - e_\lambda R(\lambda, B)\Psi)\| \\
&\leq (\sum_{n=0}^{\infty} \|M_\lambda^n\|)\frac{M + M\|\Psi\|}{(\lambda - \omega_2)} \\
&\leq \frac{M + M\|\Psi\|}{(\lambda - \omega_2)} \sum_{n=0}^{\infty} \left(\|\Psi\| + \frac{M\|\Phi\|}{\lambda - \omega_2}\right)^n \\
&= \frac{M + M\|\Psi\|}{(1 - \|\Psi\|)(\lambda - \omega_2 - \frac{M\|\Phi\|}{1-\|\Psi\|})} \\
&= \frac{L}{\lambda - \lambda_0}.
\end{aligned}$$

(iii) For $\lambda > \lambda_0$ and $u := R(\lambda, G_{B,F,\Phi})f$, we have

$$\begin{aligned}
u(t) &= e^{\lambda t}[\Psi R(\lambda, G_{B,F,\Phi})f + R(\lambda, B)(\Phi R(\lambda, G_{B,F,\Phi})f - \Psi f + f(0))] + \\
&\qquad + \int_t^0 e^{\lambda(t-\xi)} f(\xi)d\xi \quad \text{for } t \leq 0.
\end{aligned}$$

We extend u and f to functions on $\mathbb{R}$ by

$$\tilde{u}(t) := \begin{cases} u(t) & \text{for } t \leq 0 \\ e^{\lambda t} g(t) & \text{for } t > 0 \end{cases} \quad \text{and } \tilde{f}(t) := \begin{cases} f(t) & \text{for } t \leq 0 \\ -e^{\lambda t} g'(t) & \text{for } t > 0, \end{cases} \tag{3.16}$$

where we take $g(t) := u(0) + \int_0^t \varphi(\tau)d\tau$ with

$$\varphi(t) := \begin{cases} 6t[t\lambda^2 - \lambda][\lambda u(0) - \frac{1}{2}f(0)] + [\lambda t - 1]f(0) & \text{for } 0 \leq t \leq \frac{1}{\lambda}, \\ 0 & \text{for } t \geq \frac{1}{\lambda}. \end{cases}$$

Then g is continuously differentiable with compact support contained in $[0, \frac{1}{\lambda}]$ satisfying $g(0) = u(0)$, $g'(0) = -f(0)$, and

$$\|e^{\lambda t} g'(t)\| \leq 3e[(M + L)\|\Psi\| + 2M + 1]\|f\|$$

for $\lambda > \max\{2\lambda_0, \omega_2 + L\|\Phi\|\}$ and all $t \in \mathbb{R}_+$. Hence, the functions $\tilde{u}, \tilde{f}$ belong to the Banach space $C_0(\mathbb{R}, X)$ and satisfy the equation

$$\tilde{u}(t) = e^{\lambda(t-s)}\tilde{u}(s) + \int_t^s e^{\lambda(t-\xi)}\tilde{f}(\xi)d\xi \quad \text{for } t \leq s, \tag{3.17}$$

and

$$\|\tilde{f}\| \leq 3e[(M+L)\|\Psi\| + 2M + 1]\|f\|. \tag{3.18}$$

We now look at the left translation semigroup $(\tilde{T}(t))_{t\geq 0}$ on the Banach space $\tilde{E} := C_0(\mathbb{R}, X)$, i.e.,

$$(\tilde{T}(t)\tilde{f})(s) := \tilde{f}(s+t) \text{ for all } \tilde{f} \in \tilde{E},\ s \in \mathbb{R},\ \text{and } t \geq 0. \tag{3.19}$$

This semigroup is strongly continuous on $\tilde{E}$ and its generator is

$$\begin{aligned} \tilde{G}_m &:= \frac{d}{ds} \quad \text{on the domain} \\ D(\tilde{G}_m) &:= \{f \in \tilde{E} \cap C^1(\mathbb{R}, X) : f' \in \tilde{E}\} \end{aligned}$$

(see [**23**, Chap. II.2]). Furthermore, we observe that for $v, w \in \tilde{E}$ and $\lambda \in \mathbb{C}$, we have that $v \in D(\tilde{G}_m)$ and $\lambda v - \tilde{G}_m v = w$ if and only if v, w satisfy the equation (3.17). Since $\lambda \in \rho(\tilde{G}_m)$ for $\lambda > \lambda_0$, we obtain that $\tilde{u} = R(\lambda, \tilde{G}_m)\tilde{f}$ for $\lambda > \lambda_0$, where $\tilde{u}$ and $\tilde{f}$ are defined as in (3.16).
Therefore, by (3.16), we have that

$$[R(\lambda, G_{B,F,\Phi})f](t) = u(t) = \tilde{u}(t) = [R(\lambda, \tilde{G}_m)\tilde{f}](t)$$

for $t \leq 0$ and $\lambda > \max\{2\lambda_0, \omega_2 + L\|\Phi\|\} =: \omega_0$.

By induction we obtain

$$[R(\lambda, G_{B,F,\Phi})^n f](t) = [R(\lambda, \tilde{G}_m)^n \tilde{f}](t) \text{ for } t \leq 0 \text{ and } \lambda > \omega_0.$$

Using the fact that $\tilde{G}_m$ is the generator of the strongly continuous semigroup $(\tilde{T}(t))_{t\geq 0}$ on $\tilde{E}$ and by the inequality (3.18), we have

$$\begin{aligned} \|[R(\lambda, G_{B,F,\Phi})^n f](t)\| &= \|[R(\lambda, \tilde{G}_m)^n \tilde{f}](t)\| \\ &\leq \frac{\|\tilde{f}\|}{\lambda^n} \leq \frac{3e[(M+L)\|\Psi\| + 2M + 1]}{(\lambda - \omega_0)^n}\|f\| \end{aligned}$$

for all $t \leq 0,\ \lambda > \omega_0$, and all $n \in \mathbb{N}$. Therefore, putting $P := 3e[(M+L)\|\Psi\| + 2M + 1]$ we obtain

$$\|R(\lambda, G_{B,F,\Phi})^n f\| \leq \frac{P}{(\lambda - \omega_0)^n}\|f\| \quad \text{for } \lambda > \omega_0 \text{ and } n \in \mathbb{N}.$$

(iv̈) For $\lambda > \lambda_0$ we consider the operator $S : E \to E$ defined by

$$Sf := -e_\lambda R(\lambda, B)\Psi f + R(\lambda, G_{B,0})f,\ \ f \in E.$$

Observe that if its range ImS is dense in E, then we have, by (3.15), that $D(G_{B,F,\Phi}) = R(\lambda, G_{B,F,\Phi})E = (I - M_\lambda)^{-1} ImS$ is dense in E. Therefore, it is enough to verify that ImS is dense in E. Since $D(G_m)$ is dense in E, we only need to show that $\overline{ImS} \supset D(G_m) = D(\lambda - G_m)$.

In fact, for $u \in D(\lambda - G_m)$ there exists $f \in E$ such that

$$u(t) = e^{\lambda t}u(0) + \int_t^0 e^{\lambda(t-\xi)}f(\xi)d\xi \quad \text{for } t \leq 0.$$

Since the operator B is densely defined, there exists a sequence $(y_n) \subset D(B)$ such that $\lim_{n\to\infty} y_n = u(0)$. Let $(x_n) \subset X$ be a sequence such that $R(\lambda, B)x_n = y_n$.

For each $n \in \mathbb{N}$ we choose a real valued, continuous function $\alpha_n(t)$ with support contained in $[\max\{-\frac{1}{n}, -\frac{1}{n\|x_n\|}\}, 0]$ satisfying $\alpha_n(0) = 1$ and $\sup_{t\le 0} |\alpha_n(t)| \le 1$.

By the condition $\|\Psi\| < 1$, we have that the functions

$$f_n(\cdot) := (I - \alpha_n(\cdot)\Psi)^{-1}(\alpha_n(\cdot)(x_n - f(0)) + f(\cdot)), \quad n \in \mathbb{N},$$

belong to E. Moreover, these functions satisfy

$$f_n(t) = \alpha_n(t)(x_n + \Psi(f_n) - f(0)) + f(t),$$
$$f_n(0) - \Psi(f_n) = x_n,$$

and

$$\|f_n\| \le \frac{\|x_n\| + 2\|f\|}{1 - \|\Psi\|}.$$

We now put

$$u_n(t) := e^{\lambda t} R(\lambda, B)(f_n(0) - \Psi f_n) + \int_t^0 e^{\lambda(t-\xi)} f_n(\xi) d\xi.$$

Then $u_n = Sf_n$, hence $u_n \in ImS$, and for $\lambda > \lambda_0$ we obtain

$$\begin{aligned}
\|u_n(t) - u(t)\| &= \|e^{\lambda t}(R(\lambda, B)x_n - u(0)) + \int_t^0 e^{\lambda(t-\xi)}(f_n(\xi) - f(\xi))d\xi\| \\
\le\ & \|y_n - u(0)\| + \int_t^0 |\alpha_n(t)|(\|x_n\| + \|\Psi\|\|f_n\| + \|f(0)\|)dt \\
\le\ & \|y_n - u(0)\| + \int_{\max\{-\frac{1}{n}, -\frac{1}{n\|x_n\|}\}}^0 (\|x_n\| + \frac{\|\Psi\|(\|x_n\| + 2\|f\|)}{1 - \|\Psi\|} + \|f\|)dt \\
\le\ & \|y_n - u(0)\| + \frac{1}{n(1 - \|\Psi\|)} + \frac{(1 + \|\Psi\|)\|f\|}{n(1 - \|\Psi\|)} \stackrel{n\to\infty}{\longrightarrow} 0 \text{ uniformly for all } t \in \mathbb{R}_-.
\end{aligned}$$

This means that $\lim_{n\to\infty} u_n = u$. Thus, ImS is dense in E. □

The Hille-Yosida Theorem now yields the following main result.

THEOREM 3.6. *Let the difference operator F have the form (3.8) with Ψ satisfying $\|\Psi\| < 1$. Then the operator $G_{B,F,\Phi}$ generates a strongly continuous semigroup $(T_{B,F,\Phi}(t))_{t\ge 0}$ on E satisfying*

$$\|T_{B,F,\Phi}(t)\| \le P e^{\omega_0 t}, \quad t \ge 0,$$

where the constants P and ω_0 are defined as in Theorem 3.5.

We conclude this section by a result about "well-posedness" of the equation (NPDE). To this end, we denote by $t \mapsto u_t(\cdot, \varphi)$ the *classical solution* of the equation $(NPDE)$ corresponding to the initial condition $u_0 = \varphi$, i.e., $t \mapsto u_t(\cdot, \varphi)$ is continuously differentiable and satisfies the equation $(NPDE)$.

COROLLARY 3.7. *Assume that the difference operator F is of the form (3.8) such that Ψ satisfies $\|\Psi\| < 1$. Then the equation $(NPDE)$ is well-posed. More precisely, for every $\varphi \in D(G_{B,F,\Phi})$ there exists a unique classical solution $u_t(\cdot, \varphi)$ of $(NPDE)$ given by*

$$u_t(\cdot, \varphi) = T_{B,F,\Phi}(t)\varphi,$$

and for every sequence $(\varphi_n)_{n\in\mathbb{N}} \subset D(G_{B,F,\Phi})$ *satisfying* $\lim_{n\to\infty}\varphi_n = 0$*, one has*

$$\lim_{n\to\infty} u_t(\cdot,\varphi_n) = 0$$

uniformly in compact intervals.

PROOF. By Theorem 3.6, the operator $(G_{B,F,\Phi}, D(G_{B,F,\Phi}))$ defined by (3.7) is the generator of the strongly continuous semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$.

For $\varphi \in D(G_{B,F,\Phi})$ we put $u_t := T_{B,F,\Phi}(t)\varphi$. Then it is clear that $u_t \in D(G_{B,F,\Phi}) \subset D(G_m)$. We now show that u_t satisfies the equation $(NPDE)$. Indeed, we have

$$\begin{aligned}
\frac{d}{dt}Fu_t &= \lim_{h\to 0}\frac{Fu_{t+h}-Fu_t}{h} = \lim_{h\to 0}\frac{FT(t+h)\varphi - FT(t)\varphi}{h} \\
&= F\lim_{h\to 0}\frac{T(h)T(t)\varphi - T(t)\varphi}{h} = FG_{B,F,\Phi}T(t)\varphi = BFT(t)\varphi + \Phi T(t)\varphi \\
&= BFu_t + \Phi u_t.
\end{aligned}$$

For the uniqueness of the solution, we prove that, if v_t is a classical solution of $(NPDE)$ satisfying $v_0 = 0$, then $v_t = 0$ for all $t \geq 0$. In fact, since v_t satisfies $(NPDE)$ and $v_t \in D(G_m)$, we have that

$$BFv_t + \Phi v_t = \frac{d}{dt}Fv_t = \lim_{h\to 0}\frac{Fv_{t+h}-Fv_t}{h} = F\lim_{h\to 0}\frac{v_{t+h}-v_t}{h} = Fv_t'.$$

Therefore, $v_t \in D(G_{B,F,\Phi})$ satisfies the Cauchy problem

$$\begin{cases} \frac{d}{dt}v_t &= G_{B,F,\Phi}v_t \quad \text{for } t \geq 0, \\ v_0 &= 0. \end{cases}$$

Since $G_{B,F,\Phi}$ is the generator of a strongly continuous semigroup, this Cauchy problem has a unique solution $v_t = 0$ (see [**23**, Theorem II.6.7]).

Finally, the last assertion, called the *continuous dependence on the initial data* of the solutions, follows from the uniform boundedness of the strongly continuous semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$ on compact intervals. □

3. Neutral semigroups with finite delay

In this section, we study the equation $(NPDE)$ on a finite delay interval $[-r, 0]$, i.e.,

$$\begin{cases} \frac{\partial}{\partial t}Fu_t &= BFu_t + \Phi u_t \quad \text{for } t \geq 0, \\ u_0 &= \varphi \in C([-r,0],X), \end{cases} \tag{3.20}$$

where F and Φ are now operators from $C([-r,0],X)$ into X. We again assume the difference operator F to be written as in (3.8), i.e.,

$$F\varphi = \varphi(0) - \Psi\varphi, \qquad \varphi \in C([-r,0],X), \tag{3.21}$$

for some bounded linear operator $\Psi : C([-r,0],X) \to X$. However, instead of assuming Ψ to be "small", we suppose that Ψ *has no mass in 0* (see Definition 3.8 below). Our main idea is to renorm the space $C([-r,0],X)$ such that, with the new equivalent norm, the norm of Ψ is small, so we can adapt the arguments from the previous section. This idea has been used, e.g., by M. Schwarz [**71**] to study ordinary neutral functional differential equations via dissipativity conditions. We begin with the definition of "no mass in 0".

DEFINITION 3.8. A bounded linear operator $\Psi \in \mathcal{L}(C([-r,0],X),X)$ is said to have *no mass in 0* if for every $\epsilon > 0$ there exists a positive number $\delta \le r$ such that

$$\|\Psi(f)\|_X \le \epsilon \|f\|_\infty \quad \text{for all } f \in C([-r,0],X) \text{ satisfying } \operatorname{supp} f \subseteq [-\delta, 0].$$

REMARK 3.9. This definition is taken from [**71**, Definition II.2.1]. We note that, if $\Psi \in \mathcal{L}(C([-r,0],X),X)$ has the form

$$\Psi(f) = \int_{-r}^{0} [d\eta(\theta)] f(\theta)$$

for some function $\eta(\cdot)$ of bounded variation, then the above definition is equivalent to the fact that the function $\eta(\cdot)$ is nonatomic at 0 in the sense of Hale and Verduyn Lunel [**36**, Chap. 9.2] or Wu [**81**, Chap. 2.3].

We are now prepared to renorm the space $C := C([-r,0],X)$. Indeed, for each positive number ω, the new norm $\|\cdot\|_\omega$ defined by

$$\|f\|_\omega := \sup_{-r \le s \le 0} \|f(s)e^{-\omega s}\|_X, \quad f \in C, \tag{3.22}$$

is equivalent to the sup-norm. Furthermore, we denote by C_ω the space $C([-r,0],X)$ endowed with the norm $\|\cdot\|_\omega$.

LEMMA 3.10. *Let the operator $\Psi \in \mathcal{L}(C,X)$ have no mass in 0. Then there exists a positive number ω such that the norm of the operator Ψ, as a bounded linear operator from C_ω into X, is smaller than one.*

PROOF. We first prove that there exists a number $\omega > 0$ such that

$$\|\Psi(f)\|_X \le \frac{1}{2} \quad \text{for all } f \in C([-r,0],X) \text{ satisfying } \|f(s)\| \le e^{\omega s} \quad \text{for all } s \in [-r,0]. \tag{3.23}$$

Indeed, since Ψ has no mass in 0, there exists a positive number $\delta \le r$ such that $\|\Psi(f)\|_X \le \frac{1}{8}\|f\|$ if $\operatorname{supp} f \subseteq [-\delta,0]$. For this δ we take an $\omega > 0$ such that $\|\Psi\| < \frac{1}{4}e^{\delta\omega}$. Now, for a given $f \in C([-r,0],X)$ satisfying $\|f(s)\|_X \le e^{\omega s}$ for all $s \in [-r,0]$, we will prove that $\|\Psi(f)\|_X \le \frac{1}{2}$. To that purpose, we put

$$f_1(s) := \begin{cases} f(s) & \text{for } s \in [-r,-\delta] \\ f(-\delta) & \text{otherwise,} \end{cases}$$

and $f_2(s) := f(s) - f_1(s)$. Then $\operatorname{supp} f_2 \subseteq [-\delta,0]$. Therefore, we have that

$$\|\Psi(f)\|_X \le \|\Psi(f_1)\|_X + \|\Psi(f_2)\|_X \le \|\Psi\|\|f_1\|_\infty + \frac{1}{8}(\|f_1\|_\infty + \|f\|_\infty) < \frac{1}{2},$$

and (3.23) follows.

Denote by $\|\Psi\|_\omega$ the norm of Ψ as a bounded linear operator in $\mathcal{L}(C_\omega,X)$. Then, by the inequality (3.23), we have that

$$\|\Psi\|_\omega = \sup_{\|f\|_\omega \le 1} \|\Psi f\| = \sup_{\sup_{-r\le s\le 0} \|f(s)\|e^{-\omega s} \le 1} \|\Psi f\| \le \frac{1}{2} < 1.$$

□

This renorming allows us to adapt the arguments from the proof of Theorem 3.5 to prove that the operator corresponding to (3.20) is the generator of a C_0-semigroup on the space $C([-r,0],X)$. Note that the generator property of an operator is preserved when passing to an equivalent norm.

THEOREM 3.11. *Assume that the difference operator F has the form $F\varphi = \varphi(0) - \Psi\varphi$ with the bounded linear operator $\Psi : C([-r,0],X) \to X$ having no mass in 0, and let the operator B generate a strongly continuous semigroup on X. Then the operator $(G, D(G))$, defined by*

$$\begin{aligned} Gf &:= f' \quad \text{on the domain} \\ D(G) &:= \{f \in C([-r,0],X) \cap C^1([-r,0],X) : Ff \in D(B) \\ &\qquad \text{and } Ff' = BFf + \Phi f\}, \end{aligned} \tag{3.24}$$

is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $C([-r,0],X)$.

PROOF. Let C_ω be the space $C([-r,0],X)$ normed by the new norm $\|\cdot\|_\omega$ for ω as in Lemma 3.10. Then the norm of the operator Ψ, as a bounded linear operator from C_ω into X, is smaller than one. Therefore, as in Theorem 3.5, one shows that the operator $(G, D(G))$ defined by (3.24) is densely defined and satisfies the Hille-Yosida estimates, hence, it generates a strongly continuous semigroup. □

Analogously to Corollary 3.7, we have the following result about the existence, uniqueness, and continuous dependence on initial data of the solutions to NPDE (3.20).

COROLLARY 3.12. *Assume that the difference operator F is of the form (3.21) such that Ψ has no mass in 0. Then for every $\varphi \in D(G)$, there exists a unique classical solution $u_t(\cdot,\varphi)$ of the equation (3.20), given by $u_t(\cdot,\varphi) = T(t)\varphi$, where the strongly continuous semigroup $(T(t))_{t\geq 0}$ is generated by the operator G as in Theorem 3.11. Moreover, for every sequence $(\varphi_n)_{n\in\mathbb{N}} \subset D(G)$ satisfying $\lim_{n\to\infty} \varphi_n = 0$, one has $\lim_{n\to\infty} u_t(\cdot,\varphi_n) = 0$ uniformly in compact intervals.*

Having established the well-posedness of the equation (3.20), we now consider the robustness of the exponential stability of the solution semigroup. This can be done by using the constants appearing in the Hille-Yosida estimates of the operator G.

COROLLARY 3.13. *Let the assumptions of Theorem 3.11 be satisfied. In addition, let the operator B generate an exponentially stable C_0-semigroup and the norm of the operator $\Psi : C \to X$ satisfy $\|\Psi\| < 1$. Then, if the norm of the delay operator Φ is sufficiently small, the solution semigroup $(T(t))_{t\geq 0}$ generated by $(G, D(G))$ is exponentially stable.*

PROOF. We note that, in the case of the finite delay interval $[-r,0]$, the operators e_λ defined as in (3.9) are well-defined for all $\lambda \in \mathbb{C}$, and the exponent ω_2 in the exponential estimate (3.6) can be chosen as $\omega_2 := \omega_1$ with the constant M being replaced by $K := \max\{M, Me^{-\omega_1 r}\}$, where the constant ω_1 appears in Assumption 3.1. Therefore, for the equation (3.20) on the finite delay interval $[-r,0]$, we can adapt the arguments in the proof of Theorem 3.5 to obtain an analogue of Theorem 3.6. That is, the generator $(G, D(G))$ defined by (3.24) generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ satisfying

$$\|T(t)\| \leq Pe^{\omega_0 t},\ t \geq 0,$$

with the constant P defined as in Theorem 3.5 and the constant

$$\omega_0 := \max\{2\omega_1 + \frac{2K\|\Phi\|}{1-\|\Psi\|}, \omega_1 + \frac{K+K\|\Psi\|}{1-\|\Psi\|}\|\Phi\|\}.$$

Now, the assumption that $(e^{tB})_{t\geq 0}$ is exponentially stable means that $\omega_1 < 0$. Therefore, if $\|\Phi\| < \frac{-\omega_1(1-\|\Psi\|)}{K+K\|\Psi\|}$, then the solution semigroup is exponentially stable. □

REMARK 3.14. In order to show the robustness of the exponential stability of the solution semigroup as in the above corollary, we need the condition $\|\Psi\| < 1$. This is due to the fact that, in the renorming Lemma 3.10, the constant $\omega > 0$, which will appear in the exponential bound of the solution semigroup, does not depend on the operator B. Moreover, we do not have an explicit estimate for this constant. This somehow corresponds to the fact considered by Hale [**34**, Theorem 1.2] that one needs some additional conditions on the difference operator F to develop a qualitative theory of NPDE (3.20).

4. Examples

We now investigate some concrete examples.

EXAMPLE 3.15. We consider the equation

$$\begin{cases} \frac{\partial}{\partial t} F u_t &= BFu_t + \Phi u_t \quad \text{for } t \geq 0, \\ u_0 &= \varphi \in C([-r,0],X) := C. \end{cases} \tag{3.25}$$

Here, the defference operator $F : C \to X$ is defined by

$$Ff = f(0) - \int_{-r}^{0} [d\eta(\theta)] f(\theta), \quad f \in C,$$

for some function of bounded variation $\eta(\cdot)$ which is nonatomic at 0. The operator B generates an analytic semigroup. Finally, the delay operator $\Phi : C \to X$ is bounded and linear (e.g., $\Phi f := \int_{-r}^{0} \varphi(s) f(s) ds$ for $f \in C$ and some function $\varphi \in L^1([-r,0])$).

By Remark 3.9, Theorem 3.11 and Corollary 3.12 we obtain that the equation (3.25) is well-posed. We note that, a similar problem has been considered by Hale [**34**] for $X = H^1(S^1,\mathbb{R})$ and by Wu [**81**, Chap 2.3] for $X = C(S^1,\mathbb{R})$ (with $B := \frac{\partial^2}{\partial x^2}$ and the same F and Φ). They obtained the existence and uniqueness of the mild solutions to the equation (3.25).

Moreover, using Corollary 3.13 and the fact that the Spectral Mapping Theorem holds for analytic semigroups (see [**23**, Theorem IV.3.12]) we obtain that, if the spectral bound $s(B) < 0$, the total variation $\mathrm{Var}_{[-r,0]}\eta(\cdot) < 1$ and the norm $\|\Phi\|$ is sufficient small, then the solutions of the equation (3.25) are exponentially stable.

Our methods may be extended to consider the equation posed by Datko [**18**].

EXAMPLE 3.16. In [**18**, Eq. (1)], Datko considered the equation

$$\begin{cases} \frac{d}{dt}\left[u(t) - \sum_{j=1}^{m} A_j u(t-r_j)\right] &= Bu(t) + \sum_{j=1}^{m} B_j u(t-r_j) \quad \text{for } t \geq 0, \\ u(t) &= \phi(t) \quad \text{for } t \leq 0 \text{ and } \phi \in C([-r,0],X) := C, \end{cases} \tag{3.26}$$

where, the operator B generates a strongly continuous semigroup, while the operators $\{A_j\}$ and $\{B_j\}$ belong to $\mathcal{L}(X)$. Finally, the numbers $0 < r_1 < r_2 < \cdots < r_m = r$ are fixed. This equation can be formulated in more general form as

$$\begin{cases} \frac{\partial}{\partial t} F u_t &= Bu(t) + \Phi u_t \quad \text{for } t \geq 0, \\ u_0 &= \varphi \in C([-r,0],X) := C. \end{cases} \tag{3.27}$$

Here, the operator B generates a strongly continuous semigroup, while the delay operator $\Phi : C \to X$ is given by $\Phi\varphi := \sum_{j=1}^m B_j\varphi(-r_j)$ for $\varphi \in C$. Finally, the difference operator $F : C \to X$ is defined by $F = \delta_0 - \Psi$ with $\Psi\varphi := \sum_{j=1}^m A_j\varphi(-r_j)$ for $\varphi \in C$ showing that Ψ has no mass in 0.

We now consider the following operator

$$\begin{aligned} Gf &:= f' \quad \text{on the domain} \\ D(G) &:= \{f \in C^1 \cap C : f(0) \in D(B) \text{ and } F(f') = Bf(0) + \Phi f\} \end{aligned} \tag{3.28}$$

in the space C.

By the same methods as in Theorem 3.11 and Corollary 3.12, we obtain that the operator G defined above is the generator of a strongly continuous semigroup and the equation (3.27) is well-posed.

Moreover, using Corollary 3.13, we can obtain, for the equation (3.26), an explicit result on the robustness of exponential stability of the solution. Precisely, if the operators B generates an exponentially stable semigroup $(e^{tB})_{t\geq 0}$ satisfying

$$\|e^{tB}\| \leq Me^{-\omega_1 t}, \quad t \geq 0, \ \omega_1 > 0,$$

and if $\sum_{j=1}^m \|A_j\| < 1$ and $\sum_{j=1}^m \|B_j\| < \frac{-\omega_1(1-\|\Psi\|)}{M+M\|\Psi\|}$, then the solutions of the equation (3.26) are also exponentially stable.

CHAPTER 4

Neutral Partial Differential Equations with Non-autonomous Past

In this chapter, inspired by Chapter 2, we shall consider the case of *non-autonomous past*. To explain this teminology briefly, let us turn back to equation (3.3) in Chapter 3

$$(NPDE) \quad \begin{cases} \frac{\partial}{\partial t} F u_t &= BFu_t + \Phi u_t \quad \text{for } t \geq 0, \\ u_0(t) &= \varphi(t) \quad \text{for } t \leq 0. \end{cases} \tag{4.1}$$

From Theorem 3.5 we know that, if Assumption 3.1 is satisfied, then there exists a corresponding solution semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$. If we now consider the function $u : \mathbb{R}_+ \times \mathbb{R}_- \to X$ defined as

$$u(t,s) = [T_{B,F,\Phi}(t)f](s),$$

then we obtain the equality

$$\frac{\partial}{\partial t} u(t,s) = \frac{\partial}{\partial s} u(t,s),$$

which is known as the balance law between the velocity of the evolution process in the past and in the future (see [**20**, p. 39-40]). However, in many applications this balance law may not be true. An idea introduced by Brendle and Nagel [**10**] to control the unbalance is to suppose that the value of the history function is modified according to an evolution law. Consequently, this modification leads to the following system of equations

$$\frac{\partial}{\partial t} F(u(t,\cdot)) = BFu(t,\cdot) + \Phi u(t,\cdot), \qquad t \geq 0, \tag{4.2}$$

$$\frac{\partial}{\partial t} u(t,s) = \frac{\partial}{\partial s} u(t,s) + A(s)u(t,s), \qquad t \geq 0 \geq s. \tag{4.3}$$

Here, the function $u(\cdot,\cdot)$ takes values in a Banach space X and B is a linear operator on X. The difference operator F and the delay operator Φ are bounded linear operators from the space $C_0(\mathbb{R}_-, X)$ into X. Finally, $A(s)$ are (unbounded) operators on X for which the non-autonomous Cauchy problem

$$\begin{cases} \frac{dx(t)}{dt} &= -A(t)x(t), \ t \leq s \leq 0, \\ x(s) &= x_s \in X, \end{cases} \tag{4.4}$$

is well-posed with exponential bound. In particular, there exists an exponentially bounded backward evolution family $\mathcal{U} = (U(t,s))_{t\leq s\leq 0}$ solving (4.4), i.e., the solutions of (4.4) are given by $x(t) = U(t,s)x(s)$ for $t \leq s \leq 0$ (see Chapter 1, Section 5).

We refer to [**10**] for more information about "non-autonomous past". If F has the form $F\varphi = \varphi(0)$ for φ in relevant spaces, i.e., for delay PDE's with non-autonomous past, these equations have been solved using semigroup methods in the space $C_0(\mathbb{R}_-, X)$ in [**10, 27**] or in the space $L^p(\mathbb{R}_-, X)$ in [**26, 28**].

As in Section 1 of Chapter 2, we again use the theory of evolution semigroups as introduced in Chapter 1, where we defined an abstract differential operator G on $C_0(\mathbb{R}_-, X)$ (see Definition 1.23). We now use the difference operator F, the delay operator Φ (and the operator B) to define a restriction $G_{B,F,\Phi}$ of G. Then, under appropriate assumptions on F, we compute explicitly the resolvent of this restriction and show the Hille-Yosida estimates. In this way, we obtain a semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$ which solves (4.2) and (4.3) in a mild sense. Moreover, by computing the constants appearing in the Hille- Yosida estimates, we can show that exponential stability of this semigroup, hence of the solutions of (4.2) and (4.3), is robust under small perturbations of the delay operator Φ.

1. Evolution semigroups with difference and delay

In this section, for the Banach spaces X and $E := C_0(\mathbb{R}_-, X)$, we shall consider bounded linear operators $\Phi : E \to X$, $F : E \to X$, called *delay operator* and *difference operator*, respectively, and the linear operator $B : X \to X$ satisfying Assumption 3.1. We use them to define the following restriction of the operator G from Definition 1.23 in Chapter 1.

DEFINITION 4.1. The operator $G_{B,F,\Phi}$ is defined by

$$\begin{aligned} G_{B,F,\Phi} f &:= Gf \text{ on the domain} \\ D(G_{B,F,\Phi}) &:= \{f \in D(G) : Ff \in D(B) \text{ and } F(Gf) = BFf + \Phi f\}. \end{aligned} \tag{4.5}$$

As in the formula (3.8), we write F in the form

$$F\varphi := \varphi(0) - \Psi\varphi, \qquad \varphi \in E, \tag{4.6}$$

for some bounded linear operator $\Psi : E \to X$. The domain of the operator $G_{B,F,\Phi}$ can then be rewritten as

$$D(G_{B,F,\Phi}) := \{f \in D(G) : f(0) - \Psi f \in D(B) \text{ and } [Gf](0) = B(f(0) - \Psi f) + \Phi f + \Psi G f\}.$$

If the operator Ψ is "small", we can prove that $G_{B,F,\Phi}$ satisfies the Hille-Yosida estimates. In order to handle the case of "non-autonomous past", we may, in principle, use techniques similar as in Theorem 3.5 in Chapter 3. Heuristically, the first derivative $\frac{d}{dt}$ has to be replaced by the operator G. More precisely, we have to use the integral equation

$$u(t) = e^{\lambda(t-s)}U(t,s)u(s) + \int_t^s e^{\lambda(t-\xi)}U(t,\xi)f(\xi)d\xi, \quad t \leq s \leq 0,$$

instead of the integral equation

$$u(t) = e^{\lambda(t-s)}u(s) + \int_t^s e^{\lambda(t-\xi)}f(\xi)d\xi, \quad t \leq s \leq 0,$$

which has been used in the proof of Theorem 3.5 in Chapter 3 (see the equation (3.11)). Furthermore, we have to use the left translation evolution semigroup defined as in Definition 1.25 in Chapter 1 instead of the left translation semigroup defined as in Definition 3.2 in Chapter 3.

THEOREM 4.2. *Let the operator Ψ satisfy the condition $\|\Psi\| < \frac{1}{N}$ (with the constant N as in Definition 1.20 in Chapter 1), and define the operator $e_\lambda : X \to E$ by*

$$[e_\lambda x](t) := e^{\lambda t}U(t,0)x \quad \text{for } t \leq 0, \; x \in X \text{ and } Re\lambda > \omega(\mathcal{U}).$$

Then the following assertions hold.

(i) $\lambda \in \rho(G_{B,F,\Phi})$ *for all* $\lambda > \omega + \frac{K\|\Phi\|}{1-N\|\Psi\|}$ *(with the constants* ω *and* K *as in Proposition 1.26). For all such* λ *the resolvent of* $G_{B,F,\Phi}$ *satisfies*

$$\begin{aligned} R(\lambda, G_{B,F,\Phi})f &= e_\lambda[\Psi R(\lambda, G_{B,F,\Phi}) + R(\lambda, B)(\Phi R(\lambda, G_{B,F,\Phi}) - \Psi)]f + \\ &\quad + R(\lambda, G_{B,0})f \quad \text{for } f \in E, \end{aligned} \tag{4.7}$$

where the operator $G_{B,0}$ *is defined as in Definition 1.25 in Chapter 1.*

(ii) *For* $L := \frac{K+K\|\Psi\|}{1-N\|\Psi\|}$ *and* $\lambda_0 := \omega + \frac{K\|\Phi\|}{1-N\|\Psi\|}$ *we have*

$$\|R(\lambda, G_{B,F,\Phi})\| \leq \frac{L}{(\lambda - \lambda_0)} \qquad \text{for } \lambda > \lambda_0.$$

(iii) *For* $\lambda > \omega_0 := \max\{2\lambda_0, \omega + L\|\Phi\|\}$ *and* $P := 3eN[(K+L)\|\Psi\| + 2K + 1]$ *we have that*

$$\|R(\lambda, G_{B,F,\Phi})^n\| \leq \frac{P}{(\lambda - \omega_0)^n} \qquad \text{for all } n \in \mathbb{N}.$$

(iv) *The operator* $G_{B,F,\Phi}$ *is densely defined.*

PROOF. By the remarks preceding the theorem, we can see that the proofs of (i), (ii) and (iv) are similar to the proofs of (i), (ii) and (iv) in Theorem 3.5. To prove (iii) we need some more involved techniques.

(iii) For $\lambda > \lambda_0$ and $u := R(\lambda, G_{B,F,\Phi})f$, by (4.7) we have

$$u(t) = e^{\lambda t}U(t,0)[\Psi u + R(\lambda, B)(\Phi u - \Psi f + f(0))] + \int_t^0 e^{\lambda(t-\xi)}U(t,\xi)f(\xi)d\xi \quad \text{for } t \leq 0.$$

We extend u, f to functions on $\mathbb{R}$ by

$$\tilde{u}(t) := \begin{cases} u(t) & \text{for } t \leq 0 \\ e^{(\lambda-\omega_1)t}g(t) & \text{for } t > 0, \end{cases}$$

$$\tilde{f}(t) := \begin{cases} f(t) & \text{for } t \leq 0 \\ -e^{(\lambda-\omega_1)t}g'(t) & \text{for } t > 0. \end{cases}$$

Here, ω_1 is as in Definition 1.20, and $g(t) := u(0) + \int_0^t \varphi(\tau)d\tau$ with

$$\varphi(t) = \begin{cases} 6t[t(\lambda-\omega_1)^2 - \lambda + \omega_1][(\lambda-\omega_1)u(0) - \frac{1}{2}f(0)] + \\ \qquad\qquad + [(\lambda-\omega_1)t - 1]f(0) & \text{for } 0 \leq t \leq \frac{1}{\lambda-\omega_1}, \\ 0 & \text{for } t \geq \frac{1}{\lambda-\omega_1}. \end{cases}$$

Then g is continuously differentiable with compact support contained in $[0, \frac{1}{\lambda-\omega_1}]$ having the properties that $g(0) = u(0)$, $g'(0) = -f(0)$ and

$$\|e^{(\lambda-\omega_1)t}g'(t)\| \leq 3e[(K+L)\|\Psi\| + 2K + 1]\|f\|$$

for $\lambda > \max\{2\lambda_0, \omega + L\|\Phi\|\}$ and all $t \in \mathbb{R}_+$.

Hence, $\tilde{u}, \tilde{f}$ belong to $\tilde{E}$ and satisfy the equation

$$\tilde{u}(t) = e^{\lambda t}\tilde{U}_1(t,0)\tilde{u}(0) + \int_t^0 e^{\lambda(t-\xi)}\tilde{U}_1(t,\xi)\tilde{f}(\xi)d\xi \quad \text{for } t \leq 0, \tag{4.8}$$

with

$$\tilde{U}_1(t,s) := \begin{cases} U(t,s) & \text{for } t \le s \le 0, \\ e^{\omega_1 s} U(t,0) & \text{for } t \le 0 \le s, \\ e^{\omega_1 (s-t)} U(0,0) = e^{\omega_1 (s-t)} Id & \text{for } 0 \le t \le s. \end{cases}$$

Moreover,

$$\|\tilde{f}\| \le 3e[(K+L)\|\Psi\| + 2K + 1]\|f\|. \tag{4.9}$$

We now look at the left translation evolution semigroup $(\tilde{T}_1(t))_{t\ge 0}$ on the Banach space $\tilde{E} := C_0(\mathbb{R}, X)$, defined by

$$(\tilde{T}_1(t)\tilde{f})(s) := \tilde{U}_1(s, s+t)\tilde{f}(s+t) = \begin{cases} U(s,s+t)\tilde{f}(s+t) & \text{for } s \le s+t \le 0, \\ e^{\omega_1(s+t)} U(s,0)\tilde{f}(s+t) & \text{for } s \le 0 \le s+t, \\ e^{\omega_1 t}\tilde{f}(s+t) & \text{for } 0 \le s \le s+t. \end{cases}$$

As in Chapter 1, Section 1, we can see that this semigroup is strongly continuous on $\tilde{E}$ and its generator, which we denote by $\tilde{G}_1$, satisfies the following property.

For $v, w \in \tilde{E}$ and $\lambda \in \mathbb{C}$, we have that $v \in D(\tilde{G}_1)$ and $\lambda v - \tilde{G}_1 v = w$ if and only if v, w satisfy the equation (4.8).

Therefore, we obtain $\tilde{u} = R(\lambda, \tilde{G}_1)\tilde{f}$. Here we note that $\lambda \in \rho(\tilde{G})$ for $\lambda > \lambda_0 > \omega_1$. Thus, for $t \le 0$ and $\lambda > \max\{2\lambda_0, \omega + L\|\Phi\|\} =: \omega_0$, we have

$$[R(\lambda, G_{B,F,\Phi})f](t) = u(t) = \tilde{u}(t) = [R(\lambda, \tilde{G})\tilde{f}](t).$$

By induction we obtain

$$[R(\lambda, G_{B,F,\Phi})^n f](t) = [R(\lambda, \tilde{G})^n \tilde{f}](t) \text{ for } t \le 0.$$

Using the fact that $\tilde{G}_1$ is the generator of the strongly continuous semigroup $(\tilde{T}_1(t))_{t\ge 0}$ satisfying $\|\tilde{T}_1(t)\| \le N e^{\omega_1 t}$, and by inequality (4.9), we have

$$\begin{aligned} \|[R(\lambda, G_{B,F,\Phi})^n f](t)\| &= \|[R(\lambda, \tilde{G})^n \tilde{f}](t)\| \\ &\le \frac{N}{(\lambda - \omega_1)^n}\|\tilde{f}\| \le \frac{3eN[(K+L)\|\Psi\| + 2K + 1]}{(\lambda - \omega_0)^n}\|f\| \end{aligned}$$

for all $t \le 0$, $\lambda > \omega_0$, and all $n \in \mathbb{N}$. Therefore, putting $P := 3eN[(K+L)\|\Psi\| + 2K + 1]$ we obtain

$$\|R(\lambda, G_{B,F,\Phi})^n f\| \le \frac{P}{(\lambda - \omega_0)^n}\|f\| \text{ for } \lambda > \omega_0 \text{ and } n \in \mathbb{N}.$$

□

The Hille-Yosida theorem yields the following corollaries.

Corollary 4.3. *If the operator Ψ satisfies $\|\Psi\| < \frac{1}{N}$ (with the constant N as in Definition 1.20), then the operator $G_{B,F,\Phi}$ generates a strongly continuous semigroup $(T_{B,F,\Phi}(t))_{t\ge 0}$ on E satisfying*

$$\|T_{B,F,\Phi}(t)\| \le P e^{\omega_0 t},$$

where the constants P and ω_0 are defined in Theorem 4.2.

2. Well-posedness and robustness of stability

In this section we study the well-posedness of the equations (4.2) and (4.3) in a mild sense and prove the robustness of exponential stability of the solutions under small perturbations of the delay operator. We start with the definition of the mild solutions to (4.2) and (4.3).

DEFINITION 4.4. The function $u(t,s):\mathbb{R}_+\times\mathbb{R}_-\to X$ is called a *mild solution* of the equations (4.2) and (4.3) if $u(t,s)$ is continuously differentiable with respect to t, and $u(t,\cdot)\in D(G)$ satisfies:

$$\begin{cases}\frac{\partial}{\partial t}F(u(t,\cdot)) &= BFu(t,\cdot)+\Phi u(t,\cdot), \quad t\geq 0,\\ \frac{\partial}{\partial t}u(t,\cdot) &= Gu(t,\cdot), \quad t\geq 0.\end{cases} \tag{4.10}$$

REMARK 4.5. The operator G is a closure of the operator $\frac{d}{ds}+A(\cdot)$. We refer to [**74**, Theorem 1.14] or [**60**, Chap. 3] for detailed information on the relation between the operators G and $\frac{d}{ds}+A(\cdot)$. Roughly speaking, there exists a core H for G (i.e., a linear subspace of $D(G)$ which is dense in $D(G)$ with respect to the graph norm) such that $G|_H=(\frac{d}{ds}+A(\cdot))|_H$. If we impose some appropriate conditions on the operators $A(s)$ and the initial data $u(0,\cdot)$, then we may have that $u(t,\cdot)\in H$ and hence obtain the classical solutions to (4.2) and (4.3) (see [**27**] for details). Therefore, we can consider the solutions of (4.10) as mild solutions to (4.2) and (4.3).

We now come to our first about the well-posedness in a mild sense of the equations (4.2) and (4.3).

COROLLARY 4.6. *Assume that the difference operator F is of the form (4.6) with Ψ satisfying $\|\Psi\|<\frac{1}{N}$ for the constant N appearing in the Definition 1.20. Then the equations (4.2) and (4.3) are well-posed in a mild sense. More precisely, for every $\varphi\in D(G_{B,F,\Phi})$ there exists a unique mild solution $u(t,\cdot,\varphi)$ of (4.2) and (4.3) given by*

$$u(t,\cdot,\varphi)=T_{B,F,\Phi}(t)\varphi,$$

and for every sequence $(\varphi_n)_{n\in\mathbb{N}}\subset D(G_{B,F,\Phi})$ satisfying $\lim_{n\to\infty}\varphi_n=0$, one has

$$\lim_{n\to\infty}u(t,\cdot,\varphi_n)=0$$

uniformly in compact intervals.

PROOF. By Corollary 4.3, the operator $(G_{B,F,\Phi},D(G_{B,F,\Phi}))$ defined by (4.5) is the generator of the strongly continuous semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$.

For $\varphi\in D(G_{B,F,\Phi})$ we put $u(t,\cdot):=T_{B,F,\Phi}(t)\varphi$. Then it is clear that $u(t,\cdot)\in D(G_{B,F,\Phi})\subset D(G)$ and that $u(t,\cdot)$ satisfies the second equation in (4.10). We now show that $u(t,\cdot)$ satisfies the first equation in (4.10). Indeed, we have

$$\begin{aligned}\frac{\partial}{\partial t}Fu(t,\cdot) &= \lim_{h\to 0}\frac{Fu(t+h,\cdot)-Fu(t,\cdot)}{h}=\lim_{h\to 0}\frac{FT(t+h)\varphi-FT(t)\varphi}{h}\\ &= F\lim_{h\to 0}\frac{T(h)T(t)\varphi-T(t)\varphi}{h}=FG_{B,F,\Phi}T(t)\varphi=BFT(t)\varphi+\Phi T(t)\varphi\\ &= BFu(t,\cdot)+\Phi u(t,\cdot).\end{aligned}$$

For the uniqueness of the solution, we prove that, if $v(t,\cdot)$ is a mild solution of the equations (4.2) and (4.3) satisfying $v(0,\cdot)=0$, then $v(t,\cdot)=0$ for all $t\geq 0$. In fact, since

v satisfies (4.10) and $v(t,\cdot) \in D(G)$, we have that

$$\begin{aligned} BFv(t,\cdot) + \Phi v(t,\cdot) &= \frac{\partial}{\partial t} Fv(t,\cdot) = \lim_{h\to 0} \frac{Fv(t+h,\cdot) - Fv(t,\cdot)}{h} \\ &= F \lim_{h\to 0} \frac{v(t+h,\cdot) - v_t}{h} = F\frac{\partial}{\partial t} v(t,\cdot) \\ &= FGv(t,\cdot). \end{aligned}$$

Therefore, $v(t,\cdot) \in D(G_{B,F,\Phi})$ satisfies the Cauchy problem

$$\begin{cases} \frac{d}{dt} v(t,\cdot) &= G_{B,F,\Phi} v(t,\cdot) \quad \text{for } t \geq 0, \\ v(0,\cdot) &= 0. \end{cases}$$

Since $G_{B,F,\Phi}$ is the generator of a strongly continuous semigroup, this Cauchy problem has a unique solution $v_t = 0$ (see [**23**, Theorem II.6.7]).

Finally, the last assertion, called the *continuous dependence on the initial data* of the solutions, follows from the uniform boundedness of the strongly continuous semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$ on compact intervals. □

Having proved the well-posedness in a mild sense of the equations (4.2) and (4.3), we now use the constants appearing in the Hille-Yosida estimates of the operator $G_{B,F,\Phi}$ to study the robustness of the exponential stability of the solution semigroup.

COROLLARY 4.7. *If the backward evolution family $\mathcal{U}$ and the semigroup $(e^{tB})_{t\geq 0}$ are exponentially stable, $\|\Psi\| < \frac{1}{N}$, and $\|\Phi\|$ is small enough, then the semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$, and hence the mild solution of the equations (4.2) and (4.3), is also exponentially stable.*

PROOF. The assumption that $\mathcal{U}$ and $(e^{tB})_{t\geq 0}$ are exponentially stable means that, in Theorem 4.2, $\omega = \max\{\omega_1, \omega_2\} < 0$. Moreover, we have

$$\omega_0 = \max\{2\lambda_0, \omega + L\|\Phi\|\} = \max\{2\omega + \frac{2K\|\Phi\|}{1 - N\|\Psi\|}, \omega + L\|\Phi\|\}.$$

Therefore, if $\|\Phi\| < -\frac{\omega}{L}$, then the semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$ is also exponentially stable. □

3. Examples

We now consider some concrete examples and compute the "smallness" of Φ explicitly.

EXAMPLE 4.8. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. The Dirichlet Laplacian generates an analytic semigroup $(e^{t\Delta})_{t\geq 0}$ on $X := L^2(\Omega)$. We then take operators $A(s)$ as

$$A(s) := a(s)\Delta,$$

where the function $a(\cdot) \in L^1_{loc}(\mathbb{R}_-)$ satisfies $a(\cdot) \geq \gamma > 0$ for some constant γ. These operators generate a backward evolution family $(U(r,s))_{r\leq s\leq 0}$ given by

$$U(r,s) = e^{(\int_r^s a(\tau)d\tau)\Delta} \qquad \text{for } r \leq s \leq 0.$$

We then have

$$\|U(r,s)\| = e^{(\int_r^s a(\tau)d\tau)\lambda_0} \leq e^{\gamma\lambda_0(s-t)} \qquad \text{for } r \leq s \leq 0,$$

where $0 > \lambda_0$ denotes the largest eigenvalue of Δ. Therefore, we can choose, in Definition 1.20, the constants $N = 1$ and $\omega_1 = \gamma\lambda_0 < 0$. We now define the delay and difference operators by

$$\Phi f := \int_{-\infty}^{0} \varphi(s)f(s)ds \text{ for } f \in E,$$

$$\Psi f := \int_{-\infty}^{0} \psi(s)f(s)ds \text{ for } f \in E,$$

where $\varphi(\cdot)$ and $\psi(\cdot)$ belong to $L^1(\mathbb{R}_-)$. We then have

$$\|\Phi\| = \|\varphi(\cdot)\|_{L^1},$$

$$\|\Psi\| = \|\psi(\cdot)\|_{L^1}.$$

Let now B generate a semigroup $(e^{tB})_{t\geq 0}$ satisfying $\|e^{tB}\| \leq Me^{\omega_2 t}$ with $\omega_2 < 0$. From the definition of $(T_{B,0}(t))_{t\geq 0}$ we obtain

$$\|T_{B,0}(t)\| \leq Me^{\max\{\gamma\lambda_0,\ \omega_2\}t}, \qquad t \geq 0.$$

Hence, by Corollary 4.3, if $\|\psi(\cdot)\|_{L^1} < 1$ then the equations (4.2) and (4.3) are well-posed.

Moreover, in Corollary 4.7 we can choose

$$K = M \text{ and } L = \frac{M + M\|\psi(\cdot)\|_{L^1}}{1 - \|\psi(\cdot)\|_{L^1}}.$$

Therefore, if

$$\|\varphi(\cdot)\|_{L^1} < -\frac{\max\{\gamma\lambda_0, \omega_2\}}{L},$$

then the semigroup $(T_{B,F,\Phi}(t))_{t\geq 0}$ and hence the solutions of (4.2) and (4.3) are exponentially stable.

EXAMPLE 4.9. Let now the operators $(-A(s))_{s\in\mathbb{R}_-}$ satisfy the following *hyperbolic condition* (see, e.g., Pazy [**62**, Chap. V.3], Nickel [**60**, Chap. 4]).

(H1) The family $(-A(s))_{s\in\mathbb{R}_-}$ is *stable*, i.e., all operators $-A(s)$ are generators of C_0-semigroups and there exist constants $M \geq 1$ and $\omega_1 \in \mathbb{R}$ such that

$$(\omega_1, \infty) \subset \rho(-A(s)) \quad \text{for all } s \in \mathbb{R}_-$$

and

$$\left\| \prod_{j=1}^{k} R(\lambda, -A(s_j)) \right\| \leq M(\lambda - \omega_1)^{-k} \quad \text{for all } \lambda > \omega_1$$

and every finite sequence $-\infty < s_1 \leq s_2 \leq \cdots \leq s_k \leq 0$.

(H2) There exists a densely embedded subspace $Y \hookrightarrow X$, which is a core for every $-A(s)$ such that the family of the parts $(-A(s)\mid_Y)_{t\in\mathbb{R}_-}$ in Y is a stable family on the space Y.

(H3) The mapping $\mathbb{R}_- \ni s \mapsto A(s) \in \mathcal{L}(Y, X)$ is uniformly continuous.

(H4) There exists a family $(Q(s))_{s\in\mathbb{R}_-}$ of isomorphisms of Y onto X such that for every $v \in Y$, $Q(t)v$ is continuously differentiable in X on $(-\infty, 0]$ and

$$Q(s)A(s)Q(s)^{-1} = A(s) + B(s)$$

for some strongly continuous family $(B(s))_{s\in\mathbb{R}_-}$ of bounded operators on X.

The delay and difference operators are defined by

$$\Phi f := \alpha f(0) \text{ for } f \in E \text{ and some } \alpha > 0,$$

$$\Psi f := \beta f(0) \text{ for } f \in E \text{ and some } \beta > 0.$$

By Pazy [**62**, Theorem V.4.8] or Nickel [**60**, Corollary 4.16] we obtain that there exist a unique backward evolution family $(U(t,s))_{t\le s\le 0}$ solving the Cauchy problem (4.4) satisfying

$$\|U(t,s)\| \le M e^{\omega_1(s-t)} \quad \text{for } t \le s \le 0.$$

Let now the operator B generate an analytic semigroup $(e^{tB})_{t\ge 0}$ with spectral bound $s(B)$. Then, by Corollary 4.3 we have that, if $\beta < 1$ then the equations (4.2) and (4.3) are well-posed.

Moreover, by Corollary 4.7 and the Spectral Mapping Theorem for analytic semigroups we obtain that, if $\omega_1 < 0$, $s(B) < 0$ and if

$$\alpha < -\frac{(1-\beta)\max\{\omega_1, \gamma\}}{M + M\beta} \quad \text{for some } \gamma \text{ satisfying } s(B) < \gamma < 0,$$

then semigroup $(T_{B,F,\Phi}(t))_{t\ge 0}$ and hence the solutions of (4.2) and (4.3) are exponentially stable.

CHAPTER 5

Non-autonomous Neutral Partial Differential Equations

In this chapter we study the semi-linear NPDE's of the form

$$\begin{cases} \frac{\partial}{\partial t} F u_t & = B(t) F u_t + \Phi(t, u_t) \text{ for } a \leq t \leq b \\ u_a & = \phi \in C := C([-r, 0], X). \end{cases} \tag{5.1}$$

As in Section 2 of Chapter 2, we assume that the linear operators $B(t)$ generate the evolution family $(U(t,s))_{t \geq s \geq 0}$. This fact allows us to solve the equation (5.1) in a mild manner if the difference operator F has the form (4.6) with $\|\Psi\|$ being small. More precisely, we will prove that, if F satisfies the above condition, then the mild solution of NPDE (5.1) exists uniquely on the interval $[a-r, b]$ of the real line provided that the delay operator $\Phi(t, \phi)$ is Lipschitz continuous with respect to $\phi \in C$ uniformly for all $t \in [a, b]$. Analogous results for autonomous NPDE have been obtained by Hale [**34**] and Wu [**81**, Chap. 2.3] in which they used very complicated topological methods employing Kuratowski measures of non-compactness. One of the main difficulties they encounter is that one can not use the fixed point argument directly. In our method we use the techniques related to the Neumann's series of $(I - \Psi)^{-1}$ to tranform the problem into such a situation that we can apply the fixed point argument. Therefore, we need the fact that Ψ is "small" (precisely, $\|\Psi\| < 1$). However, if Ψ is not "small" but has no mass in 0, then we may use the renorming procedure to convert the case "Ψ having no mass in 0" to the case "Ψ being small" as in Section 2 of Chapter 3.

Moreover, by our estimates, we also obtain the robustness of exponential stability of the system under small perturbations of the delay operator Φ.

1. Well-posedness and stability

We begin with the result about existence, uniqueness and continuous dependence on initial data of the mild solutions to the equation (5.1).

THEOREM 5.1. *Let the difference operator F have the form (4.6) with $\|\Psi\| < 1$. Let the delay operator $\Phi : [a, b] \times C \to X$ be continuous and satisfy a Lipschitz condition*

$$\|\Phi(t, \phi) - \Phi(t, \psi)\| \leq L \|\phi - \psi\| \text{ for all } t \in [a, b],\ \phi, \psi \in C := C([-r, 0], X), \tag{5.2}$$

where L is a positive constant. Then, for given $\phi \in C$ there exists a unique continuous function $u : [a-r, b] \to X$ which solves the following initial value problem

$$\begin{cases} F u_t & = U(t, a) F \phi + \int_a^t U(t, s) \Phi(s, u_s) ds, \quad a \leq t \leq b, \\ u_a & = \phi. \end{cases} \tag{5.3}$$

Moreover, the solutions depend continuously on the initial conditions.

PROOF. Define the operator $\tilde{\Psi} : C([a-r,b],X) \to C([a-r,b],X)$ by

$$[\tilde{\Psi}u](t) = \begin{cases} \Psi(u_t) & \text{for } a \le t \le b, \\ \Psi(u_a) & \text{for } a-r \le t \le a. \end{cases}$$

Since $\|\Psi\| < 1$ we have $\|\tilde{\Psi}\| \le \|\Psi\| < 1$. Therefore, the operator $I - \tilde{\Psi}$ is invertible. For a given $\phi \in C$, we define a mapping

$$\mathcal{H}_\phi : C([a-r,b],X) \to C([a-r,b],X)$$

by

$$(\mathcal{H}_\phi u)(t) = \begin{cases} U(t,a)F\phi + \int_a^t U(t,s)\Phi(s,u_s)ds) & \text{for } a \le t \le b, \\ \phi(t-a) - \Psi(\phi) & \text{for } a-r \le t \le a. \end{cases}$$

Hence, denoting by $\|\cdot\|_\infty$ the norm in $C([a-r,b],X)$, we obtain

$$\|(\mathcal{H}_\phi u)(t) - (\mathcal{H}_\phi v)(t)\| \le PL(t-a)\|u-v\|_\infty \text{ for } a \le t \le b \tag{5.4}$$

and $(\mathcal{H}_\phi u)(t) - (\mathcal{H}_\phi v)(t) = 0$ for $a - r \le t \le a$, where $P := \sup_{a \le s \le t \le b} \|U(t,s)\|$. By induction we obtain

$$\|(\mathcal{H}_\phi^n u)(t) - (\mathcal{H}_\phi^n v)(t)\| \le \frac{(PL(t-a))^n}{n!}\|u-v\|_\infty \text{ for } a \le t \le b \tag{5.5}$$

and $(\mathcal{H}_\phi^n u)(t) - (\mathcal{H}_\phi^n v)(t) = 0$ for $a-r \le t \le a$.

We now put $\mathcal{F}_\phi := (I - \tilde{\Psi})^{-1}\mathcal{H}_\phi$. Using the Neumann's series, we then have

$$\begin{aligned} (\mathcal{F}_\phi u)(t) - (\mathcal{F}_\phi v)(t) &= \left[\left(\sum_{n=0}^\infty \tilde{\Psi}^n\right)\mathcal{H}_\phi u\right](t) - \left[\left(\sum_{n=0}^\infty \tilde{\Psi}^n\right)\mathcal{H}_\phi v\right](t) \\ &= [(\mathcal{H}_\phi u)(t) - (\mathcal{H}_\phi v)(t)] + [(\tilde{\Psi}\mathcal{H}_\phi u)(t) - (\tilde{\Psi}\mathcal{H}_\phi v)(t)] + \\ &\quad + [(\tilde{\Psi}^2\mathcal{H}_\phi u)(t) - (\tilde{\Psi}^2\mathcal{H}_\phi v)(t)] + \cdots. \end{aligned} \tag{5.6}$$

Next, we prove by induction that, for all $n \in \mathbb{N}$ we have

$$\|(\tilde{\Psi}^n\mathcal{H}_\phi u)(t) - (\tilde{\Psi}^n\mathcal{H}_\phi v)(t)\| \le \|\Psi\|^n PL(t-a)\|u-v\|_\infty \text{ for } a \le t \le b \tag{5.7}$$

and $(\tilde{\Psi}^n\mathcal{H}_\phi u)(t) - (\tilde{\Psi}^n\mathcal{H}_\phi v)(t) = 0$ for $a - r \le t \le a$.

Indeed, for $n = 0$, the claim follows from (5.4). If the claim holds for $n-1$ we prove it for n. In fact, by definition of $\tilde{\Psi}$, we have

$$(\tilde{\Psi}^n\mathcal{H}_\phi u)(t) - (\tilde{\Psi}^n\mathcal{H}_\phi v)(t) = \begin{cases} \Psi[(\tilde{\Psi}^{n-1}\mathcal{H}_\phi u)_t] - \Psi[(\tilde{\Psi}^{n-1}\mathcal{H}_\phi v)_t] & \text{for } a \le t \le b, \\ \Psi[(\tilde{\Psi}^{n-1}\mathcal{H}_\phi u)_a] - \Psi[(\tilde{\Psi}^{n-1}\mathcal{H}_\phi v)_a] & \text{for } a-r \le t \le a. \end{cases} \tag{5.8}$$

By the inductive hypothesis we have that

$$\begin{aligned} \|(\tilde{\Psi}^{n-1}\mathcal{H}_\phi u)_t(\theta) - (\tilde{\Psi}^{n-1}\mathcal{H}_\phi v)_t(\theta)\| &= \|(\tilde{\Psi}^{n-1}\mathcal{H}_\phi u)(t+\theta) - (\tilde{\Psi}^{n-1}\mathcal{H}_\phi v)(t+\theta)\| \\ &\le \|\Psi\|^{n-1}PL(t+\theta-a)\|u-v\|_\infty \\ &\le \|\Psi\|^{n-1}PL(t-a)\|u-v\|_\infty \text{ for } a \le t+\theta \le b \end{aligned}$$

and $(\tilde{\Psi}^{n-1}\mathcal{H}_\phi u)_t(\theta) - [(\tilde{\Psi}^{n-1}\mathcal{H}_\phi v)_t(\theta) = 0$ for $a - r \le t + \theta \le a$. Thus, by (5.8), we have

$$\|(\tilde{\Psi}^n\mathcal{H}_\phi u)(t) - (\tilde{\Psi}^n\mathcal{H}_\phi v)(t)\| \le \|\Psi\|^n PL(t-a)\|u-v\|_\infty \text{ for } a \le t \le b$$

and $(\tilde{\Psi}^n\mathcal{H}_\phi u)(t) - (\tilde{\Psi}^n\mathcal{H}_\phi v)(t) = 0$ for $a - r \le t \le a$. Hence, the claim follows.

From the above claim it follows that

$$\begin{aligned}\|(\mathcal{F}_\phi u)(t)-(\mathcal{F}_\phi v)(t)\| &\le \left(\sum_{n=0}^{\infty}\|\Psi\|^n\right)PL(t-a)\|u-v\|_\infty\\ &= \frac{PL(t-a)}{1-\|\Psi\|}\|u-v\|_\infty \text{ for } a\le t\le b,\end{aligned}$$

and $(\mathcal{F}_\phi u)(t)-(\mathcal{F}_\phi v)(t)=0$ for $a-r\le t\le a$. Proceeding by induction, we obtain

$$\|(\mathcal{F}_\phi^n u)(t)-(\mathcal{F}_\phi^n v)(t)\|\le \frac{(PL(t-a))^n}{(1-\|\Psi\|)^n n!}\|u-v\|_\infty \text{ for } a\le t\le b$$

and $(\mathcal{F}_\phi^n u)(t)-(\mathcal{F}_\phi^n v)(t)=0$ for $a-r\le t\le a$. Hence,

$$\|\mathcal{F}_\phi^n u-\mathcal{F}_\phi^n v\|_\infty\le \frac{(PL(b-a))^n}{(1-\|\Psi\|)^n n!}\|u-v\|_\infty.$$

For n large enough we have that $\frac{(PL(b-a))^n}{(1-\|\Psi\|)^n n!}<1$. Therefore, by the contraction principle, $\mathcal{F}_\phi$ has a unique fixed point u in $C([a-r,b],X)$. By the definition of $\mathcal{F}_\phi$, we have that u is a solution of the problem (5.3).

The uniqueness of u and the continuous dependence of u on the initial data can be proved as follows. Let v be the solution of the equation (5.1) on $[a-r,b]$ with the initial value ψ. Then,

$$u-v=\mathcal{F}_\phi u-\mathcal{F}_\psi v=((I-\tilde{\Psi})^{-1}\mathcal{H}_\phi)u-((I-\tilde{\Psi})^{-1}\mathcal{H}_\psi)v.$$

Using Neumann's series we arrive at

$$\begin{aligned}u(t)-v(t) &= [(\mathcal{H}_\phi u)(t)-(\mathcal{H}_\psi v)(t)]+[(\tilde{\Psi}\mathcal{H}_\phi u)(t)-(\tilde{\Psi}\mathcal{H}_\psi v)(t)]+\\ &\quad +[(\tilde{\Psi}^2\mathcal{H}_\phi u)(t)-(\tilde{\Psi}^2\mathcal{H}_\psi v)(t)]+\cdots \text{ for all } t\in[a-r,b]. \qquad (5.9)\end{aligned}$$

By definition of $\mathcal{H}_\phi$, the norm of the fisrt term in (5.9) can be estimated by

$$\begin{aligned}\|(\mathcal{H}_\phi u)(t)-(\mathcal{H}_\psi v)(t)\| &\le K(1+\|\Psi\|)e^{\alpha(t-a)}\|\phi-\psi\|+\\ &\quad +\int_a^t KLe^{\alpha(t-s)}\|u_s-v_s\|ds \quad \text{for } a\le t\le b, \text{ and}\\ \|(\mathcal{H}_\phi u)(t)-(\mathcal{H}_\psi v)(t)\| &\le K(1+\|\Psi\|)\|\phi-\psi\| \quad \text{for } a-r\le t\le a, \qquad (5.10)\end{aligned}$$

with the constants K and α appearing in Definition 1.1.

Again, by induction, the norm of the nth term in (5.9) can be estimated by

$$\begin{aligned}\|(\tilde{\Psi}^n\mathcal{H}_\phi u)(t)-(\tilde{\Psi}^n\mathcal{H}_\psi v)(t)\| &\le \|\Psi\|^n[K(1+\|\Psi\|)e^{\alpha(t-a)}\|\phi-\psi\|+\\ &\quad +\int_a^t KLe^{\alpha(t-s)}\|u_s-v_s\|ds] \text{ for } a\le t\le b, \text{ and}\\ \|(\tilde{\Psi}^n\mathcal{H}_\phi u)(t)-(\tilde{\Psi}^n\mathcal{H}_\psi v)(t)\| &\le \|\Psi\|^n K(1+\|\Psi\|)\|\phi-\psi\| \text{ for } a-r\le t\le a, \quad (5.11)\end{aligned}$$

Therefore,

$$\begin{aligned}\|u(t)-v(t)\| &\le \frac{1}{1-\|\Psi\|}[K(1+\|\Psi\|)e^{\omega_2(t-a)}\|\phi-\psi\|+\\ &\quad +\int_a^t KLe^{\omega_2(t-s)}\|u_s-v_s\|ds] \text{ for } a\le t\le b, \text{ and}\\ \|u(t)-v(t)\| &\le \frac{1}{1-\|\Psi\|}K(1+\|\Psi\|)\|\phi-\psi\| \text{ for } a-r\le t\le a. \qquad (5.12)\end{aligned}$$

Hence, if $\alpha \geq 0$, then, for all $a \leq t \leq b$,

$$\|u_t - v_t\| \leq \frac{1}{1 - \|\Psi\|}[M(1 + \|\Psi\|)e^{\alpha(t-a)}\|\phi - \psi\| + \int_a^t KLe^{\alpha(t-s)}\|u_s - v_s\|ds], \quad (5.13)$$

and if $\alpha < 0$, then, for all $a \leq t \leq b$,

$$\|u_t - v_t\| \leq \frac{1}{1 - \|\Psi\|}[K(1+\|\Psi\|)e^{-\alpha r}e^{\alpha(t-a)}\|\phi - \psi\| + \int_a^t KLe^{-\alpha r}e^{\alpha(t-s)}\|u_s - v_s\|ds]. \quad (5.14)$$

Now, the inequalities (5.13), (5.14) and the Gronwall's inequality imply that

$$\|u_t - v_t\| \leq \begin{cases} \frac{K(1+\|\Psi\|)}{1-\|\Psi\|}e^{(\alpha + \frac{KL}{1-\|\Psi\|})(t-a)}\|\phi - \psi\| & \text{if } \alpha \geq 0 \\ \frac{K(1+\|\Psi\|)}{1-\|\Psi\|}e^{-\alpha r}e^{(\alpha + \frac{KLe^{-\alpha r}}{1-\|\Psi\|})(t-a)}\|\phi - \psi\| & \text{if } \alpha < 0. \end{cases} \quad (5.15)$$

Therefore, the uniqueness of u and the continuous dependence of u on the initial data follow. □

Corollary 5.2. *Assume that the delay operator $\Phi : [0, \infty) \times C \to X$ is continuous and satisfies the inequality (5.2) uniformly for all $t \geq 0$ and $\phi, \psi \in C$. Let, in addition, $\Phi(t, 0) = 0$ for $t \geq 0$. Then, if the operators $B(t)$ generates an exponentially stable evolution family and the Lipschitz constant L is small enough, the solution u with initial condition $u_0 = \phi$ is also exponentially stable, i.e., there exist positive constants N and ω such that $\|u_t\| \leq Ne^{-t\omega}\|\phi\|$ for all $t \geq 0$.*

Proof. By Theorem 5.1 and the assumption of the corollary, we have that, for each initial value ϕ, the equation (5.1) has a unique solution $u(\cdot)$ defined on $[-r, \infty)$. The condition that $\Phi(t, 0) = 0$ for $t \geq 0$ implies that $v(\cdot) = 0$ is a solution of the equation (5.1) with the initial condition $\psi = 0$. Therefore, by inequality (5.15), we obtain

$$\|u_t\| \leq \begin{cases} \frac{K(1+\|\Psi\|)}{1-\|\Psi\|}e^{(\alpha + \frac{KL}{1-\|\Psi\|})(t-a)}\|\phi\| & \text{if } \alpha \geq 0, \\ \frac{K(1+\|\Psi\|)}{1-\|\Psi\|}e^{-\alpha r}e^{(\alpha + \frac{KLe^{-\alpha r}}{1-\|\Psi\|})(t-a)}\|\phi\| & \text{if } \alpha < 0, \end{cases} \quad (5.16)$$

with the constant K and α appearing in Definition 1.1 and Lipschitz constant L as in (5.2). If now the operators $B(t)$ generates an exponentially stable evolution family (i.e., $\alpha < 0$) and if the Lipschitz constant L is small enough (i.e., $L < -\frac{\alpha(1-\|\Psi\|)}{Ke^{-\alpha r}}$), then the solution $u(\cdot)$ is exponential stable. □

2. An example

We demonstrate our result by the following example.

Example 5.3. We consider the problem

$$\begin{cases} \frac{\partial}{\partial t}Fu_t(\cdot) = B(t)Fu_t(\cdot) + \Phi(t)u_t(\cdot) \text{ for } t \geq s \geq 0, \\ u_s(\cdot) = \varphi(\cdot) \in C([-r, 0], X) = C. \end{cases} \quad (5.17)$$

Here, the coefficients $B(t)$ satisfy the following *parabolic conditions* (see, e.g., Pazy [**62**, Chap. V], Schnaubelt [**23**, Chap. VI.9], Nickel [**60**, Assumption 4.1]).

(P1) The domain $D := D(B(t))$, $t \geq 0$, is dense in X and independent of t.

(P2) For $t \geq 0$, the resolvent $R(\lambda, B(t))$ exists for all $\text{Re}\lambda \geq 0$ and there exists a constant M such that

$$\|R(\lambda, B(t))\| \leq \frac{M}{|\lambda| + 1} \quad \text{for } \text{Re}\lambda \geq 0,\ t \geq 0.$$

(P3) There exist constants L and $0 < \alpha \leq 1$ such that

$$\|(B(t) - B(s))B(\tau)^{-1}\| \leq L|t - s|^{\alpha} \quad \text{for } s, t, \tau \geq 0.$$

(Concrete examples of such operators $B(t)$ can be found in [**74**, Examples 2.3, 3.10] and references therein).
The delay operator $\Phi(\cdot) : \mathbb{R}_+ \to \mathcal{L}(C([-r, 0], X), X)$ satisfies $L := \sup_{t\geq 0} \|\Phi(t)\| < \infty$. Finally, the difference operator $F \in \mathcal{L}(C([-r, 0], X), X)$ is defined by

$$Ff := f(0) - \Psi f := f(0) - \int_{-r}^{0} \psi(s) f(s) ds \quad \text{for } f \in C([-r, 0], X),$$

where $\psi(\cdot)$ belongs to $L^1([-r, 0])$ and satisfies $\|\psi(\cdot)\|_{L^1} < 1$.

By Pazy [**62**, Theorem V.6.1] (see also [**60**, Corollary 4.12]), we have that the the operators $B(t)$ generate an evolution family $(U(t, s))_{t\geq s\geq 0}$. Therefore, by Theorem 5.1, the equation (5.17) has a unique mild solution $u(\cdot)$ for each initial condition $\varphi \in C$, and the solutions depend continuously on the initial data.

If now, in addition, the operators $B(t)$ also satisfy the following condition

(P4) the operators $B(t)B(s)^{-1}$ are uniformly bounded for $t, s \geq 0$ and there exists a closed operator $B(\infty)$ with domain D such that

$$\lim_{t\to\infty} \|(B(t) - B(\infty))B(0)^{-1}\| = 0,$$

then, by Pazy [**62**, Theorem V.8.1] or Schnaubelt [**23**, Corollary VI.9.26], the evolution family $(U(t, s)))_{t\geq s\geq 0}$ generated by $B(t)$ is exponentially stable and satisfies

$$\|U(t, s)\| \leq N e^{-\nu(t-s)}, \quad t \geq s \geq 0,$$

for positive constants ν and N. Hence, by Corollary 5.2, we obtain that, the mild solution of the equation (5.17) is also exponentially stable if

$$L = \sup_{t\geq 0} \|\Phi(t)\| < \frac{\nu(1 - \|\psi(\cdot)\|_{L_1})}{N e^{\nu r}}.$$

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