On the theory of Fitting classes of finite soluble groups

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STEPHANIE REIFFERSCHEID
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PROF. P. HAUCK

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Dekan: Prof. Dr. C. Lubich
1. Berichterstatter: Prof. Dr. P. Hauck
2. Berichterstatter: Prof. Dr. K. Doerk

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Introduction

One of the most important features of the theory of finite soluble groups is the existence of results generalizing the theorem of Sylow. A prototype is Hall's theorem from 1928 that extends the scope of Sylow's theorem for finite soluble groups from p-groups to π-groups for a set π of primes. More precisely, it states that in each finite soluble group G there exists a unique conjugacy class of so-called Hall π-subgroups of G, subgroups that are maximal among all π-subgroups of G (and their order is just the π-part of the order of G). Finite soluble groups are actually characterized by the existence of Hall π-subgroups for every $\pi \subseteq \mathbb{P}$, as Hall showed in 1937. Let G denote a finite group and π a set of primes. Then it is easily seen that a Hall π-subgroup H of G possesses the following properties: (a) HN/N is a Hall π-subgroup of G/N for every normal subgroup N of G; (b) $H \cap N$ is a Hall π-subgroup of N for every subnormal subgroup N of G. Thus Hall π-subgroups are characterized by each of these properties. It is natural to ask whether it is possible to obtain similar results for other group theoretic properties than the property of being a π-group. To this, collect all groups possessing a given group theoretic property in a class $\mathfrak{F}$ (closed under isomorphisms), and call a subgroup U of a group G an $\mathfrak{F}$-maximal subgroup of G if U is maximal among all subgroups of G belonging to $\mathfrak{F}$ (thus a Hall π-subgroup of a finite soluble group G is an $\mathfrak{S}_\pi$-maximal subgroup of G where $\mathfrak{S}_\pi$ denotes the class of all finite soluble π-groups). It is not hard to see that there is no possibility to generalize all parts of the theorems of Sylow and Hall to other classes of groups than the classes $\mathfrak{S}_\pi$, not even in the universe of finite soluble groups. Trying to obtain weaker results of such type, i.e. results about the existence and conjugacy of ($\mathfrak{F}$-maximal) subgroups of G which possess either a property analogous to (a) or a property analogous to (b) for each finite soluble group G, led to the concepts of projectors and injectors, connected with Schunck and Fitting classes, respectively. This thesis is concerned with the theory of Fitting classes, i.e. classes of groups closed under both taking subnormal subgroups and forming products of normal subgroups. (Fitting classes are named after H. Fitting, who first showed in 1938 that the class of all nilpotent groups is closed under forming products of normal subgroups; evidently, this class is closed under taking subnormal subgroups.) In 1967, Fischer, Gaschütz and Hartley proved that Fitting classes $\mathfrak{F}$ of finite and soluble groups are characterized by the existence of a unique conjugacy class of so-called $\mathfrak{F}$-injectors of G – subgroups U of G such that $U \cap N$ is $\mathfrak{F}$-maximal in N for every subnormal subgroup N of G – in each finite and soluble group G. Since such a result does not hold true – in general – in the universe of finite groups, we confine ourselves in the sequel to the universe of finite and soluble groups. Thus each group considered here is supposed to be finite and soluble, and each class of groups is assumed to be contained in the class $\mathfrak{S}$ of all finite and soluble groups.

In the investigation of Fitting classes, it seems natural to restrict oneself first to Fitting classes satisfying additional conditions related to the behaviour of their injectors in each group $G \in \mathfrak{S}$ – as done for instance by Blessenohl and Gaschütz (1970), Lockett (1971), Doerk and Porta (1980) and Hauck and Kienzle (1987). In the present work we generalize these investigations in the following way: we consider non-trivial Fitting classes $\mathfrak{X}$ and $\mathfrak{F}$ such that $\mathfrak{X}$ is contained in $\mathfrak{F}$ and an $\mathfrak{X}$-injector of G satisfies a given embedding property e in G for every group $G \in \mathfrak{F}$ (in this case we call $\mathfrak{X}$ an $\mathfrak{F}_e$-class). Thus, we study such embedding properties of $\mathfrak{X}$-injectors "locally " in $\mathfrak{F}$, the global case being $\mathfrak{F} = \mathfrak{S}$.
We concentrate on the following embedding properties:

Normality

(Sub)Modularity

Normal embedding

System permutability

Our main interest concerning these relations is in the following questions:
Let e be among the embedding properties listed above.

(1) If $\mathfrak{X}$ is a non-trivial Fitting class, does there always exist a unique maximal Fitting class $\mathfrak{F}$ such that $\mathfrak{X}$ is an $\mathfrak{F}_e$-class?

(2) And vice versa, what conditions must a Fitting class $\mathfrak{F}$ satisfy to possess a unique minimal $\mathfrak{F}_e$-class?

In order to obtain an answer to the first question it seems reasonable to consider the class $\mathrm{Y}_e(\mathfrak{X})$ of all groups G such that an $\mathfrak{X}$-injector of G satisfies a given embedding property e in G. Unfortunately, in general this class is not closed under forming normal products for any of the embedding properties e listed above, and therefore can fail to be a Fitting class. So, in order to decide whether there is a unique maximal Fitting class contained in $\mathrm{Y}_e(\mathfrak{X})$ it would be helpful to have some detailed knowledge of $\mathrm{Fit}(\mathcal{S})$, the Fitting class generated by a given set $\mathcal{S}$ of groups. Regrettably, this class is very hard to deal with – for instance even the problem of finding an effective description of the Fitting class generated by the symmetric group on three elements is still unsolved. For this reason we will often confine ourselves to subgroup-closed Fitting classes (in the following we will refer to these classes as SFitting classes) and to the SFitting class generated by a given set of groups. Since the subgroup-closure of a Fitting class enforces the closure of the class under a number of further closure operations (Bryce and Cossey, 1972, 1982), it is possible to use the theory of (local) formations (see 1.3 for details) in dealing with SFitting classes. This leads to strong results concerning the SFitting class generated by a given set of groups as well as the lattice of SFitting classes. Thus in considering the above listed relations between SFitting classes we might expect stronger results than in the general case.

The basic material about classes of groups needed in the following is presented in Chapter 1. There one will find – among others – the definition of the class $\mathfrak{F}^*$, the smallest Fitting class containing a given Fitting class $\mathfrak{F}$ whose radicals respect direct products, and of the Lockett section of $\mathfrak{F}$, the collection of all Fitting classes $\mathfrak{Y}$ satisfying $\mathfrak{Y}^* = \mathfrak{F}^*$ (see 1.2). (The $\mathfrak{X}$-radical $G_\mathfrak{X}$ of a group G is defined as the unique maximal normal subgroup of G which is contained in $\mathfrak{X}$ where $\mathfrak{X}$ denotes a Fitting class.) If $\mathfrak{F}$ is a Fitting class such that $\mathfrak{F} = \mathfrak{F}^*$, then $\mathfrak{F}$ is called Lockett class.
The definition of local formations – classes of groups constructed via a family of formations, a so-called local definition – is also contained in this chapter (see 1.3). Among all possible local definitions of a local formation there is exactly one that is full and integrated (see 1.3), the so-called canonical local definition, and a number of properties of the class behaves

nicely with respect to it. We will see that the above relations – considered between SFitting classes – too are mirrowed frequently in the corresponding canonical local definitions (and vice versa).

In Chapter 2 we will study the SFitting class generated by a given set of groups as well as the lattice of all SFitting classes. As mentioned before we will need these results in investigating the above listed relations (considered between SFitting classes), but they are of interest also in their own right. Using the theory of (local) formations, we will prove that the SFitting class generated by arbitrary many SFitting classes behaves nicely with respect to intersections and certain extensions. A consequence of these results is that the collection of all SFitting classes forms a distributive lattice – a fact which has already been proved by Shemetkov and Skiba in 1989 ([20, 9.8]). Furthermore, it turns out that this lattice is atomic and that its atoms can be described explicitly.

Chapter 3 is devoted to locally normal Fitting classes, i.e. non-trivial Fitting classes $\mathfrak{X}$ and $\mathfrak{F}$ such that $\mathfrak{X}$ is contained in $\mathfrak{F}$ and that an $\mathfrak{X}$-injector of G is a normal subgroup of G for all $G \in \mathfrak{F}$. (In this situation we refer to $\mathfrak{X}$ as being normal in $\mathfrak{F}$ or being $\mathfrak{F}$-normal.) Obviously, in this case each $\mathfrak{X}$-injector coincides with $G_{\mathfrak{X}}$ for all $G \in \mathfrak{F}$.
This chapter is subdivided in two sections. In the first part, we collect the basic facts on locally normal Fitting classes – most of them proved by Hauck (1977) –, and discuss the above mentioned questions for arbitrary Fitting classes. It is a well-known fact that in this investigation we may assume without loss of generality that both classes under consideration are Lockett classes, and therefore classes that are easier to be handled than arbitrary Fitting classes (we will give a further proof of this result which can be easily transferred to other embedding properties). Nevertheless, question (1) is almost intractable even for Lockett classes, since in general the Fitting (or Lockett) class generated by a given set of groups is very hard to handle. However, we will give some conditions on Fitting classes $\mathfrak{F}_1$ and $\mathfrak{F}_2$ contained in $\mathrm{Y_n}(\mathfrak{X})$, the class of all groups G such that $G_{\mathfrak{X}}$ is $\mathfrak{X}$-maximal in G, which guarantee that the Fitting class generated by $\mathfrak{F}_1$ and $\mathfrak{F}_2$ is still contained in $\mathrm{Y_n}(\mathfrak{X})$ (where $\mathfrak{X}$ denotes a non-trivial Fitting class). Question (2), too, is open in general. It is clear that there are Fitting classes $\mathfrak{F}$ such that a smallest $\mathfrak{F}$-normal Fitting class does not exist (for instance the class of all nilpotent groups), but it is far from clear what kind of conditions a Fitting

class must satisfy to possess such a class. However, we will prove that for a number of important Fitting classes $\mathfrak{F}$ a unique minimal $\mathfrak{F}$-normal Fitting class exists and can be described explicitly.
In the second part of this chapter, we confine ourselves to the investigation of locally normal SFitting classes (i.e. SFitting classes $\mathfrak{X}$ and $\mathfrak{F}$ such that $\mathfrak{X}$ is normal in $\mathfrak{F}$). As mentioned before, this enables us to use a much more powerful theory and thus to obtain much stronger results concerning the above questions. The key to almost all results proved here is the fact that local normality between SFitting classes (satisfying some weak additional conditions) is equivalent to local normality between their corresponding canonical local definitions. From this it follows that for an arbitrary SFitting class $\mathfrak{X}$ there always exists a unique SFitting class that is maximal among all SFitting classes contained in $\mathrm{Y_n}(\mathfrak{X})$, and that $\mathfrak{X}$ is determined uniquely by this class. Furthermore, we will see that in many cases – for instance when $\mathfrak{X}$ is of bounded nilpotent length – there is an algorithm to describe this class. It turns out, too, that for each SFitting class $\mathfrak{X}$ the collection of all SFitting classes in which $\mathfrak{X}$ is normal forms a complete, distributive and atomic lattice, whose atoms can be described explicitly.
In investigating the dual class, it is possible as well to obtain satisfying results, although question (2) remains open in general. However, we prove that if $\mathfrak{F}$ is an SFitting class such that a smallest $\mathfrak{F}$-normal SFitting class exists, the collection of all $\mathfrak{F}$-normal SFitting classes forms a complete and distributive lattice, too, which, in addition, is dual atomic if $\mathfrak{F}$ is of bounded nilpotent length.

In Chapter 4 we study the remaining embedding properties listed above.
We begin with the investigation of locally (sub)modular Fitting classes, i.e. non-trivial Fitting classes $\mathfrak{X}$ and $\mathfrak{F}$ such that $\mathfrak{X}$ is contained in $\mathfrak{F}$ and an $\mathfrak{X}$-injector of G is a (sub)modular subgroup of G for every $G \in \mathfrak{F}$ (see 4.1 for the definition). In this case $\mathfrak{X}$ is said to be (sub)modular in $\mathfrak{F}$ or $\mathfrak{F}$-(sub)modular. One of the first results to emerge is that the class of all groups G such that an $\mathfrak{X}$-injector of G is a modular subgroup of G is not closed under forming direct products. This implies that the concept of locally modular Fitting classes coincides with the concept of locally normal Fitting classes – a fact which was proved already by Hauck and Kienzle (1987) for the case $\mathfrak{F} = \mathfrak{S}$. So, in order to obtain a new relation between Fitting classes, we have to weaken this embedding property; this leads us

to locally submodular Fitting classes. Although for $\mathfrak{F} = \mathfrak{S}$ this concept too coincides with local normality (Hauck and Kienzle, 1987), there exist Fitting classes $\mathfrak{X}$ and $\mathfrak{F}$ such that $\mathfrak{X}$ is submodular but not normal in $\mathfrak{F}$.
We will see that this relation as well is a relation of the corresponding Lockett sections, hence we may confine ourselves to the case that both classes are Lockett classes. Further, for a number of important Fitting classes $\mathfrak{F}$, we prove the existence of a smallest Fitting class being submodular in $\mathfrak{F}$. It turns out that in each class treated there, the smallest $\mathfrak{F}$-submodular Fitting class coincides with the smallest $\mathfrak{F}$-normal Fitting class.
That the concept of local submodularity is very close to the concept of local normality is also stressed by the fact that these concepts coincide for SFitting classes, hence an SFitting class $\mathfrak{X}$ is submodular in an SFitting class $\mathfrak{F}$ if and only if it is $\mathfrak{F}$-normal. This implies that all results shown in the second part of the third chapter remain true for locally submodular SFitting classes.

In the remaining sections of this chapter we take a look at locally normally embedded and locally permutable Fitting classes (see 4.2 for the definition). Those classes were considered by Lockett (1971) and Doerk and Porta (1987) for $\mathfrak{F} = \mathfrak{S}$, and there it turned out that the concept of strong containment (see 4.2) plays an important part in this investigation. This remains valid in the general case, and therefore we obtain that those relations, too, are relations of the corresponding Lockett sections.
Let $\mathfrak{X}$ be a non-trivial Fitting class. As mentioned above, the class $\mathrm{Y}_e(\mathfrak{X})$ in general fails to be closed under forming normal products for each of the embedding properties e treated here. Nevertheless, in case of local normality there are a number of Fitting classes $\mathfrak{X}$ such that $\mathrm{Y}_{\mathrm{n}}(\mathfrak{X})$ is a Fitting class distinct from $\mathfrak{S}$. We will see that this is impossible for local permutability, i.e., in this case the class $\mathrm{Y}_e(\mathfrak{X})$ is a Fitting class if and only if it coincides with $\mathfrak{S}$. If this also holds for the property of normal embedding remains an open problem. The special case of considering the above relations only between SFitting classes $\mathfrak{X}$ and $\mathfrak{F}$ leads to the case that $\mathfrak{F} = \mathfrak{S}$, and therefore to the investigation of Lockett, Doerk and Porta.

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Notation

All groups treated here are supposed to be finite and soluble. Further, we adhere mainly to the notation used in [9]; all terms, that differ from this are listed below.

$G \wr H$	regular wreath product of G with H
G^*	base group of $G \wr H$
$l(G)$	nilpotent length of G
$\pi(G)$	set of all prime divisors of $\lvert G\rvert$
$U \trianglelefteq\trianglelefteq G$	U is a subnormal subgroup of G
U (s)mod G	U is a (sub)modular subgroup of G
$\mathfrak{F}_1 \circ \mathfrak{F}_2$	class product of $\mathfrak{F}_1$ and $\mathfrak{F}_2$
$\mathfrak{F}_1\mathfrak{F}_2$	Fitting class product of $\mathfrak{F}_1$ and $\mathfrak{F}_2$
$\mathfrak{F}_1 * \mathfrak{F}_2$	formation product of $\mathfrak{F}_1$ and $\mathfrak{F}_2$
$\mathfrak{F}_1 \times \mathfrak{F}_2$	direct product of $\mathfrak{F}_1$ and $\mathfrak{F}_2$
$l(\mathfrak{F})$	nilpotent length of $\mathfrak{F}$
$\pi(\mathfrak{F})$	$\{p \in \mathbb{P} \mid p \in \pi(G),\ G \in \mathfrak{F}\}$
(S)Fit($\mathcal{S}$)	(S)Fitting class generated by a set $\mathcal{S}$ of groups
$S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$	$(G \mid G/G_{\mathfrak{F}_1}G_{\mathfrak{F}_2} \in \mathfrak{S}_\pi)$
$\mathfrak{F}_1 \trianglelefteq \mathfrak{F}_2$	$\mathfrak{F}_1$ is normal in $\mathfrak{F}_2$
$\mathfrak{F}_1$ (s)mod $\mathfrak{F}_2$	$\mathfrak{F}_1$ is (sub)modular in $\mathfrak{F}_2$
$\mathfrak{F}_1$ ne $\mathfrak{F}_2$	$\mathfrak{F}_1$ is normally embedded in $\mathfrak{F}_2$
$\mathrm{Y_n}(\mathfrak{X})$	$(G \mid$ an $\mathfrak{X}$-injector of G is normal in $G)$
$\mathrm{Y_{mod}}(\mathfrak{X})$	$(G \mid$ an $\mathfrak{X}$-injector of G is modular in $G)$
$\mathrm{Y_{smod}}(\mathfrak{X})$	$(G \mid$ an $\mathfrak{X}$-injector of G is submodular in $G)$
$\mathrm{Y_{ne}}(\mathfrak{X})$	$(G \mid$ an $\mathfrak{X}$-injector of G is normally embedded in $G)$
$\mathrm{Y_p}(\mathfrak{X})$	$(G \mid$ an $\mathfrak{X}$-injector of G is system permutable in $G)$

Chapter 1

Examples and basic results

In this chapter we introduce the basic concepts and results about classes of groups – in particular Fitting classes and local formations. For the proofs and further information we refer to [9].

1.1 Classes of groups and closure operations

Groups with special properties – for instance the property of being abelian or nilpotent – are collected in classes (see [9, II]).

1.1.1 Definition
A class of groups is a collection $\mathfrak{X}$ of groups with the property that if $G \in \mathfrak{X}$ and if $H \cong G$, then $H \in \mathfrak{X}$.
We will often use the term $\mathfrak{X}$-group to describe a group belonging to $\mathfrak{X}$.

Notation: If $\mathcal{S}$ is a set of groups, we use $(\mathcal{S})$ to denote the smallest class of groups containing $\mathcal{S}$, and when $\mathcal{S}= \{G\}$, we write (G) instead of $(\{G\})$.

Some examples:

$\emptyset$	the empty class.
$\mathfrak{S}$	the class of all (finite soluble) groups.
$\mathfrak{S}_\pi$	the class of all (finite soluble) π-groups where π is a set of primes. (When $\pi = \{p\}$, we write $\mathfrak{S}_p$ rather than $\mathfrak{S}_\pi$.)
$\mathfrak{N}$	the class of all (finite) nilpotent groups.

$\mathfrak{U}$ the class of all (finite) supersoluble groups.
$\mathfrak{A}$ the class of all (finite) abelian groups.

If $\mathfrak{X}$ is any class of groups and π a set of primes, we denote the class $\mathfrak{X}_\pi$ by $\mathfrak{X} \cap \mathfrak{S}_\pi$.

1.1.2 Definition

(a) A map C is called a closure operation if C assigns to each class $\mathfrak{X}$ of groups a class $\mathrm{C}\mathfrak{X}$ of groups such that the following conditions are satisfied:

(i) $\mathfrak{X} \subseteq \mathrm{C}\mathfrak{X}$.

(ii) $\mathrm{C}(\mathrm{C}\mathfrak{X}) = \mathfrak{X}$.

(iii) If $\mathfrak{X} \subseteq \mathfrak{Y}$, then $\mathrm{C}\mathfrak{X} \subseteq \mathrm{C}\mathfrak{Y}$.

(b) A class $\mathfrak{X}$ is said to be C-closed if $\mathrm{C}\mathfrak{X} = \mathfrak{X}$.
According to (a) the class $\mathrm{C}\mathfrak{X}$ is the smallest C-closed class containing $\mathfrak{X}$.

Convention: The empty class is C-closed for every closure operation C.

(c) The product AB of two closure operations A and B is defined by composition:

$$\mathrm{AB}\mathfrak{X} = \mathrm{A}(\mathrm{B}\mathfrak{X})$$

for all classes $\mathfrak{X}$.

The following list contains some of the most frequently used closure operations:

$$\begin{aligned}
\mathrm{S_n}\mathfrak{X} &= (G \mid \exists H \in \mathfrak{X} \text{ with } G \trianglelefteq\trianglelefteq H);\\
\mathrm{N_0}\mathfrak{X} &= (G \mid \exists N_i \trianglelefteq\trianglelefteq G,\ N_i \in \mathfrak{X}\ (i = 1, \ldots, r) \text{ with } G = \langle N_1, \ldots, N_r\rangle);\\
\mathrm{D_0}\mathfrak{X} &= (G \mid \exists G_i \in \mathfrak{X}\ (i = 1, \ldots, r) \text{ with } G = G_1 \times \ldots \times G_r);\\
\mathrm{Q}\mathfrak{X} &= (G \mid \exists H \in \mathfrak{X} \text{ and an epimorphism from } H \text{ onto } G);\\
\mathrm{R_0}\mathfrak{X} &= (G \mid \exists N_i \trianglelefteq G,\ G/N_i \in \mathfrak{X}\ (i = 1, \ldots, r) \text{ with } N_1 \cap \ldots \cap N_r = 1);\\
\mathrm{E}_\phi\mathfrak{X} &= (G \mid \exists N \trianglelefteq G,\ N \leq \phi(G) \text{ and } G/N \in \mathfrak{X}\ );\\
\mathrm{S_F}\mathfrak{X} &= (G \mid \exists H \in \mathfrak{X} \text{ with } G \leq H \text{ and } G/\mathrm{Core}_H(G) \in \mathfrak{N});\\
\mathrm{S}\mathfrak{X} &= (G \mid \exists H \in \mathfrak{X} \text{ with } G \leq H).
\end{aligned}$$

$\mathfrak{S}$, $\mathfrak{S}_\pi$, $\mathfrak{N}$, $\mathfrak{N}_\pi$ are C-closed for every C in the list. $\mathfrak{A}$ and $\mathfrak{U}$ are examples for classes which are Q- and $\mathrm{R_0}$-, but not $\mathrm{N_0}$-closed.

1.1.3 Definition
If $\mathfrak{X}$ and $\mathfrak{Y}$ are classes of groups, we define their class product $\mathfrak{X} \circ \mathfrak{Y}$ as follows:

$$\mathfrak{X} \circ \mathfrak{Y} = (G \mid G \text{ has a normal subgroup } N \in \mathfrak{X} \text{ with } G/N \in \mathfrak{Y}\ ).$$

We set $\mathfrak{X}^0 = (1)$ and $\mathfrak{X}^n = (\mathfrak{X}^{n-1}) \circ \mathfrak{X}$ for $n \in \mathbb{N},\ n \geq 1$.

1.1.4 Definition
Let G be a group and $\mathfrak{X}$ be a class of groups.

(a) We define

$$\pi(G) = \{p \mid p \in \mathbb{P},\ p \mid |G|\ \} \text{ and } \pi(\mathfrak{X}) = \bigcup\{\pi(X) \mid \ X \in \mathfrak{X}\ \}.$$

(b) The characteristic of $\mathfrak{X}$ is defined as follows:

$$\operatorname{Char}(\mathfrak{X}) = \{p \mid p \in \mathbb{P} \text{ and } Z_p \in \mathfrak{X}\ \}.$$

(c) We also define

$$l(\mathfrak{X}) = \begin{cases} \min\{r \in \mathbb{N} \mid \mathfrak{X} \subseteq \mathfrak{N}^r\ \} & \text{if it exists,} \\ \infty & \text{otherwise} \end{cases}$$

and call $l(\mathfrak{X})$ the nilpotent length of $\mathfrak{X}$.

1.2 Fitting classes

In this section we recall some basic definitions and facts about Fitting classes. For the proofs and further information the reader is referred to [9, IX, X].

1.2.1 Definition
(a) A Fitting class is a class of groups which is both $\mathrm{S_n}$- and $\mathrm{N_0}$-closed.

Obviously, the intersection of Fitting classes is again a Fitting class. Therefore there exists a (unique) smallest Fitting class containing a given set $\mathcal{S}$ of groups – the Fitting class generated by $\mathcal{S}$. We will denote this class by $\operatorname{Fit}(\mathcal{S})$.

(b) Let $\mathfrak{X}$ be a Fitting class and G be a group. Define the $\mathfrak{X}$-radical of G by

$$G_{\mathfrak{X}} = \langle N \mid N \trianglelefteq\trianglelefteq G, \quad N \in \mathfrak{X} \rangle.$$

Obviously, $G_{\mathfrak{X}}$ belongs to $\mathfrak{X}$ and $G_{\mathfrak{X}}$ is the unique maximal normal subgroup with this property.

Fitting classes are named after H. Fitting, who first showed in 1938 that the class of nilpotent groups is closed under forming products of normal subgroups. $\mathfrak{S}$, $\mathfrak{S}_\pi$, $\mathfrak{N}$, $\mathfrak{N}_\pi$ are examples of Fitting classes. Examples of $\mathfrak{X}$-radicals are $O_\pi(G)$ if $\mathfrak{X} = \mathfrak{S}_\pi$, and $F(G)$ if $\mathfrak{X} = \mathfrak{N}$.

The following elementary fact will be useful.

1.2.2 Remark

Let $\mathfrak{X}$ and $\mathfrak{F}$ be Fitting classes and G be a group of minimal order in $\mathfrak{X} \setminus \mathfrak{Y}$. Then G has a unique maximal normal subgroup.

Fitting classes are distinguished by the existence of some special subgroups in every group.

1.2.3 Definition

Let $\mathfrak{X}$ be a class of groups and G be a group.

(a) A subgroup U of G is called $\mathfrak{X}$-maximal in G provided that

- **(i)** $U \in \mathfrak{X}$ and
- **(ii)** if $U \leq V \leq G$ and $V \in \mathfrak{X}$, then $V = U$.

(b) An $\mathfrak{X}$-injector of G is a subgroup V of G with the property that $V \cap N$ is an $\mathfrak{X}$-maximal subgroup of N for every subnormal subgroup N of G.

We denote the (possibly empty) set of $\mathfrak{X}$-injectors of G by $\mathrm{Inj}_{\mathfrak{X}}(G)$.

Let G be a group. Hall π-subgroups of G are examples of $\mathfrak{X}$-injectors for the special case $\mathfrak{X} = \mathfrak{S}_\pi$. If $\mathfrak{X} = \mathfrak{N}$, then $\mathrm{Inj}_{\mathfrak{X}}(G)$ consists of all $\mathfrak{N}$-maximal subgroups of G containing $F(G)$ (cf. [9, IX, 4.12]).

According to Fischer, Gaschütz and Hartley (cf. [9, IX, 1.4]), Fitting classes $\mathfrak{X}$ are characterized by the existence of $\mathfrak{X}$-injectors in every group:

1.2.4 Theorem

A class $\mathfrak{X}$ is a Fitting class if and only if every group G possesses an $\mathfrak{X}$-injector. Furthermore, the $\mathfrak{X}$-injectors of G then form a single conjugacy class.

Next we gather some important properties of radicals and injectors.

1.2.5 Theorem ([9], IX, 1.1, 1.3, 1.5, 1.6)
Let $\mathfrak{X}$ be a Fitting class and G a group.

(a) *If N is a normal subgroup of G and if $V \in Inj_{\mathfrak{X}}(G)$, then $G_{\mathfrak{X}} \leq V$, $N \cap G_{\mathfrak{X}} = N_{\mathfrak{X}}$ and $N \cap V \in Inj_{\mathfrak{X}}(N)$.*

(b) $(G \times G)_{\mathfrak{X}}/G_{\mathfrak{X}} \times G_{\mathfrak{X}} \leq Z(G \times G/G_{\mathfrak{X}} \times G_{\mathfrak{X}})$.

(c) *If V is an $\mathfrak{X}$-injector of G and if $V \leq U \leq G$, then V is an $\mathfrak{X}$-injector of U.*

(d) *(*Frattini*) If $K \trianglelefteq G$ and $V \in Inj_{\mathfrak{X}}(G)$, then $G = KN_G(V \cap K)$.*

(e) *Let $N \trianglelefteq G$ and L be an $\mathfrak{X}$-subgroup of G such that $L \cap N \in Inj_{\mathfrak{X}}(N)$ and $LN = G$. Then $L \in Inj_{\mathfrak{X}}(G)$.*

1.2.6 Theorem ([9], IX, 1.7, 1.9)
Let $\mathfrak{X}$ be a Fitting class.

(a) $\pi(\mathfrak{X}) =$ *Char*$(\mathfrak{X})$.

(b) *Char*$(\mathfrak{X}) = \pi \;\Leftrightarrow\; \mathfrak{N}_\pi \subseteq \mathfrak{X} \subseteq \mathfrak{S}_\pi$.

In particular: If p is a prime such that $P \in \mathfrak{X}$ for some non-trivial p-group P, then $\mathfrak{S}_p \subseteq \mathfrak{X}$.

1.2.7 Definition
Let $\mathfrak{F}_1, \mathfrak{F}_2$ be Fitting classes. We define

$$\mathfrak{F}_1\mathfrak{F}_2 = (G \mid G/G_{\mathfrak{F}_1} \in \mathfrak{F}_2)$$

and call $\mathfrak{F}_1\mathfrak{F}_2$ the Fitting class product of $\mathfrak{F}_1$ with $\mathfrak{F}_2$.

1.2.8 Proposition ([9], IX, 1.11, 1.12)
Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ be Fitting classes.

(a) *$\mathfrak{F}_1\mathfrak{F}_2$ is a Fitting class.*

(b) *If $\mathfrak{F}_2 = Q\mathfrak{F}_2$, then $\mathfrak{F}_1\mathfrak{F}_2 = \mathfrak{F}_1 \circ \mathfrak{F}_2$.*

(c) *For any group G, the $\mathfrak{F}_2$-radical of $G/G_{\mathfrak{F}_1}$ is $G_{\mathfrak{F}_1\mathfrak{F}_2}/G_{\mathfrak{F}_1}$.*

Lockett associated to each Fitting class $\mathfrak{F}$ the Fitting class $\mathfrak{F}^*$, the smallest Fitting class containing $\mathfrak{F}$ whose radicals respect direct products. It is defined by

$$\mathfrak{F}^* \ = \ (\ G \mid (G \times G)_{\mathfrak{F}} \text{ is subdirect in } G \times G\ )$$

and possesses – among others – the following properties (cf. [9, X, 1.3, 1.4, 1.5, 1.8, 1.13, 1.32]):

1.2.9 Theorem
Let $\mathfrak{F}$ be a Fitting class and G be a group.

(a) $\mathfrak{F}^*$ *is a Fitting class.*

(b) $\mathfrak{F} \subseteq \mathfrak{F}^* = (\mathfrak{F}^*)^*$.

(c) $\mathfrak{F}_1 \subseteq \mathfrak{F}_2 \ \Rightarrow\ \mathfrak{F}_1^* \subseteq \mathfrak{F}_2^*$.

(d) *Let $\{\mathfrak{F}_i\}_{i \in I}$ be a family of Fitting classes. Then $(\bigcap_{i\in I} \mathfrak{F}_i)^* = \bigcap_{i\in I} \mathfrak{F}_i^*$.*

(e) $(G \times G)_{\mathfrak{F}} = (G_{\mathfrak{F}} \times G_{\mathfrak{F}})\langle (g, g^{-1}) \mid g \in G_{\mathfrak{F}^*}\rangle$.

(f) $G_{\mathfrak{F}^*}/G_{\mathfrak{F}}$ *is abelian.*

(g) *If $V \in \mathit{Inj}_{\mathfrak{F}}(G)$, then $V_{\mathfrak{F}}$ is an $\mathfrak{F}$-injector of G.*

A Fitting class $\mathfrak{F}$ is called a Lockett class if $\mathfrak{F} = \mathfrak{F}^*$. For each Fitting class $\mathfrak{F}$ we define $\mathfrak{F}_* = \bigcap\{\mathfrak{X} \mid \mathfrak{X}$ Fitting class and $\mathfrak{X}^* = \mathfrak{F}^*\}$ and call $\{\mathfrak{X} \mid \mathfrak{X}^* = \mathfrak{F}^*\} = \{\mathfrak{X} \mid \mathfrak{F}_* \subseteq \mathfrak{X} \subseteq \mathfrak{F}^*\}$ the Lockett section of $\mathfrak{F}$.
By definition, each Q-closed Fitting class is a Lockett class. In particular $\mathfrak{S}$, $\mathfrak{S}_\pi$, $\mathfrak{N}$, $\mathfrak{N}_\pi$ are Lockett classes. Furthermore, $\mathrm{S_F}$-closed Fitting classes (so-called Fischer classes) are Lockett classes (cf. [9, X, 1.25]).

1.2.10 Theorem ([9], X, 1.9, 1.33)
Let G_1, G_2 be groups and $\mathfrak{F}$ be a Lockett class.

(a) $(G_1 \times G_2)_{\mathfrak{F}} = (G_1)_{\mathfrak{F}} \times (G_2)_{\mathfrak{F}}$.

(b) *Let V be an $\mathfrak{F}$-injector of G. Then $V = (V \cap G_1) \times (V \cap G_2)$; in particular $V = V_1 \times V_2$ where $V_i \in \mathit{Inj}_{\mathfrak{F}}(G_i)$ for $i = 1, 2$, and every subgroup of this form is an $\mathfrak{F}$-injector of $G_1 \times G_2$.*

1.2.11 Proposition ([9], X, 1.18, 1.26)
Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ be Fitting classes.

(a) *If $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$, then $(\mathfrak{F}_1)_* \subseteq (\mathfrak{F}_2)_*$.*

(b) $(\mathfrak{F}_1\mathfrak{F}_2^*) = (\mathfrak{F}_1\mathfrak{F}_2)^*$; *in particular, the Fitting class product of Lockett classes is again a Lockett class.*

Lockett classes are characterized in various ways. In the following, we will only need the sharpened form of the so-called quasi-R_0-lemma (cf. [9, X, 1.24]):

1.2.12 Lemma
Let $\mathfrak{F}$ be a Fitting class. Then the following statements are equivalent:

(i) *$\mathfrak{F}$ is a Lockett class.*

(ii) *For all groups G with normal subgroups N_1 and N_2 such that $N_1 \cap N_2 = 1$ and $G/N_1N_2 \in \mathfrak{N}$, the following holds:*

$$G \in \mathfrak{F} \Leftrightarrow G/N_1 \in \mathfrak{F} \text{ and } G/N_2 \in \mathfrak{F}.$$

We mainly apply 1.2.12 to regular wreath products $(G_1 \times G_2) \wr H$ where G_1 and G_2 are arbitrary groups and $H \in \mathfrak{N}$. Identifying G_1 with $G_1 \times 1$ and G_2 with $1 \times G_2$, 1.2.12 implies

$$(G_1 \times G_2) \wr H \in \mathfrak{F} \Leftrightarrow G_1 \wr H \in \mathfrak{F} \text{ and } G_2 \wr H$$

provided that $\mathfrak{F}$ is a Lockett class.

The following Fitting class construction is also due to Lockett (cf [9, IX, 1.14]):

1.2.13 Definition
Let $\mathfrak{X}$ be a Fitting class and π be a set of primes. Set

$$\mathfrak{L}_\pi(\mathfrak{X}) = (G \mid \text{the } \mathfrak{X}\text{-injectors of } G \text{ have } \pi'\text{-index in } G).$$

Thus $\mathfrak{L}_\pi(\mathfrak{X})$ consists of all groups whose injectors contain a Hall π-subgroup of G.

We have

1.2.14 Theorem ([9], IX, 1.15, 1.16, X, 1.37)
Let $\mathfrak{X}$ be a Fitting class, π a set of primes, G a group and $V \in Inj_{\mathfrak{X}}(G)$.

(a) $\mathfrak{L}_\pi(\mathfrak{X})$ *is a Fitting class.*

(b) *The following statements are equivalent:*

- **(i)** $\mathfrak{X} = \mathfrak{X}\mathfrak{S}_\pi$.
- **(ii)** $\mathfrak{L}_\pi(\mathfrak{X}) = \mathfrak{S}$.

(c) $\mathfrak{L}_\pi(\mathfrak{X})^* = \mathfrak{L}_\pi(\mathfrak{X}^*)$.

(d) *Let $H \in Hall_{\pi'}(G)$ and $W = \langle V, H\rangle$. Then W is an $\mathfrak{L}_\pi(\mathfrak{X})$-injector of G if and only if $HV = VH$.*

The $\mathfrak{L}_\pi$-construction enables us to describe the $\mathfrak{X}\mathfrak{Y}$-injectors of a group.

1.2.15 Theorem ([9], IX, 1.22)
Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Fitting classes and $\pi = \pi(\mathfrak{Y})$. Let G be a group and T an $\mathfrak{X}$-injector of $G_{\mathfrak{L}_\pi(\mathfrak{X})}$. By the Frattini argument and the definition of $\mathfrak{L}_\pi(\mathfrak{X})$ there exists $G_\pi \in Hall_\pi(G)$ normalizing T. Then the following holds:

If $V/T \in Inj_{\mathfrak{Y}}(TG_\pi/T)$, then $V \in Inj_{\mathfrak{X}\mathfrak{Y}}(G)$.

In particular: Let $G \in \mathfrak{S}_\pi$. Then V is an $\mathfrak{X}\mathfrak{Y}$-injector of G if and only if $V/G_{\mathfrak{X}} \in Inj_{\mathfrak{Y}}(G/G_{\mathfrak{X}})$.

We will need this theorem especially for the description of the smallest $\mathfrak{S}_{\pi_1} \dots \mathfrak{S}_{\pi_r}$-normal Fitting class (see Chapter 3).

Let $(\mathfrak{F}_i)_{i\in I}$ be Fitting classes. In general, one knows very little about the class $\mathrm{Fit}(\mathfrak{F}_i \mid \ i \in I)$ – the smallest Fitting class containing $\mathfrak{F}_i$ for all $i \in I$. If, however, the characteristics of the classes $\mathfrak{F}_i$ are coprime, an easy description is possible.

1.2.16 Definition

(a) Let $(\mathfrak{F}_i)_{i\in I}$ be Fitting classes whose characteristics are pairwise disjoint. Then define the direct product of $\mathfrak{F}_i,\ i \in I$, as follows:

$$\prod_{i\in I} \mathfrak{F}_i := (G \mid G = G_{i_1} \times \ldots \times G_{i_{n_i}},\ G_i \in \mathfrak{F}_{i_j},\ i_j \in I,\ n_i \in \mathbb{N}).$$

As usual $\prod_{i\in\emptyset} \mathfrak{F}_i = 1$.

(b) A Fitting class $\mathfrak{F}$ is called directly decomposable if there exist non-trivial Fitting classes $\mathfrak{F}_i,\ i \in I,\ |I| > 1$, such that $\mathfrak{F} = \prod_{i\in I} \mathfrak{F}_i$.
Otherwise $\mathfrak{F}$ is said to be directly indecomposable.

Some elementary facts:

1.2.17 Remark

Let $\mathfrak{F}_i,\ i \in I$, be non-trivial Fitting classes whose characteristics are pairwise disjoint.

(a) *The direct product of Fitting (Lockett) classes is again a Fitting (Lockett) class.*
If G is a group, then $G_{\prod_{i\in I}\mathfrak{F}_i} = G_{\mathfrak{F}_{i_1}} \times \ldots \times G_{\mathfrak{F}_{i_{n_i}}}$ for suitable $i_1, \ldots, i_{n_i} \in I,\ n_i \in \mathbb{N}$.

(b) *If $\mathfrak{X}$ is a Fitting class, then $\mathfrak{X}(\prod_{i\in I} \mathfrak{F}_i)$ is the smallest Fitting class containing $\mathfrak{X}\mathfrak{F}_i$ for all $i \in I$.*
In particular, $\prod_{i\in I} \mathfrak{F}_i$ is the smallest Fitting class containing $\mathfrak{F}_i$ for all $i \in I$.

(c) *Let $\mathfrak{F}_i$ be directly indecomposable for all $i \in I$ and $\mathfrak{F} = \prod_{i\in I} \mathfrak{F}_i$. Then the direct factors are unique up to ordering.*

(d) *$\mathfrak{X}\mathfrak{Y}$ is directly indecomposable whenever $\mathfrak{X}$ and $\mathfrak{Y}$ are non-trivial Fitting classes.*

Let $(\pi_i)_{i\in I}$ be pairwise disjoint sets of primes and $\mathfrak{F} = \prod_{i\in I} \mathfrak{S}_{\pi_i}$. Then it is possible to describe the $\mathfrak{F}$-injectors of a group.

1.2.18 Theorem ([9], IX, 4.12, [15], 2.1.3)
Let π be a set of primes, $(\pi_i)_{i\in I}$ a partition of π and $\mathfrak{F} = \prod_{i\in I} \mathfrak{S}_{\pi_i}$. Then the following statements are equivalent:

(i) $V \in \mathit{Inj}_{\mathfrak{F}}(G)$.

(ii) $V = \prod_{i\in I} V_{\pi_i}$ *where* $V_{\pi_i} \in \mathit{Hall}_{\pi_i}(C_G(O_{\pi_i'}(F(G))))$.

In particular, there is a description of the $\mathfrak{N}$-injectors of a group.

For arbitrary Fitting classes $\mathfrak{F}_1,\ \mathfrak{F}_2$ it is much harder to describe $\mathrm{Fit}(\mathfrak{F}_1, \mathfrak{F}_2)$. However, there exists an upper bound for this class introduced by Hauck (cf. [9, IX, 2, A]).

1.2.19 Definition
Let $\mathfrak{F}_1,\ \mathfrak{F}_2$ be Fitting classes and let

$\pi_1 = \{p \in \mathbb{P} \mid p \mid |G/G_{\mathfrak{F}_1}|,\ G \in \mathfrak{F}_2\}$ and $\pi_2 = \{p \in \mathbb{P} \mid p \mid |G/G_{\mathfrak{F}_2}|,\ G \in \mathfrak{F}_1\}$.

Further let π be a set of primes containing $\pi_1 \cap \pi_2$. Then define

$$N_\pi(\mathfrak{F}_1,\ \mathfrak{F}_2) = (G \mid G/G_{\mathfrak{F}_1}G_{\mathfrak{F}_2} \in \mathfrak{N}_\pi).$$

Obvious:

(i) $N_\pi(\mathfrak{F}_1,\ \mathfrak{F}_2) \supseteq \mathfrak{F}_1,\ \mathfrak{F}_2$.

(ii) $N_\emptyset(\mathfrak{F}_1,\ \mathfrak{F}_2) = (G \mid G = G_{\mathfrak{F}_1}G_{\mathfrak{F}_2})$.

1.2.20 Theorem ([9], IX, 2.1)
The class $N_\pi(\mathfrak{F}_1,\ \mathfrak{F}_2)$ defined in 1.2.19 is a Fitting class.

In particular, $N_{\pi(\mathfrak{F}_1)\cap\pi(\mathfrak{F}_2)}(\mathfrak{F}_1, \mathfrak{F}_2)$ is a Fitting class.

Fitting classes and wreath products

Wreath products play an important part in the theory of Fitting classes (cf. [9, X, 2]). In this section we collect some facts needed frequently in the sequel.

1.2.21 Notation
Let G and H be groups. Then $G \wr H$ denotes the regular wreath product of G with H. The base group of $G \wr H$ is denoted by G^*.

We recall the following well-known properties of regular wreath products:

1.2.22 Lemma ([9], A, 18.8)
Let G, H be non-trivial groups.

(a) *If $L \leq H$, then $G^*L \cong G^n \wr L$ where $n = |G : H|$.*

(b) *If N is a normal subgroup of $G \wr H$ such that $G^* \cap N = 1$, then $N = 1$.*

In particular, if $1 \neq G \in \mathfrak{S}_p$ for some prime p, then $O_q(G \wr H)$ is trivial for all primes $q \neq p$.

1.2.23 Theorem ([9], A, 18.9)
Let N be a normal subgroup of a group G. Then there exists a monomorphism from G to $N \wr G/N$.

The following results – most of them are based on the work of Hauck (1977) – show the significance of regular wreath products for the theory of Fitting classes. Because the base group of a wreath product is a direct product, it is hardly surprising that Lockett classes play an important part in this investigation.

1.2.24 Lemma ([9], X, 2.1)
Let $\mathfrak{F}$ be a Lockett class and G be a group such that $G \notin \mathfrak{F}$. Then

$$(G \wr H)_{\mathfrak{F}} = (G_{\mathfrak{F}})^*$$

for each group H.

1.2.25 Theorem ([9], X, 2.7)
Let $\mathfrak{X}$ be a Fitting class, $G \in \mathfrak{X}$ and p a prime. If there exists a non-trivial p-group H such that $G \wr H \in \mathfrak{X}$, then $G \wr P \in \mathfrak{X}^*$ for all p-groups P.

1.2.26 Theorem ([9], X, 2.12)
Let $\mathfrak{X}$ be a Fitting class, let G be an $\mathfrak{X}$-group and let H be a nilpotent group. Then exactly one of the following cases holds:

(i) *$G^n \wr H \notin \mathfrak{X}^*$ for all $n \in \mathbb{N}$.*

(ii) *$G^{2n} \wr H \in \mathfrak{X}$ and $G^{2n-1} \wr H \notin \mathfrak{X}$ for all $n \in \mathbb{N}$.*

(iii) *$G^{2n} \wr H \in \mathfrak{X}$ for all $n \in \mathbb{N}$.*

In particular, if $G^2 \wr H \notin \mathfrak{X}$, then $G^2 \wr H \notin \mathfrak{X}^*$.

The following lemma will be used frequently.

1.2.27 Lemma
Let $\mathfrak{X}$ be a Fitting class, $G \in \mathfrak{X}$, and Q a non-trivial q-group such that $G^2 \wr Q \in \mathfrak{X}$ (q prime). Further let p be a prime satisfying $H = G^2 \wr Z_p \notin \mathfrak{X}$. Then $(H \wr Q)_{\mathfrak{X}} = (G^{2p})^*$ and $(G^{2p})^*Q \in Inj_{\mathfrak{X}}(H \wr Q)$.

Proof: By 1.2.26, we obtain $G^2 \wr Z_p \notin \mathfrak{X}^*$, and consequently 1.2.24 and 1.2.10 yield $(G^{2p})^* = (H \wr Q)_{\mathfrak{X}^*} = (H \wr Q)_{\mathfrak{X}} \in \mathrm{Inj}_{\mathfrak{X}}(H^*)$. According to 1.2.26, the group $(G^{2p})^*Q$ belongs to $\mathfrak{X}$, thus the assertion follows from 1.2.5(e). $\square$

By construction of the regular wreath product, the next lemma is easily proved, too.

1.2.28 Lemma
Let $\mathfrak{X}$ be a Lockett class, G be a group and p be a prime. If $F \in Inj_{\mathfrak{X}}(G \wr Z_p)$ such that $F \not\leq G^*$, then F is conjugate to $V^*Z_p \cong V \wr Z_p$ where $V \in Inj_{\mathfrak{X}}(G)$.

1.2.29 Theorem ([9], X, 2.13)
Let $\mathfrak{X}$ be a Fitting class contained in a Lockett class $\mathfrak{F}$, and let p be a prime. Assume that for each $G \in \mathfrak{X}$ there exist a natural number n and a non-trivial p-group P such that $G^n \wr P \in \mathfrak{F}$. Then $\mathfrak{X}^*\mathfrak{S}_p \subseteq \mathfrak{F}$.

In particular: Let $\mathfrak{X}$ be a Lockett class such that for each $G \in \mathfrak{X}$ there exists a non-trivial p-group P with $G \wr P \in \mathfrak{X}$. Then $\mathfrak{X}\mathfrak{S}_p = \mathfrak{X}$.

1.3 Local formations

In this section we collect some basic facts about (local) formations. For the proofs and further information we refer to [9, IV].

The closure operations $\mathrm{S_n}$ and $\mathrm{N_0}$, respectively, can be regarded as dual to the closure operations Q and $\mathrm{R_0}$, respectively. Thus, from this point of view, the theory of formations is the dual of the theory of Fitting classes (and vice versa).

1.3.1 Definition
(a) A formation is a class of groups which is both Q- and $\mathrm{R_0}$-closed.
A formation $\mathfrak{F}$ is called saturated if $\mathrm{E}_\phi\mathfrak{F} = \mathfrak{F}$ holds.

(b) Let $\mathfrak{F}$ be a formation and G be a group. We define the $\mathfrak{F}$-residual of G by

$$G^{\mathfrak{F}} = \bigcap\{N \mid \; N \trianglelefteq G,\; G/N \in \mathfrak{F}\;\}.$$

Obviously, $G^{\mathfrak{F}}$ is the (unique) smallest normal subgroup of G whose factor group belongs to $\mathfrak{F}$.

(c) A class which is both Fitting class and formation is called a Fitting formation.

Examples of saturated formations are all classes listed in 1.1 except for the class $\mathfrak{A}$, which is a formation but not saturated.
An example of an $\mathfrak{F}$-residual is $O^{\pi}(G)$ for $\mathfrak{F} = \mathfrak{S}_{\pi}$.

An elementary consequence of the definition of a saturated formation is the following description of a minimal counterexample.

1.3.2 Remark
Let $\mathfrak{X}$ and $\mathfrak{F}$ be saturated formations and let G be a group of minimal order in $\mathfrak{X} \setminus \mathfrak{F}$. Then G has a unique minimal normal subgroup and the Frattini subgroup of G is trivial (that is, G is primitive).

1.3.3 Definition
Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be formations. We define

$$\mathfrak{F}_1 * \mathfrak{F}_2 = (G \mid \; G^{\mathfrak{F}_2} \in \mathfrak{F}_1)$$

and call $\mathfrak{F}_1 * \mathfrak{F}_2$ the formation product of $\mathfrak{F}_1$ with $\mathfrak{F}_2$.

1.3.4 Proposition ([9], IV, 1.8, 1.9)
Let $\mathfrak{F}_1, \mathfrak{F}_2$ be formations.

(a) *$\mathfrak{F}_1 * \mathfrak{F}_2$ is a formation, and $\mathfrak{F}_1 * \mathfrak{F}_2$ is saturated provided that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are saturated.*

(b) *If $\mathfrak{F}_1$ is s_n-closed, then $\mathfrak{F}_1 * \mathfrak{F}_2 = \mathfrak{F}_1 \circ \mathfrak{F}_2$.*

The well-known Gaschütz-Lubeseder-Schmid-Theorem (see [9, IV, 4.6]) states that saturated formations are exactly the local formations, that is, formations introduced in 1963 by Gaschütz and constructed in the following way (cf. [9, IV, 3.2]):

1.3.5 Definition
Let f be a map which assigns to each prime p a (possibly empty) formation. Then define

$$\mathfrak{F} := \mathrm{LF}(f) := \bigcap_{p\in\pi} \mathfrak{S}_{p'} * \mathfrak{S}_p * f(p) \cap \mathfrak{S}_\pi$$

where $\pi = \{p \mid \ f(p) \neq \emptyset\}$.
$\mathfrak{F}$ is called a local formation (the class locally defined by f) and f is a local definition of $\mathfrak{F}$.

Obviously, if $\mathfrak{F} = \mathrm{LF}(f)$, then $\mathrm{Char}(\mathfrak{F}) = \{p \mid f(p) \neq \emptyset\}$.

1.3.6 Example
Let π be a set of primes. Then $\mathfrak{S}_\pi$ is a local formation. If f assigns to each prime p the class $\mathfrak{S}_\pi$ if $p \in \pi$ and the empty class otherwise, then f is a local definition of $\mathfrak{S}_\pi$.
Another example of a local formation is the class of all nilpotent groups. The function which assigns to each prime p the class $\mathfrak{S}_p$ is a local definition of $\mathfrak{N}$.

Let $\mathfrak{F}$ be a local formation. Among all possible local definitions there exists exactly one, denoted by F, such that F is integrated (that is $F(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$) and full (that is $\mathfrak{S}_p * F(p) = F(p)$ for all $p \in \mathbb{P}$) (see [9, IV, 3.7]). F is called the canonical local definition of $\mathfrak{F}$.
We collect some basic properties of local formations in the following theorem (see [9, VI, 3.5, 3.8, 3.13, 3.17]).

1.3.7 Theorem
Let $\mathfrak{F} = \mathrm{LF}(f)$ and $\mathfrak{G} = \mathrm{LF}(g)$ be non-trivial local formations with canonical local definitions F and G, respectively.

(a) *If $f(p) \subseteq \mathfrak{S}_p * g(p)$ for all $p \in \mathbb{P}$, then $\mathfrak{F} \subseteq \mathfrak{G}$.*

(b) *$\mathfrak{F} \cap \mathfrak{G} = \mathrm{LF}(f \cap g)$, where $(f \cap g)(p) = f(p) \cap g(p)$.*

(c) *$F(p) = \mathfrak{S}_p * (f(p) \cap \mathfrak{F})$ for all $p \in \mathbb{P}$.*

(d) *If $l(\mathfrak{F}) = r < \infty$, then $F \cap \mathfrak{N}^{r-1}$, defined by $(F \cap \mathfrak{N}^{r-1})(p) = F(p) \cap \mathfrak{N}^{r-1}$, is a local definition of $\mathfrak{F}$.*

In particular: $F(p) = \mathfrak{S}_p * (F(p) \cap \mathfrak{N}^{r-1})$ for all $p \in \mathbb{P}$.

(e) *The canonical local definition H of $\mathfrak{F} * \mathfrak{G}$ is given by*

$$H(p) = \begin{cases} F(p) * \mathfrak{G} & \text{if } p \in \mathit{Char}(\mathfrak{F}), \\ G(p) & \text{otherwise.} \end{cases}$$

(f) *Let* $\textsc{c} \in \{\textsc{s}, \textsc{s}_n, \textsc{n}_0\}$. *Then the following statements are equivalent:*

(i) $\mathfrak{F}$ *is* $\textsc{c}$*-closed.*

(ii) $F(p)$ *is* $\textsc{c}$*-closed for all* $p \in \mathbb{P}$.

Some examples

In this thesis we will frequently refer to some special classes. Here we describe these classes together with their canonical local definitions. Each of them is closed under every closure operation listed in 1.1, thus the different class products coincide for these classes (cf. 1.2.8 and 1.3.4). Since the present work is concerned with the theory of Fitting classes we will use here – as well as in similar situations in the following – the Fitting class product.

1.3.8 The classes $\mathfrak{S}_{\pi_1} \cdots \mathfrak{S}_{\pi_r}$
Set $\mathfrak{F} = \mathfrak{S}_{\pi_1} \cdots \mathfrak{S}_{\pi_r}$ where $\pi_i \neq \mathbb{P}, \emptyset$ are sets of primes such that $\pi_i \neq \pi_{i+1}$ for $i = 1, \ldots, r-1$. Then the canonical local definition F of $\mathfrak{F}$ is given by

$$F(p) = \begin{cases} \mathfrak{S}_{\pi_1} \cdots \mathfrak{S}_{\pi_r} & \text{if } p \in \pi_1, \\ \mathfrak{S}_{\pi_2} \cdots \mathfrak{S}_{\pi_r} & \text{if } p \in \pi_2 \setminus \pi_1, \\ \vdots & \\ \mathfrak{S}_{\pi_{r-1}} \mathfrak{S}_{\pi_r} & \text{if } p \in \pi_{r-1} \setminus (\pi_1 \cup \ldots \cup \pi_{r-2}), \\ \mathfrak{S}_{\pi_r} & \text{if } p \in \pi_r \setminus (\pi_1 \cup \ldots \cup \pi_{r-1}), \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof: We only have to show that $\mathfrak{F} = \mathrm{LF}(F)$. This will be done by induction on r.
$r = 1$ is clear. Thus assume that $r > 1$ and that the assertion holds for $k < r$. If $\mathfrak{X} = \mathfrak{S}_{\pi_2} \cdots \mathfrak{S}_{\pi_r}$, then by inductive hypothesis $\mathfrak{X} = \mathrm{LF}(X)$ where

$$X(p) = \begin{cases} \mathfrak{S}_{\pi_2} \cdots \mathfrak{S}_{\pi_r} & \text{if } p \in \pi_2, \\ \mathfrak{S}_{\pi_3} \cdots \mathfrak{S}_{\pi_r} & \text{if } p \in \pi_3 \setminus \pi_2, \\ \vdots & \\ \mathfrak{S}_{\pi_{r-1}} \mathfrak{S}_{\pi_r} & \text{if } p \in \pi_{r-1} \setminus (\pi_1 \cup \ldots \cup \pi_{r-2}), \\ \mathfrak{S}_{\pi_r} & \text{if } p \in \pi_r \setminus (\pi_1 \cup \ldots \cup \pi_{r-1}), \\ \emptyset & \text{otherwise.} \end{cases}$$

$\mathfrak{F} = \mathfrak{S}_{\pi_1}\mathfrak{X}$, thus the assertion follows from 1.3.7(e). □

1.3.9 Lattice formations

Let π be a set of primes and let $(\pi_i)_{i\in I}$ be a partition of π. Then

$$\mathfrak{F} = \prod_{i\in I} \mathfrak{S}_{\pi_i}$$

is called lattice formation belonging to $(\pi_i)_{i\in I}$. (This notation refers to the fact that, if $\pi = \mathbb{P}$, these classes are exactly the subgroup closed saturated formations $\mathfrak{F}$ such that the set of all so-called $\mathfrak{F}$-subnormal subgroups of any group forms a lattice; see [1].)
It is easily seen that the canonical local definition of $\mathfrak{F}$ is given by

$$F(p) = \begin{cases} \mathfrak{S}_{\pi_i} & \text{if } p \in \pi_i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Notice that $\mathfrak{N}_\pi$ occurs as an important special case of this construction (π any set of primes).

1.3.10 A further example

In [2], the following classes are considered: to each prime p, let $\pi(p)$ be a set of primes containing p such that the following holds: if $q \in \pi(p)$, then $\pi(p) = \pi(q)$ or $\pi(q) = \mathbb{P}$ or $\pi(p) = \mathbb{P}$. Further set $\pi = \{p \in \mathbb{P} \mid \pi(p) \neq \mathbb{P}\}$ and consider the following equivalence relation on π

$$p \sim q \Longleftrightarrow \pi(p) = \pi(q).$$

If $\hat{\pi}$ denotes a system of representatives and if $f(p) = \mathfrak{S}_{\pi(p)}$ for $p \in \mathbb{P}$, then $\mathfrak{F} = \mathrm{LF}(f)$ has the following properties ([2, Prop. 3.2, Lemma 3.3, Prop. 3.3]):

(a) $\mathfrak{F}$ is an s-closed Fitting class and the canonical local definition is given by

$$F(p) = \begin{cases} \mathfrak{S}_{\pi(p)} & \text{if } \pi(p) \neq \mathbb{P}, \\ \mathfrak{F} & \text{otherwise.} \end{cases}$$

(b) $\mathfrak{S}_{\pi'}\mathfrak{F} = \mathfrak{F}$.

(c) $\mathfrak{F} = \bigcap_{p\in\hat{\pi}}(\mathfrak{S}_{\pi'}\mathfrak{S}_{\sigma(p)}\mathfrak{S}_{\pi(p)})$ where $\sigma(p) = \bigcup_{q\in\hat{\pi},\pi(p)\neq\pi(q)} \pi(q)$.

Let p be a prime. Then the class $\mathfrak{S}_{p'}\mathfrak{S}_p$ of all p-nilpotent groups occurs as special case of this construction.

Chapter 2

Subgroup-closed Fitting classes

The subgroup-closure of a Fitting class is strong enough to guarantee the closure of the class under a number of further closure operations; this was proved in 1982 by Bryce and Cossey ([6], [8]). More precisely, they have shown that s-closed Fitting classes are saturated formations and therefore local formations. Thus, in dealing with s-closed Fitting classes, a much more powerful theory can be used than in the general case. This makes it possible to obtain strong results about the s-closed Fitting class generated by a given set of groups as well as the lattice of s-closed Fitting classes.

2.1 Fundamental results

In this section we present some fundamental results about subgroup-closed Fitting classes. For the proofs and further information we refer to [9, XI].

Recall:

2.1.1 Definition
A Fitting class $\mathfrak{F}$ is called subgroup-closed if

$$\mathfrak{F} = \mathrm{s}\mathfrak{F} = (G \mid \exists H \in \mathfrak{F} \text{ with } G \leq H).$$

If $\mathfrak{F} = \mathrm{s}\mathfrak{F}$, we call $\mathfrak{F}$ an SFitting class.

$\mathfrak{S},\ \mathfrak{S}_\pi,\ \mathfrak{N},\ \mathfrak{N}_\pi$ are examples of SFitting classes.

2.1.2 Remark ([9], X, 1.2.5)

(a) *If* $(\mathfrak{F}_i)_{i\in I}$ *is a family of SFitting classes, then* $\cap_{i\in I}\mathfrak{F}_i$ *is again an SFitting class.*

In particular, there exists a (unique) smallest SFitting class containing a given set $\mathcal{S}$ *of groups, the SFitting class generated by* $\mathcal{S}$. *We denote this class by* $\mathrm{SFit}(\mathcal{S})$.

(b) *If* $\mathfrak{F}$ *is an SFitting class, then* $\mathfrak{F} = \mathfrak{F}^*$.

The following theorem, which was proved by Bryce and Cossey in 1982, enables us to use the theory of local formations in the treatment of SFitting classes.

2.1.3 Theorem ([6], Theorem 1, [8], Theorem 1.1)
A subgroup-closed Fitting class is a saturated formation.

Therefore, a subgroup-closed Fitting class $\mathfrak{F}$ is also a local formation. Let F be the corresponding canonical local definition. By 1.3.7, $F(p)$ is again an SFitting class for all $p \in \mathbb{P}$, and if $l(\mathfrak{F}) = r < \infty$, then f , defined by $f(p) = F(p) \cap \mathfrak{N}^{r-1}$, is a local definition of $\mathfrak{F}$ as well. In this case the class $f(p)$ is an SFitting class of nilpotent length $r-1$ for all $p \in \mathbb{P}$. Thus in the above situation it is frequently possible to argue by induction on the nilpotent length of $\mathfrak{F}$.

Furthermore, we will see that it is often possible to deduce embedding properties of $F(p)$-injectors from embedding properties of $\mathfrak{F}$-injectors (where p is any prime).

2.1.4 Proposition
Let $\mathfrak{X}$ *and* $\mathfrak{F}$ *be SFitting classes with corresponding canonical local definitions* X *and* F, *respectively, and let* p *be a prime such that* $p \in \mathit{Char}(\mathfrak{X}) \cap \mathit{Char}(\mathfrak{F})$. *Then the following holds:*

If $G \in F(p)$ *and* $W \in \mathit{Inj}_{\mathfrak{X}}(\mathrm{Z}_p \wr G)$, *then* $W \cap G \in \mathit{Inj}_{X(p)}(G)$.

Proof: $G \in F(p)$, thus $H = \mathrm{Z}_p \wr G$ is contained in $\mathfrak{S}_p F(p) = F(p) \subseteq \mathfrak{F}$.

(1) $W \in \mathrm{Inj}_{X(p)}(\mathrm{Z}_p \wr G)$:
$C_H(Z_p^*) \cap G = 1$ by construction of the regular wreath product, whence $O_{p'}(U) = 1$ for all subgroups U of G containing Z_p^*. In particular $O_{p'}(W) = 1$ and thus the statement holds true (notice that $X(p) \subseteq \mathfrak{X} = \cap_{q\in\pi}\mathfrak{S}_{q'}X(q) \cap \mathfrak{S}_\pi$).

(2) $W \cap G \in \mathrm{Inj}_{X(p)}(G)$:
$X(p)$ is an SFitting class, thus $W \cap G \in X(p)$. Let N be a normal subgroup of G and U an $X(p)$-subgroup of N such that $U > (W \cap G) \cap N = W \cap N$. Then $Z_p^* U \in \mathfrak{S}_p X(p) = X(p)$ and $Z_p^* U \leq Z_p^* N \trianglelefteq Z_p^* G$. Thus $Z_p^* U > Z_p^*(W \cap N) = Z_p^* N \cap W$. This contradicts (1). □

Observe that the Q-closure of an SFitting class implies that the Fitting class product of SFitting classes is again an SFitting class.
By Bryce and Cossey (cf. [6], [8]), SFitting classes are precisely the so-called primitive saturated formations. In particular, the following statements hold true.

2.1.5 Proposition ([9], VII, 3.8, [7], 2.6)

(a) *Let $\mathfrak{X}$ be an SFitting class of bounded nilpotent length. Then there exists a countable set of classes $\mathfrak{X}_i$ such that $\mathfrak{X}_i = \mathfrak{S}_{\pi(i)_1} \cdots \mathfrak{S}_{\pi(i)_{n_i}}$ (for suitable $n_i \in \mathbb{N}$ and $\pi(i)_j \subseteq \mathbb{P}$) and*

$$\mathfrak{X} = \bigcap_{i=1}^{\infty} \mathfrak{X}_i.$$

Furthermore, if ρ is a finite set of primes, then $\mathfrak{S}_\rho$ is contained in all but a finite number of the classes $\mathfrak{X}_i$.

(b) *Let $\mathfrak{X}$ be an SFitting class contained in $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r}$, where $p_1, \ldots, p_r$ are primes satisfying $p_i \neq p_{i+1}$ for $i = 1, \ldots, r-1$. If $l(\mathfrak{X}) = r$, then $\mathfrak{X} = \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r}$.*

In particular: Let G be a group contained in $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r}$. If $l(G) = r$, then the smallest SFitting class containing G coincides with $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r}$.

2.2 On the lattice of SFitting classes

In this section we will obtain a number of results concerning the SFitting class generated by arbitrary many SFitting classes $\mathfrak{F}_i, \quad i \in I$. It turns out that this class behaves nicely with respect to intersections and certain extensions. A consequence of these results is that the lattice of SFitting classes is distributive – a fact which has already been proved by Shemetkov and Skiba in 1989 ([20]). Moreover, this lattice is atomic and it is also possible to describe the atoms explicitly.
For basic facts concerning lattice theory, the reader is referred to [12].

We recall:

2.2.1 Definition
Let $\mathcal{S}$ be a set of groups. Then

$$\mathrm{SFit}(\mathcal{S}) = \bigcap \{\mathfrak{F} \mid \mathfrak{F} \text{ SFitting class }, \ \mathfrak{F} \supseteq \mathcal{S}\}$$

denotes the SFitting class generated by $\mathcal{S}$.

Obviously, $\mathrm{SFit}(\mathcal{S}) = \cup_{i \in \mathbb{N}} (\mathrm{SN}_0)^i(\mathcal{S})$.

2.2.2 Proposition
Let $\mathfrak{F}_1, \ \mathfrak{F}_2$ be classes of groups and $\mathfrak{Y}$ be a non-trivial SFitting class. Then

$$\mathfrak{Y}\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2) = \mathrm{SFit}(\mathfrak{Y} \circ \mathfrak{F}_1, \mathfrak{Y} \circ \mathfrak{F}_2).$$

Proof:

$\subseteq$: Let G be a group contained in $\mathfrak{Y}\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)$. Then there exist a normal subgroup N of G, $N \in \mathfrak{Y}$, and a natural number i such that $G/N \in (\mathrm{SN}_0)^i(\mathfrak{F}_1 \cup \mathfrak{F}_2)$. Thus it remains to prove:

$$\mathfrak{Y} \circ (\mathrm{SN}_0)^i(\mathfrak{F}_1 \cup \mathfrak{F}_2) \subseteq \mathrm{SFit}(\mathfrak{Y} \circ \mathfrak{F}_1, \mathfrak{Y} \circ \mathfrak{F}_2)$$

for all $i \in \mathbb{N}$. This will be done by induction on i.
$i = 1$:

(a) $\mathfrak{Y} \circ \mathrm{N}_0(\mathfrak{F}_1 \cup \mathfrak{F}_2) \subseteq \mathrm{SFit}(\mathfrak{Y} \circ \mathfrak{F}_1, \mathfrak{Y} \circ \mathfrak{F}_2)$: Let G be a $\mathfrak{Y} \circ \mathrm{N}_0(\mathfrak{F}_1 \cup \mathfrak{F}_2)$-group. Then there exists a normal subgroup N of G such that $N \in \mathfrak{Y}$ and $G/N = \langle \overline{N_1}, \ldots, \overline{N_k} \rangle$ where $\overline{N_1}, \ldots, \overline{N_k}$ are subnormal subgroups of G/N such that $\overline{N_1}, \ldots, \overline{N_k} \in \mathfrak{F}_1 \cup \mathfrak{F}_2$. If $\overline{N_j} = N_j/N$, then $N_j \in \mathfrak{Y} \circ \mathfrak{F}_1 \cup \mathfrak{Y} \circ \mathfrak{F}_2 \subseteq \mathrm{SFit}(\mathfrak{Y} \circ \mathfrak{F}_1, \mathfrak{Y} \circ \mathfrak{F}_2)$ for all $j \in \{1, \ldots, k\}$, and consequently $G \in \mathrm{N}_0\mathrm{SFit}(\mathfrak{Y}\circ\mathfrak{F}_1, \mathfrak{Y}\circ\mathfrak{F}_2) = \mathrm{SFit}(\mathfrak{Y}\circ \mathfrak{F}_1, \mathfrak{Y} \circ \mathfrak{F}_2)$.

(b) $\mathfrak{Y}\circ\mathrm{SN}_0(\mathfrak{F}_1\cup\mathfrak{F}_2) \subseteq \mathrm{SFit}(\mathfrak{Y}\circ\mathfrak{F}_1, \mathfrak{Y}\circ\mathfrak{F}_2)$: Let G be a group contained in $\mathfrak{Y} \circ \mathrm{SN}_0(\mathfrak{F}_1 \cup \mathfrak{F}_2)$. By definition there exist a normal subgroup N of G, $N \in \mathfrak{Y}$, and a monomorphism from G/N to W where $W \in \mathrm{N}_0(\mathfrak{F}_1 \cup \mathfrak{F}_2)$. By 1.2.23 there exists a monomorphism from G to $N \wr G/N$ and by 1.2.22(a) there exists a monomorphism from $N \wr G/N$ to $N \wr W$. Thus we obtain $G \in \mathrm{S}(\mathfrak{Y} \circ \mathrm{N}_0(\mathfrak{F}_1 \cup \mathfrak{F}_2))$. Now the subgroup-closure of $\mathrm{SFit}(\mathfrak{Y} \circ \mathfrak{F}_1, \mathfrak{Y} \circ \mathfrak{F}_2)$ and (a) provide the assertion.

The case $i > 1$ is proved analogously.

$\supseteq$: $\mathfrak{Y}\circ\mathfrak{F}_1$ and $\mathfrak{Y}\circ\mathfrak{F}_2$ are classes of groups contained in $\mathfrak{Y}\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)$, thus this inclusion is trivial.

□

2.2.3 Proposition

Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ be saturated Fitting formations with canonical local definitions F_1, F_2. Define $F := \mathrm{SFit}(F_1, F_2)$ by $\mathrm{SFit}(F_1, F_2)(p) := \mathrm{SFit}(F_1(p), F_2(p))$. Then

$$\mathrm{LF}(\mathrm{SFit}(F_1, F_2)) = \mathrm{SFit}(\mathrm{LF}(F_1), \mathrm{LF}(F_2)) = \mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)$$

and F is the canonical local definition of $\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)$.

Proof:

$\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)$is locally defined by F: Set $\pi_i = \pi(\mathfrak{F}_i)$ $(i = 1, 2)$. If $\mathfrak{F} = \mathrm{LF}(F)$, then it remains to prove that $\mathfrak{F} = \mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)$.

$\supseteq$: Let G be a group contained in $\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2) = \mathrm{SFit}(\cap_{p\in\pi_1}\mathfrak{S}_{p'}\mathfrak{S}_pF_1(p) \cap \mathfrak{S}_{\pi_1}, \cap_{p\in\pi_2}\mathfrak{S}_{p'}\mathfrak{S}_pF_2(p) \cap \mathfrak{S}_{\pi_2})$. By 2.2.2 we conclude $G \in \mathfrak{S}_{\pi_1\cup\pi_2} \cap \mathfrak{S}_{q'}\mathfrak{S}_qF(q)$ for all $q \in \pi_1 \cup \pi_2$ and thus the assertion.

$\subseteq$: Suppose not. Let G be a group of minimal order contained in $\mathfrak{F} \setminus \mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)$. Then G has a unique minimal normal subgroup, hence there exists a prime q such that $O_q(G) \neq 1$ and $O_{q'}(G) = 1$. By 2.2.2 we obtain $G \in \mathfrak{S}_qF(q) = \mathfrak{S}_q\mathrm{SFit}(F_1(q), F_2(q)) = \mathrm{SFit}(\mathfrak{S}_qF_1(q), \mathfrak{S}_qF_2(q)) = \mathrm{SFit}(F_1(q), F_2(q)) \subseteq \mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)$, a contradiction.

F is the canonical local definition: This is an immediate consequence of 2.2.2 and the hypothesis. $\square$

2.2.4 Proposition
Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{X}$ be SFitting classes and $l(\mathfrak{X}) < \infty$. Then

$$\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2) \cap \mathfrak{X} = \mathrm{SFit}(\mathfrak{F}_1 \cap \mathfrak{X}, \mathfrak{F}_2 \cap \mathfrak{X}).$$

Proof: $\supseteq$: Obvious. $\subseteq$: By induction on $r := l(\mathfrak{X})$.

$r = 1$: $\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2) \cap \mathfrak{X} = \mathfrak{N}_{\pi(\mathfrak{F}_1)\cup\pi(\mathfrak{F}_2)} \cap \mathfrak{N}_{\pi(\mathfrak{X})} = \mathfrak{N}_{(\pi(\mathfrak{F}_1)\cap\pi(\mathfrak{X}))\cup(\pi(\mathfrak{F}_2)\cap\pi(\mathfrak{X}))} = \mathrm{SFit}(\mathfrak{F}_1 \cap \mathfrak{X}, \mathfrak{F}_2 \cap \mathfrak{X})$.

$r > 1$: Let F_1, F_2, F be the canonical local definitions belonging to $\mathfrak{F}_1$, $\mathfrak{F}_2$, and $\mathfrak{X}$, respectively, and set $f = F \cap \mathfrak{N}^{r-1}$. 1.3.7 yields $F(p) = \mathfrak{S}_p f(p)$ for all $p \in \mathbb{P}$ and f is a local definition of $\mathfrak{X}$. Furthermore, $F_i \cap F$ is the canonical local definition of $\mathfrak{F}_i \cap \mathfrak{X}$ $(i = 1, 2)$.
For each $p \in \mathbb{P}$ the class $(F_i \cap F)(p)$ is an s-closed Fitting class, consequently we obtain by 1.3.7(b), inductive hypothesis and 2.2.3

$$\begin{aligned}
\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2) \cap \mathfrak{X} &= \mathrm{LF}(\mathrm{SFit}(F_1, F_2)) \cap \mathrm{LF}(f) &=\\
\mathrm{LF}(\mathrm{SFit}(F_1, F_2) \cap f) &= \mathrm{LF}(\mathrm{SFit}(F_1 \cap f, F_2 \cap f)) &\subseteq\\
\mathrm{LF}(\mathrm{SFit}(F_1 \cap F, F_2 \cap F)) &= \mathrm{SFit}(\mathfrak{F}_1 \cap \mathfrak{X}, \mathfrak{F}_2 \cap \mathfrak{X}).
\end{aligned}$$

(Notice 1.3.7(a) and $\mathrm{SFit}(F_1(p) \cap f(p), F_2(p) \cap f(p)) \subseteq \mathrm{SFit}(F_1(p) \cap F(p), F_2(p) \cap F(p))$ for all $p \in \mathbb{P}$.)

$\square$

2.2.5 Corollary
Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{X}$ be SFitting classes. Then

$$\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2) \cap \mathfrak{X} = \mathrm{SFit}(\mathfrak{F}_1 \cap \mathfrak{X}, \mathfrak{F}_2 \cap \mathfrak{X}).$$

Proof: $\supseteq$: Obvious. $\subseteq$: Let G be a group contained in $\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2) \cap \mathfrak{X}$. Then there exists a natural number r such that $G \in \mathfrak{N}^r$. 2.2.4 yields

$$\begin{aligned}
G \in \mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2) \cap \mathfrak{X} \cap \mathfrak{N}^r &= \mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2) \cap (\mathfrak{X} \cap \mathfrak{N}^r) &=\\
\mathrm{SFit}(\mathfrak{F}_1 \cap (\mathfrak{X} \cap \mathfrak{N}^r), \mathfrak{F}_2 \cap (\mathfrak{X} \cap \mathfrak{N}^r)) &\subseteq \mathrm{SFit}(\mathfrak{F}_1 \cap \mathfrak{X}, \mathfrak{F}_2 \cap \mathfrak{X}),
\end{aligned}$$

and the proof is complete. □

2.2.6 Proposition
Let $\mathfrak{X}_1, \ldots, \mathfrak{X}_n,\ \mathfrak{Y}_1, \ldots, \mathfrak{Y}_m$ *be SFitting classes. Then*

$$\mathrm{SFit}(\bigcap_{i=1}^{n} \mathfrak{X}_i, \bigcap_{j=1}^{m} \mathfrak{Y}_j) = \bigcap_{i=1}^{n}\bigcap_{j=1}^{m} \mathrm{SFit}(\mathfrak{X}_i, \mathfrak{Y}_j).$$

Proof: Set $\mathfrak{Y} := \cap_{j=1}^{m} \mathfrak{Y}_j$. We show

(a) $\mathrm{SFit}(\cap_{i=1}^{n} \mathfrak{X}_i, \mathfrak{Y}) = \cap_{i=1}^{n} \mathrm{SFit}(\mathfrak{X}_i, \mathfrak{Y})$.

(b) $\mathrm{SFit}(\mathfrak{X}_i, \cap_{j=1}^{m} \mathfrak{Y}_j) = \cap_{j=1}^{m} \mathrm{SFit}(\mathfrak{X}_i, \mathfrak{Y}_j)$ for all $i \in \{1, \ldots, n\}$.

We prove the non-trivial inclusion of (a) by induction on n.

$n = 2$: By 2.2.5,

$$\mathrm{SFit}(\mathfrak{X}_1, \mathfrak{Y}) \cap \mathrm{SFit}(\mathfrak{X}_2, \mathfrak{Y}) = \mathrm{SFit}(\mathfrak{X}_1 \cap \mathrm{SFit}(\mathfrak{X}_2, \mathfrak{Y}), \mathfrak{Y} \cap \mathrm{SFit}(\mathfrak{X}_2, \mathfrak{Y}))$$

$$= \mathrm{SFit}(\mathrm{SFit}(\mathfrak{X}_1 \cap \mathfrak{X}_2, \mathfrak{X}_1 \cap \mathfrak{Y}), \mathfrak{Y}) \subseteq \mathrm{SFit}(\mathrm{SFit}(\mathfrak{X}_1 \cap \mathfrak{X}_2, \mathfrak{Y}), \mathfrak{Y})$$

$$= \mathrm{SFit}(\mathfrak{X}_1 \cap \mathfrak{X}_2, \mathfrak{Y}).$$

$n > 2$: By inductive hypothesis we obtain

$$\mathrm{SFit}(\cap_{i=1}^{n} \mathfrak{X}_i, \mathfrak{Y}) = \mathrm{SFit}(\cap_{i=1}^{n-1} \mathfrak{X}_i \cap \mathfrak{X}_n, \mathfrak{Y}) = \mathrm{SFit}(\cap_{i=1}^{n-1} \mathfrak{X}_i, \mathfrak{Y}) \cap \mathrm{SFit}(\mathfrak{X}_n, \mathfrak{Y})$$

$$= \cap_{i=1}^{n-1} \mathrm{SFit}(\mathfrak{X}_i, \mathfrak{Y}) \cap \mathrm{SFit}(\mathfrak{X}_n, \mathfrak{Y}) = \cap_{i=1}^{n} \mathrm{SFit}(\mathfrak{X}_i, \mathfrak{Y}).$$

(b) can be proved analogously. □

2.2.7 Proposition
Let $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be SFitting classes of bounded nilpotent length, and let* $\mathfrak{X}_i$ *and* $\mathfrak{Y}_j$*, respectively, be as in 2.1.5(a). Then*

$$\mathrm{SFit}(\mathfrak{X}, \mathfrak{Y}) = \bigcap_{i=1}^{\infty}\bigcap_{j=1}^{\infty} \mathrm{SFit}(\mathfrak{X}_i, \mathfrak{Y}_j) \cap \mathfrak{N}^r$$

where $r = \max\{l(\mathfrak{X}), l(\mathfrak{Y})\}$.

Proof:

- $\subseteq$: Since 2.1.5(a) yields $\mathfrak{X} = \cap_{i=1}^{\infty}\mathfrak{X}_i$ and $\mathfrak{Y} = \cap_{j=1}^{\infty}\mathfrak{Y}_j$, this inclusion is trivial.

- $\supseteq$: Let G be a group contained in $\cap_{i=1}^{\infty}\cap_{j=1}^{\infty}\mathrm{SFit}(\mathfrak{X}_i,\mathfrak{Y}_j)\cap\mathfrak{N}^r$. Set $\pi := \pi(G)$. $|\pi| < \infty$, thus 2.1.5(a) yields that $\mathfrak{S}_\pi \subseteq \mathfrak{X}_i,\ \mathfrak{Y}_j$ for all but a finite number of $\mathfrak{X}_i,\ \mathfrak{Y}_j$. Let $\mathfrak{X}_{i_1},\dots,\mathfrak{X}_{i_n},\ \mathfrak{Y}_{j_1},\dots,\mathfrak{Y}_{j_m}$ be these exceptions. By 2.2.6 and 2.2.4 we obtain

$$G \in \cap_{k=1}^{n}\cap_{l=1}^{m}\mathrm{SFit}(\mathfrak{X}_{i_k},\mathfrak{Y}_{j_l})\cap\mathfrak{N}^r\cap\mathfrak{S}_\pi = \mathrm{SFit}(\cap_{k=1}^{n}\mathfrak{X}_{i_k},\cap_{l=1}^{m}\mathfrak{Y}_{j_l})\cap\mathfrak{N}^r\cap\mathfrak{S}_\pi$$

$$= \mathrm{SFit}(\cap_{k=1}^{n}\mathfrak{X}_{i_k}\cap\mathfrak{N}^r\cap\mathfrak{S}_\pi,\cap_{l=1}^{m}\mathfrak{Y}_{j_l}\cap\mathfrak{N}^r\cap\mathfrak{S}_\pi) \subseteq \mathrm{SFit}(\mathfrak{X},\mathfrak{Y}).$$

$\square$

Set $\mathfrak{X} = \mathfrak{S}_{\pi_1}\cdots\mathfrak{S}_{\pi_r}$ and $\mathfrak{Y} = \mathfrak{S}_{\sigma_1}\cdots\mathfrak{S}_{\sigma_k}$, where $\pi_1,\dots,\pi_r,\sigma_1,\dots,\sigma_k$ are non-trivial sets of primes. Then $\mathrm{SFit}(\mathfrak{X},\mathfrak{Y})$ can be determined recursively.

2.2.8 Lemma
Let $\pi_1,\dots,\pi_r,\sigma_1,\dots,\sigma_r$ be non-trivial sets of primes where $r,t \geq 1$ are natural numbers, and let $\mathfrak{Y}$ be an SFitting class.

$$\mathrm{SFit}(\mathfrak{S}_{\pi_1}\dots\mathfrak{S}_{\pi_r}\mathfrak{Y},\mathfrak{S}_{\sigma_1}\dots\mathfrak{S}_{\sigma_t}\mathfrak{Y}) =$$

$$\mathfrak{S}_{\pi_1}\mathrm{SFit}(\mathfrak{S}_{\pi_2}\dots\mathfrak{S}_{\pi_r}\mathfrak{Y},\mathfrak{S}_{\sigma_1}\dots\mathfrak{S}_{\sigma_t}\mathfrak{Y})\cap\mathfrak{S}_{\sigma_1}\mathrm{SFit}(\mathfrak{S}_{\pi_1}\dots\mathfrak{S}_{\pi_r}\mathfrak{Y},\mathfrak{S}_{\sigma_2}\dots\mathfrak{S}_{\sigma_t}\mathfrak{Y}).$$

Proof:
$\subseteq$: Obvious. $\supseteq$: By induction on $r+t$:

$r+t=2$: Suppose not. Let G be a counterexample of minimal order. G has a unique minimal normal subgroup M, and $M \in \mathfrak{S}_p$ for a prime p. Thus $G \in \mathfrak{S}_p\mathrm{SFit}(\mathfrak{S}_{\pi_1}\mathfrak{Y},\mathfrak{S}_{\sigma_1}\mathfrak{Y})$.
If $p \in \pi_1\cap\sigma_1$, then 2.2.2 yields a contradiction to the choice of G.
If $p \in \pi_1\setminus\sigma_1$, then $O_{\sigma_1}(G) = 1$. Thus G belongs to $\mathfrak{S}_{\pi_1}\mathfrak{Y}$; a contradiction.
$p \in \sigma_1\setminus\pi_1$: analogously.

$r+t>2$: Suppose not. Let G be a counterexample of minimal order. Then $G \in \mathfrak{S}_p\mathrm{SFit}(\mathfrak{S}_{\pi_1}\dots\mathfrak{S}_{\pi_r}\mathfrak{Y},\mathfrak{S}_{\sigma_1}\dots\mathfrak{S}_{\sigma_t}\mathfrak{Y})$ for a suitable prime p. Arguing as above we obtain a contradiction to the choice of G.

$\square$

2.2.9 Remark
Let $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ be Fitting classes and $\mathfrak{Y}$ be a Fitting formation. Then

$$(\bigcap_{i=1}^{n} \mathfrak{X}_i)\mathfrak{Y} = \bigcap_{i=1}^{n}(\mathfrak{X}_i\mathfrak{Y})$$

Proof: Notice that $\cap_{i=1}^{n} G_{\mathfrak{X}_i} = G_{\cap_{i=1}^{n}\mathfrak{X}_i}$ and $\mathfrak{Y} = \mathrm{Q}\mathfrak{Y} = \mathrm{R}_0\mathfrak{Y}$. □

2.2.10 Lemma
Let $\pi_1, \ldots, \pi_r, \sigma_1, \ldots, \sigma_r$ be non-trivial sets of primes, $r, t \geq 1$ be natural numbers and let $\mathfrak{Y}$ be an SFitting class. Further set $\mathfrak{F}_1 = \mathfrak{S}_{\pi_1} \ldots \mathfrak{S}_{\pi_r}$ and $\mathfrak{F}_2 = \mathfrak{S}_{\sigma_1} \ldots \mathfrak{S}_{\sigma_t}$. Then

$$\mathrm{SFit}(\mathfrak{F}_1\mathfrak{Y}, \mathfrak{F}_2\mathfrak{Y}) = \mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)\mathfrak{Y}.$$

Proof: The assertion follows by induction on $r + t$ and repeated application of 2.2.9 and 2.2.8. □

2.2.11 Proposition
Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ be SFitting classes of bounded nilpotent length and let $\mathfrak{Y}$ be an arbitrary SFitting class. Then

$$\mathrm{SFit}(\mathfrak{F}_1\mathfrak{Y}, \mathfrak{F}_2\mathfrak{Y}) = \mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)\mathfrak{Y}.$$

Proof:

- $\subseteq$: Obvious.

- $\supseteq$: Let G be a group contained in $\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)\mathfrak{Y}$ and set $r = \max\{l(\mathfrak{F}_1), l(\mathfrak{F}_2)\}$. According to 2.1.5(a) there exists a countable set of classes $\mathfrak{X}_i$, $\mathfrak{Y}_j$ such that $\mathfrak{F}_1 = \cap_{i=1}^{\infty}\mathfrak{X}_i$, $\mathfrak{F}_2 = \cap_{j=1}^{\infty}\mathfrak{Y}_j$ and $\mathfrak{S}_{\pi(G)} \subseteq \mathfrak{X}_i, \mathfrak{Y}_j$ for all but a finite number of $\mathfrak{X}_i$, $\mathfrak{Y}_j$. Denote these exceptions by $\mathfrak{X}_{i_1}, \ldots, \mathfrak{X}_{i_n}, \mathfrak{Y}_{j_1}, \ldots, \mathfrak{Y}_{j_m}$. 2.2.7 yields

  $$G \in (\cap_{k=1}^{n} \cap_{l=1}^{m} \mathrm{SFit}(\mathfrak{X}_{i_k}, \mathfrak{Y}_{j_l}) \cap \mathfrak{N}^r)\mathfrak{Y},$$

  and thus we obtain by 2.2.9 and 2.2.10

  $$G \in \cap_{k=1}^{n}\cap_{l=1}^{m}\mathrm{SFit}(\mathfrak{X}_{i_k}, \mathfrak{Y}_{j_l})\mathfrak{Y}\cap\mathfrak{N}^r\mathfrak{Y} \subseteq \cap_{k=1}^{n}\cap_{l=1}^{m}\mathrm{SFit}(\mathfrak{X}_{i_k}\mathfrak{Y}, \mathfrak{Y}_{j_l}\mathfrak{Y})\cap\mathfrak{N}^r\mathfrak{Y}.$$

By 2.2.6 and a further application of 2.2.9, we obtain

$$G \in \mathrm{SFit}(\cap_{k=1}^{n}\mathfrak{X}_{i_k}\mathfrak{Y}, \cap_{l=1}^{m}\mathfrak{Y}_{j_l}\mathfrak{Y}) \cap \mathfrak{S}_{\pi(G)}$$

$$\subseteq \mathrm{SFit}((\cap_{k=1}^{n}\mathfrak{X}_{i_k})\mathfrak{Y}, (\cap_{l=1}^{m}\mathfrak{Y}_{j_l})\mathfrak{Y}) \cap \mathfrak{S}_{\pi(G)}.$$

By the choice of the classes $\mathfrak{X}_{i_k},\ \mathfrak{Y}_{j_l}$, we obtain using 2.2.2

$$G \in \mathrm{SFit}((\cap_{k=1}^{n}\mathfrak{X}_{i_k})\mathfrak{Y} \cap \mathfrak{S}_{\pi(G)}, (\cap_{l=1}^{m}\mathfrak{Y}_{j_l})\mathfrak{Y} \cap \mathfrak{S}_{\pi(G)})$$

$$\subseteq \mathrm{SFit}((\cap_{i=1}^{\infty}\mathfrak{X}_i)\mathfrak{Y}, (\cap_{j=1}^{\infty}\mathfrak{Y}_j)\mathfrak{Y}) = \mathrm{SFit}(\mathfrak{F}_1\mathfrak{Y}, \mathfrak{F}_2\mathfrak{Y}),$$

and the proof is complete.

□

2.2.12 Corollary
Let $\mathfrak{F}_1,\ \mathfrak{F}_2$ and $\mathfrak{Y}$ be SFitting classes. Then

$$\mathrm{SFit}(\mathfrak{F}_1\mathfrak{Y}, \mathfrak{F}_2\mathfrak{Y}) = \mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)\mathfrak{Y}.$$

Proof: $\subseteq$: Obvious. $\supseteq$: Let G be an $\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)\mathfrak{Y}$-group. Then G is contained in $\mathfrak{N}^r$, where r is the nilpotent length of G. By 2.2.5 and 2.2.11 this provides the assertion. □

To generalize the above results to arbitrary many SFitting classes we need the following lemma.

2.2.13 Lemma
Let $\mathfrak{X}_i,\ i \in I$, be classes of groups, $\mathfrak{X} = \cup_{i\in I}\mathfrak{X}_i$.

(a) *If $I = \{1, \ldots, n\}$, then* $\mathrm{SFit}(\mathfrak{X}_1, \ldots, \mathfrak{X}_n) = \mathrm{SFit}(\mathfrak{X}_1, \mathrm{SFit}(\mathfrak{X}_2, \ldots, \mathfrak{X}_n))$.

(b) *Let I be an arbitrary set and n a natural number. If $G \in (s\mathrm{N}_0)^n\mathfrak{X}$, then there exists a finite subset $I_n(G)$ of I such that $G \in \mathrm{SFit}(\cup_{i\in I_n(G)}\mathfrak{X}_i)$.*

Proof:

(a) Obvious.

(b) We argue by induction on n:

$n = 1$: If $G \in \mathrm{N}_0\mathfrak{X}$, then there exist subnormal subgroups $N_1, \ldots, N_k$ of G such that $G = \langle N_1, \ldots, N_k \rangle$ and $N_j \in \mathfrak{X}$ for $j = 1, \ldots, k$. Thus there exist $i_1, \ldots, i_k \in I$ such that $N_j \in \mathfrak{X}_{i_j}$. Hence $G \in \mathrm{N}_0(\cup_{j=1}^{k} \mathfrak{X}_{i_j}) \subseteq \mathrm{SFit}(\mathfrak{X}_{i_j} \mid j \in \{1, \ldots, k\})$ and $I_0(G) = \{i_1, \ldots, i_k\}$.
Let G be a group contained in $\mathrm{SN}_0\mathfrak{X}$. Then there exist a group $W \in \mathrm{N}_0\mathfrak{X}$ and a monomorphism from G to W. By hypothesis, W is contained in $\mathrm{SFit}(\mathfrak{X}_i \mid i \in I_0(W))$ where $I_0(W)$ is a finite subset of I. Hence in this case we have $I_0(G) = I_0(W)$.

$n > 1$: Analogously.

□

2.2.14 Theorem
Let $\mathfrak{Y}$ be an SFitting class.

(a) *Let $\mathfrak{F}_i,\ i \in I$, be classes of groups, then*

$$\mathfrak{Y}\mathrm{SFit}(\mathfrak{F}_i \mid i \in I) = \mathrm{SFit}(\mathfrak{Y} \circ \mathfrak{F}_i \mid i \in I).$$

(b) *Let $\mathfrak{F}_i,\ i \in I$, be SFitting classes, then*

$$\mathrm{SFit}(\mathfrak{F}_i \mid i \in I) \cap \mathfrak{Y} = \mathrm{SFit}(\mathfrak{F}_i \cap \mathfrak{Y} \mid i \in I).$$

(c) *Let $\mathfrak{F}_i,\ i \in I$, be SFitting classes, then*

$$\mathrm{SFit}(\mathfrak{F}_i \mid i \in I)\mathfrak{Y} = \mathrm{SFit}(\mathfrak{F}_i\mathfrak{Y} \mid i \in I).$$

Proof: The theorem follows by induction and 2.2.2, 2.2.5, 2.2.12 and 2.2.13.
□

Notice that by 2.2.14(b), the lattice of SFitting classes is distributive. Since by Bryce and Cossey (cf. [6], [8]) SFitting classes are precisely the so-called primitive saturated formations (or, in the terminology of Shemetkov and Skiba, the totally local formations), this has been proved already by Shemetkov and Skiba (cf. [20, 9.8]).
The following proposition enables us to obtain further information about this lattice.

2.2.15 Proposition
Let $\mathfrak{F}$ be an SFitting class of bounded nilpotent length. Then

$$\mathfrak{F} = \mathrm{SFit}(\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \mid p_i \textit{ primes}, \, p_i \neq p_{i+1} \textit{ and } \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \subseteq \mathfrak{F}).$$

Proof:

- $\supseteq$: Obvious.
- $\subseteq$: We argue by induction on $r := l(\mathfrak{F})$.
  $r = 1$ is obvious.
  $r > 1$: First, we consider the case $\mathfrak{F} = \mathfrak{S}_p(\mathfrak{F} \cap \mathfrak{N}^{r-1})$ where p is a prime.
  Set $\mathfrak{X} = \mathrm{SFit}(\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \mid p_i$ primes, $p_i \neq p_{i+1}$ and $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \subseteq \mathfrak{F})$. Suppose $\mathfrak{F}$ is not contained in $\mathfrak{X}$. $\mathfrak{F} = \mathfrak{S}_p\mathfrak{F}$, thus 2.2.2 gives $\mathfrak{X} = \mathfrak{S}_p\mathfrak{X}$. Let G be a group of minimal order contained in $\mathfrak{F} \setminus \mathfrak{X}$. Then $O_p(G) = 1$ and $G \in F(q) \subseteq \mathfrak{N}^{r-1}$ where F denotes the canonical local definition of $\mathfrak{F}$ and q a prime distinct of p. By inductive hypothesis we obtain $F(q) = \mathrm{SFit}(\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \mid p_i$ primes, $p_i \neq p_{i+1}$ and $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \subseteq F(q) \subseteq \mathfrak{F}) \subseteq \mathfrak{X}$, a contradiction to the choice of G.
  Now, let $\mathfrak{F}$ be an arbitrary SFitting class of nilpotent length r. Let G be a group of minimal order contained in $\mathfrak{F} \setminus \mathrm{SFit}(\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \mid p_i$ primes, $p_i \neq p_{i+1}$ and $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \subseteq \mathfrak{F})$, thus $O_{p'}(G) = 1$ for a suitable prime p. Hence $G \in F(p)$ where F denotes the canonical local definition of $\mathfrak{F}$. By 1.3.7 $F(p) = \mathfrak{S}_p(F(p) \cap \mathfrak{N}^{r-1})$, so the assertion holds for $F(p)$ and we obtain a contradiction to the choice of G. □

2.2.16 Corollary
Let $\mathfrak{F}$ be an SFitting class. Then

$$\mathfrak{F} = \mathrm{SFit}(\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \mid p_i \textit{ primes } p_i \neq p_{i+1} \textit{ and } \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \subseteq \mathfrak{F}).$$

Proof: 2.2.5 and 2.2.15. □

2.2.17 Definition
Let $\mathfrak{F}$ be an SFitting class. We define

$$\mathfrak{L}_{\mathfrak{F}} := (\{\mathfrak{X} \mid \mathfrak{X} \text{ SFitting class}, \, \mathfrak{F} \subseteq \mathfrak{X}\}, \, \subseteq).$$

2.2.18 Proposition
Let $\mathfrak{F} \neq \mathfrak{S}$ be an SFitting class and set $\pi = \pi(\mathfrak{F})$.

(a) *If $l(\mathfrak{F}) = r < \infty$, then the following statements are equivalent:*

- **(i)** *$\mathfrak{X}$ is an atom of $\mathfrak{L}_{\mathfrak{F}}$.*
- **(ii)** *$\mathfrak{X} = \mathfrak{F} \times \mathfrak{S}_q$ for a suitable prime $q \notin \pi$ or $\mathfrak{X} = \mathrm{SFit}(\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k}, \mathfrak{F})$ where $p_i \neq p_{i+1}$, $k \leq r+1$ and $p_1, \ldots, p_k \in \pi$ such that $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \not\subseteq \mathfrak{F}$ but $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \cap \mathfrak{N}^{k-1} \subseteq \mathfrak{F}$.*

(b) *If $l(\mathfrak{F}) = \infty$, then the following statements are equivalent:*

- **(i)** *$\mathfrak{X}$ is an atom of $\mathfrak{L}_{\mathfrak{F}}$.*
- **(ii)** *$\mathfrak{X} = \mathfrak{F} \times \mathfrak{S}_q$ for a suitable prime $q \notin \pi$ or $\mathfrak{X} = \mathrm{SFit}(\mathfrak{X}_0, \mathfrak{F})$ where $\mathfrak{X}_0$ is an atom of $\mathfrak{L}_{\mathfrak{F} \cap \mathfrak{N}^k}$ such that $\mathfrak{X}_0 \not\subseteq \mathfrak{F}$ but $\mathfrak{X}_0 \cap \mathfrak{N}^{k-1} \subseteq \mathfrak{F}$ ($k \in \mathbb{N}$ suitable).*

(c) *$\mathfrak{L}_{\mathfrak{F}}$ is atomic, that is, for every element $\mathfrak{X} \in \mathfrak{L}_{\mathfrak{F}}$, $\mathfrak{X} \neq \mathfrak{F}$, there exists an atom contained in $\mathfrak{X}$.*

Proof:
$(a), (c)$: It is obvious that classes as described in $(a)(ii)$ always exist (for otherwise, $\mathfrak{N}_\pi^{r+1} = \mathrm{SFit}(\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_{r+1}} \mid p_i \in \pi) \subseteq \mathfrak{F}$, a contradiction). Moreover $\mathfrak{F}$ is strictly contained in these classes.
We now prove the assertion by induction on r.

$r = 1$, hence $\mathfrak{F} = \mathfrak{N}_\pi$:

(1) If $\mathfrak{H} \in \mathfrak{L}_{\mathfrak{N}_\pi}$, $\mathfrak{H} \supset \mathfrak{N}_\pi$, then there exists a class $\mathfrak{X}$ as described in $(a)(ii)$ such that $\mathfrak{X} \subseteq \mathfrak{H}$:
If $\pi(\mathfrak{H}) \neq \pi$, then $\mathfrak{S}_q \subseteq \mathfrak{H}$ for some prime $q \in \pi(\mathfrak{H}) \setminus \pi$ and therefore $\mathfrak{N}_\pi \times \mathfrak{S}_q \subseteq \mathfrak{H}$.
Hence assume that $\pi(\mathfrak{H}) = \pi$ and let G be a group of minimal order contained in $\mathfrak{H} \setminus \mathfrak{N}_\pi$. Then $l(G) = 2$ and $G \in \mathfrak{S}_{p_1}\mathfrak{S}_{p_2}$ for suitable primes $p_1, p_2 \in \pi$. By 2.1.5(b), we obtain $\mathfrak{S}_{p_1}\mathfrak{S}_{p_2} \subseteq \mathfrak{H}$ and the assertion follows.

(2) The classes described in $(a)(ii)$ are atoms and each atom is of this form:
Let $\mathfrak{X}$ be a class as described in $(a)(ii)$. If $\mathfrak{X} = \mathfrak{N}_\pi \times \mathfrak{S}_q$, $q \notin \pi$, the assertion evidently holds. Thus we may assume that $\mathfrak{X} = \mathrm{SFit}(\mathfrak{N}_\pi, \mathfrak{S}_{p_1}\mathfrak{S}_{p_2})$

for suitable primes p_1, p_2. Let $\mathfrak{H}$ be a class contained in $\mathfrak{L}_{\mathfrak{N}_\pi}$ such that $\mathfrak{N}_\pi \subset \mathfrak{H} \subseteq \mathfrak{X}$. According to (1), there exist primes q_1, q_2 such that $\mathfrak{N}_\pi \subset \mathrm{SFit}(\mathfrak{N}_\pi, \mathfrak{S}_{q_1}\mathfrak{S}_{q_2}) \subseteq \mathfrak{H} \subseteq \mathfrak{X}$. If $\mathfrak{S}_{p_1}\mathfrak{S}_{p_2} \neq \mathfrak{S}_{q_1}\mathfrak{S}_{q_2}$, then we obtain $\mathfrak{S}_{p_1}\mathfrak{S}_{p_2} \cap \mathfrak{S}_{q_1}\mathfrak{S}_{q_2} \subseteq \mathfrak{N}_\pi$ and 2.2.5 yields $\mathrm{SFit}(\mathfrak{N}_\pi, \mathfrak{S}_{q_1}\mathfrak{S}_{q_2}) = \mathrm{SFit}(\mathfrak{N}_\pi, \mathfrak{S}_{p_1}\mathfrak{S}_{p_2} \cap \mathfrak{S}_{q_1}\mathfrak{S}_{q_2}) \subseteq \mathfrak{N}_\pi$, a contradiction. Consequently $\mathfrak{H} = \mathfrak{X}$ and $\mathfrak{X}$ is an atom of $\mathfrak{L}_{\mathfrak{N}_\pi}$. Now the assertion follows from (1).

$r > 1$:

First, we prove the assertion for classes of the form $\mathfrak{F} = \mathfrak{S}_p(\mathfrak{F} \cap \mathfrak{N}^{r-1})$:

(3) If $\mathfrak{H} \in \mathfrak{L}_{\mathfrak{F}}$, $\mathfrak{H} \supset \mathfrak{F}$, then there exists a class $\mathfrak{X}$ as described in $(a)(ii)$ fulfilling $\mathfrak{X} \subseteq \mathfrak{H}$:
Without loss of generality we may assume that $\pi(\mathfrak{H}) = \pi$. Let H and F, respectively, be the canonical local definitions of $\mathfrak{H}$ and $\mathfrak{F}$, respectively. Let G be a group of minimal order contained in $\mathfrak{H} \setminus \mathfrak{F}$. Since G has a unique minimal normal subgroup and $\mathfrak{S}_p\mathfrak{F} = \mathfrak{F}$, there exists a prime $q \neq p$ such that $O_q(G) \neq 1 = O_{q'}(G)$. Evidently, $F(q) \subset H(q)$. By the choice of $\mathfrak{F}$, $F(q) \subseteq \mathfrak{N}^{r-1}$; thus by inductive hypothesis there exists an atom $\mathfrak{X}_0$ of $\mathfrak{L}_{F(q)}$ as described in $(a)(ii)$ such that $\mathfrak{X}_0 \subseteq H(q)$. If $\mathfrak{X}_0 = F(q) \times \mathfrak{S}_t$ where $t \in \pi \setminus \pi(F(q))$, then $\mathfrak{S}_q\mathfrak{S}_t \subseteq H(q) \subseteq \mathfrak{H}$ and $\mathfrak{S}_q\mathfrak{S}_t$ is not contained in $\mathfrak{F}$ (for otherwise $Z_q \wr Z_t \in F(q)$, a contradiction); consequently, $\mathrm{SFit}(\mathfrak{F}, \mathfrak{S}_q\mathfrak{S}_t) \subseteq \mathfrak{H}$ is a class satisfying the condition in $(a)(ii)$ and we are finished.
Thus we may assume that $\mathfrak{X}_0 = \mathrm{SFit}(\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k}, F(q))$ where $k \leq r$ and $p_1, \ldots, p_k \in \pi(F(q))$ such that $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \not\subseteq F(q)$ but $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \cap \mathfrak{N}^{k-1} \subseteq F(q)$. If $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \not\subseteq \mathfrak{F}$, the assertion follows. Thus we may assume that $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \subseteq \mathfrak{F}$.
$q \neq p_1$, for otherwise we obtain $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \subseteq \mathfrak{S}_q F(q) = F(q)$, a contradiction.
$\mathfrak{S}_q\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \not\subseteq \mathfrak{F}$: Suppose that $Z_q \wr G$ is contained in $\mathfrak{F}$ for an arbitrary group $G \in \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k}$. Since $O_{q'}(Z_q \wr G) = 1$, this implies $Z_q \wr G \in F(q)$, a contradiction.
$\mathfrak{S}_q\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \cap \mathfrak{N}^k \subseteq \mathfrak{F}$: Suppose not. Let G be a group of minimal order contained in $\mathfrak{S}_q\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \cap \mathfrak{N}^k \setminus \mathfrak{F}$. Then G has a unique minimal normal subgroup. Since $O_q(G) = 1$ implies the contradiction $G \in \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \subseteq \mathfrak{F}$, we obtain $F(G) = O_q(G)$ and $G/O_q(G) \in \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \cap \mathfrak{N}^{k-1} \subseteq F(q)$. Consequently $G \in \mathfrak{S}_q F(q) = F(q) \subseteq \mathfrak{F}$, a

final contradiction.
Thus $\mathfrak{X} = \mathrm{SFit}(\mathfrak{F}, \mathfrak{S}_q\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k})$ is a class as required in $(a)(ii)$. Since $\mathfrak{S}_q\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \subseteq \mathfrak{S}_q H(q) = H(q)$, we have $\mathfrak{X} \subseteq \mathfrak{H}$.

(4) The classes described in $(a)(ii)$ are atoms and each atom is of this form:
Let $\mathfrak{X}$ be a class as described in $(a)(ii)$. If $\mathfrak{X} = \mathfrak{F} \times \mathfrak{S}_q$ for a suitable prime q, the assertion evidently holds. Consequently, $\mathfrak{X} = \mathrm{SFit}(\mathfrak{F}, \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k})$ for suitable primes $p_1, \ldots, p_k \in \pi$. Let $\mathfrak{H} \in \mathfrak{L}_{\mathfrak{F}}$ such that $\mathfrak{F} \subset \mathfrak{H} \subseteq \mathfrak{X}$. According to (3), there exist primes $q_1, \ldots, q_l$ as described in $(a)(ii)$ satisfying $\mathfrak{F} \subset \mathrm{SFit}(\mathfrak{F}, \mathfrak{S}_{q_1} \cdots \mathfrak{S}_{q_l}) \subseteq \mathfrak{H} \subseteq \mathfrak{X}$.
By 2.2.5 and the choice of $p_1, \ldots, p_k$, $q_1, \ldots, q_l$, we obtain $l(\mathfrak{S}_{q_1} \cdots \mathfrak{S}_{q_l} \cap \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k}) = k$ (for otherwise $\mathfrak{S}_{q_1} \cdots \mathfrak{S}_{q_l} \cap \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \subseteq \mathfrak{F}$ and consequently $\mathrm{SFit}(\mathfrak{F}, \mathfrak{S}_{q_1} \cdots \mathfrak{S}_{q_l}) = \mathrm{SFit}(\mathfrak{F}, \mathfrak{S}_{q_1} \cdots \mathfrak{S}_{q_l}) \cap \mathfrak{H} = \mathfrak{F}$; a contradiction). Hence 2.1.5(b) implies $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} = \mathfrak{S}_{q_1} \cdots \mathfrak{S}_{q_l}$ and $\mathfrak{X}$ is an atom.
Now, the assertion follows from (3).

Let $\mathfrak{F}$ be an arbitrary SFitting class of nilpotent length r:

(5) If $\mathfrak{H} \in \mathfrak{L}_{\mathfrak{F}}$, $\mathfrak{H} \supset \mathfrak{F}$, then there exists a class $\mathfrak{X}$ as described in $(a)(ii)$ satisfying $\mathfrak{X} \subseteq \mathfrak{H}$:
We may assume that $\pi(\mathfrak{H}) = \pi(\mathfrak{F})$; let H and F, respectively, be the canonical local definition of $\mathfrak{H}$ and $\mathfrak{F}$, respectively. If G is a group of minimal order contained in $\mathfrak{H} \setminus \mathfrak{F}$, then there exists a prime $p \in \pi(\mathfrak{F})$ such that $O_{p'}(G) = 1$. Thus G is contained in $H(p) \setminus F(p)$. $F(p) = \mathfrak{S}_p(F(p) \cap \mathfrak{N}^{r-1})$, hence by inductive hypothesis there exists an atom $\mathfrak{X}_0$ of $\mathfrak{L}_{F(p)}$ as required in $(a)(ii)$ fulfilling $H(p) \supseteq \mathfrak{X}_0 \supset F(p)$. We obtain a final contradiction as above.

(6) The classes described in $(a)(ii)$ are atoms and each atom is of this form:
This follows as above.

$(b), (c)$:

(1) A class $\mathfrak{X}_0$ as required in $(b)(ii)$ always exists and $\mathfrak{X}_0 \in \mathfrak{L}_{\mathfrak{F}} \setminus \{\mathfrak{F}\}$:
It is sufficient to show the existence of such a class. Assume $\mathfrak{X}_0$ does not exist. Then $\pi(\mathfrak{F}) = \mathbb{P}$ and therefore $\mathfrak{N} \subseteq \mathfrak{F}$. Since $\mathrm{SFit}(\mathfrak{X} \mid \mathfrak{X} \text{ is atom of } \mathfrak{L}_{\mathfrak{N}^i}) = \mathfrak{N}^{i+1}$, we conclude inductively $\mathfrak{N}^{i+1} \subseteq \mathfrak{F}$ for all $i \in \mathbb{N}$. This implies $\mathfrak{F} = \mathfrak{S}$; a contradiction.

(2) If $\mathfrak{H} \in \mathfrak{L}_{\mathfrak{F}}$, $\mathfrak{H} \supset \mathfrak{F}$, then there exists a class $\mathfrak{X}$ as described in $(b)(ii)$ satisfying $\mathfrak{X} \subseteq \mathfrak{H}$:
If $\pi(\mathfrak{H}) \neq \pi(\mathfrak{F})$, there is nothing to prove. Thus assume $\pi(\mathfrak{H}) = \pi(\mathfrak{F})$. As $\mathfrak{H} \supset \mathfrak{F}$, there is a natural number k such that $\mathfrak{H} \cap \mathfrak{N}^k \supset \mathfrak{F} \cap \mathfrak{N}^k$. Let k be minimal with this property. According to 2.2.18 (a), (c) there exists an atom $\mathfrak{X}_0$ of $\mathfrak{L}_{\mathfrak{F}\cap\mathfrak{N}^k}$ fulfilling $\mathfrak{F} \cap \mathfrak{N}^k \subset \mathfrak{X}_0 \subseteq \mathfrak{H} \cap \mathfrak{N}^k$. $\mathfrak{X}_0 \subseteq \mathfrak{N}^k$, thus $\mathfrak{X}_0 \not\subseteq \mathfrak{F}$, and by the choice of k we obtain $\mathfrak{F} \cap \mathfrak{N}^{k-1} \subseteq \mathfrak{X}_0 \cap \mathfrak{N}^{k-1} \subseteq \mathfrak{H} \cap \mathfrak{N}^{k-1} = \mathfrak{F} \cap \mathfrak{N}^{k-1} \subseteq \mathfrak{F}$. Consequently, $\mathfrak{X}_0$ is a class as required in $(a)(ii)$ and we are finished.

(3) The classes described in $(b)(ii)$ are atoms, and each atom is of this form:
Let $\mathfrak{X}$ be a class as described in $(b)(ii)$ and let $\mathfrak{H} \in \mathfrak{L}_{\mathfrak{F}}$ such that $\mathfrak{F} \subset \mathfrak{H} \subseteq \mathfrak{X}$. If $\mathfrak{X} = \mathfrak{F} \times \mathfrak{S}_q$, there is nothing to show. Thus we may assume that $\mathfrak{X} = \mathrm{SFit}(\mathfrak{F}, \mathfrak{X}_0)$ and $\pi(\mathfrak{X}) = \pi(\mathfrak{F})$. According to (2), there exists a class $\mathfrak{Y} = \mathrm{SFit}(\mathfrak{F}, \mathfrak{Y}_0)$ such that $\mathfrak{Y}_0$ is an atom of $\mathfrak{L}_{\mathfrak{F}\cap\mathfrak{N}^l}$ and $\mathfrak{F} \subset \mathfrak{Y} \subseteq \mathfrak{H} \subseteq \mathfrak{X}$. Hence 2.2.5 yields $\mathfrak{Y} = \mathfrak{Y} \cap \mathfrak{X} = \mathrm{SFit}(\mathfrak{F}, \mathfrak{Y}_0 \cap \mathfrak{X}_0)$.
We show that $\mathfrak{X}$ and $\mathfrak{Y}$ coincide: If $k \geq l$, then $\mathfrak{X}_0 \cap \mathfrak{Y}_0 \supseteq \mathfrak{F} \cap \mathfrak{N}^l$. $\mathfrak{Y}_0$ is an atom of $\mathfrak{L}_{\mathfrak{F}\cap\mathfrak{N}^l}$ and $\mathfrak{X}_0 \cap \mathfrak{Y}_0 \not\subseteq \mathfrak{F}$, thus we obtain $\mathfrak{X}_0 \cap \mathfrak{Y}_0 = \mathfrak{Y}_0$, and therefore $\mathfrak{Y}_0 \subseteq \mathfrak{X}_0$. Since $\mathfrak{F} \cap \mathfrak{N}^k$ is a class strictly contained in $\mathrm{SFit}(\mathfrak{F} \cap \mathfrak{N}^k, \mathfrak{Y}_0) \subseteq \mathrm{SFit}(\mathfrak{F} \cap \mathfrak{N}^k, \mathfrak{X}_0) = \mathfrak{X}_0$, and $\mathfrak{X}_0$ is an atom of $\mathfrak{L}_{\mathfrak{F}\cap\mathfrak{N}^k}$, we conclude that $\mathrm{SFit}(\mathfrak{F}\cap\mathfrak{N}^k, \mathfrak{Y}_0) = \mathfrak{X}_0$. Consequently, $\mathfrak{X} \subseteq \mathfrak{Y}$. If $k \leq l$, we obtain analogously $\mathfrak{X}_0 \subseteq \mathfrak{Y}_0$ and consequently $\mathfrak{X} = \mathfrak{Y}$. Now the assertion follows from (2).

□

2.2.19 Definition
Let $\mathfrak{F}$ be an SFitting class. We define

$$\mathfrak{L}^{\mathfrak{F}} = (\{\mathfrak{X} \mid \mathfrak{X} \text{ SFitting class, } \mathfrak{X} \subseteq \mathfrak{F}\},\ \subseteq).$$

2.2.20 Proposition
Let $\mathfrak{F}$ be an SFitting class of bounded nilpotent length and let $\{\mathfrak{F}_i\}_{i\in I}$ denote the maximal elements of the set $\{\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \mid p_i \neq p_{i+1}$ primes , $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \subseteq \mathfrak{F}\}$, thus $\mathfrak{F}_i \subseteq \mathfrak{F}_j$ implies $\mathfrak{F}_i = \mathfrak{F}_j$ for all i, j.

(a) *The following statements are equivalent:*

(i) *$\mathfrak{X}$ is a dual atom of $\mathfrak{L}^{\mathfrak{F}}$.*

(ii) $\mathfrak{X} = \mathrm{SFit}(\mathrm{SFit}(\mathfrak{F}_i \mid i \in I \setminus \{i_0\}), \mathfrak{F}_{i_0} \cap \mathfrak{N}^{k_0-1})$ *where* k_0 *denotes the nilpotent length of* $\mathfrak{F}_{i_0}$.

(b) $\mathfrak{L}^{\mathfrak{F}}$ *is dual atomic (that is, for every element* $\mathfrak{X} \in \mathfrak{L}^{\mathfrak{F}}$, $\mathfrak{X} \neq \mathfrak{F}$ *there exists a dual atom containing* $\mathfrak{X}$*).*

Proof: We prove (a) and (b) simultanously. Set $r = l(\mathfrak{F})$.

(1) The classes described in $(a)(ii)$ are elements of $\mathfrak{L}^{\mathfrak{F}}$ distinct from $\mathfrak{F}$:
$\mathfrak{F} = \mathrm{SFit}(\mathfrak{F}_i \mid i \in I)$ according to 2.2.15, thus the classes described in $(a)(ii)$ are elements of $\mathfrak{L}^{\mathfrak{F}}$. Let $\mathfrak{X}$ be such a class. Then $\mathfrak{F}_{i_0} \not\subseteq \mathrm{SFit}(\mathrm{SFit}(\mathfrak{F}_i \mid i \in I \setminus \{i_0\}), \mathfrak{F}_{i_0} \cap \mathfrak{N}^{k_0-1})$ and consequently $\mathfrak{X} \neq \mathfrak{F}$:
Assume not. Then 2.2.5 gives $\mathfrak{F}_{i_0} = \mathrm{SFit}(\mathrm{SFit}(\mathfrak{F}_i \cap \mathfrak{F}_{i_0} \mid i \in I \setminus \{i_0\}), \mathfrak{F}_{i_0} \cap \mathfrak{N}^{k_0-1})$. By the choice of $\mathfrak{F}_i$ and 2.1.5(b) this leads to $\mathfrak{F}_i \cap \mathfrak{F}_{i_0} \subseteq \mathfrak{N}^{k_0-1}$, a contradiction.

(2) If $\mathfrak{H} \in \mathfrak{L}^{\mathfrak{F}}$, $\mathfrak{H} \subset \mathfrak{F}$, then there exists a class $\mathfrak{X}$ described as in $(a)(ii)$ such that $\mathfrak{X} \subseteq \mathfrak{H}$:
Let $\{\mathfrak{H}_j\}_{j \in J}$ be the maximal elements of the set $\{\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \mid p_i \neq p_{i+1}$ primes , $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_k} \subseteq \mathfrak{H}\}$. Then $\mathfrak{H} = \mathrm{SFit}(\mathfrak{H}_j \mid j \in J)$ according to 2.2.15. Since $\mathfrak{H}$ is strictly contained in $\mathfrak{F}$, there exists an element $i_0 \in I$ such that $\mathfrak{H} \cap \mathfrak{F}_{i_0} = 1$ or $\mathfrak{H}_{j_0} \subset \mathfrak{F}_{i_0}$ (with $j_0 \in J$ suitable). In the first case we obtain $\mathfrak{H} = \mathfrak{H} \cap \mathfrak{F} = \mathrm{SFit}(\mathfrak{H} \cap \mathfrak{F}_i \mid i \in I) \subseteq \mathrm{SFit}(\mathrm{SFit}(\mathfrak{H} \cap \mathfrak{F}_i \mid i \in I \setminus \{i_0\}), \mathfrak{F}_{i_0} \cap \mathfrak{N}^{l(\mathfrak{F}_{i_0})-1})$ and we are finished.
So assume $\mathfrak{H}_{j_0} \subset \mathfrak{F}_{i_0}$ for suitable indices i_0, j_0 and set $k_0 = l(\mathfrak{F}_{i_0})$. If $l(\mathfrak{H}_{j_0}) = k_0$, then 2.1.5(b) provides a contradiction. Consequently, $\mathfrak{H}_{j_0} \subseteq \mathfrak{F}_{i_0} \cap \mathfrak{N}^{k_0-1}$ holds true. If $j_1 \in J \setminus \{j_0\}$ and $\mathfrak{H}_{j_1} \subseteq \mathfrak{F}_{i_0}$, $\mathfrak{H}_{j_1} \not\subseteq \mathfrak{N}^{k_0-1}$, then 2.1.5(b) gives $\mathfrak{H}_{j_1} = \mathfrak{F}_{i_0} \supseteq \mathfrak{H}_{j_0}$; this contradicts the choice of the $\mathfrak{H}_j$. Hence, $\mathfrak{H}_j \subseteq \mathrm{SFit}(\mathrm{SFit}(\mathfrak{F}_i \mid i \in I \setminus \{i_0\}), \mathfrak{F}_{i_0} \cap \mathfrak{N}^{k_0-1})$ for every $j \in J$ and the assertion follows.

(3) The classes described in $(a)(ii)$ are dual atoms and each dual atom is of this form:
Let $\mathfrak{X}$ be a class as described in $(a)(ii)$. Assume there is an element $\mathfrak{H} \in \mathfrak{L}^{\mathfrak{F}}$ such that $\mathfrak{X} \subseteq \mathfrak{H} \subset \mathfrak{F}$. According to (2) there is a class $\mathfrak{G}$ as described in $(a)(ii)$ satisfying $\mathfrak{X} \subseteq \mathfrak{H} \subseteq \mathfrak{G}$. If $\mathfrak{X} \neq \mathfrak{G}$, then we obtain $\mathfrak{F}_i \subseteq \mathrm{SFit}(\mathfrak{X}, \mathfrak{G}) = \mathfrak{G}$ for all $i \in I$. But this implies $\mathfrak{F} \subseteq \mathfrak{G}$, a contradiction. Hence the assertion follows. □

2.2.21 Remark
For SFitting classes whose nilpotent length is not bounded, $\mathfrak{L}^{\mathfrak{F}}$ need not be dual atomic.

Proof: Let $p_1, p_2, \ldots$ be infinitely many pairwise distinct primes and set $\mathfrak{F} = \cup_{r \geq 1} \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r}$.
We prove that, if $\mathfrak{X} \in \mathfrak{L}^{\mathfrak{F}}$ such that $\mathfrak{X} \subset \mathfrak{F}$, then there exists a class $\mathfrak{H} \in \mathfrak{L}^{\mathfrak{F}}$ satisfying $\mathfrak{X} \subset \mathfrak{H} \subset \mathfrak{F}$. In particular, there do not exist dual atoms in $\mathfrak{L}^{\mathfrak{F}}$.
$\mathfrak{X} \subset \mathfrak{F}$, so there is a natural number r satisfying $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r} \not\subseteq \mathfrak{X}$. Set $\mathfrak{H} = \mathrm{SFit}(\mathfrak{X}, \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r})$. Clearly, $\mathfrak{X}$ is contained in $\mathfrak{H}$. Assume that $\mathfrak{H} = \mathfrak{F}$. Then 2.2.5 yields $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r} \mathfrak{S}_{p_{r+1}} = \mathfrak{H} \cap \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r} \mathfrak{S}_{p_{r+1}} = \mathrm{SFit}(\mathfrak{X} \cap \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r} \mathfrak{S}_{p_{r+1}}, \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r})$. If $l(\mathfrak{X} \cap \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r} \mathfrak{S}_{p_{r+1}}) = r+1$, then it follows from 2.1.5(b) that $\mathfrak{X} \cap \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r} \mathfrak{S}_{p_{r+1}} = \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r} \mathfrak{S}_{p_{r+1}}$, a contradiction. Thus $\mathfrak{X} \cap \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r} \mathfrak{S}_{p_{r+1}} \subseteq \mathfrak{N}^r$ and whence $\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_{r+1}} \subseteq \mathfrak{N}^r$, a final contradiction. □

2.2.22 Definition
Let $\mathfrak{G}$ and $\mathfrak{F}$ be SFitting classes, $\mathfrak{G} \subseteq \mathfrak{F}$. We define

$$\mathfrak{L}^{\mathfrak{F}}_{\mathfrak{G}} = (\{\mathfrak{X} \mid \mathfrak{X} \text{ SFitting class}, \mathfrak{G} \subseteq \mathfrak{X} \subseteq \mathfrak{F}\},\ \subseteq).$$

By 2.2.18, 2.2.20 and 2.2.21 we obtain

2.2.23 Theorem
Let $\mathfrak{G}$ and $\mathfrak{F}$ be SFitting classes, $\mathfrak{G} \subset \mathfrak{F}$.

(a) *$\mathfrak{L}^{\mathfrak{F}}_{\mathfrak{G}}$ is a (complete) distributive and atomic lattice. The atoms are given as described in 2.2.18.*

(b) *If additionally $\mathfrak{F}$ is of bounded nilpotent length, then $\mathfrak{L}^{\mathfrak{F}}_{\mathfrak{G}}$ is dual atomic and the dual atoms are given as described in 2.2.20.*

(c) *$\mathfrak{L}^{\mathfrak{F}}_{\mathfrak{G}}$ need not be dual atomic.*

Finally, we give an upper bound for the SFitting class generated by two SFitting classes $\mathfrak{F}_1$, $\mathfrak{F}_2$, a result that will be needed in Chapter 3.
Evidently, $\mathfrak{F}_1\mathfrak{F}_2 \cap \mathfrak{F}_2\mathfrak{F}_1$ is an SFitting class containing $\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)$. If $\pi(\mathfrak{F}_1) \cap \pi(\mathfrak{F}_2) = \emptyset$, then equality holds, but in general, $\mathfrak{F}_1\mathfrak{F}_2 \cap \mathfrak{F}_2\mathfrak{F}_1$ is strictly larger than $\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)$ and there is a better bound for $\mathrm{SFit}(\mathfrak{F}_1,\ \mathfrak{F}_2)$.

2.2.24 Proposition
Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be SFitting classes and π be a set of primes as described in 1.2.19. Then $S_\pi(\mathfrak{F}_1, \mathfrak{F}_2) = (G \mid G/G_{\mathfrak{F}_1}G_{\mathfrak{F}_2} \in \mathfrak{S}_\pi)$ is a Fitting class and the following statements hold:

(a) $S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)\mathfrak{S}_\pi = S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$.

(b) $S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$ *is subgroup-closed.*

In particular, $\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2) \subseteq S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$.

Proof: That $S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$ is a Fitting class is proved analogously to [9, IX,2.1].

(a) Obviously, $S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$ is contained in $S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)\mathfrak{S}_\pi$.
Suppose that $S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)\mathfrak{S}_\pi \not\subseteq S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$. Let G be a group of minimal order contained in $S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)\mathfrak{S}_\pi \setminus S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$. Then G has a unique maximal normal subgroup, N, and $G_{\mathfrak{F}_i} \leq N$ $(i = 1, 2)$. $G/N \in \mathfrak{S}_\pi$, thus we obtain $G/G_{\mathfrak{F}_1}G_{\mathfrak{F}_2} \in \mathfrak{S}_\pi\mathfrak{S}_\pi = \mathfrak{S}_\pi$, a contradiction.

(b) Suppose not. Let $G \in S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$ be a group of minimal order possessing a subgroup U which is not contained in $S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$. Let U be maximal among all such subgroups.
G has a unique maximal normal subgroup N, and $NU = G$:
Assume not. Let N_1, N_2 be different maximal normal subgroups of G. $S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$ is closed under taking subnormal subgroups, hence it follows $G = N_1U = N_2U = (N_1 \cap N_2)U$ by the choice of G. This implies $U = (U \cap N_1)(U \cap N_2) \in \mathrm{N}_0 S_\pi(\mathfrak{F}_1, \mathfrak{F}_2) = S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$, a contradiction.
Since $\mathfrak{F}_i = \mathrm{s}\mathfrak{F}_i \subseteq S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$, we obtain $G_{\mathfrak{F}_i} \leq N$ $(i = 1, 2)$, and consequently $G/N \cong U/U \cap N \in \mathfrak{S}_\pi$. $N \cap U$ is a subgroup of N, so this implies $N \cap U \in S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$. Since $U \in S_\pi(\mathfrak{F}_1, \mathfrak{F}_2)\mathfrak{S}_\pi$, (a) gives a final contradiction.

□

2.2.25 Corollary
Let $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *be SFitting classes and let* π *be minimal among all sets of primes fulfilling the conditions of 1.2.19. Assume further that* $\mathfrak{F}_i\mathfrak{S}_\pi = \mathfrak{F}_i$ *for* $i = 1, 2$. *Then*

$$\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2) = S_\pi(\mathfrak{F}_1, \mathfrak{F}_2).$$

In particular, if in addition $|\pi| \geq 2$, *then* $\mathrm{Fit}(\mathfrak{F}_1, \mathfrak{F}_2) \subseteq N_\pi(\mathfrak{F}_1, \mathfrak{F}_2) \subset \mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)$.

Proof: $\subseteq$: 2.2.24. $\supseteq$: 2.2.12 yields $S_\pi(\mathfrak{F}_1, \mathfrak{F}_2) \subseteq \mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)\mathfrak{S}_\pi = \mathrm{SFit}(\mathfrak{F}_1\mathfrak{S}_\pi, \mathfrak{F}_2\mathfrak{S}_\pi) = \mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2)$. $\square$

Chapter 3

Locally normal Fitting classes

The concept of normal Fitting classes was introduced by Blessenohl and Gaschütz in 1970 ([5]). They considered non-trivial Fitting classes $\mathfrak{X}$ such that an $\mathfrak{X}$-injector of G is a normal subgroup of G (thus $G_{\mathfrak{X}}$ is $\mathfrak{X}$-maximal in G) for each group G. This concept is generalized in the following way: let $\mathfrak{X}$ and $\mathfrak{F}$ be non-trivial Fitting classes, $\mathfrak{X} \subseteq \mathfrak{F}$. Then $\mathfrak{X}$ is said to be normal in $\mathfrak{F}$ ($\mathfrak{F}$-normal) if $G_{\mathfrak{X}}$ is an $\mathfrak{X}$-maximal subgroup of G for all $G \in \mathfrak{F}$. In this investigation, it seems natural to consider the class $\mathrm{Y_n}(\mathfrak{X})$ of all groups G such that $G_{\mathfrak{X}}$ is $\mathfrak{X}$-maximal in G (for arbitrary Fitting classes $\mathfrak{X}$). Unfortunately, this class is not, in general, closed under forming normal products (Hauck, 1977), and therefore can fail to be a Fitting class. The following questions arise: (1) Does there nevertheless exist a unique maximal Fitting class in which $\mathfrak{X}$ is normal? And vice versa, (2) what conditions must a Fitting class $\mathfrak{F}$ satisfy to possess a unique minimal $\mathfrak{F}$-normal Fitting class? In this chapter, which is subdivided in two parts, we mainly discuss these problems.

The first section deals with local normality between arbitrary Fitting classes. Some basic facts – most of them proved by Hauck in 1977 (cf. [13]) – are presented. In this general setting, question (1) is almost intractable. This is caused mainly by the lack of knowledge of the Fitting class generated by two given Fitting classes $\mathfrak{F}_1$, $\mathfrak{F}_2$. Hence, it seems to be hard to decide whether or not this class is contained in $\mathrm{Y_n}(\mathfrak{X})$ provided that $\mathfrak{F}_1$, $\mathfrak{F}_2 \subseteq \mathrm{Y_n}(\mathfrak{X})$, and thus to answer question (1). However, we will give some conditions which guarantee that in this situation $\mathrm{Fit}(\mathfrak{F}_1, \mathfrak{F}_2)$ is still contained in $\mathrm{Y_n}(\mathfrak{X})$ (where $\mathfrak{X}$ denotes a non-trivial Fitting class).

Problem (2), too, remains open in general. Nevertheless, we will prove that for some important classes $\mathfrak{F}$ a unique minimal $\mathfrak{F}$-normal Fitting class exists and can furthermore be described explicitly.

In the second part of this chapter we turn our attention to locally normal SFitting classes (that is, $\mathfrak{X}$ and $\mathfrak{F}$ are SFitting classes and $\mathfrak{X}$ is normal in $\mathfrak{F}$). Using the theory of local formations, we obtain much stronger results concerning the above questions (restricted to SFitting classes). The key to almost all of these results is the fact that local normality between SFitting classes (satisfying a weak additional condition) is equivalent to local normality between their corresponding canonical local definitions. It follows from this that for an arbitrary SFitting class $\mathfrak{X}$ there exists a unique maximal SFitting class $\mathfrak{F}$ such that $\mathfrak{X}$ is normal in $\mathfrak{F}$, and moreover that this class determines $\mathfrak{X}$ uniquely. Furthermore, for many important cases we present an algorithm to describe this class. Using the results obtained in Chapter 2, we derive that the collection of all SFitting classes in which $\mathfrak{X}$ is normal forms a complete, distributive and atomic lattice, whose atoms can be described explicitly.
The second question remains open, even if we confine ourselves to SFitting classes. However, if $\mathfrak{F}$ is an SFitting class such that a (unique) smallest $\mathfrak{F}$-normal SFitting class exists, then the collection of all SFitting classes which are normal in $\mathfrak{F}$ also forms a complete and distributive lattice, which, in addition, is dual atomic, provided that $\mathfrak{F}$ is of bounded nilpotent length.

3.1 Local normality and arbitrary Fitting classes

3.1.1 Definition
Let $\mathfrak{X}$ and $\mathfrak{F}$ be non-trivial Fitting classes, $\mathfrak{X} \subseteq \mathfrak{F}$. Then $\mathfrak{X}$ is said to be normal in $\mathfrak{F}$ ($\mathfrak{X} \trianglelefteq \mathfrak{F}$) if $G_{\mathfrak{X}}$ is an $\mathfrak{X}$-maximal subgroup of G for all $G \in \mathfrak{F}$.

If $\mathfrak{X}$ is normal in $\mathfrak{F}$, we also refer to $\mathfrak{X}$ as being $\mathfrak{F}$-normal.

The following remark is obvious.

3.1.2 Remark

(a) *Each non-trivial Fitting class* $\mathfrak{F}$ *is normal in* $\mathfrak{F}\mathfrak{N}$. *In particular,* $\mathfrak{N}$ *is normal in* $\mathfrak{N}^2$.

(b) *Let* $(\mathfrak{F}_i)_{i\in I}$ *be non-trivial Fitting classes whose characteristics are pairwise disjoint. Then* $\mathfrak{F}_j$ *is normal in* $\prod_{i\in I}\mathfrak{F}_i$ *for all* $j \in I$.
In particular, $\mathfrak{S}_p$ *is normal in* $\mathfrak{N}$ *for each prime* p.

(c) *Let* $\mathfrak{X}$, $\mathfrak{F}$ *and* $\mathfrak{Y}$ *be non-trivial Fitting classes such that* $\pi(\mathfrak{X})\cap\pi(\mathfrak{Y}) = \emptyset$.
If $\mathfrak{X}$ *is normal in* $\mathfrak{F}$, *then* $\mathfrak{X}$ *is normal in* $\mathfrak{F}\mathfrak{Y}$.
In particular, $\mathfrak{X}$ *is normal in* $\mathfrak{X}\mathfrak{S}_{\pi(\mathfrak{X})'}$.

(d) *Let* $\mathfrak{X}$, $\mathfrak{F}$ *and* $\mathfrak{Y}$ *be non-trivial Fitting classes such that* $\mathfrak{X}$ *is normal in* $\mathfrak{F}$ *and* $\mathfrak{Y} \subseteq \mathfrak{F}$. *Then* $\mathfrak{X}\cap\mathfrak{Y}$ *is normal in* $\mathfrak{Y}$.

The relation of local normality is far from being transitive: According to Hauck (cf. [13, 4.3]), a Fitting class $\mathfrak{X}$ is normal in $\mathfrak{X}\mathfrak{N}^2$ precisely when it is normal in $\mathfrak{S}$ (observe that $\mathfrak{X} \trianglelefteq \mathfrak{X}\mathfrak{N} \trianglelefteq \mathfrak{X}\mathfrak{N}^2$).

Let π be a set of primes. There exist a number of characterizations of $\mathfrak{S}_\pi$-normal Fitting classes.

3.1.3 Theorem ([9], X, 3.7)
Let $\mathfrak{X}$ *be a Fitting class and* π *be a set of primes. Then the following statements are equivalent:*

(i) $\mathfrak{X}$ *is normal in* $\mathfrak{S}_\pi$.

(ii) $\mathfrak{F}^* = \mathfrak{S}_\pi$.

(iii) *For each prime* $p \in \pi$ *and* $G \in \mathfrak{X}$, *there exists a natural number* n *such that* $G^n \wr Z_p \in \mathfrak{X}$.

In particular, a Fitting class $\mathfrak{X}$ is normal in $\mathfrak{S}_\pi$ if and only if $\mathfrak{X}$ is contained in the Lockett section of $\mathfrak{S}_\pi$. According to 3.1.2(a),(b), this fails to be true for an arbitrary Fitting class $\mathfrak{F}$. Nevertheless, it is possible to confine oneself to the case that both classes are Lockett classes – as proved independly by Hauck and Laue (cf. [9, X, 3.3]) –, thus classes which are easier to handle than arbitrary Fitting classes.

We give a further proof of this result which can be easily transferred to every embedding property e of injectors such that e is invariant under epimorphisms and such that the following holds: if $W \in \mathrm{Inj}_{\mathfrak{X}^*}(G)$ satisfies the embedding property e in G, then $W_{\mathfrak{X}}$, too, satisfies this property in G for all groups G.

3.1.4 Proposition
Let $\mathfrak{X}$ and $\mathfrak{F}$ be Fitting classes, $\mathfrak{X} \subseteq \mathfrak{F}$. Then the following statements are equivalent:

(i) *$\mathfrak{X}$ is normal in $\mathfrak{F}$.*

(ii) *$\mathfrak{X}$ is normal in $\mathfrak{F}^*$.*

(iii) *$\mathfrak{X}^*$ is normal in $\mathfrak{F}^*$.*

Proof:
$(i) \Rightarrow (iii)$: Let $G \in \mathfrak{F}^*$, $V \in \mathrm{Inj}_{\mathfrak{X}^*}(G)$. According to 1.2.10 and 1.2.9 the group $(V \times V)_{\mathfrak{X}}$ is an $\mathfrak{X}$-injector of $G \times G$ and thus we obtain $(V \times V)_{\mathfrak{X}} \cap (G \times G)_{\mathfrak{F}} \trianglelefteq (G \times G)_{\mathfrak{F}}$ by assumption. Normality is invariant under epimorphisms, hence 1.2.9(e) yields that V is a normal subgroup of G and the assertion follows.

$(iii) \Rightarrow (ii)$: Let $G \in \mathfrak{F}^*$ and $V \in \mathrm{Inj}_{\mathfrak{X}}(G)$. According to 1.2.9, $V = W_{\mathfrak{X}}$ for a suitable $\mathfrak{X}^*$-injector W of G. By assumption, W is a normal subgroup of G, and thus we obtain $V = G_{\mathfrak{X}}$.

$(ii) \Rightarrow (i)$: Obvious. □

3.1.5 Definition
Let $\mathfrak{X}$ be a Fitting class. We define

$$\mathrm{Y_n}(\mathfrak{X}) = (G \mid G_{\mathfrak{X}} \text{ is } \mathfrak{X}\text{-maximal in } G).$$

If $\mathfrak{X} = \emptyset$, we set $\mathrm{Y_n}(\mathfrak{X}) = \mathfrak{S}$.

It is obvious that $\mathrm{Y_n}(\mathfrak{X})$ is closed under taking subnormal subgroups and – provided that $\mathfrak{X}$ is a Lockett class – under forming direct products. For some special classes $\mathfrak{X}$, the class $\mathrm{Y_n}(\mathfrak{X})$ is N_0-closed as well. For instance $\mathrm{Y_n}(\mathfrak{X}) = \mathfrak{X}\mathfrak{S}_{\pi'}$ if $\mathfrak{X} = \mathfrak{S}_\pi\mathfrak{S}_{\pi'}\mathfrak{S}_\pi \cdots \mathfrak{S}_{\pi'}\mathfrak{S}_\pi$ where π is an arbitrary set of primes (cf. [13, 3.2]). However, in general, $\mathrm{Y_n}(\mathfrak{X})$ fails to be a Fitting class (an example for this fact, concerning the class $\mathfrak{N}$, is presented by Hauck in [13, 3.2]).

It is to be noted that if $\mathfrak{X}$ is a Fitting class such that $\mathrm{Y_n}(\mathfrak{X})$ is $\mathrm{N_0}$-closed, then by 3.1.4 the class $\mathrm{Y_n}(\mathfrak{X})$ is a Lockett class and coincides with $\mathrm{Y_n}(\mathfrak{X}^*)$.

As mentioned above, it is an unsolved problem whether or not there always exists a unique maximal Fitting class in which a given Fitting class $\mathfrak{X}$ is normal. According to Zorn's Lemma there always exists one which is maximal among all Fitting classes contained in $\mathrm{Y_n}(\mathfrak{X})$. But in general, it seems to be hard to obtain results about the uniqueness of such a class. Nevertheless, we are able to give some conditions which guarantee that the Fitting class generated by Fitting classes $\mathfrak{F}_1,\ \mathfrak{F}_2 \subseteq \mathrm{Y_n}(\mathfrak{X})$ is still contained in $\mathrm{Y_n}(\mathfrak{X})$ (cf. 3.1.8).

The following well-known lemma (see for instance [16, proof of the main result] and [3, 1.1]) will be useful in establishing the structure of a group of minimal order contained in $\mathfrak{F} \setminus \mathrm{Y_n}(\mathfrak{X})$.

3.1.6 Lemma
Let $\mathfrak{X}$ be a Fitting class and $\mathfrak{F}$ be an s_n-closed class such that $\mathfrak{F} \not\subseteq Y_n(\mathfrak{X})$. Let G be a group of minimal order contained in $\mathfrak{F} \setminus Y_n(\mathfrak{X})$, and $V \in Inj_{\mathfrak{X}}(G)$.

(a) *G has a unique maximal normal subgroup N, $VN = G$, $V \cap N = G_{\mathfrak{X}}$ and $V/G_{\mathfrak{X}} \cong Z_p$ for a suitable prime p.*

(b) *Suppose that $\mathfrak{F}$ is s_F-closed, and let K be a normal subgroup of G such that $G_{\mathfrak{X}} \leq K < N$. Then $K \leq N_G(V)$, $N/G_{\mathfrak{X}} = F(G/G_{\mathfrak{X}})$ is a q-group (for a suitable prime $q \neq p$) and $V = PG_{\mathfrak{X}}$ ($P \in Syl_p(G)$ suitable).*

(c) *Suppose that $\mathfrak{F}$ is s_F- and Q- closed, and that $\mathfrak{X}$ is a Fitting formation. Then G has a unique minimal normal subgroup M, and N/M belongs to $\mathfrak{X}$.*

In particular, all statements listed above hold provided that $\mathfrak{X}$ and $\mathfrak{F}$ are SFitting classes.

Proof: Evidently, $G \neq 1$.

(a) Let N be a maximal normal subgroup of G. According to the choice of G, $N \cap V = N_{\mathfrak{X}}\ \mathrm{char} N \trianglelefteq G$, and thus $NV = G$.
Let M be a maximal normal subgroup of G, $M \neq N$. Since $G/N \cap M$ is abelian, $G = V(N \cap M)$ (for otherwise $V \trianglelefteq V(N \cap M) \trianglelefteq G$, a contradiction), and therefore $G = V(N \cap M) = VN = VM$. According

to [9, A, 1.2], this implies $V = V \cap NM = (V \cap N)(V \cap M) \trianglelefteq G$; this contradicts the choice of G. Hence G has a unique maximal normal subgroup and the assertion follows.

(b) $G_{\mathfrak{X}} \leq K < N$ implies $KV < G$ and $VK/K \cong V/V \cap K = V/G_{\mathfrak{X}} \cong Z_p$. Since $\mathfrak{F} = \mathrm{S_F}\mathfrak{F}$ and $V \in \mathrm{Inj}_{\mathfrak{X}}(VK)$, we obtain $K \leq N_G(V)$ by the choice of G.
Obviously, $\mathrm{F}(G/G_{\mathfrak{X}}) < G/G_{\mathfrak{X}}$. Assume that $M/G_{\mathfrak{X}} := \mathrm{F}(G/G_{\mathfrak{X}}) < N/G_{\mathfrak{X}}$. Then $V \trianglelefteq MV$ and consequently $V/G_{\mathfrak{X}} \leq C_{G/G_{\mathfrak{X}}}(M/G_{\mathfrak{X}}) \leq N/G_{\mathfrak{X}}$; this contradicts the choice of G. Thus $M = N$.
Let $\{q_1, \ldots, q_m\}$ denote the set of primes dividing $|N/G_{\mathfrak{X}}|$, and let $Q_i/G_{\mathfrak{X}} \in \mathrm{Syl}_{q_i}(N/G_{\mathfrak{X}})$. Assume that $m > 1$; as before, we obtain $V/G_{\mathfrak{X}} \leq \cap_{i=1}^{m} C_{G/G_{\mathfrak{X}}}(Q_i/G_{\mathfrak{X}}) \leq C_{G/G_{\mathfrak{X}}}(N/G_{\mathfrak{X}}) \leq N/G_{\mathfrak{X}}$, a contradiction.
Now the assertion follows.

(c) Let M be a minimal normal subgroup of G and set $(G/M)_{\mathfrak{X}} = W/M$. If $W < N$, then (b) implies $W \leq N_G(V)$ and thus $(W/M)(VM/M) = VW/M \in \mathrm{N_0}\mathfrak{X} = \mathfrak{X}$. By the choice of G, $G/M \in \mathrm{Y}_n(\mathfrak{X})$ and consequently $V \leq W \leq N$, a contradiction. Hence W is a normal subgroup of G containing N and therefore $N/M \in \mathfrak{X}$. Let M_0 be a minimal normal subgroup of G, $M_0 \neq M$. As before we obtain $N/M_0 \in \mathfrak{X}$ and thus $N \in \mathrm{R_0}\mathfrak{X} = \mathfrak{X}$, a contradiction.

□

3.1.7 Lemma
Let $\mathfrak{X}$ and $\mathfrak{F}$ be Fitting classes, $\mathfrak{X} = \mathrm{Q}\mathfrak{X}$, and let p be a prime such that $\mathfrak{S}_p\mathfrak{X} = \mathfrak{X}$. Assume further that $\mathfrak{F} \subseteq \mathrm{Y}_n(\mathfrak{X})$. Then $\mathfrak{S}_p\mathfrak{F}$ is contained in $\mathrm{Y}_n(\mathfrak{X})$.

Proof: Let G be a group contained in $\mathfrak{S}_p\mathfrak{F}$. By assumption, $G_{\mathfrak{S}_p\mathfrak{X}}/O_p(G) = (G/O_p(G))_{\mathfrak{X}}$ is an $\mathfrak{X}$-maximal subgroup of $G/O_p(G)$. The Q-closure of $\mathfrak{X}$ implies that $G_{\mathfrak{S}_p\mathfrak{X}} = G_{\mathfrak{X}}$ is an $\mathfrak{X}$-maximal subgroup of G, and the proof is complete. □

As mentioned before, the Fitting class generated by Fitting classes $\mathfrak{F}_1$, $\mathfrak{F}_2$ is difficult to handle. An upper bound for this class is the class $N_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$ cited in 1.2.19. The next result gives a condition, which guarantees that $N_\pi(\mathfrak{F}_1,\ \mathfrak{F}_2)$, and therefore $\mathrm{Fit}(\mathfrak{F}_1, \mathfrak{F}_2)$, is still contained in $\mathrm{Y_n}(\mathfrak{X})$ provided that $\mathfrak{F}_1,\ \mathfrak{F}_2 \subseteq \mathrm{Y_n}(\mathfrak{X})$.

3.1.8 Proposition
Let $\mathfrak{F}_1,\ \mathfrak{F}_2$ and $\mathfrak{X}$ be Fitting classes such that $\mathfrak{X}$ is normal in $\mathfrak{F}_i$ for $i = 1,\ 2$. Furthermore, let π be a set of primes that satisfies the conditions in 1.2.19 and such that $\mathfrak{F}_i\mathfrak{S}_p = \mathfrak{F}_i$ for all $p \in \pi$ $(i = 1,\ 2)$. Then

$$\mathrm{Fit}(\mathfrak{F}_1,\ \mathfrak{F}_2) \subseteq N_\pi(\mathfrak{F}_1,\ \mathfrak{F}_2) \subseteq Y_n(\mathfrak{X}).$$

Proof: Assume not. Let G be a group of minimal order contained in $N_\pi(\mathfrak{F}_1,\ \mathfrak{F}_2) \setminus \mathrm{Y_n}(\mathfrak{X})$. According to 3.1.6, G has a unique maximal normal subgroup N, and $G/N \cong Z_p$, $VN = G$, $V \cap N = G_{\mathfrak{X}}$ and $V = PG_{\mathfrak{X}}$ (where $V \in \mathrm{Inj}_{\mathfrak{X}}(G)$ and $P \in \mathrm{Syl}_p(V)$ suitable, p prime). $G \notin \mathfrak{F}_i$, whence $G_{\mathfrak{F}_i} < G$ for $i = 1,\ 2$, and therefore $p \in \pi$.

(1) $PG_{\mathfrak{F}_1}G_{\mathfrak{F}_2} = G$:
Assume not. $PG_{\mathfrak{F}_1}G_{\mathfrak{F}_2}$ belongs to $N_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$ and $G_{\mathfrak{X}} \le G_{\mathfrak{F}_i}$, hence, by the choice of G, we obtain $V = PG_{\mathfrak{X}} \trianglelefteq PG_{\mathfrak{F}_1}G_{\mathfrak{F}_2} \trianglelefteq \trianglelefteq G$ $(i = 1, 2)$, a contradiction.

(2) $PG_{\mathfrak{F}_1} = G$ or $PG_{\mathfrak{F}_2} = G$, and consequently $G \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \subseteq \mathrm{Y_n}(\mathfrak{X})$:
Obviously, $PG_{\mathfrak{F}_i} \in \mathfrak{F}_i\mathfrak{S}_p \subseteq N_\pi(\mathfrak{F}_1, \mathfrak{F}_2)$ and $V \le PG_{\mathfrak{F}_i}$ $(i = 1, 2)$. Suppose that $PG_{\mathfrak{F}_i}$ is a proper subgroup of G for $i = 1, 2$. Then each of the subgroups $G_{\mathfrak{F}_1},\ G_{\mathfrak{F}_2}$ and P is contained in $\mathrm{N}_G(V)$. By (1), this contradicts the choice of G.

□

3.1.9 Remark
In particular, it follows from 3.1.8 that $\mathrm{Fit}(\mathfrak{F}_1, \mathfrak{F}_2) \subseteq Y_n(\mathfrak{X})$ *provided that $\mathfrak{F}_1,\ \mathfrak{F}_2$ and $\mathfrak{X}$ are Fitting classes such that $\mathfrak{X}$ is normal in $\mathfrak{F}_i$ and a minimal set of primes as required in 1.2.19 is empty. But in this case, the result can be concluded more easily from 3.1.6.*

The dual problem – which classes $\mathfrak{F}$ do possess a unique minimal $\mathfrak{F}$-normal Fitting class? – is open as well. (Obviously, there are Fitting classes not possessing such a class, for instance the Fitting class $\mathfrak{N}$ of all nilpotent groups.) However, for some important classes we show that such a class exists, and furthermore give an explicit description of it.

We need

3.1.10 Remark ([13], 4.12, cf. [18], 2.1)
Let $\mathfrak{F}$ be a Fischer class, and let $(\mathfrak{X}_i)_{i\in I}$ be a family of Fitting classes such that $\mathfrak{X}_i$ is normal in $\mathfrak{F}$ for all $i \in I$. Then

$$\mathfrak{F} \subseteq Y_n(\bigcap_{i\in I} \mathfrak{X}_i).$$

It is an open question, whether or not this statement holds for arbitrary Lockett classes.

We defined local normality only for non-trivial Fitting classes. So, in order to solve the above problem for Fischer classes $\mathfrak{F}$, it suffices to determine when the intersection of all $\mathfrak{F}$-normal Lockett classes is non-trivial. For some types of Fitting classes this can be done:

3.1.11 Remark
Let $\mathfrak{F}$ be a Lockett class, and let $(\mathfrak{X}_i)_{i\in I}$ be the family of all $\mathfrak{F}$- normal Lockett classes.

(a) *Suppose there exists a prime p such that $\mathfrak{S}_p\mathfrak{F} = \mathfrak{F}$. Then $p \in \pi(\mathfrak{X}_i)$ for all $i \in I$; in particular $\cap_{i\in I}\mathfrak{X}_i \neq 1$.*

(b) *Let $\mathfrak{F}$ be a Q-closed Fischer class such that $|\pi(\mathfrak{F})| < \infty$. Then the following statements are equivalent:*

- **(i)** *$\cap_{i\in I}\mathfrak{X}_i \neq 1$, that is, there exists a unique minimal $\mathfrak{F}$-normal Fitting class.*
- **(ii)** *There exist no sets of primes $\pi_1,\ \pi_2$ such that $\pi_1 \cap \pi_2 = \emptyset$, $\mathfrak{F} \cap \mathfrak{S}_{\pi_1} \neq 1 \neq \mathfrak{F} \cap \mathfrak{S}_{\pi_2}$, and $\mathfrak{F} \subseteq (\mathfrak{S}_{\pi_1} \times \mathfrak{S}_{\pi_2})\mathfrak{S}_{(\pi_1\cup\pi_2)'}$.*

Proof:

(a) Let $\mathfrak{X}_i$ be an $\mathfrak{F}$-normal Lockett class. If $p \notin \pi(\mathfrak{X}_i)$, 1.2.24 implies $Z_p \wr G \in \mathfrak{S}_p\mathfrak{F} \setminus \mathrm{Y_n}(\mathfrak{X}_i)$ for an arbitrary $G \in \mathfrak{X}_i$. But by assumption, this class is empty; a contradiction.

(b) $(i) \Rightarrow (ii)$: Suppose not. Then there exist sets of primes $\pi_1,\ \pi_2$ as required above. $\mathfrak{S}_{\pi_i}$ is normal in $(\mathfrak{S}_{\pi_1} \times \mathfrak{S}_{\pi_2})\mathfrak{S}_{(\pi_1\cup\pi_2)'}$, hence $1 \neq \mathfrak{F} \cap \mathfrak{S}_{\pi_i}$ is normal in $\mathfrak{F}$ $(i = 1, 2)$, a contradiction.

$(ii) \Rightarrow (i)$: Let $\mathfrak{X}_{i_1}$ be an $\mathfrak{F}$-normal Lockett class of minimal characteristic (note that $\pi(\mathfrak{F})$ is finite). Set $\pi(\mathfrak{X}_{i_1}) = \pi_1$. We show that $\pi_1 \subseteq \pi(\mathfrak{X}_j)$ for all $j \in I$, proving the assertion:
Assume, there exists $\mathfrak{X}_{i_2}$ such that $\mathfrak{X}_{i_2}$ is $\mathfrak{F}$-normal and $\pi_1 \not\subseteq \pi(\mathfrak{X}_{i_2}) =: \pi_2$. According to 3.1.10, $\mathfrak{F}$ is contained in $\mathrm{Y_n}(\mathfrak{X}_{i_1} \cap \mathfrak{X}_{i_2})$. By the minimality of $\pi(\mathfrak{X}_{i_1})$, this implies $\mathfrak{X}_{i_1} \cap \mathfrak{X}_{i_2} = 1$, and therefore $\pi_1 \cap \pi_2 = \emptyset$.
$\mathfrak{F} \subseteq (\mathfrak{S}_{\pi_1} \times \mathfrak{S}_{\pi_2})\mathfrak{S}_{(\pi_1 \cup \pi_2)'}$:
Assume not. Let G be a group of minimal order contained in $\mathfrak{F} \setminus (\mathfrak{S}_{\pi_1} \times \mathfrak{S}_{\pi_2})\mathfrak{S}_{(\pi_1 \cup \pi_2)'}$. Then G has a unique maximal normal subgroup N, $|G/N| = q$, and a unique minimal normal subgroup M, $M \in \mathfrak{S}_p$ (where p and q are primes, $q \in \pi_1 \cup \pi_2$). We assume without loss of generality that $q \in \pi_1$. Since G/M belongs to $(\mathfrak{S}_{\pi_1} \times \mathfrak{S}_{\pi_2})\mathfrak{S}_{(\pi_1 \cup \pi_2)'}$, it follows that $G \in \mathfrak{S}_p\mathfrak{S}_{\pi_1}$. Now, $\mathfrak{F}$-normality of $\mathfrak{X}_{i_1}$ implies that $p \in \pi_1$. Consequently, G belongs to $\mathfrak{S}_{\pi_1}$; a contradiction.
Thus π_1 and π_2 are sets of primes violating condition (ii).

□

3.1.12 Remark
The hypothesis of finite characteristic in 3.1.11(b) is necessary.

Proof: Let $\{p_1,\ p_2, \ldots\}$ be the set of all primes, and set $\mathfrak{F} = \cup_{i \in \mathbb{N}} \mathfrak{S}_{p_i} \cdots \mathfrak{S}_{p_1}$. Then $\mathfrak{X}_k = \cup_{i \in \mathbb{N}, i \geq k} \mathfrak{S}_{p_i} \cdots \mathfrak{S}_{p_k}$ is $\mathfrak{F}$-normal for every $k \in \mathbb{N}$ and $\cap_{k \in \mathbb{N}} \mathfrak{X}_k = 1$. But evidently, there are no sets of primes fulfilling the conditions in 3.1.11(b)(ii). □

The following lemma is particularly useful in investigating locally normal Fitting classes.

3.1.13 Lemma
Let $\mathfrak{X}$ and $\mathfrak{F}$ be Lockett classes such that $\mathfrak{X}$ is normal in $\mathfrak{F}$. Further, let G be a group contained in $\mathfrak{X}$ and p, q $(p \neq q)$ be primes such that $G \wr Z_p \in \mathfrak{X}$ and $\mathrm{G} \wr \mathrm{Z}_q \wr Z_p \in \mathfrak{F}$. Then

$$G \wr Z_q \in \mathfrak{X}.$$

In particular, if $\mathfrak{G}$ is a Lockett class such that $\mathfrak{G}\mathfrak{S}_p \subseteq \mathfrak{X}$ and $\mathfrak{G}\mathfrak{S}_p\mathfrak{S}_q\mathfrak{S}_p \subseteq \mathfrak{F}$, then $\mathfrak{G}\mathfrak{S}_p\mathfrak{S}_q \subseteq \mathfrak{X}$.

Proof:
Suppose that $G \wr Z_q \notin \mathfrak{X}$. By 1.2.24 and repeated application of 1.2.12 we obtain

$$(\mathrm{G} \wr \mathrm{Z}_q \wr Z_p)_{\mathfrak{X}} = (G^*)^* < (G^*)^* Z_p \cong (G^*) \wr Z_p \in \mathfrak{X},$$

contradicting the $\mathfrak{F}$-normality of $\mathfrak{X}$. Now, the final assertion follows from 1.2.29. □

3.1.14 Corollary
Let $\mathfrak{G}, \mathfrak{X}$ and $\mathfrak{F}$ be Lockett classes such that $\mathfrak{G} \subseteq \mathfrak{X}$ and $\mathfrak{X}$ is normal in $\mathfrak{F}$. Further, let π_1, π_2 be sets of primes.

(a) *If $\mathfrak{G}\mathfrak{S}_{\pi_2} = \mathfrak{G}$ and $\mathfrak{G}\mathfrak{S}_{\pi_1}\mathfrak{S}_{\pi_2} \subseteq \mathfrak{F}$, then $\mathfrak{G}\mathfrak{S}_{\pi_1} \subseteq \mathfrak{X}$.*

(b) *If $\pi_1 \subseteq \pi(\mathfrak{X})$ and $\mathfrak{S}_{\pi_1} \subseteq \mathfrak{F}$, then $\mathfrak{S}_{\pi_1} \subseteq \mathfrak{X}$.*

Proof:

(a): Let G be a group contained in $\mathfrak{G}\mathfrak{S}_{\pi_1}$. Then $G \in \mathfrak{G}\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r}$ for suitable primes $p_1, \ldots, p_r \in \pi_1$. According to 3.1.13, $\mathfrak{G}\mathfrak{S}_{p_1} \subseteq \mathfrak{X}$, and repeating this argument we obtain $G \in \mathfrak{G}\mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r} \subseteq \mathfrak{X}$.

(b): Let G be a group contained in $\mathfrak{S}_{\pi_1}$. Then there exist primes $p_1, \ldots, p_r \in \pi_1$ such that $G \in \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r}$. Since $\mathfrak{S}_{p_1} \subseteq \mathfrak{X}$ and $\mathfrak{S}_{\pi_1} \subseteq \mathfrak{F}$, repeated application of (a) yields the assertion.

□

The following theorem has already been proved in [18, 1.3, 2.3].

3.1.15 Theorem

(a) *Let $\mathfrak{F}$ be a non-trivial Fitting class, $n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) *$\mathfrak{F}$ is normal in $\mathfrak{N}^{n+1}$.*

(ii) *$\mathfrak{N}^n \subseteq \mathfrak{F}^* \subseteq \mathfrak{N}^{n+1}$.*

In particular, there exists a (unique) smallest $\mathfrak{N}^{n+1}$-normal Fitting class, namely $(\mathfrak{N}^n)_*$.

(b) *Let $\pi \neq \emptyset$, $\mathbb{P}$ be a set of primes. Set $\mathfrak{F}_1 = \mathfrak{S}_\pi$, $\mathfrak{F}_2 = \mathfrak{F}_1\mathfrak{S}_{\pi'}$ and $\mathfrak{F}_n = \mathfrak{F}_{n-2}\mathfrak{S}_\sigma\mathfrak{S}_{\sigma'}$ if $n \geq 3$, where $\sigma = \pi$ if n is even and $\sigma = \pi'$ if n is odd.*
If $n \geq 2$, then $(\mathfrak{F}_{n-1})_*$ is the smallest $\mathfrak{F}_n$-normal Fitting class.

(c) *Let $\mathfrak{K}$ be an s_n-closed class of groups such that $G \wr Z_p \in \mathfrak{K}$ for all $G \in \mathfrak{K}$ and for all $p \in \pi(\mathfrak{K})$. Set $\mathfrak{F} = Fit(\mathfrak{K})^*$. Then a Fitting class $\mathfrak{X}$ is normal in $\mathfrak{F}$ if and only if $\mathfrak{X}^* = \mathfrak{F}$.*
In particular, there exists a (unique) smallest $\mathfrak{F}$-normal Fitting class, namely $\mathfrak{F}_*$.

Let $\mathfrak{F}$ be a lattice formation belonging to $(\pi_i)_{i\in I}$. Analogously to 3.1.15(a) it can be proved that $(\mathfrak{F}^n)_*$ is the smallest Fitting class which is normal in $\mathfrak{F}^{n+1}$ ($n \in \mathbb{N}$). In general, the converse does not hold true, as the Lockett class $\mathfrak{F}^n\mathfrak{S}_p$ is not normal in $\mathfrak{F}^{n+1}$ provided that $p \in \pi_i$ and $|\pi_i| > 1$.

Let $p_1, \ldots, p_r$ be primes, and set $\mathfrak{F} = \mathfrak{S}_{p_1} \cdots \mathfrak{S}_{p_r}$. According to 3.1.10 and 3.1.11, there exists a unique minimal $\mathfrak{F}$-normal Fitting class. In [18, 2.7] we presented an explicit description of this class. In the following we deal with the more general case of Fitting classes $\mathfrak{S}_{\pi_1} \cdots \mathfrak{S}_{\pi_r}$ where $\pi_1, \ldots, \pi_r$ are sets of primes.

We need the following lemma:

3.1.16 Lemma

(a) *Let π be a set of primes, and let $\mathfrak{Y}_1, \mathfrak{Y}_2$ and $\mathfrak{F}$ be Fitting classes such that $\pi(\mathfrak{F}) = \pi(\mathfrak{Y}_1) \cup \pi(\mathfrak{Y}_2)$, $\mathfrak{Y}_1 = {}_Q\mathfrak{Y}_1$ and $\pi(\mathfrak{S}_\pi\mathfrak{Y}_1) \cap \pi(\mathfrak{Y}_2) = \emptyset$. Furthermore, let G be a group of minimal order contained in $\mathfrak{F} \setminus Y_n(\mathfrak{S}_\pi\mathfrak{Y}_1 \times \mathfrak{Y}_2)$ and assume that $O_\pi(G) = 1$.*
Then $\mathfrak{S}_\pi\mathfrak{Y}_1 \times \mathfrak{Y}_2$-injectors and $\mathfrak{Y}_1 \times \mathfrak{Y}_2$-injectors of G coincide.

(b) *Let $\mathfrak{Y}_1, \ldots, \mathfrak{Y}_m$, $m > 1$, be non-trivial Fitting classes of pairwise coprime characteristic and set $\mathfrak{Y} = \prod_{i=1}^m \mathfrak{Y}_i$. Let $\mathfrak{F}$ be a Fitting class and suppose that G is a group of minimal order contained in $\mathfrak{F} \setminus Y_n(\mathfrak{Y})$.*
If $G_{\mathfrak{Y}_1}$ is a $\mathfrak{Y}_1$-maximal subgroup of G, then $C_G(G_{\mathfrak{Y}_1}) = G$.

(c) *Let $\pi_1, \ldots, \pi_n$ be pairwise disjoint sets of primes, and let $\mathfrak{Y}$ and $\mathfrak{F}$ be Fitting classes such that $\mathfrak{Y}$ is normal in $\mathfrak{F}$ or $\mathfrak{Y} = 1$, and $\pi_i \cap \pi(\mathfrak{F}) = \emptyset$ ($i = 1, \ldots, n$). Then*

$$\mathfrak{S}_{\pi_1} \times \ldots \times \mathfrak{S}_{\pi_n} \times \mathfrak{Y} \text{ is normal in } \mathfrak{S}_{\pi_1} \ldots \mathfrak{S}_{\pi_n}\mathfrak{F}$$

In particular, $\prod_{i=1}^n \mathfrak{S}_{\pi_i}$ is normal in $\mathfrak{S}_{\pi_1} \ldots \mathfrak{S}_{\pi_n}$.

Proof:

(a) Let V be an $\mathfrak{S}_\pi\mathfrak{Y}_1\times\mathfrak{Y}_2$-injector of G. According to 3.1.6, $G = NV$ and $V \cap N = G_{\mathfrak{S}_\pi\mathfrak{Y}_1\times\mathfrak{Y}_2} = G_{\mathfrak{Y}_1\times\mathfrak{Y}_2}$ where N denotes the unique maximal normal subgroup of G. $\mathfrak{Y}_1 \times \mathfrak{Y}_2 \subseteq \mathfrak{S}_\pi\mathfrak{Y}_1 \times \mathfrak{Y}_2$, thus it suffices to show that $V \in \mathfrak{Y}_1 \times \mathfrak{Y}_2$. If $V_{\mathfrak{S}_\pi} = 1$, we are finished. Otherwise, since $(V \cap N) \cap V_{\mathfrak{S}_\pi}$ is a subnormal subgroup of G, and therefore trivial by assumption, we obtain $(N \cap V) \times V_{\mathfrak{S}_\pi} = V$. Consequently $V_{\mathfrak{S}_\pi} = Z_p$ for some suitable prime $p \in \pi \cap \pi(\mathfrak{F})$ and the assertion follows.

(b) Let $V = V_1 \times \ldots \times V_m$ be a $\mathfrak{Y}$-injector of G. (Thus $V_i \in \mathfrak{Y}_i$, and by assumption $V_1 = G_{\mathfrak{Y}_1}$.) According to 3.1.6 $G = VN$ and $V \cap N = G_{\mathfrak{Y}}$, where N denotes the unique maximal normal subgroup of G. Since $C_G(G_{\mathfrak{Y}_1}) < G$ implies $V_2 \times \ldots \times V_m \leq C_G(G_{\mathfrak{Y}_1}) \leq N$ and consequently $V \leq N$, a contradiction, the assertion follows.

(c) Suppose not. Let G be a group of minimal order contained in $\mathfrak{S}_{\pi_1} \ldots \mathfrak{S}_{\pi_n}\mathfrak{F} \setminus \mathrm{Y_n}(\mathfrak{S}_{\pi_1} \times \ldots \times \mathfrak{S}_{\pi_n} \times \mathfrak{Y})$. According to 3.1.6, $G = VN$ where $V = V_1 \times \ldots \times V_n \times Y$ denotes an $\mathfrak{S}_{\pi_1} \times \ldots \times \mathfrak{S}_{\pi_n} \times \mathfrak{Y}$-injector and N the unique maximal normal subgroup of G ($V_i \in \mathfrak{S}_{\pi_i}$, $Y \in \mathfrak{Y}$).

$G_{\mathfrak{S}_{\pi_1}\ldots\mathfrak{S}_{\pi_k}} = \prod_{j=1}^{k} O_{\pi_j}(G)$ for all $k \leq n$; in particular, $V_i = O_{\pi_i}(G) \in \mathrm{Hall}_{\pi_i}(G)$ for $i = 1, \ldots n$:

Proof by induction on k:
The case $k = 1$ is obvious. Thus we assume that $k > 1$. As in (b), it follows that $C_G(O_{\pi_i}(G)) = G$ for all $i = 1, \ldots, k-1$. This implies

$$G_{\mathfrak{S}_{\pi_1}\ldots\mathfrak{S}_{\pi_k}} = H_{\pi_k}\prod_{j=1}^{k-1} O_{\pi_j}(G) = \prod_{j=1}^{k} O_{\pi_j}(G)$$

where $H_{\pi_k} \in \mathrm{Hall}_{\pi_k}(G)$.

Set $\pi = \pi(\mathfrak{Y})$. By assumption, $(G/G_{\mathfrak{S}_{\pi_1}\ldots\mathfrak{S}_{\pi_n}})_{\mathfrak{Y}}$ is a $\mathfrak{Y}$-maximal subgroup of $G/G_{\mathfrak{S}_{\pi_1}\ldots\mathfrak{S}_{\pi_n}}$ and $G_{\mathfrak{S}_{\pi_1}\ldots\mathfrak{S}_{\pi_n}\mathfrak{Y}} = HG_{\mathfrak{S}_{\pi_1}\ldots\mathfrak{S}_{\pi_n}}$ where $H \in \mathrm{Hall}_\pi(G_{\mathfrak{S}_{\pi_1}\ldots\mathfrak{S}_{\pi_n}\mathfrak{Y}})$. Using (b), we obtain $C_G(G_{\mathfrak{S}_{\pi_1}\ldots\mathfrak{S}_{\pi_n}}) = G$. Consequently H is a normal subgroup of G. By assumption, $V/G_{\mathfrak{S}_{\pi_1}\ldots\mathfrak{S}_{\pi_n}} \in \mathfrak{Y}$ and $G/G_{\mathfrak{S}_{\pi_1}\ldots\mathfrak{S}_{\pi_n}} \in \mathfrak{F} \subseteq \mathrm{Y_n}(\mathfrak{Y})$. It follows that $\prod_{i=1}^{n} O_{\pi_i}(G) \times H$ is an $\prod_{i=1}^{n} \mathfrak{S}_{\pi_i} \times \mathfrak{Y}$-maximal subgroup of G; this contradicts the choice of G.

□

We now return to the class $\mathfrak{F} = \mathfrak{S}_{\pi_1} \ldots \mathfrak{S}_{\pi_r}$ where $\pi_1, \ldots, \pi_r \neq \emptyset, \mathbb{P}$ are sets of primes such that $\pi_i \neq \pi_{i+1}$. According to 3.1.10 and 3.1.11, there exists a unique minimal $\mathfrak{F}$-normal Lockett class, which we will describe explicitly. It is obvious that, if $\pi_1 \cap (\pi_2 \cup \ldots \cup \pi_r) = \emptyset$, this class coincides with $\mathfrak{S}_{\pi_1}$. Otherwise we need the following construction:

Set

$$r_0 = \min\{i | (\cup_{j=1}^{i} \pi_j) \cap (\cup_{j=i+1}^{r} \pi_j) = \emptyset\}.$$

Without loss of generality we may assume that $r = r_0$:
Let $\mathfrak{Y}_0$ be the smallest $\mathfrak{S}_{\pi_1} \ldots \mathfrak{S}_{\pi_{r_0}}$-normal Lockett class. We show that $\mathfrak{Y}_0$ coincides with the smallest $\mathfrak{F}$-normal Lockett class. By definition of r_0, it is obvious that $\mathfrak{Y}_0$ is $\mathfrak{F}$-normal. Let $1 \neq \mathfrak{X}$ be an arbitrary $\mathfrak{F}$-normal Lockett class and $q \in \pi(\mathfrak{X})$. If $p \in \pi_1$ we conclude $Z_p \wr Z_q \in \mathfrak{F}$ and consequently $p \in \pi(\mathfrak{X})$. In particular, $\mathfrak{X} \cap \mathfrak{S}_{\pi_1} \ldots \mathfrak{S}_{\pi_{r_0}}$ is a non-trivial Lockett class being normal in $\mathfrak{S}_{\pi_1} \ldots \mathfrak{S}_{\pi_{r_0}}$. By definition of $\mathfrak{Y}_0$, this yields the assertion.

So, we may assume that $r = r_0 \geq 2$; in particular, $(\cup_{j=1}^{k} \pi_j) \cap (\cup_{j=k+1}^{r} \pi_j) \neq \emptyset$ for $k < r$.

We define

$$l_j := \max \{i \mid \pi_i \cap \pi_j \neq \emptyset\}.$$

for $j \in \{1, \ldots, r\}$.
Further set

$$\mu_0 := \max \{i < r \mid \text{there exists } j > i \text{ such that } \pi_j \cap \pi_i \neq \emptyset\}$$

and

$$I_0 := \{\mu_0 + 1, \ldots, l_{\mu_0}\}.$$

Now, we define

$$\mathfrak{Y}_0 := \mathfrak{S}_{\pi_{\mu_0}} (\prod_{i \in I_0} \mathfrak{S}_{\pi_i})$$

(Notice that $\mathfrak{Y}_0$ is a directly indecomposable Lockett class.)

If $i < r$, we set

$$\mu(i) := \max \{j \leq i \mid \text{there exists } k > j \text{ such that } \pi_j \cap \pi_k \neq \emptyset\}.$$

Set

$$\mu_1 := \mu(\mu_0 - 1) \text{ and } I_1 := \{i \mid \mu_1 < i \leq l_{\mu_1},\ \pi_i \cap \pi(\mathfrak{Y}_0) = \emptyset\},$$

and define

$$\mathfrak{Y}_1 := \begin{cases} \mathfrak{S}_{\pi_{\mu_1}}(\prod_{i\in I_1}\mathfrak{S}_{\pi_i}) \times \mathfrak{Y}_0 & \text{if } l_{\mu_1} < \mu_0, \\ \mathfrak{S}_{\pi_{\mu_1}}(\prod_{i\in I_1}\mathfrak{S}_{\pi_i} \times \mathfrak{Y}_0) & \text{otherwise.} \end{cases}$$

In the first case set $d(1,1) = 1$ and $\mathfrak{H}_{d(1,1)} = \mathfrak{S}_{\pi_{\mu_1}}(\prod_{i\in I_1}\mathfrak{S}_{\pi_i})$, and $d(1,2) = 0$ and $\mathfrak{H}_{d(1,2)} = \mathfrak{Y}_0$.
In the second case set $d(1,1) = 1$ and $\mathfrak{H}_{d(1,1)} = \mathfrak{S}_{\pi_{\mu_1}}(\prod_{i\in I_1}\mathfrak{S}_{\pi_i} \times \mathfrak{Y}_0) = \mathfrak{Y}_1$.

Suppose that $n \geq 2$ and $\mathfrak{Y}_{n-1}$ is defined. We set

$$\mu_n := \mu(\mu_{n-1} - 1) \text{ and } I_n := \{i \mid \mu_n < i \leq l_{\mu_n},\ \pi_i \cap \pi(\mathfrak{Y}_{n-1}) = \emptyset\}.$$

Let $\mathfrak{H}_{d(n-1,1)}, \ldots, \mathfrak{H}_{d(n-1,\nu(n-1))}$ be the (non-trivial) directly indecomposable factors of $\mathfrak{Y}_{n-1}$ – thus $\mathfrak{Y}_{n-1} = \prod_{m=1}^{\nu(n-1)} \mathfrak{H}_{d_{(n-1,m)}}$ for some suitable $\nu(n-1) \in \mathbb{N}$ –, ordered in the following way: set

$$d(n-1,1) = n-1 \text{ and } \mathfrak{H}_{d(n-1,1)} = \mathfrak{S}_{\pi_{\mu_{n-1}}}(\prod_{i\in I_{n-1}}\mathfrak{S}_{\pi_i} \times \prod_{m=1}^{m_0(n-2)} \mathfrak{H}_{d(n-2,m)})$$

where $m_0(n-2) := \max\{k \in \{0, 1, \ldots, \nu(n-2\} \mid l_{\mu_{n-1}} \geq \mu_{d(n-2,k)}\}$, and for $m \geq 2$

$$d(n-1,m) = d(n-2,k(m)) \text{ and } \mathfrak{H}_{d(n-1,m)} = \mathfrak{H}_{d(n-2,k(m))}$$

for some suitable $k(m) \in \{1, \ldots, \nu(n-2)\}$. Moreover, we assume that $d(n-1,1) > \ldots > d(n-1,\nu(n-1))$.

Now, we define

$$\mathfrak{Y}_n := \begin{cases} \mathfrak{S}_{\pi_{\mu_n}}(\prod_{i\in I_n}\mathfrak{S}_{\pi_i}) \times \mathfrak{Y}_{n-1} & \text{if } l_{\mu_n} < \mu_{n-1}, \\ \mathfrak{S}_{\pi_{\mu_n}}(\prod_{i\in I_n}\mathfrak{S}_{\pi_i} \times \prod_{m=1}^{m_0(n-1)}\mathfrak{H}_{d(n-1,m)}) \times \prod_{m=m_0(n-1)+1}^{\nu(n-1)} \mathfrak{H}_{d(n-1,m)} & \text{otherwise.} \end{cases}$$

In the second case set

$$m_0(n-1) = \max\ \{k \in \{1, \ldots, \nu(n-1)\} \mid l_{\mu_n} \geq \mu_{d(n-1,k)}\}.$$

Let α be the (unique) index such that $\mu_\alpha = 1$. Then $\mathfrak{Y}_\alpha$ is our candidate for the smallest $\mathfrak{F}$-normal Lockett class.

First some observations:

3.1.17 Remarks

Let the notations be as above.

(i) $\mathfrak{Y}_{n+1} \supset \mathfrak{Y}_n$ *provided that* $\mathfrak{Y}_{n+1}$ *is defined.*

In the following we assume that $\mathfrak{Y}_n$ *is directly decomposable, thus* $n \geq 1$ *and* $\mathfrak{Y}_n = \mathfrak{S}_{\pi_{\mu_n}}(\prod_{i\in I_n}\mathfrak{S}_{\pi_i} \times \prod_{m=1}^{m_0(n-1)}\mathfrak{H}_{d(n-1,m)}) \times \prod_{m=m_0(n-1)+1}^{\nu(n-1)}\mathfrak{H}_{d(n-1,m)}$, *where* $m_0(n-1) < \nu(n-1)$ *(set* $m_0(n-1) = 0$ *if* $l_{\mu_n} < \mu_{n-1}$*).*

(ii) $l_{\mu_k} < \mu_{d(n-1,m_0(n-1)+1)}$ *for all* k *such that* $n \geq k > d(n-1, m_0(n-1)+1)$: *By construction* $\prod_{m=m_0(n-1)+1}^{\nu(n-1)}\mathfrak{H}_{d(n-1,m)} = \mathfrak{Y}_{k_1}$ *for some suitable* $k_1 < n$.

$k_1 = d(n-1, m_0(n-1)+1)$:

$\leq$: $\mathfrak{H}_{d(n-1,m_0(n-1)+1)} = \mathfrak{H}_{d(k_1,1)} = \mathfrak{S}_{\pi_{\mu_{k_1}}}\mathfrak{H}_{d(k_1,1)}$ *and* $d(n-1, m_0(n-1)+1) = \max\{k \in \mathbb{N} \mid k \leq n-1$ *and* $\mathfrak{S}_{\pi_{\mu_k}}\mathfrak{H}_{d(n-1,m_0(n-1)+1)} = \mathfrak{H}_{d(n-1,m_0(n-1)+1)}\}$, *thus we have* $d(n-1, m_0(n-1)+1) \geq k_1$.

$\geq$: *If* $k > k_1$, *then* $l_{\mu_k} < \mu_{k_1}$, *for otherwise* $l_{\mu_k} > \mu_{k_1}$ *and consequently* $\mathfrak{H}_{d(k,1)} \supset \mathfrak{H}_{d(k_1,1)} = \mathfrak{H}_{d(n-1,m_0(n-1)+1)}$. *By construction,* $\mathfrak{H}_{d(k,1)}$ *is contained in* $\mathfrak{H}_{d(n,m_1)}$ *where* $\mathfrak{H}_{d(n,m_1)}$ *denotes some suitable directly indecomposable factor of* $\mathfrak{Y}_n$. *This implies* $\mathfrak{H}_{d(n,2)} = \mathfrak{H}_{d(n-1,m_0(n-1)+1)} = \mathfrak{H}_{d(k_1,1)} \subset \mathfrak{H}_{d(n,m_1)}$; *a contradiction.*
Hence $l_{\mu_k} < \mu_{k_1}$ *for all* $k > k_1$.
Since $l_{\mu_{d(n-1,m_0(n-1)+1)}} \geq \mu_{k_1}$, *this yields* $k_1 \geq d(n-1, m_0(n-1)+1)$.

Hence, we obtain $k_1 = d(n-1, m_0(n-1)+1)$ *and the assertion follows.*

(iii) *Set* $a(n) = \max\{l_{\mu_k} \mid n \geq k > d(n-1, m_0(n-1)+1)\}$. *Then (ii) implies* $\mathfrak{S}_{\pi_{\mu_n}} \ldots \mathfrak{S}_{\pi_{a(n)}} \cap \mathfrak{S}_{\pi_{a(n)+1}} \ldots \mathfrak{S}_{\pi_r} = 1$.

(iv) *By the choice of* $d(n-1, m_0(n-1))$, *assertions (i)-(iii) give:*

$$\begin{aligned} 1 &= \mathfrak{H}_{d(n,1)} \cap \mathfrak{Y}_{d(n-1,m_0(n-1)+1)} \\ &= \mathfrak{H}_{d(n,1)} \cap \mathfrak{S}_{\pi_{a(n)+1}} \ldots \mathfrak{S}_{\pi_r}. \end{aligned}$$

In particular, $\mathfrak{Y}_n$ *is well-defined and* $\mathfrak{H}_{d(n,1)} \cap \mathfrak{S}_{\pi_{\mu_n}} \ldots \mathfrak{S}_{\pi_{a(n)}} = \mathfrak{H}_{d(n,1)}$.

(v) *If* $n \geq 1$, *we set* $J_n = \{i \mid \mu_n \leq i \leq r,\ \pi_i \cap \pi(\mathfrak{Y}_n) = \emptyset\}$.

$$\{i \in J_{n-1} \text{ such that } i \leq a(n)\} = \{i \in I_n \text{ such that } i > \mu_{n-1}\}:$$

Set $A := \{i \in J_{n-1}$ such that $i \leq a(n)\}$ and $B := \{i \in I_n$ such that $i > \mu_{n-1}\}$.

$\subseteq$: Let i be an element of A. Then $\pi_i \cap \pi(\mathfrak{Y}_{n-1}) = \emptyset$ and $\mu_{n-1} < i \leq a(n)$. If $a(n) = l_{\mu_n}$, the proof is complete. Otherwise, there exists $k_0 \neq n$ such that $a(n) = l_{\mu_{k_0}}$. Assume, i does not belong to B. Then $l_{\mu_n} < i \leq a(n) \leq \max\{l_{\mu_k} \mid n-1 \geq k \geq 0\}$. Then, by construction, we obtain $\pi_i \subseteq \cup_{\mu_{n-1} \leq j \leq \max\{l_{\mu_k} \mid n-1 \geq k \geq 0\}} \pi_j = \pi(\mathfrak{Y}_{n-1})$, a contradiction.

$\supseteq$: Obvious.

3.1.18 Theorem
Let the notation be as above, and $\mathfrak{F} = \mathfrak{S}_{\pi_1} \ldots \mathfrak{S}_{\pi_r}$.

(a) *If* $r_0 = 1$, *then* $(\mathfrak{S}_{\pi_1})_*$ *is the smallest* $\mathfrak{F}$*-normal Fitting class.*

(b) *If* $r_0 \geq 2$, *then* $\mathfrak{Y} = \mathfrak{Y}_\alpha$ *is the smallest* $\mathfrak{F}$*-normal Lockett class.*

In particular, $(\mathfrak{Y})_*$ *is the smallest* $\mathfrak{F}$*-normal Fitting class.*

Proof:
(a): 3.1.14(b).
(b): As mentioned before, we may assume without loss of generality that $r = r_0$. Evidently, $\mathfrak{Y}$ is a non-trivial Lockett class.
Now we shall prove the assertion in two stages. First, we show that each $\mathfrak{F}$-normal Lockett class contains $\mathfrak{Y}$. Then, we prove that $\mathfrak{Y}$ is normal in $\mathfrak{F}$.

(I) Let $\mathfrak{X}$ be an $\mathfrak{F}$-normal Lockett class.

- $\mathfrak{Y}$ is directly indecomposable, for otherwise by construction

$$\mathfrak{Y}_\alpha = \mathfrak{H}_{d(\alpha,1)} \times \prod_{m=2}^{\nu(\alpha)} \mathfrak{H}_{d(\alpha,m)}.$$

  In particular, $a(\alpha) = \max\{l_{\mu_k} \mid \alpha \geq k > d(\alpha-1, m_0(\alpha-1)) + 1\} < r_0 = r$, and 3.1.17(iii) yields a contradiction to the choice of r_0.

- $Z_p \wr Z_q \in \mathfrak{F}$ for primes p and q such that $p \in \pi_1$ and $q \in \pi(\mathfrak{X})$; thus $\mathfrak{F}$-normality of $\mathfrak{X}$ implies $\pi_1 \subseteq \pi(\mathfrak{X})$. 3.1.14(b) yields $\mathfrak{S}_{\pi_1} \subseteq \mathfrak{X}$, and hence 3.1.14(a) provides $\mathfrak{S}_{\pi_{\mu_\alpha}} \mathfrak{S}_{\pi_i} \subseteq \mathfrak{X}$ for all $i \in I_\alpha$. Consequently, $\mathfrak{S}_{\pi_{\mu_\alpha}}(\prod_{i \in I_\alpha} \mathfrak{S}_{\pi_i})$ is contained in $\mathfrak{X}$.

If $\alpha = 0$, the proof is complete. Since $\mathfrak{Y}_\alpha$ is directly indecomposable, all we have to prove otherwise is that $\mathfrak{S}_{\pi_{\mu_\alpha}}\mathfrak{Y}_{\alpha-1} \subseteq \mathfrak{X}$.
(Notice that $\mathfrak{Y} = \mathrm{Fit}(\mathfrak{S}_{\pi_{\mu_\alpha}}(\prod_{i\in I_\alpha}\mathfrak{S}_{\pi_i}), \mathfrak{S}_{\pi_{\mu_\alpha}}\mathfrak{Y}_{\alpha-1}).$)
By construction

$$\mathfrak{Y}_{\alpha-1} = \prod_{m_1=1}^{\nu(\alpha-1)} \mathfrak{H}_{d(\alpha-1,m_1)},$$

thus, using the same argument as before, we conclude that it suffices to prove

$$\mathfrak{S}_{\pi_{\mu_\alpha}}\mathfrak{H}_{d(\alpha-1,m_1)} \subseteq \mathfrak{X} \text{ for } m_1 \in \{1,\dots,\nu(\alpha-1)\}.$$

Clearly either

$$\mathfrak{H}_{d(\alpha-1,m_1)} =$$

$$\mathfrak{S}_{\pi_{\mu_{d(\alpha-1,m_1)}}}(\prod_{i\in I_{d(\alpha-1,m_1)}} \mathfrak{S}_{\pi_i} \times \prod_{m_2=1}^{\nu(\alpha-1,m_1)} \mathfrak{H}_{d(\alpha-1,m_1,m_2)}),$$

(where $\mathfrak{H}_{d(\alpha-1,m_1,m_2)} \neq 1$ directly indecomposable for $m_2 \in \{1,\dots,\nu(\alpha-1,m_1)\}$, $\nu(\alpha-1,m_1) \in \mathbb{N}$ suitable and $d(\alpha-1,m_1,m_2) \in \mathbb{N}$ likewise defined as in the construction),

or

$$\mathfrak{H}_{d(\alpha-1,m_1)} = \mathfrak{S}_{\pi_{\mu_{d(\alpha-1,m_1)}}}(\prod_{i\in I_{d(\alpha-1,m_1)}} \mathfrak{S}_{\pi_i})$$

holds.
Choose an arbitrary $m_1 \in \{1,\dots,\nu(\alpha-1)\}$. Since $\mathfrak{Y}_\alpha$ is directly indecomposable and consequently (by construction) $l_{\mu_\alpha} \geq \mu_{d(\alpha-1,m_1)}$, we obtain $\mathfrak{S}_{\pi_{\mu_\alpha}}\mathfrak{S}_{\pi_{\mu_{d(\alpha-1,m_1)}}}\mathfrak{S}_{\pi_{l_{\mu_\alpha}}} \subseteq \mathfrak{F}$. 3.1.14(a) yields $\mathfrak{S}_{\pi_{\mu_\alpha}}\mathfrak{S}_{\pi_{\mu_{d(\alpha-1,m_1)}}} \subseteq \mathfrak{X}$, and analogously we conclude

$$\mathfrak{S}_{\pi_{\mu_\alpha}}\mathfrak{S}_{\pi_{\mu_{d(\alpha-1,m_1)}}}(\prod_{i\in I_{d(\alpha-1,m_1)}} \mathfrak{S}_{\pi_i}) \subseteq \mathfrak{X}.$$

In the second case, the statement now is proved. In the first case it remains to show that

$$\mathfrak{S}_{\pi_{\mu_\alpha}}\mathfrak{S}_{\pi_{\mu_{d(\alpha-1,m_1)}}}\mathfrak{H}_{d(\alpha-1,m_1,m_2)} \subseteq \mathfrak{X}$$

for $m_1 = 1,\dots,\nu(\alpha-1)$, $m_2 = 1,\dots,\nu(\alpha-1,m_1)$.
By iterating this process, we obtain the assertion.

(Notice that by construction, for each sequence $(m_k)_{k\geq 0}$ there exists a natural number k_0 such that

$$\mathfrak{H}_{d(m_0,\ldots,m_{k_0})} = \mathfrak{S}_{\pi_{\mu_{d(m_0,\ldots,m_{k_0})}}}(\prod_{i\in I_{d(m_0,\ldots,m_{k_0})}} \mathfrak{S}_{\pi_i}),$$

$m_0 := \alpha - 1$.
The usual argument yields

$$\mathfrak{S}_{\pi_{\mu_\alpha}}\mathfrak{S}_{\pi_{\mu_{d(m_0,m_1)}}}\ldots\mathfrak{S}_{\pi_{\mu_{d(m_0,m_1,\ldots,m_{k_0})}}}(\prod_{i\in I_{d(m_0,\ldots,m_{k_0})}} \mathfrak{S}_{\pi_i}) \subseteq \mathfrak{X}$$

and the proof is complete.)

(II) $\mathfrak{Y}$ is normal in $\mathfrak{F}$:
Recall that $J_n = \{i \mid \mu_n \leq i \leq r,\ \pi_i \cap \pi(\mathfrak{Y}_n) = \emptyset\}$ for $n = 0,\ldots,\alpha$. Now, by induction on n we prove $\mathfrak{Y}_n \times \prod_{i\in J_n} \mathfrak{S}_{\pi_i}$ is normal in $\mathfrak{S}_{\pi_{\mu_n}}\ldots\mathfrak{S}_{\pi_r}$. Since, by construction, $\pi(\mathfrak{Y}) = \pi(\mathfrak{F})$, the assertion follows.
$n = 0$:

(a) $\mathfrak{Y}_0$ is normal in $\mathfrak{S}_{\pi_{\mu_0}}\ldots\mathfrak{S}_{\pi_r}$:
It suffices to prove that $\mathfrak{Y}_0$ is normal in $\mathfrak{S}_{\pi_{\mu_0}}\ldots\mathfrak{S}_{\pi_{l_{\mu_0}}}$. Let G be a group contained in $\mathfrak{S}_{\pi_{\mu_0}}\ldots\mathfrak{S}_{\pi_{l_{\mu_0}}}$. According to the choice of μ_0 and to 3.1.16(c), $(G/G_{\mathfrak{S}_{\pi_{\mu_0}}})_{\prod_{i\in I_0}\mathfrak{S}_{\pi_i}}$ is an $\prod_{i\in I_0}\mathfrak{S}_{\pi_i}$-maximal subgroup of $G/G_{\mathfrak{S}_{\pi_{\mu_0}}}$. 1.2.15 yields that $G_{\mathfrak{S}_{\pi_{\mu_0}}(\prod_{i\in I_0}\mathfrak{S}_{\pi_i})}$ is an $\mathfrak{S}_{\pi_{\mu_0}}(\prod_{i\in I_0}\mathfrak{S}_{\pi_i})$-injector of G, and the assertion follows.

(b) $\mathfrak{Y}_0 \times \prod_{i\in J_0}\mathfrak{S}_{\pi_i}$ is normal in $\mathfrak{S}_{\pi_{\mu_0}}\ldots\mathfrak{S}_{\pi_i}$:
Assume not. Let G be a group of minimal order contained in $\mathfrak{S}_{\pi_{\mu_0}}\ldots\mathfrak{S}_{\pi_r} \setminus \mathrm{Y_n}(\mathfrak{Y}_0 \times \prod_{i\in J_0}\mathfrak{S}_{\pi_i})$.
$O_{\pi_{\mu_0}}(G) = 1$:
Suppose not. Then

$$(G/O_{\pi_{\mu_0}}(G))_{\mathfrak{Y}_0\times\prod_{i\in J_0}\mathfrak{S}_{\pi_i}} = (G/O_{\pi_{\mu_0}}(G))_{\mathfrak{Y}_0} \times (G/O_{\pi_{\mu_0}}(G))_{\prod_{i\in J_0}\mathfrak{S}_{\pi_i}}$$

is a $\mathfrak{Y}_0 \times \prod_{i\in J_0}\mathfrak{S}_{\pi_i}$-maximal subgroup of $G/O_{\pi_{\mu_0}}(G)$. By 3.1.16(b) we obtain $G = C_G(G_{\mathfrak{Y}_0}) \leq C_G(O_{\pi_{\mu_0}}(G))$. Consequently, a Hall π_i-subgroup H_i of $G_{\mathfrak{S}_{\pi_{\mu_0}}\mathfrak{S}_{\pi_i}}$ is normalized by G ($i \in J_0$). In particular,

$$G_{\mathfrak{Y}_0} \times \prod_{i\in J_0} H_i \leq G_{\mathfrak{Y}_0\times\prod_{i\in J_0}\mathfrak{S}_{\pi_i}}.$$

Since $\mathfrak{S}_{\pi_{\mu_0}}\mathfrak{Y}_0 = \mathfrak{Y}_0$ and $G/G_{\mathfrak{S}_{\pi_{\mu_0}}} \in \mathrm{Y_n}(\mathfrak{Y}_0 \times \prod_{i\in J_0}\mathfrak{S}_{\pi_i})$, we finally obtain that

$$G_{\mathfrak{Y}_0} \times \prod_{i\in J_0} H_i \text{ is a } \mathfrak{Y}_0 \times \prod_{i\in J_0} \mathfrak{S}_{\pi_i}\text{-maximal subgroup of } G.$$

This contradicts the choice of G.
Using 3.1.16(a) and (c) we obtain a final contradiction.

$n > 0$:

$l_{\mu_n} < \mu_{n-1}$: thus

$$\mathfrak{Y}_n = \mathfrak{S}_{\pi_{\mu_n}}(\prod_{i\in I_n}\mathfrak{S}_{\pi_i}) \times \mathfrak{Y}_{n-1} \text{ and } I_n = \{\mu_n+1, \dots, l_{\mu_n}\}.$$

Analogously to (a), we obtain

$$\mathfrak{S}_{\pi_{\mu_n}}(\prod_{i\in I_n}\mathfrak{S}_{\pi_i}) \trianglelefteq \mathfrak{S}_{\pi_{\mu_n}} \dots \mathfrak{S}_{\pi_{l_{\mu_n}}}.$$

$l_{\mu_n} < \mu_{n-1}$, hence in particular

$$\mathfrak{S}_{\pi_{\mu_n}} \dots \mathfrak{S}_{\pi_{l_{\mu_n}}} \cap \mathfrak{S}_{\pi_{l_{\mu_n}+1}} \dots \mathfrak{S}_{\pi_r} = 1,$$

and therefore

$$\mathfrak{S}_{\pi_{\mu_n}}(\prod_{i\in I_n}\mathfrak{S}_{\pi_i}) \trianglelefteq \mathfrak{S}_{\pi_{\mu_n}} \dots \mathfrak{S}_{\pi_r}.$$

Assume that $\mathfrak{S}_{\pi_{\mu_n}} \dots \mathfrak{S}_{\pi_r}$ is not contained in $\mathrm{Y_n}(\mathfrak{Y}_n \times \prod_{i\in J_n}\mathfrak{S}_{\pi_i})$. Let G be a minimal counterexample. If $O_{\pi_{\mu_n}}(G) = 1$, then, by 3.1.16(a), a $\mathfrak{Y}_n \times \prod_{i\in J_n}\mathfrak{S}_{\pi_i}$-injector V of G is an $\prod_{i\in I_n}\mathfrak{S}_{\pi_i} \times \mathfrak{Y}_{n-1} \times \prod_{i\in J_n}\mathfrak{S}_{\pi_i}$-injector of G as well. Now, inductive hypothesis and 3.1.16(c) yield

$$\prod_{i\in I_n}\mathfrak{S}_{\pi_i} \times \mathfrak{Y}_{n-1} \times \prod_{i\in J_n}\mathfrak{S}_{\pi_i} \trianglelefteq \mathfrak{S}_{\pi_{\mu_n+1}} \dots \mathfrak{S}_{\pi_r}.$$

Thus we conclude that V is a normal subgroup of G, a contradiction. Hence, it remains to show that $O_{\pi_{\mu_n}}(G) = 1$. Suppose that $O_{\pi_{\mu_n}}(G) > 1$. Then, by the choice of G,

$$(G/G_{\mathfrak{S}_{\pi_{\mu_n}}})_{\mathfrak{S}_{\pi_{\mu_n}}(\prod_{i\in I_n}\mathfrak{S}_{\pi_i})} \times (G/G_{\mathfrak{S}_{\pi_{\mu_n}}})_{\mathfrak{Y}_{n-1}} \times (G/G_{\mathfrak{S}_{\pi_{\mu_n}}})_{\prod_{i\in J_n}\mathfrak{S}_{\pi_i}}$$

is a $\mathfrak{Y}_n \times \prod_{i\in J_n}\mathfrak{S}_{\pi_i}$-maximal subgroup of $G/G_{\mathfrak{S}_{\pi_{\mu_n}}}$.

Let $\pi = \pi(\mathfrak{Y}_{n-1})$, $H \in \operatorname{Hall}_\pi(G_{\mathfrak{S}_{\pi_{\mu_n}}\mathfrak{Y}_{n-1}})$ and $H_i \in \operatorname{Hall}_{\pi_i}(G_{\mathfrak{S}_{\pi_{\mu_n}}\mathfrak{S}_{\pi_i}})$ $(i \in J_n)$. Since $C_G(G_{\mathfrak{S}_{\pi_{\mu_n}}}) = G$, we obtain that H, H_i are normal subgroups of G for all $i \in J_n$. (Notice that $\mathfrak{S}_{\pi_{\mu_n}}(\prod_{i\in I_n}\mathfrak{S}_{\pi_i})$ is normal in $\mathfrak{S}_{\pi_{\mu_n}}\dots\mathfrak{S}_{\pi_r}$, thus, using 3.1.16(b), we obtain $G = C_G(G_{\mathfrak{S}_{\pi_{\mu_n}}(\prod_{i\in I_n}\mathfrak{S}_{\pi_i})}) \le C_G(G_{\mathfrak{S}_{\pi_{\mu_n}}}).$)

$H \cong G_{\mathfrak{S}_{\pi_{\mu_n}}\mathfrak{Y}_{n-1}}/G_{\mathfrak{S}_{\pi_{\mu_n}}} \in \mathfrak{Y}_{n-1}$, and consequently

$$G_{\mathfrak{S}_{\pi_{\mu_n}}(\prod_{i\in I_n}\mathfrak{S}_{\pi_i})} \times H \times \prod_{i\in J_n} H_i \le G_{\mathfrak{Y}_n\times\prod_{i\in J_n}\mathfrak{S}_{\pi_i}}.$$

Since $G/G_{\mathfrak{S}_{\pi_{\mu_n}}}$ belongs to $\mathrm{Y_n}(\mathfrak{Y}_n \times \prod_{i\in J_n}\mathfrak{S}_{\pi_i})$, we finally obtain

$G_{\mathfrak{S}_{\pi_{\mu_n}}(\prod_{i\in I_n}\mathfrak{S}_{\pi_i})}\times H\times\prod_{i\in J_n} H_i$ is a $\mathfrak{Y}_n\times\prod_{i\in J_n}\mathfrak{S}_{\pi_i}$-maximal subgroup of G.

This contradicts the choice of G; hence $O_{\pi_{\mu_n}}(G) = 1$ and the proof is complete.

$l_{\mu_n} \ge \mu_{n-1}$: thus

$$\mathfrak{Y}_n = \mathfrak{S}_{\pi_{\mu_n}}(\prod_{i\in I_n}\mathfrak{S}_{\pi_i} \times \prod_{m=1}^{m_0(n-1)}\mathfrak{H}_{d(n-1,m)}) \times \prod_{m=m_0(n-1)+1}^{\nu(n-1)}\mathfrak{H}_{d(n-1,m)}.$$

Using 3.1.17(iii), we obtain

$$\mathfrak{S}_{\pi_{\mu_n}}\dots\mathfrak{S}_{\pi_{a(n)}} \cap \mathfrak{S}_{\pi_{a(n)+1}}\dots\mathfrak{S}_{\pi_r} = 1.$$

If $m \ge m_0(n-1)+1$, by construction $\mathfrak{H}_{d(n-1,m)} \subseteq \mathfrak{S}_{\pi_{a(n)+1}}\dots\mathfrak{S}_{\pi_r}$. If $m \le m_0(n-1)$, the class $\mathfrak{H}_{d(n-1,m)}$ is contained in $\mathfrak{S}_{\pi_{\mu_n}}\dots\mathfrak{S}_{\pi_{a(n)}}$ (by construction as well, cf. 3.1.17(iv)). We conclude

$$\begin{aligned}
&(\mathfrak{Y}_{n-1} \times \prod_{i\in J_{n-1}}\mathfrak{S}_{\pi_i}) \cap \mathfrak{S}_{\pi_{\mu_{n-1}}}\dots\mathfrak{S}_{\pi_{a(n)}}\\
&= \prod_{m=1}^{m_0(n-1)}\mathfrak{H}_{d(n-1,m)} \times \prod_{\substack{i\in J_{n-1}\text{ such that}\\ i\le a(n)}}\mathfrak{S}_{\pi_i}\\
&= \prod_{m=1}^{m_0(n-1)}\mathfrak{H}_{d(n-1,m)} \times \prod_{\substack{i\in I_n\text{ such that}\\ i>\mu_{n-1}}}\mathfrak{S}_{\pi_i}.
\end{aligned}$$

(Notice that $\{i \in J_{n-1}$ such that $i \leq a(n)\} = \{i \in I_n$ such that $i > \mu_{n-1}\}$ according to 3.1.17(v).)

This and the inductive hypothesis yield

$$\prod_{m=1}^{m_0(n-1)} \mathfrak{H}_{d(n-1,m)} \times \prod_{\substack{i \in I_n \text{ such that} \\ i > \mu_{n-1}}} \mathfrak{S}_{\pi_i} \trianglelefteq \mathfrak{S}_{\pi_{\mu_{n-1}}} \dots \mathfrak{S}_{\pi_{a(n)}}.$$

Using 3.1.16(c), we conclude

$$\prod_{i \in I_n} \mathfrak{S}_{\pi_i} \times \prod_{m=1}^{m_0(n-1)} \mathfrak{H}_{d(n-1,m)} \trianglelefteq \mathfrak{S}_{\pi_{\mu_n+1}} \dots \mathfrak{S}_{\pi_{a(n)}};$$

thus 1.2.15 finally yields

$$\mathfrak{S}_{\pi_{\mu_n}}(\prod_{i \in I_n} \mathfrak{S}_{\pi_i} \times \prod_{m=1}^{m_0(n-1)} \mathfrak{H}_{d(n-1,m)}) \trianglelefteq \mathfrak{S}_{\pi_{\mu_n}} \dots \mathfrak{S}_{\pi_{a(n)}}.$$

(Observe that $\pi(\prod_{i \in I_n} \mathfrak{S}_{\pi_i} \times \prod_{m=1}^{m_0(n-1)} \mathfrak{H}_{d(n-1,m)}) = \pi(\mathfrak{S}_{\pi_{\mu_n}} \dots \mathfrak{S}_{\pi_{a(n)}})$.)

Since $\mathfrak{S}_{\pi_{\mu_n}} \dots \mathfrak{S}_{\pi_{a(n)}} \cap \mathfrak{S}_{\pi_{a(n)+1}} \dots \mathfrak{S}_{\pi_r} = 1$, it follows

$$\mathfrak{S}_{\pi_{\mu_n}}(\prod_{i \in I_n} \mathfrak{S}_{\pi_i} \times \prod_{m=1}^{m_0(n-1)} \mathfrak{H}_{d(n-1,m)}) \trianglelefteq \mathfrak{S}_{\pi_{\mu_n}} \dots \mathfrak{S}_{\pi_r}.$$

Assume that $\mathfrak{S}_{\pi_{\mu_n}} \dots \mathfrak{S}_{\pi_r} \not\subseteq \mathrm{Y_n}(\mathfrak{Y}_n \times \prod_{i \in J_n} \mathfrak{S}_{\pi_i})$. Let G be a minimal counterexample.

$O_{\pi_{\mu_n}}(G) = 1$: Suppose $O_{\pi_{\mu_n}}(G) > 1$, then

$$(G/O_{\pi_{\mu_n}}(G))_{\mathfrak{Y}_n \times \prod_{i \in J_n} \mathfrak{S}_\pi} =$$
$$= (G/O_{\pi_{\mu_n}}(G))_{\mathfrak{S}_{\pi_{\mu_n}}(\prod_{i \in I_n} \mathfrak{S}_{\pi_i} \times \prod_{m=1}^{m_0(n-1)} \mathfrak{H}_{d(n-1,m)})} \times$$
$$\times (G/O_{\pi_{\mu_n}}(G))_{\prod_{m=m_0+1}^{m(n-1)} \mathfrak{H}_{d(n-1,m)}} \times (G/O_{\pi_{\mu_n}}(G))_{\prod_{i \in J_n} \mathfrak{S}_{\pi_i}}$$

is a $\mathfrak{Y}_n \times \prod_{i \in J_n} \mathfrak{S}_{\pi_i}$-maximal subgroup of $G/O_{\pi_{\mu_n}}(G)$. Now, analogously to the preceding case, we conclude

$$G_{\mathfrak{S}_{\pi_{\mu_n}}(\prod_{i \in I_n} \mathfrak{S}_{\pi_i} \times \prod_{m=1}^{m_0(n-1)} \mathfrak{H}_{d(n-1,m)})} \times H \times \prod_{i \in J_n} H_i$$

is a normal and a $(\mathfrak{Y}_n \times \prod_{i\in J_n} \mathfrak{S}_{\pi_i})$-maximal subgroup of G (where $\pi = \pi(\prod_{m=m_0(n-1)+1}^{\nu(n-1)} \mathfrak{H}_{d(n-1,m)})$, $H \in \mathrm{Hall}_\pi(G_{\mathfrak{S}_{\pi_{\mu_n}}(\prod_{m=m_0(n-1)+1}^{\nu(n-1)} \mathfrak{H}_{d(n-1,m)})})$ and $H_i \in \mathrm{Hall}_\pi(G_{\mathfrak{S}_{\pi_{\mu_n}}\mathfrak{S}_{\pi_i}}))$. This contradicts the choice of G, and consequently $O_{\pi_{\mu_n}}(G) = 1$. Using 3.1.16(a), we obtain that a $\mathfrak{Y}_n \times \prod_{i\in J_n} \mathfrak{S}_{\pi_i}$- injector V of G is an $\prod_{i\in I_n} \mathfrak{S}_{\pi_i} \times \prod_{m=1}^{\nu(n-1)} \mathfrak{H}_{d(n-1,m)} \times \prod_{i\in J_n} \mathfrak{S}_{\pi_i}$- injector of G as well. So finally, inductive hypothesis and 3.1.16(c) yield

$$\prod_{i\in I_n} \mathfrak{S}_{\pi_i} \times \prod_{m=1}^{\nu(n-1)} \mathfrak{H}_{d(n-1,m)} \times \prod_{i\in J_n} \mathfrak{S}_{\pi_i} \trianglelefteq \mathfrak{S}_{\pi_{\mu_n+1}} \dots \mathfrak{S}_{\pi_r}.$$

Consequently, V is a normal subgroup of G; this contradicts the choice of G. Thus $\mathfrak{Y}_n \times \prod_{i\in J_n} \mathfrak{S}_{\pi_i}$ is an $\mathfrak{S}_{\pi_{\mu_n}} \dots \mathfrak{S}_{\pi_r}$-normal Lockett class and the proof is complete.

Hence, $\mathfrak{Y}$ is normal in $\mathfrak{F}$ and therefore the unique minimal $\mathfrak{F}$-normal Lockett class. □

We obtain as a special case of 3.1.18

3.1.19 Corollary
Let $\pi_1, \dots, \pi_r$ be sets of primes such that $\pi_i \neq \pi_{i+1}$ and $\cap_{i=1}^r \pi_i \neq \emptyset$, and set $\mathfrak{F} = \mathfrak{S}_{\pi_1} \dots \mathfrak{S}_{\pi_r}$. Then $(\mathfrak{F})_*$ is the unique minimal $\mathfrak{F}$-normal Fitting class.

Note that for all SFitting classes treated above the smallest $\mathfrak{F}$-normal Lockett class coincides with the smallest $\mathfrak{F}$-normal SFitting class. Is this true in general? We will see in the next section that a positive answer would yield an explicit description of this class in many cases.

3.2 Local normality and SFitting classes

As mentioned before, the subgroup-closure of a Fitting class enables us to use the theory of local formations. Furthermore, local normality behaves nicely with respect to the corresponding canonical local definitions.

3.2.1 Proposition
Let $\mathfrak{X}$ and $\mathfrak{F}$ be SFitting classes with corresponding canonical local definitions X and F. Assume further that $\pi := \pi(\mathfrak{F}) \subseteq \pi(\mathfrak{X})$. Then the following statements are equivalent:

(i) $\mathfrak{F} \subseteq \mathrm{Y}_n(\mathfrak{X})$.

(ii) $F(p) \subseteq \mathrm{Y}_n(X(p))$ *for all* $p \in \pi$.

Proof:

$(i) \Rightarrow (ii)$: Let G be a group contained in $F(p)$ and set $H = \mathrm{Z}_p \wr G$ where p denotes an arbitrary element of π. $H \in \mathfrak{S}_p F(p) = F(p) \subseteq \mathfrak{F}$, thus by assumption, an $\mathfrak{X}$-injector W of H is a normal subgroup of H. On the other hand, 2.1.4 yields $W \cap G \in \mathrm{Inj}_{X(p)}(G)$, and the assertion follows.

$(ii) \Rightarrow (i)$: Suppose not and choose a group G of minimal order contained in $\mathfrak{F} \setminus \mathrm{Y}_n(\mathfrak{X})$. According to 3.1.6, G has a unique maximal and a unique minimal normal subgroup. In particular, $O_{q'}(G) = 1$ for some suitable prime $q \in \pi$. Consequently, $G \in F(q) \subseteq \mathrm{Y}_n(X(q))$ and $G_{\mathfrak{X}} = G_{X(q)}$. Let V be an $\mathfrak{X}$-injector of G. Then $O_{q'}(V) \neq 1$, for otherwise, $G_{X(q)} < V \in X(q)$; a contradiction. Let p be a prime such that $O_p(V) \neq 1$. Since $O_p(V) \cap G_{\mathfrak{X}}$ is a subnormal subgroup of G (and therefore trivial), we obtain $V = O_p(V) \times G_{\mathfrak{X}}$. This implies $O_p(V) \leq C_G(G_{\mathfrak{X}})$, and consequently $C_G(G_{\mathfrak{X}}) = G$ (for otherwise V is contained in the unique maximal normal subgroup of G; a contradiction). Since $\pi \subseteq \pi(\mathfrak{X})$, and therefore $F(G) \leq G_{\mathfrak{X}}$, this implies a final contradiction.

□

(A similar result holds for so-called strictly normal Fitting classes, cf. [4, 7.1]; I thank Prof. B. Brewster for pointing this out to me.)

3.2.2 Remark
3.2.1 need not be true for arbitrary SFitting classes $\mathfrak{X}$ and $\mathfrak{F}$.

Proof: Let p be a prime, $\mathfrak{X} = \mathfrak{S}_p$ and $\mathfrak{F} = \mathfrak{S}_{p'}\mathfrak{S}_p\mathfrak{S}_{p'}$. The canonical local definition X of $\mathfrak{X}$ is defined by

$$X(q) = \begin{cases} \mathfrak{S}_p & \text{if } q = p, \\ \emptyset & \text{otherwise,} \end{cases}$$

and, according to 1.3.8, the canonical local definition F of $\mathfrak{F}$ is given by

$$F(q) = \begin{cases} \mathfrak{S}_p\mathfrak{S}_{p'} & \text{if } q = p \\ \mathfrak{S}_{p'}\mathfrak{S}_p\mathfrak{S}_{p'} & \text{otherwise.} \end{cases}$$

Thus, $F(q) \subseteq \mathrm{Y_n}(X(q))$ for all primes q, but $\mathfrak{F} \not\subseteq \mathrm{Y_n}(\mathfrak{X})$.
(Notice that $Z_q \wr Z_p \in \mathfrak{F} \setminus \mathrm{Y_n}(\mathfrak{X})$, for an arbitrary prime $q \neq p$.) □

3.2.3 Lemma
Let $\mathfrak{X}$ and $\mathfrak{F}$ be non-trivial Fitting classes such that $\mathfrak{F} = {}_Q\mathfrak{F} \subseteq Y_n(\mathfrak{X})$. Set $\pi(\mathfrak{X}) = \pi$. Then $\mathfrak{F} \subseteq (\mathfrak{S}_\pi \cap \mathfrak{F})\mathfrak{S}_{\pi'}$.

Proof: We show that $\mathfrak{F} \subseteq \mathfrak{S}_\pi\mathfrak{S}_{\pi'}$. Suppose the contrary and choose a group G of minimal order contained in $\mathfrak{F} \setminus \mathfrak{S}_\pi\mathfrak{S}_{\pi'}$. Then G has a unique maximal normal subgroup N, and a unique minimal normal subgroup M, and N and G/M belong to $\mathfrak{S}_\pi\mathfrak{S}_{\pi'}$. Since $O_\pi(G) = 1$ and $O^{\pi'}(G) = G$, we conclude $N = M \in \mathfrak{S}_{\pi'}$; this contradicts the assumption that $\mathfrak{F} \subseteq \mathrm{Y_n}(\mathfrak{X})$. □

3.2.1 and 3.2.3 enable us to prove the existence of a unique maximal SFitting class contained in $\mathrm{Y_n}(\mathfrak{X})$ for any SFitting class $\mathfrak{X}$.

3.2.4 Proposition
Let $\mathfrak{X}$ be an SFitting class, and let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be SFitting classes of bounded nilpotent length.
Then $\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2) \subseteq Y_n(\mathfrak{X})$, *provided that* $\mathfrak{F}_1,\ \mathfrak{F}_2 \subseteq Y_n(\mathfrak{X})$.

Proof: By induction on $r := \max(l(\mathfrak{F}_1), l(\mathfrak{F}_2))$.
The cases $r = 0,\ 1$ are trivial. Thus we assume that $r > 1$ and that the assertion holds for $r - 1$. Set $\pi = \pi(\mathfrak{X})$.
According to 3.2.3 and 2.2.12, we may assume without loss of generality that $\mathfrak{F}_i \subseteq \mathfrak{S}_\pi$. Let F_1, F_2 and X, respectively, be the canonical local definitions belonging to $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{X}$, respectively. 3.2.1 yields $F_i(p) \cap \mathfrak{N}^{r-1} \subseteq \mathrm{Y}_n(X(p))$ for $i = 1, 2$ and $p \in \pi$. By inductive hypothesis this implies $\mathrm{SFit}((F_1(p) \cap \mathfrak{N}^{r-1}), (F_2(p) \cap \mathfrak{N}^{r-1})) \subseteq \mathrm{Y}_n(X(p))$. Thus, using 2.2.2 and 3.1.7 we obtain $\mathfrak{S}_p\mathrm{SFit}(F_1(p) \cap \mathfrak{N}^{r-1}, F_2(p) \cap \mathfrak{N}^{r-1}) =$

$\mathrm{SFit}(\mathfrak{S}_p(F_1(p) \cap \mathfrak{N}^{r-1}), \mathfrak{S}_p(F_2(p) \cap \mathfrak{N}^{r-1}) \subseteq \mathrm{Y}_n(X(p))$. The proof is completed by 3.2.1. □

3.2.5 Corollary
Let $\mathfrak{X}$, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be SFitting classes. Then $\mathrm{SFit}(\mathfrak{F}_1, \mathfrak{F}_2) \subseteq Y_n(\mathfrak{X})$ *provided that* $\mathfrak{F}_1,\ \mathfrak{F}_2 \subseteq Y_n(\mathfrak{X})$.

Proof: Using 2.2.5, we obtain the result by 3.2.4. □

3.2.6 Corollary
Let $\mathfrak{F}_i,\ i \in I$, and $\mathfrak{X}$ be SFitting classes such that $\mathfrak{F}_i \subseteq Y_n(\mathfrak{X})$. Then

$$\mathrm{SFit}(\mathfrak{F}_i \mid i \in I) \subseteq Y_n(\mathfrak{X}).$$

In particular, there exists a unique maximal SFitting class contained in $Y_n(\mathfrak{X})$.

Proof: 3.2.5 and 2.2.13. □

3.2.7 Definition
Let $\mathfrak{X}$ be an SFitting class. We define

$$\mathfrak{L}_{(n,\mathfrak{X})} = (\{\mathfrak{F} \mid \mathfrak{F} \text{ SFitting class, } \mathfrak{X} \text{ is normal in } \mathfrak{F}\},\ \subseteq).$$

3.2.8 Theorem
Let $\mathfrak{X}$ be an SFitting class.

(a) $\mathfrak{L}_{(n,\mathfrak{X})}$ *forms a complete, distributive and atomic lattice.*

(b) *An SFitting class $\mathfrak{H}$ is an atom of $\mathfrak{L}_{(n,\mathfrak{X})}$ if and only if it is an atom of* $\mathfrak{L}_{\mathfrak{X}}$.

Proof: According to 3.2.6, 2.2.5 and 2.2.18, it suffices to prove statement (b). Since an atom of $\mathfrak{L}_{\mathfrak{X}}$ is contained in $\mathfrak{X}\mathfrak{N} \subseteq \mathrm{Y_n}(\mathfrak{X})$, this assertion clearly holds. □

We will see later that, in general, $\mathfrak{L}_{(n,\mathfrak{X})}$ fails to be dual atomic (cf. 3.2.21).

3.2.9 Definition
Let $\mathfrak{X}$ be an SFitting class, and set $\pi = \pi(\mathfrak{X})$. We define

$$\mathfrak{Y}^{(n,\mathfrak{X})} := \mathrm{SFit}(\mathfrak{F} \mid \mathfrak{F} \subseteq \mathrm{Y}_n(\mathfrak{X}) \cap \mathfrak{S}_\pi,\ \mathfrak{F}\ \text{SFitting class}).$$

$$\overline{\mathfrak{Y}}^{(n,\mathfrak{X})} := \mathrm{SFit}(\mathfrak{F} \mid \mathfrak{F} \subseteq \mathrm{Y}_n(\mathfrak{X}),\ \mathfrak{F}\ \text{SFitting class}).$$

3.2.10 Remark
Let $\mathfrak{X}$ be an SFitting class.

(a) $\overline{\mathfrak{Y}}^{(n,\mathfrak{X})} = \mathfrak{Y}^{(n,\mathfrak{X})}\mathfrak{S}_{\pi'}$.

(b) *If $\mathfrak{S}_p\mathfrak{X} = \mathfrak{X}$, then $\mathfrak{S}_p\mathfrak{Y}^{(n,\mathfrak{X})} = \mathfrak{Y}^{(n,\mathfrak{X})}$ and $\mathfrak{S}_p\overline{\mathfrak{Y}}^{(n,\mathfrak{X})} = \overline{\mathfrak{Y}}^{(n,\mathfrak{X})}$.*

(c) $\pi(\mathfrak{X}) = \pi(\mathfrak{Y}^{(n,\mathfrak{X})})$.

Proof: The assertion follows from the definition and 3.2.3, 2.2.5 and 3.1.7. □

3.2.11 Proposition
Let $\mathfrak{F}$ and $\mathfrak{X}$ be non-trivial SFitting classes such that $\mathfrak{Y}^{(n,\mathfrak{X})} \subseteq \mathfrak{Y}^{(n,\mathfrak{F})}$ or $\overline{\mathfrak{Y}}^{(n,\mathfrak{X})} \subseteq \overline{\mathfrak{Y}}^{(n,\mathfrak{F})}$. Then

$$\mathfrak{X} \subseteq \mathfrak{F}.$$

Proof: According to 2.2.16, $\mathfrak{X} = \mathrm{SFit}(\mathfrak{X}_i \mid i \in I)$ where each $\mathfrak{X}_i$ is a product of $\mathfrak{S}_p$'s (for some primes p). Let i be an element of I, and $\mathfrak{X}_i = \mathfrak{S}_{p_1}\cdots\mathfrak{S}_{p_r}$ for suitable primes $p_1, \dots, p_r$. By induction on k, $k \le r$ we prove that $\mathfrak{X}_i \subseteq \mathfrak{F}$.
$k = 1$: If $\mathfrak{Y}^{(n,\mathfrak{X})} \subseteq \mathfrak{Y}^{(n,\mathfrak{F})}$, the definition of $\mathfrak{Y}^{(n,\mathfrak{X})}$ implies that $\pi(\mathfrak{X}) \subseteq \pi(\mathfrak{F})$, and consequently that $\mathfrak{S}_{p_1}$ is contained in $\mathfrak{F}$. Thus, we assume that $\overline{\mathfrak{Y}}^{(n,\mathfrak{X})} \subseteq \overline{\mathfrak{Y}}^{(n,\mathfrak{F})}$. Since $Z_{p_1} \wr Z_q$ belongs to $\mathfrak{X}_i\mathfrak{S}_q \subseteq \mathfrak{X}\mathfrak{S}_q \subseteq \overline{\mathfrak{Y}}^{(n,\mathfrak{X})} \subseteq \overline{\mathfrak{Y}}^{(n,\mathfrak{F})}$ for each prime $q \in \pi(\mathfrak{F})$, we conclude $p_1 \in \pi(\mathfrak{F})$ and thus $\mathfrak{S}_{p_1} \subseteq \mathfrak{F}$ in this case as well.
Thus suppose that $k > 1$ and that $\mathfrak{S}_{p_1}\cdots\mathfrak{S}_{p_{k-1}} \subseteq \mathfrak{F}$. By assumption, $\mathfrak{S}_{p_1}\cdots\mathfrak{S}_{p_k}\mathfrak{S}_{p_{k-1}} \subseteq \mathfrak{X}_i\mathfrak{S}_{p_{k-1}} \subseteq \mathfrak{X}\mathfrak{S}_{p_{k-1}} \subseteq \mathfrak{Y}^{(n,\mathfrak{X})} \subseteq \overline{\mathfrak{Y}}^{(n,\mathfrak{F})}$. Hence 3.1.13 yields the assertion, and the proof is complete. □

It follows from the previous proposition that a non-trivial SFitting class $\mathfrak{X}$ is uniquely determined by $\mathfrak{Y}^{(n,\mathfrak{X})}$ respectively $\overline{\mathfrak{Y}}^{(n,\mathfrak{X})}$:

3.2.12 Corollary
Let $\mathfrak{F}$ and $\mathfrak{X}$ be non-trivial SFitting classes. Then the following statements are equivalent:

(i) $\mathfrak{Y}^{(n,\mathfrak{F})} = \mathfrak{Y}^{(n,\mathfrak{X})}$.

(ii) $\overline{\mathfrak{Y}}^{(n,\mathfrak{F})} = \overline{\mathfrak{Y}}^{(n,\mathfrak{X})}$.

(iii) $\mathfrak{F} = \mathfrak{X}$.

Proof: $(i) \Rightarrow (ii)$, $(iii) \Rightarrow (i)$: trivial.
$(ii) \Rightarrow (iii)$: 3.2.11. □

3.2.13 Remark

(a) *The converse of 3.2.11 does not hold true: Let p_1, p_2, p_3 be pairwise distinct primes. Set $\mathfrak{X} = \mathfrak{S}_{p_1} \times \mathfrak{S}_{p_2} \times \mathfrak{S}_{p_3}$ and $\mathfrak{F} = \mathfrak{S}_{p_1}\mathfrak{S}_{p_2}\mathfrak{S}_{p_3}$. Then $\mathfrak{X} \subseteq \mathfrak{F}$. Evidently, $Z_{p_2} \wr Z_{p_1} \wr Z_{p_3} \in \mathfrak{Y}^{(n,\mathfrak{X})} \setminus \mathrm{Y}_{\mathrm{n}}(\mathfrak{F})$, hence neither $\overline{\mathfrak{Y}}^{(n,\mathfrak{X})} \subseteq \overline{\mathfrak{Y}}^{(n,\mathfrak{F})}$ nor $\mathfrak{Y}^{(n,\mathfrak{X})} \subseteq \mathfrak{Y}^{(n,\mathfrak{F})}$.*

(b) *We will see later that a corresponding result to 3.2.12 concerning the dual class does not hold true (cf. 3.2.29).*

3.2.14 Proposition
Let $\mathfrak{X}$ be an SFitting class and let X be the canonical local definition belonging to $\mathfrak{X}$. Further, set $\pi = \pi(\mathfrak{X})$, $\pi(p) = \pi(X(p))$ and $\widetilde{\pi} = \{p \in \pi \mid X(p) \neq \mathfrak{X}\}$.

(a) $\mathfrak{Y}^{(n,\mathfrak{X})} = \bigcap_{p\in\pi}((\mathfrak{S}_{p'}\mathfrak{Y}^{(n,X(p))}\mathfrak{S}_{\pi(p)'}) \cap \mathfrak{S}_\pi)$.

(b) $\mathfrak{Y}^{(n,\mathfrak{X})} = \bigcap_{p\in\widetilde{\pi}}((\mathfrak{S}_{p'}\mathfrak{Y}^{(n,X(p))}\mathfrak{S}_{\pi(p)'}) \cap \mathfrak{S}_\pi)$.

In particular, $\mathfrak{Y}^{(n,\mathfrak{X})}$ is known provided that $\mathfrak{Y}^{(n,X(p))}$ is known for all $p \in \widetilde{\pi}$.

Proof:

(a) $\subseteq$: Let H be the canonical local definition belonging to $\mathfrak{Y}^{(n,\mathfrak{X})}$. 3.2.1 yields $H(p) \subseteq \mathrm{Y}_{\mathrm{n}}(X(p))$ and hence $H(p) \subseteq \mathfrak{Y}^{(n,X(p))}\mathfrak{S}_{\pi(p)'}$ for all $p \in \pi$. Consequently

$$\mathfrak{Y}^{(n,\mathfrak{X})} = \cap_{p\in\pi}\mathfrak{S}_{p'}H(p) \cap \mathfrak{S}_\pi \subseteq \cap_{p\in\pi}((\mathfrak{S}_{p'}\mathfrak{Y}^{(n,X(p))}\mathfrak{S}_{\pi(p)'}) \cap \mathfrak{S}_\pi).$$

$\supseteq$: Suppose not. Let G be a group of minimal order contained in $\cap_{p\in\pi}((\mathfrak{S}_{p'}\mathfrak{Y}^{(n,X(p))}\mathfrak{S}_{\pi(p)'}) \cap \mathfrak{S}_\pi) \setminus \mathrm{Y_n}(\mathfrak{X})$. According to 3.1.6, G has a unique maximal normal subgroup N, and a unique minimal normal subgroup M, and $G/N \cong Z_q$, $NV = G$, $N \cap V = G_{\mathfrak{X}}$, and $M \in \mathfrak{S}_r$ (where $V \in \mathrm{Inj}_{\mathfrak{X}}(G)$ and $q,\ r \in \mathbb{P}$ suitable). In particular, $O_{r'}(G) = 1$ and consequently $G \in \mathfrak{Y}^{(n,X(r))}\mathfrak{S}_{\pi(r)'} \cap \mathfrak{S}_\pi \subseteq \mathrm{Y_n}(X(r))$ and $G_{\mathfrak{X}} = G_{X(r)}$. Thus we obtain $O_{r'}(V) \neq 1$ and therefore $V = O_t(V) \times G_{\mathfrak{X}}$ where $t \in \mathbb{P} \setminus \{r\}$ suitable. As usual, this implies $C_G(G_{\mathfrak{X}}) = G$, a contradiction.
Consequently, $\cap_{p\in\pi}((\mathfrak{S}_{p'}\mathfrak{Y}^{(n,X(p))}\mathfrak{S}_{\pi(p)'}) \cap \mathfrak{S}_\pi) \subseteq \mathrm{Y_n}(\mathfrak{X})$, and the assertion follows.

(b) Without loss of generality, we assume that $\pi \supset \widetilde{\pi}$.

$\subseteq$: (a).
$\supseteq$: Assume the contrary and let G be a group of minimal order contained in $\cap_{p\in\widetilde{\pi}}((\mathfrak{S}_{p'}\mathfrak{Y}^{(n,X(p))}\mathfrak{S}_{\pi(p)'}) \cap \mathfrak{S}_\pi) \setminus \mathfrak{Y}^{(n,\mathfrak{X})}$. Evidently, G has a unique maximal normal subgroup N, and a unique minimal normal subgroup M, and $N/M \in \mathfrak{X}$, $G/N \cong Z_t$ and $M \in \mathfrak{S}_r$ for suitable primes $t,\ r$. In particular, G belongs to $\mathfrak{S}_r\mathfrak{Y}^{(n,\mathfrak{X})}$.
Let q be a prime such that $q \in \pi\setminus \widetilde{\pi}$. We prove that G belongs to $\mathfrak{S}_{q'}\mathfrak{Y}^{(n,X(q))}\mathfrak{S}_{\pi(q)'}$ (this implying $G \in \mathfrak{Y}^{(n,\mathfrak{X})}$, a contradiction):
If $q = r$, then $\mathfrak{S}_r X(r) = \mathfrak{S}_r\mathfrak{X} = \mathfrak{X}$, and consequently $\mathfrak{S}_r\mathfrak{Y}^{(n,\mathfrak{X})} = \mathfrak{Y}^{(n,\mathfrak{X})} = \mathfrak{Y}^{(n,X(q))}$; in particular, G belongs to $\mathfrak{Y}^{(n,X(q))} \subseteq \mathfrak{S}_{q'}\mathfrak{Y}^{(n,X(q))}\mathfrak{S}_{\pi(q)'}$.
If $q \neq r$, then $r \in \mathbb{P}\setminus\{q\}$ and therefore $G \in \mathfrak{S}_{q'}\mathfrak{Y}^{(n,\mathfrak{X})} = \mathfrak{S}_{q'}\mathfrak{Y}^{(n,X(q))} \subseteq \mathfrak{S}_{q'}\mathfrak{Y}^{(n,X(q))}\mathfrak{S}_{\pi(q)'}$.

□

3.2.15 Corollary
Let $\mathfrak{F}$ and $\mathfrak{X}$ be SFitting classes such that $\mathfrak{F} = \mathfrak{S}_\tau\mathfrak{X}$ where $\tau \neq \emptyset$, $\mathbb{P}$ denotes a set of primes. Let X be the canonical local definition belonging to $\mathfrak{X}$.
If $\pi(\mathfrak{X}) = \pi$ and $\mathfrak{Y}^{(n,X(p))}$ is known for all $p \in \pi \setminus \tau$, then $\mathfrak{Y}^{(n,\mathfrak{F})}$ is known.

Proof: 1.3.7 and 3.2.14. □

3.2.16 Examples

(1) Set $\mathfrak{F} = \mathfrak{F}_r = \mathfrak{S}_{\pi_1} \cdots \mathfrak{S}_{\pi_r}$ where $\pi_1, \ldots, \pi_r$ are sets of primes. By 1.3.8 the canonical local definition is known, and thus $\mathfrak{Y}^{\mathfrak{F}_r}$ can be determined recursively: Set $\pi = \pi_1 \cup \ldots \cup \pi_r$. If $r = 1$, then $\mathfrak{Y}^{\mathfrak{F}_1} = \mathfrak{Y}^{\mathfrak{S}_{\pi_1}} = \mathfrak{S}_{\pi_1}$. Assume that $r > 1$ and $\mathfrak{Y}^{\mathfrak{F}_k}$ is known for $k < r$. Then the assertion follows from 3.2.14(b).

(2) Let π be a set of primes and $(\pi_i)_{i \in I}$ a partition of π. Let $\mathfrak{F}$ be the corresponding lattice formation. By 1.3.9 and 3.2.14(b) we obtain

$$\mathfrak{Y}^{(n,\mathfrak{F})} = \cap_{i \in I} \mathfrak{S}_{\pi_i'} \mathfrak{S}_{\pi_i} \mathfrak{S}_{\pi_i'} \cap \mathfrak{S}_\pi.$$

In particular, $\mathfrak{Y}^{(n,\mathfrak{N})} = \cap_{p \in \mathbb{P}} \mathfrak{S}_{p'} \mathfrak{S}_p \mathfrak{S}_{p'}$.

(3) Let $\mathfrak{F}$ be an SFitting class as described in 1.3.10. Then 3.2.14(b) yields

$$\mathfrak{Y}^{(n,\mathfrak{F})} = \cap_{p \in \pi} (\mathfrak{S}_{p'} \mathfrak{Y}^{(n,\mathfrak{S}_{\pi(p)})} \mathfrak{S}_{\pi(p)'}) = \cap_{p \in \widetilde{\pi}} \mathfrak{S}_{(\pi \cap \pi(p))'} \mathfrak{S}_{\pi(p)} \mathfrak{S}_{\pi(p)'}.$$

where π and $\widetilde{\pi}$ are as described in 1.3.10.

3.2.17 Remark

Let $\mathfrak{F}$ be an SFitting class of bounded nilpotent length. Then there exists an algorithm to describe $\mathfrak{Y}^{(n,\mathfrak{F})}$ (and consequently $\overline{\mathfrak{Y}}^{(n,\mathfrak{F})}$).

Proof: Set $\pi = \pi(\mathfrak{F})$ and $r = l(\mathfrak{F})$. $\mathfrak{F}$ is known, thus by [9, IV, 3.7] the corresponding canonical local definition F is known, too. By 1.3.7, $F(p) = \mathfrak{S}_p(F(p) \cap \mathfrak{N}^{r-1})$. Furthermore, 3.2.14(b) implies

$$\mathfrak{Y}^{(n,\mathfrak{F})} = \cap_{p \in \widetilde{\pi}} ((\mathfrak{S}_{p'} \mathfrak{Y}^{(n,F(p))} \mathfrak{S}_{\pi(p)'}) \cap \mathfrak{S}_\pi),$$

where $\pi(p) = \pi(F(p))$ and $\widetilde{\pi} = \{p \in \pi \mid F(p) \neq \mathfrak{F}\}$.
If $F(p;\ )$ denotes the canonical local definition belonging to $(F(p) \cap \mathfrak{N}^{r-1})$ and $\pi(p; p_1) = \pi(F(p; p_1))$ for all $p_1 \in \mathbb{P}$, then, by 3.2.14(b),

$$\mathfrak{Y}^{(n,F(p))} = \cap_{p_1 \in \pi(p) \setminus \{p\}} ((\mathfrak{S}_{p_1'} \mathfrak{Y}^{F(p;p_1)} \mathfrak{S}_{\pi(p;p_1)'}) \cap \mathfrak{S}_{\pi(p)}).$$

Observe that $l(F(p; p_1)) < l(\mathfrak{F})$. Iterating this process, we obtain a natural number $k \leq r-1$ such that $F(p, p_1, \ldots, p_{k-1}; p_k) \subseteq \mathfrak{N}$ for all $p_k \in \mathbb{P}$. In this case, $F(p, p_1, \ldots, p_{k-1}; p_k) = \mathfrak{N}_{\pi(p,p_1,\ldots,p_{k-1};p_k)}$ and consequently $\mathfrak{Y}^{F(p,p_1,\ldots,p_{k-1};p_k)}$ is known. $\square$

Let $\mathfrak{F}$ be a lattice formation. For this case we give a further description of $\mathfrak{Y}^{(n,\mathfrak{F})}$, which is frequently easier to handle than the one above.

3.2.18 Definition
Let π be a set of primes, let $(\pi_j)_{j\in J}$ be a partition of π, and let $j_1,\ j_2,\ldots$ be an ordering of J. We define

$$\mathfrak{F}^{(\{\pi_j \ | \ j\in J\})} := \mathrm{SFit}(\mathfrak{F}_\sigma \mid \sigma \in \mathrm{Sym}(J)),$$

where $\mathfrak{F}_\sigma = \cup_{j_i\in J}\mathfrak{S}_{\pi_{\sigma(j_1)}}\cdots\mathfrak{S}_{\pi_{\sigma(j_i)}}$.

If $|\pi_j| = 1$ for all $j \in J$, we write $\mathfrak{F}^\pi$ rather than $\mathfrak{F}^{(\{\pi_j \ | \ j\in J\})}$.

Note that $\mathfrak{X}^2 \subseteq \mathfrak{F}^{(\{\pi_j \ | \ j\in J\})} \subseteq \mathfrak{S}_\pi$ where $\mathfrak{X}$ denotes the lattice formation belonging to $(\pi_j)_{j\in J}$.

3.2.19 Lemma
Let $\pi,\ \sigma,\ \tau$ be sets of primes, $\pi \neq \emptyset$, and let $(\pi_j)_{j\in J}$ be a partition of π. Further, assume that $\sigma \cap \pi = \emptyset$.

(a) *If $\widetilde{\pi}_j = \pi_j \cap \tau$, then* $\mathfrak{F}^{(\{\widetilde{\pi}_j \ | \ j\in J\})} \subseteq \mathfrak{F}^{(\{\pi_j \ | \ j\in J\})}$.

(b) $\mathfrak{F}^{(\{\pi_j \ | \ j\in J\})} \subseteq \mathfrak{F}^{(\{\pi_j \ | \ j\in J\}\cup\{\sigma\})}$.

(c) *If $J_0 \subseteq J$ and $\pi_{J_0} = \cup_{i\in J_0}\pi_i$, then* $\mathfrak{F}^{(\{\pi_j \ | \ j\in J\})} \cap \mathfrak{S}_{\pi_{J_0}} = \mathfrak{F}^{(\{\pi_j \ | \ j\in J_0\})}$.

(d) $\mathfrak{S}_\sigma\mathfrak{F}^{(\{\pi_j \ | \ j\in J\})} \subseteq \mathfrak{F}^{(\{\pi_j \ | \ j\in J\}\cup\{\sigma\})}$.

(e) $\mathfrak{F}^{(\{\pi_j \ | \ j\in J\})}\mathfrak{S}_\sigma \subseteq \mathfrak{F}^{(\{\pi_j \ | \ j\in J\}\cup\{\sigma\})}$.

Proof: (a),(b) follow from the definition. (c) is a consequence of 2.2.14(b), (d) follows from 2.2.14(a), and 2.2.14(c) implies (e). □

3.2.20 Proposition
Let $\mathfrak{X}$ be a lattice formation belonging to $(\pi_j)_{j\in J}$. Then

$$\mathfrak{F}^{(\{\pi_j \ | \ j\in J\})}\mathfrak{S}_{\pi'} = \overline{\mathfrak{Y}}^{(n,\mathfrak{X})}.$$

Furthermore, if $\mathfrak{F}$ is a Q*-closed Fitting class contained in $\mathrm{Y}_n(\mathfrak{X})$, then* $\mathfrak{F} \subseteq \mathfrak{F}^{(\{\pi_j \ | \ j\in J\})}\mathfrak{S}_{\pi'}$.

In particular, $\mathfrak{F}^{\mathbb{P}}$ is the unique maximal Q*-closed Fitting class contained in* $\mathrm{Y}_n(\mathfrak{N})$.

Proof:

(1) $\mathfrak{F}^{(\{\pi_j \mid j \in J\})}\mathfrak{S}_{\pi'} = \overline{\mathfrak{Y}}^{(n,\mathfrak{X})}$:
Evidently, it is sufficient to prove that $\mathfrak{Y}^{(n,\mathfrak{F})} = \mathfrak{F}^{(\{\pi_j \mid j \in J\})} =: \mathfrak{H}$.

$\subseteq$: Suppose not. Let G be a group of minimal order contained in $\mathfrak{Y}^{(n,\mathfrak{F})} \setminus \mathfrak{H}$. Then there exists $j_0 \in J$ such that $O_{\pi'_{j_0}}(G) = 1$ and $G_{\mathfrak{F}} = O_{\pi_{j_0}}(G) \neq 1$. $G \in \mathrm{Y_n}(\mathfrak{F})$, thus 3.2.19(a) yields

$$G \in \mathfrak{S}_{\pi_{j_0}}\mathfrak{S}_{\pi'_{j_0}} \cap \mathfrak{S}_{\pi_{j_0}}\mathfrak{H} = \mathfrak{S}_{\pi_{j_0}}(\mathfrak{S}_{\pi'_{j_0}} \cap \mathfrak{H}) \subseteq \mathfrak{S}_{\pi_{j_0}}\mathfrak{F}^{(\{\widetilde{\pi}_j \mid j \in J\})}$$

($\widetilde{\pi_j} := \pi_j \cap \pi'_{j_0}$). Using 3.2.19(c),(a), we obtain $G \in \mathfrak{H}$, a contradiction.

$\supseteq$: According to 3.2.16(2), $\mathfrak{Y}^{(n,\mathfrak{F})} = \cap_{j \in J}\mathfrak{S}_{\pi'_j}\mathfrak{S}_{\pi_j}\mathfrak{S}_{\pi'_j} \cap \mathfrak{S}_{\pi}$, and obviously $\mathfrak{H}$ is contained in $\mathfrak{S}_{\pi}$. Let j be an arbitrary element of J, then by construction $\mathfrak{F}_{\sigma} \subseteq \mathfrak{S}_{\pi'_j}\mathfrak{S}_{\pi_j}\mathfrak{S}_{\pi'_j}$ for all $\sigma \in \mathrm{Sym}(J)$. This yields $\mathfrak{H} \subseteq \mathfrak{S}_{\pi'_j}\mathfrak{S}_{\pi_j}\mathfrak{S}_{\pi'_j}$ and the proof is complete.

(2) Let $\mathfrak{F}$ be a Q-closed Fitting class contained in $\mathrm{Y_n}(\mathfrak{X})$. Then $\mathfrak{F} \subseteq \mathfrak{F}^{(\{\pi_j \mid j \in J\})}\mathfrak{S}_{\pi'}$:
According to 3.2.3, it is sufficient to show that $\mathfrak{F} \cap \mathfrak{S}_{\pi} \subseteq \mathfrak{F}^{(\{\pi_j \mid j \in J\})}$. Suppose that there is a group G of minimal order contained in $(\mathfrak{F} \cap \mathfrak{S}_{\pi}) \setminus \mathfrak{F}^{(\{\pi_j \mid j \in J\})}$. G has a unique minimal normal subgroup, thus $G_{\mathfrak{X}} = O_{\pi_j}(G)$ for a suitable $j \in J$. Since G belongs to $\mathrm{Y_n}(\mathfrak{X})$, this implies $G \in \mathfrak{S}_{\pi_j}(\mathfrak{S}_{\pi'_j} \cap \mathfrak{F}^{(\{\pi_j \mid j \in J\})})$. Applying 3.2.19 we obtain a final contradiction.

□

3.2.20 enables us to prove that, in general, there are no dual atoms in $\mathfrak{L}_{(n,\mathfrak{F})}$ (cf. 3.2.8).

3.2.21 Remark
$\mathfrak{L}_{(n,\mathfrak{F})}$ *need not be dual atomic, not even if* $\mathfrak{F}$ *is of bounded nilpotent length.*

Proof: Set $\mathfrak{F} = \mathfrak{N}$ and let $p_1,\ p_2, \ldots$ be the set of all primes. According to 3.2.20, $\mathfrak{Y}^{(n,\mathfrak{F})} = \mathfrak{F}^{\mathbb{P}} = \mathrm{SFit}(\mathfrak{F}_{\sigma} \mid \sigma \in \mathrm{Sym}(\mathbb{N}))$. Let $\mathfrak{X} \neq \mathfrak{Y}^{(n,\mathfrak{F})}$ be a class belonging to $\mathfrak{L}_{(n,\mathfrak{F})}$. We show that there exists an element $\mathfrak{H}$ of $\mathfrak{L}_{(n,\mathfrak{F})}$ such that $\mathfrak{X} \subset \mathfrak{H} \subset \mathfrak{Y}^{(n,\mathfrak{F})}$:

$\mathfrak{X} \neq \mathfrak{Y}^{(n,\mathfrak{F})}$, thus there exists an element $\sigma \in \mathrm{Sym}(\mathbb{N})$ such that $\mathfrak{F}_\sigma \not\subseteq \mathfrak{X}$. In particular, $\mathfrak{S}_{p_{\sigma(1)}} \cdots \mathfrak{S}_{p_{\sigma(i)}} \not\subseteq \mathfrak{X}$ for some suitable $i \in \mathbb{N}$. Set $\mathfrak{H} = \mathrm{SFit}(\mathfrak{X}, \mathfrak{S}_{p_{\sigma(1)}} \cdots \mathfrak{S}_{p_{\sigma(i)}})$. By 3.2.5, we obtain $\mathfrak{X} \subset \mathfrak{H} \in \mathfrak{L}_{(n,\mathfrak{F})}$.
$\mathfrak{H} \subset \mathfrak{Y}^{(n,\mathfrak{F})}$: Suppose not. Then $\mathfrak{S}_{p_{\sigma(1)}} \cdots \mathfrak{S}_{p_{\sigma(i+1)}} \subseteq \mathfrak{H}$, and by 2.2.5 it follows that $\mathfrak{S}_{p_{\sigma(1)}} \cdots \mathfrak{S}_{p_{\sigma(i+1)}} = \mathrm{SFit}(\mathfrak{X} \cap \mathfrak{S}_{p_{\sigma(1)}} \cdots \mathfrak{S}_{p_{\sigma(i+1)}}, \mathfrak{S}_{p_{\sigma(1)}} \cdots \mathfrak{S}_{p_{\sigma(i)}})$. If $\mathfrak{X} \cap \mathfrak{S}_{p_{\sigma(1)}} \cdots \mathfrak{S}_{p_{\sigma(i+1)}} \not\subseteq \mathfrak{N}^i$, then 2.1.5(b) yields $\mathfrak{X} \cap \mathfrak{S}_{p_{\sigma(1)}} \cdots \mathfrak{S}_{p_{\sigma(i+1)}} = \mathfrak{S}_{p_{\sigma(1)}} \cdots \mathfrak{S}_{p_{\sigma(i+1)}}$, which is a contradiction to the choice of i. Hence we obtain $\mathfrak{S}_{p_{\sigma(1)}} \cdots \mathfrak{S}_{p_{\sigma(i+1)}} \subseteq \mathrm{SFit}(\mathfrak{X} \cap \mathfrak{S}_{p_{\sigma(1)}} \cdots \mathfrak{S}_{p_{\sigma(i+1)}}, \mathfrak{S}_{p_{\sigma(1)}} \cdots \mathfrak{S}_{p_{\sigma(i)}}) \subseteq \mathfrak{N}^i$, a final contradiction. □

Using 3.2.20, it is possible to obtain results about $\mathfrak{Y}^{(n,\mathfrak{H}\mathfrak{X})}$ in dependence on $\mathfrak{Y}^{(n,\mathfrak{X})}$ for a lattice formation $\mathfrak{X}$ and an SFitting class $\mathfrak{H}$ such that $\pi(\mathfrak{H}) \subseteq \pi(\mathfrak{X})$.

We need:

3.2.22 Lemma
Let $\mathfrak{X}$ be a lattice formation belonging to $(\pi_j)_{j\in J}$, and let $G \in Y_n(\mathfrak{X})$ be a group satisfying the following two properties:

(i) *G has a unique maximal normal subgroup N, and $N \in \mathfrak{F}^{(\{\pi_j \mid j\in J\})}$.*

(ii) *There exists an element $j \in J$ such that $\pi_j \cap \pi(G/N) \neq \emptyset \neq \pi_j \cap \pi(N)$.*

Then G belongs to $\mathfrak{F}^{(\{\pi_j \mid j\in J\})}$.

Proof:
Assume not. Let G denote a group of minimal order satisfying (i) and (ii) and belonging to $\mathfrak{S} \setminus \mathfrak{F}^{(\{\pi_j \mid j\in J\})}$. Further, let j_1 be an element of J such that $\pi_{j_1} \cap \pi(G/N) \neq \emptyset \neq \pi_{j_1} \cap \pi(N)$.
$G_\mathfrak{X} = O_{\pi_{j_0}}(G)$ for a suitable $j_0 \in J$:
Obviously, there exists $j_0 \in J$ such that $O_{\pi_{j_0}}(G) := M \neq 1$. If $\pi_{j_1} \cap \pi(N/M) \neq \emptyset$, it follows that $G/M \in \mathfrak{F}^{(\{\pi_j \mid j\in J\})}$ by the minimality of G. If $\pi_{j_1} \cap \pi(N/M) = \emptyset$, we obtain $G/M \in \mathfrak{F}^{(\{\pi_j \mid j\in J\setminus\{j_1\}\})}\mathfrak{S}_{\pi_{j_1}}$, and by 3.2.19(e) $G/M \in \mathfrak{F}^{(\{\pi_j \mid j\in J\})}$ as well. Assume that there exists another element $j_2 \in J$ such that $O_{\pi_{j_2}}(G) \neq 1$. Then we obtain $G \in \mathrm{R}_0\mathfrak{F}^{(\{\pi_j \mid j\in J\})} = \mathfrak{F}^{(\{\pi_j \mid j\in J\})}$, a contradiction to the choice of G.

Since $G \in \mathrm{Y_n}(\mathfrak{X})$, we conclude

$$G \in \mathfrak{S}_{\pi_{j_0}}(\mathfrak{S}_{\pi'_{j_0}} \cap \mathfrak{F}^{(\{\pi_j \mid j\in J\})}) \subseteq \mathfrak{S}_{\pi_{j_0}}\mathfrak{F}^{(\{\pi_j \mid j\in J\setminus\{j_0\}\})}.$$

Thus, 3.2.19(d) yields a final contradiction. □

3.2.23 Proposition

Let $\pi \neq \emptyset$ be a set of primes, let $(\pi_j)_{j\in J}$ be a partition of π and let $\mathfrak{X}$ be the corresponding lattice formation. Further, let $\mathfrak{H}$ denote an SFitting class such that $\pi(\mathfrak{H}) \subseteq \pi$, and $n \geq 1$ a natural number.
Then $\mathfrak{H}^n\mathfrak{F}^{(\{\pi_j \mid j\in J\})}\mathfrak{S}_{\pi'} = \overline{\mathfrak{Y}}^{(n,\mathfrak{H}^n\mathfrak{X})}$ is the unique maximal Q-closed Fitting class contained in $Y_n(\mathfrak{H}^n\mathfrak{X})$.

In particular, $\mathfrak{N}^n\mathfrak{F}^{\mathbb{P}}$ is the unique maximal Q-closed Fitting class contained in $Y_n(\mathfrak{N}^{n+1})$.

Proof:

(1) $\mathfrak{H}^n\mathfrak{F}^{(\{\pi_j \mid j\in J\})}\mathfrak{S}_{\pi'}$ is a Q-closed Fitting class contained in $\mathrm{Y_n}(\mathfrak{H}^n\mathfrak{X})$:
Obviously, $\mathfrak{H}^n\mathfrak{F}^{(\{\pi_j \mid j\in J\})}\mathfrak{S}_{\pi'}$ is a Q-closed Fitting class. Evidently, it is sufficient to prove that $\mathfrak{H}^n\mathfrak{X}$ is normal in $\mathfrak{H}^n\mathfrak{F}^{(\{\pi_j \mid j\in J\})}$. Let G be an element of $\mathfrak{H}^n\mathfrak{F}^{(\{\pi_j \mid j\in J\})}$, and $V/G_{\mathfrak{H}^n} \in \mathrm{Inj}_{\mathfrak{X}}(G/G_{\mathfrak{H}^n})$. By 1.2.15 we obtain $V \in \mathrm{Inj}_{\mathfrak{H}^n\mathfrak{X}}(G)$, and by 3.2.20 we are finished.

(2) Let $\mathfrak{F} \neq 1$ denote a Q-closed Fitting class contained in $\mathrm{Y_n}(\mathfrak{H}^n\mathfrak{X})$.
$\mathfrak{F} \subseteq \mathfrak{H}^n\mathfrak{F}^{(\{\pi_j \mid j\in J\})}\mathfrak{S}_{\pi'}$:
By 3.2.3 the assertion follows from $\mathfrak{F} \cap \mathfrak{S}_\pi \subseteq \mathfrak{H}^n\mathfrak{F}^{(\{\pi_j \mid j\in J\})}$. Thus, we assume that $\mathfrak{F} \cap \mathfrak{S}_\pi \not\subseteq \mathfrak{H}^n\mathfrak{F}^{(\{\pi_j \mid j\in J\})}$, and choose a minimal counterexample G. Then G has a unique maximal normal subgroup $N = G_{\mathfrak{H}^n\mathfrak{F}^{(\{\pi_j \mid j\in J\})}}$ and $G/N \in \mathfrak{S}_{\pi_{j_1}}$ for some suitable $j_1 \in J$. Suppose that $\pi_{j_1} \cap \pi(N/G_{\mathfrak{H}^n}) = \emptyset$. Then 3.2.19(e) yields $G/G_{\mathfrak{H}^n} \in \mathfrak{F}^{(\{\pi_j \mid j\in J\})}$, a contradiction to the choice of G. Thus 3.2.22 is applicable, and we obtain a final contradiction.

□

3.2.24 Remark

3.2.23 need not be true for SFitting classes $\mathfrak{H}$ of arbitrary characteristic.

Proof: Let $p,\ q,\ r$ be pairwise distinct primes. Then 3.2.14(b) implies

$$\mathfrak{Y}^{(n,\mathfrak{S}_p(\mathfrak{S}_q\times\mathfrak{S}_r))} = \mathfrak{S}_{\{p,q\}}\mathfrak{S}_r\mathfrak{S}_{\{p,q\}} \cap \mathfrak{S}_{\{p,r\}}\mathfrak{S}_q\mathfrak{S}_{\{p,r\}}.$$

Consequently, $Z_p \wr Z_q \wr Z_p \wr Z_r$ is a group belonging to $\mathfrak{Y}^{(n,\mathfrak{S}_p(\mathfrak{S}_q\times\mathfrak{S}_r))} \setminus \mathfrak{S}_p\overline{\mathfrak{Y}}^{(n,\mathfrak{S}_q\times\mathfrak{S}_r)}$. □

Whether or not 3.2.23 holds for arbitrary SFitting classes $\mathfrak{X}$ is an open question. As a weaker statement we obtain

3.2.25 Proposition

Let $\mathfrak{X}$ be an SFitting class of characteristic π, let $(\pi_i)_{i\in I}$ be a partition of π and let $\mathfrak{F}$ denote the corresponding lattice formation. Then

$$\mathfrak{F}\overline{\mathfrak{Y}}^{(n,\mathfrak{X})} = \overline{\mathfrak{Y}}^{(n,\mathfrak{F}\mathfrak{X})}.$$

In particular, $\mathfrak{N}_\pi^r\overline{\mathfrak{Y}}^{(n,\mathfrak{X})} = \overline{\mathfrak{Y}}^{(n,\mathfrak{N}_\pi^r\mathfrak{X})}$.

Proof: As usual, it suffices to prove $\mathfrak{F}\mathfrak{Y}^{(n,\mathfrak{X})} = \mathfrak{Y}^{(n,\mathfrak{F}\mathfrak{X})}$.

- $\subseteq$: Let G be a group belonging to $\mathfrak{F}\mathfrak{Y}^{(n,\mathfrak{X})}$. Then $(G/G_{\mathfrak{F}})_{\mathfrak{X}}$ is an $\mathfrak{X}$-maximal subgroup of $G/G_{\mathfrak{F}}$, and 1.2.15 implies the assertion.

- $\supseteq$: Suppose the contrary and choose a group G of minimal order contained in $\mathfrak{Y}^{(n,\mathfrak{F}\mathfrak{X})} \setminus \mathfrak{F}\mathfrak{Y}^{(n,\mathfrak{X})}$. Since G has a unique minimal normal subgroup, there exists a prime t such that $O_{t'}(G) = 1$. Let $i \in I$ such that $t \in \pi_i$. Let F and X, respectively, denote the canonical local definitions of $\mathfrak{F}$ and $\mathfrak{X}$, respectively. Then 3.2.14(b) implies

  $$\mathfrak{Y}^{(n,\mathfrak{F}\mathfrak{X})} = \cap_{q\in\pi}\mathfrak{S}_{q'}\mathfrak{Y}^{(n,F(q)\mathfrak{X})} \cap \mathfrak{S}_\pi =$$

  $$\cap_{q\in\pi}\mathfrak{S}_{q'}(\cap_{r\in\pi\setminus\pi_j(q)}\mathfrak{S}_{r'}\mathfrak{Y}^{(n,X(r))}\mathfrak{S}_{\pi(r)'}) \cap \mathfrak{S}_\pi,$$

  where $\pi(r) = \pi(X(r))$ and $\pi_j(q) = \pi_j$ such that $q \in \pi_j$.
  Consequently, we obtain $G \in \mathfrak{Y}^{(n,\mathfrak{S}_{\pi_i}\mathfrak{X})} = \cap_{r\in\pi\setminus\pi_i}\mathfrak{S}_{r'}\mathfrak{Y}^{(n,X(r))}\mathfrak{S}_{\pi(r)'} \cap \mathfrak{S}_\pi$.

  We prove that G belongs to $\mathfrak{F}(\mathfrak{S}_{p'}\mathfrak{Y}^{(n,X(p))}\mathfrak{S}_{\pi(p)'})$ for all $p \in \pi_i$ (then 3.2.14 provides a final contradiction):

Put $\{\pi(i)_1, \ldots, \pi(i)_n\} = \{\pi_j \mid \pi_j \cap \pi(G) \neq \emptyset\}$ and assume that $\pi(i)_1 = \pi_i \cap \pi(G)$. Noting that $O_{\pi_i}((G/O_{\pi_i}(G))) = 1$, 3.2.14 implies

$$G/O_{\pi_i}(G) \in (\mathfrak{S}_{\pi(i)_2} \times \ldots \times \mathfrak{S}_{\pi(i)_n})(\cap_{q\in\pi}\mathfrak{S}_{q'}\mathfrak{Y}^{(n,X(q))}\mathfrak{S}_{\pi(q)'} \cap \mathfrak{S}_\pi).$$

Thus, we conclude $\pi(i)_2 \cup \ldots \cup \pi(i)_n \subseteq \{p\}'$ where $p \in \pi_i \subseteq \pi$. Consequently $G \in \mathfrak{S}_{\pi_i}\mathfrak{S}_{p'}\mathfrak{Y}^{(n,X(p))}\mathfrak{S}_{\pi(p)'} \subseteq \mathfrak{F}(\mathfrak{S}_{p'}\mathfrak{Y}^{(n,X(p))}\mathfrak{S}_{\pi(p)'})$.

□

Whether or not a corresponding result is valid for arbitrary SFitting classes or for lattice formations $\mathfrak{F}$ such that $\pi(\mathfrak{F}) \subset \pi$, is an open question.

Concluding the investigation on $\overline{\mathfrak{Y}}^{(n,\mathfrak{X})}$ we show that, in general, $\overline{\mathfrak{Y}}^{(n,\mathfrak{X})}$ is not maximal among all Fitting classes contained in $\mathrm{Y_n}(\mathfrak{X})$.

3.2.26 Example
Let $p_1,\ p_2,\ p_3$ be pairwise distinct primes, and set $\pi = \{p_1,\ p_2,\ p_3\}$. Further, set $\mathfrak{X} = \mathfrak{N}_\pi$ and $\mathfrak{F} = (G \mid G/C_G(O_{p_1}(G)) \in \mathfrak{S}_{p_1}) \cap \mathfrak{S}_{p_3}\mathfrak{S}_{p_1}\mathfrak{S}_{p_2}\mathfrak{S}_{p_1}$. Then

$$\mathfrak{Y}^{(n,\mathfrak{X})} \subset \mathrm{Fit}(\mathfrak{Y}^{(n,\mathfrak{X})}, \mathfrak{F}) \subseteq \mathrm{Y_n}(\mathfrak{X}).$$

In particular, $\overline{\mathfrak{Y}}^{(n,\mathfrak{X})} = \mathfrak{Y}^{(n,\mathfrak{X})}\mathfrak{S}_{\pi'} \subset \mathrm{Fit}(\mathfrak{Y}^{(n,\mathfrak{X})}, \mathfrak{F})\mathfrak{S}_{\pi'} \subseteq \mathrm{Y_n}(\mathfrak{X})$.

Proof:

(1) $\mathfrak{F}$ is a Fitting class such that $\mathfrak{X}$ is normal in $\mathfrak{F}$, but $\mathfrak{F} \not\subseteq \mathfrak{Y}^{(n,\mathfrak{X})}$:

According to [9, IX, 2.5 (b)] and [9, IX, 3.6 (a)], $\mathfrak{F}$ is a Fischer class. Assume that $\mathfrak{F} \not\subseteq \mathrm{Y_n}(\mathfrak{X})$ and let G be a counterexample of minimal order. According to 3.1.6, G has a unique maximal normal subgroup N, and $G/N \cong Z_p$, $N = QG_\mathfrak{X}$, $NV = G$ and $V = PG_\mathfrak{X}$ (where $V \in \mathrm{Inj}_\mathfrak{X}(G)$, $P \in \mathrm{Syl}_p(V)$, $Q \in \mathrm{Syl}_q(G)$ for suitable primes p and q). If $p \neq p_1$, then $G \in \mathfrak{S}_{p_3}\mathfrak{S}_{p_1}\mathfrak{S}_{p_2}$, contradicting the choice of G. Thus $p = p_1$. Using 1.2.18, we obtain $P \leq C_G(O_{p_2}(G) \times O_{p_3}(G))$ and consequently $C_G(O_{p_2}(G) \times O_{p_3}(G)) = G$ (for otherwise $P \leq N$, a contradiction). Observe that $Q \leq C_G(O_p(G))$ by definition of $\mathfrak{F}$, whence $Q \leq C_G(F(G)) \leq F(G) = G_\mathfrak{X}$, a final contradiction.
According to 3.2.20, $\mathfrak{Y}^{(n,\mathfrak{X})} = \mathfrak{F}^\pi \subseteq \mathfrak{N}_\pi^3$. Since $Z_{p_3} \wr Z_{p_1} \wr Z_{p_2} \wr Z_{p_1} \in \mathfrak{F} \setminus \mathfrak{N}_\pi^3$, this implies $\mathfrak{F} \not\subseteq \mathfrak{Y}^{(n,\mathfrak{X})}$.

(2) Put $\mathfrak{F}_1 = \mathrm{SFit}(\mathfrak{S}_{p_2}\mathfrak{S}_{p_3}\mathfrak{S}_{p_1}, \mathfrak{S}_{p_3}\mathfrak{S}_{p_2}\mathfrak{S}_{p_1})$. Then $N_{p_1}(\mathfrak{F}_1, \mathfrak{F})$ is a Fitting class contained in $\mathrm{Y_n}(\mathfrak{X})$, and consequently $\mathrm{Fit}(\mathfrak{F}, \mathfrak{F}_1) \subseteq \mathrm{Y_n}(\mathfrak{X})$:

$\mathfrak{F}_1 \subseteq \mathfrak{S}_{p_3}\mathfrak{S}_{p_2}\mathfrak{S}_{p_3}\mathfrak{S}_{p_1}$ and $\mathfrak{S}_{p_3}\mathfrak{S}_{p_2} \subseteq \mathfrak{F}$, thus $G/G_{\mathfrak{F}} \in \mathfrak{S}_{\{p_1,p_3\}}$ provided that $G \in \mathfrak{F}_1$.
Since $\mathfrak{F} \subseteq \mathfrak{S}_{p_3}\mathfrak{S}_{p_1}\mathfrak{S}_{p_2}\mathfrak{S}_{p_1}$ and $\mathfrak{S}_{p_3}\mathfrak{S}_{p_1} \subseteq \mathfrak{F}_1$, we further obtain $G/G_{\mathfrak{F}_1} \in \mathfrak{S}_{\{p_1,p_2\}}$ for $G \in \mathfrak{F}$.
Using 1.2.20, we conclude that $N_{p_1}(\mathfrak{F}_1, \mathfrak{F})$ is a Fitting class containing $\mathrm{Fit}(\mathfrak{F}_1, \mathfrak{F})$.
$\mathfrak{F}$, $\mathfrak{F}_1$ are Fitting classes contained in $\mathrm{Y_n}(\mathfrak{X})$ (cf. (1) and 3.2.20), and from the definition and 2.2.12 it follows that $\mathfrak{F}\mathfrak{S}_{p_1} = \mathfrak{F}$ and $\mathfrak{F}_1 = \mathrm{SFit}(\mathfrak{S}_{p_2}\mathfrak{S}_{p_3}, \mathfrak{S}_{p_3}\mathfrak{S}_{p_2})\mathfrak{S}_{p_1}$. Hence 3.1.8 yields the assertion.

(3) $\mathrm{Fit}(\mathfrak{Y}^{(n,\mathfrak{X})}, \mathfrak{F}) \subseteq \mathrm{Fit}(\mathfrak{Y}^{(n,\mathfrak{X})}, N_{p_1}(\mathfrak{F}, \mathfrak{F}_1)) \subseteq \mathrm{Y_n}(\mathfrak{X})$:

Put $\mathfrak{H} = N_{p_1}(\mathfrak{F}, \mathfrak{F}_1)$.

(i) 3.2.20 and 2.2.12 yield

$$\mathfrak{Y}^{(n,\mathfrak{X})} \subseteq \mathrm{SFit}(\mathfrak{S}_{p_3}\mathfrak{S}_{p_2}\mathfrak{S}_{p_1}\mathfrak{S}_{\{p_2,p_3\}}, \mathfrak{S}_{p_2}\mathfrak{S}_{p_3}\mathfrak{S}_{p_1}\mathfrak{S}_{\{p_2,p_3\}})$$

$$= \mathrm{SFit}(\mathfrak{S}_{p_3}\mathfrak{S}_{p_2}\mathfrak{S}_{p_1}, \mathfrak{S}_{p_2}\mathfrak{S}_{p_3}\mathfrak{S}_{p_1})\mathfrak{S}_{\{p_2,p_3\}} = \mathfrak{F}_1\mathfrak{S}_{\{p_2,p_3\}}.$$

Consequently, $G/G_{\mathfrak{H}} \in \mathrm{Q}\mathfrak{S}_{\{p_2,p_3\}} = \mathfrak{S}_{\{p_2,p_3\}}$ for $G \in \mathfrak{Y}^{(n,\mathfrak{X})}$.

(ii) Note that $\mathfrak{F}_1 \subseteq \mathfrak{F}^{\pi} = \mathfrak{Y}^{(n,\mathfrak{X})}$ and $\mathfrak{F} \subseteq \mathfrak{S}_{p_3}\mathfrak{S}_{p_1}\mathfrak{S}_{p_2}\mathfrak{S}_{p_1} \subseteq \mathfrak{Y}^{(n,\mathfrak{X})}\mathfrak{S}_{p_1}$. Consequently, $\mathfrak{H} \subseteq \mathrm{Fit}(\mathfrak{F}_1, \mathfrak{F})\mathfrak{S}_{p_1} \subseteq \mathfrak{Y}^{(n,\mathfrak{X})}\mathfrak{S}_{p_1}$. Hence $G/G_{\mathfrak{Y}^{(n,\mathfrak{X})}}$ is contained in $\mathfrak{S}_{p_1}$ provided that $G \in \mathfrak{H}$.

Hence, by 1.2.20 and 3.1.8 we conclude that $\mathrm{Fit}(\mathfrak{Y}^{(n,\mathfrak{X})}, \mathfrak{F}) \subseteq \mathrm{Fit}(\mathfrak{Y}^{(n,\mathfrak{X})}, \mathfrak{H}) = N_{\emptyset}(\mathfrak{Y}^{(n,\mathfrak{X})}, \mathfrak{H}) \subseteq \mathrm{Y_n}(\mathfrak{X})$.

□

We bring this section to a close by studying the dual situation, namely the smallest SFitting class which is normal in $\mathfrak{F}$ (provided that it exists), and the family of all $\mathfrak{F}$-normal SFitting classes (where $\mathfrak{F}$ denotes an SFitting class).

3.2.27 Definition
Let $\mathfrak{F}$ be an SFitting class. We define

$$\mathfrak{Y}_{(n,\mathfrak{F})} = \bigcap \{\mathfrak{X} \mid \mathfrak{X} \text{ SFitting class, } \mathfrak{X} \text{ is normal in } \mathfrak{F}\}.$$

According to 3.1.10, this class is the smallest $\mathfrak{F}$-normal SFitting class provided that it is non-trivial.

3.2.28 Remark
Let $\mathfrak{F}$ be an SFitting class, $\mathfrak{Y} = \mathfrak{Y}_{(n,\mathfrak{F})}$.

(a) *Assume $\mathfrak{Y}$ is non-trivial. Then the following statements are equivalent:*

- **(i)** $\pi(\mathfrak{Y}) = \pi(\mathfrak{F})$.
- **(ii)** *There exists no set of primes π such that $\emptyset \neq \pi \subset \pi(\mathfrak{F})$ and $\mathfrak{F} \subseteq \mathfrak{S}_\pi \mathfrak{S}_{\pi'}$.*

(b) *Let $\mathfrak{Y}$ be non-trivial and set $\pi = \pi(\mathfrak{Y})$. Then $\mathfrak{Y} = \mathfrak{Y}_{(n,\mathfrak{F}\cap\mathfrak{S}_\pi)}$.*

(c) *If $\mathfrak{S}_p\mathfrak{F} = \mathfrak{F}$ for some prime p, then $\mathfrak{S}_p\mathfrak{Y}_{(n,\mathfrak{F})}$ is normal in $\mathfrak{F}$.*

Proof:

(a) $(i) \Rightarrow (ii)$: Assume to the contrary that there exists a non-empty set of primes $\pi \subset \pi(\mathfrak{F})$ such that $\mathfrak{F} \subseteq \mathfrak{S}_\pi\mathfrak{S}_{\pi'}$. Then $1 \neq \mathfrak{F} \cap \mathfrak{S}_\pi$ is normal in $\mathfrak{F}$ (since $\mathfrak{F} \cap \mathfrak{S}_\pi$ is normal in $(\mathfrak{F} \cap \mathfrak{S}_\pi)\mathfrak{S}_{\pi'}$). Consequently, $\mathfrak{F} \cap \mathfrak{S}_\pi \supseteq \mathfrak{Y}$, a contradiction.
$(ii) \Rightarrow (i)$: Suppose that $\pi(\mathfrak{Y}) \subset \pi(\mathfrak{F})$. Since 3.2.3 implies $\mathfrak{F} \subseteq \overline{\mathfrak{Y}}^{(n,\mathfrak{Y})} = \mathfrak{Y}^{(n,\mathfrak{Y})}\mathfrak{S}_{\pi(\mathfrak{Y})'}$, the set $\pi(\mathfrak{Y})$ fulfills the above conditions; a contradiction.

(b) $\mathfrak{Y}_{(n,\mathfrak{F})} = \mathfrak{Y}_{(n,\mathfrak{F})} \cap \mathfrak{S}_\pi$ is normal in $\mathfrak{F} \cap \mathfrak{S}_\pi$, and therefore $\mathfrak{Y}_{(n,\mathfrak{F}\cap\mathfrak{S}_\pi)} \subseteq \mathfrak{Y}_{(n,\mathfrak{F})}$. Since $\mathfrak{F} \subseteq (\mathfrak{F} \cap \mathfrak{S}_\pi)\mathfrak{S}_{\pi'}$ and $(\mathfrak{F} \cap \mathfrak{S}_\pi)\mathfrak{S}_{\pi'} \subseteq \mathrm{Y_n}(\mathfrak{Y}_{(n,\mathfrak{F}\cap\mathfrak{S}_\pi)})$, the converse is valid as well.

(c) If $\pi(\mathfrak{Y}_{(n,\mathfrak{F})}) = \pi(\mathfrak{F})$, then 1.2.15 yields the assertion.
Thus assume that $\pi(\mathfrak{Y}_{(n,\mathfrak{F})}) \subset \pi(\mathfrak{F})$. According to 3.1.11, $\mathfrak{Y}_{(n,\mathfrak{F})} \neq 1$, hence 3.2.28(b) yields $\mathfrak{Y}_{(n,\mathfrak{F})} = \mathfrak{Y}_{(n,\mathfrak{F}\cap\mathfrak{S}_\pi)}$ and $p \in \pi$ (where $\pi = \pi(\mathfrak{Y}_{(n,\mathfrak{F})})$). Since $\mathfrak{S}_p(\mathfrak{F} \cap \mathfrak{S}_\pi) = \mathfrak{F} \cap \mathfrak{S}_\pi$, the assertion follows.

□

We will see later (cf. 3.2.32) that $\mathfrak{Y}_{(n,\mathfrak{F})} = \mathfrak{S}_p\mathfrak{Y}_{(n,\mathfrak{F})}$ provided that $\mathfrak{S}_p\mathfrak{F} = \mathfrak{F}$.

3.2.29 Remark
Let $\mathfrak{F}$ be an SFitting class. Then, in general, $\mathfrak{Y}_{(n,\mathfrak{F})}$ fails to define $\mathfrak{F}$ uniquely.

Proof: Let $p_1,\ p_2,\ p_3$ be pairwise distinct primes, and set $\pi = \{p_1, p_2, p_3\}$. Further, put

$$\mathfrak{F}_1 = \mathrm{SFit}(\mathfrak{S}_{p_1}\mathfrak{S}_{p_2}\mathfrak{S}_{p_3}, \mathfrak{S}_{p_3}\mathfrak{S}_{p_2}\mathfrak{S}_{p_1}) \text{ and } \mathfrak{F}_2 = \mathrm{SFit}(\mathfrak{S}_{p_2}\mathfrak{S}_{p_1}\mathfrak{S}_{p_3}, \mathfrak{S}_{p_3}\mathfrak{S}_{p_1}\mathfrak{S}_{p_2}).$$

According to 3.2.20, the class $\mathfrak{N}_\pi$ is normal in $\mathfrak{F}_i\ \ (i = 1, 2)$. Moreover, it is easily seen that each $\mathfrak{F}_i$-normal Fitting class is of characteristic $\pi\ \ (i = 1, 2)$, and consequently $\mathfrak{Y}_{(n,\mathfrak{F}_1)} = \mathfrak{Y}_{(n,\mathfrak{F}_1)} = \mathfrak{N}_\pi$.
But 2.2.24 yields $\mathfrak{F}_2 \subseteq S_{p_1}(\mathfrak{S}_{p_2}\mathfrak{S}_{p_1}\mathfrak{S}_{p_3}, \mathfrak{S}_{p_3}\mathfrak{S}_{p_1}\mathfrak{S}_{p_2})$, hence $Z_{p_1} \wr Z_{p_2} \wr Z_{p_3} \in \mathfrak{F}_1 \setminus \mathfrak{F}_2$. □

3.2.30 Remark
Let $\mathfrak{F}$ be an SFitting class.

(a) *If $\mathfrak{Y}_{(n,\mathfrak{Y}^{(n,\mathfrak{F})})} \neq 1$, then $\mathfrak{Y}_{(n,\mathfrak{Y}^{(n,\mathfrak{F})})} = \mathfrak{F}$.*

(b) *A corresponding statement concerning the dual class does not hold true in general.*

Proof:
(a): Evidently, $\mathfrak{Y}_{(n,\mathfrak{Y}^{(n,\mathfrak{F})})} \subseteq \mathfrak{F}$. The converse is given by 3.2.11.
(b): Let the notation be as in 3.2.29 and set $\mathfrak{F} = \mathfrak{F}_1$. Then $\mathfrak{Y}_{(n,\mathfrak{F})} = \mathfrak{N}_\pi$, and $\mathfrak{F}$ is a proper subclass of $\mathfrak{Y}^{(n,\mathfrak{Y}_{(n,\mathfrak{F})})}$. □

3.2.31 Proposition
Let $\mathfrak{F}$ be an SFitting class such that $\mathfrak{Y}_{(n,\mathfrak{F})} \neq 1$ and $\pi(\mathfrak{F}) = \pi(\mathfrak{Y}_{(n,\mathfrak{F})}) = \pi$. Let F denote the canonical local definition belonging to $\mathfrak{F}$.

(a) $\mathfrak{Y}_{(n,\mathfrak{F})} = \bigcap_{p\in\pi} \mathfrak{S}_{p'}\mathfrak{S}_p\mathfrak{Y}_{(n,F(p))} \cap \mathfrak{S}_\pi$.

(b) *If $\tilde{\pi} = \{p \mid F(p) \neq \mathfrak{F}\}$, then $\mathfrak{Y}_{(n,\mathfrak{F})} = \bigcap_{p\in\tilde{\pi}} \mathfrak{S}_{p'}\mathfrak{S}_p\mathfrak{Y}_{(n,F(p))} \cap \mathfrak{S}_\pi$.*

Proof:

(a) Set $\mathfrak{X} = \cap_{p\in\pi}\mathfrak{S}_{p'}\mathfrak{S}_p\mathfrak{Y}_{(n,F(p))} \cap \mathfrak{S}_\pi$.
$\subseteq$: $\mathfrak{X} \neq 1$, thus it is sufficient to show that $\mathfrak{X}$ is normal in $\mathfrak{F}$. Assume the contrary and let G be a group of minimal order contained in

$\mathfrak{F} \setminus \mathrm{Y_n}(\mathfrak{X})$. According to 3.1.6, there exists a prime $q \in \pi$ such that $O_{q'}(G) = 1$ and $N/O_q(G) \in \mathfrak{X}$, where N denotes the unique maximal normal subgroup of G. In particular, N belongs to $\bigcap_{p\in\pi\setminus\{q\}} \mathfrak{S}_{p'}\mathfrak{S}_p\mathfrak{Y}_{(n,F(p))} \cap \mathfrak{S}_\pi \cap \mathfrak{S}_q\mathfrak{S}_{q'}\mathfrak{S}_q\mathfrak{Y}_{(n,F(q))}$.
$G_{\mathfrak{S}_q\mathfrak{Y}_{(n,F(q))}} = G_{\mathfrak{X}}$: $O_{q'}(G) = 1$, hence $G_{\mathfrak{X}}$ is contained in $G_{\mathfrak{S}_q\mathfrak{Y}_{(n,F(q))}}$. Set $M = G_{\mathfrak{S}_q\mathfrak{Y}_{(n,F(q))}} \cap N$. We conclude that $M \in \bigcap_{p\in\pi\setminus\{q\}} \mathfrak{S}_{p'}\mathfrak{S}_p\mathfrak{Y}_{(n,F(p))} \cap \mathfrak{S}_\pi \cap \mathfrak{S}_q\mathfrak{Y}_{(n,F(q))} \subseteq \mathfrak{X}$, and consequently $M = G_{\mathfrak{S}_q\mathfrak{Y}_{(n,F(q))}} = G_{\mathfrak{X}}$.
3.2.28(c) implies that $\mathfrak{S}_q\mathfrak{Y}_{(n,F(q))}$ is an $F(q)$-normal SFitting class. Let V be an $\mathfrak{X}$-injector of G. Since $O_{q'}(G) = 1$, we obtain $G \in F(q)$ whence $V = O_r(V) \times G_{\mathfrak{X}}$ for some suitable prime $r \neq q$. This implies $C_G(G_{\mathfrak{X}}) = G$, a contradiction.

$\supseteq$: Let H denote the canonical local definition belonging to $\mathfrak{Y}_{(n,\mathfrak{F})}$. According to 3.2.1, $H(p)$ is $F(p)$-normal, and consequently $H(p) \supseteq \mathfrak{Y}_{(n,F(p))}$ for all $p \in \pi$. This yields the assertion.

(b) $\subseteq$: (a).
$\supseteq$: We prove that $\bigcap_{p\in\tilde{\pi}} \mathfrak{S}_{p'}\mathfrak{S}_p\mathfrak{Y}_{(n,F(p))} \cap \mathfrak{S}_\pi \subseteq \bigcap_{p\in\pi} \mathfrak{S}_{p'}\mathfrak{S}_p\mathfrak{Y}_{(n,F(p))} \cap \mathfrak{S}_\pi = \mathfrak{Y}_{(n,\mathfrak{F})}$.
Let G be a group of minimal order contained in $\bigcap_{p\in\tilde{\pi}} \mathfrak{S}_{p'}\mathfrak{S}_p\mathfrak{Y}_{(n,F(p))} \cap \mathfrak{S}_\pi \setminus \mathfrak{Y}_{(n,\mathfrak{F})}$. Then G has a unique minimal normal subgroup M, $M \in \mathfrak{S}_q$ for some suitable prime q, and $G/M \in \mathfrak{Y}_{(n,\mathfrak{F})}$.
If $q \in \tilde{\pi}$, then $G \in \mathfrak{S}_{p'}\mathfrak{S}_p\mathfrak{Y}_{(n,F(p))}$ for all $p \in \pi \setminus \tilde{\pi}$ and we obtain a contradiction.
Thus, we assume that $q \in \pi \setminus \tilde{\pi}$. Since in this case $\mathfrak{Y}_{(n,F(q))} = \mathfrak{Y}_{(n,\mathfrak{F})}$, this implies $G \in \bigcap_{p\in\pi\setminus\{q\}} \mathfrak{S}_{p'}\mathfrak{S}_p\mathfrak{Y}_{(n,F(p))} \cap \mathfrak{S}_q\mathfrak{Y}_{(n,F(q))} \cap \mathfrak{S}_\pi \subseteq \mathfrak{Y}_{(n,\mathfrak{F})}$, a final contradiction.

□

3.2.32 Corollary
Let $\mathfrak{F}$ be an SFitting class and p be a prime such that $\mathfrak{S}_p\mathfrak{F} = \mathfrak{F}$. Then $\mathfrak{Y}_{(n,\mathfrak{F})} = \mathfrak{S}_p\mathfrak{Y}_{(n,\mathfrak{F})}$.

Proof: According to 3.1.11(a), the class $\mathfrak{Y}_{(n,\mathfrak{F})}$ is non-trivial. Therefore, we may assume without loss of generality that $\pi(\mathfrak{F}) = \pi(\mathfrak{Y}_{(n,\mathfrak{F})}) = \pi$. From 1.3.7(e), we conclude that $F(p) = \mathfrak{S}_p\mathfrak{F} = \mathfrak{F}$, where F denotes the canonical local definition of $\mathfrak{F}$. 3.2.31(b) yields $\mathfrak{Y}_{(n,\mathfrak{F})} = \bigcap_{q\in\pi\setminus\{p\}} \mathfrak{S}_{q'}\mathfrak{S}_q\mathfrak{Y}_{(n,F(q))} \cap \mathfrak{S}_\pi =$

$\mathfrak{S}_p(\cap_{q\in\pi\setminus\{p\}}\mathfrak{S}_{q'}\mathfrak{S}_q\mathfrak{Y}_{(n,F(q))}\cap\mathfrak{S}_\pi)=\mathfrak{S}_p\mathfrak{Y}_{(n,\mathfrak{F})}$, and the proof is complete. $\square$

3.2.33 Corollary
Let $\mathfrak{F}$ be an SFitting class such that $\pi(\mathfrak{Y}_{(n,\mathfrak{F})})=\pi(\mathfrak{F})=\pi$. Further, let F be the canonical local definition belonging to $\mathfrak{F}$. Then Y, defined by $Y(p)=\mathfrak{Y}_{(n,F(p))}$, is the canonical local definition of $\mathfrak{Y}_{(n,\mathfrak{F})}$.

Proof: According to 3.2.31(a), $\mathfrak{Y}_{(n,\mathfrak{F})}$ is locally defined by Y. $1\neq F(p)\cap\mathfrak{Y}_{(n,\mathfrak{F})}$ is normal in $F(p)\subseteq\mathfrak{F}$, hence $\mathfrak{Y}_{(n,F(p))}\subseteq\mathfrak{Y}_{(n,\mathfrak{F})}$, and Y is integrated. 3.2.32 implies that Y is full, and the proof is complete. $\square$

3.2.34 Remark

(a) *Using 3.2.31(b), we obtain an explicit description of $\mathfrak{Y}_{(n,\mathfrak{F})}$ for all classes $\mathfrak{F}=\mathfrak{S}_{\pi_1}\cdots\mathfrak{S}_{\pi_r}$ (where $\pi_1,\dots,\pi_r$ are sets of primes). The same holds for those classes described in 1.3.10.*

(b) *Let $\mathfrak{F}$ be an SFitting class of bounded nilpotent length. If it is possible to determine the characteristic of $\mathfrak{Y}_{(n,\mathfrak{F})}$ (thus, in particular, to decide whether or not $\mathfrak{Y}_{(n,\mathfrak{F})}$ is trivial), then by 3.2.31 there exists an algorithm to give an explicit description of $\mathfrak{Y}_{(n,\mathfrak{F})}$.*

3.2.35 Remark
Let $\mathfrak{F}$ be an SFitting class such that $\pi(\mathfrak{F})=\pi(\mathfrak{Y}_{(n,\mathfrak{F})})$. According to 2.2.16,

$$\mathfrak{F}=\mathrm{SFit}(\mathfrak{S}_{p_1}\cdots\mathfrak{S}_{p_r}\mid p_1,\dots p_r \text{ primes}, \mathfrak{S}_{p_1}\cdots\mathfrak{S}_{p_r}\subseteq\mathfrak{F}).$$

Hence, $\mathfrak{Y}_{(n,\mathfrak{F})}\supseteq\mathrm{SFit}(\mathfrak{Y}_{(n,\mathfrak{S}_{p_1}\cdots\mathfrak{S}_{p_r})}\mid p_1,\dots p_r$ primes, $\mathfrak{S}_{p_1}\cdots\mathfrak{S}_{p_r}\subseteq\mathfrak{F})=:\mathfrak{H}$ according to 2.2.5. In general, $\mathfrak{Y}_{(n,\mathfrak{F})}\supset\mathfrak{H}$.

Proof: Let $p_1,\ p_2,\ p_3$ be pairwise distinct primes, and set $\mathfrak{F}=\mathrm{SFit}(\mathfrak{F}_1,\mathfrak{F}_2)$ where $\mathfrak{F}_1=\mathfrak{S}_{p_1}\mathfrak{S}_{p_2}\mathfrak{S}_{p_1}$ and $\mathfrak{F}_2=\mathfrak{S}_{p_2}\mathfrak{S}_{p_1}\mathfrak{S}_{p_3}\mathfrak{S}_{p_2}$. By 3.1.18, $\mathfrak{Y}_{(n,\mathfrak{F}_1)}=\mathfrak{S}_{p_1}\mathfrak{S}_{p_2}$ and $\mathfrak{Y}_{(n,\mathfrak{F}_2)}=\mathfrak{S}_{p_2}(\mathfrak{S}_{p_1}\times\mathfrak{S}_{p_3})$. Consequently $\mathrm{SFit}(\mathfrak{Y}_{(n,\mathfrak{F}_1)},\mathfrak{Y}_{(n,\mathfrak{F}_2)})=(G\mid G=G_{\mathfrak{Y}_{(n,\mathfrak{F}_1)}}G_{\mathfrak{Y}_{(n,\mathfrak{F}_1)}})$. Now, it is easily seen that $Z_{p_1}\wr Z_{p_3}\wr Z_{p_2}$ is a group belonging to $\mathfrak{F}\setminus\mathrm{Y_n}(\mathrm{SFit}(\mathfrak{Y}_{(n,\mathfrak{F}_1)},\mathfrak{Y}_{(n,\mathfrak{F}_2)}))$. $\square$

Let $\mathfrak{F}$ be an SFitting class such that $\mathfrak{Y}_{(n,\mathfrak{F})}\neq 1$. We finally prove that the family of all SFitting classes which are normal in $\mathfrak{F}$ forms a complete and distributive lattice (in analogy to the dual situation).

3.2.36 Proposition

Let $\mathfrak{F}$ be an SFitting class of bounded nilpotent length, and let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be SFitting classes. Set $\mathfrak{X} = \mathrm{SFit}(\mathfrak{X}_1, \mathfrak{X}_2)$. Then $\mathfrak{F} \subseteq \overline{\mathfrak{Y}}^{(n,\mathfrak{X})}$ provided that $\mathfrak{F} \subseteq \overline{\mathfrak{Y}}^{(n,\mathfrak{X}_1)} \cap \overline{\mathfrak{Y}}^{(n,\mathfrak{X}_2)}$.

In particular, $\mathrm{SFit}(\mathfrak{X}_1, \mathfrak{X}_2)$ is normal in $\mathfrak{F}$ provided that $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are $\mathfrak{F}$-normal SFitting classes.

Proof: By induction on $r := l(\mathfrak{F})$: The cases $r = 0,\ 1$ are obvious. Thus we assume that $r > 1$. Set $\pi_i = \pi(\mathfrak{X}_i)$ $(i = 1, 2)$ and $\pi = \pi_1 \cup \pi_2$. Further let $\mathfrak{F} = \mathrm{LF}(F)$, $\mathfrak{X} = \mathrm{LF}(X)$ and $\mathfrak{X}_i = \mathrm{LF}(X_i)$ where F, X and X_i are full and integrated $(i = 1, 2)$.

(1) Assume that $\mathfrak{F} \subseteq \mathfrak{S}_{\pi_1 \cap \pi_1}$.
2.2.3 yields $X = \mathrm{SFit}(X_1, X_2)$, and using 3.2.1 we obtain $F(p) \cap \mathfrak{N}^{r-1} \subseteq \mathrm{Y_n}(X_i(p))$ for all $p \in \pi$. Consequently, by inductive hypothesis, $F(p) \cap \mathfrak{N}^{r-1} \subseteq \mathrm{Y_n}(X(p))$. Since $\mathfrak{S}_p X(p) = X(p)$, 3.1.7 implies $F(p) = \mathfrak{S}_p(F(p) \cap \mathfrak{N}^{r-1}) \subseteq \mathrm{Y_n}(X(p))$, and the assertion follows from 3.2.1.

(2) Assume that $\mathfrak{F} \subseteq \mathfrak{S}_\pi$.
$\mathfrak{F} \subseteq \overline{\mathfrak{Y}}^{(n,\mathfrak{X}_1)} \cap \overline{\mathfrak{Y}}^{(n,\mathfrak{X}_2)} \cap \mathfrak{S}_\pi$, thus $\mathfrak{F} \subseteq \mathfrak{Y}^{(n,\mathfrak{X}_1)}\mathfrak{S}_{(\pi_1' \cap \pi)} \cap \mathfrak{Y}^{(n,\mathfrak{X}_2)}\mathfrak{S}_{(\pi_2' \cap \pi)}$, and consequently $\mathfrak{F} \subseteq (G \mid G = G_{\mathfrak{Y}^{(n,\mathfrak{X}_1)}} G_{\mathfrak{Y}^{(n,\mathfrak{X}_2)}})$. Let G be be a group of minimal order contained in $\mathfrak{F} \setminus \mathrm{Y_n}(\mathfrak{X})$. According to 3.1.6, G has a unique maximal normal subgroup, thus in particular $G \in \mathfrak{Y}^{(n,\mathfrak{X}_1)} \cup \mathfrak{Y}^{(n,\mathfrak{X}_2)}$. Without loss of generality we assume that $G \in \mathfrak{Y}^{(n,\mathfrak{X}_1)}$. Let N denote the unique maximal normal subgroup of G and let $V \in \mathrm{Inj}_{\mathfrak{X}}(G)$. By 3.1.6, $G/N \cong Z_p$ for a prime p and $V = PG_{\mathfrak{X}}$ (where $P \in \mathrm{Syl}_p(G)$ suitable).
3.1.10 implies $\mathfrak{F} \subseteq \mathfrak{Y}^{\mathfrak{X}_1 \cap \mathfrak{X}_2}\mathfrak{S}_{(\pi_1 \cap \pi_2)'}$. Hence $G \in \mathfrak{F} \cap \mathfrak{S}_{\pi_1 \cap \pi_2}$ provided that $p \in \pi_1 \cap \pi_2$. Now, the preceding case provides a contradiction. Thus we assume that $p \in (\pi_1 \cap \pi_2)'$ and consequently $\mathfrak{S}_{\pi_2}$ is contained in $\mathfrak{S}_{p'}$ (notice that $G \in \mathfrak{Y}^{(n,\mathfrak{X}_1)} \subseteq \mathfrak{S}_{\pi_1}$).
According to 2.2.24, $\mathfrak{X}$ is contained in $S_{\pi_1 \cap \pi_2}(\mathfrak{X}_1, \mathfrak{X}_2)$. This implies $P \leq V_{\mathfrak{X}_1} V_{\mathfrak{X}_2}$ and hence $P \leq V_{\mathfrak{X}_1} = G_{\mathfrak{X}_1}$ (observe that $p \notin \pi_2$ and that $G_{\mathfrak{X}_1}$ is an $\mathfrak{X}_1$-maximal subgroup of G). Consequently, $V \leq N$, a contradiction. So, also in this case $\mathfrak{F}$ is contained in $\overline{\mathfrak{Y}}^{(n,\mathfrak{X})}$.

(3) $\mathfrak{F}$ of arbitrary characteristic.
By assumption and 3.2.3, $\mathfrak{F} \subseteq (\mathfrak{F} \cap \mathfrak{S}_\pi)\mathfrak{S}_{\pi'}$. Now it follows from (2) that $\mathfrak{F} \cap \mathfrak{S}_\pi \subseteq \overline{\mathfrak{Y}}^{(n,\mathfrak{X})}$. The assertion follows.

□

3.2.37 Corollary
Let $\mathfrak{F}$, $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be SFitting classes and set $\mathfrak{X} = \mathrm{SFit}(\mathfrak{X}_1, \mathfrak{X}_2)$. Then $\mathfrak{F} \subseteq \overline{\mathfrak{Y}}^{(n,\mathfrak{X})}$ provided that $\mathfrak{F} \subseteq \overline{\mathfrak{Y}}^{(n,\mathfrak{X}_1)} \cap \overline{\mathfrak{Y}}^{(n,\mathfrak{X}_2)}$.

In particular, $\mathrm{SFit}(\mathfrak{X}_1, \mathfrak{X}_2)$ is normal in $\mathfrak{F}$ provided that $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are $\mathfrak{F}$-normal SFitting classes.

Proof: 2.2.5 and 3.2.36. □

3.2.38 Corollary
Let $\mathfrak{F}$ and $(\mathfrak{X}_i)_{i\in I}$ be SFitting classes such that $\mathfrak{F} \subseteq Y_n(\mathfrak{X}_i)$ for all $i \in I$, and set $\mathfrak{X} = \mathrm{SFit}(\mathfrak{X}_i \mid i \in I)$. Then $\mathfrak{F} \subseteq \overline{\mathfrak{Y}}^{(n,\mathfrak{X})}$.

In particular: $\mathrm{SFit}(\mathfrak{X}_i \mid i \in I)$ is normal in $\mathfrak{F}$ provided that $\mathfrak{X}_i$ is $\mathfrak{F}$-normal for each $i \in I$.

Proof: 2.2.13(b) and 3.2.37. □

3.2.39 Definition
Let $\mathfrak{F}$ be an SFitting class. We define

$$\mathfrak{L}^{(n,\mathfrak{F})} = (\{\mathfrak{X} \mid \mathfrak{F} \text{ SFitting class, } \mathfrak{X} \text{ is normal in } \mathfrak{F}\},\ \subseteq).$$

3.2.40 Theorem
Let $\mathfrak{F}$ be an SFitting class such that $\mathfrak{Y}_{(n,\mathfrak{F})} \neq 1$. Then $\mathfrak{L}^{(n,\mathfrak{F})}$ is a complete and distributive lattice, which is dual atomic, too, provided that $\mathfrak{F}$ is of bounded nilpotent length.

Proof: This follows from 3.2.38, 3.1.10, 2.2.14(b) and 2.2.20(b); observe that $l(\mathfrak{F}) < \infty$ leads to $\mathfrak{F} \subseteq \mathfrak{H}\mathfrak{N} \subseteq \mathrm{Y_n}(\mathfrak{H})$ where $\mathfrak{H}$ denotes a dual atom of $\mathfrak{L}^{(n,\mathfrak{F})}$. □

3.2.41 Remark
Let $\mathfrak{F}$ be an SFitting class such that $\mathfrak{Y}_{(n,\mathfrak{F})} \neq 1$.

(a) *In general, $\mathfrak{L}^{(n,\mathfrak{F})}$ fails to be atomic.*

(b) *In general, $\mathfrak{L}^{(n,\mathfrak{F})}$ fails to be dual atomic.*

Proof:

(a) Let $\mathfrak{F}$ be as described in 3.1.12. Then it is easily seen that $\mathfrak{L}^{(n,\mathfrak{F})}$ does not possess any atoms.

(b) Let $\mathfrak{F} = \mathfrak{Y}^{(n,\mathfrak{N})} = \mathfrak{F}^{\mathbb{P}}$. It is easily seen, too, that $\mathfrak{L}^{(n,\mathfrak{F})}$ does not possess any dual atoms.

$\square$

3.2.42 Remark
Let $\mathfrak{F}$ be an SFitting class such that $\mathfrak{Y}_{(n,\mathfrak{F})} \neq 1$ and such that there exist atoms in $\mathfrak{L}^{(n,\mathfrak{F})}$. Then, in general, the atoms of $\mathfrak{L}^{(n,\mathfrak{F})}$ do not coincide with the atoms of $\mathfrak{L}_{\mathfrak{Y}_{(n,\mathfrak{F})}}$.

Proof: Let $\pi = \{p_1,\ p_2,\ p_3\}$ be a set of pairwise distinct primes. Then $\mathfrak{Y}^{(n,\mathfrak{N}_\pi)} = \mathrm{SFit}(\mathfrak{S}_{p_{\sigma(1)}}\mathfrak{S}_{p_{\sigma(2)}}\mathfrak{S}_{p_{\sigma(3)}} \mid \sigma \in \mathrm{S}_3)$, and $\mathfrak{N}_\pi$ is the smallest SFitting class which is normal in $\mathfrak{Y}^{(n,\mathfrak{N}_\pi)}$. It is easily seen that there exists atoms in $\mathfrak{L}^{(n,\mathfrak{N}_\pi)}$. But if $\mathfrak{H}$ is an atom of $\mathfrak{L}_{\mathfrak{N}_\pi}$ such that $\mathfrak{H} \subseteq \mathfrak{Y}^{(n,\mathfrak{N}_\pi)}$, then evidently, $\mathfrak{H}$ is not normal in $\mathfrak{Y}^{(n,\mathfrak{N}_\pi)}$ (note that, according to 2.2.18, $\mathfrak{H} = \mathfrak{S}_{p_{\sigma(1)}} \times \mathfrak{S}_{p_{\sigma(2)}}\mathfrak{S}_{p_{\sigma(3)}}$ for a suitable permutation $\sigma \in \mathrm{S}_3$). $\square$

Chapter 4

Further embedding properties

We now turn our attention to further embedding properties of injectors, all of them weakening normality.

4.1 Local (Sub)Modularity

In this section we study locally (sub)modular Fitting classes, that is, nontrivial Fitting classes $\mathfrak{X}$ and $\mathfrak{F}$, $\mathfrak{X} \subseteq \mathfrak{F}$, such that for each $G \in \mathfrak{F}$ an $\mathfrak{X}$-injector of G is a (sub)modular subgroup of G. In this investigation, one of the first results to emerge is that the class of all groups G such that an $\mathfrak{X}$-injector of G is a modular subgroup of G is not closed under forming direct products. As an immediate consequence of this fact we obtain that the concepts of local modularity and local normality coincide. (In the global case $\mathfrak{F} = \mathfrak{S}$, this has been proved already by Hauck and Kienzle, cf. [14].)
Weakening modularity in so far that transitivity holds leads to the concept of locally submodular Fitting classes. Although this, in general, defines a new relation between Fitting classes, it turns out that for Fitting classes possessing strong additional properties – for instance SFitting classes – the concept of local submodularity coincides with that of local normality, too.

Local modularity

The concept of a modular subgroup stems from the theory of lattices: the modular subgroups of a group G are precisely the modular elements in the lattice of all subgroups of G.

4.1.1 Definition
Let G be a group. A subgroup U of G is called modular (U mod G) if the following conditions are satisfied:

(i) $\langle W, U\rangle \cap V = \langle W, U \cap V\rangle$ for all $W, V \leq G$ such that $W \leq V$.

(ii) $\langle W, U\rangle \cap V = \langle U, W \cap V\rangle$ for all $W, V \leq G$ such that $U \leq V$.

The reader is referred to the book of Schmidt [19] for further information on subgroup lattices and its modular elements.
Evidently, each normal subgroup is a modular subgroup of G, but in general, the converse does not hold true: for instance, a Sylow 2-subgroup of S_3 is modular but not normal in S_3. However, the following characterization of maximal modular subgroups – i.e. subgroups of G being maximal among all modular subgroups of G – indicates that these concepts are very close to each other.

4.1.2 Lemma ([19], 5.1.2)
A subgroup U of a group G is a maximal modular subgroup of G if and only if U is a maximal normal subgroup or $G/Core_G(U)$ is a non-abelian group of order pq (for suitable primes p and q).

It is also possible to characterize arbitrary modular subgroups of a group G by the structure of the corresponding quotient group $G/\mathrm{Core}_G(U)$ (cf. [19, 5.1.14]). We will need only a weak form of this statement.

4.1.3 Theorem ([19], 5.1.14)
Let G be a group, and let U be a modular subgroup of G. Then

$$G/Core_G(U) = S_1/Core_G(U) \times \ldots \times S_r/Core_G(U) \times T/Core_G(U),$$

$r \in \mathbb{N} \cup \{0\}$, *and where for all* $i,\ j \in \{1, \ldots, r\}$

(a) $S_i/Core_G(U) \in \mathfrak{S}_{p_i}\mathfrak{S}_{q_i}$ *is a group of order* $p_i^{n_i} q_i$, *and* $Z(S_i/Core_G(U)) = 1$ *(where $q_i,\ p_i$ are (distinct) primes, $n_i \in \mathbb{N}$).*

(b) $(|S_i/Core_G(U)|, |S_j/Core_G(U)|) = 1 = (|S_i/Core_G(U)|, |T/Core_G(U)|)$ *for* $i \neq j$.

(c) $U/Core_G(U) = Q_1/Core_G(U) \times \ldots \times Q_r/Core_G(U) \times (T \cap U)/Core_G(U)$, *where $Q_i/Core_G(U)$ is a non-normal Sylow q_i-subgroup of $S_i/Core_G(U)$.*

(d) *$U \cap T$ is modular and subnormal in G.*

4.1.4 Definition

(a) Let $\mathfrak{X}$ and $\mathfrak{F}$ be non-trivial Fitting classes such that $\mathfrak{X} \subseteq \mathfrak{F}$. Then $\mathfrak{X}$ is said to be modular in $\mathfrak{F}$ ($\mathfrak{F}$-modular) if for all $G \in \mathfrak{F}$ an $\mathfrak{X}$-injector of G is a modular subgroup of G.

(b) Let $\mathfrak{X}$ be a Fitting class. We define

$$\mathrm{Y}_{\mathrm{mod}}(\mathfrak{X}) = (G \mid \text{If } V \in \mathrm{Inj}_{\mathfrak{X}}(G), \text{ then } V \text{ is a modular subgroup of } G).$$

In [14, Theorem 1] it is proved that the concepts of $\mathfrak{S}$-modularity and $\mathfrak{S}$-normality coincide. Using 4.1.3, we obtain that this is valid in general. First notice:

4.1.5 Remark

Let $\mathfrak{X}$ be a Fitting class. Assume that G is a group such that an $\mathfrak{X}$-injector U of G is a modular subgroup of G. Further let T be a subgroup of G as described in 4.1.3.

(a) *$Core_G(U) = G_{\mathfrak{X}}$ is the unique maximal subnormal $\mathfrak{X}$-subgroup of G; in particular, $T \cap U = G_{\mathfrak{X}}$.*

(b) *4.1.3 implies that $U/G_{\mathfrak{X}}$ is of square free order and $(|G/G_{\mathfrak{X}} : U/G_{\mathfrak{X}}|, |U/G_{\mathfrak{X}}|) = 1$.*

4.1.6 Proposition

Let $\mathfrak{X}$ be a non-trivial Fitting class and $G \in Y_{mod}(\mathfrak{X}) \setminus Y_n(\mathfrak{X})$. Then

$$G \times G \notin Y_{mod}(\mathfrak{X}).$$

In particular: Let $\mathfrak{X}$ and $\mathfrak{F}$ be non-trivial Fitting classes. Then $\mathfrak{X}$ is modular in $\mathfrak{F}$ if and only if $\mathfrak{X}$ is normal in $\mathfrak{F}$.

Proof: Assume to the contrary that $G \times G \in \mathrm{Y}_{\mathrm{mod}}(\mathfrak{X})$, and let $V \in \mathrm{Inj}_{\mathfrak{X}}(G \times G)$. Then $V \geq F_1 \times F_2 \geq G_{\mathfrak{X}} \times G_{\mathfrak{X}}$ for suitable F_1, $F_2 \in \mathrm{Inj}_{\mathfrak{X}}(G)$. Using 4.1.3, we obtain $|G/G_{\mathfrak{X}}| = \prod_{i=1}^{r} p_i^{n_i} q_i m$ and $|F_1/G_{\mathfrak{X}}| = |F_2/G_{\mathfrak{X}}| = q_1 \cdots q_r$ where $r \in \mathbb{N}$, $p_1, \ldots, p_r, q_1, \ldots, q_r$ pairwise distinct primes, $n_i \geq 1$, $(p_i, m) = 1 = (q_i, m)$ for all i, and $|Z(G/G_{\mathfrak{X}})| \mid m$. 1.2.5 yields $|(G \times G)_{\mathfrak{X}}/(G_{\mathfrak{X}} \times G_{\mathfrak{X}})| \mid |Z((G \times G)/(G_{\mathfrak{X}} \times G_{\mathfrak{X}}))| \mid m^2$; hence, from 4.1.3 it follows that $q_i^2 \mid |(G \times G)/(G \times G)_{\mathfrak{X}}|$ for every $i \in \{1, \ldots, r\}$. Consequently, $q_i \nmid |V/(G \times G)_{\mathfrak{X}}|$. Since $F_1 \times F_2 \leq V$, this implies $F_1 \times F_2 \leq (G \times G)_{\mathfrak{X}}$, a contradiction to $(q_i, m) = 1$. □

Submodular subgroups

In view of 4.1.6, we turn our attention to a weaker concept than modularity.

4.1.7 Definition
Let G be a group. A subgroup U of G is called submodular in G (U smod G) if there exists a series

$$U = U_1 < U_2 < \ldots < U_n = G$$

of subgroups U_i of G such that U_i is modular in U_{i+1} for $i = 1, \ldots, n-1$. Obviously, this series can be choosen in such a way that U_i is a maximal modular subgroup of U_{i+1} for every $i = 1, \ldots, n-1$.

Let G be a group. Evidently, each modular subgroup of G is submodular in G. The converse does not hold true in general, so for instance a Sylow 2-subgroup of $S_3 \times S_3$ is submodular but not modular in $S_3 \times S_3$.
Detailed analysis of submodular subgroups has been carried out by Zimmermann (cf. [21]), and almost all results needed here are taken from this work.

4.1.8 Lemma ([21], Lemma 1, Prop 1)
Let G be a group, $U,\ V \leq G$, and let N be a normal subgroup of G.

(a) *If U smod G, then $U \cap V$ smod V.*

(b) *If U smod G, then UN/N smod G/N.*

(c) *If U/N smod G/N, then U smod G.*

(d) *If $U,\ V$ smod G, then $U \cap V$ smod G.*

Observe that the join $\langle U, V \rangle$ of submodular subgroups $U,\ V$ of a group G is, in general, not submodular in G, not even when U and V are Sylow-subgroups of G (cf. [21, p. 547]): Let $G = \langle a, b \mid a^7 = b^6 = 1,\ ab = ba^3 \rangle$ be the holomorph of the cyclic group $A = \langle a \rangle$ of order 7. Set $U_1 = \langle b^2 \rangle$ and $U_2 = \langle b^3 \rangle$. Then U_i mod $AU_i \trianglelefteq G$, and consequently U_i smod G $(i = 1, 2)$. The join $\langle U_1, U_2 \rangle = \langle b \rangle$ is the maximal subgroup $\langle b \rangle$ of order 6, which is not (sub)modular in G.

The concepts of submodularity and (sub)normality, too, are very close to each other.

4.1.9 Lemma ([21], Lemma 4)
Let U be a submodular subgroup of a group G. If K denotes the unique minimal normal subgroup of U such that U/K is abelian of squarefree exponent, then K is subnormal in G.

In particular, if $U \in \mathrm{Inj}_{\mathfrak{X}}(G)$ for a Fitting class $\mathfrak{X}$, then $U/G_{\mathfrak{X}}$ is abelian of squarefree exponent.

Groups in which all Sylow subgroups are submodular can be characterized as follows:

4.1.10 Theorem ([21], Theorem 4)
Let G be a group, and let $\pi(G) = \{p_1, \ldots, p_r\}$. Assume that $p_1 > \ldots > p_r$. Then the following statements are equivalent:

(i) *The Sylow subgroups of G are submodular subgroups of G.*

(ii) *The following conditions are satisfied:*

- **(a)** *G possesses a Sylow tower $1 < P_1 < P_1P_2 < \ldots < P_1 \cdots P_r$ ($P_i \in \mathrm{Syl}_{p_i}(G)$ suitable).*
- **(b)** *If $P_j \in \mathrm{Syl}_{p_j}(G)$ such that $[P_1 \cdots P_i,\ P_j] \not\leq P_1 \cdots P_{i-1}$ for $j > i$, then $p_j \mid p_i - 1$.*
- **(c)** *$G/F(G)$ has elementary abelian Sylow subgroups.*

Locally submodular Fitting classes

Submodular Fitting classes, i.e. non-trivial Fitting classes $\mathfrak{X}$ such that for each group G an $\mathfrak{X}$-injector of G is a submodular subgroup of G, were introduced by Hauck and Kienzle in 1987 (cf. [14]). We generalize this concept:

4.1.11 Definition
Let $\mathfrak{X}$ and $\mathfrak{F}$ be non-trivial Fitting classes such that $\mathfrak{X} \subseteq \mathfrak{F}$. Then $\mathfrak{X}$ is said to be submodular in $\mathfrak{F}$ ($\mathfrak{X}$ smod $\mathfrak{F}$) if for all $G \in \mathfrak{F}$ an $\mathfrak{X}$-injector of G is a submodular subgroup of G.

If $\mathfrak{X}$ is submodular in $\mathfrak{F}$, we also refer to $\mathfrak{X}$ as being $\mathfrak{F}$-submodular.

Obviously, the relation of local normality implies that of local submodularity. The converse does not hold true in general. To prove this, we need a Fitting class constructed by Menth in [17], which we denote by $\mathfrak{M}(p,3)$ (where p is a prime such that $p \equiv 1 \bmod 3$). We will not present the (complex) definition of this class, but only the following statements needed here (cf. [17, 4.2, 4.3]):

4.1.12 Theorem
Let $\mathfrak{M}(p,3)$ be as described in [17].

(a) *$\mathfrak{M}(p,3)$ is a Fitting class such that $\mathfrak{S}_p \times \mathfrak{S}_3 \subset \mathfrak{M}(p,3) \subseteq \mathfrak{S}_p\mathfrak{S}_3 \cap \mathfrak{U}$.*

(b) *If $G \in \mathfrak{M}(p,3)$, then $G/F(G)$ is an elementary abelian 3-group.*

4.1.13 Remark
Let $\mathfrak{M}(p,3)$ be as described in [17]. Then $\mathfrak{N}\mathfrak{S}_3$ is submodular, but not normal in $\mathfrak{N}\mathfrak{M}(p,3)$.

Proof:

(1) Let G be a group contained in $\mathfrak{N}\mathfrak{M}(p,3)$. Then $F(G)P_3$ is an $\mathfrak{N}\mathfrak{S}_3$-injector of G (where $P_3 \in \mathrm{Syl}_3(G)$):
Let T be an $\mathfrak{N}$-injector of $G_{\mathfrak{L}_3(\mathfrak{N})}$. According to 1.2.15, $TP_3 \in \mathrm{Inj}_{\mathfrak{N}\mathfrak{S}_3}(G)$ for a suitable Sylow 3-subgroup P_3 of G. Set $N = G_{\mathfrak{L}_3(\mathfrak{N})}$. By 1.2.18 $T = \prod T_q$ where $T_q \in \mathrm{Syl}_q(C_N(O_{q'}(F(N))))$. In particular, T_p is a normal subgroup of $F(G)T_p$, and consequently $T_p = O_p(G)$. Obviously, $O_q(G) = T_q$ for $q \neq p, 3$. Hence we obtain $TP_3 = F(G)P_3$, and the proof is complete.

(2) $\mathfrak{N}\mathfrak{S}_3$ is submodular in $\mathfrak{N}\mathfrak{M}(p,3)$:
Let the notation be as in (1). It follows from 4.1.12 and 4.1.10 that $F(G)P_3/F(G)$ is a submodular subgroup of $G/F(G)$; thus (1) yields the assertion.

(3) $\mathfrak{N}\mathfrak{M}(p,3) \not\subseteq \mathrm{Y_n}(\mathfrak{N}\mathfrak{S}_3)$:
Suppose that $\mathfrak{N}\mathfrak{M}(p,3) \subseteq \mathrm{Y_n}(\mathfrak{N}\mathfrak{S}_3)$. By 4.1.12 there exists a group $G \in \mathfrak{M}(p,3) \setminus \mathfrak{S}_p \times \mathfrak{S}_3$. Let q be a prime $\neq p, 3$. Then (1) implies $G \cong Z_q \wr G/F(Z_q \wr G) \in \mathfrak{S}_3\mathfrak{S}_p \cap \mathfrak{S}_p\mathfrak{S}_3$, a contradiction to the choice of G.

□

The following remark is obvious.

4.1.14 Remark

(a) *Let* $\mathfrak{F}$, $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be non-trivial Fitting classes such that* $\pi(\mathfrak{X}) \cap \pi(\mathfrak{Y}) = \emptyset$. *If* $\mathfrak{X}$ *is submodular in* $\mathfrak{F}$, *then* $\mathfrak{X}$ *is submodular in* $\mathfrak{F}\mathfrak{Y}$. *In particular,* $\mathfrak{X}$ *is submodular in* $\mathfrak{X}\mathfrak{S}_{\pi(\mathfrak{X})'}$.

(b) *Let* $\mathfrak{X}$, $\mathfrak{F}$ *and* $\mathfrak{Y}$ *be non-trivial Fitting classes such that* $\mathfrak{X}$ *is submodular in* $\mathfrak{F}$ *and* $\mathfrak{X} \subseteq \mathfrak{Y} \subseteq \mathfrak{F}$. *Then* $\mathfrak{X}$ *is submodular in* $\mathfrak{Y}$.

(c) *Let* $\mathfrak{F}$ *and* $\mathfrak{X}$ *be non-trivial Fitting classes such that* $\mathfrak{X}$ *is submodular in* $\mathfrak{F}$. *Further, let* $\mathfrak{Y}$ *be an SFitting class contained in* $\mathfrak{F}$. *Then* $\mathfrak{X} \cap \mathfrak{Y}$ *is submodular in* $\mathfrak{Y}$.

Like normality, the relation of submodularity between Fitting classes is far from being transitive:

4.1.15 Proposition

Let $\mathfrak{X}$ *be a Fitting class such that* $\mathfrak{X}$ *is submodular in* $\mathfrak{X}\mathfrak{N}^2$. *Then* $\mathfrak{X}^* = \mathfrak{S}$.

In particular, a Fitting class $\mathfrak{X}$ *is submodular in* $\mathfrak{S}$ *if and only if* $\mathfrak{X}$ *is normal in* $\mathfrak{S}$.

Proof: According to 3.1.3, it is sufficient to prove that for every $G \in \mathfrak{X}$ and every prime p the group $G^2 \wr Z_p$ belongs to $\mathfrak{X}$. Assume not. We choose a group $G \in \mathfrak{X}$ of minimal order such that there exists a prime p satisfying $G^2 \wr Z_p \notin \mathfrak{X}$.
$p \in \pi(\mathfrak{X})$, in particular $G \neq 1$: Let q be a prime contained in $\pi(\mathfrak{X})$ and let Q denote a non-abelian q-group. If $p \notin \pi(\mathfrak{X})$, then $Q \in \mathrm{Inj}_{\mathfrak{X}}(Z_p \wr Q)$ and $(Z_p \wr Q)_{\mathfrak{X}} = 1$. This contradicts 4.1.9.
Let N denote a maximal normal subgroup of G, thus $G/N \cong Z_r$ for a suitable $r \in \pi(\mathfrak{X})$.
$G^2 \wr Z_r \in \mathfrak{X}$: Assume not. Then $(G^2 \wr Z_r)_{\mathfrak{X}} = (G^2)^*$. Since $N^2 \wr Z_r$ is a subnormal subgroup of $G^2 \wr Z_r$, which is contained in $\mathfrak{X}$, this is a contradiction.
Let R denote a non-abelian r-group. According to 1.2.25 and 1.2.26, the group $G^2 \wr R$ belongs to $\mathfrak{X}$. By 1.2.27, $(G^2)^p \wr R \in \mathrm{Inj}_{\mathfrak{X}}(G^2 \wr Z_p \wr R)$ and $(G^2 \wr Z_p \wr R)_{\mathfrak{X}} = (G^{2p})^*$; hence 4.1.9 implies a final contradiction. □

(That $\mathfrak{S}$-submodularity coincides with $\mathfrak{S}$-normality has been proven already by Hauck and Kienzle in 1987, cf. [14, Theorem 2].)

Submodularity is invariant under epimorphisms, and moreover, a normal subgroup of a submodular subgroup of a group G is submodular in G. The proof of the following proposition is therefore analogous to the proof of 3.1.4.

4.1.16 Proposition
Let $\mathfrak{X}$ and $\mathfrak{F}$ be Fitting classes, $\mathfrak{X} \subseteq \mathfrak{F}$. Then the following statements are equivalent:

(i) *$\mathfrak{X}$ is submodular in $\mathfrak{F}$.*

(ii) *$\mathfrak{X}$ is submodular in $\mathfrak{F}^*$.*

(iii) *$\mathfrak{X}^*$ is submodular in $\mathfrak{F}^*$.*

Consequently, when considering submodularity between Fitting classes we may assume that both classes are Lockett classes. In this case the following lemma is particulary useful (compare with 3.1.13).

4.1.17 Lemma
Let $\mathfrak{X}$ and $\mathfrak{F}$ be Lockett classes such that $\mathfrak{X}$ is submodular in $\mathfrak{F}$. Further let G be a group contained in $\mathfrak{X}$ and p, q $(p \neq q)$ be primes such that $G \wr Z_p \in \mathfrak{X}$ and $\mathrm{G} \wr \mathrm{Z}_q \wr Z_p \in \mathfrak{F}$. Then

$$G \wr Z_q \in \mathfrak{X}.$$

In particular, if $\mathfrak{G}$ is a Lockett class such that $\mathfrak{G}\mathfrak{S}_p \subseteq \mathfrak{X}$ and $\mathfrak{G}\mathfrak{S}_p\mathfrak{S}_q\mathfrak{S}_p \subseteq \mathfrak{F}$, then $\mathfrak{G}\mathfrak{S}_p\mathfrak{S}_q \subseteq \mathfrak{X}$.

Proof: Let P denote a non-abelian p-group. According to 1.2.25, $G \wr Z_p \in \mathfrak{X}$ implies that $G \wr P$ belongs to $\mathfrak{X}$. Assume that $G \wr Z_q \notin \mathfrak{X}$. Then 1.2.24 yields $(G \wr Z_q \wr P)_{\mathfrak{X}} = (G^*)^*$. Thus by 1.2.5 we obtain $(G^*)^* P \in \mathrm{Inj}_{\mathfrak{X}}(G \wr Z_q \wr P)$, what is a contradiction to 4.1.9.

□

4.1.18 Definition
Let $\mathfrak{X}$ be a Fitting class. We define

$$\mathrm{Y}_{\mathrm{smod}}(\mathfrak{X}) = (G \mid \text{If } V \in \mathrm{Inj}_{\mathfrak{X}}(G), \text{ then } V \text{ smod } G).$$

It is obvious that $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{X})$ is closed under taking subnormal subgroups. But in general, $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{X})$ is not closed under forming normal products, and thus fails to be a Fitting class.

4.1.19 Remark
Let $\mathfrak{X}$ be a Fitting class. Then $Y_{smod}(\mathfrak{X})$ need not be closed under forming normal products.

Proof: Let p and q be prime numbers such that $p \mid q-1$. Set $H = Z_p \wr Z_p = H_1H_2$ where $H_1 \cong Z_p^*$ and $H_2 \cong Z_p$. Consider the group $G = Z_q \wr H$. Then $G = \langle Z_q^* H_1, \ Z_q^* H_2 \rangle$, and $Z_q^* H_1$ and $Z_q^* H_2$ are subnormal $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{S}_p)$-subgroups of G according to 4.1.10. But evidently, $O_p(G) = 1$ and an $\mathfrak{S}_p$-injector of G is a non-abelian subgroup of G. By 4.1.9, this implies $G \notin \mathrm{Y}_{\mathrm{smod}}(\mathfrak{S}_p)$. □

4.1.20 Remark
Let $\mathfrak{X}$ be a Fitting class.

(a) *If $Y_{smod}(\mathfrak{X}) = {}_{N_0}Y_{smod}(\mathfrak{X})$, then $Y_{smod}(\mathfrak{X}) = Y_{smod}(\mathfrak{X})^* = Y_{smod}(\mathfrak{X}^*)$.*

(b) *$Y_{smod}(\mathfrak{X}^*) = {}_{D_0}Y_{smod}(\mathfrak{X}^*)$.*

Proof: (a) follows from 4.1.16. (b): Evidently, it is sufficient to prove that $G_1 \times G_2 \in \mathrm{Y}_{\mathrm{smod}}(\mathfrak{X}^*)$ provided that $G_1, \ G_2 \in \mathrm{Y}_{\mathrm{smod}}(\mathfrak{X}^*)$. Thus, let $G_1, \ G_2$ be groups belonging to $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{X}^*)$ and set $G = G_1 \times G_2$. 1.2.10 states that $V_1 \times V_2 \in \mathrm{Inj}_{\mathfrak{X}^*}(G)$ where $V_i \in \mathrm{Inj}_{\mathfrak{X}^*}(G_i)$, and that each $\mathfrak{X}^*$-injector of G is of this form ($i = 1,2$). V_i smod G_i, consequently there exists a series $V_i = D_0^i \leq D_1^i \leq \ldots \leq D_{n_i}^i = G_i$ such that D_j^i is a maximal modular subgroup of D_{j+1}^i for all $j = 1, \ldots, n_i - 1$; $i = 1,2$. We assume that $n_1 \leq n_2$ and consider the series

$$\begin{aligned} V_1 \times V_2 &= D_0^1 \times D_0^2 \leq D_0^1 \times D_1^2 \leq D_1^1 \times D_1^2 \leq \ldots \leq D_{n_1}^1 \times D_{n_1}^2 \leq \\ D_{n_1}^1 \times D_{n_1+1}^2 &\leq D_{n_1}^1 \times D_{n_1+2}^2 \leq \ldots \leq D_{n_1}^1 \times D_{n_2}^2 = G_1 \times G_2. \end{aligned}$$

Now, repeated application of 4.1.2 yields the assertion. □

Thus, in investigating locally submodular Fitting classes, we are in a similar situation as in the case of local normality. Therefore the question about the existence of a unique maximal Fitting class contained in $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{X})$ seems to be hard to attack as well. However, we will see later that for this relation, too, the special case that both classes are subgroup-closed is easier to handle (cf. 4.1.32).

For some types of Fitting classes $\mathfrak{X}$ it is possible to obtain an upper bound of $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{X})$ (compare with a result of Hauck, cf. [13, 3.3]).

4.1.21 Proposition
Let $\mathfrak{X}$ be a Fitting class and let $\pi \neq \emptyset$ denote a set of primes such that $\mathfrak{X}\mathfrak{S}_\pi = \mathfrak{X}$.

(a) $Y_{smod}(\mathfrak{X}) \subseteq \mathfrak{X}\mathfrak{S}_{\pi'} \circ \mathfrak{A}_\pi \circ \mathfrak{S}_{\pi'}$.

In particular, if $|\pi'| = 1$, *then* $Y_{smod}(\mathfrak{X}) \subseteq \mathfrak{X}\mathfrak{N}^3$.

(b) *Assume further that* $q \nmid p-1$ *for all* $q \in \pi$, $p \in \pi'$. *Then* $Y_{smod}(\mathfrak{X}) \subseteq \mathfrak{X}\mathfrak{S}_{\pi'}$.

In particular, if $\pi = 2'$, *then* $Y_{smod}(\mathfrak{X}) = Y_n(\mathfrak{X}) = \mathfrak{X}\mathfrak{S}_2$.

Proof:

(a): Let $G \in \mathrm{Y}_{\mathrm{smod}}(\mathfrak{X})$ and $V \in \mathrm{Inj}_{\mathfrak{X}}(G)$. It follows from 1.2.14 that $p \nmid |G : V|$ for all $p \in \pi$. By 4.1.9, $V/G_{\mathfrak{X}}$ is abelian. Consequently, $G/G_{\mathfrak{X}}$ has abelian Hall π-subgroups. Put $\overline{G} = G/G_{\mathfrak{X}}$. Then

$$\begin{aligned} O_\pi(\overline{G}/O_{\pi'}(\overline{G})) &= F(\overline{G}/O_{\pi'}(\overline{G})) \geq \\ C_{\overline{G}/O_{\pi'}(\overline{G})}(F(\overline{G}/O_{\pi'}(\overline{G}))) &\geq \overline{H}O_{\pi'}(\overline{G})/O_{\pi'}(\overline{G}), \end{aligned}$$

where $\overline{H} \in \mathrm{Hall}_\pi(\overline{G})$.
Thus, we conclude that $\overline{G} \in \mathfrak{S}_{\pi'} \circ \mathfrak{A}_\pi \circ \mathfrak{S}_{\pi'}$, and the assertion follows.

(b): Assume not. Let G be a group of minimal order contained in $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{X}) \setminus \mathfrak{X}\mathfrak{S}_{\pi'}$. Then G has a unique maximal normal subgroup $N = G_{\mathfrak{X}\mathfrak{S}_{\pi'}}$, and $|G/N| = q \in \pi$. If $V \in \mathrm{Inj}_{\mathfrak{X}}(G)$, then $q \nmid |G : V|$ according to 1.2.14, whence $VN = G$. By assumption, V is a submodular subgroup of G. Hence there exists a maximal modular subgroup K of G such that $V \leq K$. Since $K \nleq N$, it follows from 4.1.2 that $K = V\mathrm{Core}_G(K)$ and $K/\mathrm{Core}_G(K) \cong Z_q$.
$\mathrm{Core}_G(K) = N \cap K = \mathrm{Core}_G(K)(V \cap N) \geq G_{\mathfrak{X}}$, thus in particular $|N/\mathrm{Core}_G(K)| = p \in \pi'$. Since $G/\mathrm{Core}_G(K)$ is non-abelian, Sylow's theorem yields $q \mid p-1$, a contradiction.

$\square$

We have already seen that $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{S}_p)$ fails to be a Fitting class (in contrast to the case of local normality). Nevertheless, there is an easy description of this class.

4.1.22 Corollary
Let p be a prime. Set $\pi(p) = \{q \in \mathbb{P} \mid p \mid q-1\}$ and $\mathfrak{H}(p) = (G \mid P/O_p(G)$ is elementary abelian, $P \in Syl_p(G))$. Then

$$\mathrm{Y}_{smod}(\mathfrak{S}_p) = \mathfrak{H}(p) \cap \mathfrak{S}_p\mathfrak{S}_{\pi(p)}\mathfrak{S}_p\mathfrak{S}_{p'}.$$

Proof:

$\supseteq$: Assume not. Let G be a counterexample of minimal order and let $P \in \mathrm{Syl}_p(G)$. Then obviously $P \neq 1$. Since $G/O_p(G)$ is contained in $\mathfrak{H}(p) \cap \mathfrak{S}_p\mathfrak{S}_{\pi(p)}\mathfrak{S}_p\mathfrak{S}_{p'}$, we conclude that $O_p(G) = 1$ (for otherwise $P/O_p(G)$ smod $G/O_p(G)$, and consequently P smod G; a contradiction). In particular, P is an elementary abelian subgroup of G.
Let M denote a minimal normal subgroup of G. The choice of G implies that PM is submodular in G. Consequently $PM = G$ (for otherwise P smod PM smod G; a contradiction). Hence G belongs to $\mathfrak{S}_{\pi(p)}\mathfrak{S}_p$. Applying 4.1.10, we obtain a final contradiction.

$\subseteq$: Assume not. According to 4.1.9, G belongs to $\mathfrak{H}(p)$. Thus we may choose a group G of minimal order contained in $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{S}_p) \setminus \mathfrak{S}_p\mathfrak{S}_{\pi(p)}\mathfrak{S}_p\mathfrak{S}_{p'}$. By 4.1.21(a), $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{S}_p) \subseteq \mathfrak{S}_p\mathfrak{S}_{p'}\mathfrak{S}_p\mathfrak{S}_{p'}$. The choice of G implies $G \in \mathfrak{S}_{p'}\mathfrak{S}_p$. Set $M = O_q(G)$, $q \neq p$ prime, and let $P \in \mathrm{Syl}_p(G)$. The minimality of G implies $O_{q'}(G) = 1$. Since $G/M \in \mathfrak{S}_p\mathfrak{S}_{\pi(p)}\mathfrak{S}_p \cap \mathfrak{S}_{p'}\mathfrak{S}_p \subseteq \mathfrak{S}_{\pi(p)}\mathfrak{S}_p$, to obtain a contradiction it is sufficient to show that $q \in \pi(p)$. By assumption, P smod PM, and thus PM is a group in which all Sylow subgroups are submodular. If $[P, M] = 1$, then $P \leq C_G(M) \leq M$, a contradiction. Consequently 4.1.10 implies $p \mid q-1$, and the proof is complete.

$\square$

Assume that $\mathfrak{X}$ is a Lockett class and that p is a prime such that $\mathfrak{X}\mathfrak{S}_p = \mathfrak{X}$. In this situation the N_0-closure of $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{X})$ forces the class $\mathfrak{X}$ to be "large":

4.1.23 Proposition

Let $\mathfrak{X}$ be a Lockett class such that $\mathfrak{X}\mathfrak{S}_p = \mathfrak{X}$ for some prime p, and assume that $\mathrm{Y}_{smod}(\mathfrak{X})$ is a Fitting class.
Further define π_n recursively by $\pi_0 = \{p\}$ and $\pi_n = \{q \in \mathbb{P} \mid$ there exists $t \in \pi_{n-1}$ such that $t \mid q-1\}$, and set $\pi = \cup_{n\in\mathbb{N}\cup\{0\}}\pi_n$.
Then

$$\mathfrak{X}\mathfrak{S}_\pi = \mathfrak{X}.$$

In particular, if $p = 2$, then $\mathfrak{X} = \mathfrak{S}$.

Proof: Let G be a group contained in $\mathfrak{X}\mathfrak{S}_\pi$. Then there exists a natural number m such that $G \in \mathfrak{X}\mathfrak{S}_{\cup_{i=0}^m \pi_i}$. Since $\mathfrak{X}\mathfrak{S}_{\pi_0} = \mathfrak{X}$, it is sufficient to show that $\mathfrak{X}\mathfrak{S}_{\pi_n} = \mathfrak{X}$ provided that $\mathfrak{X}\mathfrak{S}_{\pi_{n-1}} = \mathfrak{X}$ ($n \in \mathbb{N}$).
Suppose not. Let G be a group of minimal order contained in $\mathfrak{X}\mathfrak{S}_{\pi_n}\backslash\mathfrak{X}$. Then $G_\mathfrak{X} \in \mathrm{Inj}_\mathfrak{X}(G)$ and $G/G_\mathfrak{X} \cong Z_{q_1}$ for a prime $q_1 \in \pi_n$. By definition of π_n, there exists a prime $q_2 \in \pi_{n-1}$ such that $q_2 \mid q_1 - 1$. Put $H = Z_{q_2} \wr Z_{q_2} = H_1H_2$ where H_1 denotes the base group of $Z_{q_2} \wr Z_{q_2}$ and H_2 a complement to H_1 in $Z_{q_2} \wr Z_{q_2}$. Then 4.1.10 implies that $G \wr H \in \mathrm{N}_0\mathrm{Y}_{\mathrm{smod}}(\mathfrak{X}) = \mathrm{Y}_{\mathrm{smod}}(\mathfrak{X})$. Since $(G_\mathfrak{X})^*H \in \mathrm{Inj}_\mathfrak{X}(G \wr H)$ according to 1.2.5, and $(G \wr H)_\mathfrak{X} = (G_\mathfrak{X})^*$ by 1.2.24, this contradicts 4.1.9. □

Let $\mathfrak{F}$ be a non-trivial Fitting class, and let $\mathfrak{X}_i,\ i \in I$, denote $\mathfrak{F}$-submodular Fitting classes. Whether or not $\mathfrak{F}$ is contained in $\mathrm{Y}_{\mathrm{smod}}(\cap_{i\in I}\mathfrak{X}_i)$ – and thus in particular whether or not there exists a unique minimal $\mathfrak{F}$-submodular Fitting class – is an open question. It is open even in case when $\mathfrak{F}$ is a Fischer class (or, stronger, when $\mathfrak{F}$ is an SFitting class). Compared to local normality, in this situation it seems to be harder to describe the structure of a minimal counterexample for two reasons: on the one hand submodular subgroups do not – in general – form a lattice; on the other hand, there is nothing known – in general – about the relation between $\mathfrak{X}_i$-injectors and $\cap_{i\in I}\mathfrak{X}_i$-injectors of a group G.
Nevertheless, in some important cases it is possible to obtain a positive answer to the above mentioned question. Since 3.1.13 is valid as well for local submodularity (cf. 4.1.17), the following results can be proved essentially as for the case of local normality.

4.1.24 Theorem
Let $\mathfrak{X}$ and $\mathfrak{F}$ be non-trivial Fitting classes, $n \in \mathbb{N}$.

(a) *Let $\mathfrak{F}$ be a lattice formation. Then $(\mathfrak{F}^n)_*$ is the unique minimal Fitting class which is submodular in $\mathfrak{F}^{n+1}$.*

(b) *Let $\mathfrak{F}$ be as described in 3.1.15(c). Then $\mathfrak{F}_*$ is the unique minimal Fitting class which is submodular in $\mathfrak{F}$.*

(c) *Let $\mathfrak{F} = \mathfrak{S}_{\pi_1} \cdots \mathfrak{S}_{\pi_r}$ and $\mathfrak{Y}$ be as described in 3.1.18. Then $\mathfrak{Y}_*$ is the unique minimal Fitting class which is submodular in $\mathfrak{F}$.*

In particular: Let $\mathfrak{F} = \mathfrak{N}^{n+1}$ or $\mathfrak{F}$ be as described in (b), and let $\mathfrak{X}$ be a Fitting class. Then $\mathfrak{X}$ is submodular in $\mathfrak{F}$ if and only if $\mathfrak{X}$ is normal in $\mathfrak{F}$.

Note that for each Fitting class $\mathfrak{F}$ treated above, the smallest $\mathfrak{F}$-normal Fitting class coincides with the smallest $\mathfrak{F}$-submodular Fitting class. But there are Fitting classes $\mathfrak{F}$ – for instance the class $\mathfrak{M}(p,3)$ (cf. 4.1.10 and 4.1.12) – such that a smallest $\mathfrak{F}$-normal Fitting class exists, but not a smallest $\mathfrak{F}$-submodular Fitting class.

Local submodularity and local normality

We have already seen that the concepts of local normality and local submodularity are very close to each other. Moreover, in a special case of 4.1.21(b) we gave sufficient conditions for $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{X}) = \mathrm{Y}_{\mathrm{n}}(\mathfrak{X})$ to hold. In this section we will extent reflections of this kind.

4.1.25 Remark
In general, the hypothesis of 4.1.21(b) is not sufficient to conclude that $Y_n(\mathfrak{X}) = Y_{smod}(\mathfrak{X})$.

Proof: Set $\pi = \mathbb{P} \setminus \{2,3\}$. Then π is a set of primes as required in 4.1.21(b), thus in particular $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{X}) \subseteq \mathfrak{X}\mathfrak{S}_{\pi'}$ where $\mathfrak{X} = \mathfrak{N}\mathfrak{S}_{\pi}$. However, $\mathrm{Y}_{\mathrm{n}}(\mathfrak{X}) \subset \mathrm{Y}_{\mathrm{smod}}(\mathfrak{X})$, since $Z_2 \wr Z_3 \wr Z_2 \in \mathrm{Y}_{\mathrm{smod}}(\mathfrak{X}) \setminus \mathrm{Y}_{\mathrm{n}}(\mathfrak{X})$. $\square$

The condition of 4.1.21(b) is sufficient provided that $\mathfrak{X} = \mathfrak{S}_\pi$:

4.1.26 Remark
Let $\mathfrak{X} = \mathfrak{S}_\pi,\ \pi \neq \emptyset$. Then the following statements are equivalent:

(i) $Y_n(\mathfrak{X}) = Y_{smod}(\mathfrak{X})$.

(ii) $Y_{smod}(\mathfrak{X}) = {}_{N_0}Y_{smod}(\mathfrak{X})$.

(iii) *π is a set of primes satisfying $q \nmid p-1$ for all $q \in \pi,\ p \in \pi'$.*

Proof: 4.1.23 and 4.1.21. Notice that $\mathrm{Y_n}(\mathfrak{S}_\pi) = \mathfrak{S}_\pi\mathfrak{S}_{\pi'}$. □

4.1.27 Proposition
Let $\mathfrak{X}$ be a Lockett class and π be a set of primes such that $Y_{smod}(\mathfrak{X}) = \mathfrak{X}\mathfrak{S}_{\pi'}$. If $Y_n(\mathfrak{X}) = {}_{N_0}Y_n(\mathfrak{X})$, then $Y_{smod}(\mathfrak{X}) = Y_n(\mathfrak{X})$.

Proof: Obviously, $\mathrm{Y_n}(\mathfrak{X}) \subseteq \mathrm{Y_{smod}}(\mathfrak{X})$. To prove the converse, according to 1.2.29 it is sufficient to show that $G \wr Z_p \in \mathrm{Y_n}(\mathfrak{X})$ for every $G \in \mathrm{Y_n}(\mathfrak{X})$ and every $p \in \pi'$. Assume the contrary and choose a group $G \in \mathrm{Y_n}(\mathfrak{X})$ and a prime $p \in \pi'$ such that $G \wr Z_p \notin \mathrm{Y_n}(\mathfrak{X})$; in particular $G \notin \mathfrak{X}$. By 1.2.28, $G_\mathfrak{X} \wr Z_p \in \mathrm{Inj}_\mathfrak{X}(G \wr Z_p)$, and 1.2.25 yields $G_\mathfrak{X} \wr P \in \mathfrak{X}$ where P denotes a non-abelian p-group. Since by assumption $G \wr P$ belongs to $\mathrm{Y_{smod}}(\mathfrak{X})$, 4.1.9 yields a final contradiction. □

It is an open question what conditions a Fitting class must satisfy to fulfill $\mathrm{Y_{smod}}(\mathfrak{X}) = \mathfrak{X}\mathfrak{S}_{\pi'}$. It is open, too, whether or not these Fitting classes are precisely the Fitting classes such that $\mathrm{Y_n}(\mathfrak{X}) = \mathfrak{X}\mathfrak{S}_{\pi'}$ holds (at least for π as described in 4.1.21). However, it is easily seen that the condition that $\mathfrak{X}\mathfrak{S}_\pi = \mathfrak{X}$ is not a sufficient one: Choose π and $\mathfrak{X}$ as described in 4.1.25. Then $Z_2 \wr Z_3 \wr (Z_2 \wr Z_2) \in \mathfrak{X}\mathfrak{S}_{\pi'} \setminus \mathrm{Y_{smod}}(\mathfrak{X})$.

4.1.28 Proposition
Let $\mathfrak{X}$ be a Lockett class such that $Y_n(\mathfrak{X}) = Y_{smod}(\mathfrak{X})$.

(a) *Suppose that there exists a prime p such that $\mathfrak{X}\mathfrak{S}_p = \mathfrak{X}$. Then $\mathfrak{X}\mathfrak{S}_\pi = \mathfrak{X}$ where π is a set of primes defined as in 4.1.23 corresponding to p.*

In particular, if $p = 2$, then $\mathfrak{X} = \mathfrak{S}$.

(b) *Assume further that $Y_n(\mathfrak{X}) = {}_{N_0}Y_n(\mathfrak{X})$. Then $Y_n(\mathfrak{X})\mathfrak{S}_2 = Y_n(\mathfrak{X})$.*

Proof:

(a): By induction and 1.2.29 it is sufficient to show that $G \wr Z_q \in \mathfrak{X}$ for all $q \in \pi_n$ and all $G \in \mathfrak{X}$, provided that $\mathfrak{X}\mathfrak{S}_{\pi_{n-1}} = \mathfrak{X}$ ($n \in \mathbb{N}$). Assume to the contrary that there exist a group $G \in \mathfrak{X}$ and a prime $q \in \pi_n$ such that $G \wr Z_q \notin \mathfrak{X}$. By definition of π_n, there exists a prime $t \in \pi_{n-1}$ satisfying $t \mid q-1$. By assumption and 1.2.5, $G^* \wr Z_t \in \mathrm{Inj}_{\mathfrak{X}}(G \wr Z_q \wr Z_t)$. But 4.1.10 implies that $G \wr Z_q \wr Z_t$ belongs to $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{X}) = \mathrm{Y}_{\mathrm{n}}(\mathfrak{X})$, a contradiction.

(b): Suppose not. Let $G \in \mathrm{Y}_{\mathrm{n}}(\mathfrak{X})$ be minimal with respect to $G \wr Z_2 \notin \mathrm{Y}_{\mathrm{n}}(\mathfrak{X})$. By 1.2.28 we obtain $V := G_{\mathfrak{X}} \wr Z_2 \in \mathrm{Inj}_{\mathfrak{X}}(G \wr Z_2)$. Let $N \geq G_{\mathfrak{X}}$ denote a maximal normal subgroup of G. If $N > G_{\mathfrak{X}}$, then we conclude that $N \wr Z_2 \notin \mathrm{Y}_{\mathrm{n}}(\mathfrak{X})$, a contradiction. Thus $G_{\mathfrak{X}}$ is a maximal normal subgroup of G, and consequently $G/G_{\mathfrak{X}} \cong Z_p$ for some prime $p \neq 2$. Now 4.1.10 yields $G \wr Z_2 \in \mathrm{Y}_{\mathrm{smod}}(\mathfrak{X}) = \mathrm{Y}_{\mathrm{n}}(\mathfrak{X})$, a final contradiction.

□

We close this section by listing a number of open questions.

4.1.29 Remark

(a) *What conditions must a Fitting class $\mathfrak{X}$ satisfy to fulfill $Y_{smod}(\mathfrak{X}) = Y_n(\mathfrak{X})$?*

(b) *Let $\mathfrak{X}$ be a Lockett class such that $Y_{smod}(\mathfrak{X}) = {}_{\mathrm{N}_0}Y_{smod}(\mathfrak{X})$. Does this imply $Y_n(\mathfrak{X}) = {}_{\mathrm{N}_0}Y_n(\mathfrak{X})$?*
Note that the converse does not hold: $Y_n(\mathfrak{S}_p) = {}_{\mathrm{N}_0}Y_n(\mathfrak{S}_p)$ but $Y_{smod}(\mathfrak{S}_p) \neq {}_{\mathrm{N}_0}Y_{smod}(\mathfrak{S}_p)$ where p denotes an arbitrary prime number.

(c) *Let $\mathfrak{X}$ be a Lockett class such that $Y_{smod}(\mathfrak{X}) \subseteq \mathfrak{X}\mathfrak{S}_{\pi'}$ for a suitable set of primes π. Assume further that $Y_n(\mathfrak{X}) = {}_{\mathrm{N}_0}Y_n(\mathfrak{X})$.*
Does this imply $Y_n(\mathfrak{X}) = Y_{smod}(\mathfrak{X})$?

(d) *Let $\mathfrak{X}$ be a Lockett class such that $Y_{smod}(\mathfrak{X}) = \mathfrak{X}\mathfrak{S}_{\pi'}$ for some set of primes π.*
Does this imply $Y_{smod}(\mathfrak{X}) = Y_n(\mathfrak{X})$?

(e) *Is there a Fitting class $\mathfrak{X}$ such that $Y_{smod}(\mathfrak{X}) = {}_{N_0}Y_{smod}(\mathfrak{X})$ and $Y_n(\mathfrak{X}) \subset Y_{smod}(\mathfrak{X})$ holds?*

Local submodularity and SFitting classes

Our aim in this section is to prove that the concepts of local submodularity and local normality between Fitting classes coincide provided that both classes are SFitting classes. Whether or not it is sufficient for this fact to require the subgroup closure of the larger class, remains an open question.

4.1.30 Lemma
Let $\mathfrak{X}$ be a Fitting class and set $\pi(\mathfrak{X}) = \pi$. If $\mathfrak{F}$ is an SFitting class contained in $Y_{smod}(\mathfrak{X})$, then $\mathfrak{F} \subseteq \mathfrak{S}_\pi \mathfrak{S}_{\pi'}$.

Proof: Assume not. Let G be a group of minimal order contained in $\mathfrak{F} \setminus \mathfrak{S}_\pi \mathfrak{S}_{\pi'}$. Then G has a unique maximal normal subgroup N, and a unique minimal normal subgroup M, and N and G/M belong to $\mathfrak{S}_\pi \mathfrak{S}_{\pi'}$. Moreover, $O_\pi(G) = 1$ and $O^{\pi'}(G) = G$, thus we obtain $G \in \mathfrak{S}_q \mathfrak{S}_p$ and $l(G) = 2$ (where $q \in \pi'$ and $p \in \pi$ are primes). Consequently, 2.1.5(b) implies $\mathfrak{S}_q \mathfrak{S}_p = \mathrm{SFit}(G) \subseteq \mathfrak{F} \subseteq \mathrm{Y}_{\mathrm{smod}}(\mathfrak{X})$, a contradiction to 4.1.9. $\square$

4.1.31 Theorem
Let $\mathfrak{F}$ be an SFitting class of bounded nilpotent length. Assume further that $\mathfrak{F}$ is contained in $Y_{smod}(\mathfrak{X})$ for some SFitting class $\mathfrak{X}$. Then

$$\mathfrak{F} \subseteq Y_n(\mathfrak{X}).$$

Proof: By induction on $r := l(\mathfrak{F})$. The cases $r = 0,\ 1$ are obvious.
$r > 1$: According to 4.1.30, we may assume that $\pi(\mathfrak{F}) \subseteq \pi(\mathfrak{X})$. Let X and F, respectively, denote the canonical local definitions belonging to $\mathfrak{X}$ and $\mathfrak{F}$, respectively. Then analogously to the proof of 3.2.1, we obtain $F(p) \cap \mathfrak{N}^{r-1} \subseteq F(p) \subseteq \mathrm{Y}_{\mathrm{smod}}(X(p))$ for every $p \in \pi$. By inductive hypothesis this implies $F(p) \cap \mathfrak{N}^{r-1} \subseteq \mathrm{Y}_{\mathrm{n}}(X(p))$, and consequently $F(p) = \mathfrak{S}_p(F(p) \cap \mathfrak{N}^{r-1}) \subseteq \mathrm{Y}_{\mathrm{n}}(X(p))$. 3.2.1 completes the proof. $\square$

4.1.32 Corollary
Let $\mathfrak{F}$ be an SFitting class. Assume further that $\mathfrak{F}$ is contained in $Y_{smod}(\mathfrak{X})$ for some SFitting class $\mathfrak{X}$. Then

$$\mathfrak{F} \subseteq Y_n(\mathfrak{X}).$$

In particular: Let $\mathfrak{X}$ and $\mathfrak{F}$ be SFitting classes. Then $\mathfrak{X}$ is submodular in $\mathfrak{F}$ if and only if $\mathfrak{X}$ is normal in $\mathfrak{F}$. Furthermore, there exists a unique maximal SFitting class contained in $Y_{smod}(\mathfrak{X})$, and this class coincides with $\overline{\mathfrak{Y}}^{(n,\mathfrak{X})}$.

Proof: 2.2.5 and 4.1.31. □

4.2 Local normal embedding and local permutability

In this section we turn our attention to two further embedding properties of injectors – both of them being considerably weaker than normality/submodularity. These embedding properties – with respect to all (finite soluble) groups – were introduced by Lockett (cf. [15]), and studied in detail by Doerk and Porta (cf. [10]).

In the following, we will need a number of further concepts and results taken mainly from [9, I, 4, 5, 7] and [9, IX, 3].

Fundamental facts and auxiliary results

In investigating locally normally embedded and locally permutable Fitting classes the concept of a Hall system plays an important role.

4.2.1 Definition
Let G be a group, U a subgroup of G and K a normal subgroup of G.

(a) A Hall system of G is a set Σ of Hall subgroups of G satisfying the following properties:

- **(i)** For each $\pi \subseteq \mathbb{P}$, the set Σ contains exactly one π-subgroup.
- **(ii)** If $H,\ K \in \Sigma$, then $HK = KH$ (i.e. H and K permute).

For a Hall system Σ we set $\Sigma K/K := \{HK/K \mid H \in \Sigma\}$ and $\Sigma \cap U := \{H \cap U \mid H \in \Sigma\}$.

(b) Let Σ be a Hall system of G, and let U be a subgroup of G. We say that Σ reduces into U ($\Sigma \searrow U$) if $U \cap \Sigma$ is a Hall system of U.

Using Hall's theorem we obtain

4.2.2 Proposition ([9], I, 4.4, 4.16)
Let G be a group.

(a) *There exist Hall systems of G.*

(b) *Let U be a subgroup of G. Then there exists a Hall system Σ of G such that $\Sigma \searrow U$.*

In many cases, it is sufficient to consider a suitable "basis" of a Hall system:

4.2.3 Definition
Let G be a group. A set B consisting of pairwise permutable Sylow p-subgroups of G, exactly one for each $p \in \pi(G)$, together with the identity subgroup, is called a Sylow basis of G.

4.2.4 Lemma ([9], I, 4.8)
Each Hall system Σ of a group G contains a unique Sylow basis B_Σ and each Sylow basis B can be extended to a unique Hall system Σ_B. (In this case we say that B generates Σ_B.)

We will further need the concept of a system normalizer, a subgroup N of a group G, that – under certain circumstances – can be regarded as "controlling" an $\mathfrak{X}$-injector of G (where $\mathfrak{X}$ denotes a suitable Fitting class). In the context of (locally) permutable Fitting classes, system normalizers play an important role.

4.2.5 Definition
Let G be a group. A subgroup U of G is called system normalizer if there exists a Hall system Σ of G such that

$$U = N_G(\Sigma) := \{g \in G \mid H = H^g \text{ for each } H \in \Sigma\}.$$

In this case we also refer to U as the normalizer of Σ.

4.2.6 Theorem ([9], IX, 3.16)
Let $\mathfrak{X}$ be a Fitting class, and let K be a normal subgroup of a group G such that $K \in \mathrm{Y}_n(\mathfrak{X})$ and $G/K \in \mathfrak{N}$. Further, let V denote an $\mathfrak{X}$-maximal subgroup of G with $V \geq K_{\mathfrak{X}} =: W$. If Σ is a Hall system of G reducing into V and if $D = N_G(\Sigma)$, then $V = (DW)_{\mathfrak{X}}$.

Locally normally embedded Fitting classes

In this section we consider non-trivial Fitting classes $\mathfrak{X}$ and $\mathfrak{F}$ such that $\mathfrak{X} \subseteq \mathfrak{F}$ and that for each group $G \in \mathfrak{F}$ an $\mathfrak{X}$-injector of G is a normally embedded subgroup of G. We will see that local normal embedding is a property considerably weaker than local normality/submodularity. Nevertheless, it turns out that this relation, too, is a relation between the corresponding Lockett sections.

4.2.7 Definition
Let G be a group and U be a subgroup of G.

(a) If p is a prime, we say that U is p-normally embedded in G (U p-ne G) if a Sylow p-subgroup U_p of U is a Sylow p-subgroup of some normal subgroup of G, that is, $U_p \in \mathrm{Syl}_p(\langle U_p^G \rangle)$.

(b) U is called normally embedded in G (U ne G) if U is p-normally embedded in G for all primes p.

Typical examples of normally embedded subgroups of a group G are Hall subgroups of a normal subgroup of G.

4.2.8 Proposition ([9], I, 7.3, 7.8)
Let G be a group.

(a) *If $K \trianglelefteq G$ and U ne G, then UK/K ne G/K.*

(b) *Let U and V be normally embedded subgroups of G into which a given Hall system Σ reduces. Then $UV = VU$, and $U \cap V$ and UV are normally embedded subgroups of G into which Σ reduces.*

4.2.9 Definition
Let $\mathfrak{X}$ and $\mathfrak{F}$ be non-trivial Fitting classes such that $\mathfrak{X} \subseteq \mathfrak{F}$. Then $\mathfrak{X}$ is said to be normally embedded in $\mathfrak{F}$ ($\mathfrak{X}$ ne $\mathfrak{F}$) if an $\mathfrak{X}$-injector of G is a normally embedded subgroup of G for all $G \in \mathfrak{F}$.

If $\mathfrak{X}$ is normally embedded in $\mathfrak{F}$, we also refer to $\mathfrak{X}$ as $\mathfrak{F}$-normally embedded.

$\mathfrak{S}$-normally embedded Fitting classes have been studied in detail by Lockett [15] and Doerk and Porta [10] (cf. [9, IX, 3]). By [9, IX, 3.4(a)], each Fischer class – thus in particular each SFitting class – is an $\mathfrak{S}$-normally embedded Fitting class, and according to [9, IX, 2.9, 3.7], the class $\mathfrak{Z}^3 = (G \mid \mathrm{Soc}_3(G) \leq Z(G))$ is a Lockett class which is not normally embedded in $\mathfrak{S}$.

4.2.10 Remark

(a) *Let $\mathfrak{X}$, $\mathfrak{F}$ and $\mathfrak{Y}$ be non-trivial Fitting classes such that $\mathfrak{X}$ ne $\mathfrak{F}$ and $\mathfrak{X} \subseteq \mathfrak{Y} \subseteq \mathfrak{F}$. Then $\mathfrak{X}$ ne $\mathfrak{Y}$.*

(b) *Let $\mathfrak{X}$ and $\mathfrak{F}$ be non-trivial Fitting classes such that $\mathfrak{X}$ ne $\mathfrak{F}$. Further, let $\mathfrak{Y}$ be an SFitting class contained in $\mathfrak{F}$. Then $\mathfrak{X} \cap \mathfrak{Y}$ ne $\mathfrak{Y}$.*

Evidently, this relation, too, fails to be transitive: since any Fitting class $\mathfrak{X}$ is normally embedded in $\mathfrak{X}\mathfrak{N}$, transitivity of local normal embedding would imply that all Fitting classes are normally embedded in $\mathfrak{S}$, a contradiction. However, local normal embedding is a relation considerably weaker than local normality/submodularity, since, evidently, neither does $\mathfrak{X}$ ne $\mathfrak{X}\mathfrak{N}^2$ imply that $\mathfrak{X}^* = \mathfrak{S}$, nor does a corresponding statement to 4.1.17 hold true in general. Moreover, let n be an arbitrary natural number. Then there exist a Fitting class $\mathfrak{X}$ and a group G such that $l(V/G_{\mathfrak{X}}) = n$ where V denotes an $\mathfrak{X}$-injector of G: Let $\pi \neq \emptyset$ be a set of primes, and p a prime contained in π'. Choose a group $H \in \mathfrak{S}_\pi =: \mathfrak{X}$ with $l(H) = n$. Then $G = Z_p \wr H$ is a group as required.

Nevertheless, the property of normal embedding is another invariant of Lockett sections. To prove this we need

4.2.11 Definition

Let $\mathfrak{X}$ and $\mathfrak{F}$ be Fitting classes, and let $\mathfrak{H}$ be an arbitrary class of groups. Then $\mathfrak{X}$ is said to be $\mathfrak{H}$-strongly contained in $\mathfrak{F}$ ($\mathfrak{X}$ $\mathfrak{H}$-$\ll$ $\mathfrak{F}$) if an $\mathfrak{F}$-injector of G contains an $\mathfrak{X}$-injector of G for all $G \in \mathfrak{H}$.
If $\mathfrak{X}$ is $\mathfrak{S}$-strongly contained in $\mathfrak{F}$ we write $\mathfrak{X} \ll \mathfrak{F}$ rather than $\mathfrak{X}$ $\mathfrak{S}$-$\ll \mathfrak{F}$ and say that $\mathfrak{X}$ is strongly contained in $\mathfrak{F}$.

4.2.12 Lemma ([15], proof of 3.3.1, 3.3.6)

Let $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{F}$ be Fitting classes, and let G be a group.

(a) *If $N \trianglelefteq G$ and $V \cap N$ p-ne N, then $V \cap N$ p-ne G where $V \in Inj_{\mathfrak{X}}(V)$ and $p \in \mathbb{P}$.*

(b) *If $V \in Inj_{\mathfrak{Y}}(G)$ and $W \in Inj_{\mathfrak{X}}(V)$ such that V p-ne G and W p-ne V, then W p-ne G.*
In particular, if $\mathfrak{X} \ll \mathfrak{Y}$ and $\mathfrak{X}$ ne $\mathfrak{Y}$ ne $\mathfrak{F}$, then $\mathfrak{X}$ ne $\mathfrak{F}$.

Consequently,

4.2.13 Remark

(a) *A Fitting class $\mathfrak{X}$ is normally embedded in $\mathfrak{S}$ provided that it is normally embedded in $\mathfrak{S}_{\pi(\mathfrak{X})}$.*

(b) *Let $\mathfrak{F}$, $\mathfrak{X}$ and $\mathfrak{Y}$ be non-trivial Fitting classes such that $\pi(\mathfrak{X}) \cap \pi(\mathfrak{Y}) = \emptyset$. If $\mathfrak{X}$ is normally embedded in $\mathfrak{F}$, then $\mathfrak{X}$ is normally embedded in $\mathfrak{F}\mathfrak{Y}$.*
In particular, $\mathfrak{X}$ is normally embedded in $\mathfrak{X}\mathfrak{S}_{\pi(\mathfrak{X})'}$.

Using 4.2.12, we now obtain analogously to 3.1.4

4.2.14 Proposition
Let $\mathfrak{X}$ and $\mathfrak{F}$ be Fitting classes, $\mathfrak{X} \subseteq \mathfrak{F}$. Then the following statements are equivalent:

(i) *$\mathfrak{X}$ is normally embedded in $\mathfrak{F}$.*

(ii) *$\mathfrak{X}$ is normally embedded in $\mathfrak{F}^*$.*

(iii) *$\mathfrak{X}^*$ is normally embedded in $\mathfrak{F}^*$.*

(In case $\mathfrak{F} = \mathfrak{S}$ this has already been proved by Doerk and Porta, cf. [9, X, 1.38].)

4.2.15 Definition
Let $\mathfrak{X}$ be a Fitting class. We define

$$\mathrm{Y}_{\mathrm{ne}}(\mathfrak{X}) = (G \mid \text{If } V \in \mathrm{Inj}_{\mathfrak{X}}(G),\ \text{then } V \text{ ne } G).$$

Obviously, the class $\mathrm{Y}_{\mathrm{ne}}(\mathfrak{X})$ is closed under taking subnormal subgroups, and – provided that $\mathfrak{X} = \mathfrak{X}^*$ – under forming direct products as well.
In general, $\mathrm{Y}_{\mathrm{ne}}(\mathfrak{X})$ is not closed under forming normal products.

4.2.16 Remark
$\mathfrak{X} = \mathfrak{Z}^3 = (G \mid Soc_3(G) \leq Z(G))$ is a Fitting class such that $Y_{ne}(\mathfrak{X}) \neq \mathrm{N}_0 Y_{ne}(\mathfrak{X})$.

Proof: Assume to the contrary that $\mathrm{Y}_{\mathrm{ne}}(\mathfrak{X}) = \mathrm{N}_0\mathrm{Y}_{\mathrm{ne}}(\mathfrak{X})$; then 4.2.14 yields $\mathrm{Y}_{\mathrm{ne}}(\mathfrak{X}) = \mathrm{Y}_{\mathrm{ne}}(\mathfrak{X})^*$. We prove that this implies $G \wr Z_p \in \mathrm{Y}_{\mathrm{ne}}(\mathfrak{X})$ for every $G \in \mathrm{Y}_{\mathrm{ne}}(\mathfrak{X})$ and every prime p, and, consequently, by 1.2.29, $\mathrm{Y}_{\mathrm{ne}}(\mathfrak{X}) = \mathfrak{S}$, a contradiction.

By a result proved independly by Lockett and Frantz (cf. [9, IX, 4.19]), the radicals and injectors of $\mathfrak{X}$ are known: $G_{\mathfrak{X}} = C_G(\mathrm{Soc}_3(G))$ and $\mathrm{Inj}_{\mathfrak{X}}(G) = \{C_G(C_{\mathrm{Soc}_3(G_{\mathfrak{X}})}(G_3)) \mid G_3 \in \mathrm{Syl}_3(G)\}$ for every group G.
Let G be a group contained in $\mathrm{Y}_{\mathrm{ne}}(\mathfrak{X})$, and let p be a prime; then $G \wr Z_p \in \mathrm{Y}_{\mathrm{n}}(\mathfrak{X})$:

$p = 3$: Let V be an $\mathfrak{X}$-injector of $G \wr Z_p$. If $V \leq G^*$, there is nothing to prove. Thus we may assume that $V \not\leq G^*$. Then 1.2.28 yields $V \cong F^* Z_p$ for a suitable $F \in \mathrm{Inj}_{\mathfrak{X}}(G)$. If $q \neq p$, then evidently V q-ne $G \wr Z_p$. Since according to the above mentioned description of V, a Sylow 3-subgroup of V is a Sylow 3-subgroup of $G \wr Z_p$ as well, we obtain that V 3-ne $G \wr Z_p$.

$p \neq 3$: Put $H := G \wr Z_p$ and assume that $H \notin \mathrm{Y}_{\mathrm{ne}}(\mathfrak{X})$. Let $V \in \mathrm{Inj}_{\mathfrak{X}}(H)$. Since $p \neq 3$, $G_3^* = H_3 \in \mathrm{Syl}_3(H)$ (where $G_3 \in \mathrm{Syl}_3(G)$). Since $G \notin \mathfrak{X}$, it follows from 1.2.24 that $H_{\mathfrak{X}} = G_{\mathfrak{X}}^* \leq G^*$, and consequently that $G^* \geq \mathrm{Soc}_3(H_{\mathfrak{X}}) \geq (\mathrm{Soc}_3(G_{\mathfrak{X}}))^*$. If $V = C_H(C_{\mathrm{Soc}_3(H_{\mathfrak{X}})}(G_3^*)) \not\leq G^*$, then there exists an element $(x_1, \ldots, x_p; z) \in C_H(C_{\mathrm{Soc}_3(H_{\mathfrak{X}})}(G_3^*)) \leq C_H(C_{(\mathrm{Soc}_3(G_{\mathfrak{X}}))^*}(G_3^*)) = C_H((C_{\mathrm{Soc}_3(G_{\mathfrak{X}})}(G_3))^*)$ such that $z \neq 1$. By construction of the regular wreath product this implies $C_{\mathrm{Soc}_3(G_{\mathfrak{X}})}(G_3) = 1$, a contradiction.
Hence we obtain $V \leq G^*$, and consequently V ne H; this final contradiction completes the proof.

□

It is an open question, whether or not $\mathrm{Y}_{\mathrm{ne}}(\mathfrak{X}) = \mathrm{N}_0\mathrm{Y}_{\mathrm{ne}}(\mathfrak{X})$ implies that $\mathfrak{X}$ is normally embedded in $\mathfrak{S}$ in case that $\mathfrak{X}$ is an arbitrary Lockett class.

4.2.17 Remark

(a) *For this relation, too, it is an open problem whether or not there exists a unique maximal Fitting class contained in $Y_{ne}(\mathfrak{X})$.*
The special case of considering local normal embedding between SFitting classes only, leads to $\mathfrak{S}$-normally embedded Fitting classes, and consequently to the investigations of Lockett and Doerk and Porta.

(b) *Whether or not the intersection of $\mathfrak{F}$-normally embedded Fitting classes is still $\mathfrak{F}$-normally embedded – provided that it is non-trivial – is an open question as well; it is open even for the case $\mathfrak{F} = \mathfrak{S}$.*

Locally permutable Fitting classes

Like normal embedding, local permutability is a property considerably weaker than local normality/submodularity. However, we will see that this property, too, is an invariant of Lockett sections.

4.2.18 Definition
Let G be a group and U a subgroup of G.
Let Σ denote a Hall system of G. Then U is called Σ-permutable if

$$UH = HU$$

for all $H \in \Sigma$.
We say that U is system permutable if there exists a Hall system Σ of G such that U is Σ-permutable.

To obtain Σ-permutability of a subgroup U of a group G, it is sufficient to require that U permutes with the corresponding Sylow basis.

4.2.19 Proposition ([9], I, 4.26)
Let Σ be a Hall system of a group G with corresponding Sylow basis B. Then a subgroup U is Σ-permutable if and only if $UH = HU$ for every $H \in B$.

Obviously, each normal subgroup of a group G is system permutable in G. According to [9, I, 7.10], each normally embedded subgroup, too, is a system permutable subgroup of G.

4.2.20 Proposition ([9], I, 4.25, 4.29)
Let Σ be a Hall system of a group G, and let U and V be Σ-permutable subgroups of G.

(a) *Σ reduces into U.*

(b) *For all $K \trianglelefteq G$, the quotient group UK/K is a $\Sigma K/K$-permutable subgroup of G/K.*

(c) *If N denotes a normal subgroup of G containing U, then U is a $\Sigma \cap N$-permutable subgroup of N.*

(d) *$U \cap V$ and $\langle U, V\rangle$ are Σ-permutable subgroups of G.*

In particular, if $K \trianglelefteq G$, then $U \cap K$ and UK are Σ-permutable subgroups of G.

4.2.21 Definition

(a) Let $\mathfrak{X}$ and $\mathfrak{F}$ be non-trivial Fitting classes such that $\mathfrak{X} \subseteq \mathfrak{F}$. Then $\mathfrak{X}$ is said to be permutable in $\mathfrak{F}$ if an $\mathfrak{X}$-injector of G is a system permutable subgroup of G for all $G \in \mathfrak{F}$.

If $\mathfrak{X}$ is permutable in $\mathfrak{F}$, we also refer to $\mathfrak{X}$ as being $\mathfrak{F}$-permutable.

(b) Let $\mathfrak{X}$ be a Fitting class. We define

$$\mathrm{Y_p}(\mathfrak{X}) = (G \mid \text{If } V \in \mathrm{Inj}_{\mathfrak{X}}(G), \text{ then } V \text{ is system permutable in } G).$$

According to [9, I, 7.10], a normally embedded subgroup of a group G is system permutable in G. In particular, every Fischer class – and consequently every SFitting class – is an $\mathfrak{S}$-permutable Fitting class. In 1972, Dark published an example of a Fitting class which is not permutable in $\mathfrak{S}$ (cf. [9, IX, 5.19]).

Evidently,

4.2.22 Remark

(a) *Let $\mathfrak{X}$, $\mathfrak{F}$ and $\mathfrak{Y}$ be non-trivial Fitting classes such that $\mathfrak{X}$ is permutable in $\mathfrak{F}$ and $\mathfrak{X} \subseteq \mathfrak{Y} \subseteq \mathfrak{F}$. Then $\mathfrak{X}$ is permutable in $\mathfrak{Y}$.*

(b) *Let $\mathfrak{X}$ and $\mathfrak{F}$ be non-trivial Fitting classes such that $\mathfrak{X}$ is permutable in $\mathfrak{F}$. Further, let $\mathfrak{Y}$ be an SFitting class contained in $\mathfrak{F}$. Then $\mathfrak{X} \cap \mathfrak{Y}$ is permutable in $\mathfrak{Y}$.*

The following lemma is due to Lockett.

4.2.23 Lemma ([9], IX, 3.18)

Let $\mathfrak{X}$ be a Fitting class, and Σ be a Hall system of a group G which reduces into an $\mathfrak{X}$-injector V of G. Further, let K be a normal subgroup of G with $G/K \in \mathfrak{S}_\pi$ (where π is a set of primes), and let $H \in \Sigma \cap \mathrm{Hall}_\pi(G)$. Then the following statements are equivalent:

(i) $VH = HV$.

(ii) $(V \cap K)(H \cap K) = (H \cap K)(V \cap K)$.

In particular, 4.2.23 yields

4.2.24 Corollary
Let $\mathfrak{X}$ be a Fitting class, and Σ be a Hall system of a group G which reduces into an $\mathfrak{X}$-injector V of G. Further, let K be a normal subgroup of G with $G/K \in \mathfrak{S}_p$ (where p is a prime).
Then $V \cap K$ is Σ-permutable in G provided that $V \cap K$ is Σ-permutable in K.

Proof: It is sufficient to show that $Q(V \cap K) = (V \cap K)Q$ where $Q \in \mathrm{Syl}_q(G) \cap \Sigma$ and q is a prime. If $q \neq p$, there is nothing to prove. If $q = p$, then 4.2.23 implies that $VQ = QV$. Since $V = V_q(V \cap K)$ and $\Sigma \searrow V$, this completes the proof ($V_q \in \mathrm{Syl}_q(V)$). $\square$

Let $\mathfrak{F}$ be a Fitting class. Analogously to a result of Lockett (cf. [9, IX, 3.19]) it can be proved that $\mathfrak{F}$-permutable Fitting classes are precisely those Fitting classes $\mathfrak{X} \subseteq \mathfrak{F}$ such that for each group $G \in \mathfrak{F}$ an $\mathfrak{X}$-injector V of G is "controlled" by a system normalizer, i.e. $V \leq N_G(\Sigma)(V \cap G^{\mathfrak{N}})$ (where Σ denotes a Hall system of G such that $\Sigma \searrow V$).

As mentioned for locally normal embedded Fitting classes, the fact that a Fitting class $\mathfrak{X}$ is permutable in $\mathfrak{X}\mathfrak{N}^2$ does not imply the $\mathfrak{S}$-normality of $\mathfrak{X}$. To the contrary:

4.2.25 Remark
Each non-trivial Fitting class $\mathfrak{X}$ is permutable in $\mathfrak{X}\mathfrak{N}^2$.

Proof: Let G be a group contained in $\mathfrak{X}\mathfrak{N}^2$, let $V \in \mathrm{Inj}_{\mathfrak{X}}(G)$, and set $N = G_{\mathfrak{X}\mathfrak{N}}$. Then $V \cap N = N_{\mathfrak{X}}$ and 4.2.6 is applicable. Consequently, $V \leq D(N \cap V)$ and hence $V = (N \cap V)(D \cap V)$ (where $D = N_G(\Sigma)$ and Σ denotes a Hall system of G which reduces into V). This implies the assertion, since $D \leq N_G(H)$ for every $H \in \Sigma$ and $N \cap V$ is a normal subgroup of G. $\square$

An elementary but useful consequence of 1.2.14 is

4.2.26 Lemma
Let $\mathfrak{X}$ be a Fitting class and let $\mathfrak{H}$ be an arbitrary class of groups. Then $\mathfrak{H}$ is contained in $Y_p(\mathfrak{X})$ if and only if $\mathfrak{X}$ is $\mathfrak{H}$-$\ll\mathfrak{L}_\pi(\mathfrak{X})$ for every $\pi \subseteq \mathbb{P}$.

Our next aim is to prove that also this relation is a relation of the corresponding Lockett sections. Keeping 4.2.26 in mind, we obtain analogously to a result of Doerk and Porta (cf. [9, X, 1.39]):

4.2.27 Proposition
$Y_p(\mathfrak{X}^*) \subseteq Y_p(\mathfrak{X})$ where $\mathfrak{X}$ denotes an arbitrary Fitting class.

Since system permutability is an invariant of epimorphisms, we now obtain that local permutability, too, is a property of the corresponding Lockett sections (compare with 3.1.4).

4.2.28 Proposition
Let $\mathfrak{X}$ and $\mathfrak{F}$ be Fitting classes, $\mathfrak{X} \subseteq \mathfrak{F}$. Then the following statements are equivalent:

(i) *$\mathfrak{X}$ is permutable in $\mathfrak{F}$.*

(ii) *$\mathfrak{X}$ is permutable in $\mathfrak{F}^*$.*

(iii) *$\mathfrak{X}^*$ is permutable in $\mathfrak{F}^*$.*

In particular: If $Y_p(\mathfrak{X}) = {\rm N}_0 Y_p(\mathfrak{X})$, then $Y_p(\mathfrak{X}) = Y_p(\mathfrak{X})^* = Y_p(\mathfrak{X}^*)$.

According to 4.2.20, the class $\mathrm{Y_p}(\mathfrak{X})$ is closed under taking subnormal subgroups, and, evidently, $\mathrm{Y_p}(\mathfrak{X})$ is closed under forming direct products provided that $\mathfrak{X} = \mathfrak{X}^*$. In general, $\mathrm{Y_p}(\mathfrak{X})$ fails to be ${\rm N}_0$-closed:

4.2.29 Proposition
Let $\mathfrak{X}$ be a Lockett class, let $G \in Y_p(\mathfrak{X})$ and let p be a prime. Then

$$G \wr Z_p \in Y_p(\mathfrak{X}).$$

In particular, $Y_p(\mathfrak{X}) = {\rm N}_0 Y_p(\mathfrak{X})$ if and only if $Y_p(\mathfrak{X}) = \mathfrak{S}$.

Proof: Suppose that the first assertion holds true. Then, by 4.2.28, $\mathrm{Y_p}(\mathfrak{X}) = \mathrm{Y_p}(\mathfrak{X})^* = \mathrm{Y_p}(\mathfrak{X}^*)$. Hence we may assume that $\mathfrak{X} = \mathfrak{X}^*$. 1.2.29 implies that $\mathrm{Y_p}(\mathfrak{X}) = \mathfrak{S}$, and the additional remark is valid as well.
To prove the first assertion we put $H = G \wr Z_p$. Let F be an $\mathfrak{X}$-injector of H. If $F \leq G^*$, then it follows from 4.2.24 that F is system permutable in H, and we are finished. Thus $F \not\leq G^*$, and according to 1.2.28 we may assume that $F = V^* Z_p$ (where $V \in \mathrm{Inj}_{\mathfrak{X}}(G)$). By assumption, V^* is a system permutable subgroup of G^*. Let $\Sigma = \{G^*_\pi \mid \pi \subseteq \pi(G)\}$ denote

a corresponding Hall system of G^*. Then by construction of the regular wreath product, $Z_p \leq N_H(U^*)$ for every subgroup U of G. Consequently, F permutes with $G^*_\pi Z_p \in \mathrm{Hall}_\pi(H)$ where π is a set of primes containing p. If $\pi \subseteq \mathbb{P} \setminus \{p\}$, then $G^*_\pi \in \mathrm{Hall}_\pi(H)$, and F permutes with G^*_π. Observe further that $\Sigma_0 := \{G^*_\pi Z_p \mid \pi \subseteq \mathbb{P},\ p \in \pi\} \cup \{G^*_\pi \mid \pi \subseteq \mathbb{P},\ p \notin \pi\}$ forms a Hall system of H; hence the proof is complete. □

4.2.30 Remark
Let $\mathfrak{X}$ and $\mathfrak{F}$ be non-trivial Fitting classes.

(a) *Also for local permutability, it is an open question whether or not there exists a unique maximal Fitting class contained in $\mathrm{Y}_p(\mathfrak{X})$.*

(b) *The intersection of $\mathfrak{F}$-permutable Fitting classes is, in general, not permutable in $\mathfrak{F}$, not even in case that $\mathfrak{F} = \mathfrak{S}$, cf. [9, IX, 3.14].*

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Zusammenfassung in deutscher Sprache

Einer der wichtigsten Sätze in der Theorie der endlichen Gruppen ist der Satz von Sylow (1872), der im Universum der endlichen auflösbaren Gruppen auf verschiedene Weisen verallgemeinert wurde. Ein Prototyp dafür ist Halls Satz von 1928, der die Aussagen des Satzes von Sylow für endliche auflösbare Gruppen von p-Gruppen auf π-Gruppen erweitert, wobei π eine beliebige Primzahlmenge sei. In jeder endlichen auflösbaren Gruppe G existiert also genau eine Konjugiertenklasse sogenannter π-Hallgruppen von G, maximaler π-Untergruppen von G, deren Ordnung gerade der π-Teil der Ordnung von G ist. Wie 1937 von Hall gezeigt wurde, sind endliche auflösbare Gruppen durch die Existenz von π-Hallgruppen für jede Primzahlmenge π bereits ausgezeichnet. Ist G eine endliche Gruppe und π eine Primzahlmenge, so ist leicht zu sehen, daß eine π-Hallgruppe H von G folgende Eigenschaften besitzt: (a) HN/N ist eine π-Hallgruppe von G/N für jeden Normalteiler N von G; (b) $H \cap N$ ist eine π-Hallgruppe von N für jeden Subnormalteiler N von G. Insbesondere sind π-Hallgruppen von G durch jede dieser Eigenschaften charakterisiert. Es ist nun naheliegend, zu fragen, ob entsprechende Aussagen auch für andere gruppentheoretische Eigenschaften gelten, und falls ja, wodurch sich diese auszeichnen. Wir fassen dazu zunächst alle endlichen Gruppen mit einer gegebenen gruppentheoretischen Eigenschaft in einer (unter Isomorphismen abgeschlossenen) Klasse $\mathfrak{F}$ zusammen, und nennen eine Untergruppe U einer Gruppe G eine $\mathfrak{F}$-maximale Untergruppe von G, falls U unter allen in $\mathfrak{F}$ liegenden Untergruppen von G maximal ist (eine π-Hallgruppe einer endlichen auflösbaren Gruppe G ist also eine $\mathfrak{S}_\pi$-maximale Untergruppe von G, wobei $\mathfrak{S}_\pi$ die Klasse aller endlichen auflösbaren π-Gruppen bezeichne). Es ist nicht schwer zu sehen, daß es nicht möglich ist, den Satz von Sylow in voller Stärke auf andere als die Gruppenklassen $\mathfrak{S}_\pi$ zu verallgemeinern. Versuche, schwächere Aussagen von dieser Form im Universum der endlichen auflösbaren Gruppen zu erhalten, also Fragen nach der Existenz und Konjugiertheit von $\mathfrak{F}$-maximalen Untergruppen in jeder endlichen auflösbaren Gruppe, die entweder eine (a) entsprechende oder eine (b) entsprechende Eigenschaft besitzen, führten zur Theorie der Schunck- und Fittingklassen. In der vorliegenden Arbeit beschäftigen wir uns mit Fittingklassen, also mit Gruppenklassen, die bezüglich der Bildung von Normalteilern und normaler Produkte abgeschlossen sind. (Fittingklassen sind nach H. Fitting benannt, der 1938 zeigte, daß die Klasse aller endlichen nilpotenten Gruppen bezüglich der Bildung normaler Produkte abgeschlossen ist; offensichtlich ist diese Klasse auch bzgl. der Bildung von Subnormalteilern abgeschlossen.) Wie 1967 von Fischer, Gaschütz und Hartley bewiesen wurde, sind Fittingklassen $\mathfrak{F}$ endlicher auflösbarer Gruppen dadurch ausgezeichnet, daß in jeder endlichen auflösbaren Gruppe G genau eine Konjugiertenklasse sogenannter $\mathfrak{F}$-Injektoren existiert, Untergruppen U von G derart, daß für jeden Subnormalteiler N von G die Untergruppe $F \cap N$ eine $\mathfrak{F}$-maximale Untergruppe von N ist. Da eine solche Aussage für beliebige endliche Gruppen im allgemeinen falsch ist, werden wir uns im folgenden auf das Universum der endlichen auflösbaren Gruppen beschränken; jede hier betrachtete Gruppe sei also endlich und auflösbar und jede Gruppenklasse in der Klasse $\mathfrak{S}$ aller endlichen und auflösbaren Gruppen enthalten.

Bei der Untersuchung von Fittingklassen liegt es nahe, sich zunächst auf solche mit gewissen Zusatzeigenschaften zu beschränken, was unter anderem von Blessenohl und Gaschütz (1970), Lockett (1971), Doerk und Porta (1980) und Hauck und Kienzle (1987) getan wurde, die Fittingklassen untersuchten, deren Injektoren in jeder Gruppe $G \in \mathfrak{S}$ gewissen Einbettungskriterien genügen. In der vorliegenden Arbeit werden diese Untersuchungen verallgemeinert. Wir betrachten nicht-triviale Fittingklassen $\mathfrak{X}$ und $\mathfrak{F}$, so daß $\mathfrak{X}$ in $\mathfrak{F}$ enthalten ist und daß für jede Gruppe $G \in \mathfrak{F}$ die $\mathfrak{X}$-Injektoren von G einem gegebenen Einbettungskriterium e genügen. In diesem Fall nennen wir $\mathfrak{X}$ eine $\mathfrak{F}_e$-Klasse. Wir untersuchen also Einbettungseigenschaften von $\mathfrak{X}$-Injektoren "lokal" in $\mathfrak{F}$, wobei der globale Fall $\mathfrak{F} = \mathfrak{S}$ sei.
Wir werden dabei Fragen zu folgenden Einbettungseigenschaften behandeln:

Normalität

(Sub)Modularität

Normale Einbettung

Systemvertauschbarkeit

Bei der Untersuchung obiger Relationen konzentrieren wir uns auf folgende Fragestellungen:

(1) Existiert für jede nicht-triviale Fittingklasse $\mathfrak{X}$ eine (eindeutig bestimmte) größte Fittingklasse $\mathfrak{F}$, so daß $\mathfrak{X}$ eine $\mathfrak{F}_e$-Klasse ist?

(2) Für welche Fittingklassen $\mathfrak{F}$ existiert umgekehrt eine (eindeutig bestimmte) kleinste $\mathfrak{F}_e$-Klasse?

Bei der Untersuchung der ersten Fragestellung ist es dabei naheliegend, die Klasse $\mathrm{Y}_e(\mathfrak{X})$ all derjenigen Gruppen G zu betrachten, in denen die $\mathfrak{X}$-Injektoren von G dem Einbettungskriterium e genügen. Bedauerlicherweise bildet diese Klasse bei allen oben aufgezählten Einbettungseigenschaften im allgemeinen keine Fittingklasse. Um dennoch zu Aussagen über die Existenz einer größten in $\mathrm{Y}_e(\mathfrak{X})$ enthaltenen Fittingklasse zu gelangen, wäre es hilfreich, mehr über das Fittingklassenerzeugnis beliebiger Klassen – die kleinste Fittingklasse, die eine gegebene Gruppenklasse enthält – zu wissen. Leider ist dieses im allgemeinen nur sehr schwer zugänglich, so ist z.B. die Fittingklasse, die von der symmetrischen Gruppe auf drei Elementen – der kleinsten in diesem Zusammenhang nicht-trivialen Gruppe – erzeugt wird, trotz intensiver Bemühungen noch nicht explizit beschreibbar. Beschränkt man sich auf die Untersuchung untergruppenabgeschlossener Fittingklassen (im folgenden SFittingklassen genannt), so sind jedoch starke Aussagen möglich. Dies liegt im wesentlichen daran, daß der Untergruppenabschluss einer Fittingklasse bereits eine Reihe von weiteren Abschlüssen erzwingt (Bryce und Cossey, 1972, 1982), und es folglich bei der Untersuchung von SFittingklassen möglich ist, neben der Theorie der Fittingklassen auch die der (lokal erklärten) Formationen zu verwenden (zu Definition und Eigenschaften derselben vgl. 1.3). Aus diesem Grund sind bei der Betrachtung obiger Relationen zwischen SFittingklassen auch deutlich stärkere Aussagen zu erwarten als für beliebige Fittingklassen.

Wir beginnen diese Arbeit mit einem in die Theorie der Fittingklassen und (lokal erklärten) Formationen einführenden Kapitel, in dem Definitionen und grundlegende Resultate zur Verfügung gestellt werden. Hier findet sich auch die Definition der zu einer Fittingklasse $\mathfrak{F}$ assoziierten Klasse $\mathfrak{F}^*$, der kleinsten $\mathfrak{F}$ enthaltenden Fittingklasse, deren Radikale sich direkten Produkten anpassen, und des Lockettabschnittes zu $\mathfrak{F}$, der Gesamtheit aller Fittingklassen $\mathfrak{Y}$ mit $\mathfrak{Y}^* = \mathfrak{F}^*$. (Dabei ist das $\mathfrak{X}$-Radikal einer Gruppe G der eindeutig bestimmte größte in einer gegebenen Fittingklasse $\mathfrak{X}$ enthaltene Normalteiler von G; dieser existiert nach der Definition von Fittingklassen.) Fällt $\mathfrak{F}$ mit $\mathfrak{F}^*$ zusammen, so wird $\mathfrak{F}$ Lockettklasse genannt.
In diesem Kapitel findet sich auch die Definition von lokal erklärten Formationen, das heißt von Gruppenklassen, die durch eine sogenannte lokale Erklärung gegeben sind (vgl. 1.3). Ist $\mathfrak{F}$ eine lokal erklärte Formation, so ist eine lokale Erklärung von $\mathfrak{F}$ im allgemeinen nicht eindeutig bestimmt, es existiert jedoch genau eine, die voll und inklusiv ist (vgl. 1.3), die sogenannte kanonische lokale Erklärung von $\mathfrak{F}$. Kanonisch deshalb, da sich viele Eigenschaften der Klasse auf sie übertragen lassen. Es wird sich herausstellen, daß auch obige Relationen zwischen SFittingklassen häufig bereits in den zugehörigen kanonischen lokalen Erklärungen widergespiegelt werden (und umgekehrt).

Das zweite Kapitel ist der Untersuchung des SFittingklassenerzeugnisses – der kleinsten SFittingklasse, die eine gegebene Klasse enthält – sowie des Verbandes der SFittingklassen gewidmet. Wie bereits geschildert, sind die hier erzielten Resultate hilfreich bei der Untersuchung obiger Relationen zwischen SFittingklassen. Sie beanspruchen aber auch für sich allein genommen ein gewisses Interesse. Mit Hilfe der Theorie der lokal erklärten Formationen werden wir hier unter anderem zeigen, daß das SFittingklassenerzeugnis beliebig vieler SFittingklassen verträglich ist bezüglich gewissen Erweiterungen sowie bezüglich der Schnittbildung. Letzteres bedeutet insbesondere, daß die Gesamtheit aller zwischen SFittingklassen $\mathfrak{X}$ und $\mathfrak{F}$, $\mathfrak{X} \subseteq \mathfrak{F}$, liegenden SFittingklassen einen distributiven Verband bilden – ein Faktum, das bereits von Shemetkov und Skiba (1989) gezeigt wurde. Es wird sich weiter herausstellen, daß dieser Verband auch atomar ist, also stets minimale Elemente existieren, und die Atome insofern explizit beschreibbar sind, als daß sie erzeugende SFittingklassen angegeben werden können.

Im dritten Kapitel werden wir uns mit lokal normalen Fittingklassen beschäftigen, also mit nicht-trivialen Fittingklassen $\mathfrak{X}$ und $\mathfrak{F}$ derart, daß $\mathfrak{X}$ in $\mathfrak{F}$ enthalten ist und die $\mathfrak{X}$-Injektoren in jeder Gruppe $G \in \mathfrak{F}$ normal liegen, also mit dem $\mathfrak{X}$-Radikal von G zusammenfallen. Wir unterteilen dieses Kapitel in zwei Abschnitte. Im ersten Teil werden wir zunächst Grundlagen über lokal normale Fittingklassen aufführen (ein wesentlicher Teil davon geht auf Hauck (1977) zurück), und anschließend oben aufgeführte Fragen (1) und (2) für beliebige Fittingklassen diskutieren. Wie aus der Literatur bekannt ist, kann man sich bei der Betrachtung dieser Relation auf den Fall zurückziehen, daß beide Klassen Lockettklassen sind (wir werden einen weiteren Beweis für diese Aussage angeben, der sich leicht auch auf andere Einbettungseigenschaften übertragen läßt). Dieses Resultat ist insofern erfreulich als daß sich Lockettklassen im allgemeinen wesentlich leichter behandeln lassen als beliebige Fittingklassen. Dennoch ist Problem (1) in diesem allgemeinen Rahmen kaum zu bearbeiten, da das Fittingklassenerzeugnis nur sehr schwer greifbar ist und damit auch die Frage, ob mit in $\mathrm{Y_n}(\mathfrak{X})$ enthaltenen Fittingklassen $\mathfrak{F}_1$, $\mathfrak{F}_2$ auch die von diesen erzeugte Fittingklasse in $\mathrm{Y_n}(\mathfrak{X})$ liegt, nur sehr schwer zu beantworten ist. (Hierbei bezeichne $\mathrm{Y_n}(\mathfrak{X})$ für eine Fittingklasse $\mathfrak{X}$ die Klasse all derjenigen Gruppen G, in denen das $\mathfrak{X}$-Radikal bereits $\mathfrak{X}$-maximal in G ist.) Wir werden jedoch Bedingungen angeben, unter denen die von $\mathfrak{F}_1$ und $\mathfrak{F}_2$ erzeugte Fittingklasse wieder in $\mathrm{Y_n}(\mathfrak{X})$ liegt.
Auch die dazu duale Frage (2) ist i.a. weiter offen. Es ist offensichtlich, daß es Fittingklassen $\mathfrak{F}$ gibt, für die keine kleinste in $\mathfrak{F}$ normale Fittingklasse existiert (dies ist z.B. bei der Klasse aller nilpotenten Gruppen der Fall). Es scheint aber schwieriger zu sein, diejenigen Klassen zu charakterisieren, die eine solche besitzen. Wir werden obige Frage jedoch für eine Reihe wichtiger Fittingklassen $\mathfrak{F}$ positiv beantworten und dabei die kleinste $\mathfrak{F}$-normale Fittingklasse auch explizit beschreiben.
Im zweiten Abschnitt wird lokale Normalität zwischen SFittingklassen untersucht. Wie bereits erwähnt, ist hier eine wesentlich stärkere Theorie einsetzbar als bei beliebigen Fittingklassen, was zu deutlich befriedigenderen Antworten auf obige Fragen führt. Das liegt i.w. daran, daß sich – unter einer weiteren schwachen Voraussetzung an die Charakteristiken von $\mathfrak{X}$ und $\mathfrak{F}$ (also an die Menge aller Primzahlen p, für die die zyklische Gruppe der Ordnung p in der jeweiligen Klasse liegt) – zeigen läßt, daß $\mathfrak{X}$ genau dann normal in $\mathfrak{F}$ liegt, wenn die zugehörigen kanonischen lokalen Erklärungen ineinander normal liegen für jede Primzahl p.

So werden wir sehen, daß für jede SFittingklasse $\mathfrak{X}$ eine größte in $\mathrm{Y}_{\mathrm{n}}(\mathfrak{X})$ enthaltene SFittingklasse existiert, welche $\mathfrak{X}$ zudem eindeutig bestimmt. In vielen Fällen läßt sich diese Klasse auch explizit beschreiben. Weiter bildet auch die Gesamtheit aller SFittingklassen, in denen eine gegebene SFittingklasse normal liegt, einen vollständigen, distributiven und atomaren Verband. Auch hier sind die Atome insoweit beschreibbar, als daß sie erzeugende Fittingklassen explizit angegeben werden können.
Bei dem dazu dualen Problem, der Frage nach der Existenz einer kleinsten in einer SFittingklasse $\mathfrak{F}$ normalen SFittingklasse, sind ebenfalls befriedigende Aussagen möglich, auch wenn Frage (2) i.a. offen bleibt. Existiert jedoch für eine SFittingklasse $\mathfrak{F}$ eine eindeutig bestimmte minimale in $\mathfrak{F}$ normale Fittingklasse, so bildet auch die Gesamtheit aller SFittingklassen, die in $\mathfrak{F}$ normal liegen, einen vollständigen distributiven und unter gewissen Umständen auch dual atomaren Verband.

Die verbleibenden oben angegebenen Einbettungseigenschaften werden im vierten und letzten Kapitel behandelt.
Wir beginnen mit der Untersuchung lokal (sub)modularer Fittingklassen, also nicht-trivialer Fittingklassen $\mathfrak{X}$ und $\mathfrak{F}$, derart daß $\mathfrak{X}$ in $\mathfrak{F}$ enthalten ist und daß für alle Gruppen $G \in \mathfrak{F}$ die $\mathfrak{X}$-Injektoren von G (sub)modulare Untergruppen von G sind (zur Definition von (sub)modularen Untergruppen siehe 4.1). Als eines der ersten Ergebnisse zeigt sich bei der Betrachtung lokal modularer Fittingklassen, daß die Klasse all derjenigen Gruppen, in denen die $\mathfrak{X}$-Injektoren modular liegen, nicht abgeschlossen ist bezüglich der Bildung direkter Produkte. Als unmittelbare Folgerung daraus erhalten wir, daß der Begriff der lokal modularen Fittingklassen bereits mit dem der lokal normalen Fittingklassen übereinstimmt, ein Faktum, das für $\mathfrak{F} = \mathfrak{S}$ bereits von Hauck und Kienzle (1987) gezeigt wurde. Um zu einer neuen Relation zwischen Fittingklassen zu gelangen, muß also eine etwas schwächere Einbettungseigenschaft gefordert werden – die der Submodularität. Hier läßt sich zeigen, daß Fittingklassen $\mathfrak{X}$ und $\mathfrak{F}$ existieren, so daß $\mathfrak{X}$ submodular, aber nicht normal in $\mathfrak{F}$ ist. Wir werden sehen, daß auch lokale Submodularität eine Eigenschaft der zugehörigen Lockettabschnitte ist , daß man sich also bei der Untersuchung dieser Relation ebenso auf den Fall zurückziehen kann, daß beide Klassen Lockettklassen sind. Auch hier stehen die Fragen nach der Existenz einer größten in $\mathrm{Y}_{\mathrm{smod}}(\mathfrak{X})$ enthaltenen Fittingklasse und nach der Existenz einer

kleinsten in $\mathfrak{F}$ submodularen Fittingklasse im Mittelpunkt, wobei ähnliche Probleme wie bei lokaler Normalität auftreten. Dennoch werden wir für einige spezielle Fittingklassen $\mathfrak{F}$ letztere Frage positiv beantworten und die entsprechende Klasse auch explizit beschreiben. Es stellt sich dabei heraus, daß sie in allen diesen Fällen mit der kleinsten $\mathfrak{F}$-normalen Fittingklasse zusammenfällt.
Daß das Konzept lokal submodularer Fittingklassen sehr eng mit dem lokaler Normalität zusammenhängt, zeigt sich sowohl darin, daß beide Konzepte für $\mathfrak{F} = \mathfrak{S}$ übereinstimmen (Hauck, Kienzle, 1987), als auch in der Tatsache, daß diese Relationen für SFittingklassen zusammenfallen, daß also eine SFittingklasse $\mathfrak{X}$ genau dann submodular in einer SFittingklasse $\mathfrak{F}$ ist, wenn $\mathfrak{X}$ bereits normal in $\mathfrak{F}$ liegt. Dies hat zur Folge, daß alle im dritten Kapitel für lokal normale SFittingklassen gezeigten Resultate ihre Gültigkeit behalten.

In den folgenden beiden Abschnitten dieses Kapitels werden wir abschließend lokal normal eingebettete und lokal vertauschbare Fittingklassen betrachten (zur Definition vgl. 4.2). Diese Relationen wurden für $\mathfrak{F} = \mathfrak{S}$ bereits von Lockett (1971) und Doerk und Porta (1980) untersucht. Dabei hat sich herausgestellt, daß der Begriff des starken Enthaltenseins (vgl. 4.2) in diesem Zusammenhang eine wichtige Rolle spielt. Wir werden sehen, daß das unter gewissen Umständen im lokalen Fall ebenfalls richtig ist, womit sich insbesondere zeigen läßt, daß man sich auch bei diesen Einbettungseigenschaften auf die Untersuchung der jeweiligen Lockettklassen zurückziehen kann.
Wie oben erwähnt, ist die Klasse $\mathrm{Y}_e(\mathfrak{X})$ im allgemeinen für keine der in der vorliegenden Arbeit untersuchten Einbettungseigenschaften eine Fittingklasse. Im lokal normalen Fall existieren jedoch eine Reihe von Fittingklassen, für die diese Klasse eine von $\mathfrak{S}$ verschiedene Fittingklasse ist. Wie sich herausstellen wird, ist dies für lokal vertauschbare Fittingklassen ausgeschlossen, das heißt die Klasse all derjenigen Gruppen, in denen die Injektoren systemvertauschbar sind, ist genau dann eine Fittingklasse, wenn sie bereits mit der Klasse aller (endlichen auflösbaren) Gruppen übereinstimmt. Ob dies für lokal normal eingebettete Fittingklassen ebenfalls gilt, ist offen.
Die gesonderte Untersuchung obiger Relationen für SFittingklassen erübrigt sich hier insofern, als daß SFittingklassen bereits normal eingebettet (und damit vertauschbar) in $\mathfrak{S}$ sind, und diese Untersuchung damit mit der von Lockett, Doerk und Porta zusammenfällt.

Hiermit erkläre ich, die Arbeit selbständig verfaßt und keine anderen als die angegeben Quellen und Hilfsmittel benutzt zu haben.