Bi–Continuous Semigroups on Spaces with Two Topologies: Theory and Applications

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Introduction

Problems worthy of attack
prove their worth by hitting back.
P. Hein [1]

Between 1942 and 1950, E. Hille [Hil42], [Hil48], K. Yosida [Yos48] and many others created the theory of strongly continuous semigroups on Banach spaces in order to treat initial value problems for partial differential equations. By now, their theory is well established, and its applications reach well beyond the classical field of partial differential equations.
However, from the very beginning many situations occured in which the corresponding semigroup is not strongly continuous or the underlying space is not a Banach space. In order to deal with such phenomena, already E. Hille and R. S. Phillips [HP57] introduced a whole range of semigroups on Banach spaces having weaker continuity properties. On the other hand, I. Miyadera [Miy59], H. Komatsu [Kom64], T. Kōmura [Kōm68], S. Ōuchi [Ōuc73], K. Yosida [Yos74], and others generalized the theory to strongly continuous semigroups on locally convex spaces. It seems, however, that both theories have found relatively few applications.
In contrast and motivated by concrete applications, many authors considered semigroups on Banach spaces which are strongly continuous for a topology weaker than the norm topology. We mention, e.g., *adjoint semigroups* (e.g., [BR79], [Nee92]) or *implemented semigroups* as occuring in [BR79, Section 3.2]. Motivated by stochastic differential equations on Banach spaces, S. Cerrai [Cer94] introduced *weakly continuous semigroups* which were subsequently applied to transition semigroups like the (infinite dimensional) Ornstein–Uhlenbeck semigroup (see, e.g., [DPZ92]). Finally, we mention work by J. R. Dorroh and J. W. Neuberger (e.g., [DN93], [DN96]) who "linearized" a flow $(\phi_t)_{t\geq 0}$ on a metric space Ω and introduced its *Lie generator* as

[1] Danish poet and scientist (1905–1996)

the generator of a linear operator semigroup on $C_b(\Omega)$ which is strongly continuous with respect to the finest locally convex topology agreeing with the compact–open topology on norm bounded sets.
To treat these semigroups, generation theorems and approximation results have been developed.
The aim of this thesis is to put these individual results into a general framework. To that purpose, we propose the concept of *bi–continuous semigroups* on spaces with two topologies. We show that these semigroups allow, as in the case of C_0–semigroups, a systematic theory including Hille–Yosida and Trotter–Kato type theorems. A long series of applications shows the flexibility and strength of our theory.

In Chapter 1 we consider Banach spaces endowed with an additional locally convex Hausdorff topology τ which is coarser than the norm topology and such that the topological dual $(X,\tau)'$ is *norming* for $(X,\|\cdot\|)$. On such spaces we define *bi–continuous semigroups* $(T(t))_{t\geq 0}$ as semigroups consisting of bounded linear operators which are *locally bi–equicontinuous for* τ (see Definition 1.2) and such that the orbit maps

$$\mathbb{R}_+ \ni t \mapsto T(t)x \in X$$

are τ–continuous. For such a bi–continuous semigroup $(T(t))_{t\geq 0}$ we show the existence of its τ–Laplace transform

$$(1) \qquad R(\lambda)x := \int_0^\infty e^{-\lambda t}T(t)x dt = \|\cdot\| - \lim_{a\to\infty}\left(\tau\text{–}\int_0^a e^{-\lambda t}T(t)xdt\right), \quad x \in X.$$

From $R(\lambda)$ we obtain the *generator* of $(T(t))_{t\geq 0}$ as a Hille–Yosida operator defined on a τ–dense subspace of the Banach space X. Finally, the relation between bi–continuous semigroups, integrated semigroups and Hille–Yosida operators yields a characterization of the generator of a bi–continuous semigroup in form of a generalized Hille–Yosida theorem (Theorem 1.28).
In Chapter 2 we study the convergence of sequences of bi–continuous semigroups. We use our results from Chapter 1 in order to establish approximation theorems of Trotter–Kato type. Based on these results, we then obtain an explicit formula for bi–continuous semigroups in form of a generalization of the Chernoff Product Formula (Proposition 2.9). We use this formula to state the Post–Widder Inversion Formula for bi–continuous semigroups in terms of the powers of the resolvent

of its generator (Corollary 2.10). Finally, we show that under *stability* and *consistency* conditions on two bi–continuous semigroups, the closure of the sum of their generators is a generator and the perturbed semigroup can be represented by the Lie–Trotter Product Formula (Corollary 2.11).
In order to check the applicability of our approach, we discuss in Chapter 3 a series of examples of the previous results.
First, we give a survey on locally equicontinuous semigroups as treated, e.g., in [Kom64], [Kōm68], [Ōuc73], K. Yosida [Yos74], which can be viewed as bi–continuous semigroups in many concrete situations.
In Section 3.2, we reproduce results by J. R. Dorroh and J. W. Neuberger [DN93], [DN96] by verifying that semigroups on $C_b(\Omega)$ which are induced by flows are bi–continuous for the topology of compact convergence. In particular, we give a simplified proof for their generation theorem for such semigroups on $C_b(\Omega)$ and give conditions implying the Lie–Trotter Product Formula for this class of semigroups (Proposition 3.8). This formula is then illustrated by an example (Example 3.9).
In Section 3.3 we concentrate on the Ornstein–Uhlenbeck semigroup which has been intensively studied by many authors, e.g., [DPZ92],[CDP93], [Cer94], [CG95], [DPL95], [Pri99], [TZ]. Using the results by S. Cerrai [Cer94] we show that the Ornstein–Uhlenbeck semigroup on $C_b(H)$, H Hilbert space, is bi–continuous. Hence, our Hille-Yosida Theorem and our approximation results apply. Further, based on joint work with A. Albanese [AK00], we show that the Lie–Trotter Product Formula holds for these semigroup on $C_b(\mathbb{R}^n)$ if we take a locally convex topology finer than the compact–open topology.
In Section 3.4 we look at *implemented semigroups* on Banach spaces of bounded linear operators which have been studied, e.g., in [GN81], [Pho91], [ARS94], [PS98], [Alb99], [Alb]. We show that these semigroups fit into the theory of bi–continuous semigroups by using the strong operator topology on $\mathcal{L}(X, Y)$, X, Y Banach spaces. Moreover, we state the Lie–Trotter Product Formula for these semigroups.
Finally, we look at *adjoint semigroups* on the topological dual X' of a Banach space X assuming that the corresponding semigroup on X is strongly continuous. Every such adjoint semigroup is bi–continuous with respect to the weak* topology. Moreover, we characterize adjoint semigroups, which are bi–continuous with respect to the Mackey topology on X'.
For the reader's convenience, the Appendix contains some results on Laplace trans-

form methods for evolution equations which are needed in Chapter 1.

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Chapter 1

Bi–continuous semigroups, generators and resolvents

As mentioned in the introduction, for many applications of operator semigroups strong continuity with respect to the norm of a Banach space is a too strong requirement. Instead, a "weaker" strong continuity with respect to some locally convex topology holds in many interesting cases (see Chapter 3). We take this observation as motivation to introduce *bi–continuous semigroups* on a Banach space X.

1.1 Bi–continuous semigroups

In order to define bi–continuous semigroups, we assume that our underlying space X satisfies the following conditions.

Assumptions 1.1. *Let $(X, \|\cdot\|)$ be a Banach space with topological dual X', and let τ be a locally convex topology on X with the following properties.*

1. *The space (X, τ) is sequentially complete on $\|\cdot\|$–bounded sets, i.e., every $\|\cdot\|$–bounded τ–Cauchy sequence converges in (X, τ).*

2. *The topology τ is Hausdorff and coarser than the $\|\cdot\|$–topology.*

3. *The space $(X, \tau)'$ is norming for $(X, \|\cdot\|)$, i.e.,*

$$\|x\| = \sup\{| < x, \phi > | \mid \phi \in (X, \tau)' \text{ and } \|\phi\|_{(X,\|\cdot\|)'} \leq 1\} \quad \text{for all } x \in X.$$

Comment. The notion of a *norming space* was introduced by J. Lindenstrauss and L. Tzafriri in [LT70, p. 29]. Clearly, Assumption 1.1.3 implies that $(X,\tau)'$ separates the points of X. On the other hand, separation of points does not imply that $(X,\tau)'$ is norming for $(X,\|\cdot\|)$. As a simple example, let $X := (l^\infty, \interleave\cdot\interleave)$ be the space of norm bounded, real sequences endowed with the norm $\interleave\cdot\interleave$ defined as $\interleave x\interleave := 2\|x\|_\infty := 2\sup_{n\in\mathbb{N}}|x_n|$, $x := (x_n)_{n\in\mathbb{N}} \in l^\infty$. Additionally, we take the weak topology $\sigma(l^\infty, l^1)$, where $l^1 \subset l^{\infty\prime}$ denotes the space of absolutely summable sequences. It is easy to see that $(l^\infty, \sigma(l^\infty, l^1))$ is sequentially complete on $\interleave\cdot\interleave$–bounded sets, and l^1 separates the points of l^∞. Now, we take $x := (1,1,...) \in l^\infty$ and suppose that l^1 is norming for $(l^\infty, \interleave\cdot\interleave)$. Then

$$\begin{aligned} 2 = \interleave x\interleave &= \sup\{|<x,y>| :\ y \in l^1, \|y\|_{(l^\infty,\interleave\cdot\interleave)'} \leq 1\} \\ &\leq \sup\{|<x,y>| :\ y \in l^1, \|y\|_{(l^\infty,\|\cdot\|_\infty)'} \leq 1\} \\ &= 1, \end{aligned}$$

which is a contradiction to our assumption.

In the following we denote by $\mathcal{L}(X)$ the space of bounded linear operators on $(X,\|\cdot\|)$, and P_τ denotes a family of seminorms inducing the locally convex topology τ on X. Since τ is coarser than the $\|\cdot\|$–topology, we assume without loss of generality that $p(x) \leq \|x\|$ for all $x \in X$ and $p \in P_\tau$. For the definition of *bi–continuous semigroups*, we require a specific relation between the semigroup operators and the τ–topology.

Definition 1.2. *An operator family $\{T(t) :\ t \geq 0\} \subseteq \mathcal{L}(X)$ is called* **(globally) bi–equicontinuous** *if for every $\|\cdot\|$–bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ which is τ–convergent to $x \in X$ we have*

$$\tau\text{-}\lim_{n\to\infty}(T(t)(x_n - x)) = 0$$

uniformly for all $t \geq 0$.
It is called **locally bi–equicontinuous** *if for every $t_0 \geq 0$ the subset $\{T(t) :\ 0 \leq t \leq t_0\}$ is bi–equicontinuous.*

Definition 1.3. *An operator family $\{T(t) :\ t \geq 0\} \subseteq \mathcal{L}(X)$ is called a* **bi–continuous semigroup** *(with respect to τ and of type ω) if the following conditions hold.*

(i) $T(0) = Id$ and $T(t+s) = T(t)T(s)$ for all $s,t \geq 0$.

(ii) *The operators $T(t)$ are exponentially bounded, i.e., $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t \geq 0$ and some constants $M \geq 1$ and $\omega \in \mathbb{R}$.*

(iii) *$(T(t))_{t\geq 0}$ is strongly τ–continuous, i.e., the map*

$$\mathbb{R}_+ \ni t \longmapsto T(t)x \in X$$

is τ–continuous for each $x \in X$.

(iv) *$(T(t))_{t\geq 0}$ is locally bi–equicontinuous.*

For a bi–continuous semigroup $(T(t))_{t\geq 0}$ we call

$$\omega_0 := \omega_0(T(\cdot)) := \inf\{\omega \in \mathbb{R} : \text{there exists } M \geq 1 \text{ such that } \|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0\} \tag{1.1}$$

its **growth bound**. We call $(T(t))_{t\geq 0}$ **bounded** if we can take $\omega = 0$ in Definition 1.3(ii), and **contractive** if $\omega = 0$ and $M = 1$ is possible.
Clearly, every strongly continuous semigroup on a Banach space is a bi–continuous semigroup with respect to $\tau = \|\cdot\|$ (see [EN00, Ch. I, Def. 5.1]). We now list interesting examples of bi–continuous semigroups most of which will be discussed in detail in Chapter 3 below.

- evolution semigroups as in [CL99], [EN00, Ch. VI, Sec. 9b] but defined on the space $C_b(\mathbb{R}, X)$ (e.g., [Sch96, Sec. 5.3, Thm. 5.6]),
- semigroups canonically extended from X to the sequence space $l^\infty(X)$ as in [NP00],
- semigroups induced by flows (e.g., [DN96], see Section 3.2 below),
- the Ornstein–Uhlenbeck semigroup on $C_b(H)$ (e.g., [DPZ92], [Cer94], [DPL95], see Section 3.3 below),
- adjoint semigroups (e.g., [Nee92], see Section 3.5 below), and
- implemented semigroups (e.g., [BR79, Section 3.2], [ARS94], see Section 3.4 below).

We now state some important consequences of the above definition.

Proposition 1.4. *Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup of type ω on X. Then the following properties hold.*

(a) For every $a \geq 0$ and $\lambda \in \mathbb{C}$ there exists the operator $R_a(\lambda) : X \longrightarrow X$ defined as

$$R_a(\lambda)x := \int_0^a e^{-\lambda t}T(t)xdt \tag{1.2}$$

for all $x \in X$. The integral has to be understood as a τ–Riemann integral (sometimes denoted by τ–$\int_0^a e^{-\lambda t}T(t)xdt$).

(b) The rescaled semigroup $(e^{-\alpha t}T(t))_{t\geq 0}$ is globally bi–equicontinuous for every $\alpha > \omega$.

Proof. Assertion (a) is an immediate consequence of Assumptions 1.1 and Definition 1.3. Indeed, for fixed $\lambda \in \mathbb{C}$ and $a \geq 0$, $\xi_x(\cdot) := e^{-\lambda \cdot}T(\cdot)x$ is a uniformly τ–continuous X–valued function on the interval $[0, a]$ for all $x \in X$. Therefore, the Riemann sums $S(\xi_x(\cdot), \Delta)$ defined as

$$S(\xi_x(\cdot), \Delta) := \sum_{k=1}^{n} \xi_x(t'_k)(t_k - t_{k-1}),$$
$$\Delta : \quad 0 = t_0 \leq t'_1 \leq t_1 \leq ... \leq t'_n \leq t_n = a,$$

form a τ–Cauchy net. Taking

$$S(\xi_x(\cdot), \Delta_n) := \frac{a}{n}\sum_{k=1}^{n} \xi_x(a\frac{k}{n}), \quad n = 1, 2, ...$$

we obtain an equivalent $\|\cdot\|$–bounded τ–Cauchy sequence. Since (X, τ) is sequentially complete on $\|\cdot\|$–bounded sets, $S(\xi_x(\cdot), \Delta_n)$ converges, and hence $\xi_x(\cdot) = e^{-\lambda \cdot}T(\cdot)x$ is Riemann integrable for all $x \in X$ (cf. [Kom64, Prop. 1.1]).

To prove property (b), let $\alpha > \omega, \epsilon > 0, p \in P_\tau$, and $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a $\|\cdot\|$–bounded sequence which is τ–convergent to $x \in X$. Then there exists $t_0 \geq 0$ such that

$$\begin{aligned}
\sup_{t>t_0} p(e^{-\alpha t}T(t)(x_n - x)) &\leq \sup_{t>t_0} e^{-\alpha t}\|T(t)(x_n - x)\| \\
&\leq \sup_{t>t_0} e^{(\omega-\alpha)t}M(\|x_n\| + \|x\|) \\
&\leq \frac{\epsilon}{2}
\end{aligned}$$

for all $n \in \mathbb{N}$.

Further, by Definition 1.3(iv), there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{0\le t\le t_0} p(e^{-\alpha t}T(t)(x_n - x)) \le \frac{\epsilon}{2}$$

for all $n \ge n_0$. Therefore,

$$\begin{aligned}\sup_{t\ge 0} p(e^{-\alpha t}T(t)(x_n - x)) &\le \sup_{0\le t\le t_0} p(e^{-\alpha t}T(t)(x_n - x)) + \sup_{t>t_0} p(e^{-\alpha t}T(t)(x_n - x))\\ &\le \frac{\epsilon}{2} + \sup_{t>t_0} e^{-\alpha t}\|T(t)(x_n - x)\|\\ &\le \epsilon\end{aligned}$$

for all $n \ge n_0$. $\square$

Remark 1.5. (a) The semigroup law, the τ–continuity of the map $t \mapsto T(t)x$ at 0 and local bi–equicontinuity imply the τ–continuity at every point in $\mathbb{R}_+$. To see this, let $t_0 > 0$, $x \in X$ and $p \in P_\tau$. Then $(T(1/n)x)_{n\in\mathbb{N}} \subseteq X$ is a $\|\cdot\|$–bounded sequence which is τ–convergent to x by the τ–continuity at 0. By Definition 1.3(ii),(iv), we obtain that $p(T(t)(T(1/n)x - x)) \to 0$ uniformly for $0 \le t \le t_0$ as $n \to \infty$, and therefore $p(T(t_0+h)x - T(t_0)x)$ converges to 0 as $h \searrow 0$, and by the same argument as $h \nearrow 0$ which implies the continuity at t_0.
(b) In [Kōm68, Prop. 1.1] (cf. [Sch80, Ch. III, Thm. 4.2]) it is shown that on a barreled[1] locally convex vector space (X, τ) conditions (i)–(iii) in Definition 1.3 automatically imply that $(T(t))_{t\ge 0}$ is **locally equicontinuous**, i.e., for any fixed $t_0 > 0$ and for any continuous seminorm $p \in P_\tau$ there exists a continuous seminorm $q \in P_\tau$ such that

$$p(T(t)x) \le q(x)$$

for all $x \in X$ and uniformly for $0 \le t \le t_0$. Therefore, condition (iv) in Definition 1.3 is satisfied automatically.

In the following we give first an example of a bi–continuous semigroup which is not locally equicontinuous in the sense of the definition given in Remark 1.5(b). Further,

[1]A locally convex vector space is **barreled** if each absorbing, absolutely convex and closed subset is a neighborhood of zero.

we show that the translation semigroup on $C_b(\mathbb{R})$ is not bi–continuous with respect to the topology of pointwise convergence. However, bi–continuity holds if we use the topology of uniform convergence on compact intervals.

Examples 1.6. (a) Let X be the space $C_b(\mathbb{R})$ endowed with the supremum norm $\|\cdot\|_\infty$ and the topology τ_c of uniform convergence on compact subsets of $\mathbb{R}$. Clearly, $(C_b(\mathbb{R}),\tau_c)$ is sequentially complete on $\|\cdot\|_\infty$–bounded sets, τ_c is coarser than the $\|\cdot\|_\infty$–topology, Hausdorff, and, since the topological dual $(C_b(\mathbb{R}),\tau_c)'$ contains the point measures, it is norming for $(C_b(\mathbb{R}),\|\cdot\|_\infty)$. On this space we consider the diffusion semigroup $(T(t))_{t\geq 0}$ defined as

$$T(t)f(x)=\int_{\mathbb{R}} f(y)\mathcal{N}(x,t)dy,\quad x\in\mathbb{R}, f\in C_b(\mathbb{R}), t>0,$$

where $\mathcal{N}(x,t)$ denotes the Gauss measure with mean x and variance t defined via the probability density $g_{x,t}$ on $\mathbb{R}$ defined as

$$g_{x,t}(y):=\frac{1}{\sqrt{4\pi t}}e^{\frac{-(y-x)^2}{4t}}$$

for all $y\in\mathbb{R}$. This semigroup is a bi–continuous semigroup with respect to τ_c. In fact, $(T(t))_{t\geq 0}$ is a contraction semigroup on $C_b(\mathbb{R})$ and for $f\in C_b(\mathbb{R}), \epsilon>0$, and a compact subset $K\subseteq\mathbb{R}$ there exists $\delta_{\epsilon,K}>0$ such that $|y|<\delta_{\epsilon,K}$ implies $|f(x+y)-f(x)|\leq\epsilon$ for all $x\in K$. Therefore, by the Chebyshev inequality (see [Bau92, Ch. II, Lemma 20.1]), we have

$$\begin{aligned}
&\sup_{x\in K}|T(t)f(x)-f(x)|\\
&\qquad\leq \sup_{x\in K}\int_{\{|y|<\delta_{\epsilon,K}\}}|f(x+y)-f(x)|\mathcal{N}(0,t)dy+2\|f\|_\infty\frac{1}{\sqrt{4\pi t}}\int_{\{|y|\geq\delta_{\epsilon,K}\}}e^{\frac{-|y|^2}{4t}}dy\\
&\qquad\leq \epsilon+t\frac{2\|f\|_\infty}{\delta_{\epsilon,K}^2},
\end{aligned}$$

which yields the strong τ–continuity of $(T(t))_{t\geq 0}$ at 0. Next, we show that $(T(t))_{t\geq 0}$ is locally bi–equicontinuous. To that purpose, let $K\subset\mathbb{R}$ be compact, $t_0\geq 0$, and $\epsilon>0$. Then there exists a compact subset $K_\epsilon\subseteq\mathbb{R}$ such that

$$\mathcal{N}(x,t)(K_\epsilon)\geq 1-\epsilon$$

uniformly for $x\in K$ and $0\leq t\leq t_0$.

Fix a function $f \in C_b(\mathbb{R})$ and a $\|\cdot\|_\infty$–bounded sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_b(\mathbb{R})$ which is τ_c–convergent to f. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{x\in K_\epsilon} |f_n(x) - f(x)| \leq \epsilon$$

for all $n \geq n_0$. Thus

$$\begin{aligned}
&\sup_{x\in K}|T(t)(f_n(x) - f(x))| \\
&\quad \leq \sup_{x\in K} \int_{K_\epsilon} |f_n(y) - f(y)|\mathcal{N}(x,t)dy + \sup_{x\in K} \int_{\complement K_\epsilon} |f_n(y) - f(y)|\mathcal{N}(x,t)dy \\
&\quad \leq \epsilon + (\|f_n\|_\infty + \|f\|_\infty)\epsilon
\end{aligned}$$

uniformly for $0 \leq t \leq t_0$, and hence $(T(t))_{t\geq 0}$ is locally bi–equicontinuous.
By Definition 1.3 it follows that $(T(t))_{t\geq 0}$ is bi–continuous.
However, $(T(t))_{t\geq 0}$ is not locally equicontinuous in the sense of the definition given in Remark 1.5 (cf. Definition 3.1 in Chapter 3). Suppose the contrary, then for every $t_0 \geq 0$ and $K \subset \mathbb{R}$ compact there would exist a compact subset $K_0 \subset \mathbb{R}$ such that

$$\sup_{x\in K} |T(t)f(x)| \leq \sup_{x\in K_0} |f(x)|$$

for all $f \in C_b(\mathbb{R})$ and uniformly for $0 \leq t \leq t_0$. This must also be true for any function $0 < g \in C_b(\mathbb{R})$ such that $g(x) = 0$ for all x in the interval $[a,b]$ containing K_0. Therefore $\sup_{x\in K_0} |g(x)| = 0$, but

$$\sup_{x\in K} |T(t)g(x)| = \sup_{x\in K} |\int_{\mathbb{R}\setminus[a,b]} g(y)\mathcal{N}(x,t)dy| > 0$$

because of the strict positivity of the integrand. This is a contradiction to our assumption.
(b) Let X be the space $C_b(\mathbb{R})$ endowed with the supremum norm $\|\cdot\|_\infty$ and the topology τ_p of pointwise convergence. By the same argument as in (a) $(C_b(\mathbb{R}), \tau_p)$ satisfies Assumptions 1.1. We consider the (left)translation semigroup $(T(t))_{t\geq 0}$ defined as

$$T(t)f(x) := f(x+t), \quad x \in \mathbb{R}, f \in C_b(\mathbb{R}), t \geq 0.$$

It does not satisfy the property of local bi–equicontinuity. This can be easily seen by taking a sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_b(\mathbb{R})$ defined for $n \geq 2$ as

$$f_n(x) := \begin{cases} \max\left(1 - n^2|x - \frac{1}{n}|, 0\right) & \text{if } x \in [0,1], \\ 0 & \text{else.} \end{cases}$$

Then $(f_n)_{n\in\mathbb{N}}$ is a $\|\cdot\|_\infty$–bounded sequence, and $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in \mathbb{R}$. However, since $T(\frac{1}{n})f_n(0) = 1$ for all $n \geq 2$, the semigroup $(T(t))_{t\geq 0}$ is not locally equicontinuous. Hence $(T(t))_{t\geq 0}$ is not bi–continuous with respect to τ_p. However, if we take the finer topology τ_c as in (a), then it is easy to see that conditions (i)–(iv) in Definition 1.3 hold, hence $(T(t))_{t\geq 0}$ is bi–continuous with respect to τ_c.

1.2 Generators and resolvents

We now assume that $(T(t))_{t\geq 0}$ is a bi–continuous semigroup on X, where X satisfies Assumptions 1.1. Since the space $(X,\tau)'$ is norming for $(X, \|\cdot\|)$, we obtain that the τ–Laplace transform $R(\lambda)$ defined as in (1) becomes a $\|\cdot\|$–bounded operator satisfying the Hille–Yosida estimates. This observation will lead us to the generator of a bi–continuous semigroup whose resolvent coincides with $R(\lambda)$.
First, we collect some elementary properties of these resolvents. We remark that the results of Lemma 1.7, Proposition 1.9 and 1.12 have already appeared in [Alb99, Ch. 2] in the context of implemented semigroups.
For $\omega \in \mathbb{R}$ we set $\Lambda_\omega := \{\lambda \in \mathbb{C} : Re\lambda > \omega\}$.

Lemma 1.7. *Let $(T(t))_{t\geq 0}$ be a bi–continuous semigroup on X. Then the following properties hold.*

(a) Let $\lambda \in \mathbb{C}$ and $a \geq 0$. Then $R_a(\lambda) \in \mathcal{L}(X)$ and

$$R(\lambda) := \lim_{a\to\infty} R_a(\lambda) \tag{1.3}$$

for $\lambda \in \Lambda_{\omega_0}$ exists with respect to the operator norm and satisfies the estimate

$$\|R(\lambda)\|_{\mathcal{L}(X)} \leq \frac{M}{Re\lambda - \omega}$$

for all $\lambda \in \Lambda_\omega$, $\omega > \omega_0$, and some constant $M \geq 1$.

(b) For every $x \in X$ we have

$$\tau\text{–}\lim_{\omega<\lambda\to\infty} \lambda R(\lambda)x = x.$$

Proof. (a) Let $a \geq 0$, $\lambda \in \mathbb{C}$, and $x \in X$. Since $(X,\tau)'$ is norming for $(X, \|\cdot\|)$, we have with $\Phi := \{\phi \in (X,\tau)' : \ \|\phi\|_{(X,\|\cdot\|)'} \leq 1\}$ that

$$\begin{aligned}
\|R_a(\lambda)x\| &= \sup_{\phi\in\Phi} | < \int_0^a e^{-\lambda t} T(t)x dt, \phi > | \\
&= \sup_{\phi\in\Phi} | \int_0^a e^{-\lambda t} < T(t)x, \phi > dt| \\
&\leq M\|x\| \int_0^a e^{-(Re\lambda - \omega)t} dt \\
&\leq \frac{M}{Re\lambda - \omega}\|x\|.
\end{aligned}$$

Therefore assertion (a) holds.
(b) Let $x \in X, p \in P_\tau$, and $\epsilon > 0$. There exists $\delta_\epsilon > 0$ such that $0 \leq t < \delta_\epsilon$ implies $p(T(t)x - x) < \epsilon$. Thus, we have

$$\begin{aligned}
p(\lambda R(\lambda)x - x) &= p\left(\int_0^\infty \lambda e^{-\lambda t} T(t)x dt - \int_0^\infty \lambda e^{-\lambda t} x dt\right) \\
&\leq p\left(\int_0^{\delta_\epsilon} \lambda e^{-\lambda t}(T(t)x - x) dt\right) + p\left(\int_{\delta_\epsilon}^\infty \lambda e^{-\lambda t}(T(t)x - x) dt\right) \\
&=: T_1 + T_2 \ .
\end{aligned}$$

For the term T_2 we obtain with Φ as above that

$$\begin{aligned}
T_2 &\leq \sup_{\phi\in\Phi} | < \int_{\delta_\epsilon}^\infty \lambda e^{-\lambda t}(T(t)x - x) dt, \phi > | \\
&\leq \sup_{\phi\in\Phi} \int_{\delta_\epsilon}^\infty \lambda e^{-\lambda t} | < T(t)x - x, \phi > | dt \\
&\leq \|x\| \left[M \frac{\lambda}{\lambda - \omega} e^{(\omega - \lambda)\delta_\epsilon} + e^{-\lambda\delta_\epsilon} \right],
\end{aligned}$$

which converges to zero as λ tends to infinity. For the term T_1 we obtain

$$T_1 \leq \int_0^{\delta_\epsilon} \lambda e^{-\lambda t} p(T(t)x - x) dt \leq \epsilon \int_0^\infty \lambda e^{-\lambda t} dt = \epsilon,$$

which concludes the proof. □

To the operators $(R(\lambda))_{\lambda\in\Lambda_{\omega_0}}$ we can now associate an operator whose resolvent coincides with $(R(\lambda))_{\lambda\in\Lambda_{\omega_0}}$. To that purpose, we first give some basic results of pseudoresolvents which will be needed to further develop our theory (see [EN00, Ch. III, Sec. 4.a]).

Definition 1.8. *Let X be a Banach space, $\Lambda \subset \mathbb{C}$, and consider operators $\mathcal{J}(\lambda) \in \mathcal{L}(X)$ for each $\lambda \in \Lambda$. The family $\{\mathcal{J}(\lambda) : \lambda \in \Lambda\}$ is called a* **pseudoresolvent** *if*

$$\mathcal{J}(\lambda) - \mathcal{J}(\mu) = (\mu - \lambda)\mathcal{J}(\lambda)\mathcal{J}(\mu) \tag{RE}$$

holds for all $\mu, \lambda \in \Lambda$.

By the resolvent equation (RE) we obtain the following elementary properties of pseudoresolvents (see [EN00, Ch. III, Prop. 4.6]).

Proposition 1.9. *Let $\{\mathcal{J}(\lambda) : \lambda \in \Lambda\}$ be a pseudoresolvent on a Banach space X. Then $\mathcal{J}(\lambda)\mathcal{J}(\mu) = \mathcal{J}(\mu)\mathcal{J}(\lambda)$, $ker\mathcal{J}(\lambda) = ker\mathcal{J}(\mu)$ and $rg\mathcal{J}(\lambda) = rg\mathcal{J}(\mu)$ hold for all $\lambda, \mu \in \Lambda$.*
Moreover, the following assertions are equivalent.

(i) There exists a closed operator $(A, D(A))$ such that $\Lambda \subset \rho(A)$ and $\mathcal{J}(\lambda) = R(\lambda, A)$ for all $\lambda \in \Lambda$.

(ii) $ker\mathcal{J}(\lambda) = \{0\}$ for some/all $\lambda \in \Lambda$.

For the following, we recall that X satisfies Assumptions 1.1.

Definition 1.10. *A subset $M \subseteq X$ is called* **bi–dense** *if for every $x \in X$ there exists a $\|\cdot\|$–bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq M$ which is τ–convergent to x.*

With the above definition a particular case of Proposition 1.9 is stated in the following corollary.

Corollary 1.11. *Let $\{\mathcal{J}(\lambda) : \lambda \in \Lambda\}$ be a pseudoresolvent on X and assume that Λ contains an unbounded sequence $(\lambda_n)_{n\in\mathbb{N}}$. If*

$$\tau\text{–}\lim_{n\longrightarrow\infty} \lambda_n\mathcal{J}(\lambda_n)x = x \quad \textit{for all } x \in X, \tag{1.4}$$

then $\{\mathcal{J}(\lambda) : \lambda \in \Lambda\}$ is a resolvent.
In particular, (1.4) holds if $rg\mathcal{J}(\lambda)$ is bi–dense,

$$\|\lambda_n\mathcal{J}(\lambda_n)\| \le M \tag{1.5}$$

for some constant $M \ge 0$ and all $n \in \mathbb{N}$, and the family $\{\lambda_n\mathcal{J}(\lambda_n) : n \in \mathbb{N}\}$ is bi–equicontinuous.

Proof. For $x \in ker\mathcal{J}(\lambda)$ we have $x \in ker\mathcal{J}(\lambda_n)$ for all $n \in \mathbb{N}$ and $\tau - \lim_{n\to\infty} \lambda_n \mathcal{J}(\lambda_n)x = 0$. Since τ is Hausdorff, it follows $x = 0$. Applying Proposition 1.9 the first assertion holds. Next, estimate (1.5) and the resolvent equation imply $\lim_{n\to\infty} \|(\lambda_n \mathcal{J}(\lambda_n) - Id)\mathcal{J}(\mu)\| = 0$ for fixed $\mu \in \Lambda$. Therefore, we have

$$\| \cdot \|\text{–} \lim_{n\to\infty} \lambda_n \mathcal{J}(\lambda_n) y = y$$

for all $y \in rg\mathcal{J}(\mu)$. Let $x \in X$, $\epsilon > 0$, and $p \in P_\tau$. Since $rg\mathcal{J}(\mu)$ is bi–dense, there exists a $\| \cdot \|$–bounded sequence $(y_k)_{k\in\mathbb{N}} \subseteq rg\mathcal{J}(\mu)$ and $k_0 \in \mathbb{N}$ such that

$$p(y_k - x) \leq \frac{\epsilon}{3}$$

for all $k \geq k_0$. The bi–equicontinuity of the family $\{\lambda_n \mathcal{J}(\lambda_n) : \ n \in \mathbb{N}\}$ implies that there exists $\tilde{k}_0 \geq k_0$ such that

$$p(\lambda_n \mathcal{J}(\lambda_n)(y_k - x)) \leq \frac{\epsilon}{3}$$

for all $k \geq \tilde{k}_0$ and uniformly for $n \in \mathbb{N}$. Thus, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} & p(\lambda_n \mathcal{J}(\lambda_n)x - x) \\ & \qquad \leq p(\lambda_n \mathcal{J}(\lambda_n)(x - y_{\tilde{k}_0})) + \|\lambda_n \mathcal{J}(\lambda_n) y_{\tilde{k}_0} - y_{\tilde{k}_0}\| + p(y_{\tilde{k}_0} - x) \\ & \qquad \leq \epsilon \end{aligned}$$

for all $n \geq n_0$. □

Let now $(T(t))_{t\geq 0}$ be a bi–continuous semigroup on X and ω_0 its growth bound as defined in (1.1). Applying the results above to the corresponding operators $R(\lambda)_{\lambda\in\Lambda_{\omega_0}}$ defined in (1.3), we obtain the following.

Proposition 1.12. *The family of operators* $(R(\lambda))_{\lambda\in\Lambda_{\omega_0}}$ *is a resolvent.*

Proof. Let $\lambda \neq \mu \in \Lambda_{\omega_0}$. We assume without loss of generality that $\lambda > \mu$ and obtain

$$\begin{aligned}
&\frac{R(\mu)x - R(\lambda)x}{\lambda - \mu} \\
&\quad = \int_0^\infty e^{(\mu-\lambda)t} dt\ R(\mu)x - \int_0^\infty \frac{e^{(\mu-\lambda)t}}{(\lambda-\mu)} e^{-\mu t} T(t)x\ dt \\
&\quad = \int_0^\infty e^{(\mu-\lambda)t} \int_0^\infty e^{-\mu s} T(s)x\ ds\ dt - \left[\frac{e^{(\mu-\lambda)t}}{\lambda-\mu} \int_0^t e^{-\mu s} T(s)x\ ds \right]_{t=0}^{t=\infty} \\
&\qquad - \int_0^\infty e^{(\mu-\lambda)t} \int_0^t e^{-\mu s} T(s)x\ ds\ dt \\
&\quad = \int_0^\infty e^{(\mu-\lambda)t} \int_t^\infty e^{-\mu s} T(s)x\ ds\ dt \\
&\quad = \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\mu s} T(s)T(t)x\ ds\ dt \\
&\quad = R(\lambda)R(\mu)x
\end{aligned}$$

for all $x \in X$. Therefore, $(R(\lambda))_{\lambda\in\Lambda_{\omega_0}}$ is a pseudoresolvent by Definition 1.8. By Lemma 1.7(b) we obtain the injectivity of the operators $R(\lambda)$. In fact, for $x \in ker R(\lambda)$ and an unbounded sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \Lambda_{\omega_0} \cap \mathbb{R}_+$, we have $x \in ker R(\lambda_n)$ for all $n \in \mathbb{N}$ and $\tau\text{–}\lim_{n\to\infty} \lambda_n R(\lambda_n)x = 0$. Since τ is Hausdorff, it follows $x = 0$. Proposition 1.9 concludes the proof. ☐

We observe that, by Proposition 1.12, the map $\Lambda_{\omega_0} \ni \lambda \mapsto R(\lambda) \in \mathcal{L}(X)$ is holomorphic and

$$\frac{d^k}{d\lambda^k} R(\lambda)x = (-1)^k k! R(\lambda)^{k+1} x \tag{1.6}$$

for all $x \in X, k \in \mathbb{N}$, and $\lambda \in \Lambda_{\omega_0}$ (see [EN00, Ch. IV, Prop. 1.3]).
The above observations allow the definition of the generator of a bi–continuous semigroup.

Definition 1.13. *The* **generator** $A : D(A) \subseteq X \longrightarrow X$ *of a bi–continuous semigroup* $(T(t))_{t\geq 0}$ *on* X *is the unique operator on* X *such that its resolvent* $R(\lambda, A)$ *is*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x dt \tag{1.7}$$

for all $\lambda \in \Lambda_{\omega_0}$ *and* $x \in X$.

As a first consequence, we obtain that these generators satisfy the Hille–Yosida estimates.

Proposition 1.14. *Let $(A, D(A))$ be the generator of a bi–continuous semigroup $(T(t))_{t\geq 0}$ of type ω on X. Then we have*

$$\frac{d^k}{d\lambda^k}R(\lambda, A)x = (-1)^k \int_0^\infty t^k e^{-\lambda t}T(t)x dt \tag{1.8}$$

for all $x \in X, k \in \mathbb{N}$ and $\lambda \in \Lambda_{\omega_0}$. In particular, there exists for each $\omega > \omega_0$ a constant $M \geq 1$ such that

$$\|R(\lambda, A)^k\| \leq \frac{M}{(Re\lambda - \omega)^k}$$

for all $k \in \mathbb{N}$ and $\lambda \in \Lambda_\omega$.

Proof. Let $x \in X$, $\lambda \in \Lambda_\omega$, $\omega > \omega_0$, and $\Phi := \{\phi \in (X, \tau)' : \ \|\phi\|_{X'} \leq 1\}$. Since the space $(X, \tau)'$ is norming for $(X, \|\cdot\|)$, for every $\mu \in \Lambda_\omega$ we have

$$\begin{aligned}
&\left\| \frac{R(\mu)x - R(\lambda)x}{\mu - \lambda} + \int_0^\infty te^{-\lambda t}T(t)xdt \right\| \\
&\qquad \leq \sup_{\phi\in\Phi} \int_0^\infty \left| \frac{e^{-\mu t} - e^{-\lambda t}}{\mu - \lambda} + te^{-\lambda t} \right| |< T(t)x, \phi > | dt \\
&\qquad \leq M\|x\| \int_0^\infty \left| \frac{e^{-\mu t} - e^{-\lambda t}}{\mu - \lambda} + te^{-\lambda t} \right| e^{\omega t} dt,
\end{aligned} \tag{1.9}$$

which converges to zero as μ tends to λ as a consequence of Lebesgue's dominated convergence theorem. Via induction we obtain the desired equality. Further, (1.6) and (1.8) imply

$$\begin{aligned}
\|R(\lambda)^k x\| &\leq \sup_{\phi\in\Phi} \frac{1}{(n-1)!} \int_0^\infty t^{k-1} |e^{-\lambda t} < T(t)x, \phi > | dt \\
&\leq \frac{M}{(n-1)!}\|x\| \int_0^\infty t^{k-1} e^{(\omega - Re\lambda)t} dt \\
&= \frac{M}{(Re\lambda - \omega)^k}\|x\|
\end{aligned} \tag{1.10}$$

for all $x \in X$ and $\lambda \in \Lambda_\omega$. □

Proposition 1.14 says that generators of bi–continuous semigroups are *Hille–Yosida* operators (see [EN00, Ch. II, Def. 3.22] and Section 1.3 below).

Following, e.g., [Yos74, Ch. IX, Sec. 3], [Kōm68, p. 260], [DS57, Ch. VIII, Def. 1.6], [EN00, Ch. II, Def. 1.2], another way to introduce the generator $(A_\tau, D(A_\tau))$ of a bi–continuous semigroup $(T(t))_{t\geq 0}$ would be to define

$$\begin{aligned} &A_\tau x := \tau\text{–}\lim_{t\searrow 0} \frac{T(t)x - x}{t} \\ &\text{for all } x \in D(A_\tau) := \{x \in X :\ \tau\text{–}\lim_{t\searrow 0} \frac{T(t)x - x}{t} \text{ exists in } X\}. \end{aligned} \tag{1.11}$$

In general, i.e., if we are not in the setting of bi–continuous semigroups, the operators A_τ and A, defined as in Definition 1.13, do not coincide. This can be seen in the following example.

Example 1.15. Let $C(\mathbb{R})$ be the space of continuous functions endowed with the compact-open topology τ_c. We consider the multiplication semigroup $(T_q(t))_{t\geq 0}$ defined as

$$T_q(t)f := e^{tq} f, \quad t \geq 0, f \in C(\mathbb{R}),$$

for some function $q \in C(\mathbb{R})$. It can be easily verified that $(T_q(t))_{t\geq 0}$ is strongly τ_c–continuous, and its generator is given by

$$A_{\tau_c} f = \tau_c\text{–}\lim_{t\searrow 0} \frac{T_q(t)f - f}{t} = q \cdot f \quad \text{ for all } f \in D(A_{\tau_c}) = C(\mathbb{R}).$$

However, if q is an unbounded function, the integral $\int_0^\infty e^{-\lambda t} T_q(t)x dt$ does not always exist and the operator A cannot be defined as in Definition 1.13.

However, in the setting of bi–continuous semigroups, the operators $(A_\tau, D(A_\tau))$ and $(A, D(A))$ coincide. To see this, we first look at the following fundamental properties of the operator $(A_\tau, D(A_\tau))$ (cf. [Kōm68, Prop. 1.2, 1.4]).

Proposition 1.16. *Let $(T(t))_{t\geq 0}$ be a bi–continuous semigroup on X and $(A_\tau, D(A_\tau))$ as above. Then the following properties hold.*

(a) If $x \in D(A_\tau)$, then $T(t)x \in D(A_\tau)$ for all $t \geq 0$, $T(t)x$ is continuously differentiable in t with respect to the topology τ, and

$$\frac{d}{dt} T(t)x = A_\tau T(t)x = T(t)A_\tau x$$

for all $t \geq 0$.

(b) *An element* $x \in X$ *belongs to* $D(A_\tau)$ *and* $A_\tau x = y$ *if and only if*

$$T(t)x - x = \int_0^t T(s)y\,ds \tag{1.12}$$

for all $t \geq 0$.

(c) *The operator* $(A_\tau, D(A_\tau))$ *is* **bi–closed**, *i.e., for all sequences* $(x_n)_{n\in\mathbb{N}} \subseteq D(A_\tau)$ *with* $(x_n)_{n\in\mathbb{N}}$ *and* $(A_\tau x_n)_{n\in\mathbb{N}}$ $\|\cdot\|$*–bounded,* $x_n \xrightarrow{\tau} x \in X$ *and* $A_\tau x_n \xrightarrow{\tau} y \in X$ *we have* $x \in D(A_\tau)$ *and* $A_\tau x = y$.

Proof. (a) If $x \in D(A_\tau)$, then for $t \geq 0$ we have

$$T(t)A_\tau x = \lim_{h\searrow 0} \frac{T(t+h)x - T(t)x}{h} = \lim_{h\searrow 0} \frac{(T(h) - Id)T(t)x}{h},$$

which shows that $T(t)x \in D(A_\tau)$ and the right derivative $\frac{d^+}{dt}T(t)x$ exists. Thus we have

$$\frac{d^+}{dt}T(t)x = A_\tau T(t)x = T(t)A_\tau x.$$

Let now $\phi \in (X, \tau)'$. Then

$$\frac{d^+}{dt} < T(t)x, \phi > = < \frac{d^+}{dt}T(t)x, \phi > = < T(t)A_\tau x, \phi >,$$

which implies the continuity in t of $\frac{d^+}{dt} < T(t)x, \phi >$. Therefore, applying Dini's Lemma from [Yos74, p. 239], $< T(t)x, \phi >$ is differentiable in t and

$$\frac{d}{dt} < T(t)x, \phi > = < T(t)A_\tau x, \phi > .$$

Since $(T(t))_{t\geq 0}$ is bi–continuous, the integral $\int_0^t T(s)A_\tau x\,ds$ exists in X, and we obtain

$$\begin{aligned} < T(t)x - x, \phi > &= \int_0^t \frac{d}{ds} < T(s)x, \phi > ds \\ &= \int_0^t < T(s)A_\tau x, \phi > ds \\ &= < \int_0^t T(s)A_\tau x\,ds, \phi > . \end{aligned}$$

Hence,

$$T(t)x - x = \int_0^t T(s)A_\tau x ds \tag{1.13}$$

for all $t \geq 0$, and $T(t)x$ is differentiable in t with

$$\frac{d}{dt}T(t)x = T(t)A_\tau x.$$

(b) Let $x \in D(A_\tau)$ and $A_\tau x = y$. Then equation (1.13) yields the assertion. On the other hand, let $x \in X$ and $x = T(t)x - \int_0^t T(s)yds$ for all $t \geq 0$. Then

$$\tau\text{–}\lim_{t \searrow 0} \frac{T(t)x - x}{t} = \tau\text{–}\lim_{t \searrow 0} \frac{1}{t} \int_0^t T(s)y = y,$$

and hence $x \in D(A_\tau)$ and $A_\tau x = y$.
(c) Let $(x_n)_{n\in\mathbb{N}} \subseteq D(A_\tau)$ be a $\|\cdot\|$–bounded sequence which is τ–convergent to $x \in X$ and $(A_\tau x_n)_{n\in\mathbb{N}} \subseteq X$ be $\|\cdot\|$–bounded and τ–convergent to $y \in X$. Then, by Theorem 1.17 and assertion (b), we obtain

$$T(t)x_n - x_n = \int_0^t T(s)A_\tau x_n ds$$

for all $t \geq 0$. Using the local bi–equicontinuity of $(T(t))_{t\geq 0}$, we have

$$T(t)x - x = \int_0^t T(s)yds.$$

Therefore, again by Theorem 1.17 and assertion (b), $x \in D(A_\tau)$ and $A_\tau x = y$, i.e., $(A_\tau, D(A_\tau))$ is bi–closed. □

Theorem 1.17. *Let $(T(t))_{t\geq 0}$ be a bi–continuous semigroup on X with generator $(A, D(A))$ and define $(A_\tau, D(A_\tau))$ as in (1.11). Then $A = A_\tau$.*

Proof. We show first that $A \subset A_\tau$. For $x \in X$ and $\lambda \in \Lambda_{\omega_0}$ we have

$$\frac{T(h) - Id}{h} R(\lambda, A)x = \frac{(e^{\lambda h} - 1)}{h} \int_0^\infty e^{-\lambda t}T(t)xdt - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t}T(t)xdt,$$

which converges to $\lambda R(\lambda, A)x - x = AR(\lambda, A)x$ as $h \searrow 0$. Thus $A \subset A_\tau$.
On the other hand, for $x \in D(A_\tau)$ we define $y := (\lambda - A_\tau)x$. By Proposition 1.16 we have

$$A_\tau \int_0^\infty e^{-\lambda t}T(t)xdt = \int_0^\infty e^{-\lambda t}T(t)A_\tau xdt.$$

Therefore, we obtain

$$R(\lambda, A)y = (\lambda - A_\tau) \int_0^\infty e^{-\lambda t} T(t)x dt = (\lambda - A_\tau) R(\lambda, A)x = x,$$

and hence $A_\tau \subset A$. □

Proposition 1.18. *Let $(A, D(A))$ be the generator of a bi–continuous semigroup $(T(t))_{t \geq 0}$ of type ω on X. Then the following properties hold.*

(a) The generator $(A, D(A))$ is **bi–closed***.*

(b) The domain of A is **bi–dense** *(see Definition 1.10) in X.*

(c) Let $D \subseteq D(A)$ be a bi–dense subset in X. Then $R(\lambda, A)D, \lambda > \alpha > \omega$, is bi–dense in $D(A)$.

(d) The subspace $X_0 := \overline{D(A)}^{\|\cdot\|} \subseteq X$ is $(T(t))_{t \geq 0}$–invariant and $(T(t)_{|X_0})_{t \geq 0}$ is the strongly continuous semigroup on X_0 generated by the part [2] of A in X_0.

Proof. Assertion (a) follows directly from Proposition 1.16(c) and Theorem 1.17.
(b) Let $x \in X$. By Lemma 1.7 the sequence $(x_n)_{n \in \mathbb{N}} \subseteq D(A)$ defined as

$$x_n := \begin{cases} nR(n, A)x & \text{if } n > \omega \ , \\ 0 & \textit{else}, \end{cases}$$

is $\| \cdot \|$–bounded and τ–convergent to x.
To prove (c), let $x \in D(A), \lambda > \alpha > \omega, \epsilon > 0$, and $p \in P_\tau$. There exists $z \in X$ such that $R(\lambda, A)z = x$. Since D is bi–dense in X, there exists a $\| \cdot \|$–bounded sequence $(y_n)_{n \in \mathbb{N}} \subseteq D$ and $n_0 \in \mathbb{N}$ such that

$$p(y_n - z) \leq \epsilon$$

for all $n \geq n_0$. Further, the sequence $(R(\lambda, A)y_n)_{n \in \mathbb{N}}$ is $\| \cdot \|$–bounded, and, by the bi–continuity of $(T(t))_{t \geq 0}$, we obtain that there exists $\tilde{n}_0 \geq n_0$ such that

$$p(R(\lambda, A)y_n - x) = p(R(\lambda, A)(y_n - z)) \leq \int_0^\infty e^{-(\lambda - \alpha)t} p(e^{-\alpha t} T(t)(y_n - z)) dt \leq \epsilon$$

for all $n \geq \tilde{n}_0$.

[2] The **part** of A in $Y \hookrightarrow X$ is the operator $A_|$ defined as $A_| y := Ay$ with domain $D(A_|) := \{D(A) \cap Y : \ Ay \in Y\}$.

Assertion (d) is a consequence of Proposition 1.14 and [EN00, Ch. II, Cor. 3.21].

□

Combining Formulas (1.6) and (1.8) we obtain the following additional properties of the powers of the resolvent operators $R(\lambda, A)$, $\lambda \in \Lambda_{\omega_0}$ (cf. [Cer94, Prop. 3.5] in the context of her "weakly continuous semigroups on $C_{ub}(H)$").

Proposition 1.19. *Let $(T(t))_{t\geq 0}$ be a bi–continuous semigroup of type ω on X with generator $(A, D(A))$. Then the operators $\lambda^k R(\lambda, A)^k$, $\lambda > \omega_0$, $k \in \mathbb{N}$, have the following properties.*

(a) $\tau\text{–}\lim_{\lambda\to\infty} \lambda^k R(\lambda, A)^k x = x$ *for all $x \in X$ and $k \in \mathbb{N}$.*

(b) Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a $\|\cdot\|$–bounded sequence which is τ–convergent to $x \in X$. Let $\alpha > \omega$. Then

$$\tau\text{–}\lim_{n\to\infty} (\lambda - \alpha)^k R(\lambda, A)^k (x_n - x) = 0 \tag{1.14}$$

uniformly for $k \in \mathbb{N}$ and $\lambda > \alpha$.

Proof. (a) Without loss of generality we suppose that $(T(t))_{t\geq 0}$ is $\|\cdot\|$–bounded. Let $x \in X, k \in \mathbb{N}, \epsilon > 0$, and $p \in P_\tau$. There exists $\delta_\epsilon > 0$ such that $0 \leq t < \delta_\epsilon$ implies $p(T(t)x - x) < \frac{\epsilon}{2}$. Formula (1.6) and (1.8) imply that there exists $\lambda_0 > \omega_0$ such that

$$\begin{aligned}
&p(\lambda^k R(\lambda, A)^k x - x) \\
&\quad \leq p\left(\frac{\lambda^k}{(k-1)!}\int_0^{\delta_\epsilon} t^{k-1}e^{-\lambda t}(T(t)x - x)dt\right) \\
&\qquad + p\left(\frac{\lambda^k}{(k-1)!}\int_{\delta_\epsilon}^{\infty} t^{k-1}e^{-\lambda t}(T(t)x - x)dt\right) \\
&\quad \leq \frac{\epsilon}{2}\frac{\lambda^k}{(k-1)!}\int_0^{\delta_\epsilon} t^{k-1}e^{-\lambda t}dt + (1+M)\|x\|\frac{\lambda^k}{(k-1)!}\int_{\delta_\epsilon}^{\infty} t^{k-1}e^{-\lambda t}dt \\
&\quad \leq \frac{\epsilon}{2} + (1+M)\|x\|\left\{\frac{\lambda^k}{(k-1)!}\delta_\epsilon^{k-1}e^{-\lambda\delta_\epsilon} + ... + \lambda\delta_\epsilon e^{-\lambda\delta_\epsilon} + e^{-\lambda\delta_\epsilon}\right\} \\
&\quad \leq \epsilon
\end{aligned}$$

for all $\lambda \geq \lambda_0$ and some constant $M \geq 1$.
(b) Let $\epsilon > 0$, $p \in P_\tau$, and $\alpha > \omega$. By Proposition 1.4(b) we obtain that the rescaled

semigroup $(e^{-\alpha t}T(t))_{t\geq 0}$ is globally bi–equicontinuous. Hence, by Formula (1.6) and (1.8), there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} p((\lambda-\alpha)^k R(\lambda,A)^k(x_n-x)) &\leq \frac{(\lambda-\alpha)^k}{(k-1)!}\int_0^\infty t^{k-1}e^{-(\lambda-\alpha)t}p(e^{-\alpha t}T(t)(x_n-x))dt \\ &\leq \epsilon \end{aligned}$$

for all $n \geq n_0$ and uniformly for $k \in \mathbb{N}$ and $\lambda > \alpha$. □

In the following we introduce the notion of a *bi–core* for a linear operator. This terminology will be useful for the approximation theory treated in Chapter 2.

Definition 1.20. *A subspace D of the domain of a linear operator $A : D(A) \subseteq X \longrightarrow X$ is called a* **bi–core** *for A if for all $x \in D(A)$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ such that $(x_n)_{n\in\mathbb{N}}$ and $(Ax_n)_{n\in\mathbb{N}}$ are $\|\cdot\|$–bounded, and $\lim_{n\to\infty} x_n = x$ with respect to the topology induced by the family $\tilde{P}_\tau$ of continuous seminorms defined as $\tilde{p}(x) := p(x) + p(Ax)$ for all $x \in D(A)$ and $p \in P_\tau$.*

The following is a criterion for subspaces to be a bi–core for the generator of a bi–continuous semigroup analogous to [EN00, Ch. II, Prop. 1.7].

Proposition 1.21. *Let $(A, D(A))$ be the generator of a bi–continuous semigroup $(T(t))_{t\geq 0}$ and D be a subspace of $D(A)$ which is invariant under the semigroup $(T(t))_{t\geq 0}$. If for every $x \in X$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ such that $(x_n)_{n\in\mathbb{N}}$ and $(Ax_n)_{n\in\mathbb{N}}$ are $\|\cdot\|$–bounded and τ–$\lim_{n\to\infty} x_n = x$, then D is a bi–core for A.*

Proof. Let $x \in D(A)$. By assumption there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ which is τ–convergent to x and $(x_n)_{n\in\mathbb{N}}$ and $(Ax_n)_{n\in\mathbb{N}}$ are $\|\cdot\|$–bounded. By Proposition 1.16(a) for each $n \in \mathbb{N}$ the map $\mathbb{R}_+ \ni s \mapsto T(s)x_n \in D$ is continuous with respect to the system of seminorms $\tilde{P}_\tau$ occuring in Definition 1.20. It follows that $\int_0^t T(s)x_n ds$, being a Riemann integral, belongs to the closure of D with respect to the topology induced by the family of seminorms $\tilde{P}_\tau$. Similarly, the $\tilde{P}_\tau$–continuity of $\mathbb{R}_+ \ni s \mapsto T(s)x$ for $x \in D(A)$ and Proposition 1.16(b) imply for $\tilde{p} \in \tilde{P}_\tau$ that

$$\begin{aligned} \tilde{p}\left(\frac{1}{t}\int_0^t T(s)xds - x\right) &= p\left(\frac{1}{t}\int_0^t T(s)xds - x\right) \\ &\quad + p\left(\frac{1}{t}\int_0^t T(s)Axds - Ax\right) \end{aligned}$$

converges to zero as t tends to zero, and

$$\tilde{p}\left(\frac{1}{t}\int_0^t T(s)x_n ds - \frac{1}{t}\int_0^t T(s)x ds\right)$$

converges to zero as n tends to infinity for each $t > 0$. Therefore, for every $\epsilon > 0$ there exists $t > 0$ and $n \in \mathbb{N}$ such that

$$\tilde{p}\left(\frac{1}{t}\int_0^t T(s)x_n ds - x\right) \leq \epsilon,$$

and hence $x \in \overline{D}^{\tilde{P}_\tau}$. □

1.3 Hille-Yosida operators

In Proposition 1.14 we showed that bi–continuous semigroups on a Banach space X are generated by Hille–Yosida operators. Such operators generate strongly continuous semigroups on the closure of their domains (see Proposition 1.18(d)) and, e.g., from the result of R. Nagel and E. Sinestrari [NS70], we conclude that the original space X is a closed subspace of the extrapolated Favard space F_0 (see below for the definition of F_0).

First, we briefly recall the definition of a Hille–Yosida operator (see [EN00, Ch. II, 3.22]) and construct the associated *Sobolev tower* (see [NS70], [NNR96] for more details).

Definition 1.22. *An operator $(A, D(A))$ on a Banach space X is called a* **Hille–Yosida operator** *(of type ω) if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and*

$$\|R(\lambda, A)^k\| \leq \frac{M}{(\lambda - \omega)^k}$$

for all $k \in \mathbb{N}, \lambda > \omega$ and some $M \geq 1$.

For the following construction we assume (without loss of generality) that $\omega < 0$. It is well known (see [EN00, Ch. III, Cor. 3.21]) that the part A_0 of A in $X_0 := \overline{D(A)}^{\|\cdot\|_X}$ is the generator of a strongly continuous semigroup $(T_0(t))_{t \geq 0}$ on X_0. To this semigroup and its generator we associate the following spaces:

(i) the domain space $X_1 := D(A_0)$ with norm $\|x\|_1 := \|A_0 x\|$ for $x \in X_0$.

(ii) the extrapolation space $X_{-1} := (X_0, \|\cdot\|_{-1})^{\sim}$ with $\|x\|_{-1} := \|A_0^{-1}x\|$ for $x \in X_0$.

(iii) the Favard space F_1 of $(T_0(t))_{t\geq 0}$ defined as

$$F_1 := \{x \in X_0 : \sup_{t>0} \frac{1}{t}\|T_0(t)x - x\| < \infty\}$$

with norm

$$\|x\|_{F_1} := \sup_{t>0} \frac{1}{t}\|T_0(t)x - x\| \ .$$

By continuity the semigroup $(T_0(t))_{t\geq 0}$ can be extended to a strongly continuous semigroup $(T_{-1}(t))_{t\geq 0}$ on $X_{-1} := (X_0, \|\cdot\|_{-1})^{\sim}$ and its generator A_{-1} is an extension of A with domain $D(A_{-1}) = X_0$. Therefore, we define

(iv) the extrapolated Favard space F_0 with respect to $(T_{-1}(t))_{t\geq 0}$ in analogy to (iii).

The semigroup $(T_{-1}(t))_{t\geq 0}$ leaves F_1 and F_0 invariant, but is not strongly continuous on these Banach spaces. We collect these facts in the following diagram, and call it the **Sobolev tower** associated to the Hille–Yosida operator A (see [NS70]).

Proposition 1.23. *For a Hille–Yosida operator $(A, D(A))$ on a Banach space X and with the above definitions one has the following situation.*

$$\begin{tikzcd}
X_{-1} \arrow[rrr, "T_{-1}(t)"] & & & X_{-1} \arrow[ddd] \\
& F_0 \arrow[ul] \arrow[r] & F_0 \arrow[ur] \arrow[d] & \\
& X \arrow[u] \arrow[r] & X & \\
X_0 \arrow[uuu] \arrow[ur] \arrow[rrr, "T_0(t)"] & & & X_0 \arrow[ul] \arrow[ddd] \\
& F_1 \arrow[ul] \arrow[r] & F_1 \arrow[ur] \arrow[d] & \\
& D(A) \arrow[u] \arrow[r] & D(A) & \\
D(A_0) \arrow[uuu] \arrow[ur] \arrow[rrr, "T_1(t)"] & & & D(A_0) \arrow[ul]
\end{tikzcd}$$

Moreover, one has the inclusions

$$D(A_0) \subseteq D(A) \subseteq F_1 \hookrightarrow X_0 \subseteq X \subseteq F_0 \hookrightarrow X_{-1} \ .$$

As a consequence the space X is always sandwiched between X_0 and the extrapolated Favard space F_0.
A certain converse of this statements also holds and follows directly from the definitions.

Corollary 1.24. *Under the above assumptions let Y be a closed subspace of F_0 containing X_0. Then the part of A_{-1} in Y is a Hille–Yosida operator on Y.*

Corollary 1.25. *If X is reflexive, then every Hille–Yosida operator on X is already the generator of a strongly continuous semigroup. In particular, every bi–continuous semigroup on X is already strongly continuous for the norm topology.*

Proof. By [EN00, Ch. II, Cor. 5.21] we obtain $F_1 = D(A_0)$ and $F_0 = X_0$. Hence, by Proposition 1.23, the semigroup $(T_{-1}(t)_{|X})_{t\geq 0} = (T_0(t))_{t\geq 0}$ is strongly continuous. □

In general, the space X need not be invariant under $(T_{-1}(t))_{t\geq 0}$ (see [Nee92, Example 3.1.18]) and therefore, there is no semigroup on it. However, X is $(T_{-1}(t))_{t\geq 0}$–invariant if and only if $D(A)$ is $(T_0(t))_{t\geq 0}$–invariant. For instance, this is fulfilled if $X = F_0$.
At this point it may be interesting to look for topologies on X for which $(T_{-1}(t)_{|X})_{t\geq 0}$ becomes continuous. In Section 3.5 we will give some answers to this problem.

1.4 Integrated semigroups

In this section we collect some results concerning integrated semigroups on Banach spaces and their relation to strongly continuous semigroups and Hille–Yosida operators, respectively. Integrated semigroups were introduced by W. Arendt in [Are87a]. For further informations we refer to the book of W. Arendt et al. [ABHN] and the references therein.
First, we recall the definition of the generator of an integrated semigroup.

Definition 1.26. *We call an operator A on a Banach space X the* **generator of an integrated semigroup** *if there exists a strongly continuous function $F : \mathbb{R}_+ \longrightarrow$*

$\mathcal{L}(X)$ such that $\omega := \inf\{\lambda \in \mathbb{R} : \int_0^\infty e^{-\lambda t} F(t) x dt$ exists for all $x \in X\} < \infty$, $(\omega, \infty) \subseteq \rho(A)$ and

$$R(\lambda, A)x = \lambda \int_0^\infty e^{-\lambda t} F(t) x dt \tag{1.15}$$

for all $x \in X$ and $\lambda > \omega$.

If instead of (1.15) the equality

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} F(t) dt, \quad \lambda > \omega,$$

holds, then $(F(t))_{t \geq 0}$ is a strongly continuous semigroup on X(see [ABHN, Thm. 3.1.5]).
Mainly as a consequence of Widder's Theorem A.3, a Hille–Yosida operator is always the generator of an integrated semigroup with the following additional properties (see [ABHN, Section 3.3]).

Proposition 1.27. *Let A be a Hille–Yosida operator of type ω on X and denote $X_0 = \overline{D(A)}^{\|\cdot\|}$. Then there exists an integrated semigroup F on X possessing the following properties.*

(a) The map $\mathbb{R}_+ \ni t \mapsto F(t)x \in X$ is continuously differentiable with respect to the norm for all $x \in D(A)$. The operator family $(F'(t)_{|X_0})_{t \geq 0}$ is a strongly continuous semigroup on X_0.

(b) The integrated semigroup F is given by

$$F(t) = \lim_{k \to \infty} (-1)(k+1) \int_{\frac{k}{t}}^\infty s^k R(s, A)^{k+2} ds \tag{1.16}$$

for all $t > 0$, $F(0) = 0$, and $\|F(t+h) - F(t)\| \leq M \int_t^{t+h} e^{\omega r} dr$ for all $t, h \geq 0$.

Proof. Since $(A, D(A))$ is a Hille–Yosida operator, we are able to apply Widder's Theorem A.3 to the function

$$(\omega, \infty) \ni \lambda \mapsto R(\lambda, A) \in \mathcal{L}(X).$$

Therefore, we obtain that there exists a Lipschitz continuous function $F : \mathbb{R}_+ \longrightarrow \mathcal{L}(X)$ satisfying $F(0) = 0$, $\|F(t+h) - F(t)\| \leq M \int_t^{t+h} e^{\omega r} dr$ for all $t, h > 0$ and some constant $M \geq 1$, and

$$R(\lambda, A)x = \int_0^\infty \lambda e^{-\lambda t} F(t) x dt \tag{1.17}$$

for all $\lambda > \omega$ and $x \in X$. Thus, F is an integrated semigroup with generator $(A, D(A))$. Applying the approximation formula for integrated semigroups from [HN93] (see Theorem A.5), we obtain assertion (b).
By [EN00, Ch. III, Cor. 3.21]) the part A_0 of A in X_0 is the generator of a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on X_0. On the other hand, applying Lemma A.4, we obtain

$$F(t)x - tx = \int_0^t F(s)Axds \tag{1.18}$$

for all $x \in D(A)$. Therefore, the map $\mathbb{R}_+ \ni t \mapsto F(t)x \in X$ is continuously differentiable with respect to the norm for all $x \in D(A)$, and integration by parts yields

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} F'(t)xdt \tag{1.19}$$

for all $\lambda > \omega$ and $x \in D(A)$. Note that $F'(t)$ has a bounded extension to X_0 by part (b). Thus $F'(t)$ coincides with $T_0(t)$ on X_0. □

1.5 A generation theorem

We are now able to state the desired relation between bi–continuous semigroups, integrated semigroups and Hille–Yosida operators in form of a generalized Hille–Yosida theorem for bi–continuous semigroups. This theorem puts in a general framework the Hille–Yosida type theorems due to S. Cerrai for *weakly continuous semigroups* ([Cer94], see Section 3.3), due to J. R. Dorroh and J. W. Neuberger for *semigroups induced by flows* ([DN96], see Section 3.2), and due to O. Bratelli and D. W. Robinson for *adjoint semigroups* ([BR79, Thm. 3.1.10], see Section 3.5).
Let X satisfy Assumptions 1.1.

Theorem 1.28. *Let $A : D(A) \subseteq X \longrightarrow X$ be a linear operator and denote $X_0 = \overline{D(A)}^{\|\cdot\|}$. Then the following assertions are equivalent.*

(a) $(A, D(A))$ *generates a bi–continuous semigroup* $(T(t))_{t\geq 0}$ *(of type ω) on X.*

(b) $(A, D(A))$ *is a bi–densely defined Hille–Yosida operator of type ω, and the family $\{(s-\alpha)^k R(s, A)^k : k \in \mathbb{N}, s > \alpha\}$ is bi–equicontinuous for every $\alpha > \omega$.*

(c) There exists an integrated semigroup F satisfying the following conditions.

(i) $F(\cdot)x \in C^1(\mathbb{R}_+, (X, \|\cdot\|))$ *for all* $x \in D(A)$, $(F'(t)_{|X_0})_{t\geq 0}$ *exists and is a strongly continuous semigroup on* X_0.

(ii) $F(\cdot)x \in C^1(\mathbb{R}_+, (X, \tau))$ *for all* $x \in X$.

(iii) The operator family $(F'(t))_{t\geq 0}$ *is locally bi–equicontinuous.*

(iv) $F'(t)$, $t \geq 0$, *is exponentially bounded on* X.

(v) For all $\lambda > \omega$ *and* $x \in X$ *we have*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} F'(t)x dt.$$

Proof. $(b) \Rightarrow (c)$ First, we assume $\omega < 0$.
As a consequence of Proposition 1.27 there exists an integrated semigroup F satisfying assertion (i).
To prove (ii) we consider, for $x \in X$ and $t \geq 0$, the sequence $(D_m(x,t))_{m\in\mathbb{N}} \subseteq X$ defined as

$$D_m(x,t) := \frac{F(t+1/m)x - F(t)x}{1/m}$$

for all $m \in \mathbb{N}$. As a consequence of Proposition 1.27(b) this sequence is $\|\cdot\|$–bounded. It remains to prove that it is a τ–Cauchy sequence. Let $x \in X, \epsilon > 0, p \in P_\tau$. By the assumptions there exists a $\|\cdot\|$–bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(A)$ which is τ–convergent to x and there exists $n_0 \in \mathbb{N}$ such that

$$p(s^k R(s,A)^k (x - x_n)) \leq \frac{\epsilon}{3} \tag{1.20}$$

for all $n \geq n_0$ and uniformly for $s \geq 0$ and $k \in \mathbb{N}$. Applying estimate (1.20) and

Proposition 1.27(b) we obtain

$$
\begin{aligned}
p&\left(\frac{F(t+1/m)-F(t)}{1/m}(x-x_n)\right)\\
&\qquad= p\left(\lim_{k\to\infty} m\left[(k+1)\int_{\frac{k}{t}}^{\infty} s^k R(s,A)^{k+2}(x-x_n)ds\right.\right.\\
&\qquad\qquad\left.\left. -\,(k+1)\int_{\frac{k}{t+1/m}}^{\infty} s^k R(s,A)^{k+2}(x-x_n)ds\ \right]\right)\\
&\qquad= p\left(m\lim_{k\to\infty}(k+1)\left[\int_{\frac{k}{t+1/m}}^{\frac{k}{t}} s^k R(s,A)^{k+2}(x-x_n)ds\right]\right) \qquad (1.21)\\
&\qquad\le \frac{\epsilon}{3} m\lim_{k\to\infty}(k+1)\left[-\frac{1}{s}\right]_{s=\frac{k}{t+1/m}}^{\frac{k}{t}}\\
&\qquad= \frac{\epsilon}{3}\lim_{k\to\infty}(1+\frac{1}{k})\\
&\qquad= \frac{\epsilon}{3}
\end{aligned}
$$

for all $n \ge n_0$ and uniformly for $m \in \mathbb{N}$ and $t \ge 0$.
Since (i) is valid, $(D_m(x_n,t))_{m\in\mathbb{N}}$ is a τ–Cauchy sequence for all $n \in \mathbb{N}$. Therefore, there exists $m_0 \in \mathbb{N}$ such that

$$
\begin{aligned}
p&\left(\frac{F(t+1/m)x-F(t)x}{1/m}-\frac{F(t+1/l)x-F(t)x}{1/l}\right)\\
&\le p\left(\frac{F(t+1/m)-F(t)}{1/m}(x-x_{n_0})\right)\\
&\quad + p\left(\frac{F(t+1/m)x_{n_0}-F(t)x_{n_0}}{1/m}-\frac{F(t+1/l)x_{n_0}-F(t)x_{n_0}}{1/l}\right)\\
&\quad + p\left(\frac{F(t+1/l)-F(t)}{1/l}(x-x_{n_0})\right)\\
&\le \epsilon
\end{aligned}
$$

for all $m, l \ge m_0$. Since (X,τ) is sequentially complete on $\|\cdot\|$–bounded sets, the map $(t \mapsto F(t)x)$ is differentiable with respect to τ for all $x \in X$.
Before proving that its derivative is continuous we show assertion (iii).
Clearly, by estimate (1.21), the operator family $\{F'(t) : t \ge 0\}$ is globally bi–equicontinuous.

Next, we show the τ–continuity of $F'(\cdot)x$ for all $x \in X$. To that purpose, let $x \in X$, $t_0 \geq 0$, $\epsilon > 0$, and $p \in P_\tau$. Further, let $(x_n)_{n\in\mathbb{N}} \subseteq D(A)$ be a $\|\cdot\|$–bounded sequence which is τ–convergent to $x \in X$. Since $F'(\cdot)x$ is τ–continuous for all $x \in D(A)$, we obtain that there exists $\delta_\epsilon > 0$ depending on n such that $|t - t_0| \leq \delta_\epsilon$ implies

$$p(F'(t)x_n - F'(t_0)x_n) \leq \frac{\epsilon}{3}.$$

Again with estimate (1.21) there exists $n_0 \in \mathbb{N}$ such that $|t - t_0| \leq \delta_\epsilon$ implies

$$\begin{aligned} p(F'(t)x - F'(t_0)x) &\leq p(F'(t)(x - x_{n_0})) + p(F'(t)x_{n_0} - F'(t_0)x_{n_0}) \\ &\quad + p(F'(t_0)(x_{n_0} - x)) \\ &\leq \epsilon. \end{aligned}$$

Therefore, assertion (ii) is shown.

Property (iv) holds by Proposition 1.27(b) and the fact that $(X, \tau)'$ is norming for $(X, \|\cdot\|)$. In fact, for $\Phi := \{\phi \in (X, \tau)' : \ \|\phi\| \leq 1\}$ we have

$$\begin{aligned} \|F'(t)x\| &= \sup_{\phi\in\Phi} | < \tau\text{–} \lim_{h\to 0^+} D_{h,t}x, \phi > | \\ &= \sup_{\phi\in\Phi} | \lim_{h\to 0^+} < D_{h,t}x, \phi > | \\ &\leq M\|x\| \lim_{h\searrow 0} \frac{1}{h} \int_t^{t+h} e^{\omega s} ds \\ &= M e^{\omega t} \|x\| \end{aligned}$$

for all $x \in X$.

To prove (v), we note first that the sequential completeness of (X, τ) on $\|\cdot\|$–bounded sets implies that

$$\tau - \int_0^a e^{-\lambda t} F'(t) x dt$$

exists for all $a \geq 0, \lambda > \omega$, and $x \in X$.

Since $(X, \tau)'$ is norming for $(X, \|\cdot\|)$, we obtain as in the proof of Lemma 1.7(a) that

$$\tau - \int_0^\infty e^{-\lambda t} F'(t) x dt$$

exists.

It remains to prove that

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} F'(t)x dt \tag{1.22}$$

for all $x \in X$ and $\lambda > \omega$. To show this, let $x \in X$, $\epsilon > 0$. By assumption there exists a $\|\cdot\|$–bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(A)$ and $n_0 \in \mathbb{N}$ such that

$$p(R(\lambda, A)(x - x_n)) \leq \frac{\epsilon}{2}$$

for all $n \geq n_0$. Since F is an integrated semigroup, we obtain by Proposition A.1

$$\begin{aligned} &p\left(R(\lambda, A)x - \int_0^\infty e^{-\lambda t} F'(t)x dt\right) \\ &\leq p(R(\lambda, A)(x - x_{n_0})) + p\left(\int_0^\infty e^{-\lambda t} F'(t)(x_{n_0} - x) dt\right) \\ &\leq \epsilon. \end{aligned}$$

Therefore, assertion (v) holds.
By a rescaling argument we obtain the desired properties (i)–(v) for arbitrary $\omega \in \mathbb{R}$.
$(c) \Rightarrow (a)$ We define first

$$T(t)x := F'(t)x = \tau\text{--}\lim_{h \searrow 0} \frac{F(t+h)x - F(t)x}{h}\ , \quad x \in X, t \geq 0.$$

By assumption (c) it remains to show that the semigroup law on X holds. By (i) the semigroup law already holds on X_0. Further, let $x \in X$ and $t, s \geq 0$. By the same arguments as in the proof of Proposition 1.7(b) and equality (iv) there exists a $\|\cdot\|$–bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(A)$ which is τ–convergent to x such that

$$T(t+s)x_n = T(t)T(s)x_n$$

for all $t, s \geq 0$ and $n \in \mathbb{N}$, and

$$\tau\text{--}\lim_{n\to\infty} T(t+s)x_n = T(t+s)x.$$

Since $(T(s)x_n)_{n\in\mathbb{N}}$ is $\|\cdot\|$–bounded and τ–convergent, we also obtain

$$\tau\text{--}\lim_{n\to\infty} T(t)T(s)x_n = T(t)T(s)x.$$

The topology τ is Hausdorff, therefore the limit is unique and the semigroup law holds for $(T(t))_{t\geq 0}$. By the same way we obtain $T(0) = Id$.

$(a) \Rightarrow (b)$ If we have a bi–continuous semigroup $(T(t))_{t\geq 0}$, assertion (b) follows directly from Proposition 1.14, Proposition 1.18 and Proposition 1.19. $\square$

This characterization of the generators of bi–continuous semigroups will play an essential role in the following chapter to establish approximation results for bi–continuous semigroups. A concrete application of Theorem 1.28 is given in Section 3.2.

Chapter 2

Approximation of bi–continuous semigroups

In this chapter we study the convergence of sequences of bi–continuous semigroups $(T_k(t))_{t\geq 0}$ on a Banach space X with two topologies (see Assumptions 1.1). To that purpose, we need to impose stability conditions on $(T_k(t))_{t\geq 0}$ which are the basis for generalized Trotter–Kato theorems. Results on approximation theory on locally convex spaces can be found, e.g., in [Yos74] for *equicontinuous semigroups*, [Buc68] for semigroups on Fréchet spaces, and [Ōuc73], [AK00] for *locally equicontinuous semigroups*. For the classical results on C_0–semigroups we refer to [Dav80], [Gol85], [Paz92], [EN00] and the references therein. We present first a version in which we obtain the convergence of $(T_k(t))_{t\geq 0}$ to a bi–continuous semigroup $(T(t))_{t\geq 0}$ by assuming that $R(\lambda, A_k)$ is pointwise $\|\cdot\|$–convergent to the resolvent of the generator A of $(T(t))_{t\geq 0}$ on a $\|\cdot\|$–dense subset of $\overline{D(A)}^{\|\cdot\|}$. A second approximation theorem, more valuable for the applications, permits us to conclude that an operator A is the generator of a bi–continuous semigroup only by assuming that a sequence $(A_k)_{k\in\mathbb{N}}$ of generators converges to it.

We then use our results to obtain a Chernoff Product Formula which, in the C_0–case, goes back to [Che68]. From this formula we then deduce the Post–Widder Inversion Formula representing a bi–continuous semigroup in terms of the resolvents of its generator.

A classical result of S. Lie around 1900 says that for $n \times n$–matrices A and B the exponential of their sum is

$$e^{t(A+B)} = \lim_{k\to\infty} \left(e^{A_{t/k}} e^{B_{t/k}}\right)^k$$

for all $t \in \mathbb{R}$. The extension of this formula to generators A and B of strongly continuous semigroups on Banach spaces was first considered by H. F. Trotter [Tro59], and then, e.g., by P. R. Chernoff [Che68], [Che74]. In this thesis we obtain such a Lie–Trotter Product Formula for bi–continuous semigroups as a consequence of the Chernoff Product Formula.
Moreover, in combination with Section 3.3, we give an answer to a question recently asked by G. Da Prato at the Trento EVEQ 2000 conference in the context of semigroups corresponding to evolution equations for convex gradient systems (see [DP00]).

2.1 Generalized Trotter–Kato theorems

We now investigate the relation between the convergence of bi–continuous semigroups, their resolvents and generators. To that purpose, we first introduce the notion of *uniformly bi–continuous* semigroups which will play the role of the stability condition essential for the subsequent approximation theory.
As in Chapter 1 we assume that the underlying space X satisfies Assumptions 1.1, P_τ denotes a family of seminorms inducing the locally convex topology τ on X, assuming, without loss of generality, that $p(x) \leq \|x\|$ for all $x \in X$ and $p \in P_\tau$.

Definition 2.1. *Let $(T_k(t))_{t\geq 0}, k \in \mathbb{N}$, be bi–continuous semigroups on X. They are called* ***uniformly bi–continuous*** *(of type ω) if the following conditions hold.*

(i) $\|T_k(t)\| \leq Me^{\omega t}$ *for all* $t \geq 0$ *and* $k \in \mathbb{N}$ *and some constants* $M \geq 1, \omega \in \mathbb{R}$.

(ii) $(T_k(t))_{t\geq 0}$ *are locally bi–equicontinuous uniformly for* $k \in \mathbb{N}$*, i.e., for every* $t_0 \geq 0$ *and for every* $\|\cdot\|$*–bounded sequence* $(x_n)_{n\in\mathbb{N}} \subseteq X$ *which is* τ*–convergent to* $x \in X$ *we have that*

$$\tau\text{–}\lim_{n\to\infty} (T_k(t)(x_n - x)) = 0$$

uniformly for $0 \leq t \leq t_0$ *and* $k \in \mathbb{N}$.

Lemma 2.2. *Let $(T_k(t))_{t\geq 0}, k \in \mathbb{N}$, be uniformly bi–continuous semigroups (of type ω) with generators A_k. Then, for every $\|\cdot\|$–bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ which is τ–convergent to $x \in X$ and for every $\alpha > \omega$, we have*

$$\tau\text{-}\lim_{n\to\infty} e^{-\alpha t} T_k(t)(x_n - x) = 0 \tag{2.1}$$

uniformly for all $k \in \mathbb{N}$ and $t \geq 0$, and

$$\tau\text{-}\lim_{n\to\infty} (\lambda - \alpha)^l R(\lambda, A_k)^l (x_n - x) = 0 \tag{2.2}$$

uniformly for $l, k \in \mathbb{N}$ and $\lambda > \alpha$.

Proof. Assertion (2.1) is an easy consequence of Proposition 1.4(b) and Definition 2.1. To prove the second assertion, let $x \in X$, $\epsilon > 0$, $p \in P_\tau$, $\alpha > \omega$, and $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a $\|\cdot\|$–bounded sequence which is τ–convergent to x. Combining (2.1) and Formula (1.6) from Chapter 1, we obtain, in analogy to the proof of Proposition 1.19(b), that there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} p((\lambda-\alpha)^l R(\lambda, A_k)^l (x_n - x)) &\leq \frac{(\lambda-\alpha)^l}{(l-1)!} \int_0^\infty t^{l-1} e^{-(\lambda-\alpha)t} p(e^{-\alpha t} T_k(t)(x_n - x)) dt \\ &\leq \epsilon \end{aligned}$$

for all $n \geq n_0$ and uniformly for $k, l \in \mathbb{N}$ and $\lambda > \alpha$. Therefore, assertion (2.2) holds. $\square$

We are now able to state the generalization of what in [EN00, Ch.III, Sec. 4] is called the First Trotter–Kato Approximation Theorem.

Theorem 2.3. *Let $(T_k(t))_{t\geq 0}, k \in \mathbb{N}$, and $(T(t))_{t\geq 0}$ be uniformly bi–continuous semigroups (of type ω) on X with generators A_k and A, respectively, and let D be a $\|\cdot\|$–dense subset of $\overline{D(A)}^{\|\cdot\|}$.*
If

$$R(\lambda_0, A_k)x \xrightarrow{\|\cdot\|} R(\lambda_0, A)x$$

for all $x \in D$ and some $\lambda_0 > \alpha > \omega$ as $k \to \infty$, then

$$T_k(t)x \xrightarrow{\tau} T(t)x$$

for all $x \in X$ as $k \to \infty$. Moreover, the convergence is uniform for t in compact intervals of $\mathbb{R}_+$.

Proof. Let $t_0 \geq 0$. The first step is to show that

$$\tau\text{–}\lim_{k\to\infty}(T_k(t) - T(t))R(\lambda_0, A)y = 0 \tag{2.3}$$

for all $y \in D$ and uniformly for $0 \leq t \leq t_0$. The proof is similar to the one given in [Paz92, Thm. 3.4.2]. For $y \in D$, $0 \leq t \leq t_0$ and $p \in P_\tau$, we have

$$\begin{aligned}
&p((T_k(t) - T(t))R(\lambda_0, A)y)\\
&\quad\leq p(T_k(t)(R(\lambda_0, A) - R(\lambda_0, A_k))y)\\
&\qquad + p(R(\lambda_0, A_k)(T_k(t) - T(t))y) + p((R(\lambda_0, A_k) - R(\lambda_0, A))T(t)y)\\
&\quad\leq \|T_k(t)(R(\lambda_0, A) - R(\lambda_0, A_k))y\|\\
&\qquad + p(R(\lambda_0, A_k)(T_k(t) - T(t))y) + \|(R(\lambda_0, A_k) - R(\lambda_0, A))T(t)y\|\\
&\quad=: a_k(t) + b_k(t) + c_k(t).
\end{aligned} \tag{2.4}$$

Since $\|T_k(t)\| \leq C$ for all $0 \leq t \leq t_0$ and some constant $C > 0$, it follows that $a_k(t) \to 0$ uniformly on $[0, t_0]$ as $k \to \infty$. Further, the $\|\cdot\|$–continuity of the map $t \mapsto T(t)y$ by Proposition 1.27 implies that the set $\{T(t)y :\ 0 \leq t \leq t_0\} \subseteq D(A)$ is $\|\cdot\|$–compact. By the $\|\cdot\|$–density of D in $\overline{D(A)}^{\|\cdot\|}$, we obtain, by [Sch80, Ch. III, Thm. 4.5], $c_k(t) \to 0$ uniformly on $[0, t_0]$ as $k \to \infty$.
It remains to prove that $b_k(t) \to 0$ uniformly on $[0, t_0]$ as $k \to \infty$. To show this, we consider, for each $t \in [0, t_0], k \in \mathbb{N}$, and $y \in D$, the map

$$[0, t] \ni s \mapsto T_k(t - s)R(\lambda_0, A_k)T(s)R(\lambda_0, A)y \in D(A_k)$$

which is τ–differentiable in $[0, t]$, and its derivative is given by

$$\begin{aligned}
[0, t] \ni s \mapsto &-T_k(t - s)A_kR(\lambda_0, A_k)T(s)R(\lambda_0, A)y\\
&+ T_k(t - s)R(\lambda_0, A_k)T(s)AR(\lambda_0, A)y \in X.
\end{aligned}$$

Consequently, for each $t \in [0, t_0]$ and $y \in D$ we have

$$\begin{aligned}
&p(R(\lambda_0, A_k)(T_k(t) - T(t))R(\lambda_0, A)y)\\
&\quad\leq \int_0^t p(T_k(t - s)[-A_kR(\lambda_0, A_k)T(s) + R(\lambda_0, A_k)T(s)A]R(\lambda_0, A)y)ds\\
&\quad= \int_0^t p(T_k(t - s)\{[Id - \lambda_0R(\lambda_0, A_k)]R(\lambda_0, A)\\
&\qquad\qquad + R(\lambda_0, A_k)[\lambda_0R(\lambda_0, A) - Id]\}T(s)y)ds\\
&\quad\leq C\int_0^{t_0} \|[R(\lambda_0, A) - R(\lambda_0, A_k)]T(s)y\|ds,
\end{aligned}$$

which converges to zero uniformly on $[0, t_0]$ as $k \to \infty$ by assumption and the same compactness argument as above. Therefore,

$$R(\lambda_0, A_k)(T_k(t) - T(t))y \xrightarrow{\tau} 0$$

for all $y \in R(\lambda_0, A)D$ uniformly on $[0, t_0]$ as $k \to \infty$.
By (2.4) and Proposition 1.18(c) we obtain

$$\tau\text{–}\lim_{k\to\infty} (T_k(t) - T(t))R(\lambda_0, A)y = 0$$

for all $y \in D$ and uniformly for $0 \le t \le t_0$. It follows that

$$T_k(t)x \xrightarrow{\tau} T(t)x$$

for all $x \in R(\lambda_0, A)D$ and uniformly for $0 \le t \le t_0$.
Now, let $x \in X$, $\epsilon > 0$ and $p \in P_\tau$. Since $D(A)$ is bi–dense in X and D is $\|\cdot\|$–dense in $D(A)$, there exists a $\|\cdot\|$–bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ which is τ–convergent to x. By the stability condition from Definition 2.1(ii) there exists $n_0 \in \mathbb{N}$ such that

$$p((T_k(t) - T(t))(x_n - x)) \le \epsilon$$

for all $n \ge n_0$ and uniformly for $k \in \mathbb{N}$ and $0 \le t \le t_0$. Applying Proposition 1.18(c), there exists a sequence $(y_n)_{n\in\mathbb{N}} \subseteq D$ such that $(R(\lambda_0, A)y_n)_{n\in\mathbb{N}} \subseteq D(A)$ is $\|\cdot\|$–bounded and τ–convergent to $x_{n_0} \in D$. Thus, there exists $\tilde{n}_0 \ge n_0$ such that

$$p((T_k(t) - T(t))(x - x_{n_0})) + p((T_k(t) - T(t))(x_{n_0} - R(\lambda_0, A)y_n)) \le \frac{2\epsilon}{3}$$

for all $n \ge \tilde{n}_0$ and uniformly for $k \in \mathbb{N}$ and $0 \le t \le t_0$. By (2.3) there exists $k_0 \in \mathbb{N}$ such that

$$p((T_k(t) - T(t))R(\lambda_0, A)y_{\tilde{n}_0}) \le \frac{\epsilon}{3}$$

for all $k \ge k_0$ and uniformly for $0 \le t \le t_0$, and hence

$$\begin{aligned} p(T_k(t)x - T(t)x) &\le p((T_k(t) - T(t))(x - x_{n_0})) \\ &\quad + p((T_k(t) - T(t))(x_{n_0} - R(\lambda_0, A)y_{\tilde{n}_0})) \\ &\quad + p((T_k(t) - T(t))R(\lambda_0, A)y_{\tilde{n}_0}) \\ &\le \epsilon \end{aligned}$$

for all $k \ge k_0$ and uniformly for $0 \le t \le t_0$. □

Remark 2.4. 1) Let $(T_k(t))_{t\geq 0}, k \in \mathbb{N}$, and $(T(t))_{t\geq 0}$ be uniformly bi–continuous semigroups on X with generators A_k and A, respectively. If

$$T_k(t)x \xrightarrow{\tau} T(t)x$$

for all $x \in X$ as $k \to \infty$ uniformly for t in compact intervals of $\mathbb{R}_+$, then for $\lambda > \alpha > \omega$, $p \in P_\tau$ and $x \in X$, we obtain

$$p(R(\lambda, A_k)x - R(\lambda, A)x) \leq \int_0^\infty e^{-\lambda t} p(T_k(t)x - T(t)x)dt,$$

which converges to zero as $k \to \infty$ by Lebesgue's dominated convergence theorem.
2) Let $(T_k(t))_{t\geq 0}, k \in \mathbb{N}$, and $(T(t))_{t\geq 0}$ be uniformly bi–continuous semigroups on X with generators A_k and A, respectively, such that

($\star$) for each $t_0 > 0$ and $p \in P_\tau$ there exists $q \in P_\tau$ such that $p(T_k(t)x) \leq q(x)$ for all $0 \leq t \leq t_0$, $x \in X$ and uniformly for $k \in \mathbb{N}$.

Then we obtain as in the proof of Theorem 2.3 that the τ–convergence of $R(\lambda_0, A_k)x$ to $R(\lambda_0, A)x$ for all $x \in X$ implies that $T_k(t)x \xrightarrow{\tau} T(t)x$ for all $x \in X$ as $k \to \infty$ and uniformly for t in compact intervals of $\mathbb{R}_+$ (cf. [Yos74, Ch. IX, Thm. 12.1], [AK00, Thm. 15]).
It remains an open question if condition ($\star$) is redundant.

Before proving the second approximation result we restate Proposition III. 4.4 from [EN00] replacing the norm convergence by τ–convergence.

Lemma 2.5. *Let $(T_k(t))_{t\geq 0}, k \in \mathbb{N}$, be uniformly bi–continuous semigroups (of type ω) on X with generators $(A_k, D(A_k))$. If*

$$\tau\text{–}\lim_{k\to\infty} R(\lambda_0, A_k)x$$

exists for all $x \in X$ and some $\lambda_0 > \omega$, then

$$R(\lambda)x := \tau\text{–}\lim_{k\to\infty} R(\lambda, A_k)x$$

exists for all $x \in X$ and $Re\lambda > \omega$ and defines a pseudoresolvent.

Proof. By [EN00, Ch. IV, Prop. 1.3] $R(\lambda, A_k)x$ has a power series expansion around λ_0 given by

$$R(\lambda, A_k)x = \sum_{j\geq 0}(\lambda-\lambda_0)^j R(\lambda_0, A_k)^{j+1}x,$$

where the series converges in X uniformly for $k \in \mathbb{N}$ if $|\lambda - \lambda_0| \leq \epsilon(\lambda_0 - \omega)$ for each $0 < \epsilon < 1$. We now show that $\tau - \lim_{k\to\infty} R(\lambda, A_k)x$ exists for all λ with $|\lambda-\lambda_0| \leq \epsilon(\lambda_0-\omega)$. Since X is sequentially complete on $\|\cdot\|$–bounded sets, it is sufficient to prove that $\{R(\lambda, A_k)x\}_{k\in\mathbb{N}}$ is a τ–Cauchy sequence in X. To that purpose, let $p \in P_\tau$ and $\epsilon > 0$. There exists $h_0 \in \mathbb{N}$ such that

$$p\left(\sum_{j>h_0}(\lambda-\lambda_0)^j (R(\lambda_0, A_k)^{j+1} - R(\lambda_0, A_l)^{j+1})x\right) \leq \frac{\epsilon}{2}$$

for all $k, l \geq h_0$. Further, using Lemma 2.2, $\{R(\lambda_0, A_k)^j x\}_{k\in\mathbb{N}}$ is a τ–Cauchy sequence in X for all $j \in \mathbb{N}$, and hence there exists $n_0 \geq h_0$ such that

$$\begin{aligned}
&p(R(\lambda, A_k)x - R(\lambda, A_l)x) \\
&\quad = p\left(\sum_{j\geq 0}(\lambda-\lambda_0)^j (R(\lambda_0, A_k)^{j+1} - R(\lambda_0, A_l)^{j+1})x\right) \\
&\quad \leq \sum_{j=0}^{h_0}(\lambda-\lambda_0)^j p(R(\lambda_0, A_k)^{j+1}x - R(\lambda_0, A_l)^{j+1}x) + \frac{\epsilon}{2} \\
&\quad \leq \epsilon
\end{aligned}$$

for all $k, l \geq n_0$.
Furthermore, the above considerations imply that the set of all $\lambda \in \mathbb{C}$ with $Re\lambda > \omega$ such that $\lim_{k\to\infty} R(\lambda, A_k)x$ exists for all $x \in X$ is open and relatively closed (cf. [EN00, Ch. III, Prop. 4.4]). Therefore, it coincides with the set $\{\lambda \in \mathbb{C} : Re\lambda > \omega\}$. Clearly, the resolvent equation (RE)[1] also holds for the operators $R(\lambda), Re\lambda > \omega$. This concludes the proof. □

In the following main result we extend the above considerations by giving conditions such that the convergence of $R(\lambda_0, A_k)x$ to some limit implies that this limit is the resolvent of a suitable generator. Moreover, we obtain that the convergence of a sequence of generators to an operator already implies that it is a generator.

[1]see Definition 1.8

Theorem 2.6. *Let $(T_k(t))_{t\geq 0}, k \in \mathbb{N}$, be uniformly bi–continuous semigroups (of type ω) on X with generators $(A_k, D(A_k))$. Consider for some $\lambda_0 > \alpha > \omega$ the following assertions.*

(a) There exists a bi–densely defined operator $(A, D(A))$ such that $A_k x \xrightarrow{\|\cdot\|} Ax$ for all x in a bi–core[2] D for A and $rg(\lambda_0 - A)$ is bi–dense in X.

(b) There exists an operator $R \in \mathcal{L}(X)$ such that $R(\lambda_0, A_k)x \xrightarrow{\|\cdot\|} Rx$ for all x in a subset of rgR which is bi–dense in X.

(c) There exists a bi–continuous semigroup $(T(t))_{t\geq 0}$ with generator $(B, D(B))$ such that

$$T_k(t)x \xrightarrow{\tau} T(t)x$$

for all $x \in X$ and uniformly for t in compact intervals of $\mathbb{R}_+$.

Then, the implications

$$(a) \Longrightarrow (b) \Longrightarrow (c)$$

hold.
In particular, if (a) holds, then the bi–closure $\overline{A}^\tau$ of A is equal to B.

Before proving the theorem we state the following simple lemma.

Lemma 2.7. *Let $(C, D(C))$ be a bi–continuous operator, i.e., for all sequences $(x_n)_{n\in\mathbb{N}} \subseteq D(C)$ with $(x_n)_{n\in\mathbb{N}}$ and $(Cx_n)_{n\in\mathbb{N}}$ $\|\cdot\|$–bounded and $x_n \xrightarrow{\tau} x \in X$ we have $Cx_n \xrightarrow{\tau} Cx$. If C is bi–closed, then its domain $D(C)$ is bi–closed.*

We are now able to prove Theorem 2.6.
Proof. $(a) \Rightarrow (b)$ Take $y := (\lambda_0 - A)x, x \in D$. Then

$$\begin{aligned} R(\lambda_0, A_k)y &= R(\lambda_0, A_k)[(\lambda_0 - A_k)x - (\lambda_0 - A_k)x + (\lambda_0 - A)x] \\ &= x + R(\lambda_0, A_k)[A_k x - Ax], \end{aligned} \tag{2.5}$$

which converges to $x =: Ry$ as $k \to \infty$. Moreover, rgR contains D. Since D is a bi–core for A and $D(A)$ is bi–dense in X, we have that D is bi–dense in X,

[2] see Definition 1.20

and hence rgR is bi–dense in X. It remains to show that the operator R can be $\|\cdot\|$–continuously extended to X. To that purpose, we show that for $x \in X$ the $\|\cdot\|$–bounded sequence $(R(\lambda_0, A_k)x)_{k\in\mathbb{N}}$ is a τ–Cauchy sequence. Let $x \in X$, $p \in P_\tau$, and $\epsilon > 0$. Since $rg(\lambda_0 - A)$ is bi–dense in X, there exists a sequence $(z_n)_{n\in\mathbb{N}} \subseteq D(A)$ and $n_0 \in \mathbb{N}$ such that

$$p(x - (\lambda_0 - A)z_n) \leq \frac{\epsilon}{2}$$

for all $n \geq n_0$. Further, since D is a bi–core for A, there exists a $\|\cdot\|$–bounded sequence $y_n := (\lambda_0 - A)x_n, x_n \in D, n \in \mathbb{N}$, and $\tilde{n}_0 \geq n_0$ such that

$$p(x - y_n) \leq p(x - (\lambda_0 - A)z_{n_0}) + p((\lambda_0 - A)z_{n_0} - y_n) \leq \epsilon$$

for all $n \geq \tilde{n}_0$. By Lemma 2.2, there exists $\hat{n} \in \mathbb{N}$ such that

$$p(R(\lambda_0, A_k)(x - y_n)) \leq \frac{\epsilon}{3} \tag{2.6}$$

for all $n \geq \hat{n}$ and uniformly for $k \in \mathbb{N}$. With equation (2.5) there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} &p(R(\lambda_0, A_k)x - R(\lambda_0, A_l)x) \\ &\quad\leq p(R(\lambda_0, A_k)(x - y_{n_0})) + \|R(\lambda_0, A_k)y_{n_0} - R(\lambda_0, A_l)y_{n_0}\| \\ &\qquad + p(R(\lambda_0, A_l)(y_{n_0} - x)) \\ &\quad\leq \epsilon \end{aligned} \tag{2.7}$$

for all $k, l \geq k_0$. Thus,

$$\tau\text{–}\lim_{k\to\infty} R(\lambda_0, A_k)x =: Rx$$

exists for all $x \in X$. Since $(X, \tau)'$ is norming for $(X, \|\cdot\|)$, we have for
$\Phi := \{\phi \in (X, \tau)' : \ \|\phi\|_{(X,\|\cdot\|)'} \leq 1\}$ that

$$\|Rx\| = \|\tau\text{–}\lim_{k\to\infty} R(\lambda_0, A_k)x\| = \sup_{\phi\in\Phi} |\lim_{k\to\infty} < R(\lambda_0, A_k)x, \phi > | \leq \frac{1}{\lambda_0 - \omega}\|x\|$$

for all $x \in X$, and therefore $R \in \mathcal{L}(X)$.
$(b) \Rightarrow (c)$ As in (2.6) and (2.7), we obtain that $\tau\text{–}\lim_{k\to\infty} R(\lambda_0, A_k)x$ exists for all $x \in X$ and for some $\lambda_0 > \alpha > \omega$. Without loss of generality we assume $\alpha = 0$. By Lemma 2.5 we obtain that

$$R(\lambda)x := \tau\text{–}\lim_{k\to\infty} R(\lambda, A_k)x$$

exists for all $x \in X$ and $\lambda > 0$ and defines a pseudoresolvent. This pseudoresolvent has a bi–dense range $rgR(\lambda) = rgR$ in X, and

$$\|\lambda R(\lambda)\| \leq M$$

for all $\lambda > 0$ and some constant $M \geq 1$. Since

$$R(\lambda)^l x = \tau\text{–}\lim_{k\to\infty} R(\lambda, A_k)^l x$$

for all $x \in X$ and $l \in \mathbb{N}$, we have

$$\|\lambda^l R(\lambda)^l\| \leq M$$

for all $\lambda > 0$ and $l \in \mathbb{N}$. Let now $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a $\|\cdot\|$–bounded sequence which is τ–convergent to $x \in X$. Further, let $p \in P_\tau$, $\epsilon > 0$. By Lemma 2.2 there exists $n_0 \in \mathbb{N}$ such that

$$p(\lambda^l R(\lambda, A_k)^l (x - x_n)) \leq \epsilon$$

for all $n \geq n_0$ and uniformly for $k, l \in \mathbb{N}$ and $\lambda > 0$, and hence

$$p(\lambda^l R(\lambda)^l (x - x_n)) \leq \epsilon$$

for all $n \geq n_0$ and uniformly for $\lambda > 0$ and $l \in \mathbb{N}$. Applying Corollary 1.11, we obtain the existence of a bi–densely defined operator $(B, D(B))$ such that $R(\lambda) = R(\lambda, B)$ for all $\lambda > 0$. It satisfies

$$\|\lambda^l R(\lambda, B)^l\| \leq M$$

for all $\lambda > 0$ and $l \in \mathbb{N}$, and the operator family $\{\lambda^l R(\lambda, B)^l : \lambda > 0, l \in \mathbb{N}\}$ is bi–equicontinuous. Applying Theorem 1.28 we obtain that $(B, D(B))$ generates a bi–continuous semigroup $(T(t))_{t\geq 0}$. We can now apply Theorem 2.3 in order to conclude that

$$T_k(t)x \xrightarrow{\tau} T(t)x$$

for all $x \in X$ as $k \to \infty$. Moreover, the convergence is uniform for t in compact intervals of $\mathbb{R}_+$.

In the final step, we show that condition (a) implies $\overline{A}^\tau = B$. Since $R(\lambda_0, B) = R$,

we have $R(\lambda_0, B)(\lambda_0 - A)x = x$ for all $x \in D$, and hence $D \subseteq D(B)$. We now take $x \in D(A)$. Since D is a bi–core for A, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D \subseteq D(B)$ such that $(x_n)_{n\in\mathbb{N}}$ and $(Ax_n)_{n\in\mathbb{N}}$ are $\|\cdot\|$–bounded, $x_n \xrightarrow{\tau} x$ and $Ax_n \xrightarrow{\tau} Ax$. Since $Bx_n = Ax_n$ for every $n \in \mathbb{N}$, we obtain $Bx_n \xrightarrow{\tau} Ax$. By Proposition 1.18(a) the operator B is bi–closed. Therefore, $x \in D(B)$ and $Bx = Ax$. Thus, we have shown that $D(A) \subseteq D(B)$ and $Bx = Ax$ for all $x \in D(A)$.
Moreover, $(\lambda_0 - A)^{-1}$ exists and its bi–closure $(\lambda_0 - \overline{A}^{\tau})^{-1}$ is contained in $R(\lambda_0, B)$. Since $R(\lambda_0, B)$ is bi–continuous, we obtain that $(\lambda_0 - \overline{A}^{\tau})^{-1}$ is bi–continuous. Further, the domain $D((\lambda_0 - \overline{A}^{\tau})^{-1})$ contains the range $rg(\lambda_0 - A)$ which is bi–dense in X by assumption. Applying Lemma 2.7, this implies that $D((\lambda_0 - \overline{A}^{\tau})^{-1}) = X$. Consequently, we obtain $R(\lambda_0, B) = R(\lambda_0, \overline{A}^{\tau})$, and therefore $\overline{A}^{\tau} = B$. □

2.2 Approximation formulas

We apply our approximation results to obtain a generalization of the Chernoff Product Formula to bi–continuous semigroups. Based on this formula we easily obtain the Post–Widder Formula representing a bi–continuous semigroup by products of the resolvents of its generator. Moreover, this leads to the Lie–Trotter Product Formula for a bi–continuous semigroup whose generator is the sum of two generators. First, we recall Lemma III.5.1 from [EN00].

Lemma 2.8. *Let $S \in \mathcal{L}(X)$. Assume that $\|S^m\| \leq M$ for all $m \in \mathbb{N}$ and some constant $M \geq 1$. Then we have*

$$\|e^{n(S-Id)}x - S^n x\| \leq \sqrt{n}\|Sx - x\|$$

for all $x \in X$ and $n \in \mathbb{N}$.

Proposition 2.9. *Let $V : \mathbb{R}_+ \longrightarrow \mathcal{L}(X)$ satisfy the following conditions.*

(i) $V(0) = Id$.

(ii) $\|V(t)^m\| \leq Me^{m\omega t}$ *for all* $t \geq 0, m \in \mathbb{N}$, *and some constants* $M \geq 1$ *and* $\omega \in \mathbb{R}$.

(iii) The operator family $\{V(t)^m : t \geq 0\}$ *is locally bi–equicontinuous uniformly for* $m \in \mathbb{N}$.

(iv) $Ax := \|\cdot\|\text{–}\lim_{s\searrow 0} \frac{V(s)x-x}{s}$ *exists for all* $x \in D \subseteq X$, *where* D *and* $(\lambda_0 - A)D$ *are bi–dense subsets in* X *for some* $\lambda_0 > \alpha > \omega$.

Then the bi–closure $\overline{A}^{\tau}$ *of* A *generates a bi–continuous semigroup* $(T(t))_{t\geq 0}$ *which is given by the* **Chernoff Product Formula**, *i.e.,*

$$T(t)x = \tau\text{–}\lim_{k\to\infty} \left[V(\tfrac{t}{k})\right]^k x$$

for all $x \in X$ *and uniformly for* t *in compact intervals of* $\mathbb{R}_+$.

Proof. Without loss of generality we assume $\omega = 0$. For $s > 0$ we define

$$A_k := \frac{V(\frac{s}{k}) - Id}{\frac{s}{k}} \in \mathcal{L}(X), \quad k \in \mathbb{N},$$

and observe that $A_k x \stackrel{\|\cdot\|}{\longrightarrow} Ax$ for all $x \in D$ as $k \to \infty$. Since the operators A_k are bounded, the semigroups $(e^{tA_k})_{t\geq 0}$ are uniformly continuous and satisfy

$$\|e^{tA_k}\| \leq e^{-\frac{tk}{s}} \sum_{m=0}^{\infty} \frac{\left(\frac{tk}{s}\right)^m}{m!} \|[V(\tfrac{s}{k})]^m\| \leq M$$

for all $t \geq 0$ and some $M \geq 1$. Moreover, by assumption (iii), for every $\|\cdot\|$–bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ which is τ–convergent to $x \in X$ we have for $\epsilon > 0$ and $p \in P_\tau$ that there exists $n_0 \in \mathbb{N}$ such that

$$p([V(\tfrac{s}{k})]^m (x_n - x)) \leq \epsilon$$

for all $n \geq n_0$ and uniformly for $k, m \in \mathbb{N}$. Hence,

$$p(e^{tA_k}(x_n - x)) \leq e^{-\frac{tk}{s}} \sum_{m=0}^{\infty} \frac{\left(\frac{tk}{s}\right)^m}{m!} p([V(\tfrac{s}{k})]^m (x_n - x)) \leq \epsilon$$

for all $n \geq n_0$ and uniformly for $k \in \mathbb{N}$ and $t \geq 0$. Therefore, the semigroups $(e^{tA_k})_{t\geq 0}$ are bi–equicontinuous uniformly for $k \in \mathbb{N}$, and hence, according to Definition 2.1, uniformly bi–continuous. Consequently, the assumptions of Theorem 2.6(a) are satisfied, and we obtain that the bi–closure $\overline{A}^\tau$ of A generates a bi–continuous semigroup $(T(t))_{t\geq 0}$ such that

$$e^{tA_k}x \xrightarrow{\tau} T(t)x \tag{2.8}$$

for all $x \in X$ as $k \to \infty$, where the convergence is uniform for t in compact intervals of $\mathbb{R}_+$. Further, by Lemma 2.8, we have

$$\|e^{tA_k}y - V(\tfrac{t}{k})^k y\| \leq \frac{tM}{\sqrt{k}}\|A_k y\| \tag{2.9}$$

for all $y \in D$. This converges to zero as $k \to \infty$ uniformly for t in compact intervals of $\mathbb{R}_+$.

Finally, let $x \in X, 0 \leq t \leq t_0, p \in P_\tau$ and $\epsilon > 0$. By assumption (iii) and (iv) there exists a $\|\cdot\|$–bounded sequence $(y_n)_{n\in\mathbb{N}} \subseteq D$ which is τ–convergent to x, and there exists $n_0 \in \mathbb{N}$ such that

$$p(T(t)(x - y_n)) + p(V(\tfrac{t}{k})^k(y_n - x)) \leq \frac{\epsilon}{2} \tag{2.10}$$

for all $n \geq n_0$, uniformly for $k \in \mathbb{N}$ and $0 \leq t \leq t_0$. Combining (2.8)–(2.10) we can conclude that there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} p(T(t)x - V(\tfrac{t}{k})^k x) &\leq p(T(t)(x - y_{n_0})) + p(T(t)y_{n_0} - V(\tfrac{t}{k})^k y_{n_0}) \\ &\quad + p(V(\tfrac{t}{k})^k(y_{n_0} - x)) \\ &\leq \frac{\epsilon}{2} + \|e^{tA_k}y_{n_0} - V(\tfrac{t}{k})^k y_{n_0}\| \\ &\leq \epsilon \end{aligned}$$

for all $k \geq k_0$ uniformly for $0 \leq t \leq t_0$. Therefore, $(T(t))_{t\geq 0}$ is given by the Chernoff Product Formula and the proof is complete. □

Corollary 2.10. *For every bi–continuous semigroup $(T(t))_{t\geq 0}$ of type ω on X with generator $(A, D(A))$ the* **Post–Widder Inversion Formula** *holds, i.e.,*

$$T(t)x = \tau\text{–}\lim_{k\to\infty} \left[\tfrac{k}{t}R(\tfrac{k}{t}, A)\right]^k x \tag{2.11}$$

for all $x \in X$ and uniformly for t in compact intervals of $\mathbb{R}_+$.

Proof. Let us assume without loss of generality that $(T(t))_{t\geq 0}$ is of type $\omega < 0$ and define

$$V(t) := \begin{cases} Id & \text{if } t = 0, \\ \frac{1}{t}R(\frac{1}{t}, A) & \text{if } t \in (0, -\frac{1}{\omega}), \\ 0 & \text{if } t \geq -\frac{1}{\omega}. \end{cases}$$

Then, the map $V : \mathbb{R}_+ \longrightarrow \mathcal{L}((X, \|\cdot\|))$ possess the following properties.

(i) $V(0) = Id$.

(ii) $\|V(t)^m\| \leq \frac{M}{(1-t\omega)^m} \leq M$ for all $t > 0, m \in \mathbb{N}$, and some constant $M \geq 1$.

(iii) Let $t_0 \geq 0$, $p \in P_\tau$, and $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a $\|\cdot\|$–bounded sequence which is τ–convergent to $x \in X$. Then, by Proposition 1.19(b), we obtain

$$p(V(t)^m(x - x_n)) = p(\left[\tfrac{1}{t}R(\tfrac{1}{t}, A)\right]^m (x - x_n)) \to 0$$

uniformly for $m \in \mathbb{N}$ and $t > 0$ as $n \to \infty$.

(iv) Since the part A_0 of A in $\overline{D(A)}^{\|\cdot\|}$ is the generator of $(T(t))_{t\geq 0}$ restricted to $\overline{D(A)}^{\|\cdot\|}$ (cf. Chapter 1, Section 1.3), we obtain that $D(A_0)$ and $(\lambda_0 - A)D(A_0)$ are bi–dense in X. Moreover, for $x \in D(A_0)$ we have

$$\|\cdot\|\text{–}\lim_{t\searrow 0} \frac{V(t)x - x}{t} = \|\cdot\|\text{–}\lim_{t\searrow 0} \tfrac{1}{t}R(\tfrac{1}{t}, A)Ax = Ax.$$

Therefore, taking $D := D(A_0)$, we obtain that assumptions (i)–(iv) of Proposition 2.9 are fulfilled. Thus, $(T(t))_{t\geq 0}$ is given by

$$T(t)x = \tau\text{–}\lim_{k\to\infty} \left[V(\tfrac{t}{k})\right]^k x = \tau\text{–}\lim_{k\to\infty} \left[\tfrac{k}{t}R(\tfrac{k}{t}, A)\right]^k x$$

for all $x \in X$ and uniformly for t in compact intervals of $\mathbb{R}_+$. □

The next corollary states that under *stability* and *consistency* conditions on two bi–continuous semigroups generated by A and B, respectively, the closure of the sum of A and B is a generator and the perturbed semigroup can be represented by the so–called Lie–Trotter Product Formula. For results on the classical Lie–Trotter Product Formula for C_0–semigroups we refer to [Tro59], [Gol70], [Kat78], [Che74], [EN00], [KW00], [KW] and the references therein.

Corollary 2.11. *Let* $(T(t))_{t\geq 0}$ *and* $(S(t))_{t\geq 0}$ *be bi–continuous semigroups on* X *with generators* $(A, D(A))$ *and* $(B, D(B))$*, respectively. Assume the following stability conditions.*

(i) $\|[T(\frac{t}{m})S(\frac{t}{m})]^m\| \leq Me^{\omega t}$ *for all* $t \geq 0$*, and some constants* $M \geq 1$, $\omega \in \mathbb{R}$.

(ii) The operator family $\{[T(\frac{t}{m})S(\frac{t}{m})]^m : t \geq 0\}$ *is locally bi–equicontinuous uniformly for* $m \in \mathbb{N}$.

Consider the sum $A+B$ *on a subspace* $D \subseteq D(A_0) \cap D(B_0)$*, where* $D(A_0)$ *and* $D(B_0)$ *denote the domains of the parts* A_0 *and* B_0 *of* A *and* B *in* $\overline{D(A)}^{\|\cdot\|}$ *and* $\overline{D(B)}^{\|\cdot\|}$*, respectively, and assume that* D *and* $(\lambda_0 - A - B)D$ *are bi–dense in* X *for some* $\lambda_0 > \alpha > \omega$*. Then the bi–closure of* $A + B$ *exists and generates a bi–continuous semigroup* $(U(t))_{t\geq 0}$ *given by the* **Lie–Trotter Product Formula***, i.e.,*

$$U(t)x = \tau\text{-}\lim_{k\to\infty} \left[T(\tfrac{t}{k})S(\tfrac{t}{k})\right]^k x, \tag{2.12}$$

where the limit exists for all $x \in X$ *and uniformly for* t *in compact intervals in* $\mathbb{R}_+$.

Proof. We define $V(t) := T(t)S(t)$ for $t \geq 0$. Then

$$\|\cdot\|\text{-}\lim_{t\searrow 0} \frac{V(t)x - x}{t} = Ax + Bx$$

for all $x \in D$. The assertion is now a consequence of Proposition 2.9. $\square$

Concrete examples are given in Section 3.2, 3.3.2 and 3.4 below.

Chapter 3

Applications

In this chapter we apply the general results obtained in the previous chapters to classes of semigroups which do not fit into the classical theory of C_0–semigroups. Here, we concentrate on such examples for which, due to their importance, different theories, including Hille–Yosida type theorems, had been developed. For instance, we retrieve the theory of *semigroups induced by flows* introduced by J. R. Dorroh and J. W. Neuberger, e.g. in [DN93], the theory of *weakly continuous semigroups* introduced by S. Cerrai and F. Gozzi, e.g. in [Cer94], [CG95] to treat the Ornstein–Uhlenbeck semigroup on $C_b(H)$, and the theory of *adjoint semigroups*, as studied systematically in [Nee92].
Our aim is to verify that these semigroups are bi–continuous. This will lead to a unification, and therefore to a more systematic treatment of these semigroups. Moreover, we will obtain, even in these special cases, new approximation results.
Finally, we mention the theory of π*–semigroups* on $C_b(H)$, H a Hilbert space, due to E. Priola [Pri99], which does not completely fit into our theory of bi–continuous semigroups. However, it seems that there is no example known of a π–semigroup which is not already a bi–continuous semigroup for some canonical topology on $C_b(H)$.

3.1 A survey on locally equicontinuous semigroups

In order to deal with phenomena in which the underlying space is not a Banach space, L. Schwartz [Sch57] already in 1957 generalized the classical Hille–Yosida theorem to semigroups on complete locally convex spaces. Thereafter, many authors studied operator semigroups on locally convex spaces and tried to develop a systematical theory parallel to the one in Banach spaces (e.g. [Kom64], [Kōm68], [Ōuc73], [Yos74]). The theory divides basically into two classes of semigroups: equicontinuous and locally equicontinuous semigroups, respectively, and it suffices to suppose that the underlying vector space is sequentially complete.

Definition 3.1. *Let (X, τ) be a sequentially complete locally convex space (where P_τ is a family of seminorms inducing τ) and $\{T(t): \ t \geq 0\}$ be a family of continuous linear operators on X. The family $(T(t))_{t\geq 0}$ is called an* **equicontinuous semigroup** *if*

(i) $T(0) = Id$ *and* $T(t+s) = T(t)T(s)$ *for all* $s, t \geq 0$.

(ii) $(T(t))_{t\geq 0}$ *is strongly* τ*–continuous, i.e.,* $\mathbb{R}_+ \ni t \mapsto T(t)x \in X$ *is* τ*–continuous for all* $x \in X$.

(iii) For each seminorm $p \in P_\tau$ *there exists* $q \in P_\tau$ *such that*

$$p(T(t)x) \leq q(x)$$

for all $t \geq 0$ *and* $x \in X$.

The family $(T(t))_{t\geq 0}$ is called **quasi–equicontinuous** *if $\{e^{-\alpha t}T(t) : \ t \geq 0\}$ is equicontinuous for some $\alpha > 0$, and it is called* **locally equicontinuous** *if instead of (iii) the following condition is satisfied.*

(iii') For each $t_0 > 0$ *the subset* $\{T(t): \ 0 \leq t \leq t_0\}$ *is equicontinuous.*

The ***generator*** $(A, D(A))$ *of* $(T(t))_{t\geq 0}$ *is defined as*

$$Ax := \lim_{t \searrow 0} \frac{T(t)x - x}{t} \tag{3.1}$$

with domain $D(A) := \{x \in X : \ \lim_{t\searrow 0} \frac{T(t)x - x}{t} \text{ exists}\}$.

Starting from the results of L. Schwartz, J. Miyadera [Miy59] developed a theory for semigroups on Fréchet spaces including a generation result for quasi–equicontinuous semigroups. Following his approach, A. B. Buche [Buc68] added to this theory certain approximation results. Parallel, H. Komatsu [Kom64], S. Ôharu [Ôha66], R. T. Moore [Moo71a], [Moo71b], and K. Yosida [Yos74] generalized the theory to sequentially complete spaces. Finally, we mention a paper by Y. H. Choe [Cho85] in which so–called calibrations are used for a reformulation of the Hille–Yosida theorem for equicontinuous semigroups.

This theory depends heavily on the fact that for an equicontinuous semigroup $(T(t))_{t\geq 0}$ with generator A on a sequentially complete locally convex space X, its Laplace transform defined as

$$\tau\text{--}\int_0^\infty e^{-\lambda t}T(t)x dt$$

exists for all $x \in X$ and $\lambda \in \mathbb{C}$ with $Re\lambda > 0$ and coincides with $(\lambda - A)^{-1}x$.

On the other hand, if equicontinuity fails, this integral representation may no longer hold. For instance, let $(T(t))_{t\geq 0}$ be the left translation semigroup on the space $C(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ endowed with the topology of uniform convergence on bounded subsets. Then $(T(t))_{t\geq 0}$ is not equicontinuous, its Laplace transform does not exist for any $\lambda \in \mathbb{C}$, and its generator given by $A = \frac{d}{dx}$ has empty resolvent set.

To avoid these difficulties, T. Kōmura [Kōm68] introduced the "generalized resolvent" for such locally equicontinuous semigroups. She obtained a Hille–Yosida theorem for locally equicontinuous semigroups on sequentially complete locally convex spaces by giving conditions on such a generalized resolvent (see [Kōm68, Thm.3]).

Since this type of resolvent is difficult to treat, S. Ōuchi [Ōuc73] used "asymptotic resolvents" instead to state a simplified generation theorem of Hille–Yosida type [Ōuc73, Thm. 2.1]. Following this approach, C. Grosu proved in [Gro86] a version of the Trotter–Kato theorem for continuous (or, in some cases, even bounded) generators of locally equicontinuous semigroups in Fréchet–Schwartz spaces.

In a recent work with A. Albanese [AK00] we prove Trotter–Kato theorems and the Lie–Trotter product formula for locally equicontinuous semigroups in the general setting of sequentially complete locally convex spaces with no additional assumption on the generator. Therefore, we substantially improve the results of C. Grosu [Gro86] and add a missing piece to the theory of locally equicontinuous semigroups.

For the perturbation theory of locally equicontinuous semigroups we refer to [Dem73] and [Dem74] where perturbations by relatively bounded operators are treated. Further, continuous perturbations are considered by L. Wenzel [Wen85] and an application to the Boltzmann equation on a space of distributions and on a L^p_{loc}–space, respectively, can be found in [Wen87] and [Wen89].

Although the theory of operator semigroups on locally convex spaces is well established, it seems that there are only very few serious applications of this theory. In addition, no systematic qualitative theory has been developed, certainly due to the lack of an appropriate spectral theory.
However, in many concrete situation the underlying space is indeed a Banach space $(X, \|\cdot\|)$, but has an additional locally convex topology τ satisfying Assumptions 1.1 from Chapter 1. Moreover, the equicontinuous (resp. locally equicontinuous) semigroups to be considered on (X, τ) are exponentially norm bounded. This leads us to bi–continuous semigroups as treated in Chapter 1 and 2.

Proposition 3.2. *Let $(X, \|\cdot\|)$ be a Banach space, τ be a locally convex topology on X satisfying Assumptions 1.1, and let $\{T(t) : t \geq 0\} \subseteq \mathcal{L}(X)$ be a locally equicontinuous semigroup on (X, τ) such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and some constants $M \geq 1$, $\omega \in \mathbb{R}$. Then $(T(t))_{t\geq 0}$ is a bi–continuous semigroup on X with respect to τ.*

The proof is a trivial consequence of Definition 1.3 and Definition 3.1. In addition, by Lemma 1.7, the resolvent of the generator $(A, D(A))$ of such a semigroup is given by

$$R(\lambda, A)x = \tau\text{–}\int_0^\infty e^{-\lambda t} T(t)x dt$$

for all $x \in X$ and $Re\lambda > \omega$ for some $\omega \in \mathbb{R}$, and is a norm bounded operator. In particular, the theory developed in Chapter 1 and 2 applies.
In the following, however, we concentrate on more concrete situations occuring in the literature.

3.2 Semigroups induced by flows

Around 1970, J. W. Neuberger [Neu72], [Neu73] and D. L. Lovelady [Lov75] started a theory of nonlinear, jointly continuous semigroups (flows) on a metric space Ω in terms of their *Lie generator*. Following S. Lie, the main idea was to define an associated linear semigroup, the semigroup induced by the flow, on the Banach space X of bounded, continuous functions on Ω. This semigroup, in general, is not strongly continuous for the supremum norm. However, there exists a locally convex topology on X such that strong continuity holds. To treat this class of semigroups, J. W. Neuberger proved a generation theorem and a representation formula.
20 years later, in joint work by J. R. Dorroh and J. W. Neuberger [DN93], [DN96] a complete characterization of the Lie generator of a jointly continuous flow on a complete, separable, metric space Ω was established, and an exponential formula in terms of the powers of the resolvents of its Lie generator was found. The main step towards these results was, using a result by F. D. Sentilles [Sen72], to find the "right" locally convex topology on the space $C_b(\Omega)$ of bounded, continuous functions on Ω, for which the associated linear semigroup becomes strongly continuous. More recently, in [DN00] and [Neu00] even the adjoint of such a semigroup has been considered on the topological dual $(C_b(\Omega),\beta)'$, β defined as below.

For our purpose it is now essential to note that the space $C_b(\Omega)$ is a Banach space for the supremum norm $\|\cdot\|_\infty$, but has an additional locally convex topology β defined as the finest locally convex topology on $C_b(\Omega)$ agreeing with the compact–open topology τ_c on $\|\cdot\|_\infty$–bounded sets. By a result of F. D. Sentilles [Sen72] a sequence in $C_b(\Omega)$ converges with respect to β if and only if it is $\|\cdot\|$–bounded and convergent with respect to τ_c.
Using the norm and the β–topology we show that these Dorroh–Neuberger semigroups fit into the theory of bi–continuous semigroups. Further, we give a short proof of the generation result stated in [DN96] using our Theorem 1.28 and the Post–Widder Formula given in Corollary 2.10.
As a first step, we verify that the space $C_b(\Omega)$ endowed with $\|\cdot\|_\infty$ and β satisfies Assumptions 1.1. Clearly, the topology β is coarser than the $\|\cdot\|_\infty$–topology, is Hausdorff, sequentially complete on $\|\cdot\|_\infty$–bounded sets, and the topological dual $(C_b(\Omega),\beta)'$ contains the point measures which implies that the space $(C_b(\Omega),\beta)'$ is

norming for $(C_b(\Omega), \|\cdot\|_\infty)$. Therefore, Assumptions 1.1 are fulfilled.
We now give some basic definitions taken from [DN96] and [Nag86, Ch. B.II, Sec. 3].

Definition 3.3. *a) A* **jointly continuous flow** *(or semigroup of jointly continuous transformations) on a topological space Ω is a mapping $\phi : (t,x) \mapsto \phi_t(x)$ from $\mathbb{R}_+ \times \Omega \longrightarrow \Omega$ such that*

(i) $\phi_0 = id$, $\phi_t\phi_s = \phi_{t+s}$ *for all* $t,s \geq 0$.

(ii) The mapping $(t,x) \mapsto \phi_t(x)$ is jointly continuous from $\mathbb{R}_+ \times \Omega$ into Ω.

b) The **Lie generator** *of a jointly continuous flow ϕ is the linear operator A in $C_b(\Omega)$ consisting of all ordered pairs (f,g) such that $f,g \in C_b(\Omega)$ and*

$$g(x) = \lim_{t \searrow 0} \frac{f(\phi_t(x)) - f(x)}{t}$$

for all $x \in \Omega$.
c) A linear operator $(A, D(A))$ on $C_b(\Omega)$ is called a **derivation** *if $f,g \in D(A)$ implies $fg \in D(A)$ and $A(fg) = fAg + gAf$.*
d) The (linear) semigroup $(T(t))_{t\geq 0}$ on $C_b(\Omega)$ defined as

$$T(t)f := f \circ \phi_t \tag{3.2}$$

for all $f \in C_b(\Omega)$ and $t \geq 0$ is called the **semigroup induced by the (jointly continuous) flow ϕ**.

It is clear that $\{T(t) : \ t \geq 0\} \subseteq \mathcal{L}(C_b(\Omega))$ and $\|T(t)\| \leq 1$ for all $t \geq 0$.
Moreover, $(T(t))_{t\geq 0}$ is a semigroup which, if Ω is compact, is strongly continuous on $(C_b(\Omega), \|\cdot\|_\infty)$ (see [Nag86, Ch. B.II, Lemma 3.2]). In the general case, the following continuity properties hold (see [DN93, Thm. 2.1, 2.2]).

Proposition 3.4. *Let $(T(t))_{t\geq 0}$ be the semigroup induced by the jointly continuous flow ϕ on $C_b(\Omega)$. Then $(T(t))_{t\geq 0}$ has the following properties.*

(i) $(T(t))_{t\geq 0}$ is strongly β–continuous.

(ii) $(T(t))_{t\geq 0}$ is locally equicontinuous with respect to β in the sense of Definition 3.1.

As an immediate consequence of Proposition 3.2 and Proposition 3.4 we obtain the following.

Proposition 3.5. *Let ϕ be a jointly continuous flow on Ω and $(T(t))_{t\geq 0}$ be the semigroup induced by ϕ on $C_b(\Omega)$. Then $(T(t))_{t\geq 0}$ is a bi–continuous contraction semigroup with respect to the topology β.*

We note that, by Theorem 1.17, the generator $(A, D(A))$ of $(T(t))_{t\geq 0}$ is given by

$$\begin{aligned} &Af = \beta\text{–}\lim_{t\searrow 0} \frac{T(t)f - f}{t} \\ &\text{for all } f \in D(A) = \{f \in C_b(\Omega) :\ \beta\text{–}\lim_{t\searrow 0} \frac{T(t)f - f}{t} \text{ exists in } C_b(\Omega)\}. \end{aligned} \tag{3.3}$$

This generator $(A, D(A))$ coincides with the Lie generator of the flow ϕ (see [DN96, Prop. 2.4]).

Proposition 3.6. *Let ϕ be a jointly continuous flow and let $f, g \in C_b(\Omega)$. Then the following assertions are equivalent.*

(i) $g(x) = \lim_{t\searrow 0} \frac{f(\phi_t(x))-f(x)}{t}$ *for all* $x \in \Omega$.

(ii) $g = \beta\text{–}\lim_{t\searrow 0} \frac{f\circ\phi_t - f}{t}$.

Proof. Since β is finer than the topology of pointwise convergence on $C_b(\Omega)$, assertion (ii) implies (i).
$(i) \Rightarrow (ii)$ Since ϕ is a jointly continuous flow, the semigroup $(T(t))_{t\geq 0}$ is well defined on $C_b(\Omega)$, and, by Proposition 3.5, bi–continuous with respect to β. Let $(A, D(A))$ denote its generator. Applying Proposition 1.16(b) and Theorem 1.17, we obtain that

$$T(t)f - f = \beta\text{–}\int_0^t T(s)g\,ds \tag{3.4}$$

for all $t \geq 0$. This implies (ii). □

We can now apply Theorem 1.28 to obtain a generation theorem for bi–continuous contraction semigroups induced by jointly continuous flows. Moreover, by Proposition 3.6, this yields a characterization of the Lie generator of a jointly continuous flow thereby reproving the results in [DN96, Thm. 3.1, 3.2].

Theorem 3.7. *Let $A : D(A) \subseteq C_b(\Omega) \longrightarrow C_b(\Omega)$ be a linear operator. Then the following assertions are equivalent.*

(a) The operator A is the Lie generator of a jointly continuous flow ϕ on Ω.

(b) $(A, D(A))$ is a derivation and generates a bi–continuous contraction semigroup $(T(t))_{t\geq 0}$ on $C_b(\Omega)$ with respect to β which is induced by a jointly continuous flow ϕ.

(c) $(A, D(A))$ is a bi–densely defined Hille–Yosida operator of type 0 and a derivation, and the family $\{(s-\alpha)^k R^k(s, A) : \ k \in \mathbb{N}, s \geq \alpha\}$ is bi–equicontinuous for every $\alpha > 0$.

Furthermore, if A is the Lie generator of a jointly continuous flow ϕ, then

$$f \circ \phi_t = \beta\text{–}\lim_{k\to\infty} \left[\tfrac{k}{t} R(\tfrac{k}{t}, A)\right]^k f \tag{3.5}$$

for all $f \in C_b(\Omega)$ and uniformly for t in compact intervals of $\mathbb{R}_+$.

Proof. $(a) \Rightarrow (b)$ It is easy to see that the Lie generator of a jointly continuous flow ϕ on Ω is a derivation. Defining $T(t)f := f \circ \phi_t$ for all $f \in C_b(\Omega)$ and $t \geq 0$, we obtain, by Proposition 3.5, that $(T(t))_{t\geq 0}$ is a bi–continuous contraction semigroup on $C_b(\Omega)$ with respect to β. Let $(B, D(B))$ denote its generator. Combining (3.3) and Proposition 3.6, we have $A = B$. Therefore, (b) holds.
$(b) \Rightarrow (c)$ If (b) holds, Theorem 1.28 leads to assertion (c).
$(c) \Rightarrow (a)$ By Theorem 1.28 we obtain that there exists a bi–continuous semigroup $(T(t))_{t\geq 0}$ on $C_b(\Omega)$ with respect to β. We show that each operator $T(t)$ is a multiplicative homomorphism, i.e., we have

$$T(t)(fg) = T(t)f \cdot T(t)g \tag{3.6}$$

for all $t > 0$ and $f, g \in C_b(\Omega)$. Indeed, for $f, g \in D(A)$ we define a function $\eta : [0, t] \longrightarrow C_b(\Omega)$ by

$$\eta(s) := T(t-s)[T(s)f \cdot T(s)g],$$

which is differentiable with respect to β by Proposition 1.16. Then $\eta(0) = T(t)(f \cdot g)$ and $\eta(t) = T(t)f \cdot T(t)g$. Since A is a derivation, we have $\eta'(s) = 0$ for $s \in [0, t]$. Hence, $\eta(0) = \eta(t)$ and $T(t)(fg) = T(t)f \cdot T(t)g$. The bi–density of $D(A)$ yields

equation (3.6) for all $f, g \in C_b(\Omega)$.
Since $(T(t))_{t\geq 0}$ consists of multiplicative homomorphisms, there exists, by [DN93, Prop. 1.6], for each $x \in \Omega$ a unique $y \in \Omega$ such that $T(t)f(x) = f(y)$ for all $f \in C_b(\Omega)$. Thus, $(T(t))_{t\geq 0}$ is a contraction semigroup, and, for every $t \geq 0$, we can define a flow ϕ from Ω into Ω as

$$f(\phi_t(x)) := T(t)f(x)$$

for all $f \in C_b(\Omega), x \in \Omega$. This flow is jointly continuous from $\mathbb{R}_+ \times \Omega \to \Omega$ by Theorem 3.3 stated in [DN93]. Further, by Theorem 1.17, the generator of $(T(t))_{t\geq 0}$ is given by

$$Af = \beta\text{–}\lim_{t\searrow 0} \frac{f \circ \phi_t - f}{t},$$

and, by Proposition 3.6, this is equivalent to the fact that A is the Lie generator of the flow ϕ. This concludes the proof of the equivalent assertions.
Finally, if A is the Lie generator of a jointly continuous flow ϕ, then implication $(a) \Rightarrow (b)$ and the Post–Widder Formula in Corollary 2.10 imply the approximation of the semigroup $(T(t))_{t\geq 0}$ by the powers of the resolvents of A as stated in (3.5).

□

Using the Lie–Trotter Product Formula stated in Proposition 2.11, we obtain the following product formula for two semigroups induced by flows.

Proposition 3.8. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be bi–continuous semigroups on $C_b(\Omega)$ which are induced by jointly continuous flows ϕ and ψ, and generated by $(A, D(A))$ and $(B, D(B))$, respectively. Suppose that for every $t_0 > 0$ and compact subset $K \subseteq \Omega$ there exists a compact subset $\tilde{K} \subseteq \Omega$ such that*

$$\left[\psi_{t/m}\phi_{t/m}\right]^m K \subseteq \tilde{K} \tag{3.7}$$

for all $t \in [0, t_0]$ and $m \in \mathbb{N}$. If there exists a subset $D \subseteq D(A_0) \cap D(B_0)$, where $D(A_0)$ and $D(B_0)$ denote the domains of the parts A_0 and B_0 of A and B in $\overline{D(A)}^{\|\cdot\|}$ and $\overline{D(B)}^{\|\cdot\|}$, respectively, such that D and $(\lambda_0 - A - B)D$ are bi–dense in $C_b(\Omega)$, then the bi–closure of $A+B$ generates a bi–continuous semigroup $(U(t))_{t\geq 0}$ on $C_b(\Omega)$ which is induced by a jointly continuous flow ξ such that

$$U(t)f = f \circ \xi_t = \beta\text{–}\lim_{m\to\infty} f \circ \left[\psi_{t/m}\phi_{t/m}\right]^m$$

for all $f \in C_b(\Omega)$ and uniformly for t in compact intervals of $\mathbb{R}_+$.

Proof. First, we recall that, by [Sen72], the β–convergence of a $\|\cdot\|$–bounded sequence in $C_b(\Omega)$ is equivalent to its τ_c–convergence. Clearly, the semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ consist of contractions, hence

$$\|[T(\tfrac{t}{m})S(\tfrac{t}{m})]^m\| \leq 1 \tag{3.8}$$

for all $t \geq 0$ and $m \in \mathbb{N}$.
Next, we show that the operator family $\{[T(\frac{t}{m})S(\frac{t}{m})]^m : \ t \geq 0\}$ is locally bi–equicontinuous uniformly for $m \in \mathbb{N}$. Let $t_0 \geq 0$ and $(f_n)_{n\in\mathbb{N}} \subseteq C_b(\Omega)$ be a $\|\cdot\|_\infty$–bounded sequence which is β–convergent to $f \in C_b(\Omega)$. Therefore, $(f_n)_{n\in\mathbb{N}}$ is τ_c–convergent to f. Hence, for each compact subset $K \subset \Omega$ there exists a compact subset $\tilde{K} \subset \Omega$ such that

$$\begin{aligned}\sup_{x\in K} |[T(\tfrac{t}{m})S(\tfrac{t}{m})]^m(f_n(x) - f(x))| &= \sup_{x\in K} |f_n([\psi_{t/m}\phi_{t/m}]^m(x)) - f([\psi_{t/m}\phi_{t/m}]^m(x))| \\ &\leq \sup_{x\in \tilde{K}} |f_n(x) - f(x)|,\end{aligned}$$

which converges to zero uniformly for $t \in [0, t_0]$ and $m \in \mathbb{N}$ as $n \to \infty$. With estimate (3.8) this implies that $\{[T(\frac{t}{m})S(\frac{t}{m})]^m : \ t \geq 0\}$ is locally bi–equicontinuous uniformly for $m \in \mathbb{N}$.
Thus, we are able to apply Corollary 2.11 and obtain a bi–continuous semigroup $(U(t))_{t\geq 0}$ on $(C_b(\Omega), \beta)$ generated by the bi–closure of $A + B$ such that the Lie–Trotter Product Formula holds. Furthermore, using the same arguments as in the proof of Theorem 3.7, we obtain the existence of a jointly continuous flow ξ on Ω such that $U(t)f = f \circ \xi_t$ for all $f \in C_b(\Omega)$. □

Example 3.9. To illustrate the above proposition, we consider on $C_b(\mathbb{R})$ the operators $(A, D(A))$ and $(B, D(B))$ defined as

$$Af(x) := x^{2/3} f'(x)$$

for all $x \in \mathbb{R}$ and $f \in D(A) := \{f \in C_b(\mathbb{R}) : \ Af \in C_b(\mathbb{R})\}$, and

$$Bf(x) := xf'(x)$$

for all $x \in \mathbb{R}$ and $f \in D(B) := \{f \in C_b(\mathbb{R}) : \ Bf \in C_b(\mathbb{R})\}$. Then, the operator $(A, D(A))$ generates the semigroup $(T(t))_{t\geq 0}$ given by

$$T(t)f(x) = f((x^{1/3} + t/3)^3) =: (f \circ \phi_t)(x)$$

for all $x \in \mathbb{R}, f \in C_b(\mathbb{R})$ and $t \geq 0$, which is bi–continuous with respect to β and is induced by the jointly continuous flow ϕ with $\phi_t(x) = (x^{1/3} + t/3)^3$.
The operator $(B, D(B))$ generates the semigroup $(S(t)_{t\geq 0}$ given by

$$S(t)f(x) = f(e^t x) =: (f \circ \psi_t)(x)$$

for all $x \in \mathbb{R}, f \in C_b(\mathbb{R})$ and $t \geq 0$, which is bi–continuous with respect to β and induced by the jointly continuous flow ψ with $\psi_t(x) = e^t x$.
By induction we obtain

$$[\psi_{t/m}\phi_{t/m}]^m(x) = \left[e^{\frac{t}{3}} x^{\frac{1}{3}} + \frac{t}{3m} \sum_{k=1}^{m} e^{\frac{kt}{3m}} \right]^3 \tag{3.9}$$

for all $x \in \mathbb{R}, t \geq 0$ and $m \in \mathbb{N}$. Since $\frac{t}{3m}\sum_{k=1}^m e^{\frac{kt}{3m}} \leq \frac{t}{3} e^{\frac{t}{3}}$ for all $m \in \mathbb{N}$ and $t \geq 0$, the products $[\psi_{t/m}\phi_{t/m}]^m$ map a fixed compact subset $K \subseteq \mathbb{R}$ into some fixed compact subset $\tilde{K}$ of $\mathbb{R}$. Moreover, the Schwartz space $\mathcal{S}(\mathbb{R})$ is contained in $C_0^1(\mathbb{R})$, the space of continuously differentiable functions vanishing at infinity, and therefore $\mathcal{S}(\mathbb{R}) \subseteq D(A_0) \cap D(B_0)$. Furthermore, by solving an ordinary differential equation, there exists $\lambda_0 > 0$ such that $(\lambda_0 - A - B)\mathcal{S}(\mathbb{R})$ is bi–dense in $C_b(\mathbb{R})$. Thus, the assumptions of Proposition 3.8 are fulfilled, and we obtain that the bi–closure of $A + B$ generates a semigroup $(U(t))_{t\geq 0}$ on $C_b(\Omega)$ which is induced by a jointly continuous flow ξ such that

$$U(t)f = f \circ \xi_t = \beta\text{–}\lim_{m\to\infty} f \circ [\psi_{t/m}\phi_{t/m}]^m$$

for all $f \in C_b(\Omega)$ and uniformly for t in compact intervals of $\mathbb{R}_+$.
Moreover, in this case the flow ξ is given explicitly by

$$\xi_t(x) = e^t \left(x^{1/3} + 1 - e^{-t/3}\right)^3$$

for all $t \geq 0$ and $x \in \mathbb{R}$. In fact, using formula (3.9), we obtain

$$\frac{t}{3m} \sum_{k=1}^{m} e^{\frac{kt}{3m}} = \frac{t\left(e^{\frac{t(m+1)}{3m}} - e^{\frac{t}{3m}}\right)}{3m\left(e^{\frac{t}{3m}} - 1\right)}$$

for all $t \geq 0$ and $m \in \mathbb{N}$, which converges to $e^{\frac{t}{3}} - 1$ as $m \to \infty$ for all $t \geq 0$. Therefore,

$$\lim_{m\to\infty} [\psi_{t/m}\phi_{t/m}]^m(x) = e^t(x^{1/3} + 1 - e^{-t/3})^3$$

for all $t \geq 0$ and $x \in \mathbb{R}$.

3.3 The Ornstein–Uhlenbeck semigroup

In this section we start from the stochastic equation

$$\begin{cases} dX(t) = AX(t)dt + dW(t), \\ X(0) = x, \end{cases} \tag{3.10}$$

where A is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on a separable Hilbert space H, W is an H–valued Q–Wiener process for a self–adjoint, positive, bounded linear operator Q on H, and $x \in H$.
We now look at the Ornstein–Uhlenbeck semigroup (or transition semigroup) corresponding to equation (3.10) which is given by

$$P(t)f(x) = \mathbb{E}(f(X(t,x)))$$

for all $f \in C_b(H)$, $x \in H$ and $t \geq 0$, and has been studied by many authors, e.g., [DPZ92],[CDP93], [Cer94], [CG95], [DPL95], [Pri99], [TZ].
Our aim is to show that the Ornstein–Uhlenbeck semigroup on $C_b(H)$ is bi–continuous with respect to the compact–open topology. Further, we apply Corollary 2.11 and obtain that the Ornstein–Uhlenbeck semigroup on $C_b(\mathbb{R}^n)$ is given by the Lie–Trotter Product Formula with respect to a locally convex topology finer than the compact–open topology.

3.3.1 The Ornstein–Uhlenbeck semigroup on $C_b(H)$

Let A be the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on a separable Hilbert space H, Q be a self–adjoint, positive, bounded linear operator on H. Further, let $Q(t)$, $t \geq 0$, be the so–called covariance operator defined by

$$Q(t)x := \int_0^t S(r)QS(r)'x dr$$

for all $x \in H$ and $t \geq 0$, where $(S(t)')_{t\geq 0}$ is the adjoint semigroup of $(S(t))_{t\geq 0}$. Suppose that each $Q(t)$ is of trace class[1] operator. Then the Gaussian measures $\mathcal{N}(S(t)x, Q(t))$ with mean $S(t)x$ and covariance $Q(t)$ exist for all $t \geq 0$ and $x \in H$

[1] A bounded linear operator Q on a separable Hilbert space H is called **trace class** if $tr|Q| := \sum_{n\geq 1}\langle \phi_n, |A|\phi_n\rangle < \infty$, where $\{\phi_n : \ n \geq 1\}$ is an orthonormal basis for H (cf. [RS72, p. 207]).

(see [DPZ92, Ch. I, Sec. 2.3.2]). The Ornstein–Uhlenbeck semigroup on $C_b(H)$, the space of bounded and continuous functions on H, is defined as

$$P(t)f(x) := \mathbb{E}(f(X(t,x))) = \int_H f(y)\mathcal{N}(S(t)x, Q(t))dy$$

for all $f \in C_b(H)$, $x \in H$ and $t \geq 0$.

In general, $(P(t))_{t\geq 0}$ is not strongly continuous on $C_b(H)$ with respect to the supremum norm (see [Cer94], [DPL95]). However, using the results stated in [Cer94], it turns out that this semigroup is bi–continuous on $C_b(H)$ with respect to the compact–open topology.

Proposition 3.10. *Under the above assumptions, the Ornstein–Uhlenbeck semigroup $(P(t))_{t\geq 0}$ is a bi–continuous contraction semigroup with respect to the compact–open topology τ_c on $C_b(H)$.*

Proof. It is easy to see that $C_b(H)$ endowed with the supremum norm $\|\cdot\|$ and the compact–open topology τ_c satisfies Assumptions 1.1. Clearly, $(P(t))_{t\geq 0}$ consists of bounded linear operators on $C_b(H)$ and $\|P(t)\| \leq 1$ for all $t \geq 0$. By the same way as in [Cer94, Proposition 6.2] we show now the strong τ_c–continuity of $(P(t))_{t\geq 0}$. Let $f \in C_b(H)$, $K \subseteq H$ compact, and $\epsilon > 0$. Since f is uniformly continuous on K and $(S(t))_{t\geq 0}$ strongly continuous on H, there exists $\delta(K) > 0$ and $t_0 \geq 0$ such that

$$\sup_{x\in K} |f(y + S(t)x) - f(y)| \leq \frac{\epsilon}{2}$$

for all $\|y\| \leq \delta(K)$ and $t \leq t_0$. Thus, there exists $\tilde{t} \leq t_0$ such that

$$\begin{aligned}
\sup_{x\in K} |P(t)f(x) - f(x)| &\leq \sup_{x\in K} |\int_{\|y\|\leq\delta(K)} |f(y + S(t)x) - f(y)|\mathcal{N}(0, Q(t))dy \\
&\quad + 2\|f\|_\infty \int_{\|y\|>\delta(K)} \mathcal{N}(0, Q(t))dy \\
&\leq \frac{\epsilon}{2} + \frac{2\|f\|_\infty}{\delta(K)^2} \operatorname{Tr} Q(t) \\
&\leq \epsilon
\end{aligned}$$

for all $t \leq \tilde{t}$.

It remains to show that $(P(t))_{t\geq 0}$ is locally bi–equicontinuous. Let $K \subseteq H$ be a compact set, $t_0 \geq 0$, $\epsilon > 0$, and $(f_n)_{n\in\mathbb{N}} \subseteq C_b(H)$ be a $\|\cdot\|$–bounded sequence which is τ_c–convergent to $f \in C_b(H)$. The tightness of the family of probability measures

$$\{\mathcal{N}(S(t)x, Q(t)) : \ 0 \leq t \leq t_0, x \in K\}$$

proved, e.g., in [Cer94, Lemma 6.3] implies that there exists a compact subset $\tilde{K} \subseteq H$ such that

$$1 - \mathcal{N}(S(t)x, Q(t))(\tilde{K}) \leq \epsilon$$

for all $0 \leq t \leq t_0$ and $x \in K$. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}\sup_{x\in K} |P(t)(f_n(x) - f(x))| &\leq \sup_{x\in K} \int_{\tilde{K}} |f_n(y) - f(y)| \mathcal{N}(S(t)x, Q(t)) dy \\ &\quad + \sup_{x\in K} \int_{\complement\tilde{K}} |f_n(y) - f(y)| \mathcal{N}(S(t)x, Q(t)) dy \\ &\leq \epsilon + (\|f_n\|_\infty + \|f\|_\infty)\epsilon\end{aligned}$$

for all $n \geq n_0$ and uniformly for $0 \leq t \leq t_0$. This concludes the proof. □

3.3.2 The Lie–Trotter Product Formula for the Ornstein–Uhlenbeck semigroup on $C_b(\mathbb{R}^n)$

In this subsection we show that the Lie–Trotter Product Formula given in Corollary 2.11 can be applied to the Ornstein–Uhlenbeck semigroup on $C_b(\mathbb{R}^n)$. This is joint work with A. Albanese [AK00].

Let $A := (a_{ij})$ be a symmetric, positive definite matrix, and $B := (b_{ij}) \in \mathcal{L}(\mathbb{R}^n)$. In this case, the generator of the Ornstein–Uhlenbeck semigroup can be written as

$$\begin{aligned}\mathcal{O}f(x) &:= \sum_{i,j=1}^{n} a_{ij} D_{ij} f(x) + \sum_{i,j=1}^{n} b_{ij} x_j D_i f(x) \\ &=: \langle \nabla, A\nabla f(x)\rangle + \langle Bx, \nabla f(x)\rangle \\ &=: \mathcal{A}f(x) + \mathcal{B}f(x)\end{aligned} \tag{3.11}$$

for all $f \in \mathcal{S}(\mathbb{R}^n), x \in \mathbb{R}^n$, $\nabla := (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ (cf. [DPL95]). The Ornstein–Uhlenbeck semigroup $(\mathcal{P}(t))_{t\geq 0}$ generated by $\mathcal{O}$ has the following representation (see [DPL95]):

$$(\mathcal{P}(t)f)(x) = \begin{cases} \frac{1}{(2\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^n} e^{-\frac{\langle Q_t^{-1} y, y\rangle}{2}} f(e^{tB}x - y) dy, & \text{if } t > 0, \\ f(x), & \text{if } t = 0, \end{cases} \tag{3.12}$$

for all $f \in C_b(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, where $Q_t := \int_0^t e^{sB} A e^{sB'} ds$.
From (3.11) we see that the generator $\mathcal{O}$ is the sum of two simpler generators $\mathcal{A}$ and $\mathcal{B}$. However, since $(\mathcal{P}(t))_{t\geq 0}$ is not strongly continuous on $(C_b(\mathbb{R}^n), \|\cdot\|_\infty)$, Lie–Trotter's Product Formula in its classical formulation does not apply. It is our aim to show that $(\mathcal{P}(t))_{t\geq 0}$ still can be obtained via a Lie–Trotter Product Formula from the semigroups generated by $\mathcal{A}$ and $\mathcal{B}$.
As a first step, we show that $C_b(\mathbb{R}^n)$ can be endowed with a suitable locally convex Hausdorff topology τ finer than the compact–open topology such that $(C_b(\mathbb{R}^n), \tau)$ becomes sequentially complete on $\|\cdot\|_\infty$–bounded sets. Then, this space satisfies Assumptions 1.1, and $(\mathcal{P}(t))_{t\geq 0}$ is bi–continuous on it.
To show this, we construct the appropriate topology τ by taking a family P_τ of seminorms on $C_b(\mathbb{R}^n)$ generating a locally convex topology τ on $C_b(\mathbb{R}^n)$ such that the inclusion maps

$$(C_b(\mathbb{R}^n), \|\cdot\|_\infty) \hookrightarrow (C_b(\mathbb{R}^n), \tau) \hookrightarrow (C_b(\mathbb{R}^n), \tau_c)$$

are continuous, where τ_c denotes the compact–open topology on $C_b(\mathbb{R}^n)$. (This construction of P_τ is similar to the one given in [Gig93, Sec. 2].) Denote by $C_0(\mathbb{R}^n)$ the space of continuous functions vanishing at infinity and let

$$\Gamma := \{\gamma \in C_0(\mathbb{R}^n) : \ \gamma > 0, \lim_{\|x\|\to\infty} \|x\|^2 \gamma(x) \text{ exists in } \mathbb{R}\}.$$

Clearly, Γ is nonempty. Indeed, each function defined as

$$\gamma(x) := \begin{cases} l & \text{if } \|x\| \leq r, \\ \frac{lr^2}{\|x\|^2} & \text{if } \|x\| > r, \end{cases} \tag{3.13}$$

with $l, r > 0$ arbitrary, belongs to Γ. Moreover, if $(D_m)_{m\in\mathbb{N}}$ is an exhaustion of $\mathbb{R}^n$ (i.e., $\overline{D}_m$ is compact, $\overline{D}_m \subset D_{m+1}$ for all $m \in \mathbb{N}$, and $\bigcup_{m=0}^\infty D_m = \mathbb{R}^n$), and $(\gamma_m)_{m\in\mathbb{N}} \subseteq C_0(\mathbb{R}^n)$ such that

$$0 \leq \gamma_m \leq 1 \quad \text{on} \quad \mathbb{R}^n, \ \gamma_m = 1 \quad \text{on} \quad \overline{D}_{m-1} \text{ and } \ \gamma_m = 0 \quad \text{on} \quad \mathbb{R}^n \setminus D_m$$

for all $m \in \mathbb{N}$, then each function defined as

$$\gamma_0(x) := \sum_{m=1}^\infty \frac{1}{2^{l_m}} \gamma_m(x) \tag{3.14}$$

for $x \in \mathbb{R}^n$ belongs to Γ too, where $(l_m)_{m\in\mathbb{N}}$ is an increasing sequence of integers such that $l_m \geq \max\{m, d(0, D_m)\}$ and $l_m \in \mathbb{N}$ for all $m \in \mathbb{N}$.
Furthermore, we have the following property.

(i) Let A be a non–zero, real matrix and $\gamma \in \Gamma$. For each $s > 0$ the function $\tilde{\gamma}_s$ defined as

$$\tilde{\gamma}_s(x) := \sup_{0\leq t\leq s} \gamma(e^{-t\|A\|}x), \ x \in \mathbb{R}^n,$$

belongs to Γ and satisfies $\gamma \leq \tilde{\gamma}_s$.

We now consider the family of seminorms $P_\tau := \{p_\gamma : \ \gamma \in \Gamma\}$ on $C_b(\mathbb{R}^n)$ defined as

$$p_\gamma(f) := \sup_{x\in\mathbb{R}^n} \gamma(x)|f(x)| \quad \text{for all } f \in C_b(\mathbb{R}^n).$$

Clearly, P_τ generates a locally convex topology τ coarser than the topology of uniform convergence on $\mathbb{R}^n$. Since for each $\gamma \in \Gamma$ there exists $M := \sup_{x\in\mathbb{R}^n} \gamma(x) > 0$ such that

$$p_\gamma(f) = \sup_{x\in\mathbb{R}^n} \gamma(x)|f(x)| \leq M\|f\|_\infty$$

for all $f \in C_b(\mathbb{R}^n)$, the inclusion map $(C_b(\mathbb{R}^n), \|\cdot\|_\infty) \hookrightarrow (C_b(\mathbb{R}^n), \tau)$ is continuous. Also the inclusion map $(C_b(\mathbb{R}^n), \tau) \hookrightarrow (C_b(\mathbb{R}^n), \tau_c)$ is continuous. Indeed, for each $m \in \mathbb{N}$ there exists $\gamma \in \Gamma$, where γ is given as in (3.13) with $l = 1$ and $r = m$, such that

$$\sup_{\|x\|\leq m} |f(x)| \leq \sup_{x\in\mathbb{R}^n} \gamma(x)|f(x)| = p_\gamma(f)$$

for all $f \in C_b(\mathbb{R}^n)$.
Moreover, by repeating the proof in [Gig93, Prop. 2.3, 2.4] with minor changes and using functions γ defined as in (3.14), we obtain the following.

(ii) The space $C_0(\mathbb{R}^n)$ is bi–dense in $(C_b(\mathbb{R}^n), \tau)$.

(iii) The space $(C_b(\mathbb{R}^n), \tau)$ is sequentially complete on $\|\cdot\|_\infty$–bounded sets.

Hence $C_b(\mathbb{R}^n)$ satisfies Assumptions 1.1.
We now show that the closure of $(\mathcal{A}, \mathcal{S}(\mathbb{R}^n))$ and $(\mathcal{B}, \mathcal{S}(\mathbb{R}^n))$ from (3.11) are generators of bi–continuous semigroups on $C_b(\mathbb{R}^n)$ with respect to τ.

Proposition 3.11. *The semigroup* $(\mathcal{S}(t))_{t\geq 0}$ *given by*

$$(\mathcal{S}(t)f)(x) = f(e^{tB}x) \quad \text{for } t \geq 0,\ f \in C_b(\mathbb{R}^n),\ x \in \mathbb{R}^n \tag{3.15}$$

is bi–continuous on $C_b(\mathbb{R}^n)$ *with respect to* τ *and its generator coincides with the bi–closure of the operator*

$$\mathcal{B}f(x) := \sum_{i,j=1}^{n} b_{ij}x_j D_i f(x) = \langle Bx, \nabla f(x)\rangle$$

defined for every $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. It is easy to see that $\mathcal{S}(t) \in \mathcal{L}(C_b(\mathbb{R}^n))$ and $\|\mathcal{S}(t)\| \leq 1$ for all $t \geq 0$. Further, for each compact subset $K \subset \mathbb{R}^n$ we have

$$\lim_{t\searrow 0} \sup_{x\in K} \|e^{tB}x - x\| = 0. \tag{3.16}$$

Let $f \in C_b(\mathbb{R}^n)$, $M := \sup_{x\in\mathbb{R}^n} |f(x)|$, $\gamma \in \Gamma$ and $\epsilon > 0$. There exists $r > 0$ such that $0 < \gamma(x) < \frac{\epsilon}{4M}$ for all $x \in \mathbb{R}^n$ with $\|x\| > r$. Thus,

$$\sup_{\|x\|>r} \gamma(x)|f(e^{tB}x) - f(x)| \leq \frac{\epsilon}{4M} \sup_{\|x\|>r} |f(e^{tB}x) - f(x)| \leq \frac{\epsilon}{4M}2M = \frac{\epsilon}{2} \tag{3.17}$$

for all $t \geq 0$. Now, let $K := \{x \in \mathbb{R}^n :\ \|x\| \leq r\}$ and $0 < d := \max_{x\in K} \gamma(x) < \infty$. Then, by (3.16), there exists $\delta > 0$ such that

$$\sup_{\|x\|\leq r} |f(e^{tB}x) - f(x)| < \frac{\epsilon}{2d}$$

for all $t \in]0, \delta[$, and hence

$$\sup_{\|x\|\leq r} \gamma(x)|f(e^{tB}x) - f(x)| \leq d \sup_{\|x\|\leq r} |f(e^{tB}x) - f(x)| < d\frac{\epsilon}{2d} = \frac{\epsilon}{2} \tag{3.18}$$

for all $t \in]0, \delta[$.
Combining (3.17) and (3.18), we obtain

$$\tau\text{–}\lim_{t\searrow 0} \mathcal{S}(t)f = f \tag{3.19}$$

for all $f \in C_b(\mathbb{R}^n)$.
Next, we show the local equicontinuity of $(\mathcal{S}(t))_{t\geq 0}$. Let $s > 0$ and $\gamma \in \Gamma$ and take $\tilde{\gamma}_s(y) := \sup_{0\leq t\leq s} \gamma(e^{-tB}y)$, $y \in \mathbb{R}^n$, so that $\tilde{\gamma} \in \Gamma$ by (i).

Then we have

$$p_\gamma(\mathcal{S}(t)f) = \sup_{x\in\mathbb{R}^n} \gamma(x)|f(e^{tB}x)| \le \sup_{y\in\mathbb{R}^n} \tilde{\gamma}_s(y)|f(y)| = p_{\tilde{\gamma}_s}(f) \tag{3.20}$$

for all $f \in C_b(\mathbb{R}^n)$ and $0 \le t \le s$. Combining (3.19), (3.20) and Proposition 3.2, we obtain that $(\mathcal{S}(t))_{t\ge 0}$ is bi–continuous.
Let $(\tilde{\mathcal{B}}, D(\tilde{\mathcal{B}}))$ be the generator of $(\mathcal{S}(t))_{t\ge 0}$. For $f \in \mathcal{S}(\mathbb{R}^n)$ we consider $g := f - \mathcal{B}f \in \mathcal{S}(\mathbb{R}^n)$. Then, using Lemma 1.7 and integration by parts, we obtain

$$\begin{aligned} R(1,\tilde{\mathcal{B}})g(x) &= \int_0^\infty e^{-t} f(e^{tB}x)dt - \int_0^\infty e^{-t}\langle Be^{tB}x, \nabla f(e^{tB}x)\rangle dt \\ &= f(x). \end{aligned}$$

Therefore, $\mathcal{S}(\mathbb{R}^n) \subset D(\tilde{\mathcal{B}})$. On the other hand, $\mathcal{S}(\mathbb{R}^n)$ is invariant under $(\mathcal{S}(t))_{t\ge 0}$ and bi–dense in $(C_b(\mathbb{R}^n), \tau)$. So, it is a bi–core by Proposition 1.21. This completes the proof. $\square$

Proposition 3.12. *The semigroup $(\mathcal{T}(t))_{t\ge 0}$ given by*

$$(\mathcal{T}(t)f)(x) = \begin{cases} ((2\pi t)^{n/2}(\det A)^{1/2})^{-1} & \\ \quad \cdot \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(x-y),(x-y)\rangle\right) f(y)dy & \text{if } t > 0, \\ f(x) & \text{if } t = 0 \end{cases}$$

for all $t \ge 0$, $f \in C_b(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ is bi–continuous on $C_b(\mathbb{R}^n)$ with respect to τ and its generator coincides with the bi–closure of the operator

$$\mathcal{A}f(x) := \frac{1}{2}\sum_{i,j=1}^n a_{ij}D_{ij}f(x) = \langle \nabla, A\nabla f(x)\rangle$$

defined for every $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Clearly, $\mathcal{T}(t) \in \mathcal{L}(C_b(\mathbb{R}^n))$ and $\|\mathcal{T}(t)\| \le 1$ for all $t \ge 0$. We now prove the local equicontinuity of $(\mathcal{T}(t))_{t\ge 0}$ with respect to τ. Let $\gamma \in \Gamma$, $f \in C_b(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$.

Then

$$
\begin{aligned}
&\gamma(x)|\mathcal{T}(t)f(x)| \\
&\qquad \le \frac{\gamma(x)}{(2\pi t)^{n/2}(\det A)^{1/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t} < \langle A^{-1}(x-y),(x-y)\rangle\right) \\
&\qquad\quad \cdot (1+\|y\|^2)\frac{|f(y)|}{1+\|y\|^2}dy \\
&\qquad \le \frac{\gamma(x)}{(2\pi t)^{n/2}(\det A)^{1/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(x-y),(x-y)\rangle\right)(1+\|y\|^2)dy \\
&\qquad\quad \cdot \sup_{z\in\mathbb{R}^n} \frac{|f(z)|}{1+\|z\|^2} \\
&\qquad \le \gamma(x)(1+\|x\|^2+nt\|A\|^2) \sup_{z\in\mathbb{R}^n} \frac{|f(z)|}{1+\|z\|^2} \\
&\qquad \le M(1+t) \sup_{z\in\mathbb{R}^n} \frac{|f(z)|}{1+\|z\|^2},
\end{aligned}
$$

where $M := 2\max\{M_\gamma, n\|A\|^2\}$ with $M_\gamma := \sup_{x\in\mathbb{R}^n}(1+\|x\|^2)\gamma(x) < \infty$. Put $\tilde{\gamma}(z) := \frac{M}{1+\|z\|^2}$, $z \in \mathbb{R}^n$, so that $\tilde{\gamma} \in \Gamma$. It follows that

$$
p_\gamma(\mathcal{T}(t)f) \le (1+t)p_{\tilde{\gamma}}(f) \tag{3.21}
$$

for all $f \in C_b(\mathbb{R}^n)$. Therefore, $(\mathcal{T}(t))_{t\ge0}$ is locally equicontinuous on $(C_b(\mathbb{R}^n),\tau)$. Since $(\mathcal{T}(t))_{t\ge0}$ is strongly continuous on $(C_0(\mathbb{R}^n),\|\cdot\|_\infty)$ (see [CDP93]), we have

$$
\tau\text{–}\lim_{t\searrow 0} \mathcal{T}(t)f = f
$$

for all $f \in C_0(\mathbb{R}^n)$, since τ is coarser than $\|\cdot\|_\infty$. Now, let $f \in C_b(\mathbb{R}^n)$, $\gamma \in \Gamma$, and $\epsilon > 0$. Since $C_0(\mathbb{R}^n)$ is dense in $(C_b(\mathbb{R}^n),\tau)$, there exists $f_0 \in C_0(\mathbb{R}^n)$ such that

$$
p_{\tilde{\gamma}}(f-f_0) < \frac{\epsilon}{4}\ ,
$$

where $\tilde{\gamma} \in \Gamma$ is taken as in the above inequality (3.21). Moreover, there exists $0 < \delta_\epsilon < 1$ such that

$$
p_\gamma(\mathcal{T}(t)f_0 - f_0) < \frac{\epsilon}{4}
$$

for all $0 < t < \delta_\epsilon$. Therefore,

$$
\begin{aligned}
p_\gamma(\mathcal{T}(t)f - f) &\le p_\gamma(\mathcal{T}(t)(f-f_0)) + p_\gamma(\mathcal{T}(t)f_0 - f_0) + p_\gamma(f_0 - f) \\
&\le (1+t)p_{\tilde{\gamma}}(f-f_0) + \frac{\epsilon}{4} + p_{\tilde{\gamma}}(f_0 - f) \\
&< \epsilon
\end{aligned}
$$

for all $0 < t < \delta_\epsilon$.

Hence,

$$\tau\text{–}\lim_{t\searrow 0} \mathcal{T}(t)f = f$$

for all $f \in C_b(\mathbb{R}^n)$. By Proposition 3.2 we obtain that $(\mathcal{T}(t))_{t\geq 0}$ is bi–continuous. Let $(\tilde{\mathcal{A}}, D(\tilde{\mathcal{A}}))$ be the generator of $(\mathcal{T}(t))_{t\geq 0}$. Note that $(\mathcal{T}(t))_{t\geq 0}$ is strongly continuous on $C_0(\mathbb{R}^n)$ with the $\|\cdot\|$–closure of $(\mathcal{A}, \mathcal{S}(\mathbb{R}^n))$ as its generator. Further, $\mathcal{S}(\mathbb{R}^n)$ is invariant under $(\mathcal{T}(t))_{t\geq 0}$, and hence $\mathcal{S}(\mathbb{R}^n) \subset D(\tilde{\mathcal{A}})$. On the other hand, $\mathcal{S}(\mathbb{R}^n)$ is bi–dense in $(C_b(\mathbb{R}^n), \tau)$. So, it is a bi–core by Proposition 1.21. This completes the proof. □

With the previous propositions we are now able to approximate $(\mathcal{P}(t))_{t\geq 0}$ by the Lie–Trotter products of $(\mathcal{T}(t))_{t\geq 0}$ and $(\mathcal{S}(t))_{t\geq 0}$.

Theorem 3.13. *Let $(\mathcal{T}(t))_{t\geq 0}$ and $(\mathcal{S}(t))_{t\geq 0}$ be the bi–continuous semigroups on $C_b(\mathbb{R}^n)$ given in Proposition 3.11 and 3.12 and generated by $(\mathcal{A}, D(\mathcal{A}))$ and $(\mathcal{B}, D(\mathcal{B}))$, respectively. Then the Ornstein–Uhlenbeck semigroup on $C_b(\mathbb{R}^n)$ given by (3.12) is bi–continuous with respect to τ, generated by the bi–closure of $\mathcal{A}+\mathcal{B}$, and represented by the Lie–Trotter product formula, i.e.,*

$$\mathcal{P}(t)f = \tau\text{–}\lim_{n\to\infty} \left[\mathcal{T}(\tfrac{t}{n})\mathcal{S}(\tfrac{t}{n})\right]^n f$$

for all $f \in C_b(\mathbb{R}^n)$ and uniformly for t in compact intervals of $\mathbb{R}_+$.

Proof. Clearly, $\|[T(t)S(t)]^m\| \leq 1$ for all $t \geq 0, m \in \mathbb{N}$. Let $m \in \mathbb{N}$, $t \geq 0$, $f \in C_b(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$. Then

$$\begin{aligned}
&[\mathcal{T}(t)\mathcal{S}(t)]^m f(x) = \\
&\frac{1}{(2\pi t)^{nm/2}(\det A)^{m/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(x-y_1), (x-y_1)\rangle\right) dy_1 \\
&\quad \prod_{i=2}^{m-1} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(e^{tB}y_{i-1}-y_i), (e^{tB}y_{i-1}-y_i)\rangle\right) dy_i \\
&\quad \cdot \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(e^{tB}y_{m-1}-y_m), (e^{tB}y_{m-1}-y_m)\rangle\right) f(e^{tB}y_m) dy_m.
\end{aligned}$$

Put $l_t := (2\pi t)^{n/2}(\det A)^{1/2}$ and let $\gamma \in \Gamma$.

It follows that

$$
\begin{aligned}
&\gamma(x)\left|[\mathcal{T}(t)\mathcal{S}(t)]^m f(x)\right| \\
&\leq \frac{\gamma(x)}{l_t^m}\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(x-y_1),(x-y_1)\rangle\right)dy_1 \\
&\qquad \prod_{i=2}^{m-1}\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(e^{tB}y_{i-1}-y_i),(e^{tB}y_{i-1}-y_i)\rangle\right)dy_i \\
&\qquad \cdot\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(e^{tB}y_{m-1}-y_m),(e^{tB}y_{m-1}-y_m)\rangle\right) \\
&\qquad\qquad \cdot(1+\|y_m\|^2)\frac{|f(e^{tB}y_m)|}{1+\|y_m\|^2}dy_m \\
&\leq \frac{\gamma(x)}{l_t^m}\sup_{z\in\mathbb{R}^n}\frac{|f(e^{tB}z)|}{1+\|z\|^2} \\
&\quad \cdot\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(x-y_1),(x-y_1)\rangle\right)dy_1 \\
&\quad \cdot\prod_{i=2}^{m-1}\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(e^{tB}y_{i-1}-y_i),(e^{tB}y_{i-1}-y_i)\rangle\right)dy_i \\
&\quad \cdot\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(e^{tB}y_{m-1}-y_m),(e^{tB}y_{m-1}-y_m)\rangle\right) \\
&\qquad\qquad \cdot(1+\|y_m\|^2)dy_m.
\end{aligned}
\tag{3.22}
$$

Now, fix $s>0$ and put $\tilde{\gamma}_0(z) := \sup_{z\in\mathbb{R}^n}\frac{1}{1+\|e^{-tB}z\|^2}$, $z\in\mathbb{R}^n$, so that $\tilde{\gamma}_0\in\Gamma$ by (i). Then, by (3.22), we obtain that for $0\leq t\leq s$

$$
\begin{aligned}
&\gamma(x)\left|[\mathcal{T}(t)\mathcal{S}(t)]^m f(x)\right| \\
&\leq \frac{\gamma(x)}{l_t^m}\sup_{z\in\mathbb{R}^n}\tilde{\gamma}_0(z)|f(z)| \\
&\quad \cdot\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(x-y_1),(x-y_1)\rangle\right)dy_1 \\
&\quad \prod_{i=2}^{m-1}\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(e^{tB}y_{i-1}-y_i),(e^{tB}y_{i-1}-y_i)\rangle\right)dy_i \\
&\quad \cdot\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(e^{tB}y_{m-1}-y_m),(e^{tB}y_{m-1}-y_m)\rangle\right)(1+\|y_m\|^2)dy_m
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sup_{z\in\mathbb{R}^n} \tilde{\gamma}_0(z)|f(z)| \\
&\quad\cdot \Bigg(\gamma(x) + \frac{\gamma(x)}{l_t^m}\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(x-y_1),(x-y_1)\rangle\right) dy_1 \\
&\qquad\qquad \prod_{i=2}^{m-2}\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(e^{tB}y_{i-1}-y_i),(e^{tB}y_{i-1}-y_i)\rangle\right) dy_i \\
&\qquad\qquad \cdot\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(e^{tB}y_{m-2}-y_{m-1}),(e^{tB}y_{m-2}-y_{m-1})\rangle\right) \\
&\qquad\qquad\qquad \cdot (\|e^{tB}y_{m-1}\|^2 + nt\|A\|^2) l_t dy_{m-1}\Bigg) \\
&\leq \sup_{z\in\mathbb{R}^n} \tilde{\gamma}_0(z)|f(z)| \\
&\quad\cdot \Bigg(\gamma(x)(1+nt\|A\|^2) + \frac{\gamma(x)e^{2t\|B\|}}{l_t^{m-1}}\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(x-y_1),(x-y_1)\rangle\right) dy_1 \\
&\quad \prod_{i=2}^{m-2}\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(e^{tB}y_{i-1}-y_i),(e^{tB}y_{i-1}-y_i)\rangle\right) dy_i \\
&\quad \cdot\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2t}\langle A^{-1}(e^{tB}y_{m-2}-y_{m-1}),(e^{tB}y_{m-2}-y_{m-1})\rangle\right) \|y_{m-1}\|^2 dy_{m-1}\Bigg) \\
&\leq \sup_{z\in\mathbb{R}^n} \tilde{\gamma}_0(z)|f(z)|\, \gamma(x)(1+mnt\|A\|^2 + e^{2mt\|B\|}\|x\|^2),
\end{aligned}
$$

and hence

$$
\gamma(x)\,|[\mathcal{T}(t)\mathcal{S}(t)]^m f(x)| \leq \sup_{z\in\mathbb{R}^n} \tilde{\gamma}_0(z)|f(z)|\, \gamma(x)(1+mnt\|A\|^2 + e^{2mt\|B\|}\|x\|^2). \tag{3.23}
$$

Take $w := \max\{2\|B\|, 1\}$ which is independent of γ, s and f, and $M := 2\max\{M_\gamma, n\|A\|^2\}$ with $M_\gamma := \sup_{x\in\mathbb{R}^n}(1+\|x\|^2)\gamma(x) < \infty$. It follows by (3.23) that there exists $\tilde{\gamma} := M\tilde{\gamma}_0 \in \Gamma$ such that

$$
p_\gamma\left([\mathcal{T}(t)\mathcal{S}(t)]^m f\right) \leq e^{mwt} p_{\tilde{\gamma}}(f)
$$

for all $f \in C_b(\mathbb{R}^n)$, $0 \leq t \leq s$ and $m \in \mathbb{N}$. Since γ and s were arbitrary, we conclude that there exists $w \in \mathbb{R}_+$ such that for $\gamma \in \Gamma$ and $s > 0$ there exists $\tilde{\gamma} \in \Gamma$ such that

$$
p_\gamma\left(\left[\mathcal{T}(\tfrac{t}{m})\mathcal{S}(\tfrac{t}{m})\right]^m f\right) \leq e^{wt} p_{\tilde{\gamma}}(f)
$$

for all $f \in C_b(\mathbb{R}^n)$, $0 \le t \le s$, and $m \in \mathbb{N}$.
Therefore, the family $\{[\mathcal{T}(\frac{t}{m})\mathcal{S}(\frac{t}{m})]^m : t \ge 0\}$ is locally bi–equicontinuous uniformly for $m \in \mathbb{N}$.
As stated in (ii) at the beginning of this subsection, the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is bi–dense in $(C_b(\mathbb{R}^n), \tau)$. Moreover, it is a subset of $D(\mathcal{A}_0) \cap D(\mathcal{B}_0)$, where $D(\mathcal{A}_0)$ and $D(\mathcal{B}_0)$ denote the domains of the parts $\mathcal{A}_0$ and $\mathcal{B}_0$ of $\mathcal{A}$ and $\mathcal{B}$ in $C_0(\mathbb{R}^n)$. On $C_0(\mathbb{R}^n)$ the Ornstein–Uhlenbeck semigroup $(\mathcal{P}(t))_{t\ge0}$ is strongly continuous and is represented by the Lie–Trotter Product Formula (see [KW, Prop. 12]). In particular, its generator coincides with $\mathcal{A} + \mathcal{B}$ restricted to $\mathcal{S}(\mathbb{R}^n)$. Hence, by the invariance of the Schwartz space under $(\mathcal{P}(t))_{t\ge0}$, we obtain that $(\lambda - \mathcal{A} - \mathcal{B})\mathcal{S}(\mathbb{R}^n)$ is bi–dense in $(C_b(\mathbb{R}^n), \tau)$ for $\lambda > 0$. Applying Corollary 2.11 we obtain that the bi–closure of $\mathcal{A} + \mathcal{B}$ generates the bi–continuous semigroup $(\mathcal{P}(t))_{t\ge0}$ on $C_b(\mathbb{R}^n)$ given by the Lie–Trotter product formula

$$\mathcal{P}(t)f = \tau\text{–}\lim_{m\to\infty} \left[\mathcal{T}(\tfrac{t}{m})\mathcal{S}(\tfrac{t}{m})\right]^m f$$

for all $f \in C_b(\mathbb{R}^n)$ and uniformly for t in compact intervals of $\mathbb{R}^n$. □

3.4 Implemented semigroups

In this section we look at so–called *implemented semigroups*, which are of interest, e. g., in the context of operator equations of the form

$$AX + XB = Y,\ X \in \mathcal{X},$$

where $\mathcal{X} := \mathcal{L}(X, Y)$, X, Y Banach spaces, and A and B are generators of strongly continuous semigroups $(T(t))_{t\ge0}$ and $(S(t))_{t\ge0}$ on X and Y, respectively. Such equations have been studied, e.g., in [GN81], [Pho91], [ARS94], [PS98], [CL99, Sec. 4.4], [Alb99], [Alb]. Moreover, implemented semigroups are the natural semigroups on operator algebras such as $\mathcal{L}(H)$, H Hilbert space, or more general $C^\star$–algebras (see [BR79]).
On the Banach space $\mathcal{X} = \mathcal{L}(X, Y)$ endowed with the operator norm $\|\cdot\|_{\mathcal{X}}$ we consider the operator family $\{\mathcal{U}(t) :\ t \ge 0\}$ defined as

$$\mathcal{U}(t)X := T(t)XS(t), \quad t \ge 0,\ X \in \mathcal{X}. \tag{3.24}$$

Clearly, $\mathcal{U}(t) \in \mathcal{L}(\mathcal{X})$ for all $t \geq 0$ and $\mathcal{U}(0) = Id_{\mathcal{X}}$, $\mathcal{U}(t+s) = \mathcal{U}(t)\mathcal{U}(s)$ for all $t, s \geq 0$. The semigroup $(\mathcal{U}(t))_{t\geq 0}$ given in (3.24) is called **implemented semigroup** on $\mathcal{X}$ with implementing semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$.
The following lemma gives the relation between the growth bound of the implemented semigroup and its implementing semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ (see [Alb99, Lemma 4.4.1]).

Lemma 3.14. *The growth bound $\omega_0(\mathcal{U}(\cdot))$ of $(\mathcal{U}(t))_{t\geq 0}$ is given by*

$$\omega_0(\mathcal{U}(\cdot)) = \omega_0(T(\cdot)) + \omega_0(S(\cdot)),$$

where $\omega_0(T(\cdot))$ and $\omega_0(S(\cdot))$ denote the growth bound of the semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, respectively.

Proof. Clearly, $\|\mathcal{U}(t)X\| \leq \|T(t)\| \; \|S(t)\| \; \|X\|$ for all $X \in \mathcal{X}$.
On the other hand, let $\Phi := \{\phi \in (Y, \|\cdot\|)' : \; \|\phi\| \leq 1\}$, then we have

$$\begin{aligned}
\|S(t)\| \; \|T(t)\| &= \sup\{| < S(t)y, \phi > | : \; y \in Y \text{ with } \|y\|_Y \leq 1 \text{ and } \phi \in \Phi\} \; \|T(t)\| \\
&= \sup\{\|T(t) < S(t)y, \phi > x\| : \|y\|_Y \leq 1, \|x\|_X \leq 1, \phi \in \Phi\} \\
&\leq \sup_{\|X\|_{\mathcal{X}} \leq 1} \|T(t)XS(t)\|_{\mathcal{X}} \\
&= \|\mathcal{U}(t)\|,
\end{aligned}$$

and hence $\|S(t)\| \; \|T(t)\| = \|\mathcal{U}(t)\|$ for all $t \geq 0$. Therefore, applying [EN00, Ch. IV, Prop. 2.2], we obtain

$$\omega_0(\mathcal{U}(\cdot)) = \lim_{t\to\infty} \frac{1}{t} \log(\|T(t)\| \; \|S(t)\|) = \omega_0(T(\cdot)) + \omega_0(S(\cdot)).$$

□

If the implementing semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are uniformly continuous, one can verify that the corresponding implemented semigroup is also uniformly continuous. However, if the implementing semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are only strongly continuous, the implemented semigroup is not strongly continuous in general.

Example 3.15. On $X = Y = L^1(\mathbb{R}, \mathbb{C})$ we define $(T(t))_{t\in\mathbb{R}}$ as the left translation group defined as $T(t)f(s) := f(s+t)$ for all $s, t, \in \mathbb{R}$ and $f \in X$. The group $(S(t))_{t\in\mathbb{R}}$

is taken as $S(t)f(s) := \overline{T(t)f(s)}$ for all $f \in X$ and $s, t \in \mathbb{R}$.
Now, we consider the operator $X \in \mathcal{L}(X)$ defined as $Xf := \chi_{[0,1]}f$ for all $f \in X$, and choose a real–valued sequence $(f_n)_{n\in\mathbb{N}} \subseteq X$ such that $\|f_n\|_1 = 1$ and the support of f_n is contained in $[0, 1/n]$ for all $n \in \mathbb{N}$. We then have

$$\|\mathcal{U}(\tfrac{1}{n})Xf_n - Xf_n\|_1 = \|\chi_{[1/n,1+1/n]}f_n - \chi_{[0,1]}f_n\|_1 = \|f_n\|_1 = 1.$$

Thus, the implemented semigroup is not strongly continuous (cf. [Alb99, Beispiel 1.2.1]).

However, if we endow $\mathcal{X}$ with the strong operator topology τ_{stop}, which is induced by a family of seminorms $P_{\tau_{stop}} := \{p_x : x \in X\}$ where p_x is defined as $p_x(X) := \|Xx\|_Y$ for all $X \in \mathcal{X}$, then the implemented semigroups fit into the theory of bi–continuous semigroups.
To show this, we verify first that the space $\mathcal{X}$ satisfies Assumptions 1.1. Clearly, the strong operator topology τ_{stop} is coarser than the norm topology and Hausdorff. Further, $(\mathcal{X}, \tau_{stop})$ is sequentially complete on norm bounded sets. Indeed, let $(X_n)_{n\in\mathbb{N}}$ be a $\|\cdot\|_{\mathcal{X}}$–bounded τ_{stop}–Cauchy–sequence in $\mathcal{X}$, i.e., there exists $M \geq 0$ such that $\sup_{n\in\mathbb{N}} \|X_n\| \leq M$ and for $x \in X$ and for $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|X_n x - X_m x\| = p_x(X_n - X_m) < \epsilon$ for all $n, m \geq n_0$. Therefore, $(X_n x)_{n\in\mathbb{N}}$ is a Cauchy–sequence in $(X, \|\cdot\|_X)$, and hence $Xx := \lim_{n\to\infty} X_n x$ and X is a linear operator from X into Y. Further, $\|Xx\| = \lim_{n\to\infty} \|X_n x\| \leq \sup_{n\in\mathbb{N}} \|X_n\| \, \|x\| \leq M\|x\|$. Finally, since the mappings $\psi_{\phi,x} : X \mapsto < Xx, \phi >$ belong to the dual $(\mathcal{X}, \tau_{stop})'$ with $\|\psi_{\phi,x}\|_{\mathcal{X}'} \leq 1$ for all $x \in X$ with $\|x\|_X \leq 1$ and $\phi \in Y'$ with $\|\phi\|_{Y'} \leq 1$, the dual $(\mathcal{X}, \tau_{stop})'$ is norming for $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$.

Proposition 3.16. *Let $(\mathcal{U}(t))_{t\geq 0}$ be a semigroup on $\mathcal{X}$ implemented by strongly continuous semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ on Banach spaces X and Y, respectively. Then the implemented semigroup $(\mathcal{U}(t))_{t\geq 0}$ is a bi–continuous semigroup with respect to τ_{stop}.*

Proof. As mentioned above, $(\mathcal{U}(t))_{t\geq 0}$ consists of bounded linear operators on $\mathcal{X}$ and the semigroup law holds. Now, let $t_0 \geq 0$, $X \in \mathcal{X}$, $y \in Y$, and $h \in \mathbb{R}$ such that $t_0 - h \geq 0$. We then have

$$\begin{aligned}
\mathcal{U}(t_0 + h)Xy - \mathcal{U}(t_0)Xy &= T(t_0 + h)XS(t_0 + h)y - T(t_0)XS(t_0)y \\
&= T(t_0 + h)X(S(t_0 + h)y - S(t_0)y) \\
&\quad + (T(t_0 + h) - T(t_0))XS(t_0)y,
\end{aligned}$$

which converges to zero as $h \to 0$ with respect to the norm on X.
Moreover, for every $t_0 > 0$ and every $\|\cdot\|_{\mathcal{X}}$–bounded sequence $(X_n)_{n\in\mathbb{N}} \subseteq \mathcal{X}$ which is τ_{stop}–convergent to $X \in \mathcal{X}$, it follows that for all $p_x \in P_{\tau_{stop}}$ there exists a constant $M \geq 0$ such that

$$\begin{aligned}\sup_{0\leq t\leq t_0} p_x(\mathcal{U}(t)(X_n - X)) &= \sup_{0\leq t\leq t_0} \|T(t)(X_n - X)S(t)x\|_Y \\ &\leq M \sup_{0\leq t\leq t_0} \|(X_n - X)S(t)x\|_Y.\end{aligned} \tag{3.25}$$

By the strong continuity of the semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ and the fact that $X_n - X$ converges uniformly on compact sets, it follows that (3.25) converges to zero as $n \to \infty$, i.e., $(\mathcal{U}(t))_{t\geq 0}$ is locally bi–equicontinuous. This completes the proof. □

As a consequence we obtain that the characterization theorem from Chapter 1 and the approximation theorems from Chapter 2 hold for implemented semigroups. For a detailed description of their generators (and domains) we refer to [Alb]. Here, we only state the Lie–Trotter Product Formula for implemented semigroups.

Proposition 3.17. *Let $(\mathcal{U}(t))_{t\geq 0}$ and $(\mathcal{V}(t))_{t\geq 0}$ be implemented semigroups on $\mathcal{X}$ generated by $(\mathcal{A}, D(\mathcal{A}))$ and $(\mathcal{B}, D(\mathcal{B}))$, respectively. satisfying the following stability condition.*

$$\|[\mathcal{U}(t)\mathcal{V}(t)]^m\| \leq Me^{\omega t} \tag{3.26}$$

for all $t \geq 0, m \in \mathbb{N}$, and some constants $M \geq 1$, $\omega \in \mathbb{R}$. Consider the sum $\mathcal{A} + \mathcal{B}$ on a subspace $D \subseteq D(\mathcal{A}_0) \cap D(\mathcal{B}_0)$, where $D(\mathcal{A}_0)$ and $D(\mathcal{B}_0)$ denote the domains of the parts $\mathcal{A}_0$ and $\mathcal{B}_0$ of $\mathcal{A}$ and $\mathcal{B}$ in $\overline{D(\mathcal{A})}^{\|\cdot\|}$ and $\overline{D(\mathcal{B})}^{\|\cdot\|}$, respectively, and assume that D and $(\lambda_0 - \mathcal{A} - \mathcal{B})D$ are bi–dense in X for some $\lambda_0 > \alpha > \omega$. Then the bi–closure of $\mathcal{A} + \mathcal{B}$ exists and generates a bi–continuous semigroup $(\mathcal{W}(t))_{t\geq 0}$ with respect to τ_{stop} given by

$$\mathcal{W}(t)X = \tau_{stop}\text{–}\lim_{k\to\infty} \left[\mathcal{U}(\tfrac{t}{k})\mathcal{V}(\tfrac{t}{k})\right]^k X, \tag{3.27}$$

where the limit exists for all $X \in \mathcal{X}$ and uniformly for t in compact intervals in $\mathbb{R}_+$.

Proof. Let $(T_V(t))_{t\geq 0}$ and $(S_V(t))_{t\geq 0}$ denote the implementing semigroups of $(\mathcal{V}(t))_{t\geq 0}$, $t_0 \geq 0$ and $(X_n)_{n\in\mathbb{N}} \subseteq \mathcal{X}$ be a $\|\cdot\|_{\mathcal{X}}$–bounded sequence which is τ_{stop}–convergent to $X \in \mathcal{X}$. It follows that for all $p_x \in P_{\tau_{stop}}$ there exist constants

$C, \tilde{C}, M \geq 0$ such that

$$\begin{aligned}
&\sup_{0\le t\le t_0} p_x([\mathcal{U}(\tfrac{t}{m})\mathcal{V}(\tfrac{t}{m})]^m(X_n - X)) \\
&\qquad \le \sup_{0\le t\le t_0} \|[\mathcal{U}(\tfrac{t}{m})\mathcal{V}(\tfrac{t}{m})]^{m-1}\mathcal{U}(\tfrac{t}{m})\|\|\mathcal{V}(\tfrac{t}{m})(X_n - X)\| \\
&\qquad \le \sup_{0\le t\le t_0} Me^{\frac{t\omega}{m}}\|\mathcal{U}(\tfrac{t}{m})\| \sup_{0\le t\le a} \|\mathcal{V}(\tfrac{t}{m})(X_n - X)\| \\
&\qquad \le C \sup_{0\le t\le t_0} \|T_V(t)(X_n - X)S_V(t)x\|_Y \\
&\qquad \le \tilde{C} \sup_{0\le t\le t_0} \|(X_n - X)S(t)x\|_Y,
\end{aligned}$$

which converges to zero as $n \to \infty$ uniformly for all $m \in \mathbb{N}$.
Therefore, the assumptions of Corollary 2.11 are satisfied and the assertion holds.

□

For instance, the stability condition in (3.26) is fulfilled if $\omega_0(\mathcal{U}(\cdot)) = \omega_0(T(\cdot)) + \omega_0(S(\cdot)) \le 0$.

3.5 Adjoint semigroups

The general theory of adjoint semigroups was initiated by W. Feller [Fel53] and R. S. Phillips [Phi55]. In [BR79, Sec. 3.1.2-3.1.5] a generation result and an approximation theory has been developed, and we refer to the book of J. van Neerven [Nee92] where a systematic exposition of the theory of adjoint semigroups is given.

3.5.1 Bi–continuous adjoint semigroups

Let X be a Banach space and X' its topological dual. We denote by $\sigma(X, X')$ the weak topology on X and by $\sigma(X', X)$ the weak* topology on X'. For a strongly continuous semigroup $(T(t))_{t\ge0}$ on X its adjoint semigroup $(T(t)')_{t\ge0}$, consisting of all adjoint operators $T(t)'$ on the dual X', is not strongly continuous in general. An example is provided by the left translation group on $L^1(\mathbb{R})$. Its adjoint is the right translation group on $L^\infty(\mathbb{R})$ which is not strongly continuous (see [EN00, Ch. I, Sec. 4.c]).
However, in our first result we show that every adjoint semigroup $(T(t)')_{t\ge0}$ is bi–continuous with respect to $\sigma(X', X)$, hence the results from Chapter 1 and 2 apply.

To show this we note that X' endowed with the norm $\|\cdot\|_{X'}$ and the topology $\sigma(X', X)$ satisfies Assumptions 1.1.

Proposition 3.18. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on X. Then $(T(t)')_{t\geq 0}$ is a bi–continuous semigroup on X' with respect to $\sigma(X', X)$.*

Proof. Clearly, $(T(t)')_{t\geq 0}$ is a semigroup which is exponentially bounded. Since $(T(t))_{t\geq 0}$ is strongly continuous on X, we have

$$|< T(t)'x' - T(s)'x', x >| = |< x', T(t)x - T(s)x >| \leq \|x'\| \, \|T(t)x - T(s)x\|$$

for all $x \in X$, $x' \in X'$ and $t, s \geq 0$, and hence $(T(t)')_{t\geq 0}$ is $\sigma(X', X)$–continuous. Let now $t_0 \geq 0$ and $(x'_n)_{n\in\mathbb{N}} \subseteq X'$ be a $\|\cdot\|$–bounded sequence which is $\sigma(X', X)$–convergent to $x' \in X'$. Then,

$$|< T(t)'(x'_n - x'), x >| = |< x'_n - x', T(t)x >|$$

for all $x \in X$ which converges to zero as $n \to \infty$ uniformly for $0 \leq t \leq t_0$ by the compactness of $\{T(t)x : \ 0 \leq t \leq t_0\}$. Therefore, $(T(t)')_{t\geq 0}$ is a bi–continuous semigroup with respect to $\sigma(X', X)$. $\square$

3.5.2 A characterization of Mackey–continuous semigroups on dual spaces

It is a natural question to look for topologies on X' coarser than the norm topology but finer than the weak* topology for which the adjoint semigroup $(T(t)')_{t\geq 0}$ still is continuous. A natural candidate for this purpose is the **Mackey topology** $\tau(X', X)$ on X'. This topology is defined by the family P_τ of seminorms p_S on X' with

$$p_S(x') := \sup_{x\in S} |< x', x >|$$

for all $x' \in X'$ and $S \in \Sigma$, the family of all (convex, see [Sch80, Ch. II, 4.3]) $\sigma(X, X')$–compact subsets of X. By the Mackey–Arens theorem [Sch80, Ch. IV, Thm. 3.2] this topology is the finest locally convex topology on X' such that all $\tau(X', X)$–continuous functionals belong to X.

Proposition 3.19. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X and $\tau(X', X)$ the Mackey topology on the dual space X'. Then the following assertions are equivalent.*

(a) *For every $S \in \Sigma$*

$$\bigcup_{0 \le t \le 1} T(t)S \in \Sigma. \tag{3.28}$$

(b) *The adjoint semigroup $(T(t)')_{t\geq 0}$ is $\tau(X', X)$–continuous on the dual space X'.*

Proof. $(a) \Rightarrow (b)$ First, we prove that the subspace of X' defined by

$$E := \{x' \in X' : \tau - \lim_{t \downarrow 0}(T(t)'x' - x') = 0\}$$

is $\sigma(X', X)$–dense in X' and therefore τ–dense in X' since all $\tau(X', X)$–continuous functionals belong to X.
Let $x' \in X'$ and $0 < r \le 1$. We define a linear form x'_r on X by

$$< x'_r, x > = \frac{1}{r}\int_0^r < T(s)'x', x > ds \quad \text{ for } x \in X \ .$$

Then x'_r is bounded and therefore $x'_r \in X'$. Furthermore, the set

$$D := \{x'_r : 0 < r \le 1, x' \in X'\}$$

is $\sigma(X', X)$–dense in X' because $(T(t)')_{t\geq 0}$ is $\sigma(X', X)$–continuous. Moreover, for $S \in \Sigma$ there exists a constant $M \ge 1$ such that

$$\begin{aligned}
(3.29) \qquad & \sup_{x\in S} | < T(t)'x'_r - x'_r, x > | \\
&= \sup_{x \in S} |\frac{1}{r} T(t)' \int_0^r < T(s)'x', x > ds - \frac{1}{r}\int_0^r < T(s)'x', x > ds| \\
&\le \sup_{x\in S}\left\{ |\frac{1}{r}\int_r^{t+r} < T(s)'x', x > ds| + |\frac{1}{r}\int_0^t < T(s)'x', x > ds| \right\} \\
(3.30) \qquad &\le \frac{2t}{r} M \|x'\| \sup_{x\in S}\|x\|,
\end{aligned}$$

where $\sup_{x\in S}\|x\| < \infty$. This implies that (3.30) converges to zero as $t \searrow 0$. Thus $D \subseteq E$ which implies that E is τ–dense in X'.
Finally, we prove the τ–continuity of $(T(t)')_{t\geq 0}$ on X'. Let $S \in \Sigma$ and $\epsilon > 0$. By assumption (3.28) we have

$$S \subseteq \bigcup_{0\le t\le 1} T(t)S =: S_1 \in \Sigma \ .$$

Since E is $\sigma(X',X)$–dense in X', there exists a linear form $x_0' \in E$ such that

$$\sup_{x\in S_1} | < x' - x_0', x > | \leq \epsilon \ .$$

Moreover, since $x_0' \in E$, there exists $t_0 > 0$ such that

$$\sup_{x\in S} | < T(t)'x_0', x > - < x_0', x > | \leq \epsilon$$

for each $0 \leq t \leq t_0$. Hence,

$$\begin{aligned}
&\sup_{x\in S} | < T(t)'x', x > - < x', x > | \\
&\quad = \sup_{x\in S} | < T(t)'x', x > - < T(t)'x_0', x > \\
&\qquad + < T(t)'x_0', x > - < x_0', x > + < x_0' - x', x > | \\
&\quad \leq \sup_{x\in S} | < T(t)'x', x > - < T(t)'x_0', x > | + 2\epsilon \\
&\quad = \sup_{x\in S} | < x' - x_0', T(t)x > | + 2\epsilon \\
&\quad \leq \sup_{x\in S_1} | < x' - x_0', x > | + 2\epsilon \\
&\quad \leq 3\epsilon \ .
\end{aligned}$$

$(b) \Rightarrow (a)$ Let $S \subseteq X$ be relatively weakly compact. By the Eberlein–Šmulian theorem ([DS57, Chap. V.6.1, Thm. 1]) it suffices to show that $S_1 := \bigcup_{0\leq t\leq 1} T(t)S$ is weakly sequentially compact. Let $(x_n)_{n\in\mathbb{N}} \subseteq S_1$ such that $x_n = T(t_n)y_n$ with $(t_n)_{n\in\mathbb{N}} \subseteq [0,1]$ and $(y_n)_{n\in\mathbb{N}} \subseteq S$. Since $[0,1]$ is compact and S is relatively weakly compact, by passing to subsequences, we have that $t_{n_k} \to t_0$ for some $t_0 \in [0,1]$, and $y_{n_k} \to y_0$ with respect to the weak topology for some $y_0 \in S$. To simplify matters, we denote these subsequences by $(t_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$. We claim that $T(t_n)y_n \to T(t_0)y_0$ as $n \to \infty$ with respect to the weak topology. To that purpose, we observe that

$$T(t_n)y_n - T(t_0)y_0 = T(t_n)y_n - T(t_0)y_n + T(t_0)y_n - T(t_0)y_0 \tag{3.31}$$

for all $n \in \mathbb{N}$. Since $T(t_0)$ is norm, and hence weakly continuous, we obtain

$$\sigma(X,X')\text{–}\lim_{n\to\infty} T(t_0)y_n = T(t_0)y_0. \tag{3.32}$$

Moreover, $H := \{y_m : \ m \in \mathbb{N}\}$ is a relatively weakly compact subset of X. Therefore, by assumption, we have

$$\sup_{y \in H} | < T(t_n)y - T(t_0)y, x' > | = \sup_{m \in \mathbb{N}} | < T(t_n)y_m - T(t_0)y_m, x' > |,$$

which converges to zero as $n \to \infty$, and this implies that

$$\lim_{n \to \infty} < T(t_n)y_n - T(t_0)y_n, x' >= 0.$$

Combining (3.31) and (3.32) concludes the proof. □

Take now the adjoint $(T(t)')_{t \geq 0}$ of a strongly continuous semigroup $(T(t))_{t \geq 0}$ and assume that it is $\tau(X', X)$–continuous. It is then locally bi–equicontinuous with respect to $\tau(X', X)$. In fact, take $t_0 \geq 0$, $(x'_n)_{n \in \mathbb{N}} \subseteq X'$ a $\| \cdot \|_{X'}$-bounded sequence which is $\tau(X', X)$–convergent to $x \in X'$, and $S \in \Sigma$. We then obtain, by Proposition 3.19, that $S_{t_0} := \bigcup_{0 \leq t \leq t_0} T(t)S \in \Sigma$, and therefore

$$\sup_{x \in S} | < T(t)'(x'_n - x'), x > | = \sup_{x \in S} | < x'_n - x, T(t)x > | = \sup_{x \in S_{t_0}} | < x'_n - x, x > | \ ,$$

which converges to zero as $n \to \infty$ uniformly for $0 \leq t \leq t_0$.
Since evidently X' with $\tau(X', X)$ satisfies Assumptions 1.1, we obtain the following result.

Proposition 3.20. *Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup on a Banach space X and take $\tau(X', X)$ the Mackey topology on the dual space X'. If the adjoint semigroup $(T(t)')_{t \geq 0}$ is $\tau(X', X)$–continuous, then it is a bi–continuous semigroup with respect to $\tau(X', X)$.*

Remark 3.21. a) If τ is any locally convex topology on X' which is given by a saturated[2] family Σ of relatively $\sigma(X, X')$–compact subsets of X, one shows, as in the proof of Proposition 3.19, that the adjoint semigroup $(T(t)')_{t \geq 0}$ is τ–continuous on X' if and only if

$$\bigcup_{0 \leq t \leq 1} T(t)S \in \Sigma$$

[2] A family $\emptyset \neq \Sigma$ of bounded subsets of a locally convex space is called **saturated** if it contains arbitrary subsets and all scalar multiples of each of its members, and it contains the closed, convex, circled hull of the union of each finite subfamily (see [Sch80, Ch. III, p. 81])

for all $S \in \Sigma$.
b) Condition (3.28) is trivially satisfied if τ is the $\sigma(X', X)$–topology or if τ is the locally convex topology of uniform convergence on sets formed by the ranges of all null sequences in X (cf. [Sch80, p. 151]).

As an example we show the Mackey–continuity of the translation group on $L^\infty(\mathbb{R})$.

Example 3.22. Let $(L(t))_{t\in\mathbb{R}}$ be the left translation group on $L^1(\mathbb{R})$. Then the right translation group $(R(t))_{t\in\mathbb{R}}$ is bi–continuous on $L^\infty(\mathbb{R})$ with respect to the Mackey topology $\tau(L^\infty, L^1)$.
By Proposition 3.19, it suffices to show that assumption (3.28) is satisfied with respect to the Mackey topology, i.e., for Σ the family of all $\sigma(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$–compact subsets of $L^1(\mathbb{R})$. To show this, we first recall that, by a result of Dunford–Pettis (see [DU77, p. 76]), a subset S of $L^1(\mathbb{R})$ is relatively weakly compact if and only if S is $\|\cdot\|$–bounded and

$$(3.33) \qquad \begin{cases} \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that} \\ \forall \Omega \subset \mathbb{R} \text{ with Lebesgue measure } \lambda(\Omega) < \delta \text{ one has} \\ \int_\Omega |f(s)| ds < \epsilon \quad \forall f \in S \ . \end{cases}$$

Let $S \subseteq L^1(\mathbb{R})$ be relatively weakly compact. Then S is $\|\cdot\|$–bounded and hence

$$\|L(t)f\| = \|f\| \leq \sup_{f\in S} \|f\| < \infty$$

for all $t \in \mathbb{R}$ and $f \in S$. This means that

$$S_1 := \bigcup_{0\leq t\leq 1} L(t)S$$

is $\|\cdot\|$–bounded. It remains to show that S_1 satisfies condition (3.33). Since S is relatively weakly compact and the Lebesgue measure is translation invariant, we have

$$\begin{cases} \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that} \\ \forall \Omega \subset \mathbb{R} \text{ with } \lambda(\Omega) < \delta : \\ \int_\Omega |L(t)f(s)| ds = \int_\Omega |f(s+t)| ds = \int_{\Omega - t} |f(s)| ds < \epsilon \quad \forall f \in S \ \forall t \in \mathbb{R} \ . \end{cases}$$

This completes the proof.

Remark 3.23. If we take the multiplication semigroup $(T(t))_{t\geq 0}$ on $L^1(\mathbb{R})$ generated by $A = M_{iq}$ for $q : \mathbb{R} \longrightarrow \mathbb{R}$ a measurable and locally integrable function, then, the adjoint semigroup $(T(t)')_{t\geq 0}$ is bi–continuous on $L^\infty(\mathbb{R})$ with respect to the Mackey topology $\tau(L^\infty, L^1)$. In fact, by the same way as in Example 3.22, for a relatively weakly compact subset $S \subset L^1(\mathbb{R})$ we have

$$\begin{cases} \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that} \\ \forall \Omega \subset \mathbb{R} \text{ with } \lambda(\Omega) < \delta : \\ \int_\Omega |T(t)f(s)|ds = \int_\Omega |e^{iq(s)} f(s)|ds = \int_\Omega |f(s)|ds < \epsilon \quad \forall f \in S \ \forall t \in \mathbb{R} \ , \end{cases}$$

which yields the assertion.

Since the geometric condition (3.28) stated in Proposition 3.28 is hard to verify, we look for conditions on the semigroup $(T(t))_{t\geq 0}$ implying (3.28). It is an easy consequence of the Eberlein–Šmulian theorem ([DS57, Chap. V.6.1, Thm. 1]) that if the map $(t, x) \mapsto T(t)x$ is weakly sequentially jointly continuous, we obtain that

$$\bigcup_{0\leq t\leq 1} T(t)S$$

is relatively weakly compact for every relatively weakly compact subset $S \subseteq X$. By Proposition 3.19 this implies the Mackey continuity of $(T(t)')_{t\geq 0}$ in X'. Furthermore, it turns out that the local bi–equicontinuity of $(T(t))_{t\geq 0}$ with respect to $\sigma(X, X')$ already implies that the map $(t, x) \mapsto T(t)x$ is weakly sequentially jointly continuous.

Proposition 3.24. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. If $(T(t))_{t\geq 0}$ is locally bi–equicontinuous with respect to $\sigma(X, X')$, then the adjoint semigroup $(T(t)')_{t\geq 0}$ on X' is bi–continuous with respect to the Mackey topology.*

Proof. By the previous considerations it remains to show that the local bi–equicontinuity of $(T(t))_{t\geq 0}$ with respect to $\sigma(X, X')$ implies that the map $(t, x) \mapsto T(t)x$ is jointly weakly sequentially continuous. To that purpose, let $(t_n)_{n\in\mathbb{N}} \subseteq [0, 1]$ be a sequence which converges to $t_0 \in [0, 1]$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges to $x_0 \in X$ with respect to $\sigma(X, X')$. By the principle of uniform boundedness the sequence $(x_n)_{n\in\mathbb{N}}$ is $\|\cdot\|$–bounded. Let $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $t_n \in [\epsilon - t_0, \epsilon + t_0]$ for all $n \geq n_0$. Therefore, we have, by the local bi–equicontinuity of $(T(t))_{t\geq 0}$ with respect to $\sigma(X, X')$, that there exists $\tilde{n}_0 \geq n_0$ such that

$$\sup_{t_n\in[\epsilon - t_0, \epsilon + t_0]} | < T(t_n)(y_n - y_0), x' > | \leq \frac{\epsilon}{2}$$

for all $x' \in X'$ and $n \geq \tilde{n}_0$. Hence, using the strong $\sigma(X, X')$–continuity of $(T(t))_{t\geq 0}$, we obtain

$$\begin{aligned} &| < T(t_n)y_n - T(t_0)y_0, x' > | \\ &\qquad \leq | < T(t_n)(y_n - y_0), x' > | + | < T(t_n)y_0 - T(t_0)y_0, x' > | \\ &\qquad \leq \epsilon \end{aligned}$$

for all $x' \in X'$ and $n \geq \tilde{n}_0$. This concludes the proof. $\square$

Appendix A

Laplace transform methods

In this appendix we collect some results needed for the generalized Hille–Yosida Theorem 1.28. For a systematic treatment and much more informations we refer to the monograph to [ABHN].
In the following let X be a Banach space.

Proposition A.1. *Let $F : [a, b] \longrightarrow X$ and $g : [a, b] \longrightarrow \mathbb{C}$. If F is an antiderivative of an L^1–function f and $g \in C([a, b])$, then $\int_a^b g(t)dF(t)$ is equal to the Bochner integral $\int_a^b g(t)f(t)dt$. If F is continuous and g absolutely continuous, then $\int_a^b F(s)dg(s)$ is equal to the Bochner integral $\int_a^b F(s)g'(s)ds$.*

To state the main result of the Laplace–Stieltjes transform theory, we define for functions $r \in C^\infty((0, \infty), X)$ the norm

$$\|r\|_W := \sup_{k\in\mathbb{N}} \sup_{\lambda>0} \|\lambda^{k+1} \frac{1}{k!} r^{(k)}(\lambda)\|$$

and the Widder space

$$C_W^{(\infty)}((0, \infty), X) := \{r \in C^\infty((0, \infty), X) : \|r\|_W < \infty\}.$$

This is a Banach space, and we obtain the following result due to D. V. Widder [Wid36] in the numerical case and to W. Arendt [Are87b] in the vector–valued case (see also [BN94]).

Theorem A.2. *The Laplace–Stieltjes transform*

$$\mathcal{L}_S \ : Lip_0([0, \infty), X) \longrightarrow C_W^{(\infty)}((0, \infty), X)$$

defined as

$$\mathcal{L}_S\ F(\lambda) := \int_0^\infty e^{-\lambda t} dF(t)$$

for $F \in Lip_0([0,\infty), X)$ *is an isometric isomorphism.*

To adapt this results to functions with exponential growth, we define, for $\omega \in \mathbb{R}$, the Banach space

$$\begin{aligned} Lip_\omega([0,\infty), X) := \{F : [0,\infty) \to X \mid\ & F(0) = 0 \text{ and} \\ & \|F(t+h) - F(t)\| \le M \int_t^{t+h} e^{\omega s} ds \\ & \text{for all } t, h > 0 \text{ and some } M > 0\} \end{aligned}$$

endowed with the norm

$$\|F\|_{Lip(\omega)} := \inf\{M\ :\ \|F(t+h) - F(t)\| \le M \int_t^{t+h} e^{\omega s} ds \text{ for all } t, h > 0\},$$

and $C_W((\omega,\infty), X)$ the space of all functions $r \in C^\infty((\omega,\infty), X)$ with the norm

$$\|r\|_W := \sup_{k\in\mathbb{N}} \sup_{\lambda>\omega} \|(\lambda-\omega)^{k+1} \tfrac{1}{k!} r^{(k)}(\lambda)\| < \infty.$$

By using a "shift" procedure we obtain the following reformulation of Widder's Theorem (cf. [ABHN, Thm. 2.5.1]).

Theorem A.3. *Let* $M > 0$, $\omega \in \mathbb{R}$, *and* $r \in C^\infty((\omega,\infty), X)$. *Then the following assertions are equivalent.*

(i) $\|(\lambda-\omega)^{k+1} \frac{1}{k!} r^{(k)}(\lambda)\| \le M$ *for all* $\lambda > \omega$ *and* $k \in \mathbb{N}$.

(ii) There exists $F : [0,\infty) \longrightarrow X$ *satisfying* $F(0) = 0$ *and*

$$\|F(t+h) - F(t)\| \le M \int_t^{t+h} e^{\omega r} dr \tag{A.1}$$

for all $t, h > 0$ *such that*

$$r(\lambda) = \int_0^\infty e^{-\lambda t} dF(t)$$

for all $\lambda > \omega$.

Lemma A.4. *Let A be a Hille–Yosida operator on a Banach space X. Then there exists a function $F \in Lip_\omega([0,\infty), \mathcal{L}(X))$ such that*

$$F(t)x - tx = \int_0^t F(s)Axds \tag{A.2}$$

for all $x \in D(A)$.

Proof. Since A is a Hille–Yosida operator, we are able to apply Widder's Theorem A.3 to the function

$$(\omega, \infty) \ni \lambda \mapsto R(\lambda, A) \in \mathcal{L}(X).$$

Therefore, we obtain a function $F \in Lip_\omega([0,\infty), \mathcal{L}(X))$ such that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} dF(t)$$

for all $\lambda > \omega$. Hence, by Proposition A.1, we obtain

$$R(\lambda, A)x = \| \cdot \| - \int_0^\infty \lambda e^{-\lambda t} F(t)xdt \tag{A.3}$$

for all $\lambda > \omega$ and $x \in X$. Now, let $x \in D(A)$. By the closedness of A, equation (A.3), and integration by parts, we have

$$\begin{aligned} \lambda^2 \int_0^\infty e^{-\lambda t} txdt = x &= \lambda R(\lambda, A)x - R(\lambda, A)Ax \\ &= \lambda^2 \int_0^\infty e^{-\lambda t} F(t)xdt - \lambda \int_0^\infty e^{-\lambda t} F(t)Axdt \\ &= \lambda^2 \int_0^\infty e^{-\lambda t} [F(t)x - \int_0^t F(s)Axds]dt. \end{aligned}$$

Thus, by the uniqueness of the Laplace transform [ABHN, Ch. I, Thm. 1.5.3], we obtain the desired formula (A.2). □

Finally, we mention an approximation result for functions in $Lip_\omega([0,\infty), X)$ (see [HN93, Thm. 2.7]).

Theorem A.5. *Let $M, \omega \geq 0$ and $r(\cdot) \in C_W^{(\infty)}((\omega,\infty), X)$ satisfying Widder's growth condition $\sup_{\lambda > \omega} |\frac{1}{k!}(\lambda - \omega)^{k+1} r^{(k)}(\lambda)| \leq C$ for some $C > 0$ and all $k \in \mathbb{N}$.*

Then

$$\begin{aligned} F(t) :&= \lim_{k\to\infty} (-1)^k \tfrac{1}{k!} \left(\tfrac{k}{t}\right)^{k+1} \left(\tfrac{r(\lambda)}{\lambda}\right)^{(k)}_{\lambda=\frac{k}{t}} \\ &= \lim_{k\to\infty} \sum_{i=0}^{k} (-1)^i \tfrac{1}{i!} \left(\tfrac{k}{t}\right)^i r^{(i)}\left(\tfrac{k}{t}\right) \\ &= \lim_{k\to\infty} (-1)^k \tfrac{1}{k!} \int_{\frac{k}{t}}^{\infty} s^k r^{(k+1)}(s)ds \end{aligned}$$

exists for all $t > 0$, $\lim_{t\searrow 0} F(t) = 0$, $\|F(t+h) - F(t)\| \le Mhe^{\omega(t+h)}$ *for all* $t, h \ge 0$ *and*

$$r(\lambda) = \lambda \int_0^\infty e^{-\lambda t} F(t)dt$$

for all $\lambda > \omega$.

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1. A. Albanese, F. Kühnemund, *Trotter–Kato approximation theorems for locally equicontinuous semigroups*, preprint.
2. F. Kühnemund, R. Nagel, F. Räbiger, U. Schlotterbeck, R. Schnaubelt, *Die Mischung macht's! – Ein internationales Internetseminar*, DMV–Mitteilungen **3** (2000), 47–49.
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Zusammenfassung in deutscher Sprache

Das Ziel der vorliegenden Arbeit ist es, Klassen von Halbgruppen auf Banachräumen, die nur für eine schwächere Topologie als die Normtopologie stark stetig sind, in einer umfassenden Theorie zu behandeln. Hierzu führen wir das Konzept der *bi–stetigen Halbgruppen* ein.
Das erste Kapitel ist der Charakterisierung bi–stetiger Halbgruppen durch ein Hille–Yosida–Theorem gewidmet.
Im zweiten Kapitel studieren wir die Konvergenz von Folgen bi–stetiger Halbgruppen und entwickeln verallgemeinerte Trotter–Kato–Theoreme. Basierend auf diesen Resultaten erhalten wir anschließend eine explizite Darstellung bi–stetiger Halbgruppen in Form einer Chernoff–Produktformel. Diese Resultate werden dann benutzt, um die Post–Widder–Umkehrformel für bi–stetige Halbgruppen zu beweisen. Schließlich zeigen wir, dass unter „Stabilitäts"–und „Konsistenzbedingungen" an zwei bi–stetige Halbgruppen der Abschluss der Summe ihrer Generatoren wieder ein Generator ist, und die gestörte Halbgruppe durch die Lie–Trotter–Produktformel dargestellt werden kann.
Um die Stärke unseres Zugangs zu demonstrieren, diskutieren wir im dritten Kapitel Beispiele und Anwendungen der in den ersten beiden Kapiteln entwickelten Resultate. Zuerst geben wir einen Überblick über lokalgleichstetige Halbgruppen (siehe z.B. [Kom64], [Kōm68], [Ōuc73], K. Yosida [Yos74]), die in konkreten Situationen als bi–stetige Halbgruppen angesehen werden können. Basierend auf Arbeiten von J. R. Dorroh und J. W. Neuberger [DN93], [DN96] verifizieren wir in Abschnitt 3.2, dass Halbgruppen auf $C_b(\Omega)$, die von Flüssen induziert werden, bi–stetig sind. Insbesondere geben wir einen vereinfachten Beweis des Erzeugungstheorems und Bedingungen an, welche die Lie–Trotter–Produktformel für diese Halbgruppen implizieren.

Danach konzentrieren wir uns in Abschnitt 3.3 auf die Ornstein–Uhlenbeck–Halbgruppe, die intensiv z.B. in [DPZ92],[CDP93], [Cer94], [CG95], [DPL95], [Pri99]und [TZ] studiert wurde. Wir zeigen, dass diese Halbgruppe auf $C_b(H)$, H Hilbert, bi–stetig ist, wobei H ein Hilbertraum ist. Basierend auf einer gemeinsamen Arbeit mit A. Albanese [AK00] gelingt es, diese Halbgruppen auf $C_b(\mathbb{R}^n)$, versehen mit einer lokalkonvexen Topologie, die feiner ist als die kompakt–offene Topologie, durch die Lie–Trotter–Produktformel zu repräsentieren.
In Abschnitt 3.4 betrachten wir *implementierte Halbgruppen* auf $\mathcal{L}(X,Y)$, wobei X, Y Banachräume sind, wie sie z.B. in [BR79], [GN81], [Pho91], [ARS94], [PS98], [Alb99] und [Alb] auftreten. Wir zeigen, dass diese für die starke Operatortopologie in die Theorie der bi–stetigen Halbgruppen passen.
In Abschnitt 3.5 betrachten wir *adjungierte Halbgruppen* (siehe [BR79], [Nee92]) auf dem topologischen Dual X' eines Banachraumes X unter der Annahme, dass die ursprüngliche Halbgruppe auf X stark stetig ist. Unter diesen Annahmen ist jede solche adjungierte Halbgruppe bi–stetig bezüglich der schwach*–Topologie. Weiter charakterisieren wir Mackey–stetige adjungierte Halbgruppen, die bi–stetig bezüglich der Mackey–Topologie auf X' sind.