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Scientific Breakthroughs, Innovation Clusters and Stochastic Growth Cycles

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Abstract

We develop a dynamic stochastic general-equilibrium model of science, education and innovation to explain the simultaneous emergence of innovation clusters and stochastic growth cycles. Firms devote human-capital resources to research activities in order to invent higher quality products. The technological requirements in climbing up the quality ladders increase over time but this hampering effect is compensated for by an improving qualification of researchers allowing for a sustainable process of innovation and scale-invariant growth. Jumps in human capital, triggered by scientific breakthroughs, induce innovation clusters across industries and generate long-run growth cycles.

Keywords: Science, education, innovation clusters, stochastic growth cycles

JEL Classification: C61, E32, O33

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1 Introduction

For a long time, growth and business cycles have been investigated separately in macroeconomics. While growth theory has focused on characterizing the long-run growth path, business cycle theory has considered the growth trend as exogenous and studied the detrended cyclical development. Nowadays it is well-known that trend and cycles may be influenced by the same explanatory factors and therefore are closely related to each other. The modern R&D-based growth theory builds decisively on stochastic innovation processes and hence obviously suggests a unified treatment of innovation, growth and cycles as it was pointed out by Schumpeter (1939) nearly 75 years ago.

In their seminal work on R&D-based growth theory Grossman, Helpman (1991a,b), Aghion, Howitt (1992, 1998), Stokey (1995) and Segerstrom (1998) have pointed out the role of research and development and stochastic innovation in generating growth. While the first two generations of the R&D-based growth models, namely endogenous growth models characterized by a scale-effect of the labor force, and the semi-endogenous growth models, introduced to eliminate this scale effect, have neglected human-capital accumulation, the endogenous scale-invariant growth models of the third generation have introduced education and skill acquisition as suggested by Lucas (1988) in the framework of Schumpeterian growth theory (e.g., Arnold 2002, Stadler 2003, 2012, Strulik 2005). This contemporary framework proves particularly appropriate to additionally account for scientific discoveries affecting the human capital of workers and researchers in an analytically tractable way.[1] Furthermore, such an augmented model explains the emergence of innovation clusters (e.g., Iyigun 2006) and long-run stochastic growth cycles (e.g., Jovanovic, Rob 1990, Corriveau 1994, Stein 1997, Matsuyama 1999, and Francois 1999) in a unified treatment based on the intertemporal relations between science, education and innovation.

In the Schumpeterian literature, growth cycles are usually triggered by the so-called general purpose technologies influencing most (or even all) industries of an economy (see, e.g., Bresnahan, Trajtenberg 1995, Bresnahan 2010). With a few exceptions (see, e.g., Li 2001) these growth models do not distinguish between science and technology. However, even if there are no clear borders, it is conventional wisdom

[1] Carlaw, Lipsey (2006, 2011), for example, present a more complex model which is, however, no longer analytically tractable and hence can be solved only with the help of numerical techniques.

that fundamentally new technologies are usually based on some path breaking scientific discoveries: The construction of the steam engine and the locomotive was not enabled until at least the simplest thermodynamic laws of gases were known. The internal combustion motor relies decisively on accumulated knowledge in chemistry about the oxidation of fuel. The expansion of network electricity, enabled by dynamos, transformers, and electric motors, was based on applications of the electrodynamic laws of direct and alternating current. The information and communication technologies of the ongoing electronic revolution build on the knowledge of electromagnetic waves. Computers are based on fundamental principles of digital operations developed in the abstract field of mathematical logic. The miniaturizing transistor technology leading to personal computers, notebooks and smartphones, and the laser technology leading to the modern audio and video systems were not enabled until the fundamentals of quantum mechanics have been learned.

This paper takes into account that scientific discoveries, human-capital accumulation and technological innovations are closely linked to each other in driving economic growth cycles. Human capital rises continuously as a result of skill acquisition in the educational sector but discontinuously as a by-product of discoveries in the (natural) sciences such as mathematics and informatics, physics and chemistry as well as biology and medicine. When scientific breakthroughs occur, the abilities of researchers in many (if not all) industries improve and generate temporarily higher innovation rates leading to clusters of innovation across industries.[2] Following each upturn, the innovation rates decline again until the next important scientific discovery occurs. This process repeats itself over an infinite time horizon thereby generating long-run cycles in economic growth. By capturing these intertemporal relations, the presented model goes far beyond the predecessor models considering only one single cycle in isolation (see, e.g., Helpman, Trajtenberg 1998, Petsas 2003).

The remainder of the paper is organized as follows. In Section 2 the dynamic stochastic general-equilibrium model of science, education and innovation is presented. In Section 3 we derive the balanced path of endogenous scale-invariant growth. Section 4 studies the stochastic appearance of innovation clusters and growth cycles. Section 5 concludes.

[2]For supportive evidence see, e.g., Kleinknecht (1987), for a methodological discussion see, e.g., Silverberg, Verspagen (2003).

2 The Model

According to the latest generation of scale-invariant R&D-based growth models we consider an economy consisting of a continuum of firms, each producing a differentiated consumer good. The goods are sold to households who have preferences for quantity and quality. Improvements of product qualities appear stochastically with intensities increasing in the innovative efforts of firms. Households are endowed with human capital which accumulates continuously by education and discontinuously as a by-product of scientific discoveries. Thus, there are three engines of economic growth which are closely linked to each other: science, education and technological innovation.

2.1 Consumer Spending and Education of Households

There is a fixed measure of representative dynastic households indexed on the interval $[0,1]$ which supply human-capital services. They share identical preferences and maximize discounted utility

$$U(C) = \int_0^\infty e^{-\rho t} \ln C \; dt \; , \tag{1}$$

where $\rho > 0$ is the constant discount rate and

$$C = \left[\int_0^1 q(j)^{1-\alpha} x(j)^\alpha \; dj \right]^{1/\alpha} , \quad 0 < \alpha < 1 \; , \tag{2}$$

is a quality-augmented Dixit-Stiglitz consumption index which measures instantaneous utility. It reflects the households' preferences for quantity $x(j)$ and quality $q(j)$ of the demanded products available in a continuum of industries j on the interval $[0,1]$.[3] According to these preferences, the elasticity of substitution between any two products across industries is given by $1/(1-\alpha)$.

For each household, the utility maximization problem can be solved in two steps. The first step is to solve the across-industry optimization problem at each point of

[3] This specification of the consumption index is widely used in the R&D-based growth literature (see, e.g., Thompson, Waldo 1994, Dinopoulos, Thompson 1998, Li 2003, and Segerstrom 2007).

time. Maximizing the consumption index (2) subject to the budget constraint

$$I = \int_0^1 p(j)x(j)\ dj\ ,$$

where I is consumer spending and $p(j)$ is the price of product j, yields the individual demand function

$$x(j) = \frac{q(j)p(j)^{-\frac{1}{1-\alpha}} I}{\int_0^1 q(j)p(j)^{-\frac{\alpha}{1-\alpha}}\ dj} \tag{3}$$

for product j. By aggregating expenditure $p(j)q(j)$ over all industries $j \in [0,1]$, the consumption index (2) can be rewritten as

$$C = I/p^C\ , \tag{4}$$

where $p^C \equiv \left[\int_0^1 q(j)p(j)^{-\frac{\alpha}{1-\alpha}}\ dj\right]^{-\frac{1-\alpha}{\alpha}}$ is the price index of the consumer goods.

The second step is to solve the dynamic optimization problem by maximizing discounted utility. Households devote human capital H to education, production and R&D. By investing the share $(1-\theta) \in (0,1)$ in education, they raise their human capital according to the stochastic differential equation

$$dH = \kappa(1-\theta)H\ dt + (\phi H)\ ds\ . \tag{5}$$

The deterministic part of this differential equation corresponds to the well-known Uzawa-Lucas function where κ $(> \rho)$ denotes the effectiveness of the educational system. The stochastic part is governed by a Poisson process s with arrival rate k indicating jumps in human capital from H to $(1+\phi)H$, triggered by scientific discoveries. As long as s does not jump, $ds = 0$, human capital accumulates continuously. When s jumps, $ds = 1$, human capital increases at rate ϕ.[4]

The dynamic budget constraint of each household is

$$dA = [rA + w\theta H - I]dt\ , \tag{6}$$

[4] In the derivation of our main results, we treat ϕ and k as exogenously given parameters. However, the analysis can be generalized by treating ϕ as a random variable. According to some empirical evidence on scientific progress the arrival rate k is assumed to be constant over time, but it can easily be endogenized, for example in terms of educational effectiveness κ.

where A denotes the value of asset holdings, r is the nominal interest rate and w is the nominal wage rate for each unit of human capital employed either in production or in R&D. It is convenient to choose human capital as numéraire, i.e. to normalize the wage rate to $w = 1$. Thus, each household maximizes its discounted utility (1), given (4), subject to the human-capital accumulation function (5) and the budget constraint (6). Dynamic optimization leads to the stochastic Keynes-Ramsey rule (see equation (A.12) in the Appendix)

$$dI/I = (\kappa - \rho)dt + \phi\, ds\ . \tag{7}$$

The larger the effectiveness of education κ and the lower the discount rate ρ, the higher is the continuous growth rate of consumer spending. In the moment of a scientific discovery, when human capital jumps at rate ϕ, consumer spending jumps at the same rate.

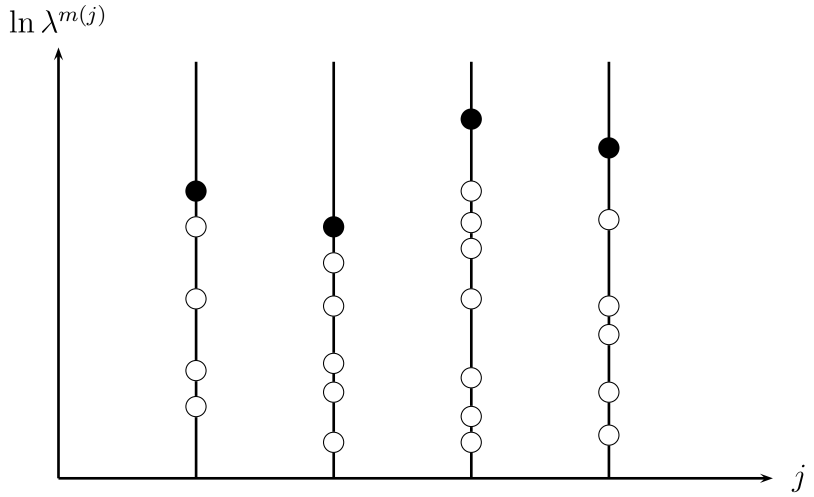
2.2 Price Competition of Incumbent Firms

In each industry, the products' quality grades are arrayed along the rungs of a quality ladder. Each new generation of products provides a quality level, λ times higher than the previous one, where the sizes of the technological jumps realized by innovative firms are independently drawn form a probability distribution on the support $\lambda \in [1; \infty)$. Following a suggestion recently made by Minniti, Parello and Segerstrom (2013), we assume that the random variable λ follows a Pareto distribution with c.d.f. $G(\lambda) = 1 - \lambda^{-1/\beta}, \beta \in (0, 1)$, and p.d.f.

$$g(\lambda) = (1/\beta)\lambda^{-(1+\beta)/\beta}\ , \tag{8}$$

where the scale parameter is normalized to 1. The distribution has mean $1/(1 - \beta)$ and (in case of $\beta < 1/2$) variance $\beta^2/[(1 - 2\beta)(1 - \beta)^2]$ such that the inverse shape parameter β can be interpreted as an indicator of technological dispersion and heterogeneity of incumbent firms. Treating λ as a random variable implies asymmetric industry-specific quality ladders and, hence, a stochastic evolution of the industrial structure of the economy (see Figure 1). In our view, these industry-specific and stochastic quality ladders capture the essential characteristics of innovation dynamics and structural change much better than the symmetric and deterministic ladders used in former R&D-based growth models.

Figure 1: Stochastic Evolution of Industries



All consumer goods are produced subject to a constant returns to scale technology with human capital as single input. In all industries, production of one output unit requires one unit of human capital, independently of the quality, i.e. $x(j) = H_x(j)$. Therefore, each firm has a constant marginal cost equal to the normalized wage rate $w = 1$, and the supplier of the highest quality of product j maximizes the flow of profits

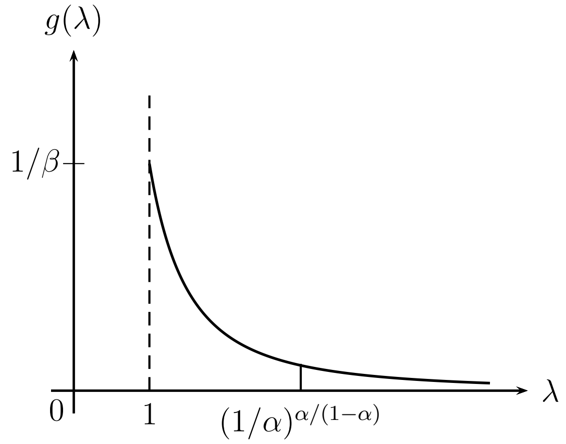
$$\pi(j) = (p(j) - 1)x(j)$$

with $x(j)$ specified in (3). The price-setting behavior of the leading firms depends on the underlying industry structure which is characterized by the quality differences of products. In case of drastic innovations, the price decisions of the quality leaders are constrained by competition from the producers of substitute goods supplied in the other industries. According to the demand functions (3), the price elasticity of demand is $-1/(1 - \alpha)$, implying monopolistic competition where equilibrium prices are $p(j) = 1/\alpha$. In case of non-drastic innovations the leading firms charge limit prices since, as can be shown, each industry leader is exactly one step ahead. With innovation sizes $\lambda(j)$ as the realizations of the Pareto distributed random variable λ, both strategies of price setting will be observed in different industries at the same time. According to the consumption index (2), industry-specific equilibrium prices are

$$p(j) = \begin{cases} \lambda(j)^{\frac{1-\alpha}{\alpha}} & \text{if} \quad \lambda(j) \;\leq\; (1/\alpha)^{\frac{\alpha}{1-\alpha}} \\ 1/\alpha & \text{if} \quad \lambda(j) \;>\; (1/\alpha)^{\frac{\alpha}{1-\alpha}} \;. \end{cases} \tag{9}$$

Figure 2 illustrates the different price-setting regimes depending on whether the realized value $\lambda(j)$ exceeds or falls short of the threshold level $(1/\alpha)^{\alpha/(1-\alpha)}$. Due to this industry heterogeneity, incumbent firms produce at different price-cost margins and hence earn different profits.

Figure 2: Price-Setting Regimes Depending on Innovation Size



2.3 Patent Races between Potential Entrants

The quality of consumer goods can be upgraded by a sequence of product innovations, each building on its predecessors. The opportunity of realizing profits drives potential entrants to engage in R&D in order to develop higher quality products. The first firm to invent the next higher quality product is granted an infinitely-lived patent. Competition therefore takes the form of an endless sequence of patent races between an arbitrary number of potential entrants.[5] Each firm may target its R&D efforts at any of the continuum of top-of-the-line products, i.e. it may engage in any

[5]It can be shown that incumbent firms have no incentive to engage in R&D activities aimed at an improvement of the own product. Of course, this property may no longer hold if the quality leaders have an advantage in doing R&D (see, e.g., Etro 2004 and Denicolo, Zanchettin 2010, 2012).

industry $j \in [0, 1]$. If a firm undertakes R&D at intensity $h(j)$ for an infinitely small time interval dt, it will succeed in taking the next step up the quality ladder for the targeted product j with probability $h(j)dt$. This implies that the number of realized innovations in each industry follows a Poisson process with arrival rate $h(j)$. The innovation rate $h(j)$ is assumed to depend proportionally on the amount of human capital $H_h(j)$ devoted to R&D activities in order to improve the product quality in industry j such that

$$h(j) = \frac{H_h(j)}{\mu q(j)} \ . \tag{10}$$

The inverse of the parameter μ relates the productivity of human capital in R&D relative to its (normalized) productivity in production and is assumed to be equal across industries. However, the quality variable $q(j)$ in the denominator reflects a hampering effect of the number and sizes of industry-specific innovations made in the past by indicating that the realization of further innovations becomes more and more difficult.

2.4 The Stock Market

The expected discounted profits of an entrant winning the next patent race are represented by the stock market values $V(j)$. To participate in a patent race, firms have to employ human-capital resources in their research labs. According to (10), a firm devoting $H_h(j)$ units of human capital to R&D at a cost of $H_h(j)$ for a small time interval dt attains the expected stock market value $V(j)$ with probability $(H_h(j))/(\mu q(j))dt$. R&D expenditure is financed by issuing equity claims to households which pay out nothing in the case that the research effort fails, but entitle the claimants to the flow of dividend payments $\pi(j)$ if the effort succeeds. If we ignore the uninteresting case where firms undertake no R&D at all, free entry into each patent race implies

$$V(j) = \mu q(j) \ . \tag{11}$$

As the quality $q(j)$ in industry j increases stochastically over time, the realization of innovations becomes progressively more difficult and the reward of innovations must increase correspondingly to induce innovative activities by challengers. Since there is a continuum of industries and the returns from participating in patent races are independently distributed across firms and industries, each investor minimizes risk by holding a diversified portfolio of stocks. Absence of arbitrage opportunities implies that the expected return on equities of innovators must equal the return on an equal size investment in a riskless bond, i.e.

$$rV(j) = E\pi(j) + \dot{V}(j) - h(j)V(j) \ , \tag{12}$$

where the right-hand side describes the expected rate of return on equities of innovators, consisting of the expected dividend rate, the expected capital gains and the risk of losing the dividends due to another entrant's quality innovation in the future. The expected value of the uncertain flow of profits realized by the entering firm winning the next patent race in industry j is

$$E\pi(j) = E[(p(j) - 1)\lambda x(j)]$$

By substituting $x(j)$ from (3), $p(j)$ from (9), and accounting for the p.d.f. (8) of the random variable λ one obtains

$$E\pi(j) = \frac{E\left[\lambda p(j)^{-\alpha/(1-\alpha)} - \lambda p(j)^{-1/(1-\alpha)}\right] \ q(j)I}{\int_0^1 q(j)p(j)^{-\alpha/(1-\alpha)} \ dj} \ = \ \xi I q(j)/Q \ , \tag{13}$$

where

$$\xi \equiv \frac{(1+\beta)\beta[(1-\beta) + \beta\alpha^{\alpha/[(1-\alpha)\beta]}]}{(1-\beta)(\alpha/(1-\alpha)+\beta)[1+\beta\alpha^{\alpha(1+\beta)/[(1-\alpha)\beta]}]}$$

is a constant parameter, depending on the exogenously given parameters α and β, and where $Q = \int_0^1 q(j) \ dj$ is the average quality of the top-of-the-line consumer goods. By substituting (11) and (13) in (12) the no-arbitrage equation can be rewritten as

$$r = \xi I/(\mu Q) - h(j) \ ,$$

implying that the innovation rate is the same across all industries and, hence, in the aggregate economy, i.e. $h(j) = h \ \ \forall \ \ j$. An important implication of equal innovation rates is that the employment levels of workers in R&D vary stochastically across industries. In industries that have experienced more innovations in the past more human-capital resources are devoted to R&D. In the aggregate, however, as will be shown in the Sections 3 and 4, the employment of researchers is constant over time. This is an appealing feature of our model since it implies that we do not have to care about intersectoral mobility of workers.

2.5 The Dynamics of Innovation and Quality Growth

Since the quality of product j jumps from $q(j)$ to $\lambda q(j)$ whenever an innovation occurs, and the innovation rates h are equal across industries, the time derivative of average quality can be derived, using the law of large numbers, as

$$\begin{aligned}\dot{Q} &= \int_0^1 (\lambda - 1) q(j) h \; dj \\ &= \int_0^1 \left(\int_1^\infty \lambda g(\lambda) d\lambda - 1 \right) q(j) h \; dj \\ &= [\beta/(1-\beta)] h Q \; ,\end{aligned}$$

such that its growth rate

$$\dot{Q}/Q = [\beta/(1-\beta)] h \tag{14}$$

depends proportionally on the innovation rate h.

2.6 The Market for Human Capital

The labor market is perfectly competitive. The share $(1-\theta)$ of human capital is devoted to education. The remaining amount of human capital is devoted either to production or to R&D. It follows from (3), (8), and (9) that the aggregate demand for human capital in the production sector is

$$\int_0^1 H_x(j) \; dj = \int_0^1 x(j) dj = \frac{\int_0^1 q(j) p(j)^{-1/(1-\alpha)} dj}{\int_0^1 q(j) p(j)^{-\alpha/(1-\alpha)} dj} \; I = \frac{\alpha(1+\beta)}{\alpha+\beta} \; I \; ,$$

and from (10) that the aggregate demand in the research sector amounts to

$$\int_0^1 H_h(j)\, dj = \int_0^1 \mu h q(j)\, dj = \mu h Q \ .$$

Thus, full employment of workers' human capital implies that

$$H = (1-\theta)H + \frac{\alpha(1+\beta)}{\alpha+\beta} I + \mu h Q \ . \tag{15}$$

This labor market clearing condition will be used in Section 3 to analyze the balanced growth equilibrium for a given level of scientific knowledge and in Section 4 to study growth cycles in response to scientific breakthroughs.

3 The Balanced Growth Equilibrium

First, we solve the model for a balanced growth path as an equilibrium path, i.e. an equilibrium path in which all aggregate variables grow at constant rates over time. In such a setting (5) and (7) together imply a constant steady-state growth rate of human capital

$$\dot{H}/H = \kappa(1-\theta) = \kappa - \rho \ , \tag{16}$$

so that the share of human capital devoted to education is

$$1 - \theta = 1 - \rho/\kappa \ .$$

Obviously, this share is constant over time and depends negatively on the discount rate and positively on the effectiveness of the educational system.[6] From (14), (15), and (16) we derive

$$\dot{Q}/Q = [\beta/(1-\beta)]h = \dot{H}/H = \kappa - \rho \ , \tag{17}$$

such that the steady-state innovation rate is endogenized by

$$h^* = (\kappa - \rho)(1-\beta)/\beta \ . \tag{18}$$

[6] It can be shown that public expenditure, financed by a lump-sum tax, is appropriate to promote education, either by subsidizing education or by improving the effectiveness of education (see, e.g., Stadler 2012).

This innovation rate depends neither on the exogenous labor force as in first-generation endogenous growth models, nor on the exogenously given growth rate of the labor force as in the semi-endogenous growth models of the second generation, but instead is endogenously determined in terms of educational and technological parameters. More precisely, it depends positively on educational effectiveness κ but negatively on the discount rate ρ and technological dispersion β. The realization of innovations becomes more and more difficult as technology evolves, but researchers compensate for this deterioration by increasing their skills in the educational sector.

The consumption index

$$C = Q^{1/\alpha-1}\alpha I$$

grows, according to (7), (14), and (18) at the constant rate

$$\dot{C}/C = (1/\alpha - 1)[\beta/(1-\beta)]h^* + (\kappa - \rho) = (1/\alpha)(\kappa - \rho) ,$$

where the effectiveness of education is the driving force. This result classifies our model as an endogenous scale-invariant growth model of the third generation. While the technological development in any particular industry evolves stochastically, the economy at the aggregate level experiences smooth and non-random time paths of the macroeconomic variables.

4 Innovation Clusters and Growth Cycles

We now consider a certain point in time when a scientific breakthrough occurs, i.e. when H jumps to the level $(1+\phi)H$. Since consumer spending I jumps at the same rate as human capital does (see the Appendix) and since average quality Q cannot increase discontinuously, it is clear that the innovation rates h, being equal across industries, jump at the same rate as well, so that the share of workers employed in the R&D sector remains constant. A cluster of innovations across industries occurs since innovation dynamics accelerate economy-wide. Higher innovation rates go along with faster improvements of product qualities. As argued e.g. by David (1990), it takes a while before entrants are able to introduce new products which incorporate improvements based on a scientific discovery. It further follows from the human-capital market clearing condition (15) that, until the next scientific breakthrough occurs, the sum of the growth rates of h and Q must equal the growth rate of H, implying that

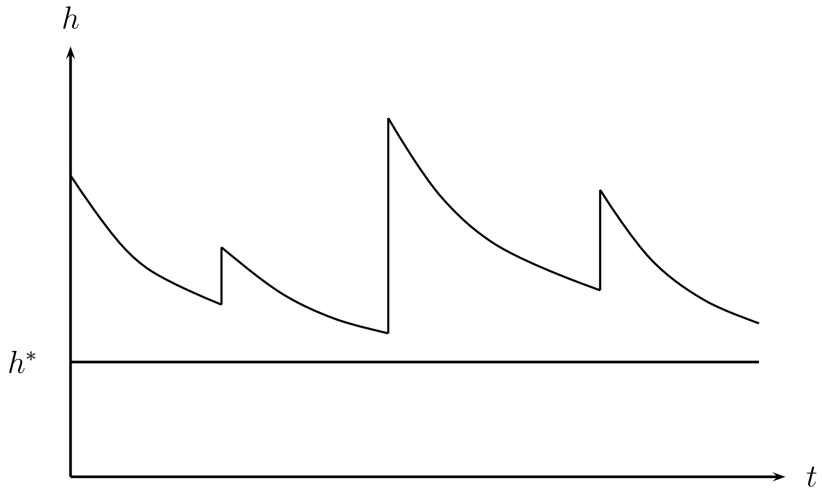
$$\dot{h}/h + [\beta/(1-\beta)]h = \kappa - \rho \ .$$

Rearranging this equation by using the expression (18) for the steady-state innovation rate h^* leads to the nonlinear first-order differential equation

$$\dot{h}/h = (\kappa - \rho) - [\beta/(1-\beta)]h = -\ [\beta/(1-\beta)](h - h^*) \quad < \quad 0 \ ,$$

describing a declining time path of the innovation rates, which monotonically converges to the steady-state value h^*. But as soon as the next scientific breakthrough succeeds, the innovation rates jump again and initialize a new growth cycle. Figure 3 illustrates such a process of stochastic innovation and growth cycles with random jump sizes.

Figure 3: Stochastic Innovation and Growth Cycles



5 Summary and Conclusion

The paper has extended the Schumpeterian growth literature by analyzing the intertemporal relations between science, education and innovation to explain the stochastic emergence of innovation clusters and growth cycles. Compared to the endogenous growth models of the first generation, the scale effect of the labor force is removed by the assumption that in each industry technological requirements increase from innovation to innovation. Compared to the semi-endogenous growth models of the second generation, exogenous growth of the labor force is replaced by an endogenous improvement of worker qualification by education. In our endogenous scale-invariant growth model of the third generation, human capital and qualification improve not only continuously as a result of skill acquisition in the educational sector, but also discontinuously as a by-product of scientific discoveries. The jumps in human capital, triggered by scientific breakthroughs, induce clusters of innovation across industries and generate long-run cycles of R&D-based growth.

The Poisson process of scientific breakthroughs leads to growth cycles similar to those resulting in the real-business-cycles theory (see, e.g., Stadler 1990). Due to the inertia of the innovation process, the adjustment of the growth rates is comparatively slow indicating long swings sometimes referred to as Kondratieff cycles. However, in contrast to earlier interpretations of such long waves, the cycles triggered in our model are intrinsically stochastic and therefore hard to predict.

Appendix: Households' Dynamic Stochastic Optimization

Consider the intertemporal utility maximization problem

$$\max\ E_0 \int_0^\infty e^{-\rho t}\, ln\ (I/p_C)\ dt$$

subject to the dynamic budget constraint

$$dA = [rA + \theta H - I]\ dt$$

and to the human-capital accumulation

$$dH = \kappa(1-\theta)H\ dt + (\phi H)\ ds\ ,$$

where ds is the increment of a Poisson process $s(t)$ with arrival rate k. This implies that $s(0) = 0$ and that the random variables $s(t + dt) - s(t)\ \forall\ t \in [0, \infty)$ are independent and Poisson distributed with mean $k\ dt$. The boundary conditions are given by

$$A(0) = A_0\ , \quad H(0) = H_0\ .$$

According to the general procedure as shown, e.g., in Malliaris, Brock (1982, chap. 2.12), the Hamilton-Jacobi-Bellman (HJB) equation of this autonomous optimization problem reads

$$\rho J(A,H) = \max_{I,\theta} \{ ln\ (I/p^C) + J_A(A,H)[rA + \theta H - I] + J_H(A,H)\kappa(1-\theta)H \\ + k[J(\tilde{A},\tilde{H}) - J(A,H)] \} \qquad \text{(A.1)}$$

where $J(A,H)$ is the current-value function and $J_A(A,H)$ and $J_H(A,H)$ are the current-value costate variables of A and H. As long as s does not jump ($ds = 0$), the state variables A and H evolve continuously. When s jumps ($ds = 1$ and $dt = 0$), however, human capital jumps from H to $\tilde{H} = (1+\phi)H$ and assets jump from A to $\tilde{A}$ where the latter jump rate is yet unknown.

The maximization in (A.1) with respect to θ leads to the first-order condition

$$J_A(A,H) = \kappa J_H(A,H) \;, \tag{A.2}$$

maximization with respect to consumer spending I to the first-order condition

$$I = 1/J_A(A,H) \;. \tag{A.3}$$

To determine the evolution of the costate variables J_A and J_H, we have to partially differentiate the HJB equation with respect to the state variables, to derive the total differentials dJ_A and dJ_H by using Ito's Lemma and to finally insert the partial derivatives into these expressions.

Partially differentiating the HJB equation (A.1) with respect to A, using the envelope theorem, gives

$$\begin{aligned}\rho J_A(A,H) =& rJ_A(A,H) + J_{AA}(A,H)[rA + \theta H - I] + J_{HA}(A,H)\kappa(1-\theta)H \\ &+ k[J_A(\tilde{A},\tilde{H}) - J_A(A,H)] \;,\end{aligned} \tag{A.4}$$

partially differentiating it with respect to H gives

$$\begin{aligned}\rho J_H(A,H) =& \theta J_A(A,H) + J_{AH}(A,H)[rA + \theta H - I] + J_H(A,H)\kappa(1-\theta) \\ &+ J_{HH}(A,H)\kappa(1-\theta)H + k[(J_H(\tilde{A},\tilde{H}) - J_H(A,H)] \;.\end{aligned} \tag{A.5}$$

By applying Itô's Lemma we derive the costates' total differentials

$$\begin{aligned}dJ_A(A,H) =& J_{AA}(A,H)[rA + \theta H - I]dt \\ &+ J_{AH}(A,H)\kappa(1-\theta)Hdt + [J_{\tilde{A}}(\tilde{A},\tilde{H}) - J_A(A,H)]ds\end{aligned} \tag{A.6}$$

and

$$\begin{aligned}dJ_H(A,H) =& J_{HA}(A,H)[rA + \theta H - I]dt \\ &+ J_{HH}(A,H)\kappa(1-\theta)H + [J_{\tilde{H}}(\tilde{A},\tilde{H}) - J_H(A,H)]ds \;.\end{aligned} \tag{A.7}$$

Substituting $J_{AA}(A,H)$ and $J_{AH}(A,H) = J_{HA}(A,H)$ from (A.4) in (A.6) as well as $J_{AH}(A,H)$ and $J_{HH}(A,H)$ from (A.5) in (A.7) gives

$$dJ_A(A,H) =(\rho - r)J_A(A,H)$$

$$- k[J_A(\tilde{A},\tilde{H}) - J_A(A,H)]dt + [J_{\tilde{A}}(\tilde{A},\tilde{H}) - J_A(A,H)]ds \qquad (A.8)$$

and

$$dJ_H(A,H) = (\rho - \kappa(1-\theta))J_H(A,H)dt - \theta J_A(A,H)dt$$
$$- k[J_H(\tilde{A},\tilde{H}) - J_H(A,H)]dt + [J_{\tilde{H}}(\tilde{A},\tilde{H}) - J_H(A,H)]ds \ ,$$

where the latter equation can be rewritten, by using (A.2), as

$$dJ_A(A,H) = (\rho - \kappa)J_A(A,H)dt - k[J_A(\tilde{A},\tilde{H}) - J_A(A,H)]dt$$
$$+ [J_{\tilde{A}}(\tilde{A},\tilde{H}) - J_A(A,H)]ds \ . \qquad (A.9)$$

Obviously, the state variables A and H jump at the same rate implying that $\tilde{A} = (1+\phi)A$. Hence, a comparison of (A.8) and (A.9) shows that the interest rate is determined by

$$r = \kappa \qquad (A.10)$$

and thus remains constant over time. To derive the evolution of consumer spending I, we apply Ito's Lemma to (A.3) to obtain

$$dI = -(1/J_A(A,H)^2)\left\{(\rho-\kappa)J_A(A,H) - k[J_A(\tilde{A},\tilde{H}) - J_A(A,H)]\right\}dt$$
$$+ [1/J_{\tilde{A}}(\tilde{A},\tilde{H}) - 1/J_A(A,H)]ds \ .$$

Substitute J_A from (A.3) and divide through by I to find the stochastic Keynes-Ramsey rule

$$dI/I = \left\{(\kappa-\rho)\ dt + k[(1+\phi)I/\tilde{I} - 1]\right\}dt + (\tilde{I}/I - 1)\ ds \ . \qquad (A.11)$$

It remains to determine the rate of jump from I to $\tilde{I}$. An obvious solution is $J(A,H) = \varepsilon + (1/\rho)ln(A + H/\kappa)$ with some constant ε implying $J_A(A,H) = (1+\phi)J_{\tilde{A}}(\tilde{A},\tilde{H}) = 1/[\rho(A+H/\kappa)]$. Thus the costate variable jumps from $J_A(A,H)$ to $J_{\tilde{A}}(\tilde{A},\tilde{H}) = J_A(A,H)/(1+\phi)$ and consumer spending from $I = 1/J_A(A,H)$ to $\tilde{I} = (1+\phi)I$, respectively, such that $(1+\phi)I/\tilde{I} = 1$. This simplifies the stochastic Keynes-Ramsey rule (A.11) to

$$dI/I = (\kappa - \rho)\ dt + \phi\ ds \ . \qquad (A.12)$$

As long as s does not jump ($ds = 0$), consumer spending grows at the constant rate $\kappa - \rho$. When s jumps ($ds = 1$ and $dt = 0$), however, consumer spending jumps from I to $\tilde{I} = (1+\phi)I$.

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