Quantum Stochastic Calculus using Infinitesimals

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To

Claudia, Paul

and Yolanda

Preface

> 94. Aber mein Weltbild habe ich nicht, weil ich mich von seiner Richtigkeit überzeugt habe; auch nicht, weil ich von seiner Richtigkeit überzeugt bin. Sondern es ist der überkommene Hintergrund, auf welchem ich zwischen wahr und falsch unterscheide.
>
> L. WITTGENSTEIN, *Über Gewißheit*

The first time I began thinking about a doctoral thesis was while finishing my diploma thesis. That was about a special question in the so-called modular theory of von-Neumann algebras. But as I wanted also to go into the field of nonstandard analysis I looked for a combination of both fields. I searched at the university Internet sites for someone who could supervise me within these fields: nonstandard analysis and operator algebras. Finally I found my supervisor Manfred Wolff. As an impetus he gave me some ideas how to construct continuous tensor products of operator algebras using nonstandard analytical methods, especially continuous tensor products of the ring of 2×2-matrices with itself. In the stages of research I drifted more and more away from continuous tensor products and ended up with "A Nonstandard Approach to Quantum Stochastics". (Initially I considered that as subtitle.) However, in my opinion this is in some sense very close to the original ideas of Manfred Wolff. Indeed, I believe that a quantum stochastic process is nothing else than one element of the continuous tensor product of the ring of 2×2-matrices. Unfortunately there isn't time and place to discuss this here but probably we will make up for that in some article.

On technical grounds a doctoral thesis is written to take one's doctor's degree. The author should show that he is able to do scientific research and that he's a full member of the scientific community. Thus the thesis has one author. But in fact there are many fathers of this thesis. In first place I thank my 'Doktorvater' Manfred Wolff. He always encouraged me and was always interested in my work. Without his ideas, comments and remarks this thesis wouldn't have come into existence. He gave me stimulus and the ambience to discern true and false while doing my work.
I am deeply grateful to Horst Osswald for the opportunity of visiting Munich twice to discuss with him my nonstandard ideas of quantum stochastic calculus. Conversely, he provided me with useful facts on his nonstandard approach to abstract Wiener spaces, much related to my own work. I am also grateful to Eduard Y. Emel'yanov. He corrected several of my erroneous beliefs on external and internal objects. Nigel J. Cutland invited me for one month to the University of Hull. It was a real pleasure for me to talk with him.

This cleared up many of my nonstandard analytical problems and even much of the general structure of my thesis. During the stay in England I visited twice Martin J. Lindsay at University of Nottingham. We had two days full of discussion and I owe him much for bringing me on the path regarding quantum stochastic calculus. Also Hans Maassen had undoubtly a strong influence on my work. We met once at University of Aarhus where I presented him my ideas in the very early stages of my thesis. He encouraged me to work out these ideas. At a second meeting at University of Tübingen he proposed to me some fruitful applications. (But I have to confess that I didn't follow his proposals.)
Many thanks go also to the participants of our weekly seminar Vielteilchenphysik (Many-Particle-Physics), especially to Burkhard Kümmerer for his interest in my work and to Jürgen Hellmich with whom I had several interesting discussions on Bochner integration and other vector integrals. There are many people which supported me with preprinted material, ideas, hints and moral strengthening, among them: R.L. Hudson, S.J. Wills, R.F. Streater, T.V. Panchapagesan, D.A. Ross, T. Lindstrøm, W. Lyantse, T. Kudryk, Siu-Ah Ng, Josef Berger and my colleague Jürgen Schweizer. I am also grateful to Michael Brunet for correcting my English.
Finally, I am most indebted to my family, to my wife for giving me so much time to write this thesis in the course of many nights, to my son for many hours of playing with him, and to both and to my daughter for showing me every day that there are more important things in life than mathematics.

Tübingen, November 2001

Martin Leitz-Martini

German Summary

Diese Doktorarbeit hat ihre Wurzeln in einer Randbemerkung die Meyer mehrfach in seinen Arbeiten zum *Baby-Fockraum* gemacht hat (siehe [Mey93a, S. 83] und [Mey86, Mey87]). Meyer führt eine endliche Zeitachse $T = \{1, \cdots, H\}$ ein. Danach nimmt er die Potenzmenge von T und betrachtet Funktionen F auf dieser Potenzmenge. Er definiert für jeden Zeitpunkt $s \in T$ *Baby-Erzeugungs-* und *Baby-Vernichtungsoperatoren.* Die Summe über diese Operatoren ergibt dann den entsprechenden Prozeß. Oder anders gesagt, die für jeden Zeitpunkt s definierten Operatoren sind die Zuwächse der Prozesse. Ferner führt Meyer einen *Baby-Anzahloperator* ein. Meyers Randbemerkung in allen drei Arbeiten ist nun, daß man mit Methoden der Nichtstandard-Analysis den Baby-Fockraum und die Baby-Operatoren zu einem Modell des bekannten Bosonen-Fockraums über $L^2([0, 1])$ mit Erzeugungs-, Anzahl- und Vernichtungsoperatorprozeß machen können sollte. Mit dieser Doktorarbeit wird diese Randbemerkung zu einer gültigen mathematischen Aussage. Wir nehmen den Baby-Fockraum und interpretieren ihn als hyperendlich-dimensionalen internen Fockraum in einem Nichtstandard-Modell. Dazu müssen wir allerdings erstens das richtige Maß auf der internen Potenzmenge Γ der hyperendlichen Zeitachse T einführen und zweitens die Definition des Baby-Vernichtungsoperators zur Zeit $s \in T$ korrigieren und mit dem Zeitzuwachs $\frac{1}{H}$ skalieren.

In der englischen Einleitung führen wir das Thema dieser Doktorarbeit wie gerade eben auf die heuristischen Ideen von Meyer zurück. Wir gehen kurz auf die verwendeten Mittel der Nichtstandard-Analysis ein, um danach diese Arbeit innerhalb des quantenstochastischen Kalküls einzuordnen, insbesondere unsere nicht-linearen quantenstochastischen Differentialgleichungen von der nicht-linearen Quanten-Wechselwirkung abzuheben [ALV98, AV97, CJO98]. Ferner grenzen wir diese Arbeit gegen zwei nichtstandard-analytisch quantentheoretische Artikel ab, Gudder [Gud94], der „Nichtstandard-Fockräume“ einführt, und Yamashita [Yam98], der „Hyperendlich-dimensionale Darstellungen der kanonischen Kommutatorrelation“ betrachtet. Ferner betrachten wir die Arbeit im Lichte der zentralen Grenzwerte von endlichen Spin-Systemen (Quanten-Bernoulli-Zufallsläufen [Bia90, Bia91]) zu Bosonen-Fockraum-Erzeugungs- und Vernichtungsoperatoren [Mab95, GL98] und des Grenzwertes eines Baby-Fockraums mit abzählbar unendlicher Zeitachse [Att00]. Wir erwähnen zwei Artikel [Mey91, Par98], die beide den Baby-Fockraum als eigenständiges mathematisches Objekt behandeln. Danach kommt in der englischen Einleitung, wie auch jetzt, ein Überblick über den Inhalt dieser Doktorarbeit.

In Kapitel 1 geben wir ohne Beweise einen kurzen Überblick über den *quantenstochastischen Kalkül.* Wir orientieren uns zum Teil an Lindsays Artikel [Lin90]. Das Gewicht legen wir dabei etwas auf den *Kern-Zugang* zum quantenstochastischen Kalkül. Für eine vollständige Darstellung verweisen wir auf die Literatur [Par92, Mey93a, Bia95, Hud] und [Maa85, LM88, Lin98] für den Kern-Zugang.

In Kapitel 2 entwickeln wir einen quantenstochastischen Kalkül für die hyperendliche Zeitachse T: den *internen quantenstochastischen Kalkül.*
In Abschnitt 1 konstruieren wir den *symmetrischen Maßraum* $(\Gamma, \mathfrak{A}, m)$ über der hyperendlichen Zeitachse T und definieren den *internen Fockraum* $\mathcal{F}_{\text{int}}$ als den Raum aller internen Funktionen auf Γ. Da Γ hyperendlich ist ist $\mathcal{F}_{\text{int}}$ hyperendlich-dimensional. Ferner führen wir die üblichen wahrscheinlichkeitstheoretischen Begriffe ein, wie z.B. Unabhängigkeit, Erwartungswert, Adaptiertheit, und weitere.
In Abschnitt 2 definieren wir für jeden Zeitpunkt $s \in T$ die *fundamentalen Operatoren* $a_s^\bullet, a_s^+, a_s^\circ, a_s^-$, den ‚infinitesimalen' Zeit-, Erzeugungs-, Anzahl- und Vernichtungsoperator. Wir führen die entsprechenden Prozesse $A_s^\sharp$ als die Summe der infinitesimalen Operatoren ein, so daß diese gerade der Zuwachs des Prozesses werden. Für die $a_s^\sharp$s beweisen wir eine *interne Itô Tabelle*, die bis auf vier zusätzliche infinitesimale Einträge mit der bekannten Itô Tabelle übereinstimmt. Weiterhin betrachten wir auf dem internen Fockraum die Analoga zu Brownscher Bewegung und Poisson Prozeß.
In Abschnitt 3 definieren wir interne *2-Argument-Kernoperatoren* und beweisen einige ihrer Eigenschaften. Die Entsprechung zu den 2-Argument-Kernen im Standard-Zugang wären distributionelle 2-Argument-Kerne aus dem Weißen-Rauschen-Zugang zum quantenstochastischen Kalkül [Hua93, Oba97]. Wir werden jedoch den Nichtstandard-Zugang zur Analysis des Weißen Rauschens nicht beschreiten [Ng00]. Daher führen wir interne *3-Argument-Kernoperatoren* ein und zeigen einige Tatsachen, wie z.B. die Gestalt der Kernfunktion eines Podukts von zwei Operatoren. Dadurch wird das *Wickprodukt* der entsprechenden Kernfunktionen definiert. Außerdem definieren wir interne *quantenstochastische Integrale* als hyperendliche Summen von adaptierten internen Operatoren mit den fundamentalen Prozessen als Integratoren. Wir berechnen die Gestalt der Kernfunktion der Integrale, die genauso wie im Zeit-kontinuierlichen Fall aussieht.
In Abschnitt 4 wenden wir den entwickelten internen Kalkül an, um eine interne *Clark-Ocone-Formel* und ein internes *Quantenmartingal-Darstellungstheorem* zu beweisen. Da jeder interne Operator ein Kernoperator ist, können wir für jedes interne Quantenrauschen, d.h. für Erzeugungs-, Anzahl- und Vernichtungsprozess, eine entsprechende *adaptierte (stochastische) Ableitung* definieren und der Beweis ist dann ein einfaches kombinatorisches Argument. Eine weitere Anwendung ist das Studium von internen *quantenstochastischen Differentialgleichungen.* Hier geben wir explizit die Lösung als interner Kernoperator für eine Gleichung mit nicht-linearen Rauschentermen.

Kapitel 3 und 4 ziehen die Verbindung zwischen internem und gewöhnlichem Kalkül.
In Abschnitt 1 von Kapitel 3 konstruieren wir eine *Standarteil-Abbildung* st von einer geeigneten Teilmenge Γ_{st} des symmetrischen Maßraums $(\Gamma, \mathfrak{A}, m)$ auf den symmetrischen Maßraum $(\mathcal{P}_{\text{fin}}, \mathfrak{B}, \Lambda)$ über $[0, 1]$. Die Konstruktion geschieht dabei auf solche Weise, daß das induzierte Maß $m_L \circ \text{st}^{-1}$ auf $\mathcal{P}_{\text{fin}}$ gerade mit dem ursprünglichen Maß Λ übereinstimmt. Danach erweitern wir diese Standardteil-Abbildung auf Γ^n. Weiterhin führen wir entsprechend verschiedener Topologien auf dem Hilbertraum $\mathcal{K}$ und den linearen Operatoren $\mathcal{B}(\mathcal{K})$ auf $\mathcal{K}$ verschiedene Standardteil-Abbildungen auf ${}^*\mathcal{K}$ und ${}^*\mathcal{B}(\mathcal{K})$ ein.

In Abschnitt 2 zeigen wir die Existenz von *Liftings* für Funktionen auf $[0,1]^n$ und $\mathcal{P}_{\text{fin}}^n$. Im komplexwertigen Fall ist das für $\mathcal{P}_{\text{fin}}$ eine geschickte Anwendung der bekannten Resultate für $[0,1]^n$ (siehe etwa [OS, chapter 5] oder [Cut00]). Dann führen wir einen Anfangshilbertraum $\mathcal{K}$ ein und betrachten $\mathcal{K}$-wertige und $\mathcal{B}(\mathcal{K})$-wertige Funktionen. Wir haben ein bekanntes Lifting-Resultat für $\mathcal{K}$-wertige Bochner quadratintegrierbare Funktionen und ein neues Lifting-Resultat für $\mathcal{B}(\mathcal{K})$-wertige Funktionen, die *wop-meßbar* sind. Dies ist erstaunlich, da $\mathcal{B}(\mathcal{K})$ in der schwachen Operator-Topologie nicht einmal erst-abzählbar ist. Die Beweise zu beiden Resultaten werden in den Anhang verlagert.
In Abschnitt 3 definieren wir drei Arten der Darstellung eines Fockraum-Operators durch einen internen Operator. Erstens die *strikte Darstellung*. Dabei identifizieren wir Operatoren mit ihrer Kernfunktion und nennen einen internen Operator eine strikte Darstellung eines Standard-Operators falls die interne Kernfunktion ein Lifting der Standard-Kernfunktion ist. Ganz offensichtlich kann es strikte Darstellungen nur für Kernoperatoren geben. Andererseits hat nach den Lifting-Resultaten des letzten Abschnitts auch jede Kernfunktion ein Lifting, folglich jeder Kernoperator eine strikte Darstellung. Als nächstes gibt es die *starke Darstellung*. Wir fixieren einen Definitionsbereich von Exponentialvektoren ϕ und nennen einen internen Operator K eine starke Darstellung des Fockraum-Operator k, falls $\mathsf{K}\pi_\Phi$ ein Lifting von $\mathsf{k}\pi_\phi$ ist für ein SL^2-Lifting Φ von ϕ. Die *schwache Darstellung* wird ganz ähnlich definiert, nur daß diesmal $\langle \pi_\Psi, \mathsf{K}\pi_\Phi \rangle \approx \langle \pi_\psi, \mathsf{k}\pi_\phi \rangle$ für alle SL^2-Liftings Ψ, Φ von ψ, ϕ gefordert wird. Wir erweitern diese Darstellungsarten auf Operatorprozesse und zeigen, daß die internen Zeit-, Erzeugungs-, Anzahl- und Vernichtungsprozesse starke Darstellungen der entsprechenden Standard-Prozesse sind. Außerdem ist die Erwartungswertbildung im Vakuumzustand verträglich mit allen drei Arten der internen Darstellung von Operatoren und Operatorprozessen.

Im letzten Kapitel, in Kapitel 4, betrachten wir quantenstochastische Differentialgleichungen und starke und schwache Darstellungen.
In Abschnitt 1 untersuchen wir den Zusammenhang zwischen der linearen quantenstochastischen Differentialgleichung mit konstanten Koeffizienten und ihrem internen Gegenpart. Wir zeigen, daß die interne Kern-Lösung eine strikte Darstellung der Standard-Kern-Lösung ist. Die dabei gewonnenen Ideen erlauben es uns, aus der internen Kern-Lösung der internen quantenstochastischen Differentialgleichung mit nicht-linearen Rauschentermen eine Standard-Kern-Lösung zu konstruieren, welche die entsprechende Standard-Gleichung für die Kernfunktion löst. Ferner zeigen wir, daß die so gewonnene Standard-Kernfunktion auf einem geeigneten Definitionsbereich einen Operator definiert.
In Abschnitt 2 beweisen wir die internen Analoga von *erster* und *zweiter fundamentaler Formel* und von der *fundamentalen Abschätzung* des quantenstochastischen Kalküls. Ferner führen wir die Eigenschaft *S-Integrierbarkeit* für interne adaptierte Operatorprozesse ein. Mit Hilfe der fundamentalen Abschätzungen und der ersten fundamentalen Formeln können wir zeigen, daß ein adaptierter Operatorprozeß der eine S-integrierbare adaptierte starke Darstellung besitzt, selbst integrierbar ist.
In Abschnitt 3 zeigen wir, daß beschränkte Operatoren eine starke Darstellung, sowie wop-meßbare Operatorprozesse, beschränkte Operatorprozesse und beschränkte Martingale eine entsprechende schwache Darstellung besitzen. Wir geben einige Hinweise, wie das eingesetzt werden könnte, um vorhersagbare Darstellungen von Operatoren zu geben

(vgl. [Att96b]). Außerdem erwarten wir dadurch in Zukunft ein besseres Verständnis des Quantenmartingal-Darstellungstheorems [PS86, Att94, Att99].

Nach Kapitel 4 schließen wir einen Schlußteil an, in dem wir den Verlauf dieser Doktorarbeit rekapitulieren und einen Ausblick auf zukünftige Anwendungen und Entwicklungen geben. Im Anhang schließlich führen wir in Abschnitt 1 das Bochner-Integral ein und stellen ein Lifting-Theorem für Bochner-integrierbare Funktionen bereit. In Abschnitt 2 beweisen wir ein neues Lifting-Resultat. Für einen separablen Hilbertraum $\mathcal{K}$ zeigen wir, daß eine wop-meßbare $\mathcal{B}(\mathcal{K})$-wertige Funktion auf einem hyperendlichen Maßraum ein Lifting bezüglich der schwachen Operator-Topologie besitzt.

Zum Verstehen dieser Arbeit sollten bezüglich des quantenstochastischen Kalküls keine Voraussetzungen nötig sein, da wir ja in Kapitel 1 die wichtigsten Ideen darlegen. Für ein besseres Verständnis empfehlen wir jedoch die Bücher [Bia95, Mey93a, Hud, Par92] und speziell für den Kern-Kalkül zusätzlich die Artikel [LM88, Maa85]. Lesenswert sind auch der Originalartikel von Hudson und Parthasarathy [HP84] und der Artikel von Attal [Att98] über den Bezug von klassischer stochastischer Analysis und quantenstochastischem Kalkül.
Im Gebiet der Nichtstandard-Analysis setzen wir Kenntnisse auf dem Niveau der Kapitel 2, 3 und 5 im Buch [LW00] herausgegeben von Loeb und Wolff voraus. Insbesondere sollte Loebs Konstruktion eines Standard-Maßraumes aus einem internen Maßraum gut verstanden sein (Originalartikel [Loe75]) und wie man dadurch mit der n-dimensionalen Zeitachse $(T^n, {}^*\mathcal{P}(T^n), \mu^n)$ mit normiertem Zählmaß μ^n den n-dimensionalen Lebesgue-Maßraum $([0,1]^n, \mathcal{B}^n\lambda^n)$ nachbilden kann. Ferner empfehlen wir als einführende Literatur den Artikel von Lindstrøm [Lin88], Cutlands Buch [Cut00] und Kapitel 1-3 in Albeverio et. al. [AFHKL86]. Selbstverständlich gibt es noch viele weitere einführende Texte zur Nichtstandard-Analysis. Wir möchten nur die Bücher [HL85, ACH97] und als deutschsprachiges Buch [LR94] erwähnen.

Contents

Introduction

38. Das Wissen in der Mathematik. Man muß sich hier immer wieder an die Unwichtigkeit eines ›inneren Vorgangs‹ oder ›Zustands‹ erinnern und fragen »Warum soll er wichtig sein? Was geht er mich an?« Interessant ist es, wie wir die mathematischen Sätze *gebrauchen.*

L. Wittgenstein, *Über Gewißheit*

If we ask for the origin of a thesis then we can give the trivial answer: it is written by the author. If we ask for the roots then the answer is much less trivial. But for this thesis we have undoubtly one root:

> We are now ready to describe in a heuristic way the relation between Fock space and finite spin systems. According to T. Lindstrøm, a rigorous discussion is possible using non-standard analysis, but I do not think there is anything published on this subject. This section is not meant as serious mathematics, and pretends only to make formal computations easier.
>
> P.-A. Meyer, *Quantum Probability for Probabilists,* p. 83

Also in his articles [Mey86] and [Mey87] about the toy Fock space Meyer suggests that a nonstandard analyst should be able to recover the standard Fock space and the usual operators from the toy Fock space and the toy creators and toy annihilitors. If we speak in this work of Fock space we mean always the Boson (or symmetric) Fock space. But neither Tom Lindstrøm nor any other nonstandard analysts have done the work needed to convert some easy formal computations into the serious mathematics of quantum stochastic calculus. So it was a longstanding task to develop a nonstandard approach to quantum stochastic calculus. With this thesis we have taken up this task, and as we hope, had some success in this endeavor. Therefore we bring together *nonstandard analysis* and *quantum stochastic calculus.*

Nonstandard analysis (as it is used here) traces back to Robinson [Rob66] where he gives the mathematical foundation for infinite large and infinitesimal numbers. Since then many heuristic ideas with infinitesimals have become reality using the real infinitesimals of nonstandard analysis. But not only numbers, also sets can be handled more conveniently by nonstandard analysis. Remarkable here is the use of hyperfinite, that is formally finite, sets to model infinite standard sets. In this way one has for every finite standard measure space a hyperfinite internal measure space such that using Loeb's construction [Loe75] the internal measure space contains an 'approximate' image of the standard space. That means every standard measurable set can be approximated up to a Loeb nullset with an internal measurable set. But the so-called Loeb spaces are interesting in their own right since they're rich standard measure spaces where a weak solution implies the existence of a strong solution for stochastic differential equations [Kei84]. In Chapter 3 we use Loeb's construction to make the toy Fock space into a standard Fock space such that the usual Boson Fock space can be identified as a subspace. The applications of nonstandard methods are now widespread over many mathematical and physical fields [ALW95]. With this thesis we add the field of quantum stochastic calculus.

The primer of quantum stochastic calculus is the article by Hudson and Parthasarathy [HP84]. There they develop an extension of stochastic analysis to operator-valued stochastic analysis on Fock space. Here the integrators, time, creation, number and annihilation process, as well as the integrands are operator valued processes. In this thesis we use the more measure-oriented approach with the Guichardet space. The main idea is to construct the symmetric measure space directly for the underlying Lebesgue measure space $[0, 1]$ [Gui72]. Then the L^2-space, the Guichardet space, is isomorphic to the usual Fock space. Thus we will use equivalently the names Guichardet space and Fock space. Most operators on Fock space have then a representation by a kernel function. This can be seen as a Fock expansion of the operator in quantum noises, creation, number and annihilation process. In the kernel picture many ideas of classical stochastic analysis carry directly over to quantum stochastic calculus. An application is the use of quantum stochastic differential equations in quantum physics of irreversible systems [Dav76, Fri85, Hud96]. Normally the quantum stochastic differential equations considered are linear, and the interaction is linear as well. For nonlinear interactions, that is powers of singular white noise operators, we refer to Accardi, Volovich with Lu [ALV98, AV97] and Chung, Ji, Obata [CJO98]. In Chapter 4, Subsection 1.2, we treat the case of an equation with nonlinear noise terms with nonstandard methods.

Nowadays there are many publications in both fields, nonstandard analysis and quantum stochastic calculus. But we have found only two sources which contribute to both fields jointly. There is an article by Gudder on "Nonstandard Fock spaces" [Gud94] (cf. also [Gud96]) and an article by Yamashita on "Hyperfinite-dimensional representations of canonical commutation relation" [Yam98] (cf. also [YO00]).
Gudder constructs in the article [Gud94] a nonstandard Fock space $\Gamma(\mathcal{H})$ over some standard Hilbert space $\mathcal{H}$ in the following way. Let $\mathcal{H}_n = \oplus_{k \le n} \mathcal{H}^{\widehat{\otimes} k}$ be the space with n or fewer particles. Here $\widehat{\otimes}$ is the symmetric tensor product. Then the sequence $(\mathcal{H}_n)_{n \in \mathbb{N}}$ modulo an ultrafilter on $\mathbb{N}$ defines an internal hypercomplex Hilbert space $\Gamma(\mathcal{H})$. Afterwards Gudder gives imbeddings for all ${}^*\mathcal{H}_n$ into $\Gamma(\mathcal{H})$ in a natural way. Thus he has the domain of finite particle vectors modelled by $\Gamma(\mathcal{H})$. Of course, ${}^*\mathcal{F}$ for $\mathcal{F} = \oplus_{n \in \mathbb{N}} \mathcal{H}^{\widehat{\otimes} n}$ is much bigger than $\Gamma(\mathcal{H})$ as Gudder points out. Compared to our internal Fock space we see that we take in some sense also a sequence $\widetilde{\mathcal{H}}_0 \subset \widetilde{\mathcal{H}}_1 \subseteq \cdots$ like Gudder but of internal Hilbert spaces $\widetilde{\mathcal{H}}_n$. Then we extend by saturation to $(\widetilde{\mathcal{H}}_n)_{n \in {}^*\mathbb{N}}$ and set $\Gamma(\widetilde{\mathcal{H}}) = \mathcal{H}_H$ for some $H \in {}^*\mathbb{N}_\infty$. For the case $\mathcal{H} = L^2([0, 1])$ we have $\widetilde{\mathcal{H}} = {}^*L^2(T)$ and we approximate already the underlying Hilbert space $\mathcal{H}$. We think that our approach, using the sym-

metric spaces over $[0,1]$ and T, is more convenient since we have to deal later only with hyperfinite sums and not with integrals against *-Lebesgue measure like Gudder must do.

In the article [Yam98] Yamashita shows how to construct on the internal Hilbert space ${}^*\mathbb{C}^H, H \in {}^*\mathbb{N}_\infty$ internal $H \times H$-matrices Q and P such that $[Q,P]x \approx \mathrm{i}x$ for every $x \in S \subseteq {}^*\mathbb{C}^H$ of an appropriate, possibly external subspace S. Further he supposes $Qx \in S, Px \in S$ or all $x \in S$. In part IV of the article Yamashita gives an example of what he calls the hyperfinite para-Fermi representation. Indeed he constructs operators $\widetilde{A}^+$ and $\widetilde{A}^-$ such that $Q = \frac{1}{\sqrt{2}}(\widetilde{A}^+ + \widetilde{A}^-)$ and $P = \frac{1}{\mathrm{i}\sqrt{2}}(\widetilde{A}^+ - \widetilde{A}^-)$ fulfill the approximate canonical commutation relation. We sketch only the main idea of Yamashita since he uses a rather complicated construction and confusing notation. Roughly speaking, he takes the matrices $\alpha^\pm = \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2)$ and $\alpha^\circ = \frac{1}{2}(\mathbb{1} + \sigma_3)$ where $\sigma_j, j = 1,2,3$ are the Pauli matrices. He defines the operators $\widetilde{\alpha}_k^\sharp = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \alpha^\sharp \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$ on ${}^*\mathbb{C}^H$ where $\alpha^\sharp$ is in the kth place. Then $\widetilde{A}^- = \sum_{k \le H} \widetilde{\alpha}_k^-$ and $\widetilde{A}^+ = \sum_{k \le H} \widetilde{\alpha}_k^+$ give an approximate representation of the canonical commutation relation. As is well-known, in the finite case the matrices $\widetilde{\alpha}_s^+$ and $\widetilde{\alpha}_s^-$ are representations of the toy creators a_s^+ and toy annihilators a_s^-. By transfer we have this for the internal operators too. Further, by the internal quantum Itô table (see equation (2.2.8), p. 32) for the corresponding processes holds $[A_t^-, A_t^+] = A_t^\bullet - \frac{1}{H}A_t^\circ$. This shows that on a certain external subspaces of the internal Fock space we have constructed a hyperfinite representation of the canonical commutation relation in the sense of Yamashita. We see indeed that ours is also a para-Fermi representation since with the Poisson bracket $\{A_t^-, A_t^+\} = A_t^\bullet + \frac{1}{H}A_t^\circ = \mathbb{1}_t$.

For a standard finite set the construction above is known as the quantum Bernoulli random walk which was invented and studied by Biane [Bia90, Bia91]. For the central limit of the quantum Bernoulli walk we refer to Mabrouki [Mab95]. Later de Giosa and Lu have shown that we can get as limit creation and annihilation operators on interacting q-Fock space [GL98]. In some sense our nonstandard approach is the limit for $q = 1$ to Boson creation and annihilation process. For yet another standard treatment for the limit of quantum random walks we refer to the articles by Parthasarathy with Lindsay [Par88, LP88].

As we indicated before the quantum Bernoulli random walk is one relization of the creation, number and annihilation process on toy Fock space. Taking a countable infinite time-line Attal showed how to approximate the Boson Fock space over $\mathbb{R}_+$ with this ‘infinite’ toy Fock space [Att00]. The infiniteness forces Attal to work in the countable infinite tensor product of the 2×2-matrices with stabilizing sequence $(\mathbb{1}_n)_{n\in\mathbb{N}}$ (cf. [AW69, AW66, GS85]). In our nonstandard approach we would approximate this also with a hyperfinite time-line $\widetilde{T} = \{\frac{k}{H} : 0 \le k < H^2\}$ and thus avoid the direct use of countable infinite tensor products. In this work we take a finite discrete time-line T and its associated toy Fock space and interpret this as hyperfinite time-line and internal Fock space in a nonstandard model. In this sense we later approximate the Fock space over $[0,1]$ with the ‘finite’ toy Fock space over T.

Two articles, one by Meyer [Mey91] and one by Parthasarathy [Par98], show that the toy Fock space has some interest in its own right. But it seems to be the case that this thesis is the first work which takes the toy Fock space seriously as a nonstandard model for the standard Fock space. We give now a summary of the content of the thesis.

In Chapter 1 we give a short overview of the standard *quantum stochastic calculus*. Since there are good introductions to the subject [Par92, Mey93a, Bia95, Hud] we don't give the proofs but sketch chiefly the main ideas. We put some weight on the *kernel approach* to quantum stochastic calculus [Maa85, LM88, Lin90, Lin98].

In Chapter 2 we develop a quantum stochastic calculus for the discrete hyperfinite timeline T: the *internal quantum stochastic calculus*.
In Section 1 we construct the *symmetric measure space* $(\Gamma, \mathfrak{A}, m)$ over $(T, \mathcal{A} = {}^*\mathcal{P}(T), \mu)$ and define the *internal Fock space* $\mathcal{F}_{\text{int}}$ as the space of all internal functions on Γ. Because Γ is hyperfinite we see that $\mathcal{F}_{\text{int}}$ is hyperfinite-dimensional. Also we introduce in this internal setting the usual probabilistic concepts as independence, expectation, adaptedness, and others.
In Section 2 we define for each time instant $s \in T$ the *fundamental operators* $a_s^\sharp, \sharp \in \{\bullet, +, \circ, -\}$, the 'infinitesimal' time, creation, number and annihilation process. In fact the $a_s^\sharp$s are the increment processes of the corresponding 'large' processes $A_s^\sharp$. We prove for the $a_s^\sharp$s an *internal quantum Itô formula* which gives the known Itô table up to four infinitesimal entries. Comparing our definition with that of the usual toy Fock space [Mey86] we have rectified the definition of the annihilator a_s^- and given it the right scaling with $\frac{1}{H}$. We study also the analogues of Brownian motion and Poisson process on the internal Fock space.
In Section 3 we define internal *2-argument kernel operators* and prove some of their properties. Looking at the standard calculus our 2-argument kernel should be connected to the distributional 2-argument kernels in the white noise approach to quantum stochastic calculus [Hua93, Oba97]. But we don't want to go into the nonstandard approach to white noise analysis which is a field in development [Ng00]. Henceforth we introduce internal *3-argument kernel operators* and show several facts, including the kernel function of the product of two operators. This gives the (internal) *Wick product* of the corresponding single kernel functions. We define also internal *quantum stochastic integrals* for adapted internal operators as hyperfinite sums against the fundamental processes as integrators. We calculate the kernel function of an integral and this turns out to be the same as in the continuous in time setting.
In Section 4 we apply the developed internal calculus firstly to prove a *Clark-Ocone formula* for internal operators and a *martingale representation theorem* for internal operator martingales. Since every internal operator and operator process has an internal kernel function we can introduce for each internal quantum noise, i.e. fundamental creation, number, annihilation process, a corresponding *adapted derivative* and the proofs become a simple combinatorial question. The martingale representation theorem in the standard case is by far not as simple as here [PS86, Mey93b]. Secondly we give explicitly the kernel solutions to a linear internal *quantum stochastic differential equation* with constant coefficients and to one with nonlinear noise terms.

Chapters 3 and 4 connect the internal quantum stochastic calculus to the standard quantum stochastic calculus.
In Section 1 of Chapter 3 we define a *standard part map* st on an appropriate subset Γ_{st} of the internal measure space $(\Gamma, \mathfrak{A}, m)$ onto the symmetric measure space $(\mathcal{P}_{\text{fin}}, \mathfrak{B}, \Lambda)$ over $([0,1], \mathcal{B}, \lambda)$. The construction is carried out in such a way that the induced mea-

sure $m_L \circ \mathrm{st}^{-1}$ on $\mathcal{P}_{\mathrm{fin}}$ is precisely the original measure Λ. We extend this standard part map to Γ^n and define standard part maps for ${}^*\mathcal{K}$ and ${}^*\mathcal{B}(\mathcal{K})$ corresponding to the various topologies on $\mathcal{K}$ and $\mathcal{B}(\mathcal{K})$.
In Section 2 we treat the question of liftings for functions on $[0,1]^n$ and $\mathcal{P}_{\mathrm{fin}}^n$. For $\mathbb{C}$-valued functions this is in the $\mathcal{P}_{\mathrm{fin}}^n$-case in a certain sense just a clever application of the known results for $[0,1]^n$ (see [OS, Chapter 5] and [Cut00]). Including an initial Hilbert space $\mathcal{K}$, that means looking at $\mathcal{K}$-valued and $\mathcal{B}(\mathcal{K})$-valued functions, then the problem is far less simple. We include a known lifting theorem for $\mathcal{K}$-valued Bochner square integrable functions [BO] and a new lifting theorem for $\mathcal{B}(\mathcal{K})$-valued functions which are *woply measurable*. The proofs of both results are postponed to the appendix.
In Section 3 we define three concepts on how to represent a standard Fock space operator by an internal operator on the internal Fock space. For the first, *strict representation*, the results of the preceding section apply. We say that an internal operator represents strictly a standard kernel operator if the internal kernel function is a lifting of the standard kernel function. Clearly this works only for kernel operators but on the other hand every kernel operator has a strict representation by the lifting theorem. For the second, *strong representation*, we fix the 'bounded' exponential domain and liftings of the exponential domain. Then we say that an internal operator K represents strongly a standard operator k if for all bounded square integrable ϕ on $[0,1]$ the vector $\mathsf{K}\pi_\Phi$ is a SL^2-lifting of $\mathsf{k}\pi_\phi$ for some SL^2-lifting Φ of ϕ. The third, *weak representation*, works the same way but with the condition $\langle \pi_\Psi, \mathsf{K}\pi_\Phi\rangle \approx \langle \pi_\psi, \mathsf{k}\pi_\phi\rangle$ for all SL^2-liftings Ψ, Φ of ψ, ϕ. Further, we extend these representations to operator processes and show that our internal time, creation, number and annihilation process is a strong representation of the respective standard process in quantum stochastic calculus. We prove also that the vacuum expectation is compatible with all three ways of representation.

In the last chapter, in Chapter 4, we take a closer look on two topics: quantum stochastic differential equations, and strong and weak representations.
In Section 1 we treat quantum stochastic differential equations in the kernel picture. We show that our internal kernel solution obtained in Chapter 2 Subsection 4.2 is a strict representation of the standard kernel solution to the linear equation with constant coefficients. Surprisingly we convert the internal kernel solution of the nonlinear equation to a standard kernel function. We show that this standard kernel defines a reasonable operator process. Then we prove that in the kernel interpretation of quantum stochastic differential equations this kernel solves an equation with nonlinear noise terms.
In Section 2 we prove the internal analogues of the *first* and *second fundamental formula* and the *fundamental estimation* of quantum stochastic calculus. We define *S-integrability* for internal adapted operator processes. Using the fundamental estimations and the first fundamental formulas we show that an adapted operator process with an S-integrable adapted strong representation is integrable.
In Section 3 we prove the existence of strong representations for bounded operators and that of weak representations for wop-measurable processes, bounded processes and bounded martingales. In the latter cases the weak representation can be chosen to be adapted respectively to be a martingale. We give an idea in which way this should be useful for investigating predictable representations of operators (cf. [Att96b]). We expect also

new insights into the quantum martingale representation theorem [PS86, Att94, Att99].

In the Conclusion we recapitulate what we have done in this thesis. We give an outlook on further applications and sketch possible extensions of the methods developed in this thesis. In the Appendix in Section 1 we introduce the Bochner integral. That provides us with the lifting theorem for Bochner integrable functions. For a separable standard Hilbert space $\mathcal{K}$ we show in Section 2 that a woply measurable $\mathcal{B}(\mathcal{K})$-valued function on a hyperfinite measure space has a lifting with respect to the weak operator topology.

Let us finally say something about the prerequisites for reading this thesis. Since in the first chapter we give the main ideas of quantum stochastic calculus the thesis should be understandable without knowledge of this subject. But for a better understanding we recommend the texts [Bia95, Mey93a, Hud, Par92] and [LM88, Maa85, Mey93a] for the kernel approach. Also the original article by Hudson and Parthasarathy [HP84] is readable and an article by Attal [Att98] where he shows the close connections between classical and quantum stochastic calculus. In nonstandard analysis the thesis requires knowledge at the level of chapters 2, 3 and 5 in the book [LW00] edited by Loeb and Wolff. We recommend as well Lindstrøm's article [Lin88], Cutland's book [Cut00] and the first three chapters in Albeverio et. al. [AFHKL86]. Of course there are many other introductory texts on nonstandard analysis, like [HL85, ACH97, BO] for example.

Chapter 1

Standard Quantum Stochastic Calculus

In this Chapter we give a short overview of the one-dimensional quantum stochastic calculus in the kernel picture of Maassen. The results are well known and we give them without proof. Quantum stochastic calculus on Boson Fock space traces back to the famous article of Hudson and Parthasarathy [HP84]. A general introduction is given by Parthasarathy [Par92] or Meyer [Mey93a] and a shorter one in French by Biane [Bia95]. There is also a forthcoming book by Hudson [Hud]. The kernel approach to quantum stochastic calculus was initiated by Maassen [Maa85] together with Lindsay [LM88, LM92] and further developed by Lindsay [Lin93b, Lin90, Lin98] and Attal [AL99, Att98]. One advantage of the kernel approach is its combinatorial feature. This allows some nice calculations. Another is the measure theoretic background that makes this approach very fruitful for a nonstandard treatment.

1 Standard Guichardet Space

In this Section we show the construction of the (standard) Guichardet space. This is a general construction for arbitrary measure spaces and in terms of category theory it is a functor that maps a category to the 'symmetric category' of that category. This is described by Guichardet in his book [Gui72]. For our purpose we restrict ourselves to a special case of this functorial construction.

Let $([0,1],\mathcal{B},\lambda)$ be the Lebesgue space. We introduce some notation.

Notation 1.1.1

$$\begin{aligned}
\mathcal{P}_{\mathrm{fin},n} &= \{\omega \subset [0,1] : |\omega| = n\}, \quad n \in \mathbb{N}, \\
\mathcal{P}_{\mathrm{fin}} &= \cup_{n\in\mathbb{N}} \mathcal{P}_{\mathrm{fin},n} = \{\omega \subset [0,1] : |\omega| \text{ is finite}\}, \\
(t_1, \cdots, t_n)_{\neq} &= (t_1, \cdots t_n) \text{ with } t_i \neq t_j \text{ for } i \neq j, \\
[0,1]^n_{\neq} &= \{(t_1,\cdots,t_n)_{\neq} : (t_1,\cdots,t_n) \in [0,1]^n\}.
\end{aligned}$$

We refer to these sets as the *n-finite power set*, the *finite power set*, the *n-tuple in general position*, the *set of n-tuples in general position* of the interval $[0,1]$. $\triangleleft$

For $n = 0$ we define on $[0,1]^n_{\neq} = [0,1]^n = \{\emptyset\}$ as measure the Dirac measure $\lambda^0(\,\cdot\,) = \delta_{\{\emptyset\}}(\,\cdot\,)$. We endow the finite power set with a natural measure that comes from the n-dimensional Lebesgue spaces. For this we introduce the homomorphisms

$$\varphi_n : [0,1]^n_{\neq} \longrightarrow \mathcal{P}_{\mathrm{fin},n} : (t_1, \cdots, t_n)_{\neq} \longmapsto \{t_1, \cdots, t_n\}$$

and glue them together to a single homomorphism

$$\varphi : \cup_{n\in\mathbb{N}}[0,1]^n_{\neq} \longrightarrow \mathcal{P}_{\mathrm{fin}} : \omega \longmapsto \{t : t \in \omega\}\,.$$

Definition 1.1.2 *A subset $B \subseteq \mathcal{P}_{\mathrm{fin}}$ is* measurable *if $\varphi^{-1}(B \cap \mathcal{P}_{\mathrm{fin},n})$ is λ^n-measurable in $[0,1]^n$ for each $n \in \mathbb{N}$. We denote the σ-Algebra of all measurable sets by $\mathfrak{B}$. For $B \in \mathfrak{B}$ the measure Λ is defined by*

$$\Lambda(B) \;=\; \sum_{n\in\mathbb{N}} \frac{1}{n!}\lambda^n(\varphi^{-1}(B \cap \mathcal{P}_{\mathrm{fin},n}))\,. \qquad \triangleleft$$

By definition we have $\Lambda(\mathcal{P}_{\mathrm{fin}}) = \mathrm{e}$. Furthermore, if we introduce the set of sets in general position

$$\mathcal{P}^n_{\mathrm{fin},\neq} \;=\; \{(\sigma_1, \cdots, \sigma_n) \in \mathcal{P}^n_{\mathrm{fin}} : \sigma_i \cap \sigma_j = \emptyset \;\text{ if }\; i \neq j\}$$

then the complement has zero product measure: $\Lambda^n(\mathcal{P}^n_{\mathrm{fin}} \setminus \mathcal{P}^n_{\mathrm{fin},\neq}) = 0$. This follows from the fact that in each n-dimensional Lebesgue space the diagonals are sets of measure zero.

Definition 1.1.3 *By the* (standard) Guichardet space *or* (Boson) Fock space $\mathcal{F}$ *over* $[0,1]$ *we mean the space of all complex valued square integrable functions on* $(\mathcal{P}_{\mathrm{fin}}, \mathfrak{B}, \Lambda)$*:*

$$\mathcal{F} \;=\; L^2_{\mathbb{C}}(\mathcal{P}_{\mathrm{fin}}, \mathfrak{B}, \Lambda)\,. \qquad \triangleleft$$

Remark 1.1.4 Normally as Boson Fock space is referred to the space $\oplus_{n\in\mathbb{N}}\mathcal{H}^{\widehat{\otimes} n}$ with $\widehat{\otimes}$ the symmetric tensor product and $\mathcal{H} = L^2_{\mathbb{C}}([0,1], \mathcal{B}, \lambda)$. But this space is isomorphic to $\mathcal{F}$ as a Hilbert space.

Corresponding to the direct sum decomposition

$$L^2_{\mathbb{C}}([0,1], \mathcal{B}, \lambda) \equiv L^2_{\mathbb{C}}([0,t], \mathcal{B}, \lambda) \oplus L^2_{\mathbb{C}}([t,1], \mathcal{B}, \lambda)$$

we have for each $t \in [0,1]$ the tensor decomposition $\mathcal{F} \equiv \mathcal{F}_{t]} \otimes \mathcal{F}_{[t}$. More general we have for arbitrary $k \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1$ the decomposition

$$\mathcal{F} \equiv \otimes_{l=0}^{k} \mathcal{F}_{[t_l, t_{l+1}]}$$

whereby $\mathcal{F}_{[t_l, t_{l+1}]}$ is the Guichardet space over $[t_l, t_{l+1}]$.

Definition 1.1.5 *Let* $\psi \in L^2_{\mathbb{C}}([0,1],\mathcal{B},\lambda)$ *The* coherent vector *or* product vector *of* ψ *is the function*

$$\pi_\psi : \mathcal{P}_{\text{fin}} \longrightarrow \mathbb{C} : \sigma \longmapsto \prod_{s\in\sigma} \psi(s)\,.$$

The coherent vector $\omega = \pi_0$ *to the function* $\psi \equiv 0$ *is called the* vacuum state. *We denote by* $\overline{\mathcal{E}}$ *the linear span of the set* $\mathcal{E} = \{\pi_\psi : \psi \in L^2_{\mathbb{C}}([0,1],\mathcal{B},\lambda), \psi$ *is bounded*$\}$. ◁

It is well known that $\mathcal{E}$ is total in $\mathcal{F}$. A remarkable lemma for the Guichardet space is the so-called integral-sum lemma. It shows that iterated integrals in Guichardet space can be reduced to ordinary sums.

Lemma 1.1.6 ($\sum\!\!\!\!\!\int$-lemma)
Let $f : \mathcal{P}_{\text{fin}} \times \mathcal{P}_{\text{fin}} \to \mathbb{C}$ *be a measurable respectively integrable function. Define a function* $g : \mathcal{P}_{\text{fin}} \to \mathbb{C}$ *by*

$$g(\sigma) \;=\; \sum_{\alpha\subseteq\sigma} f(\alpha, \sigma\setminus\alpha)\,.$$

Then g *is measurable resp. integrable and in the latter case we have*

$$\int_{\mathcal{P}_{\text{fin}}} g(\sigma)d\sigma \;=\; \int_{\mathcal{P}_{\text{fin}}}\int_{\mathcal{P}_{\text{fin}}} f(\alpha,\beta)d\alpha d\beta\,.$$

This lemma holds also for functions $f : \mathcal{P}^n_{\text{fin}} \to \mathbb{C}$ and then $g : \mathcal{P}_{\text{fin}} \to \mathbb{C}$ is defined by $g(\sigma) = \sum_{\sigma_1\dot\cup\cdots\dot\cup\sigma_n=\sigma} f(\sigma_1,\cdots,\sigma_n)$. Furthermore instead of $\mathbb{C}$ we can take an arbitrary Banach space.

2 Kernel Operators and Integration

In this Section we introduce kernel operators and integration theory against adapted processes of kernel operators. First we consider 2-argument kernel operators then 3-argument kernel operators. We follow more or less the exposition by Lindsay [Lin90].

Definition 1.2.1 *Let* $k : \mathcal{P}^2_{\text{fin}} \to \mathbb{C}$ *be a function and* $f \in \mathcal{F}$ *a vector. Then we say that* k *defines a* 2-argument kernel operator k *by the equation*

$$\mathsf{k}f(\sigma) \;=\; \sum_{\alpha\subseteq\sigma}\int_{\mathcal{P}_{\text{fin}}} k(\alpha,\beta) f((\sigma\setminus\alpha)\cup\beta)d\beta$$

whenever the right hand side is defined for every $f \in \overline{\mathcal{E}}$. ◁

Thus as common subset of the domain of every operator we take the set of coherent vectors to bounded functions. The 2-argument kernel operators are very fundamental and give also a connection to the white noise kernel operators and the white noise approach to quantum stochastic calculus. This was initiated by Huang [Hua93] and developed by

Obata [Oba96, Oba95, Oba97, Oba99]. Furthermore we can take the kernel function k with values in an arbitrary Banach space. In this way one includes an initial space in the calculus.

The product $\mathsf{g} = \mathsf{kh}$ (if it exists) of two 2-argument kernel operators k and h is given by the following formula for the kernel function g:

$$g(\sigma,\tau) \;=\; \sum_{\gamma\dot{\cup}\alpha=\sigma}\sum_{\rho\dot{\cup}\beta=\tau}\int_{\mathcal{P}_{\mathrm{fin}}} k(\gamma,\rho\cup\eta)h(\alpha\cup\eta,\beta)d\eta\,. \tag{1.2.1}$$

Lindsay [Lin90] has shown that the map of kernel functions to operators is injective in the sense that if the operator k vanishes on the coherent vectors then the kernel function k is zero almost everywhere. Unfortunately the finite rank operators can't be 2-argument kernel operators for a scalar kernel function [Lin90]. Also the number operator can't be expressed as a 2-argument kernel operator as is well known. So Meyer [Mey93a] was led to introduce 3-argument kernel operators.

Definition 1.2.2 *Let $k : \mathcal{P}_{\mathrm{fin}}^3 \to \mathbb{C}$ be a function and $f \in \mathcal{F}$ a vector. Then k defines a* (3-argument) kernel operator k *by the equation*

$$\begin{aligned}\mathsf{k}f(\sigma) \;&=\; \sum_{\alpha\subseteq\sigma}\sum_{\substack{\gamma\subseteq\sigma\\ \gamma\cap\alpha=\emptyset}}\int_{\mathcal{P}_{\mathrm{fin}}} k(\alpha,\gamma,\beta)f((\sigma\setminus\alpha)\cup\beta)d\beta \\ &=\; \sum_{\sigma_1\dot{\cup}\sigma_2\dot{\cup}\sigma_3=\sigma}\int_{\mathcal{P}_{\mathrm{fin}}} k(\sigma_1,\sigma_2,\beta)f(\sigma_2\cup\sigma_3\cup\beta)d\beta\,. \end{aligned} \tag{1.2.2}$$

Note that since the set of sets in general position has full measure it is sufficient to know the kernel function on disjoint subsets. Also this equation makes only sense under certain conditions on f and k. As before we suppose in general that as common core at least the product vectors are contained in the domain of our operator. Thus the equation (1.2.2) are said to define a kernel operator if the right hand side converges for an arbitrary $f \in \overline{\mathcal{E}}$.

Definition 1.2.3 *An* operator process $(\mathsf{k}_s)_{s\in[0,1]}$ *is given by a family of kernel functions* $(k_s)_{s\in[0,1]}$. *The process* $(\mathsf{k}_s)_{s\in[0,1]}$ *is said to be* adapted *if the family of kernel functions fulfills the following condition:*

$$\sigma\cup\rho\cup\tau \not\subset [0,t[\quad\Longrightarrow\quad k_t(\sigma,\rho,\tau) = 0\,. \qquad \triangleleft$$

Note that we use a stronger form of adaptedness which is sometimes called *non-anticipation*. We exclude the present information in our filtration. Next we introduce the three fundamental processes of quantum stochastic calculus. For these three processes and the time process as integrators one develops the quantum stochastic integration theory. This is the extension of classical stochastic analysis and Itô integration.

Definition 1.2.4 *Write $\sigma \setminus s$ for $\sigma \setminus \{s\}$ and $\sigma \cup s$ for $\sigma \cup \{s\}$ if $\sigma \in \mathcal{P}_{\text{fin}}$ and $s \in [0,1]$. Let $f \in \mathcal{F}$. Then the action of the fundamental processes on f is given by*

$$\begin{array}{rcll} \mathsf{a}_t^\bullet f(\sigma) &=& \int_0^t f(\sigma)ds\,, & \text{time process,}\\ \mathsf{a}_t^+ f(\sigma) &=& \sum_{\substack{s<t\\ s\in\sigma}} f(\sigma\setminus s)\,, & \text{creation process,}\\ \mathsf{a}_t^\circ f(\sigma) &=& \sum_{\substack{s<t\\ s\in\sigma}} f(\sigma)\,, & \text{number process,}\\ \mathsf{a}_t^- f(\sigma) &=& \int_0^t f(\sigma\cup s)ds\,, & \text{annihilation process.} \end{array}$$ ◁

Notation 1.2.5 For a kernel operator k and its kernel function k we write $\mathsf{k} \mathrel{\widehat{=}} k$ for the identification of the operator with its kernel. ◁

The kernels of the four fundamental processes are given by

$$\begin{array}{rcl} \mathsf{a}_t^\bullet &\widehat{=}& t \text{ for } (\sigma,\rho,\tau) = (\emptyset,\emptyset,\emptyset),\ \ 0,\ \text{otherwise},\\ \mathsf{a}_t^+ &\widehat{=}& 1 \text{ for } (\sigma,\rho,\tau) = (\{s\},\emptyset,\emptyset) \text{ and } s<t,\ \ 0,\ \text{otherwise},\\ \mathsf{a}_t^\circ &\widehat{=}& 1 \text{ for } (\sigma,\rho,\tau) = (\emptyset,\{s\},\emptyset) \text{ and } s<t,\ \ 0,\ \text{otherwise},\\ \mathsf{a}_t^- &\widehat{=}& 1 \text{ for } (\sigma,\rho,\tau) = (\emptyset,\emptyset,\{s\}) \text{ and } s<t,\ \ 0,\ \text{otherwise}. \end{array}$$

The last three kernels contribute only on the first level $\mathcal{P}_{\text{fin},1}$ of Guichardet space and the time process is a multiple of the identity. The fundamental processes are adapted by definition.

In the usual approach to quantum stochastic calculus the integrals against the fundamental processes are defined by Riemann sums:

$$\int_0^t \mathsf{k}_s d\mathsf{a}_s^\sharp \;=\; \lim_{n\to\infty} \sum_{k=0}^{n-1} \mathsf{k}_{\frac{kt}{n}} (\mathsf{a}^\sharp_{\frac{(k+1)t}{n}} - \mathsf{a}^\sharp_{\frac{kt}{n}})$$

where $(\mathsf{k}_t)_{t\in[0,1]}$ is an adapted operator process. We should note that in the tensor decomposition the integrand and the increments are defined on different parts of the tensor decomposition of the Fock space. Thus the ordinary product is merely a tensor product of operators. In the language of kernels integration becomes very simple:

Proposition 1.2.6 *Let $(k_s)_{s\in[0,1]}$ be the kernel function of an adapted operator process. Then the* integral kernel function *$\int_0^t k_s d\mathsf{a}_s^\sharp(\sigma,\rho,\tau)$ is given by*

$$\int_0^t k_s d\mathsf{a}_s^\bullet(\sigma,\rho,\tau) \;=\; \begin{cases} 0 & \textit{if}\ \ t \le \max(\sigma\cup\rho\cup\tau),\\ \int_{\max(\sigma\cup\rho\cup\tau)}^t k_s(\sigma,\rho,\tau)ds & \textit{if}\ \ t > \max(\sigma\cup\rho\cup\tau), \end{cases}$$

$$\int_0^t k_s d\mathsf{a}_s^+(\sigma,\rho,\tau) \;=\; \begin{cases} 0 & \textit{if}\ \ t \le \max(\sigma\cup\rho\cup\tau),\\ k_{\max\sigma}(\sigma\setminus\max\sigma,\rho,\tau) & \textit{if}\ \ t > \max(\sigma\cup\rho\cup\tau) = \max\sigma\,, \end{cases}$$

$$\int_0^t k_s d\mathsf{a}_s^\circ(\sigma,\rho,\tau) \;=\; \begin{cases} 0 & \textit{if}\ \ t \le \max(\sigma\cup\rho\cup\tau),\\ k_{\max\rho}(\sigma,\rho\setminus\max\rho,\tau) & \textit{if}\ \ t > \max(\sigma\cup\rho\cup\tau) = \max\rho\,, \end{cases}$$

$$\int_0^t k_s d\mathsf{a}_s^-(\sigma,\rho,\tau) \;=\; \begin{cases} 0 & \textit{if}\ \ t \le \max(\sigma\cup\rho\cup\tau),\\ k_{\max\tau}(\sigma,\rho,\tau\setminus\max\tau) & \textit{if}\ \ t > \max(\sigma\cup\rho\cup\tau) = \max\tau\,. \end{cases}$$

Of course for the integrals to exist we have to assume certain conditions on the integrand. But then it is possible to define multiple stochastic integrals. Using the notation $d\mathsf{a}^\sharp_\sigma = \prod_{s\in\sigma} d\mathsf{a}^\sharp_s$ one can also look for a sensible meaning to

$$\int_{\mathcal{P}^3_{\text{fin}}} k(\alpha,\gamma,\beta) d\mathsf{a}^+_\alpha d\mathsf{a}^\circ_\gamma d\mathsf{a}^-_\beta \tag{1.2.3}$$

where k is a Banach space valued function. Actually, evaluated formally on vectors $f \in \mathcal{F}$ this expression gives the defining equation (1.2.2) for 3-argument kernel operators. Thus the convergence of the triple integral is defined for the correspondingly defined kernel operators. In this sense we look at equation (1.2.3) as a Fock expansion of an operator in annihilator, number and creation processes.

As we have four fundamental processes one can think about operators defined by 4-argument kernel functions. The formal expression of such an operator would be

$$\mathsf{h} \;=\; \int_{\mathcal{P}^4_{\text{fin}}} k(\gamma,\sigma,\rho,\tau) d\mathsf{a}^\bullet_\gamma d\mathsf{a}^+_\alpha d\mathsf{a}^\circ_\rho d\mathsf{a}^-_\tau \,.$$

Since the choice of the kernel function is highly non-unique in that case, we don't proceed further. As a remark it should be noted that the time integral kernel of an adapted 4-argument kernel operator is given by

$$\int_0^t k_s d\mathsf{a}^\bullet_s(\gamma,\sigma,\rho,\tau) \;=\; \begin{cases} 0 & \text{if} \quad t \le \max(\gamma\cup\sigma\cup\rho\cup\tau), \\ k_{\max\gamma}(\gamma\setminus\max\gamma,\sigma,\rho,\tau) & \text{if} \quad t > \max(\gamma\cup\sigma\cup\rho\cup\tau) = \max\gamma, \end{cases}$$

whereas the other integrals are 'the same' since the variable γ is not involved.

3 Quantum Stochastic Differential Equations

In the normal approach to quantum stochastic calculus we have three fundamental propositions that enable us to solve quantum stochastic differential equations. The first one is in some sense the first part of a quantum Itô isometry.

Proposition 1.3.1 (**1. F**undamental **F**ormula)

$$\left\langle \pi_\psi, \int_0^t \mathsf{k}_s d\mathsf{a}^\sharp_s \pi_\phi \right\rangle \;=\; \int_0^t \langle \pi_\psi, \mathsf{k}_s\pi_\phi\rangle\, \mathfrak{y}^\sharp_s ds\,, \qquad \mathfrak{y}^\sharp_s = \begin{cases} 1 & \textit{if} \quad \sharp = \bullet\,, \\ \overline{\psi(s)} & \textit{if} \quad \sharp = +\,, \\ \overline{\psi(s)}\phi(s) & \textit{if} \quad \sharp = \circ\,, \\ \phi(s) & \textit{if} \quad \sharp = -\,. \end{cases}$$

Then we have the so-called 2. fundamental formula. This is the quantum version of the Itô product formula and gives an algorithm on how to compute the product of two quantum stochastic integrals. This is summarized in the quantum Itô table which is an extension of the formal "$(dB_t)^2 = dt$" relation for Brownian motion.

Proposition 1.3.2 (**2. F**undamental **F**ormula)
Let $\mathsf{m}_t = \int_0^t \mathsf{x}_s d\mathsf{a}_s^\sharp$ *and* $\mathsf{n}_t = \int_0^t \mathsf{y}_s d\mathsf{a}_s^\natural$*. Then*

$$\begin{aligned} \langle \mathsf{n}_t \pi_\phi, \mathsf{m}_t \pi_\psi \rangle \quad = \quad & \int_0^t \langle \mathsf{y}_s \pi_\phi, \mathsf{m}_s \pi_\psi \rangle\, \mathfrak{y}_s^{\star\natural} ds \;+\; \int_0^t \langle \mathsf{n}_s \pi_\phi, \mathsf{x}_s \pi_\psi \rangle\, \mathfrak{y}_s^{\sharp} ds \\ & \;+\; \int_0^t \langle \mathsf{y}_s \pi_\phi, \mathsf{x}_s \pi_\psi \rangle\, \mathfrak{z}_s^{\star\natural,\sharp} ds \,. \end{aligned}$$

Here we adopt the notation $\star\bullet = \bullet$, $\star + = -$, $\star\circ = \circ$, $\star - = +$. In differential form this reads as follows:

$$d(\mathsf{n}_t \mathsf{m}_t) = d\mathsf{n}_t \cdot \mathsf{m}_t + \mathsf{n}_t \cdot d\mathsf{m}_t + d\mathsf{n}_t \cdot d\mathsf{m}_t$$

whereby $\mathfrak{z}_s^{\natural,\sharp}$ is defined by the following first table and for the differential form we have the quantum Itô table:

$\natural \backslash \sharp$	$\bullet$	$+$	$\circ$	$-$
$\bullet$	0	0	0	0
$+$	0	0	0	0
$\circ$	0	$\mathfrak{y}_s^+$	$\mathfrak{y}_s^\circ$	0
$-$	0	$\mathfrak{y}_s^\bullet$	$\mathfrak{y}_s^-$	0

$\cdot$	$d\mathsf{a}^\bullet$	$d\mathsf{a}^+$	$d\mathsf{a}^\circ$	$d\mathsf{a}^-$
$d\mathsf{a}^\bullet$	0	0	0	0
$d\mathsf{a}^+$	0	0	0	0
$d\mathsf{a}^\circ$	0	$d\mathsf{a}^+$	$d\mathsf{a}^\circ$	0
$d\mathsf{a}^-$	0	$d\mathsf{a}^\bullet$	$d\mathsf{a}^-$	0

This is the heart of quantum stochastic calculus. It extends the well-known formal relation "$dB_t^2 = dt$" of classical stochastic analysis. In our nonstandard approach we will get a similar quantum Itô table but with additional four 'infinitesimal' entries. These infinitesimal entries are the second-order correction terms in Meyer's finite toy Fock space.

Next we mention a proposition which is essential for applying the Picard iteration method to show the existence of solutions of quantum stochastic differential equations.

Proposition 1.3.3 (**F**undamental **E**stimation)
Let ψ *be a bounded square integrable function. Then*

$$\left\| \int_0^t \mathsf{k}_s d\mathsf{a}_s^\sharp \pi_\psi \right\|^2 \;\leq\; C(\psi) \int_0^t \left\| \mathsf{k}_s \pi_\psi \right\|^2 ds \tag{1.3.4}$$

where the constant C *depends only on* ψ*.*

This proposition FE ensures the existence of iterated quantum stochastic integration. Necessary for this is the restriction of the exponential vectors to ψs which are bounded. For an infinite time-line the boundedness condition is normally substituted by local boundedness. Then the constant C would depend also on the particular t. On the other hand the fundamental estimation tells us that if an adapted operator process $(\mathsf{k}_s)_{s\in[0,1]}$ is such that for all bounded $\psi \in L^2([0,1])$ one has

$$\int_0^1 \|\mathsf{k}_s \pi_\psi\|^2 ds \;<\; \infty$$

then for $\sharp \in \{\bullet, +, \circ, -\}$ the process $(\int_0^t \mathsf{k}_s d\mathsf{a}_s^\sharp)_{s\in[0,1]}$ is well-defined as an adapted operator process on $\overline{\mathcal{E}}$ given by the first fundamental formula. In this point of view the fundamental estimation together with the first fundamental formula serve as a substitute for the classical Itô isometry. Thus we are encouraged to call the first fundamental formula and the fundamental estimation the first, respectively second, part of the *quantum Itô isometry*.

For quantum stochastic differential equations one introduces an initial Hilbert space $\mathcal{K}$. This is the observed system. The Fock space $\mathcal{F}$ models then the noise space and the quantum stochastic differential equation resides on the space $\mathcal{K} \otimes \mathcal{F}$. Every operator on $\mathcal{F}$ or $\mathcal{K}$ is to be understood as an operator on $\mathcal{K} \otimes \mathcal{F}$ by ampliation. The next theorem proves the existence and uniqueness for quantum stochastic differential equations with constant coefficients.

Theorem 1.3.4 *Let $\ell, \ell^\sharp \in \mathcal{B}(\mathcal{K})$ be given. Then the quantum stochastic differential equation*

$$\mathsf{x}_t \;=\; \ell + \sum_{\sharp\in\{\bullet,+,\circ,-\}} \int_0^t \ell^\sharp \mathsf{x}_s d\mathsf{a}^\sharp, \quad t \in [0,1]$$

has a unique solution x_t and for this solution the following estimation holds:

$$\sup\{\|\mathsf{x}_s(b \otimes \pi_\psi)\| \;:\; \|b\| \le 1,\; 0 \le s \le t\} \;<\; \infty$$

for all $t \in [0,1]$ and every $\psi \in L^2([0,1])$, ψ bounded.

Note that we adopt the convention that the 'coefficients' are to the left of the process. In our nonstandard approach we will also write the differentials as operators that act on the left of the integrated process. Actually in the adapted calculus the left and the right integration is equal. The proof of the theorem is given by Hudson and Parthasarathy [HP84] using a Picard iteration argument and for this argument the fundamental estimation (1.3.4) is used.

Now we will give an interpretation of this quantum stochastic differential equation in the language of the kernel calculus. Then we give an explicit solution.

In terms of the kernel calculus operators k on $\mathcal{K} \otimes \mathcal{F}$ are given by kernel functions $k : \mathcal{P}_{\text{fin}}^3 \to \mathcal{B}(\mathcal{K})$. The action of such an operator on elementary tensors is given by

$$\mathsf{k}(b \otimes f(\sigma)) \;=\; \sum_{\sigma_1 \dot\cup \sigma_2 \dot\cup \sigma_3 = \sigma} \int_{\mathcal{P}_{\text{fin}}} k(\sigma_1, \sigma_2, \beta) b \otimes f(\sigma_2 \cup \sigma_3 \cup \beta) d\beta\,. \qquad (1.3.5)$$

Certainly this definition has meaning only for certain kernel functions on the appropriate domain. An operator k that acts only on the initial Hilbert space has a kernel function k with $k(\sigma, \rho, \tau) = \ell \in \mathcal{B}(\mathcal{K})$ for $\sigma = \rho = \tau = \emptyset$ and zero elsewhere.

Theorem 1.3.5 *As before let* $\ell, \ell^\sharp \in \mathcal{B}(\mathcal{K})$, $\sharp \in \{\bullet, +, \circ, -\}$. *Suppose* $(\mathsf{l}_t)_{t\in[0,1]}$ *be an adapted kernel process with kernel function* $l_t(\emptyset,\emptyset,\emptyset) = \ell$. *We identify the constant kernel processes* $(\mathsf{l}^\sharp_t)_{t\in[0,1]} = \ell^\sharp$ *with its kernel functions*

$$l^\sharp_t(\sigma,\rho,\tau) = \begin{cases} \ell^\sharp & \text{for } \sigma = \rho = \tau = \emptyset\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

Then the corresponding quantum stochastic differential equation for the kernel function is

$$k_t \;=\; l_0 + \sum_\sharp \int_0^t l^\sharp_s k_s \mathsf{d}\mathsf{a}^\sharp_s, \quad t \in [0,1]$$

and has as solution the adapted kernel function

$$k_t(\sigma,\rho,\tau) = \mathrm{e}^{\ell^\bullet(t-t_n)}\Pi(t_n)\mathrm{e}^{\ell^\bullet(t_n-t_{n-1})}\Pi(t_{n-1})\cdots\Pi(t_1)\mathrm{e}^{\ell^\bullet(t_1-t_0)}\ell$$

where $\sigma\cup\rho\cup\tau = \{t_1 < t_2 < \cdots < t_n\} \subset [0,t[$ *and* $t_0 = 0$ *and* Π *is given by*

$$\Pi(t) \;=\; \begin{cases} \ell^+ & \text{if } t \in \sigma\,, \\ \ell^\circ & \text{if } t\in\rho\,, \\ \ell^- & \text{if } t\in\tau\,. \end{cases}$$

Note: *the function* Π *depends actually on the fixed triple* (σ,ρ,τ) *and if convenient we will show this dependence by* $\Pi(t) = \Pi_{\sigma,\rho,\tau}(t)$.

Proof: By the definition of the kernel processes it is enough to look for functions on $\mathcal{P}^3_{\text{fin},\neq}$ since $\mathcal{P}^3_{\text{fin},\neq}$ is a set of full measure in $\mathcal{P}^3_{\text{fin}}$. Thus we suppose the triple (σ,ρ,τ) to be disjoint. Now by proposition 1.2.6 we know how the integration acts on kernel functions. Let $\sigma\cup\rho\cup\tau = \{t_1 < \cdots < t_n\} \subset [0,t[$.

$$\begin{aligned} \int_0^t l^\bullet_s k_s \mathsf{d}\mathsf{a}^\bullet(\sigma,\rho,\tau) &= \int_{t_n}^t \ell^\bullet \mathrm{e}^{\ell^\bullet(s-t_n)}\Pi(t_n)\mathrm{e}^{\ell^\bullet(t_n-t_{n-1})}\Pi(t_{n-1})\cdots\Pi(t_1)\mathrm{e}^{\ell^\bullet(t_1-t_0)}\ell ds \\ &= \int_{t_n}^t \ell^\bullet \mathrm{e}^{\ell^\bullet(s-t_n)} ds \cdot \Pi(t_n)\mathrm{e}^{\ell^\bullet(t_n-t_{n-1})}\Pi(t_{n-1})\cdots\Pi(t_1)\mathrm{e}^{\ell^\bullet(t_1-t_0)}\ell \\ &= [\mathrm{e}^{\ell^\bullet(t-t_n)} - 1]\Pi(t_n)\mathrm{e}^{\ell^\bullet(t_n-t_{n-1})}\Pi(t_{n-1})\cdots\Pi(t_1)\mathrm{e}^{\ell^\bullet(t_1-t_0)}\ell\,. \end{aligned} \tag{1.3.6}$$

On the other hand we know by disjointness that $\max(\sigma,\rho,\tau)$ is either $\max\sigma$ or $\max\rho$ or $\max\tau$. To begin with we suppose $\max(\sigma,\rho,\tau) = \max\sigma = t_n$. Then only the creation integral contributes.

$$\begin{aligned} \int_0^t l^+_s k_s \mathsf{d}\mathsf{a}^+(\sigma,\rho,\tau) &= \ell^+ k_{t_n}(\sigma\setminus t_n,\rho,\tau) \\ &= \ell^+\mathrm{e}^{\ell^\bullet(t_n-t_{n-1})}\Pi(t_{n-1})\cdots\Pi(t_1)\mathrm{e}^{\ell^\bullet(t_1-t_0)}\ell\,. \end{aligned}$$

In the other two cases we get a similar result and in general we have

$$\sum_{\sharp\neq\bullet}\int_0^t l^\sharp_s k_s \mathsf{d}\mathsf{a}^\sharp(\sigma,\rho,\tau) \;=\; \Pi_{\sigma,\rho,\tau}(t_n)\mathrm{e}^{\ell^\bullet(t_n-t_{n-1})}\Pi(t_{n-1})\cdots\Pi(t_1)\mathrm{e}^{\ell^\bullet(t_1-t_0)}\ell\,.$$

Combining this equation with equation (1.3.6) we get

$$\begin{aligned}\sum_{\sharp}\int_0^t l_s^{\sharp}k_s\mathsf{d}\mathsf{a}^{\sharp}(\sigma,\rho,\tau) &= [\mathrm{e}^{\ell^\bullet(t-t_n)}-1]\Pi(t_n)\mathrm{e}^{\ell^\bullet(t_n-t_{n-1})}\Pi(t_{n-1})\cdots\Pi(t_1)\mathrm{e}^{\ell^\bullet(t_1-t_0)}\ell \\ &\quad + \Pi_{\sigma,\rho,\tau}(t_n)\mathrm{e}^{\ell^\bullet(t_n-t_{n-1})}\Pi(t_{n-1})\cdots\Pi(t_1)\mathrm{e}^{\ell^\bullet(t_1-t_0)}\ell \\ &= \mathrm{e}^{\ell^\bullet(t-t_n)}\Pi(t_n)\mathrm{e}^{\ell^\bullet(t_n-t_{n-1})}\Pi(t_{n-1})\cdots\Pi(t_1)\mathrm{e}^{\ell^\bullet(t_1-t_0)}\ell \\ &= k_t(\sigma,\rho,\tau)\,.\end{aligned}$$

This accomplishes the proof. □

In our hyperfinite setting used in the next Chapter we will prove an analogous formula for the corresponding internal quantum stochastic differential equation. We use only a (hyper)finite combinatorial argument. In Chapter 4 we connect the two solutions.

We close this Chapter now. For a more complete representation of quantum stochastic calculus we refer the reader to the text books mentioned at the beginning of the chapter.

Chapter 2

Internal Quantum Stochastic Calculus

The ideas of this chapter were inspired by P.A. Meyer's toy fock space [Mey86, Mey87, Mey93a]. The main difference between Meyer and the approach used here is that we construct what is in reality a hyperfinite version of the Guichardet space with the appropriate weighted measure instead of a space of hyperfinite many Bernoulli random variables with normalized counting measure. The problem of taking limits to get the continuous time processes from the discrete ones as it was indicated by Meyer in [Mey86] disappears since we use the right measure on the toy Fock space and the correct definition of the annihilation operator.

In the first section we construct an internal Guichardet space. We prove several properties which are useful in Chapter 3 where the standard Guichardet space is identified as an external subspace of the internal Guichardet space. In the second section we define the fundamental processes, prove an internal quantum Itô formula and study the hyperfinite analogues of Brownian motion and Poisson process. Furthermore, it is seen that there is a duality between the pair annihilator process, time process and the pair creator process, number process. In the third section we introduce internal operators which are in fact kernel operators. So we can exploit the full power of the kernel approach to quantum stochastic calculus, converting also the integrals into hyperfinite sums. In the fourth section we prove internal versions of a quantum Clark-Ocone formula and the quantum martingale representation theorem. Further, we study internal quantum stochastic differential equations. We show that we have a specific nonstandard solution to an equation with nonlinear noise terms if we look at the equation in the interpretation as an equation for internal kernel functions of some process.

1 Internal Guichardet Space

In this section we study in the first subsection the hyperfinite time-line. In the second subsection we construct the internal symmetric measure space over the hyperfinite time-line and study its properties. We introduce the internal Fock space. Then in the last subsection we give probabilistic concepts to get a probabilistic interpretation of the internal Fock space respectively of internal operators on the internal Fock space.

1.1 The Hyperfinite Time-Line T

Let $N \in {}^*\mathbb{N}$ be an unlimited natural number and set $H = N!$. This H is fixed through the whole work. Further let

$$T \;=\; \{0, \frac{1}{H}, \frac{2}{H}, \cdots, \frac{H-1}{H} \approx 1\}$$

be a hyperfinite (bounded) time-line. On T we take as measure algebra $\mathcal{A}$ the internal power set ${}^*\mathcal{P}(T)$ of T and as measure μ the normalized internal counting measure: $\mu(A) = \frac{|A|}{H}$ for every $A \in \mathcal{A}$. It is clear that $(T, \mathcal{A}, \mu)$ is a hyperfinite internal probability space and it is well known that the corresponding Loeb space $(T, L(\mathcal{A}), \mu_L)$ contains an image of the Lebesgue measure space $([0,1], \mathcal{B}, \lambda)$.

For later purposes we introduce the subsets $T^n_{\neq}$ and $T^n_{\not\approx}$ of n-tuples in general and approximately general position

$$T^n_{\neq} = \{(t_1, \cdots, t_n) \in T^n : t_i \neq t_j \;\text{ for }\; i \neq j\},$$
$$T^n_{\not\approx} = \{(t_1, \cdots, t_n) \in T^n : t_i \not\approx t_j \;\text{ for }\; i \neq j\}$$

for $0 < n \leq H$ and for $n = 0$ we define $T^0_{\neq} = T^0_{\not\approx} = T^0 = \{\emptyset\}$ with the measure algebra $\mathcal{A}^0 = \{\{\emptyset\}, \emptyset\}$ and the Dirac measure $\mu^0(\,\cdot\,) = \delta_{\{\emptyset\}}(\,\cdot\,)$.

Taking on T^n the product normalized internal counting measure μ^n and the product measure algebra that comes from $(T, \mathcal{A}, \mu)$ one obtains the following result on $T^n_{\neq}$:

Proposition 2.1.1 *$T^n_{\neq}$ is measurable. For every $n \in \mathbb{N}$ it holds $\mu^n(T^n \setminus T^n_{\neq}) \approx 0$ and for all $n \in {}^*\mathbb{N}, n \leq H$ it is $\frac{1}{n!}\mu^n(T^n \setminus T^n_{\neq}) \approx 0$.*

PROOF: For $n = 0$ and $n = 1$ one has $T^n_{\neq} = T^n$. This means that $T^n \setminus T^n_{\neq} = \emptyset$ and $\mu^0(\emptyset) = \mu^1(\emptyset) = 0$. So let $n \geq 2$. Obviously $T^n_{\neq}$ is measurable since it is internal by Keisler's internal definition principle. Introducing for fixed $i \neq j \in \{1, \cdots, n\}$ the sets

$$G_{ij} = \{(t_1, \cdots, t_n) \in T^n : t_i = t_j\}$$

one has $T^n \setminus T^n_{\neq} = \cup_{i \neq j} G_{ij}$. But each G_{ij} has the same measure as the set $G_i = \{(t_1, \cdots, t_n) \in T^n : t_i = 0\}$. Clearly $\mu^n(G_i) = \frac{H^{k-1}}{H^k} = \frac{1}{H}$. Suppose now $n \in \mathbb{N}$. Since one has $\binom{n}{2}$ different sets G_{ij} one gets

$$\mu^n(T^n \setminus T^n_{\neq}) \;=\; \mu^n(\cup_{i \neq j} G_{ij}) \;\leq\; \sum_{i \neq j} \mu^n(G_{ij}) \;=\; \binom{n}{2}\frac{1}{H} \;=\; \frac{n!}{2!(n-2)!}\frac{1}{H} \approx 0$$

because $\frac{n!}{2!(n-2)!}$ is finite and $\frac{1}{H}$ is infinitesimal. For infinite n we have

$$\frac{1}{n!}\mu^n(T^n \setminus T^n_{\neq}) \;\approx\; \frac{1}{n!} \;\approx\; 0\,. \qquad \Box$$

Corollary 2.1.2 *Let $0 < n \in \mathbb{N}$ and fix some $s \in T$. Define the set T_s^n by*

$$T_s^n = \cup_{k=1}^n \{(s_1, \cdots, s_n) \in T^n : s_k = s\}.$$

Then T_s^n is measurable and $\mu^n(T_s^n) \approx 0$.

PROOF: By Keisler's internal definition principle all sets T_s^n are internal and thus measurable.

$$\begin{aligned} \mu^n(T_s^n) &= \mu^n(\cup_{k=1}^n \{(s_1, \cdots, s_n) \in T^n : s_k = s\}) \\ &\leq \sum_{k=1}^n \mu^n(\{(s_1, \cdots, s_n) \in T^n : s_k = s\}) = \frac{n}{H} \approx 0. \end{aligned}$$

□

Since later we work in the corresponding Loeb space, we prove that also the infinitesimal thickened diagonals have Loeb measure zero.

Proposition 2.1.3 *Let $n \in \mathbb{N}$, $n \geq 2$. For every $\varepsilon \in \mathbb{R}_+$ there exists a $\delta \in \mathbb{R}_+$ such that the set*

$$T_\delta^n = \{(t_1, \cdots, t_n) \in T^n : \exists i, j \in \{1, \cdots, n\} \; (i \neq j \wedge t_j - \delta < t_i < t_j + \delta)\}$$

has measure less than ε.

PROOF: Fix $i \neq j$ and define the sets $G_{ij}^\delta = \{(t_1, \cdots, t_n) \in T^n : t_j - \delta < t_i < t_j + \delta\}$. Further choose $\delta \in [0, 1[\cap \mathbb{Q}$ which means that $\delta = \frac{k}{H}$ for some $k \in {}^*\mathbb{N}, k < H$. We obtain

$$\mu^n(G_{ij}^\delta) = \mu^{n-2}(T^{n-2})\mu^2(\{(t_1, t_2) \in T^2 : t_2 - \delta < t_1 < t_2 + \delta\}.$$

Since $\mu^{n-2}(T^{n-2}) = 1$ and $T_\delta^n \subseteq \cup_{i \neq j} G_{ij}^\delta$ it is sufficient to prove the case $n = 2$. By the choice of $\delta \in \mathbb{Q}$ we get

$$\mu^2(T_\delta^2) = \mu^2(\{(t_1, t_2) \in T^2 : t_2 - \frac{k}{H} < t_1 < t_2 + \frac{k}{H}\}) \leq \frac{H \cdot 2k}{H^2} = \frac{2k}{H} = 2\delta.$$

Thus with $\delta < \frac{\varepsilon}{2}$ we obtain the result. And for general n we may choose $\delta < \frac{\varepsilon}{2 \cdot n!}$ which is certainly sufficient. □

This shows that the set $T_\approx^n = T^n \setminus T_{\not\approx}^n$ has outer measure zero. In other words the set $T_{\not\approx}^n$ of n-tuples in approximately general position has full Loeb measure. Using the same trick as in the proposition we can prove the following corollary.

Corollary 2.1.4 *Let $n \in \mathbb{N}$, $n \geq 1$. Fix some $t \in T$ and define the set*

$$T_{\delta t}^n = \{(t_1, \cdots, t_n) \in T^n : \exists i \in \{1, \cdots, n\} \; (t - \delta < t_i < t + \delta)\}.$$

Then for every $\varepsilon \in \mathbb{R}_+$ there exists a $\delta \in \mathbb{R}_+$ such that $\mu^n(T_{\delta t}^n) < \varepsilon$.

PROOF: Introducing the sets $G_i^{\delta t} = \{(t_1, \cdots, t_n) \in T^n : t - \delta < t_i < t + \delta\}$ we reduce the case to $n = 1$ by the same argument as in the proposition. Further, for $T_{\delta t}$ a similar argument applies. Choosing again $\delta \in [0, 1[\cap \mathbb{Q}$ we obtain $\mu(T_{\delta t}) < 2\delta$. □

Fix now some $t \in [0, 1]$. Then a nice consequence of this corollary is that the external set

$$T^n_{\mathrm{st}=t} = \{(t_1, \cdots, t_n) \in T^n : \exists i \in \{1, \cdots, n\} \ (\mathrm{st}(t_i) = t)\}$$

has Loeb measure zero. Here st is the usual standard part map.

Based on the internal probability space $(T, \mathcal{A}, \mu)$ we construct in the next subsection the symmetric measure space Γ of T.

1.2 The Symmetric Measure Space over T

Let $(T, \mathcal{A}, \mu)$ be the hyperfinite time-line as introduced before. Now we construct the symmetric measure space Γ of T. Take Γ to be the set of all hyperfinite internal subsets of T. Actually Γ coincides with $\mathcal{A}$ but this is only an effect of the hyperfinite setting. For each $n \leq H$ one has the subset $\Gamma_n = \{A \in \Gamma : |A| = n\}$ of Γ and Γ is the disjoint union of the Γ_n, i.e. $\Gamma = \dot{\cup}_{n \leq H} \Gamma_n$.

For $n \neq 0$ we define the internal homomorphisms

$$\varphi_n \ : \ T^n_{\neq} \longrightarrow \Gamma_n \ : \ (t_1, \cdots, t_n) \longmapsto \{t_1, \cdots, t_n\}$$

and for $n = 0$ we define φ_0 to be the identity from $T^0_{\neq}$ onto Γ_0. Thus for each element $\{t_1, \cdots, t_n\} \in \Gamma_n$ we obtain

$$\varphi_n^{-1}(\{t_1, \cdots, t_n\}) = \{(t_{\pi(1)}, \cdots, t_{\pi(n)}) : \pi \in S(n)\} \subseteq T^n_{\neq} \subseteq T^n$$

whereby $S(n)$ denotes the internal automorphism group of $\{1, \cdots, n\}$. Since Γ is the disjoint union of the Γ_n one can combine these homomorphisms to a single homomorphism φ from $\dot{\cup}_{n \leq H} T^n_{\neq}$ onto Γ.

For the following definition we identify $T^n_{\neq}$ as a subset of T^n via the natural inclusion. In this sense the inverse φ^{-1} of the homomorphism φ is a map from ${}^*\mathcal{P}(\Gamma)$ into $\dot{\cup}_{n \leq H} {}^*\mathcal{P}(T^n)$. (Actually as an inverse φ^{-1} is a map from $\mathcal{P}(\Gamma)$ into $\dot{\cup}_{n \leq H} \mathcal{P}(T^n)$.) If we copy the definition of measurability in the standard case by saying that an internal subset $M \subseteq \Gamma$ is measurable if and only if $\varphi^{-1}(M \cap \Gamma_n)$ is a measurable subset of T^n for each $n \leq H, n \in {}^*\mathbb{N}$, then it turns out that every internal subset $M \subseteq \Gamma$ is measurable. Since Γ_n is internal and the diagonals $T^n_{\neq}$ in T^n are measurable we know that $\varphi^{-1}(M \cap \Gamma_n)$ is internal. Thus we define a measure on ${}^*\mathcal{P}(\Gamma)$.

Definition 2.1.5 *For $M \in \mathfrak{A}$ the measure m of M is defined by*

$$m(M) \ = \ \sum_{n \leq H} \frac{1}{n!} \mu^n(\varphi^{-1}(M \cap \Gamma_n)) \ = \ \sum_{n \leq H} \frac{|M \cap \Gamma_n|}{H^n} \tag{2.1.1}$$

where we denote by $\mathfrak{A} = {}^*\mathcal{P}(\Gamma)$ the algebra of all measurable sets. ◁

From the definition and proposition 2.1.1 it is clear that $m(\Gamma) \approx \mathrm{e}$.

Proposition 2.1.6 *For each standard $\varepsilon > 0$ there exist a standard $l \in \mathbb{N}$ such that $m(\cup_{k\le n\le H}\Gamma_n) < \varepsilon$ for all standard $k \ge l$ in $\mathbb{N}$.*

Proof: For each $n \in {}^*\mathbb{N}$, $n \le H$ it is $m(\Gamma_n) = \frac{1}{n!}\mu^n(T^n_{\neq})$ and since $\mu^n(T^n_{\neq}) \le 1$ it follows $m(\Gamma_n) \le \frac{1}{n!}$. We obtain

$$m(\cup_{k\le n\le H}\Gamma_n) \ \le \ \sum_{k\le n\le H}\frac{1}{n!} \quad \text{for all } \ k \in \mathbb{N}.$$

But the right hand side is less than the tail of the (nonstandard) exponential expansion and tends to zero up to an infinitesimal as k becomes larger in $\mathbb{N}$. Thus choosing a large standard l we are done. □

This shows that the external set $\Gamma_\infty = \Gamma \setminus \cup_{n\in\mathbb{N}}\Gamma_n$ has Loeb measure zero.

Corollary 2.1.7 *Fix some $s \in T$. Define $E_s = \{\sigma \in \Gamma : s \in \sigma\}$. Then E_s is measurable and $m(E_s) \approx 0$.*

Proof: That E_s is measurable is clear since E_s is internal by Keisler's internal definition principle. We obtain

$$m(E_s) \ = \ \sum_{n\le H}\frac{|E_s\cap\Gamma_n|}{H^n} \ = \ \frac{1}{H}\sum_{1\le n\le H}\binom{H}{n-1}H^{-(n-1)} \ \le \ \frac{1}{H} \ \approx \ 0\,. \qquad □$$

Likewise as for T^n the set $T^n_{\neq}$ we define for Γ^n and Γ^n_k the sets $\Gamma^n_{\neq}$ and $\Gamma^n_{k,\neq}$ through

$$\begin{aligned}
\Gamma^n_{\neq} &= \{(\sigma_1,\cdots,\sigma_n) \in \Gamma^n : \sigma_i \cap \sigma_j = \emptyset \text{ for } i \ne j\},\\
\Gamma^n_{k,\neq} &= \{(\sigma_1,\cdots,\sigma_n) \in \Gamma^n_k : \sigma_i \cap \sigma_j = \emptyset \text{ for } i \ne j\}.
\end{aligned}$$

Taking the product measure on Γ^n a similar result as for $T^n \setminus T^n_{\neq}$ holds for $\Gamma^n \setminus \Gamma^n_{\neq}$:

Proposition 2.1.8 *For every standard $n \in \mathbb{N}$ it is $m^n(\Gamma^n \setminus \Gamma^n_{\neq}) \approx 0$.*

Proof: Fix a standard $n \in \mathbb{N}$. Then

$$\Gamma^n \setminus \Gamma^n_{\neq} = \{(\sigma_1,\cdots,\sigma_n) \in \Gamma^n : \sigma_i \cap \sigma_j \ne \emptyset \text{ for some } i \ne j \in \{1,\cdots,n\}\},$$

whence $\Gamma^n \setminus \Gamma^n_{\neq}$ is internal by Keisler's internal definition principle and thus measurable. Since we can choose in $\binom{n}{2}$ ways two pieces out of n pieces and using the product measure one gets

$$\begin{aligned}
m^n(\Gamma^n \setminus \Gamma^n_{\neq}) \ &\le \ \binom{n}{2}m^n(\{(\sigma_1,\cdots,\sigma_n) \in \Gamma^2 \times \Gamma^{n-2} : \sigma_1 \cap \sigma_2 \ne \emptyset\}\\
&= \ \binom{n}{2}m^2(\{(\sigma_1,\sigma_2) \in \Gamma^2 : \sigma_1 \cap \sigma_2 \ne \emptyset\})m^{n-2}(\Gamma^{n-2})\\
&\lessapprox \ \binom{n}{2}m^2(\{(\sigma_1,\sigma_2) \in \Gamma^2 : \sigma_1 \cap \sigma_2 \ne \emptyset\})\mathrm{e}^{n-2}\\
&= \ \binom{n}{2}m^2(\Gamma^2 \setminus \Gamma^2_{\neq})\mathrm{e}^{n-2}. \qquad (2.1.2)
\end{aligned}$$

Now as n is finite so $\binom{n}{2}\mathrm{e}^{n-2}$ is finite and it is sufficient to prove the case for $n=2$. First remark that

$$\begin{aligned} \Gamma^2 \setminus \Gamma^2_{\neq} &= \{(\sigma_1,\sigma_2) \in \Gamma^2 : \sigma_1 \cap \sigma_2 \neq \emptyset\} \\ &= \cup_{k,l\leq H}\{(\sigma_1^k,\sigma_2^l) \in \Gamma_k \times \Gamma_l : \sigma_1^k \cap \sigma_2^l \neq \emptyset\} \\ &= \cup_{k,l\leq H}\{(\{s_1^1,\cdots,s_1^k\},\{s_2^1,\cdots,s_2^l\}) \in \Gamma_k \times \Gamma_l : s_1^i = s_2^j \\ &\qquad \text{for some } \; i \in \{1,\cdots,k\},\, j \in \{1,\cdots,l\}\}. \end{aligned} \tag{2.1.3}$$

Let $k,l>0$. Since $\Gamma_k \times \Gamma_l$ is naturally homomorphic to $T^k_{\neq} \times T^l_{\neq} \subseteq T^{k+l}$ one calculates

$$\begin{aligned} & m^2(\cup_{0<k,l\leq H}\{(\{s_1^1,\cdots,s_1^k\},\{s_2^1,\cdots,s_2^l\}) \in \Gamma_k \times \Gamma_l : s_1^i = s_2^j \;\text{ for some }\; i,j\}) \\ &\leq \sum_{0<k,l\leq H} \frac{1}{k!l!}\mu^k \times \mu^l(\{((s_1^1,\cdots,s_1^k),(s_2^1,\cdots,s_2^l)) \in T^k_{\neq} \times T^l_{\neq} : s_1^i = s_2^j \text{ for some } i,j\}) \\ &= \sum_{0<k,l\leq H} \frac{1}{k!l!}\mu^k \times \mu^l(\{(s_1,\cdots,s_k,s_{k+1},\cdots,s_{k+l}) \in T^{k+l} \cap (T^k_{\neq} \times T^l_{\neq}) : s_i = s_j \\ &\qquad \text{for some } i \in \{1,\cdots,k\}, j \in \{k+1,\cdots,k+l\}\}) \\ &= \sum_{0<k,l\leq H} \frac{1}{k!l!}\binom{k}{1}\binom{l}{1}\mu^k \times \mu^l(\{(s_1,\cdots,s_k,s_{k+1},\cdots,s_{2k}) \in T^{k+l} \cap (T^k_{\neq} \times T^l_{\neq}) : s_1 = s_{k+1}\}) \\ &= \sum_{0<k,l\leq H} \frac{1}{(k-1)!(l-1)!}\mu^k \times \mu^l(\{(s_1,\cdots,s_k,s_{k+1},\cdots,s_{2k}) \in T^{k+l} \cap (T^k_{\neq} \times T^l_{\neq}) : s_1 = 0\}) \\ &= \sum_{0<k,l\leq H} \frac{1}{(k-1)!(l-1)!}\mu^k(\{(s_1,\cdots,s_k) \in T^k_{\neq} : s_1 = 0\}\mu^l(\{(s_{k+1},\cdots,s_{k+l}) \in T^l_{\neq}\}) \\ &\leq \sum_{0<k,l\leq H} \frac{1}{(k-1)!(l-1)!H}\,. \end{aligned}$$

For $k,l=0$ it is $\Gamma_0 = \Gamma_{0,\neq} = \{\emptyset\}$. This means $\Gamma_0 \setminus \Gamma_{0,\neq} = \emptyset$. Thus $m(\Gamma_0^2 \setminus \Gamma^2_{0,\neq}) = 0$ and the product measure vanishes on every product with such a set. So setting $\frac{1}{(-1)!} = 0$ we can extend the sum to $k,l=0$. Together with formulae (2.1.3) and the preceding calculation one gets

$$m^2(\Gamma^2 \setminus \Gamma^2_{\neq}) \;\leq\; \sum_{k,l\leq H} \frac{1}{(k-1)!(l-1)!H} \;\approx\; \frac{\mathrm{e}^2}{H} \;\approx\; 0\,.$$

The general case follows by formula (2.1.2):

$$m^n(\Gamma^n \setminus \Gamma^n_{\neq}) \;\approx\; \binom{n}{2}\frac{\mathrm{e}^n}{H} \;\approx\; 0\,. \qquad \square$$

A more important result is that the set

$$\Gamma_{\not\approx} = \{\sigma \in \Gamma : \forall s \in \sigma \forall t \in \sigma\ (s \neq t \rightarrow s \not\approx t)\}$$

has full Loeb measure. But since this set is external we show that the complement can be approximated by internal sets of arbitrary small measure. The internal sets are given by

$$\Gamma^\delta = \{\sigma \in \Gamma : \exists s \in \sigma \exists t \in \sigma\ (s \neq t \wedge t - \delta < s < t + \delta)\}.$$

Proposition 2.1.9 *For every $\varepsilon \in \mathbb{R}_+$ there exists a $\delta \in \mathbb{R}_+$ such that $m(\Gamma^\delta) < \varepsilon$.*

PROOF: By proposition 2.1.6 choose some $l \in \mathbb{N}$ such that $m(\cup_{l \le n \le H} \Gamma_n) < \frac{\varepsilon}{2}$. Then we get

$$m(\Gamma^\delta) = \sum_{n \le H} m(\Gamma_n \cap \Gamma^\delta) = \sum_{n<l} m(\Gamma_n \cap \Gamma^\delta) + \sum_{l \le n \le H} m(\Gamma_n \cap \Gamma^\delta) \le \sum_{n<l} m(\Gamma_n \cap \Gamma^\delta) + \frac{\varepsilon}{2}.$$

On the other hand applying proposition 2.1.3 with $\frac{\varepsilon}{2l}$ we choose some δ such that

$$\sum_{n<l} m(\Gamma_n \cap \Gamma^\delta) = \sum_{n<l} \frac{1}{n!} \mu^n(T_\delta^n) < \sum_{n<l} \frac{\varepsilon}{2l} \le \frac{\varepsilon}{2}.$$

Combining these together we obtain the result. □

As before in the case of the time-line (cf. proposition 2.1.3, corollary 2.1.4) we have a similar corollary to this proposition. The proof mimics the proof of the proposition but uses corollary 2.1.4 instead of proposition 2.1.3.

Corollary 2.1.10 *Let $t \in T$. Then for every $\varepsilon \in \mathbb{R}_+$ there exists a $\delta \in \mathbb{R}_+$ such that the set*

$$\Gamma^{\delta t} = \{\sigma \in \Gamma : \exists s \in \sigma\ (t - \delta < s < t + \delta)\}$$

has measure less than ε.

Fix now some $t \in [0, 1]$. Then as consequence of the corollary we have that the following external subset of Γ is a Loeb nullset:

$$\Gamma_{\mathrm{st}=t} = \{\sigma \in \Gamma : \exists s \in \sigma\ (\mathrm{st}(s) = t)\}.$$

We define some important external subsets of Γ.

Definition 2.1.11
The set of points in approximately general position*:*

$$\Gamma_{\not\approx} = \{\sigma \in \Gamma : \forall s, t \in \sigma\ (s \neq t \to s \not\approx t)\}.$$

The set of points of finite length *(or just* finite *points):*

$$\Gamma_{\mathrm{fin}} = \{\sigma \in \Gamma : |\sigma| \in \mathbb{N}\} = \cup_{n \in \mathbb{N}} \Gamma_n.$$

The set of points of infinite length *(or just* infinite *points):*

$$\Gamma_\infty = \Gamma \setminus \Gamma_{\mathrm{fin}}.$$

The set of nearstandard points *(the defining index given by some abuse of notation):*

$$\Gamma_{\mathrm{st}} \;=\; \Gamma_{\mathrm{fin}} \cap \Gamma_{\not\approx}\,.$$

For every $t \in [0,1]$ the set of nearstandard points not intersecting the monad of t:

$$\Gamma_{\mathrm{st}\notin\mathrm{t}} \;=\; \Gamma_{\mathrm{st}} \setminus \Gamma_{\mathrm{st}=t} \;=\; \{\sigma \in \Gamma_{\mathrm{st}} : \forall s \in \sigma\ (\mathrm{st}(s) \neq t)\}\,. \qquad \lhd$$

Note that since $\Gamma_{\not\approx}$ and Γ_{fin} are sets of full Loeb measure also Γ_{st} has full Loeb measure. The set Γ_{st} is that external subset on which we will define in chapter 3 an appropriate standard part map to the symmetric measure space $(\mathcal{P}_{\mathrm{fin}}, \mathfrak{B}, \Lambda)$ over the Lebesgue space $([0,1], \mathcal{B}, \lambda)$. Since $\Gamma_{\mathrm{st}=t}$ is a Loeb nullset also $\Gamma_{\mathrm{st}\notin\mathrm{t}}$ has full Loeb measure. This set becomes interesting when looking at (adapted) processes and their standard counterparts.

Notation 2.1.12 We define for $n \in \mathbb{N}$ the following sets

$$\Gamma^{[n]}_{\not\approx} \;=\; \{(\sigma_1, \cdots, \sigma_n) \in \Gamma^n : \sigma_1 \cup \cdots \cup \sigma_n \in \Gamma_{\not\approx}\}\,,$$

$$\Gamma^{[n]}_{\mathrm{st}} \;=\; \Gamma^{[n]}_{\not\approx} \cap \Gamma^n_{\mathrm{fin}} \quad \text{and} \quad \Gamma^{[n]}_{\mathrm{st}\notin\mathrm{t}} \;=\; \Gamma^{[n]}_{\mathrm{st}} \cap \Gamma^n_{\mathrm{st}\notin\mathrm{t}}\,. \qquad \lhd$$

Corollary 2.1.13
$\Gamma^n \setminus \Gamma^{[n]}_{\not\approx}$, $\Gamma^n \setminus \Gamma^{[n]}_{\mathrm{st}}$ *and* $\Gamma^n \setminus \Gamma^{[n]}_{\mathrm{st}\notin\mathrm{t}}$ *have Loeb measure zero.*

PROOF: Looking at propositions 2.1.9, 2.1.6, corollary 2.1.10 and the proof of proposition 2.1.8 this is obvious. □

The most important lemma in standard quantum stochastic calculus is the so-called integral-sum lemma (see lemma 1.1.6). The next proposition is the hyperfinite version of the integral-sum lemma.

Proposition 2.1.14 (hyperfinite $\not\int$-lemma) *Let $F : \Gamma^2_{\neq} \to {}^*\mathbb{C}$ be an internal function and define $G : \Gamma \to {}^*\mathbb{C}$ by $G(\sigma) = \sum_{\alpha\subseteq\sigma} F(\alpha, \sigma\setminus\alpha)$. Then*

$$\sum_{\sigma\in\Gamma} G(\sigma)m(\sigma) \;=\; \sum_{(\alpha,\tau)\in\Gamma^2_{\neq}} F(\alpha,\tau)m(\alpha)m(\tau)\,.$$

PROOF: First note that for $\sigma \cap \tau = \emptyset$ we have

$$m(\sigma)m(\tau) \;=\; \frac{1}{H^{|\sigma|}}\frac{1}{H^{|\tau|}} \;=\; \frac{1}{H^{|\sigma\cup\tau|}} \;=\; m(\sigma\cup\tau)\,.$$

Thus setting $\alpha \cup \tau = \sigma$ or $\tau = \sigma \setminus \alpha$ we get

$$\sum_{(\alpha,\sigma)\in\Gamma^2_{\neq}} F(\alpha,\sigma)m(\alpha)m(\sigma) \;=\; \sum_{\sigma\in\Gamma}\sum_{\alpha\subseteq\sigma} F(\alpha,\sigma\setminus\alpha)m(\alpha)m(\sigma\setminus\alpha) \;=\; \sum_{\sigma\in\Gamma} G(\sigma)m(\sigma)\,. \qquad \square$$

1.3 Tensor Independence and Other Probabilistic Concepts

Over the measure space $(\Gamma, \mathfrak{A}, m)$ one can build the internal function space $\mathcal{F}_{\text{int}} = {}^*L^2_{^*\mathbb{C}}(\Gamma, \mathfrak{A}, m)$ as the space of all internal functions from Γ into ${}^*\mathbb{C}$ equipped with the scalar product $\langle F, G\rangle = \sum_{\sigma\in\Gamma} \overline{F(\sigma)}G(\sigma)m(\sigma)$.

Definition 2.1.15 *The internal function space* $\mathcal{F}_{\text{int}} = {}^*L^2_{^*\mathbb{C}}(\Gamma, \mathfrak{A}, m)$ *is called the* internal Guichardet space *or* internal Fock space. $\lhd$

The internal Fock space is an internal version of the Fock space over $L^2([0,1], \mathcal{B}, \lambda)$. Later, suitable external subspaces of functions that are Loeb integrable will be isolated. A natural property of the internal Guichardet space is important: for every $t \in T$ the space $\mathcal{F}_{\text{int}}$ splits into a part $\mathcal{F}_{\text{int},t)}$ and a part $\mathcal{F}_{\text{int},[t}$ such that $\mathcal{F}_{\text{int}} = \mathcal{F}_{\text{int},t)} \otimes \mathcal{F}_{\text{int},[t}$.

Notation 2.1.16 Let $t \in T$. Then every $\sigma \in \Gamma$ can be divided into two parts:

$$\sigma_{t)} = \{s \in \sigma : s < t\} \quad \text{and} \quad \sigma_{[t} = \{s \in \sigma : s \geq t\}\,.$$

According to this we define for $A \in \mathfrak{A}$

$$\begin{aligned} A_{t)} &= \{\sigma_{t)} : \sigma \in A\}\,, & A_{[t} &= \{\sigma_{[t} : \sigma \in A\}\,,\\ \Gamma_{t)} &= \{\sigma_{t)} : \sigma \in \Gamma\}\,, & \Gamma_{[t} &= \{\sigma_{[t} : \sigma \in \Gamma\}\,,\\ \mathfrak{A}_{t)} &= \{A_{t)} : A \in \mathfrak{A}\}\,, & \mathfrak{A}_{[t} &= \{A_{[t} : A \in \mathfrak{A}\}\,, \end{aligned}$$

and the Fock spaces of all internal functions

$$\mathcal{F}_{\text{int},t)} = {}^*L^2_{^*\mathbb{C}}(\Gamma_{t)}, \mathfrak{A}_{t)}, m) \quad \text{and} \quad \mathcal{F}_{\text{int},[t} = {}^*L^2_{^*\mathbb{C}}(\Gamma_{[t}, \mathfrak{A}_{[t}, m)\,.$$

Note that $\emptyset$ is an element of both $\Gamma_{t)}$ and $\Gamma_{[t}$ for every t. $\lhd$

We identify the spaces $\mathcal{F}_{\text{int},t)}$ and $\mathcal{F}_{\text{int},[t}$ as natural subspaces of $\mathcal{F}_{\text{int}}$ for saying $F \in \mathcal{F}_{\text{int}}$ belongs to $\mathcal{F}_{\text{int},t)}$ if $F(\sigma) = 0$ for $\sigma \notin \Gamma_{t)}$ ($\sigma \not\subseteq [0,t)$) or belongs to $\mathcal{F}_{\text{int},[t}$ if $F(\sigma) = 0$ for $\sigma \notin \Gamma_{[t}$ ($\sigma \not\subseteq [t,1)$).

Definition 2.1.17 *For each subset* $A \in \mathfrak{A}$ *of* Γ *the* characteristic function $\chi_A(\sigma)$ *is defined by*

$$\chi_A(\sigma) \;=\; \begin{cases} 1 & \text{if } \sigma \in A\,,\\ 0 & \text{if } \sigma \notin A\,. \end{cases}$$

Further for elements $\tau \in \Gamma$, *i.e.* $\tau \subseteq T$ *we define the* delta functions $\delta_\tau = \chi_{\{\tau\}}$. *Note that one has for* $A = \emptyset$ *and* $A = \{\emptyset\}$ *two different characteristic functions. If* $A = \{\emptyset\}$ *respectively* $\tau = \emptyset$ *then* $\Omega = \chi_{\{\emptyset\}} = \delta_\emptyset$ *is called the* vacuum vector. $\lhd$

Remark 2.1.18 The δ_τ form an orthogonal system in $\mathcal{F}_{\text{int}}$. Since the internal cardinality of $\{\delta_\tau : \tau \in \Gamma\}$ is the internal dimension of $\mathcal{F}_{\text{int}}$ it is a basis. So each internal function $F \in \mathcal{F}_{\text{int}}$ can be expanded $F = \sum_{\tau\in\Gamma} c_\tau\delta_\tau$. But notice that $\|\delta_\tau\| \approx 0$ unless $\tau = \emptyset$.

We denote by $[s,t]$ (for $s < t \in T$) the set $\{k \in T : s \leq k \leq t\}$. In a similar way the intervals (s,t), $(s,t]$ and $[s,t)$ are defined.

Proposition 2.1.19 *Let $A \in {}^*\mathcal{P}(\Gamma)$. Then A is isomorphic to $A_{t)} \times A_{[t}$ and $\mathcal{F}_{\text{int}}$ is isometrically isomorphic to $\mathcal{F}_{\text{int},t)} \otimes \mathcal{F}_{\text{int},[t}$.*

Proof: Just take the restriction to A of the map

$$\mathrm{d}_t : \Gamma \longrightarrow \Gamma_{t)} \times \Gamma_{[t} : \sigma \longmapsto (\sigma_{t)}, \sigma_{[t})$$

and note that the inverse map is given by

$$\mathrm{d}_t^{-1} : \Gamma_{t)} \times \Gamma_{[t} \longrightarrow \Gamma : (\sigma, \tau) \longmapsto \sigma \dot{\cup} \tau.$$

Of course d_t is injective and surjective and so we have an isomorphism of Γ onto $\Gamma_{t)} \times \Gamma_{[t}$ respectively of A onto $A_{t)} \times A_{[t}$. This proves the first part.
By the previous remark we see that the sets $\{\delta_\tau : \tau \in \Gamma_{t)}\}$ and $\{\delta_\sigma : \sigma \in \Gamma_{[t}\}$ are bases for the spaces $\mathcal{F}_{\text{int},t)}$, respectively $\mathcal{F}_{\text{int},[t}$. Thus $\{\delta_\tau \otimes \delta_\sigma : \tau \in \Gamma_{t)}, \sigma \in \Gamma_{[t}\}$ is a basis for $\mathcal{F}_{\text{int},t)} \otimes \mathcal{F}_{\text{int},[t}$ and every $D \in \mathcal{F}_{\text{int},t)} \otimes \mathcal{F}_{\text{int},[t}$ has an expansion $D = \sum_{\tau \in \Gamma_{t)}, \sigma \in \Gamma_{[t}} d_{(\tau,\sigma)} \delta_\tau \otimes \delta_\sigma$.
Suppose that $K = \sum_{\sigma \in \Gamma} k_\sigma \delta_\sigma$ is the expansion of $K \in \mathcal{F}_{\text{int}}$. We define the map

$$\Phi_t : \mathcal{F}_{\text{int},t)} \otimes \mathcal{F}_{\text{int},[t} \longrightarrow \mathcal{F}_{\text{int}} : D \longmapsto K$$

where $K(\sigma) = k_\sigma = d_{(\sigma_{t)}, \sigma_{[t})} = D(\sigma_{t)}, \sigma_{[t})$. Clearly this map is bijective. The remainder follows from

$$\|\delta_\sigma\|^2 = \frac{1}{H^{|\sigma|}} = \frac{1}{H^{|\sigma_{t)}|} H^{|\sigma_{[t}|}} = \|\delta_{\sigma_{t)}} \otimes \delta_{\sigma_{[t}}\|^2 .$$

Thus the map Φ_t is isometric and gives a Hilbert space isomorphism from $\mathcal{F}_{\text{int},t)} \otimes \mathcal{F}_{\text{int},[t}$ onto $\mathcal{F}_{\text{int}}$. □

Remark 2.1.20 (tensor factorization property)
Actually, a more sophisticated reasoning shows that $\mathcal{F}_{\text{int}} \equiv \otimes_{t \in T} \mathcal{F}_{\text{int},t}$ where $\mathcal{F}_{\text{int},t}$ is the internal Fock space at the single time instant $t \in T$. Correspondingly, the vacuum vector factorizes in a hyperfinite tensor product: $\Omega \equiv \otimes_{t \in T} \Omega_t$ with $\Omega_t = \delta_\emptyset$ for $\emptyset \in \Gamma_t = \Gamma_{t]} \cap \Gamma_{[t}$.
For the internal Banach space ${}^*\mathcal{B}(\mathcal{F}_{\text{int}})$ we have also ${}^*\mathcal{B}(\mathcal{F}_{\text{int}}) \equiv \otimes_{t \in T} {}^*\mathcal{B}(\mathcal{F}_{\text{int},t})$.

Besides Ω_t the space $\mathcal{F}_{\text{int},t}$ only contains functions F with $F(\sigma_{t)}) = 0$ and $F(\sigma_{(t}) = 0$.
Indeed, $\mathcal{F}_{\text{int},t}$ is the ${}^*\mathbb{C}$-linear span of the two elements Ω_t and $\delta_{\{t\}}$. Thus $\mathcal{F}_{\text{int},t}$ is isomorphic to ${}^*\mathbb{C}^2$ for every $t \in T$. We introduce the notation $\delta^\tau = \otimes_{t \in T} \delta_t^\tau$ where $\delta_t^\tau = \delta_{\{t\}}$ if $t \in \tau$ and $\delta_t^\tau = \Omega_t$ if $t \notin \tau$. Then the isomorphism between $\mathcal{F}_{\text{int}}$ and $\otimes_{t \in T} \mathcal{F}_{\text{int},t}$ is described as $\delta_\tau \stackrel{\equiv}{\longleftrightarrow} \delta^\tau$ since $\{\delta^\tau : \tau \in \Gamma\}$ is an orthogonal base in $\otimes_{t \in T} \mathcal{F}_{\text{int},t}$. Obviously, we have $\mathcal{F}_{\text{int}} \equiv \otimes_{t \in T} {}^*\mathbb{C}^2$ and also ${}^*\mathcal{B}(\mathcal{F}_{\text{int}}) \equiv \otimes_{t \in T} {}^*\mathcal{B}({}^*\mathbb{C}^2)$.

Definition 2.1.21 *For each internal function $\Psi : T \to {}^*\mathbb{C}$ we define the function $\pi_\Psi(\sigma)$ by $\pi_\Psi(\sigma) = \prod_{s \in \sigma} \Psi(s)$. For $\sigma = \emptyset$ the right hand side as an empty product is defined to be* 1. *The function π_Ψ is called the* coherent/exponential vector *corresponding to Ψ.* ◁

Proposition 2.1.22 *Let $\Psi \equiv 0$. Then $\pi_\Psi = \Omega$.*

PROOF: By definition $\pi_\Psi(\emptyset) = 1$ for every Ψ. And since Ψ is constant zero $\pi_\Psi(\sigma) = \prod_{s\in\sigma} \Psi(s) = 0$ for every $\sigma \neq \emptyset$. Thus $\pi_\Psi = \delta_\emptyset = \Omega$. □

In Chapter 3 Section 2.1 we need a little more. Namely that every element δ_τ can be expressed by a linear combination of coherent vectors.

Proposition 2.1.23 *Let $\rho \subset T$ be internal and write π_ρ for the coherent vector of the characteristic function of ρ. Suppose that $\tau = \{t_1 < \cdots < t_n\}$. Then*

$$\delta_\tau \quad = \quad \sum_{k=0}^{n} (-1)^k \sum_{1\le i_1<\cdots<i_k\le n} \pi_{\tau\setminus\{t_{i_1},\cdots,t_{i_k}\}} \tag{2.1.4}$$

where we set $\pi_\emptyset = \pi_0 = \delta_\emptyset$ as in the previous proposition.

PROOF: By the previous proposition we can leave out $\tau = \emptyset$. So we assume τ to be non-void. First note that

$$\pi_\rho(\sigma) \quad = \quad \begin{cases} 1 & \text{if } \sigma \subseteq \rho\,, \\ 0 & \text{if } \sigma \not\subseteq \rho\,. \end{cases}$$

With this in mind we see immediately that for $\sigma \not\subseteq \tau$ equation (2.1.4) is fulfilled since all terms are equal to zero. Suppose $\sigma = \emptyset$. Then $\delta_\tau(\emptyset) = 0$. But $\pi_{\tau\setminus\{t_{i_1},\cdots,t_{i_k}\}}(\emptyset) = 1$ for every choice of $\{t_{i_1}, \cdots, t_{i_k}\}$ out of τ. We get

$$\sum_{k=0}^{n} (-1)^k \sum_{1\le i_1<\cdots<i_k\le n} \pi_{\tau\setminus\{t_{i_1},\cdots,t_{i_k}\}} \quad = \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \quad = \quad 0\,. \tag{2.1.5}$$

Now suppose $\sigma \neq \tau$ but $\sigma = \{s_1, \cdots, s_m\} \subset \{t_1, \cdots, t_n\} = \tau$. Then

$$\pi_{\tau\setminus\{t_{i_1},\cdots,t_{i_k}\}}(\sigma) = 0 \quad \text{if} \quad \{t_{i_1}, \cdots, t_{i_k}\} \cap \sigma \neq \emptyset\,. \tag{2.1.6}$$

Thus a similar calculation as (2.1.5) shows that the right hand side of equation (2.1.4) is zero and equals the left hand side. For $\sigma = \tau$ we have $\delta_\tau(\tau) = 1 = \pi_\tau(\tau)$ and obviously all other terms in the sum are zero by equation (2.1.6). □

Using the vacuum vector we transfer the classical probabilistic concepts to the internal Fock space. Note that here "random variables" are operators on Fock space. A "stochastic process" is then just a family of operators indexed with the time-line. As a matter of fact, we later make the restriction to internal operators, since these operators behave well.

Definition 2.1.24 *For every (internal) operator X on $\mathcal{F}_{\text{int}} = {}^*L^2_{{}^*\mathbb{C}}(\Gamma, \mathfrak{A}, m)$ the* expectation in the vacuum state *or shortly* expectation $\mathbb{E}(\,\cdot\,)$ *is defined by*

$$\mathsf{X} \longmapsto \mathbb{E}(\mathsf{X}) \quad = \quad \langle \Omega, \mathsf{X}\Omega \rangle.$$

The variance $\mathbb{V}(\mathsf{X})$ *of* X *is defined by* $\mathbb{V}(\mathsf{X}) = \mathbb{E}((\mathsf{X} - \mathbb{E}(\mathsf{X}))^2)$. ◁

Next we define the quadratic variation of a process.

Definition 2.1.25 *The* quadratic variation process *or for short* quadratic variation $[\mathsf{X}]_t$ *of a process* X_t *is defined by the equation*

$$[\mathsf{X}]_t \quad = \quad \int_0^t d\mathsf{X}_s \cdot d\mathsf{X}_s \quad = \quad \sum_{s<t} (\mathsf{X}_{s+\frac{1}{H}} - \mathsf{X}_s)^2. \qquad \triangleleft$$

The following two definitions give the concept of the characteristic distribution of a process and the concept of the stationarity of its increments.

Definition 2.1.26
The characteristic distribution *of a process* X_t *is the function* $\varphi(y) = \mathbb{E}(\mathrm{e}^{\mathrm{i}y\mathsf{X}_t})$. $\triangleleft$

Definition 2.1.27 *An operator process* X_t *has* stationary *increments if for all* $s < t \in T$ *and all* $h \in T$ *such that* $t + h \in T$ *one has*

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}y(\mathsf{X}_t - \mathsf{X}_s)}) \quad = \quad \mathbb{E}(\mathrm{e}^{\mathrm{i}y(\mathsf{X}_{t+h} - \mathsf{X}_{s+h})}) \, . \qquad \triangleleft$$

Now we define vector processes in the internal Guichardet Space and adaptedness.

Definition 2.1.28
A vector process *is an internal family of vectors* $\{F_t\}_{t\in T}$ *in* $\mathcal{F}_{\mathrm{int}}$ *indexed by* T. *A vector process is said to be* adapted *iff* $(\tau \not\subseteq [0,t) \Rightarrow F_t(\tau) = 0)$. *This means* $F_t \in \mathcal{F}_{\mathrm{int},t)}$. $\triangleleft$

Before we close this section we want to discuss the fundamental concept of independence in the internal setting. In remark 2.1.20 we have seen that the internal Fock space, and in particular the vacuum vector, factorize in a hyperfinite tensor product. This factorization property gives rise to the following definition.

Definition 2.1.29 *Let* I *be an internal set and* $(\mathcal{B}_i)_{i\in I}$ *a family of internal subalgebras of* ${}^*\mathcal{B}(\mathcal{F}_{\mathrm{int}})$. *We say that this family is* tensor independent, *for short* independent, *if*

(1) *the algebras* $\mathcal{B}_i$ *are pairwise commuting (that is, if* $i \neq j$ *then* $[b_i, b_j] = 0$ *for all* $b_i \in \mathcal{B}_i$, $b_j \in \mathcal{B}_j$*),*

(2) *for each hyperfinite internal subset* $J \subseteq I$ *and all* $b_j \in \mathcal{B}_j$, $j \in J$ *it holds*

$$\mathbb{E}(\prod_{j\in J} b_j) \quad = \quad \prod_{j\in J} \mathbb{E}(b_j), \quad \textit{that is,} \quad \langle \Omega, \prod_{j\in J} b_j \Omega \rangle \quad = \quad \prod_{j\in J} \langle \Omega, b_j \Omega \rangle \, .$$

A family $(\mathsf{X}_i)_{i\in I}$ *of internal operators is* independent *if the corresponding internal* $W^\star$*-algebras* $\mathcal{W}^\star(\mathsf{X}_i)$ *generated by* X_i *are independent.* $\triangleleft$

Note that this notion of independence is in fact the *-transfered notion of *Boson independence* of standard quantum stochastic calculus (cf. Schürmann [Sch93, Section 1.3]). There are several other concepts of independence in non-commutative probability (see for example [Sch95, Spe97, Len98, Seo97]). The name tensor independence is suggested by the following fact. Take $I = T$, the hyperfinite time-line, and assume that the family $(\mathcal{A}_t)_{t\in T}$ of internal subalgebras of $^*\mathcal{B}(\mathcal{F}_{\text{int}})$ commutes pairwise. Furthermore, assume that for each $t \in T$ every $a_t \in \mathcal{A}_t$ is of the form

$$a_t \;=\; \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \widehat{a}_t \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \quad \text{for some} \quad \widehat{a}_t \in {}^*\mathcal{B}(\mathcal{F}_{\text{int},t}) \tag{2.1.7}$$

where $\widehat{a}_t$ is in the t th place of the tensor product. Then we have the following proposition.

Proposition 2.1.30
Let $(\mathcal{A}_t)_{t\in T}$ be as above described. Then $(\mathcal{A}_t)_{t\in T}$ is tensor independent.

Proof: We only need to prove property (2) of definition 2.1.29. Take $t_1, \ldots, t_k \in T$, $k \in {}^*\mathbb{N}$, $k \leq H$ and $a_{t_i} \in \mathcal{A}_{t_i}$, $i = 1, \ldots, k$. Further, set $\overline{a}_t = \widehat{a}_{t_i}$ if $t = t_i$ and $\overline{a}_t = \mathbb{1}$ otherwise. Then using the factorization property of the vacuum vector we obtain:

$$\begin{aligned}
\langle \Omega, \prod_{i=1}^{k} a_{t_i}\Omega\rangle &= \langle \Omega, \prod_{i=1}^{k}(\mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \widehat{a}_{t_i} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1})\Omega\rangle \\
&= \langle \Omega, \otimes_{t\in T}\overline{a}_t\Omega\rangle \;=\; \langle \otimes_{t\in T}\Omega_t, (\otimes_{t\in T}\overline{a}_t)(\otimes_{t\in T}\Omega_t)\rangle \\
&= \prod_{t\in T}\langle \Omega_t, \overline{a}_t\Omega_t\rangle \;=\; \prod_{i=1}^{k}\langle \Omega_{t_i}, \widehat{a}_{t_i}\Omega_{t_i}\rangle\,.
\end{aligned}$$

On the other hand we have

$$\prod_{i=1}^{k}\langle \Omega, a_{t_i}\Omega\rangle \;=\; \prod_{i=1}^{k}\langle \otimes_{t\in T}\Omega_t, (\mathbb{1} \otimes \cdots \otimes \widehat{a}_{t_i} \otimes \cdots \otimes \mathbb{1})(\otimes_{t\in T}\Omega_t)\rangle \;=\; \prod_{i=1}^{k}\langle \Omega_{t_i}, \widehat{a}_{t_i}\Omega_{t_i}\rangle\,. \quad \square$$

By the previous proposition we see that internal operators which act non-trivially on different parts in the tensor factorization, are tensor independent. Thus the tensor factorization of the internal Fock space and its internal operators provides us with the fundamental concept of independence. In the next section we define time, creation, number and annihilation operators and the corresponding fundamental processes on the internal Guichardet space. It turns out that these fundamental processes have independent increments.

2 The Fundamental Quantum Processes and the Classical Processes

In the first subsection of this section we define the fundamental internal quantum processes. These become our integrators. We prove an internal quantum Itô formula. Furthermore, we give a realization of the increment processes such that the independence of the increments is immediately evident. In the second and third subsection we study the internal analogues of operator-valued Brownian motion and Poisson process on Fock space.

2.1 Internal Time, Creation, Number and Annihilation Process, an Internal Itô Formula, and Independence in Time

In the following discussion the functions F always are elements of the internal Guichardet space $\mathcal{F}_{\text{int}} = {}^*L^2_{*\mathbb{C}}(\Gamma, \mathfrak{A}, m)$ and s or t are elements of the time-line T. If $\sigma \in \Gamma$ and $s \in T$ we use $\sigma \cup s$ as shorthand notation for $\sigma \cup \{s\}$ and $\sigma \setminus s$ for $\sigma \setminus \{s\}$.

Definition 2.2.1 *Let $s \in T$ be fixed. We define the* creators a_s^+ *and the* annihilators a_s^- *as operators on $\mathcal{F}_{\text{int}}$. For $F \in \mathcal{F}_{\text{int}}$ the action of a_s^+ and a_s^- on F is defined by*

$$a_s^+ F(\sigma) = \begin{cases} 0 & \textit{for } s \notin \sigma, \\ F(\sigma \setminus s) & \textit{for } s \in \sigma, \end{cases} \qquad a_s^- F(\sigma) = \begin{cases} \frac{1}{H} F(\sigma \cup s) & \textit{for } s \notin \sigma, \\ 0 & \textit{for } s \in \sigma. \end{cases}$$

The number operators a_s° *and the* time operators $a_s^\bullet$ *are defined by*

$$a_s^\circ F(\sigma) = H a_s^+ a_s^- F(\sigma) = \begin{cases} 0 & \textit{for } s \notin \sigma, \\ F(\sigma) & \textit{for } s \in \sigma, \end{cases}$$

$$a_s^\bullet F(\sigma) = a_s^- a_s^+ F(\sigma) = \begin{cases} \frac{1}{H} F(\sigma) & \textit{for } s \notin \sigma, \\ 0 & \textit{for } s \in \sigma. \end{cases} \qquad \triangleleft$$

These operators are our fundamental operators defined for each element of the time-line T. For some calculations it is more convenient to introduce another notation.

Notation 2.2.2 For fixed $s \in T$ we introduce two functions $\in_s$ and $\notin_s$ on Γ:

$$\in_s(\sigma) = \begin{cases} 0 & \text{for } s \notin \sigma, \\ 1 & \text{for } s \in \sigma, \end{cases} \qquad \notin_s(\sigma) = \begin{cases} 1 & \text{for } s \notin \sigma, \\ 0 & \text{for } s \in \sigma, \end{cases}$$

as somehow special characteristic functions on Γ that distinguish for every $\sigma \in \Gamma$ whether s is in σ or not. $\triangleleft$

So the fundamental operators can be written as

$$a_s^\bullet F(\sigma) = \frac{1}{H} F(\sigma) \notin_s(\sigma), \qquad a_s^+ F(\sigma) = F(\sigma \setminus s) \in_s(\sigma),$$
$$a_s^\circ F(\sigma) = F(\sigma) \in_s(\sigma), \qquad a_s^- F(\sigma) = \frac{1}{H} F(\sigma \cup s) \notin_s(\sigma).$$

Proposition 2.2.3
The operators $a_s^\bullet$ and a_s° are self-adjoint and the adjoint of a_s^+ is a_s^- and vice-versa.

Proof: For a_s° we have

$$\langle G, a_s^\circ F \rangle = \sum_{\sigma \in \Gamma} \overline{G(\sigma)} F(\sigma) \in_s(\sigma) m(\sigma) = \sum_{\sigma \in \Gamma} \overline{G(\sigma) \in_s(\sigma)} F(\sigma) m(\sigma) = \langle a_s^\circ G, F \rangle .$$

The same calculation with $\frac{1}{H}\not\in_s(\sigma)$ in the place of $\in_s(\sigma)$ shows the result for $a_s^\bullet$.

$$\begin{aligned}
\langle a_s^+ G, F\rangle &= \sum_{\sigma\in\Gamma} \overline{G(\sigma\setminus s)\in_s(\sigma)}F(\sigma)m(\sigma) &=& \sum_{\substack{\sigma\in\Gamma\\ s\in\sigma}} \overline{G(\sigma\setminus s)}F(\sigma)m(\sigma)\\
&= \sum_{\substack{\sigma\in\Gamma\\ s\notin\sigma}} \overline{G(\sigma)}F(\sigma\cup s)m(\sigma\cup s) &=& \sum_{\sigma\in\Gamma} \overline{G(\sigma)}F(\sigma\cup s)\not\in_s(\sigma)m(s)m(\sigma)\\
&= \sum_{\sigma\in\Gamma} \overline{G(\sigma)}F(\sigma\cup s)\not\in_s(\sigma)\frac{1}{H}m(\sigma) &=& \langle G, a_s^- F\rangle\,.
\end{aligned}$$

This proves the adjointness of a_s^+ and a_s^-. □

The processes corresponding to the fundamental operators are easily defined as the sums of the 'infinitesimal' operators. Summation is always over the time-line and for convention we use the symbol $\sharp$ as fill-in for the symbols $\bullet, +, \circ, -$.

Definition 2.2.4 *Let* $\sharp \in \{\bullet, +, \circ, -\}$. *The* time, creation, number *and* annihilation process *are defined according to the following scheme:*

$$A_t^\sharp = \sum_{s<t} a_s^\sharp, \qquad t\in T\,. \qquad \lhd$$

Note that the endpoint t is not contained in the sum. The explicit action of these processes on an element $F\in\mathcal{F}_{\text{int}}$ is described by the formulas

$$\begin{aligned}
A_t^\bullet F(\sigma) &= \sum_{s<t}\frac{1}{H}F(\sigma)\not\in_s(\sigma), & A_t^+ F(\sigma) &= \sum_{s<t} F(\sigma\setminus s)\in_s(\sigma),\\
A_t^\circ F(\sigma) &= \sum_{s<t} F(\sigma)\in_s(\sigma), & A_t^- F(\sigma) &= \sum_{s<t}\frac{1}{H}F(\sigma\cup s)\not\in_s(\sigma)\,.
\end{aligned}$$

Because we want to develop an integration theory against these processes we define the integrators $dA_t^\sharp$ to be the *forward increment*:

Definition 2.2.5 *The* integrator $dA_t^\sharp$ *of the process* $A_t^\sharp$ *is the operator* $a_t^\sharp$ *according to the defining scheme for the forward increment:*

$$dA_t^\sharp = A_{t+\frac{1}{H}}^\sharp - A_t^\sharp, \qquad t\in T\setminus\{\frac{H-1}{H}\}. \qquad \lhd$$

Now we establish an internal version of the famous quantum Itô table. It will be seen that there are an additional four non-zero but infinitesimal entries. This shows the duality of the pair (creator, number operator) to the pair (annihilator, time operator) in the discrete time-line. Thus we calculate the 16 multiplications for a single time-element $s\in T$. For this let $F\in\mathcal{F}_{\text{int}}$ be an arbitrary function.

First note that $(\in_s)^2 = \in_s$ and $(\notin_s)^2 = \notin_s$. This shows immediately

$$\begin{aligned}
a_s^\circ a_s^\circ F(\sigma) &= F(\sigma)(\in_s(\sigma))^2 = a_s^\circ F(\sigma),\\
a_s^\bullet a_s^\bullet F(\sigma) &= (\frac{1}{H})^2 F(\sigma)(\notin_s(\sigma))^2 = \frac{1}{H} a_s^\bullet F(\sigma),\\
a_s^\circ a_s^+ F(\sigma) = a_s^\circ F(\sigma\setminus s)\in_s(\sigma) &= F(\sigma\setminus s)(\in_s(\sigma))^2 = a_s^+ F(\sigma),\\
a_s^\bullet a_s^- F(\sigma) = a_s^\bullet \frac{1}{H} F(\sigma\cup s)\notin_s(\sigma) &= (\frac{1}{H})^2 F(\sigma\cup s)(\notin_s(\sigma))^2 = \frac{1}{H} a_s^- F(\sigma)\,.
\end{aligned}$$

Next we observe that $\in_s(\sigma\setminus s) = 0$ and $\notin_s(\sigma\cup s) = 0$. Thus we have

$$\begin{aligned}
a_s^+ a_s^+ F(\sigma) = a_s^+ F(\sigma\setminus s)\in_s(\sigma) &= F((\sigma\setminus s)\setminus s)\in_s(\sigma\setminus s)\in_s(\sigma) = 0,\\
a_s^- a_s^- F(\sigma) = a_s^+ \frac{1}{H} F(\sigma\cup s)\notin_s(\sigma) &= (\frac{1}{H})^2 F((\sigma\cup s)\cup s)\notin_s(\sigma\cup s)\notin_s(\sigma) = 0,\\
a_s^- a_s^\bullet F(\sigma) = a_s^- \frac{1}{H} F(\sigma)\notin_s(\sigma) &= (\frac{1}{H})^2 F(\sigma\cup s)\notin_s(\sigma\cup s)\notin_s(\sigma) = 0,\\
a_s^+ a_s^\circ F(\sigma) = a_s^+ F(\sigma)\in_s(\sigma) &= F(\sigma\setminus s)\in_s(\sigma\setminus s)\in_s(\sigma) = 0\,.
\end{aligned}$$

Further, for all $\sigma \in \Gamma$ it is $\in_s(\sigma)\notin_s(\sigma) = \notin_s(\sigma)\in_s(\sigma) = 0$ and this shows

$$\begin{aligned}
a_s^\circ a_s^- F(\sigma) = a_s^\circ \frac{1}{H} F(\sigma\cup s)\notin_s(\sigma) &= \frac{1}{H} F(\sigma\cup s)\notin_s(\sigma)\in_s(\sigma) = 0,\\
a_s^\circ a_s^\bullet F(\sigma) = a_s^\circ \frac{1}{H} F(\sigma)\notin_s(\sigma) &= \frac{1}{H} F(\sigma)\notin_s(\sigma)\in_s(\sigma) = 0,\\
a_s^\bullet a_s^+ F(\sigma) = a_s^\bullet F(\sigma\setminus s)\in_s(\sigma) &= \frac{1}{H} F(\sigma\setminus s)\in_s(\sigma)\notin_s(\sigma) = 0,\\
a_s^\bullet a_s^\circ F(\sigma) = a^\bullet F(\sigma)\in_s(\sigma) &= \frac{1}{H} F(\sigma)\in_s(\sigma)\notin_s(\sigma) = 0\,.
\end{aligned}$$

By definition it is $a_a^- a_s^+ = a_s^\bullet$ and $a_s^+ a_s^- = \frac{1}{H} a_s^\circ$. As for all $\sigma \in \Gamma$ it is $\in_s(\sigma\cup s) = 1$ and $\notin_s(\sigma\setminus s) = 1$ we have

$$\begin{aligned}
a_s^- a_s^\circ F(\sigma) = a_s^- F(\sigma)\in_s(\sigma) &= \frac{1}{H} F(\sigma\cup s)\in_s(\sigma\cup s)\notin_s(\sigma) = a_s^- F(\sigma),\\
a_s^+ a_s^\bullet F(\sigma) = a_s^+ \frac{1}{H} F(\sigma)\notin_s(\sigma) &= \frac{1}{H} F(\sigma\setminus s)\notin_s(\sigma\setminus s)\in_s(\sigma) = \frac{1}{H} a_s^+ F(\sigma)\,.
\end{aligned}$$

Finally, all possible multiplications are summarized in the following table

$\cdot$	$a_s^\bullet$	a_s^+	a_s°	a_s^-
$a_s^\bullet$	$\frac{1}{H} a_s^\bullet$	0	0	$\frac{1}{H} a_s^-$
a_s^+	$\frac{1}{H} a_s^+$	0	0	$\frac{1}{H} a_s^\circ$
a_s°	0	a_s^+	a_s°	0
a_s^-	0	$a_s^\bullet$	a_s^-	0

(2.2.8)

Comparing this with the standard quantum Itô table one sees that there are in addition the four entries $\frac{1}{H} a_s^\sharp$, $\sharp \in \{\bullet, +, \circ, -\}$. But as H is an infinite natural number these entries do not count while going back to the standard world and we obtain the usual quantum Itô table.

The commutators are easily seen to be of the following form:

$$\begin{aligned}
[a_s^\sharp, a_t^\natural] &= 0 \ \text{for} \ s \neq t, \ \sharp, \natural \in \{\bullet, +, \circ, -\}, \\
[a_s^-, a_s^+] &= a_s^\bullet - \frac{1}{H} a_s^\circ, \\
[a_s^\circ, a_s^+] &= a_s^+, \qquad [a_s^+, a_s^\bullet] = \frac{1}{H} a_s^+, \\
[a_s^-, a_s^\circ] &= a_s^-, \qquad [a_s^\bullet, a_s^-] = \frac{1}{H} a_s^-,
\end{aligned}$$

and all other commutators are zero. Up to the terms with infinitesimal amount these are again the usual commutators. For the operator processes this results in the commutation relations using ordinary calculations with finite sums:

$$\begin{aligned}
[A_s^-, A_t^+] &= A_{s\wedge t}^\bullet - \frac{1}{H} A_{s\wedge t}^\circ, \\
[A_s^\circ, A_t^+] &= A_{s\wedge t}^+, \qquad [A_s^+, A_t^\bullet] = \frac{1}{H} A_{s\wedge t}^+, \\
[A_s^-, A_t^\circ] &= A_{s\wedge t}^-, \qquad [A_s^\bullet, A_t^-] = \frac{1}{H} A_{s\wedge t}^-,
\end{aligned}$$

whereby $s \wedge t$ denotes the minimum of s and t.

Let $\mathcal{A}_t$ be the internal $W^\star$-algebra generated by the operators $a_t^\bullet$, a_t^+, a_t° and a_t^-. Now we will show that the family $(\mathcal{A}_t)_{t\in T}$ is independent in the sense of definition 2.1.29. By proposition 2.1.30 we only need to prove that each $a_t^\sharp$ is of the form

$$a_t^\sharp = 1\!\!1 \otimes \cdots \otimes 1\!\!1 \otimes \widehat{a}_t^\sharp \otimes 1\!\!1 \otimes \cdots \otimes 1\!\!1 \tag{2.2.9}$$

where $\widehat{a}_t$ is in the tth place of the tensor product. We will do this using the isomorphic spaces $\mathcal{F}_{\text{int}}$, $\otimes_{t\in T}\mathcal{F}_{\text{int},t}$ and $\otimes_{t\in T}{}^*\mathbb{C}^2$. First, we construct the isomorphism $\mathcal{F}_{\text{int},t} \equiv {}^*\mathbb{C}^2$ explicitly by

$$\Omega_t \overset{\equiv}{\longleftrightarrow} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \delta_{\{t\}} \overset{\equiv}{\longleftrightarrow} \begin{pmatrix} \frac{1}{\sqrt{H}} \\ 0 \end{pmatrix}.$$

We define the following matrices on ${}^*\mathbb{C}^2$:

$$\widehat{c}_t^+ = \begin{pmatrix} 0 & \frac{1}{\sqrt{H}} \\ 0 & 0 \end{pmatrix}, \quad \widehat{c}_t^- = \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{H}} & 0 \end{pmatrix}, \quad \widehat{c}_t^\circ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \widehat{c}_t^\bullet = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{H} \end{pmatrix}$$

and use the same symbol for the corresponding operator on $\mathcal{F}_{\text{int},t}$. Then we have

$$\begin{aligned}
&\widehat{c}_t^+\Omega_t = \delta_{\{t\}}, \quad \widehat{c}_t^+\delta_{\{t\}} = 0, \quad \widehat{c}_t^-\Omega_t = 0, \quad \widehat{c}_t^-\delta_{\{t\}} = \frac{1}{H}\Omega_t, \\
&\widehat{c}_t^\circ\Omega_t = 0, \quad \widehat{c}_t^\circ\delta_{\{t\}} = \delta_{\{t\}}, \quad \widehat{c}_t^\bullet\Omega_t = \frac{1}{H}\Omega_t, \quad \widehat{c}_t^\bullet\delta_{\{t\}} = 0\,.
\end{aligned}$$

We define $c_t^\sharp = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \widehat{c}_t^\sharp \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$ where $\widehat{c}_t^\sharp$ is at place t in the hyperfinite tensor product. Then the operators $c_t^\sharp$ act on an element δ^τ of the base of $\otimes_{t\in T}\mathcal{F}_{\mathrm{int},t}$ in the following way:

$$\begin{aligned} c_t^+\delta^\tau &= \begin{cases} \delta^{\tau\cup t}, & t\notin\tau, \\ 0, & t\in\tau, \end{cases} & c_t^-\delta^\tau &= \begin{cases} 0, & t\notin\tau, \\ \frac{1}{H}\delta^{\tau\setminus t}, & t\in\tau, \end{cases} \\ c_t^\circ\delta^\tau &= \begin{cases} 0, & t\notin\tau, \\ \delta^\tau, & t\in\tau, \end{cases} & c_t^\bullet\delta^\tau &= \begin{cases} \frac{1}{H}\delta^\tau, & t\notin\tau, \\ 0, & t\in\tau. \end{cases} \end{aligned}$$

We claim that $a_t^\sharp \stackrel{\equiv}{\longleftrightarrow} c_t^\sharp$ under the isomorphism between $\mathcal{F}_{\mathrm{int}}$ and $\otimes_{t\in T}\mathcal{F}_{\mathrm{int},t}$. Let $F = \sum_{\tau\in\Gamma} b_\tau\delta_\tau$ be the expansion of $F\in\mathcal{F}_{\mathrm{int}}$ in the base $\{\delta_\tau : \tau\in\Gamma\}$ and $G = \sum_{\tau\in\Gamma} b_\tau\delta^\tau$ the corresponding element in $\otimes_{t\in T}\mathcal{F}_{\mathrm{int},t}$. By definition 2.2.1 we see that

$$\begin{aligned} a_t^+F(\sigma) &= \begin{cases} 0, & t\notin\sigma, \\ b_{\sigma\setminus t}, & t\in\sigma, \end{cases} & a_t^-F(\sigma) &= \begin{cases} \frac{1}{H}b_{\sigma\cup t}, & t\notin\sigma, \\ 0, & t\in\sigma, \end{cases} \\ a_t^\circ F(\sigma) &= \begin{cases} 0, & t\notin\sigma, \\ b_\sigma, & t\in\sigma, \end{cases} & a_t^\bullet F(\sigma) &= \begin{cases} \frac{1}{H}b_\sigma, & t\notin\sigma, \\ 0, & t\in\sigma. \end{cases} \end{aligned}$$

On the other hand, for $c_t^\sharp G(\sigma)$ we obtain

$$\begin{aligned} c_t^+G(\sigma) &= c_t^+\sum_{\tau\in\Gamma} b_\tau\delta^\tau(\sigma) = \sum_{\substack{\tau\in\Gamma\\ t\notin\tau}} b_\tau\delta^{\tau\cup t}(\sigma) = \sum_{\substack{\tau\in\Gamma\\ t\in\tau}} b_{\tau\setminus t}\delta^\tau(\sigma) = \begin{cases} 0, & t\notin\sigma, \\ b_{\sigma\setminus t}, & t\in\sigma, \end{cases} \\ c_t^-G(\sigma) &= c_t^-\sum_{\tau\in\Gamma} b_\tau\delta^\tau(\sigma) = \sum_{\substack{\tau\in\Gamma\\ t\in\tau}} b_\tau\frac{1}{H}\delta^{\tau\setminus t}(\sigma) = \sum_{\substack{\tau\in\Gamma\\ t\notin\tau}} \frac{1}{H}b_{\tau\cup t}\delta^\tau(\sigma) = \begin{cases} \frac{1}{H}b_{\sigma\cup t}, & t\notin\sigma, \\ 0, & t\in\sigma, \end{cases} \\ c_t^\circ G(\sigma) &= c_t^\circ\sum_{\tau\in\Gamma} b_\tau\delta^\tau(\sigma) = \sum_{\substack{\tau\in\Gamma\\ t\in\tau}} b_\tau\delta^\tau(\sigma) = \begin{cases} 0, & t\notin\sigma, \\ b_\sigma, & t\in\sigma, \end{cases} \\ c_t^\bullet G(\sigma) &= c_t^\bullet\sum_{\tau\in\Gamma} b_\tau\delta^\tau(\sigma) = \sum_{\substack{\tau\in\Gamma\\ t\notin\tau}} \frac{1}{H}b_\tau\delta^\tau(\sigma) = \begin{cases} \frac{1}{H}b_\sigma, & t\notin\sigma, \\ 0, & t\in\sigma. \end{cases} \end{aligned}$$

This proves that $c_t^\sharp$ is a realization of $a_t^\sharp$. But since $c_t^\sharp$ is by construction of the form

$$c_t^\sharp = \mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\widehat{c}_t^\sharp\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}$$

we obtain that $a_t^\sharp$ is of the form

$$a_t^\sharp = \mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\widehat{a}_t^\sharp\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}$$

where $c_t^\sharp$ respectively $a_t^\sharp$ is in the tth place of the tensor product. Thus using proposition 2.1.30 we have proved the following result.

Proposition 2.2.6 *Let $\mathcal{A}_t$ be the internal $W^\star$-algebra generated by the four operators a_t^+, a_t^-, a_t° and $a_t^\bullet$. Then the family $(\mathcal{A}_t)_{t\in T}$ is independent.*

As an immediate consequence we obtain the following corollary.

Corollary 2.2.7
The operator processes A_t^+, A_t^-, A_t° and $A_t^\bullet$ have independent increments.

Of course this assertion also holds for linear combinations of these four processes.

2.2 Brownian Motion on the Internal Guichardet Space

Notation 2.2.8 Let $\theta \in \mathbb{R}$ and

$$b_s^\theta \quad = \quad \mathrm{e}^{\mathrm{i}\theta} a_s^+ + \mathrm{e}^{-\mathrm{i}\theta} a_s^-$$

be the sum of the creator and the annihilator and

$$B_t^\theta \quad = \quad \sum_{s<t} b_s^\theta$$

the corresponding operator process. ◁

In the following sub-subsection we show that B_t^θ is a Brownian motion and for different θs these Brownian motions do not commute. In the second sub-subsection we introduce for $\theta = 0$ integration against $b_t = b_t^0 = a_t^+ + a_t^-$ for adapted vector processes and this is Itô integration on the internal Guichardet space (cf. [LM]).

2.2.1 B_t^θ is a Brownian Motion

Now we collect some basic facts of Brownian motion and show that they are true for our introduced process $(B_t^\theta)_{t\in T}$. To begin with we calculate the quadratic variation process.

Proposition 2.2.9 *The quadratic variation of B_t^θ is*

$$[B^\theta]_t = A_t^\bullet + \frac{1}{H} A_t^\circ.$$

PROOF:

$$\begin{aligned} [B^\theta]_t &= \sum_{s<t} (B_{s+\frac{1}{H}}^\theta - B_s^\theta)^2 \quad = \quad \sum_{s<t} (b_s^\theta)^2 \quad = \quad \sum_{s<t} (\mathrm{e}^{\mathrm{i}\theta} a_s^+ + \mathrm{e}^{-\mathrm{i}\theta} a_s^-)^2 \\ &= \sum_{s<t} (\mathrm{e}^{2\mathrm{i}\theta} a_s^+ a_s^+ + \mathrm{e}^{-2\mathrm{i}\theta} a_s^- a_s^- + a_s^+ a_s^- + a_s^- a_s^+) \\ &= \sum_{s<t} (a_s^\bullet + \frac{1}{H} a_s^\circ) \quad = \quad A_t^\bullet + \frac{1}{H} A_t^\circ. \end{aligned}$$ □

Proposition 2.2.10 *The expectation of $B_t^\theta B_s^\theta$ is* $\min\{t, s\}$.

PROOF:

$$\begin{aligned}\mathbb{E}(B_t^\theta B_s^\theta) &= \sum_{t'<t}\sum_{s'<s}\langle\Omega, (\mathrm{e}^{\mathrm{i}\theta}a_{t'}^+ + \mathrm{e}^{-\mathrm{i}\theta}a_{t'}^-)(\mathrm{e}^{\mathrm{i}\theta}a_{s'}^+ + \mathrm{e}^{-\mathrm{i}\theta}a_{s'}^-)\Omega\rangle \\ &= \sum_{t'<t}\sum_{s'<s}\left(\, \mathrm{e}^{2\mathrm{i}\theta}\langle\Omega, a_{t'}^+a_{s'}^+\Omega\rangle + \mathrm{e}^{-2\mathrm{i}\theta}\langle\Omega, a_{t'}^-a_{s'}^-\Omega\rangle + \langle\Omega, a_{t'}^+a_{s'}^-\Omega\rangle + \langle\Omega, a_{t'}^-a_{s'}^+\Omega\rangle\,\right) \\ &= \sum_{t'<t}\sum_{s'<s}\langle\Omega, a_{t'}^-a_{s'}^+\Omega\rangle.\end{aligned}$$

In the last step we use the fact that $a_s^-\Omega = 0$. Thus it remains to evaluate the last term. Since we have $a_s^+\Omega = \delta_{\{s\}}$ we obtain

$$\begin{aligned}\langle\Omega, \sum_{t'<t}\sum_{s'<s}a_{t'}^-a_{s'}^+\Omega\rangle &= \langle\Omega, \sum_{t'<t}\sum_{s'<s}a_{t'}^-\delta_{\{s'\}}\rangle = \langle\Omega, \sum_{t'<\min\{t,s\}}a_{t'}^-\delta_{\{t'\}}\rangle \\ &= \sum_{t'<\min\{t,s\}}\frac{1}{H}\langle\Omega,\Omega\rangle = \min\{t,s\},\end{aligned}$$

where in the third and second to last steps we used the identity $a_t^-\delta_{\{s\}} = \delta_{ts}\frac{1}{H}\Omega$. □

Corollary 2.2.11 *Let $s \le t \le s' \le t'$ or $s' \le t' \le s \le t$. Then the expectation of $(B_t^\theta - B_s^\theta)(B_{t'}^\theta - B_{s'}^\theta)$ is zero.*

PROOF: Suppose $s < t \le s' < t'$. Using the preceding proposition and

$$(B_t^\theta - B_s^\theta)(B_{t'}^\theta - B_{s'}^\theta) = B_t^\theta B_{t'}^\theta + B_s^\theta B_{s'}^\theta - B_t^\theta B_{s'}^\theta - B_s^\theta B_{t'}^\theta$$

it follows

$$\begin{aligned}\mathbb{E}((B_t^\theta - B_s^\theta)(B_{t'}^\theta - B_{s'}^\theta)) &= \mathbb{E}(B_t^\theta B_{t'}^\theta) + \mathbb{E}(B_s^\theta B_{s'}^\theta) - \mathbb{E}(B_t^\theta B_{s'}^\theta) - \mathbb{E}(B_s^\theta B_{t'}^\theta) \\ &= t + s - t - s = 0\,.\end{aligned}$$

Similarly the other part is proved. □

This shows that the process B_t^θ has uncorrelated increments since $\mathbb{E}(B_t^\theta) = 0$ (see proposition 2.2.15). Now we calculate the characteristic distribution of the infinitesimal increments. It turns out that this distribution is independent of the chosen increment.

Lemma 2.2.12 $\mathbb{E}(\mathrm{e}^{\mathrm{i}yb_s^\theta}) = \cos(\frac{y}{\sqrt{H}})$ *for every $s \in T$.*

PROOF: We recall that the annihilator a_s^- destroys the vacuum state Ω and that $\langle\Omega, a_s^+\Omega\rangle = \langle\delta_\emptyset, \delta_{\{s\}}\rangle = 0$. So the only possibility leading to an inner product $\langle\Omega, \prod a_s^\sharp\Omega\rangle$ that is not zero for an arbitrary finite product of creators and annihilators is $\langle\Omega, (a_s^-a_s^+)^k\Omega\rangle = \frac{1}{H^k}$. Thus we obtain

$$\begin{aligned}\mathbb{E}(\mathrm{e}^{\mathrm{i}yb_s^\theta}) &= \langle\Omega, \mathrm{e}^{\mathrm{i}yb_s^\theta}\Omega\rangle = \sum_k\langle\Omega, \frac{(\mathrm{i}y)^k}{k!}(\mathrm{e}^{\mathrm{i}\theta}a_s^+ + \mathrm{e}^{-\mathrm{i}\theta}a_s^-)^k\Omega\rangle \\ &= \sum_k\frac{(\mathrm{i}y)^{2k}}{(2k)!}\langle\Omega, (a_s^-a_s^+)^k\Omega\rangle = \sum_k\frac{(-1)^k y^{2k}}{(2k)!H^k} = \cos(\frac{y}{\sqrt{H}})\,.\end{aligned}$$

□

Proposition 2.2.13 *The process B_t^θ has independent and stationary increments.*

PROOF: The independence of the increments is a consequence of proposition 2.2.6 (or corollary 2.2.7). The stationarity follows immediately from

$$\begin{aligned} B_{t+h}^\theta - B_{s+h}^\theta &= \sum_{t'<t} b_{t'}^\theta + \sum_{t\le t'<t+h} b_{t'}^\theta - \sum_{s'<s} b_{s'}^\theta - \sum_{s\le s'<s+h} b_{s'}^\theta \\ &= B_t^\theta - B_s^\theta + \sum_{t\le t'<t+h} b_{t'}^\theta - \sum_{s\le s'<s+h} b_{s'}^\theta . \end{aligned}$$

and the fact that for $h = \frac{k}{H} \in T$ we have

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}y\sum_{t\le t'<t+h} b_{t'}^\theta}) = \mathbb{E}(\prod_{t\le t'<t+h} \mathrm{e}^{\mathrm{i}y b_{t'}^\theta}) = \prod_{t\le t'<t+h} \mathbb{E}(\mathrm{e}^{\mathrm{i}y b_{t'}^\theta}) = (\cos(\frac{y}{\sqrt{H}}))^k$$

since the increments are independent. A similar calculation shows the same result for the last term. Thus the last two terms cancel and the result follows. □

Corollary 2.2.14 *Let $s, t \in T, s \le t$. Then $\mathbb{E}((B_t^\theta - B_s^\theta)^2) = t - s$ and $\mathbb{E}([B^\theta]_t) = t$.*

PROOF: By proposition 2.2.10 we have

$$\begin{aligned} \mathbb{E}((B_t^\theta - B_s^\theta)^2) &= \mathbb{E}(B_t^\theta B_t^\theta) + \mathbb{E}(B_s^\theta B_s^\theta) - \mathbb{E}(B_t^\theta B_s^\theta) - \mathbb{E}(B_s^\theta B_t^\theta) \\ &= t + s - s - s \ = \ t - s\,. \end{aligned}$$

$$\mathbb{E}([B^\theta]_t) = \mathbb{E}(\sum_{s<t}(B_{s+\frac{1}{H}}^\theta - B_s^\theta)^2) = \sum_{s<t}\mathbb{E}((B_{s+\frac{1}{H}}^\theta - B_s^\theta)^2) = \sum_{s<t}\frac{1}{H} = t\,. \quad \Box$$

Proposition 2.2.15 *The expectation of B_t^θ is zero and the variance of B_t^θ is t.*

PROOF:

$$\begin{aligned} \mathbb{E}(B_t^\theta) &= \langle \Omega, \sum_{s<t}(\mathrm{e}^{\mathrm{i}\theta} a_s^+ + \mathrm{e}^{-\mathrm{i}\theta} a_s^-)\Omega\rangle \\ &= \sum_{s<t} (\ \mathrm{e}^{\mathrm{i}\theta}\langle\Omega, a_s^+\Omega\rangle + \mathrm{e}^{-\mathrm{i}\theta}\langle\Omega, a_s^-\Omega\rangle\ ) = \sum_{s<t} \mathrm{e}^{\mathrm{i}\theta}\langle\Omega, \delta_{\{s\}}\rangle = 0\,. \end{aligned}$$

Using proposition 2.2.10 and the definition of the variance one obtains

$$\mathbb{V}(B_t^\theta) = \mathbb{E}((B_t^\theta - \mathbb{E}(B_t^\theta))^2) = \mathbb{E}(B_t^\theta B_t^\theta) = t\,. \quad \Box$$

Theorem 2.2.16 *The characteristic distribution of B_t^θ is approximately given by*

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}y B_t^\theta}) \approx \mathrm{e}^{-\frac{y^2}{2}\cdot t}\,.$$

PROOF: For the independence of the increments we have

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}yB_t^\theta}) \;=\; \mathbb{E}(\mathrm{e}^{\mathrm{i}y\sum_{s<t} b_s^\theta}) \;=\; \mathbb{E}(\prod_{s<t} \mathrm{e}^{\mathrm{i}yb\theta_s}) \;=\; \prod_{s<t} \mathbb{E}(\mathrm{e}^{\mathrm{i}yb\theta_s}) \;=\; (\cos(\frac{y}{\sqrt{H}}))^k$$

since t is of the form $t = \frac{k}{H}$ for some $k \in {}^*\mathbb{N}$ with $k < H$. Using the Taylor expansion for the cosine and $k = Ht$ we get

$$(\cos(\frac{y}{\sqrt{H}}))^k \;=\; (1 - \frac{y^2}{2H} + O(\frac{y^4}{H^2}))^{Ht} \;\approx\; \mathrm{e}^{\frac{-y^2 t}{2}}\,. \qquad \square$$

The preceding theorem and the other results proved in this part show that B_t^θ is a realization of the Brownian motion on the internal Guichardet space. The attraction of the nonstandard setting is that we can prove the results by elementary algebraic calculations and some finite combinatorics. In the next theorem we show that for different θs in general the corresponding Brownian motions do not commute.

Theorem 2.2.17 *The commutator of B_t^θ and B_t^ρ is $2\mathrm{i}\,\sin(\rho-\theta)(A_t^\bullet - \frac{1}{H}A_t^\circ)$.*

PROOF:

$$\begin{aligned}
[B_t^\theta, B_t^\rho] \;&=\; \left(\sum_{s<t}(\mathrm{e}^{\mathrm{i}\theta}a_s^+ + \mathrm{e}^{-\mathrm{i}\theta}a_s^-)\right)\left(\sum_{s'<t}(\mathrm{e}^{\mathrm{i}\rho}a_{s'}^+ + \mathrm{e}^{-\mathrm{i}\rho}a_{s'}^-)\right)\\
&\quad - \left(\sum_{s'<t}(\mathrm{e}^{\mathrm{i}\rho}a_{s'}^+ + \mathrm{e}^{-\mathrm{i}\rho}a_{s'}^-)\right)\left(\sum_{s<t}(\mathrm{e}^{\mathrm{i}\theta}a_s^+ + \mathrm{e}^{-\mathrm{i}\theta}a_s^-)\right)\\
&=\; \sum_{s<t}(\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\rho}a_s^+a_s^- + \mathrm{e}^{-\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\rho}a_s^-a_s^+ - \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\mathrm{i}\theta}a_s^+a_s^- - \mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\theta}a_s^-a_s^+)\\
&=\; \sum_{s<t}\left(\,(\mathrm{e}^{\mathrm{i}(\theta-\rho)} - \mathrm{e}^{-\mathrm{i}(\theta-\rho)})a_s^+a_s^- + (\mathrm{e}^{-\mathrm{i}(\theta-\rho)} - \mathrm{e}^{\mathrm{i}(\theta-\rho)})a_s^-a_s^+\,\right)\\
&=\; -2\mathrm{i}\,\sin(\theta-\rho)\sum_{s<t}(a_s^-a_s^+ - a_s^+a_s^-) \;=\; 2\mathrm{i}\,\sin(\rho-\theta)(A_t^\bullet - \frac{1}{H}A_t^\circ)\,. \qquad \square
\end{aligned}$$

Corollary 2.2.18 *For $\rho = \theta + k\pi$, $k \in \mathbb{N}$ it is $[B_t^\theta, B_t^\rho] = 0$ and for $\rho = \theta + \frac{2k+1}{2}\pi$, $k \in \mathbb{N}$ it is $[B_t^\theta, B_t^\rho] = 2\mathrm{i}(A_t^\bullet - \frac{1}{H}A_t^\circ)$.*

Whence for $\rho = \theta + \frac{2k+1}{2}\pi$, $k \in \mathbb{N}$ we get a pair of conjugate operator processes on the internal Fock space.

2.2.2 The Internal Itô Integral

In this sub-subsection we show how one obtains an internal version of Itô integration. The calculations can be done formally since all sums are hyperfinite. The resulting formula has to be compared with Maassen's formula for Itô integration on Fock space which is structurally the same [LM]. We fix $\theta = 0$ and set $b_s = b_s^0$ and $B_t = B_t^0$.

The action of the increment $dB_s = b_s$ on an element F of $\mathcal{F}_{\text{int}}$ is given by

$$b_s F(\sigma) \quad = \quad \begin{cases} \frac{1}{H} F(\sigma \cup s) & \text{if } s \notin \sigma, \\ F(\sigma \setminus s) & \text{if } s \in \sigma. \end{cases}$$

The definition of the Itô integral is now just the formal application of the process B_s to an adapted vector process F_s.

Definition 2.2.19 *Let F_s be an adapted internal vector process. Define the* internal Itô integral *in the following way:*

$$\int_0^t dB_s \cdot F_s(\sigma) \quad = \quad \sum_{0 \leq s < t} b_s F_s(\sigma)\,. \qquad \triangleleft$$

Note that the upper boundary of the integral is not contained in the sum of the right hand side.

Proposition 2.2.20 *It is*

$$\int_0^t dB_s \cdot F_s(\sigma) \quad = \quad \begin{cases} 0 & \textit{for } t \leq \max\sigma, \\ F_{\max\sigma}(\sigma \setminus \max\sigma) & \textit{for } t > \max\sigma. \end{cases}$$

PROOF:

$$\int_0^t dB_s \cdot F_s(\sigma) \quad = \quad \sum_{0 \leq s < t} b_s F_s(\sigma) \quad = \quad \sum_{\substack{s<t \\ s \notin \sigma}} \frac{1}{H} F_s(\sigma \cup s) \; + \; \sum_{\substack{s<t \\ s \in \sigma}} F_s(\sigma \setminus s).$$

The first term is zero since $F_s(\sigma \cup s) = 0$ for every $s \in T$ because of the adaptedness of F_s. For the same reason $F_s(\sigma \setminus s)$ is zero unless $s = \max\sigma$. But since we sum $s < t$ the second term is zero for $t \leq \max\sigma$. The remaining possibility for non-zeroness is that $t > \max\sigma$ and $s = \max\sigma$. So we have

$$\int_0^t dB_s \cdot F_s(\sigma) \quad = \quad \begin{cases} 0 & \text{for } t \leq \max\sigma, \\ F_{\max\sigma}(\sigma \setminus \max\sigma) & \text{for } t > \max\sigma. \end{cases} \qquad \square$$

Remark: For the empty set the calculation is done in the same way with the convention that an empty sum has value zero. So one has $\int_0^t dB_s \cdot F_s(\emptyset) = 0$.

Corollary 2.2.21
The Itô integral of an adapted internal process is an adapted internal process.

Thus the internal Itô integral can be put again into the integration machinery and iterated internal Itô integration is well-defined.

2.3 Poisson Process on the Internal Guichardet Space

In this subsection we demonstrate that one even has a realization of the Poisson process on the internal Guichardet space. This is done by a combinatorial proof that the defined process has the correct distribution function.

Notation 2.2.22 Let $z \in \mathbb{C}$. Then define p_s^z by

$$p_s^z \;=\; a_s^\circ + z a_s^+ + \bar{z} a_s^- + |z|^2 a_s^\bullet$$

and the corresponding process by $P_t^z = \sum_{s<t} p_s^z$. ◁

We show now that the characteristic distribution of the process P_t^z is approximately the function $\varphi(y) = \exp(|z|^2(\mathrm{e}^{\mathrm{i}y} - 1))$. This is done in three steps. First we calculate the action of the infinitesimal process on the vacuum vector $\Omega = \delta_\emptyset$.

Lemma 2.2.23 *For $k \geq 1$ we have*

$$(p_s^z)^k\Omega = (a_s^\circ + z a_s^+ + \bar{z} a_s^- + |z|^2 a_s^\bullet)^k \delta_\emptyset \;=\; \sum_{n=0}^{k-1} \binom{k-1}{n} \left[z \left(\frac{|z|^2}{H}\right)^n \delta_{\{s\}} + \left(\frac{|z|^2}{H}\right)^{n+1} \delta_\emptyset \right].$$

PROOF: The proof is done by induction over k. The case for $k = 1$ is clear:

$$(a_s^\circ + z a_s^+ + \bar{z} a_s^- + |z|^2 a_s^\bullet)\delta_\emptyset \;=\; a_s^\circ\delta_\emptyset + z a_s^+\delta_\emptyset + \bar{z} a_s^-\delta_\emptyset + |z|^2 a_s^\bullet\delta_\emptyset \;=\; z\delta_{\{s\}} + \frac{|z|^2}{H}\delta_\emptyset\,.$$

Thus we prove the step from k to $k+1$. Suppose that the formula is valid for some $k \geq 1$. Recalling the relations

$$a_s^\circ\delta_\emptyset = a_s^-\delta_\emptyset = a_s^+\delta_{\{s\}} = a_s^\bullet\delta_{\{s\}} = 0\,,$$

then we obtain

$$\begin{aligned}
(p_s^z)^{k+1}\Omega &= p_s^z(a_s^\circ + z a_s^+ + \bar{z} a_s^- + |z|^2 a_s^\bullet)^k \delta_\emptyset \\
&= (a_s^\circ + z a_s^+ + \bar{z} a_s^- + |z|^2 a_s^\bullet) \left[\sum_{n=0}^{k-1} \binom{k-1}{n} \left(z \left(\frac{|z|^2}{H}\right)^n \delta_{\{s\}} + \left(\frac{|z|^2}{H}\right)^{n+1} \delta_\emptyset \right) \right]
\end{aligned}$$

$$= \sum_{n=0}^{k-1} \binom{k-1}{n} \left[z \left(\frac{|z|^2}{H}\right)^n \delta_{\{s\}} + z \left(\frac{|z|^2}{H}\right)^{n+1} \delta_{\{s\}} + \frac{\bar{z}z}{H} \left(\frac{|z|^2}{H}\right)^n \delta_\emptyset + \frac{|z|^2}{H} \left(\frac{|z|^2}{H}\right)^{n+1} \delta_\emptyset \right]$$

$$\begin{aligned}
&= \sum_{n=0}^{k-1} \binom{k-1}{n} \left[z \left(\frac{|z|^2}{H}\right)^n \delta_{\{s\}} + \left(\frac{|z|^2}{H}\right)^{n+1} \delta_\emptyset \right] \\
&\quad + \sum_{n=0}^{k-1} \binom{k-1}{n} \left[z \left(\frac{|z|^2}{H}\right)^{n+1} \delta_{\{s\}} + \left(\frac{|z|^2}{H}\right)^{n+2} \delta_\emptyset \right] \\
&= \sum_{n=0}^{k-1} \binom{k-1}{n} \left[z \left(\frac{|z|^2}{H}\right)^n \delta_{\{s\}} + \left(\frac{|z|^2}{H}\right)^{n+1} \delta_\emptyset \right]
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{n=1}^{k} \binom{k-1}{n-1} \left[z \left(\frac{|z|^2}{H} \right)^n \delta_{\{s\}} + \left(\frac{|z|^2}{H} \right)^{n+1} \delta_\emptyset \right] \\
= & \binom{k-1}{0} \left[z\delta_{\{s\}} + \left(\frac{|z|^2}{H} \right) \delta_\emptyset \right] + \binom{k-1}{k-1} \left[z \left(\frac{|z|^2}{H} \right)^k \delta_{\{s\}} \left(\frac{|z|^2}{H} \right)^{k+1} \delta_\emptyset \right] \\
& + \sum_{n=1}^{k-1} \left(\binom{k-1}{n-1} + \binom{k-1}{n} \right) \left[z \left(\frac{|z|^2}{H} \right)^n \delta_{\{s\}} + \left(\frac{|z|^2}{H} \right)^{n+1} \delta_\emptyset \right] \\
= & \sum_{n=0}^{k} \binom{k}{n} \left[z \left(\frac{|z|^2}{H} \right)^n \delta_{\{s\}} + \left(\frac{|z|^2}{H} \right)^{n+1} \delta_\emptyset \right] .
\end{aligned}
$$

In the last step we used the identity $\binom{k-1}{n-1} + \binom{k-1}{n} = \binom{k}{n}$. This proves the lemma. □

Corollary 2.2.24

$$
\langle \Omega, (p_s^z)^k \Omega \rangle \;=\; \sum_{n=0}^{k-1} \binom{k-1}{n} \left(\frac{|z|^2}{H} \right)^{n+1} \;=\; \frac{|z|^2}{H} \left(1 + \left(\frac{|z|^2}{H} \right) \right)^{k-1} .
$$

Proof: Using $\langle \Omega, \Omega \rangle = 1$ and $\langle \Omega, \delta_{\{s\}} \rangle = 0$ the first part follows immediately from the lemma. The second equality follows by

$$
\sum_{n=0}^{k-1} \binom{k-1}{n} \left(\frac{|z|^2}{H} \right)^{n+1} \;=\; \frac{|z|^2}{H} \sum_{n=0}^{k-1} \binom{k-1}{n} \left(\frac{|z|^2}{H} \right)^n . \qquad \square
$$

We note that for $k = 0$ it is $\langle \Omega, (p_s^z)^k \Omega \rangle = 1$. Next we calculate the distribution function of the infinitesimal process p_s^z.

Proposition 2.2.25
The characteristic distribution of the infinitesimal process p_s^z is given by

$$
\mathbb{E}(\mathrm{e}^{\mathrm{i}yp_s^z}) \;=\; 1 + \frac{1}{H} \left(\frac{|z|^2}{1 + \frac{|z|^2}{H}} \right) \left(\exp(\mathrm{i}y(1 + \frac{|z|^2}{H})) - 1 \right) .
$$

Proof: Using the preceding corollary one calculates

$$
\begin{aligned}
\mathbb{E}(\mathrm{e}^{\mathrm{i}yp_s^z}) \;=\; \langle \Omega, \mathrm{e}^{\mathrm{i}yp_s^z} \Omega \rangle \;&=\; \sum_{k \geq 0} \frac{(\mathrm{i}y)^k}{k!} \langle \Omega, (p_s^z)^k \Omega \rangle \\
&=\; 1 + \sum_{k>0} \left[\frac{(\mathrm{i}y)^k}{k!} \frac{|z|^2}{H} \left(1 + \left(\frac{|z|^2}{H} \right) \right)^{k-1} \right] \\
&=\; 1 + \frac{\frac{|z|^2}{H}}{1 + \frac{|z|^2}{H}} \sum_{k>0} \frac{\left(\mathrm{i}y \left(1 + \frac{|z|^2}{H} \right) \right)^k}{k!}
\end{aligned}
$$

$$= \quad 1 + \frac{1}{H}\left(\frac{|z|^2}{1+\frac{|z|^2}{H}}\right)\left(\exp(\mathrm{i}y(1+\frac{|z|^2}{H})) - 1\right). \qquad \square$$

Now we are prepared to calculate the characteristic distribution of the process P_t^z.

Theorem 2.2.26 *The characteristic distribution of P_t^z is approximately given by*

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}yP_t^z}) \quad \approx \quad \exp(|z|^2 t(\mathrm{e}^{\mathrm{i}y} - 1)).$$

PROOF: We use the same trick as in the calculation of the characteristic distribution of B_t^θ. Recall that the increments of the process for different times are independent (see proposition 2.2.6 or corollary 2.2.7). This implies that the characteristic distribution of the sum of the increments is the product of the characteristic distribution of the increments. Since $t \in T$ we have $t = \frac{k}{H}$ for some $k \in {}^*\mathbb{N}$ with $k < H$. We obtain

$$\begin{aligned}
\mathbb{E}(\mathrm{e}^{\mathrm{i}yP_t^z}) &= \mathbb{E}(\mathrm{e}^{\mathrm{i}y\sum_{s<t}p_s^z}) = \mathbb{E}(\prod_{s<t}\mathrm{e}^{\mathrm{i}yp_s^z}) = \prod_{s<t}\mathbb{E}(\mathrm{e}^{\mathrm{i}yp_s^z}) \\
&= \prod_{s<t}\left[1 + \frac{1}{H}\left(\frac{|z|^2}{1+\frac{|z|^2}{H}}\right)\left(\exp(\mathrm{i}y(1+\frac{|z|^2}{H})) - 1\right)\right] \\
&= \left[1 + \frac{1}{H}\left(\frac{|z|^2}{1+\frac{|z|^2}{H}}\right)\left(\exp(\mathrm{i}y(1+\frac{|z|^2}{H})) - 1\right)\right]^{Ht} \\
&\approx \exp(|z|^2 t(\mathrm{e}^{\mathrm{i}y} - 1))
\end{aligned}$$

using the continuity of the exponential function. $\square$

This theorem shows that the process P_t^z has approximately the characteristic distribution of the Poisson process with intensity $|z|^2$. Thus in the sense of nonstandard analysis P_t^z is a realization of the Poisson process on the internal Guichardet space. Using this realization of the Poisson process one can develop an integration theory for adapted vector processes against the Poisson process. We omit the parameter z.

The action of the increment $dP_s = p_s$ on an element F of $\mathcal{F}_{\mathrm{int}}$ is given by

$$p_s F(\sigma) \quad = \quad \begin{cases} \frac{|z|^2}{H}F(\sigma) + \frac{\bar{z}}{H}F(\sigma \cup s) & \text{if } s \notin \sigma, \\ F(\sigma) + zF(\sigma \setminus s) & \text{if } s \in \sigma. \end{cases}$$

Definition 2.2.27 *Let F_s be an adapted internal vector process. Define the* internal Poisson integral *in the following way:*

$$\int_0^t dP_s \cdot F_s(\sigma) \quad = \quad \sum_{0 \le s < t} p_s F_s(\sigma). \qquad \triangleleft$$

Using similar considerations as in the calculation for the internal Itô integral one obtains for the internal Poisson integral the following proposition.

Proposition 2.2.28 *Let F_s be an adapted internal vector process. Then the Poisson integral of F_s is given by*

$$\int_0^t dP_s \cdot F_s(\sigma) \quad = \quad \begin{cases} 0 & \text{for } t \le \max\sigma, \\ zF_{\max\sigma}(\sigma \setminus \max\sigma) + |z|^2 \sum_{\max\sigma < s < t} F_s(\sigma)\frac{1}{H} & \text{for } t > \max\sigma, \end{cases}$$

where for $t = \max\sigma + \frac{1}{H}$ the empty sum has value zero.

Thus the Poisson integral consists of a Brownian part weighted with z (see proposition 2.2.20) and a jump part weighted by the intensity $|z|^2$ of the Poisson process. In fact by the adaptedness of the integrated process the sum $\sum_{\max\sigma<s<t}$ can be extended to a full sum $\sum_{0\le s<t}$.

3 Internal Kernel Operators

In this section we develop an internal version of Maassen's kernel calculus. For a standard treatment we refer to [Mey93a, Maa85, Lin98]. As we will see every internal Fock space operator is a kernel operator. We introduce internal adapted operator processes and define an internal quantum stochastic integral against the fundamental processes as integrators. Up to some infinitesimal correction terms the obtained formulas are the same as in the standard continuous in time setting. Because in this section we work only with internal objects we will usually omit the predicate internal.

3.1 2-Argument Kernels

In this subsection we take a look at 2-argument kernel operators. In some sense these are the most fundamental operators and with them one can represent many of the operators in Boson Fock space in quantum stochastic calculus. In fact using the notation of white noise analysis every operator can be represented by a 2-argument kernel distribution. In addtition, since we work here in a formally finite setting, every internal operator on the internal Fock space can be represented by an internal 2-argument kernel function. For the 2-argument kernel operators the normal ordering of an product of a_t^+s and a_t^-s is important. That means that all creation operators are to the left of every annihilation operator. In a similiar way, as for exponential vectors, we introduce the following notation for the fundamental operators and lift them as operators indexed by $\Gamma = {}^*\mathcal{P}(T)$.

Notation 2.3.1 Let $\sharp \in \{\bullet, +, \circ, -\}$. For every $\sigma \in \Gamma$ the operator $a_\sigma^\sharp$ is defined by

$$a_\sigma^\sharp \quad = \quad \prod_{s\in\sigma} a_s^\sharp.$$

Further, we define $\notin_\sigma$ and $\in_\sigma$ by

$$\notin_\sigma \quad = \quad \prod_{s\in\sigma} \notin_s \qquad \text{and} \qquad \in_\sigma \quad = \quad \prod_{s\in\sigma} \in_s . \qquad \triangleleft$$

This notation makes sense since σ is a set so that the elements are different and the elementary operators in $\prod_{s\in\sigma} a_s^\sharp$ commute.

Definition 2.3.2 *Let $K : \Gamma\times\Gamma \to {}^*\mathbb{C}$ be an internal function. Then a (normally ordered)* 2-argument kernel operator K *is given by*

$$\mathsf{K} \;=\; \sum_{\sigma,\tau\in\Gamma} K(\sigma,\tau) a_\sigma^+ a_\tau^- \,. \qquad \triangleleft$$

First of all we calculate the kernel function of the product of two operators. This is complicated since in the expansion the base $\{a_\sigma^+ a_\tau^-\}_{\sigma,\tau\in\Gamma}$ consists of normally ordered operators. Thus in the product we have to commute operators to obtain a normally ordered product.

Proposition 2.3.3 *Let $\mathsf{K} = \sum_{\gamma,\rho\in\Gamma} K(\gamma,\rho) a_\gamma^+ a_\rho^-$ and $\mathsf{L} = \sum_{\alpha,\beta\in\Gamma} L(\alpha,\beta) a_\alpha^+ a_\beta^-$ be two 2-argument kernel operators. Then the kernel function G of the product $\mathsf{G} = \mathsf{KL}$ is "approximately" given by*

$$G(\sigma,\tau) \;=\; \sum_{\gamma\dot\cup\alpha=\sigma}\ \sum_{\rho\dot\cup\beta=\tau}\ \sum_{\substack{\eta \\ \eta\cap\alpha=\eta\cap\rho=\emptyset}} K(\gamma,\rho\cup\eta) L(\alpha\cup\eta,\beta) \left(\frac{1}{H}\right)^{|\eta|}$$

Here by "approximately" we mean that the set where the function differs from the given one is a set of infinitesimal measure.

Proof:

$$\begin{aligned}
\mathsf{KL} &= \left(\sum_{\gamma,\rho\in\Gamma} K(\gamma,\rho) a_\gamma^+ a_\rho^-\right)\left(\sum_{\alpha,\beta\in\Gamma} L(\alpha,\beta) a_\alpha^+ a_\beta^-\right) \\
&= \sum_{\gamma,\rho,\alpha,\beta\in\Gamma} K(\gamma,\rho) L(\alpha,\beta) a_\gamma^+ a_\rho^- a_\alpha^+ a_\beta^- \\
&= \sum_{\gamma,\rho,\alpha,\beta\in\Gamma} K(\gamma,\rho) L(\alpha,\beta) a_\gamma^+ a_{\rho\setminus\alpha}^- a_{\rho\cap\alpha}^\bullet a_{\alpha\setminus\rho}^+ a_\beta^- \\
&= \sum_{\gamma,\rho,\alpha,\beta\in\Gamma} K(\gamma,\rho) L(\alpha,\beta) a_\gamma^+ a_{\alpha\setminus\rho}^+ a_{\rho\cap\alpha}^\bullet a_{\rho\setminus\alpha}^- a_\beta^- \\
&= \sum_{\gamma,\rho,\alpha,\beta\in\Gamma} K(\gamma,\rho) L(\alpha,\beta) \left(\frac{1}{H}\right)^{|\rho\cap\alpha|} a_\gamma^+ a_{\alpha\setminus\rho}^+ a_{\rho\setminus\alpha}^- a_\beta^- \\
&= \sum_{\sigma,\tau\in\Gamma}\left(\sum_{\gamma\dot\cup\alpha=\sigma}\ \sum_{\rho\dot\cup\beta=\tau}\ \sum_{\substack{\eta \\ \eta\cap\alpha=\eta\cap\rho=\emptyset}} K(\gamma,\rho\cup\eta) L(\alpha\cup\eta,\beta)\left(\frac{1}{H}\right)^{|\eta|}\right) a_\sigma^+ a_\tau^- \,.
\end{aligned}$$

In the second to last step we have used $a_s^+ a_s^\bullet = \frac{1}{H} a_s^+ = \frac{1}{H} a_s^+ + a_s^\bullet a_s^+$ and the fact that $a_s^\bullet$ acts on vectors almost everywhere as multiplication by $\frac{1}{H}$. That means the difference set is a set of infinitesimal measure. Thus it can be written as $\frac{1}{H}$ in front of the a^+s. $\square$

For 2-argument operator processes it is characteristic that integration with respect to creation or annihilation process is best understood as integration in normal ordering (cf. [HS81]). Thus integration with respect to the creation process should be defined by multiplication with the differential from the left instead integration with respect to the annihilation process by multiplication with the differential from the right. So defined the creation integration behaves formally like the annihilation creation. If the annihilation integration is also defined by left multiplication then some infinitesimal correction terms occur which disappear when going back to the standard world.

Definition 2.3.4 *Let* $\mathsf{K}_t = \sum_{\sigma,\tau\in\Gamma} K_t(\sigma,\tau)a^+_\sigma a^-_\tau$ *be a 2-argument operator process. Then this process is* adapted *if the process kernel function fulfills the following condition*

$$\max(\sigma\cup\tau) \;\geq\; t \quad\Longrightarrow\quad K_t(\sigma,\tau) \;=\; 0\,. \qquad \lhd$$

Next we calculate the integration with respect to the annihilation process and the creation process. As before integration is defined as left multiplication by the increments:

$$\int_0^t dA^\sharp_s \mathsf{K}_s \;=\; \sum_{0\leq s<t} a^\sharp_s \mathsf{K}_s \;=\; \sum_{\sigma,\tau\in\Gamma}\sum_{0\leq s<t} K_t(\sigma,\tau)a^\sharp_s a^+_\sigma a^-_\tau\,.$$

But for the annihilation process we define also the right multiplicative integration and for the number process the left-and-right integration:

Notation 2.3.5

$$R-\int_0^t dA^-_s\mathsf{K}_s \;=\; \int_0^t \mathsf{K}_s dA^-_s \;=\; \sum_{\sigma,\tau\in\Gamma}\sum_{0\leq s<t} K_s(\sigma,\tau)a^+_\sigma a^-_\tau a^-_s\,,$$

$$LR-\int_0^t dA^\circ_s\mathsf{K}_s \;=\; LR-\int_0^t dA^+_s\mathsf{K}_s H dA^-_s \;=\; \sum_{\sigma,\tau\in\Gamma}\sum_{0\leq s<t} K_s(\sigma,\tau)Ha^+_s a^+_\sigma a^-_\tau a^-_s\,.$$

This notation is suggested by the normal ordering of the infinitesimal operators. $\lhd$

Where it seems convenient we write the integrand and integrator in a way similar to the defined order and omit R or LR before the integral. Note that as for the Itô integral the upper boundary of the integral is not contained in the defining hyperfinite sum. Formally the annihilation and the creation integration acts on the kernel function as the Brownian motion integration on vector processes. The result is summarized in the next proposition. But let us first introduce the following notation.

Notation 2.3.6 Let $K(\sigma,\tau)$ be a kernel function and $\mathsf{K} = \sum_{\sigma,\tau\in\Gamma} K(\sigma,\tau)a^+_\sigma a^-_\tau$ its operator. Then we write as identification of the operator with its kernel function

$$\mathsf{K} \;\widehat{=}\; K(\sigma,\tau)\,.$$

If not otherwise mentioned the variables of a kernel function are meant to be σ and τ. $\lhd$

Proposition 2.3.7 *Let $K_s(\sigma,\tau)$ be an adapted internal kernel function and K_s the corresponding operator. Then the kernel functions of the integrated process are given according to the following equations.*

$$\int_0^t dA_s^+\mathsf{K}_s \;\widehat{=}\; \begin{cases} 0 & \text{if } \; t \le \max(\sigma\cup\tau) \; \text{ or } \; \max\sigma = \max\tau\,, \\ K_{\max\sigma}(\sigma\setminus\max\sigma,\tau) & \text{if } \; t > \max(\sigma\cup\tau) = \max\sigma \ne \max\tau\,. \end{cases}$$

$$R-\int_0^t dA_s^-\mathsf{K}_s \;\widehat{=}\; \begin{cases} 0 & \text{if } \; t \le \max(\sigma\cup\tau) \; \text{ or } \; \max\tau = \max\sigma\,, \\ K_{\max\tau}(\sigma,\tau\setminus\max\tau) & \text{if } \; t > \max(\sigma\cup\tau) = \max\tau \ne \max\sigma\,. \end{cases}$$

$$LR-\int_0^t dA_s^\circ\mathsf{K}_s \;\widehat{=}\; \begin{cases} 0 & \text{if } \; t \le \max(\sigma\cup\tau) \; \text{ or } \; \max\sigma \ne \max\tau\,, \\ K_{\max(\sigma\cup\tau)}(\sigma\setminus\max\sigma,\tau\setminus\max\tau)H & \\ & \text{if } \; t > \max(\sigma\cup\tau) = \max\sigma = \max\tau\,. \end{cases}$$

Proof: This time we prove the annihilation case since the creation case follows in a similar way.

$$\begin{aligned} R-\int_0^t dA_s^-\mathsf{K}_s &= \sum_{\sigma,\tau\in\Gamma}\sum_{s<t} K_s(\sigma,\tau)a_\sigma^+a_\tau^-a_s^- \;=\; \sum_{\sigma,\tau\in\Gamma}\sum_{\substack{s<t\\ s\notin\tau}} K_s(\sigma,\tau)a_\sigma^+a_{\tau\cup s}^- \\ &= \sum_{\sigma,\tau\in\Gamma}\sum_{\substack{s<t\\ s\in\tau}} K_s(\sigma,\tau\setminus s)a_\sigma^+a_\tau^- \;=\; \begin{cases} 0 \;\text{ for } t\le\max\tau \;\text{ or }\; \max\tau\ne\max(\sigma\cup\tau) \\ \qquad\qquad\qquad\qquad \text{or }\; \max\tau = \max\sigma\,, \\ \sum_{\sigma,\tau\in\Gamma} K_{\max\tau}(\sigma,\tau\setminus\max\tau)a_\sigma^+a_\tau^- \quad \text{otherwise}\,. \end{cases} \end{aligned}$$

For the number-case one has the following calculation.

$$\begin{aligned} LR-\int_0^t dA_s^\circ\mathsf{K}_s &= \sum_{\sigma,\tau\in\Gamma}\sum_{s<t} K_s(\sigma,\tau)Ha_s^+a_\sigma^+a_\tau^-a_s^- \\ &= \sum_{\sigma,\tau\in\Gamma}\sum_{\substack{s<t\\ s\notin\sigma\cup\tau}} K_s(\sigma,\tau)Ha_{\sigma\cup s}^+a_{\tau\cup s}^- \;=\; \sum_{\sigma,\tau\in\Gamma}\sum_{\substack{s<t\\ s\in\sigma\cap\tau}} K_s(\sigma\setminus s,\tau\setminus s)Ha_\sigma^+a_\tau^- \\ &= \begin{cases} 0 \quad \text{for } \; t\le\max(\sigma\cup\tau) \;\text{ or }\; \max\sigma\ne\max\tau\,, \\ \sum_{\sigma,\tau\in\Gamma} K_{\max(\sigma\cup\tau)}(\sigma\setminus\max\sigma,\tau\setminus\max\tau)Ha_\sigma^+a_\tau^- \quad \text{otherwise}\,. \end{cases} \end{aligned}$$ □

First note that every integral defines an adapted internal kernel process. Second, for the number integral to exist in the standard world the kernel function must be an infinitesimal of order $\frac{1}{H}$ on the intersection of the arguments, and finite elsewhere. Otherwise the number integral of an adapted kernel process becomes a process with several infinite values. In the next proposition we show that the left integral of the annihilator process differs from the right integral only by terms of infinitesimal order in the operators. A similar result is true for the left integral of the number process and the left-and-right integral. For this we integrate an arbitrary but not necessarily adapted kernel process.

Proposition 2.3.8 *Let $K_s(\sigma,\tau)$ be an internal kernel function (not necessarily adapted). Then*

$$\int_0^t dA_s^-\mathsf{K}_s - \int_0^t \mathsf{K}_s dA_s^- \;=\; \sum_{\sigma,\tau\in\Gamma}\sum_{\substack{s<t\\ s\in\sigma}} K_s(\sigma,\tau)\frac{1}{H}a_{\sigma\setminus s}^+a_\tau^-\,,$$

$$\int_0^t dA_s^\circ \mathsf{K}_s - \int_0^t dA_s^+ \mathsf{K}_s H dA_s^- \quad = \quad \sum_{\sigma,\tau\in\Gamma} \sum_{\substack{s<t \\ s\in\sigma}} K_s(\sigma,\tau) a_\sigma^+ a_\tau^- \,.$$

PROOF: First we see that the action of a_s^- on a kernel $K_s(\sigma,\tau)a_\sigma^+ a_\tau^-$ is

$$a_s^- K_s(\sigma,\tau) a_\sigma^+ a_\tau^- \quad = \quad K_s(\sigma,\tau) \not\in_s(\sigma) a_\sigma^+ a_s^- a_\tau^- + K_s(\sigma,\tau) \in_s(\sigma) \frac{1}{H} a_{\sigma\setminus s}^+ a_\tau^- \,.$$

Taking the sums over (σ,τ) and s the left hand side is the (left) integral while on the right hand side the contribution of the first term is just the right multiplicative integral. Thus we obtain for the difference of the integrals:

$$\int_0^t dA_s^- \mathsf{K}_s - \int_0^t \mathsf{K}_s dA_s^- \quad = \quad \sum_{\sigma,\tau\in\Gamma} \sum_{\substack{s<t \\ s\in\sigma}} K_s(\sigma,\tau) \frac{1}{H} a_{\sigma\setminus s}^+ a_\tau^- \,.$$

For the number operator we obtain with $a_s^\circ = H a_s^+ a_s^-$

$$a_s^\circ K_s(\sigma,\tau) a_\sigma^+ a_\tau^- \quad = \quad K_s(\sigma,\tau) H \not\in_s(\sigma) a_s^+ a_\sigma^+ a_s^- a_\tau^- + K_s(\sigma,\tau) \in_s(\sigma) a_\sigma^+ a_\tau^- \,.$$

Thus the same consideration as before leads to

$$\int_0^t dA_s^\circ \mathsf{K}_s - \int_0^t dA_s^+ \mathsf{K}_s H dA_s^- \quad = \quad \sum_{\sigma,\tau\in\Gamma} \sum_{\substack{s<t \\ s\in\sigma}} K_s(\sigma,\tau) a_\sigma^+ a_\tau^- \,. \qquad \square$$

Remark 2.3.9

(1) For a kernel function such that $K_s(\sigma,\tau)\frac{1}{H}$ is infinitesimal for all $\sigma,\tau \in \Gamma$ and every $s \in T$ one has

$$\sum_{\sigma,\tau\in\Gamma} \sum_{\substack{s<t \\ s\in\sigma}} K_s(\sigma,\tau) \frac{1}{H} a_{\sigma\setminus s}^+ a_\tau^- \quad \approx \quad 0 \,.$$

Thus the left annihilator integral is approximately the right annihilator integral. For a kernel function such that $K_s(\sigma,\tau)H$ is finite the left-and-right number integral is approximately the (left) number integral.

(2) For an adapted kernel function we see immediately that the correction term between the integrals is zero. Thus for adapted integrands the values of the left-integral, the right-integral and the left-right-integral are the same, independent of the integrator.

We close this Section and go to the next, to 3-argument kernel operators. Since in Chapter 3 Section 3 we develop the concept of strict representation of standard operators, that is, lifting of the kernel function, 3-argument kernels are more convenient. In the standard case they give rise to more operators. Using only 2-argument kernels we would have to develop a representation for distributions and connect this to the white noise approach to quantum stochastic calculus.

3.2 3-Argument Kernels

In this section we consider 3-argument kernels. These are the kernels which in Chapter 3 Section 3 we intend to connect to standard operators.

Definition 2.3.10 *Let $K : \Gamma^3_{\neq} \to {}^*\mathbb{C}$ be an internal function. Then a (normally ordered)* 3-argument kernel operator K *is given by*

$$\mathsf{K} \;=\; \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma,\rho,\tau)a^+_\sigma a^\circ_\rho a^-_\tau \,. \qquad \triangleleft$$

Since the complement of $\Gamma^3_{\neq}$ has outer measure zero in Γ^3 we restrict our analysis to kernel functions defined on $\Gamma^3_{\neq}$. In the internal setting 3-argument and 2-argument kernels are related by the following formula

$$K(\sigma,\rho,\tau) \;=\; \frac{1}{H}K(\widetilde{\sigma},\widetilde{\tau}) \quad \text{with} \quad \sigma=\widetilde{\sigma}\setminus\widetilde{\tau},\ \rho=\widetilde{\sigma}\cap\widetilde{\tau} \ \text{ and } \ \tau=\widetilde{\tau}\setminus\widetilde{\sigma}\,.$$

But going back to the standard world this relation does not make any sense in standard quantum stochastic calculus. One must use the white noise approach, which is analytically much more complicated. Another advantage of taking $\Gamma^3_{\neq}$ instead of Γ^3 is that every internal operator has then a unique expansion in terms of the elements $\{a^+_\sigma a^\circ_\rho a^-_\tau : (\sigma,\rho,\tau)\in\Gamma^3_{\neq}\}$.

Proposition 2.3.11 *Every internal operator K and every internal operator process $(\mathsf{K}_t)_{t\in T}$ on $\mathcal{F}_{\mathrm{int}}$ has a unique expansion*

$$\mathsf{K}_t \;=\; \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K_t(\sigma,\rho,\tau)a^+_\sigma a^\circ_\rho a^-_\tau$$

with K an internal kernel function, respectively $(K_t)_{t\in T}$ an internal family of internal kernel functions. That is $\{a^+_\sigma a^\circ_\rho a^-_\tau : (\sigma,\rho,\tau)\in\Gamma^3_{\neq}\}$ is a basis for internal operators and operator processes.

Proof: By transfer of the finite dimensional case (cf. Meyer [Mey93a, p. 17]) we see that $\{a^+_\sigma a^\circ_\rho a^-_\tau : (\sigma,\rho,\tau)\in\Gamma^3_{\neq}\}$ is a basis for all internal operators K on $\mathcal{F}_{\mathrm{int}}$. But the result holds also for (discrete time) processes in the finite dimensional case and thus by transfer for all internal operator processes on $\mathcal{F}_{\mathrm{int}}$. □

Just as before we can explicitly compute the kernel of the product of two operators.

Proposition 2.3.12 *Let*

$$\mathsf{K} \;=\; \sum_{(\alpha,\beta,\gamma)\in\Gamma^3_{\neq}} K(\alpha,\beta,\gamma)a^+_\alpha a^\circ_\beta a^-_\gamma \quad \textit{and} \quad \mathsf{L} \;=\; \sum_{(\zeta,\eta,\vartheta)\in\Gamma^3_{\neq}} L(\zeta,\eta,\vartheta)a^+_\zeta a^\circ_\eta a^-_\vartheta$$

be two 3-argument kernel operators. Then the kernel function of the product KL *at* (σ,ρ,τ) *is given by*

$$\mathsf{KL} \;\widehat{=}\; \sum_{\kappa \text{ (disjoint)}} \sum_{\substack{\alpha\dot\cup\zeta_1\dot\cup\zeta_2=\sigma\\ \beta\cup\eta=\rho\\ \vartheta\dot\cup\gamma_1\dot\cup\gamma_2=\tau}} K(\alpha\cup\zeta_1\cup\zeta_2,\beta\cup\zeta_2,\gamma_1\cup\gamma_2\cup\kappa)\cdot$$
$$\cdot L(\zeta_1\cup\zeta_2\cup\kappa,\eta\cup\gamma_2,\gamma_1\cup\gamma_2\cup\vartheta)a^\bullet_\kappa$$

where the sum over κ *is such that* $(\kappa,\sigma,\rho,\tau)\in\Gamma^4_{\neq}$.

Proof: We do not indicate at every step of the proof which summation indices are disjoint from which other ones. This should be intuitivly clear.

$$\begin{aligned}
\mathsf{KL} &= \left(\sum_{\alpha,\beta,\gamma} K(\alpha,\beta,\gamma)a^+_\alpha a^\circ_\beta a^-_\gamma\right)\left(\sum_{\zeta,\eta,\vartheta} L(\zeta,\eta,\vartheta)a^+_\zeta a^\circ_\eta a^-_\vartheta\right)\\
&= \sum_{\substack{\alpha,\beta,\gamma\\ \zeta,\eta,\vartheta}} K(\alpha,\beta,\gamma)L(\zeta,\eta,\vartheta)a^+_\alpha a^\circ_\beta a^-_\gamma a^+_\zeta a^\circ_\eta a^-_\vartheta\\
&= \sum_{\substack{\alpha,\beta,\gamma\\ \zeta,\eta,\vartheta}} K(\alpha,\beta,\gamma)L(\zeta,\eta,\vartheta)a^+_\alpha a^\circ_\beta a^+_{\zeta\setminus\gamma} a^\bullet_{\zeta\cap\gamma} a^-_{\gamma\setminus\zeta} a^\circ_\eta a^-_\vartheta\,.
\end{aligned}$$

Now since $(\alpha,\beta,\gamma),(\zeta,\eta,\vartheta)\in\Gamma^3_{\neq}$ the set $\zeta\cap\gamma$ is disjoint of $\alpha,\beta,\eta,\vartheta$. Setting $\kappa=\zeta\cap\gamma$ we get the following sum where κ is disjoint of all other sets:

$$\sum_\kappa \sum_{\substack{\alpha,\beta,\gamma\\ \zeta,\eta,\vartheta}} K(\alpha,\beta,\gamma\cup\kappa)L(\zeta\cup\kappa,\eta,\vartheta)a^\bullet_\kappa a^+_\alpha a^\circ_\beta a^+_\zeta a^-_\gamma a^\circ_\eta a^-_\vartheta\,.$$

Thus recalling the relations $a^\circ_s a^+_s = a^+_s$ and $a^-_s a^\circ_s = a^-_s$ we obtain

$$\sum_\kappa \sum_{\substack{\alpha,\beta,\gamma\\ \zeta,\eta,\vartheta}} K(\alpha,\beta,\gamma\cup\kappa)L(\zeta\cup\kappa,\eta,\vartheta)a^\bullet_\kappa a^+_\alpha a^+_\zeta a^\circ_{\beta\setminus\zeta} a^\circ_{\eta\setminus\gamma} a^-_\gamma a^-_\vartheta\,.$$

Then by the relations $a^+_s a^+_s = a^-_s a^-_s = 0$ we can assume that $\alpha\cap\zeta=\gamma\cap\vartheta=\emptyset$ and obtain

$$\sum_\kappa \sum_{\substack{\alpha,\beta,\gamma\\ \zeta,\eta,\vartheta\\ \alpha\cap\zeta=\gamma\cap\vartheta=\emptyset}} K(\alpha,\beta,\gamma\cup\kappa)L(\zeta\cup\kappa,\eta,\vartheta)a^\bullet_\kappa a^+_{\alpha\cup\zeta} a^\circ_{\beta\setminus\zeta} a^\circ_{\eta\setminus\gamma} a^-_{\gamma\cup\vartheta}\,.$$

Now we set $\zeta=(\zeta\cap\beta)\dot\cup\zeta_2=\zeta_1\dot\cup\zeta_2$ and $\gamma=(\gamma\cap\eta)\dot\cup\gamma_2=\gamma_1\dot\cup\gamma_2$ and shift the summations. Using the relation $a^\circ_s a^\circ_s = a^\circ_s$ the result is

$$\sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} \sum_{\kappa \text{ disjoint}} \sum_{\substack{\alpha\dot\cup\zeta_1\dot\cup\zeta_2=\sigma\\ \beta\cup\eta=\rho\\ \vartheta\dot\cup\gamma_1\dot\cup\gamma_2=\tau}} K(\alpha\cup\zeta_1\cup\zeta_2,\beta\cup\zeta_2,\gamma_1\cup\gamma_2\cup\kappa)\cdot$$
$$\cdot L(\zeta_1\cup\zeta_2\cup\kappa,\eta\cup\gamma_2,\gamma_1\cup\gamma_2\cup\vartheta)a^\bullet_\kappa a^+_\sigma a^\circ_\rho a^-_\tau\,. \qquad \square$$

Remark 2.3.13 Note that the partition of ρ is not disjoint. But we can rewrite the kernel in the even more convenient form with all partitions disjoint. Further we substitute $a_\kappa^\bullet$ by $\not\in_\kappa(\sigma\cup\tau)\left(\frac{1}{H}\right)^{|\kappa|}$ since on vectors of the internal Fock space this is essentially the action of that operator. Thus we obtain for the kernel function of the product KL:

$$\sum_{\kappa \text{ disjoint}} \sum_{\substack{\sigma_1\dot\cup\sigma_2\dot\cup\sigma_3=\sigma\\ \rho_1\dot\cup\rho_2\dot\cup\rho_3=\rho\\ \tau_1\dot\cup\tau_2\dot\cup\tau_3=\tau}} K(\sigma_1,\rho_1\cup\rho_2\cup\sigma_2,\tau_1\cup\tau_2\cup\kappa)\cdot$$
$$\cdot L(\sigma_2\cup\sigma_3\cup\kappa,\rho_2\cup\rho_3\cup\tau_2,\tau_3)\not\in_\kappa(\sigma\cup\tau)\left(\frac{1}{H}\right)^{|\kappa|}. \quad (2.3.10)$$

Definition 2.3.14 *Equation (2.3.10) gives a product on the space of kernel functions. We call this product* Wick product *and denote it by $K\diamond L$. Thus $\mathsf{KL}\mathrel{\widehat{=}}K\diamond L(\sigma,\rho,\tau)$.* ◁

We see that the non-commutative product of operators gives a non-commutative product of kernel functions. The name "Wick product" is normally taken for the resulting kernel function of the product of two Brownian random variables. In this case the random variables are taken in the chaos expansion of iterated stochastic integrals and the product is calculated with the formal $(dB_t)^2 = dt$ rule. Since the kernels of our operators are Fock expansions of these operators we feel free to use the same name for the kernel function of the product of two operators.

Definition 2.3.15 *A* kernel process K_t *is given by an internal family $(K_t)_{t\in T}$ of 3-argument internal kernel functions. A process is* adapted *if the family fulfills the following condition*

$$\max(\sigma\cup\rho\cup\tau)\ge t \implies K_t(\sigma,\rho,\tau)=0\,.$$

For $\sharp\in\{\bullet,+,\circ,-\}$ (left) integration *of an adapted process is given by*

$$\int_0^t dA_s^\sharp\mathsf{K}_s = \sum_{0\le s<t} a_s^\sharp\mathsf{K}_s = \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}}\sum_{0\le s<t} K_s(\sigma,\rho,\tau)a_s^\sharp a_\sigma^+ a_\rho^\circ a_\tau^-\,. \quad ◁$$

Proposition 2.3.16 *Let $K_s(\sigma,\rho,\tau)$ be an adapted internal function and K_s the corresponding operator process. Then the kernel functions of the integrated process are given at (σ,ρ,τ) according to the following equations:*

$$\int_0^t dA_s^\bullet\mathsf{K}_s \mathrel{\widehat{=}} \begin{cases} 0 & \text{if } t\le\max(\sigma\cup\rho\cup\tau),\\ \sum_{\max(\sigma\cup\rho\cup\tau)<s<t}K_s(\sigma,\rho,\tau)\not\in_s(\sigma\cup\tau)\frac{1}{H} & \text{if } t>\max(\sigma\cup\rho\cup\tau),\end{cases}$$
$$\int_0^t dA_s^+\mathsf{K}_s \mathrel{\widehat{=}} \begin{cases} 0 & \text{if } t\le\max(\sigma\cup\rho\cup\tau),\\ K_{\max\sigma}(\sigma\setminus\max\sigma,\rho,\tau) & \text{if } t>\max\sigma=\max(\sigma\cup\rho\cup\tau),\end{cases}$$
$$\int_0^t dA_s^\circ\mathsf{K}_s \mathrel{\widehat{=}} \begin{cases} 0 & \text{if } t\le\max(\sigma\cup\rho\cup\tau),\\ K_{\max\rho}(\sigma,\rho\setminus\max\rho,\tau) & \text{if } t>\max\rho=\max(\sigma\cup\rho\cup\tau),\end{cases}$$
$$\int_0^t dA_s^-\mathsf{K}_s \mathrel{\widehat{=}} \begin{cases} 0 & \text{if } t\le\max(\sigma\cup\rho\cup\tau),\\ K_{\max\tau}(\sigma,\rho,\tau\setminus\max\tau) & \text{if } t>\max\tau=\max(\sigma\cup\rho\cup\tau).\end{cases}$$

PROOF: The proof is a variation of the arguments as in the 2-argument case. But note that the 3-argument kernel functions are defined only for disjoint (σ, ρ, τ). Thus $\max(\sigma, \rho, \tau)$ is either $\max \sigma$ or $\max \rho$ or $\max \tau$. For the creation case note that by $a_s^+ a_s^+ = 0$ we get:

$$\begin{aligned}
\int_0^t dA_s^+ \mathsf{K}_s &= \sum_{0 \le s < t} \sum_{(\sigma,\rho,\tau) \in \Gamma^3_{\neq}} K_s(\sigma, \rho, \tau) a_s^+ a_\sigma^+ a_\rho^\circ a_\tau^- \\
&= \sum_{0 \le s < t} \sum_{\substack{(\sigma,\rho,\tau) \in \Gamma^3_{\neq} \\ s \notin \sigma}} K_s(\sigma, \rho, \tau) a_{\sigma \cup s}^+ a_\rho^\circ a_\tau^- \\
&= \sum_{0 \le s < t} \sum_{\substack{(\sigma,\rho,\tau) \in \Gamma^3_{\neq} \\ s \in \sigma}} K_s(\sigma \setminus s, \rho, \tau) a_\sigma^+ a_\rho^\circ a_\tau^- \\
&= \sum_{\substack{(\sigma,\rho,\tau) \in \Gamma^3_{\neq} \\ \max \sigma = \max(\sigma \cup \rho \cup \tau) < t}} K_{\max \sigma}(\sigma \setminus \max \sigma, \rho, \tau) a_\sigma^+ a_\rho^\circ a_\tau^- \\
&\mathrel{\widehat{=}} \begin{cases} K_{\max \sigma}(\sigma \setminus \max \sigma, \rho, \tau) & \text{if} \quad \max \sigma = \max(\sigma \cup \rho \cup \tau) < t, \\ 0, \quad \text{otherwise.} \end{cases}
\end{aligned}$$

In the last step we have used the adaptedness of the kernel function. For the number case we apply $a_s^\circ a_s^+ = a_s^+$ and $a_s^\circ a_s^\circ = a_s^\circ$ and several times the adaptedness condition.

$$\begin{aligned}
\int_0^t dA_s^\circ \mathsf{K}_s &= \sum_{0 \le s < t} \sum_{(\sigma,\rho,\tau) \in \Gamma^3_{\neq}} K_s(\sigma, \rho, \tau) a_s^\circ a_\sigma^+ a_\rho^\circ a_\tau^- \\
&= \sum_{0 \le s < t} \left(\sum_{\substack{(\sigma,\rho,\tau) \in \Gamma^3_{\neq} \\ s \in \sigma}} K_s(\sigma, \rho, \tau) a_\sigma^+ a_\rho^\circ a_\tau^- + \sum_{\substack{(\sigma,\rho,\tau) \in \Gamma^3_{\neq} \\ s \notin \sigma}} K_s(\sigma, \rho, \tau) a_\sigma^+ a_s^\circ a_\rho^\circ a_\tau^- \right) \\
&= \sum_{0 \le s < t} \sum_{\substack{(\sigma,\rho,\tau) \in \Gamma^3_{\neq} \\ s \notin \sigma}} K_s(\sigma, \rho, \tau) a_\sigma^+ a_s^\circ a_\rho^\circ a_\tau^- \\
&= \sum_{0 \le s < t} \left(\sum_{\substack{(\sigma,\rho,\tau) \in \Gamma^3_{\neq} \\ s \notin \sigma \wedge s \in \rho}} K_s(\sigma, \rho, \tau) a_\sigma^+ a_\rho^\circ a_\tau^- + \sum_{\substack{(\sigma,\rho,\tau) \in \Gamma^3_{\neq} \\ s \notin \sigma \wedge s \notin \rho}} K_s(\sigma, \rho, \tau) a_\sigma^+ a_{\rho \cup s}^\circ a_\tau^- \right) \\
&= \sum_{0 \le s < t} \sum_{\substack{(\sigma,\rho,\tau) \in \Gamma^3_{\neq} \\ s \notin \sigma \wedge s \in \rho}} K_s(\sigma, \rho \setminus s, \tau) a_\sigma^+ a_\rho^\circ a_\tau^- \\
&= \sum_{\substack{(\sigma,\rho,\tau) \in \Gamma^3_{\neq} \\ \max \rho = \max(\sigma \cup \rho \cup \tau) < t}} K_{\max \rho}(\sigma, \rho \setminus \max \rho, \tau) a_\sigma^+ a_\rho^\circ a_\tau^- \\
&\mathrel{\widehat{=}} \begin{cases} K_{\max \rho}(\sigma, \rho \setminus \max \rho, \tau) & \text{if} \quad \max \rho = \max(\sigma \cup \rho \cup \tau) < t, \\ 0, \quad \text{otherwise.} \end{cases}
\end{aligned}$$

The proof for the annihilation case follows the same scheme but this time one uses $a_s^- a_s^+ = a_s^\bullet$, $a_s^- a_s^\circ = a_s^-$ and $a_s^- a_s^- = 0$ and of course the adaptedness of the kernel function.

$$
\begin{aligned}
\int_0^t dA_s^- \mathsf{K}_s &= \sum_{0\le s<t} \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K_s(\sigma,\rho,\tau) a_s^- a_\sigma^+ a_\rho^\circ a_\tau^- \\
&= \sum_{0\le s<t} \left(\sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ s\in\sigma}} K_s(\sigma,\rho,\tau) a_s^\bullet a_{\sigma\setminus s}^+ a_\rho^\circ a_\tau^- + \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ s\notin\sigma}} K_s(\sigma,\rho,\tau) a_\sigma^+ a_s^- a_\rho^\circ a_\tau^- \right) \\
&= \sum_{0\le s<t} \left(\sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ s\notin\sigma\wedge s\in\rho}} K_s(\sigma,\rho,\tau) a_\sigma^+ a_s^- a_{\rho\setminus s}^\circ a_\tau^- + \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ s\notin\sigma\wedge s\notin\rho}} K_s(\sigma,\rho,\tau) a_\sigma^+ a_\rho^\circ a_s^- a_\tau^- \right) \\
&= \sum_{0\le s<t} \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ s\notin\sigma\wedge s\notin\rho\wedge s\notin\tau}} K_s(\sigma,\rho,\tau) a_\sigma^+ a_\rho^\circ a_{\tau\cup s}^- \\
&= \sum_{0\le s<t} \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ s\notin\sigma\wedge s\notin\rho\wedge s\in\tau}} K_s(\sigma,\rho,\tau\setminus s) a_\sigma^+ a_\rho^\circ a_\tau^- \\
&= \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max\tau=\max(\sigma\cup\rho\cup\tau)<t}} K_{\max\tau}(\sigma,\rho,\tau\setminus\max\tau) a_\sigma^+ a_\rho^\circ a_\tau^- \\
&\mathrel{\widehat{=}} \begin{cases} K_{\max\tau}(\sigma,\rho,\tau\setminus\max\tau) & \text{if} \quad \max\tau=\max(\sigma\cup\rho\cup\tau)<t, \\ 0, \quad \text{otherwise.} \end{cases}
\end{aligned}
$$

For the time case we use the adaptedness of the integrand and the rules $a_s^\bullet a_s^+ = a_s^\bullet a_s^\circ = 0$ and $a_s^\bullet a_s^- = \frac{1}{H} a_s^-$.

$$
\begin{aligned}
\int_0^t dA_s^\bullet \mathsf{K}_s &= \sum_{0\le s<t} \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K_s(\sigma,\rho,\tau) a_s^\bullet a_\sigma^+ a_\rho^\circ a_\tau^- \\
&= \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} \sum_{0\le s<t} K_s(\sigma,\rho,\tau) a_s^\bullet a_\sigma^+ a_\rho^\circ a_\tau^- \\
&= \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} \sum_{\max(\sigma\cup\rho\cup\tau)<s<t} K_s(\sigma,\rho,\tau) a_s^\bullet a_\sigma^+ a_\rho^\circ a_\tau^- \\
&= \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} \sum_{\max(\sigma\cup\rho\cup\tau)<s<t} K_s(\sigma,\rho,\tau) \notin_s(\sigma\cup\rho) \frac{1}{H} a_\sigma^+ a_\rho^\circ a_\tau^- \\
&\mathrel{\widehat{=}} \sum_{\max(\sigma\cup\rho\cup\tau)<s<t} K_s(\sigma,\rho,\tau) \notin_s(\sigma\cup\rho) \frac{1}{H},
\end{aligned}
$$

and an empty sum is zero. □

This proposition suggests defining the integration directly on the level of kernel functions. Thus we are going to define integration of an adapted 3-argument kernel process function with respect to the integrators in the following way. Note that by adaptedness of the kernel function the summation in the time case can begin at time zero.

Definition 2.3.17

$$\begin{aligned}
\int_0^t dA_s^{\bullet} K_s(\sigma,\rho,\tau) &= \sum_{0\le s<t} K_s(\sigma,\rho,\tau) \notin_s(\sigma\cup\rho)\frac{1}{H}\,. \\
\int_0^t dA_s^{+} K_s(\sigma,\rho,\tau) &= \begin{cases} K_{\max\sigma}(\sigma\setminus\max\sigma,\rho,\tau) & \textit{for} \quad \max\sigma = \max(\sigma\cup\rho\cup\tau) < t\,, \\ 0, \quad \textit{otherwise.} \end{cases} \\
\int_0^t dA_s^{\circ} K_s(\sigma,\rho,\tau) &= \begin{cases} K_{\max\rho}(\sigma,\rho\setminus\max\rho,\tau) & \textit{for} \quad \max\rho = \max(\sigma\cup\rho\cup\tau) < t\,, \\ 0, \quad \textit{otherwise.} \end{cases} \\
\int_0^t dA_s^{-} K_s(\sigma,\rho,\tau) &= \begin{cases} K_{\max\tau}(\sigma,\rho,\tau\setminus\tau) & \textit{for} \quad \max\tau = \max(\sigma\cup\rho\cup\tau) < t\,, \\ 0, \quad \textit{otherwise.} \end{cases}
\end{aligned}$$

Here K is in all cases an adapted internal kernel function. ◁

Since $\Gamma^3_{\neq}$ is measurable with respect to the internal measure we can extend the 3-argument kernel functions to Γ^3 by setting $K(\sigma,\rho,\tau)=0$ if $(\sigma,\rho,\tau)\in\Gamma^3\setminus\Gamma^3_{\neq}$. Owing to $m^3(\Gamma^3\setminus\Gamma^3_{\neq})\approx 0$ this is consistent with the concept of standard 3-argument kernels. All results obtained in this section then remain valid (with the obvious modification).

4 Clark-Ocone Formula, Martingale Representation and Quantum Stochastic Differential Equations

This Section provides us with two simple applications in the internal calculus. In the first Subsection we first prove an internal Clark-Ocone formula. This corresponds to the predictable representation of random variables in the classical case. Secondly we derive the internal quantum martingale representation theorem. Indeed, this is a consequence of the internal Clark-Ocone formula applied to a martingale. In the second Subsection we consider internal quantum stochastic differential equations. These are in fact hyperfinite difference equations. We show that an equation with constant coefficients but nonlinear noise terms has a specific internal kernel solution.

4.1 Internal Clark-Ocone Formula and Internal Martingale Representation Theorem

In this Subsection we prove a Clark-Ocone formula and the martingale representation theorem in the internal setting. Since all sets involved are hyperfinite there are no problems regarding convergence of series and domains of operators. We derive the formulas for 3-argument kernel operators. We define for each internal quantum noise an adapted derivative.

Definition 2.4.1 *For each of the symbols* $\{+, \circ, -\}$ *we define the* adapted derivatives $(\mathbb{D}_t^+)_{t\in T}, (\mathbb{D}_t^\circ)_{t\in T}$ *and* $(\mathbb{D}_t^-)_{t\in T}$ *as internal operators acting on an internal operator* K *by*

$$\begin{aligned}
\mathbb{D}_t^+\mathsf{K} &= \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\rho\cup\tau)<t}} K(\sigma\cup t,\rho,\tau)a_\sigma^+a_\rho^\circ a_\tau^- \,,\\
\mathbb{D}_t^\circ\mathsf{K} &= \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\rho\cup\tau)<t}} K(\sigma,\rho\cup t,\tau)a_\sigma^+a_\rho^\circ a_\tau^- \,,\\
\mathbb{D}_t^-\mathsf{K} &= \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\rho\cup\tau)<t}} K(\sigma,\rho,\tau\cup t)a_\sigma^+a_\rho^\circ a_\tau^- \,. \qquad \lhd
\end{aligned}$$

These derivatives map every internal operator to an adapted internal operator process. As a preliminary result we calculate the expectation of a kernel operator.

Lemma 2.4.2 $\quad \mathbb{E}(\mathsf{K}) \;=\; K(\emptyset,\emptyset,\emptyset).$

PROOF: $\quad \mathbb{E}(\mathsf{K}) \;=\; \langle\Omega, \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma,\rho,\tau)a_\sigma^+a_\rho^\circ a_\tau^-\Omega\rangle \;=\; K(\emptyset,\emptyset,\emptyset).$ □

Now we can derive a version of the Clark-Ocone formula in the hyperfinite setting. The result is analogous to the standard result but defined for every internal quantum random variable. The proof follows the author's idea for the proof in the classical case in discrete finite time [LM00].

Theorem 2.4.3 *Let* $\mathsf{K} = \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma,\rho,\tau)a_\sigma^+a_\rho^\circ a_\tau^-$ *be an internal 3-argument kernel operator. Then we have*

$$\mathsf{K} \;=\; \mathbb{E}(\mathsf{K})\mathbb{1} + \int_0^1 dA_t^+\mathbb{D}_t^+\mathsf{K} + \int_0^1 dA_t^\circ\mathbb{D}_t^\circ\mathsf{K} + \int_0^1 dA_t^-\mathbb{D}_t^-\mathsf{K}$$

where $\mathbb{1} = a_\emptyset^+a_\emptyset^\circ a_\emptyset^-$ *is the identity operator on* $\mathcal{F}_{\mathrm{int}}$.

PROOF: By definition of the integrals we get

$$\begin{aligned}
\int_0^1 dA_t^+\mathbb{D}_t^+\mathsf{K} &= \sum_{t\in T}\sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\rho\cup\tau)<t}} K(\sigma\cup t,\rho,\tau)a_{\sigma\cup t}^+a_\rho^\circ a_\tau^-\\
&= \sum_{t\in T}\sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max\sigma=\max(\sigma\cup\rho\cup\tau)=t}} K(\sigma,\rho,\tau)a_\sigma^+a_\rho^\circ a_\tau^-
\end{aligned}$$

and similar results for the number and annihilation case apply. The final result is

$$\int_0^1 dA_t^+\mathbb{D}_t^+\mathsf{K} + \int_0^1 dA_t^\circ\mathbb{D}_t^\circ\mathsf{K} + \int_0^1 dA_t^-\mathbb{D}_t^-\mathsf{K} \;=$$

$$\sum_{t\in T}\Bigg[\sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max\sigma=\max(\sigma\cup\rho\cup\tau)=t}} + \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max\rho=\max(\sigma\cup\rho\cup\tau)=t}} + \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max\tau=\max(\sigma\cup\rho\cup\tau)=t}}\Bigg]K(\sigma,\rho,\tau)a^+_\sigma a^\circ_\rho a^-_\tau =$$

$$\sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\setminus\{(\emptyset,\emptyset,\emptyset)\}} K(\sigma,\rho,\tau)a^+_\sigma a^\circ_\rho a^-_\tau = \mathsf{K}-\mathbb{E}(\mathsf{K})\mathbb{1}\,. \qquad \square$$

Next we show, analogously to the internal Clark-Ocone formula, how to express every internal quantum martingale as integral against the fundamental processes. First we define what we mean by a martingale in this setting.

Definition 2.4.4 *An internal kernel process M_t is called a* martingale *if it is adapted and the internal kernel function M_t has the following additional property:*

$$\forall s\in T\,\forall t\in T\,\forall(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\left(\max(\sigma\cup\rho\cup\tau)<\min(s,t)\implies M_t(\sigma,\rho,\tau)=M_s(\sigma,\rho,\tau)\right).$$

This property is called the martingale property*.* ◁

We note that the fundamental processes are martingales. Now we intend to prove the martingale representation theorem for internal martingales. We give explicitly the representing adapted processes in terms of the original martingale. But a trivial little lemma is needed first. Recall that M_t is an internal kernel operator for each fixed $t\in T$. Thus we can apply the adapted derivatives. In the lemma we show that the resulting process is independent of t for large enough t.

Lemma 2.4.5 *Let M_t be an internal martingale. Further let $\mathbb{D}^\sharp_s$, $\sharp\in\{+,\circ,-\}$ be the adapted derivative operators as defined above. Then $\mathbb{D}^\sharp_s\mathsf{M}_t=0$ if $t\le s$ and*

$$\forall s\in T\,\forall t_1\in T\,\forall t_2\in T\left(s<t_1\wedge s<t_2\Longrightarrow \mathbb{D}^\sharp_s\mathsf{M}_{t_1}=\mathbb{D}^\sharp_s\mathsf{M}_{t_2}\right).$$

Proof: Suppose $t\le s$. Assuming that $\sharp=+$ we have

$$\mathbb{D}^+_s\mathsf{M}_t = \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\rho\cup\tau)<s}} M_t(\sigma\cup s,\rho,\tau)a^+_\sigma a^\circ_\rho a^-_\tau = 0$$

by adaptedness of the kernel function. In the other two cases a similar calculation gives the same result. This proves the first assertion. For the second we again prove only the creation case since the calculation for the other cases are similar. Thus suppose that $s<t_1$ and $s<t_2$. By the martingale property we obtain

$$\begin{aligned}\mathbb{D}^+_s\mathsf{M}_{t_1} &= \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\rho\cup\tau)<s}} M_{t_1}(\sigma\cup s,\rho,\tau)a^+_\sigma a^\circ_\rho a^-_\tau\\ &= \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\rho\cup\tau)<s}} M_{t_2}(\sigma\cup s,\rho,\tau)a^+_\sigma a^\circ_\rho a^-_\tau = \mathbb{D}^+_s\mathsf{M}_{t_2}\end{aligned}$$

This proves the second assertion. $\square$

Theorem 2.4.6 *Let M_t be an internal martingale. Then there exist three adapted kernel operator processes K_s^+, K_s° and K_s^- such that*

$$\mathsf{M}_t \;=\; \mathbb{E}(\mathsf{M}_0)\mathbb{1} + \int_0^t dA_s^+\mathsf{K}_s^+ + \int_0^t dA_s^\circ\mathsf{K}_s^\circ + \int_0^t dA_s^-\mathsf{K}_s^- \,.$$

Furthermore, the adapted processes are given by $\mathsf{K}_s^\sharp = \mathbb{D}_s^\sharp\mathsf{M}_t$ $(t > s)$ independent of t.

PROOF: Define the three adapted processes by the formula $\mathsf{K}_s^\sharp = \mathbb{D}_s^\sharp\mathsf{M}_t$ for some $t > s$. By the lemma this definition is independent of the particular chosen t. (We could take for example $t = \max T$.) Thus we can define the processes $\mathsf{K}_s^\sharp$ for all $s \in T \setminus \{\frac{H-1}{H}\}$. First note that by the martingale property we have

$$\mathbb{E}(\mathsf{M}_t) \;=\; M_t(\emptyset,\emptyset,\emptyset) \;=\; M_0(\emptyset,\emptyset,\emptyset) \;=\; \mathbb{E}(\mathsf{M}_0)$$

for every $t \in T$. We show now explicitly that for the defined processes the integrals on the right hand side equal $\mathsf{M}_t - \mathbb{E}(\mathsf{M}_0)\mathbb{1}$. For the creation case we obtain that

$$\begin{aligned}
\int_0^t dA_s^+\mathbb{D}_s^+\mathsf{M}_t \;&=\; \sum_{0\le s<t} a_s^+ \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\tau\cup\rho)<s}} M_t(\sigma\cup s,\rho,\tau)a_\sigma^+a_\rho^\circ a_\tau^- \\
&=\; \sum_{0\le s<t} \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\tau\cup\rho)<s}} M_t(\sigma\cup s,\rho,\tau)a_{\sigma\cup s}^+a_\rho^\circ a_\tau^- \\
&=\; \sum_{0\le s<t} \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\tau\cup\rho)=\max\sigma=s}} M_t(\sigma,\rho,\tau)a_\sigma^+a_\rho^\circ a_\tau^-
\end{aligned}$$

and similar results for the other cases. Summing up this gives

$$\int_0^t dA_s^+\mathbb{D}_s^+\mathsf{M}_t + \int_0^t dA_s^\circ\mathbb{D}_s^\circ\mathsf{M}_t + \int_0^t dA_s^-\mathbb{D}_s^-\mathsf{M}_t$$

$$= \sum_{0\le s<t}\Bigg[\sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\tau\cup\rho)=\max\sigma=s}} + \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\tau\cup\rho)=\max\rho=s}} + \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\tau\cup\rho)=\max\tau=s}}\Bigg] M_t(\sigma,\rho,\tau)a_\sigma^+a_\rho^\circ a_\tau^-$$

$$= \sum_{0\le s<t} \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\tau\cup\rho)=s}} M_t(\sigma,\rho,\tau)a_\sigma^+a_\rho^\circ a_\tau^- \;=\; \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\setminus\{(\emptyset,\emptyset,\emptyset)\}\\ \max(\sigma\cup\tau\cup\rho)<t}} M_t(\sigma,\rho,\tau)a_\sigma^+a_\rho^\circ a_\tau^-$$

$$= \sum_{\substack{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}\\ \max(\sigma\cup\tau\cup\rho)<t}} M_t(\sigma,\rho,\tau)a_\sigma^+a_\rho^\circ a_\tau^- - M_t(\emptyset,\emptyset,\emptyset)a_\emptyset^+a_\emptyset^\circ a_\emptyset^- \;=\; \mathsf{M}_t - \mathbb{E}(\mathsf{M}_0)\mathbb{1}\,.$$

This proves the internal martingale representation theorem. □

4.2 Internal Quantum Stochastic Differential Equations

In this subsection we replace our time process $A_t^\bullet = \sum_{s<t} a_s^\bullet$ by a strongly equivalent process $\widetilde{A}_t^\bullet = \sum_{s<t} \widetilde{a}_s^\bullet$ where $\widetilde{a}_s^\bullet$ is just multiplication by the time increment $\frac{1}{H}$. Strong equivalence is explained in section 3.1 of chapter 3. Roughly speaking, strongly equivalent operators are the same when going back to the standard world. To indicate that we have $\widetilde{a}_s^\bullet$ in the sum instead of $a_s^\bullet$ we use the notation $\sharp \in \{\widetilde{\bullet}, +, \circ, -\}$ or just $\widetilde{\sharp}$.

Also for this section we introduce an initial space. That is, we take our internal kernel functions as ${}^*\mathcal{B}(\mathcal{K})$-valued internal functions for some separable (standard) Hilbert space $\mathcal{K}$. As a matter of fact, this doesn't effect the proved assertions about ${}^*\mathbb{C}$-valued kernel functions. The only difference is that such kernel functions act on ${}^*\mathcal{K}$-valued internal function. But to rescue the concept of product vectors we take functions of the type $x \otimes \pi_\Phi$ for standard $x \in {}^*\mathcal{K}$ and $\Phi \in SB^2(T)$. The action of a 3-argument kernel operator K with kernel function $K : \Gamma^3_{\neq} \to {}^*\mathcal{B}(\mathcal{K})$ is then given by

$$\mathsf{K}x \otimes \pi_\Phi(\alpha) \;=\; \sum_{\sigma \dot\cup \rho \dot\cup \beta = \alpha} \; \sum_{\substack{\tau \\ \tau \cap (\rho \cup \beta) = \emptyset}} K(\sigma,\rho,\tau) x \otimes \pi_\Phi(\rho \cup \beta \cup \tau) \left(\frac{1}{H}\right)^{|\tau|} .$$

We take K as given by the Fock expansion

$$\mathsf{K} \;=\; \sum_{(\sigma,\rho,\tau) \in \Gamma^3_{\neq}} K(\sigma,\rho,\tau) \otimes a_\sigma^+ a_\rho^\circ a_\tau^- .$$

Note that operators L that act only on the initial Hilbert space ${}^*\mathcal{K}$ have kernels of the following form

$$L(\sigma,\rho,\tau) \;=\; \begin{cases} L & \text{if } \sigma = \rho = \tau = \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

for some $L \in {}^*\mathcal{B}(\mathcal{K})$. On the other hand every operator $L \in {}^*\mathcal{B}(\mathcal{K})$ gives us a constant internal adapted kernel process L_t by setting

$$L_t(\sigma,\rho,\tau) \;=\; \begin{cases} L & \text{if } \sigma = \rho = \tau = \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

independent of the particular $t \in T$.

Lemma 2.4.7 *Suppose that* L *is a constant internal kernel operator and* K *some kernel operator. Then the product kernel function is given by*

$$\mathsf{LK} \;\widehat{=}\; L(\emptyset,\emptyset,\emptyset) K(\sigma,\rho,\tau) .$$

PROOF: This is a corollary to proposition 2.3.12. □

In the notation using the Wick product this shows that $L \diamond K(\sigma,\rho,\tau) \;=\; LK(\sigma,\rho,\tau)$ where L is the single value which the operator L attains.

Also unchanged is the action of the fundamental operators on a kernel. We calculate the action of a_s^+, a_s°, a_s^- on a kernel function according to the multiplication rules of the internal quantum Itô table. Then we summarize the result in a lemma.

$$\begin{aligned}
a_s^+\mathsf{K} &= \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma,\rho,\tau)a_s^+a_\sigma^+a_\rho^\circ a_\tau^- = \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma,\rho,\tau)\notin_s(\sigma)a^+_{\sigma\cup s}a_\rho^\circ a_\tau^- \\
&= \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma\setminus s,\rho,\tau)\in_s(\sigma)a_\sigma^+a_\rho^\circ a_\tau^- .
\end{aligned}$$

$$\begin{aligned}
a_s^\circ\mathsf{K} &= \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma,\rho,\tau)a_s^\circ a_\sigma^+a_\rho^\circ a_\tau^- \\
&= \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma,\rho,\tau)\in_s(\sigma)a_\sigma^+a_\rho^\circ a_\tau^- + \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma,\rho,\tau)\notin_s(\sigma)a_\sigma^+a_s^\circ a_\rho^\circ a_\tau^- \\
&= \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} \Big[K(\sigma,\rho,\tau)\in_s(\sigma) + K(\sigma,\rho,\tau)\notin_s(\sigma)\in_s(\rho)\Big]a_\sigma^+a_\rho^\circ a_\tau^- \\
&\quad + \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma,\rho,\tau)\notin_s(\sigma)\notin_s(\rho)a_\sigma^+a^\circ_{\rho\cup s}a_\tau^- \\
&= \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} \Big[K(\sigma,\rho,\tau)\Big(\in_s(\sigma)+\in_s(\rho)\Big) + K(\sigma,\rho\setminus s,\tau)\in_s(\rho)\Big]a_\sigma^+a_\rho^\circ a_\tau^- .
\end{aligned}$$

$$\begin{aligned}
a_s^-\mathsf{K} &= \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma,\rho,\tau)a_s^-a_\sigma^+a_\rho^\circ a_\tau^- \\
&= \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma,\rho,\tau)\in_s(\sigma)\frac{1}{H}a^+_{\sigma\setminus s}a_\rho^\circ a_\tau^- + \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma,\rho,\tau)\notin_s(\sigma)a_\sigma^+a_s^-a_\rho^\circ a_\tau^- \\
&= \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma\cup s,\rho,\tau)\notin_s(\sigma)\frac{1}{H}a_\sigma^+a_\rho^\circ a_\tau^- + \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma,\rho,\tau)\notin_s(\sigma)\in_s(\rho)a_\sigma^+a^\circ_{\rho\setminus s}a_s^-a_\tau^- \\
&\quad + \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} K(\sigma,\rho,\tau)\notin_s(\sigma)\notin_s(\rho)a_\sigma^+a_\rho^\circ a_s^-a_\tau^- \\
&= \sum_{(\sigma,\rho,\tau)\in\Gamma^3_{\neq}} \Big[K(\sigma\cup s,\rho,\tau)\notin_s(\sigma)\frac{1}{H} + K(\sigma,\rho\cup s,\tau\setminus s)\in_s(\tau) + K(\sigma,\rho,\tau\setminus s)\in_s(\tau)\Big]a_\sigma^+a_\rho^\circ a_\tau^- .
\end{aligned}$$

Lemma 2.4.8 *Let K be the kernel of some internal operator. Then the action of a_s^+, a_s°, a_s^- on K is given by:*

$$\begin{aligned}
a_s^+K(\sigma,\rho,\tau) &= K(\sigma\setminus s,\rho,\tau)\in_s(\sigma), \\
a_s^\circ K(\sigma,\rho,\tau) &= \Big[K(\sigma,\rho,\tau)\Big(\in_s(\sigma)+\in_s(\rho)\Big) + K(\sigma,\rho\setminus s,\tau)\in_s(\rho)\Big], \\
a_s^-K(\sigma,\rho,\tau) &= \Big[K(\sigma\cup s,\rho,\tau)\notin_s(\sigma)\frac{1}{H} + K(\sigma,\rho\cup s,\tau\setminus s)\in_s(\tau) + K(\sigma,\rho,\tau\setminus s)\in_s(\tau)\Big].
\end{aligned}$$

Thus on the level of kernel functions we can take this as definition.

We will see later that in the above lemma only the contributions $K(\sigma \setminus s, \rho, \tau) \in_s(\sigma)$, $K(\sigma, \rho \setminus s, \tau) \in_s(\rho)$ and $K(\sigma, \rho, \tau \setminus s) \in_s(\tau)$ are important. Thus in some sense the fundamental operators act as singular white noise operators on kernel functions. The next theorem is a existence result for a differential equation in the hyperfinite setting. We need some more notation. We set $T = \{t_0, \ldots, t_{H-1}\}$. Thus $t_k = \frac{k}{H}$ and $t_k - t_l = t_{k-l}$ for $k \geq l$. Further we use the notation

$$\prod_{n}^{k=0} t_k \quad = \quad t_n \cdots t_0$$

to indicate that the product is written in reversed order from right to left.

Theorem 2.4.9 *Identify $L_0, L^{\tilde{\bullet}}, L^+, L^\circ, L^- \in {}^*\mathcal{B}(\mathcal{K})$ with the kernel functions of the constant internal adapted operator processes they define. Then the internal stochastic differential equation*

$$K_t \quad = \quad L_0 + \sum_{\tilde{\sharp}} \sum_{s<t} a_s^{\tilde{\sharp}} L^{\tilde{\sharp}} \diamond K_s \quad = \quad L_0 + \sum_{\tilde{\sharp}} \sum_{s<t} a_s^{\tilde{\sharp}} L^{\tilde{\sharp}} K_s$$

has as solution the kernel function

$$K_t(\sigma, \rho, \tau) \quad = \quad \begin{cases} \prod_n^{k=0} \Pi_{\sigma,\rho,\tau}(t_k) L_0 & \textit{if} \;\; \max(\sigma \cup \rho \cup \tau) < t\,, \\ 0, & \textit{otherwise}\,. \end{cases}$$

Here the function $\Pi_{\sigma,\rho,\tau}(t_k)$ is defined for $(\sigma, \rho, \tau) \in \Gamma^3_{\neq}$ by

$$\Pi_{\sigma,\rho,\tau}(t_k) \quad = \quad \begin{cases} 1 + \frac{1}{H} L^{\tilde{\bullet}} & \textit{if} \;\; t_k \notin \sigma \cup \rho \cup \tau\,, \\ L^+ & \textit{if} \;\; t_k \in \sigma\,, \\ L^\circ & \textit{if} \;\; t_k \in \rho\,, \\ L^- & \textit{if} \;\; t_k \in \tau\,. \end{cases}$$

Proof: We need only write the differential equation as hyperfinite difference equation. By induction we get:

$$\begin{aligned} K_{t_{n+1}}(\sigma, \rho, \tau) \quad &= \quad \Big[(1 + \frac{1}{H} L^{\tilde{\bullet}} + a^+_{t_n} L^+ + a^\circ_{t_n} L^\circ + a^-_{t_n} L^-) K_{t_n}\Big](\sigma, \rho, \tau) \\ &= \quad \Big[\prod_{n}^{k=0} (1 + \frac{1}{H} L^{\tilde{\bullet}} + a^+_{t_k} L^+ + a^\circ_{t_k} L^\circ + a^-_{t_k} L^-) L_0\Big](\sigma, \rho, \tau)\,. \end{aligned}$$

Now suppose that $(\sigma, \rho, \tau) \in \Gamma^3_{\neq}$ is given. Then since L_0 is constant we see that by the rules of lemma 2.4.8 the triple (σ, ρ, τ) must be reduced to $(\emptyset, \emptyset, \emptyset)$. Thus by the action of a_s^+, a_s° and a_s^- there must occur the operators $\{a_s^+,\, s \in \sigma\}$, $\{a_s^\circ,\, s \in \rho\}$ and $\{a_s^-,\, s \in \tau\}$ but no more. Calculating the product we see that at the points $s \in \sigma \cup \rho \cup \tau$ of the time line only the action of the indicated operators contribute. This has as first effect that if $\max(\sigma \cup \rho \cup \tau) \geq t$ then the product is zero since $L_0(\sigma, \rho, \tau)$ can not be made to $L_0(\emptyset, \emptyset, \emptyset)$. (Note that the product runs only up to $t - \frac{1}{H}$.) On the other hand if $t > \max(\sigma \cup \rho \cup \tau)$

then on the points $s \in \{t_0, \ldots, t_n = t - \frac{1}{H}\} \setminus (\sigma \cup \rho \cup \tau)$ of the relative complement of $\sigma \cup \rho \cup \tau$ we have as the only non-trivial acting operator $1 + \frac{1}{H} L^{\widetilde{\bullet}}$. We note that by lemma 2.4.7 the Wick multiplication of kernels is here only the multiplication by $L^{\widetilde{\sharp}}$ since these kernels are constant. Combining the results we have proved the assertion of the theorem. $\square$

For almost all nearstandard (especially finite) (σ, ρ, τ) this solution turns out to be approximately the solution given in proposition 1.3.5 in the introductory Chapter on the standard kernel calculus. Now we take a nonlinear differential equation and solve this explicitly. Exactly we leave the term in the time process linear but take the input to the quantum noises as nonlinear. The coefficients remain constant.

Let $n_+, n_\circ, n_- \in {}^*\mathbb{N}$ be finite and suppose that we have internal operators $L_k^\sharp \in {}^*\mathcal{B}(\mathcal{K})$, $k = 1, \ldots, n_\sharp$, $\sharp \in \{+, \circ, -\}$. Then we build the internal polynomials

$$P^\sharp \diamond K \;=\; P^\sharp(K) \;=\; L^\sharp_{n_\sharp} K^{\diamond n_\sharp} + \cdots + L^\sharp_1 K, \;\; \sharp \in \{+, \circ, -\},$$

where the product is the Wick product. Further we set $P^{\widetilde{\bullet}}(K) = L^{\widetilde{\bullet}} K$ for some $L^{\widetilde{\bullet}} \in {}^*\mathcal{B}(\mathcal{K})$. We note that the polynomials are only defined if $K^{\diamond n}$ exists for all relevant n. To avoid parenthesis we introduce also the notation $P^\sharp \diamond$ to indicate that the polynomial $P^\sharp$ acts on everything which stands to the right of $P^\sharp$.

Lemma 2.4.10 *Let $K \in {}^*\mathcal{B}(\mathcal{K})$. Suppose that*

$$K(\sigma, \rho, \tau) \;=\; \begin{cases} K & \text{if } \sigma = \rho = \tau = \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

and that P is some polynomial as described above. Then

$$P \diamond K(\sigma, \rho, \tau) \;=\; \begin{cases} P(K) & \text{if } \sigma = \rho = \tau = \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, this gives

$$P \diamond K(\sigma, \rho, \tau) \;=\; \Big[L_n \Big(K(\emptyset, \emptyset, \emptyset)\Big)^{n-1} + \cdots + L_2 K(\emptyset, \emptyset, \emptyset) + L_1\Big] K(\sigma, \rho, \tau)\,.$$

Proof: By lemma 2.4.7 and an inductive argument we have

$$K^{\diamond n}(\sigma, \rho, \tau) \;=\; (K(\emptyset, \emptyset, \emptyset))^{n-1}\, K(\sigma, \rho, \tau)\,.$$

Thus the assertion follows immediately. $\square$

Theorem 2.4.11 *Suppose that we have polynomials $P^\sharp$, $\sharp \in \{+, \circ, -\}$ and linear $P^{\widetilde{\bullet}}$ as above. Further let $L_0 \in {}^*\mathcal{B}(\mathcal{K})$. Then the internal quantum stochastic differential equation*

$$K_t \;=\; L_0 \,+\, \sum_{\widetilde{\sharp}} \sum_{s<t} a_s^{\widetilde{\sharp}} P^{\widetilde{\sharp}} \diamond K_s$$

has a solution given by

$$K_t(\sigma,\rho,\tau) \;=\; \begin{cases} \prod_n^{k=0} \Pi_{\sigma,\rho,\tau}(t_k) L_0 & \textit{if} \;\; \max(\sigma\cup\rho\cup\tau) < t\,, \\ 0, & \textit{otherwise}\,. \end{cases}$$

Here the function $\Pi_{\sigma,\rho,\tau}(t_k)$ *is defined for* $(\sigma,\rho,\tau)\in\Gamma^3_{\neq}$ *by*

$$\Pi_{\sigma,\rho,\tau}(t_k) \;=\; \begin{cases} 1+\frac{1}{H}\tilde{L^{\bullet}} & \textit{if} \;\; t_k \notin \sigma\cup\rho\cup\tau\,, \\ P^{+}\diamond & \textit{if} \;\; t_k\in\sigma\,, \\ P^{\circ}\diamond & \textit{if} \;\; t_k\in\rho\,, \\ P^{-}\diamond & \textit{if} \;\; t_k\in\tau\,. \end{cases}$$

Proof: As in the linear case we get a hyperfinite difference equation:

$$K_{t_{n+1}}(\sigma,\rho,\tau) \;=\; \Big[\prod_n^{k=0}(1+\frac{1}{H}\tilde{L^{\bullet}} + a^{+}_{t_k}P^{+}\diamond + a^{\circ}_{t_k}P^{\circ}\diamond + a^{-}_{t_k}P^{-}\diamond)L_0\Big](\sigma,\rho,\tau)\,.$$

Using lemma 2.4.10 we see that at each step k of the application of the product the resulting operator is an operator on the initial space ${}^*\mathcal{K}$. Thus the same argument as in the linear case applies. This proves the theorem. $\square$

The former proved theorem 2.4.9 is a special case of this theorem. Going back to the standard world (if possible) the given solution would become a solution to a (standard) quantum stochastic differential equation where the process comes nonlinear in the noise terms. As we see in the structure of the differential equation we have to assure only that the coefficients and the initial condition are constant. Further, $\tilde{L^{\bullet}}$ should be the generator of a semigroup such that for each time also the appropriate powers of that semigroup exist. To our knowledge nonlinear quantum stochastic differential equation have not yet been studied in the literature. The motivation for our approach to converting the internal solution to a solution of the standard differential equation comes from Keisler's infinitesimal approach to classical stochastic analysis (cf. [Kei84]).

Chapter 3

Standard Part Maps, Liftings and Representations of Operators

This chapter is the first of two chapters where we connect the internal quantum stochastic calculus to the standard quantum stochastic calculus. In the first section we define several standard part maps. In particular one from Γ onto $\mathcal{P}_{\text{fin}}$. This map becomes measurable and so we identify the measure space $(\mathcal{P}_{\text{fin}}, \mathfrak{B}, \Lambda)$ as a subspace of the Loeb measure space $(\Gamma, L(\mathfrak{A}), m_L)$. In the second section we study liftings for functions on $[0,1]$ and $\mathcal{P}_{\text{fin}}$. If the functions are operator-valued we use the lifting result given in Section 2 of the Appendix. In the third section we introduce three ways of representing a standard operator by an internal operator. The first strict representation is just a lifting of the kernel functions and applies only to standard kernel operators. The second strong representation is related to the strong operator topology. We take a lifting of an exponential vector. Then we check if the internal operator applied to the lifting gives a lifting of the standard operator applied to the exponential vector. The third weak representation works a similar way but we evaluate it in the scalar product. Hence it is related to the weak operator topology.

1 Standard Part Maps

In this section we define some standard part maps. In the first subsection we give an appropriate standard part map for the Loeb space of Γ. In the second subsection we extend this standard part map to Γ^n. Actually the case in the first subsection is included as special case in the second subsection for $n = 1$. But since in this special case one has more powerful constructions and less technical proofs we develop it in its own right. As usual we denote by $\mathcal{B}(\mathcal{K})$ the Banach space of all bounded linear operators on some Hilbert space $\mathcal{K}$. For later purposes we define also standard part maps for ${}^*\mathcal{B}(\mathcal{K})$ and ${}^*\mathcal{K}$.

1.1 Standard Part Map for Γ and Loeb Space of Γ

In Section 1.2 of Chapter 2 we introduced the symmetric measure space $(\Gamma, \mathfrak{A}, m)$ over the hyperfinite time-line $(T, \mathcal{A}, \mu)$. These are both finite internal measure spaces and we can build up the corresponding Loeb spaces $(\Gamma, L(\mathfrak{A}), m_L)$ and $(T^n, L(\mathcal{A}^n), \mu_L^n)$, $n \in \mathbb{N}$. Actually we should write $(\mu^n)_L$ but to avoid too many parenthesis we omit them. In nonstandard analysis it is well known that the inverse standard part map st^{-1} is a measure preserving homomorphism of the completed n-dimensional Lebesgue space $([0,1]^n, \mathcal{B}^n, \lambda^n)$ to the Loeb space $(T^n, L(\mathcal{A}^n), \mu_L^n)$. This is the content of the next proposition which is essentially proposition 5.2.7(1) in Osswald's contribution [OS, Chaper 5] of the book [LW00]. But first we define the standard part map component-wise.

Definition 3.1.1 *Let* $\mathrm{st} : T \to [0,1]$ *be the restriction of the standard part map* $\mathrm{st} : {}^*\mathbb{R}_{\mathrm{fin}} \to \mathbb{R}$ *to* T*. Then we extend* st *to* T^n *by*

$$\mathrm{st} \ : \ T^n \longrightarrow [0,1]^n \quad : \quad (t_1, \cdots, t_n) \longmapsto (\mathrm{st}(t_1), \cdots, \mathrm{st}(t_n)) \qquad \triangleleft$$

Proposition 3.1.2 *Fix* $n \in \mathbb{N}$*. A subset* $B \subseteq [0,1]^n$ *is Lebesgue measurable iff* $\mathrm{st}^{-1}(B)$ *is Loeb measurable. In that case we have*

$$\lambda^n(B) \ = \ \mu_L^n(\mathrm{st}^{-1}(B)).$$

The proof is given in [OS] proposition 5.2.7 and for the case $n = 1$ this is the result in Albeverio et al. [AFHKL86] proposition 3.2.5. Our goal is to use these results for all $n \in \mathbb{N}$ simultaneously and generalize it to the Loeb space of the symmetric measure space $(\Gamma, \mathfrak{A}, m)$ over $(T, \mathcal{A}, \mu)$.

To this end we intend to extend the standard part map in an appropriate way for the symmetric space Γ over T and show that the inverse standard part map is a measure preserving homomorphism of the (completed) symmetric measure space $(\mathcal{P}_{\mathrm{fin}}, \mathfrak{B}, \Lambda)$ over $([0,1], \mathcal{B}, \lambda)$ into the Loeb space of Γ. Recall that the constructions of $(\mathcal{P}_{\mathrm{fin}}, \mathfrak{B}, \Lambda)$ in Section 1 of Chapter 1 and that of $(\Gamma, \mathfrak{A}, m)$ in Section 2.1.2 are similar. The homomorphism φ^{-1} that sends sets to tuples in general position is by necessity a general map that is defined on the (hyper)finite power set of an arbitrary (internal) set. Thus this homomorphism is the same in both cases and it is justified to represent it by a single symbol φ. But the whole construction depends essentially only on this homomorphism φ respectively its inverse φ^{-1}.

We had defined two subspaces of Γ by

$$\Gamma_{\mathrm{fin}} \ = \ \cup_{n \in \mathbb{N}} \Gamma_n \ \text{ and } \ \Gamma_{\not\approx} \ = \ \{\sigma \in \Gamma : \forall s, t \in \sigma (s \neq t \to s \not\approx t\}$$

and the propositions 2.1.6 and 2.1.9 show that their complements in Γ are sets of Loeb measure zero. The set of near standard points in Γ was the intersection of these two sets. The next definition makes clear why.

Definition 3.1.3 *The* set of nearstandard points Γ_{st} *in* Γ *is defined by* $\Gamma_{\text{st}} = \Gamma_{\text{fin}} \cap \Gamma_{\not\approx}$. *The* standard part map st *is (partially) defined by*

$$\text{st} \;:\; \Gamma_{\text{st}} \longrightarrow \mathcal{P}_{\text{fin}} \quad:\quad \sigma \longmapsto \text{st}(\sigma) = \{\text{st}(s) : s \in \sigma\}\,.$$

For any set $A \in \mathcal{P}(\Gamma)$ *we define*

$$\text{st}(A) \;=\; \{\text{st}(\sigma) : \sigma \in A \cap \Gamma_{\text{st}}\}. \qquad \lhd$$

Note that by definition every point in $\Gamma \setminus \Gamma_{\text{st}}$ has not a standard part. Furthermore, subsets of Γ are mapped only partially pointwise to subsets of $\mathcal{P}_{\text{fin}}$. Some might wonder at why we do not take the standard part map as $\text{st}(\sigma) = \{\text{st}(s) : s \in \sigma\}$ for every $\sigma \in \Gamma$. But this would result in sets B such that $m_L(B) \neq \Lambda(\text{st}(B))$. For example taking for some finite n the set

$$B \;=\; \{\sigma = \{s_1, \cdots, s_n\} \in \Gamma_n : \text{st}(\sigma) = \{\text{st}(s_1), \cdots, \text{st}(s_n)\} = \{t\}, t \in [0,1]\}$$

we would have $\text{st}(B) = \{\{t\} : t \in [0,1]\}$ and $\varphi^{-1}(\text{st}(B)) = [0,1]$. This implies that $\Lambda(\text{st}(B)) = \lambda(\varphi^{-1}(\text{st}(B))) = 1$. But owing to $B \subseteq \Gamma \setminus \Gamma_{\not\approx}$ we have $m_L(B) = 0$. To exclude such pathological sets we restrict our standard part map to Γ_{st}. We will see that Γ_{st} is rich enough to make the standard part map an $L(\mathfrak{A})$-$\mathfrak{B}$-measurable map. Before we collect in the next proposition some basic facts about this standard part map we need the following definition and an important lemma.

Definition 3.1.4 *Let* $A \subseteq T^n$. *Then* A *is said to be* symmetric *if* $A \subseteq T^n_{\neq}$ *and*

$$(s_1, \cdots, s_n) \in A \Longrightarrow \forall \pi \in S(n)((s_{\pi(1)}, \cdots, s_{\pi(n)}) \in A)$$

where $S(n)$ *is the automorphism group of* $\{1, \cdots, n\}$. *For arbitrary* $B \subseteq T^n_{\neq}$ *the* symmetric set generated by B *is given by* $\varphi^{-1}(\varphi(B))$. $\lhd$

Remark 3.1.5 Note that for $B \subseteq T^n_{\neq}$ one has $B \subseteq \varphi^{-1}(\varphi(B))$ and if B is internal then $\varphi^{-1}(\varphi(B))$ is internal. Further a set $B \subseteq T^n_{\neq}$ is symmetric iff $B = \varphi^{-1}(\varphi(B))$. Indeed the set of all internal symmetric subsets of T^n is an internal subalgebra of ${}^*\mathcal{P}(T^n_{\neq})$ (complement is taken with respect to $T^n_{\neq}$).

Lemma 3.1.6 *A set* $A \subseteq \mathcal{P}(\Gamma)$ *is* m_L*-measurable iff* $\varphi^{-1}(A \cap \Gamma_n)$ *is* μ^n_L*-measurable for every* $n \in \mathbb{N}$. *Furthermore if* $A \subseteq \mathcal{P}(\Gamma)$ *is* m_L*-measurable then*

$$m_L(A) \;=\; \sum_{n \in \mathbb{N}} \frac{1}{n!} \mu^n_L(\varphi^{-1}(A \cap \Gamma_n))\,.$$

PROOF: Suppose $A \in L(\mathfrak{A})$. Then $A \cap \Gamma_n \in L(\mathfrak{A})$ since $\Gamma_n \in \mathfrak{A}$. For every standard $\varepsilon \in \mathbb{R}_+$ there exist $A_i, A_o \in \mathfrak{A}$ with

$$A_i \subseteq A \cap \Gamma_n \subseteq A_o \quad \text{and} \quad m(A_o \setminus A_i) < \varepsilon\,. \tag{3.1.1}$$

We can assume that $A_o \subseteq \Gamma_n$. (Otherwise take $\tilde{A}_o = A_o \cap \Gamma_n \in \mathfrak{A}$ and this fulfills condition (3.1.1).) But then we have

$$\varphi^{-1}(A_i) \subseteq \varphi^{-1}(A \cap \Gamma_n) \subseteq \varphi^{-1}(A_o) \in \mathcal{A}^n = {}^*\mathcal{P}(T^n)$$

where also $\varphi^{-1}(A_i) \in {}^*\mathcal{P}(T^n)$. By construction we get

$$\mu^n(\varphi^{-1}(A_o) \setminus \varphi^{-1}(A_i)) = n! m(A_o \setminus A_i) < n!\varepsilon$$

and this argument holds for every $n \in \mathbb{N}$. Thus $\varphi^{-1}(A \cap \Gamma_n)$ is μ_L^n-measurable for every $n \in \mathbb{N}$. Now let be $A \subseteq \mathcal{P}(\Gamma)$ and assume that $\varphi^{-1}(A \cap \Gamma_n) \in L(\mathcal{A}^n)$ for every $n \in \mathbb{N}$. We decompose $A = (A \cap \Gamma_{\text{fin}}) \dot\cup (A \cap \Gamma_\infty)$. By proposition 2.1.6 we know that $m_L(\Gamma_\infty) = 0$ and since the Loeb measure is complete it follows $m_L(A \cap \Gamma_\infty) = 0$. We have to show that $A \cap \Gamma_{\text{fin}}$ is m_L-measurable. By $\Gamma_{\text{fin}} = \dot\cup_{n\in\mathbb{N}} \Gamma_n$ it is

$$A \cap \Gamma_{\text{fin}} \;=\; A \cap \dot\cup_{n\in\mathbb{N}} \Gamma_n \;=\; \dot\cup_{n\in\mathbb{N}} (A \cap \Gamma_n)$$

and it is sufficient to prove for $n \in \mathbb{N}$ that $A \cap \Gamma_n$ is m_L-measurable. Fix $n \in \mathbb{N}$. Then by assumption $\varphi^{-1}(A \cap \Gamma_n)$ is μ_L^n-measurable. Thus for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ there exist $B_i, B_o \in \mathcal{A}^n$ with

$$B_i \subseteq \varphi^{-1}(A \cap \Gamma_n) \subseteq B_o \;\text{ and }\; \mu^n(B_o \setminus B_i) < \varepsilon\,.$$

Because $\varphi^{-1}(A \cap \Gamma_n)$ is symmetric we can assume that B_i and B_o are symmetric. Then $\varphi(B_i), \varphi(B_o) \subseteq \Gamma_n$ are internal and measurable and

$$\begin{aligned} m(\varphi(B_o) \setminus \varphi(B_i)) &= \frac{1}{n!}\mu^n(\varphi^{-1}(\varphi(B_o) \setminus \varphi(B_i))) \\ &= \frac{1}{n!}\mu^n(\varphi^{-1}(\varphi(B_o)) \setminus \varphi^{-1}(\varphi(B_i)))) \\ &= \frac{1}{n!}\mu^n(B_o \setminus B_i) \;\le\; \frac{1}{n!}\varepsilon\,. \end{aligned}$$

Because $\varphi(B_i) \subseteq A \cap \Gamma_n \subseteq \varphi(B_o)$ we see that $A \cap \Gamma_n$ is m_L-measurable. Thus A is m_L-measurable. The second assertion follows by the definition

$$m(B) \;=\; \sum_{n \in {}^*\mathbb{N}, n \le H} \frac{1}{n!}\mu^n(\varphi^{-1}(A \cap \Gamma_n))$$

of the internal measure and $m_L(\Gamma_\infty) = 0$. $\square$

Corollary 3.1.7 *Let $A \subset T$ be μ_L-measurable with $\mu_L(A) = 0$. Then*

$$\mathcal{P}_{\text{int}}(A) \;=\; \{B \subseteq A,\; B \;\textit{internal}\}$$

is m_L-measurable and $m_L(\mathcal{P}_{\text{int}}(A)) = 0$.

PROOF: $\mathcal{P}_{\mathrm{int}}(A) = \dot\cup_{n\in\mathbb{N}}\mathcal{P}_{\mathrm{int},n}(A)\dot\cup\mathcal{P}_{\mathrm{int},\infty}(A)$ and $\mathcal{P}_{\mathrm{int},n}(A) \subset \Gamma_n$ where by $\mathcal{P}_{\mathrm{int},n}(A)$ we denote the internal subsets of cardinality $n \in \mathbb{N}$ and by $\mathcal{P}_{\mathrm{int},\infty}(A)$ the internal subsets of infinite cardinality. Then since $\mathcal{P}_{\mathrm{int},\infty}(A) \subset \Gamma_\infty$ and by completeness of m_L we have $m_L(\mathcal{P}_{\mathrm{int},\infty}(A)) = 0$. Using that $\varphi^{-1}(\mathcal{P}_{\mathrm{int},n}(A)) \subseteq A^n$ we have for finite n by the previous lemma

$$m_L(\mathcal{P}_{\mathrm{int},n}(A)) \;=\; \frac{1}{n!}\mu_L^n(\varphi^{-1}(\mathcal{P}_{\mathrm{int},n}(A)) \;\le\; \frac{1}{n!}\mu_L^n(A^n) \;=\; 0\,.$$ □

Proposition 3.1.8

(1) st *is defined on a Loeb measurable set of full measure.*

(2) st *preserves levels. That means if* $\sigma \in \Gamma_{\mathrm{st}} \cap \Gamma_n$ *then* $\mathrm{st}(\sigma) \in \mathcal{P}_{\mathrm{fin},n}$ *for every* $n \in \mathbb{N}$. *This implies also* $\mathrm{st}(\Gamma_n) = \mathcal{P}_{\mathrm{fin},n}$ *for finite* n *and* $\mathrm{st}(\Gamma) = \mathcal{P}_{\mathrm{fin}}$.

(3) $B \subseteq \mathcal{P}_{\mathrm{fin}}$ *is* Λ*-measurable iff* $\mathrm{st}^{-1}(B)$ *is* m_L*-measurable. Thus* st *is a measurable map.*

PROOF: As above mentioned by proposition 2.1.6 and 2.1.9 it follows immediately that $m_L(\Gamma_{\mathrm{st}}) = m_L(\Gamma)$. This shows (1). For (2) we recall that by the definition of $\Gamma_{\not\approx}$ we have for every $\sigma \in \Gamma_{\mathrm{st}}$ that whenever $s \approx t$ for $s, t \in \sigma$ then $s = t$. Thus the elements of σ are in approximately general position and we have $\mathrm{st}(s) \neq \mathrm{st}(t)$ for arbitrary elements $s \neq t$ in σ. This shows $|\sigma| = |\mathrm{st}(\sigma)|$. But then clearly $\mathrm{st}(\Gamma_n) = \mathcal{P}_{\mathrm{fin},n}$ for finite n and also $\mathrm{st}(\Gamma) = \mathrm{st}(\Gamma_{\mathrm{st}}) = \mathcal{P}_{\mathrm{fin}}$.
Now let $B \subseteq \mathcal{P}_{\mathrm{fin}}$. Note that by (2) one has $\mathrm{st}^{-1}(\mathcal{P}_{\mathrm{fin},n}) = \Gamma_{\mathrm{st}} \cap \Gamma_n$.

$$\begin{aligned}
\mathrm{st}^{-1}(B) &= \{\sigma \in \Gamma_{\mathrm{st}} : \mathrm{st}(\sigma) \in B\} \\
&= \dot\cup_{n\in\mathbb{N}}\{\{s_1, \cdots, s_n\} \in \Gamma_{\mathrm{st}} \cap \Gamma_n : \{\mathrm{st}(s_1), \cdots, \mathrm{st}(s_n)\} \in B\} \\
&= \dot\cup_{n\in\mathbb{N}}\mathrm{st}^{-1}(B \cap \mathcal{P}_{\mathrm{fin},n}).
\end{aligned}$$

By definition 1.1.2 and lemma 3.1.6 subsets B of $\mathcal{P}_{\mathrm{fin}}$ and A of Γ are (Loeb-)measurable if and only if $\varphi^{-1}(B \cap \mathcal{P}_{\mathrm{fin},n})$ respectively $\varphi^{-1}(A \cap \Gamma_n)$ is (Loeb-)measurable for every $n \in \mathbb{N}$. So we can restrict our analysis to $B \subseteq \mathcal{P}_{\mathrm{fin},n}$ for some finite n. Then we get

$$\begin{aligned}
\varphi^{-1}(\mathrm{st}^{-1}(B)) &= \varphi^{-1}(\{\{s_1, \cdots, s_n\} \in \Gamma_{\mathrm{st}} \cap \Gamma_n : \{\mathrm{st}(s_1), \cdots, \mathrm{st}(s_n)\} \in B\}) \\
&= \{(s_1, \cdots, s_n) \in T^n_{\not\approx} : \{\mathrm{st}(s_1), \cdots, \mathrm{st}(s_n)\} \in B\} \\
&= \{(s_1, \cdots, s_n) \in T^n_{\neq} : \varphi^{-1}(\{\mathrm{st}(s_1), \cdots, \mathrm{st}(s_n)\}) \in \varphi^{-1}(B) \subseteq [0,1]^n_{\neq}\} \\
&= \{(s_1, \cdots, s_n) \in T^n : (\mathrm{st}(s_1), \cdots, \mathrm{st}(s_n)) \in \varphi^{-1}(B) \subseteq [0,1]^n\} \\
&= \mathrm{st}^{-1}(\varphi^{-1}(B)).
\end{aligned}$$

Thus applying proposition 3.1.2 we obtain the result. □

We finish with a corollary that subsumes the essence of this section:

Corollary 3.1.9 *Let* $B \subseteq \mathcal{P}_{\mathrm{fin}}$. *Then*

$$B \in \mathfrak{B} \;\Leftrightarrow\; \mathrm{st}^{-1}(B) \in L(\mathfrak{A})\,.$$

In that case we have

$$\Lambda(B) \;=\; m_L(\mathrm{st}^{-1}(B))\,.$$

PROOF: It remains to prove the last equality. Using proposition 3.1.2, lemma 3.1.6 and the same calculation as in the last proof we get

$$\begin{aligned}\Lambda(B) &= \sum_{n\in\mathbb{N}} \frac{1}{n!}\lambda^n(\varphi^{-1}(B\cap\mathcal{P}_{\mathrm{fin},n})) = \sum_{n\in\mathbb{N}} \frac{1}{n!}\mu_L^n(\mathrm{st}^{-1}(\varphi^{-1}(B\cap\mathcal{P}_{\mathrm{fin},n})))\\ &= \sum_{n\in\mathbb{N}} \frac{1}{n!}\mu_L^n(\varphi^{-1}(\mathrm{st}^{-1}(B\cap\mathcal{P}_{\mathrm{fin},n})))\\ &= \sum_{n\in\mathbb{N}} \frac{1}{n!}\mu_L^n(\varphi^{-1}(\mathrm{st}^{-1}(B)\cap\Gamma_n)) = m_L(\mathrm{st}^{-1}(B)).\end{aligned}$$ □

Thus the Loeb measure space $(\Gamma, L(\mathfrak{A}), m_L)$ contains an image of the standard measure space $(\mathcal{P}_{\mathrm{fin}}, \mathfrak{B}, \Lambda)$.

1.2 Standard Part Maps for Γ^n, $^*\mathcal{K}$ and $^*\mathcal{B}(\mathcal{K})$

In this subsection we extend the standard part map to the space Γ^n. Also we define standard part maps for the space $^*\mathcal{B}(\mathcal{K})$ of all internal linear operators of the internal Hilbert space $^*\mathcal{K}$ (the star of some standard Hilbert space $\mathcal{K}$) and for $^*\mathcal{K}$ itself.

1.2.1 Extending the Standard Part Map to Γ^n

Recall the following definition:

$$\Gamma_{\mathrm{st}}^{[n]} = \Gamma_{\mathrm{st}}^n \cap \Gamma_{\not\approx}^{[n]} \quad \text{and} \quad \Gamma_{\not\approx}^{[n]} = \{(\sigma_1,\cdots,\sigma_n)\in\Gamma^n : \sigma_1\cup\cdots\cup\sigma_n \in \Gamma_{\not\approx}\}.$$

By corollary 2.1.13 we know that this two subsets of Γ^n have full Loeb measure. We abbreviate again (by abuse of notation) $m_L^n = (m^n)_L$ leaving out the parenthesis. So we conclude

$$m_L^n(\Gamma_{\mathrm{st}}^{[n]}) = m_L^n(\Gamma_{\not\approx}^{[n]}) = m_L^n(\Gamma^n). \tag{3.1.2}$$

We extend the standard part map on Γ to Γ^n by the following definition.

Definition 3.1.10
Let $n\in\mathbb{N}$. The standard part st *map for Γ^n is (partially) defined by*

$$\mathrm{st} : \Gamma_{\mathrm{st}}^{[n]} \longrightarrow \mathcal{P}_{\mathrm{fin}}^n : (\sigma_1,\cdots,\sigma_n)\longmapsto(\mathrm{st}(\sigma_1),\cdots,\mathrm{st}(\sigma_n)).$$

For any set $A\in\mathcal{P}(\Gamma^n)$ we define

$$\mathrm{st}(A) = \{\mathrm{st}(\sigma_1,\cdots,\sigma_n) : (\sigma_1,\cdots,\sigma_n)\in A\cap\Gamma_{\mathrm{st}}^{[n]}\}.$$ ◁

Incidently st maps $\Gamma_{\mathrm{st}}^{[n]}$ onto

$$\mathcal{P}^n_{\mathrm{fin},\neq} \;=\; \{(\sigma_1,\cdots,\sigma_n)\in\mathcal{P}^n_{\mathrm{fin}} : \sigma_i\cap\sigma_j=\emptyset \text{ if } i\neq j\}$$

the set of sets in general position. But this is a subset of $\mathcal{P}^n_{\mathrm{fin}}$ of full measure. As before we can conclude that

Proposition 3.1.11 *The standard part map is defined on a Loeb measurable set of full measure and* $\mathrm{st}(\Gamma^n) = \mathcal{P}^n_{\mathrm{fin},\neq}$. *Furthermore* st *is a measurable map and for arbitrary* $B\subseteq\mathcal{P}^n_{\mathrm{fin}}$ *we have*

$$B\;\; \Lambda^n\text{-}measurable \iff \mathrm{st}^{-1}(B)\;\; m_L^n\text{-}measurable\,.$$

In that case it holds

$$\Lambda^n(B) \;=\; m_L^n(\mathrm{st}^{-1}(B))\,.$$

Proof: The first part is clear by the remarks above. Then since the standard part map is a map between measurable sets of full measure we can assume that $B\subseteq\mathcal{P}^n_{\mathrm{fin},\neq}$. Now assume that $B=B_1\times\cdots\times B_n$ is a Λ^n-nullset. But then at least one B_k is a Λ-nullset and we see that $\mathrm{st}^{-1}(B)=\mathrm{st}^{-1}(B_1)\times\cdots\times\mathrm{st}^{-1}(B_n)$ is an $(m_L)^n$-nullset. Because m_L^n is a complete extension of $(m_L)^n$ it is also an m_L^n-nullset. But in this case also the converse is true. If $\mathrm{st}^{-1}(B_1)\times\cdots\times\mathrm{st}^{-1}(B_n)$ is an m_L^n-nullset then it is also an $(m_L)^n$-nullset and for at least one B_k the set $\mathrm{st}^{-1}(B_k)$ is an m_L-nullset. But then B_k is a Λ-nullset and $B_1\times\cdots\times B_n$ is a Λ^n-nullset. Since the σ-algebra $\mathfrak{B}^n$ is generated by sets of the form $B_1\times\cdots\times B_n$ and both measure spaces are complete the proof is complete. □

This proposition shows that the Loeb measure space $(\Gamma^n, L(\mathfrak{A}^n), m_L^n)$ contains an image of the completed standard measure space $(\mathcal{P}^n_{\mathrm{fin}},\mathfrak{B}^n,\Lambda^n)$ as external subspace.

1.2.2 Standard Part Maps for ${}^*\mathcal{K}$ and ${}^*\mathcal{B}(\mathcal{K})$

We introduce on ${}^*\mathcal{K}$ two standard part maps corresponding to the norm and weak (= weak*) topology. The weak topology on $\mathcal{K}$ is the locally convex topology given by the family of seminorms $x\mapsto p_y(x)=|\langle y,x\rangle|$, $y\in\mathcal{K}$.

Definition 3.1.12 *Let* $x\in{}^*\mathcal{K}$. *Then we say that* x *is*

(1) norm infinitesimal *if* $\|x\|\approx 0$.

(2) weak infinitesimal *if* $\langle y,x\rangle\approx 0$ *for all* $y\in\mathcal{K}$.

In both cases we say that $z\in{}^*\mathcal{K}$ *is norm/weak* nearstandard *if there exists a standard* $y\in\mathcal{K}$ *and some norm/weak infinitesimal* $x\in{}^*\mathcal{K}$ *such that* $z=y+x$. *Then* y *is called the* standard part *of* z *and we write* $\mathrm{st}(z)=y$ *in the norm case and* $\mathrm{st}_\star(z)=y$ *in the weak case.* ◁

Generally a weak infinitesimal is not a norm infinitesimal. For example if $\mathcal{K} = \ell^2(\mathbb{N})$ and $(e_i)_{i\in{}^*\mathbb{N}}$ is the basis for ${}^*\ell^2(\mathbb{N})$ which extends the canonical base of $\ell^2(\mathbb{N})$ then for some infinite $H \in {}^*\mathbb{N}_\infty$ the element e_H has norm 1 but is weak infinitesimal since $\langle y, e_H\rangle \approx 0$ for all standard $y \in \mathcal{K}$. As in the case of the Hilbert space itself we note that corresponding to the different topologies on $\mathcal{B}(\mathcal{K})$ we have different kinds of infinitesimals and concepts of nearstandardness for ${}^*\mathcal{B}(\mathcal{K})$.

Definition 3.1.13 *Let $Q \in {}^*\mathcal{B}(\mathcal{K})$ be some internal operator. We say that Q is*

(1) norm infinitesimal *if $Qx \approx 0$ for all $x \in {}^*\mathcal{K}$ with $\|x\|$ finite.*

(2) strong infinitesimal *if $Qx \approx 0$ for all $x \in \mathcal{K}$.*

(3) weak infinitesimal *if $\langle y, Qx\rangle \approx 0$ for all $x, y \in \mathcal{K}$.*

In either case we say that some $W \in {}^*\mathcal{B}(\mathcal{K})$ is norm/strong/weak nearstandard *if there exists an operator $L \in \mathcal{B}(\mathcal{K})$ and a norm/strong/weak infinitesimal operator Q such that $W = {}^*L + Q$. Then L is the* standard part *of W in the respective topology and we denote the* standard part maps *by $\mathrm{st}_\infty(W)$ (norm), $\mathrm{st}_s(W)$ (strong) and $\mathrm{st}(W)$ (weak).* $\lhd$

Obviously we have

$$\text{norm infinitesimal} \Rightarrow \text{strong infinitesimal} \Rightarrow \text{weak infinitesimal}\,.$$

To illustrate that the inclusions are proper we give two examples of a strong but not norm and a weak but not strong infinitesimal.
Let $\mathcal{K} = \ell^2(\mathbb{N})$. Then ${}^*\mathcal{K} = {}^*\ell^2(\mathbb{N})$ and we have an orthonormal base $(e_i)_{i\in{}^*\mathbb{N}}$ in ${}^*\mathcal{K}$ that extends the standard base $(e_i)_{i\in\mathbb{N}}$ of $\mathcal{K}$. Now fix an infinite $H \in {}^*\mathbb{N}$. Then the operator $W(x) = \langle e_H, x\rangle e_1$ has norm 1 but is strongly infinitesimal since $\langle e_H, x\rangle \approx 0$ for all standard $x \in \mathcal{K}$. On the other hand the adjoint of W given by $W^\star(x) = \langle e_1, x\rangle e_H$ has norm 1 and is not strongly infinitesimal but weakly infinitesimal by the same argument.

Definition 3.1.14 *Let $W \in {}^*\mathcal{B}(\mathcal{K})$. Then W is* S-bounded *if there is an $M \in \mathbb{N}$ such that*

$$\forall x \in {}^*\mathcal{K}\,(\|Wx\| \;\leq\; M\|x\|)\,. \qquad \lhd$$

Now we show that the (weak) standard part can be obtained rather explicitly for an S-bounded operator. But first we note a very useful lemma, see for example Pedersen [Ped89, Lemma 3.2.2].

Lemma 3.1.15 *Let $\mathcal{K}$ be a Hilbert space. There is a bijective, isometric correspondence between operators in $\mathcal{B}(\mathcal{K})$ and bounded sesquilinear forms on $\mathcal{K}$, given by $L \mapsto b_L$, where*

$$b_L(y, x) \quad = \quad \langle y, Lx\rangle\ .$$

Corollary 3.1.16 *Let $W \in {}^*\mathcal{B}(\mathcal{K})$ be an S-bounded operator. Then the map*

$$B_W : \mathcal{K} \times \mathcal{K} \longrightarrow \mathbb{C} \ : \ (y,x) \longmapsto \mathrm{st}(\langle y, Wx\rangle)$$

defines a bounded sesquilinear form on $\mathcal{K}$.

PROOF: That B_W is a sesquilinear form is trivial. The boundedness follows by

$$|\langle y, Wx\rangle| \quad \leq \quad \|y\| \, \|W\| \, \|x\|$$

and the S-boundedness of W. □

Let $W \in {}^*\mathcal{B}(\mathcal{K})$ be S-bounded. Then by the lemma there corresponds a unique operator $L_W \in \mathcal{B}(\mathcal{K})$ to the sesquilinear form B_W of the corollary. This L_W is in fact the standard part of W with respect to the weak operator topology. If $W = {}^*L$ for some $L \in \mathcal{B}(\mathcal{K})$ then W is S-bounded and of course $L = \mathrm{st}(W) = L_W$.

2 Liftings

In this section we connect the Fock space to the internal Fock space. This is done by showing that every function in the Fock space has a square S-integrable lifting. A similar assertion holds for measurable kernel functions. In the first subsection we show also some consistency results concerning exponential vectors. In the second subsection we include an initial Hilbert space. The corresponding lifting theorems for Bochner square integrable functions and woply measurable operator valued functions are proved in the Appendix.

2.1 Liftings for the Pairs $([0,1], T)$ and $(\mathcal{P}_{\mathrm{fin}}, \Gamma)$

In the context of Loeb spaces we have two concepts of measurability that are needed. In this section let $(\Omega, \mathfrak{D}, \nu)$ be an arbitrary hyperfinite internal measure space with $\nu(\Omega)$ finite and $(\Omega, L(\mathfrak{D}), \nu_L)$ the corresponding Loeb space. Further $\mathbb{B}$ is a standard Hausdorff topological space. We collect some basic facts about functions on Loeb spaces and for this we follow slightly the exposition of Cutland [Cut00]. We specialize some of the results to our context.

Definition 3.2.1
Let $(\mathbb{B}, \tau)$ be a topological space and $\mathcal{B}(\tau)$ the Borel algebra generated by τ. A function $f : \Omega \to \mathbb{B}$ is called Loeb measurable *if it is $L(\mathfrak{D})$-$\mathcal{B}(\tau)$-measurable. An internal function $F : \Omega \to {}^*\mathbb{B}$ is called* *-measurable *if it is $\mathfrak{D}$-${}^*\mathcal{B}(\tau)$-measurable.* ◁

The connection of these two concepts of measurability is given by the following proposition. This is a variation of theorem 1.27 in [Cut00] and the proof for functions with values in a second countable space is given by Anderson [And82].

Proposition 3.2.2 *Let $\mathbb{B}$ be a second countable Hausdorff topological space. Then a function $f : \Omega \to \mathbb{B}$ is Loeb measurable iff there exist a *-measurable internal function $F : \Omega \to {}^*\mathbb{B}$ with $f(\omega) \approx F(\omega)$ for ν_L-almost all $\omega \in \Omega$.*

This result was extended by Ross [Ros90, Ros96] to arbitrary metric spaces using the special model axiom. Furthermore this proposition leads to the following definition (where we have specialized Ω to the space T^n resp. Γ).

Definition 3.2.3 *Let $\mathbb{B}$ be a Hausdorff topological space.*

(1) *Let $n \in \mathbb{N}$ and $\phi : [0,1]^n \to \mathbb{B}$ be a function. Then a lifting Φ of ϕ is a *-measurable internal function $\Phi : T^n \to {}^*\mathbb{B}$ such that*

$$\mathrm{st}(\Phi(t_1, \cdots, t_n)) \;=\; \phi(\mathrm{st}(t_1), \cdots, \mathrm{st}(t_n)) \quad \text{for } \mu_L^n\text{-a.a. } (t_1, \cdots, t_n) \in T^n \,.$$

(2) *Let $f : \mathcal{P}_{\text{fin}} \to \mathbb{B}$ be a function. Then a lifting of f is a *-measurable internal function $F : \Gamma \to {}^*\mathbb{B}$ such that*

$$\mathrm{st}(F(\sigma)) \;=\; f(\mathrm{st}(\sigma)) \quad \text{for } m_L\text{-almost all } \sigma \in \Gamma \,.$$

In both cases we write $\Phi \approx \phi$ resp. $F \approx f$. ◁

Note that in our case here actually every internal function is *-measurable because we work on hyperfinite measure spaces with the internal power set as measure algebra. This kind of lifting is sometimes called a *two-legged* (or *bipedal*) lifting since the corresponding commutative diagram is a square (has two legs). The next proposition shows that the construction of exponential vectors is compatible with the concept of a lifting.

Proposition 3.2.4 *Let $\mathbb{B}$ be a Hausdorff topological space, $\phi : [0,1] \to \mathbb{B}$ a function and Φ a lifting of ϕ. Then π_Φ is a lifting of π_ϕ.*

PROOF: Set $A = \{t \in T : \mathrm{st}(\Phi(t)) \neq \phi(\mathrm{st}(t))\}$. Then since Φ is lifting of ϕ we have $\mu_L(A) = 0$. By corollary 3.1.7 the set $\mathcal{P}_{\text{int}}(A)$ of all internal subsets of A is m_L-measurable and $m_L(\mathcal{P}_{\text{int}}(A)) = 0$. Now let be $\sigma \in \Gamma_{\text{st}} \setminus \mathcal{P}_{\text{int}}(A)$. Then if $\sigma = \{s_1, \cdots, s_n\}$ we have

$$\begin{aligned} \mathrm{st}(\pi_\Phi(\sigma)) &= \mathrm{st}(\Phi(s_1) \cdots \Phi(s_n)) \quad = \quad \mathrm{st}(\Phi(s_1)) \cdots \mathrm{st}(\Phi(s_n)) \\ &= \phi(\mathrm{st}(s_1)) \cdots \phi(\mathrm{st}(s_n)) \quad = \quad \pi_\phi(\mathrm{st}(\sigma)) \end{aligned}$$

and clearly $m_L(\Gamma_{\text{st}} \setminus \mathcal{P}_{\text{int}}(A)) = m_L(\Gamma)$. Thus $\mathrm{st}(\pi_\Phi(\sigma)) = \pi_\phi(\mathrm{st}(\sigma))$ m_L-almost everywhere and π_Φ is a lifting of π_ϕ. □

In the next definition and propositions we collect some additional basic facts about integration in Loeb spaces found in Sections 1.4 and 1.5.1 of Cutland's book [Cut00]. We leave now the general concept and go to the case $\mathbb{B} = \mathbb{C}$.

Definition 3.2.5

(1) *An internal function $\Phi : T^n \to {}^*\mathbb{C}$ is* S-integrable *if*

(a) $\sum_{t\in T^n} |\Phi(t)|\mu^n(t)$ *is finite,*

(b) $\forall B \in {}^*\mathcal{P}(T^n)\ (\mu^n(B) \approx 0 \Longrightarrow \sum_{t\in B} |\Phi(t)|\mu^n(t) \approx 0)$.

(2) *An internal function* $F : \Gamma \to {}^*\mathbb{C}$ *is* S-integrable *if*

(a) $\sum_{\sigma\in\Gamma} |F(\sigma)|m(\sigma)$ *is finite,*

(b) $\forall B \in \mathfrak{A}\ (m(B) \approx 0 \Longrightarrow \sum_{\sigma\in B} |F(\sigma)|m(\sigma) \approx 0)$.

(3) *We say that an internal function* Φ *or* F *is* SL^p *for* $p > 0$ *if* $|\Phi|^p$ *resp.* $|F|^p$ *is S-integrable.* $\lhd$

Proposition 3.2.6 *Let* $\phi : [0,1]^n \to \mathbb{C}$ *be a function. Then*

(1) ϕ *is Lebesgue-measurable iff it has a lifting* Φ.

(2) ϕ *is Lebesgue-integrable iff it has an S-integrable lifting* Φ. *In that case we have:*

$$\int_{[0,1]^n} \phi(t)d\lambda^n(t) \quad = \quad \mathrm{st}\left(\sum_{t\in T^n} \Phi(t)\mu^n(t)\right).$$

The assertions of this proposition are stated in theorems 1.39 and 1.40 in Cutland's book [Cut00]. The proof for the n-dimensional case is given by Osswald [OS, Proposition 5.2.7 (2)]. We cite a general result of Loeb measure theory [Cut00, Theorem 1.36].

Proposition 3.2.7 *A Loeb measurable function* $f : \Omega \to \mathbb{C}$ *is* ν_L*-integrable iff there is an S-integrable function* $F : \Omega \to {}^*\mathbb{C}$ *such that* $f(\omega) \approx F(\omega)$ *for* ν_L*-almost all* $\omega \in \Omega$.

The next proposition is an easy consequence of this.

Proposition 3.2.8 *Let* $f : \mathcal{P}_{\text{fin}} \to \mathbb{C}$. *Then*

(1) f *is* Λ*-measurable iff it has a lifting* $F : \Gamma \to {}^*\mathbb{C}$.

(2) $f \in L^1_{\mathbb{C}}(\mathcal{P}_{\text{fin}}, \Lambda)$ *iff it has an S-integrable lifting* F. *In that case we have*

$$\int_{\mathcal{P}_{\text{fin}}} f(\sigma)d\Lambda(\sigma) \quad = \quad \mathrm{st}\left(\sum_{\sigma\in\Gamma} F(\sigma)m(\sigma)\right).$$

PROOF: The first assertion is a corollary of proposition 3.2.2 and the second of proposition 3.2.7 using proposition 3.1.8 in both cases. For the formula of the integral we apply lemma 3.1.6, respectively the fact that for every infinite $n \in {}^*\mathbb{N}$, $n \leq H$ we have $m(\cup_{n\leq k\leq H}\Gamma_k) \approx 0$. $\square$

Note that by lemma 3.1.6 we could have used the result for liftings of functions on $[0,1]^n$ simultaneously for all $n \in \mathbb{N}$ and prove the preceding proposition.

Corollary 3.2.9 $k : \mathcal{P}^3_{\text{fin}} \to \mathbb{C}$ *is measurable iff it has a lifting* $K : \Gamma^3 \to {}^*\mathbb{C}$.

In the next proposition we show the compatibility of the concept of S-integrability and the construction of exponential vectors. But first we notice a useful combinatorial identity which immediately follows by transfer of the distributive law.

Lemma 3.2.10 *Let $F : T \to {}^*\mathbb{C}$ be an internal function. Then we have for every $n \in {}^*\mathbb{N}$ and all internal $A_1, \cdots, A_n \subseteq T$ that*

$$\sum_{s_1 \in A_1} \cdots \sum_{s_n \in A_n} \prod_{k=1}^{n} F(s_k) \quad = \quad \prod_{k=1}^{n} \left(\sum_{s_k \in A_k} F(s_k) \right) .$$

Proposition 3.2.11 *Let $\Phi \in SL^p(T)$, $p > 0$, p standard. Then $\pi_\Phi \in SL^p(\Gamma)$.*

PROOF: We prove the case $p = 1$ since the other cases are the same. Let $B \in \mathfrak{A}$ with $m(B) \approx 0$. We have to show that $\sum_{\sigma \in B} |\pi_\Phi(\sigma)| m(\sigma) \approx 0$. By definition we know that

$$m(B) \quad = \quad \sum_{n \leq H} \frac{1}{n!} \mu^n(\varphi^{-1}(B \cap \Gamma_n)) \quad \approx \quad 0 . \tag{3.2.3}$$

Thus $\emptyset \notin B$ otherwise we would have $m(B) \geq \mu^0(\emptyset) = 1$. So we obtain the disjoint union $B = \cup_{0<n\leq H} B \cap \Gamma_n$. In addition equation (3.2.3) tells us that $\mu^n(\varphi^{-1}(B \cap \Gamma_n)) \approx 0$ for each $n \in \mathbb{N}$. For the moment we assume that $B_n^1 \times \cdots \times B_n^n = \varphi^{-1}(B \cap \Gamma_n) \subseteq T^n$. Note that since B is internal each B_n^k $(n \leq H,\, k \leq n)$ is internal. Furthermore we see that for every finite $n \in \mathbb{N}$

$$\mu^n(B_n^1 \times \cdots \times B_n^n) \approx 0 \Leftrightarrow \exists k \in \{1, \cdots, n\} \, (\mu(B_n^k) \approx 0) .$$

By S-integrability of Φ we obtain for this B_n^k and for T

$$\sum_{s \in B_n^k} |\Phi(s)| \mu(s) = \varepsilon_n \approx 0 \quad \text{and} \quad \sum_{s \in T} |\Phi(s)| \mu(s) < M \tag{3.2.4}$$

for some standard $M \in \mathbb{N}$. We show that we have $\sum_{\sigma \in B} |\pi_\Phi(\sigma)| m(\sigma) < \varepsilon$ for every $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$. This obviously implies $\sum_{\sigma \in B} |\pi_\Phi(\sigma)| m(\sigma) \approx 0$.
Let $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$. Then choose some $m \in \mathbb{N}$ such that $\sum_{m<n\leq H} \frac{M^n}{n!} < \frac{\varepsilon}{2}$. Using lemma 3.2.10 and equation (3.2.4) we calculate:

$$\begin{aligned}
\sum_{\sigma \in B} |\pi_\Phi(\sigma)| m(\sigma) \quad &= \quad \sum_{0<n\leq H} \sum_{\sigma \in B \cap \Gamma_n} |\pi_\Phi(\sigma)| m(\sigma) \quad = \quad \sum_{0<n\leq H} \sum_{\sigma \in B \cap \Gamma_n} \left(\prod_{s \in \sigma} |\Phi(s)| \right) m(\sigma) \\
&= \quad \sum_{0<n\leq H} \sum_{(s_1, \cdots, s_n) \in \varphi^{-1}(B \cap \Gamma_n)} |\Phi(s_1)| \cdots |\Phi(s_n)| \frac{1}{n!} \mu^n((s_1, \cdots, s_n)) \\
= \quad \sum_{0<n\leq H} \frac{1}{n!} \sum_{(s_1, \cdots, s_n) \in B_n^1 \times \cdots \times B_n^n} &\prod_{k=1}^{n} \left(|\Phi(s_k)| \frac{1}{H} \right) \quad = \quad \sum_{0<n\leq H} \frac{1}{n!} \prod_{k=1}^{n} \left(\sum_{s_k \in B_n^k} |\Phi(s_k)| \frac{1}{H} \right) \\
&\leq \quad \sum_{0<n\leq m} \frac{M^{n-1} \varepsilon_n}{n!} + \sum_{m<n\leq H} \frac{M^n}{n!} \quad < \quad \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad = \quad \varepsilon .
\end{aligned}$$

Since ${}^*\mathcal{P}(T^n)$ is generated by ${}^*\mathcal{P}(T)^n$ the general case follows by decomposition of the set $\varphi^{-1}(B \cap \Gamma_n)$ into internal rectangles. □

The next corollary recovers the exponential relation in the nonstandard setting. We give an explicit proof despite the fact that one could prove this relation using the standard relation and the concept of SL^2-liftings.

Corollary 3.2.12 *Let $\Phi, \Psi \in SL^2(T)$. Then $\langle \pi_\Phi, \pi_\Psi \rangle \approx \exp\langle \Phi, \Psi \rangle$.*

Proof: Note that $\mu^n(T \setminus T^n_{\neq}) \approx 0$ by proposition 2.1.1.

$$\begin{aligned}
\langle \pi_\Phi, \pi_\Psi \rangle &= \sum_{n \le H} \sum_{\sigma \in \Gamma_n} \prod_{s \in \sigma} \overline{\Phi(s)} \Psi(s) m(\sigma) \\
&= \sum_{n \le H} \sum_{(s_1, \cdots, s_n) \in T^n_{\neq}} \overline{\Phi(s_1)} \cdots \overline{\Phi(s_n)} \Psi(s_1) \cdots \Psi(s_n) \frac{1}{n!} \mu^n((s_1, \cdots, s_n)) \\
&\approx \sum_{n \le H} \frac{1}{n!} \sum_{(s_1, \cdots, s_n) \in T} \overline{\Phi(s_1)} \Psi(s_1) \cdots \overline{\Phi(s_n)} \Psi(s_n) \left(\frac{1}{H} \right)^n \\
&= \sum_{n \le H} \frac{1}{n!} \prod_{k=1}^n \left(\sum_{s_k \in T} \overline{\Phi(s_k)} \Psi(s_k) \frac{1}{H} \right) \;=\; \sum_{n \le H} \frac{1}{n!} \left(\sum_{s \in T} \overline{\Phi(s)} \Psi(s) \frac{1}{H} \right)^n \\
&= \sum_{n \le H} \frac{\langle \Phi, \Psi \rangle^n}{n!} \;\approx\; \exp\langle \Phi, \Psi \rangle\,.
\end{aligned}$$

□

Corollary 3.2.13
Let $\Phi, \Psi \in SL^2(T)$ be liftings of $\phi, \psi \in L^2([0,1])$. Then $\langle \pi_\Phi, \pi_\Psi \rangle \approx \langle \pi_\phi, \pi_\psi \rangle$.

Next we introduce two natural topologies, one on the space of all internal functions, another on the space of all square S-integrable functions.

Definition 3.2.14 *The* internal topology *on ${}^*L^2(\Gamma)$ is the topology given by the internal norm*

$$\left(\sum_{\sigma \in \Gamma} |F(\sigma)|^2 m(\sigma) \right)^{\frac{1}{2}}, \; F \in {}^*L^2(\Gamma)\,.$$

The ${}^{\mathrm{st}}L^2$-topology *on $SL^2(\Gamma)$ is the topology generated by the standard semi-norm*

$$\mathrm{st} \left(\sum_{\sigma \in \Gamma} |F(\sigma)|^2 m(\sigma) \right)^{\frac{1}{2}}, \; F \in SL^2(\Gamma))\,. \qquad \triangleleft$$

It is well known in standard Fock space theory that the set of exponential vectors is total in the Fock space. Since by proposition 2.1.23 every internal function can be expanded in a hyperfinite linear combination of exponential vectors, it is a trivial fact that the exponential vectors $\{\pi_\Phi : \Phi \in {}^*L^2(T)\}$ are *-total in ${}^*L^2(\Gamma)$ in the internal topology. The case of SL^2-functions is a little bit more difficult. First note that by proposition 3.2.11 we have $\pi_\Phi \in SL^2(\Gamma)$ for every $\Phi \in SL^2(T)$. Thus the set $\{\pi_\Phi : \Phi \in SL^2(T)\}$ is contained in $SL^2(\Gamma)$. But for this set being total the space $SL^2(\Gamma)$ is too big. The ${}^{\mathrm{st}}L^2$-topology on $SL^2(\Gamma)$ gives an equivalence relation $F \approx G \Leftrightarrow_{\mathrm{def}} \mathrm{st}\left(\sum_{\sigma\in\Gamma} |F(\sigma) - G(\sigma)|^2 m(\sigma)\right)^{\frac{1}{2}} = 0$. Then by Anderson [And76, Theorem 11] $SL^2(\Gamma)/\approx$ is isomorphic to $L^2(\Gamma, m_L)$. On the other hand, we have the same for functions on T, namely that $SL^2(T)/\approx$ is isomorphic to $L^2(T, \mu_L)$. Indeed, the ${}^{\mathrm{st}}L^2$-topology does not distinguish between functions in a single equivalence class. But the standard Hilbert space $L^2(\Gamma, m_L)$ is much bigger than the Fock space over the standard Hilbert space $L^2(T, \mu_L)$. The reason for this is that $L^2(\Gamma, m_L)$ corresponds to the direct sum over $\mathbb{N}$ of the Loeb spaces $L^2(T^n, (\mu^n)_L)$ whereas the Fock space over $L^2(T, \mu_L)$ is the direct sum of the Loeb spaces $L^2(T^n, (\mu_L)^n)$. It is well known that $L^2(T^n, (\mu_L)^n)$ is strictly contained in $L^2(T^n, (\mu^n)_L)$. For an exact characterization when the Loeb product space is strictly bigger than the product of the Loeb spaces see [OS, proposition 7.4.5] of Sun's contribution in [LW00]. Thus by the standard result on the totality of exponential vectors we can only conclude that $\{\pi_\Phi : \Phi \in SL^2(T)\}$ is total in $\oplus_{n\in\mathbb{N}} SL^2(T)^{\otimes n}$ in the ${}^{\mathrm{st}}L^2$-topology (see [Par92, Proposition 19.4, Corollary 19.5] or [Mey93a, IV (1) 3, page 58]). This space corresponds to a subspace in $SL^2(\Gamma)$. Since $\{\pi_\phi : \phi \in L^2([0,1])\}$ is total in the standard Fock space this subspace certainly contains all liftings of functions of the standard Fock space.

We need also a definition which gives us the analogue of standard boundedness in non-standard analysis.

Definition 3.2.15 *An internal function $\Phi : T \to {}^*\mathbb{C}$ is called* S-bounded *if there exists a standard number $M \in \mathbb{N}$ such that*

$$\sup_{t\in T} |\Phi(t)| \;<\; M\,.$$

A function $\phi : [0,1] \to \mathbb{C}$ is bounded *if there exists a number $M \in \mathbb{N}$ such that*

$$\sup_{t\in[0,1]} |\phi(t)| \;<\; M\,. \qquad \lhd$$

Proposition 3.2.16
A bounded function $\phi : [0,1] \to \mathbb{C}$ has an S-bounded lifting $\Phi : T \to {}^*\mathbb{C}$.

Proof: Let ϕ be bounded and $B \in \mathbb{C}$ a ball with radius M such that the range of ϕ lays in B. If we regard $\phi : [0,1] \to B \subset \mathbb{C}$ as a function into B then it has by the usual lifting theorem a lifting $\Phi : T \to {}^*B \subset {}^*\mathbb{C}$. Apparently Φ is S-bounded with bound M. □

Notation 3.2.17

$$\begin{aligned} B^2([0,1]) &= \{\phi \in L^2_{\mathbb{C}}([0,1]) : \phi \text{ is bounded}\} \\ SB^2(T) &= \{\Phi \in SL^2_{{}^*\mathbb{C}}(T) : \Phi \text{ is S-bounded}\}\,. \end{aligned}$$

If the underlying space is clear we write only B^2 and SB^2. $\lhd$

Note that by the proposition above we have for every $\phi \in B^2$ a lifting $\Phi \in SB^2$.

2.2 Liftings Including an Initial Space

In this subsection we need some of the results of the Appendix. We suppose $\mathcal{K}$ to be a separable Hilbert space in the standard universe. As before we denote by $\mathcal{B}(\mathcal{K})$ the set of all bounded linear operators on $\mathcal{K}$.

We want to include an initial Hilbert space and look for liftings for the space $\mathcal{K} \otimes L^2(\mathcal{P}_{\text{fin}})$. We identify this space with $L^2(\mathcal{P}_{\text{fin}}, \mathcal{K}) = \int_{\mathcal{P}_{\text{fin}}}^{\oplus} d\Lambda\mathcal{K}$ (the latter is a notation for the so-called *direct integral* of $\mathcal{K}$) since this space is naturally isomorphic to the former one. In particular, by integrability of $\mathcal{K}$-valued functions we always mean Bochner integrability. Thus we must look for liftings of L^2-functions $f : \mathcal{P}_{\text{fin}} \to \mathcal{K}$.

We include a general result due to Osswald [BO]:

Theorem 3.2.18 *Let $\mathbb{B}$ be a separable Banach space and $(\Omega, \mathfrak{D}, \nu)$ a hyperfinite internal measure space with $\nu(\Omega)$ finite. Then an $L(\mathfrak{D})$-measurable function $f : \Omega \to \mathbb{B}$ is Bochner ν_L-integrable iff it has an S-integrable lifting $F : \Omega \to {}^*\mathbb{B}$. In that case it holds*

$$\int_\Omega f(\omega) d\nu_L(\omega) \quad = \quad \text{st}\left(\sum_{\omega\in\Omega} F(\omega)\nu(\omega)\right) .$$

The proof of this theorem is essentially that of [BO] (cf. also [LO97, Zim98]) and we postpone it to the Appendix. There the theorem is numbered as theorem A.1.14. The separability assumption ensures that $\mathcal{K}$ is a second countable Hausdorff topological space in the norm topology. Thus we can apply the previous proposition and proposition 3.2.2 and conclude:

Corollary 3.2.19 *Let* st $: {}^*\mathcal{K} \to \mathcal{K}$ *be the standard part map for the norm topology. Further, let $f : \mathcal{P}_{\text{fin}} \to \mathcal{K}$ be a function. Then*

(1) *f is Λ-$\|\cdot\|$-measurable iff it has a lifting $F : \Gamma \to {}^*\mathcal{K}$.*

(2) *If f, g are measurable then $f, g \in L^2(\mathcal{P}_{\text{fin}}, \mathcal{K})$ iff they have SL^2-liftings F, G. In that case we have*

$$\begin{aligned}
\int_{\mathcal{P}_{\text{fin}}} \|f(\sigma)\|^2 d\Lambda(\sigma) &= \text{st}\left(\sum_{\sigma\in\Gamma} \|F(\sigma)\|^2 m(\sigma)\right) , \\
\int_{\mathcal{P}_{\text{fin}}} f(\sigma) d\Lambda(\sigma) &= \text{st}\left(\sum_{\sigma\in\Gamma} F(\sigma) m(\sigma)\right) \quad \textit{and} \\
\int_{\mathcal{P}_{\text{fin}}} \langle f(\sigma), g(\sigma)\rangle d\Lambda(\sigma) &= \text{st}\left(\sum_{\sigma\in\Gamma} \langle F(\sigma), G(\sigma)\rangle m(\sigma)\right) .
\end{aligned}$$

We include a proof of the first and third equality in the Appendix under proposition A.1.15.

Now we extend this result to kernel functions $k : \mathcal{P}_{\text{fin}}^3 \to \mathcal{B}(\mathcal{K})$ but using the weak operator topology on $\mathcal{B}(\mathcal{K})$. We introduce the concept of woply measurable functions.

Definition 3.2.20 *A function $k : \mathcal{P}^3_{\text{fin}} \to \mathcal{B}(\mathcal{K})$ is said to be* woply measurable *(or* measurable in the weak operator topology*) if for all $y, x \in \mathcal{K}$ the functions*

$$k_{yx} : \mathcal{P}^3_{\text{fin}} \longrightarrow \mathbb{C} : (\sigma, \rho, \tau) \longmapsto \langle y, k(\sigma, \rho, \tau)x \rangle$$

are Lebesgue measurable. $\triangleleft$

But since $\mathcal{B}(\mathcal{K})$ is not second countable in the weak operator topology we can't apply the quoted result. To apply this result we may restrict to the case where the function k is essentially bounded. This concept is made clear in the next definition.

Definition 3.2.21 *Let $k : \mathcal{P}^3_{\text{fin}} \to \mathcal{B}(\mathcal{K})$ be a function. Then we say that k is* essentially bounded *if there exists a measurable set $B \subseteq \mathcal{P}^3_{\text{fin}}$ of full measure such that*

$$\sup\{\|k(\sigma, \rho, \tau)\| : (\sigma, \rho, \tau) \in B\} < M$$

for some fixed $M \in \mathbb{R}$. Further we say that k is bounded *if $B = \mathcal{P}^3_{\text{fin}}$.* $\triangleleft$

Thus an essentially bounded function k takes its values almost surely in a norm bounded subset $B(M)$ of $\mathcal{B}(\mathcal{K})$. But one knows that every norm closed bounded subset of $\mathcal{B}(\mathcal{K})$ is separable, compact and metrizable in the weak operator topology and thus second countable [Ped89, p. 172]. So we can again apply proposition 3.2.2 and obtain

Corollary 3.2.22 *Let $\mathrm{st} : {}^*\mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ be the above defined (weak) standard part map. Further, let $k : \mathcal{P}^3_{\text{fin}} \to \mathcal{B}(\mathcal{K})$ be an essentially bounded function. Then k is Λ^3-wop-measurable iff it has a lifting $K : \Gamma^3 \to {}^*\mathcal{B}(\mathcal{K})$. Since k is essentially bounded we can take K to be S-bounded m_L-almost everywhere.*

Proof: Just regard k as function $k : \mathcal{P}^3_{\text{fin}} \to B(M)$ where $B(M)$ is a norm bounded subset of $\mathcal{B}(\mathcal{K})$. $\square$

The next result extends this for arbitrary woply measurable functions k.

Theorem 3.2.23 *Let $k : \mathcal{P}^3_{\text{fin}} \to \mathcal{B}(\mathcal{K})$ be a woply measurable function. Then k has a lifting $K : \mathcal{P}^3_{\text{fin}} \to {}^*\mathcal{B}(\mathcal{K})$ for the weak operator topology.*

Proof: This is an application of theorem A.2.7 in the Appendix. $\square$

A consequence of this theorem is that every woply measurable kernel function k of some standard kernel operator k has a lifting K in this sense. But the lifting K defines an internal kernel operator K. Thus in the sense of kernels every standard kernel operator is representable by an internal operator. In the next section we call this a strict representation of the standard operator.

3 Internal Representations of Fock Space Operators

In this section we develop three concepts for the representation of a Fock space operator by an internal kernel operator. The first one, the strict representation is just doing a lifting of the standard kernel function. In this way we obtain an internal representation of the standard kernel calculus of quantum stochastics. The second is concerned with strong representations, i.e. we fix a certain set of test vectors and liftings of them and say that an internal kernel operator represents a Fock space operator if both act the same way on the test vectors respectively their liftings. In the third one, the weak representation, we investigate under what conditions the scalar products of the operators with certain test vectors and their liftings are the same. Of course we have to make clear what we mean by sameness.

We fix our sets of test vectors:

Notation 3.3.1

$$\begin{aligned} \mathcal{E} &= \{\pi_\psi : \psi \in B^2_{\mathbb{C}}([0,1], \mathcal{B}, \lambda)\} \\ \mathfrak{E} &= \{\pi_\Psi : \Psi \in SB^2_{^*\mathbb{C}}(T, \mathcal{A}, \mu)\} \end{aligned}$$

We denote by $\overline{\mathcal{E}}$ the $\mathbb{C}$-linear span of $\mathcal{E}$ and by $\overline{\mathfrak{E}}$ the $^*\mathbb{C}_{\text{fin}}$-linear span of $\mathfrak{E}$. Here $^*\mathbb{C}_{\text{fin}}$ denotes the set of elements of finite norm in $^*\mathbb{C}$. ◁

We know that each $f \in \mathcal{E}$ has at least one lifting $F \in \mathfrak{E}$. Next we calculate explicitly the action of an internal kernel operator K with kernel function K on an element $F \in \mathcal{F}_{\text{int}}$.

$$\begin{aligned}
\mathsf{K}F(\alpha) &= \sum_{\sigma,\rho,\tau} K(\sigma,\rho,\tau) a^+_\sigma a^\circ_\rho a^-_\tau F(\alpha) \\
&= \sum_{\sigma,\rho,\tau} K(\sigma,\rho,\tau) a^+_\sigma a^\circ_\rho F(\alpha\cup\tau) \notin_\tau(\alpha) \left(\frac{1}{H}\right)^{|\tau|} \\
&= \sum_{\sigma,\rho,\tau} K(\sigma,\rho,\tau) a^+_\sigma F(\alpha\cup\tau) \notin_\tau(\alpha) \in_\rho(\alpha) \left(\frac{1}{H}\right)^{|\tau|} \\
&= \sum_{\sigma,\rho,\tau} K(\sigma,\rho,\tau) F((\alpha\cup\tau)\setminus\sigma) \notin_{\tau\setminus\sigma}(\alpha) \in_\rho(\alpha\setminus\sigma) \in_\sigma(\alpha) \left(\frac{1}{H}\right)^{|\tau|} \\
&= \sum_{\sigma\subseteq\alpha} \sum_{\rho\subseteq(\alpha\setminus\sigma)} \sum_{\substack{\tau \\ \tau\cap(\alpha\setminus\sigma)=\emptyset}} K(\sigma,\rho,\tau) F((\alpha\setminus\sigma)\cup\tau) \left(\frac{1}{H}\right)^{|\tau|} \\
&= \sum_{\sigma\dot\cup\rho\dot\cup\beta=\alpha} \sum_{\substack{\tau \\ \tau\cap(\rho\cup\beta)=\emptyset}} K(\sigma,\rho,\tau) F(\rho\cup\beta\cup\tau) \left(\frac{1}{H}\right)^{|\tau|}.
\end{aligned} \tag{3.3.5}$$

This formula is the nonstandard version of the defining formula (1.2.2) in definition 1.2.2 for standard kernel operators in Chapter 1. For internal kernel operators it is always the case that $\mathfrak{E}$ lies in the domain of the kernel operator. We see that the kernel defines a standard operator only if $\mathsf{K}\pi_\Phi(\alpha)$ is in $SL^2(\Gamma)$ for $\Phi \in SB^2(T)$. Furthermore, we have to ensure that for every $\Phi \in \mathfrak{E}$ the sum

$$\sum_{\substack{\tau \\ \tau\cap(\rho\cup\beta)=\emptyset}} K(\sigma,\rho,\tau)\pi_\Phi(\rho\cup\beta\cup\tau)\left(\frac{1}{H}\right)^{|\tau|}$$

makes sense for m_L^3-almost all triples $(\sigma,\rho,\beta) \in \Gamma^3$. In particular, that this sum is finite. For this reason we may assume that K is $SL^2(\Gamma)$ in the third argument. Then the sum is nothing else then the SL^2 scalar product of the functions $\overline{K(\sigma,\rho,\,\cdot\,)}$ and $\pi_\Phi(\rho\cup\beta\cup\,\cdot\,)$. We take a slightly weaker assumption on K.

Assumption A: *In the following we assume that the internal kernel functions K representing standard operators are such that the function*

$$\tau \quad\longmapsto\quad K(\sigma,\rho,\tau)\pi_\Phi(\rho\cup\beta\cup\tau)$$

is S-integrable for m_L^3-almost all $(\sigma,\rho,\beta) \in \Gamma^3$ and for all $\Phi \in SB^2(T)$. (The "m_L^3-almost-all" here is independent of the particular Φ.) $\triangleleft$

3.1 Strict Representation

Definition 3.3.2 *Let K be an internal kernel operator and k a standard kernel operator. Then we say that K is a* strict representation *of k if the corresponding internal kernel function $K : \Gamma^3_{\neq} \to {}^*\mathbb{C}$ of K is a lifting of the kernel function $k : \mathcal{P}^3_{\text{fin}} \to \mathbb{C}$ of k.* $\triangleleft$

Sometimes it is convenient not to take the original lifting of the kernel function but another kernel function which is close to the lifting. Thus we weaken the concept of a strict representation. We introduce an (external) equivalence relation on the set of kernel operators which is related to strong representations (cf. the next subsection).

Definition 3.3.3 *We say that two internal operators L and K are* strongly equivalent *(with respect to SB^2) if for every $\Phi \in SB^2$ we have that*

$$\mathsf{L}\pi_\Phi(\sigma) \quad\approx\quad \mathsf{K}\pi_\Phi(\sigma) \;\; \textit{for } m_L\textit{-almost all } \; \sigma \in \Gamma\,.$$

Then an internal operator L is a strict equivalent representation *of a standard operator k if there exists a strongly equivalent internal kernel operator K of L such that K is a strict representation of k.* $\triangleleft$

The following corollary is very useful for quantum stochastic differential equations since it allows one to replace the time process by a modified time process.

Corollary 3.3.4 *The internal operator process $A_t^\bullet = \sum_{s<t} a_s^\bullet$ is strongly equivalent to the internal 3-argument kernel operator $\widetilde{A}_t^\bullet$ with internal kernel function*

$$\widetilde{a}_t^\bullet(\sigma,\rho,\tau) = \begin{cases} t & \textit{if } (\sigma,\rho,\tau)=(\emptyset,\emptyset,\emptyset), \\ 0, & \textit{otherwise}\,. \end{cases}$$

Furthermore, if we introduce for $s \in T$ and for every $F \in \mathcal{F}_{\mathrm{int}}$ the infinitesimal operators $\widetilde{a}^\bullet_s F(\sigma) = \frac{1}{H} F(\sigma)$ (multiplication by the time increment) then $\widetilde{A}^\bullet_t = \sum_{s<t} \widetilde{a}^\bullet_s$.

Proof: Fix some $\sigma \in \Gamma_{\mathrm{st}}$ and $\Phi \in SB^2(T)$. Then $\pi_\Phi \in SL^2(\Gamma)$. We have

$$\begin{aligned} A^\bullet_t \pi_\Phi(\sigma) &= \sum_{s<t} \frac{1}{H} \pi_\Phi(\sigma) \not\in_s(\sigma) \approx \sum_{s<t} \frac{1}{H} \pi_\Phi(\sigma) \not\in_s(\sigma) + \sum_{s<t} \frac{1}{H} \pi_\Phi(\sigma) \in_s(\sigma) \\ &= \sum_{s<t} \frac{1}{H} \pi_\Phi(\sigma) = t\pi_\Phi(\sigma) = \widetilde{A}^\bullet_t \pi_\Phi(\sigma)\,. \end{aligned}$$

Since Γ_{st} has full Loeb measure we see that $A^\bullet_t$ is strongly equivalent to $\widetilde{A}^\bullet_t$. The remaining assertion is clear by construction. □

Since later we will be interested in strong kernel representations and we regard strong equivalent operators in most situations as similar representations we substitute in such contexts $A^\bullet_t$ by $\widetilde{A}^\bullet_t$. Obviously the modified internal time process $\widetilde{A}^\bullet_t$ is a totally strict representation of the standard time process $\mathsf{a}^\bullet_t$ in the sense of the following definition.

Definition 3.3.5 *We say that an internal process $(\mathsf{K}_t)_{t\in T}$ is a* strict representation *of the process $(\mathsf{k}_t)_{t\in[0,1]}$ if the kernel function $K : \Gamma^3_{\neq} \times T \to {}^*\mathbb{C}$ is a lifting (with respect to $(m \times \mu)_L$) of the kernel function $k : \mathcal{P}^3_{\mathrm{fin}} \times [0,1] \to \mathbb{C}$. Further, we say that $(\mathsf{K}_t)_{t\in T}$ is a* totally strict representation *if for all $t \in T$ the kernel function $K_t : \Gamma^3_{\neq} \to {}^*\mathbb{C}$ is a lifting of $k_{\mathrm{st}(t)} : \mathcal{P}^3_{\mathrm{fin}} \to \mathbb{C}$.* ◁

Including an initial space we see that all assertions of this section remain valid when we replace $(\mathbb{C}, {}^*\mathbb{C})$ by $(\mathcal{B}(\mathcal{K}), {}^*\mathcal{B}(\mathcal{K}))$. The only difference is that we usually take $\mathcal{B}(\mathcal{K})$ not as a normed space but as a locally convex topological space in the weak operator topology. In the following we suppose that $\mathcal{B}(\mathcal{K})$ is endowed with the weak operator topology.

Proposition 3.3.6 *Suppose we have a constant process k with kernel function*

$$k_t(\sigma,\rho,\tau) = \begin{cases} L \in \mathcal{B}(\mathcal{K}) & \text{if } \sigma=\rho=\tau=\emptyset, \\ 0, & \text{otherwise}. \end{cases}$$

Then the constant internal process K given by the kernel function

$$k_t(\sigma,\rho,\tau) = \begin{cases} {}^*L \in {}^*\mathcal{B}(\mathcal{K}) & \text{if } \sigma=\rho=\tau=\emptyset, \\ 0, & \text{otherwise} \end{cases}$$

is a totally strict representation of k.

Proof: The standard part of ${}^*L$ apparently is L, independently of the topology. Then if $(\sigma,\rho,\tau) \neq (\emptyset,\emptyset,\emptyset)$ we have independently of $t \in T$

$$\mathrm{st}(K_t(\sigma,\rho,\tau)) = 0 = k_{\mathrm{st}(t)}(\mathrm{st}(\sigma,\rho,\tau)) \quad \text{if} \quad (\sigma,\rho,\tau) \in \Gamma^{[3]}_{\mathrm{st}}\,.$$

Obviously $\mathrm{st}(\emptyset,\emptyset,\emptyset) = (\emptyset,\emptyset,\emptyset)$. This gives independently of $t \in T$

$$\mathrm{st}(K_t(\emptyset,\emptyset,\emptyset)) = L = k_{\mathrm{st}(t)}(\emptyset,\emptyset,\emptyset).$$

□

Given some $n \in \mathbb{N}$ and operators $L_i \in \mathcal{B}(\mathcal{K})$, $i = 1, \cdots, n$ we can build the polynomial $p(z) = \sum_{i=1}^n L_i z^i$. We regard p as a map from $\mathcal{B}(\mathcal{K})$ into $\mathcal{B}(\mathcal{K})$ by

$$p \, : \, \mathcal{B}(\mathcal{K}) \longrightarrow \mathcal{B}(\mathcal{K}) \, : \, k \longmapsto p(k) \; = \; \sum_{i=1}^{n} L_i k^i \, .$$

Further, we convert p into a kernel process operator by setting

$$p_t((\sigma,\rho,\tau);z) \quad = \quad \begin{cases} p(z) & \text{if } \sigma=\rho=\tau=\emptyset, \\ 0, & \text{otherwise}. \end{cases} \tag{3.3.6}$$

Definition 3.3.7 *We say that an internal polynomial $P : {}^*\mathcal{B}(\mathcal{K}) \to {}^*\mathcal{B}(\mathcal{K})$ is a* representation *of a polynomial $p : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ if for all nearstandard $K \in {}^*\mathcal{B}(\mathcal{K})$ we have* $\mathrm{st}(P(K)) = p(\mathrm{st}(K))$. *By abuse of notation we write in this situation* $\mathrm{st}(P) = p$. $\triangleleft$

Proposition 3.3.8 *Let $p(z) = \sum_{i=1}^n L_i z^i$. Then the internal polynomial*

$$P \, : \, {}^*\mathcal{B}(\mathcal{K}) \longrightarrow {}^*\mathcal{B}(\mathcal{K}) \, : \, K \longmapsto P(K) \; = \; \sum_{i=1}^{n} {}^*L_i K^i$$

is a representation of $p : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$.

Proof: Obviously since $\mathrm{st}({}^*L) = L$ and the standard part map is multiplicative on the set of S-bounded operators in ${}^*\mathcal{B}(\mathcal{K})$. $\square$

By converting polynomials in 'constant processes' we obtain the following corollary. The proof uses the preceding proposition and the same considerations as in proposition 3.3.6.

Corollary 3.3.9 *Let $p(z) = \sum_{i=1}^n L_i z^i$ and $P(z) = \sum_{i=1}^n {}^*L_i z^i$ be a representation of p (as in the proposition). Further, let $p_t((\sigma,\rho,\tau);z)$ be as in equation (3.3.6). Then the constant internal kernel process operator*

$$P_t((\sigma,\rho,\tau);z) \quad = \quad \begin{cases} P(z) & \textit{if } \sigma=\rho=\tau=\emptyset, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.3.7}$$

is a totally strict representation *of $p_t((\sigma,\rho,\tau);z)$ in the sense that for all nearstandard $K \in \mathcal{B}(\mathcal{K})$, all $t \in T$ and m_L^3-almost all $(\sigma,\rho,\tau) \in \Gamma^3$ we have*

$$\mathrm{st}(P_t((\sigma,\rho,\tau);K)) \quad = \quad p_{\mathrm{st}(t)}(\mathrm{st}(\sigma,\rho,\tau);\mathrm{st}(K)) \, .$$

If K is itself the kernel of an adapted internal kernel process K and the totally strict representation of some standard kernel process k with kernel function k then the preceding equation extends to

$$\mathrm{st}(P_t((\sigma,\rho,\tau);K_t(\sigma,\rho,\tau))) \quad = \quad p_{\mathrm{st}(t)}(\mathrm{st}(\sigma,\rho,\tau);k_{\mathrm{st}(t)}(\mathrm{st}(\sigma,\rho,\tau))) \, .$$

For only strict representations the equations hold solely for μ_L-almost all $t \in T$.

These results are useful regarding quantum stochastic differential equations which are nonlinear in the noise terms. For the hyperfinite case we have introduced the notation $P\diamond$ for $P(\,\cdot\,)$ to emphasize that P acts as operator to the right. Using the same notation for p we can subsume the situation described above in the single equation

$$\mathrm{st}(P\diamond) \;=\; \mathrm{st}(P)\diamond \;=\; p\diamond\,.$$

Concerning the question whether a standard operator has a strict representation we see that by definition it must be a kernel operator. But for these operators we can conclude with the help of theorem 3.2.23 and corollary 3.2.9 that

Proposition 3.3.10 *Every standard kernel operator has a strict representation.*

Thus the concept of strict representations goes so far as to extend the part of standard kernels of quantum stochastic calculus. But unfortunately not every standard operator has a so-called Maassen-Meyer kernel. For this reason we develop in the next section a kind of representation which is better suited to Fock space operators defined on exponential vectors.

3.2 Strong Representation

We consider operators that are defined only on the Fock space.

Definition 3.3.11 *Let* K *be an internal operator and* k *a Fock space operator. Then we say that* K *is a* strong representation *of* k *if*

$$\mathsf{K}\pi_\Phi \quad \textit{is a } SL^2\textit{-lifting of} \quad \mathsf{k}\pi_\phi \quad \textit{for all } \phi \in B^2 \textit{ and some lifting } \Phi \in SB^2 \textit{ of } \phi. \qquad \lhd$$

Note that by proposition 2.3.11 every internal operator is an internal kernel operator. Thus every strong representation is a representation by an internal kernel operator.

Proposition 3.3.12 *Let* $\Phi, \Psi \in SB^2(T)$ *be liftings of* $\phi \in B^2([0,1])$ *and suppose that* $\mathsf{K}\pi_\Phi \approx \mathsf{k}\pi_\phi$ *as strong representation. Then also* $\mathsf{K}\pi_\Psi \approx \mathsf{k}\pi_\phi$.

Proof: By definition we have

$$\mathsf{K}\pi_\Phi(\alpha) \;\approx\; \mathsf{k}\pi_\phi(\mathrm{st}(\alpha)) \;\text{ for } m_L\text{-almost all } \alpha \in \Gamma\,. \tag{3.3.8}$$

We show that

$$\mathsf{K}\pi_\Psi(\alpha) \;\approx\; \mathsf{K}\pi_\Phi(\alpha) \;\text{ for } m_L\text{-almost all } \alpha \in \Gamma\,.$$

Because $m_L(\Gamma_{\mathrm{st}}) = m_L(\Gamma)$ we restrict our analysis to Γ_{st}. Fix some $\alpha \in \Gamma_{\mathrm{st}}$ such that equation (3.3.8) holds. Then $|\alpha| \in \mathbb{N}$ by definition of Γ_{st}. We denote by K the internal kernel function of the operator K. Then the explicit formula for $\mathsf{K}\pi_\Phi(\alpha)$ is given by

$$\mathsf{K}\pi_\Phi(\alpha) \;=\; \sum_{\sigma\dot\cup\rho\dot\cup\beta=\alpha} \;\sum_{\substack{\tau \\ \tau\cap(\rho\cup\beta)=\emptyset}} K(\sigma,\rho,\tau)\pi_\Phi(\rho\cup\beta\cup\tau)\left(\frac{1}{H}\right)^{|\tau|}$$

Because for our finite α the first sum is a finite sum we can fix also some triple (σ, ρ, β) with $\sigma \dot\cup \rho \dot\cup \beta = \alpha$ and such that $\tau \mapsto K(\sigma, \rho, \tau)\pi_\Phi(\rho \cup \beta \cup \tau)$ is S-integrable. (Eventually we should take another $\alpha \in \Gamma_{\mathrm{st}}$.) It is sufficient to show that the second sum varies only infinitesimally if we replace Φ by Ψ. We see that

$$E = \{\tau \in \Gamma : \tau \cap (\rho \cup \beta) \neq \emptyset\} \subseteq \cup_{s \in \rho \cup \beta} E_s$$

and by corollary 2.1.7 $m_L(E_s) = 0$. Then also $m_L(E) = 0$ and the complement is a set of full measure. Further since $\pi_\Phi, \pi_\Psi \in SL^2(\Gamma)$ also the maps

$$\tau \mapsto \pi_\Phi(\rho \cup \beta \cup \tau) \quad \text{and} \quad \tau \mapsto \pi_\Psi(\rho \cup \beta \cup \tau)$$

are SL^2 for m_L^2-almost all $(\rho, \beta) \in \Gamma^2_{\neq}$. By general assumption we can ensure that for m_L^3-almost (σ, ρ, β) the maps

$$\tau \mapsto K(\sigma, \rho, \tau)\pi_\Phi(\rho \cup \beta \cup \tau) \quad \text{and} \quad \tau \mapsto K(\sigma, \rho, \tau)\pi_\Psi(\rho \cup \beta \cup \tau) \tag{3.3.9}$$

are simultaneously S-integrable. On the other hand we have for m_L^2-almost all $(\rho, \beta) \in \Gamma^2_{\neq}$ that

$$\pi_\Phi(\rho \cup \beta \cup \tau) \;\approx\; \pi_\Psi(\rho \cup \beta \cup \tau) \quad \text{for } m_L\text{-almost all } \tau \in \Gamma$$

since Ψ and Φ are liftings of the same standard function ϕ. So we obtain by adapting corollary 3.2.13 and because of $m(\Gamma) \approx m(\Gamma \setminus E)$ that

$$\sum_{\substack{\tau \\ \tau \cap (\rho \cup \beta) = \emptyset}} K(\sigma, \rho, \tau)\pi_\Phi(\rho \cup \beta \cup \tau) \left(\frac{1}{H}\right)^{|\tau|} \;\approx\; \sum_{\substack{\tau \\ \tau \cap (\rho \cup \beta) = \emptyset}} K(\sigma, \rho, \tau)\pi_\Psi(\rho \cup \beta \cup \tau) \left(\frac{1}{H}\right)^{|\tau|}.$$

This shows that

$$\forall \alpha \in \Gamma_{\mathrm{st}} \cap C \left(\mathsf{K}\pi_\Psi(\alpha) \;\approx\; \mathsf{k}\pi_\Phi(\alpha)\right)$$

where C is the set on which equation (3.3.8) holds and both terms in equation (3.3.9) are S-integrable. This proves the proposition. □

Remark 3.3.13 Since the assumption A on page 80 assures that we have simultaneously for all $\Psi \in SB^2$ that $\tau \mapsto K(\sigma, \rho, \tau)\pi_\Psi(\rho \cup \beta \cup \tau)$ is S-integrable for m_L^3-almost all $(\sigma, \rho, \beta) \in \Gamma^3$ we see that we have actually the following stronger result:

If $\mathsf{K}\pi_\Phi \approx \mathsf{k}\pi_\phi$ holds for one lifting Φ of ϕ then it holds for all.

Proposition 3.3.14
Every strict and every strict equivalent representation is a strong representation.

PROOF: Suppose K to be a strict representation of k, that is K is a lifting of k. Further choose $\phi \in B^2$ and a lifting $\Phi \in SB^2$ of ϕ. Suppose that $\alpha \in \Gamma_{\mathrm{st}}$. Then

$$\begin{aligned} \mathsf{K}\pi_\Phi(\alpha) &= \sum_{\sigma \dot\cup \rho \dot\cup \beta = \alpha} \sum_{\substack{\tau \\ \tau \cap (\rho \cup \beta) = \emptyset}} K(\sigma, \rho, \tau)\pi_\Phi(\rho \cup \beta \cup \tau) \left(\frac{1}{H}\right)^{|\tau|} \\ &\approx \sum_{\sigma \dot\cup \rho \dot\cup \beta = \mathrm{st}(\alpha)} \int_{\tau \in \mathcal{P}_{\mathrm{fin}}} k(\sigma, \rho, \tau)\pi_\phi(\rho \cup \beta \cup \tau) d\Lambda(\tau) \;=\; \mathsf{k}\pi_\phi(\alpha)\,. \end{aligned}$$

since by $|\alpha| = |\mathrm{st}(\alpha)| \in \mathbb{N}$ the sum is finite and adapting corollary 3.2.13. For a strict equivalent representation L of k we have by definition that $\mathsf{L}\pi_\Phi(\alpha) \approx \mathsf{K}\pi_\Phi(\alpha)$ for m_L-almost all $\alpha \in \Gamma$ and the result follows by the same calculation. □

Naturally the set of strong representations of some standard operator is stable under strong equivalence.

Corollary 3.3.15 *Suppose that* K *is a strong representation of* k *and* L *is strongly equivalent to* K*. Then* L *is a strong representation of* k*.*

PROOF: Fix $\phi \in B^2$. Then for all liftings $\Phi \in SB^2$ of ϕ it is $\mathsf{L}\pi_\Phi \approx \mathsf{K}\pi_\Phi \approx \mathsf{k}\pi_\phi$. □

Proposition 3.3.16 *Assume we have two internal operators* K_1 *and* K_2 *that are strong kernel representations of the same standard operator* k*. Then we have for the corresponding internal kernel functions that*

$$K_1(\sigma,\rho,\tau) \quad \approx \quad K_2(\sigma,\rho,\tau) \ \textit{ for } m_L^3\textit{-almost all } \ (\sigma,\rho,\tau) \in \Gamma^3 \,.$$

PROOF: By definition we have

$$\mathsf{K}_1\pi_\Phi(\alpha) \quad \approx \quad \mathsf{K}_2\pi_\Phi(\alpha) \quad \approx \quad \mathsf{k}\pi_\phi(\mathrm{st}(\alpha)) \ \text{ for } m_L\text{-almost all } \ \alpha \in \Gamma \,.$$

Fix some $\alpha \in \Gamma_{\mathrm{st}}$ such that this equation holds. Hence we get

$$\sum_{\sigma\dot\cup\rho\dot\cup\beta=\alpha} \ \sum_{\substack{\tau \\ \tau\cap(\rho\cup\beta)=\emptyset}} K_1(\sigma,\rho,\tau)\pi_\Phi(\rho\cup\beta\cup\tau)\left(\frac{1}{H}\right)^{|\tau|} \quad \approx$$
$$\sum_{\sigma\dot\cup\rho\dot\cup\beta=\alpha} \ \sum_{\substack{\tau \\ \tau\cap(\rho\cup\beta)=\emptyset}} K_2(\sigma,\rho,\tau)\pi_\Phi(\rho\cup\beta\cup\tau)\left(\frac{1}{H}\right)^{|\tau|} \,. \tag{3.3.10}$$

Now suppose that there exists a measurable set $A \subseteq \Gamma^3$ of positive measure, say $m_L^3(A) = a \in \mathbb{R}_+$, such that

$$K_1(\sigma,\rho,\tau) \quad \not\approx \quad K_2(\sigma,\rho,\tau) \ \text{ for all } \ (\sigma,\rho,\tau) \in A \,.$$

By Loeb measure theory exists an internal set $B \subseteq A$ with $m^3(B) \approx m_L^3(B) = b \in \mathbb{R}_+$. Since B is internal and has positive real measure, we can assume it to be of the form $B = B_1 \times B_2 \times B_3$. By construction we get $m(B_3) \geq b_3 \in \mathbb{R}_+$. By the totality of the π_ϕs and since every π_ϕ has a lifting π_Φ there is some $F \in \overline{\mathfrak{E}}$ (the ${}^*\mathbb{C}_{\mathrm{fin}}$-linear span of the π_Φs) and some finite triple (σ,ρ,β) such that

$$\sum_{\tau\in B_3} K_1(\sigma,\rho,\tau)F(\rho\cup\beta\cup\tau)\left(\frac{1}{H}\right)^{|\tau|} \quad \not\approx \quad \sum_{\tau\in B_3} K_2(\sigma,\rho,\tau)F(\rho\cup\beta\cup\tau)\left(\frac{1}{H}\right)^{|\tau|}$$

on the set B_3. But for the chosen α this is a contradiction to equation (3.3.10). □

We turn now to strong representations of operator processes.

Definition 3.3.17 *An internal operator process* $(\mathsf{K}_t)_{t\in T}$ *is a* strong representation *of an operator process* $(\mathsf{k})_{t\in[0,1]}$ *if for* μ_L*-almost all* $t \in T$ *the operator* K_t *is a strong representation of* $\mathsf{k}_{\mathrm{st}(t)}$. *The representation is called* total *if this holds for all* $t \in T$. $\lhd$

First note that by proposition 2.3.11 again every strong representation is a representation by an internal kernel process. Secondly note that in the definition above the almost-all-part of the formula $\mathsf{K}_t\pi_\Phi \approx \mathsf{k}_{\mathrm{st}(t)}\pi_\phi$ (see definition 3.3.11) depends not only on the chosen ϕ but also on $t \in T$. We formulate the following proposition.

Proposition 3.3.18
Every (totally) strict representation is a (totally) strong representation.

PROOF: This follows immediately with proposition 3.3.14 using the same proof with K_t in place of K and $k_{\mathrm{st}(t)}$ in place of k. $\square$

We need the next two results in Section 2 of Chapter 4 where we prove internal versions of the fundamental formulas of quantum stochastic calculus.

Proposition 3.3.19 *Let* $(\mathsf{K}_s)_{s\in T}$ *be a totally strong (strong) representation of* $(\mathsf{k}_s)_{s\in[0,1]}$. *Then* $(\notin_s\mathsf{K}_s)_{s\in T}$ *is a totally strong (strong) representation of* $(\notin_s\mathsf{k}_s)_{s\in[0,1]}$.

PROOF: Fix $\phi \in B^2$ and a lifting $\Phi \in SB^2$ of ϕ. Then $\mathsf{K}_s\pi_\Phi(\sigma) \approx \mathsf{k}_{\mathrm{st}(s)}\pi_\phi(\mathrm{st}(\sigma))$ for m_L-almost all $\sigma \in \Gamma$ (and μ_L-almost all $s \in T$). Take in addition $\sigma \in \Gamma_{\mathrm{st}\notin\mathrm{t}}$ where $t = \mathrm{st}(s)$. Then $\notin_s(\sigma) = \notin_{\mathrm{st}(s)}(\mathrm{st}(\sigma))$ for all $\sigma \in \Gamma_{\mathrm{st}\notin\mathrm{t}}$. But $\Gamma_{\mathrm{st}\notin\mathrm{t}}$ is a set of full Loeb measure and so (for μ_L-almost all $s \in T$):

$$\notin_s(\sigma)\mathsf{K}_s\pi_\Phi(\sigma) \quad\approx\quad \notin_{\mathrm{st}(s)}(\mathrm{st}(\sigma))\mathsf{k}_{\mathrm{st}(s)}\pi_\phi(\mathrm{st}(\sigma))$$

for m_L-almost all $\sigma \in \Gamma$. $\square$

Corollary 3.3.20 *If* K *is a totally strong (strong) representation of* k *then*

$$\langle\pi_\Psi, \notin_s\mathsf{K}_s\pi_\Phi\rangle \quad\approx\quad \langle\pi_\psi, \mathsf{k}_{\mathrm{st}(s)}\pi_\phi\rangle \quad \textit{for } (\mu_L\textit{-almost}) \textit{ all } s \in T.$$

PROOF: The set $\{\sigma \in \mathcal{P}_{\mathrm{fin}} : \notin_{\mathrm{st}(s)}(\sigma) = 0\}$ is a Λ-nullset. $\square$

We show now that the fundamental internal processes are totally strong representations of the corresponding standard processes.

Proposition 3.3.21 *Let* $\sharp \in \{\bullet, +, \circ, -\}$. *Then the fundamental process* $A_t^\sharp$ *is a totally strong representation of the fundamental process* $\mathsf{a}_t^\sharp$. *The same is true for the modified time process* $\widetilde{A}_t^\bullet$.

PROOF: Fix $t \in [0,1]$ and some $\tilde{t} \in \mathrm{st}^{-1}(t)$. We first note that by corollary 2.1.9 the set $\Gamma_{\mathrm{st}\notin\mathrm{t}}$ of all nearstandard σ not intersecting the monad of t has full Loeb measure. Moreover for each $\sigma \in \Gamma_{\mathrm{st}\notin\mathrm{t}}$ there exists no $s \in \sigma$ with $s \in \mathrm{st}^{-1}(t)$. But this shows that

$s < \tilde{t}$ iff $\mathrm{st}(s) < t$ and $s \geq \tilde{t}$ iff $\mathrm{st}(s) \geq t$ for all $s \in \sigma$. Thus for each $\sigma \in \Gamma_{\mathrm{st}\notin\mathrm{t}}$ we can conclude in the cases of the creation and the number operator process:

$$\begin{aligned}
(A^+_{\tilde{t}}\pi_\Phi)(\sigma) &= \sum_{\substack{s<\tilde{t}\\ s\in\sigma}} \pi_\Phi(\sigma \setminus s) \approx \sum_{\substack{s<t\\ s\in\mathrm{st}(\sigma)}} \pi_\phi(\mathrm{st}(\sigma)\setminus s) = (\mathsf{a}^+_t\pi_\phi)(\mathrm{st}(\sigma))\,;\\
(A^\circ_{\tilde{t}}\pi_\Phi)(\sigma) &= \sum_{\substack{s<\tilde{t}\\ s\in\sigma}} \pi_\Phi(\sigma) \approx \sum_{\substack{s<t\\ s\in\mathrm{st}(\sigma)}} \pi_\phi(\mathrm{st}(\sigma)) = (\mathsf{a}^\circ_t\pi_\phi)(\mathrm{st}(\sigma))
\end{aligned}$$

since π_Φ is a lifting of π_ϕ. For the annihilation and time cases we need that π_Φ is SL^2 and $m(E_s) \approx 0$ (see corollary 2.1.7). Again with $\sigma \in \Gamma_{\mathrm{st}\notin\mathrm{t}}$ we obtain

$$\begin{aligned}
(A^-_{\tilde{t}}\pi_\Phi)(\sigma) &= \sum_{\substack{s<\tilde{t}\\ s\notin\sigma}} \frac{1}{H}\pi_\Phi(\sigma\cup s) \approx \sum_{s<\tilde{t}} \frac{1}{H}\pi_\Phi(\sigma\cup s)\\
&\approx \int_0^t \pi_\phi(\mathrm{st}(\sigma)\cup s)ds = (\mathsf{a}^-_t\pi_\phi)(\mathrm{st}(\sigma))\,;\\
(A^\bullet_{\tilde{t}}\pi_\Phi)(\sigma) &= \sum_{\substack{s<\tilde{t}\\ s\notin\sigma}} \frac{1}{H}\pi_\Phi(\sigma) \approx \sum_{s<\tilde{t}} \frac{1}{H}\pi_\Phi(\sigma) \approx \int_0^t \pi_\phi(\mathrm{st}(\sigma))ds = (\mathsf{a}^\bullet_t\pi_\phi)(\mathrm{st}(\sigma))\,.
\end{aligned}$$

Further $\widetilde{A}^\bullet_t$ is strongly equivalent to $A^\bullet_t$. This completes the proof. □

It is well known in quantum stochastic calculus that the process $\mathsf{b}^\theta_t = \mathrm{e}^{\mathrm{i}\theta}\mathsf{a}^+_t + \mathrm{e}^{-\mathrm{i}\theta}\mathsf{a}^-_t$ is a realization of Brownian motion on Fock space. Also $\mathsf{p}^z_t = \mathsf{a}^\circ_t + z\mathsf{a}^+_t + \bar{z}\mathsf{a}^-_t + |z|^2\mathsf{a}^\bullet_t$ gives a Poisson process on Fock space. The preceding proposition shows that our internal Brownian motion B^θ_t and internal Poisson process P^z_t are totally strong representations of the respective standard processes.

3.3 Weak Representation

In this subsection we introduce a third concept of representation of Fock space operators by internal operators.

Definition 3.3.22 *Let K be an internal operator and k a (standard) Fock space operator. We say that K is a* weak representation *of k if $\langle\pi_\Psi, \mathsf{K}\pi_\Phi\rangle$ is a lifting of $\langle\pi_\psi, \mathsf{k}\pi_\phi\rangle$ for all $\psi, \phi \in B^2$ and all liftings $\Psi, \Phi \in SB^2$ of ψ, ϕ. An internal operator process K is a* weak representation *of a standard process k if for m_L-almost all $t \in T$ the operator K_t is a weak representation of the operator $\mathsf{k}_{\mathrm{st}(t)}$. As before, we say that the representation is* total *if this holds for all $t \in T$.* ◁

Proposition 3.3.23 *Every strong representation is a weak representation. The same is true for (total) representations of processes.*

Proof: Suppose that K is a totally strong (strong) representation of k. Fix some $\phi \in B^2$. Then by definition and with proposition 3.3.12 we have for (μ_L-almost) all $t \in T$

that $\mathsf{K}_t\pi_\Phi \approx \mathsf{k}_{\mathrm{st}(t)}\pi_\phi$ for all liftings $\Phi \in SB^2$ of ϕ. Since we know that $\mathsf{K}_t\pi_\Phi$ is SL^2 by definition we have for every $\psi \in B^2$ and (μ_L-almost) all $t \in T$ that

$$\langle \pi_\Psi, \mathsf{K}_t\pi_\Phi \rangle \quad \approx \quad \langle \pi_\psi, \mathsf{k}_{\mathrm{st}(t)}\pi_\phi \rangle$$

for all liftings $\Psi, \Phi \in SB^2$ of ψ, ϕ. Taking $\mathsf{K}_t = \mathsf{K}$ for all $t \in T$ proves the assertion for operators. □

The next proposition connects the vacuum expectation in the internal Fock space to that in the Fock space. This is necessary to transport the stochastic concepts from the nonstandard universe to the standard universe. As we will see this gives us the possibility of calculating characteristic distributions by some hyperfinite combinatorics and only in the end do we apply the right identifications and get the characteristic distribution of some standard process.

Proposition 3.3.24
Let K be a strict, strong or weak representation of k. Then $\mathrm{st}(\mathbb{E}(\mathsf{K})) = \mathbb{E}(\mathsf{k})$.

Proof: Since the function $\Phi \equiv 0$ is a lifting of $\phi \equiv 0$ we see that the vacuum state $\Omega = \pi_0$ in the internal Fock space is a lifting of the vacuum state $\omega = \pi_0$ in the standard Fock space. Then for weak and thus also for strong and strict representations it is clear that

$$\mathbb{E}(\mathsf{K}) \;=\; \langle \Omega, \mathsf{K}\Omega \rangle \;\approx\; \langle \omega, \mathsf{k}\omega \rangle \;=\; \mathbb{E}(\mathsf{k})\,. \qquad \square$$

Corollary 3.3.25 *Let the internal operator process K be a totally strict, totally strong or totally weak representation of the standard operator process k. Then for all $t \in T$ we have* $\mathrm{st}(\mathbb{E}(\mathsf{K}_t)) = \mathbb{E}(\mathsf{k}_{\mathrm{st}(t)})$. *If the representation is not total then this holds for μ_L-almost all $t \in T$.*

This corollary has nice consequences. Using proposition 3.3.21 and our results of sections 2.2 and 2.3 of chapter 2 we can argue now vice-versa. Firstly since the internal Brownian motion $B_t^\theta = \mathrm{e}^{\mathrm{i}\theta}A_t^+ + \mathrm{e}^{-\mathrm{i}\theta}A_t^-$ is a totally strong representation of $\mathsf{b}_t^\theta = \mathrm{e}^{\mathrm{i}\theta}\mathsf{a}_t^+ + \mathrm{e}^{-\mathrm{i}\theta}\mathsf{a}_t^-$ we know now that b_t^θ has characteristic distribution $\exp(-\frac{y^2}{2}\cdot t)$ (theorem 2.2.16). Thus it is a Brownian motion on the standard Fock space. Secondly since $P_t^z = A_t^\circ + zA_t^+ + \overline{z}A_t^- + |z|^2 A_t^\bullet$ is a totally strong representation of $\mathsf{p}_t^z = \mathsf{a}_t^\circ + z\mathsf{a}_t^+ + \overline{z}\mathsf{a}_t^- + |z|^2\mathsf{a}_t^\bullet$ we see that p_t^z has characteristic distribution $\exp(|z|^2 t(\mathrm{e}^{\mathrm{i}y} - 1))$ (theorem 2.2.26). Thus it is a Poisson process on the standard Fock space.

Chapter 4

Internal and Standard Objects

In this Chapter we connect certain standard objects to certain internal objects. In the first Section we show how the internal solution of the linear quantum stochastic differential equation with constant coefficients is a representation of the standard solution. Following this we construct a standard kernel solution for the quantum stochastic differential equation with nonlinear noise terms from the 'nonlinear' internal solution. In the second Section we define S-integrability for internal operator processes. Then we show that if an adapted standard process has an S-integrable strong representation then it is an integrable process. In the third Section we prove existence of strong representations for bounded operators, bounded processes and bounded martingales.

1 QSDEs as Hyperfinite Difference Equations

In this Section we look at quantum stochastic differential equations and hyperfinite difference equations. We show that in a certain sense the solution of the linear quantum stochastic differential equation with bounded constant coefficients can be obtained by solving the corresponding hyperfinite difference equation. We extend this method to show the existence of a solution of a certain quantum stochastic differential equation with nonlinear noise terms. The language we use is the language of Maassen-Meyer kernels and strict representations. Thus we obtain in reality a kernel solution of the nonlinear equation.

1.1 The Linear Case

In this Subsection we connect the solution of the hyperfinite difference equation in theorem 2.4.9, Section 4.2 of Chapter 2 to the kernel solution of the standard quantum stochastic differential equation in theorem 1.3.5, Section 3 of Chapter 1. Specifically, we show that the solution obtained by hyperfinite combinatorics is a totally strict representation of the kernel solution in the standard case. In fact, the essential part of the proof is showing that for $t_k = \frac{k}{H} \in T$ and some S-bounded $K \in {}^*\mathcal{B}(\mathcal{K})$ we have $(1 + \frac{K}{H})^k \approx \exp(\mathrm{st}(t_k)\mathrm{st}(K))$. But for that we need some preparatory results.

Lemma 4.1.1 *Fix $m \in \mathbb{N}$. Let $K_1, \ldots, K_m \in {}^*\mathcal{B}(\mathcal{K})$ be nearstandard in the weak operator topology. Then $K_1 \cdots K_m$ is nearstandard in this topology and*

$$\mathrm{st}(K_1 \cdots K_m) \;=\; \mathrm{st}(K_1) \cdots \mathrm{st}(K_m)\,.$$

Proof: For $i = 1, \ldots, m$ we have $L_i \in \mathcal{B}(\mathcal{K})$ and weak infinitesimals $Q_i \in {}^*\mathcal{B}(\mathcal{K})$ such that $K_i = {}^*L_i + Q_i$. We show that $\mathrm{st}(K_1 \cdots K_m) = L_1 \cdots L_m$. We get for the product

$$K_1 \cdots K_m \;=\; {}^*L_1 \cdots {}^*L_m + \{\text{ terms with some } Q_k \text{ somewhere }\}\,.$$

For example if $m = 2$ we have

$$K_1 K_2 \;=\; {}^*L_1 {}^*L_2 + {}^*L_1 Q_2 + Q_1 {}^*L_2 + Q_1 Q_2\,.$$

Note that for $y \in \mathcal{K}$ we have ${}^*L_i y \in \mathcal{K}$. Thus $\langle x, Q_k {}^*L_i y\rangle \approx 0$ for all $x, y \in \mathcal{K}$. Also

$$\langle x, {}^*L_i Q_k y\rangle \;=\; \langle {}^*L_i^\star x, Q_k y\rangle \;\approx\; 0$$

for all $x, y \in \mathcal{K}$. Obviously this extends to arbitrary products of ${}^*L_i$s and Q_ks if at least one Q_k appears. But then for all $x, y \in \mathcal{K}$

$$\langle x, K_1 \cdots K_m y\rangle \;\approx\; \langle x, {}^*L_1 \cdots {}^*L_m y\rangle\,.$$

since the terms with Q_k somewhere become infinitesimal. This shows

$$\mathrm{st}(K_1 \cdots K_m) \;=\; L_1 \cdots L_m \;=\; \mathrm{st}(K_1) \cdots \mathrm{st}(K_m). \qquad \square$$

Corollary 4.1.2 *Fix some $m \in \mathbb{N}$. Suppose that $K \in {}^*\mathcal{B}(\mathcal{K})$ is nearstandard in the weak operator topology. Then $\mathrm{st}(K^m) = \mathrm{st}(K)^m$.*

Proposition 4.1.3 *Let $H \in {}^*\mathbb{N}_\infty$ and $K \in {}^*\mathcal{B}(\mathcal{K})$ be S-bounded. Then $(1 + \frac{K}{H})^H$ is S-bounded and*

$$\mathrm{st}\left(\left(1 + \frac{K}{H}\right)^H\right) \;=\; \exp(\mathrm{st}(K)) \;\; \textit{(in the weak operator topology).}$$

Proof: First note that $0 \leq \frac{H!}{(H-l)!H^l} < 1$ for all ${}^*\mathbb{N} \ni l \leq H$ and for finite $l \in \mathbb{N}$ it is $\frac{H!}{(H-l)!H^l} \approx 1$. Since K is S-bounded we have $\|K\| \leq M \in \mathbb{N}$. This gives us

$$\left\|\left(1 + \frac{K}{H}\right)^H\right\| \;=\; \left\|\sum_{l=0}^{H} \binom{H}{l} \left(\frac{K}{H}\right)^l\right\| \;=\; \left\|\sum_{l=0}^{H} \frac{H!}{(H-l)!l!} \left(\frac{K}{H}\right)^l\right\|$$

$$= \;\left\|\sum_{l=0}^{H} \frac{H!}{(H-l)!H^l} \frac{K^l}{l!}\right\| \;\leq\; \sum_{l=0}^{H} \frac{H!}{(H-l)!H^l} \frac{\|K\|^l}{l!} \;\leq\; \sum_{l=0}^{H} \frac{\|K\|^l}{l!}$$

$$\leq\; \sum_{l=0}^{H} \frac{M^l}{l!} \;\leq\; \exp(M) + 1\,.$$

Thus $(1+\frac{K}{H})^H$ is S-bounded and has a standard part (in the weak operator topology). Denote by $\mathrm{st}(K)$ the standard part of K. Then by definition $\langle x, Ky\rangle \approx \langle x, \mathrm{st}(K)y\rangle$ for all $x, y \in \mathcal{K}$. We show now that

$$\left\langle x, \left(1+\frac{K}{H}\right)^H y\right\rangle \approx \langle x, \exp(\mathrm{st}(K))y\rangle$$

for all $x, y \in \mathcal{K}$. This proves $\mathrm{st}((1+\frac{K}{H})^H) = \exp(\mathrm{st}(K))$. Fix some $x, y \in \mathcal{K}$. We obtain

$$\begin{aligned}
\left\langle x, \left(1+\frac{K}{H}\right)^H y\right\rangle &= \sum_{l=0}^{H} \frac{H!}{(H-l)!H^l l!}\langle x, K^l y\rangle \\
&= \sum_{l=0}^{r} \frac{H!}{(H-l)!H^l l!}\langle x, K^l y\rangle + \sum_{l=r+1}^{H} \frac{H!}{(H-l)!H^l l!}\langle x, K^l y\rangle \\
&\approx \sum_{l=0}^{r}\langle x, \frac{\mathrm{st}(K)^l}{l!}y\rangle + \sum_{l=r+1}^{H} \frac{H!}{(H-l)!H^l l!}\langle x, K^l y\rangle \\
&= \langle x, \sum_{l=0}^{r}\frac{\mathrm{st}(K)^l}{l!}y\rangle + \sum_{l=r+1}^{H} \frac{H!}{(H-l)!H^l l!}\langle x, K^l y\rangle
\end{aligned}$$

for every $r \in \mathbb{N}$. We show now that the second term tends to an infinitesimal if $r \in \mathbb{N}$ goes to infinity in $\mathbb{N}$. We calculate

$$\begin{aligned}
\left|\sum_{l=r+1}^{H} \frac{H!}{(H-l)!H^l l!}\langle x, K^l y\rangle\right| &\leq \sum_{l=r+1}^{H} \frac{|\langle x, K^l y\rangle|}{l!} \\
\leq \sum_{l=r+1}^{H} \frac{\|x\|\,\|y\|\,\|K\|^l}{l!} &\leq \|x\|\,\|y\| \sum_{l=r+1}^{H} \frac{M^l}{l!}\,.
\end{aligned}$$

Since x and y have finite norm the last term approaches zero up to some infinitesimal as r becomes larger (in $\mathbb{N}$). By continuity of the inner product we conclude for all $x, y \in \mathcal{K}$

$$\left\langle x, \left(1+\frac{K}{H}\right)^H y\right\rangle \approx \left\langle x, \sum_{l\in\mathbb{N}} \frac{\mathrm{st}(K)^l}{l!} y\right\rangle = \langle x, \exp(\mathrm{st}(K))y\rangle. \qquad \square$$

Let $t_{k_1}, t_{k_2} \in T = \{t_0, \ldots t_{H-1}\}$ with $t_{k_1} < t_{k_2}$. Note that we have exactly $k_2 - k_1 - 1$ points in T that lie between t_{k_1} and t_{k_2}. Under these conditions we have the following proposition.

Proposition 4.1.4 *Let $K \in {}^*\mathcal{B}(\mathcal{K})$ be S-bounded. Then*

$$\mathrm{st}\left(\left(1+\frac{K}{H}\right)^{k_2-k_1-1}\right) = \exp\Big((\mathrm{st}(t_{k_2}) - \mathrm{st}(t_{k_1}))\mathrm{st}(K)\Big)\,.$$

PROOF:

$$\begin{aligned}\Big(1+\frac{K}{H}\Big)^{k_2-k_1-1} &= \Big(1+\frac{K}{H}\Big)^{H(t_{k_2}-t_{k_1}-\frac{1}{H})} \approx \Big(\exp(\mathrm{st}(K))\Big)^{t_{k_2}-t_{k_1}-\frac{1}{H}} \\ &= \exp\Big((t_{k_2}-t_{k_1}-\frac{1}{H})\mathrm{st}(K)\Big) \approx \exp\Big((\mathrm{st}(t_{k_2})-\mathrm{st}(t_{k_1}))\mathrm{st}(K)\Big)\end{aligned}$$

where we have used the general spectral calculus and the continuity of the map $t \mapsto \exp(tL)$ in the weak operator topology for some $L \in \mathcal{B}(\mathcal{K})$. □

We show now that our hyperfinite solution to the internal quantum stochastic differential equation

$$K_t = {}^*L_0 + \sum_{\widetilde{\sharp}} \sum_{s<t} a_s^{\widetilde{\sharp}}\, {}^*L^{\widetilde{\sharp}} \diamond K_s = {}^*L_0 + \sum_{\widetilde{\sharp}} \sum_{s<t} a_s^{\widetilde{\sharp}}\, {}^*L^{\widetilde{\sharp}} K_s$$

is a totally strict representation of the kernel solution to the standard quantum stochastic differential equation

$$\mathsf{k}_t = L_0 + \sum_{\sharp} \int_0^t L^{\sharp} \mathsf{k}_s d\mathsf{a}_s^{\sharp} .$$

We recall that the respective solutions are given in theorems 2.4.9 and 1.3.5.

Theorem 4.1.5 *Let $L_0, L^\sharp \in \mathcal{B}(\mathcal{K})$, $\sharp \in \{\widetilde{\bullet}, +, \circ, -\}$. Then the adapted internal kernel function*

$$K_t(\sigma,\rho,\tau) = \begin{cases} \prod_n^{k=0} \Pi_{\sigma,\rho,\tau}(t_k)^* L_0 & \text{if } \max(\sigma\cup\rho\cup\tau) < t, \\ 0, & \text{otherwise,} \end{cases}$$

with $\Pi_{\sigma,\rho,\tau}(t_k)$ defined (for $(\sigma,\rho,\tau) \in \Gamma^3_{\neq}$) through

$$\Pi_{\sigma,\rho,\tau}(t_k) = \begin{cases} 1 + \frac{1}{H}{}^*L^{\widetilde{\bullet}} & \text{if } t_k \notin \sigma\cup\rho\cup\tau, \\ {}^*L^+ & \text{if } t_k \in \sigma, \\ {}^*L^\circ & \text{if } t_k \in \rho, \\ {}^*L^- & \text{if } t_k \in \tau, \end{cases}$$

is a totally strict representation of the standard kernel function

$$k_t(\sigma,\rho,\tau) = \mathrm{e}^{L^{\widetilde{\bullet}}(t-t_n)}\Pi(t_n)\mathrm{e}^{L^{\widetilde{\bullet}}(t_n-t_{n-1})}\Pi(t_{n-1})\cdots\Pi(t_1)\mathrm{e}^{L^{\widetilde{\bullet}}(t_1-t_0)}L_0$$

where $\sigma\cup\rho\cup\tau = \{t_1 < t_2 < \cdots < t_n\} \subset [0,t[$ and $t_0 = 0$ and Π is given by

$$\Pi(t) = \begin{cases} L^+ & \text{if } t \in \sigma, \\ L^\circ & \text{if } t \in \rho, \\ L^- & \text{if } t \in \tau, \end{cases}$$

and k_t is zero if $\max(\sigma\cup\rho\cup\tau) \geq t$.

PROOF: By definition 3.3.5 we have to prove that for all $\tilde{t} \in T$ it is $K_{\tilde{t}}(\sigma,\rho,\tau) \approx k_{\mathrm{st}(\tilde{t})}(\mathrm{st}(\sigma,\rho,\tau))$ for m_L^3-almost all $(\sigma,\rho,\tau) \in \Gamma^3$. Choose some $t \in [0,1]$ and $\tilde{t} \in \mathrm{st}^{-1}(t)$. Then for arbitrary $(\sigma,\rho,\tau) \in \Gamma^{[3]}_{\mathrm{st}\notin\mathrm{t}}$ we know that σ,ρ and τ are finite, approximately pairwise disjoint and do not intersect the monad of t. Further, $\Gamma^{[3]}_{\mathrm{st}\notin\mathrm{t}}$ is a subset of Γ^3 of full Loeb measure. Take $(\sigma,\rho,\tau) \in \Gamma^{[3]}_{\mathrm{st}\notin\mathrm{t}}$. Then we have

$$\max(\sigma \cup \rho \cup \tau) \geq \tilde{t} \quad \Longleftrightarrow \quad \max(\mathrm{st}(\sigma) \cup \mathrm{st}(\rho) \cup \mathrm{st}(\tau)) \geq t$$

and if this is the case both kernel functions vanish. On the other hand we have

$$\max(\sigma \cup \rho \cup \tau) < \tilde{t} \quad \Longleftrightarrow \quad \max(\mathrm{st}(\sigma) \cup \mathrm{st}(\rho) \cup \mathrm{st}(\tau)) < t\,.$$

In this case suppose that

$$\sigma \cup \rho \cup \tau \;=\; \{t_{k_{i_1}} < \cdots < t_{k_{i_n}}\} \;\subset\; \{t_0, \ldots, \tilde{t} - \frac{1}{H} = t_{k_t}\}\,.$$

Then using proposition 4.1.4 and lemma 4.1.1 we have

$$\begin{aligned}
K_{\tilde{t}}(\sigma,\rho,\tau) \;&=\; (1 + \frac{{}^*\widetilde{L^\bullet}}{H})^{k_t - k_{i_n} - 1}\Pi(t_{k_{i_n}})(1 + \frac{{}^*\widetilde{L^\bullet}}{H})^{k_{i_n} - k_{i_{n-1}} - 1}\cdot \\
&\qquad\qquad \cdot\Pi(t_{k_{i_{n-1}}}) \;\cdots\; \Pi(t_{k_{i_1}})(1 + \frac{{}^*\widetilde{L^\bullet}}{H})^{k_{i_1} - 0 - 1}{}^*L_0 \\
&\approx\; \exp((t - \mathrm{st}(t_{k_{i_n}}))\widetilde{L^\bullet}\Pi(\mathrm{st}(t_{k_{i_n}}))\exp((\mathrm{st}(t_{k_{i_n}}) - \mathrm{st}(t_{k_{i_{n-1}}}))\widetilde{L^\bullet})\cdot \\
&\qquad\qquad \cdot\Pi(\mathrm{st}(t_{k_{i_{n-1}}})) \;\cdots\; \Pi(\mathrm{st}(t_{k_{i_1}}))\exp((\mathrm{st}(t_{k_{i_1}}) - 0)\widetilde{L^\bullet})L_0 \\
&=\; k_t(\mathrm{st}(\sigma,\rho,\tau))
\end{aligned}$$

where $\Pi(\,\cdot\,) = \Pi_{\sigma,\rho,\tau}(\,\cdot\,)$ refers to the internal choice function and $\Pi(\mathrm{st}(\,\cdot\,))$ to the standard choice function. This proves that $K_{\tilde{t}}$ is a totally strict representation of k_t. □

If we look at theorem 4.1.5 we see that more follows. In the proof of the theorem it is implicitly shown that $K_t(\sigma,\rho,\tau)$ is almost surely t-S-continuous in the following sense.

Definition 4.1.6 *We say that a process kernel K is* almost surely t-S-continuous *if for all $t \in [0,1]$ there is a set $M(t) \subset \Gamma^3$ of full Loeb measure such that*

$$\begin{aligned}
&\forall r \in T\, \forall s \in T\, \forall(\sigma,\rho,\tau) \in M(t)\, \forall(\alpha,\beta,\gamma) \in M(t) \\
&\qquad (\mathrm{st}(r) = \mathrm{st}(s) = t \wedge \mathrm{st}(\alpha,\beta,\gamma) = \mathrm{st}(\sigma,\rho,\tau) \;\Longrightarrow\; K_r(\sigma,\rho,\tau) \approx K_s(\alpha,\beta,\gamma))\,.
\end{aligned}$$

We say that this holds uniformly *if $M(t) = M$ for all $t \in T$.* ◁

For t-S-continuous internal kernels the following standard process kernel is well-defined:

$$k_t(\mathrm{st}(\sigma,\rho,\tau)) \;=\; \begin{cases} \mathrm{st}(K_{\tilde{t}}(\sigma,\rho,\tau)) & \text{if } (\sigma,\rho,\tau) \in M(t),\ \tilde{t} \in \mathrm{st}^{-1}(t), \\ 0, & \text{otherwise,} \end{cases} \tag{4.1.1}$$

and by construction K is a totally strict and thus also a totally strong representation of k. So we conclude in our case:

Corollary 4.1.7 *Take the internal kernel of the hyperfinite quantum stochastic differential equation and define according to equation (4.1.1) a standard kernel function. If the internal operator is S-integrable (see definition 4.2.7) then the standard kernel defines an operator process which solves the standard quantum stochastic differential equation.*

PROOF: This is an application of theorem 4.2.9 and the fact that with K_s also $L\mathsf{K}_s$ is S-integrable for every S-bounded $L \in {}^*\mathcal{B}(\mathcal{K})$. $\square$

That the linear quantum stochastic differential equation has a unique solution is well-known and one finds a proof in the standard text books on quantum stochastic calculus. Despite this fact we see that we have not used a Picard iteration method but a crude combinatorial argument to solve the equation. Then the work to do was to 'push down' the internal solution to the standard solution.

1.2 The Nonlinear Case

The most common normal generalization is to take linear quantum stochastic differential equations with unbounded coefficients (cf. [Fag98, FW99, FW00]). We go in a different direction and take an equation with nonlinear noise terms:

$$\mathsf{k}_t \;=\; L_0 + \int_0^t L^{\tilde{\bullet}}\mathsf{k}_s ds + \int_0^t p^+(\mathsf{k}_s)d\mathsf{a}_s^+ + \int_0^t p^\circ(\mathsf{k}_s)d\mathsf{a}_s^\circ + \int_0^t p^-(\mathsf{k}_s)d\mathsf{a}_s^- \quad (4.1.2)$$

where the $p^\sharp$s are appropriate polynomials. It seems to be the case that such equations have not yet been studied in the literature. Next we construct a (standard) solution for this quantum stochastic differential equation with nonlinear noise terms by 'pushing down' the corresponding hyperfinite solution. We use the notation of Section 4.2 and corollary 3.3.9. Take $n_+, n_\circ, n_- \in \mathbb{N}$, operators $L_k^\sharp \in \mathcal{B}(\mathcal{K})$, $k = 1, \ldots, n_\sharp$, $\sharp \in \{+, \circ, -\}$ and $L^{\tilde{\bullet}}, L_0 \in \mathcal{B}(\mathcal{K})$. Then we build $p^\sharp(z) = \sum_{k=1}^{n_\sharp} L_k^\sharp z^k$, $P^\sharp(z) = \sum_{k=1}^{n_\sharp} {}^*L_k^\sharp z^k$, $p^{\tilde{\bullet}}(z) = L^{\tilde{\bullet}} z$ and $P^{\tilde{\bullet}}(z) = {}^*L^{\tilde{\bullet}} z$. If we interpret these polynomials as constant coefficient functions then $P_t((\sigma, \rho, \tau); z)$ is a totally strict representation of $p_t((\sigma, \rho, \tau); z)$ in the sense of corollary 3.3.9. Let for $(\sigma, \rho, \tau) \in \Gamma^3_{\neq}$

$$K_t(\sigma, \rho, \tau) \;=\; \begin{cases} \prod_n^{k=0} \Pi_{\sigma,\rho,\tau}(t_k) L_0 & \text{if } \max(\sigma \cup \rho \cup \tau) < t, \\ 0, & \text{otherwise,} \end{cases}$$

be the solution of the hyperfinite difference equation of theorem 2.4.11 where the function $\Pi_{\sigma,\rho,\tau}(t_k)$ is defined through

$$\Pi_{\sigma,\rho,\tau}(t_k) \;=\; \begin{cases} 1 + \frac{1}{H}{}^*L^{\tilde{\bullet}} & \text{if } t_k \notin \sigma \cup \rho \cup \tau, \\ P^+ \diamond & \text{if } t_k \in \sigma, \\ P^\circ \diamond & \text{if } t_k \in \rho, \\ P^- \diamond & \text{if } t_k \in \tau. \end{cases}$$

Proposition 4.1.8 *Fix some $(\sigma, \rho, \tau) \in \Gamma^3_{\neq} \cap \Gamma^3_{\text{fin}}$. Then the operator $K_t(\sigma, \rho, \tau) \in {}^*\mathcal{B}(\mathcal{K})$ is S-bounded for all $t \in T$. Moreover, the bound depends only on the cardinality of $\sigma \cup \rho \cup \tau$, i.e. $\|K_t(\sigma, \rho, \tau)\| \le C(|\sigma \cup \rho \cup \tau|)$.*

PROOF: If $t \leq \max(\sigma \cup \rho \cup \tau)$ then $K_t(\sigma,\rho,\tau) = 0$ and K_t is trivially S-bounded. Thus let $t > \max(\sigma \cup \rho \cup \tau)$. We set $n = \max\{n_+, n_\circ, n_-\}$ and $M = \max\{\|L_k^{\tilde{\sharp}}\|, \|L_0\|, 1\}$. By propositions 4.1.3 and 4.1.4 we conclude that for all $t \in T$

$$\left\| \left(1 + \frac{{}^*L^{\tilde{\bullet}}}{H}\right)^{Ht} \right\| \quad \leq \quad E \tag{4.1.3}$$

for some finite $E > 1$. Further, we note that if some operator $L \in \mathcal{B}(\mathcal{K})$ is S-bounded with bound $S > 1$ then

$$\|P^\sharp(L)\| \quad \leq \quad \sum_{k=1}^{n_\sharp} \|L_k\| \, \|L\|^k \quad \leq \quad nMS^n \,.$$

If we introduce the function $J(z) = EnMz^n$ then it is easy to see that

$$\|K_t(\sigma,\rho,\tau)\| \quad \leq \quad J^{|\sigma\cup\rho\cup\tau|}(EM)$$

where the right hand side is finite because we apply only a finite number of an operator $P^\sharp$ in the defining product of the kernel function. □

Now fix some $t \in [0,1]$. Then for all $\tilde{t}_1, \tilde{t}_2 \in \mathrm{st}^{-1}(t)$ and all $(\sigma_1,\rho_1,\tau_1), (\sigma_2,\rho_2,\tau_2) \in \Gamma^{[3]}_{\mathrm{st}\notin\mathrm{t}}$ we have

$$\mathrm{st}(\sigma_1,\rho_1,\tau_1) \;=\; \mathrm{st}(\sigma_2,\rho_2,\tau_2) \quad\Longrightarrow\quad \mathrm{st}(K_{\tilde{t}_1}(\sigma_1,\rho_1,\tau_1)) \;=\; \mathrm{st}(K_{\tilde{t}_2}(\sigma_2,\rho_2,\tau_2))\,.$$

In other words the kernel function is almost surely t-S-continuous. Thus we can define the standard kernel function k for every $t \in [0,1]$ and all $(\sigma,\rho,\tau) \in \Gamma^{[3]}_{\mathrm{st}}$ by

$$k_t(\mathrm{st}(\sigma,\rho,\tau)) \quad = \quad \begin{cases} \mathrm{st}(K_{\tilde{t}}(\sigma,\rho,\tau)) & \text{if } (\sigma,\rho,\tau) \in \Gamma^{[3]}_{\mathrm{st}\notin\mathrm{t}},\ \tilde{t} \in \mathrm{st}^{-1}(t), \\ 0, & \text{otherwise}\,. \end{cases} \tag{4.1.4}$$

Thus k is well-defined and by construction we have for every $\tilde{t} \in T$ that $K_{\tilde{t}}$ is a lifting of $k_{\mathrm{st}(\tilde{t})}$. Modulo the consideration whether k defines an operator process k we can therefore say that K is a totally strict representation of k. In fact we have a bit more since we know explicitly that for every $t \in [0,1]$ the whole set $\Gamma^{[3]}_{\mathrm{st}\notin\mathrm{t}}$ is our set of full Loeb measure where $K_{\tilde{t}}$ is a lifting of k_t ($\mathrm{st}(\tilde{t}) = t$).

Proposition 4.1.9 *Let $K_{\tilde{t}}$ and k_t be as before. Then for $\sharp \in \{\tilde{\bullet}, +, \circ, -\}$ we have that $\sum_{s<\tilde{t}} a_s^\sharp K_s$ is a totally strict representation of $\int_0^t k_s \mathrm{d}\mathsf{a}_s^\sharp$.*

PROOF: The creation, number and annihilation cases are similar. So we give only the proof for the creation case. By propositions 1.2.6 and 2.3.16 we have

$$\sum_{s<\tilde{t}} a_s^+ K_s(\sigma,\rho,\tau) \quad = \quad \begin{cases} 0 & \text{if } \tilde{t} \leq \max(\sigma\cup\rho\cup\tau), \\ K_{\max\sigma}(\sigma\setminus\max\sigma,\rho,\tau) & \text{if } \tilde{t} > \max\sigma = \max(\sigma\cup\rho\cup\tau), \end{cases}$$

$$\int_0^t k_s \mathrm{d}\mathsf{a}_s^+(\sigma,\rho,\tau) \quad = \quad \begin{cases} 0 & \text{if } t \leq \max(\sigma\cup\rho\cup\tau), \\ k_{\max\sigma}(\sigma\setminus\max\sigma,\rho,\tau) & \text{if } t > \max\sigma = \max(\sigma\cup\rho\cup\tau)\,. \end{cases}$$

We have already observed that for every $t \in [0,1]$ and $\tilde{t} \in \mathrm{st}^{-1}(t)$

$$K_{\tilde{t}}(\sigma,\rho,\tau) \;\approx\; k_t(\mathrm{st}(\sigma,\rho,\tau)) \quad \text{for all } (\sigma,\rho,\tau) \in \Gamma^{[3]}_{\mathrm{st}\notin\mathrm{t}}\,. \tag{4.1.5}$$

Fix now $t \in [0,1]$, some $\tilde{t} \in \mathrm{st}^{-1}(t)$ and take $(\sigma,\rho,\tau) \in \Gamma^{[3]}_{\mathrm{st}\notin\mathrm{t}}$. Then $\max(\sigma\cup\rho\cup\tau) \geq \tilde{t} \Leftrightarrow \max(\mathrm{st}(\sigma)\cup\mathrm{st}(\rho)\cup\mathrm{st}(\tau)) \geq t$ and in this case both kernel functions are zero. On the other hand if $\max(\sigma\cup\rho\cup\tau) < \tilde{t}$ then $\max(\sigma\cup\rho\cup\tau) \neq \max\sigma \Leftrightarrow \max(\mathrm{st}(\sigma)\cup\mathrm{st}(\rho)\cup\mathrm{st}(\tau)) \neq \max\mathrm{st}(\sigma)$ and again both kernel functions are zero. Thus let $\max(\sigma\cup\rho\cup\tau) = \max\sigma < \tilde{t}$ and $\max(\mathrm{st}(\sigma)\cup\mathrm{st}(\rho)\cup\mathrm{st}(\tau)) = \max\mathrm{st}(\sigma) < t$. Then clearly for all such $(\sigma,\rho,\tau) \in \Gamma^{[3]}_{\mathrm{st}\notin\mathrm{t}}$ we have $(\sigma\setminus\max\sigma,\rho,\tau) \in \Gamma^{[3]}_{\mathrm{st}\notin\max\sigma}$. This shows

$$K_{\max\sigma}(\sigma\setminus\max\sigma,\rho,\tau) \;\approx\; k_{\mathrm{st}(\max\sigma)}(\mathrm{st}(\sigma)\setminus\mathrm{st}(\max\sigma),\mathrm{st}(\rho),\mathrm{st}(\tau))$$

owing to equation (4.1.5). Hence $\sum_{s<\tilde{t}} a_s^+ K_s$ is a totally strict representation of $\int_0^t k_s d\mathsf{a}_s^+$. Evidently for the number and annihilation case a similar proof applies. It remains to prove the time case. Again by propositions 1.2.6 and 2.3.16 we get (with the minor modification from $\bullet$ to $\tilde{\bullet}$):

$$\sum_{s<\tilde{t}} a^{\tilde{\bullet}} K_s(\sigma,\rho,\tau) \;=\; \begin{cases} 0 & \text{if } \tilde{t} \leq \max(\sigma\cup\rho\cup\tau)\,, \\ \sum_{\max(\sigma\cup\rho\cup\tau)<s<\tilde{t}} K_s(\sigma,\rho,\tau)\frac{1}{H} & \text{if } \tilde{t} > \max(\sigma\cup\rho\cup\tau)\,, \end{cases}$$

$$\int_0^t k_s d\mathsf{a}_s^{\bullet}(\sigma,\rho,\tau) \;=\; \begin{cases} 0 & \text{if } t \leq \max(\sigma\cup\rho\cup\tau)\,, \\ \int_{\max(\sigma\cup\rho\cup\tau)}^t k_s(\sigma,\rho,\tau)ds & \text{if } t > \max(\sigma\cup\rho\cup\tau)\,. \end{cases}$$

We take again $(\sigma,\rho,\tau) \in \Gamma^{[3]}_{\mathrm{st}\notin\mathrm{t}}$ and we see that if $\max(\sigma\cup\rho\cup\tau) \geq \tilde{t}$ then both kernel functions are zero. We set $\tilde{r} = \max(\sigma\cup\rho\cup\tau)$ and $r = \max(\mathrm{st}(\sigma)\cup\mathrm{st}(\rho)\cup\mathrm{st}(\tau))$. For $s > r$ and $\tilde{s} > \tilde{r}$ we have

$$\begin{aligned} K_{\tilde{s}}(\sigma,\rho,\tau) &= \left(1+\frac{{}^*L^{\tilde{\bullet}}}{H}\right)^{H(\tilde{s}-\tilde{r}-\frac{1}{H})} \Pi(\tilde{r})K_{\tilde{r}}((\sigma,\rho,\tau)\setminus\tilde{r})\,, \\ k_s(\mathrm{st}(\sigma,\rho,\tau)) &= \exp((s-r)^*L^{\tilde{\bullet}})\Pi(r)k_r(\mathrm{st}(\sigma,\rho,\tau)\setminus r) \end{aligned}$$

where $(\sigma,\rho,\tau)\setminus\tilde{r}$ means that we subtract $\tilde{r}$ of this set in (σ,ρ,τ) where it is an element of and the same for $\mathrm{st}(\sigma,\rho,\tau)\setminus r$. Hence by the previous calculation (cf. equation (4.1.5)) it is sufficient to prove that

$$\sum_{\tilde{r}<\tilde{s}<\tilde{t}} \left(1+\frac{{}^*L^{\tilde{\bullet}}}{H}\right)^{H(\tilde{s}-\tilde{r}-\frac{1}{H})} \frac{1}{H} \;\approx\; \int_r^t \exp((s-r)L^{\tilde{\bullet}})ds\,.$$

By proposition 4.1.4 we already know that $\mathrm{st}\left(1+\frac{{}^*L^{\tilde{\bullet}}}{H}\right)^{H(\tilde{s}-\tilde{r}-\frac{1}{H})} = \exp((s-r)L^{\tilde{\bullet}})$. We show that $\left(1+\frac{{}^*L^{\tilde{\bullet}}}{H}\right)^{H(\tilde{s}-\tilde{r}-\frac{1}{H})}$ is S-integrable which proves the proposition. With the help of proposition 4.1.3 we obtain

$$\left\| \sum_{\tilde{r}<\tilde{s}<\tilde{t}} \left(1+\frac{{}^*L^{\tilde{\bullet}}}{H}\right)^{H(\tilde{s}-\tilde{r}-\frac{1}{H})} \frac{1}{H} \right\| \;\leq\; \sum_{\tilde{r}<\tilde{s}<\tilde{t}} \left\| \left(1+\frac{{}^*L^{\tilde{\bullet}}}{H}\right)^{H(\tilde{s}-\tilde{r}-\frac{1}{H})} \right\| \frac{1}{H} \;\leq\; \sum_{\tilde{r}<\tilde{s}<\tilde{t}} E\frac{1}{H}$$

where E is the same S-bound as in equation (4.1.3) in the proof of proposition 4.1.8. But the integrand on the right hand side is certainly S-integrable on T. This accomplishes the proof of the proposition. $\square$

Corollary 4.1.10 *The assertion of the preceding proposition holds also if we replace $K_{\tilde{t}}$ by $P^{\tilde{\sharp}} \diamond K_{\tilde{t}}$ and k_t by $p^{\tilde{\sharp}} \diamond k_t$.*

Proof: Evident, since by corollary 3.3.9 (cf. also lemma 2.4.10) and if $\tilde{t} \in \mathrm{st}^{-1}(t)$ by equation (4.1.5) for all $(\sigma, \rho, \tau) \in \Gamma^{[3]}_{\mathrm{st}\notin \mathrm{t}}$

$$(P^{\tilde{\sharp}}_{\tilde{t}}((\sigma,\rho,\tau); K_{\tilde{t}}(\sigma,\rho,\tau))) \;\approx\; p^{\tilde{\sharp}}_{t}(\mathrm{st}(\sigma,\rho,\tau); k_t(\mathrm{st}(\sigma,\rho,\tau)))\,. \qquad \square$$

We recall the standard quantum stochastic differential equation (4.1.2)

$$\mathsf{k}_t \;=\; L_0 \,+\, \int_0^t \widetilde{L^{\bullet}\mathsf{k}_s} ds \,+\, \int_0^t p^+(\mathsf{k}_s) d\mathsf{a}^+_s \,+\, \int_0^t p^{\circ}(\mathsf{k}_s) d\mathsf{a}^{\circ}_s \,+\, \int_0^t p^-(\mathsf{k}_s) d\mathsf{a}^-_s$$

with the notation as above (confer also page 83). On the level of kernel functions we draw the conclusion that

$$k_t \;\approx\; K_{\tilde{t}} \;=\; {}^*L_0 + \sum_{\tilde{\sharp}} \sum_{s<\tilde{t}} a^{\sharp}_s P \diamond K_s \;\approx\; L_0 + \sum_{\tilde{\sharp}} \int_0^t p^{\sharp} \diamond k_s d\mathsf{a}^{\sharp}_s$$

which shows that k_t is a kernel solution to the quantum stochastic differential equation (4.1.2). It remains to prove that the kernel function $k_t(\sigma,\rho,\tau)$ defines for every $t \in [0,1]$ a standard operator process on a certain domain. As we will see, we have to restrict our exponential domain and take the intersection with the finite particle domain. We recall equation (1.3.5) which gives the action of a standard kernel on an elementary tensor $b \otimes f$ with $b \in \mathcal{K}$ and $f \in \mathcal{F}$:

$$\mathsf{k}(b \otimes f(\sigma)) \;=\; \sum_{\sigma_1 \dot\cup \sigma_2 \dot\cup \sigma_3 = \sigma} \int_{\mathcal{P}_{\mathrm{fin}}} k(\sigma_1, \sigma_2, \beta) b \otimes f(\sigma_2 \cup \sigma_3 \cup \beta) d\beta\,. \qquad (4.1.6)$$

We said in the first Chapter that this expression makes only sense for certain k, f and b such that the integral over $\beta \in \mathcal{P}_{\mathrm{fin}}$ is defined. We show now that the standard kernel k constructed as standard part of the nonstandard kernel solution K to the hyperfinite quantum stochastic difference equation really defines an operator process on the following domain:

Definition 4.1.11 *The set $\mathcal{E}_{\mathrm{fin}}$ of 'bounded' exponential vectors restricted to the* finite particle space *is given by*

$$\mathcal{E}_{\mathrm{fin}} \;=\; \{\pi_\psi \chi_{\mathcal{P}_{\mathrm{fin},n}} : \psi \in B^2([0,1]), n \in \mathbb{N}\}\,.$$

Here $\chi_{\mathcal{P}_{\mathrm{fin},n}}$ is the characteristic function of $\mathcal{P}_{\mathrm{fin},n}$. We denote by $\overline{\mathcal{E}}_{\mathrm{fin}}$ the complex linear span of $\mathcal{E}_{\mathrm{fin}}$. $\triangleleft$

We use the following proposition which is a modified form of the first part of proposition 3.1 in Lindsay [Lin93a].

Proposition 4.1.12 *Suppose that some kernel function $k : \mathcal{P}_{\text{fin}}^3 \to \mathcal{B}(\mathcal{K})$ fulfills for each $n \in \mathbb{N}$*

$$\int_{\mathcal{P}_{\text{fin},n}} \|k(\sigma,\rho,\tau)\| d\tau < \infty \quad \textit{for } \Lambda^2\textit{-almost all } (\sigma,\rho) \in \mathcal{P}_{\text{fin}}^2$$

then k defines by equation (4.1.6) an operator k on $\mathcal{K} \otimes \overline{\mathcal{E}}_{\text{fin}}$.

Proof: Suppose that $f \in \overline{\mathcal{E}}_{\text{fin}}$. If $f = \pi_\psi \chi_{\mathcal{P}_{\text{fin},n}}$ then $\|f\| \le M^n \chi_{\mathcal{P}_{\text{fin},n}}$ where $M > 1$ is a bound for ψ. Since a general f is the finite linear combination of such elementary $\pi_\psi \chi_{\mathcal{P}_{\text{fin},n}}$ we see that for every $f \in \overline{\mathcal{E}}_{\text{fin}}$ there exists an $M > 1$ and an $n \in \mathbb{N}$ such that $\|f\| \le M \chi_{\cup_{k=0}^n \mathcal{P}_{\text{fin},k}}$. But then we conclude for Λ^3-almost all $(\sigma_1,\sigma_2,\sigma_3) \in \mathcal{P}_{\text{fin}}^3$ that

$$\begin{aligned}
\left\| \int_{\mathcal{P}_{\text{fin}}} k(\sigma_1,\sigma_2,\beta) b \otimes f(\sigma_2 \cup \sigma_3 \cup \beta) d\beta \right\| &\le \int_{\mathcal{P}_{\text{fin}}} \|k(\sigma_1,\sigma_2,\beta) b \otimes f(\sigma_2 \cup \sigma_3 \cup \beta)\| d\beta \\
&\le \int_{\mathcal{P}_{\text{fin}}} \|k(\sigma_1,\sigma_2,\beta)\| \, \|b\| \, \|f(\sigma_2 \cup \sigma_3 \cup \beta)\| d\beta \\
&\le \|b\| M \int_{\cup_{k=0}^n \mathcal{P}_{\text{fin},k}} \|k(\sigma_1,\sigma_2,\beta)\| d\beta \;<\; \infty
\end{aligned}$$

by the assumption on k. Thus with the help of equation (4.1.6) k gives an almost everywhere defined operator k on $\mathcal{K} \otimes \overline{\mathcal{E}}_{\text{fin}}$. □

Proposition 4.1.13 *Let k be defined as in equation (4.1.4). Then k defines an operator process on the domain $\mathcal{K} \otimes \overline{\mathcal{E}}_{\text{fin}}$.*

Proof: We have already seen that by construction K is a totally strict representation of k. Also by proposition 4.1.8 we know that $K_{\tilde{t}}(\sigma,\rho,\tau)$ is S-bounded m_L^3-almost everywhere with bound $C = C(|\sigma \cup \rho \cup \tau|)$. But then for all $t \in [0,1]$ the kernel function k_t is also bounded Λ^3-almost everywhere with bound $M = M(|\sigma \cup \rho \cup \tau|)$. This shows that

$$\int_{\mathcal{P}_{\text{fin},n}} \|k_t(\sigma,\rho,\tau)\| d\Lambda(\tau) \;\le\; \int_{\mathcal{P}_{\text{fin},n}} M(|\sigma \cup \rho \cup \tau|) d\Lambda(\tau) \;=\; M \int_{\mathcal{P}_{\text{fin},n}} 1 d\Lambda(\tau) \;<\; \infty$$

for Λ^2-almost all $(\sigma,\rho) \in \mathcal{P}_{\text{fin}}^2$. Thus the kernel satisfies for all $t \in [0,1]$ the assumption of proposition 4.1.12 and hence defines an operator process $(\mathsf{k}_t)_{t\in[0,1]}$ on $\mathcal{K} \otimes \overline{\mathcal{E}}_{\text{fin}}$. □

In the end of this section we subsume our discussion of quantum stochastic differential equations with nonlinear noise terms by the following theorem. The proof is just a combination of proposition 4.1.9, corollary 4.1.10 and proposition 4.1.13.

Theorem 4.1.14
Let $L^\bullet, L_0 \in \mathcal{B}(\mathcal{K})$, $n_+, n_\circ, n_- \in \mathbb{N}$ and $L_k^\sharp \in \mathcal{B}(\mathcal{K})$, $k = 1,\ldots,n_\sharp$, $\sharp \in \{+,\circ,-\}$. We build $p^\sharp(z) = \sum_{k=1}^{n_\sharp} L_k^\sharp z^k$. Then the quantum stochastic differential equation

$$\mathsf{k}_t \;=\; L_0 \;+\; \int_0^t L^\bullet \mathsf{k}_s ds \;+\; \int_0^t p^+(\mathsf{k}_s) d\mathsf{a}_s^+ \;+\; \int_0^t p^\circ(\mathsf{k}_s) d\mathsf{a}_s^\circ \;+\; \int_0^t p^-(\mathsf{k}_s) d\mathsf{a}_s^-$$

has a solution $(\mathsf{k}_t)_{t\in[0,1]}$ on the domain $\mathcal{K} \otimes \overline{\mathcal{E}}_{\text{fin}}$.

2 Internal and Quantum Stochastic Integration

The first fundamental formula (proposition 1.3.1), the second fundamental formula (proposition 1.3.2) and the fundamental estimation (proposition 1.3.3) of Chapter 1 are the important tools in the development of the standard theory of quantum stochastic calculus. In this Section we will prove the internal versions of these propositions and use them to connect the internal integration to the standard quantum stochastic integration. Before proving the internal analogue to the first fundamental formula we include a lemma.

Lemma 4.2.1 *Suppose that* K *is an adapted internal process with kernel function* K*. Then for every function* $F \in \mathcal{F}_{\rm int}$ *we have*

$$\mathsf{K}_s F \in_s \;=\; \in_s \mathsf{K}_s F \quad \textit{and} \quad \mathsf{K}_s F \notin_s \;=\; \notin_s \mathsf{K}_s F\,.$$

Proof: By equation (3.3.5) it is enough to prove that for all $\alpha \in \Gamma$ we have

$$\begin{aligned} K_s(\sigma,\rho,\tau)F(\rho\cup\beta\cup\tau)\in_s(\rho\cup\beta\cup\tau) &= K_s(\sigma,\rho,\tau)F(\rho\cup\beta\cup\tau)\in_s(\alpha) \quad (4.2.7)\\ K_s(\sigma,\rho,\tau)F(\rho\cup\beta\cup\tau)\notin_s(\rho\cup\beta\cup\tau) &= K_s(\sigma,\rho,\tau)F(\rho\cup\beta\cup\tau)\notin_s(\alpha) \quad (4.2.8)\end{aligned}$$

if $\sigma\dot\cup\rho\dot\cup\beta = \alpha$ and $\tau \cap (\rho\cup\beta) = \emptyset$. Thus fix some $\alpha \in \Gamma$ and suppose first that $s \in \alpha$. If $s \in \sigma$ or $s \in \rho$ then in equation (4.2.7) both sides are zero by the adaptedness of K_s. But if $s \in \beta$ then $\in_s(\rho\cup\beta\cup\tau) = \in_s(\alpha) = 1$ and again both sides of equation (4.2.7) are equal. For equation (4.2.8) we have $\notin_s(\rho\cup\beta\cup\tau) = \notin_s(\alpha) = 0$ if $s \in \rho$ or $s \in \beta$. And if $s \in \sigma$ then by the adaptedness of K_s both sides are zero. Secondly suppose that $s \notin \alpha$. If $s \notin \tau$ then $\in_s(\rho\cup\beta\cup\tau) = \in_s(\alpha) = 0$ in equation (4.2.7) and $\notin_s(\rho\cup\beta\cup\tau) = \notin_s(\alpha) = 1$ in equation (4.2.8). But if $s \in \tau$ then in both equations both sides equal zero by the adaptedness of K_s. Thus for all cases equations (4.2.7) and(4.2.8) are valid and this proves the assertion of the lemma. □

Proposition 4.2.2
Let $\Phi, \Psi \in {}^*L^2(T)$ *and* K_s *be an adapted internal process. Then we have*

$$\left\langle \pi_\Psi, \int_0^t dA_s^\sharp \mathsf{K}_s \pi_\Phi \right\rangle = \sum_{s<t} \mathfrak{Y}_s^\sharp \left\langle \pi_\Psi, \notin_s \mathsf{K}_s \pi_\Phi \right\rangle \frac{1}{H}\,, \qquad \mathfrak{Y}_s^\sharp = \begin{cases} 1 & \text{if } \sharp = \bullet\,,\\ \overline{\Psi(s)} & \text{if } \sharp = +\,,\\ \overline{\Psi(s)}\Phi(s) & \text{if } \sharp = \circ\,,\\ \Phi(s) & \text{if } \sharp = -\,. \end{cases}$$

Proof: We first prove the creation case. Note that the increment commutes with the integrand by the adaptedness of the integrand and that a_s^+ and a_s^- are mutually adjoint.

$$\begin{aligned}\langle \pi_\Psi, \int_0^t dA_s^+ \mathsf{K}_s \pi_\Phi\rangle &= \sum_{s<t} \langle \pi_\Psi, a_s^+ \mathsf{K}_s \pi_\Phi\rangle = \sum_{s<t} \langle a_s^- \pi_\Psi, \mathsf{K}_s \pi_\Phi\rangle\\ &= \sum_{s<t}\sum_{\sigma\in\Gamma} \frac{1}{H} \overline{\pi_\Psi(\sigma\cup s)\notin_s(\sigma)} \mathsf{K}_s \pi_\Phi(\sigma) m(\sigma)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{s<t} \overline{\Psi(s)} \sum_{\sigma\in\Gamma} \overline{\pi_\Psi(\sigma)} \not\in_s(\sigma) \mathsf{K}_s \pi_\Phi(\sigma) m(\sigma) \frac{1}{H} \\
&= \sum_{s<t} \overline{\Psi(s)} \langle \pi_\Psi, \not\in_s \mathsf{K}_s \pi_\Phi \rangle \frac{1}{H}\,.
\end{aligned}
$$

The annihilation case uses the same argument together with lemma 4.2.1.

$$
\begin{aligned}
\langle \pi_\Psi, \int_0^t dA_s^- \mathsf{K}_s \pi_\Phi \rangle &= \sum_{\sigma\in\Gamma} \overline{\pi_\Psi(\sigma)} \sum_{s<t} \mathsf{K}_s a_s^- \pi_\Phi(\sigma) m(\sigma) \\
&= \sum_{s<t} \sum_{\sigma\in\Gamma} \overline{\pi_\Psi(\sigma)} \mathsf{K}_s \pi_\Phi(\sigma\cup s) \not\in_s(\sigma) \frac{1}{H} m(\sigma) \\
&= \sum_{s<t} \Phi(s) \sum_{\sigma\in\Gamma} \overline{\pi_\Psi(\sigma)} \not\in_s(\sigma) \mathsf{K}_s \pi_\Phi(\sigma) m(\sigma) \frac{1}{H} \\
&= \sum_{s<t} \Phi(s) \langle \pi_\Psi, \not\in_s \mathsf{K}_s \pi_\Phi \rangle \frac{1}{H}\,.
\end{aligned}
$$

And for the number case we have

$$
\begin{aligned}
\langle \pi_\Psi, \int_0^t dA_s^\circ \mathsf{K}_s \pi_\Phi \rangle &= \sum_{s<t} \langle \pi_\Psi, \mathsf{K}_s a_s^\circ \pi_\Phi \rangle = \sum_{s<t} \sum_{\substack{\sigma\in\Gamma \\ s\in\sigma}} \overline{\pi_\Psi(\sigma)} \mathsf{K}_s \pi_\Phi(\sigma) m(\sigma) \\
&= \sum_{s<t} \sum_{\substack{\sigma\in\Gamma \\ s\notin\sigma}} \overline{\pi_\Psi(\sigma\cup s)} \mathsf{K}_s \pi_\Phi(\sigma\cup s) m(\sigma\cup s) \\
&= \sum_{s<t} \overline{\Psi(s)} \Phi(s) \sum_{\sigma\in\Gamma} \overline{\pi_\Psi(\sigma)} \not\in_s(\sigma) \mathsf{K}_s \pi_\Phi(\sigma) m(\sigma) m(s) \\
&= \sum_{s<t} \overline{\Psi(s)} \Phi(s) \langle \pi_\Psi, \not\in_s \mathsf{K}_s \pi_\Phi \rangle \frac{1}{H}\,.
\end{aligned}
$$

The time case is now an easy exercise:

$$
\langle \pi_\Psi, \int_0^t dA_s^\bullet \mathsf{K}_s \pi_\Phi \rangle = \sum_{s<t} \sum_{\sigma\in\Gamma} \overline{\pi_\Psi(\sigma)} \not\in_s(\sigma) \mathsf{K}_s \pi_\Phi(\sigma) \frac{1}{H} m(\sigma) = \sum_{s<t} \langle \pi_\Psi, \not\in_s \mathsf{K}_s \pi_\Phi \rangle \frac{1}{H}\,. \qquad \square
$$

If K is a (totally) strong representation of some integrable standard process k then the relations of the proposition above are almost the same as the familiar relations of the first fundamental formula. For example, if we take the internal Brownian motion (to the parameter $\theta = 0$) then we get the 'isometry' relation

$$
\langle \pi_\Psi, \int_0^t dB_s \mathsf{K}_s \pi_\Phi \rangle = \sum_{s<t} (\overline{\Psi(s)} + \Phi(s)) \langle \pi_\Psi, \not\in_s \mathsf{K}_s \pi_\Phi \rangle \frac{1}{H}\,.
$$

And for the internal Poisson process (with parameter $z = 1$) this becomes

$$
\langle \pi_\Psi, \int_0^t dP_s \mathsf{K}_s \pi_\Phi \rangle = \sum_{s<t} (\overline{\Psi(s)} \Phi(s) + \overline{\Psi(s)} + \Phi(s) + 1) \langle \pi_\Psi, \not\in_s \mathsf{K}_s \pi_\Phi \rangle \frac{1}{H}\,.
$$

But also the second fundamental formula (proposition 1.3.2) has its hyperfinite counterpart which we formulate and prove now.

Proposition 4.2.3 *Suppose that* Y_s, X_s *are adapted internal processes and set* $\mathsf{N}_t = \int_0^t dA_s^\natural \mathsf{Y}_s$ *and* $\mathsf{M}_t = \int_0^t dA_s^\sharp \mathsf{X}_s$. *Then for all* $\Psi, \Phi \in {}^*L^2(T)$ *we have*

$$\begin{aligned} \langle \mathsf{N}_t \pi_\Psi, \mathsf{M}_t \pi_\Phi \rangle &= \sum_{s<t} \mathfrak{Y}_s^{\star\natural} \langle \mathsf{Y}_s \pi_\Psi, \notin_s \mathsf{M}_s \pi_\Phi \rangle \frac{1}{H} + \sum_{s<t} \mathfrak{Y}_s^{\sharp} \langle \mathsf{N}_s \pi_\Psi, \notin_s \mathsf{X}_s \pi_\Phi \rangle \frac{1}{H} \\ &\quad + \sum_{s<t} \mathfrak{Z}_s^{\star\natural,\sharp} \langle \mathsf{Y}_s \pi_\Psi, \notin_s \mathsf{X}_s \pi_\Phi \rangle \frac{1}{H} \end{aligned}$$

where we adopt the notation $\star\bullet = \bullet$, $\star + = -$, $\star\circ = \circ$, $\star - = +$ *and* $\mathfrak{Z}_s^{\natural,\sharp}$ *is defined by the following table:*

$\natural \backslash \sharp$	$\bullet$	$+$	$\circ$	$-$
$\bullet$	0	0	0	0
$+$	0	0	0	0
$\circ$	0	$\mathfrak{Y}_s^+$	$\mathfrak{Y}_s^\circ$	0
$-$	0	$\mathfrak{Y}_s^\bullet$	$\mathfrak{Y}_s^-$	0

PROOF:

$$\langle \mathsf{N}_t \pi_\Psi, \mathsf{M}_t \pi_\Phi \rangle = \left\langle \int_0^t dA_s^\natural \mathsf{Y}_s \pi_\Psi, \int_0^t dA_s^\sharp \mathsf{X}_s \pi_\Phi \right\rangle = \sum_{s_1<t} \sum_{s_2<t} \left\langle a_{s_1}^\natural \mathsf{Y}_{s_1} \pi_\Psi, a_{s_2}^\sharp \mathsf{X}_{s_2} \pi_\Phi \right\rangle$$

$$= \sum_{s_1<s_2<t} \left\langle a_{s_1}^\natural \mathsf{Y}_{s_1} \pi_\Psi, a_{s_2}^\sharp \mathsf{X}_{s_2} \pi_\Phi \right\rangle + \sum_{s_2<s_1<t} \left\langle a_{s_1}^\natural \mathsf{Y}_{s_1} \pi_\Psi, a_{s_2}^\sharp \mathsf{X}_{s_2} \pi_\Phi \right\rangle + \sum_{s<t} \left\langle a_s^\natural \mathsf{Y}_s \pi_\Psi, a_s^\sharp \mathsf{X}_s \pi_\Phi \right\rangle .$$

Since the processes are adapted they commute with the increments. We calculate firstly the last term:

$$\sum_{s<t} \left\langle a_s^\natural \mathsf{Y}_s \pi_\Psi, a_s^\sharp \mathsf{X}_s \pi_\Phi \right\rangle = \sum_{s<t} \left\langle \mathsf{Y}_s \pi_\Psi, a_s^{\star\natural} a_s^\sharp \mathsf{X}_s \pi_\Phi \right\rangle .$$

Using the internal adjointness relations in proposition 2.2.3 and the internal Itô table (2.2.8) on page 32 we can calculate the product $a_s^{\star\natural} a_s^\sharp$. We write $a_s^{\star\natural,\sharp}$ for this product. If we apply the same tricks as in the proof of the internal version of the first fundamental formula we obtain

$$\sum_{s<t} \left\langle \mathsf{Y}_s \pi_\Psi, a_s^{\star\natural,\sharp} \mathsf{X}_s \pi_\Phi \right\rangle = \sum_{s<t} \sum_{\sigma\in\Gamma} \overline{\mathsf{Y}_s \pi_\Psi(\sigma)} \mathsf{X}_s a_s^{\star\natural,\sharp} \pi_\Phi(\sigma) = \sum_{s<t} \mathfrak{Z}_s^{\star\natural,\sharp} \langle \mathsf{Y}_s \pi_\Psi, \notin_s \mathsf{X}_s \pi_\Phi \rangle \frac{1}{H} .$$

Secondly we calculate the first term:

$$\begin{aligned} \sum_{s_1<s_2<t} \left\langle a_{s_1}^\natural \mathsf{Y}_{s_1} \pi_\Psi, a_{s_2}^\sharp \mathsf{X}_{s_2} \pi_\Phi \right\rangle &= \sum_{s_2<t} \left\langle \sum_{s_1<s_2} a_{s_1}^\natural \mathsf{Y}_{s_1} \pi_\Psi, \mathsf{X}_{s_2} a_{s_2}^\sharp \pi_\Phi \right\rangle \\ = \sum_{s_1<t} \sum_{\sigma\in\Gamma} \overline{\sum_{s_1<s_2} a_{s_1}^\natural \mathsf{Y}_{s_1} \pi_\Psi(\sigma)} \mathsf{X}_{s_2} a_{s_2}^\sharp \pi_\Phi(\sigma) &= \sum_{s<t} \mathfrak{Y}_s^\sharp \left\langle \int_0^s dA_{\tilde{s}}^\natural \mathsf{Y}_{\tilde{s}} \pi_\Psi, \notin_s \mathsf{X}_s \pi_\Phi \right\rangle \frac{1}{H} \end{aligned}$$

using again the same tricks as in the proof of the internal version of the first fundamental lemma. The second term follows then by a similar calculation.

$$\sum_{s_2<s_1<t} \left\langle a^\natural_{s_1} \mathsf{Y}_{s_1} \pi_\Psi, a^\sharp_{s_2} \mathsf{X}_{s_2} \pi_\Phi \right\rangle \;=\; \sum_{s<t} \mathfrak{Y}^{\star\natural}_s \left\langle \mathsf{Y}_s \pi_\Psi, \notin_s \int_0^s dA^\sharp_{\tilde{s}} \mathsf{X}_{\tilde{s}} \pi_\Phi \right\rangle \frac{1}{H}\,.$$

Combining the three terms we have the assertion of the proposition. □

The fundamental estimation in the hyperfinite setting is now an easy consequence of the second fundamental formula. But first we make the following two useful observations.

Corollary 4.2.4 *Let X_s and Y_s be adapted internal processes. Then we have for all $\Psi, \Phi \in {}^*L^2(T)$ that*

$$\sum_{s_1<s_2<t} \left\langle a^\natural_{s_1} \mathsf{Y}_{s_1} \pi_\Psi, a^\sharp_{s_2} \mathsf{X}_{s_2} \pi_\Phi \right\rangle \;=\; \sum_{s_2<t} \sum_{s_1<s_2} \mathfrak{Y}^\sharp_{s_2} \mathfrak{Y}^{\star\natural}_{s_1} \left\langle \notin_{s_1} \mathsf{Y}_{s_1} \pi_\Psi, \notin_{s_2} \mathsf{X}_{s_2} \pi_\Phi \right\rangle \frac{1}{H} \frac{1}{H}\,.$$

Lemma 4.2.5 *For all $F \in \mathcal{F}_{\text{int}}$ it is* $\|\notin_s F\| \;\le\; \|F\|$ *.*

Proof: $\|\notin_s F\|^2 \;=\; \sum_{\sigma\in\Gamma} \notin_s(\sigma)^2 |F(\sigma)|^2 \;\le\; \sum_{\sigma\in\Gamma} |F(\sigma)|^2 \;=\; \|F\|^2\,.$ □

Proposition 4.2.6 *Suppose that X_s is adapted. Let $|\Phi(s)| < M$ for all $s \in T$ and some $M > 1, M \not\approx 1$. Then we have for all $t \in T$:*

$$\left\| \int_0^t dA^\sharp_s \mathsf{X}_s \pi_\Phi \right\|^2 \;\lessapprox\; 4M^4 \sum_{s<t} \|\mathsf{X}_s \pi_\Phi\|^2 \frac{1}{H}\,.$$

Proof:

$$\begin{aligned}
\left\| \int_0^t dA^\sharp_s \mathsf{X}_s \pi_\Phi \right\|^2 \;&=\; \left| \left\langle \int_0^t dA^\sharp_s \mathsf{X}_s \pi_\Phi, \int_0^t dA^\sharp_s \mathsf{X}_s \pi_\Phi \right\rangle \right| \\
&= \Big| \sum_{s_1<s_2<t} \mathfrak{Y}^\sharp_{s_2} \mathfrak{Y}^{\star\sharp}_{s_1} \left\langle \notin_{s_1} \mathsf{X}_{s_1} \pi_\Phi, \notin_{s_2} \mathsf{X}_{s_2} \pi_\Phi \right\rangle \frac{1}{H}\frac{1}{H} \\
&\quad + \sum_{s_2<s_1<t} \mathfrak{Y}^{\star\sharp}_{s_1} \mathfrak{Y}^\sharp_{s_2} \left\langle \notin_{s_1} \mathsf{X}_{s_1} \pi_\Phi, \notin_{s_2} \mathsf{X}_{s_2} \pi_\Phi \right\rangle \frac{1}{H}\frac{1}{H} \;+\; \sum_{s<t} \mathfrak{Z}^{\star\sharp,\sharp}_s \left\langle \notin_s \mathsf{X}_s \pi_\Phi, \notin_s \mathsf{X}_s \pi_\Phi \right\rangle \frac{1}{H} \Big| \\
&\le \sum_{s_1<s_2<t} |\mathfrak{Y}^\sharp_{s_2}|\, |\mathfrak{Y}^{\star\sharp}_{s_1}|\, |\langle \notin_{s_1} \mathsf{X}_{s_1} \pi_\Phi, \notin_{s_2} \mathsf{X}_{s_2} \pi_\Phi \rangle| \frac{1}{H}\frac{1}{H} \\
&\quad + \sum_{s_2<s_1<t} |\mathfrak{Y}^{\star\sharp}_{s_1}|\, |\mathfrak{Y}^\sharp_{s_2}|\, |\langle \notin_{s_1} \mathsf{X}_{s_1} \pi_\Phi, \notin_{s_2} \mathsf{X}_{s_2} \pi_\Phi \rangle| \frac{1}{H}\frac{1}{H} \;+\; \sum_{s<t} \mathfrak{Z}^{\star\sharp,\sharp}_s\, |\langle \notin_s \mathsf{X}_s \pi_\Phi, \notin_s \mathsf{X}_s \pi_\Phi \rangle| \frac{1}{H} \\
&\le 2 \sum_{s_1<s_2<t} M^4 \|\notin_{s_1} \mathsf{X}_{s_1} \pi_\Phi\|\, \|\notin_{s_2} \mathsf{X}_{s_2} \pi_\Phi\| \frac{1}{H}\frac{1}{H} + \sum_{s<t} M^4 \|\notin_s \mathsf{X}_s \pi_\Phi\|^2 \frac{1}{H} \\
&\le 2 \sum_{s_2<t} \sum_{s_1<t} M^4 \frac{1}{2} (\|\mathsf{X}_{s_1} \pi_\Phi\|^2 + \|\mathsf{X}_{s_2} \pi_\Phi\|^2) \frac{1}{H}\frac{1}{H} + \sum_{s<t} M^4 \|\mathsf{X}_s \pi_\Phi\|^2 \frac{1}{H}
\end{aligned}$$

$$\begin{aligned} &= t\sum_{s_1<t} M^4\|\mathsf{X}_{s_1}\pi_\Phi\|^2\frac{1}{H} + t\sum_{s_2<t} M^4\|\mathsf{X}_{s_2}\pi_\Phi\|^2\frac{1}{H} + \sum_{s<t} M^4\|\mathsf{X}_s\pi_\Phi\|^2\frac{1}{H} \\ &\leq 3M^4\sum_{s<t}\|\mathsf{X}_s\pi_\Phi\|^2\frac{1}{H}\,. \end{aligned}$$

If we take four times the constant M^4 then surely we have a strict inequality. Thus

$$\left\|\int_0^t dA_s^\sharp \mathsf{X}_s\pi_\Phi\right\|^2 \;<\; 4M^4\sum_{s<t}\|\mathsf{X}_s\pi_\Phi\|^2\frac{1}{H}$$

and in addition the two terms are not infinitesimally close to each other. □

If we would take an infinite time-line we see by the same proof that the constant depends on t:

$$\left\|\int_0^t dA_s^\sharp \mathsf{X}_s\pi_\Phi\right\|^2 \;<\; (2tM(t)^4 + M(t)^2 + 1)\sum_{s<t}\|\mathsf{X}_s\pi_\Phi\|^2\frac{1}{H}$$

where $M(t) > 1$ is a local bound for Φ on $[0,t]$. Then $M(t)^2$ is a sharper bound for $|\mathfrak{Z}_s^{\star\sharp,\sharp}|$.

Now we connect via the fundamental estimations and the first fundamental formulas the standard integrable processes to internal processes. For this we introduce the concept of an S-integrable process. It is well known that every standard process k such that $\int_0^1\|\mathsf{k}_s\pi_\phi\|^2 ds$ is finite for all $\phi\in B^2([0,1])$ defines an adapted process $\int_0^t \mathsf{k}_s d\mathsf{a}_s^\sharp$ (see the remark after proposition 1.3.3 page 13). This motivates our next definition.

Definition 4.2.7 *Let $(\mathsf{K}_s)_{s\in T}$ be an adapted internal operator process. Then we say that $(\mathsf{K}_s)_{s\in T}$ is* S-integrable *if for all $\Phi\in SB^2(T)$ the function*

$$G_\Phi \;:\; T\longrightarrow {}^*\mathbb{R} \;:\; s\longmapsto \|\mathsf{K}_s\pi_\Phi\|^2$$

is S-integrable. ◁

Lemma 4.2.8 *Let the adapted internal operator process K be S-integrable. Suppose that K is the (totally) strong representation of some standard operator process k. Then for every lifting $\Phi\in SB^2$ of $\phi\in B^2$ the function $G(s)=\|\mathsf{K}_s\pi_\Phi\|$ is a lifting of $g(s)=\|\mathsf{k}_s\pi_\phi\|$ and $g\in L^2([0,1])$.*

PROOF:

$$|G(s)|^2 \;=\; \langle \mathsf{K}_s\pi_\Phi, \mathsf{K}_s\pi_\Phi\rangle \;\approx\; \langle \mathsf{k}_{\mathrm{st}(s)}\pi_\phi, \mathsf{k}_{\mathrm{st}(s)}\pi_\phi\rangle \;=\; |g(\mathrm{st}(s))|^2$$

for μ_L-almost all $s\in T$ (for all $s\in T$) and thus G is a lifting of g. That g is L^2 follows immediately since G was supposed to be SL^2. □

Theorem 4.2.9 *Let the S-integrable adapted internal operator process K be the totally strong or strong representation of the adapted operator process k. Then k is integrable. In this case we have for every $t\in[0,1]$ and $\tilde{t}\in \mathrm{st}^{-1}(t)$ that*

$$\langle \pi_\psi, \int_0^t \mathsf{k}_s d\mathsf{a}_s^\sharp \pi_\phi\rangle \;\approx\; \langle \pi_\Psi, \int_0^{\tilde{t}} dA_s^\sharp \mathsf{K}_s\pi_\Phi\rangle$$

for all $\psi,\phi\in B^2([0,1])$ and all liftings $\Psi,\Phi\in SB^2(T)$ of ψ,ϕ.

PROOF: Fix some $t \in [0,1]$ and $\tilde{t} \in \mathrm{st}^{-1}(t)$. Further choose $\psi, \phi \in B^2([0,1])$ and liftings $\Psi, \Phi \in SB^2(T)$. With the fundamental estimations and the preceding lemma it is immediate that k is integrable. It remains to prove the approximate equality. By the first fundamental lemmas it is sufficient to show that

$$\int_0^t \mathfrak{y}_s^\sharp \langle \pi_\psi, \mathsf{k}_s \pi_\phi \rangle ds \;\approx\; \sum_{s<\tilde{t}} \mathfrak{Y}_s^\sharp \langle \pi_\Psi, \notin_s \mathsf{K}_s \pi_\Phi \rangle \frac{1}{H} \,.$$

Note that from proposition 3.3.19 we know that $\notin_s \mathsf{K}_s$ is a strong representation of $\notin_{\mathrm{st}(s)} \mathsf{k}_{\mathrm{st}(s)}$. Obviously we have $\langle \pi_\psi, \mathsf{k}_s \pi_\phi \rangle = \langle \pi_\psi, \notin_s \mathsf{k}_s \pi_\phi \rangle$ since $\{\sigma \in \mathcal{P}_{\mathrm{fin}} : \notin_s(\sigma) = 0\}$ is a set of measure zero. Thus also $\int_0^t \mathfrak{y}_s^\sharp \langle \pi_\psi, \mathsf{k}_s \pi_\phi \rangle ds = \int_0^t \mathfrak{y}_s^\sharp \langle \pi_\psi, \notin_s \mathsf{k}_s \pi_\phi \rangle ds$ and we will prove that

$$\int_0^t \mathfrak{y}_s^\sharp \langle \pi_\psi, \notin_s \mathsf{k}_s \pi_\phi \rangle ds \;\approx\; \sum_{s<\tilde{t}} \mathfrak{Y}_s^\sharp \langle \pi_\Psi, \notin_s \mathsf{K}_s \pi_\Phi \rangle \frac{1}{H} \,.$$

We have the following inequality

$$\left| \sum_{s<\tilde{t}} \mathfrak{Y}_s^\sharp \langle \pi_\Psi, \notin_s \mathsf{K}_s \pi_\Phi \rangle \frac{1}{H} \right| \;\le\; \sum_{s<\tilde{t}} \left| \mathfrak{Y}_s^\sharp \langle \pi_\Psi, \notin_s \mathsf{K}_s \pi_\Phi \rangle \right| \frac{1}{H} \;\le\; \|\pi_\Psi\| \sum_{s<\tilde{t}} |\mathfrak{Y}_s^\sharp| \, \|\mathsf{K}_s \pi_\Phi\| \frac{1}{H}$$

where the right hand side is finite since $\|\pi_\Psi\|$ is finite and $|\mathfrak{Y}_s^\sharp|$ and $\|\mathsf{K}_s \pi_\Phi\|$ are square S-integrable on T. Moreover, we see that the function $s \mapsto \mathfrak{Y}_s^\sharp \langle \pi_\Psi, \notin_s \mathsf{K}_s \pi_\Phi \rangle$ is S-integrable. By assumption $\mathfrak{Y}_s^\sharp$ is a SL^2-lifting of $\mathfrak{y}_s^\sharp$ and by corollary 3.3.20 $\langle \pi_\Psi, \notin_s \mathsf{K}_s \pi_\Phi \rangle$ is a lifting of $\langle \pi_\psi, \notin_s \mathsf{k}_s \pi_\phi \rangle$. Thus $\mathfrak{Y}_s^\sharp \langle \pi_\Psi, \notin_s \mathsf{K}_s \pi_\Phi \rangle$ is an S-integrable lifting of $\mathfrak{y}_s^\sharp \langle \pi_\psi, \notin_s \mathsf{k}_s \pi_\phi \rangle$ and the assertion follows by proposition 3.2.6. □

The theorem tells us that strong representations transport the property of S-integrability to integrability of the represented standard process. This provides us with a tool for checking the integrability of a process by showing that it has an S-integrable strong representation. In the next Section we will show that bounded operators admit a strong representation. For processes we will prove the weaker result that every bounded process admits a suitable weak representation.

3 Existence of Strong and Weak Representations

This Section is intended to show the existence of strong and weak representations. In a certain sense these are kinds of lifting results since the notion of a strong or weak representation extends the usual notion of a lifting of a function to operators acting on functions. In the first subsection we consider bounded operators. We prove that every bounded Fock space operator has a strong representation. In the second subsection we prove the existence of weak representations for bounded operator-valued processes, adapted processes and martingales. Some of the results are inspired by Chapter 5 in Albeverio et. al. [AFHKL86]. Concerning closed unbounded operators and their strong representation we refer the reader to Albeverio et. al. [AFHKL86, Chapter 5] and their detailed statements on hyperfinite representations of closed unbounded forms.

3.1 Strong Representation of Bounded Fock Space Operators

In this subsection we show that every bounded operator k on Fock space has a strong representation K where K is S-bounded. But first, we quote two results of Anderson [And82, corollary 6.6, theorem 3.7] which are useful in general. The second one is normally called "Anderson's nonstandard version of Lusin's theorem" and one finds a proof in Albeverio et. al. [AFHKL86, corollary 3.4.9]. The first one is a consistency result for extensions of functions to nonstandard internal functions.

Proposition 4.3.1 *Let $(\mathcal{M}, \mathfrak{M}, \mathfrak{m})$ be a complete standard measure space and $\mathbb{B}$ a separable Banach space. Fix some standard p, $1 \leq p < \infty$. If $f \in L^p_{\mathbb{B}}(\mathcal{M}, \mathfrak{M}, \mathfrak{m})$ then ${}^*f \in SL^p_{{}^*\mathbb{B}}({}^*\mathcal{M}, {}^*\mathfrak{M}, {}^*\mathfrak{m})$.*

Actually Anderson formulates the theorem for $\mathbb{B} = \mathbb{R}$ but since p-Bochner integrability of some $f : \mathcal{M} \to \mathbb{B}$ is characterized by the integrability of $\mathcal{M} \ni m \mapsto \|f(m)\|^p$ and analogously for p-Bochner S-integrability, the result extends easily to vector-valued functions (see proposition A.1.9 and theorem A.1.14). The next proposition gives a situation where ${}^*f$ is a lifting of f, which in general not need to be the case.

Proposition 4.3.2 *Let $(\mathcal{M}, \mathfrak{M}, \mathfrak{m})$ be a complete Radon measure space and $\mathbb{B}$ a second countable Hausdorff topological space. If $f : \mathcal{M} \to \mathbb{B}$ is $\mathfrak{M}$-measurable then*

$$\mathrm{st}({}^*f(m)) \quad = \quad f(\mathrm{st}(m)) \quad \text{for } {}^*\mathfrak{m}_L\text{-almost all } m \in {}^*\mathcal{M}\,.$$

That is ${}^*f$ is a lifting of f with respect to ${}^*\mathfrak{m}_L$.

By construction our complete measure space $(\mathcal{P}_{\mathrm{fin}}, \mathfrak{B}, \Lambda)$ is identified as the completion of the countable direct sum of the complete Radon measure spaces $([0,1]^n, \mathcal{B}^n, \lambda^n)$ and thus itself Radon. So both results quoted apply in our situation and we get:

Corollary 4.3.3 *Let $\mathcal{K}$ be a separable Hilbert space. If $f \in L^2_{\mathcal{K}}(\mathcal{P}_{\mathrm{fin}}, \mathfrak{B}, \Lambda)$ then ${}^*f \in SL^2_{{}^*\mathcal{K}}({}^*\mathcal{P}_{\mathrm{fin}}, {}^*\mathfrak{B}, {}^*\Lambda)$ and ${}^*f$ is a lifting of f with respect to the Loeb measure ${}^*\Lambda_L$.*

Definition 4.3.4 *Let $\mathcal{H}$ be a standard Hilbert space and ${}^*\mathcal{H}$ its nonstandard extension. Then a closed internal subspace $\mathcal{S} \subseteq {}^*\mathcal{H}$ is said to be* S-dense *if for all $x \in \mathcal{H}$ there exists a $y \in \mathcal{S}$ such that $\|x - y\| \approx 0$. If we speak of an S-dense subspace, we always assume that the space is internal and closed.* $\lhd$

Lemma 4.3.5 *Let $\mathcal{S} \subseteq {}^*\mathcal{H}$ be an S-dense subspace and $P : {}^*\mathcal{H} \to \mathcal{S}$ the orthogonal projection onto $\mathcal{S}$. Then for every $x \in \mathcal{H}$ we have $\|Px - x\| \approx 0$.*

Proof: Let $\widehat{P} : \widehat{\mathcal{H}} \to \widehat{\mathcal{S}}$ be the nonstandard hull of the projection P where $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{S}}$ are the nonstandard hulls of $\mathcal{H}$ respectively $\mathcal{S}$. Then by the S-denseness of $\mathcal{S}$ we have $\mathcal{H} \subset \widehat{\mathcal{S}}$. Thus $\widehat{P}x = x$ for all $x \in \mathcal{H}$. But this means that $Px \approx x$ for all $x \in \mathcal{H}$ which is nothing else than $\|Px - x\| \approx 0$ for all $x \in \mathcal{H}$. $\square$

Proposition 4.3.6 *Let $\mathcal{S} \subseteq {}^*\mathcal{H}$ be an S-dense subspace and $P : {}^*\mathcal{H} \to \mathcal{S}$ the orthogonal projection onto $\mathcal{S}$. Then $\|Py - y\| \approx 0$ for all nearstandard $y \in {}^*\mathcal{H}$.*

PROOF: Since y is nearstandard there exists an $x \in \mathcal{H}$ with $\|x - y\| \approx 0$. By the continuity of the projection we obtain $\|Py - Px\| \approx 0$. And using the lemma it is $\|Px - x\| \approx 0$. Thus we conclude

$$\|Py - y\| \;\leq\; \|Py - Px\| \;+\; \|Px - x\| \;+\; \|x - y\| \;\approx\; 0\,. \qquad \square$$

We introduce the weak Hilbert space topology. That is some $y \in {}^*\mathcal{H}$ is *weakly nearstandard* if there exists an $x \in \mathcal{H}$ such that $\langle z, y\rangle \approx \langle z, x\rangle$ for all $z \in \mathcal{H}$. Then we have the following proposition.

Proposition 4.3.7 *Let $\mathcal{S} \subseteq {}^*\mathcal{H}$ be S-dense and $P : {}^*\mathcal{H} \to \mathcal{S}$ the orthogonal projection onto $\mathcal{S}$. Suppose that $y \in {}^*\mathcal{H}$ is weakly nearstandard and has finite norm. Then*

$$\langle z, Py\rangle \;\approx\; \langle z, y\rangle \quad \textit{for all } z \in \mathcal{H}\,.$$

PROOF: We show that $\langle z, Py - y\rangle \approx 0$ for all $z \in \mathcal{H}$. Since y is weakly nearstandard there exists an $x \in \mathcal{H}$ such that $\langle z, x - y\rangle \approx 0$ for all $z \in \mathcal{H}$. Using lemma 4.3.5 we have

$$|\langle z, Px - x\rangle| \;\leq\; \|z\|\,\|Px - x\| \;\approx\; 0$$

since $\|z\|$ is finite. We obtain

$$\langle z, Py - y\rangle \;=\; \langle z, Py - Px\rangle \;+\; \langle z, Px - x\rangle \;+\; \langle z, x - y\rangle \;\approx\; \langle z, P(y - x)\rangle\,.$$

It remains to prove that $\langle z, P(y - x)\rangle = \langle Pz, y - x\rangle \approx 0$. Since P is the projection onto an S-dense subspace there exists a $w \in \mathcal{H}$ such that $\|Pz - w\| \approx 0$. By the weak nearstandardness of y to x it is $\langle w, y - x\rangle \approx 0$. On the other hand y has finite norm. Thus $\|y - x\| \leq \|y\| + \|x\|$ is finite. Using this we obtain

$$|\langle Pz - w, y - x\rangle| \;\leq\; \|Pz - w\|\,\|y - x\| \;\approx\; 0$$

and conclude

$$\langle Pz, y - x\rangle \;=\; \langle Pz - w, y - x\rangle \;+\; \langle w, y - x\rangle \;\approx\; 0\,. \qquad \square$$

The next proposition shows that every bounded operator on $\mathcal{H}$ can be represented by an S-bounded operator on an S-dense subspace $\mathcal{S} \subseteq {}^*\mathcal{H}$.

Proposition 4.3.8 *Let $\mathcal{S} \subseteq {}^*\mathcal{H}$ be an S-dense subspace. If $k \in \mathcal{B}(\mathcal{H})$ then there is an S-bounded $K \in {}^*\mathcal{B}(\mathcal{S})$ such that for all $x \in \mathcal{H}$ and $y \in \mathcal{S}$ it is $kx = \mathrm{st}(Ky)$ whenever $\|x - y\| \approx 0$.*

Proof: Let P be the orthogonal projection of ${}^*\mathcal{H}$ onto $\mathcal{S}$. Clearly P and also ${}^*k$ are S-bounded. We define the S-bounded operator $K = P\,{}^*k$. If $z \in {}^*\mathcal{H}$ is nearstandard then using proposition 4.3.6 it is $\|{}^*kz - P\,{}^*kz\| \approx 0$ since ${}^*k$ takes nearstandard elements to nearstandard elements, and $\mathcal{S}$ is S-dense in ${}^*\mathcal{H}$. But then we have for all $x \in \mathcal{H}$ and $y \in \mathcal{S}$ with $\|x - y\| \approx 0$ that

$$\begin{aligned}\|kx - Ky\| &= \|kx - P\,{}^*ky\| \;\leq\; \|kx - {}^*ky\| + \|{}^*ky - P\,{}^*ky\| \\ &\leq \|k\|\,\|x-y\| + \|{}^*ky - P\,{}^*ky\| \;\approx\; 0\,.\end{aligned}$$

This shows $\mathrm{st}(Ky) = kx$. □

We note that the operator K is obtained by extending k to ${}^*k$ and then taking the orthogonal projection P onto the S-dense subspace. That is $K = P\,{}^*k = P\,{}^*kP$. Now we want to apply this result to our situation with $\mathcal{H} = \mathcal{F} = L^2_{\mathcal{K}}(\mathcal{P}_{\mathrm{fin}}, \Lambda)$ and $\mathcal{S} = \mathcal{F}_{\mathrm{int}} = {}^*L^2_{{}^*\mathcal{K}}(\Gamma, m)$. The problem is that $\mathcal{F}_{\mathrm{int}}$ is not a subspace of ${}^*\mathcal{F}$. We construct an internal isomorphism such that $\mathcal{F}_{\mathrm{int}}$ is isomorphic to an internal S-dense subspace of ${}^*\mathcal{F}$. Furthermore, the isomorphism is such that the integral is preserved.

By definition we have that ${}^*\mathcal{F}$ is the internal space of all internal square ${}^*\Lambda$-integrable functions on ${}^*\mathcal{P}_{\mathrm{fin}}([0,1])$. And ${}^*\mathcal{P}_{\mathrm{fin}}([0,1])$ is the internal set of all hyperfinite subsets of ${}^*[0,1]$, i.e. ${}^*\mathcal{P}_{\mathrm{fin}}([0,1]) = \cup_{n\in{}^*\mathbb{N}}\{\sigma \subseteq {}^*[0,1] : |\sigma| = n,\ \sigma \text{ internal}\}$. Thus since $T \subset {}^*[0,1]$ we have also that $\Gamma = {}^*\mathcal{P}(T) \subset {}^*\mathcal{P}_{\mathrm{fin}}([0,1])$. We see that for $n \leq H$ it is ${}^*\mathcal{P}_n(T) \subset {}^*\mathcal{P}_n([0,1])$ and ${}^*\mathcal{P}_n(T) = \emptyset$ for $n > H$. We define on ${}^*\mathcal{P}_{\mathrm{fin}}([0,1])$ the internal relation $\sim$ by:

$$\sigma \sim \tau \quad \Leftrightarrow_{\mathrm{def}} \quad |\sigma| = |\tau| \wedge \forall s \in \sigma \forall t \in \tau\,(s \leq t \wedge t - s < \frac{1}{H})\,.$$

Note that this relation is not symmetric. We denote by $\widetilde{\sigma}$ the class:

$$\widetilde{\sigma} \;=\; \{\tau \in {}^*\mathcal{P}_{\mathrm{fin}}([0,1]) \;:\; \sigma \sim \tau\}\,.$$

and see in particular that $\widetilde{\emptyset} = \{\emptyset\}$. Obviously for $\sigma, \tau \in \Gamma_n$ one has either $\sigma = \tau$ or $\exists s \in \sigma \exists t \in \tau(|s - t| \geq \frac{1}{H})$. Thus for all $\sigma, \tau \in \Gamma$ we have $\sigma \neq \tau \Leftrightarrow \widetilde{\sigma} \cap \widetilde{\tau} = \emptyset$ and $\widetilde{\sigma} \cap \widetilde{\tau} \neq \emptyset \Rightarrow \widetilde{\sigma} = \widetilde{\tau} \wedge \sigma = \tau$. Now suppose that $n \leq H$ and $\sigma \in \Gamma_n$. Then it is easy to show that ${}^*\Lambda(\widetilde{\sigma}) = \left(\frac{1}{H}\right)^n = m(\{\sigma\})$ and so the map ${}^*\mathcal{P}(\Gamma) \ni \{\sigma\} \mapsto \widetilde{\sigma} \in {}^*\mathcal{P}(\mathcal{P}_{\mathrm{fin}}([0,1]))$ extends to a measure preserving map. Geometrically, we identify points $\sigma \in {}^*\mathcal{P}_n(T) \equiv T^n$ with 'halfopen cubes' with edge length $\frac{1}{H}$ in ${}^*\mathcal{P}_n([0,1]) \equiv {}^*[0,1]^n$ such that σ is this vertex of the cube which points to the origin. This allows us to define for each $F \in {}^*L^2_{{}^*\mathcal{K}}(\Gamma)$ the following function $\widetilde{F} \in {}^*L^2_{{}^*\mathcal{K}}(\mathcal{P}_{\mathrm{fin}})$:

$$\widetilde{F}(\sigma) \;=\; \begin{cases} F(\tau), & \exists \tau \in \Gamma\,(\sigma \in \widetilde{\tau}), \\ 0, & \text{otherwise}\,. \end{cases}$$

We see that $\widetilde{F}(\sigma)$ is constant on each class $\widetilde{\tau}$, $\tau \in \Gamma$ and in particular $\widetilde{F}(\sigma) = 0$ if $\sigma \in \cup_{n>H}{}^*\mathcal{P}([0,1])$. On the other hand, if we take a function $G \in {}^*L^2_{{}^*\mathcal{K}}(\mathcal{P}_{\mathrm{fin}})$ which is constant on each class $\widetilde{\tau}$, $\tau \in \Gamma$ and zero on $\cup_{n>H}{}^*\mathcal{P}([0,1])$ then obviously there exists a unique $F \in {}^*L^2_{{}^*\mathcal{K}}(\Gamma)$ such that $G = \widetilde{F}$. Thus we have shown the first part of the following proposition:

Proposition 4.3.9 *The above constructed map $\tilde{\imath} : \mathcal{F}_{\mathrm{int}} \longrightarrow \mathcal{F} : F \longmapsto \tilde{\imath}(F) = \widetilde{F}$ defines an internal isometric isomorphism $\tilde{\imath}$ onto an S-dense internal subspace $\widetilde{\mathcal{F}}_{\mathrm{int}}$ of $\mathcal{F}$.*

Proof: That the map is an internal isomorphism is clear by construction. The isometry property follows by the fact that ${}^*\mathcal{P}(\Gamma) \ni \{\sigma\} \mapsto \widetilde{\sigma} \in {}^*\mathcal{P}(\mathcal{P}_{\mathrm{fin}}([0,1]))$ is measure preserving. It remains to prove that $\widetilde{\mathcal{F}}_{\mathrm{int}}$ is S-dense in $\mathcal{F}$. Take some $f \in \mathcal{F} = L^2_{\mathcal{K}}(\mathcal{P}_{\mathrm{fin}}, \Lambda)$. Then by corollary 4.3.3 ${}^*f \in {}^*L^2_{{}^*\mathcal{K}}({}^*\mathcal{P}_{\mathrm{fin}}, {}^*\Lambda)$ is a SL^2-lifting of f. On the other hand there is a SL^2-lifting $F \in {}^*L^2_{{}^*\mathcal{K}}(\Gamma, m)$ which extends by $\tilde{\imath}(F) = \widetilde{F}$ to an SL^2-lifting of f with respect to ${}^*\Lambda_L$. But then we get

$$\mathrm{st}(\widetilde{F}(\sigma)) \;=\; f(\mathrm{st}(\sigma)) \;=\; \mathrm{st}({}^*f(\sigma)) \quad \text{for } {}^*\Lambda_L\text{-almost all } \sigma \in {}^*\mathcal{P}_{\mathrm{fin}}([0,1])$$

and hence $\|\widetilde{F} - {}^*f\| \approx 0$. □

Theorem 4.3.10 *Every bounded Fock space operator k has a strong representation K on the internal Fock space. We can choose K S-bounded.*

Proof: By proposition 4.3.9 we have that $\tilde{\imath}({}^*L^2_{{}^*\mathcal{K}}(\Gamma, m)) = \tilde{\imath}(\mathcal{F}_{\mathrm{int}}) = \widetilde{\mathcal{F}}_{\mathrm{int}}$ is an S-dense subspace of ${}^*L^2({}^*\mathcal{P}_{\mathrm{fin}}, {}^*\Lambda) = {}^*\mathcal{F}$. With the aid of proposition 4.3.8 we see that every $\mathsf{k} \in \mathcal{B}(\mathcal{F})$ has an S-bounded representation $\widetilde{\mathsf{K}} \in {}^*\mathcal{B}(\widetilde{\mathcal{F}}_{\mathrm{int}})$. But then also $\mathsf{K} \in {}^*\mathcal{B}(\mathcal{F}_{\mathrm{int}})$ defined by $\mathsf{K}F = \widetilde{\mathsf{K}}\widetilde{F}$ for $F \in \mathcal{F}_{\mathrm{int}}$ is an S-bounded representation of k in the following sense:

$$\mathrm{st}(\mathsf{K}F) \;=\; \mathsf{k}f \;\text{ whenever }\; \|\widetilde{F} - {}^*f\| \;\approx\; 0\,.$$

This is by corollary 4.3.3 equivalent to

$$\|\widetilde{\mathsf{K}F} - {}^*(\mathsf{k}f)\| \;\approx\; 0 \;\text{ whenever } F \text{ is a } SL^2\text{-lifting of } f$$

which is again by corollary 4.3.3 equivalent to

$$\mathsf{K}F \text{ is a } SL^2\text{-lifting of } \mathsf{k}f \text{ whenever } F \text{ is a } SL^2\text{-lifting of } f.$$

Thus K is a strong representation of k. □

We know that every internal operator K on $\mathcal{F}_{\mathrm{int}}$ is an internal kernel operator with some internal kernel function K (see proposition 2.3.11). Hence for the internal adapted derivatives $\mathbb{D}^+, \mathbb{D}^\circ, \mathbb{D}^-$ of Section 4.1 in Chapter 2 the adapted internal processes $(\mathbb{D}^+_t\mathsf{K})_{t\in T}$, $(\mathbb{D}^\circ_t\mathsf{K})_{t\in T}$ and $(\mathbb{D}^-_t\mathsf{K})_{t\in T}$ are well-defined internal kernel operator processes. Using theorem 2.4.3 we have an internal Clark-Ocone representation of every bounded operator. If we assume that $(\mathbb{D}^+_t\mathsf{K})_{t\in T}, (\mathbb{D}^\circ_t\mathsf{K})_{t\in T}$ and $(\mathbb{D}^-_t\mathsf{K})_{t\in T}$ are S-integrable and strong representations of some adapted standard processes $\mathsf{k}^+_t, \mathsf{k}^\circ_t$ and k^-_t then using theorem 4.2.9 we can convert this internal Clark-Ocone formula to a predictible representation for the bounded operator k:

$$\begin{aligned}\mathsf{k} \;\approx_s\; \mathsf{K} \;&=\; \mathbb{E}(\mathsf{K})\mathbb{1} \;+\; \int_0^1 dA^+_t\mathbb{D}^+_t\mathsf{K} \;+\; \int_0^1 dA^\circ_t\mathbb{D}^\circ_t\mathsf{K} \;+\; \int_0^1 dA^-_t\mathbb{D}^-_t\mathsf{K}\\ &\approx_w\; \mathbb{E}(\mathsf{k})\mathbb{1} \;+\; \int_0^1 \mathsf{k}^+_t\,d\mathsf{a}^+_t \;+\; \int_0^1 \mathsf{k}^\circ_t\,d\mathsf{a}^\circ_t \;+\; \int_0^1 \mathsf{k}^-_t\,d\mathsf{a}^-_t\end{aligned}$$

where $\approx_s$ denotes the strong representation relation and $\approx_w$ the weak representation relation. Thus with respect to the weak operator topology generated by exponential vectors of bounded L^2-functions the operator k admits a predictable representation in the case assumed.

3.2 Weak Representation of Processes, Adapted Processes and Martingales

In this subsection we assume that all processes k are at least woply measurable. Now we will prove the existence of a weak representation for a processes with values in the space of bounded Fock space operators. We feel confident that one could prove also the existence of a strong representation under the assumption that the process is measurable in the strong operator topology. But here we only treat the case under the weak assumption of wop measurability.

Theorem 4.3.11 *Every woply measurable bounded Fock space operator process* $(\mathsf{k}_t)_{t\in[0,1]}$ *has a weak representation* $(\mathsf{K}_t)_{t\in T}$ *on the internal Fock space. Then* K_t *is S-bounded for* μ_L*-almost all* $t \in T$.

PROOF: Suppose that $\mathsf{k} : [0,1] \ni t \mapsto \mathsf{k}_t \in \mathcal{B}(\mathcal{F})$ is a woply measurable operator process. Then by the lifting theorem A.2.7 in Section 2 of the Appendix there exists a lifting $\widetilde{\mathsf{K}} : T \ni t \mapsto \widetilde{\mathsf{K}}_t \in {}^*\mathcal{B}(\mathcal{F})$ such that $\widetilde{\mathsf{K}}_t$ is S-bounded for μ_L-almost all $t \in T$. Now fix some $t \in T$ such that $\widetilde{\mathsf{K}}_t$ is S-bounded and $\mathrm{st}(\widetilde{\mathsf{K}}_t) = \mathsf{k}_{\mathrm{st}(t)}$. That means

$$\langle {}^*g, \widetilde{\mathsf{K}}_t\, {}^*f\rangle \;\approx\; \langle g, \mathsf{k}_{\mathrm{st}(t)} f\rangle \quad \text{for all } f, g \in \mathcal{F}\,. \tag{4.3.9}$$

In particular this holds for every fixed $f \in \mathcal{F}$. Thus $\widetilde{\mathsf{K}}_t\, {}^*f$ is weakly nearstandard in ${}^*\mathcal{F}$. But since ${}^*f$ has finite norm and $\widetilde{\mathsf{K}}_t$ is S-bounded also $\widetilde{\mathsf{K}}_t\, {}^*f$ has finite norm. By the S-denseness of $\widetilde{\mathcal{F}}_{\mathrm{int}}$ and using proposition 4.3.7 we conclude that

$$\langle {}^*g, P\widetilde{\mathsf{K}}_t\, {}^*f\rangle \;\approx\; \langle {}^*g, \widetilde{\mathsf{K}}_t\, {}^*f\rangle \quad \text{for all } f, g \in \mathcal{F} \tag{4.3.10}$$

where $P : {}^*\mathcal{F} \to \widetilde{\mathcal{F}}_{\mathrm{int}}$ is the orthogonal projection onto $\widetilde{\mathcal{F}}_{\mathrm{int}}$. Suppose now that $F, G \in \mathcal{F}_{\mathrm{int}}$ are SL^2-liftings of f, g. By corollary 4.3.3 also ${}^*f$ and ${}^*g$ are SL^2-liftings of f and g, hence $\|{}^*g - \widetilde{G}\| \approx 0$ and $\|{}^*f - \widetilde{F}\| \approx 0$ where $\widetilde{F} = \tilde{\imath}(F)$ and $\widetilde{G} = \tilde{\imath}(G)$ with the isometric isomorphism $\tilde{\imath}$ of proposition 4.3.9. Since P and $\widetilde{\mathsf{K}}_t$ are S-bounded we obtain

$$\langle \widetilde{G}, P\widetilde{\mathsf{K}}_t\widetilde{F}\rangle \;\approx\; \langle {}^*g, P\widetilde{\mathsf{K}}_t\, {}^*f\rangle \quad \text{for all } f, g \in \mathcal{F} \text{ and all } SL^2\text{-liftings } F, G \in \mathcal{F}_{\mathrm{int}}\,. \tag{4.3.11}$$

Now define

$$\mathsf{K} \;=\; \tilde{\imath}^{-1} P\widetilde{\mathsf{K}}\tilde{\imath} \;:\; T \longrightarrow {}^*\mathcal{B}(\mathcal{F}_{\mathrm{int}}) \;:\; t \longmapsto \mathsf{K}_t \;=\; \tilde{\imath}^{-1} P\widetilde{\mathsf{K}}_t\tilde{\imath}\,.$$

Then $(\mathsf{K}_t)_{t\in T}$ is internal since $\widetilde{\mathsf{K}}$ was internal and also P, $\tilde{\imath}$ and $\tilde{\imath}^{-1}$ are internal. Since $\tilde{\imath}$ is an internal isometric isomorphism we get $\langle G, \mathsf{K}_t F\rangle = \langle \widetilde{G}, P\widetilde{\mathsf{K}}_t\widetilde{F}\rangle$ for all $F, G \in \mathcal{F}_{\mathrm{int}}$ and all $t \in T$. Combining this with equations (4.3.9), (4.3.10) and (4.3.11) gives for μ_L-almost all $t \in T$

$$\langle G, \mathsf{K}_t F\rangle \;\approx\; \langle g, \mathsf{k}_{\mathrm{st}(t)} f\rangle \quad \text{for all } f, g \in \mathcal{F} \text{ and all } SL^2\text{-liftings } F, G \in \mathcal{F}_{\mathrm{int}}\,.$$

Thus K is a weak representation of k. □

To extend the result of the preceding theorem to adapted processes we must define exactly what we mean by adaptedness and being a martingale for a Fock space operator process k. To be precise we define both properties with respect to the exponential domain $\mathcal{E}$. Before we give the definition we want to introduce the following notation and prove a proposition.

Notation 4.3.12 For each $\phi \in L^2([0,1])$ and each $t \in [0,1]$ we set $\phi_{t)} = \phi\chi_{[0,t)}$ and $\phi_{[t} = \phi\chi_{[t,1]}$. Analogous for $\Phi \in {}^*L^2(T)$ and $\tilde{t} \in T$ we set $\Phi_{\tilde{t})} = \Phi\chi_{[0,\tilde{t})}$ and $\Phi_{[\tilde{t}} = \Phi\chi_{[\tilde{t},1-\frac{1}{H}]}$ where this time the characteristic functions are defined on T. ◁

Obviously if $\phi \in B^2([0,1])$ then $\phi_{t)}, \phi_{[t} \in B^2([0,1])$ and similarly $\Phi_{\tilde{t})}, \Phi_{[\tilde{t}} \in SB^2(T)$ if $\Phi \in SB^2(T)$. Furthermore, one has $\pi_{\phi_{t)}} \in \mathcal{F}_{t)}$, $\pi_{\phi_{[t}} \in \mathcal{F}_{[t}$, $\pi_{\Phi_{\tilde{t})}} \in \mathcal{F}_{\mathrm{int},\tilde{t})}$ and $\pi_{\Phi_{[\tilde{t}}} \in \mathcal{F}_{\mathrm{int},[\tilde{t}}$.

Proposition 4.3.13 *For each $t \in [0,1]$ and $\tilde{t} \in T$ with $\mathrm{st}(\tilde{t}) = t$ the function $\Phi_{\tilde{t})}$ is an SL^2-lifting of $\phi_{t)}$ and $\Phi_{[\tilde{t}}$ is an SL^2-lifting of $\phi_{[t}$ if Φ is an SL^2-lifting of ϕ.*

Proof: Since $\{\mathrm{st}(s) : s \in T, 0 \le s < \tilde{t}\} = [0,t]$ we see that for μ_L-almost all $s \in T$ one has $\chi_{[0,\tilde{t})}(s) = \chi_{[0,t)}(\mathrm{st}(s))$ and $\chi_{[\tilde{t},1-\frac{1}{H}]}(s) = \chi_{[t,1]}(\mathrm{st}(s))$. Thus $\Phi_{\tilde{t})}$ and $\Phi_{[\tilde{t}}$ are SL^2-liftings of $\phi_{t)}$ and $\phi_{[t}$. □

As an application of propositions 3.2.4 and 3.2.11 we have the following corollary.

Corollary 4.3.14 *$\pi_{\Phi_{\tilde{t})}}$ is an SL^2-lifting of $\pi_{\phi_{t)}}$ and $\pi_{\Phi_{[\tilde{t}}}$ is an SL^2-lifting of $\pi_{\phi_{[t}}$ if Φ is an SL^2-lifting of ϕ and $\mathrm{st}(\tilde{t}) = t$.*

But then we have also that $\pi_\phi = \pi_{\phi_{t)}} \otimes \pi_{\phi_{[t}}$ and $\pi_\Phi = \pi_{\Phi_{\tilde{t})}} \otimes \pi_{\Phi_{[\tilde{t}}}$. Recall the notation $\tau_{\tilde{t})}$ and $\tau_{[\tilde{t}}$ for $\tau \in \Gamma$ and $\tilde{t} \in T$. We use the same notation for $\sigma \in \mathcal{P}_{\mathrm{fin}}$ and $t \in [0,1]$:

$$\sigma_{t)} \;=\; \{s \in \sigma : s < t\} \quad \text{and} \quad \sigma_{[t} \;=\; \{s \in \sigma : s \ge t\}\,.$$

Definition 4.3.15 *An operator process k is said to be* adapted *if for all $\phi \in B^2([0,1])$ it is*

$$\mathsf{k}_t\pi_{\phi_{t)}} \;\in\; \mathcal{F}_{t)} \quad \textit{and} \quad \mathsf{k}_t\pi_\phi \;=\; (\mathsf{k}_t\pi_{\phi_{t)}}) \otimes \pi_{\phi_{[t}}\,. \tag{4.3.12}$$

An adapted operator process is said to be a martingale *if whenever $s \le t$ we have*

$$\langle \pi_{\psi_{s)}}, \mathsf{k}_t\pi_{\phi_{s)}}\rangle \;=\; \langle \pi_{\psi_{s)}}, \mathsf{k}_s\pi_{\phi_{s)}}\rangle$$

for all $\phi, \psi \in B^2([0,1])$. ◁

Let $P_{t)}$ be the orthogonal projection from $\mathcal{F}$ onto $\mathcal{F}_{t)}$. Then for a bounded operator process the two conditions for adaptedness are equivalent to the following condition:

$$\mathsf{k}_t \;=\; P_{t)}\mathsf{k_t}|_{\mathcal{F}_{t)}} \otimes \mathbb{1}_{[t} \tag{4.3.13}$$

where $\mathbb{1}_{[t}$ is the identity on $\mathcal{F}_{[t}$ (cf. [AL97, p. 18]). Further the martingale condition is for bounded martingales equivalent to the following equation (cf. [AL97, p. 19]):

$$P_{s)}\mathsf{k}_t|_{\mathcal{F}_{s)}} \otimes \mathbb{1}_{[s} \;=\; P_{s)}\mathsf{k}_s|_{\mathcal{F}_{s)}} \otimes \mathbb{1}_{[s} \;=\; \mathsf{k}_s \quad \text{for all } s \leq t \in [0,1]\,.$$

It follows that we have

$$\mathsf{k}_t \;=\; P_{t)}\mathsf{k}_1|_{\mathcal{F}_{t)}} \otimes \mathbb{1}_{[t} \quad \text{for all } t \in [0,1]\,. \tag{4.3.14}$$

Suppose now that we have an internal kernel process K which fulfills the conditions similar to in definition 4.3.15, namely:

$$\mathsf{K}_t\pi_{\Phi_{t)}} \;\in\; \mathcal{F}_{\mathrm{int},t)} \quad \text{and} \quad \mathsf{K}_t\pi_{\Phi} \;=\; (\mathsf{K}_t\pi_{\Phi_{t)}}) \otimes \pi_{\Phi_{[t}} \tag{4.3.15}$$

where $\Phi \in SB^2(T)$ and $t \in T$.

Proposition 4.3.16 *Suppose that an internal kernel process* K *satisfies the conditions of the preceding equation (4.3.15). Then the kernel function* K *is adapted:*

$$\max(\sigma \cup \rho \cup \tau) \geq t \implies K_t(\sigma,\rho,\tau) = 0\,.$$

PROOF: The first condition means that if $\alpha \neq \alpha_{t)}$ then it must be $(\mathsf{K}_t\pi_{\Phi_{t)}})(\alpha) = 0$. For the kernel we have

$$(\mathsf{K}_t\pi_{\Phi_{t)}})(\alpha) \;=\; \sum_{\sigma\dot\cup\rho\dot\cup\beta=\alpha} \; \sum_{\substack{\tau \\ \tau\cap(\rho\dot\cup\beta)=\emptyset}} K_t(\sigma,\rho,\tau)\pi_{\Phi_{t)}}(\rho\cup\beta\cup\tau)m(\tau)\,.$$

If $\alpha \neq \alpha_{t)}$ then there exists an $s \in \alpha$ with $s \geq t$. If $s \in \rho \cup \beta$ then $\pi_{\Phi_{t)}}$ is zero. But if $s \in \sigma$ it must be $K_t(\sigma,\rho,\tau) = 0$ hence

$$\max\sigma \geq t \implies K_t(\sigma,\rho,\tau) = 0\,. \tag{4.3.16}$$

The second condition reads in the kernel language as

$$\begin{aligned}
(\mathsf{K}_t\pi_{\Phi})(\alpha) \;&=\; \sum_{\sigma\dot\cup\rho\dot\cup\beta=\alpha} \; \sum_{\substack{\tau \\ \tau\cap(\rho\dot\cup\beta)=\emptyset}} K_t(\sigma,\rho,\tau)\pi_{\Phi}(\rho\cup\beta\cup\tau)m(\tau) \\
&\overset{!}{=}\; \sum_{\widetilde\sigma\dot\cup\widetilde\rho\dot\cup\widetilde\beta=\alpha_{t)}} \; \sum_{\substack{\tau \\ \tau\cap(\widetilde\rho\dot\cup\widetilde\beta)=\emptyset}} K_t(\sigma,\rho,\tau)\pi_{\Phi_{t)}}(\rho\cup\beta\cup\tau)m(\tau)\cdot\pi_{\Phi_{[t}}(\alpha_{[t}) \\
&=\; (\mathsf{K}_t\pi_{\Phi_{t)}}) \otimes \pi_{\Phi_{[t}}\,.
\end{aligned}$$

In particular for $\alpha = \alpha_{t)}$ we see that $\alpha_{[t} = \emptyset$ and $\pi_{\Phi_{[t}}(\alpha_{[t}) = 1$. But this implies that the two double sums must be equal and this can only be the case if

$$\max \tau \geq t \implies K_t(\sigma, \rho, \tau) = 0\,. \tag{4.3.17}$$

So from equations (4.3.16) and (4.3.17) we conclude that $K_t(\sigma, \rho, \tau) = 0$ if $\max(\sigma \cup \tau) \geq t$. We can now rewrite the second condition:

$$\begin{aligned}
(\mathsf{K}_t \pi_\Phi)(\alpha) &= \sum_{\sigma \dot\cup \rho \dot\cup \beta = \alpha} \sum_{\substack{\tau \\ \tau \cap (\rho \cup \beta) = \emptyset}} K_t(\sigma, \rho, \tau) \pi_\Phi(\rho \cup \beta \cup \tau) m(\tau) \\
&= \sum_{\sigma_{t)} \dot\cup \rho_{t)} \dot\cup \beta_{t)} = \alpha_{t)}} \sum_{\rho_{[t} \dot\cup \beta_{[t} = \alpha_{[t}} \sum_{\substack{\tau,\, \max \tau < t \\ \tau \cap (\rho \cup \beta) = \emptyset}} K_t(\sigma, \rho_{t)} \cup \rho_{[t}, \tau) \pi_\Phi(\rho_{t)} \cup \rho_{[t} \cup \beta_{t)} \cup \beta_{[t} \cup \tau) m(\tau) \\
&= \sum_{\sigma_{t)} \dot\cup \rho_{t)} \dot\cup \beta_{t)} = \alpha_{t)}} \sum_{\rho_{[t} \dot\cup \beta_{[t} = \alpha_{[t}} \sum_{\substack{\tau,\, \max \tau < t \\ \tau \cap (\rho_{t)} \cup \beta_{t)}) = \emptyset}} K_t(\sigma, \rho_{t)} \cup \rho_{[t}, \tau) \pi_{\Phi_{t)}}(\rho_{t)} \cup \beta_{t)} \cup \tau) m(\tau) \cdot \pi_{\Phi_{[t}}(\rho_{[t} \cup \beta_{[t}) \\
&\overset{!}{=} \sum_{\widetilde\sigma \dot\cup \widetilde\rho \dot\cup \widetilde\beta = \alpha_{t)}} \sum_{\substack{\tau \\ \tau \cap (\rho \cup \beta) = \emptyset}} K_t(\sigma, \rho, \tau) \pi_{\Phi_{t)}}(\rho \cup \beta \cup \tau) m(\tau) \cdot \pi_{\Phi_{[t}}(\alpha_{[t}) \;=\; (\mathsf{K}_t \pi_{\Phi_{t)}}) \otimes \pi_{\Phi_{[t}}\,.
\end{aligned}$$

We see that equality is possible if and only if the sum over partitions of $\alpha_{[t}$ reduces to the trivial partition $\beta_{[t} = \alpha_{[t}$ and all other terms must vanish. But this means that $K_t(\sigma, \rho_{t)} \cup \rho_{[t}, \tau)$ must be zero if $\rho_{[t} \neq \emptyset$ hence $K_t(\sigma, \rho, \tau) = 0$ if $\max \rho \geq t$. Taking the three arguments together we conclude

$$\max(\sigma \cup \rho \cup \tau) \geq t \implies K_t(\sigma, \rho, \tau) = 0\,. \qquad \square$$

Proposition 4.3.17 *Suppose that an adapted internal kernel process* K *satisfies for all* $\Phi, \Psi \in SB^2(T)$

$$\langle \pi_{\Psi_{s)}}, \mathsf{K}_t \pi_{\Phi_{s)}} \rangle = \langle \pi_{\Psi_{s)}}, \mathsf{K}_s \pi_{\Phi_{s)}} \rangle \tag{4.3.18}$$

whenever $s \leq t$, $s, t \in T$. *Then the kernel function has the martingale property:*

$$\forall s \in T\, \forall t \in T\, \forall (\sigma, \rho, \tau) \in \Gamma^3_{\neq} \Big(\max(\sigma \cup \rho \cup \tau) < \min(s, t) \implies K_t(\sigma, \rho, \tau) = K_s(\sigma, \rho, \tau) \Big)\,.$$

Proof: For the kernel function the condition in equation (4.3.18) reads as

$$\sum_{\substack{\alpha \in \Gamma \\ \max \alpha < s}} \overline{\pi_{\Psi_{s]}}(\alpha)} \sum_{\sigma \dot\cup \rho \dot\cup \beta = \alpha} \sum_{\substack{\tau \\ \tau \cap (\rho \cup \beta) = \emptyset}} K_t(\sigma, \rho, \tau) \pi_{\Phi_{s)}}(\rho \cup \beta \cup \tau) m(\tau) =$$

$$\sum_{\substack{\alpha \in \Gamma \\ \max \alpha < s}} \overline{\pi_{\Psi_{s]}}(\alpha)} \sum_{\sigma \dot\cup \rho \dot\cup \beta = \alpha} \sum_{\substack{\tau \\ \tau \cap (\rho \cup \beta) = \emptyset}} K_s(\sigma, \rho, \tau) \pi_{\Phi_{s)}}(\rho \cup \beta \cup \tau) m(\tau)$$

since $\overline{\pi_{\Psi_{s]}}(\alpha)} = 0$ if $\max \alpha \geq s$. But this also means that we sum only over such (σ, ρ, τ) that $\max(\sigma \cup \rho \cup \tau) < s$. This shows that for $s \leq t$ we have

$$\max(\sigma \cup \rho \cup \tau) < s \implies K_t(\sigma, \rho, \tau) = K_s(\sigma, \rho, \tau)$$

and the kernel function $(K_t)_{t \in T}$ has the martingale property. $\square$

Lemma 4.3.18 *Let $(P_{t)})_{t\in[0,1]}$ be the familiy of othogonal projections $P_{t)} : \mathcal{F} \to \mathcal{F}_{t)}$ and $(\mathbb{P}_{\tilde{t})})_{\tilde{t}\in T}$ be the internal family of the orthogonal internal projections $\mathbb{P}_{\tilde{t})} : \mathcal{F}_{\text{int}} \to \mathcal{F}_{\text{int},\tilde{t})}$. Then for all $f \in \mathcal{F}$ and all SL^2-liftings $F \in \mathcal{F}_{\text{int}}$ of f we have*

$$\mathbb{P}_{\tilde{t})}F(\sigma) \;\approx\; P_{t)}f(\text{st}(\sigma)) \quad \textit{for } m_L\textit{-almost all } \sigma \in \Gamma$$

whenever $\text{st}(\tilde{t}) = t, \tilde{t} \in T, t \in [0,1]$.

PROOF: If $\tau \neq \tau_{t)}$ we have $P_{t)}f(\tau) = 0$ and similarly $\mathbb{P}_{\tilde{t})}F(\sigma) = 0$ if $\sigma \neq \sigma_{\tilde{t})}$. Take $\sigma \in \Gamma_{\text{st}\notin\text{t}}$ and suppose that $\sigma \neq \sigma_{\tilde{t})}$. Then also $\text{st}(\sigma) \neq \text{st}(\sigma)_{t)}$ and both functions are zero. But if $\sigma = \sigma_{\tilde{t})}$ then $\text{st}(\sigma) = \text{st}(\sigma)_{t)}$ and we have

$$\mathbb{P}_{\tilde{t})}F(\sigma_{\tilde{t})}) \;=\; F(\sigma_{\tilde{t})}) \;\approx\; f(\text{st}(\sigma)_{t)}) \;=\; P_{t)}f(\text{st}(\sigma)_{t)})$$

since F is a lifting of f. □

Theorem 4.3.19 *Let k be an adapted bounded woply measurable Fock space operator. Then k has a weak representation K such that the corresponding kernel function K is adapted. If in addition k is a martingale then we can choose K such that K satisfies the martingale property.*

PROOF: By theorem 4.3.11 there is a weak representation $\widetilde{\mathsf{K}}$ of k. Let $P_{t)}$ be the orthogonal projection from $\mathcal{F}$ onto $\mathcal{F}_{t)}$. By equation (4.3.13) one has $\mathsf{k}_t = P_{t)}\mathsf{k}_{\mathsf{t}}|_{\mathcal{F}_{t)}} \otimes \mathbb{1}_{[t}$. Now let $\mathbb{P}_{\tilde{t})}$ be the orthogonal internal projection from Γ onto $\Gamma_{\tilde{t})}$. We define

$$\mathsf{K}_{\tilde{t}} \;=\; \mathbb{P}_{\tilde{t})}\widetilde{\mathsf{K}}_{\tilde{t}}|_{\Gamma_{\tilde{t})}} \otimes \mathbb{1}_{[\tilde{t}}\,.$$

Clearly by the preceding lemma K is internal and a weak representation of k. Furthermore, for all $\Phi \in SB^2$ and $t \in T$ we have

$$\mathsf{K}_t\pi_{\Phi_{t)}} \;\in\; \mathcal{F}_{\text{int},t)} \quad\text{and}\quad \mathsf{K}_t\pi_{\Phi} \;=\; (\mathsf{K}_t\pi_{\Phi_{t)}}) \otimes \pi_{\Phi_{[t}}\,.$$

With the help of proposition 4.3.16 we conclude that the corresponding kernel function K is adapted.
If k is a martingale we already know that there is a weak representation $\widetilde{\mathsf{K}}$ such that $\widetilde{K}$ is adapted. By equation (4.3.14) we have $\mathsf{k}_t = P_{t)}\mathsf{k}_1|_{\mathcal{F}_{t)}} \otimes \mathbb{1}_{[t}$. Choose some element $\tilde{1} \in T$ with $\text{st}(\tilde{1}) = 1$ such that $\widetilde{\mathsf{K}}_{\tilde{1}}$ is a weak representation of k_1. Then for $\tilde{t} \in T$ we define the internal process

$$\mathsf{K}_{\tilde{t}} \;=\; \begin{cases} \mathbb{P}_{\tilde{t})}\widetilde{\mathsf{K}}_{\tilde{1}}|_{\Gamma_{\tilde{t})}} \otimes \mathbb{1}_{[\tilde{t}} & \text{for } \tilde{t} < \tilde{1}\,, \\ \mathbb{P}_{\tilde{t})}\widetilde{\mathsf{K}}_{1-\frac{1}{H}}|_{\Gamma_{\tilde{t})}} \otimes \mathbb{1}_{[\tilde{t}} & \text{for } \tilde{t} \geq \tilde{1}\,. \end{cases}$$

Again using lemma 4.3.18 K is internal and a weak representation of k. But for all $\Phi \in SB^2$ and $s \leq t \in T$ we also have

$$\langle \pi_{\Psi_{s)}}, \mathsf{K}_t\pi_{\Phi_{s)}} \rangle \;=\; \langle \pi_{\Psi_{s)}}, \mathsf{K}_s\pi_{\Phi_{s)}} \rangle\,.$$

With proposition 4.3.17 we obtain that the corresponding kernel function K has the martingale property. $\square$

In Section 4.1 of Chapter 2 we have shown that every internal martingale admits a martingale representation (see theorem 2.4.6). Now take the weak representation K of some bounded martingale k. If we again choose $T \ni \tilde{1} \approx 1$ carefully then the internal adapted processes

$$\mathsf{H}_t^+ = \mathbb{D}_t^+ \mathsf{K}_{\tilde{1}}\,, \quad \mathsf{H}_t^\circ = \mathbb{D}_t^\circ \mathsf{K}_{\tilde{1}}\,, \quad \mathsf{H}_t^- = \mathbb{D}_t^- \mathsf{K}_{\tilde{1}}$$

are well-defined for all $t \in T, t < \tilde{1}$. Concerning the standard quantum martingale representation theorem we can say the following. If the processes $\mathsf{H}_t^+, \mathsf{H}_t^\circ, \mathsf{H}_t^-$ are S-integrable, and if we can show that they are the strong representations of some adapted standard processes $\mathsf{h}_t^+, \mathsf{h}_t^\circ, \mathsf{h}_t^-$ then the bounded martingale k has by theorem 4.2.9 the representation:

$$\begin{aligned} \mathsf{k}_t \;\approx_w\; \mathsf{K}_{\tilde{t}} \;&=\; \mathbb{E}(\mathsf{K}_0)\mathbb{1} + \int_0^{\tilde{t}} dA_s^+ \mathsf{H}_s^+ \;+\; \int_0^{\tilde{t}} dA_s^\circ \mathsf{H}_s^\circ \;+\; \int_0^{\tilde{t}} dA_s^- \mathsf{H}_s^- \\ &\approx_w\; \mathbb{E}(\mathsf{k}_0)\mathbb{1} \;+\; \int_0^t \mathsf{h}_s^+ d\mathsf{a}_s^+ \;+\; \int_0^t \mathsf{h}_s^\circ d\mathsf{a}_s^\circ \;+\; \int_0^t \mathsf{h}_s^- d\mathsf{a}_s^- \end{aligned}$$

where $\approx_w$ denotes the relation for weak representation. We see that in such a case the bounded standard martingale is representable:

$$\mathsf{k}_t \;=\; \mathbb{E}(\mathsf{k}_0)\mathbb{1} \;+\; \int_0^t \mathsf{h}_s^+ d\mathsf{a}_s^+ \;+\; \int_0^t \mathsf{h}_s^\circ d\mathsf{a}_s^\circ \;+\; \int_0^t \mathsf{h}_s^- d\mathsf{a}_s^-$$

where the equality is in the weak operator topology on $\mathcal{F}$ generated by exponential vectors of bounded L^2-functions.

Conclusion

> 217. Wer annähme, daß *alle* unsre Rechnungen unsicher seien und daß wir uns auf keine verlassen können (mit der Rechtfertigung, daß Fehler überall möglich sind), würden wir vielleicht für verrückt erklären. Aber können wir sagen, er sei im Irrtum? Reagiert er nicht einfach anders: wir verlassen uns darauf, er nicht; wir sind sicher, er nicht.
>
> L. Wittgenstein, *Über Gewißheit*

"First say what you will do. Then do it. And after it say what you have done." This dictum attributed to Bertrand Russell is meant as a short guidance for scientific writings. Since we are now in the last stage we have to take a look at the work which was done. And of course complementarily we should look as well at that what was not done but could be done in the future.

Following from Meyer's heuristic ideas about the toy Fock space we constructed an internal Fock space in Chapter 2. Time, creation, number and annihilation process were defined on this Fock space and an internal Itô formula proved. We included sections on Brownian motion and Poisson process occuring on the internal Fock space. Then we introduced internal kernel operators and internal (stochastic) integration. The Chapter closed with a Section on the Clark-Ocone formula, the martingale representation theorem and quantum stochastic differential equations in the internal setting. In the course of Chapter 2 we proved several statements which were later useful in connecting this internal calculus to the standard calculus.
But if we look at the proofs themselves we see that they were constructed from more or less simple combinatorial arguments. This is not so surprising because in the case of the internal calculus we worked in a formally finite setting. So far most arguments were valid in finite discrete mathematics except some 'approximate' arguments. In finite discrete mathematics those would constitute a heuristic handling with infinitesimal and infinitely large quantities. We assume that most mathematicians sometimes use such finite conbinatorics and heuristics to intuit ideas that should be the result of a calculation in the continuous time setting.

Eventually in Chapter 3 we gave these heuristic arguments a solid mathematical foundation connecting the internal Fock space to the standard Fock space. First we showed that the underlying measure spaces are related by Loeb's construction. We concluded that every function in the standard Fock space has a square S-integrable lifting in the internal Fock space. Based on this we introduced three ways of representing a Fock space operator

by an internal operator.
The strict representation applies only to standard kernel operators. Nevertheless we were able to apply strict representations to quantum stochastic differential equations for the kernel function. Namely in subsection 1.2 of chapter 4 we constructed out of the internal kernel solution of an equation with nonlinear noise terms a standard kernel function such that the internal one was a strict representation of the standard one. We showed that the standard kernel function defines a reasonable operator and solves a quantum stochastic differential equation with nonlinear noise terms.
For weak representations we showed compatibility with the expectation in the vacuum state. Thus characteristic distributions can be calculated by choosing a weak representation and carrying out a hyperfinite calculation. Furthermore we proved upward compatibility: every strong representation is a weak representation. Since the internal fundamental processes proved to be strong representations of the corresponding standard processes we could apply this result and obtain the characteristic distributions of Brownian motion and Poisson process on Fock space by hyperfinite combinatorics. We anticipate that this method will be very useful in the future. Furthermore, in Section 3.2 of Chapter 4 we proved existence of weak representations for bounded adapted processes and martingales such that the kernel of the representing process is adapted respectively has the martingale property.
The strong representation was important for integration theory. We demonstrated in Chapter 4 Section 2 that S-integrability of an adapted strong representation implies integrability of the represented adapted process. This gives a criterium for checking integrability by checking that the process admits an S-integrable strong representation. In Section 3.1 we proved the existence of a strong representation for a bounded operator. Considering the internal Clark-Ocone formula we can try to construct the predictible representation for a Fock space operator using a strong representation. Further, using a weak representation of a martingale the same idea applies of course with the internal martingale representation theorem. We feel confident that this will give new insight into the quantum martingale representation theorem.

A possible extension of this infinitesimal approach to quantum stochastic calculus lies in multiple Fock space calculus. A natural idea would be to take hyperfinite multiplicity by starting with $T \times \{1, \dots, N\}, N \in {}^*\mathbb{N}_\infty$ and then to determine the right 'limit' measure. This should become a nonstandard model for standard Fock space with countable multiplicity. With the same starting point we could even think of a 'Loeb' Fock space with uncountable multiplicity.
Another extension would be to look at quantum stochastic differential equations with unbounded coefficients. But this needs a deep understanding of unbounded operators and their nonstandard hulls. That is itself a field in development [Wol, Kru90, GZ98, KP00]. What seems more advantageous to us is using reasonable nonstandard hulls of operators as coefficients and solving quantum stochastic differential equations directly by kernel functions on the Loeb space. This could end up by replacing the initial Hilbert space by some nonstandard hull Hilbert space and the Fock space by a 'Loeb' Fock space, that is, by the L^2-space over the Loeb space of the symmetric measure space over T.

With this thesis we have now created a new doorway to quantum stochastic calculus. But as the considerations above plainly show we have made just one or two small steps on the way behind the door. Nevertheless we feel confident that in the future nonstandard analysis in the field of quantum stochastic calculus will become as fruitful as it is already in so many other fields.

Appendix

This Appendix will provide us with two lifting results. In the first Section we introduce the Bochner integral and prove a known lifting theorem for Bochner integrable functions. In the second Section we prove a new lifting result for woply measurable functions. That is, we lift functions with values in $\mathcal{B}(\mathcal{K})$ endowed with the weak operator topology. This is remarkable since $\mathcal{B}(\mathcal{K})$ is not even first countable in the weak operator topology. But since we assume $\mathcal{K}$ to be separable the predual space (for the wop topology) is separable and this is sufficient for the proof.

1 Bochner Integral

In this Section we introduce some known facts about the Bochner integral. We follow more or less Section IV.7 of Takesaki's book [Tak79] and connect these results to those in the book by Diestel and Uhl [DU77]. Then we present a lifting theorem for Bochner integrable functions due to Osswald. As general assumption let $\mathbb{B}$ be a Banach space and $(\mathcal{M}, \mathfrak{M}, \mathfrak{m})$ a complete standard measure space where $\mathcal{M}$ is a locally compact topological space and $\mathfrak{m}$ is Radon.

Definition A.1.1 *Let $\mathbb{B}$ be a Banach space. A function $f : \mathcal{M} \to \mathbb{B}$ is said to be* $\mathfrak{m}$-measurable *(or simply* measurable*) if for any compact set $K \subseteq \mathcal{M}$ and any $\varepsilon \in \mathbb{R}_+$ there exists a compact set $K_\varepsilon \subseteq K$ with $\mathfrak{m}(K \setminus K_\varepsilon) < \varepsilon$ and f is continuous on K_ε.* $\lhd$

A characterization of measurable functions is given by the next proposition. One finds the proof in [Tak79, p. 253]. Note the fine difference between the notation $\mathbb{B}^\star$ for the dual space and ${}^*\mathbb{B}$ for the nonstandard $*$-image of $\mathbb{B}$.

Proposition A.1.2
A $\mathbb{B}$-valued function f is measurable iff the following two conditions hold:

(1) *For each $b^\star \in \mathbb{B}^\star$ the function $\mathcal{M} \ni m \longmapsto \langle b^\star, f(m)\rangle \in \mathbb{C}$ is measurable.*

(2) *For each compact subset $K \subseteq \mathcal{M}$ there exists a measurable set $K_0 \subseteq K$ such that $\mathfrak{m}(K \setminus K_0) = 0$ and $f(K_0)$ is separable.*

Let $\mathcal{L}$ be some linear space of $\mathbb{C}$-valued measurable functions on $\mathcal{M}$. Then we identify the algebraic tensor product $\mathcal{L} \otimes \mathbb{B}$ with a space of $\mathbb{B}$-valued functions by the identification:

$$\sum_{i=1}^{n} f_i \otimes b_i \quad \longleftrightarrow \quad \sum_{i=1}^{n} f_i(\,\cdot\,) b_i\,.$$

If $\mathcal{L}$ is the space of functions with compact support then the tensor product is identified with the $\mathbb{B}$-valued functions with compact support and finite dimensional range. Further if $\mathcal{L}$ is the space of constant functions with compact support then the tensor product results in the $\mathbb{B}$-valued 'step' functions. For both cases we define a seminorm on this space by

$$\|g\|_p = \left(\int_{\mathcal{M}} \|g(m)\|^p d\mathfrak{m}(m) \right)^{\frac{1}{p}} \text{ for } \mathbb{B}\text{-valued } g \text{ with compact support.}$$

We denote by $L^p_{\mathbb{B}}(\mathcal{M}, \mathfrak{m})$, $1 \leq p < \infty$ the completion of the tensor product $\mathcal{L} \otimes \mathbb{B}$ where $\mathcal{L}$ are the functions with compact support. As usual we identify functions which are equal $\mathfrak{m}$-almost everywhere. We have the following proposition in Takesaki's book [Tak79, proposition 7.4].

Proposition A.1.3

$$L^p_{\mathbb{B}}(\mathcal{M}, \mathfrak{m}) \quad = \quad \{ f : \mathcal{M} \to \mathbb{B} : \|f\|_p < \infty \}, \quad 1 \leq p < \infty\,.$$

In the literature there is also the following definition. But note that the measure space is assumed to be finite.

Definition A.1.4 *Let $(\mathcal{M}, \mathfrak{M}, \mathfrak{m})$ be a finite measure space and $\mathbb{B}$ a Banach space. A function $f : \mathcal{M} \to \mathbb{B}$ is called* simple *if $f = \sum_{k=1}^{n} b_k \chi_{M_k}$ for some $b_k \in \mathbb{B}$ and $M_k \in \mathfrak{M}$ and χ_{M_k} is the characteristic function of M_k. A function $f : \mathcal{M} \to \mathbb{B}$ is called* strongly measurable *if there exists a sequence of simple functions $(f_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} \|f_n(m) - f(m)\| = 0$ $\mathfrak{m}$-almost everywhere.* $\lhd$

In the book [DU77] there appears the following proposition as Pettis' measurability theorem on page 42. This connects the two definitions of measurability.

Proposition A.1.5 *A function $f : \mathcal{M} \to \mathbb{B}$ is strongly measurable iff the following two conditions hold:*

(1) *For each $b^\star \in \mathbb{B}^\star$ the function $\mathcal{M} \ni m \longmapsto \langle b^\star, f(m) \rangle \in \mathbb{C}$ is measurable.*

(2) *There exists a set $M_0 \in \mathfrak{M}$ with $\mathfrak{m}(M_0) = \mathfrak{m}(\mathcal{M})$ and $f(M_0)$ is separable.*

Corollary A.1.6 *For a finite measure space the notions of $\mathfrak{m}$-measurability and strong measurability are equivalent.*

Definition A.1.7 *A function $f : \mathcal{M} \to \mathbb{B}$ is called* weakly measurable *if for each $b^\star \in \mathbb{B}^\star$ the function $\mathcal{M} \ni m \longmapsto \langle b^\star, f(m)\rangle \in \mathbb{C}$ is measurable. This gives the concept of* weak measurability*.* $\triangleleft$

We see from the proposition above that a function is (strongly) measurable iff it is weakly measurable and essentially separable valued.

From now on we suppose that our measure space $(\mathcal{M}, \mathfrak{M}, \mathfrak{m})$ is finite. For a simple function $f = \sum_{k=1}^n b_k \chi_{M_k}$ and $M \in \mathfrak{M}$ we define the integral by

$$\int_M f d\mathfrak{m} \;=\; \sum_{k=1}^n b_k \mathfrak{m}(M \cap M_k)\,.$$

The following is a definition of Bochner integrability.

Definition A.1.8 *A measurable function $f : \mathcal{M} \to \mathbb{B}$ is* Bochner integrable *if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \int_{\mathcal{M}} \|f_n(m) - f(m)\| d\mathfrak{m}(m) \;=\; 0\,.$$

In that case we define the Bochner integral *for each $M \in \mathfrak{M}$ by*

$$\int_M f d\mathfrak{m} \;=\; \lim_{n\to\infty} \int_M f_n d\mathfrak{m}\,. \qquad \triangleleft$$

A characterization of Bochner integrable functions is given by the next proposition [DU77, p. 45].

Proposition A.1.9
A measurable function $f : \mathcal{M} \to \mathbb{B}$ is Bochner integrable iff $\int_{\mathcal{M}} \|f\| d\mathfrak{m} < \infty$.

For $1 \le p < \infty$ we generalize this notion to *p-Bochner integrability* in the obvious way and we see that for finite measure spaces the space of p-Bochner integrable functions is nothing else than the formerly introduced space $L^p_{\mathbb{B}}(\mathcal{M}, \mathfrak{m})$ (cf. proposition A.1.3). For Bochner integrable functions one has a dominated convergence theorem [DU77, p. 45].

Theorem A.1.10 *Let $f : \mathcal{M} \to \mathbb{B}$ be a function and $(f_n)_{n\in\mathbb{N}}$ a sequence of Bochner integrable functions such that*

$$\lim_{n\to\infty} \mathfrak{m}(\{m \in \mathcal{M} : \|f_n(m) - f(m)\| \ge \varepsilon\}) \;=\; 0 \quad \textit{for every } \varepsilon > 0\,.$$

Furthermore, let g be an integrable real-valued function such that $\|f_n\| \le g$ $\mathfrak{m}$-almost everywhere. Then f is Bochner integrable and for every $M \in \mathfrak{M}$

$$\lim_{n\to\infty} \int_M f_n d\mathfrak{m} \;=\; \int_M f d\mathfrak{m} \quad \textit{and} \quad \lim_{n\to\infty} \int_M \|f_n - f\| d\mathfrak{m} \;=\; 0\,.$$

As corollary we obtain an equivalent definition of Bochner integrability.

Corollary A.1.11 *A function $f : \mathcal{M} \to \mathbb{B}$ is Bochner integrable iff there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that*

(1) $\lim_{n\to\infty} \mathfrak{m}(\{m \in \mathcal{M} : \|f_n(m) - f(m)\| \geq \varepsilon\}) \ = \ 0$ *for every* $\varepsilon > 0$;

(2) $\lim_{n,m\to\infty} \int_{\mathcal{M}} \|f_n - f_m\| d\mathfrak{m} \ = \ 0$.

Now we apply this to prove a lifting theorem for Bochner integrable functions. For this purpose let $(\Omega, \mathfrak{O}, \nu)$ be a hyperfinite finite internal measure space and $(\Omega, L(\mathfrak{O}), \nu_L)$ the corresponding Loeb space. We assume that $\mathfrak{O}$ is a ${}^*\sigma$-algebra. Then we have the notion of ν_L-simple functions and Bochner ν_L-integrable functions. On the other hand there is the notion of $*$-simple functions:

Definition A.1.12 *A function $F : \Omega \to {}^*\mathbb{B}$ is called* $*$-simple *if there exist an $N \in {}^*\mathbb{N}$, sets $O_i \in \mathfrak{O}$ and elements $b_i \in {}^*\mathbb{B}$ such that $F(\omega) = \sum_{i=1}^{N} b_i \chi_{O_i}(\omega)$.* ◁

Since we suppose that Ω and then also $\mathfrak{O}$ are hyperfinite we see that in fact every $*$-measurable function is a $*$-simple function. By transfer we have then also the notion of Bochner ν-integrability. As in the complex valued case we introduce the concept of S-integrability.

Definition A.1.13
A $*$-measurable function $F : \Omega \to {}^*\mathbb{B}$ is called S-Bochner integrable *(or for short* S-integrable*) if*

(1) $\int_\Omega \|F(\omega)\| d\nu(\omega)$ *is finite;*

(2) *if* $\nu(O) \approx 0$ *then* $\int_O \|F(\omega)\| d\nu(\omega) \approx 0$.

We denote $SL^p(\Omega) = \{F : \Omega \to {}^*\mathbb{B} : \|F\|^p$ *is S-integrable*$\}$. ◁

The next theorem and its proof is taken from the lecture notes of a course given by Osswald at the University of Munich [BO].

Theorem A.1.14 *A ν_L-measurable function $f : \Omega \to \mathbb{B}$ is Bochner integrable iff it has an S-integrable lifting $F : \Omega \to {}^*\mathbb{B}$. In that case one has*

$$\int_\Omega f(\omega) d\nu_L(\omega) \quad = \quad \mathrm{st}\left(\sum_{\omega\in\Omega} F(\omega)\nu(\omega)\right) .$$

Proof: Let f be Bochner-integrable. First assume that $f = \sum_{k=1}^{n} b_k \chi_{O_k}$ is a simple function. Choose internal $\widetilde{O}_k \in \mathfrak{O}$ with $\nu_L(\widetilde{O}_k \triangle O_k) = 0$. Then $F = \sum_{k=1}^{n} {}^*b_k \chi_{\widetilde{O}_k}$ is internal and $\mathfrak{O}$-measurable. Since $\cup_{k=1}^{n} \widetilde{O}_k \triangle O_k$ is a Loeb nullset we have on the complement of this set that $f(\omega) \approx F(\omega)$. Moreover, F is S-integrable because obviously $\sup_{\omega\in\Omega} \|F(\omega)\|$ is finite. We obtain

$$\int_\Omega f d\nu_L = \sum_{k=1}^{n} b_k \nu_L(O_k) \approx \sum_{k=1}^{n} {}^*b_k \nu(\widetilde{O}_k) = \sum_{\omega\in\Omega} F(\omega)\nu(\omega)\,.$$

Now take arbitrary f. Then there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ converging to f in measure and $\lim_{n,m\to\infty} \int_\Omega \|f_n - f_m\| d\nu_L = 0$. Each f_n has an S-integrable lifting F_n such that

$$\lim_{n,m\to\infty} \mathrm{st}\left(\sum_{\omega\in\Omega} \|F_n(\omega) - F_m(\omega)\|\nu(\omega)\right) \;=\; 0\,. \tag{A.1.19}$$

Since we assume $\mathbb{B}$ to be a separable normed space $\mathbb{B}$ is second countable and thus f has a measurable lifting G by the well-known lifting result. Furthermore, for each $\varepsilon \in \mathbb{R}_+$ we have

$$\begin{aligned} \lim_{n\to\infty} \mathrm{st}(\nu(\{\omega\in\Omega : \|F_n(\omega) - G(\omega)\| \ge 2\varepsilon\})) \;&\le \\ \lim_{n\to\infty} \nu_L(\{\omega\in\Omega : \|f_n(\omega) - f(\omega)\| \ge \varepsilon\}) \;&=\; 0\,. \end{aligned} \tag{A.1.20}$$

By saturation (or countable comprehension) there exists an internal extension $(F_n)_{n\in{}^*\mathbb{N}}$ with each F_n $\mathfrak{O}$-measurable.
Fix some $\varepsilon \in \mathbb{R}_+$. By equation (A.1.20) and the nonstandard criterium of convergence of a sequence there exists an $N_\varepsilon \in {}^*\mathbb{N}_\infty$ such that $\nu(\{\omega : \|F_M(\omega) - G(\omega)\| \ge \varepsilon\}) \approx 0$ for all infinite $M \le N_\varepsilon$. Now define for each $k \in \mathbb{N}$ the sets $N_{\varepsilon,k} = \{n \in {}^*\mathbb{N} : k \le n \le N_\varepsilon\}$. Then the system $\mathcal{N} = \{N_{\varepsilon,k} : \varepsilon \in \mathbb{R}_+,\, k \in \mathbb{N}\}$ has cardinality less or equal than $\aleph_1$ and fulfills the finite intersection property. Since each $N_{\varepsilon,k}$ is internal by saturation there exists an $N \in \cap\mathcal{N}$ with $N \in {}^*\mathbb{N}_\infty$ and $N \le N_\varepsilon$ for every $\varepsilon \in \mathbb{R}_+$. Thus we get $\nu(\{\omega : \|F_M(\omega) - G(\omega)\| \ge \varepsilon\}) \approx 0$ for all infinite $M \le N$ and every $\varepsilon \in \mathbb{R}_+$. This shows that F_M is a lifting of f for every $M \in {}^*\mathbb{N}_\infty$ with $M \le N$.
Now by equation (A.1.19) we see that $\sum_{\omega\in\Omega} \|F_M(\omega) - F_{M'}(\omega)\|\nu(\omega) \approx 0$ for all $M, M' \in {}^*\mathbb{N}_\infty$. But this ensures the existence of a subsequence $(F_{n(k)})_{k\in\mathbb{N}}$ of $(F_n)_{n\in\mathbb{N}}$ such that $\sum_{\omega\in\Omega} \|F_M(\omega) - F_{n(k)}(\omega)\|\nu(\omega) < \frac{1}{k}$ for every infinite $M \le N$ and all $k \in \mathbb{N}$. But every $F_{n(k)}$ is internal and S-integrable which shows that F_M is S-integrable for infinite $M \le N$. Thus for some fixed infinite $M \le N$ the internal function $F = F_M$ is an S-integrable lifting of f and we obtain

$$\begin{aligned} \left\| \int_\Omega f d\nu_L - \mathrm{st}\left(\sum_{\omega\in\Omega} F(\omega)\nu(\omega)\right)\right\| \;&\le \\ \left\| \int_\Omega f d\nu_L - \mathrm{st}\left(\sum_{\omega\in\Omega} F_{n(k)}(\omega)\nu(\omega)\right)\right\| \;&+\; \mathrm{st}\left(\sum_{\omega\in\Omega} \|F_{n(k)}(\omega) - F(\omega)\|\nu(\omega)\right) \overset{k\to\infty}{\longrightarrow} 0\,. \end{aligned}$$

This shows $\int_\Omega f d\nu_L = \mathrm{st}\left(\sum_{\omega\in\Omega} F(\omega)\nu(\omega)\right)$.

Now we prove the converse. Assume that f has an S-integrable lifting F. Then $F \approx f$ ν_L-almost everywhere. But this means that for every $n \in \mathbb{N}$ there exists an $O_n \in \mathfrak{O}$ such that $\nu(O_n) < \frac{1}{n}$ and $F(\omega) \approx f(\omega)$ for all $\omega \in \Omega \setminus O_n$. The sequence $(O_n)_{n\in\mathbb{N}}$ can be taken as decreasing. Fix some $n \in \mathbb{N}$. Then the set $\{F(\omega) : \omega \in \Omega \setminus O_n\}$ is an internal set of nearstandard points of ${}^*\mathbb{B}$ and thus by a well known result in nonstandard topology the set

$$K_n \;=\; \{f(\omega) : \omega \in \Omega \setminus O_n\} \;=\; \{\mathrm{st}(F(\omega)) : \omega \in \Omega \setminus O_n\}$$

is compact. We take a covering of K_n by a finite collection of standard open balls $\{B_i : 1 \le i \le m\}$, each B_i having radius $r_i < \frac{1}{2n}$ and with center $b_i \in \mathbb{B}$. Then

$$\{F(\omega) : \omega \in \Omega \setminus\ O_m\} \subseteq \cup_{i=1}^m {}^*B_i\,.$$

We construct now for each $n \in \mathbb{N}$ a simple function as follows.
For $\omega \in O_n$ set $f_n(\omega) = 0$. For $\omega \notin O_n$ set $f_n(\omega) = b_k$ if $F(\omega) \in {}^*B_k$ and $F(\omega) \notin \cup_{i=1}^{k-1} {}^*B_i$. This defines f_n on Ω and we obtain for every $\omega \in \Omega \setminus O_n$ that

$$\|f(\omega) - f_n(\omega)\| \;\le\; \|f(\omega) - \mathrm{st}(F(\omega))\| \,+\, \|\mathrm{st}(F(\omega)) - f_n(\omega)\| \;<\; \frac{1}{n}\,.$$

This gives

$$\nu_L(\{\omega : \|f_n(\omega) - f(\omega)\| \ge \frac{1}{n}\}) \;\le\; \nu_L(O_n) \;=\; \mathrm{st}(\nu(O_n)) \;\le\; \frac{1}{n}\,.$$

Thus (f_n) converges in measure to f.
Now fix $\varepsilon \in \mathbb{R}_+$. Since F is S-integrable there is a $\delta \in \mathbb{R}_+$ such that

$$\sum_{\omega\in O} \|F(\omega)\|\nu(\omega) < \varepsilon \quad \text{for every } O \in \mathfrak{O} \text{ with } \nu(O) < \delta\,.$$

Then choose some $n_0 \in \mathbb{N}$ with $\frac{1}{n_0} < \delta$. It follows for each $n \in \mathbb{N}$, $n \ge n_0$ that

$$\begin{aligned}\int_\Omega \|f_n - F\| d\nu \;&=\; \int_{O_n} \|f_n - F\| d\nu + \int_{\Omega\setminus O_n} \|f_n - F\| d\nu \\ &=\; \int_{O_n} \|F\| d\nu + \int_{\Omega\setminus O_n} \|f_n - F\| d\nu \;=\; \varepsilon + \frac{1}{n}\nu(\Omega)\end{aligned}$$

where we have used $f_n|_{O_n} \equiv 0$, $\mu(O_n) < \delta$ and $\|F(\omega) - f_n(\omega)\| < \frac{1}{n}$ for $\omega \in \Omega \setminus O_n$. Therefore for $n, m \in \mathbb{N}$, $n, m \ge n_0$ we obtain

$$\int_\Omega \|f_n - f_m\| d\nu_L \;=\; \mathrm{st}\left(\int_\Omega \|f_n - f_m\| d\nu\right) \;\le\; 2\varepsilon + \left(\frac{1}{n} + \frac{1}{m}\right)\mathrm{st}(\nu(\Omega))\,.$$

This shows that $\lim_{n,m\to\infty} \int_\Omega \|f_n - f_m\| d\nu_L = 0$. □

Now we are ready to use this result in our case. We take as standard space of Bochner square integrable functions the Fock space $L^2_{\mathcal{K}}(\mathcal{P}_{\mathrm{fin}}, \Lambda)$ with initial Hilbert space $\mathcal{K}$. Then this is nothing else then the completion of $\mathcal{K} \otimes L^2_{\mathbb{C}}(\mathcal{P}_{\text{fin}}, \Lambda)$. The inner product and the norm in this space are given by

$$\begin{aligned} \langle f, g \rangle &= \int_{\mathcal{P}_{\text{fin}}} \langle f(\sigma), g(\sigma) \rangle d\Lambda(\sigma) \quad \text{for} \quad f, g \in L^2_{\mathcal{K}}(\mathcal{P}_{\text{fin}}, \Lambda)\,, \\ \|f\| &= \left(\int_{\mathcal{P}_{\text{fin}}} \|f(\sigma)\|^2 d\Lambda(\sigma) \right)^{\frac{1}{2}} \quad \text{for} \quad f \in L^2_{\mathcal{K}}(\mathcal{P}_{\text{fin}}, \Lambda)\,. \end{aligned}$$

In the literature this is sometimes also called the *direct integral* of Hilbert spaces and written $\int^{\oplus}_{\mathcal{P}_{\text{fin}}} \mathcal{K} d\Lambda$. By the previous theorem and by the properties of our standard part map (see proposition 3.1.8) we have the following proposition.

Proposition A.1.15 *A Λ-measurable function $f : \mathcal{P}_{\text{fin}} \to \mathcal{K}$ is in $L^2_{\mathcal{K}}(\mathcal{P}_{\text{fin}}, \Lambda)$ iff it has an SL^2-integrable lifting $F : \Gamma \to {}^*\mathcal{K}$ with respect to the norm topology. In this case we have*

$$\begin{aligned} \int_{\mathcal{P}_{\text{fin}}} \|f(\sigma)\|^2 d\Lambda(\sigma) &= \operatorname{st}\left(\sum_{\sigma \in \Gamma} \|F(\sigma)\|^2 m(\sigma) \right)\,, \\ \int_{\mathcal{P}_{\text{fin}}} f(\sigma) d\Lambda(\sigma) &= \operatorname{st}\left(\sum_{\sigma \in \Gamma} F(\sigma) m(\sigma) \right) \quad \textit{and} \\ \int_{\mathcal{P}_{\text{fin}}} \langle f(\sigma), g(\sigma) \rangle d\Lambda(\sigma) &= \operatorname{st}\left(\sum_{\sigma \in \Gamma} \langle F(\sigma), G(\sigma) \rangle m(\sigma) \right)\,. \end{aligned}$$

with $g \in L^2_{\mathcal{K}}(\mathcal{P}_{\text{fin}}, \Lambda)$ and G an SL^2-lifting of g.

Proof: It remains to prove the first equality. Then the third equality follows by polarization. Since f is Bochner square integrable we know that the numerical function $k : \mathcal{P}_{\text{fin}} \to \mathbb{R} : \sigma \mapsto \|f(\sigma)\|$ is square integrable. Then by the usual lifting result there is a SL^2-lifting $K : \Gamma \to {}^*\mathbb{R}$ such that

$$\int_{\mathcal{P}_{\text{fin}}} k(\sigma)^2 d\Lambda(\sigma) \approx \sum_{\sigma \in \Gamma} K(\sigma)^2 m(\sigma)\,.$$

Let F be the 'Bochner' lifting of f. Then $f(\operatorname{st}(\sigma)) \approx F(\sigma)$ m_L-almost everywhere on Γ and also $\|f(\operatorname{st}(\sigma))\|^2 \approx \|F(\sigma)\|^2$ m_L-almost everywhere. But this shows that $K(\sigma)^2 \approx \|F(\sigma)\|^2$ m_L-almost everywhere and since both functions are SL^2 we have

$$\sum_{\sigma \in \Gamma} \|F(\sigma)\|^2 m(\sigma) \approx \sum_{\sigma \in \Gamma} K(\sigma)^2 m(\sigma)\,. \qquad \square$$

2 A Lifting Theorem for $\mathcal{B}(\mathcal{K})$

Let $(\Omega, \mathfrak{D}, \nu)$ be an internal hyperfinite measure space and $(\Omega, L(\mathfrak{D}), \nu_L)$ the corresponding Loeb space. Suppose that $\nu(\Omega)$ is finite and $\mathfrak{D}$ is a ${}^*\sigma$-algebra. Further, let $\mathcal{K}$ be a separable Hilbert space. We prove in this section a lifting theorem for functions $f : \Omega \to \mathcal{B}(\mathcal{K})$ where $\mathcal{B}(\mathcal{K})$ is given the weak operator topology.

Definition A.2.1 *A function $f : \Omega \to \mathcal{B}(\mathcal{K})$ is called* woply measurable *(or* measurable in the weak operator topology*) if for all $x, y \in \mathcal{K}$ the functions*

$$f_{xy} \;:\; \Omega \longrightarrow \mathbb{C} \;:\; \omega \longmapsto \langle x, f(\omega) y\rangle$$

are Loeb measurable. We speak also of wop measurability*.* ◁

We could have called this measurability also weak measurability but this is in conflict with the traditional notion where weak measurability refers to measurability with respect to the topology induced by the dual Banach space.

Equivalently we have the following characterization of wop measurability: A function $f : \Omega \to \mathcal{B}(\mathcal{K})$ is woply measurable iff for every finite rank operator T the functions

$$f_T \;:\; \Omega \longrightarrow \mathbb{C} \;:\; \omega \longmapsto \mathsf{tr}(Tf(\omega))$$

are Loeb measurable. (Here $\mathsf{tr}(T)$ denotes the trace of the operator T.) Now the finite rank operators are dense (with respect to the trace norm) in the Banach space of the trace class operators. But the trace class operators $\mathcal{T}(\mathcal{K})$ are the predual of $\mathcal{B}(\mathcal{K})$ via the bilinear form $\mathsf{tr}(TK)$ (for $T \in \mathcal{T}(\mathcal{K})$ and $K \in \mathcal{B}(\mathcal{K})$). This shows that every woply measurable function f is also weak* measurable in the following sense.

Definition A.2.2 *A function $f : \Omega \to \mathcal{B}(\mathcal{K})$ is* weak* measurable *if for every trace class operator $T \in \mathcal{T}(\mathcal{K})$ the functions*

$$f_T \;:\; \Omega \longrightarrow \mathbb{C} \;:\; \omega \longmapsto \mathsf{tr}(Tf(\omega))$$

are Loeb measurable. ◁

Proposition A.2.3 *Every woply measurable function $f : \Omega \to \mathcal{B}(\mathcal{K})$ is weak* measurable.*

Proof: Choose some arbitrary $T \in \mathcal{T}(\mathcal{K})$. Then there exists a sequence of finite rank operators $(T_n)_{n\in\mathbb{N}}$ that converges in trace norm to T. This shows that for each $\omega \in \Omega$ we have $\mathsf{tr}(T_n f(\omega)) \stackrel{n\to\infty}{\longrightarrow} \mathsf{tr}(Tf(\omega))$. Hence $\mathsf{tr}(Tf(\,\cdot\,))$ is measurable as a pointwise limit of measurable functions. □

Next we prove a lifting theorem for woply measurable functions. Two preparatory lemmas are needed.

Lemma A.2.4 *Suppose that $f : \Omega \to \mathcal{B}(\mathcal{K})$ is woply measurable with $f(\Omega) \subseteq B$ for some norm bounded subset $B \subset \mathcal{B}(\mathcal{K})$. Then f has a lifting $F : \Omega \to {}^*\mathcal{B}(\mathcal{K})$ with respect to the weak operator topology. Furthermore, we have $F(\Omega) \subseteq {}^*B$.*

Proof: Since $f(\Omega) \subseteq B$ we can restrict to B. But as a norm bounded set B is metrizable and thus second countable in the weak operator topology. By the well known lifting theorem (see proposition 3.2.2) $f : \Omega \to B$ has a lifting $F : \Omega \to {}^*B \subseteq {}^*\mathcal{B}(\mathcal{K})$. □

Lemma A.2.5 *Let $f : \Omega \to \mathcal{B}(\mathcal{K})$ be woply measurable and B some closed norm bounded ball in $\mathcal{B}(\mathcal{K})$. Set $A = f^{-1}(B)$ and χ_A the characteristic function of A. Then $f\chi_A$ is woply measurable.*

PROOF: B is compact in the weak operator topology since it is norm bounded. Thus B is weakly closed, weakly separable and convex. Hence $A = f^{-1}(B) \in L(\mathfrak{D})$ by wop measurability of f. This shows the measurability of χ_A. Since pointwise

$$\langle x, f(\omega)\chi_A(\omega)y\rangle \;=\; \chi_A(\omega)\langle x, f(\omega)y\rangle \quad \text{for all } \; x, y \in \mathcal{K}$$

we see that $f\chi_A$ is woply measurable. □

Notation A.2.6 Let $f : \Omega \to \mathcal{B}(\mathcal{K})$ be a function. We denote by

$$\operatorname{supp}(f) \;=\; \{\omega \in \Omega : f(\omega) \neq 0\}$$

the *non-nullset* of f. Note that $\operatorname{supp}(f)$ is not necessarily closed. ◁

Theorem A.2.7
Every woply measurable function $f : \Omega \to \mathcal{B}(\mathcal{K})$ has a lifting $F : \Omega \to {}^*\mathcal{B}(\mathcal{K})$, i.e.

$$f(\omega) \;=\; \operatorname{st}(F(\omega)) \;\textit{ for } \nu_L\textit{-almost all } \;\omega \in \Omega$$

where the standard part is taken with respect to the weak operator topology. In addition, $F(\omega)$ is S-bounded for ν_L-almost all $\omega \in \Omega$.

PROOF: First set for each $n \in \mathbb{N}$

$$B_n \;=\; \{T \in \mathcal{B}(\mathcal{K}) : \|T\| \le n\}$$

and $S_0 = B_0$ and $S_{n+1} = B_{n+1} \setminus B_n$. Then by the previous lemma we see that

$$A_n \;=\; f^{-1}(S_n) \;=\; f^{-1}(B_n) \setminus f^{-1}(B_{n-1}) \in L(\mathfrak{D})$$

and thus $f_n = f\chi_{A_n}$ is woply measurable. Since $\mathcal{B}(\mathcal{K}) = \dot{\cup}_{n\in\mathbb{N}} S_n$ we have $\Omega = \cup_{n\in\mathbb{N}} A_n$ where $A_n \cap A_m = \emptyset$ if $n \neq m$. (It can happen that $A_n = \emptyset$ for some but not all $n \in \mathbb{N}$.) Then $f = \sum_{n\in\mathbb{N}} f_n$ and $f_n(\Omega) \subseteq S_n \cup \{0\} \subseteq B_n \subset \mathcal{B}(\mathcal{K})$. By lemma A.2.4 we have a lifting $\widetilde{F}_n : \Omega \to {}^*B_n \subset {}^*\mathcal{B}(\mathcal{K})$ of f_n for each $n \in \mathbb{N}$. By Loeb measure theory there are internal sets $\widetilde{E}_n \in \mathfrak{D}$ with $\nu_L(\widetilde{E}_n \bigtriangleup A_n) = 0$. Setting $E_0 = \widetilde{E}_0$ and $E_{n+1} = \widetilde{E}_{n+1} \setminus (E_0 \cup \cdots \cup E_n)$ we have also $E_n \cap E_m = \emptyset$ for $n \neq m$ and $\nu_L(E_n \bigtriangleup A_n) = 0$ continues to hold. Now set $F_n = \widetilde{F}_n \chi_{E_n}$. Then F_n is a lifting for $f_n = f\chi_{A_n}$ and in addition $\operatorname{supp}F_n \cap \operatorname{supp}F_m = \emptyset$ if $n \neq m$. Define

$$N_n \;=\; \{\omega \in \Omega : F_n(\omega) \not\approx f_n(\omega)\}\,.$$

By construction N_n is a Loeb nullset. Extend by comprehension the sequence $(F_n)_{n\in\mathbb{N}}$ to an internal sequence $(F_n)_{n\in{}^*\mathbb{N}}$ in such a way that each F_n is *-measurable. Set

$$G \;=\; \{k \in {}^*\mathbb{N} : \forall n, m \in {}^*\mathbb{N}\, (n \le k \wedge m \le k \wedge n \neq m \Rightarrow \operatorname{supp}F_n \cap \operatorname{supp}F_m = \emptyset\}\,.$$

Then by Keisler's internal definition principle G is internal and by construction $\mathbb{N} \subseteq G$. Thus by saturation there exists some $M \in {}^*\mathbb{N}_\infty$ such that $\{0, 1, \cdots, M\} \subseteq G$. Now define $F = \sum_{k=0}^{M} F_k$ for this M. Then F is internal, *-measurable and $\mathrm{supp} F_k \cap \mathrm{supp} F_l = \emptyset$ for all $k, l \in {}^*\mathbb{N}$ with $k < l \leq M$. We show that F is a lifting for f. Define $N = \cup_{n \in \mathbb{N}} N_n$, $D = \cup_{n \in \mathbb{N}} A_n \triangle E_n$ and $V = \Omega \setminus (N \cup D)$. Then obviously $\nu_L(V) = \nu_L(\Omega)$. Let $\omega \in V$. Since $\Omega = \dot{\cup}_{n \in \mathbb{N}} A_n$ there exists an $m \in \mathbb{N}$ such that $\omega \in A_m$. By definition of V this implies $\omega \in E_m$. But then we have

$$f(\omega) \;=\; f_m(\omega) \;\approx\; F_m(\omega) \;=\; F(\omega)$$

and this proves that F is a lifting of f in the weak operator topology of $\mathcal{B}(\mathcal{K})$. Further, for ν_L-almost all $\omega \in \Omega$ we have $F(\omega) = F_m(\omega)$ for some $m \in \mathbb{N}$ and $F_m(\omega)$ is S-bounded with bound m. Thus $F(\omega)$ is S-bounded for ν_L-almost all $\omega \in \Omega$. $\square$

Corollary A.2.8 *Every weak* measurable function $f : \Omega \to \mathcal{B}(\mathcal{K})$ has a lifting $F : \Omega \to {}^*\mathcal{B}(\mathcal{K})$. Further, we can take the standard part on ${}^*\mathcal{B}(\mathcal{K})$ with respect to the weak* topology.*

Proof: Obviously the proof carries over since every norm bounded ball in $\mathcal{B}(\mathcal{K})$ is also weak* compact and thus weak* closed. Further, on bounded subsets the weak operator and weak* topology coincide and by proposition A.2.3 the concepts of wop measurability and weak* measurability are the same. But the predual of $\mathcal{B}(\mathcal{K})$ is separable also if $\mathcal{K}$ is separable. (The predual is the Banach space of trace class operators in the trace norm.) $\square$

Indeed the assertion of theorem A.2.7 is valid if we replace $\mathcal{B}(\mathcal{K})$ with an arbitrary dual Banach space $\mathbb{B}^\star$ in the weak* topology such that $\mathbb{B}$ is separable. The given proof applies with the obvious changes to this situation.

Bibliography

[ABS87] S. Albeverio, Ph. Blanchard, and L. Streit, editors. *Stochastic Processes – Mathematics and Physics II. Proceedings, Bielefeld 1985*, LNM 1250, Berlin, 1987. Springer.

[Acc91] L. Accardi, editor. *Quantum Probability & Related Topics, Vol. VI*, QP-PQ VI, Singapore, 1991. World Scientific.

[Acc93] L. Accardi, editor. *Quantum Probability & Related Topics, Vol. VIII*, QP-PQ VIII, Singapore, 1993. World Scientific.

[Acc98] L. Accardi, editor. *Quantum Probability Communications, Vol.X*, QP-PQ X, Singapore, 1998. World Scientific.

[ACH97] L.O. Arkeryd, N.C. Cutland, and C.W. Henson, editors. *Nonstandard Analysis. Theory and Applications*, NATO ASI Series C: Mathematical and Physical Sciences, Vol. 493, Dordrecht, 1997. Kluwer Acad. Publ.

[ACM86] S. Albeverio, G. Casati, and D. Merlin, editors. *Stochastic Processes in Classical and Quantum Systems, Proceedings Ascona (CH) 1985*, LNP 262, Berlin, 1986. Springer.

[AFHKL86] S. Albeverio, J.E. Fenstad, R. Høegh-Krohn, and T. Lindstrøm. *Nonstandard Methods in Stochastic Analysis and Mathematical Physics.* Academic Press, Orlando, Florida, 1986.

[AH98] L. Accardi and C.C. Heyde, editors. *Probability towards 2000 - Papers from the Symposium held at Columbia University, New York, October 2-6, 1995*, New York, 1998. Springer.

[AL97] S. Attal and J.M. Lindsay. Quantum stochastic calculus with maximal operator domains. Preprint, 1997. To appear in Annals of Probability. Identical to [AL99] `http://www.maths.nottingham.ac.uk/personal/jmlsjw/QPweb/papers.html`.

[AL99] S. Attal and J.M. Lindsay. Quantum stochastic calculus - A maximal formulation. *Prépubl. Inst. Fourier, Grenoble*, 456, 1999. Identical to [AL97] `http://www-fourier.ujf-grenoble.fr/prepublications.html`.

[ALV98] L. Accardi, Y.-G. Lu, and I. Volovich. Non-linear extensions of classical and quantum stochastic calculus and essentially infinite dimensional analysis. In Accardi and Heyde [AH98], pages 1–33.

[ALW95] S.A. Albeverio, W.A.J. Luxemburg, and M.P.H. Wolff, editors. *Advances in Analysis, Probability and Mathematical Physics. Contributions of Nonstandard Analysis*, Mathematics and Its Applications 314, Dordrecht, 1995. Kluwer Academic Publ.

[AMY90] J. Azéma, P.A. Meyer, and M. Yor, editors. *Séminaire de Probabilités XXIV 1988/89*, LNM 1426, Berlin, 1990. Springer.

[AMY91] J. Azéma, P.A. Meyer, and M. Yor, editors. *Séminaire de Probabilités XXV*, LNM 1485, Berlin, 1991. Springer.

[AMY93] J. Azéma, P.A. Meyer, and M. Yor, editors. *Séminaire de Probabilités XXVII*, LNM 1557, Berlin, 1993. Springer.

[And76] R.M. Anderson. A non-standard representation for Brownian motion and Itô integration. *Israel J. Math.*, 25:15–46, 1976.

[And82] R.M. Anderson. Star-finite representations of measure spaces. *Trans. AMS*, 271:667–687, 1982.

[Att94] S. Attal. An algebra of non-commutative bounded semimartingales: square and angle quantum brackets. *J. Funct. Anal.*, 124:292–332, 1994.

[Att96a] S. Attal. Classical and quantum stochastic calculus. *Prépubl. Inst. Fourier, Grenoble*, 358, 1996. Preprint Version of [Att98], `http://www-fourier.ujf-grenoble.fr/prepublications.html`.

[Att96b] S. Attal. Non-commutative chaotic expansion of Hilbert-Schmidt operators on Fock space. *Commun. Math. Phys.*, 175:43–62, 1996.

[Att98] S. Attal. Classical and quantum stochastic calculus. In Accardi [Acc98], pages 1–52. Available also as Preprint [Att96a].

[Att99] S. Attal. The structure of the quantum semimartingale algebras. *Prépubl. Inst. Fourier, Grenoble*, 457, 1999. `http://www-fourier.ujf-grenoble.fr/prepublications.html`.

[Att00] S. Attal. Approximating the Fock space with the toy Fock space. *Prépubl. Inst. Fourier, Grenoble*, 495, 2000. `http://www-fourier.ujf-grenoble.fr/prepublications.html`.

[AV97] L. Accardi and I.V. Volovich. Quantum white noise with singular non-linear interaction. Preprint, 1997. arXiv: quant-ph/9704029 16 Apr 1997.

[AW66] H. Araki and E.J. Woods. Complete Boolean algebras of type I factors. *Publ. RIMS, Ser. A*, 2,2:157–242, 1966.

[AW69] H. Araki and E.J. Woods. A classification of factors. *Publ. RIMS, Ser. A*, 3:51–130, 1969.

[AW85] L. Accardi and W.v. Waldenfels, editors. *Quantum Probability and Applications II. Proceedings, Heidelberg 1984*, LNM 1136, Berlin, 1985. Springer.

[AW88] L. Accardi and W.v. Waldenfels, editors. *Quantum Probability and Applications III. Proceedings, Oberwolfach 1987*, LNM 1303, Berlin, 1988. Springer.

[BD95] P. Biane and R. Durrett. *Lectures on Probability Theory,* Ecole d'Eté de Probabilités de Saint-Flour XXIII-1993, ed. P. Bernard. LNM 1608. Springer, Berlin, 1995.

[Bia90] P. Biane. Marches de Bernoulli quantiques. In Azéma et al. [AMY90], pages 329–344.

[Bia91] P. Biane. Some properties of quantum Bernoulli random walks. In Accardi [Acc91], pages 193–203.

[Bia95] P. Biane. Calcul stochastique non-commutatif. In Biane and Durrett [BD95], pages 1–96.

[BO] J. Berger and H. Osswald. An introduction to malliavin calculus in abstract Wiener space. Unpublished manuscript of a lecture held by H. Osswald at University of Munich, Wintersemester 2000/2001.

[CJO98] D.M. Chung, U.C. Ji, and N. Obata. Higher powers of quantum noises in terms of integral kernel operators. *Inf. Dim. Anal. Quant. Prob. Rel. Top.*, 1:533–559, 1998.

[Cut88] N. Cutland, editor. *Nonstandard Analysis and Its Applications*, LMS Student Texts 10, Cambridge, 1988. Cambridge Univ. Press.

[Cut00] N. Cutland. *Loeb Measures in Practice: Recent Advances.* LNM 1751. Springer, Berlin, Heidelberg, 2000.

[Dav76] E.B. Davies. *Quantum Theory of Open Systems.* Academic Press, London, 1976.

[DU77] J. Diestel and J.J. Uhl. *Vector Measures.* Mathematical Surveys, No. 15. AMS, Providence, Rhode Island, 1977.

[Fag98] F. Fagnola. Quantum stochastic differential equations. Copy of a lecture note, June 1998. Summerschool on Quantum Probability, Grenoble.

[Fri85] A. Frigerio. Construction of stationary quantum Markov processes through quantum stochastic calculus. In Accardi and Waldenfels [AW85], pages 207–222.

[FW99] F. Fagnola and S.J. Wills. Solving quantum stochastic differential equations with unbounded coefficients. Preprint, 1999.

[FW00] F. Fagnola and S.J. Wills. Mild solutions of quantum stochastic differential equations. *Elect. Comm. in Probab.*, 5:158–177, 2000. `http://www.math.washington.edu/~ejpecp`.

[G$^+$90] B. Grigelionis et al., editors. *Proceedings of the Vth International Vilnius Conference on Probability and Mathematical Statistics, 1989*, volume 2 of *Probability Theory and Mathematical Statistics*, Utrecht, 1990. VSP/Mosklas.

[GL98] M. de Giosa and Y.G. Lu. From quantum Bernoulli process to creation and annihilation operators in interacting q-Fock space. *Japan. J. Math.*, 24:149–167, 1998.

[GS85] T. Giordano and G. Skandalis. On infinite tensorproducts of factors of type I_2. *Ergodic Theory Dyn. Syst.*, 5:565–586, 1985.

[Gud94] S. Gudder. Nonstandard Fock spaces. *Int. J. Theoret. Physics*, 33:865–912, 1994.

[Gud96] S. Gudder. Hyperfinite quantum random walks. *Chaos, Solitons & Fractals*, 7:669–679, 1996.

[Gui72] A. Guichardet. *Symmetric Hilbert Spaces and Related Topics.* LNM 261. Springer, Berlin, 1972.

[GZ98] E.I. Gordon and M.Y. Zdorovenko. Nonstandard hulls of unbounded symmetric operators. *Siberian Math. J.*, 39:446–459, 1998.

[HL85] A.E. Hurd and P.A. Loeb. *An Introduction to Nonstandard Real Analysis.* Academic Press, Orlando, 1985.

[HP84] R.L. Hudson and K.R. Parthasarathy. Quantum Itô's formula and stochastic evolutions. *Comm. Math. Physics*, 93:311–323, 1984.

[HS81] R.L. Hudson and R.F. Streater. Itô's formula is the chain rule with Wick ordering. *Physics Letters*, 86A(5):277–279, 1981.

[Hua93] Z. Huang. Quantum white noises - white noise approach to quantum stochastic calculus. *Nagoya Math. J.*, 129:23–42, 1993.

[Hud] R.L. Hudson. An Introduction to Quantum Probability with Quantum Stochastic Calculus. Forthcoming book.

[Hud96] R.L. Hudson. Quantum stochastic calculus, evolutions and flows. *Proc. Symp. Pure Math.*, 59:81–91, 1996.

[Kei84] H.J. Keisler. *An infinitesimal approach to stochastic analysis.* Memoirs AMS Vol. 48 No. 297. AMS, Providence, Rhode Island, 1984.

[KP00] J. Kellner and H. Ploss. F-products and nonstandard hulls for strongly continuous one-parameter semigroups of linear operators. Preprint, 2000. arXiv: math.FA/0005130v1 12 May 2000.

[Kru90] A. Krupa. On various generalizations of the notion of an $\mathcal{F}$-power to the case of unbounded operators. *Bull. Pol. Acad. Sci. Math.*, 38:159–166, 1990.

[Len98] R. Lenczewski. Unification of independence in quantum probability. *Inf. Dim. Anal. Quantum Prob. Rel. Topics*, 1:383–405, 1998.

[Lin88] T. Lindstrøm. An invitation to nonstandard analysis. In Cutland [Cut88], pages 1–105.

[Lin90] J.M. Lindsay. On set convolutions and integral-sum kernel operators. In Grigelionis et al. [G^+90], pages 105–123.

[Lin93a] J.M. Lindsay. The kernel of a Fock space operator I. In Accardi [Acc93], pages 271–280.

[Lin93b] J.M. Lindsay. Quantum and non-causal calculus. *Prob. Th. Rel. Fields*, 97:65–80, 1993.

[Lin98] J.M. Lindsay. Integral-sum kernel operators. Copy of a lecture note, June 1998. Summerschool on Quantum Probability, Grenoble.

[LM] J.M. Lindsay and H. Maassen. The stochastic calculus of Bose noise. Must be a former version of [LM92].

[LM88] J.M. Lindsay and H. Maassen. An integral kernel approach to noise. In Accardi and Waldenfels [AW88], pages 192–208.

[LM92] J.M. Lindsay and H. Maassen. Stochastic calculus for quantum Brownian motion of nonminimal variance – an approach using integral-sum kernel operators. *CWI Syllabi*, 32, 1992. Math. Centrum, Centrum Wisk. Inform., Amsterdam.

[LM00] M. Leitz-Martini. A discrete Clark-Ocone formula. *MaPhySto Research Report*, 29, July 2000. `http://www.maphysto.dk`.

[LO97] P.A. Loeb and H. Osswald. Nonstandard integration theory in topological vector lattices. *Monatshefte f. Mathe.*, 124:53–82, 1997.

[Loe75] P.A. Loeb. Conversion from nonstandard to standard measure spaces and applications in probability theory. *Trans. AMS*, 211:113–122, 1975.

[LP88] J.M. Lindsay and K.R. Parthasarathy. The passage from random walk to diffusion in quantum probability, II. *Sankhya, Ser. A*, 50:151–170, 1988.

[LR94] D. Landers and L. Rogge. *Nichtstandard-Analysis*. Springer, Berlin, 1994.

[LW00] P.A. Loeb and M. Wolff, editors. *Nonstandard Analysis for the Working Mathematician*. Mathematics and Its Applications 510. Kluwer Academic Publisher, Dordrecht, 2000.

[Maa85] H. Maassen. Quantum Markov processes on Fock space described by integral kernels. In Accardi and Waldenfels [AW85], pages 361–374.

[Mab95] M. Mabrouki. Limit theorems for the spin process. *Contemporary Mathematics*, 183:245–259, 1995.

[Mey86] P.A. Meyer. A finite approximation to Boson Fock space. In Albeverio et al. [ACM86], pages 405–410.

[Mey87] P.A. Meyer. Fock space and probability theory. In Albeverio et al. [ABS87], pages 160–170.

[Mey91] P.A. Meyer. Application du "bébé Fock" au modèle d'Ising. In Azéma et al. [AMY91], pages 52–60.

[Mey93a] P.-A. Meyer. *Quantum Probability for Probabilists*. LNM 1538. Springer, Berlin, 1993.

[Mey93b] P.A. Meyer. Représentation de martingales d'opérateurs d'après Parthasarathy-Sinha. In Azéma et al. [AMY93], pages 97–105.

[Ng00] S.-A. Ng. White noise analysis in a nonstandard model. Preprint, 2000.

[Oba95] N. Obata. Generalizations of quantum stochastic processes on Fock space. *Publ. RIMS*, 31:667–702, 1995.

[Oba96] N. Obata. White noise approach to quantum martingales. In Watanabe et al. [WFPS96], pages 379–386.

[Oba97] N. Obata. Integral kernel operators on Fock space - generalizations and applications to quantum dynamics. *Acta Appl. Math.*, 47:49–77, 1997.

[Oba99] N. Obata. White noise operator theory and applications. Trento Summer School 99, July 1999.

[OS] H. Osswald and Y. Sun. Measure and probability theory and applications. Part III in [LW00], pages 137-257.

[Par88] K.R. Parthasarathy. The passage from random walk to diffusion in quantum probability. *J. Appl. Prob.*, 25A:231–245, 1988. Celebration Volume in Applied Probability, ed. J. Gani, Applied Probability Trust, Sheffield.

[Par92] K.R. Parthasarathy. *An Introduction to Quantum Stochastic Calculus.* Monographs in Mathematics Vol. 85. Birkhäuser, Basel, 1992.

[Par98] K.R. Parthasarathy. Notes on the symmetric group action in a toy Fock space. In Accardi [Acc98], pages 125–133.

[Ped89] G.K. Pedersen. *Analysis Now.* GTM 118. Springer, New York, 1989.

[PS86] K.R. Parthasarathy and K.B. Sinha. Stochastic integral representation of bounded martingales in Fock space. *J. Funct. Anal.*, 67:126–151, 1986.

[Rob66] A. Robinson. *Nonstandard Analysis.* North-Holland, Amsterdam, 1966.

[Ros90] D. Ross. Lifting theorems in nonstandard measure theory. *Proc. AMS*, 109:809–822, 1990.

[Ros96] D. Ross. Unions of Loeb nullsets. *Proc. AMS*, 124:1883–1888, 1996.

[Sch93] M. Schürmann. *White Noise on Bialgebras.* LNM 1544. Springer, Berlin, 1993.

[Sch95] M. Schürmann. Direct sums of tensor products and non-commutative independence. *J. Funct. Anal.*, 133:1–9, 1995.

[Seo97] Y. Seo. Independence in non-commutative probability. *Math. Japon.*, 46:445–450, 1997.

[Spe97] R. Speicher. On universal products. *Fields Inst. Commun.*, 12:257–266, 1997.

[Tak79] M. Takesaki. *Theory of Operator Algebras I.* Springer, New York, 1979.

[WFPS96] S. Watanabe, M. Fukushima, Y.V. Prohorov, and A.N. Shirayev, editors. *Probability Theory and Mathematical Statistics (Tokyo, 1995)*, Singapore, 1996. World Scientific Publishing.

[Wol] M.P.H. Wolff. Functional analysis. Part II in [LW00], pages 97-136.

[Yam98] H. Yamashita. Hyperfinite-dimensional representations of canonical commutation relation. *J. Mathem. Phys.*, 39 (5):2682–2692, 1998.

[YO00] H. Yamashita and M. Ozawa. Nonstandard representations of the canonical commutation relations. *Rev. Mathem. Physics*, 12:1407–1427, 2000.

[Zim98] G.B. Zimmer. An extension of the Bochner integral generalizing the Loeb-Osswald integral. *Math. Proc. Cambridge Phil. Soc.*, 123:119–131, 1998.

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