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On Modified Mellin Transforms, Gauss-Laguerre Quadrature, and the Valuation of American Call Options

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Abstract

We extend a framework based on Mellin transforms and show how to modify the approach to value American call options on dividend paying stocks. We present a new integral equation to determine the price of an American call option and its free boundary using modified Mellin transforms. We also show how to derive the pricing formula for perpetual American call options using the new framework. A result due to Kim (1990) regarding the optimal exercise price at expiry is also recovered. Finally, we apply Gauss-Laguerre quadrature for the purpose of an efficient and accurate numerical valuation.

Keywords: Modified Mellin transform, American call option, Integral representation.

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1 Introduction

Analytical pricing of European-style derivatives has been made possible by the seminal results of Black and Scholes (1973) and Merton (1973). However, many of today's most common derivatives are American-style and are therefore subject to early exercise. The main difficulty in valuing these derivatives analytically is the presence of the early exercise boundary that specifies the conditions under which the contract should be exercised optimally prior to maturity. The optimal exercise policy is not known ex ante and must be determined simultaneously as part of the underlying valuation problem. This fact makes the pricing and hedging of American-style derivatives interesting and challenging.

The large literature on numerical methods for American option pricing comprises finite difference and element methods, penalty methods, binomial trees and simulation techniques. Brennan and Schwartz (1978) initially proposed a finite difference scheme for the purpose of pricing American options. The approach was refined and extended in various ways and is still in the focus of current interest (Zhao et al. (2007), Tangman et al. (2008), Khaliq et al. (2008) and Hu et al. (2009) among others). Cox et al. (1979) used a binomial tree lattice for an accurate valuation which still enjoys great popularity. Extensions of the initial work can be found in Leisen and Reimer (1996), Leisen (1998) or Chang and Palmer (2007). Moreover, Monte Carlo methods were modified to solve the forward-simulation-backward-induction valuation problem and to provide accurate American option prices (Boyle et al. (1997), Broadie and Glasserman (1997) and Longstaff and Schwartz (2001)).

Besides numerical methods one can distinguish two main categories of analytical pricing approaches. These approaches can be used to derive different but mathematically equivalent formulations of the American option pricing problem. The first method, similar to the solution of the Stefan's problem from physics, expresses the price of the American option as the solution of a partial differential equation (PDE). The PDE formulation goes back to Merton (1973) who first gives an economic interpretation although McKean (1965) presents a first solution of the free boundary problem in form of an integral expression. Many alternative methods based on the PDE approach were proposed for the purpose of pricing the American option and the free boundary by approximation. These methods include the works of Barone-Adesi and Whaley (1987), Geske and Johnson (1984), Bunch and Johnson (1992), Allegretto et al. (1995) or Ju and Zhong (1999).

The second set of methods comes from probability theory. It focuses on expressing the current price of an American option as a discounted expectation of the specific option's pay-off under the risk-neutral measure. This optimal stopping characterization is perhaps the most intuitive description of the problem. A complete formulation goes back to Bensoussan (1984) and Karatzas (1988). See also Myneni (1992) for further references.

At the beginning of the 1990s a breakthrough was achieved by characterizing the price of an American option as the sum of the corresponding European option plus an early exercise premium. These integral representations due to Kim (1990), Jacka (1991), and Carr et al. (1992) are exact solutions and were the starting point of new approximations for the American option price and/or the free boundary. Huang et al. (1996) use Richardson extrapolation to solve the integral expression. Ju (1998) approximates the early exercise boundary by a piece-wise exponential function and Bunch and Johnson (2000) derive expressions for the early exercise boundary using a new characterization of the option's price in terms of its time derivative.

Other popular methods are those of Broadie and Detemple (1996), Carr (1998) and Ingersoll (1998). Broadie and Detemple (1996) provide a pricing method based on a lower and upper bound. Carr (1998) determines accurate prices using a randomization approach whereas Ingersoll (1998) approximates American options using barrier derivatives. Broadie and Detemple (2004) and Detemple (2006) give excellent overviews of existing tools and

methods.
The purpose of this article is to extend a framework originally suggested by Panini and Srivastav (2004) and develop a new method for characterizing American call option prices and exercise boundaries using a modified version of the Mellin transform.
The rest of the paper is organized as follows. In Section 2 we develop a pricing formula for European call options to demonstrate the new framework. In a second step this formula will be used to decompose the American call into the early exercise premium and its European counterpart. This is done in Section 3. Here we present a new integral representation of the American call option and its free boundary. Section 4 is devoted to further analysis and applications. We show how to use the new framework to derive the valuation formula for perpetual American call options on dividend-paying stocks. Theoretical results due to Kim (1990) regarding the optimal exercise price at expiry are also recovered. In Section 5 we make some numerical experiments. More precisely, we apply Gauss-Laguerre quadrature for the purpose of valuation of American call options and compare our results to other existing approaches. Section 6 concludes.

2 The European Call Option

In a first step we develop a valuation formula for European call options which will be used in the next section to decompose the American call price.
In our economy the dynamics of the asset price $S_t, t \in [0, T]$, are given by the stochastic differential equation (SDE):

$$dS_t = (r - q)\, S_t\, dt + \sigma\, S_t\, dW_t \,, \tag{2.1}$$

with initial value $S_0 \in (0, \infty)$, and where r is the riskless interest rate, q is the dividend yield, $\sigma > 0$ is the volatility, and W_t is a one-dimensional Brownian motion.

A European call option is an option that can be only exercised at maturity and has a linear payoff given by the difference between the terminal asset price and the strike price of the option

$$C^E(S,T) = \max(S(T) - X, 0)\,. \tag{2.2}$$

Standard arbitrage arguments show that any derivative $V = V(S,t)$ written on S must satisfy the partial differential equation (PDE) (see for example Wilmott et al. (1993)):

$$\frac{\partial V}{\partial t} + (r-q)S\,\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2} - rV = 0\,. \tag{2.3}$$

This is the fundamental PDE due to Black/Scholes and Merton whose solutions depend on boundary and terminal conditions. If V is a European call option, i.e. $V(S,t) = C^E(S,t)$, we have the conditions

$$\lim_{S\to\infty} C^E(S,t) = \infty \qquad \text{on} \quad [0,T)\,, \tag{2.4}$$

$$C^E(S,T) = \theta(S) = \max(S(T) - X, 0) \qquad \text{on} \quad [0,\infty)\,, \tag{2.5}$$

and

$$C^E(0,t) = 0 \qquad \text{on} \quad [0,T)\,. \tag{2.6}$$

The celebrated solution is known as the (extended) Black-Scholes-Merton valuation formula and is given by

$$C^E(S,t) = Se^{-q(T-t)}N(d_1(S,X,T-t)) - Xe^{-r(T-t)}N(d_2(S,X,T-t)) \tag{2.7}$$

where

$$d_1(S,X,T-t) = \frac{\ln\frac{S}{X} + (r-q+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\,, \tag{2.8}$$

$$d_2(S,X,T-t) = d_1(S,X,T-t) - \sigma\sqrt{T-t}\,, \tag{2.9}$$

and $N(x)$ denotes the cumulative standard normal distribution function at x.

The objective of this section is to derive a valuation formula for European call options using Mellin transform techniques. Recall that for a locally Lebesgue integrable function $f(x)$ defined over positive reals the Mellin transform $M(f(x), \omega)$ is defined by the equation

$$M(f(x), \omega) := \widetilde{f}(\omega) = \int_0^\infty f(x)\, x^{\omega-1}\, dx.$$

The Mellin transform is a complex valued function defined on a vertical strip in the ω-plane, whose boundaries are determined by the asymptotic behavior of $f(x)$ as $x \to 0^+$ and $x \to \infty$. The largest strip (a, b) in which the integral converges is called the fundamental strip. The conditions $f(x) = O(x^u)$ for $x \to 0^+$ and $f(x) = O(x^v)$ for $x \to \infty$ when $u > v$, guarantee the existence of $M(f(x), \omega)$ in the strip $(-u, -v)$. Thus, the existence is granted for locally integrable functions, whose exponent of the order at 0 is strictly larger than the exponent of the order at infinity. Conversely, if $f(x)$ is an integrable function with fundamental strip (a, b), then if c is such that $a < c < b$ and $f(c + it)$ is integrable, the equality

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{f}(\omega)\, x^{-\omega}\, d\omega$$

holds almost everywhere. Moreover, if $f(x)$ is continuous, then the equality holds everywhere on $(0, \infty)$. For a proof see for example Titchmarsh (1986) or Sneddon (1972). The functions $f(x)$ and $\widetilde{f}(\omega)$ are called a Mellin transform pair. Since $C^E(S, t) = O(1)$ for $S \to 0^+$ and $C^E(S, t) = O(S)$ for $S \to \infty$ we propose the modified Mellin transform for call options defined by

$$M(C^E(S, t), -\omega) = \widetilde{C}^E(\omega, t) := \int_0^\infty C^E(S, t)\, S^{-(\omega+1)}\, dS\,, \tag{2.10}$$

where $1 < Re(\omega) < \infty$. Conversely, the inverse of the modified Mellin transform is given by

$$C^E(S, t) = M^{-1}\big(\widetilde{C}^E(\omega, t)\big) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{C}^E(\omega, t)\, S^{\omega}\, d\omega\,, \tag{2.11}$$

with $1 < c < \infty$.
Applying the modified Mellin transform to PDE (2.3) gives

$$\frac{\partial \widetilde{C}^E(\omega,t)}{\partial t} + \frac{1}{2}\sigma^2 Q(\omega)\, \widetilde{C}^E(\omega,t) = 0 \tag{2.12}$$

where

$$Q(\omega) = \omega^2 - \omega(1-\kappa_2) - \kappa_1\,, \tag{2.13}$$

and $\kappa_1 = \frac{2r}{\sigma^2}$ and $\kappa_2 = \frac{2(r-q)}{\sigma^2}$. The general solution of this ODE is given by

$$\widetilde{C}^E(\omega,t) = c(\omega)\cdot e^{-\frac{1}{2}\sigma^2 Q(\omega)t} \tag{2.14}$$

where $c(\omega)$ a constant depending on the boundary conditions. The terminal condition gives

$$c(\omega) = \widetilde{\theta}(\omega,t)\cdot e^{\frac{1}{2}\sigma^2 Q(\omega)T} \tag{2.15}$$

where

$$\widetilde{\theta}(\omega,t) = \widetilde{\theta}(\omega) = X^{-\omega+1}\Big(\frac{1}{\omega-1} - \frac{1}{\omega}\Big) \tag{2.16}$$

is the modified Mellin transform of the terminal condition (2.5). Finally, using (2.11), we see that the price of a European call option equals

$$\begin{aligned} C^E(S,t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{C}^E(\omega,t)\, S^\omega\, d\omega \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{\theta}(\omega,t)\cdot e^{\frac{1}{2}\sigma^2 Q(\omega)(T-t)}\, S^\omega\, d\omega \end{aligned} \tag{2.17}$$

with $(S,t) \in (0,\infty)\times[0,T)$, $c \in (1,\infty)$ a constant, $\{\omega \in \mathbb{C}\,|\, 1 < Re(\omega) < \infty\}$, and $\widetilde{\theta}(\omega,t)$ and $Q(\omega)$ as defined in equations (2.16) and (2.13), respectively. The next proposition summarizes the results and gives the connection to the BSM-formula.

Proposition 2.1 *Equations (2.17) and (2.7) are equivalent.*

PROOF: First, using (2.16), observe that

$$\begin{aligned} C^E(S,t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} S\Big(\frac{S}{X}\Big)^{\omega-1}\frac{1}{\omega-1}e^{\frac{1}{2}\sigma^2 Q(\omega)(T-t)}\,d\omega \\ &\quad -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} X\Big(\frac{S}{X}\Big)^{\omega}\frac{1}{\omega}e^{\frac{1}{2}\sigma^2 Q(\omega)(T-t)}\,d\omega\,. \end{aligned}$$

Now write $\omega = c+iy, 1 < c < \infty$ and $\zeta = \frac{1}{2}\sigma^2(T-t)$ to get

$$C^E(S,t) = I_1(S,X,T-t) - I_2(S,X,T-t)\,,$$

with

$$I_1(S,X,T-t)) = Se^{-r(T-t)+\zeta c^2+c(\alpha-2c\zeta)-\ln(S/X)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{c-1-iy}{(c-1)^2+y^2}e^{-\zeta y^2+iy\alpha}\,dy\,,$$

where we have set

$$\alpha = \ln\Big(\frac{S}{X}\Big) + \zeta(2c+\kappa_2-1).$$

Similarly,

$$I_2(S,X,T-t) = Xe^{-r(T-t)+\zeta c^2+c(\alpha-2c\zeta)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{c-iy}{c^2+y^2}e^{-\zeta y^2+iy\alpha}\,dy\,.$$

Using Euler's theorem for the complex valued exponential function $e^{ix} = \cos(x) + i\sin(x)$ we can simplify further and get

$$I_1(S,X,T-t) = Xe^{-r(T-t)+\zeta c^2+c(\alpha-2c\zeta)}\frac{1}{\pi}\int_0^{\infty} e^{-\zeta y^2}\frac{(c-1)\cos(\alpha y)+y\sin(\alpha y)}{(c-1)^2+y^2}\,dy\,,$$

and

$$I_2(S,X,T-t) = rXe^{-r(T-t)+\zeta c^2+c(\alpha-2c\zeta)}\frac{1}{\pi}\int_0^{\infty} e^{-\zeta y^2}\frac{c\cos(\alpha y)+y\sin(\alpha y)}{c^2+y^2}\,dy\,,$$

where we have used that $\cos(x)$ and $\sin(x)$ are even and odd functions, respectively. From Gradshteyn and Ryzhik (2007), p. 504 we have: For $a > 0$, $Re(\beta) > 0$, and $Re(\gamma) > 0$:

$$\begin{aligned} \int_0^{\infty} e^{-\beta x^2}\sin(ax)\frac{x\,dx}{\gamma^2+x^2} &= -\frac{\pi}{4}e^{\beta\gamma^2}\Big[2\sinh a\gamma + e^{-\gamma a}\Phi\Big(\gamma\sqrt{\beta}-\frac{a}{2\sqrt{\beta}}\Big) \\ &\quad -e^{\gamma a}\Phi\Big(\gamma\sqrt{\beta}+\frac{a}{2\sqrt{\beta}}\Big)\Big] \end{aligned} \tag{2.18}$$

and

$$\int_0^\infty e^{-\beta x^2}\cos(ax)\frac{dx}{\gamma^2+x^2} = \frac{\pi}{4\gamma}e^{\beta\gamma^2}\Big[2\cosh a\gamma - e^{-\gamma a}\Phi\Big(\gamma\sqrt{\beta}-\frac{a}{2\sqrt{\beta}}\Big) - e^{\gamma a}\Phi\Big(\gamma\sqrt{\beta}+\frac{a}{2\sqrt{\beta}}\Big)\Big] \qquad (2.19)$$

where $\Phi(x)$ is the error function defined by

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt\,.$$

Inserting $\beta = \zeta$,$a = \alpha$, $\gamma = c - 1$ and $\gamma = c$, respectively, and simplifying gives

$$\begin{aligned} I_1(S,X,T-t) &= Xe^{-r(T-t)+\zeta c^2+c(\alpha-2c\zeta)}\frac{1}{2}e^{\zeta(c-1)^2} \\ &\quad\cdot\Big(\cosh((c-1)\alpha)-\sinh((c-1)\alpha)-e^{-(c-1)\alpha}\Phi\Big((c-1)\sqrt{\zeta}-\frac{\alpha}{2\sqrt{\zeta}}\Big)\Big) \\ &= Xe^{-r(T-t)+\zeta c^2+c(\alpha-2c\zeta)}e^{\zeta(c-1)^2-(c-1)\alpha}\frac{1}{2}\Big(1-\Phi\Big((c-1)\sqrt{\zeta}-\frac{\alpha}{2\sqrt{\zeta}}\Big)\Big)\,, \end{aligned}$$

where in the last step we have used the relation $\cosh(x) - \sinh(x) = e^{-x}$. In the same manner we obtain for $I_2(S,X,T-t)$

$$I_2(S,X,T-t) = rXe^{-r(T-t)+\zeta c^2+c(\alpha-2c\zeta)}e^{\zeta c^2-c\alpha}\frac{1}{2}\Big(1-\Phi\Big(c\sqrt{\zeta}-\frac{\alpha}{2\sqrt{\zeta}}\Big)\Big)\,.$$

Now, the exponentials can be simplified further to get

$$I_1(S,X,T-t) = Se^{-q(T-t)}\frac{1}{2}\Big(1-\Phi\Big((c-1)\sqrt{\zeta}-\frac{\alpha}{2\sqrt{\zeta}}\Big)\Big)\,,$$

and

$$I_2(S,X,T-t) = rXe^{-r(T-t)}\frac{1}{2}\Big(1-\Phi\Big(c\sqrt{\zeta}-\frac{\alpha}{2\sqrt{\zeta}}\Big)\Big)\,.$$

The final step in our proof is to use the connection between the error function $\Phi(x)$ and the normal distribution function $N(x)$ given by the relation

$$\Phi(x) = 2N(\sqrt{2}x) - 1\,,$$

and observing that

$$\frac{\alpha}{\sqrt{2\zeta}} - (c-1)\sqrt{2\zeta} = \frac{\ln\left(\frac{S}{X}\right) + \zeta(\kappa_2 + 1)}{\sigma\sqrt{T-t}} = d_1(S, X, T-t)\,,$$

and

$$\frac{\alpha}{\sqrt{2\zeta}} - c\sqrt{2\zeta} = \frac{\ln\left(\frac{S}{X}\right) + \zeta(\kappa_2 - 1)}{\sigma\sqrt{T-t}} = d_2(S, X, T-t)\,.$$

This completes the proof. □

3 The American Call Option

The main difference between European and American options is that an American option can be exercised by its holder at any time before and including expiry. This early exercise feature creates a free boundary problem and makes the valuation mathematically more complex. It became a prominent problem in finance and applied mathematics throughout the last thirty years. Nevertheless, analytical closed-form solutions turned out to be rare except in very few cases.

The free boundary is given by the critical stock price $S^*(t)$ which specifies the conditions under which the option should be exercised prematurely. Formally, it can be defined as an optimal solution of a problem of first passage through a boundary, see for example Bunch and Johnson (2000). The set of critical stock prices is a function of time and separates the domain $(0, \infty) \times [0, T)$ into a continuation region and an exercise region. At any time $t \in [0, T]$ it is optimal to exercise the option prematurely and receive the payoff $S(t) - X$ if $S^*(t) \leq S(t) < \infty$. On the other hand, it is optimal to hold the option if $0 < S(t) < S^*(t)$. Then the option price is the solution to the fundamental BSM PDE from (2.3). Following Kwok (1998) we extend the domain of the PDE by setting $C^A(S,t) = S(t) - X$ for $S^*(t) \leq S(t) < \infty$. Then $C^A = C^A(S,t)$ satisfies the non-homogeneous PDE:

$$\frac{\partial C^A}{\partial t} + (r-q)\,S\,\frac{\partial C^A}{\partial S} + \frac{1}{2}\sigma^2\,S^2\,\frac{\partial^2 C^A}{\partial S^2} - rC^A = f \qquad (3.1)$$

with

$$f = f(S,t) = \begin{cases} rX - qS & \text{for} \quad S^*(t) \leq S(t) < \infty \\ 0 & \text{for} \quad 0 < S(t) < S^*(t) \end{cases} \tag{3.2}$$

on $(0,\infty) \times [0,T)$. Furthermore, we have the boundary conditions

$$\lim_{S\to\infty} C^A(S,t) = \infty \qquad \text{on} \quad [0,T)\,, \tag{3.3}$$

$$C^A(S,T) = \theta(S) = \max(S(T) - X, 0) \qquad \text{on} \quad [0,\infty) \tag{3.4}$$

and

$$C^A(0,t) = 0 \qquad \text{on} \quad [0,T). \tag{3.5}$$

Arbitrage arguments show that the option's price must also satisfy the "smooth pasting conditions" at $S^*(t)$ (see Wilmott et al. (1993)):

$$C^A(S^*,t) = S^*(t) - X \qquad \text{and} \qquad \frac{\partial C^A}{\partial S}\Big|_{S(t)=S^*(t)} = 1. \tag{3.6}$$

The modified Mellin transform of (3.1) is given by

$$\frac{\partial \widetilde{C}^A(\omega,t)}{\partial t} + \frac{1}{2}\sigma^2 Q(\omega)\, \widetilde{C}^A(\omega,t) = \widetilde{f}(\omega,t) \tag{3.7}$$

where

$$\widetilde{f}(\omega,t) = \frac{rX}{\omega}(S^*(t))^{-\omega} - \frac{q}{\omega-1}(S^*(t))^{-\omega+1}\,, \tag{3.8}$$

and $Q(\omega)$ is defined in equation (2.13). The general solution to this non-homogeneous ODE is given by

$$\begin{aligned} \widetilde{C}^A(\omega,t) &= c(\omega)e^{-\frac{1}{2}\sigma^2 Q(\omega)t} - \int_t^T \widetilde{f}(\omega,t)e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\,dx \\ &= \widetilde{\theta}(\omega)e^{\frac{1}{2}\sigma^2 Q(\omega)(T-t)} \\ &\quad + \int_t^T \frac{q}{\omega-1}(S^*(x))^{-\omega+1}e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\,dx \\ &\quad - \int_t^T \frac{rX}{\omega}(S^*(x))^{-\omega}e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\,dx\,, \end{aligned}$$

where $Q(\omega)$ is defined in equation (2.13) and $\widetilde{\theta}(\omega)$ is the terminal condition given in equation (2.16). Once again, the application of the modified Mellin inversion yields

$$
\begin{aligned}
C^A(S,t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{\theta}(\omega)\cdot e^{\frac{1}{2}\sigma^2 Q(\omega)(T-t)}\, S^{\omega}\, d\omega \\
&+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{qS^*(x)}{\omega-1}\Big(\frac{S}{S^*(x)}\Big)^{\omega} e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\, dx d\omega \\
&-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{rX}{\omega}\Big(\frac{S}{S^*(x)}\Big)^{\omega} e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\, dx d\omega. \qquad (3.9)
\end{aligned}
$$

Notice that the first term in equation (3.9) is the European call price from (2.17) and the last two terms capture the early exercise premium. Therefore, we finally arrive at the new integral representation

$$
\begin{aligned}
C^A(S,t) &= C^E(S,t) \\
&+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{qS^*(x)}{\omega-1}\Big(\frac{S}{S^*(x)}\Big)^{\omega} e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\, dx d\omega \\
&-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{rX}{\omega}\Big(\frac{S}{S^*(x)}\Big)^{\omega} e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\, dx d\omega. \qquad (3.10)
\end{aligned}
$$

where $(S,t)\in(0,\infty)\times[0,T)$, $c\in(1,\infty)$, $\{\omega\in\mathbb{C}\,|\,1<Re(\omega)<\infty\}$, and

$$
Q(\omega) = \omega^2 - \omega(1-\kappa_2) - \kappa_1
$$

with $\kappa_1 = \frac{2r}{\sigma^2}$ and $\kappa_2 = \frac{2(r-q)}{\sigma^2}$. The free boundary is given by

$$
\begin{aligned}
S^*(t) - X &= C^E(S^*(t),t) \\
&+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{qS^*(x)}{\omega-1}\Big(\frac{S^*(t)}{S^*(x)}\Big)^{\omega} e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\, dx d\omega \\
&-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \frac{rX}{\omega}\Big(\frac{S^*(t)}{S^*(x)}\Big)^{\omega} e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\, dx d\omega. \qquad (3.11)
\end{aligned}
$$

The following proposition holds[1]:

[1]For a survey of integral representations for American call options see Chiarella et al. (2004).

Proposition 3.1 *Equation (3.10) is equivalent to the following integral representation derived by Kim (1990)*

$$\begin{aligned} C^A(S,\tau) &= C^E(S,\tau) \\ &+ \int_0^\tau qS\,e^{-q(\tau-\xi)}\,N(d_1(S,S^*(\xi),\tau-\xi))\,d\xi \\ &- \int_0^\tau rX\,e^{-r(\tau-\xi)}\,N(d_2(S,S^*(\xi),\tau-\xi))\,d\xi \end{aligned} \tag{3.12}$$

where $\tau = T-t, S = S(\tau), S \le S^*(\tau),$ *and*

$$d_1(x,y,t) = \frac{\ln\frac{x}{y} + (r-q-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\,,$$

$$d_2(x,y,t) = d_1(x,y,t) - \sigma\sqrt{t}.$$

PROOF: A direct proof of the equivalence is similar to that one presented in the previous section so we just give the main idea. Set $\tau = T-t$ and $\xi = \tau - x$ and write for the American call price in (3.10)

$$C^A(S,\tau) = C^E(S,\tau) + \int_0^\tau I_1(\xi)\,d\xi - \int_0^\tau I_2(\xi)\,d\xi\,, \tag{3.13}$$

with

$$I_1(\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{qS^*(\tau-\xi)}{\omega-1}\Big(\frac{S(\tau)}{S^*(\tau-\xi)}\Big)^\omega e^{\frac{1}{2}\sigma^2 Q(\omega)\xi} d\omega \tag{3.14}$$

and

$$I_2(\xi) = \frac{rX}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{\omega}\Big(\frac{S(\tau)}{S^*(\tau-\xi)}\Big)^\omega e^{\frac{1}{2}\sigma^2 Q(\omega)\xi} d\omega\,. \tag{3.15}$$

Now, with $\omega = c+iy, 1 < c < \infty$ and $\zeta = \frac{1}{2}\sigma^2\xi$ we have

$$I_1(\xi) = qS^*(\tau-\xi)e^{-r\xi+\zeta c^2+c(\alpha-2c\zeta)}\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{c-1-iy}{(c-1)^2+y^2}e^{-\zeta y^2+iy\alpha}\,dy\,, \tag{3.16}$$

where we have set

$$\alpha = \ln\Big(\frac{S(\tau)}{S^*(\tau-\xi)}\Big) + \zeta(2c+\kappa_2-1). \tag{3.17}$$

Similarly,

$$I_2(\xi) = rXe^{-r\xi+\zeta c^2+c(\alpha-2c\zeta)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{c-iy}{c^2+y^2}e^{-\zeta y^2+iy\alpha}\,dy\,. \tag{3.18}$$

From now on the argumentation goes along the same lines as in the proof from the previous section and straightforward calculations establish the result. □

4 Further Analysis and Applications

In this section we extend our analysis of the new integral representation for the American call option and its free boundary. As special cases of equations (3.10) and (3.11), respectively, we recover theoretical properties of the option's price and the free boundary using the new approach. First we consider the perpetual American call option initially studied by Samuelson (1965) and Merton (1973). We derive the closed-form expressions for the free boundary and the price of the option. Next, we show how the new framework can be used to recover a theoretical result derived by Kim (1990) regarding the optimal exercise price of American call options at expiry.

Proposition 4.1 *If $T \to \infty$ the free boundary of the perpetual American call option is given by*

$$S^*_\infty = X\frac{\omega_1}{\omega_1-1}\,, \tag{4.1}$$

where

$$\omega_1 = \frac{1-\kappa_2}{2} + \frac{\sqrt{(1-\kappa_2)^2+4\kappa_1}}{2}\,, \tag{4.2}$$

and the closed-form solution for the perpetual American call option equals

$$C^A_\infty(S,t) = \left(\frac{S}{S^*_\infty}\right)^{\omega_1}(S^*_\infty - X)\,. \tag{4.3}$$

PROOF: The roots of $Q(\omega)$ defined in (2.13) are given by

$$\omega_{1/2} = \frac{1-\kappa_2}{2} \pm \frac{\sqrt{(1-\kappa_2)^2+4\kappa_1}}{2}\,.$$

Thus, we have $Q(\omega) = (\omega - \omega_1)(\omega - \omega_2)$ with $-\kappa_1 \leq \omega_2 \leq 0$ and $1 \leq \omega_1 < \infty$. The limiting cases $\omega_1 = 1$ and $\omega_2 = -\kappa_1$ are special roots for $q = 0$. We will determine the unknown critical stock price $S^*(t)$ using the second smooth pasting condition from equation (3.6).

Notice, that for the valuation formula (3.10) to hold as $T \to \infty$, it is necessary that $Re(Q(\omega)) < 0$, i.e. $1 < Re(\omega) < \omega_1$.

Using the second smooth pasting condition we obtain as $T \to \infty$

$$1 = \frac{\partial C^A}{\partial S}\Big|_{S=S^*} = \frac{\partial C^E}{\partial S}\Big|_{S=S^*} + \frac{\partial C_1}{\partial S}\Big|_{S=S^*} + \frac{\partial C_2}{\partial S}\Big|_{S=S^*} \tag{4.4}$$

where the free boundary $S^* = S^*_\infty$ is now independent of time, and C_1 and C_2 denote the second and third term in the valuation formula (3.10), respectively.

The first summand in (4.4) is the delta of a European call option on a dividend-paying stock and equals

$$\frac{\partial C^E}{\partial S} = e^{-q(T-t)} N(d_1(S, X, T-t))$$

with $d_1(S, X, T-t)$ given in (2.8). It follows[2] that as $T \to \infty$

$$\frac{\partial C^E}{\partial S}\Big|_{S=S^*_\infty} \to 0.$$

Now consider the C_1 term. The limit $T \to \infty$ gives

$$\frac{\partial C_1}{\partial S} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_t^\infty \frac{q\omega}{\omega - 1} \Big(\frac{S}{S^*_\infty}\Big)^{\omega-1} e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\, dx\, d\omega\,.$$

Therefore

$$\frac{\partial C_1}{\partial S}\Big|_{S=S^*_\infty} = \frac{\kappa_2 - \kappa_1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\omega}{(\omega-1)(\omega-\omega_1)(\omega-\omega_2)}\, d\omega\,. \tag{4.5}$$

[2]Note that this is not true if $q = 0$. In this case we have

$$\frac{\partial C^E}{\partial S}\Big|_{S=S^*_\infty} \to 1.$$

Similarly, the C_2 term is determined as

$$\frac{\partial C_2}{\partial S} = -\frac{rX}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_{t}^{\infty}\frac{1}{S}\Big(\frac{S}{S^*_\infty}\Big)^{\omega} e^{\frac{1}{2}\sigma^2 Q(\omega)(x-t)}\,dx\,d\omega\,,$$

and we have

$$\frac{\partial C_2}{\partial S}\Big|_{S=S^*_\infty} = \kappa_1\frac{X}{S^*_\infty}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{(\omega-\omega_1)(\omega-\omega_2)}\,d\omega\,. \tag{4.6}$$

An application of the residue theorem (see Freitag and Busam (2000)) gives

$$\frac{\partial C_1}{\partial S}\Big|_{S=S^*_\infty} = (\kappa_2-\kappa_1)\Big(\frac{1}{(1-\omega_1)(1-\omega_2)} + \frac{\omega_2}{(\omega_2-1)(\omega_2-\omega_1)}\Big) \tag{4.7}$$

and

$$\frac{\partial C_2}{\partial S}\Big|_{S=S^*_\infty} = \kappa_1\frac{X}{S^*_\infty}\frac{1}{(\omega_2-\omega_1)}\,. \tag{4.8}$$

Finally, we get for the critical stock price

$$S^*_\infty = X\frac{\kappa_1}{\omega_2+\kappa_1} = X\frac{\omega_1}{\omega_1-1}\,. \tag{4.9}$$

Now, the perpetual American call can be expressed as

$$\begin{aligned} C^A_\infty(S,t) &= \frac{\kappa_2-\kappa_1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Big(\frac{S}{S^*_\infty}\Big)^{\omega}\frac{S^*_\infty}{(\omega-1)(\omega-\omega_1)(\omega-\omega_2)}\,d\omega \\ &\quad +\kappa_1\frac{X}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Big(\frac{S}{S^*_\infty}\Big)^{\omega}\frac{1}{\omega(\omega-\omega_1)(\omega-\omega_2)}\,d\omega\,. \end{aligned}$$

Another application of the residue theorem gives us the closed-form solution for the perpetual American call option:

$$\begin{aligned} C^A_\infty(S,t) &= \Big(\frac{S}{S^*_\infty}\Big)^{\omega_1}\frac{X}{\omega_1-1} \\ &= \Big(\frac{S}{S^*_\infty}\Big)^{\omega_1}(S^*_\infty - X)\,. \end{aligned}$$

This completes the proof. □

Remark 4.2 *Note that for $q=0$ the critical stock price of the perpetual American call option becomes infinite and $C^A_\infty(S,t) = S(t)$.*

Next, we show that Kim's result concerning the behavior of the free boundary at expiry is a special case of (3.11). The first part of the proof partially follows Chiarella et al. (2004).

Proposition 4.3 *If $t \to T$ it follows from equation (3.11) that*

$$\lim_{t \to T} S^*(t) = \max\left(X, \frac{r}{q} X\right). \tag{4.10}$$

PROOF: Change the time variable in (3.11), $\tau = T - t$, to obtain

$$\begin{aligned}
&S^*(\tau) - X = C^E(S^*(\tau), \tau) \\
&+\frac{1}{2\pi i}\int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{qS^*(x)}{\omega - 1}\left(\frac{S^*(\tau)}{S^*(x)}\right)^\omega e^{\frac{1}{2}\sigma^2 Q(\omega)(\tau - x)}\, d\omega dx \\
&-\frac{1}{2\pi i}\int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{rX}{\omega}\left(\frac{S^*(\tau)}{S^*(x)}\right)^\omega e^{\frac{1}{2}\sigma^2 Q(\omega)(\tau - x)}\, d\omega dx\,.
\end{aligned}$$

Straightforward manipulations give an implicit equation for $S^*(\tau)$:

$$\frac{S^*(\tau)}{X} = \frac{1 - e^{-r\tau}\, N(d_2(S^*(\tau), X, \tau)) - r \cdot I_1(\tau)}{1 - e^{-q\tau}\, N(d_1(S^*(\tau), X, \tau)) - q \cdot I_2(\tau)} \tag{4.11}$$

where

$$I_1(\tau) = \frac{1}{2\pi i}\int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{1}{\omega}\left(\frac{S^*(\tau)}{S^*(x)}\right)^\omega e^{\frac{1}{2}\sigma^2 Q(\omega)(\tau - x)}\, d\omega dx \tag{4.12}$$

and

$$I_2(\tau) = \frac{1}{2\pi i}\int_0^\tau \int_{c-i\infty}^{c+i\infty} \frac{1}{\omega - 1}\left(\frac{S^*(\tau)}{S^*(x)}\right)^{\omega - 1} e^{\frac{1}{2}\sigma^2 Q(\omega)(\tau - x)}\, d\omega dx\,. \tag{4.13}$$

Notice first that the critical stock price satisfies $S^*(\tau) \geq X, \forall \tau > 0$. To find the value $S^*(0^+) = \lim_{\tau \to 0^+} S^*(\tau)$, in a first step we evaluate the limits involving d_1 and d_2. We have

$$\lim_{\tau \to 0^+} d_1(S^*(\tau), X, \tau) = \begin{cases} 0 & \text{for} \quad S^*(0^+) = X \\ \infty & \text{for} \quad S^*(0^+) > X\,. \end{cases}$$

Similarly,

$$\lim_{\tau\to 0^+} d_2(S^*(\tau),X,\tau)=\begin{cases} 0 & \text{for} \quad S^*(0^+)=X \\ \infty & \text{for} \quad S^*(0^+)>X\,. \end{cases}$$

Hence, if $\lim_{\tau\to 0^+} S^*(\tau)=X$ then

$$\lim_{\tau\to 0^+} N(d_1(S^*(\tau),X,\tau))=\lim_{\tau\to 0^+} N(d_2(S^*(\tau),X,\tau))=\frac{1}{2}$$

and

$$\lim_{\tau\to 0^+}\frac{S^*(\tau)}{X}=\frac{\frac{1}{2}-r\lim_{\tau\to 0^+} I_1(\tau)}{\frac{1}{2}-q\lim_{\tau\to 0^+} I_2(\tau)}\,.$$

It is easily verified that both expressions $I_1(\tau)$ and $I_2(\tau)$ tend to zero as $\tau\to 0^+$. As a result we have $\lim_{\tau\to 0^+} S^*(\tau)=X$ being a possible solution. In the second case where

$$\lim_{\tau\to 0^+} S^*(\tau)>X\,,$$

the implicit equation for $S^*(\tau)$ reads

$$\lim_{\tau\to 0^+}\frac{S^*(\tau)}{X}=\frac{r}{q}\cdot\lim_{\tau\to 0^+}\frac{I_1(\tau)}{I_2(\tau)}\,. \tag{4.14}$$

But

$$I_1(\tau)=\int_0^\tau \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\omega}\Big(\frac{S^*(\tau)}{S^*(x)}\Big)^\omega e^{\frac{1}{2}\sigma^2 Q(\omega)(\tau-x)}\,d\omega dx$$

and a simple application of the residue theorem (see Freitag and Busam (2000)) shows that the inner integral equals

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{\omega}\Big(\frac{S^*(\tau)}{S^*(x)}\Big)^\omega e^{\frac{1}{2}\sigma^2 Q(\omega)(\tau-x)}\,d\omega=e^{-r(\tau-x)} \tag{4.15}$$

and thus

$$I_1(\tau)=\frac{1}{r}\Big(1-e^{-r\tau}\Big)\,. \tag{4.16}$$

In the same manner we apply the residue theorem to the second integral to get

$$I_2(\tau)=\frac{1}{q}\Big(1-e^{-q\tau}\Big)\,. \tag{4.17}$$

Obviously, the above calculations can be used to prove the limits in the first case, i.e. for $\lim_{\tau\to 0^+} S^*(\tau) = X$, as well. Putting the results together we arrive at

$$\lim_{\tau\to 0^+} \frac{S^*(\tau)}{X} = \frac{r}{q} \cdot \lim_{\tau\to 0^+} \frac{\frac{1}{r}\Big(1 - e^{-r\tau}\Big)}{\frac{1}{q}\Big(1 - e^{-q\tau}\Big)} = \lim_{\tau\to 0^+} \frac{1 - e^{-r\tau}}{1 - e^{-q\tau}} . \tag{4.18}$$

Now, use the rule of d'Hospital to establish the second assertion. Recalling that the result holds only when $S^*(0^+) > X$, it follows that $r > q$. Combining both results confirms Kim's formula. □

5 Numerical Experiments

In this section we show how to use Gauss-Laguerre quadrature for an efficient and accurate pricing of American call options. From (3.13), (3.16) and (3.18) we have

$$C^A(S,\tau) = C^E(S,\tau) + \int_0^\tau I_1(\xi)\,d\xi - \int_0^\tau I_2(\xi)\,d\xi\,, \tag{5.1}$$

with

$$I_1(\xi) = qS^*(\tau-\xi)e^{-r\xi-\zeta c^2+c\alpha}\frac{1}{\pi}\int_0^\infty e^{-\zeta y^2}\frac{(c-1)\cos(\alpha y)+y\sin(\alpha y)}{(c-1)^2+y^2}\,dy\,, \tag{5.2}$$

and

$$I_2(\xi) = rXe^{-r\xi-\zeta c^2+c\alpha}\frac{1}{\pi}\int_0^\infty e^{-\zeta y^2}\frac{c\cos(\alpha y)+y\sin(\alpha y)}{c^2+y^2}\,dy\,, \tag{5.3}$$

where again we have set

$$\alpha = \ln\Big(\frac{S(\tau)}{S^*(\tau-\xi)}\Big) + \zeta(2c+\kappa_2-1). \tag{5.4}$$

From Gradshteyn and Ryzhik (2007), p. 228 and p. 229 we have:

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}\big(a\sin(bx) - b\cos(bx)\big)}{a^2+b^2} \tag{5.5}$$

and

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}\big(a\cos(bx)+b\sin(bx)\big)}{a^2+b^2} \tag{5.6}$$

so the equations for $I_1(\xi)$ and $I_2(\xi)$ become, respectively:

$$\begin{aligned} I_1(\xi) &= qS^*(\tau-\xi)e^{-r\xi-\zeta c^2+c\alpha}\frac{1}{\pi}\Big(\int_0^\infty\int_0^\infty e^{-\zeta y^2}e^{-(c-1)x}\cos(\alpha y)\cos(xy)\,dxdy \\ &\quad+\int_0^\infty\int_0^\infty e^{-\zeta y^2}e^{-(c-1)x}\sin(\alpha y)\sin(xy)\,dxdy\Big)\,, \end{aligned}$$

and

$$\begin{aligned} I_2(\xi) &= rXe^{-r\xi-\zeta c^2+c\alpha}\frac{1}{\pi}\Big(\int_0^\infty\int_0^\infty e^{-\zeta y^2}e^{-cx}\cos(\alpha y)\cos(xy)\,dxdy \\ &\quad+\int_0^\infty\int_0^\infty e^{-\zeta y^2}e^{-cx}\sin(\alpha y)\sin(xy)\,dxdy\Big)\,. \end{aligned}$$

Now, we use product rules for the sine and cosine function, respectively,

$$\sin(x)\sin(y) = \frac{1}{2}\big(\cos(x-y)-\cos(x+y)\big)$$

$$\cos(x)\cos(y) = \frac{1}{2}\big(\cos(x-y)+\cos(x+y)\big)$$

to obtain

$$\begin{aligned} I_1(\xi) &= A_1\frac{1}{\pi}\Big(\int_0^\infty \frac{1}{2}e^{-(c-1)x}\int_0^\infty e^{-\zeta y^2}\big(\cos(y(\alpha-x))+\cos(y(\alpha+x))\big)\,dydx \\ &\quad+\int_0^\infty \frac{1}{2}e^{-(c-1)x}\int_0^\infty e^{-\zeta y^2}\big(\cos(y(\alpha-x))-\cos(y(\alpha+x))\big)\,dydx\Big)\,, \end{aligned}$$

and

$$\begin{aligned} I_2(\xi) &= A_2\frac{1}{\pi}\Big(\int_0^\infty \frac{1}{2}e^{-cx}\int_0^\infty e^{-\zeta y^2}\big(\cos(y(\alpha-x))+\cos(y(\alpha+x))\big)\,dydx \\ &\quad+\int_0^\infty \frac{1}{2}e^{-cx}\int_0^\infty e^{-\zeta y^2}\big(\cos(y(\alpha-x))-\cos(y(\alpha+x))\big)\,dydx\Big) \end{aligned}$$

where we have set

$$A_1 = qS^*(\tau-\xi)e^{-r\xi-\zeta c^2+c\alpha}$$

and

$$A_2 = rXe^{-r\xi-\zeta c^2+c\alpha}\,.$$

Again, from Gradshteyn and Ryzhik (2007), p. 488 we have for $Re(\beta) > 0$:

$$\int_0^\infty e^{-\beta x^2}\cos(bx)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}e^{b^2/4\beta}\,, \tag{5.7}$$

and the last equations for I_1 and I_2 can be simplified to

$$I_1(\xi) = A_1\frac{1}{2\sqrt{\pi\zeta}}\int_0^\infty e^{-(c-1)x}e^{-\frac{(\alpha-x)^2}{4\zeta}}\,dx \tag{5.8}$$

and

$$I_2(\xi) = A_2\frac{1}{2\sqrt{\pi\zeta}}\int_0^\infty e^{-cx}e^{-\frac{(\alpha-x)^2}{4\zeta}}\,dx\,. \tag{5.9}$$

Finally, observe that the integrals can be approximated accurately using Gauss-Laguerre quadrature

$$\begin{aligned}\int_0^\infty e^{-(c-1)x}e^{-\frac{(\alpha-x)^2}{4\zeta}}\,dx &= \frac{1}{c-1}\int_0^\infty e^{-x}f\big(\frac{x}{c-1}\big)\,dx \\ &\approx \frac{1}{c-1}\sum_{i=1}^n \omega_i f\big(\frac{x_i}{c-1}\big)\,,\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\int_0^\infty e^{-cx}e^{-\frac{(\alpha-x)^2}{4\zeta}}\,dx &= \frac{1}{c}\int_0^\infty e^{-x}f\big(\frac{x}{c}\big)\,dx \\ &\approx \frac{1}{c}\sum_{i=1}^n \omega_i f\big(\frac{x_i}{c}\big)\,,\end{aligned} \tag{5.11}$$

where f equals

$$f(x) = e^{-\frac{(\alpha-x)^2}{4\zeta}} \tag{5.12}$$

and ω_i and $x_i, i = 1, 2, ...n$, correspond to the weights and abscissa of the Gauss-Laguerre quadrature. As a final result we have the following approximation for the American call option:

$$C^A(S,\tau) = C^E(S,\tau) + \int_0^\tau I_1(\xi)\,d\xi - \int_0^\tau I_2(\xi)\,d\xi\,, \tag{5.13}$$

with

$$I_1(\xi) = qS^*(\tau - \xi)e^{-r\xi - \zeta c^2 + c\alpha} \frac{1}{2(c-1)\sqrt{\pi\zeta}} \sum_{i=1}^{n} \omega_i f\big(\frac{x_i}{c-1}\big) \tag{5.14}$$

and

$$I_2(\xi) = rXe^{-r\xi - \zeta c^2 + c\alpha} \frac{1}{2c\sqrt{\pi\zeta}} \sum_{i=1}^{n} \omega_i f\big(\frac{x_i}{c}\big)\,, \tag{5.15}$$

with $1 < c < \infty$,$\zeta = 1/2\sigma^2\xi$, and α and f given in equations (5.4) and (5.12), respectively. The weights $\omega_i, i = 1, ..., n$, are determined by

$$\begin{aligned} \omega_i &= \frac{1}{x_i(L_n'(x_i))^2} \\ &= \frac{x_i}{(n+1)^2(L_{n+1}(x_i))^2}\,, \end{aligned}$$

with $L_n(x)$ the n-th Laguerre polynomial defined by

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\big(e^{-x}x^n\big)\,.$$

The integrals in equation (5.13) are determined using the trapezoidal rule. Additionally, in equation (5.13) we assume that the critical stock price $S^*(\tau)$ is known for all τ. The calculation is performed using equation (3.11) where the complex integrals are approximated recursively using an n-point Gauss-Laguerre scheme and the time integral is evaluated using the trapezoidal rule. As a specific numerical example, we value a six months American call option with strike price $X = 100$. The parameters (r, q, σ) are varied from $(0.03, 0.07, 0.2)$ (top) to $(0.03, 0.07, 0.4)$ (center) to $(0.07, 0.03, 0.3)$ (bottom). For the valuation we use a 16-point Gauss-Laguerre scheme combined with a 300 time step approximation of the time integral. Furthermore we fix the parameter $c = 4$. The results are shown in Table 1. We compare our results to nine other numerical and analytical approaches known in the literature. The "True" value is based on a binomial tree method with $N = 10000$ time steps. The following approaches represent the method proposed by Barone-Adesi and Whaley (1987) (BAW), the four-point method of Geske and Johnson (1984) (GJ4), the modified two-point Geske-Johnson approach of Bunch and Johnson (1992) (BJ2), the four-point schemes of Huang et al. (1996) (HSY4), the lower and upper bound approximation of Broadie and Detemple (1996) (LUBA), the four-point randomization method of Carr (1998) (RAN4), the three-point multi-piece exponential boundary approximation of Ju (1998) (EXP3), an approximation of Ju and Zhong (1999) (JZ), and the procedure based on Gauss-Laguerre quadrature of this article (GL), respectively. The calculations show that the new method provides comparable results. The accuracy is convincing and the absolute deviations from the "true" value are negligible. Moreover, since the numerical approximation of our integral solution is easy to implement, we suggest the new framework as a capable alternative to existing methods.

Table 1: Comparison of American call option prices computed ten different ways.

S	True	BAW	GJ4	BJ2	HSY4	LUBA	RAN4	EXP3	JZ	GL
80	0.2194	0.2300	0.2191	0.2186	0.2199	0.2195	0.2188	0.2196	0.2216	0.2185
90	1.3864	1.4050	1.3849	1.3818	1.3898	1.3862	1.3802	1.3872	1.3857	1.3851
100	4.7825	4.7821	4.7851	4.7862	4.8044	4.7821	4.7728	4.7837	4.7682	4.7835
110	11.0978	11.0409	11.0889	11.2553	11.0686	11.0976	11.0893	11.0993	11.0794	11.1120
120	20.0004	20.0000	20.0073	20.0000	20.0531	20.0000	20.0000	20.0005	20.0000	20.0000
80	2.6889	2.7108	2.6864	2.6827	2.6897	2.6893	2.6787	2.6899	2.6871	2.6788
90	5.7223	5.7416	5.7212	5.7163	5.7361	5.7231	5.7113	5.7237	5.7110	5.7195
100	10.2385	10.2417	10.2451	10.2351	10.2752	10.2402	10.2205	10.2404	10.2143	10.2265
110	16.1812	16.1520	16.1831	16.2107	16.2012	16.1817	16.1629	16.1831	16.1456	16.1756
120	23.3598	23.2883	23.3419	23.4771	23.3288	23.3574	23.3389	23.3622	23.3211	23.3828
80	1.6644	1.6645	1.6644	1.6644	1.6644	1.6644	1.6604	1.6644	1.6644	1.6644
90	4.4947	4.4950	4.4946	4.4947	4.4947	4.4947	4.4959	4.4947	4.4947	4.4947
100	9.2504	9.2513	9.2509	9.2506	9.2506	9.2506	9.2513	9.2506	9.2507	9.2506
110	15.7977	15.7988	15.7973	15.7975	15.7975	15.7975	15.7994	15.7975	15.7977	15.7980
120	23.7061	23.7086	23.7082	23.7062	23.7062	23.7062	23.7027	23.7062	23.7066	23.7060

6 Conclusion

We have extended a technique proposed by Panini and Srivastav (2004) and introduced a modified version of Mellin transforms for the purpose of valuing American call options. Using the new framework we have derived a new integral representation for European and American call options on dividend-paying stocks. To emphasize the generality of our results, we have shown the equivalence of the new integral representation and a classical integral characterization due to Kim (1990). Additionally, we have recovered important theoretical properties of American call options using the new method. Finally, we have proposed Gauss-Laguerre quadrature for an accurate pricing and showed that the numerical scheme is a good alternative to other approaches existing in the literature.

The analysis presented in this paper is based on the price process due to Black/Scholes and Merton. The valuation formulas for the American call option and its free boundary may be used to derive new approximations of the option's price and the free boundary. Also, the method can be extended to value more complex European- and American-styled derivatives. Extensions to other stochastic price processes and multi-factor models are left to further research.

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