PROOF AND TRUTH

An anti-realist perspective

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For Chiara

iv

Contents

Acknowledgments

The present work is the result of my doctoral studies, carried out partially at Siena University and partially at Tübingen University.

In fall 2007, I was enrolled as a Ph.D. student at Siena University. The research proposal submitted for the admission was based on my master thesis, 'Negation. A meaning-theoretical investigation', which was written under the supervision of Gabriele Usberti and discussed in June 2007 at Siena University.

The proposal was based on two ideas which appeared, in a rather confused way, in my master thesis. One was that the need of a notion of truth in Dummett and Prawitz' anti-realist perspective is due to the role played by implication in the definition of validity. The other was that, in a proof-theoretic setting, the notions of proof and refutation are connected with the roles of sentences as (respectively) assumptions and conclusions in derivations.

Although in the research proposal I was stating the will of developing an alternative and fully-fledged proof-theoretic semantic picture, what I achieved in these three years is, if anything, a clarification of these two ideas.

Short before discussing my master thesis, in spring 2007, I had the opportunity to meet Peter Schroeder-Heister, who had been invited to give two lectures in Pisa, by Enrico Moriconi. It turned out that the work on negation I was carrying out was of some interest to him. One year later, I was moving to Tübingen.

The work with Peter and his 'team' contributed in a unique way to my growth. With Peter, Thomas Piecha and Bartosz Więckowski I always had the possibility of presenting and discussing (most of the times in English) my ongoing research, even when my ideas were not completely clear to myself. Peter's lectures 'on the fly'—no need of slides (sometimes not even of notes),

just a marker for the whiteboard—will be an example to which I will try to attain. Tübingen has been (and still is) an extremely stimulating environment into which I could find myself. If I were not in Tübingen, I could not achieve the few certainties I now have about the ground concepts of proof-theoretic semantics (for the uncertainties I will never menage to clarify I am the only to be blamed). I am glad of having had the possibility of sharing with Peter the excitement in discussing the subtleties of the definition of validity.

I tribute the most significance influence on the development of the work on negation to Heinrich Wansing, who I met for the first time at Hejnice, Czech Republic, at the 'Logica 2008' conference. The result of my work is far-off from the clarity and precision characterizing Heinrich' own. Nonetheless, I hope that he may acknowledge that the time he spent reading, commenting and also discussing, whenever we met, my ideas on negation and refutations was not wasted. I thank him especially for the respect shown for my ideas, irrespective of my attempts at criticizing his.

I also wish to thank Enrico, for having read and commented draft material that eventually ended up into the thesis, for having invited me to present my ongoing research at Pisa university philosophy department and, afterward, for having offered me the possibility of publishing some of these ideas in a collection edited by Carlo Marletti. But above all, for the kindness he always manifested to me in all the occasions in which we met.

Chapter two and four of the thesis, which respectively present, in a concise manner, Dummett and Prawitz' proof-theoretic semantics and, more extensively, the possible characterizations of a notion of refutation in proof-theoretic terms are the result of all this.

A previous version of the third chapter, dealing with the paradox of deduction and sketching the ideas for a proof-theoretic characterization of truth- and assertion-conditions, was discussed in winter 2007 in Siena with Gabriele.

From September 2009 to April 2010, Peter spent a sabbatical semester in Paris. I profited of his absence for spending six month in Siena, where I lived (shortly but happily) with my girlfriend Chiara in the countryside. During this time, I occupied myself almost exhaustively with the issue of truth.

As a 'Christmas' present' I undertook the enterprise of reading (extensive

parts of) Dummett's 'Frege. Philosophy of language'. Although demanding, reading it was definitely worth. It helped me a lot in giving the thesis its actual form. I have been fascinated by the structural analogies between Dummett's reconstruction of Frege's thought and the architecture of the proof-theoretic anti-realism Dummett later developed. I felt like I found some confirmation to the way in which I was framing the issue of truth in relation to anti-realism. I hope I was not wrong.

It is hard to explain in words how much I owe to Gabriele. When I first took his course in philosophy of language, it was about eight years ago, I felt as if I was introduce in a world I wished I never had to leave. He taught me how to walk by myself into it. And, thanks to him, I discovered the pure joy deriving from the understanding of conceptual problems and of their solutions, the very same joy I felt, as a child, in discovering the world. I thank him for all the love he poured into teaching.

The thesis could never be what it is without all the never-ending discussions I had with Gabriele. Surely, some more would have helped! I thank him again, as all the rage to which philosophical discussion may lead, especially in a conceited person as I am, has been often tempered by his patience— sometimes also by his delicious recipes and a few glasses of red wine. I feel as if all the good that can be found in this pages goes, someway or another, back to him. The bad, I'm afraid, is all mine!

I am glad and honored that all the people who followed the development of the thesis accepted to take part to the defense. I warmly thank them for being with me in the conclusion of this route.

Thanks to all my friends—most of them are somewhere far away—especially to Umberto and Peppoquaresima for all that we did and will share. Thanks to the whole Lummerland, for being my home.

Finally, I thank Chiara and all my family, my parents, my sister, my nephew and my niece, my grandfathers—a thought for Nonna Titti, that left us before all this could start, and for Nonna Lola, that slowly faded away while all this was ending. Your love is always with me. And I wish you know that mine is always with you.

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Summary

In the first chapter, we discuss the role that the notion of truth plays in Frege's and Tarski's semantic conceptions. The issue is approached by discussing Dummett's idea that the notion of truth arises from the one of the correctness of an assertion.

According to Dummett, for a propositional language, whose logical constants are interpreted through the standard two-valued truth-tables, it is possible to give a compositional account of speakers' competence purely in terms of the notion of the correctness of an assertion.

In order to show how the notion of a sentence possessing a truth-value must be distinguished by the assertion of a sentence being correct or incorrect, Dummett considers two alternative three-valued interpretations of negation and implication.

The outcome of the analysis of these cases is that a notion of truth as distinct from the one of an assertion being correct arises only in presence of certain linguistic devices. Their distinguishing semantics characteristic is that of producing logically complex sentences, the correctness of the assertions of which is not fully determined by the the correctness of the assertion of their sub-sentences. For the condition, under which an assertion of these sentence is correct, to be determined, one must ascribe to their sub-sentences the possession of a truth-value, that is a semantic feature going beyond the condition of correctness of an assertion.

We argue that, in a first-order language, the need of defining truth in terms of the notion of satisfaction, which is yielded by the presence of quantifiers, is structurally analogous to the need of a notion of truth as distinct from the one of correctness of an assertion.

In the light of Dummett's own claim that predicates in Frege play the role of open formulas in Tarksi, the need of defining truth through satisfaction is equivalent to a robust conception of the semantic role of predicates. We register a dual attitude of Dummett towards Frege's ascription of reference to predicates. On the one hand, he argues that such ascription is needed to endow quantifiers with their appropriate meaning. On the other hand, he identifies the ascription of concepts to predicates as their semantic correlates with the introduction of a realist element in the overall semantic picture and hence Dummett expresses the will of developing a semantic picture free from this realist trait. We argue, *contra* Dummett, that this identification is erroneous.

In the second chapter, we present the idea of a proof-theoretic semantics. As a truth-theoretic semantics defines a truth-predicate that applies to sentence, so a proof-theoretic semantics defines a predicate of validity applying to argumentations. As true sentences denotes the truth-value True, so valid argumentations denotes proofs.

The possession of a valid closed argumentation (i.e. an argumentation in which the conclusion depends on no assumptions) having a sentence A as conclusion is taken to warrant the assertion of A. Verificationism, the theory of meaning based on the proof-theoretic semantics we consider, aims at giving an account of speakers competence in terms of the notion of validity as applies to closed argumentations, i.e. of a notion of validity as correctness of an assertion.

If a language contains implication-like operators, it looks as if this goal cannot be attained. For, in order to characterize the condition of validity of a closed argumentations, having a sentence governed by implication as conclusion, one has to introduce a notion of validity applying to 'open' argumentations, i.e. argumentations in which the possibility of asserting the conclusion depends on the possibility of asserting some assumption.

We argue that this situation is analogous to the one yielded by quantifiers in Frege-Tarksi's style semantics that was discussed in the previous chapter. That is, implication forces one to introduce a notion of validity for argumentations which is distinct from the correctness of the assertion of their conclusions.

As we saw, the corresponding claim in the truth-based approach—that quantifiers require the introduction a notion of truth as distinct from the one of an assertion being correct—was taken by Dummett as the source of realism. As he aims at an anti-realist conception of meaning, it is no surprise that he is scared of the consequences, the presence of implication leads to.

As a result, Dummett tries to reduce the semantic contribution of open argumentations to the one of their 'closed instances'. In the truth-based perspective, this would correspond to the denial of the need of introducing concepts as the semantic correlates of predicates.

We argue that Dummett's strategy is unsatisfactory as the notion of 'reduction procedure', involved in the specification of the 'closed instances' of open argumentations, is inherently vague.

Furthermore, by comparing verificationism and intuitionism, which is taken by Dummett as the paradigm of anti-realism, Dummett's strategy may be further criticized. The intuitionistic account of implication is based on the notion of method, i.e. of constructive function. And in intuitionism, this notion is taken as a primitive one. Hence, Dummett's fear, that an irreducible notion of function (represented by the need of ascribing validity to open argumentations) would lead to realism, turns out to be ill-founded.

In the third chapter, we discuss the role played by the notion of truth in the anti-realist account. In the first two chapters, we suggested an analogy between the need of ascribing validity to open argumentations and the need of introducing a notion of truth as distinct from the one of the correctness of an assertion.

In this chapter, we try to investigate whether this claim has a philosophical content, by providing a positive answer to the question: are there grounds to connect the notion of open validity with the notion of truth?

We first show that some kind of notion of truth is what the anti-realist needs to cope with the so-called paradox of deduction. The analysis of the paradox yields to distinguish between the truth of a sentence and the truth of a sentence being recognized. In terms of these conceptual couple, we reconsider the relationship between truth and assertion in an anti-realist perspective.

Grounds are provided for revising the connection between validity and assertion. We claim that the notion of the assertion of a sentence being correct is primarily connected only with the *canonical* means of establishing a sentence, i.e. only with valid closed *canonical* argumentations and not with 'simply' valid closed argumentations . The possession of valid closed (non-canonical) argumentations of conclusion A does not always put one in the position of asserting A. The possession of a closed non-canonical argumentation is equated with the mere truth of a sentence. As a result, we have that, as it is natural to expect, truth is not a sufficient condition for the correctness of an assertion.

These characterizations of truth and assertion clash with the intuition that a speaker in possession of a closed non-canonical argumentation for a sentence is entitled to assert it. We provide grounds for accepting this counter-intuitive consequence, by characterizing the notion of an assertion being correct in the context of (at least) two speakers, one of the two challenging the assertion made by other. We defend the thesis that the correctness of an assertion is to be equated with the possession of closed *canonical* argumentations, by characterizing correctness as the assertion being unchallengeable.

Finally, we remark that the possibility of establishing a sentence by indirect means, i.e. through valid closed non-canonical argumentations, is conceptually dependent on the practice of establishing logical relationship of dependence among sentences, i.e. on the availability of a notion of validity as applying to open argumentations. That is, the notion of a closed valid non-canonical argumentation arises only in presence of a notion of validity applying to open argumentations. Thus, the connection between open validity and truth is spelled out.

In the fourth chapter, we discuss the possibility of characterizing in the proof-theoretic-semantics a notion of refutation.

In intuitionistic logic, a refutation of a sentence is a method of turning proofs of the sentence into proofs of the absurdity. But if the sentence can actually be refuted, then there is no proof of the sentence to which the method could possibly be applied. This suggests that the notion of method should be

defined independently of what the method is expected to do, i.e. its yielding proofs of the absurdity provided proofs of the refuted sentences. This further suggests that Dummett's account of the validity of open argumentations in terms of that of their closed instances is untenable.

We develop an original characterization of refutations starting from an informal inductive specification of the condition of refutations of logically complex sentences. We develop in a systematic manner this idea by formulating a 'refutation-theoretic' semantics, in which a predicate of validity is defined in such a way that an argumentation is valid if it denotes a refutation of its main assumption. A sub-structural logic, called dual-intuitionistic logic, stands to this semantics in the same relationship in which intuitionistic logic stands to the proof-theoretic semantics developed in chapter 2. Dual-intuitionistic logic derivations admit derivations with many conclusions and only one assumption. All notions developed in chapter 2 have their corresponding (dual) one in the framework developed. In particular, the distinctions canonical/non-canonical and closed/open argumentations. In the refutation based perspective, elimination rules have priority over introductions and the (only) assumption over the (many) conclusions.

As the proof-theoretic semantics had the drawback that in presence of implication a notion of validity distinct from the correctness of an assertion was needed, so it happens in the refutation case, in presence of a dual connective. One must introduce a notion of validity as distinct from the denial of a sentence being correct, the latter being the dual notion of the correctness of an assertion in the refutation setting. In this case as well, this further notion of validity applies to open argumentations, in opposition to the notion of validity as the correctness of the denial of a sentence, which applies to closed ones (where closed argumentations are now those in which the possibility of refuting the assumption is independent of the possibility of refuting any of the conclusions).

In the last chapter, we indicate the ingredients that an anti-realist approach to meaning should incorporate, in order to avoid the difficulties we registered.

The core of an alternative view should be a different conception of the rela-

tionship between categorical and hypothetical notions.

Dummett's efforts to reduce the notion of validity as it applies to open argumentations to the notion of validity as it applies to closed ones failed. Dummett was afraid that the acceptance of an irreducible notion of open validity would amount to the introduction of some realist feature in the semantics. But in intuitionism, the notion of method, which corresponds to the notion of open validity, is taken as primitive. And this does not deprive intuitionism of its anti-realist character.

Negation suggests that methods should not be characterized by what they are supposed to do (yielding certain proofs when applied to other proofs), but rather by some structural features of them since, in some cases, methods actually do nothing.

Analogously, we propose to develop a conception of open argumentations which does not try to reduce their validity to the one of their instances, but defines it directly. The notion of validity applying to closed argumentations would be recovered as a limit case.

This perspective, would be strongly analogous to Tarksi's semantics, in which the notion of satisfaction of open formulas is directly defined and as a limit case one gets a semantic treatment for closed formulas, i.e. sentences.

By ascribing priority to the notions of closed validity, we were driven to two distinct semantic pictures, the one based on the notion of proof, the other on the notion of refutation. In both semantics picture we had to introduce a notion of validity applying to open argumentations, i.e. to argumentations not denoting (respectively) proofs or refutations. The ascription of priority to open argumentations could yield a unified perspective, in which both categorical notions can be recovered as limit cases of a primitive hypothetical notion.

Chapter 1

Realism

Anti-realism, in philosophy of language, is the enterprise of developing a theory of meaning in which bivalence, the thesis that every sentence is either true of false, fails.

The paradigm of an anti-realist theory of meaning is the intuitionistic explanation of the meaning of logical constants. The explanation is in terms of proof-conditions and, as such, it opposes the traditional meaning explanation in terms of truth-conditions.

Michael Dummett, one of the most prominent advocates of anti-realism, aims at further developing the intuitionistic picture, originally tailored for mathematical languages only, to get a systematic account of a whole, possibly natural, language. To do this, Dummett enriched, sometimes readjusted, the intuitionistic picture with further elements, whose origins are not always easy to establish.

We argue that a crucial ingredient of Dummett's views should be traced back to the way in which he reconstructs the truth-theoretic account of meaning. While the truth-theoretic approach is usually identified with Tarski's work (and the subsequent development), Frege's doctrines receive a certain predilection in Dummett's philosophical picture.

1.1 Tarski's truth-definitions

Tarski was the first who succeeded in defining the notion of truth. We will be concerned with the definitions of truth he gave for a propositional and a first-order interpreted language.[1]

1.1.1 The propositional case (I)

We consider an extremely simple propositional language. Its vocabulary consists of two individual constants 'a' and 'b'; two unary predicate 'R' and 'S'; two binary relations 'H' and 'L'; the following logical constants: '$\neg$','$\wedge$','$\vee$' and '$\rightarrow$'.

Atomic sentences are obtaining by opportunely joining together the individual constants with predicates and relations, e.g. 'Ra', 'aLb', 'bHb'.

Logically complex sentences are obtained by combining atomic sentences by means of logical constants. More precisely:

Definition 1 (Sentences)

- *Atomic sentences are sentences;*

- *if A is a sentence then '$\neg A$' is a sentence;*

- *if A and B are sentences, then '$A \wedge B$, '$A \vee B$ and '$A \rightarrow B$ are sentences.*

The denotation of non-logical constants is given with a list: 'a' denotes Ada; 'b' denotes Bert; 'R' denotes the set of red-hair people; 'S' denotes the set of short hair people; 'H' denotes the set of ordered couples of people, the first member of which hates the second one; 'L' denotes the set of ordered couples of people, the first member of which loves the second one.

The truth of a sentence is so defined:

Definition 2 (Truth for a propositional language)

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[1]The brief presentation of Tarski's truth-definitions we give below follows Casalegno's (1997, Ch. 4). For the moment, we do not wish to consider the semantic apparatus of models needed to account for alternative interpretations of non-logical constants, as we are interested in the philosophical goal of inquiring the feasibility of a semantics for a language capable of being used, hence interpreted.

- *A sentence $\alpha\beta$, with α a predicate and β an individual constant, is true iff the denotation of β belongs to the denotation of α; a sentence of the form $\alpha\beta\gamma$, with α and γ individual constants and β a relation, is true iff the ordered couple, whose first member is the denotation of α and its second member is the denotation of γ, belongs to the denotation of β;*

- *a sentence of the form '$\neg A$' is true iff A is not true;*

- *sentences of the form '$A \wedge B$, '$A \vee B$' and '$A \rightarrow B$' are true iff (resp.) both A and B are true, A or B are true, A is not true or B is true.*

1.1.2 The first-order case (I)

A first-order language is richer than a propositional one. It contains, beyond the symbols of the language previously described, an infinite set of variables 'x_0', 'x_1', … and a new logical constant: '$\forall$'.

Both individual constants and variables are referred to as terms.

In this language, it is not possible to directly define the notion of sentence. Instead, the notion of formula is defined. Atomic formulas are obtaining by linking together terms with predicates and relations, e.g. 'Rx_1', '$x_{123}Ha$'. The notion of formula is defined as follows:

Definition 3 (Formulas)

- *Atomic formulas are formulas;*

- *if A is a formula then '$\neg A$' is a formula;*

- *if A and B are formulas, then '$A \wedge B$, '$A \vee B$ and '$A \rightarrow B$ are formulas;*

- *if A is a formula and x a variable, then $\forall x A$ is a formula.*

A sentence is a formula with no free variables, i.e. whose variables (if any) are all bound by a quantifier.[2]

As the syntax of the language is more complex, so is the semantics. Actually, the truth of sentences cannot be directly defined by induction on the complexity of sentences. This is due to the fact that, in general, a quantified

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[2]For the definition of 'free' and 'bound variable' see, for instance, van Dalen (1994, §2.3).

sentence is not obtained from a simpler sentence, but from a simpler open formula —i.e. a formula with at least one free variable.

Tarski's solution consists in introducing the notion of an assignment of values to the variables. An assignment is a function that assigns an object to each variable.

The notion of denotation is then relativized to assignments as follows. The denotation of non-logical constants relative to an assignment g coincides with the non-relativized denotation we described in the previous section. The denotation of a variable v relative to g is the value of the assignment for the variable, $g(v)$.

At this point, we can define the notion of satisfaction of a formula by an assignment as follow:

Definition 4 (Satisfaction of a formula by an assignment) *A formula A is satisfied by an assignment g iff*

- *A is of the form $\alpha\beta$, with α a predicate and β a term, and the denotation of β relative to g belongs to the denotation of α; A is of the form $\alpha\beta\gamma$, with α and γ terms and β a relation, and the ordered couple, whose first member is the denotation of α relative to g and its second member is the denotation of γ relative to g, belongs to the denotation of β;*

- *A is of the form '$\neg B$' and B is not satisfied by g;*

- *A is of the form '$B \wedge C$, '$B \vee C$' and '$B \rightarrow C$' and (resp.) both B and C are satisfied by g, B is satisfied by g or C is satisfied by g, B is not satisfied by g or C is satisfied by g;*

- *A is of the form '$\forall x_i B$' and B is satisfied by all assignments h, such that $h(x_j) = g(x_j)$, for all $j \neq i$.*

Taken a sentence α, α is true if it satisfied by all possible assignments.[3]

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[3]It is actually indifferent to define 'is true' as satisfaction by all assignments or by any assignment, since the predicate applies to sentences and sentences are either satisfied by all assignments or by none. This depends on the fact that the definition of satisfaction is shaped in such a way that, for all assignments h such that $h(x_i) = g = (x_i)$ for all i such that x_i is a free variable of a formula A, A is satisfied by h iff A is satisfied by g. As sentence are formulas with no occurrences of free variables, then a sentence is either satisfied by all assignments or by none.

1.2 Dummett's semantic theories

According to Dummett (1991, ch. 1), a semantic theory for a language consists in a mapping of syntactic expressions defined in the language onto semantic values. The mapping has to be specified in a compositional way, that is by induction on the logical complexity of the syntactic expressions. That is, the mapping of a complex onto a certain semantic value must be uniquely and exhaustively determined by which semantic values its components are mapped onto. To linguistic expressions of different categories, different kinds of entities are assigned.

The general characterization of semantic values is the following:

> 'The semantic value of an expressions is that feature of it that goes to determine the truth of every sentence in which it occurs.' (Dummett 1991, p. 24)

This characterization of semantic values goes back to Frege's so-called 'context principle', according to which:

> 'it is only in the context of a proposition that words have any meaning." (Frege 1884, §62)

Dummett takes the principle as amounting to a certain priority of the semantic value of sentences over the one of other linguistic expressions, although he does not give a precise formulation of what this should actually mean.[4]

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[4]Compositionality and principle of context are in slight tension with each other. For, looking at atomic sentences, the question is whether priority is to be assigned to sentences themselves or to the expressions composing them. For Dummett:

> 'In a certain sense, therefore, sentences have a certain priority over other linguistic expressions: a sentence is determined by as true under certain conditions, which conditions are derivable from the way in which the sentence is constructed out of its constituents words; and the senses of the words relate solely to this determination of the truth-conditions of the sentences in which the words may occur. Of course, looked at in one way, the word has a sense independently of any particular sentence in which it occurs: but its sense is something relating entirely to the occurrence of the word in a sentence[.]' (Dummett 1973a, pp. 194–195)

A more detailed reflection on the role of the context principle in the economy of Dummett's picture is surely of great interest, but we do not undertake the task here. We address further remarks in section 1.3, especially note 11 on page 11 and later, in the conclusion of the chapter, in section 1.6.

So, a semantic theory is an assignments of semantic values to expressions that must satisfy two requirements: (i) the assignment must be compositional; (ii) the assignment of semantic values to sentences must have a certain priority.

Tarski's truth-definitions yield the definition of the predicate 'is true' to be applied to sentence. Can the definitions be viewed as yielding a semantic theory in Dummett's sense?

1.2.1   The propositional case (II)

In the propositional case, Tarski defined the truth of sentences in terms of the denotation of non-logical constants. We claim that the notion of denotation comes pretty near to Dummett's own notion of semantic values. That is, we can take objects as the semantic values of individuals, sets (of objects and of ordered pairs of objects) as the semantic values of (resp.) predicates and relations.[5] In Tarski's presentation, there is no semantic correlate for sentences. Nonetheless, we can introduce the notion of truth-values, to be conceived as the kind of entities to be assigned to sentences.[6]

Tarski's truth-definition can then be seen as an inductive specification of the semantic values of logically complex sentences in terms of the one of their components. The components of atomic sentences are individual predicative and relational constants. The components of logically complex sentences are simpler sentences. So, Dummett's first requirement, compositionality, is satisfied.

In order to show that Tarski's truth-definition for a propositional language can be taken as a semantic theory, we have to check whether Dummett's second requirement, the priority of sentences, can also be met. According to our presentation, the semantic correlates of names, predicates and relations have priority over those of sentences, since the latter ones are defined in terms of

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[5] Dummett's notion is nearer to Frege's *Bedeutungen* than to Tarksi's 'denotation'. More precisely, the semantic values of predicates and relations are functions rather than sets. See, for instance, Dummett (1991, p. 31). Cf. section 1.5.2.

[6] The idea of truth-values as the semantic correlates of sentences is of course due to Frege, who, in his later writings at least, came to treat truth-values as special objects, and hence sentences as special names for them. For a recent survey on the conception of truth-values, see Shramko and Wansing (2010).

the former ones. Hence, Tarksi's truth-definition for a propositional language does not seem to match Dummett's characterization.

But as Casalegno observes:

'The truth-conditions of every atomic sentence could be specified without mentioning the relation of denotation. For example, instead of saying that the sentence '$R(a)$' is true iff the denotation of 'a' belongs to the denotation of 'P', one can simply say that '$R(a)$' iff Ada is red-haired. Since atomic sentence are finite in number, the first clause of definition 2 can be substituted with another equivalent clause in which the concept of denotation does not figure... The use of the concept of denotation is here an advantage only insofar as it allows to specify the truth-conditions of atomic sentences in a concise way.' (Casalegno 1997, p. 98–99)

In other words, one can simply list which atomic sentences are true. Then, the truth of all other sentences can be defined in terms of the primitive notion of truth for atomic sentences. In this way, the priority to be assigned to sentences in the architecture of a semantic theory is saved.

The semantic values of individual constants and of predicates can then be characterized in general as the contribution they give to the determination of the sentences in which they figure as true (actually, just of the *atomic* sentences, since the determination of all other sentences as true is defined in terms of that of atomic sentences).

In this way, Tarksi's truth-definition for a propositional language can be viewed as a semantic theory in Dummett's sense, since both requirements he imposes on them (compositionality and context principle) are satisfied.

1.2.2 The first-order case (II)

Can Tarski's truth-definition for a first-order language be also viewed as yielding a semantic theory in Dummett's sense?

First of all we have to find the appropriate semantic values that must be ascribed to expressions of distinct linguistic categories in such a way as to obtain a compositional assignment.

By looking at the definition of truth for a first-order language, it looks as if we cannot reason as we did in the propositional case, that is, we cannot just introduce truth-values as the semantic values of sentences. For, it would not be possible to determine the truth-values of quantified sentences in terms of those of their sub-sentences, as quantified sentences have no immediate sub-sentences at all.

Actually, it is the very syntactic notion of sentence that cannot be defined by straightforward induction. In order to cope with this problem, Tarski, in place of the notion of sentence, considered the syntactic notion of formula and introduced the relation of satisfaction of a formula by an assignment, defined by induction on the complexity of formulas. In terms of this notion, he defined the predicate 'is true', as applying to the formulas that are satisfied by all assignments.

As the notion of truth for a propositional language, so the notion of satisfaction is defined in terms of the notion of denotation. The difference with respect to the propositional case is that, in the first-order case, the notion of denotation is relativized to assignments.

For non-logical constants, this relativization does not modify anything, since individual constants denote the same objects relative to all assignments and predicates and relations the same sets. Concerning variables, they denote different objects depending on which assignment we are considering. But we could possibly conceive a variable v as having a unique denotation independent of the assignment we are considering—namely, a function from assignments to objects that, when applied to an assignment g, gives as value the object assigned by g to v, $g(v)$; analogously, non-logical constants may be viewed as denoting constant functions, giving the same value for all assignments.

We propose to take as denotation of non-logical constants those that we took in the propositional case (objects, sets and relations). The denotation of a variable v is the function that applied to an assignment g gives as value $g(v)$.

At this point, we can think of the semantic values onto which formulas are to be mapped as classes of assignments. An atomic formula $\alpha\beta$, with β a term and α a predicative constant, is mapped onto the class constituted by all assignments g, such that the value of the denotation of β for g belongs to the

denotation of α. (Analogously for atoms constituted by a relational constant.)

The class of assignments onto which a logically complex formula is mapped is uniquely determined by the classes of assignments onto which its components are. A conjunctive formula is mapped onto the intersection of the classes onto which its conjuncts are mapped; disjunction is mapped onto the union of the classes onto which its disjuncts are mapped; etc. A quantified formula '$\forall x_i B$' is mapped onto the class of assignments g such that for all assignments h, such that $h(x_j) = g(x_j)$ for all $j \neq i$, h belongs to the class of assignments onto which B is mapped.

In this way, we have a compositional assignment of semantic values (although of a rather bizarre sort) to the different kinds of expressions.

We now turn to the other requirement Dummett imposes on semantic theories, namely the priority of sentences. It is not so clear what the requirement amounts to in this case. It may either be taken in a stronger sense, as a priority that sentences should have over expressions of all other categories (open formulas included). Or it may be taken in a weaker sense: since open formulas play in the definition of satisfaction a role analogous to that of sentences in the truth-definition for a propositional language, the principle of context may be taken, in the first-order case, as a priority that open formulas should have over other categories of expressions.

We first discuss the weak interpretation of the requirement.

In the definition of truth for a propositional language, the prior introduction of the notion of denotation could be avoided. It was just a shortcut allowing the uniform expression of the truth-conditions of all atomic sentences together.

Also, in the definition of satisfaction, we can think of replacing the first clause of the definition in with a set of lists, each one of these specifying which assignments satisfy each atomic formula.[7]

In analogy with the propositional case, the semantic values of individual constants, predicates, relations and, in this case, variables can be characterized

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[7]More precisely, this is not exactly the same as in the propositional case. In that case, the replacement could take place if the *language* contained a finite number of *constants*. Here, we are capable of performing the replacement if the *domain* contains a finite number of *objects*. While the finiteness of the number of individual constants has a certain plausibility in a natural language, the finiteness in the number of objects in the domain has less plausibility.

as 'that feature that goes to determine the semantic value of the' open formu-
las in which they occur. So, Tarksi's semantics can be viewed as a semantic
theory in Dummett's sense, taken the weak interpretation of the requirement
concerning the priority of sentences.

On the other hand, if we interpret the requirement concerning the prior-
ity of sentences in the stronger sense, it is not possible to take Tarski's truth-
definition for a first-order language as a semantic theory in Dummett's sense.
For, the assignment of semantic values to sentences is just a special case of the
assignment of semantic values to open formulas.

By now, it is not yet clear the significance of the clash between Dummett's
formulation of what is a semantic theory (when the context principle is inter-
preted in the stronger sense)[8] and Tarski's definition of truth for a first-order
language, which in turn is the core of the truth-theoretic approach to the theory
of meaning.

We will try to spell it out in the rest of the chapter.

1.3   Truth and assertion

According to Dummett (1976), a theory of meaning is an explication of lan-
guage functioning describing what is known by competent speakers. The ex-
planation is naturally articulated in two parts: a theory of force and another
part, whose 'core' is basically a semantic theory.[9] The theory of force explains
the way in which sentences are used in different speech acts (like commands,

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[8]In the next section, in describing Dummett's idea of the general architecture of a theory of
meaning, it will be cleared that the context principle must actually be taken in its stronger inter-
pretation. Cf. note 11 on page 11.

[9]Cf. Dummett (1976, p. 40): here Dummett actually calls it 'theory of truth' (that 'would be
better called "the theory of reference", since... the conditions under which sentences are true [are
inferred from axioms that] assign references of the appropriate kind to [the individual] words
[composing them].'). Later (p. 66) when he comes to reject the truth-theoretic theory of meaning
Dummett says that 'we shall have to construct a semantics which does not take, as its basic notion,
that of an objectively determined truth-value at all.' By also comparing the ideas presented in
Dummett (1991, ch. 1) and Dummett (1998a), it appears as if the theory of truth is a semantic
theory (among other possible ones) which takes truth-values as the semantic values of sentences.
Hence, the choice of the terminology.

questions and so on). Among the different speech acts, the theory selects one of them as basic and tries to explain all others in terms of this one. The task of the semantic theory is to provide an interpretation of the language, so that the basic features of the linguistic act selected by the theory of force are accounted for.

Usually, a theory of meaning selects assertion as the most basic linguistic act.[10] And a basic fact concerning assertion is speaker's acceptance and rejection of assertions made by other speakers. The notion of the correctness of an assertion may be introduced in order to qualify such attitudes. The semantics is now expected to give an account of the notion of the correctness of an assertion.

The significance of the assignment of priority to sentences, stated by the context principle, arises by considering the connection between the semantic theory and the theory of meaning. In particular, the utterance of sentences amounts to the performance of a linguistic act. On the other hand, the utterance of primitive expressions (individual, predicative and relational constants) does not, 'save for cases in which, as in the answer to some questions, the remainder of the sentence is understood from the context' (Dummett 1973a, p. 194). This is the source of Dummett's thesis according to which sentences play a distinctive role, in opposition to the primitive expressions composing them (individual, predicative and relational constants):

> 'Indeed, it is certainly part of the content of [Frege's context principle] that sentences play a special role in language: that since it is by means of them alone than anything can be *said*, that is, any linguistic act (of assertion, question, command, etc,) can be performed, the sense of any expression less than a complete sentence must consist only in the contribution it makes to determining the content of a sentence in which it may occur.'[11] (Dummett 1973a, p. 495)

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[10] An alternative choice will be discussed in chapter 4.

[11] From this remark it is clear that, among the two possible interpretation of the context principle in a first-order language we suggested in the previous sub-section, Dummett would opt for the stronger one. For, the utterance of an open formulas cannot be used to perform any linguistic act. Cf. note 8 on page 10.

The assertion of a given sentence may either be correct or incorrect. No further alternative is available. Whenever a speaker has grounds for asserting a sentence then the assertion is correct. It is incorrect otherwise.[12]

According to Dummett, the notion of truth arises from the one of the correctness of an assertion, i.e. of justified assertion. The way in which he argues in favor of this claim, heavily relies on the thesis that the semantic value of an expression consists in the contribution it gives to the truth and falsity of the sentences in which they figure:

> 'When we seek to characterize the semantic role of sentences as used, not on their own, but as constituents of more complex sentences, we are concerned with them in the same way as with other kinds of sentence-components, which are incapable of being used on their own. Our basic assumption is that the semantic role of any expression consists in the contribution which it makes in determining the conditions for the truth and the falsity of any sentence of which it forms part, where the truth of a sentence is equated with the correctness of the corresponding assertion and its falsity to the incorrectness of the assertion. We have, therefore to inquire after the way in which a subordinate sentence may contribute to determining the condition for the correct assertibility of a complex sentence in which it occurs. There is no *a priori* ground for assuming that this contribution will be determined solely by the assertibility-condition for the subordinate sentence: in the case of natural language, it plainly is not.' (Dummett 1973a, pp. 420–421)

1.3.1   The propositional case (III)

Tarski's truth-definition for the propositional language represents a privileged case, in which 'the contribution to the determination of the condition for the correct assertibility of a complex sentence' is actually 'determined solely by

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[12]Alternately one may take an assertion as ruling out certain states of affairs, so that when one of such states is recognized as holding the assertion must be withdrawn as incorrect. In any case the notions of correctness and incorrectness are taken to be mutually exclusive and jointly exhaustive. Cf. Dummett (1973a, ch. 10) remarks on the difference between assertions and bets.

the assertibility-conditions for the subordinate sentence[s]'. That is, the notion of truth coincides with the one of the correctness of an assertion.

According to Dummett, it is in presence of certain logical operators that we have to distinguish between the notions of truth of a sentence as the correctness of its assertion and as the contribution it gives to the correctness of the assertion of the sentences in which it figures as a component.

To clarify the need of introducing the distinction, Dummett considers two examples of many-valued semantics for a propositional language.

The first example, which is the more intuitively-compelling one, is aimed at showing that a distinction must be drawn between a notion of falsity as incorrectness of an assertion and a notion of falsity required to account for the conditions of assertion of more complex sentences.

Dummett supposes that the language allows for the formation of non-denoting singular terms. By accepting the idea that the assertion of sentences containing a non-denoting term is always incorrect, it is intuitive to expect negation to operate in such a way that both the assertions of such sentences and of their negation are incorrect.

Under this interpretation of negation, the correctness of the assertion of the negation of a sentence, $\neg A$, cannot be defined in terms of the incorrectness of the assertion of the sentence A itself. If the sentence A contains a non-denoting term, its assertion is incorrect but also the one of its negation.

A proper account of negation can be attained in this context by introducing a notion of falsity as distinct from the one of the incorrectness of an assertion. Once such a notion of falsity, as distinct from the mere incorrectness of an assertion is introduced, one can specify the correctness of the assertion of $\neg A$ in terms of A's being false in this new sense.

That is, in order to be in the position of giving a compositional account of a propositional language according to the lines envisaged, one must ascribe to speakers not only the capability of recognizing assertions as correct or incorrect, but also of being able to distinguish the different ways in which an assertion may be incorrect.

In Dummett's terms, while the notion of truth coincides with that of the correctness of an assertion, we get two notion of falsity, only one of which co-

inciding with the one of the incorrectness of an assertion. We can call false$_1$ and false$_2$ the two ways in which a sentence fails to be true, a sentence being false$_1$ when its negation is true and false$_2$ simply when its assertion is incorrect.[13]

As in this case we are forced to introduce a concept of falsity distinct from the mere incorrectness of an assertion, so one can very easily conceive a semantic framework such that, in order to give a compositional account of negation, one has to distinguish between the notion of truth as the correctness of an assertion and a notion of truth needed to account for the correctness of the assertion of more complex sentences. As an example, Dummett considers natural language indicative conditionals and tries to give substance to the intuition, pretty common among speakers, that the negation of an indicative conditional is not true when the conditional antecedent is false.

> 'The suggestion that indicative conditionals should be construed as
> being neither true nor false when their antecedents were false have
> a [certain] plausibility... in a language which displayed the follow-
> ing features: there was in the language a unary sentential operator
> "Non ..." which for the most part simply reversed the assertibility
> condition of a sentence to which it is applied, but which, when ap-
> plied to a conditional "If A, then B" yielded a sentence having the
> same assertibility-conditions of as "If A, then non B". In that case,
> in order to interpret the negation operator [in a compositional man-
> ner], we would need to distinguish, for conditionals, two modes of
> truth: a conditional would be true$_1$ when it was true in virtue of the
> truth of both antecedent and consequent' (Dummett 1973a, p. 422)

that is, when its negation was false; the conditional would be true$_2$ when its

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[13]Dummett presents this example in a slightly different manner. He considers a three-valued interpretation of logical constants. Irrespective of the way the other truth-tables are extended, he extends the truth-table for negation with a third line, stating that the negation of a sentence $\neg A$ has the third truth-value, whenever the sentence A has it. Dummett, says that a sentence is false$_1$ when its negation is true and false$_2$ otherwise. By doing this, a sentence is false$_2$ when it has the third truth-value and not when its assertion is incorrect (as in our reformulation of the example). A sentence is false$_2$ in our sense when it is either false$_1$ or false$_2$ in Dummett's sense. Our modification of Dummett's example is innocuous and helps—hopefully—in clarifying the point at stake.

assertion was correct.

In the suggested setting, in order to achieve a compositional account of negation, we need to distinguish, for indicative conditionals, the notion of truth as correctness of an assertion from a more substantial notion of truth.[14]

This example (and dually the one we considered before, concerning the notions of incorrectness and falsity) leads Dummett to the thesis according to which the source of the concept of truth is the role of sentences in compound formulas (in particular, the role of sentences as antecedent of conditionals).[15]

Hence, it is exactly because we cannot give a compositional account of how the predicate 'is true' applies to complex sentences formed by means of certain operator that a notion of truth distinct from the correctness of an assertion must be introduced.

1.3.2 The first-order case (III)

Can analogous remarks be applied to the first-order case?

Dummett's thesis is that the notion of truth can serve at least two different purposes: on the one hand, we have a notion of truth as correctness of an assertion; on the other hand, we have a notion of truth to be ascribed to sentences in order to account for the correctness of the assertions of other sentences in which the former ones figure as components. It is the fact that sentences figure as components of more complex ones that forces us to ascribe them a semantic property, their being true, that goes beyond our being entitled to assert them.

Although Dummett does explicitly claim it, an analogy, although weak,

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[14]Again, we slightly modified Dummett's example (cf. note 13). The quotation ends like this: 'but true_2 when it was true in virtue of the falsity of the antecedent. The negation operator could then be seen as taking a true_1 sentence into a false one, a false sentence into a true_1 one and a true_2 sentence into one which was still true_2.' As before, in Dummett's original formulation, a sentence is true_2 when it is neither true nor false and not, as in our reformulation, when its assertion is correct. A sentence is true_2 in our sense when it is either true_1 or true_2 in Dummett's sense.

[15]Such a thesis is one of the recurrent themes of his writings. The thesis is used in different places for different purposes. Without trying a full evaluation of the different nuances that the thesis gets, we just cite the most significant sources: Dummett (1990, 1991, 1994, 1998a, 1998b). Prawitz also endorses the thesis, especially w.r.t. sentences as antecedent of conditionals: cf. Prawitz (1987, 1994, 1998b, 1998c). A neat presentation of the thesis, avoiding the conceptual complications in which both Dummett and Prawitz are often entangled, can be found in Brandom (1976).

may be drawn between the propositional cases discussed in the previous sub-section and the first-order case. In the first-order case, an account of the truth-conditions of quantified sentences cannot be attained in terms of the truth-conditions of the component sentences. A distinct semantic notion must be introduced, namely the relation of satisfaction. If we equate the truth of quantified sentences with the notion of truth as the correctness of an assertion, we have that in order to account for the correctness of the assertions of quantified sentences we need to introduce a further semantic feature to be ascribed to the expressions composing them. Since the expressions composing them are open formulas, such a feature is not a notion of truth distinct from the correctness of an assertion, but the relation of satisfaction.

In the examples discussed above, in order to account for the notion of truth (as the correctness of an assertion) as applying to complex sentences, we needed a a distinct notion of truth as applying to their components. What we have here is that, in order to account for how the notion of truth (as correctness of an assertion) applies to quantified sentences, we need the notion of satisfaction, to be applied to their sub-formulas.

Although the analogy is rather loose, we believe it to be the starting point of a line of reasoning, implicit in Dummett thought, that has many consequences for his conception of anti-realism.

In the remaining part of the chapter, we will try to trace in Dummett's analysis of Frege's views on quantifiers a few hints reinforcing our ascription of this implicit analogy to him.

1.4   Slipping into realism

Before doing that, we throw light on the shift of focus to which this analysis may yield.

We ascribed to Dummett the idea of an analogy, according to which the need of characterizing truth in terms of satisfaction corresponds to the need of introducing a notion of truth distinct from the one of correctness of an assertion.

The idea of truth as arising from the correctness of assertions may just look

like a way of rephrasing the fact that in a first-order setting the notion of truth must be defined in terms of satisfaction. But this way of rephrasing the problem is not so innocuous. We quote Dummett:

> 'Within a realist theory of meaning, sentences are regarded as having objective truth-conditions, which obtain independently of our recognition of their truth-values, and, in general, independently even of the means available to us of recognizing them. One reason why such a conception appears so plausible is that the notion of truth is born in the first place, out of the necessity to distinguish between it and the epistemic notion of justifiability: and this necessity is in turn imposed by the requirements for understanding certain kinds of compound sentences.' (Dummett 1973a, p. 451)

The argument is not compelling. One may agree on the idea that certain kinds of compounds sentences yield the introduction of a semantic notion distinct from the notion of correctness of an assertion. In chapter 3, we will argue that there are indeed reasons to refer to such a feature as a notion of truth. But it is far from clear why the notion of truth so obtained has to do with epistemic transcendence.

Nonetheless, Dummett seems to rely on this line of reasoning. As a result, the need of defining truth of sentences in terms of satisfaction of formulas is equated by him to the introduction of a realist element in the picture.

We will see that this 'ambiguity'—not to say confusion—is pervasive. We will first try to track its origins in Dummett's reconstruction of Frege; in the next chapter, we will see how it projects on the formulation of Dummett's own anti-realistic perspective.

1.5   Exegesis of Dummett's exegesis of Frege

In Frege, there is no actual statement of a semantic theory in a systematic way. Nonetheless, all elements that led to later precise formulations can be found in his work.

What distinguishes Frege from Tarski is the use the latter made of the notion of formula. Formulas are syntactic expressions of the same logical type as

sentence: in particular, sentences form the sub-class of formulas containing no free variable.

In Frege, there are no expressions of the same logical type as sentences, acting like open formulas. For, an open formula is neither true not false, i.e. it structurally does not serve the purpose of expressing a thought. Hence, it has a semantic role which is radically different from the one of a sentence. As a result, if anything as an open formula is to be considered in Frege's terms, it must belong to another logical category. Actually, Frege made no use of free variables and hence of the notion of open formula.

Nonetheless, the role played by open formulas in Tarski is essentially the same role that predicates have in Frege's framework.

In the next sub-sections we will show that the identification of open formulas and predicates is very natural.

In the light of this identification, the supposed realist features yielded by the satisfaction relation will have a correlate in the analysis of predicates and of their semantic role.

1.5.1   Frege on the category of predicates

According to Frege, apart from primitive predicates, the general category of predicates contains entities which are 'produced'. The procedure, by which a new predicate can be obtained, is described by Dummett as follows: taken a sentence, one can obtain an n-ary predicate by removing $m_i \geq 1$ $(1 \leq i \leq n)$ occurrences of n $(n \geq 1)$ singular terms from a sentence.

As an example, we can consider the sentence 'Oedipus loved his mother' which, in a language not containing possessive pronouns, sounds as:

'Oedipus loved Oedipus' mother'

The sentence is constituted by two occurrences of the singular term 'Oedipus', the primitive predicates 'ξ love ζ' and what we take to be a primitive functional expression 'ξ' mother'.

From this sentence, it is possible to extract several different complex predicates, according to the way in which we remove the several occurrences of the several singular terms from the sentence. We give a few examples:

- by removing the first occurrence of the name 'Oedipus', we obtain the predicate 'ξ loves Oedipus' mother';

- by removing the occurrence of the (complex) singular term 'Oedipus' mother' we obtain the predicate 'Oedipus loves ξ';

- by removing first the first occurrence of the name 'Oedipus' and then the second one (i.e. disregarding the fact that they are occurrences of the same name), we obtain the predicate 'ξ loves ζ' mother';

- by simultaneously removing both occurrences of the name 'Oedipus', we obtain the predicate 'ξ loves ξ' mother'.

The 'incompleteness' of predicates emerges by comparing the last two complex predicates that can be extracted from the sentence. The two predicates cannot be identified with any piece of syntactic material from which the sentence is constituted since, under this respect, they cannot be distinguished. It is only by looking at the process of extraction that we can account for the difference between the two predicates.[16]

Hence, the only way to identify a complex predicate is by making reference to the process by which it has been extracted from a sentence. In more suggestive terms, predicates have slots which are constitutive of them, that is, they are incomplete expressions.

Dummett stresses that, by just looking at primitive predicates, there is nothing that compels us to take predicates as incomplete in this sense.[17] For, slots are unnecessary to the identification of primitive predicates.

'It is true that, in order to give an account of the rules governing the formation of atomic sentences, we must explain the 'valen-

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[16]Note that this feature of complex predicates does not depend on the availability of n-ary relations ($n \geq 2$) together with the language lack of pronouns. As an alternative example not relying on this, consider the two predicates 'ξ is blond and ζ is tall' and 'ξ is blond and ξ is tall'. For predicates not containing logical constants cf. note 19.

[17]Yet, there is another sense in which primitive predicates also are incomplete, but under this respect they are no more incomplete than singular terms. While sentences are complete (in the sense that their utterance may constitute a move in a language-game) both singular terms and predicates are not. Cf. the priority of sentences in connection with the context principle in Dummett's quotation in section 1.3.

cies' belonging to the different words—which expressions can, and which cannot, be juxtaposed, and when we have a whole sentence and when only a fragment of one. It is also true...that, in stating these rules, it is to the simple predicates...that we must assign slots into which singular terms have to be fitted, rather than ascribing to the singular terms slots into which the predicates...have to be fitted. But this does not make the simple predicates incomplete in the sense that Frege intended when he spoke of incomplete expressions. We might say that, in the case of simple predicates, the slots are external to them, whereas in the case of complex predicates, they are internal. That is, we can know what linguistic entity, considered just as a sequence of phonemes or of printed letters, a simple predicate is, without knowing anything about the slots it carries with it: the slot consists merely in the predicate's being subject to a certain rule about how it can be put together with a term to form a sentence. But the complex predicate cannot be so much as recognized unless we know what slots it carries: they are internal to its very being.' (Dummett 1973a, p. 32–33)

1.5.1.1   The reason for introducing complex predicates

Complex predicates are not the only kind of incomplete expression envisaged by Frege. On the contrary, we can conceive a potentially infinite hierarchy of expressions grounded on the logical type of sentences and singular terms.[18]

'We start with the two types of complete expressions: 'proper names' and 'sentences'; These two types are both to be considered as of

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[18]Dummett follows Frege in referring to singular terms as 'proper names' and to individual constants as 'simple proper names'. We stick to the more usual terminology, but we warn the reader that, since in Frege no free variable is admitted, all singular terms 'aim' at referring to an object of the domain. Of course, the presence of functional expressions allows for the formation of singular terms lacking a referent. But the lack of an object as referent for such singular terms is radically different from the lack of an object as referent for terms containing free variables. Actually, this may be viewed as a deep reason that led Frege to treat expressions lacking referent as problematic. In a framework in which we have free variables there is nothing surprising in expressions lacking a referent.

level 0. Given a type of expression, of some level n, we can intro-
duce, as an $(n + 1)$th-level type of incomplete expression, all those
expressions which can be derived from a complete expression, of
one or the other of the two types of complete expression, by the
omission of one or more occurrences of some one expression of
given n-th level type.' (Dummett 1973a, p. 44)

Dummett stresses that:

'The entire hierarchy... would, however, be pointless. There are two
primary reasons for recognizing a given type of incomplete expres-
sions. First, that there are in the language simple signs belonging
to this type.' (Dummett 1973a, p. 48)

This happens for both predicates and first-order quantifiers (i.e. for first and
second-level type of expressions). But although the presence of primitive signs
belonging to a certain type requires us to accept such a type, it does not compel
us to recognize the complex expressions of that category, which are all possible
expressions belonging to the category, that could be formed according to the
procedure suggested in the above quotation.

For, it is just

'the second reason that makes it mandatory to recognize the exis-
tence of the entire type, simple and complex: namely that the type
in question constitutes the kind of expression to which a given sin-
gle expression can be attached to form a sentence... From this point
of view, the use of the universal quantifier makes it necessary to rec-
ognize the type of expression, considered strictly as an incomplete
expression, to which it can be attached, namely first-level predi-
cates.' (Dummett 1973a, pp. 48–49)

It is because we have linguistic devices that allow us to produce sentences
from predicates (i.e. quantifiers) that we are prompted to recognize the entire
type of first-level predicates; that is, to recognize, beyond primitive predicates,
complex predicative patters within sentences. On the other hand, since we do
not have (as long as we restrict ourselves to a first-order language) primitive

expressions of third level (i.e. expressions that, when applied to quantifiers as arguments, yield sentences by binding the argument place(s) for predicates in the quantifier), there is no need to recognize the entire type of second-level expressions, beyond the primitive expressions belonging to it.

Summing up, while in order to characterize the formation rules of atomic sentences we just need the notion of primitive predicate, in order to characterize the formation rules of complex sentences (i.e. quantified sentences) we need to introduce a more substantial notion of predicate, the notion of complex 'incomplete' predicate.

In section 1.3.2, we argued that Dummett implicitly states an analogy between the need of defining truth in terms of satisfaction and the need of a notion of truth distinct from the correctness of an assertion to account for the conditions of correctness of the assertions of complex sentences. The need of introducing a more substantial notion of predicate, in order to account for the formation of quantified sentences, is a further ring in this chain of analogies. It is true that, what we discussed earlier was the need of introducing more substantial *semantic* features; while now we discussed the need of introducing a *syntactic* notion of predicate more substantial than the one of simple predicate. But the difference will soon disappear: we will argue in section 1.5.2 that the need of introducing complex predicates to account for quantified sentences has a semantic counterpart as well.

1.5.1.2   A few definitions

Before considering the semantics of predicates, we first sketch how the ideas discussed can be presented in a more systematic fashion.

With Dummett's own words:

> 'given a basic fund of atomic sentences, all other sentences can be regarded as being formed by means of a sequence of operations, which are of three kinds: the application of sentential operators to sentences to form new sentences; the omission from a sentence of one or more occurrences of a proper name to form a one-place predicate; the application of a quantifier to a one-place predicate to form a sentence.' (Dummett 1973a, p. 16)

Consider a language whose vocabulary consists of individual constants, (one- and two-place) primitive predicates, the logical connectives ∧, ∨, → and ¬, the universal quantifier ∀ and an infinite set of variables.

Definition 5 (Atomic sentences)

- *If c is an individual constant and $P(\xi)$ a one-place primitive predicate then $P(c)$ is an atomic sentence;*

- *if c, d are individual constants and $R(\xi, \zeta)$ a two-place primitive predicate then $R(c, d)$ is an atomic sentence;*

- *nothing else is an atomic sentence*

Definition 6 (Sentences and Complex predicates)

- *Atomic sentences are sentences;*

- *if A and B are sentences so are $A \wedge B$, $A \vee B$ and $A \to B$;*

- *if A is a sentence so is $\neg A$;*

- *if $A(c)$ is a sentence and c an individual constant occurring in it, $A(\xi)$ is a (one-place) complex predicate obtained by removing some (possibly all) occurrences of c from A;*

- *if $A(\xi)$ is a one-place complex predicate, $\forall x A(x)$ is a sentence obtained by prefixing the predicate with the quantifier followed by a variable and inserting the variable in the slot of the predicate;*

- *nothing else is a sentence nor a complex predicate.*

Greeks letters represent the slots of predicates and hence do not properly speaking belong to the language. The definitions would require a more precise specification of the substitutions operations implicit in the handling of slots.

Although primitive predicates used in the first two definition are schematically represented exactly as the complex predicates of the third definition, the Greek letters representing their slots are only in the latter case constitutive of the predicate.

Concerning the third definition, when we first produce complex predicates from quantifier-free sentences, we will obtain (among others[19]) the primitive predicates we started with. The difference is that this time they count as complex predicates, in the sense that the slots are 'constitutive of their very being'.

By iterating the inductive steps, we get complex predicates containing more and more quantifiers from which we produce sentences with more and more quantifiers. The sentences produced by this procedure are exactly those produced by defining sentences as formulas with no free variables, in the usual Tarski-style definition.[20]

1.5.2   Concepts: the semantic of predicates

1.5.2.1   Frege's semantic picture

According to Dummett, in spite of the differences at the syntactic level, Frege's and Tarski's conceptions of the semantics for a first-order language are not really different:

> '[Frege's] account is not exactly that given in a modern textbook of
> predicate logic [i.e. the one given by Tarski]: but it is essentially so.
> Modern symbolism usually differs from Frege's in allowing as well-

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[19]The possibility of obtaining a set of complex predicates not containing logical constants larger that the one of the primitive predicates depends on whether we have primitive n-ary relations ($n \geq 2$). For example. the relation 'to love' gives rise to both the predicate 'ξ loves ψ' and 'ξ loves ξ'. Cf. note 16 above.

[20]Yet, as Dummett (1973a, pp. 26–27) himself remarks, the possibility of drawing such a sharp distinction between the process of formation of atomic sentences and the one of more complex sentences is due to the lack of a higher-order functional expressions such as the description operator. Such an operator (which is of type 2 in the hierarchy described in 1.5.1.1) yields, when applied to predicates whatsoever complex, what Dummett (following Frege) calls complex proper names. It could then be possible to produce atomic sentences by applying, say, one-place predicates to the complex proper names obtained by means of the new operator. But the formation of a complex proper name required the prior formation of a complex predicate, which in turn was extracted from a complex sentence. Hence, the distinction between the process of formation of atomic sentences and of complex ones would be blurred. By considering a language not equipped with such expressions, Dummett highlights more properly the conceptual difference between primitive and complex predicates—i.e. the fact that the introduction of complex ones is forced primarily by the presence of quantifiers.

formed expressions containing variables, known as 'free variables', not bounded by any quantifier. Nothing exactly corresponding to a free variable appears in Frege's symbolism. A Greek letter [used by Frege] is not properly part of the symbolic language: it is merely a device for indicating where the argument-place of a predicate occurs. It is usual, in modern symbolism, to use the very same letters as free variables as may be used as bound variables. In this case, the statement of the formation rules (rules for the construction of sentences) can be stated very simply, by describing first the formation of 'open [formulas]', expressions like sentences save by containing free variables): no separate operation of forming a predicate has to be stipulated, but only the operation of prefixing a quantifier to an open sentence, thus converting into a bound variable the variable identical in form to the variable attached to the new quantifier. A sentence can then be specified as being a formula with no free variables.

This simplification is, however, more or less illusory. When the truth-conditions of sentences are explained, this has to be done inductively, and, since sentences have to be constructed not only out of other sentences but out of open [formulas], what in fact has to be defined is the truth or falsity of an open [formula] relative to some assignment to the free variables of individuals from the domain of the bound variables. In regard to any given open [formula], such an assignment confers upon the free variables occurring in it the effective status of individual constant...Moreover, those clauses in the inductive stipulation of the truth-conditions of open [formulas] which relate to the quantifiers are stated in terms of truth-conditions of the open [formula] to which the quantifier is prefixed which result from holding fixed the assignments to the free variables other than the one which becomes bound by the new quantifier, and allowing the assignment to the latter free variable to run through all individuals in the domain. This stipulation corresponds to considering the predicate which results from removing every occurrence

of the 'free variable' due to be bound by the new quantifier, and asking of which individuals it is true; the notion of a one-place predicate's being *true of* a given individual being explained in the same way as with Frege.' (Dummett 1973a, pp. 16–17)

According to Dummett, for Frege,

'A one-place predicate is true of a given individual just in case the sentence which results from inserting a name of that individual in the argument-place (gap) of the predicate is true. A sentence formed by this predicate and the [universal quantifier] is true just in case the predicate is true of every individual.' (Dummett 1973a, p. 11)

These remarks suggest the following tentative definition:

Definition 7 (True and true-of)

Assuming as known which atomic sentences are true

- *$A \wedge B$, $A \vee B$, $A \to B$ and $\neg A$ are true iff (respectively) both A is true and B is true, either A is true or B is true, A is not true or B is true, A is not true;*

- *a predicate $A(\xi)$ is true of a given individual iff c is a name of that individual and $A(c)$ is true;*

- *$\forall x A(x)$ is true iff $A(\xi)$ is true of every individual.*

According to Dummett, for such an account to work, one has to make an assumption, namely that

'whenever we understand the truth-conditions for any sentence containing (one or more) occurrences of a [singular term], we likewise understand what it is for any arbitrary object to satisfy the predicate which results from removing (those occurrences of) the [singular term] from the sentence, irrespective of whether we have or can form, in our language a name for that object.' (Dummett 1973a, p. 17)

This assumptions has a corresponding one in Tarski's framework. With Dummett words:

'this [i.e. Frege's assumption] is precisely the assumption which underlies the explanation of the truth-conditions of quantified sentences which is framed in terms of 'free variables' [i.e. Tarski's]. Under this explanation, we are supposed to be considering an open [formula] $A(x)$ as having certain determinate truth-conditions relative to some particular assignment of an object in the domain of the variables to the free variable 'x': and now, in order to take the step necessary to grasp the truth-conditions of the quantified sentence '$\forall x A(x)$', we have to consider the truth-conditions for $A(x)$ which result from assigning each object in the domain to the free variable 'x' in turn. Thus it is assumed that, simultaneously with our grasp of the truth-conditions of '$A(x)$' under the assignment to 'x' with which we started out, we also understand its truth-conditions under every other assignment to that free variable which could be made. And this assumption is precisely the same as Frege's.' (Dummett 1973a, pp. 17–18)

Dummett ascribes great importance to this assumption. In order to see why, we follow him, in his attempt to spell out its content. As a result, we will suggest that definition 7 is misleading, exactly because it tends to hide the significance of the assumption on which it relies.

1.5.2.2 Given and arbitrary objects

According to Dummett, the weight of this assumption emerges as soon as we consider a non-denumerable domain of the bounded variables. In such a case, we cannot pretend to have a name for every object of the domain.

Clearly, when considering a given object, we can think of extending the language by introducing a constant for that object. And so, a predicate being true of a given object can always be reduced to a sentence being true, the sentence obtained by filling the predicate slot with a name for the object.

'But this is not the nub of the question:... [t]o imagine a situation in which we are considering whether or not the predicate is true of some 'given' object is to treat a case in which it is merely accidental

that the language lacks a name of the object in question.' (Dummett 1973a, p. 19)

In other terms, it is not the nub of the question

'precisely because we have smuggled in, by means of the world 'given', the presupposition that we have some quite determinate object in mind. For an object to be 'given' to us, we must have some means of referring to it or of indicating which object it is that it is in question.' (Dummett 1973a, p. 18)

On the other hand,

'[...W]hat is in question is, rather, whether, we can assume that from the knowledge of the truth-conditions of '$A(c)$' we can derive a knowledge of the conditions under which the predicate '$A(\xi)$' will be true of all the objects of the domain, when we do not and could not have the means of referring to each of those objects.' (Dummett 1973a, p. 19)

That is, as Dummett himself says (cf. p. 78 and, later, p. 188) 'the conditions under which the predicate is true of an arbitrary object'.

1.5.2.3   The ascription of reference to predicates

Hence, an account of quantifiers requires the introduction of the notion of 'arbitrary object', as opposed to the one of 'given object'. According to Dummett, this need does not involve any further semantic thesis concerning singular terms, but rather predicates.

To appreciate this, it may be useful to recall the idea of semantics as a mapping of expressions onto semantic values.

Definition 7, we extracted from Dummett's remarks on Frege, is a simultaneous definition of 'being true' and 'being true of a given object'. Which semantic picture arises from this definition? The most natural one is the following. Sentences are mapped onto truth-values. Predicates are mapped onto truth-values as well, but only *given an assignment of objects* for their gaps.

That is, we can think of the semantic values as objects, among which we find the truth-values, the True and the False. Syntactic expressions of any type are mapped onto objects. While singular terms and sentences are mapped onto them *directly*, predicates are mapped onto truth-values only *indirectly*, that is given an assignment of objects to fill their gaps.

But this picture is incomplete exactly because it disregards the role of the assumption concerning arbitrary objects in the explanation of the semantics of quantified sentences. In particular, from this picture, it looks as if the semantic role played by a predicate '$P(\xi)$' could be completely reduced to that of all sentences of the form '$P(a)$'. But the resulting account of quantified sentences would be adequate only in presence of a denumerable domain of objects.

In the chapter 'The reference of incomplete expressions', Dummett discusses the issue of whether it is necessary to introduce concepts as the referents of incomplete expressions. At first, the only reason to be committed to concepts seems to be second-order quantification (i.e. quantification over predicates), according to Quine's doctrine 'to be is to be the value of a bounded variable'. But by pushing the issue further, Dummett comes to ascribe to Frege the need of introducing concepts not because of second-order quantification (which, besides, Frege would have readily accepted), but because of *first*-order quantification.

> '[In an account of the sense of atomic sentences], the notion of reference for predicates did not work at all: the sense of a predicate consisted in the way in which its application to objects is determined, and there is nothing in the way of the identification of a concept as the referent of the predicate to correspond to the identification of an object as the referent of a name.
>
> It may be retorted that confining ourselves to atomic sentences is precisely the cause of the trouble. In determining the truth-value of an atomic sentence, we had, in general, to identify an object as the referent of the name, and then determine whether the predicate applied to that object. The parallel case, in which we should expect an identification of a concept as referent of a predicate to be required, would be in determining the truth-value of a sentence of the next higher level: that is a sentence obtained by filling the argument-

place of a second-level predicate (of a quantifier, say) with a first-level one. In that case, we should expect to have to identify a concept as the referent of the first-level predicate, and then determine whether the second-level predicate was true of it. And it is certainly the case that the truth-value of such sentence depends on the whole extension of the first-level predicate, in a sense in which that of an atomic sentence does not... The truth-value of a sentence 'For every x, $P(x)$' depends upon the extension of the predicate '$P(\xi)$', which, since we are talking of quantified sentence, we regard as, in general, complex. Thus in order to recognize the quantified sentence as true or as false, it is not in all cases enough to understand, or recognize the truth-value of, all sentences in the language of the form '$P(a)$': this would be adequate only if we had an assurance that the language contained a name for every object. In general, what we must know is what it is for the predicate '$P(\xi)$' to be true of an arbitrary object; it is rather natural to characterize this as knowing what concept '$P(\xi)$' stands for, as opposed to knowing how '$P(\xi)$' applies to the objects nameable in the language.' (Dummett 1973a, p. 241–242)

So, in order to give an account of quantified sentences, one has to enrich the semantic picture we sketched, by introducing also concepts, i.e. functions, alongside with objects among the realm of semantic values. We will refer to this by saying that a more substantial notion of reference must be ascribed to predicates.

It should now be clear in which sense definition 7 is misleading. The clause for quantified sentences makes reference to every object of the domain, i.e. to arbitrary objects, and not simply to all objects nameable in language, i.e. all given objects. On the other hand, from the clause for predicates one can extract only the conditions under which a predicate is true of given objects. The semantic role played by predicates in the specification of the truth-conditions of quantified sentences goes beyond the semantic role played by the predicate in all sentences obtained by filling its slot with a singular term.

This is indeed the significance of Tarski truth-definition. In Tarski's terms, we have that the notion of satisfaction can be reduced to the truth of sentences

only if the number of objects of the domain is denumerable. In such a case, we can think of extending the language with a constant for every object; and for every assignment we will have a sentence corresponding to the formula under that assignment. But, in the general case of a non-denumerable domain, such a characterization does not work. As a result, apart from the restricted cases in which the domain is denumerable, there seems to be something irreducible in the notion of satisfaction.[21]

We can rephrase Dummett's argument as follows. As we saw, it is the presence of first-order quantifiers that forces us to recognize the entire type of first-level predicates. In the light of the actual discussion, the presence of quantifiers not only has consequences for predicates from the syntactic perspective, but from the semantic one as well.

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[21] Although Dummett does not states it explicitly, it is clear that definition 7 yields an account of quantifiers that is essentially equivalent to the so-called 'substitutional' interpretation of quantifiers, first introduced by Marcus (1962). In particular, instead of the last two clauses of definition 7, one could simply give Marcus' clause, according to which $\forall x A$ is true iff

'Every substitution instance of A is true' (Marcus 1962, p. 252)

where, '[b]y a substitution instance of A is meant the result of replacing 'x' in 'A' by the value of x', values for variables being names.

We believe that Dummett intentionally avoids discussing the issue in terms of substitutional vs ontic quantification. As he himself states, the substitutional interpretation is often presented as a way of avoiding ontological commitments to the objects forming the domain of quantification.

> 'But, substitutional quantification is not a genuine alternative to 'ontic' quantification, let alone one which provides an escape from otherwise troublesome problems. If it so happens that there is, for the intended range of the quantified variable, a expression in the language corresponding to every element of the range, then of course it must be the case that the universally quantified statement is true if and only if every permissible instance of it is true... But this in no way relieves us of the responsibility for assigning a reference to the expressions the insertion of which in the argument-place of the incomplete expression yields the instances of the quantified sentence, if we are to provide an adequate semantics for the language: for we need to do this in order to state the truth-conditions of those instances.' (Dummett 1973a, p. 526)

Dummett's reasoning shows that, *contra* the advocates of the substitutional account, the point is not that ontic quantification involves of an ontological commitments to objects while substitutional quantification does not; the difference concerns, rather, the need of introducing a more substantial notion of reference for predicates in order to attain an appropriate account of quantification.

By considering only atomic (or quantifier-free) sentences there is no need to make reference to complex predicates in accounting for the syntactic process of their formation. Not only: the primitive predicates' semantic role can be conceived as exhausted by the contribution they give to the truth-conditions of atomic sentences, i.e. to their being true of given objects.

On the other hand, when the language contains quantifiers, we need to accept the whole category of predicates in order to get a proper account of the formation rules for complex sentences. Furthermore, when predicates are used to fill the gap of a quantifier, we need to make reference to the functions denoted by predicates, in order to account for the truth-condition of quantified sentences.

Hence, Frege's choice of a ascribing referent to incomplete expression reflects a deep philosophical insight, namely the fact that we cannot develop the semantics only by taking objects and truth-values as semantic values. At some point, we need the notion of function defined for any arbitrary object of the domain, and not simply for the objects we can name.

In a sense, as Dummett himself observes, what the assumption shared by Frege and Tarski amounts to is the functions denoted by predicates being total, i.e. their yielding a value for all objects of the domain. Dummett expresses the point in a cumbersome way:

> 'We are tacitly assuming that all the primitive predicates of the language are defined for all the objects of the domains of the variables: and, if this is so, then the same must be true for all the complex predicates as well.' (Dummett 1973a, p. 19)

We take the passage to mean that the need of totality for the functions denoted by complex predicates can be satisfied by the stipulation that 'primitive predicates and relational expressions can be considered as defined over *objects* and not over ways in which objects are presented' (Dummett 1973a, p. 242), that is, over all objects of the domain and not simply over all objects we can name. But such a stipulation is only required because of the presence of quantifiers. In a propositional language, neither do we need to recognize complex predicates nor to ascribe to primitive predicates a semantic role going beyond that of the sentences that can be formed by filling their gaps with names for objects.

1.6 Dummett's qualms about realism

So, an account of first-order quantification induces the need of a notion of arbitrary objects as opposed to the one of given object. And this amounts to ascribing reference to predicates.

In section 1.4, we argued that Dummett implicitly endorses an analogy that leads him to see the source of realism in the need of defining truth in terms of satisfaction and, in turn, denotation. As we tried to argue, the ascription of reference to predicate is what in Frege corresponds to the ascription of priority to satisfaction in Tarski's. If we right, we may expect Dummett to connect the ascription of reference to predicate with realism. And as he advocates for anti-realism, we may expect him to argue against such ascription.

This is actually so. Despite the care with which Dummett emphasizes the weight of the ascription of reference to predicates, he stresses that there is a dis-analogy with the need of ascribing reference to names (and sentences).Actually, he comes very close to doubting that there is a real need of ascribing reference to predicates at all.

> 'The notion of identifying a concept as the referent of a predicate...
> is thus not wholly devoid of content; but the content seems thin...
> And just for this reason, Frege's attribution of reference to incom-
> plete expressions appears in the end unjustified.'

After the long discussion of the issue, it seems on the contrary that Frege's ascription of reference to concepts is perfectly well justified. Rather, it is Dummett's suggestion that we can get rid of them that appears unjustified.

The need of discarding the ascription of concepts to predicates is yielded on the one hand by the acceptance of the analogy between the priority of satis-faction and the need of a notion of truth as distinct from the correctness of an assertion, together with the confusion we stressed in section 1.4, leading Dummett to equate the need of a semantic notion distinct from the the correctness of an assertion with a concession to realism.

A further hint in this direction is the clash that Dummett acknowledges between the need of ascribing a more substantial notion of reference to predicates and the priority assigned to sentences in virtue of the context principle.

Although it is not completely clear in which sense is this clash to be taken, due
to a certain vagueness in the statement of what the priority of sentences should
amount to (cf. section 1.2), one can actually find textual evidence showing that
Dummett does implicitly endorse this idea:

> 'There is, indisputably, a considerable tension between Frege's re-
> alism and the doctrine of meaning only in context: the question is
> whether it is a head-on collision. For incomplete expressions, Frege
> held that the application to them of the notion of reference could
> be effected only by analogy: but we saw that, while the application
> could be defended, the analogy broke down at a crucial point...
>
> [I]f the 'context' doctrine is taken in the very strong sense in
> which Frege appears to take it in *Grundlagen*, ...then it seems to
> provide a way of dispensing with reference altogether.' (Dummett
> 1973a, p. 500)

In the next chapter, we try to show that the will of expunging a genuine
notion of function from the semantic framework is a *Leitmotif* of the proof-
theoretic semantics, the core of an anti-realist theory of meaning of verifica-
tionist inspiration.

Chapter 2

Proof-theoretic Semantics

An alternative to Tarski's semantics is the so-called proof-theoretic semantics. As Tarski's semantics is the core of a truth-theoretic approach to meaning, so the proof-theoretic semantics is the core of an alternative theory of meaning, so-called neo-verificationist (henceforth simply verificationist).

A proof-theoretic semantics, as it has been presented by Schroeder-Heister (2006), defines the semantic predicate 'valid'. Unlike 'true', which applies to sentences, the predicate 'valid' applies to argumentations.

Hence, the language for which a proof-theoretic semantics can be proposed, must be equipped with a set of inference rules. We consider inference rules in natural deduction format. The general form of an inference rule is:

$$\frac{\begin{array}{ccc} [A_{1_1}]^k & \dots & [A_{m_1}]^k \\ & \vdots \; \pi_1 & \\ B_1 & & \end{array} \quad \dots \quad \begin{array}{ccc} [A_{1_n}]^k & \dots & [A_{m_n}]^k \\ & \vdots \; \pi_n & \\ & B_n & \end{array}}{C} \; R^k$$

where C is the conclusion of the rule, $B_1, \dots, B_n$ ($n \geq 0$) are the premises of the rule. For $n > 0$, if, for some n, $m_n > 0$ then all A_{i_j} ($1_1 \leq i_j \leq m_n$) are the assumptions discharged by the rule.[1]

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[1] More precisely, assumptions are sets of occurrences of sentences. Sets of occurrences of the same sentence discharged by different rule applications count as different assumptions. Undischarged occurrences of the same sentence count as one single assumption. We will speak loosely of assumptions discharged by rules throughout the work, although actually we should speak of rule applications discharging occurrences of sentences. The index k links sets of discharged occur-

An argumentation is a chain of rules such that the conclusion of each rule (except one) is (one of) the premise(s) of another rule. The rule whose conclusion is not (one of) the premise(s) of any rule is the last rule of the argumentation. Its conclusion is the conclusion of the argumentation. The premises of the rules that are not the conclusion of any other rule are the assumptions of the argumentation. For $n > 0$, a sub-argumentation π_i ($1 \leq i \leq n$) of an argumentation π is an argumentation having the i-th premise of the last rule of π as conclusion. If all assumptions of an argumentation are discharged by some rule application, then the argumentation is 'top-closed'. It is 'top-open' otherwise.[2]

As we said, usually a proof-theoretic semantics is presented as a definition of validity for argumentations. We claim that, as a truth-theoretic semantics, so a proof-theoretic semantic can be presented in terms of a mapping of syntactic expressions onto semantic values. The syntactic expressions are argumentations and the semantic values are proofs.

As in a truth-theoretic perspective, the proof-theoretic semantics at the core of a verificationist theory of meaning aims at accounting for the basic features of assertion.[3] The starting point for defining the notion of V-validity[4] is that

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rences of sentences with the rule application. (This notation will be used throughout the work.) If a rule allows to discharge occurrences of sentences of a certain form, it is meant that applications of the rules may discharge all, some of even none occurrences of the sentences of that form.

[2]We speak of 'top-closed', instead of simply 'closed', argumentations since we will introduce in chapter 4 the notion of 'bottom-closed argumentation'. Throughout this chapter and whenever the context clearly disambiguates between the two notions, we simply speak of 'closed' argumentations. Analogous remarks apply to the notion of 'top-open' argumentation.

[3]More precisely, we should say 'in the way in which Dummett conceives a truth-theoretic theory of meaning'. Actually, for a realist, meaning is equated to truth-conditions and the semantics has to specify the truth-conditions of logically complex sentences in terms of that of their components, without any reference to the notion of an assertion being correct. It is only Dummett that is willing to present a realist theory of meaning as primarily aiming at accounting for the correctness of assertion of logically complex sentences. From his perspective, the notion of truth must be introduced only in presence of certain logical operator, the correctness of the assertions of the sentences governed by which cannot be characterized in terms of the correctness of the assertion of their sub-sentences.

[4]In chapter 4, a different notion of validity will be defined: we will refer to the verificationist one as 'V-validity' and to the other notion of validity as 'F-validity'. We simply speak of 'validity'

proofs are what warrant assertions and argumentations that are interpreted as proofs are V-valid.[5]

Without further delay, we give the definition of validity. The concepts involved in the definition will be explained in the following sub-sections.

Definition 8 (Verificationist V-validity$_{<S,J>}$) *Given an atomic system S and a set of reduction procedure J,*

- *top-closed argumentations in S are V-valid$_{<S,J>}$;*

- *top-closed I-canonical argumentations are V-valid$_{<S,J>}$ if their immediate sub-argumentations are V-valid$_{<S,J>}$;*

- *top-closed argumentations are V-valid$_{<S,J>}$ if they J-reduce to V-valid$_{<S,J>}$ top-closed I-canonical argumentations;*

- *top-open argumentations are V-valid$_{<S,J>}$ if the result of substituting V-valid$_{<S',J'>}$ ($S' \geq S$ and $J' \geq J$) top-closed argumentations for the undischarged assumptions yields V-valid$_{<S',J'>}$ top-closed argumentations.[6]*

2.1 Atomic systems

The definition of validity is relative to atomic systems. The notion of atomic system has been introduced by Prawitz (1971) as follows.

> By an atomic system, I shall understand a system determined by a
> set of descriptive constants, (i.e. individual, operational and pred-
> icative constants) and a set of inference rules for atomic sentences
> with these constants (i.e. both the premises and the conclusion are

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when the context disambiguates opportunely.

[5]We warn the reader that the relationship between valid argumentations and proofs is not as smooth as expected. For closed argumentation, the idea works flawlessly: a closed argumentation is valid iff it denotes a proof—i.e. if it warrants the assertion of its conclusion. The complication depends on the need of applying the predicate 'valid' to open argumentations as well, although intuitively they do not warrant the assertion of their conclusion. A great deal of the chapter will be devoted to a detailed analysis of the reasons for (and the consequences of) this need.

[6]This is Schroeder-Heister's (2006, §5) 'Definition of S–Validity for arguments'.

to be atomic formulas of this kind). A rule may lack premisses and is then called an axiom...

In many contexts, it is not essential how the rules of a system **S** are specified and we make no restriction of that kind. Of special interest, however, are the Post systems where the inference rules are determined as the instances of a finite number of schemata of the form

$$\frac{A_1 \quad A_2 \quad \ldots \quad A_n}{B}$$

where A_1, A_2, A_n and B are atomic formulas. One may also require that B contains no parameter that does not figure in some A_i.
Prawitz (1971, §II.1.3.5)

The literature is very poor of remarks on what atomic systems are and of why they must be taken into consideration.[7] We believe it reasonable to claim that atomic systems play the same role of models in Tarski's truth-definition.[8] Essentially, they tell which atomic sentences are provable. The truth-value of atomic sentences yielded by the model onto which descriptive constants are interpreted is the basis of Tarski's truth-definition. So, closed argumentations in atomic systems are the building blocks of the definition of validity. The first clause of the definition of validity claims that closed argumentations in **S** are valid 'by definition'.

2.2   Canonical argumentations

An introduction rule for a logical constant $*$ is, roughly, an inference rule having a sentence governed by $*$ as conclusion. The set of introduction rules for

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[7]Besides Prawitz's remark just quoted, other few remarks in the philosophical literature concerning atomic systems can be found in Schroeder-Heister (2006, §3.3), Dummett (1991, ch. 11) and Tennant (1987, ch. 10). In logic programming, the specification of relationships of inferential kind among atoms is a well established practice.

[8]This idea, briefly suggested by Tranchini (2009, 2010), can be traced back to Schroeder-Heister's (2008, §3) idea of a common pattern shared by model-theoretic and proof-theoretic characterization of logical consequence. Cf. below, section 2.7, where the relationship of atomic systems to models is more properly stated.

∗, intuitively specifies the conditions under which a sentence having the relevant constant as principal operator can be introduced as the conclusion of an argumentation.

In general, verificationism focuses on the role of sentences as conclusions of deductive processes: assertion is the linguistic act selected by the theory of force and the semantic theory tells us that the sentences that can be asserted are the *conclusions* of some argumentations. As a consequence, introduction rules are taken to fix the meaning of the logical constants.

To say that introduction rules fix the meaning of logical constants has two components.[9]

The first one is that each rule expresses a *sufficient* condition for establishing a sentence governed by the logical constant in questions. That is, if the premises of the rule have been established, so is the conclusion. This feature is codified in the clause of the definition of validity concerning I-canonical argumentations. An I-canonical argumentation is an argumentation ending with an introduction rule.[10] For a closed I-canonical argumentation to be valid, one requires only its immediate sub-argumentation(s) to be valid.

As the atomic system intuitively codifies all the information available on them, I-canonical argumentations having atomic sentences as conclusions are just argumentations in the atomic system S.

To say that a set of introduction rules for a logical constant ∗ is 'meaning-fixing', a further requirement has to be met. Namely, that rules jointly express a *necessary* condition for establishing a sentence governed by ∗. Of course, a sentence governed by ∗ can be established in other ways as well (that is not through I-canonical argumentations). Nonetheless, to say that introduction rules fix the meaning of connectives means that whenever a sentence is not established by introduction, it could have been so established.[11]

As introduction rules fix the meaning of logical constants, canonical argumentations represent the most direct means of establishing a sentence. Non-

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[9]Cf. Prawitz (1985, §2.1).

[10]We introduce the notion of 'I-canonical' (instead of simply 'canonical') argumentations as we will introduce in section 4.3.3 the notion of 'E-canonical' argumentation. We will simply speak of 'canonical' argumentations whenever the context disambiguates between the two notions.

[11]This is Dummett's (1991, ch. 13–14) so-called 'fundamental assumption'.

canonical argumentations represent, in general, indirect means of doing it. The relationship between direct and indirect ways of establishing sentences is then expressed as follows: whenever a sentence is established by indirect means, it could have been directly established.

The expression 'could have been directly established' is explained by defining so-called reduction procedures. Reduction procedures are 'rewriting' operation on argumentations, that globally allow to re-arrange every valid closed argumentation into one ending with an introduction rule.

More precisely, taken a non-introduction inference rule, a reduction procedure associated to the rule specifies the following: how to replace a closed argumentation π obtained by substituting closed canonical argumentations for the premises of the rule, with a closed argumentation π' having the same conclusion of π and as assumptions (a subset of) the assumptions of π which does not make use of the inference rule at stake.[12]

Concerning atoms, as a special case of the definitions of validity and of I-canonicity, we get that closed argumentations having atomic conclusion are valid iff they reduce to closed argumentations in S, which in turn are valid by definition.

Unfortunately, it is not clear which are the conditions that a reduction procedure has to satisfy. From what we said, it is clear that a reduction procedure is a function that, applied to an argumentation of conclusion A as argument, gives an argumentation of conclusion A as value. But this is not enough. For

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[12]In the case of introduction and elimination rules, the possibility of specifying such procedures is referred to as 'harmony'. The notion of harmony is introduced by Dummett, who gives different and non-equivalent characterizations of it. Informally, he presents it as a requirement on 'different aspects of the meaning of expressions'. More formally, sometimes as requirement on reduction procedures, namely that they should globally yield normalization; other times as a criterion according to which the language obtained by adding a logical constant $*$ should be a conservative extension of the $*$-free fragment. In the literature, arguments have been given for taking harmony as normalization and not as conservativity, for instance by Read (2000). But then, if we have harmony, the fundamental assumptions trivially holds, since it is implied by normalization. On the other hand, Dummett (1991, ch. 13) claims that the 'fundamental assumption' *must be assumed* (hence its name) in order to show that rules are in harmony. The issue is far from clear. We believe that at the core of it lays some kind of confusion. We do not undertake the issue of clarifying it. We simply avoid speaking of both harmony and of the fundamental assumption, framing the whole issue in terms of reduction procedures.

not every function of this kind can be accepted as a genuine reduction procedure, on pain of trivializing the definition of validity Consider a function associating to any intuitively invalid argumentation of conclusion A an intuitively valid argumentation of conclusion A. If this function were to count as a reduction procedure, then the intuitively invalid argumentation would be treated by the semantics on a par with the intuitively valid one. Nonetheless, no clear criterion for being a reduction procedure has been given so far. The consequence of this will be discussed in more detail in the next sections.

2.2.1 Validity and assertion: a first approximation

The definition of validity rests on the idea that valid closed argumentations denote proofs of their conclusion, where a proof of a sentence is what warrants its assertion.

Hence, with a slogan, the verificationist theory of meaning having the proof-theoretic semantics as its core could be characterized as follows:

(C) To know the meaning of a sentence is to know the conditions under which a closed argumentation, having the sentence as conclusion, is valid.

That is, to know the conditions under which the assertion of the sentence is correct.

According to **(C)**, we would get an account of meaning purely in terms of assertion, or—to use a terminology resembling the one of the previous chapter—without calling into question any notion of validity as distinct from the correctness of an assertion.

We characterized the meaning-fixing role of introduction rules as follows: whenever a sentence is established by means of a non-canonical argumentation, it could have been established by canonical means. Consider a logical constant $*$. Its meaning is fixed by the introduction rules for $*$. As we said, the rules express sufficient conditions for establishing sentences governed by $*$ (briefly, $*$-sentences). Not only, they also jointly express a necessary condition for establishing such sentences. For, if any $*$-sentence is established not by means of an introduction rule, it could have been so established.

But then, the possibility of establishing a $*$-sentence by non-canonical means

should be irrelevant to the characterization of the meaning of $*$-sentences. In particular, it should be plausible to characterize knowledge of meaning of a $*$-sentence as knowledge of the conditions under which canonical argumentations for $*$-sentences are valid.

More in general, knowledge of meaning should be characterized in terms of the conditions of validity of canonical argumentations and not of simply closed argumentation. This would amount to replacing **(C)** with the following principle:

(CC) To know the meaning of a sentence is to know the conditions under which a closed canonical argumentation, having the sentence as conclusion, is valid.

In this way, not only would we not need any notion of validity going beyond the correctness of an assertion to characterize meaning; in a certain sense, we would not even need the 'full' notion of the correctness of an assertion to do that, but only a specific feature of it, namely that of the correctness of an assertion based on the possession of canonical argumentations.

Prawitz (1985, p. 167) states this point as follows:

> The condition for asserting a sentence A is then knowledge of a closed valid argument for A. Since the only specific thing we need to know about a sentence A in order to know what is meant by a valid argument for A is what the canonical argumentations for A are, the latter notion is a more central feature of a sentence than the condition under which it is correctly asserted.[13]

It is worth remarking that the possibility of replacing **(C)** with **(CC)** is not only a pleasant possibility. On the contrary, it is a necessary step if a compositional account has to be achieved. The reason is that a closed non-canonical argumentation for a sentence may end with whichever kind of inference. In particular, with inferences in which the conclusion may be of lower logical

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[13]Actually, one should mention that to know what is meant by a V-valid$_{<S,J>}$ closed argumentation for A, one needs to also know, beyond the structure of a canonical argument, which justification procedures are available in J.

complexity than the premises. Due to cases such as this, it is clear that we cannot define the notion of valid closed argumentation of conclusion A in terms of valid closed argumentations having the sub-sentences of A as conclusions. This means that the notion with which the correctness of an assertion of a sentence A is identified—the possession of a valid closed argumentation of conclusion A—cannot be defined in a compositional way.

Of course, if the replacement of **(C)** with **(CC)** is to solve the problem, introduction rules must satisfy what Dummett (1991, p. 258) calls 'complexity condition', according to which

> the conclusion will be of higher logical complexity than any of the
> premises and than any discharged assumption.

Summing up, it looks as if, whenever the introduction rules satisfies the complexity condition, by replacing principle **(C)** with **(CC)**, we get an account of meaning in terms of the conditions of an assertion begin (canonically) correct.

2.3 Closed and open argumentations

Unfortunately, this picture, pretending to show that an account of understanding can be achieved in terms of the notion of the (canonical) correctness of an assertion, fails.

In particular, to account for speaker's competence, we must introduce a notion of validity which is distinct from the notion of validity as the correctness of an assertion. Namely, a notion of validity that applies to open argumentations.

Why does the notion of validity as applying to closed canonical argumentations not suffice to give an account of knowledge of meaning?

This depends on the fact that we cannot always specify the validity of a closed canonical argumentation having a sentence A as conclusion only in terms of the validity of closed canonical argumentations for the sub-sentences of A.

Open argumentations of course do not allow the assertion of their conclusions. But even if an open argumentation does not allow the assertion of its

conclusion, it may still serve as a constituent of a more complex argumentation, whose conclusions can be asserted, independently of the possibility of asserting the assumption of the open argumentation from which we started. Typically, implication is the device by means of which an open argumentation is taken into a closed one, having an implication (of the assumption and the conclusion of the open argumentation) as conclusion:[14]

$$\cfrac{\begin{array}{c}[A]^n\\\vdots\\B\end{array}}{A \to B} \; I \to^n$$

As a consequence, we need to distinguish among open argumentations those that can be used to obtain valid closed argumentations from those that cannot. That is, we need to apply the semantic predicate 'valid' to open argumentations as well.

A canonical argumentation for an implication is valid if and only if its sub-argumentation is valid, where, in general, the sub-argumentation can be an open one.

2.3.1 The gap between validity and assertion

The need of making reference to the validity of open argumentations, in defining the one of closed argumentations, suggests that a theory of meaning construed on principle **(CC)** would be unsatisfactory.

According to **(CC)**, speakers' competence consists in knowledge of the condition of validity of closed canonical argumentations. In the case of implication, knowledge of the condition of validity of a closed canonical argument having $A \to B$ as conclusion depends not only on knowledge of the condition of validity of closed canonical argumentations of the sub-sentences, but also on knowledge of the condition of validity of (possibly) open argumentations having B as conclusion and A among the assumptions.

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[14]For discharge policies, cf. note 1. Note that, of course implication introduction may be applied without discharging any occurrence of the assumption of the open argumentation. Nonetheless, the need of applying the predicate 'valid' to open argumentations arises by considering the cases in which the application of the rule does discharge all occurrences of the (only) assumption of the open argumentation.

Mastery of meaning of logically complex sentences would not merely depend on mastery of meaning of logically simpler ones. For, it would also depend on other features of the component sentences, namely the condition of validity of open argumentations in which they figure as conclusions.

The possession of valid closed (no matter if canonical or not) argumentations having A as conclusion amounts to the assertion of A being correct. This notion cannot be defined in a compositional manner, so we tried to account for it in terms of the notion of canonical argumentations. But also the latter cannot be autonomously defined in a compositional manner. So, we have that the correctness of assertions of logically complex sentences cannot be specified only in terms of that of their sub-sentences. The thesis, that the notion of the correctness of an assertion (not even when taken as canonical correctness) is not enough to characterize knowledge of meaning, is thus justified.

To stress the analogy with the relationship between truth and assertion we developed in the previous chapter, we called the notion of validity applying to closed argumentations a notion of validity as correctness of an assertion.

As in the case of truth, so here a further notion of validity is required in order to explain what it is for a closed argumentation to be valid. Namely, a notion of validity applying to open argumentations. The need for this further notion of validity is induced by implication, since closed argumentations for implications are obtained from open ones.

Actually, the role played by implication strongly resembles the one of quantifiers in Tarski's semantics for a first-order language.[15] As we saw, in order to get an account of truth as the correctness of an assertion of quantified sentences, we need to make reference to the notion of satisfaction applying to open formulas. Analogously, in order to get an account of validity as correctness of an assertion for implications, we need to make reference to the notion of validity as applying to open argumentations. (These considerations are summed up in table 2.1.)

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[15]The connection is clear if one thinks of derivations in a natural deductions system as terms of a simply-typed lambda calculus, that is modulo the so-called Curry-Howard isomorphism. Then, both implication and universal quantifiers can be viewed as variables binding operators; and the functional character of open argumentations and predicates can be fully appreciated. Cf. also note 19 on page 104 in chapter 3.

Table 2.1: Open-ness and assertion

In the	
Truth-theoretic Semantics	Proof-theoretic Semantics
an account of	
Quantifiers	Implication
requires the introduction of the notion of	
Satisfaction of Open Formulas	Valid Open Argumentation
that is of a semantic notion distinct from the correctness of an assertion	

2.4 Dummett's fear of realism

In a first-order language, the need of defining truth in terms of satisfaction makes it hard to view the truth-theoretic semantics as a mapping of sentences onto truth-values (cf. section 1.2.2).

Dummett's tentative account, as we reconstructed it, was that the semantic role of predicates could possibly be explained by saying that predicates denotes truth-values, given objects as arguments. That is, sentences directly denote truth-values; predicates denotes truth-values indirectly, given objects for their slots (cf. section 1.5.2.1).

Dummett himself was forced to reject this proposal as inadequate. A presentation of the truth-theoretic perspective among these lines hides the real significance of the notion of formula and of satisfaction or, which is the same, of concept. For, in general, the role played by the notion of satisfaction of a formula by an assignment, or of concept denoted by a predicate, in the expla-

nation of quantifiers cannot be taken up by the notion of truth of sentences.

In order to attain an appropriate account of quantifiers, a more substantial semantic role has to be ascribed to predicates/open formulas. A way of stating this point is that the semantic role of predicates is their being true or false of *arbitrary* objects and not simply their being true or false of any *given* object (cf. section 1.5.2.3).

Dummett, for reason that falls short of being compelling (cf. sections 1.4 and 1.6), takes this need as essentially amounting to the realist feature of Tarski and Frege's semantic picture.

As a result, it was no surprise that Dummett assumed a dual attitude with respect to the status of concepts. For, although he stressed the significance of ascribing to predicates a more substantial semantic role in order to properly understand Frege, he was willing to discard such an ascription as unneeded, for the theoretical goal of developing a theory of meaning. We quote again his opinion on the whole matter

> 'The notion of identifying a concept as the referent of a predicate
> . . . is thus not wholly devoid of content; but the content seems thin
> . . . And just for this reason, Frege's attribution of reference to incomplete expressions appears in the end unjustified.' (Dummett 1973a,
> p. 243)

As this attribution amounts, for Dummett, to the realist character of Tarski and Frege's semantic perspective, it is no surprise that Dummett is ready to discard any corresponding semantic thesis in the proof-theoretic setting, since a proof-theoretic semantics should serve as a basis for an anti-realist theory of meaning. In the proof-theoretic framework, the need of ascribing validity to open argumentations, as well as to closed ones, is indeed structurally analogous to the need of a more substantial attribution of reference to predicates, in the truth-theoretic semantics.

The analogy between the truth- and proof-theoretic cases suggests that Dummett fears that the introduction of a notion of validity as applying to open argumentations may yield realist consequences in the latter setting as well.

But then, it is natural to expect Dummett to find a way to avoid such ascription, or at least to endorse a view according to which the semantic role

played by open argumentations can be somehow reduced to the one of closed
(canonical) ones.

We will argue (in section 2.5.2) that Dummett tries to solve this supposed
problem by reducing the semantic role of valid open argumentations to the one
of their closed instances. We will then suggest (in section 2.6) that this strategy
is viable only if a precise formulation of the notion of reduction procedures
can be given. Finally, we argue (in section 2.7.2) that the very existence of the
problem is questionable.

2.5   Dummett's 'anti-realist' move

We believe that Dummett envisages the possibility of avoiding the supposed
realist consequences deriving from the need of ascribing validity to open argu-
mentations as follows: if such a notion of validity could be somehow reduced
to the one of closed argumentations, then the need of ascribing validity to open
argumentations would not amount to the introduction of a notion of validity
distinct from the correctness of an assertion.

2.5.1   What kind of reduction is possible?

The verificationist milestone is that introduction rules fix the meaning of logical
constants and hence canonical argumentations should play a distinctive role in
meaning specification, in the light of the connection between I-canonicity and
introduction rules.

This idea is perfectly sound in the implication-free case, where closed canon-
ical argumentations and their validity can be defined in a compositional
manner. This is what Dummett (1991, ch. 11, pp. 254–255) does with his
'proof-theoretic justifications of second grade'. He defines canonical argumen-
tations for an implication-free fragment so that they are constituted only by
application of introductions rules and of rules of the atomic system. In this
way, closed canonical argumentations turn out to be valid by definition. And,
given a set of reduction procedure, the validity of closed non-canonical argu-
mentation is reduced to the one of closed canonical ones.

But the problem is exactly whether, in dealing also with implication, the

distinctive role of canonical argumentation is still tenable. According to Dummett:

> 'We shall no longer be able to define canonical arguments in such a way that a canonical argument is automatically valid ...When the canonical argument involves an appeal to introduction rules that discharge one of the [assumptions] of their premiss or premisses, ...we now have simultaneously to define "valid canonical argument" and "valid (arbitrary) argument".' Dummett (1991, p. 260)

Dummett's 'proof-theoretic justifications of third grade', dealing also with implication, correspond to our definition of V-validity$_{<S,J>}$.[16]

Also Schroeder-Heister, seems to endorse the possibility of ascribing priority to closed canonical argumentations when he comments the definition of validity as follows:

> In S−validity, closed canonical derivations are self-justifying, carrying the burden of semantic justification.

but in the note following this comment he clarifies that:

> This does not mean that the S−validity of closed and open derivations is defined separately. These two cases occur intertwined in the same derivation. This is due to the fact that the immediate sub-derivation of a closed canonical derivation of $A \rightarrow B$ is a derivation of B from the assumption A.

Hence, the reduction of open argumentation to closed argumentation—that we believe Dummett is willing to propose—cannot be conceived as one notion being logically defined in terms of the other. The two notions are indeed defined by simultaneous induction.

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[16]Actually the notions are not identical. Dummett does not consider atomic systems nor reduction procedures (cf. note 20 on page 54) and he allows non-introductory steps only in what he calls 'critic arguments', that is argumentations that have as conclusions the premise of an implication introduction rule (cf. note 16 in chapter 3, on page 101).

2.5.2   The semantic status of open argumentations

In the first chapter, we discussed the relationship between Tarski's truth-definitions and Dummett's idea of a semantic theory as a mapping of syntactic expressions onto semantic values.

We presented the proof-theoretic perspective as consisting in a definition of validity (the analogous of Tarski's truth-definitions). But we also briefly suggested that, intuitively, a proof-theoretic semantics can be taken as a mapping of certain complex syntactic expressions, argumentations, onto semantic values, proofs.

The intuitive idea of valid argumentations denoting proofs is smooth in the case of closed argumentations. A closed argumentations represents a proof of its conclusion iff it is valid; and a speaker in possession of a valid closed argumentation is in the position of asserting its conclusion.

The need of ascribing validity to open argumentations makes the picture less clear. For, an open argumentations, although valid, does not stand for a proofs of its conclusion; of course, a speaker in possession of an open argumentation is not warranted in asserting its conclusion.

Nonetheless, the definition of validity suggests how the idea of mapping valid argumentations onto proofs can be accounted for. The intuitive idea behind the clause for open argumentation of the definition of validity is the following: the predicate 'valid' applies to an open argumentation of conclusion B and assumption A

$$
\begin{array}{c}
A \\
\vdots \\
B
\end{array}
$$

iff whenever the sentence A can be asserted so B can be. We said that a sentence can be asserted if we have a proof of it. Hence, we can say that an open argumentation is V-valid if the result of substituting an argumentation, representing a proof of the assumption, to the assumption yields an argumentation representing a proof of the conclusion.

Since the predicate valid is meant to apply to closed argumentations if they represent proofs of their conclusions, we get that an open argumentation is valid iff the result of replacing its undischarged assumptions with closed valid argumentations for them yields a closed argumentation which is valid.

In terms of mapping of syntactic expressions onto semantic values, at this point we can say that closed valid argumentations directly denote proofs of their conclusions; and that, although open argumentations do not directly represent proofs, they can be taken as representing proofs of their conclusions *given proofs of their undischarged assumptions*.

Such an idea is strongly analogous to the supposed (and then rejected) treatment of predicates given by Dummett (see section 1.5.2.3 of previous chapter). We argued that Dummett was implicitly proposing to treat predicates as being mapped onto truth-values given objects for their slots. That is, as their semantic role to be exhausted by their being true of *given* objects and not of *arbitrary* objects.

In the proof-theoretic case, open argumentations can be viewed as being mapped onto proofs of their conclusions given proofs for their undischarged assumptions. In the clause for open argumentation in the definition of validity, we do not actually make reference to proofs for the undischarged assumptions, but to closed valid argumentations having the assumptions as conclusions. This exactly corresponds to Dummett's supposed picture (cf. definition 7 on page 26), where a predicate is true of an individual iff the sentence obtained by filling the predicate slot with a name for it is true. In both cases Dummett does not make references to the semantic values (objects and proofs, respectively) but to the linguistic expressions they denote (names and closed valid argumentations, respectively).

That is, the account of open valid argumentations codified in the definition of validity treats them as yielding valid closed argumentations (i.e. proofs) of their conclusions for any *given* proof for their assumptions, not for any *arbitrary* proof.

In the case of predicates, this picture had to be rejected. A proper understanding of quantifier could be achieved only by denying that the role of predicate could be reduced to the one of the sentences obtained by filling their slot with names—or, equivalently, to their being true or false of given objects.

What about the analogous analysis of open argumentations?

Dummett (wrongly) identifies the notion of arbitrary objects required by an account of quantifiers amounted to the realist feature of the truth-theoretic

semantics. Hence, Dummett, in order to get an anti-realist picture, takes the suggested treatment of open argumentations for granted.[17]

Even if Dummett worries about realism were justified, the will of avoiding the introduction of a realist element in the semantic picture is *per se* not enough to show that a proper understanding of the validity of open argumentation can be achieved without having to deal with the notion of arbitrary proof. One needs independent grounds showing that the suggested account, while in the case of predicates it yields an unsatisfactory account of quantifiers, when applied to open argumentations it yields a proper understanding of implication. In chapter 4, in section 4.2.1 we will actually provide an argument showing that the account of the validity of open argumentations suggested by Dummett is not adequate.

By now, we want to stress that Dummett's proposed account relies on the notion of reduction procedure. As result, the vagueness of the latter notion projects on Dummett's proposed account as well. This is shown in the next section.

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[17]Actually, it looks as if Dummett (1991) does not take anymore the 'substitutional' account problematic even in the case of predicates/open formulas. Although speaking of a programmatic interpretation, he argues that,

> 'it does not use any notion relating to closed formulas other than that of truth...: that is to say, it states the condition for the truth of a complex formulas... directly in terms of the truth of its constituents formulas.... (If as an alternative device alternative to using the notion of satisfaction of an open formula by a sequence of elements of the domain, we assume the language either to contain a name for every element, or to be expanded so as to do so, we can extend this formulation from sentential to predicate logic, saying that a programmatic interpretation states the conditions for the truth of a quantified formula in terms of the truth of its instances).' (Dummett 1991, p. 62)

And later on,

> 'The easiest way to handle free variables is to assume that the language contains a constant term for each element of the domain.' (Dummett 1991, p. 259)

2.6 The 'extensions business'

We will discuss later whether an anti-realist view has to restrict himself to given objects and given proofs in the account of quantifiers and implication. By now we will concede this to Dummett and we will show that, anyway, his supposed reduction of the semantic value of open argumentations to closed ones is hardly tenable.

Roughly, Dummett's supposed reduction relies on the fact that, by considering all closed instances of an open argumentation, one is considering all possible given proofs for the undischarged assumptions of the argumentation.

The closed instances of an open argumentation are obtained by 'plugging' closed valid argumentations having the undischarged assumptions as conclusions on the top of the open argumentations.

But a point is worth mentioning, namely that validity is defined relative to an atomic system and a set of reduction procedures. Even accepting Dummett's will of getting rid of arbitrary proofs, in order to check the validity of an open argumentation relative to an atomic system S and a set of reduction procedures J, one has to consider all possible given proofs for the undischarged assumptions of the argumentation. But to do this, one has to consider the closed instances of an open argumentation obtained by plugging closed argumentations that are valid relative to any possible extension of both the atomic system S and the set of reductions J.[18] In the light of this, it is at least doubtful that there is any reasonable sense in which closed proofs have priority over open ones.

If one disregards the 'extensions business'[19], than it is clear in which sense closed argumentations have priority over open ones: one could claim that a given open argumentation is recognized as valid on the basis of a certain set

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[18]The need of referring to extensions in the clause for open argumentations is usually argued for in terms of a requirement of monotonicity that validity should satisfy. Without extensions, one cannot rule out that an open argumentation that is valid with respect to a certain $< S, J >$ may cease to be valid when either S or J are extended. It is not clear how tight is the relation between monotonicity and the reason for introducing extensions we are suggesting.

[19]I owe the use of the phrase 'extensions business' to Schroeder-Heister, who used it in informal discussions to refer to this issue.

of closed argumentations already recognized as such. By means of this open argumentation, new closed valid argumentations can be produced. In their terms, further open argumentations are recognized as valid, etc.

But since we have to care also of the extensions and since the notion of reduction procedure is inherently vague, it is hard to see how an open argumentation can ever be recognized as valid, since to do that, we have to consider the closed argumentations valid with respect to all possible extensions of the set of reductions (as well as of the atomic system).

Usberti (1995) expresses a radical judgment on the issue:

> 'The clause of the definition[, that] makes reference to the extensions of the set J ..., makes the whole definition highly impredicative.' (Usberti 1995, p. 75)

We leave the issue of impredicativity undecided, due to the vague character of the notion of predicativity itself. Nothing prohibits that, given a precise formulation of the notion of reduction procedure, the definition may turn out to be predicative (although surely in some loose sense, because of its high complexity). And from the consideration developed up to now, Dummett could actually achieve an account in which the semantic role of open argumentations is exhausted by that of their closed instances.[20]

Summing up, the definition of validity codifies the idea that the role of open argumentations is exhausted by that of their closed instances. In the case of predicate, the supposed and then rejected account proposed by Dummett amounted to treating predicate as denoting functions defined for every object nameable in language, i.e. for any given object. Analogously, in the case of open argumentations, the account codified by the definition of validity amounts to treating open argumentations as denoting function defined for

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[20]It must be noted that Dummett does not explicitly mention reduction procedures. Rather he uses less precise formulations, such as the following:

> An arbitrary argument is valid if we can effectively transform any supplementation
> of it into a closed canonical argument with the same final conclusion.

The vagueness of the notion of possibility, expressed by the phrase 'we can effectively transform', is essentially the same as the one of reduction procedure. In the next chapter, we will come back to the philosophical role of this notion of possibility in the economy of Dummett's picture.

every proof that can be represented by means of a valid argumentation, i.e. for any *given* proof. By now, without committing ourselves to judge the adequacy of Dummett's 'anti-realist' move, we remark that his characterization of all given proofs is unsatisfactory, as it relies on the vague notion of reduction procedure.

In addition to the argument showing that this strategy is not adequate (which will be provided in chapter 4), we want to argue that Dummett's fear of falling in a realist position is unrelated to the issue of given and arbitrary objects.

To do this, we will consider the use the intuitionists made of the notion of hypothetical construction. We claim that this notion is the intuitionistic correlate of the one of arbitrary proof. Intuitionism represents for Dummett a paradigm of anti-realism. Hence, intuitionists' acceptance of a notion structurally identical to the one of arbitrary proof suggests that Dummett's fear is unjustified.

2.7 A constructive semantics

There is a very strong correspondence between the introduction rules of the natural deduction system for intuitionistic logic **NJ**—whose rules are listed in table 2.2—and the clauses of the so-called Brouwer-Heyting-Kolmogorov (**BHK**) informal semantics for intuitionistic logic.

The latter defines the notion 'the construction c proves A' as follows:

- the construction c proves $A \land B$ iff c is of the form $< d, e >$ and d proves A and e proves B;

- the construction c proves $A \lor B$ iff c is of the form $< i, d >$ with i either 0 or 1 and if $i = 0$, then d proves A and if $i = 1$ then d proves B;

- the construction c proves $A \to B$ iff c is a general method of construction such that applied to a hypothetical construction a that proves A, $c(a)$ proves B.

In the intuitionistic perspective, the **BHK** clauses confer meaning to logical constants. On the other hand, in a verificationist perspectives, introduction

rules confer meaning to logical constants.

Table 2.2: NJ rules

$$\frac{A_1 \quad A_2}{A_1 \wedge A_2} \; I\wedge \qquad\qquad\qquad \frac{A_1 \wedge A_2}{A_i} \; E_i\wedge$$

$$\frac{A_i}{A_1 \vee A_2} \; I\vee \qquad\qquad \frac{A_1 \vee A_2 \quad \overset{[A]^n}{\underset{C}{\vdots}} \quad \overset{[B]^n}{\underset{C}{\vdots}}}{C} \; E\vee^n$$

$$\frac{\overset{[A]^n}{\underset{\overset{\vdots}{B}}{}}}{A \to B} \; I\to^n \qquad\qquad \frac{A \to B \quad A}{B} \; E\to$$

$$\frac{\bot}{C}$$

$$i = 1, \; 2$$

Concerning discharge policies, cf. note 1 on page 35.

2.7.1 Proof-theoretic semantics and intuitionistic logic

The notion of V-validity$_{<S,J>}$, in short $\vDash_{<S,J>}$, allows the definition of a notion of logical consequence in all analogous to Tarksi's. This notion of logical consequence can be formulated by claiming that B is a logical consequence of A, $A \vDash B$, iff for all atomic systems S and set of reduction J, there is a reduction j that transform every V-valid$_{<S,J>}$ closed argumentation π of conclusion A into

a V-valid $_{<S,J>}$ closed argumentation $j(\pi)$ of conclusion B:[21]

$$\forall <S,J> \; \exists j \; (\vDash_{<S,J>} \overset{\vdots\;\pi}{A} \; \Rightarrow \vDash_{<S,J>} \overset{\vdots\;j(\pi)}{A} \quad)$$

This notion of logical consequence—or, as we may call it—of universal validity, corresponds to validity in the empty atomic system and empty set of reduction, $< \varnothing, \varnothing >$. This depends on the reference to the extensions of S and J made in the clause for open argumentations of the definition of validity. An open argumentation π of conclusion B from assumption A is valid in the empty atomic system and the empty set of reduction, in short

$$\vDash_{<\varnothing,\varnothing>} \overset{\overset{A}{\vdots\;\pi}}{B}$$

iff for all atomic system $S \geq \varnothing$ and set of reductions $J \geq \varnothing$ there is a reduction procedure j that takes any valid$_{<S,J>}$ closed argumentations π' of A into a valid$_{<S,J>}$ closed argumentations $j(\pi')$ of B, that is iff

$$\forall <S,J> \; \exists j \; (\vDash_{<S,J>} \overset{\vdots\;\pi'}{A} \; \Rightarrow \vDash_{<S,J>} \overset{\vdots\;j(\pi')}{B} \quad)$$

So B is a logical consequence of A if the open argument having B as conclusion from assumption A is valid relative to $< \varnothing, \varnothing >$. As a special case, logical validity is validity in the empty atomic system.[22]

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[21] This clearly correspond to Tarski's notion of logical consequence, in that, for Tarski, B is a logical consequence of A iff for all models $\mathscr{M}$ if A is true in $\mathscr{M}$ then B is true in $\mathscr{M}$:

$$A \vDash B \text{ iff } \forall \mathscr{M} \, (\vDash_{\mathscr{M}} A \Rightarrow \vDash_{\mathscr{M}} B)$$

The difference between the two views is that proof-theoretic consequence makes reference to the procedure j. So, the couple $< S, J >$ (and not just atomic system, as we tentatively suggested in section 2.1) in the proof-theoretic setting serves the same theoretical purpose of models in Tarski. Cf. note 8 on page 38.

Schroeder-Heister's (2006, §7) characterization of logical consequence is the following:

$$\forall <S,J> \; \exists j \; (\vDash_{<S,J>} \overset{\vdots\;\pi}{A} \; \Rightarrow \vDash_{<S,J \cup j>} \overset{\overset{\vdots\;\pi}{A}}{B} \quad)$$

Our characterization is slightly stricter than Schroeder-Heister's, but has the advantage of displaying more clearly the correspondence between models $\mathscr{M}$ and couples $< S, J >$.

[22] This is the main difference between the role of couples constituted by atomic systems and set

In the light of the correspondence between **BHK** clauses and **NJ** introduc-
tion rules, one may expect that universal validity based on **NJ** introduction
rules could provide a possible semantic interpretation of intuitionistic logic.
That is, one expects that B is derivable from Γ in an intuitionistic formal sys-
tem, e.g. the natural deduction system **NJ**, iff the inference of B from Γ is V-
valid in $< \varnothing, \varnothing >$ given **NJ** introduction rules, $\Gamma \vDash_{<\varnothing,\varnothing>} B$.

In logic, completeness theorems show that all rules validated by a certain
semantic notion of logical consequence are derivable from the primitive rules
of a given formal system.

Completeness, of the natural deduction system **NJ** with respect to the proof-
theoretic notion of validity (the so-called 'Prawitz' conjecture'), risks either to
be a vague question or to be a trivial one.

As we already remarked, the definition of validity relies on the notion of re-
duction and there is no general characterization of what a reduction procedure
is. As a result, the conjecture has no clear mathematical content.

Restricting oneself to a specific set of reduction procedures, one can prove
the validity of a certain set of inference rules. Typically, given the reductions
procedures used in establishing normalization results, one can prove the valid-
ity of **NJ** elimination rules. But without further reductions, it is not possible to
show that arbitrary inferences are valid, that would make the semantics some-
what uninteresting.

A possibility could be that of enriching the basic set of reductions used in
normalization results, by allowing to replace any application of an arbitrary

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of reductions for validity and models for truth: there is no particular model such that logical truth
correspond to truth in that particular model.

inference rule with a derivation of that rule in **NJ**.[23] For example,

$$
\cfrac{\cfrac{\vdots\ \pi}{A \to (B \to C)}}{B \to (A \to C)}\ R \qquad \text{would reduce to} \qquad \cfrac{\cfrac{\cfrac{\cfrac{\vdots\ \pi}{A \to (B \to C)}\quad [A]^1}{B \to C}\ E \to \quad [B]^2}{C}\ E \to}{\cfrac{A \to C}{B \to (A \to C)}\ I \to^1}\ I \to^2
$$

In a natural deduction system, a rule is shown to be derivable by providing an open derivation having the premises of the rule as assumptions and the conclusion of the rule as conclusion. In the proof-theoretic semantic picture, if the open argumentation is valid, then one can claim that the inference rule is valid as well. But the point is that if this were the only kind of reductions available, then completeness would be trivial: the rules semantically valid would be those derivable in the formal system (provided its primitive rules are sound).

In order to make completeness an interesting question, it must be possible to 'directly' validate arbitrary inferences. But we are back to the question of how an arbitrary reduction procedure should look like.[24]

2.7.2 Constructive methods vs open argumentations

Because of the vagueness of the notion of reduction procedure, Dummett's analysis of the validity of open argumentations in terms of the validity of their closed instances was unsatisfactory.

But, as foretold, we doubt the very need of reducing the validity of open argumentations to the validity of closed ones. Dummett's reason for being afraid that a notion of validity applying to open argumentations was a concession to the realist is ill-founded. We traced back this reason to the unjustified equivalence between the need of a semantic notion distinct from the correctness of assertion and realism (cf. sections 1.4 and 1.6 in the previous chapter).

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[23]We use the term 'derivation' only for formal argumentations, i.e. argumentations produced within a specific formal system. The notion of validity can of course be also applied to derivations, although as these remarks should make clear, its significance is best appreciated in the broader context of argumentations.

[24]An exhaustive and more detailed description of these issues is given by Schroeder-Heister (2006, §5 and §7).

The analogy between the **BHK** clause for implication and the **NJ** implication introduction rule suggest to equate the intuitionistic notion of method with the proof-theoretic notion of open argumentation.[25]

In spite of the analogy, it seems that the two notions receive a sensibly different treatment in the two pictures. While Dummett tries to equate the semantic role of valid argumentations with that of their closed instances, intuitionists, on the other hand, take the notion of method as primitive. The difference is best appreciated by looking at the issue in terms of 'given' vs 'arbitrary proofs'.

2.7.2.1 Dummett and realism

Dummett explicitly ascribes (Dummett 1973a, pp. 55–58) to Frege the claim that the recognition of singular terms, as a linguistic category, has priority over the recognition of objects, as a kind of entities.

On the basis of this observation, Dummett is willing to endorse a further claim, namely that the very notion of object could be thought of as logically dependent on the one of singular term. That is, a full explanation of the notion of object could be given by claiming that objects are the denotations of singular terms. Language would come before objects. One could argue that by learning a language one learns how to pick out objects from an otherwise undetermined reality. No prior conception of objects should be required in order to account for how the mastery of a language is acquired. On the contrary, an account of how we learn to identify objects and to deal with them would be exhaustively provided once an account of the functioning of their names was given.

But due to the presence of quantifiers, this further claim has to be rejected. In Dummett's words,

> 'The notion of an object plays within Frege's semantics a twofold
> role. On the one hand, objects are the referents of proper names: the
> truth-conditions of sentences containing proper names, in particu-
> lar, of atomic sentences, are to be explained in terms of the relation
> of reference between proper names and the objects for which they
> stand. Equally, of course, objects are what predicates are true and

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[25]This is indeed the basis of the Curry-Howard isomorphism. Cf. note 15 on page 45.

false of. While we are concerned only solely with atomic sentences and combinations of these by means of the sentential operators, we need have the conception of a predicate's being true or false of an object... only for simple predicates.... It must be extended to complex predicates when we come to the second of the roles played by the notion of an object, namely the account of quantification: objects are required to compose the domains of quantifiers, that is, the ranges of the individual variables which can be bound by quantifiers.' (Dummett 1973a, p. 474)

Probably, Dummett sees the need of arbitrary objects required to account for quantifiers as a concession to the realist, in the sense that, in the notion of object, there would be something going beyond the possibility of naming them.

On the other hand, when he comes to proofs and argumentation, he seems to presuppose that all proofs must be *given* in some way, that is through some particular argumentation. So, Dummett, by equating the semantic role of open argumentation with that of their closed instances, is denying that the notion of proof characterized in terms of the notion of validity goes beyond the linguistic means we have to refer to proofs. All proofs are given in terms of valid argumentations.

2.7.2.2 Intuitionism in a nutshell

But such an argument is not compelling.

As Dummett himself remarks, making reference to arbitrary objects is not to advance a semantic thesis on singular terms, but rather on predicates. That is, it amounts to the introduction of functions, i.e. concepts, as the referents of predicates.

Looked at in this terms, Dummett's will of keeping out of the picture arbitrary objects looks like a concession to a formalistic perspective, which is alien to the intuitionistic view. What Dummett is proposing, it is to reduce the notion of function to a list, stating which values correspond to different arguments.

On the other hand, intuitionists actually do take the notion of method as primitive. Together with that, they admit the notion of arbitrary object, used in the explanation of the universal quantifier, as well as the notion of hypo-

thetical construction, whose role in the explanation of implication is analogous to the one of the notion of arbitrary object in the explanation of the universal quantifier. In listing the basic ingredients of the intuitionistic conception of arithmetic, Heyting is pretty clear.

> So far we have needed the notions of a natural number, of a hypo-
> thetical construction of a natural number and of a general method
> of construction to be applied to a hypothetical construction.
> (Heyting 1974, p. 81)

Hence in the intuitionistic perspective, a construction for $A \to B$ is a general method of construction that applied to a hypothetical construction for A yields a hypothetical constructions of B.

This would corresponds, in the proof theoretic setting, to say that a valid closed canonical argumentation of conclusion $A \to B$ is constituted by a valid open argumentation of conclusion B from assumption A, whose validity consists in its yielding an 'arbitrary' proofs of B when applied to an 'arbitrary' proof of A. On the other hand, the characterization of open validity codified in the definition of validity avoids the notion of 'arbitrary' proof, by considering only proofs 'given' through some argumentation.

In section 4.2.1 of chapter 4, we will actually give an argument showing that the notion of validity so characterized is inadequate.

But already now, by comparing Dummett's approach with the intuitionist one, it is natural to ask whether Dummett, scared of falling in a realist position, is gone too far in refusing to ascribe to the validity of open argumentation the autonomous role that in intuitionism is accorded to the notion of method.

Chapter 3

Anti-realist truth

In the first two chapters, we developed a twofold criticism of Dummett.

First, we argued that, not only in the truth-based semantic pictures, as Dummett himself points out, but also in the proof-based one, the notion of an assertion being correct is not autonomous. In order to characterize this notion for certain types of logically complex sentences, a semantic feature irreducible to the correctness of an assertion—in one case, satisfaction of open formulas by an arbitrary assignment; in the other case, a notion of validity applying to open argumentations—must be ascribed to the component expressions. Dummett's attempt to reduce the validity of open argumentations to that of its closed instances is at least vague. We will provide in chapter 4 an ultimate argument that it is also inadequate.

Secondly, we argued that Dummett wrongly identifies such need with a concession to realism: the identification is far from being compelling.

In this chapter, we will argue that, however, Dummett is not wrong in referring to this need as the introduction of some kind of notion of truth as distinct from the correctness of an assertion.

By doing this, we do not want to claim that the semantic feature going beyond assertion is ineradicably realist in character. By showing that it is plausible to refer to this semantic feature as a notion of truth, we implicitly restate Dummett and Prawitz' claim that the disagreement between realism and anti-realism does not concern the need of introducing a notion of truth in the

meaning-theoretical framework, but rather it has to do with how the notion of truth is to be conceived.[1]

In order to achieve an at-least-consistent account, we will have to move apart from Dummett's own way of stating the issues. For, Dummett's idea, that the need of a notion of truth commits to realism, pushes him towards conceptual choices that fall short of being consistent with the views he himself advocates. As this misconceived idea has been dispelled, we are in the position of giving a more transparent picture.

In this chapter, we will first present a new argument for introducing a notion of truth, as an essential ingredient of the solution to the so-called paradox of deduction.

Then we will state the conditions for a sentence to be (respectively) correctly assertible and true, in terms of the proof-theoretic notions developed in the previous chapter.

Finally, we will show in which sense the notion of truth, discussed in relation to the paradox of deduction, relates to the semantic feature distinct from assertion discussed in the previous chapter.

3.1   The paradox of deduction

According to Dummett,

> 'The existence of deductive inference is problematic because of the tension between what seems necessary to account for its legitimacy and what seems necessary to account for its usefulness. For it to be legitimate, the process of recognizing the premisses as true must already have accomplished whatever is needed for the recognition of the truth of the conclusion; for it to be useful, a recognition of its truth need not actually have been accorded to the conclusion when it was accorded to the premisses.' (Dummett 1973b, p. 297)

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[1]See, for instance, Prawitz (1987, §4.4).

The legitimacy Dummett is claiming for is nothing but validity. Given Tarski's characterization of the validity of an inference[2]:

(∗) A deductive inference is valid if it preserves the truth from the premises to the conclusion.

then Dummett's problem with deduction can be rephrased as a tension between the validity of an inference and its usefulness, that is between the fact that truth is transmitted from the premises to the conclusion and that the recognition of the truth of the conclusion is not yet achieved when the truth of the premises is recognized.

Realists take (∗) as a definition of inference validity in terms of truth. That is, truth is an independently defined notion to which inference validity is to be reduced. Their problem is that of giving a sound account of what is truth recognition.

Anti-realists of the proof-theoretic tradition obviously reject (∗) as a definition of validity, the latter being defined independently of truth (cf. in chapter 2 definition 8 on page 37). Hence, they must look for a different account of the paradox.

As an anticipation, we will argue that for anti-realists, (∗) is not to be rejected but, rather, it can be taken as an adequacy condition to be imposed on the anti-realist notion of truth. Actually, we will show that for anti-realists, truth is to be defined in terms validity.

3.2 Two unsatisfactory solutions

As we saw in chapter 2, according to anti-realists, the meaning of sentences is fixed by certain kinds of inferences. Previously, we took the idea of inferences fixing the meaning of sentences as being both clear and unquestionable. Now, we will first try to give a philosophical explanation of what it is meant by it. As a result, we will state the need of distinguishing between different types of inferences, those that do 'fix the meaning' and those that do not. Finally, we see

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[2]The validity of an inference, in the truth-theoretic approach, amounts to the conclusion being a logical consequence of the premises. Cf. in chapter 2 note 21 on page 21.

how this need is reflected in the architecture of the proof-theoretic semantics presented in the previous chapter.

3.2.1   Inferences as definitions

By saying that inferences fix the meaning of the logical operators, we suggest that the following is to be understood:

(MFI) Whenever a competent speaker accepts the premises as true she will also accept the conclusion as true, when presented with it.

We can think of the content of **(MFI)** as the proof-theoretic counterpart of what happens in the truth-theoretic case when the meaning of a given expression is given by means of a definition. Consider the case of 'bachelor' being defined as 'not married': if a speaker knows the meaning of 'bachelor' than it is not possible that she assents to the sentence:

> Luca is not married.

and not to

> Luca is a bachelor.

That is, it is not possible that a competent speaker recognizes the truth of the first sentence without recognizing the truth of the second one, when presented with it.

Clearly, the definition can be taken as warranting inferences from sentences of the first kind to sentences of the second kind. Hence, the inferences that are taken to fix the meaning of a logical operators will be acknowledged as valid by definition. For, if speakers understanding of the meaning of a logical operator consists in the mastery of some deductive inferences, it is not possible that speakers know the meaning of the operators without accepting these inferences as valid.[3]

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[3]We take, as Dummett does, the development of a theory of meaning as a highly theoretical enterprise. Of course this does not mean that the specification of meanings is an arbitrary choice. For, any specification has to satisfy several constraints such as articulation, molecularity, compositionality, manifestability and so on. Nonetheless, it is possible in principle that different theories of meaning satisfy all such requirements. Hence the claim that a given inference in valid by definition

So, for this kind of inferences, we have that a speaker cannot recognize the truth of the premise without also recognizing the truth of the conclusion.

But at this point we face the problem stressed by Dummett: if all inferences were needed to fix the meaning of logical operators, then all inferences would be such that the whenever the truth of the premises is recognized so is the truth of the conclusion. In other words, no inference would be useful. This seems to be actually the traditional way—Dummett (1973b, 1991) ascribes it to J. S. Mill—of accounting for the validity of deductive inference, but at the price of treating it as *petitio principii*.

Hence, in order to warrant the usefulness of deductive inference we have to allow for inferences which are valid even if they do not fix the meaning of the sentence.

3.2.2   A Wittgensteinian perspective

But this seems to be no easy task. If in justifying validity one risks to repudiate the aspect of deductive inference making it fruitful, it is easy to fall in the opposite error: once warranted usefulness, being incapable of accounting for inference validity. According to Dummett, Wittgenstein comes close to this, when he holds that in accepting a new proof of a statement, we are modifying its meaning.

To clarify the point, one can consider the proof that a cylinder intersects a plane in an ellipse. The proof is an example of what is meant by fruitfulness of deduction: it provides a new criterion for recognizing something as an ellipse. Now we turn to the other aspect of deduction and we ask on which basis the proof is to be accepted as valid. According to Wittgenstein, there is no further jury, being entitled to settle the matter, beyond the linguistic (and in this case mathematical) community. That is, the decision, of accepting a given proof as valid or not, is a matter of agreement in the community and there is no base on which social practice can be criticized. The idea of feeling the correctness of a proof to be imposed on us is, accordingly, a misconceived illusion. In

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is not an empirical statement that can be proved or rejected by, say, asking speakers. Rather, it is a theoretical statements: it is possible that an inference is valid by definition relatively to a given theory of meaning but not relatively to another one.

particular, it is not on the basis of meaning specifications that we acknowledge some inferential procedures as valid. On the contrary, it is the acceptance of a given set of inferential procedures that gives meaning to sentences. And as we accept new inferences and start using them, meanings change.

But, Dummett contests, are we sure that accepting new proofs is always a modification of meanings? Considering our example, are we sure that

> 'the adoption of the new criterion for its application modifies the meaning that we attach to the predicate 'ellipse'[?] To speak of our accepting something new as a ground for applying a predicate as a modification of its meaning would not be, in itself, to go beyond what is banal, save in the use of the world 'meaning': to give substance to the thesis, we have to construe the modification as consisting, not merely in our acceptance of the new criterion, but in the possibility of its yielding a a different extension for the predicate from that yielded by the old criteria.' (Dummett 1973b, pp. 300–301)

Dummett's remark is crucial to fully grasp the significance of Wittgenstein's position and at the same time to see in which direction an alternative solution can be found in order to preserve both aspects of deduction.

In fact, the following situation is envisaged. Suppose some means to establish sentences are given. Then, when we face a new inference, two possibilities are open.

Either the new inference allows us to establish sentences in cases in which, by using only the inferences previously available, it was not. In this case, the community's acceptance of the new inference would constitute a modification of the meaning of the sentences.

Or the new inference allows to formulate new criteria, for establishing sentences, which are equivalent (or possibly simply faithful) to the previous ones. In this case it seems more natural to claim that no meaning modification takes place in accepting the inference. As we will see, Dummett construes his own position as grounded on the idea that it is the very recognition of this fact (the faithfulness to the previously established practices) that prompts the community to smoothly accept the new inference.

In a sense, Wittgenstein's position (as Dummett reconstruct it) amounts to the claim that, in general, the behavior of the linguistic community, when it comes to the decision of accepting or not some new inference form, is so motley, to make it senseless to ask for some general criteria (like the one suggested) to which it should conform. In Dummett's words, for Wittgenstein:

> 'We speak as we want to speak, and our practice, in respect to the whole of our language, determines the meaning of each sentence belonging to it... It is not, therefore, that there is something which must hold good of deductive inference, if it has to be justified, but which, because we should thereby be trapped in a vicious circle, we are unable to demonstrate, but must simply assume: rather, there is no condition whatever which a form of inference can be required to satisfy, and therefore nothing to be shown.' (Dummett 1973b, p. 304)

3.3 From Holism to Molecularity

According to Dummett's presentation, Wittgenstein's position resorts to an holistic conception of meaning. The idea of an holistic relationship between meaning and inference is presented by Dummett as follows:

> 'the meaning of an individual sentence is characterized by the totality of all possible ways that exist within language for establishing its truth, including ones which involve deductive inference; we therefore cannot fully explain the meaning of an individual sentence without giving an account of the entire language of which it forms part, and in particular, of all type of inference which might lead to it as conclusion.' (Dummett 1973b, p.302)

3.3.1 Static and dynamic holism

In the wittgensteinian perspective suggested, if we look at how the behavior of the community evolves in time, we see that the community constantly revises the set of accepted inference rules, by rejecting some of them and accepting

new ones. From this perspective, we have that meanings change progressively according to the development of the linguistic practice. The fruitfulness of inferential practice consists in the evolution of the knowledge heritage of the community, but there is no stable notion of validity admissible in such a view. As the set of accepted inferences constantly changes, so does the extension of the concept of validity.

Even if Dummett does not explicitly claim it, this conception of the relationship between meaning and deduction is strongly connected to the opposing view according to which deduction is nothing but *petitio principii*.

In fact, if we imagine to take a picture of the linguistic community at a given moment, we can describe it in the following terms. A given set of inferences is accepted by the community. The mastery of all these inferences (together with some other practical abilities) constitutes speakers' linguistic competence. In other words, all inferences accepted by the community, at the very instant in which the picture is taken, contribute to fix the meaning expressions have at that very instant. Consider now a given competent speaker, that is a speaker that knows the meaning of all expressions. He will master each of the inferences accepted in the community. And as these are part of her competence, that is, they are constitutive of the meaning of the expressions of the language, they count as meaning-fixing ones.

Hence, if we 'freeze' the evolution flow of the linguistic community, we can describe the frozen picture by saying that all inferences globally fix the meaning of all sentences. That is, all inferences are meaning-fixing. The claim that all inferences are meaning-fixing is exactly the view we analyzed in section 3.2.1. In accordance with the account given there, the application of some inference rule on the part of some member of the community would be trivial, in so far as each speaker acknowledges the inference in such a way that she cannot recognize the truth of the premises without in doing so recognizing the truth of the conclusion as well.

Then, this global way in which all inferences simultaneously fix all meanings accords perfectly with the holistic conception of the relationship between meaning and inference suggested by Dummett. So, both views we considered so far, and not only Wittgenstein's, rely on a form of holism.

At first, the resulting descriptions seemed to be antithetic. In fact, each of the two rejects, respectively, one of the two aspects of deduction: either fruitfulness, as all inferences are trivially recognized as valid by speakers; or (a stable notion of) validity, as there are no objective grounds for deciding whether a new inference should be accepted or not.

Nonetheless, the two pictures fit together quite well, in the sense that they are the two faces of the same coin. For at each instant of time, the community accepts a given set of inferences, that fix the meaning of each expression. By simply considering that moment, all inferences, being meaning-fixing, appear to be deprived of their usefulness. On the other hand, as time passes, the set of accepted inferences changes and so do meanings. Fruitfulness is essentially constituted by the resulting continuous change in the acceptance and rejection of certain forms of inference. Hence, the two views can be respectively presented as the static and the dynamic way of describing an holistic conception of the relationship between meaning and inference.

An hopefully illuminating comparison is the following: we can imagine the behavior of a linguistic community evolving in time as printed on a film. If we consider the film in its entirety, Wittgenstein's description applies: the behavior of the community is so motley that there is—apparently—no way of characterizing the acceptance of new inferences by means of general rules, hence the possibility of a notion of validity is lost. On the other hand, if we look at a single frame of the film, we can describe it in terms of the opposing view: the practices photographed fix globally the meanings at that moment and hence all inferences are valid by definition and consequently deprived of their usefulness.

An advocate of holism may argue that, by claiming as we did that the two positions criticized by Dummett are just the two faces of holism, we have actually shown that holism can account for both aspects of deductive inference. Nonetheless, there are still reasons for being unsatisfied. In particular, the notions of validity and usefulness so characterized are definitely at odd with our intuitions. On the one hand, no general notion of validity of an inference rule is available: the most that can be said is that the validity of an inference in a given frame, that is at a given instance of time, merely amounts to the fact that

the inference is accepted at that time. On the other hand, by just looking at a
single frame, that is at single instance, no inference is fruitful: the most that can
be said is that an inference is fruitful if it is accepted in one frame and rejected
in a subsequent one (or *viceversa*), that is fruitfulness is equated to the factual
possibility of the community coming to reject previously established practices.

3.3.2   Molecularism: the golden mean

According to Dummett, it is only when holism is rejected that the possibility of
a new picture, in which both aspects can be properly accounted for, appears.

   This emerges as soon as we reconsider the analysis Dummett gives of the
geometrical theorem we quoted above. As Dummett remarks, it is not always
the case that by accepting a new inference (in this case the proof of the theorem)
we modify the meaning of the expressions of our language. Namely, when the
new inference provides criteria for establishing sentences which are faithful to
those already accepted. That is, whenever a sentence established by means of
the new inference could have already been established without it, or in other
words when the set of inferences obtained by accepting the new one constitutes
a conservative extension of the previous set.

   For Dummett this amounts to the possibility of a

> 'molecular conception of language under which each sentence pos-
> sesses an individual content which may be grasped without a knowl-
> edge of the entire language. Such a conception requires that we can
> imagine each sentence as retaining its content, as being used in ex-
> actly the same way as we now use it, even when belonging to some
> extremely fragmentary language, containing only the expressions
> which occur in it and others, of the same or of lower complexity,
> whose understanding is necessary to the understanding of these
> expressions: in such a fragmentary language, sentences of greater
> logical complexity than the given one would not occur. Our lan-
> guage would then be a conservative extension of the fragmentary
> language: we could not establish, by its use, any sentence of the
> fragmentary language which could not already be established in

that fragmentary language. The rules of inference which are applied in our language are, on such molecular view, justified precisely by this fact, the fact, namely, that they remain faithful to the individual contents of the sentences which occur in any deduction carried out in accordance with them.' (Dummett 1973b, pp. 302–303)

This suggests the idea that not all inferences are actually needed to specify the meaning of sentences, but only a subset of them; these inferences will be plausibly claimed to be valid by definition. Other inferences will be said to be valid in virtue of their being faithful to them. On the basis of the possibility of finding inferences of this latter kind, deduction can be said to be fruitful: the enrichment of the set of inferential practices accepted by a linguistic community gives rise to new criteria for accepting sentences as true.

If we consider Dummett's proposal in terms of the 'film' image suggested in the previous subsection, molecularism can be seen as a more fine-grained level of analysis.

As we saw, the two conceptions of meaning described in section 3.2 amount to an analysis of the community practices either frame by frame, or as a continuum not further analyzable. In both cases, the meaning of each sentence depends on all possible ways of establishing it and as a result the account of the meaning of a single sentence depends on the one of the whole language in which arguments for it can be given. From the static perspective, because all inferences fix simultaneously all meanings; from the dynamic one, because the acceptance of new inferences in no way can be said to be faithful to the previously established practices.

Dummett is willing to challenge the dramatic consequence this latter claim has. In fact, Dummett is ready to accept that *sometimes* the acceptance of new inferential procedures constitutes a modification of the meanings. But, according to him, this does not happen *always*. In particular, most times speakers come to accept new inferences exactly because their introduction yields an extension of the practices which is faithful to the previously accepted ones.

That is, in some cases the acceptance of new inferences will yield non-conservative extensions of the existing practices. And in cases such as these,

a modification in the meaning of sentences will be acknowledged. In particular, the inferences in question will be constitutive of the 'new' meanings of the sentences that could not have been previously established. Nonetheless, during the time in which no such inferences are admitted, meanings are stable and the possibility of accounting for both validity and fruitfulness of deductive inference is open.

In fact, if speakers accept a new inference only if it yields a conservative extension of the practices, it is natural to claim that meanings do not change with the acceptance of the new inference. Furthermore, the new inference is accepted exactly *because* it is faithful to the meaning of the expressions as previously established: it is in this sense that the inference can be said to be valid. Finally, the possibility of coming to accepting new inferences as valid in this way constitutes the fruitfulness of deduction: the crucial difference between holism and molecularism is the following. In a molecular conception, a notion of fruitfulness is available which is different from the one simply consisting in revising the existing practices. And so is a notion of validity of an inference different from the mere acceptance of an inference at a given time.

So, molecularism can be seen as a third way between the 'frame by frame' and the 'all in one' account of the community history. In fact, Dummett can be taken as claiming that we can think of the film as partitioned into strips. Such strips can be characterized as follows: during each one, the newly admitted inferential procedures are only those that yield a conservative extension of those accepted at the beginning of the strip. Whenever the community comes to accept an inference violating the requirement of conservative extension, we can think of a new strip of the film beginning, in which expressions have different meanings from those they had in the previous strip.[4]

If this reconstruction is faithful, then Dummett's picture can be compared

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[4]This way of presenting the matter suggests the idea that in a language, the meanings of the expressions can change in time. Alternatively, if one is willing to claim that the notion of language should not admit meaning changes, than one can rephrase the whole picture by: a) reconstructing Wittgenstein as claiming that, at each frame, not only meanings, but also the very language spoken by the community changes; b) characterizing molecularism as the claim that the community does speak the same language in different (adjacent) frames belonging to the same strip and that whenever a new strip begins, the community starts speaking a new language.

with Kuhn's (1962) conception of scientific progress. During normal periods (corresponding to the film strips) the settled paradigm acts as giving shared standards of correctness (corresponding to meanings and the consequent notion of validity), that due to a revolution (the change from one strip to another) may change. Obviously, epistemic advance is not only due to changes of paradigm (to our changes in meanings), but takes place within paradigms as well. In particular, it is only within a paradigm that we have a shared notion of objective correctness of the practices, just like for Dummett it is only when meanings are not constantly changing that we have an objective criterion to judge the acceptability of new forms of inferences the community may encounter.[5]

3.4 A proof-theoretic solution to the paradox

In the reasoning followed, an implicit connection was acknowledge by Dummett between inferences and 'means of establishing sentences'.

The connection is so tight that, according to Dummett,

> 'Another way of expressing the perplexity to which the existence of deductive inference gives rise is by asking how it can come about that we have an *indirect* means for recognizing the truth of a statement. (Dummett 1973b, pp. 297–298)

where the indirect means of recognizing the truth of sentences are opposed the direct ones. These are so characterized:

> 'The direct means of verifying the statement is that which corresponds, step by step, with the internal structure of the statements, in accordance with that model of meaning for the statements and its constituents expressions which is being employed.' (Dummett 1973b, p. 312)

The opposition between direct and indirect means of establishing sentences, or between direct and indirect evidence, is construed by Dummett as sort of

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[5]In section 3.9 we will give some hints on how the comparison with Kuhn's picture can be pushed further.

generalization of the opposition between meaning-fixing and non-meaning fix-
ing inferences.

The connection between inferences, a sub-set of which fixes the meaning of
sentences, and the means of recognizing sentences as true is not so immediate.
In particular, a sentence is not recognized as true through a single inference.
In general, several inferences may be required for recognizing the truth of a
sentence. Furthermore, not only inferences may be involved in the process of
recognition.

So the notion of evidence is a generalization of the notion of inference in
two sense: both because evidence may be constituted by several inferential
steps and because it may also be constituted by non-inferential processes.

3.4.1   Direct evidence and objects: a Fregean point

But what is to be meant by direct evidence? The term direct, in relation to the
means of establishing sentences was used by the logical positivists of Vienna's
circle to frame a condition of meaningfulness on sentences, the so-called prin-
ciple of verification: 'Only sentences capable of being verified or falsified by
direct observation are meaningful.'

As a result, they had to face a problem with sentences, such as, for instances,
those expressing logical principles. Being not verifiable nor falsifiable by ob-
servation, they should be counted as meaningless. A 'special status' has to be
accorded to all such such sentences, in order to make them meaningful in spite
of their being not directly verifiable.

The source of the problem is, from Dummett's perspective, the error of at-
taching the adjective 'direct' to a particular mode of verifying sentences, con-
ceived as privileged in some sense. According to logical positivists, one has
to start from sense-data to build up evidence for sentences, sense-data being
what subjects have the most direct epistemic access to (in the sense of Rus-
sell's knowledge by acquaintance). According to Dummett, this way of char-
acterizing direct evidence is completely misconceived. For, it is not the case
that certain types of epistemic recognition capacity—perception, rather than,
say, religious illumination—are 'better' than others; and that the 'best' among
these should be labeled 'direct' and then used in order to specify a criterion

of significance of sentences, in such a way that if a sentence is neither verified nor falsified by means of the selected recognition capacity, then it is meaningless. Rather, different types of sentences are associated with different kinds of practices of recognition. And certain means of establishing a sentence are more direct than others, not because they rely on a certain human faculty rather than another, but because it is in their terms that the very meaning of the sentence is specified.

For Dummett,

> 'it is this insight which is one of the great contributions of Quine's celebrated essay "Two Dogmas of empiricism", and is there expressed by means of the image of language as an articulated structure of interconnected sentences, upon which experience impinges only at the periphery. The impact of experience may have the eventual effect of inducing us to assign (new) truth-values to sentences in the interior structure: but this impact will be mediated by truth-value assignments to other sentences which lie upon a path from the periphery, where the impact is initially felt, to the more centrally located sentences. This metaphor presumably represent the entirely correct conception that, save for the peripheral sentence, the process of establishing a statement as true does not consist in a sequence of bare sense-perceptions, as on the logical-positivist model of the process of verification, but in the drawing of inferences (which need not, of course, all be strictly deductive) whose ultimate premises will be based on observation. It is inherent in the meaning of such sentences as 'The earth goes round the sun' or 'Plague is transmitted by rats' that it cannot be used as a direct report of observation (and thus is not, in Quine's image, located at the periphery of the linguistic structure), but it can be established only on the basis of reasoning which takes its departure from what can can be directly observed. In extremes cases, for instances a numerical equation or a statement of the validity of a a schema of first-order predicate logic, it is intrinsic to the meaning of the statement that it is to be be established by purely linguistic operations, with-

out appeal to observation at all (save the minimum necessary for
the manipulation of the symbols themselves).' (Dummett 1973b, p.
298)

This interpretation of Quine makes Dummett's verificationism radically
distinct from the logical positivist one. The theoretical move made by Dum-
mett is analogous to one made by Frege. Namely, it is the proof-theoretic coun-
terpart of Frege's views on names and objects. In Frege's picture, in which
objects are (primarily) the linguistic correlates of names, the traditional prob-
lems arising from the distinction between particular and universal disappears.
According to Dummett, in Frege's terms this distinction becomes the one be-
tween concrete and abstract objects. This distinction reflects the several prac-
tices that govern the use of singular terms belonging to distinct types. For in-
stance, distinct kinds of singular terms will be associated with different criteria
of identity—consider names of persons as opposed to, say, names of rivers—
or with different systematic polysemous traits—consider names of rivers as
opposed to names of books. The range of possible ways in which we name ob-
jects with singular terms is a *continuum*, at one end of which we find ostension,
by means of which the 'most' concrete objects can be referred to. In general, the
means by which we refer to other kinds of objects with singular terms may be
not even loosely analogous to the practice of associating a name with an object
by means of ostension. But this does not deprive the objects named of their
status of objects.

Analogously, only in extreme cases the most direct means of establishing
a sentence will merely consists in certain collections of sense-data. In general,
the most direct means of establishing a sentence, even atomic sentences, will
consists in 'reasoning which takes its departure from what can be directly ob-
served'. But this does not make this kind evidence any less 'direct'. The range
of possible ways in which different kinds of sentences are directly established
represent a *continuum* analogous to the one of names.

In the case of singular terms, in Frege's thought there is no room for the
issue of looking for 'logically proper' names to be distinguished by spurious
names; so, in the case of evidence, in Dummett there is no issue of identifying
types of evidence with which we are directly acquainted, that should be then

taken as the base of a criterion of meaningfulness.

3.4.2 Direct and indirect evidence

According to Dummett, a general characterization of direct evidence is only possible in terms of Quine's net: what is common to the direct means of establishing sentences is that they proceed from the periphery toward the interior of the net, in accordance with the meaning of sentences, being

> 'determined by the links between it and other statements adjacent to it in the direction of the periphery, and their meanings in turn by the links that connect them with further sentences yet closer to the periphery, and so on until we reach the observation statements which lie at the periphery itself.' (Dummett 1973b, p. 299)

The distinctive feature of indirect evidence is also given in terms of Quine's net:

> 'it at least appears that chains of deductive reasoning occur which involve, either as premisses or as steps in the proof, statements which lie deeper in the interior than does the conclusion of the argument; even that the conclusion may, on occasion, be a peripheral statement. In any such case, the conclusion of the deductive argument is being established indirectly, that is by a process our understanding of which is not immediately involved in our grasp of the meaning of the statement.' (Dummett 1973b, p. 299)

The definition of validity of argumentations presented in the previous chapter may be viewed as yielding a model for evidence faithful to these characterizations.

In particular, valid closed canonical argumentations represent the essential features of direct evidence, being structured in accordance with the meaning of their conclusion.

In the case of logically complex sentences, they end with an introduction rule. In this way, the connection between direct ways of establishing sentences and meaning-fixing inferences is explained.

Closed non-canonical argumentations can be viewed as a representation of indirect evidence. Since their last step is not an introduction rule, it may happen that the the premises of the last rule applied are of higher logical complexity than the conclusion of the argumentation. Hence, non-canonical argumentations reflects the characteristic feature of indirect evidence, 'its involving sentences that lie deeper in the interior than does the conclusion'.

This characterization of the relationship between direct and indirect evidence as modeled on the couple canonical/non-canonical argumentations also yields a characterization of the way in which the validity of non-meaning-fixing inference is to be understood. In section 3.3 we said that, a 'new' inference can be accepted as valid if its acceptance yields a conservative extension of the inferential practices, that is iff sentences established by means of it, could have been established without. But this comes very close to the requirement according to which the inferential rules must be shaped so that valid closed non-canonical argumentations for sentences reduces to valid closed canonical ones.[6]

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[6]This seems to be a third characterization of harmony (cf. note 12 on page 40). The requirement of conservative extension suggested here is not that the deductive system resulting by the introduction of, say, a new connective $*$ should be a conservative extension of the $*$-free fragment. Rather, the criterion envisaged here is that the system 'containing' both the direct and indirect means of establishing sentences should be a conservative extension of the one containing only what is necessary to directly establishing sentences. Tennant's (1987, ch. 10) refers to the two formulations of the criteria of conservative extension as (respectively) the 'Burgess-Grandy' and the 'Prawitz' interpretation. It is important to note that the 'Prawitz' interpretation is however stronger than the (subsequently formulated) version of harmony as normalization, late alone too much strong. According to Tennant, the 'Prawitz' interpretation, when restricted to logical constants, amounts to the deductive system **NJ** (consisting of both introduction and elimination rules, cf. table 2.2 on page 56) being a conservative extension of the one containing only introduction rules. But this criterion is violated. This is, of course, due to implication. As an example Tennant (1987, ch. 10) gives the following:

$$\frac{\dfrac{[A \wedge B]^1}{A}\ E\wedge}{(A \wedge B) \to A}\ I\to^1$$

The conclusion of the derivation could not be derived if the deductive system did not contained conjunction elimination $E\wedge$. We do commit ourselves neither to an evaluation of which is the real criterion of harmony Dummett has in mind; nor of which is the right criterion that allows to account for all the different meaning-theoretical features depending on it; nor of whether the two 'Dummett's' conception and the 'right' conception match. Cf. also note 9 on page 83.)

The notion of possibility expressed by the phrase 'could have been established' is then analyzed in terms of the notion of reduction procedure.[7] So, to say that the sentence could have been established by direct means is interpreted as the possession of a procedure, effective in principle, which transforms the indirect evidence for the sentence into the direct one. In a sense, indirect evidence is itself (or very naturally suggests) the method, as the example of the theorem on ellipses suggests. In that case, the theorem consists in a method for showing that any possible figure obtained in the specified way satisfies the defining equation of ellipses.

3.5 Truth and its recognition

3.5.1 Truth *versus* truth-recognition preservation

As Dummett remarks,

> for there to have been an epistemic advance, it is essential that the recognition of the truth of the premise did not involve an explicit recognition of that of the conclusion' (Dummett 1973b, p. 313)

It is exactly because the recognition of the truth of their premises *does* involve the recognition of the truth of their conclusion that meaning-fixing inference do not yield epistemic advance.

We can contrast the role played by truth-recognition in meaning-fixing inference with the feature by means of which valid inferences are usually characterized:

> '[to say that] the rules of inferences we ordinarily employ are in fact valid [is to say] that they are justified in the sense that truth is preserved as we pass from the premises to conclusion.' (Dummett 1973b, p. 311)

At this point, it should appear at least reasonable to characterize the two features of deduction in terms of the couple constituted by the two notions: truth and truth-recognition.

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[7]Cf. footnote 20 in previous chapter, on page 54.

Valid inferences are those that preserve the truth in passing from premises to conclusion. This is to be understood as follow. Whenever one is in possession of (direct or indirect) evidence for the premises, she is in possession of a method to obtain direct evidence for the conclusion. The characterization applies both to meaning-fixing inferences and to not meaning-fixing ones.

For a non-meaning fixing inference, given evidence for its premise one is in possession of indirect evidence for its conclusion, exactly because the conclusion is established by means of a non meaning-fixing inference. And, as we saw, indirect evidence is (or immediately yields) a method to obtain direct evidence.

For meaning-fixing inferences, to be in possession of direct evidence for the premises of a meaning-fixing inference *is* already to be in possession of direct evidence for the conclusion, the meaning of the conclusion being specified exactly through the inference. Obviously, as indirect evidence is defined as a method to obtain direct evidence, then direct evidence can itself be seen as a very special kind of indirect evidence: the method to recover direct evidence from it is very simple, just doing nothing.[8] That is, meaning-fixing inferences preserve truth as well.

In proof-theoretic terms, valid inferences preserve truth, in the sense that whenever one is in possession of evidence (of any kind) for the premises, she is in possession of evidence for the conclusion, even though not necessarily of direct kind. That is, in both case the result of plugging closed valid argumentations for the premises yields a valid closed (though not necessarily canonical) argumentation for the conclusion.

The distinctive feature of meaning-fixing inferences is, we will say, that they 'preserve' truth-recognition in passing from premises to conclusion. That is, given evidence for their premises one is possession of direct evidence for its conclusion. A non-meaning fixing inference does not preserve truth-recognition because even when one is in possession of direct evidence for the premises, she only has indirect evidence for the conclusion.

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[8]In terms of validity of argumentations, a valid closed canonical argumentation may be viewed as a valid closed non-canonical argumentation. The reduction procedure that takes it into a canonical argumentation is just the identity function.

In proof-theoretic terms, consider a valid non-introduction inference. The result of plugging valid closed canonical argumentations for its premises, does not yield a valid closed canonical argumentation for its conclusion, but simply a valid closed argumentation. On the other hand, taken an introduction rule, the result of plugging valid closed canonical argumentations for its premises yields a valid closed canonical argumentation of its conclusion.

Hence, both meaning-fixing and non-meaning-fixing valid inferences preserve truth. So, the difference between the two kinds of inferences is whether they preserve truth-recognition or not. This is actually in line with the intuition that meaning-fixing inferences, being valid, must share some property with non-meaning fixing valid ones.[9]

In the light of this, it should now appear quite natural to say that valid inferences preserve truth from an anti-realist standpoint as well (keeping in mind that by saying this we are not reducing validity to truth-preservation but the other way around). So we take conditional (∗) as justified. In the remaining of the paper, proceeding from the notion of truth-preservation, we try to access the very notion of truth.

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[9]We did not consider what happens when evidence for the premises of a meaning-fixing inference is of indirect kind, that is when, say, closed non-canonical argumentations for the premises are plugged on top of an introduction rule . If we take seriously the second quotation of section 3.4, on page 75, then one should maintain that direct evidence—i.e. canonical argumentations—is the one constituted *only* by meaning-fixing inferences. Hence, if one has only indirect evidence—non-canonical argumentations—for the premises of a meaning-giving inference, she should not be said to be in possession of direct evidence—of a canonical argumentation—for the conclusion. On the possibility of strengthening the notion of canonicity in this sense, cf. footnote 16 below. Note that, although the notion of canonical argumentation may be restricted so to avoid the presence of reduction segments from it, it is impossible to forbid the application of elimination rules from canonical argumentations. This is due to implication (cf. note 6 on page 80 and the derivation therein). In general, the most direct way of establishing sentences may contain applications of elimination rules. In other words, the presence of implication entangles the meaning-fixing inferences with the others in an inextricable way. For a detailed comment on most Dummett early formulations of harmony, see Tennant (1987, ch. 10); for Dummett's (1991) later formulations, see Read (2000, §§1.3–1.4). Cf. also note 12 on page 40.

3.5.2   An exegetical remark

In the formulation of the solution to the paradox of deduction we gave, the point that seems strikingly counter-intuitive (beside the bad-sounding phrase 'truth-recognition preservation') is that not all valid inferences are such that the truth of the conclusion is recognized when the one of the premise is. Nonetheless it should be clear enough that in the light of the presentation of the matter, this seems the more sound way of expressing the point.

Dummett himself avoids such a rude terminology, but at the price of inducing some confusion. In fact, we can reconsider the passage quoted in the opening of section 3.5.1. Dummett is considering the proof that whoever passes through all Königsberg bridges must cross one at least twice.

> 'For there to have been epistemic advance, it is essential that the recognition of the truth of the premisses did not involve an explicit recognition of that of the conclusion . . . For the demonstration to be cogent, on the other hand, it is necessary that the passages from step to step involve a recognition of *truth* at each line. [But] this recognition of truth . . . cannot *constitute* the truth of the statement so recognized: it must be the recognition of a property which is in accordance with the content of the sentence as given by the preferred model of meaning. It is quite different with a direct demonstration. The truth of a conjunction, for instance, simply consists in the truth of the premises from which it was inferred by and-introduction [i.e. by an inference from the conjuncts to the conjunction], and so the recognition that it is true is not the recognition of a property which it had independently of the possibility of inferring it in that way.' (Dummett 1973b, p. 313)

These claims are almost the opposite of what we are arguing for. From this passage one may extract that valid inferences are those that involve the recognition of the truth in passing from premises to conclusion, and that meaning-fixing inferences are those that constitute the truth of the conclusion. Hence, here it seems that Dummett is construing the difference between the two kinds of inferences not in terms of the couple truth-recognition/truth. On the contrary, it

rather seems that the difference is to be characterized as truth-recognition/truth-constitution.

The tension between this passage and the rest of the paper is surprising since, as we saw, the notion of truth-recognition has been connected throughout Dummett's reasoning with meaning-fixing inferences and not to simply valid ones.

Indubitably, there is an evident tension between saying that, on the one hand, epistemic advance requires that the recognition of the truth of the premise does not involve the one of the conclusion; and, on the other hand, for the sake of validity, 'the passages from step to step involve a recognition of truth at each line'.

As Dummett is speaking of a whole demonstration, one may argue (and actually Dummett seems to rely on the intuition that) in all kinds of single step inferences truth-recognition is transmitted from premises to the conclusion, but in global argumentations truth-recognition is not transmitted from the assumptions to the conclusion. This point is quite interesting and could be argued for, but definitely not in this context, in the light of the property of validity according to which an open (whatever complex) argument from assumptions Γ and conclusion A receives the same treatment as the corresponding one step inferences from Γ to A: both are valid if one can is in possession of a method (i.e. a reduction procedure) that transforms every valid closed canonical argumentation for the premises/assumptions into valid closed canonical argumentations for the conclusion. Hence, if Dummett is willing to argue in this direction, he has to deeply rethink the whole presentation of the matter.

Furthermore, Dummett introduces here the notion of 'truth constitution' that should characterize meaning-fixing inferences, but he does not give any sort of hint on how is this phrase to be intended.

3.6 Dummett on truth and assertion

A theory of meaning, as Dummett (1976) conceives it, is structured in two parts (cf. section 1.3 of chapter 1). Its core is a semantic theory, which consists in an inductive definition of a notion which is taken to be the central one of the

theory of meaning. The other part is the theory of force, which gives an explication of the way in which sentences are used in different ways, that is, the way in which their utterances may constitute different speech acts (assertions, commands, questions and so on).

The theory of force and the semantic theory are connected by the link between the central notion of the theory with the practice of assertion. Assertion is (usually) taken as the most important speech act, in terms of which all others have to be explained. The central notion defined by the semantics, in turn is used to explain the practice of assertion.

We took this picture as applying also to an anti-realist theory of meaning based on the proof-theoretic semantics presented in chapter 2. In particular, there we argued that the correctness of an assertion is to be identified with the possession of closed valid argumentations. In order to get a proper account of this notion, a further notion of validity, applying to open argumentations was needed.

We will now argue that this way of presenting the architecture of the anti-realist theory of meaning clashes with the solution to the paradox of deduction we just gave. As a result we will propose (in section 3.7) a different picture of the relationship between the proof-theoretic notions and assertion. Although less intuitive, the picture we will propose will account for the different facets of the relationship between truth and assertion.

3.6.1   The need of a gap

The role of a semantic theory is presented by Dummett in a quite concise form in the following passage. Problems emerge as soon as we compare it with the foregoing presentation of the solution of the paradox of inference and with other passages coming from papers devoted to the notion of truth.

> 'We must now say that a semantic theory is one specifying, in accordance with its composition, the condition for an assertion of a sentence to be *right*, in a a sense to be explained. To be correct or incorrect is intrinsic to assertion: an utterance that is not to be assessed as correct or incorrect is not an assertion. But there are more or less objective standards of correctness ... By an assertion being

right, in the special case intended, is meant that it is correct in the
most objective sense recognized by those who accept the semantic
theory. A realist must hold that there is one grade of objectivity
higher than a speaker's being justified in virtue of possessing ob-
jectively cogent grounds for what he said, namely his statement
being *true*. A Platonist believes that truth may attach to a mathe-
matical statement independently of our knowledge or capacity to
know; so a true assertion is correct in a sense more objective than
that of *our presently* possessing a proof; for him, therefore, an asser-
tion's being *right* consists in the asserted statement's being true. But
intuitionists recognize no such notion of truth capable in principle
of transcending our ability to recognize it. For them, there is no
more objective notion of a mathematical assertion's being correct
than that a valid proof of it is available, and it is in that therefore,
that, for them, its rightness consists.' (Dummett 1998a, p. 12).

From this passage we can extract the following definition of the correctness
of the assertion of a mathematical sentence (in the strict sense of an assertion
being right) for intuitionists: the availability of a proof of it.[10] In chapter 2,
we argued that a proof of A is represented by a valid closed (non-necessarily-
canonical) argumentation having A as conclusion. Hence:

(1) The assertion of A is correct if and only if we are in possession of a valid
closed argumentation of conclusion A.

A few pages after the passage just quoted, Dummett claims:

'The appropriate conception of truth for constructivist mathemati-
cians is that a mathematical statement is true if either we have a

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[10]There is a slight inconsistency between Dummett's terminology and ours. While Dummett
uses the expression 'valid proof', we ascribe validity to argumentations iff they represent proofs
(cf. chapter 2). In our terms, there is no issue of validity relating to proofs. If proofs are taken as in-
tuitionistic constructions, this accords with Brouwer's doctrine, according to which constructions
are correct in virtue of their very existence. On the other hand, one must be careful that argumenta-
tions (i.e. the linguistic presentation of proofs) do not contain errors. So we distinguish Dummett's
'valid proofs' into intuitionistic 'proofs' and proof-theoretic 'valid closed argumentations'.

> proof of it or have an effective means of construct one.' (Dummett
> 1998a, p. 15)

In the light of the foregoing discussion, we know that to have effective means
to construct a proof of a sentence is to have indirect evidence for it. In turn,
indirect evidence is characterized in terms of valid closed argumentations. So,
from this passage we can extract the following definition of truth:

(2) A is true if and only if we are in possession of a valid closed argumenta-
tion of conclusion A.

This seems to contradict Dummett's claim on the difference in attitudes be-
tween a constructivist and a realist. In the passage quoted, Dummett argued
that only the realist can claim that the most objective notion of the correctness
of an assertion is the sentence asserted being true. But, in the light of (1) and
(2), not only the realist, but also the anti-realist can claim that an assertion being
correct is the asserted sentence being true.[11]

Furthermore, there seems to be another problem connected with the identi-
fication of the correctness of an assertion with the truth of the asserted sentence,
a problem which Prawitz points at in the following passage:

> 'For an assertion to be correct, it is not sufficient that the asserted
> proposition be true, the speaker must also have sufficient grounds
> for believing it to be true.' (Prawitz 1998c, p. 24)

That is, the notion of an assertion being correct must be related to the grounds
we have for making it, that is to our recognizing the truth of the asserted sen-
tence, rather than to just its truth.

The significance of this fact is much greater as soon as the need of account-
ing for both features of deduction induces a gap between truth and its recog-
nition. For, since the notion of the correctness of an assertion is primarily con-
nected with truth-recognition, we must also introduce a gap between the truth-

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[11]As we remarked in the previous chapters, Dummett's thesis that only the realist introduces a
notion going beyond the correctness of an assertion is wrong. Both because also the anti-realist has
to introduce one in order to cope with implication and because the introduction of such a notion
does not automatically commit to realism.

conditions of a sentence and the conditions of the correctness of its assertion.[12]

3.6.2 Dummett's solution

Once recognized the problem, Dummett adjusts his position saying:

> 'In my paper (Dummett 1998a) I characterized the notion of truth
> as attaching to a mathematical statement if "we either have a proof
> of it or have an effective means of constructing one". I failed how-
> ever to spell out the meaning I intended this formula to convey; and
> I compound this mistake by incorrectly answering a question that
> Dag [Prawitz] asked in the discussion, failing to perceive its drift.
> He asked me first to repeat the formula I have just quoted; when I
> did, he said something like, "the same condition for truth and for
> correct assertion, then?", to which I quite wrongly answered "Yes".
> I had however intended my formula to go beyond warranted as-
> sertibility. I intended to allow as true a statement for which we
> have an effective procedure that will *in fact* yield a positive result
> even if we do not know this. For example, a statement that a cer-
> tain large number prime is decidable, and may, when we apply the
> decision procedure, turn out to be true. I was making the tacit as-
> sumption that it is already determinate how the decision procedure
> will turn out, because there is no room for any play in the process
> of applying it. Hence, if it would turn out that the number is prime,
> the statement that it is prime is, on the definition I gave, true even
> though we have at present no proof that it is, and may never have
> one, though we possess what is in fact an effective means of con-
> structing one. This differentiates a statement's being true from our
> being entitled to assert it.' (Dummett 1998b, pp. 122–123)

In this passage, Dummett presents a method with the following properties:
if it is applied, it produces direct evidence for a sentence; but until it is applied
we do not know whether the direct evidence it will yield is in support of the

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[12]Note that all this equally applies to the realist perspective as well. That is, the realist too would
not claim that the assertion of a sentence *A* is correct simply when *A* is true.

sentence itself or of its negation. The example is meant to show that, in general, the possession of a method is not a sufficient condition for the assertion of sentences, while it surely is a sufficient condition for their truth.

As we saw in section 3.4.2, the notion of method is used in characterizing indirect evidence. In particular, the possession of indirect evidence is equated to the possession of a method to obtain direct evidence.

In this case, we have a method to obtain direct evidence for a sentence even if we do not see it (in the sense that we do not know if the direct evidence we will obtain is for the sentence itself or for its negation). But a method to obtain direct evidence is indirect evidence. Hence we are forced to say that we have indirect evidence for a sentence even if we do not see it. If we consider Dummett's example, considered in these terms, yields the conclusion that the possession of indirect evidence for a sentence is not always sufficient for asserting a it.

This has prompted Dummett to give up (or at least to relax) the tight relationship between indirect evidence and methods. We claim that the result is an overall tension in Dummett's picture. We will see in the next sections how a sounder picture may be attained. Namely, by accepting the *prima facie* counterintuitive claim that indirect evidence does nor warrant the correctness of an assertion. We conclude this section by clarifying in which sense Dummett's choice yields a certain tension in his views.

By rejecting the identification of effective methods with indirect evidence, Dummett is introducing a notion of method distinct from the one of indirect evidence in order to deal with truth. As we remarked, the notion of method in the anti-realist theory of meaning fulfills the role played by the (much less constrained) notion of possibility in the realist one: namely it 'measures' the gap between truth and its recognition. So to introduce a notion of method not to be identified with indirect evidence is a way of widening the gap. The result is a reformulation of (2) as follows:

(2′) A is true if we are in possession of a method to obtain a proof of it.

the notion of method at stake being accounted for independently of the one of indirect evidence.

The gain of this choice is that we can maintain the intuition that the possession of evidence for a sentence (being it either direct or not) is a sufficient condition for the correctness of its assertion. Indirect evidence continues to be a sufficient condition for the correctness of an assertion, but the possession of a method would not be anymore. That is, principle (1), governing the correctness of assertion, can be maintained.

Nonetheless, we hold that by pursuing this direction, Dummett is betraying the analysis of the paradox of deduction given in his former writings.

As Prawitz correctly remarked, the correctness of assertions has to do with the grounds we have for recognizing the truth of sentences. But the correctness of an assertion is characterized in terms of principle (1), that is in terms of evidence, direct or indirect. Hence, it seems that both kind of evidence are flattened against the notion of truth-recognition.

On the other hand, to account for the two contrasting features of deduction we attached truth-recognition only to the meaning-fixing inferences, the constituents of direct evidence. Simply valid inferences, those that constitute indirect evidence, preserve only truth.

Hence, the ultimate result of endorsing principle (1) seems to be explicitly in tension with the account of deduction we gave in the first part of the chapter. In particular, we end up with two distinct notions of 'method': one governs the relationship between direct and indirect evidence; the other governs the one between truth and its recognition. But while the former one is characterized, although vaguely, in terms of reduction procedures, how is the latter one to be characterized?[13]

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[13]An alternative possibility is the one pursued by Prawitz (1998a, 1998b, 1998c). Instead of speaking of methods as the *explicans* of the notion of truth, Prawitz introduces the idea of a realm of proofs existing or not quite independently of our means of recognizing them. So that, the principle governing truth is formulated as follows:

(2″) *A* is true if and only if a proof of *A* exists.

He insists that the existence of a proof can still be taken as the possibility of recognizing the truth of the sentence, but the notion of possibility at stake strongly resemble the classical one. The only difference is that realists postulates facts instead of proofs, the latter ones being epistemic entities and the former ones being not. Just like Dummett, so Prawitz is willing to maintain principle (1) governing assertion. Hence, the tension we register in Dummett can be also found (and in a sense is even stronger) in Prawitz' approach.

3.7   An alternative way

We suggest an alternative direction, namely the one arising by not abandoning the identification of indirect means of establishing sentences with methods for obtaining direct evidence and by interpreting truth as the possession of methods/indirect means. Hence, in order to distinguish between truth and correctness of assertion, we do not modify principle (2) governing truth; but instead principle (1), governing assertion.

As Prawitz stressed, there is a deep connection between truth-recognition and correctness of assertion. In a sense, the need of a keeping truth and its recognition apart, which emerged in analyzing the paradox of deduction, has its counterpart in the need of keeping correctness of assertion distinct from truth. To account for the gap between truth and its recognition we introduced the distinction between direct and indirect evidence. Why cannot we use this distinction to account for the gap between truth and correct assertion as well?

As we saw, in analyzing the paradox of deduction, the notions of truth and truth-recognition emerge as what is preserved, respectively, by valid and meaning-fixing inferences. As the two kinds of inferential procedures are connected respectively with the notions of direct and indirect evidence, the natural proposal for the characterization of truth is actually principle (2). On the other hand, we identify methods to obtain direct evidence with indirect evidence, which are modeled with closed non-canonical argumentations. Thus, Dummett's example leads us to amend principle (1) governing the correctness of assertion as follows:

(3)   The assertion of A is correct if we are in possession of a valid closed *canonical* argumentation of conclusion A.

3.7.1   Tarskian worries

Actually, there are strong grounds to call a notion such as the one defined by (2) a notion of truth, namely the fact that it satisfies Tarski adequacy condition: that is, all instances of the scheme:

(T) A is true if and only if A.

are derivable from the definition.

For any instance of the scheme, according to the (intuitionistic) meaning of implication, we have to provide:

(a) A method that takes direct evidence for the antecedent into direct evidence for the consequent

(b) A method that takes direct evidence for the consequent into direct evidence for the antecedent

Now, evidence for the antecedent of any instance of (T) is, according to (2), indirect evidence for A. But indirect evidence for A is a method for obtaining direct evidence for A. So (a) is satisfied since evidence for the antecedent is itself a method for obtaining direct evidence for the consequent.

Requirement (b) is satisfied in a more obvious way. If we already have direct evidence for A (evidence of the consequent), we have *a fortiori* a method to obtain direct evidence for A: just doing nothing!

In the light of this, we have that the notion of truth characterized by (2) is enough to meet Tarski's adequacy condition. As we saw, the notion of truth resulting by (2) is enough to justify the fruitfulness of deduction. That is, the notion of truth characterized by (2) is enough for satisfying conditional ($*$), when this is taken as a further adequacy condition to be imposed on the notion of truth. So, Dummett's choice of modifying (2) is quite unjustified, as it introduces a notion of truth heavier than the one which is actually needed.[14]

3.7.2 Challenging assertions

Nonetheless, to modify principle (1) in favor of (3) seems definitely counterintuitive: in this way, we are linking the correctness of an assertion only with direct evidence, in the sense that the possession of indirect evidence is no more a sufficient condition for asserting sentences. That is, our proposal consists

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[14]As we are treating 'A is true' as 'it is possible to recognize the truth of A', we are actually interpreting (T) as the principle of knowability (K) '$A \leftrightarrow \diamond A$'. Our argument in favor of (T) is analogous to Martino and Usberti's (1994) argument in favor of (K), which amounts to take (K) as expressing a condition of 'transparency' on intuitionistic constructions.

in treating only valid closed *canonical* argumentations as warrants for assertions. But indirect evidence, i.e. valid closed (non-canonical) argumentations, is a variety of evidence. So why does the possession of evidence (even if only indirect) would not put one in the position of asserting a sentence?

We will give two reasons for motivating this choice, one in this sub-section and the other one in sub-section 3.8.2.

A first reason arises from the answer given to the paradox of deduction. In solving it, we distinguished between two kinds of inferences, meaning-fixing and non meaning-fixing ones. Only meaning-fixing inferences cannot but be accepted as valid. Speakers' competence does not require the recognition of non-meaning-fixing inferences as valid.

As we saw, the distinction between direct and indirect evidence arises from the one between the two kinds of rule. This suggests that one may challenge indirect evidence for a sentence. In the sense that if a speaker asserts a sentence on the basis of indirect evidence for it, a hearer is not forced to accept the assertion by simply being presented with the indirect evidence. In particular, the hearer may ask for a justification on the basis of which she should accept the evidence presented as valid. On the other hand, it seems that direct evidence cannot be properly challenged. With the words of Prawitz,

> 'when an indirect verification is challenged we usually try to support it by further evidence, in the end we may supply the direct evidence if possible. A challenge of a direct verification, on the other hand, makes us suspect that the challenger does not know the meaning of the relevant expressions, and is therefore typically met by language teaching.' (Prawitz 1998c, p. 28)

More in general, in the light of the picture emerging from the molecular view of meaning, one may understand the meaning of a sentence and yet may not have a proper understanding of some indirect way of establishing it: the indirect procedures in general involve sentences of higher logical complexity (or that lie deeper in Quine's net) than the one established. That is sentences, the mastery of which requires concepts (or simply linguistic skills) the speaker may not have, though being competent on the meaning of the sentence established.

Presented under this light, the fact that indirect evidence is not sufficient for warranting assertion becomes much more intuitive and at the same time brings us toward a very sound picture of the general architecture of the theory of meaning.

The acceptance and rejection of assertions made by speakers is a social practice. The notion of the correctness of an assertion aims at accounting for this practice. The introduction of a distinction between direct and indirect means of establishing sentences is a way of rationally reconstructing it as being based on objective criteria. In particular, the assertion of a sentence on the basis of direct evidence cannot be challenged within the linguistic community. As direct evidence for sentences is structured according to the meaning of the sentence, whoever challenges the assertion made on its base cannot be a competent speaker, and hence falls outside the community. On the other hand, a speaker has the right to challenge an assertion based on indirect evidence, as her competence does not put her automatically in the position of recognizing the evidence as such. But suppose she is presented with a justification for the indirect evidence, that is it is shown her that the indirect evidence is faithful to the direct one, in terms of which the meaning of the sentence is characterized. In this case, she will not be able to resist to the acceptance of the evidence as a ground for the assertion. That is, she will fell compelled by the validity as imposing on her.

If we recall the 'film' image in terms of which we characterized the different positions on the relationship between meaning and inference, we have that the account sketched here properly matches the molecular view according to which the film is partitioned in strips, during which the only inferences acceptable are those faithful to the meanings.

As we argued, the film picture allows for changes in meanings, which take place at the border between adjacent strips. These situations can be described in terms of warrants for assertions as follow. As the grounds for correct assertion depend on the meaning assigned to the expressions, we have that when a change in meaning takes place, new grounds for correctness are established. Obviously, for such a thing to happen, there must be at first some disagreement on the meaning to be assigned to some expressions within the community,

which prompts the revision. This disagreement will be manifested in practice by some speakers rejecting assertions made on the base of direct evidence. In fact, the assertion being rejected highlights exactly the fact that the hearer does not assign the same meaning as the speaker to the sentence (otherwise she could not reject the assertion).

Once meaning are taken as fixed, an account of the conditions of the correctness of assertion is possible, according to the lines sketched: principle (3) states only a sufficient condition for correct assertion, but not a necessary one. In fact sometimes the possession of indirect evidence (that is of a method) is enough, namely, when it is recognized as such by the relevant members of the community (at least speaker and hearers).

In particular, suppose that a speaker (or the community as a whole) accepts a new procedure M for establishing a given sentence A, recognizing it to be faithful to the meaning of A. It seems natural to claim that even if the acceptance of the new procedure does not modify the meaning of the sentence, it modifies the conditions of assertion of the sentence. As the speaker knows that the application of the procedure will yield direct evidence for the sentence, she feels entitled to assert the sentence whenever the conditions for applying the procedure are satisfied.

If we ideally look at the moment in which the community settles the meaning of a sentence A, we could perhaps strengthen (3) to a bi-conditional:

(3′) The assertion of A is correct if *and only if* we have valid closed canonical
        argumentations for A.

In fact, as the meaning of A has just been fixed, speakers has not yet encountered any new procedure that could turn out to be an indirect mean of establishing A. That is, the only evidence for the sentence (recognized as such by speakers) *is* the one which is actually used to give the sentence its meaning, that is canonical evidence. That is, speakers would be entitled to assert A *only* if in possession of direct evidence for it.

But of course, speakers behavior will be correctly described by (3), as soon as they start recognizing new procedures of establishing the sentence. In a sense, as time passes, the conditions of asserting a sentence widen from the mere possession of direct evidence, by also allowing the possession of indirect

evidence as a sufficient ground for assertion. But as indirect evidence is not always recognized as such by us, this process never leads to (1): it is natural to expect that we can always come to new indirect ways of establishing sentences which we have not already recognized as such and hence the possession of which (though making the sentence true) does not presently allow us to recognize the truth of the sentence, i.e. to correctly assert it.

3.8 Open and non-canonical argumentations

In the first chapter, we saw that Dummett proposes to identify the notion of truth with the semantic feature going beyond the correctness of an assertion. In the second chapter, we argued that the notion of validity as applying to open argumentations has an analogous role.

Now we proposed to identify the truth of a sentence with the possession of closed non-canonical argumentations.

Hence, one may read this twofold attitude as an inconsistency. Either truth is to be connected with open validity or with closed non-canonical validity.

In this section, we argue that closed non-canonical validity is conceptually dependent on open validity. By doing this, we show that no inconsistency is at stake. Furthermore, we give a more substantial content to the thesis that the need of open validity is analogous to the introduction of a notion of truth as distinct from the correctness of an assertion.

3.8.1 Predicates and inferences

The source of the connection of open argumentations with closed non-canonical ones can be found in an analogous connection that Dummett traces in Frege, namely the one between complex predicates and the 'different analysis' that can be given of a certain sentence.

We first state Dummett's point on Frege and then we turn back to the connection between open validity and closed non-canonical validity.

We quote again Dummett:

'Once the notion of a complex predicate has had to be introduced [to account for the formation of quantified sentences (cf. section

1.5.1.1), then it also becomes necessary to recognize the complex predicate as occurring in the sort of sentence from which it was formed: it is this which gives rise to what people have in mind when they speak about different, equally legitimate, logical analyses of one and the same sentence. But it is important to notice for what purpose this sort of "analysis" is needed. The representation of "Brutus killed Caesar" as composed of a (complex) one-place predicate "ξ killed Caesar" and, in its argument-place, the name "Brutus", is required only in order to state the general principle to which we are appealing when we recognize an inference from this sentence, together with, say, "Anyone who killed Caesar is an honorable man", to the conclusion, "Brutus is an honorable man". We need this representation of the sentence, that is to say, in giving an account of inferences in which it and also some sentence involving the attachment of the sign of generality to the complex predicate in question both figure. The representation of the sentence as consisting of "Brutus" and "ξ killed Caesar" is quite irrelevant to any explanation of the way in which the sense of the expression is determined from that of its constituents. An inference may easily arise in which it is necessary in order to explain the general scheme of inference appealed to, to consider the sentence "If Brutus killed Caesar, then Brutus's wife hated Brutus" as composed of the name "Brutus" and the predicate "If ξ killed Caesar, then Brutus's wife hated ξ". But the possibility of giving such an "analysis" of the sentence has no bearing on the process by which we form the sentence, or on that by which we come to grasp its sense. We understand the sentence by reference to the process by which it was formed, namely as being put together out of two atomic sentences joined by the connective "if", those atomic sentences having in turn been constructed by linking singular terms and relational expressions; the thought that one might recognize the complex predicate cited above as occurring within that sentence could be utterly remote from the mind of someone who had the firmest grasp upon

its meaning.' (Dummett 1973a, pp. 28–29)

Consider the sentence '7 + 0 = 7'. The sense of the sentence is determined uniquely by the sense of the component expressions. The thought expressed by this sentence is, roughly, that the result of summing up 7 and 0 is equal to 7. In characterizing the sense of the sentence, there is no need to make reference to the complex predicates that can be traced into it.

The recognition of the complex predicates, as we saw in chapter 1, is induced by the presence of quantifiers. But once complex predicates are introduced, they serve a further purpose, namely that of accounting for what Dummett calls the different 'analyses' that can be given of a sentence.

By tracing in the sentence '7 + 0 = 7' different one-place predicates (we do not take relations into account), the sentence can be alternatively analyzed as obtained by applying:

the predicate '$\xi + 0 = 7$' to '7'

the predicate '$7 + 0 = \xi$' to '7'

the predicate '$\xi = 7$' to '7 + 0'

the predicate '$\xi + 0 = 7$' to '7'

the predicate '$\xi + 0 = \xi$' to '7'

To the different analyses, there may correspond several inferences having the sentence as conclusion. As an example, an inference by universal instantiation allows us to establish the sentence from the arithmetical law that 0 is the neuter element of sum, '$\forall x(x + 0 = x)$'. The recognition into the sentence of the last complex predicative pattern, among the suggested ones, is what puts one in the position of recognizing the validity of the inference.

Dummett stresses that, to grasp the sentence meaning, one is not required to recognize neither the complex patterns contained into it, nor the validity of all inferences in which the sentence may figure. But when presented with such an inference, if the speaker recognizes it as valid, she will do this by recognizing the corresponding complex pattern into the sentence.

Clearly, the sense of the sentence also suggests a route to the determination of the truth-value of the sentence, a route which is structured in accordance with the process by which the sentence has been formed. In this case, the route

consists in first applying sum to the denotation of '7' and '0' and then compar-
ing the result with the denotation of '7'.

So, the mastery of the meaning of a sentence suggests one way in which the
sentence could be established. But many more means of doing this arise in con-
nection with the different analyses that we can give of a sentence. And these,
in turn, depend on the possibility of recognizing complex predicates within the
sentence.

The speaker, by mastering the meaning of a sentence, is not compelled to
have already recognized all possible ways of establishing it, but just one. In-
deed, it looks as if it would be impossible to characterize knowledge of mean-
ing in a compositional manner, if knowledge of all possible ways of establish-
ing a sentence was included into speaker's competence.

For, by looking again at the route to the determination of the truth-value
suggested by the last of our analyses of the sentence '7 + 0 = 7', we realize that
such process passed through the establishment of a sentence, whose logical
complexity was higher than that of the sentence we wanted to establish: the
universally quantified sentence '$\forall x(x + 0 = x)$'. Hence, this possible route to
the truth-value of the sentence requires mastery of concepts, whose complexity
is higher than the one of those involved in the specification of the meaning of
the sentence.

In contrast with this, the way of establishing the sentence suggested by its
sense is structured according to the sentence meaning; that is, it is a direct
way of establishing the sentence, as opposed to the indirect ways arising from
analyses involving the recognition of complex predicates into the sentence.

3.8.2   Assertion, open validity and truth

If the interpretation of Dummett's passage is correct, then he acknowledges a
tight connection between the recognition of complex predicates and different
analyses of sentences, which are the basis for according validity to arbitrary
inference in which sentences figure.

In the previous chapters, we stressed the analogy in the role played by
predicates/open formulas in the truth-theoretic semantics and open argumen-
tations in the proof-theoretic one. In the light of the tight connection just con-

sidered between different analyses of sentences and indirect ways of establishing them it should be plausible to establish an analogous relationship between open argumentations and closed non-canonical argumentations.

We will indeed argue that the notion of valid closed non-canonical argumentation is strongly dependent on the one of open argumentation, so much that it may be natural to claim that such a notion arises only in presence of a notion of validity as applying to open argumentations.

To fully appreciate this, it is important to consider the model after which the relationship between canonical and non-canonical argumentations comes from, namely that between normal and non-normal derivations in natural deduction systems.[15]

Normal derivations in natural deduction systems are those that do not display any pattern constituted by an application of an introduction rule followed immediately by an application of (one of) the corresponding elimination rule(s). In the case of conjunction, one of the patterns in question is the following:

$$\cfrac{\cfrac{\begin{matrix}\vdots \ \pi_1 & \vdots \ \pi_2 \\ A & B\end{matrix}}{A \wedge B} \, I\wedge}{A} \, E\wedge$$

The pattern represents intuitively a certain redundancy in the derivation. For, supposing π_1 and π_2 to be closed derivations of conclusion A and B respectively, then of course the conclusion of the $E\wedge$ rule was already established by π_1. Normalization theorems show that every pattern of this kind can be eliminated from derivations. A corollary of normalization, for the intuitionistic natural deduction system **NJ**, is that the normal derivations so obtained have the property that, if they are closed, then they end with an application of an introduction rule. In the light of this, the notion of closed canonical argumentation bears a strong resemblance with the notion of closed normal derivation and the relationship between closed non-canonical and closed canonical argumentations represents the counterpart of formal normalizability.[16]

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[15]For the use of the term 'derivation' as opposed to 'argumentation' cf. note 23 on page 59.

[16]Indeed the notion of closed I-canonical argumentation we used in the definition of validity—which is essentially due to Prawitz (1971) and has been recently restated by Schroeder-Heister (2006)—is weaker than the one of closed normal derivation, since we just require the argumen-

We will argue that closed non-canonical argumentations are conceptually dependent on open argumentations by considering the relationship between closed non-normal and open derivations.

Obviously, a derivation may fail to be normal and closed either because it is non-normal or because it is open (or both).

If we take a closer look at the first alternative, that is at closed non-normal derivations, it may be natural to ask how such argumentations do arise at all. In particular, there seems to be no reason to produce a derivation like e.g. the one ending with an instance of the redundant pattern constituted by $I\wedge$ and $E\wedge$. For, suppose one is trying to establish A. Once in possession of the closed derivation π_1, there is no reason for which she should go on introducing and then again eliminating the conjunction.

A proper understanding of the reason why we do have to cope with closed non-canonical derivations is achieved as soon as we consider a situation in which a deductive relation is established between a logically complex sentence and a more simpler one, according to which the simpler sentence follows from the more complex, as for instance in the conjunction elimination rule:

$$\frac{A\wedge B}{A}\ E\wedge$$

The recognition of such a connection does not establish the conclusion A, but is taken to warrant the assertion of A whenever the assertion of $A\wedge B$ is warranted. Suppose then that a closed normal derivation for $A\wedge B$ is provided. Such a derivation, together with the elimination rule, provides a closed non-canonical derivation for the conclusion of the elimination, i.e. for A.

---

tation to be closed and its last step being an introduction. That is, its sub-argumentations may contain reducible patterns. Dummett (1991) actually proposes a more stringent notion of canonical argumentation, called by Prawitz (2006) 'hereditary canonical', which is nearer to the notion of closed normal derivation, in forbidding redundancies in what Dummett calls the 'main stem' of the argumentation. Still, redundancies may occur in the so-called 'critical sub-argumentations' of hereditary canonical argumentations, i.e. in those sub-argumentations leading to the premises of implication introduction rules. A stricter correspondence could be gained by also forbidding redundancies in critical sub-argumentations. More on the different notions of canonical argumentations in note 18. For, further remarks on the relationship between the notion of 'computable derivation' (used in establishing normalization results) and the semantic notion of 'valid argumentation', see Schroeder-Heister (2006, §3.2).

In general, it is only because we have the concept of establishing a deductive relationship between sentences

$$
\begin{array}{c}
A \\
\vdots \\
B
\end{array}
$$

that the notion of closed non-canonical derivation arises. The recognition of such a deductive relationship amounts to being in the position of asserting the conclusion whenever we are in the position of asserting the assumption. By replacing the assumptions of the open derivation with closed normal derivation having the assumptions as conclusions, one may obtain a derivation containing a redundant pattern: typically, when the assumptions of the open argumentation are the premises of an elimination rule. By just looking at the closed non-normal derivation of B so obtained, one have the impression that the direct production of an argumentation of this kind is irrational, due to its patent redundancy. The significance of coping with such a derivation arises only from the fact that we may have first established the relationship between A and B and then obtained a closed normal derivation of A. That is, although there is no point in dealing with closed non-normal derivations *per se*, we may be forced to deal with them, as an intermediate state from an open derivation and a closed normal one.[17]

Analogous considerations apply to the case of closed non-canonical argumentations as well: that is, the ascription of validity to them is a by-product of the need of ascribing validity to open argumentations.

As we remarked, it is only in presence of implication that a compositional account of meaning requires us to introduce a notion of validity that applies to open argumentations. But this means that it is only in presence of implication that the notion of closed non-canonical argumentation have any significance at all.

In section 2.2.1, we considered two possible principles connecting meaning and validity, one connecting meaning to closed validity and one connecting meaning with closed canonical validity. We remarked that the need of ascribing validity not only to closed argumentations, but also to open ones, is what

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[17]In mathematical practice, these consideration apply to the use of lemmas in proving theorems.

makes the account of meaning arising from both principles non-compositional.

Another way of stating the same point is by claiming that the two principles yield a compositional account of meaning only if the language does not contain an implication-like operator.

For such a language, no need of ascribing validity to open argumentations would arise. But then, there is also no need of coping with closed non-canonical argumentations. As a result, the two theses discussed in section 2.2.1 tend to collapse in an implication-free language. The presence of implication induces the recognition of validity to open argumentations, and in turn the need of coping with closed non-canonical argumentations. Hence it is implication what forces us to distinguish between the two theses but, at the same time, is what make the two theses yield a non-compositional picture of meaning.[18]

In chapter 1, following Dummett, we saw how the presence of quantifiers forces the recognition of the full category of first-order predicates or, which is the same, the introduction of open formulas. In the previous sub-section, we quoted a passage in which Dummett stresses that once complex predicates/open formulas have been introduced for an account of quantification, they give rise to several different analyses of sentences. These alternative analyses are required in order to account for the validity of inferences involving the sentence as well as sentences obtained by attaching the quantifier to the different complex predicates figuring in the sentence. But these alternative analyses have no bearing on the meaning of the sentence, which depends exhaustively on the one of the primitive expressions from which it is composed or, as we may say, depends exhaustively on the 'canonical analysis' of the sentence.

Analogously, implication forces us to recognize a notion of validity applying to open argumentations: as the quantifier turns open formulas into sentences, so implication may be used to obtain closed canonical argumentations from open ones. The term 'open' is not used by mere chance in both cases. Indeed, undischarged assumptions in open argumentations work exactly like slots in predicates, i.e. like free variables in open formulas.[19]

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[18]These considerations are reflected in Dummett's (1991, ch. 11) distinction between 'the justification of logical laws of second grade' and 'of third grade'. Cf. section 2.5.1, on page 48.

[19] The connection between open formulas and open argumentations is best viewed from a type-

Once the notion of valid open argumentation is introduced for this reason, one will be forced to cope with valid closed-non canonical argumentation as well. Valid closed non-canonical argumentations result by fulfilling the open slots of valid open argumentations represented by their undischarged assumptions with valid closed argumentations, having the assumptions as conclusions.

This is analogous to the fact that the sentence 'Gabriele is Italian and Peter is German' can be canonically analyzed as obtained by joining two sentences by means of conjunction or non-canonically analyzed as being obtained by applying the complex predicate 'ξ is Italian and Peter is German' to the name 'Gabriele'. The only difference is that, in the proof-theoretic case, reduction procedures are needed to show that a canonical and a non-canonical argumentation (when the latter reduces to the former) are just two different presentations of the same proof.

Once the conceptual dependence of closed non-canonical argumentations on open argumentations is acknowledged, it is possible to give a more precise content to the thesis that the semantic feature to which the presence of implication gives rise is indeed a notion of truth.

For, the presence of implication yields a notion of validity as applying to open argumentations. This one, in turns, allows for the validity of closed non-canonical argumentations, that is for sentences being indirectly established. Hence, to treat in accordance with (3) sentences indirectly established as merely true, that is not always assertible, gives more coherence to Dummett's picture.

3.9 A tentative evaluation

The acceptance of principle (3) is hence justified. It is true that, *contra* (3), indirect evidence intuitively warrants assertions. Nonetheless, the characterization of an assertion being correct along the line we suggested—i.e. an assertion made by a speaker is correct if it is unassailable by the hearer—offers a sound interpretation of (3).

Furthermore, according to (1) and (3), the possession of indirect evidence

theoretic perspective. Cf. note 15 on page 45 in chapter 2.

Table 3.1: Open-ness and non-canonicity

In the

Truth-theoretic Semantics	Proof-theoretic Semantics

the presence of

Quantifier	Implication

induces the recognition of

Complex Predicates/Open Formulas	Valid Open Arguments

whose presence allows for

Different Analyses of	Several Ways of Establishing

the same sentence

---

for a sentence, i.e. a valid closed non-canonical argumentation having the sentence as conclusion, although suffices for the truth of the sentence, does not suffice for its assertion. That is, the possession of valid closed non-canonical argumentations represents a sentence being true, where the notion of truth is distinct from the correctness of an assertion. As the availability of valid closed non-canonical argumentations is conceptually dependent on the availability of a notion of validity applying to open argumentations for sentences, the notion of validity as applying to open argumentation yields the introduction of a notion of truth as distinct from an assertion being correct.

Dummett, also in this context, links this issue with the question of realism:

the justifiability of deductive inferences—the possibility of display-

ing it as both valid and useful—requires *some* gap between truth
and its recognition; that is it requires us to travel some distance,
however small, along the path to realism, by allowing that a state-
ment may be true when things are such as to make it possible for
us to recognize it as true, even though we have not accorded it
such recognition. Of course from a realist standpoint, the gap is
much wider: the most that can be said, from that standpoint, is that
the truth of a statement involves the possibility in principle that it
should be, or should have been, recognized as true by a being—not
necessarily a human being—appropriately situated and with suffi-
cient perceptual and intellectual powers.' (Dummett 1973b, p. 314)

But as we already remarked, it is far from clear that a relationship between
the need of introducing a notion of truth as distinct from the correctness of an
assertion and realism exists.

In order to appreciate the supposed difference between a realist and an anti-
realist position, a clear explanation of the notion of possibility on which the
anti-realist relies is crucial. But as we saw, this is exactly the point where we
find a gap in Dummett and Prawitz' picture. For, as we argued, the notion
of possibility at stake is explained in terms of the notion of reduction proce-
dures. And this notion, apart from a very restricted span of cases, is in general
extremely vague.

As a result, we leave undecided the question concerning realism. On the
other hand, we claim that a notion of truth as distinct from the correctness
of an assertion is required in the proof-theoretic setting as well. Dummett's re-
duction of the notion of validity as applying to open argumentations—which is
the source of the notion of truth—to the one of correctness of an assertion—i.e.
to the notion of validity as applying to closed canonical argumentations—has
been declared both vague (due to the reference to reduction procedures) and,
possibly, unnecessary (from comparing the primary character of the notion of
method in intuitionism). In the next chapter, it will turn out that, irrespective
of the possibility of making it more precise, it is to be rejected as inadequate.

Chapter 4

Falsificationism

In this chapter, a rather specific issue will be inquired, namely how a notion of refutation, to be conceived as the dual of the notion of proof at the core of the proof-theoretic semantics, should be characterized.

At first, this theme appears as rather independent from the topic discussed in the previous chapters, that is, the question of which is the right conception and role for truth in Dummett and Prawitz' proof-theoretic setting.

Nonetheless, in analyzing the possible characterization of refutations in **NJ**, the natural deduction system for intuitionistic logic, further trouble for Dummett's picture will emerge. In particular, the possibility of reducing the semantic role of open argumentations to the one of their closed instances, which already in chapter 2 was found unconvincing, will be further criticized.

We will argue for the development of a sort of refutation-theoretic semantics, in which refutation takes the role of proof as central notion. The picture is in all symmetrical to the proof-theoretic one, which we inquired in the previous chapters. So much, that it is also flawed by the very same problems.

The interest of such an analysis will fully appear in the last chapter, where we will sketch a possible direction in which solutions to the problems of the proof-theoretic picture may be framed. We will argued that one important element for the solution may consist in the unification of the proof- and refutation-theoretic perspectives.

4.1   Refutations: an informal adequacy condition

As we said, in this chapter, we wish to characterize, in proof-theoretic terms, a notion of refutation, to be conceived as a primitive semantics alternative to the one of proof.[1]

Just as the **BHK** clauses can be viewed from a proof-theoretic perspective as an informal condition to be imposed on the notion of proof characterized by the proof-theoretic semantics, analogous clauses have been proposed for the notion of refutation by Lopez-Escobar (1972).

The idea is the following:

> 'in addition of having the concept of "a construction c proves a formula B" there is at hand the concept of "a construction d refutes a formula C".' (Lopez-Escobar 1972, p. 362)

The suggested clauses for the notion 'a construction c refutes a formula C' are the following:

- the construction c refutes $A \wedge B$ iff c is of the form $< i,\ d >$ with i either 0 or 1 and if $i = 0$, then d refutes A and if $i = 1$ then d refutes B;

- the construction c refutes $A \vee B$ iff c is of the form $< d,\ e >$ and d refutes A and e refutes B;

- the construction c refutes $A \rightarrow B$ iff c is of the form $< d,\ e >$ and d proves A and e refutes B;

Just as the **BHK** clauses, Lopez-Escobar's (henceforth **LE**) intuitively specify the canonical ways in which a sentence is refuted.[2]

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[1] The approach we develop is hence clearly distinct from the treatment of refutations originated from Łucasiewicz work, see, among others, Skura (1992, 1995) and Goranko (1994). While they present 'refutability systems' characterizing the class of non-theorems of several propositional logical systems, we will consider a few logical systems and discuss whether they are capable of grasping a notion of refutation satisfying certain intuitive constraints. A more detailed comparison between this approach and their may be developed in subsequent research. For further works more directly related to our present aim, cf. note 11.

[2] The characterization we will propose of the notion of refutation is different from Lopez-Escobar's. Nonetheless we agree with him on the informal conditions that the notion has to satisfy. We compare in detail his approach with our in section 4.4.3.

Once a notion of refutation is available, the linguistic negation operator can be viewed as the hinge between the notions of proof and refutation. That is, if both the notion of proof and refutation are available, we can expect the following two principles to hold:

($\mathbf{N_P}$) A proof of the negation of A is a refutation of A

($\mathbf{N_R}$) A refutation of the negation of A is a proof of A

We will consider three different notions of refutation (as well as three different negation operators) and we will discuss them in relation to their soundness to **LE** clauses and the two principles governing negation.

4.2 Refutations as open derivations

4.2.1 Intuitionistic negation

The usual way of doing proof-theoretic semantics does not make reference to refutations. As a result, to characterize the meaning of negation is not an easy task. For, due to the absence of a notion of refutation, the only natural characterization of the proof-conditions of the negation of a sentence is the following: the negation of A, $\neg A$, is proved when it is impossible to prove A. The difficulty arises since proofs are represented by derivations which are constituted by applications of rules.[3] And as rules specify how to produce new proofs, it is not clear how to characterize the impossibility of obtaining, or the 'absence' of, proofs.

The solution offered by the **BHK** semantics is the following:

- the construction c proves $\neg A$ iff c is a general method of construction such that for any construction a proving A, $c(a)$ proves $\bot$;

provided that

- there are no constructions proving $\bot$.

---

[3]Throughout the chapter we will always make reference to derivations in given formal systems. Hence, we avoid the term 'argumentations'. Cf. note 23 on page 59.

That is, the sentence ⊥ is introduced and by definition there is no construction proving it, i.e. ⊥ counts as an absurdity. The absence of proofs of any other sentence is defined as follows: if there were a proof of the sentence, then it would be possible to produce a proof of the absurd sentence ⊥.

At the syntactic level, ¬A is defined, in full analogy with the **BHK** clause, as $A \to \bot$. So we have two rules for negation (an introduction and an elimination), which are nothing more than special cases of the implication rules.

$$\frac{\begin{array}{c}[A]^n \\ \vdots \\ \bot\end{array}}{\neg A} \, I\neg^n \qquad \frac{A \quad \neg A}{\bot} \, E\neg$$

Such a characterization grasps all properties of intuitionistic negation except the fact that no construction proves the absurdity. So, we have to add further rules to fix the intended meaning of ⊥. The natural deduction system for intuitionistic logic **NJ** is obtained by adding the following rule, so called *ex falso quodlibet*:

$$\frac{\bot}{A}$$

Once the absurd sentence has been derived, it is possible to derive everything. As the *ex falso* formally seizes intuitionistic logic, it is natural to think of it as grasping the intended meaning of ⊥.

The proof-theoretic semantics presented in chapter 2 is not sensibly modified by the introduction of the new connective: to account for it, the *ex falso* rule is restricted (without loss of generality) to atomic conclusions, and atomic systems are extended with the restricted *ex falso*.[4]

Nonetheless, the presence of ⊥ shows the limit of the proof-theoretic semantics, at least of the notion of validity proposed by Dummett and Prawitz. As we remarked in section 2.5.2 Dummett is willing to interpret open derivations as proofs of the conclusion given an assignment of proofs for the open assumptions. But then, we have that V-valid open derivations of conclusion ⊥ from

---

[4]Indeed, this solution—due to Prawitz (1971)—is not as innocuous as it appears. For further development, see Dummett (1991, ch. 13, pp. 295 and ff.) and the criticism raised against his supposed solution, informally by Hand (1999), and recently, in a technically concise manner, by Sandqvist (2009).

assumption A should be evaluated on proofs of $\bot$, given proofs of A. But, by definition, there is no proof of $\bot$ and, given consistency, no proof of A.

The validity of an open argumentation is equated, in the definition of validity, with that of its instances. But in the cases in which no instances can be available, not even in any possible extension either of the atomic system or of the set of reductions, then the open argumentation is automatically valid. And open derivations having $\bot$ as conclusion patently demonstrate that there are indeed such argumentations.

The gap between the account of open argumentations in terms of their closed instances and of intuitionistic methods in terms of hypothetical constructions is here evident, due to the absence of closed instances of open argumentations having $\bot$ as conclusion. That is, these cases show that the notion of 'proof' goes beyond the notion of 'proof we can refer to by means of valid closed derivations'. As in the case of objects, where the need of arbitrary object was acknowledge by the presence of universal quantifier, so here, a full account of implication (in particular of negation which is defined through it) requires an analogous notion, that we can identify with the intuitionist one of hypothetical construction. Just as the need of arbitrary objects was equated with a more substantial notion of reference to predicates, so here that the need of hypothetical constructions amounts to the ascription to open argumentation of a more substantial role.

We do not confront ourselves with this issue now, but we will reconsider it in the next chapter. In this one, we will concentrate on the possibility of characterizing a genuine notion of refutation.

4.2.2 Refutations in NJ

In spite of these problems, open derivations of $\bot$ in **NJ** are sometimes referred to as refutations. That is in **NJ** :

Thesis 1 *Refutations of A are represented by V-valid top-open derivations having $\bot$ as conclusion and A as the only open assumption.*

The notion of refutation of thesis 1 is strongly connected to the clauses proposed by Lopez-Escobar (1972) for characterizing the notion of refutation. The

Table 4.1: Refutations through **NJ** elimination rules

Given a refutation of A we get a refutation of $A \wedge B$

$$
\begin{array}{c}
A \\
\vdots \\
\bot
\end{array}
\qquad\qquad
\begin{array}{c}
\dfrac{A \wedge B}{A} \\
\vdots \\
\bot
\end{array}
$$

Given two refutations of A and B we get a refutation of $A \vee B$

$$
\begin{array}{cc}
A & B \\
\vdots & \vdots \\
\bot & \bot
\end{array}
\qquad\qquad
\dfrac{\quad\quad [A] \quad [B]\quad}{\quad A \vee B \quad \vdots \quad \vdots \quad}
\begin{array}{c}
\\
\bot \quad \bot \\
\hline
\bot
\end{array}
$$

Given a proof of A and a refutation of B we get a refutation of $A \rightarrow B$

$$
\begin{array}{cc}
 & B \\
\vdots & \vdots \\
A & \bot
\end{array}
\qquad\qquad
\begin{array}{c}
\quad\quad \vdots \\
\dfrac{A \rightarrow B \quad A}{B} \\
\vdots \\
\bot
\end{array}
$$

---

idea is roughly that as introduction rules correspond to the usual **BHK** 'proof clauses', elimination rules correspond to the **LE** 'refutation clauses'. In table 4.1 we show how refutations for logically complex sentences can be produced by applying elimination rules to refutations of their sub-sentences.[5]

---

[5]Tennant (1999) proposed a notion of refutation very similar to this one. Wansing (1999) criticizes it in several respects. We agree with him in all respects, apart from one. Section 4.2.3 can be seen as an answer to one of the criticism raised by Wansing against Tennant. We reassess, although in a different manner, the main criticism of Wansing in section 4.2.4, that highlights an asymmetry between the notion of proof and the one of refutation characterized by thesis 1. Finally, the way in which we introduced the notion of proof (viz. as a semantic notion) suggests a further criticism against this characterization of refutations, that is presented in section 4.2.5.

As an example, we can consider conjunction. The clause states that to refute a conjunction we need a refutation of either of the conjuncts. This is represented by the fact that if we have a derivation of conclusion $\perp$ from A (or B) as the only open assumption, we can produce a derivation having $\perp$ as conclusion having $A \wedge B$ as the only open assumption.

Furthermore, as an open derivation of conclusion $\perp$ from assumption A intuitively represents a method transforming proofs of A into proofs of the absurdity, by defining refutations according to thesis 1, the **BHK** negation clause can be reformulated as principle $\mathbf{N_P}$.

4.2.3 Refutations and assumptions

There seems to be something counter-intuitive in thesis 1, namely the fact that refutations of sentences are represented by derivations in which the refuted sentences figures as *assumption* instead of *conclusion*. In the case of proofs, the availability of certain derivations allows us to conclude that some sentences are proved, namely the conclusions of the derivations. Analogously, we expect that in the case of refutations, the availability of certain derivations allows us to conclude that some sentences are refuted. And it is natural to expect that the sentences in questions are the conclusions of the derivations. On the contrary, thesis 1 claims that the refuted sentences are the assumptions of the derivations.

This criticism is flawed by an ambiguity in the use of the word 'conclusion'. Conclusions, as we defined this terms, are the sentences that lay at the bottom of derivations. Analogously assumptions, as we defined this terms, are the sentences that lay at the top of derivations. Derivations, being syntactic objects, *per se* do not tell us anything about sentences being proved or not. So 'being an assumption' and 'being a conclusion' are simply predicates that attach to sentences if they lay in some special positions in the formal derivations.

It is the semantics that interprets the formal objects in such a way that given a V-valid closed derivation having A as conclusion, we can 'conclude' that A is proved. But this latter use of 'conclusion' is, at least in general, different from the merely syntactic one we adopted. A sentence A is the conclusion (in the sense we adopted) of a derivation if and only if it lays at the bottom of

the derivation; on the other hand, that A is proved is a *metalinguistic* conclusion that we draw from the semantic account of the object language in which derivations are produced.

So, this latter use of the term 'conclusion' is not in conflict with the fact that in a refutation of A, A is the assumption. For, being an assumption is a syntactic property of the sentence; on the other hand, the fact that from the availability of a derivation of $\bot$ having A as assumption we 'conclude' that A is refuted has to do with how we interpret derivations. In particular, this idea is in no way in conflict with the fact that from the availability of a V-valid derivation having A as only assumption and $\bot$ as conclusion, we can 'conclude' that the sentence A is refuted. This 'conclusion' is a meta-linguistic conclusion that we draw from the availability of the derivation in the object language, together with the rules governing the interpretation of derivations provided by the semantic theory, in particular, from the definition of the notion of refutation provided by thesis 1. Such a notion of conclusion has nothing to do with the technical notion of conclusion used to label the sentences laying at the bottom of deductive trees.

As far as we can see, the criticism is a result of the idea that deduction has to do with judgments. That is, from the idea that the conclusion of a derivation is not a sentence, but the fact that a sentence is proved. As a consequence, it is natural to introduce refutations in such a way that we may have derivations having as conclusion that sentences are refuted.

But the fact that deduction has to do with judgments is a highly debatable thesis. Even if it is the way in which some formalisms are shaped (e.g. in Martin-Löf's type theory or in Frege's *Begriffschrift*), it is dubious that it is the best option in a natural deduction setting. In particular, the way in which we presented natural deduction and its proof-theoretic semantics in chapter 2 does not presuppose this.

Hence, the choice of characterizing refutations of sentences as represented by formal objects having the sentences on the top is not problematic. Nonetheless there are more substantial reasons of dissatisfaction with the notion of refutation characterized by thesis 1.

4.2.4   Thesis 1 and Lopez-Escobar's clauses

Even if by elimination rules we can come close to defining in **NJ** a notion of refutation satisfying **LE** clauses, we cannot be completely faithful to them.

For, in the case of conjunction, it is true that we can produce a derivation representing a refutation for the conjunction starting from one representing a refutation of either of the conjuncts with the elimination rule. But the converse does not hold: it is not the case that whenever we are in possession of a derivation standing for a refutation of a conjunction, we are also in possession of a derivation standing for a refutation of one of the two conjuncts. For we can easily refute $A \wedge \neg A$:

$$\frac{\dfrac{A \wedge \neg A}{A} \quad \dfrac{A \wedge \neg A}{\neg A}}{\bot}$$

but we cannot extract from it a refutation for either of the conjuncts.

As we said, the usual **BHK** proof clauses are taken to specify the *canonical* ways of establishing a sentence governed by a given logical operator. Hence, if a closed derivation is V-valid, it reduces to a V-valid closed I-canonical one that 'directly' represents a proof. If we look at the disjunction clause, we have that any V-valid closed derivation of $A \vee B$ reduces to one ending with an application of $I\vee$, having as sub-derivation a V-valid closed derivation of either of the disjuncts.

On the other hand, the refutation of $A \wedge \neg A$ is irreducible but still we cannot extract from it a refutation for either of the conjuncts. If we take **LE** clauses as specifying the canonical ways of refuting logically complex sentences, we have that the notion of refutation we defined is irreducibly non-canonical.

In a sense, it is not surprising that in the verificationist framework based on **NJ**, we cannot get a full symmetry between proofs and the notion of refutation characterized by thesis 1. The connection of refutation and elimination rules and the subsidiary role played by elimination rules naturally yields an indirect notion of refutation.

Concerning negation, we said that intuitionistic negation satisfies principle $N_\mathbf{P}$. Due to the asymmetry between proofs and refutations, we have that it does not satisfy $N_\mathbf{R}$: for a refutation of $\neg A$ yields a proof of $\neg\neg A$, which is in general not equivalent to a proof of A.

4.2.5   A syntactic notion

Beside this, there is another reason of dissatisfaction with the notion of refutation characterized by thesis 1.

By introducing this notion of refutation we do not modify the semantics given in chapter 2. That is, the notion of refutation of thesis 1 can be taken as a syntactic notion, i.e. as a sub-class of derivations. Semantically, refutations are evaluated on proofs of the absurdity given an assignment of proofs for the open assumptions.[6] Hence, the real semantic notion is still the one of proof. The notion of refutation is a mere embellishment.

Hence, elimination rules are still semantically interpreted as proofs of their conclusions given proofs of their assumptions. The role of eliminations as specifying the conditions of refutations of sentences is accidental, as a result of the introduction of the (syntactic) notion of refutation.

This subsidiary role of refutations also emerges by reconsidering the general architecture of the verificationist theory of meaning. As we are in a verificationist theory of meaning, assertion is the linguistic act selected by the theory of force to be explained by the semantic theory. All other linguistic acts are explained uniformly in terms of assertion. In particular, the linguistic act of denial, which stands intuitively to refutations as assertion stands to proofs, is governed by the following principle of the theory of force:

 (D) The denial of a sentence is correct iff the assertion of its negation is

This means that if we have a valid closed (canonical) derivation of A, we can directly assert it; but if we have a valid refutation of A, we cannot directly deny it. First we have to produce a valid closed (canonical) derivation of $\neg A$ by $I\neg$. Such a derivation warrants the correct assertion of $\neg A$, which in turns allows the correct denial of A. I.e. the availability of a refutation of A only indirectly allows the denial of A.

So, from a very abstract perspective, we can say that in verificationism, the direct notion of proof as verification goes together with (or allows the definition of) an indirect notion of refutation.

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[6]At this point, it hard to make sense of such formulations. Nonetheless, this is the way the semantics, as presented by Dummett and Prawitz, should work.

4.3 Falsificationism

The notion of refutation considered so far is strongly reduced to the one of proof. For, a refutation of A, i.e. an open derivation having $\bot$ as conclusion and A as the only assumption, represents a proof of $\bot$ (given a proof of A).

This notion of refutation is merely syntactical, i.e. it is of no use for developing the semantics of the language. The semantics is based on the definition of V-validity, where V-valid derivations represent proofs.

What we like of the notion arising from thesis 1 is that the duality proof-refutation can be connected to the basic opposition in the proof-theoretic framework, the one between introduction and elimination rules. As introduction rules are primarily connected with proofs, refutations are connected with elimination rules. Introductions specify the condition for introducing a sentence as conclusion of derivations. Eliminations specify the consequences we can draw from a sentence, hence are connected with the use of sentences as assumptions. This suggest a richer analogy between proofs-introductions-conclusions on one side and refutations-eliminations-assumptions on the other.

Nonetheless, we saw the problems of the notion arising from thesis 1 (cf. sections 4.2.4 and 4.2.5). In the next sections, we will try to develop an alternative to standard proof-theoretic semantics in which the notion of proof is replaced by a notion of refutation resembling the one of thesis 1, but that does not suffer of its problems. We will stress the full symmetry between this new notion of refutation and the one of proof.

4.3.1 An alternative interpretation of deduction

As we saw, the **BHK** clauses, specifying the proof-condition of logically complex sentences, are the core of the verificationist theory of meaning and the definition of V-validity 8 encodes it in the proof-theoretic semantics presented in chapter 2. As verificationism aims at interpreting a full deductive language in terms of the notion of proof, we develop the idea of interpreting a full deductive language in terms of the notion of refutation.

To do this, we take the **LE** clauses as primitive and we 'forget' the **BHK** ones, just like in the verificationist theory of meaning only the **BHK** clauses are

encoded in the semantic notion of V-validity. We will present an alternative notion of validity of derivations encoding the intuitive content of the **LE** clauses, F-validity: F-valid derivations will represent refutations instead of proofs.

In verificationism, assertion is the basic linguistic act. As a consequence, we focus on the conclusions of derivations, in the light of the connection between the role of sentences as conclusions of derivations and assertion: the conclusion of V-valid top-closed (canonical) derivations can be asserted. This goes together with the view according to which assumptions are nothing but 'placeholders' for valid closed derivations, codified in the definition of V-validity.[7]

The alternative proof-theoretic semantics, that we wish to develop, can be viewed as the core of a theory of meaning alternative to verificationism, falsificationism.[8] Falsificationism can be conceived as selecting denial as the most important speech act. The task of the semantic theory is that of accounting for the correct denial of sentences, where the denial of a sentence is correct if and only if we have a refutation of it. In analogy with verificationism, the semantic theory develops a definition of F-validity that applies to the derivations that denotes refutations.

As we saw considering thesis 1, elimination rules are strongly connected with the **LE** clauses, just like introduction rules with the **BHK** clauses. Introduction rules specify the conditions for introducing a sentence as conclusion in a derivation. Hence, their distinctive role and their connection with assertion.

Elimination rules specify the consequences that can be drawn from a sentence. Hence they are naturally connected with the role of sentences as assumptions in derivations. This was reflected by thesis 1 in that, in refutations, the

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[7]The expression 'placeholders', to describe the treatment of undischarged assumption in the definition of validity was introduced by Schroeder-Heister (2004).

[8]The idea of a falsificationist theory of meaning and/or of a theory of meaning in which elimination rules fix the meaning of logical constant has been sketched by both Dummett and Prawitz in several places: see, among others, Dummett (1976, §5), (1991, Ch.13) and Prawitz (1987, §6), (2007, §3 and §4). Both authors suggest (sometimes) that the development of such a theory may lead to a logic 'which is neither classical nor intuitionistic'(Dummett 1976, p.83), but it looks as if they have not a settled opinion. Further confusion is due to the will of distinguishing between pragmatism and falsificationism, the former based on intuitionistic logic, the latter not necessarily. We believe the two theories to be the very same, and our presentation the most sound to the intuitions leading Dummett and Prawitz to their own formulations.

sentences refuted are the assumptions of the derivations representing them. We want to keep this connection between refutations and assumptions and eventually refine it. As a result, we have that the notion of F-validity must be shaped so that the assumptions of F-valid derivations are those that can be correctly denied.

According to such an interpretation, assumptions rather than conclusions are what we focus on in deduction, in the sense that assumptions are connected to the linguistic act selected by the theory. Thus, the meaning of logical constants is fixed by elimination rules as they typically specify which are the immediate consequence of a sentence governed by a given logical operator.

This prompts an inversion in the interpretation of the deductive relation. For, it seems natural to take conclusions as place-holders for refutations. That is, the definition of F-validity must be shaped so that valid open derivation can be interpreted as refutations of the assumptions given refutations of the conclusions. In particular, we want an open derivation of conclusion B and assumption A to be interpreted as follows: given a refutation of B, we get a refutation of A.

As inference rules are very simple open derivations, in verificationism they were interpreted as means of producing proofs of the conclusions from proofs of the assumptions. On the other hand, in a falsificationist framework they will be interpreted as means of producing refutations of the assumptions from refutations of the conclusions. Elimination rules will play the distinctive role of producing refutations of complex sentences from refutations of their sub-sentences.[9]

---

[9]The idea of Popperian flavor at the core of falsificationism is that, in accepting a sentence, a speaker must also be ready to accept all its consequences. Whenever one of its consequences turns out to be unacceptable, so too must the sentence upon which it depends be rejected. An analogy can be developed between proof-search and what we may call refutation-search. Proof-search activity (in a natural deduction setting) starts from the conclusion of the derivation and goes up to find assumptions already established (i.e. atoms derivable in the atomic system) or that can be discharged by some of the inferences performed on the path to the conclusion. Refutation-search on the other hand, starts from the assumption and goes down looking for consequences of it that must be refuted. The analogy can be pushed further in the light of the interpretation of inference rules in the two theories: in verificationism, proof-search is bottom-up and rules are interpreted top-down (i.e. given proofs of the premises we get proofs of the conclusions); in falsificationism,

To avoid confusion, we stress again that falsificationism arises as an alternative interpretation of a deductive language. A deductive language is constituted by an inductive definition of the notion of sentence and an inductive definition of the notion of derivation (in general from assumptions Γ to conclusions Δ). The latter one is given in terms of rules of inference that simply tell how to produce more complex derivations starting from simpler ones. How derivations are to be interpreted is a matter of the semantics, or more in general of the theory of meaning, selected.

The idea is that the very same deductive language could be interpreted both ways. Nonetheless, the development of a theory of meaning on verificationist grounds tends to select intuitionistic logic as the most adequate to the meaning-theoretical intuitions.[10] As we saw in section 4.2.4, the notion of refutation definable in **NJ** is not faithful to the **LE** clauses. This suggests that the falsificationist interpretation of deduction, as well as the verificationist one, tends to select a specific logic as most adequate for the semantics in terms of refutations. We claim that the development of a falsificationist perspective can be more naturally brought out in the so-called dual-intuitionistic logic.[11]

Just as we have done for verificationism, we will first present the falsificationist interpretation of a negation-free language.

---

refutation-search is top-down and rules are interpreted bottom-up (i.e. given refutations of the conclusions we get refutations of the premises).

[10]The claim is indeed highly debatable. Nonetheless, it has a certain plausibility if one sticks to the natural deduction formalism. Remarks on the intuitionistic flavor of natural deduction are given, e.g., by Garson (2001).

[11]Dual-intuitionistic logic has been studied from an algebraic point of view (Czemark 1977, Goodman 1981) and in the sequent calculus framework (Urbas 1996). The idea of interpreting derivations in dual-intuitionistic logic as refutations is suggested by Shramko (2005) and the idea of using it to refine Dummett's (unsatisfactory) falsificationist view is proposed by Miller (2006, Ch. 13). The present work aims at an enrichment of the cited ones, by presenting a natural deduction system for this logic and developing a semantic based on the notion of refutation in the proof-theoretic semantics spirit. I thank one of the referees for bringing the work of Miller to my attention.

Table 4.2: **NDJ** rules

$$\frac{A_1 \wedge A_2}{A_i} \; E\wedge \qquad\qquad \frac{A \wedge B \quad \begin{matrix} C \\ C \\ \vdots \\ [A]^n \end{matrix} \quad \begin{matrix} C \\ \vdots \\ [B]^n \end{matrix}}{} \; I\wedge^n$$

$$\frac{A \vee B}{A \quad B} \; E\vee \qquad\qquad \frac{A_i}{A_1 \vee A_2} \; I\vee$$

$$\frac{B \prec A}{A} \; E\prec^n \qquad\qquad \frac{A}{B \quad B \prec A} \; I\prec$$
$$\vdots$$
$$[B]^n$$

$$\frac{\neg A}{\top} \; E\neg^n \qquad\qquad \frac{\top}{A \quad \neg A} \; I\neg$$
$$\vdots$$
$$[A]^n$$

$$\frac{A}{\top}$$

As usual $i = 1, \; 2$ and square brackets stand for equivalence classes of occurrences of formulas

4.3.2 Dual-intuitionistic logic

In sequent calculus, we obtain the intuitionistic system **LJ** by restricting classical sequents (and inference rules accordingly) of the system **LK** to at most one sentence in the succedent of sequents. In the light of the correspondence between succedent in sequents and conclusions in natural deduction derivations, the restriction is a the formal counterpart of the verificationist meaning-theoretical intuition according to which conclusions are the core of deductive

processes.

Dual-intuitionistic logic, on the other hand, is obtained by restricting classical sequents to at most one sentence in the antecedent. In the light of the correspondence between sentences in the antecedents of sequents and assumption, this restriction seems to be a good way of formally grasping the falsificationist idea that assumptions are the core of deductive processes.

In the natural deduction framework, a proper formulation can be obtained by means of trees branching downward instead of upward. In table 4.2 we give the rules of the full natural deduction system for dual-intuitionistic logic **NDJ**.

4.3.2.1   Comment on the rules

Conjunction elimination and disjunction introduction are the very same in both **NJ** and **NDJ**. As they have just one assumption and one conclusion, they are unaffected by the shift from the single-conclusion multiple-assumption setting to the single-assumption multiple-conclusion one. Nonetheless, they now receive a different interpretation: the first one claims that given a refutation of either of two sentence, one can obtain a refutation of the conjunction of the two; the second one that from a refutation of a disjunction one can extract a refutation of each disjunct.

The unusual presentation of conjunction introduction and disjunction elimination is explained as follows. Suppose we have two deductions having as assumption (respectively) C and D and among the conclusions (respectively) A and B

$$
\begin{array}{cc}
C & D \\
\vdots & \vdots \\
A & B
\end{array}
$$

Intuitively, we would like to apply the $\wedge$-introduction rule to obtain a derivation having among the conclusions the sentence $A \wedge B$:

$$
\begin{array}{cc}
C & D \\
\vdots & \vdots \\
\dfrac{A \quad B}{A \wedge B} & I\wedge
\end{array}
$$

But the problem is that the resulting derivation has no unique assumption: how can we choose between C and D? The solution to this problem is to restrict the application of the introduction rule to derivations having the same

assumption. So the rules tell us that once we have two refutations of C having (respectively) A and B among the conclusions, we can obtain a refutation of C having $A \wedge B$ among the conclusions.

This exactly corresponds to the fact that in **NJ**, in formulating the $E\vee$ rule, we have to require that the sub-derivations of the minor premises have the same conclusion: suppose on the contrary we have a derivation of conclusion C from assumption A and of conclusion D from assumption B. The application of the rule wouldn't be possible, since we wouldn't be able to select the conclusion:

$$\frac{A \vee B \quad \overset{\displaystyle [A] \atop \displaystyle \vdots}{C} \quad \overset{\displaystyle [B] \atop \displaystyle \vdots}{D}}{?} \; E\vee$$

Just as in **NJ**, in **NDJ** we incorporate the restriction on assumptions directly in the rule.

Dually, in a multiple-conclusions framework there is no need of presenting the disjunction elimination rule by making reference to minor premises, as the multiple-conclusions rule makes perfectly sense. To refute a disjunction, one needs a refutation for each of the two disjuncts.

4.3.2.2   The duality between NJ and NDJ

NDJ can be seen as obtained by re-writing **NJ** upside-down and exchanging the connectives with their duals. More precisely, we define an isomorphism * so that given a derivation π in **NDJ** having A as assumption and Γ as conclusions, π^* is a derivation in **NJ** having Γ^* as assumptions and A^* as conclusion[12]:

Definition 9 (Mapping from NDJ to NJ) *First we map sentences on sentences:*

- *If A is atomic, then $A^* = A$*

- *If $A = B \wedge C$, then $A^* = B^* \vee C^*$*

- *If $A = B \vee C$, then $A^* = B^* \wedge C^*$*

- *If $A = B {\prec} C$, then $A^* = B^* \rightarrow C^*$*

---

[12]The analogous result for sequent calculus is Urbas (1996) Theorem 3.1

- *If $A = \dashv B$, then $A^* = \neg B^*$*

- *If $A = \top$, then $A^* = \bot$*

Then we can map derivations on derivations:

- *If π consists of A alone, then π^* consists of A^* alone;*

- *if π begins with a conjunction elimination, then π^* ends with a disjunction introduction;*

- *if π begins with a conjunction introduction, then π^* ends with a disjunction elimination;*

- *if π begins with a disjunction elimination, then π^* ends with a conjunction introduction;*

- *if π begins with a disjunction introduction, then π^* ends with a conjunction elimination;*

- *if π begins with a co-implication elimination, then π^* ends with an implication introduction;*

- *if π begins with a co-implication introduction, then π^* ends with an implication elimination;*

- *if π begins with the $\top$ rule, then π^* ends with the $\bot$ rule.*

Theorem 1 *The mapping $*$ is an isomorphism (i.e. $\pi^{* \, *-1} = \pi$): π is an **NJ**-derivation of conclusion A and assumptions Γ iff π^* is an **NDJ**-derivation of assumption A^* and conclusions Γ^**

Proof Trivial from the mapping.   ∎

While derivations in **NDJ** are mapped onto derivations in **NJ**, it does not happen so for sub-derivations. Rather, given a derivation π of assumption A and conclusions Γ, the portions of π that are mapped onto the sub-derivations of π^* are the derivations having as conclusions the conclusions of π and as assumptions (respectively) the conclusions of the rule having the assumption of π as premise. Thus, it appears natural to refer to these portion of π as π 'sub-derivations'.

That is, we can distinguish between two notions of sub-derivations: the usual ones, we will refer to as *top-sub-derivations* of π, are the derivations having as assumptions a sub-set of π assumptions and as conclusion one of the premises of the rule having the conclusion of π as conclusion; *bottom-sub-derivations* are the derivations having as conclusions a sub-set of π conclusions and as assumption one of the conclusions of the rule having the assumption of π as premise. We will always speak of sub-derivations, the context making clear which of the two notions we are referring to.

4.3.2.3   Co-implication: the dual-deduction theorem

In intuitionistic logic, implication is the operator that 'internalizes' the deducibility relation as the deduction theorems shows:

Theorem 2 (Deduction Theorem)

$$\Gamma \vdash_{NJ} A \text{ iff } \vdash_{NJ} \bigwedge \Gamma \rightarrow A$$

If Γ consists of only one sentence, the theorem explains the connection between implication and top-open derivations: a proof of an implication is a method that takes proofs of the antecedent into proofs of the consequent.

In dual-intuitionistic logic, co-implication[13] plays this role as the following theorem shows:

Theorem 3 (Dual-deduction Theorem)

$$A \vdash_{NDJ} \Gamma \text{ iff } \bigvee \Gamma \prec A \vdash_{NDJ}$$

Proof Suppose $\Gamma = B_1 \ldots B_n$. Given an open derivation of assumption A and conclusions $B_1 \ldots B_n$

$$
\begin{array}{c}
A \\
\vdots \\
B_1 \quad \ldots \quad B_n
\end{array}
$$

---

[13]A *caveat*: co-implication, as we define it here is the converse of Urbas'. For the history of the notation, an the confusion concerning the proper way of rendering the idea of an operator dual to implication, cf. Schroeder-Heister (2009) and Wansing (2009). Our choice is motivated by the isomorphism presented in section 4.3.2.2: the resulting rules of **NDJ** are exactly the rules of **NJ** turned upside-down, with ∨ and ∧ (and *viceversa*), → and ≺, ¬ and ⫞, ⊥ and ⊤ exchanged.

by substituting each conclusion with a derivation constituted only by applications of $I\vee$ having $B_1 \vee \ldots \vee B_n$ as conclusion and then applying $E\prec$ we get

$$\frac{(B_1 \vee \ldots \vee B_n)\prec A}{A}$$

$$\vdots$$

$$\frac{B_1}{B_1 \vee B_2} \qquad\qquad \frac{B_n}{B_{n-1} \vee B_n}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$[B_1 \vee \ldots \vee B_n] \quad \ldots \quad [B_1 \vee \ldots \vee B_n]$$

that is a derivation of assumption $(B_1 \vee \ldots \vee B_n)\prec A$ with no conclusions.

Conversely, suppose we have a derivation of assumption $(B_1 \vee \ldots \vee B_n)\prec A$ with no conclusions, by applying first $I\prec$ and then a chain of $E\vee$, we get:

$$\frac{A}{(B_1 \vee \ldots \vee B_n)\prec A \quad \dfrac{B_1 \vee \ldots \vee B_n}{B_1 \quad B_2 \vee \ldots \vee B_n}}$$

$$\vdots$$

$$B_{n-1} \ldots B_n$$

that is a derivation of assumption A and conclusions $B_1 \ldots B_n$ (as the derivation of assumption $(B_1 \vee \ldots \vee B_n)\prec A$ has no conclusion *ex hypothesis*). ∎

As implication is an 'assumption-discharge' device, so co-implication appears as a 'conclusion-discharge' device. The duality naturally suggests to introduce a distinction between *bottom-closed* and *bottom-open* derivations, a derivation being bottom-closed when it has no undischarged conclusions.

According to the isomorphism, bottom-closed derivations in **NDJ** are mapped onto top-closed derivations in **NJ** and *viceversa*.

Furthermore, if Γ consists of only one sentence, the theorem explains the connection between co-implication and bottom-open derivations: a refutation of a co-implication is a method that takes refutations of the antecedent into refutations of the consequent.

4.3.3   Harmony

In verificationism, the fact that introduction rules fix the meaning of logical connectives, does not mean that a sentence governed by the constant C can be established only by C-introduction. The distinction between I-canonical and

non-I-canonical derivations explains this fact: a closed non-I-canonical derivation is V-valid if it reduces (modulo a given set of reduction procedures) to a V-valid closed I-canonical one.

In falsificationism, we have that elimination rules specify only the most direct way in which logically complex sentences can be refuted. This suggests to introduce an analogous distinction between E-canonical and non-E-canonical derivations, where this time an E-canonical derivation is a derivation beginning with an elimination rule. We will simply speak of canonical derivation, when the context disambiguates between the two.

The reduction procedure should then allow to reduce any (bottom-)closed refutation into an E-canonical one. In table 4.3 we specify the reductions associated to the patterns constituted by an introduction rule followed by the corresponding introduction rule.

In analogy with verificationism, a normalization result, warranting that the rules presented are in harmony, can be established. In particular, one can read

Table 4.3: Reduction of maximal sentences

$$\frac{\dfrac{A_i}{A_1 \vee A_2}\ I\vee}{A_1 \quad A_2}\ E\vee$$
$$\vdots \quad \vdots \qquad\qquad \dashrightarrow \qquad \begin{matrix}A_i \\ \vdots\end{matrix}$$

$$\frac{A_1 \wedge A_2}{A_i}\ E\wedge \qquad \frac{C}{\ \ C \quad\ \ C}\ I\wedge^n \qquad\qquad \begin{matrix}C\\ \vdots\\ A_i\\ \vdots\end{matrix}$$
$$\vdots \qquad [A_1]^n \quad [A_2]^n \qquad \dashrightarrow$$

$$\frac{B \quad \dfrac{A}{B\prec A}\ I\prec}{A}\ E\prec^n \qquad\qquad \begin{matrix}A\\ \vdots\\ B\\ \vdots\end{matrix}$$
$$\vdots \qquad [B]^n \qquad\qquad \dashrightarrow$$

$$i = 1,\ 2$$

Prawitz (1971, §2) to obtain a normalization theorem for **NDJ**, according to the isomorphism. To give an idea of how things look like, we state the reduction of maximal sentences in table 4.3, as they obtain from **NJ** reductions. We do not state reductions of maximal segments, but again they can be obtained from the isomorphism.

4.3.4   F-validity

At this point, we can present the definition of F-validity for derivations, the crucial notion of the proof-theoretic semantics at the core of the falsificationist theory of meaning. As we have done for the verificationist notion, we relativize the definition to atomic systems. Again the atomic system codifies the available information involving atomic sentences. In this case, the rules of atomic systems have the form

$$\frac{B}{A_1, \ldots, A_n}$$

The limit cases with no conclusions accounts for the atoms speakers are entitled to deny on the basis of extra-deductive evidence available. The applications of rules with no conclusions produce bottom-closed derivations of atoms.

Definition 10 (F-validity$_S$) *Given an atomic system S[14]*

- *bottom-closed derivations in S are F-valid$_S$;*

- *bottom-closed E-canonical derivations are F-valid$_S$ if their immediate sub-derivations are F-valid$_S$;*

- *bottom-closed non-canonical derivations are F-valid$_S$ if they reduce to F-valid$_S$ closed canonical derivations;*

- *bottom-open derivations are F-valid$_S$ if the result of substituting F-valid$_{S'}$ ($S' > S$) bottom-closed derivations for the open conclusions yields F-valid$_{S'}$ bottom-closed derivations;*

---

[14]The definition is relative only to the atomic system, instead of both the atomic system and the set of reduction procedures, simply because we are dealing with a specific formal system, i.e. **NDJ**, whose set of reduction procedures is fixed. This definition, corresponding to Schroeder-Heister's (2006, §3.3) definition 2 of S-validity, can be easily converted into a more general definition of validity for argumentations (relative to both atomic system and sets of reductions) that would be the dual of definition 8.

4.3.5 Negation in Falsificationism

In the verificationist perspective, we have that the negation of a sentence is proved when we show that it is impossible to obtain a proof of the sentence. In analogy with this, in falsificationism a dual notion of negation can be introduced. The intuitive characterization of the meaning of this operator is obtained by exchanging proofs with refutations: we have a refutation of the negation of a sentence when we show that it is impossible to obtain a refutation of the sentence.

It is not immediate how to embody in a deductive system the idea of the impossibility of obtaining certain deductive patterns. In verificationism, the following solution is adopted. A constant standing for an unprovable sentence is introduced. Whenever the unprovable sentence is obtainable from a given sentence, this means that the the latter one cannot be proved as well. The absurdity constant $\bot$ plays exactly this role. Whenever we have an open derivation having $\bot$ as conclusion and only one sentence as open assumption, we claim the sentence to be not provable; that is, we are in the position of asserting the negation of that sentence.

Adapting this solution to the falsificationist framework, we obtain the following: first of all we introduce a constant $\top$ standing for a sentence that cannot be refuted. If we can produce a derivation having $\top$ as assumption and a sentence A as the only conclusion, then we can say that it is impossible to obtain a refutation of A, otherwise we would have a refutation of $\top$:

$$
\begin{array}{c}
\top \\
\vdots \\
A
\end{array}
$$

According to the meaning that negation has in falsificationism, we are then entitled to deny the negation of A, $\neg A$. We can seize this with the following elimination rule for negation:

$$
\begin{array}{c}
\dfrac{\neg A}{\top} \ \ E{\neg}^n \\
\vdots \\
[A]^n
\end{array}
$$

To this elimination corresponds the following multiple-conclusion introduc-

tion rule:

$$\frac{\top}{A \quad \neg A} \; I\neg$$

which reads as follows: given a refutation of A and of $\neg A$ we would have a refutation of $\top$.[15]

In verificationism, the meaning of $\bot$ is specified with the *ex falso* rule, claiming that if we had a proof of the absurdity, we would have a proof of any sentence. Naturally, we can introduce an analogous rule in order to characterize our new constant, claiming that if we had a refutation of $\top$, we would have a refutation of everything:

$$\frac{A}{\top}$$

4.3.6 Proofs in NDJ

In verificationism, we saw how the notion of open derivation having $\bot$ as conclusion can be taken as a notion of refutation. Analogously, in falsificationism we can think of a sort of notion of proof as characterized by means of the notion of open refutation having $\top$ as assumption: a proof of A is *a derivation having $\top$ as assumption and A as the only conclusion*.

We speak only of a 'sort' of notion of proof: for, the criticism raised against the notion of refutation as open derivation of the absurdity considered in section 4.2.4 and 4.2.5 can be adapted to this notion as well.

In particular, this notion of proof is not faithful to the **BHK** proof clauses. In other words, it is weaker than the intuitionistic one in the sense that it allows more sentences to be 'proved'. As an example, it may be worth considering the following:

$$\frac{\dfrac{\top}{\dfrac{A}{A \vee \neg A} \; I\vee \quad \dfrac{\neg A}{A \vee \neg A} \; I\vee}}{} \; I\neg$$

which is a derivation having the excluded middle as conclusion and $\top$ as assumption. The derivation is irreducible, but it is a proof of a disjunction despite neither of the disjuncts is proved.

---

[15]In analogy with what happens in verificationism, the rules can be seen as a special case of the co-implication rules. That is, in falsificationism, $\neg A =_{def} A \prec \top$.

Due to the falsificationist reading of deductive processes, this notion of proof is indirect also from the abstract perspective of the architecture of the theory of meaning. As falsificationism considers denial the only basic linguistic act to be explained by the semantic theory, the correctness of an assertion is uniformly derived within the theory of force from the correctness of a denial, in terms of the following principle:

(A) The assertion of A is correct iff the denial of $\dashv A$ is correct

In fact, the derivation considered does not directly allow the assertion of $A \vee \dashv A$: first we have to obtain a deduction allowing the denial of the negation of the excluded middle by means of negation elimination:

$$
\cfrac{\cfrac{\dashv(A \vee \dashv A)}{\cfrac{\cfrac{A}{[A \vee \dashv A]^1} \ I\vee \quad \cfrac{\dashv A}{[A \vee \dashv A]^1} \ I\vee}{\top}} \ I\dashv}{} \ E\dashv^1
$$

This derivation warrants the denial of $\dashv(A \vee \dashv A)$ which in turn allows the assertion of the excluded middle.

Finally, as the falsificationist definition of F-validity stresses, a proof of A is interpreted as a refutation of $\top$ given a refutation of A rather than as a proof (in the genuinely semantic sense) of A. The notion of proof is merely syntactically defined and it plays no real semantic role.

That is, the notion of proof that we can characterize in falsificationism is an indirect notion of proof.

So, in falsificationism we can claim that the direct notion of refutation allows the definition of an indirect notion of proof.

In the light of this notion of proof, dual-intuitionistic negation satisfies principle $\mathbf{N_R}$. On the other hand, it does not satisfy principle $\mathbf{N_P}$. A proof of the co-negation of A yields by $E\dashv$ a refutation of $\dashv\dashv A$, which is in general not enough to yield a refutation of A (just like the provability of $\neg\neg A$ does not warrant the provability of A in **NJ**).

Again this is no surprise. As in the verificationist framework based on **NJ** we cannot get a full symmetry between proofs and refutations, so in the falsificationist framework based on **NDJ**. The connection of proofs to introduction rules and the subsidiary role played by introductions naturally yields an indirect notion of proof.

4.3.7   A remark on classical logic

A deductive system for classical logic can be obtained by extending **NJ** with the classical *reductio ad absurdum* rule. Glivenko (1929) showed that intuitionistic logic shares the negative fragment with classical logic: a negated sentence is provable in intuitionistic logic if and only if it is also in classical logic. As a consequence, the two deductive systems share the refutability notions.

Theorem 4 (Glivenko) *There is a derivation of $\bot$ from assumption A in **NJ** iff there is one in **NK***

Proof The left to right direction is trivial. For the right to left one, classical proofs can be displayed as using at most one application of classical *reductio* as the last step of the proof. Hence, given a classical proof of $\neg A$, it will be constituted either by a refutation of A or by a refutation of $\neg\neg A$ (depending on the last rule of the deduction being a negation introduction or a classical *reductio*). In the latter case, it is easy to obtain a refutation of A by plugging a derivation of conclusion $\neg\neg A$ from assumption A at the top of the refutation. ∎

Classical logic can be also presented as an extension of dual-intuitionistic logic. The system **NDK** is obtained by extending **NDJ** with the dual of the rule of classical *reductio*:

$$\frac{\begin{array}{c}\dfrac{A}{\top}\\ \vdots \\ {[\neg A]^n}\end{array}}{} \; n$$

We define a translation K from sentences in **NDK** to sentences in **NK**:

Definition 11 (Translation from NDK to NK)

- *If A is atomic, then $A^K = A$*

- *If $A = B \wedge C$, then $A^K = B^K \wedge C^K$*

- *If $A = B \vee C$, then $A^K = B^K \vee C^K$*

- *If $A = \neg B$, then $A^K = \neg B^K$*

- *If $A = B \prec C$, then $A^K = \neg B^K \wedge C^K$*

The inverse translation K^{-1} is obvious, apart from the implication clause which we explicitly state:

- *If $A = B \rightarrow C$, then $A^{K^{-1}} = \neg B^{K^{-1}} \vee C^{K^{-1}}$*

The isomorphism between **NJ** and **NDJ** can be easily extended to **NK** and **NDK** by extending the mapping $*$ to $*^K$, by adding a clause for the dual *reductio* rule:

- if π begins with the dual *reductio* rule, then π^{*^K} ends with the *reductio* rule.

Hence, there is an **NDK** derivation of conclusions Γ from assumption A iff there is an **NK** derivation of assumptions Γ^* and conclusion A^*.

Furthermore, **NK** and **NDK** share the provability and refutability notions, that is

Theorem 5 (Equivalence of NDK and NK) *There is an **NDK** derivation of conclusion A from assumption $\top$ iff there is an **NK** top-closed derivation of conclusion A^K; there is an **NDK** bottom-closed derivation of assumption A iff there is an **NK** derivation of conclusion $\bot$ from assumption A^K.*

Proof We prove only the the left to right direction of the first claim. A proof of the other direction and of the dual claim can be obtain by a dual reasoning.

We define a mapping $\neg$ from **NDK** to **NK** by modifying $*^K$ in the basic clause for sentences as follows:

- if A is atomic, then $A^\neg$ is $\neg A$

Hence, if there is an **NDK** bottom-closed derivation of assumption $\dashv A$, there is an **NK** top-closed derivation of conclusion $(\dashv A)^\neg$. Since $(\dashv A)^\neg$ is $\neg A^\neg$, we complete the proof showing by induction on the number of logical constants in A that $\neg A^\neg$ is classically equivalent to A^K.

- If A is atomic, then $\neg A^\neg = \neg\neg A \leftrightarrow A = A^K$

- If $A = B \wedge C$, then $\neg A^\neg = \neg(B^\neg \vee C^\neg) \leftrightarrow \neg B^\neg \wedge \neg C^\neg$. By applying the induction hypothesis to $\neg B^\neg$ and $\neg C^\neg$, we get that $\neg B^\neg \wedge \neg C^\neg = B^K \wedge C^K = (B \wedge C)^K$.

- If $A = \neg B$, then $\neg A^{\neg} = \neg(\neg B)^{\neg} = \neg\neg B^{\neg}$. By applying the induction hypothesis to $\neg B^{\neg}$, we get that $\neg\neg B^{\neg} = \neg B^K = (\neg B)^K$.

- If $A = B \prec C$, then $\neg A^{\neg} = \neg(B^{\neg} \rightarrow C^{\neg}) \leftrightarrow \neg(\neg B^{\neg} \vee C^{\neg}) \leftrightarrow \neg\neg B^{\neg} \wedge \neg C^{\neg}$. By applying the induction hypothesis to $\neg B^{\neg}$ and $\neg C^{\neg}$, we get that $\neg A^{\neg} = \neg B^K \wedge C^K =_{def} (B \prec C)^K$.   ■

In the light of $*$ and $*^K$, a dual-Glivenko theorem holds for **NDJ** and **NDK**: there is a bottom closed derivation of $\neg A$ in **NDK** iff there is one in **NDJ**. As a consequence, dual-intuitionistic logic shares the positive fragment with classical logic; that is, they share the provability notions (cf. Czemark (1977, pp.472–473) and Urbas (1996, Theorem 2.1))

Theorem 6 *There is an **NDJ** derivation of assumption $\top$ and conclusion A as only conclusion iff there is an **NDK** one.*

Proof The left to right direction is trivial. For the other direction, one can reason in analogy with theorem 4.   ■

As a consequence of this and of theorem 5, we have the following:

Theorem 7 (Dual-Glivenko) *There is an **NDJ** derivation of conclusion A from assumption $\top$ iff there is an **NK** top-closed derivation of conclusion A^K.*

Proof The theorem follows directly from the result just established.   ■

Hence, the notions of proof and refutation grasped by classical logic are equivalent (respectively) to the indirect notion of proof definable in **NDJ** and to the indirect notion of refutation definable in **NJ**. That is, in classical logic the availability of a proof of a sentence amounts to the impossibility of falsifying it, i.e. to the impossibility of being in the position of correctly denying it. And the availability of a refutation of a sentence amounts to the impossibility of verifying the sentence, i.e. to the impossibility of being in the position of correctly asserting it. Recently, Restall (2005, p.10) proposed an interpretation of classical sequent calculus according to these lines:

> If $X \vdash Y$ then it is incoherent to assert all of X and deny all of Y.
> This has a number of special cases worth spelling out:

- If $A \vdash$ then it is incoherent to assert A.

- If $A, \ B \vdash$ then it is incoherent to assert both A and B.

- If $\vdash B$ then it is incoherent to deny B.

- If $\vdash A, \ B$ then it is incoherent to deny both A and B.

- If $A \vdash B$ then it is incoherent to assert A and deny B.

This interpretation actually validates all rules for classical logic. And Restall concludes from this that it is not possible to reject classical logic on proof-theoretical basis.

In the light of what we said, it is clear that Restall's account makes perfectly sense. Nonetheless, an independent account of assertion and denial has to be given. And as we saw, it is in terms of those that we called the two direct notions of proof and refutation that it is possible in verificationism and falsificationism to account (respectively) for the practices of assertion and denial.

In other words, the two notions, by means of which Restall (correctly) analyzes classical logic, are two indirect notions: the impossibility of asserting a sentence is to be understood on the background of the verificationist account of assertion by means of intuitionistic logic; and the impossibility of denying a sentence is to be understood on the background of the falsificationist account of denial by means of dual-intuitionistic logic.

Analogous remarks apply also to the 'anti-realist account of classical consequence' proposed by Rumfitt (2007), which is grounded on analogous ideas, even if not presented in a proof-theoretic fashion.

4.3.8 A direct notion of refutation

We presented the falsificationist interpretation of **NDJ** as based on the notion of F-validity. The resulting notion of refutation can be characterized with the following:

Thesis 2 *Refutations are represented by F-valid bottom-closed derivations of assumption A*

The notion is not flawed by the criticism raised against the notion of refutation characterized by thesis 1. For it is a genuinely semantic notion, in terms

of which the full language of **NDJ** is interpreted.

If we look at the **LE** clauses, we have that the problem presented in section 4.2.4 is solved: as F-valid closed derivations reduce to F-valid closed E-canonical ones, it is not possible that a conjunction is refuted without either of the conjuncts being refuted as well.

Nonetheless, as we have not presented rules for implication in **NDJ**, one may doubt the notion of refutation resulting from thesis 2 being adequate.[16] The reason for the omission is the following theorem.

Theorem 8 (Indefinability of $\rightarrow$ in NDJ) *No connective $*$, such that there is an F-valid derivation of conclusion $A * B$ from assumption $\top$ iff there is an F-valid derivation of conclusion A from assumption B, can be introduced in* **NDJ***.*

Proof Urbas's (1996) proof of theorem 5.4 can be naturally adapted to the natural deduction setting.   ∎

The deduction theorems presented in section 4.3.2.3 shows that implication is the device that represents the verificationist reading of deducibility and co-implication the falsificationist one. The impossibility of defining a direct notion of proof in verificationism and a direct notion of refutation in falsificationism is reflected by the theorem.

This is highlighted by the fact that we lack a clear intuition on the proof-conditions of co-implication. Analogously, while we have a strong intuition on the conditions of refutation of conjunction and disjunction, it is not so trivial that the conditions of refutation of implication should be those codified in the **LE** clause.

In analogy with Urbas (1996), a 'sort of' implication and co-implication can be defined anyway, obtaining (respectively) an enriched intuitionistic system **NJ**$^{\prec}$ and an enriched dual-intuitionistic system **NDJ**$^{\rightarrow}$). As implicitly stated in definition 11 on page 134, implication in **NDJ** can be defined through disjunction and co-negation: $A \rightarrow B =_{def} \neg A \vee B$; dually, in **NJ** $B \prec A =_{def} \neg A \wedge B$.

---

[16]Furthermore, there is no standard **LE** clause for co-implication. The reason for this is will emerge more clearly in section 4.4 where the **LE** clauses will be presented in the context of the original motivation of their introduction

Pure rules—in the sense of Dummett (1991, Ch.12), i.e. rules that do involve just one connective—are the following:

$$\prec \text{ in } \mathbf{NJ}:$$

$$\frac{A\prec B \quad A}{\bot} \ E1{\prec} \qquad \frac{A\prec B \quad \overset{[B]}{\underset{\vdots}{C}}}{C} \ E2{\prec} \qquad \frac{\overset{[A]}{\underset{\vdots}{\bot}} \quad B}{A\prec B} \ I{\prec}$$

$$\rightarrow \text{ in } \mathbf{NDJ}:$$

$$\frac{\top}{A \rightarrow B \quad A} \ I1 \rightarrow \qquad \frac{C}{\underset{[B]}{\overset{\vdots}{A \rightarrow B \quad C}}} \ I2 \rightarrow \qquad \frac{A \rightarrow B}{\underset{[A]}{\overset{\vdots}{\top \quad B}}} \ E \rightarrow$$

We briefly comment the rules of implication in **NDJ**.

The elimination rule for implication in **NDJ** is actually sound to the corresponding **LE** clause. For, if we have an F-valid derivation of conclusion A from assumption $\top$ (i.e. a proof, in the indirect sense, of A) and an F-valid bottom-closed derivation of assumption B (representing a refutation of A), we can obtain an F-valid bottom-closed derivation of assumption $A \rightarrow B$ representing a refutation of $A \rightarrow B$.

Concerning the introduction rules for implication in **NDJ**, a more natural $I \rightarrow$ would be the following:

$$\frac{\top}{\underset{[B]}{\overset{\vdots}{A \rightarrow B \quad A}}} \ I{*} \rightarrow$$

which could be read as follow: given a derivation having A as assumption and B as conclusion, we get a derivation having $\top$ as assumption and $A \rightarrow B$ as the only conclusion (B being discharged). That is, from an open derivation of conclusion B from assumption A, we get a proof (in the indirect sense of **NDJ**) of $A \rightarrow B$.

The rule is derivable from $I1 \to$ and $I2 \to$ as follows:

$$\dfrac{A \to B \quad \dfrac{\dfrac{\top}{A} \quad \dfrac{A \to B \quad A}{A} \, I1 \to^1}{}}{ } \, I2 \to$$
$$\vdots$$
$$[B]^1$$

But the converse doesn't hold, see Urbas (1996, p.442). While $I1 \to$ is just the special case of $I* \to$ in which no occurrence of the conclusion B is discharged, only a weaker form of $I2 \to$ can be derived from $I* \to$. In particular, given an open derivation of assumption C and and conclusion B, one can actually obtain a derivation of conclusion $A \to B$, but only from assumption $\neg\neg C$ (instead of C, as $I2 \to$ wants):

$$\dfrac{A \to B \quad \dfrac{\dfrac{\dfrac{\neg\neg C}{\top}}{A}}{ } \, 1}{} \, 2$$
$$\dfrac{\top}{C \quad [\neg C]^2}$$
$$\vdots$$
$$[B]^1$$

With the full strength of the $I2 \to$ rule, we have that, if the bottom-sub-derivation of assumption B consists of the formula B alone, this rule allows to obtain a derivation of assumption B and conclusion $A \to B$, corresponding to the possibility of introducing an implication without discharging any occurrence of the antecedent:

$$\dfrac{B}{A \to B \quad [B]} \, I2 \to$$

On the other hand, the weaker form derivable from the $I* \to$ would not allow the possibility of such applications: in particular, one could only get derivations of conclusion $A \to B$ and assumption $\neg\neg B$ (instead of assumption B).

As a consequence, by defining $A \to B$ with the weaker $I* \to$ rule, it would not be possible to extend the Dual-Glivenko theorem to $\mathbf{NDJ}^\to$. For example, the classically (and intuitionistically) provable sentence $A \to (B \to A)$ would not be provable in $\mathbf{NDJ}^\to$ (in the sense that it could not be possible to obtain a derivation of assumption $\top$ and $A \to (B \to A)$ as conclusion).

On the other hand, implication defined with the stronger $I1 \to$ and $I2 \to$ is such that $A \to B$ is provable in $\mathbf{NDJ}$ iff $A^\neg \to B^\neg$ is provable in $\mathbf{NK}$, that is, it allows the extension of the dual-Glivenko theorem to the full language.

Dual considerations apply to co-implication. By opportunely adding and removing implication and co-implication rules from **NJ** and **NDJ**, one gets all six systems inquired by Urbas (1996).

4.4 Refutations and direct negation

The fact that intuitionistic negation and the notion of refutation, characterized by thesis 1, do not satisfy **LE** clauses and both principles N_P and N_R is no surprise. For the **LE** clauses and the consequent principles governing negation were introduced in the literature by authors willing to propose an alternative notion of negation more 'constructive' than the usual intuitionistic one.

We now present the way in which these authors dealt with both the notions of negation and refutation. We will raise a criticism against their approach showing that, rather than introducing a genuinely semantic notion of refutation, they actually use a negation, stronger than both the intuitionistic and the dual-intuitionistic one, to mimic the semantic notion of refutation of thesis 2 within an enriched verificationist framework.

Since Nelson (1949), through Lopez-Escobar (1972), up to Wansing (1999), the **LE** clauses have not been used to give an interpretation alternative to the usual one (as we did), but on the contrary to enrich the constructivist conception, by introducing the direct notion of refutation *in addition* to the one of proof.

In order to deal with both semantic notions within the same interpretation, a new 'direct' negation operation ~ is added to the language of **NJ** and with its help the notion of refutation is to be represented in language.

The intuitive idea is that while proofs of A are represented by V-valid top-closed derivations having A as conclusion, refutations are represented by V-valid top-closed derivations having ~ A as conclusion. That is:

Thesis 3 *Refutations of A are represented by V-valid top-closed derivations of conclusions ~ A.*

In order to get a notion of refutation, as characterized by thesis 3, sound to the **LE** clauses, the rules of introduction and elimination of the direct negation

operator are 'split' into several rules governing the introduction and elimination of the direct negation of sentences according to their main logical operator. The resulting system, **N3** is obtained by adding to **NJ** the rules for ~ that we present in table 4.4.

Direct negation properly acts as hinge between the notion of proof and refutation; that is, it satisfies both N_P and N_R. This is shown by the semantic clauses for direct negation completing the **BHK** proof clauses and the **LE** refutation clauses for logically complex sentences:

- the construction c proves ~ A iff c refutes A;

- the construction c refutes ~ A iff c proves A.

Clearly, the two semantic notions are defined by simultaneous induction.

Table 4.4: **N3** direct negation rules

$$\frac{\sim A \quad \sim B}{\sim (A \vee B)} \ I \sim \vee \qquad\qquad \frac{\sim (A_1 \vee A_2)}{\sim A_i} \ E \sim \vee$$

$$\frac{\sim A_i}{\sim (A_1 \wedge A_2)} \ I \sim \wedge \qquad\qquad \frac{\sim (A \wedge B) \quad \overset{[\sim A]}{\underset{\vdots}{C}} \quad \overset{[\sim B]}{\underset{\vdots}{C}}}{C} \ E \sim \wedge$$

$$\frac{A \quad \sim B}{\sim (A \to B)} \ I \sim \to \qquad \frac{\sim (A \to B)}{A} \ E^1 \sim \to \qquad \frac{\sim (A \to B)}{\sim B} \ E^2 \sim \to$$

$$\frac{A}{\sim\sim A} \ I \sim\sim \qquad\qquad \frac{\sim\sim A}{A} \ E \sim\sim$$

4.4.1 An asymmetry

The following is an objection to the notion of refutation characterized by thesis 3. Even if there is a full symmetry between the two notions at the semantic level, there is an asymmetry in the way in which they are represented in language. For, direct negation acts as an explicit linguistic marker of refutations, while there is no corresponding marker for proofs.

One could answer the objection by claiming that there actually is a marker for proof, the assertion sign $\vdash$, which is simply left implicit in the formulation of the rules. The direct negation sign can be read as sort of denial sign, so that $\sim A$ reads 'A can be denied'. Hence, we would have a symmetry between the assertion sign marking proofs and the direct negation (denial) sign marking refutations.

But this counter-objection misses the point since even accepting the idea of a hidden assertion sign in the formulation of rules, there would be an asymmetry. The direct negation is a logical constant, while the assertion sign is not. While the direct negation signs does interact with other logical constants, the assertion sign does not. In particular, if one wishes to make the assertion sign explicit in the **NJ** rules, he should explicitly write it in the formulation of the direct negation rules of table 4.4 as well.

On the other hand, the notion of refutation characterized by thesis 2 does not suffer of this problem, as there is no connective that marks refutations: if we compare derivations representing (respectively) a proof and a refutation of A, it is A itself that figures in both. In one case it is the conclusion of the derivation, in the other one it is the assumption. Hence, in the case of thesis 2, the opposition between a proof and a refutation of A has not to do with whether A or $\sim A$ is the conclusion of the derivation, but with whether A is the assumption or the conclusion of the derivation.

4.4.2 Refutation through direct negation

This objection shows that it is doubtful that by means of thesis 3 we are really introducing a notion of refutation distinct from the one of proof at the semantic level. In particular, as a refutation of A is represented in language by a deriva-

tion of conclusion ~ A, a more tight connection between syntax and semantics can be presented by slightly modifying the informal semantic account.

The **LE** clauses can be rephrased by substituting 'refutes A' with 'proves ~ A'. This reformulation states the proof-conditions of the direct negation of logically complex sentences instead of the refutation condition of logically complex sentences:

- the construction c proves ~ $(A \wedge B)$ iff c is of the form $< i,\ d >$ with i either 0 or 1 and if $i = 0$, then d proves ~ A and if $i = 1$ then d proves ~ B;

- the construction c proves ~ $(A \vee B)$ iff c is of the form $< d,\ e >$ and d proves ~ A and e proves ~ B;

- the construction c proves ~ $(A \rightarrow B)$ iff c is of the form $< d,\ e >$ and d proves A and e proves ~ B;

- the construction c proves ~~ A iff c proves A

Instead of adding **LE** clauses to the usual **BHK** ones, we can add their reformulation. In this way, we could get rid of the notion 'the construction c refutes A' ending up with the notion 'the construction c proves A' alone. But nothing would change in the way in which derivations are interpreted. Indeed, the interpretation would look even more convincing. For, we would have that, if a construction proves A, it is represented by a derivation ending with A and, if it proves ~ A, it is represented by a derivation ending with ~ A.[17]

Of course, the modified clauses are a trivial consequence of the original ones of Lopez-Escobar. Nonetheless, the modified clauses reflect much more properly what happens in syntax than the original ones. In other words, the distinction between proofs and refutations is a mere embellishment. For one could start from the modified clauses, that is just with the semantic notion of proof (i.e. of construction proving a sentence).

Then a sort of distinction between 'proofs' and 'refutations' could be given in syntactical terms: if a derivation has a non-negated sentence A as conclusion, then it is a 'proof' of A. If the derivation has a negated sentence ~ A as

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[17]To get a semantics for the full language, we would then need a specification of the proof-conditions of atoms *and* of the direct negation of atoms. But this of course can be done by opportunely modifying the notion of atomic system.

conclusion, then it is simultaneously a 'proof' of $\sim A$ and a 'refutation' of A. But also this syntactic distinction is quite illusory: apart from derivations of non-negated sentences, that belong only to the class of 'proofs', all other derivations belong to both classes.

This syntactic notion of proof is of course distinct from the semantic one: the latter one is to be identified with the one of construction, where all constructions prove sentences (independently of their being negated sentences or not). Their linguistic presentations can be called 'proofs' or 'refutations'. But at the semantic level, we have only proofs.

This is reflected by the proof-theoretic semantics for a language such as **N3**. For, no modification is required in the definition of V-validity 8 given in chapter 2. What changes is simply that we have a richer set of introduction rules (the set of introduction rules of **N3** contains also the introduction rules for direct negated sentences) and a richer set of reduction procedures (Consecutive applications of introduction and elimination rules for direct negated sentences yields obvious reduction patterns.). Again, we end up with a notion of V-validity that applies to the derivations that represent the semantic values, proofs.

Wansing (1999) seems to agree (to some extent) on this point even if his way of stating it is far from clear. With his own words:[18]

> The interaction between proofs, negation and refutations developed above does not have direct proofs and refutations as a disjoint class. Instead, the difference between proofs and refutations is an intentional one: what may be regarded as a refutation of something may be viewed as a proof of something else. If this something is A, the something else is $\sim A$

Hence, rather than semantic, the distinction between proofs and refutations emerging from thesis 3 is nothing but a syntactical distinction between derivations having a non-negated or negated sentence as conclusions. At the semantic level, we simply have constructions that can be represented by sev-

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[18]Wansing uses the term 'disproof' while we used 'refutation', according to the terminology of Dummett and Prawitz. For homogeneity, we replace 'disproof' with 'refutation' in the quotation.

eral different derivations having a sentence, its double negation and so on as conclusion, which in turn can be called proofs or refutations at will.

Hence, the claim that we introduce a notion of refutation to have a better account of negation is overestimated. What this proposal amounts to is to introduce a direct negation operator, whose meaning is given by enriching the **BHK** semantics with a set of clauses stating the proof-conditions of the direct negation of logically complex sentences (i.e. the modified **LE** clauses). Proofs and refutations are defined as syntactical notions, i.e. as sub-classes of derivations (overlapping to a great extent).

4.4.3   Direct negation vs dual-intuitionism

In this chapter we introduced a genuinely semantic notion of refutation, independently of the syntactic notion of direct negation, according to thesis 2.

Nonetheless, if we look at the verificationist interpretation of **NJ** and the falsificationist one of **NDJ**, one may regret the fact that in neither framework we have the direct notion of proof interacting with the direct notion of refutation.

If compared with the way of treating the notions of proof and refutation in **N3**, the drawback of our presentation is that the two notions can be grasped only by developing two alternative semantic interpretations of two deductive languages, i.e. proofs and refutations do not interact. On the other hand, as we already remarked, by means of direct negation it is actually possible to keep the two notions together, although at the price of leaving the distinction between them devoid of semantic content. In particular, we reconstructed this approach as being based only on the semantic notion of proof (i.e. of construction) but allowing to introduce a distinction at the syntactic level between 'proofs' and 'refutations'.

As **N3** is a conservative extension of **NJ**, we have the possibility of defining the indirect notion of refutation as open proof of the absurdity. That is, we can keep both thesis 1 and 3, where the former one characterizes an indirect notion of refutation, the latter one the direct one. It is natural to complete the picture by considering an indirect notion of proof as well, where an indirect proof of A is a derivation of conclusion $\bot$ and the direct negation of A, $\sim A$ as only assumption. Just as for the direct notion, also the difference between indi-

rect proofs and refutations is 'intentional' (cf. section 4.4). A given derivation is both an indirect refutation of a sentence and an indirect proof of its direct negation.

To clear up the idea, we sketch in table 4.5 our conception of the relationship between direct and indirect proofs and refutations grasped through **NJ** and **NDJ** and the one emerging from the 'direct negation approach'. The table on the one hand makes clear the differences. On the other hand, it suggests to define a correspondence between the two views.

Table 4.5: **NJ** and **NDJ** approach vs **N3** approach

NJ and **NDJ** approach

	Direct	Indirect
Proofs of A	$\vdash_{NJ} A$	$\top \vdash_{NDJ} A$
Refutations of A	$A \vdash_{NDJ}$	$A \vdash_{NJ} \bot$

N3 approach

	Direct	Indirect
Proofs of A	$\vdash_{N3} A$	$\sim A \vdash_{N3} \bot$
Refutations of A	$\vdash_{N3} \sim A$	$A \vdash_{N3} \bot$

With $\Gamma \vdash_L \Delta$ we mean that there is a derivation of conclusions Δ and assumptions Γ in the deductive system L

According to us, the notion of refutation (both direct and indirect) is connected with the role of sentences as assumptions of derivations. This choice has the pleasant result that both proofs and refutations of A are represented by derivations in which A is the sentence at stake (either in assumption position or in conclusion position).

In the **N3** setting, the role of sentences as assumptions is connected with the indirect notions (both the one of proof and refutation). This is natural since according to the definition of validity for **N3**, assumptions play the subsidiary role of being place-holders for valid closed derivations.

On the other hand, in our approach, the distinction between direct and indirect notions depends on the underlying assumption on the deducibility relation. If the deductive system is a single-conclusion one, then priority is assigned to proofs and assumptions play as subsidiary role. If the system is a single-assumption one, then priority is assigned to refutations and conclusions play a subsidiary role.

We believe the notion of refutation characterized in **NDJ** by thesis 2 to be conceptually prior to the one characterized in **N3** by thesis 3, since the former is a genuinely semantic notion, rather than a syntactic one defined by means of direct negation as the latter.

Nonetheless, the notion of refutation characterized by thesis 3, can actually be seen as a way of mimicking our semantic notion of refutation by means of the semantic notion of proof (construction) alone.

Schematically, the idea is that by turning upside down derivations in **NDJ**$^{\rightarrow}$ and adding a negation in front of all sentences, we get derivations in **N3** of conclusion $\sim A$:

$$
\begin{array}{ccc}
A & & \vdots \\
\vdots & & \\
\vdots & \rightarrow & \sim A
\end{array}
$$

If we look at **N3** direct negation rules for conjunction and disjunction, they actually look like **NDJ** rules for conjunction and disjunction turned upside-down with all sentences directly negated.

Nonetheless, by applying systematically the idea of turning upside-down **NDJ**$^{\rightarrow}$ rules, we do not get exactly those for **N3**. First, **N3** lacks co-implication; but even if co-implication rules could be added to **N3**, the rules for directly negated implication in **N3** would still differ from those obteined by turning

up-side down those of **NDJ**$^\rightarrow$.

Hence, a proper characterization of the intuitive relationship between the notion of refutation of thesis 2 and thesis 3 would require a more detailed comparison between the two systems. As this would lead us too far from our scope, we content ourselves with the sketch we gave.

4.5 Final considerations

In this chapter, we presented three different ways in which a notion of refutation can be introduced into the so called proof-theoretic semantics setting. The first one, defining refutations as **NJ** open derivations of $\perp$, is unsatisfactory because it results in an ineradicably indirect notion. By defining refutations as the semantic values of **NDJ** F-valid bottom-closed derivations, we have a characterized a genuinely semantic and direct notion of refutation but by developing a semantic framework in which the notion of proof enters the picture only as an unsatisfactory indirect notion. The characterization of a direct notion of refutation in terms of direct negation has been criticized since not genuinely semantic.

We will suggest in the conclusions that the program of developing an alternative to standard proof-theoretic semantics may yield to a setting in which both semantic notions of proof and refutation can be accounted for.

Chapter 5

Concluding Remarks

According to Schroeder-Heister (2008), both truth- and proof-based approaches to semantics share what he calls the two dogmas of standard semantics: (i) categorical notions (truth, proof) have priority over hypothetical notions (consequence); (ii) consequence is defined as transmission of the categorical concept (truth, proof). For this reason, he claims that standard proof-theoretic semantics is not a real alternative to Tarski's style semantics.

Although we agree to a great extent with Schroeder-Heister's conclusions, in the light of the considerations we developed, we would like to suggest a more fine-grained account of the relationship between categorical and hypothetical notions in both frameworks.

In particular, we argue that it is doubtful whether the first dogma is embodied in either of the two semantic approaches. We conclude, in agreement with Schroeder-Heister, by calling for an alternative proof-theoretic semantics in which both dogmas are rejected.

5.1   Truth

Following Dummett (1973a, 1991), a semantics is a mapping of syntactic expressions onto semantic values. Expressions of different logical types are mapped onto entities of distinct categories.

As we saw (cf. chapter 1 section 1.2.1) Tarski's truth-definition for a propositional language corresponds to a semantics in which truth-values are as-

signed to sentences.

The relation 'B is a logical consequence of A' is defined in terms of truth: B is a consequence of A iff B is true whenever A is.[1] In symbols,

$$(5.1) \qquad\qquad A \vDash B =_{def} \vDash A \Rightarrow \vDash B$$

Also in the case of consequence, the semantic notion can be presented as being involved in assignments of semantic values to expressions. The claim that B is a consequence of A amounts to the claim that whenever A is mapped onto the truth-value the True, so is B. This is essentially the transmission view of consequence.

So far, both Schroeder-Heister's dogmas are encapsulated in the semantics. But, this is no more so, as soon as we consider the first-order case.

As we saw, the set of sentences of a first order language cannot be inductively defined and so the truth-predicate cannot be. In this case, Tarski's defines the relation of satisfaction, holding between assignments and formulas. The truth-predicate is defined in terms of satisfaction: A is true iff for all assignments σ, σ satisfies A. In symbols,

$$(5.2) \qquad\qquad \vDash A =_{def} \sigma \vDash A, \text{ for all } \sigma$$

Again, consequence is defined by 5.1 in terms of truth.

Although the relation of truth to consequence is unaltered, we argue that the notion of satisfaction modifies the equilibrium between categorical and hypothetical notions.

For, in a sense, also the notion of satisfaction has a hypothetical nature. This should be clear in the light of the correspondence—we discussed throughout the work—between the role played by open formulas in the truth-conditional semantic picture and the one played by open argumentations in the proof-theoretic one.

Very roughly, when we consider the consequence claim $A \vDash B$ we say that B is true under A; when we consider a satisfaction claim $\alpha \vDash B$ we say that B

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[1] For simplicity, we consider consequence claims as relating couples of sentences, instead of sentences and sets of sentences. The considerations we develop do not depend in any way upon this restriction. The universal quantification over models, involved in the notion of consequence, is left implicit. Cf. note 21 on page 57.

is true under α.

Although the hypothetical notion of consequence is defined in terms of the categorical notion of truth, the latter one must be defined in terms of a further hypothetical notion, the one of satisfaction.

So, Schroeder-Heister first dogma does not seem to be endorsed by Tarski's semantics for a first-order language.

5.2 Proof

The core of the proof-theoretic semantics consists in the definition of the predicate of validity applying to argumentations (Prawitz 1971, Schroeder-Heister 2006). In order to view the semantics as an assignment of semantic values to linguistic expressions, we suggested to take proofs as the semantic values to be assigned to V-valid (closed) argumentations.

In a Tarskian setting consequence is defined as transmission of truth; so here consequence is defined as transmission of provability: B is a consequence of A iff B is provable whenever A is.

Both proofs and consequence claims are represented by means of valid argumentations: proofs are represented by closed valid argumentations; consequence claims by open valid argumentations.[2]

Roughly, the skeleton of the definition of validity is the following:

- Closed (canonical) argumentations are valid iff their immediate sub-argumentations are valid.

- Open argumentations are valid iff the result of substituting closed (canonical) valid argumentations for their open assumptions yields (argumentations that reduce to) closed valid (canonical) argumentations for their conclusions.

Apparently, we have a primitive categorical notion of validity of closed (canonical) argumentations and a derivative hypothetical one of validity of open argumentations. But as we remarked, if the language contains implication, the picture arising from a proof-theoretic semantics is not so smooth.

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[2]We leave out of the discussion the notions of canonical argumentation and of reduction.

For, the characterization of this logical constant is the following: we prove an implication if the consequent of the implication is a consequence of the antecedent. That is, the hypothetical notion of consequence is needed to specify the categorical one of proof.

More precisely, the immediate sub-argumentation of a closed valid argumentation for an implication is an open valid argumentation having the consequent as conclusion and the antecedent as assumption: the definition of validity actually is a definition by simultaneous induction of closed and open validity.

The situation resembles the one of Tarski's semantics for a first-order language, where the presence of quantifiers makes the categorical notion of truth depend on the hypothetical one of satisfaction.

Despite the similarity, there is a crucial difference. Tarski's semantics starts from the notion of satisfaction and defines truth in terms of it. The proof-theoretic approach, on the other hand, refrains from defining the categorical notion in terms of the hypothetical one. The characterization of the validity of open argumentations in terms of that of their closed instances was a clumsy way of trying to preserve the priority of the categorical notion.

But are there substantive grounds for this choice?

5.3   Dummett's 'anti-realist' reason

In discussing Frege, Dummett (1973a, ch. 1–2) stresses that in a propositional language the only predicates we have to take into account are the primitive ones, needed for explaining the formation of atomic sentences. Only when we have quantifiers we need to consider complex predicates beyond the primitive ones, since we can form sentences not only by attaching a quantifier to the primitive predicates, but also to any predicate obtained by removing occurrences of a name from sentences of arbitrary complexity.

The introduction of the notion of complex predicate has also a semantic significance. In order to account for the truth of an atomic sentences we have to consider the primitive predicates constituting them as denoting functions giving the Truth or the False when applied to objects denoted by the names

available in the vocabulary. On the other hand, in order to account for the truth of quantified sentences we need to consider the functions denoted by the predicates as applying to *arbitrary* objects of the domain.

According to Dummett, it is such a notion of arbitrary object the source of Frege's realist conception of meaning. To the notion of arbitrary object, Dummett opposes the notion of *given* object, that is of an object presented to us in some particular manner.

An analogous distinction is naturally framed in the proof-theoretic setting, between 'given' proofs, i.e. proofs given through argumentations, and 'arbitrary' proofs, which we identified with intuitionistic 'hypothetical constructions'.

We ascribed to Dummett the will of avoiding the introduction of arbitrary proofs in the proof-theoretic framework, as that for him would amount to a concession to realism. The definition of validity of open argumentations in terms of that of their closed instances is a way of dealing only with proofs given through some argumentation.

5.4   Rejecting the priority of the categorical notion

Unfortunately for Dummett, not only the solution relies on the vague notion of reduction procedure; but it is also inadequate, as witnessed by open valid argumentations having ⊥ as conclusion, constituting canonical argumentations for negations. In these cases, since there are no possible closed instances of the open argumentations having ⊥ as conclusion, the account of validity of open argumentations in terms of that of their instances is simply to be rejected.

Furthermore, contrary to Dummett's opinion, there is nothing intrinsically realist in accepting the intuitionistic notion of hypothetical construction. Dummett himself takes intuitionism as the paradigm of anti-realism. And in intuitionism the notion of method is taken as primitive, together with the one of hypothetical construction.

Hence, the possibility of an alternative to Tarksi's truth-conditional semantics is not to be sought in a semantic framework in which no notions of hypothetical construction and/or of arbitrary object are available.

A possibility worth inquiring is that of developing a proof-theoretic semantics in which the hypothetical notion of consequence is directly defined, in analogy with Tarski's definition of the relation of satisfaction.

The categorical notion of proof could be then defined in terms of this primitive notion of consequence, in analogy with the definition of truth in terms of satisfaction.

5.5   Methods

It is not a settled matter how the notion of method, in relation to the notion of hypothetical construction, is to be conceived, even among intuitionists (van Atten 2009). In particular, in the case of implications with a false, possibly contradictory antecedent, it is not clear to what does the reference to hypothetical constructions commits. What is the meaning of saying that a method gives a hypothetical construction of an absurdity provided a hypothetical construction of a contradictory sentence?

A construction for $A \rightarrow B$ is a method that warrants the possibility of providing a proof of B given a proof of A. But what does the possession of the method amounts to? The possibility of producing proofs of the conclusion given proofs of the assumptions is what the method does, but, plausibly, it is not what the method is.

Concepts are functions, which surely yield the True and the False when applied to objects, but that have certain properties independently of their actual application or, say, that go beyond all their possible applications to given objects.

This line of reasoning is actually analogous to Dummett's remarks of the need of a more substantial notion of reference for predicates required for an account of quantifiers. As we observed, for Dummett, to say that predicate must be conceived as being true of arbitrary objects—and only of given ones— it is not to commit oneself to the introduction of a new kind of objects. Rather amounts to ascribing to predicates a more substantial notion of reference, i.e. introducing a genuine notion of concept, or function.

Analogously, the talk of hypothetical construction, as opposed to actual

constructions, can be thought of as a way of referring to the properties characterizing the method, which do go beyond what the method actually does.

Thus, a tentative answer is that the method does what it does in virtue of its 'internal structure', of the steps out of which it is composed. That is, the method has certain properties, in virtue of which it effectively provides proofs of B given proofs of A. But these properties are possessed by the method even if no proof A is (or even can logically be) available.

The proof-theoretic account, on the contrary, reduces the validity of open argumentations to that of closed ones. In the case of a single inferential step, the definition works in the same manner—since an inference rule is just a very simple open argumentation. As a result the validity of an inference rule is reduced to that of the closed argumentations in which it figures. This fact, explicitly remarked by Prawitz (1985, p. 169), yields to the following situation: it is not the validity of each single steps what warrants the validity of an argumentation, but the other way around. That is, an argumentation is not correct because each of its steps is, but rather the validity of an inferential step is warranted by the fact that all argumentations in which this step figure are valid.

This is, in a sense, the opposite of the interpretation we are suggesting of the intuitionistic picture. For, the steps out of which a method is constituted are not correct in virtue of the method yielding certain results when applied to certain constructions. On the contrary, we are arguing that the fact that method yields certain results has to be analyzed in terms of the steps out of which the method is constituted.

5.6 Hypothetical First

So, the basic characteristics of an alternative to standard proof-theoretic should be the following.

First, the notion of consequence should not be defined as transmission of the categorical concept of provability. On the contrary, consequence should be directly defined.

The 'global' notion of validity of an argumentation should be displayed as being dependent on the correctness of the 'local' inferential steps out of which

argumentations are constituted—and not, as in Prawitz definition of validity, the other way around.

The validity of open argumentations should be taken as primitive and the validity of closed argumentations defined in its terms. This would be analogous to the way in which the truth of sentences (i.e. closed formulas) is defined in terms of the satisfaction of open formulas by assignments.

An interesting result could be the possibility of defining in terms of the same notion of valid open argumentation both the categorical notions of proof and refutation. As we saw, by assigning priority to the categorical notions, proofs and refutations cannot be grasped in the same framework, but in two distinct ones. In each of the two, a notion of open validity is required in order to properly account for closed validity. Taking open validity as primitive could possibly allow for the definition of both notions of proof and refutation in the same setting.

The development of such an alternative framework is left open as a challenge for further meaning-theoretical investigations.

Bibliography

Auxier and Hahn (eds): 2007, *The Philosophy of Michael Dummett*, Open Court.

Brandom, R.: 1976, Truth and Assertibility, *The journal of philosophy* **73**, 137–149.

Casalegno, P.: 1997, *Filosofia del Linguaggio*, La nuova italia scientifica, Roma.

Czemark, J.: 1977, A remark on Gentzen's calculus of sequents, *Notre Dame Journal of Formal Logic* **18**(3), 471–474.

Dummett, M.: 1973a, *Frege Philosophy of language*, Duckworth, London. Second edition, 1981.

Dummett, M.: 1973b, The justification of Deduction, *Truth and Other Enigmas*, Clarendon Press.

Dummett, M.: 1976, What is a theory of meaning?(II), *in* G. Evans and J. McDowell (eds), *Truth and Meaning*, Oxford univerity press. *now in* Dummett, *The Seas of Language*, 1996, pp.34–93.

Dummett, M.: 1990, The source of the concept of truth, *in* Boolos (ed.), *Meaning and Method: Essays in honour of Hilary Putnam*, Cambridge University Press.

Dummett, M.: 1991, *The Logical Basis of Metaphysics*, Duckworth, London.

Dummett, M.: 1994, Reply to Prawitz, *in* McGuinness and Olivieri (1994).

Dummett, M.: 1998a, Is the concept of truth needed for semantics?, *in* Martinez et al. (1998).

Dummett, M.: 1998b, Truth from a constructive standpoint, *Theoria* **64**.

Frege, G.: 1884, *Die Grundlagen der Aritmetik*. engl. trans. by J. L. Austin, Basil Blackwell, 1953.

Gabbay, D. M. and Wansing, H. (eds): 1999, *What is negation?*, Kluwer Academic Publishers.

Garson, J. W.: 2001, Natural semantics: why natural deduction is intuitionistic, *Theoria* **67**(2), 114–139.

Gethmann, C. F. (ed.): 2009, *Akten des XXI Deutschen Kongresses für Philosophie (Essen, 15.-19.9.2008), Deutsches Jahrbuch Philosophie, Band 3*, Felix Meiner Verlag, Hamburg.

Glivenko, V.: 1929, Sur quelques points de la logique de M. Brouwer', *Academie Royale de Belgique, Bulletins de la classe des sciences* **5**(15), 183–188.

Goodman, N.: 1981, The logic of contradiction, *Zeitschrift für Logik und Grundlagen der Mathematik* **27**, 119–126.

Goranko, V.: 1994, Refutation systems in modal logic, *Studia Logica* **53**(2), 299–324.

Hand, M.: 1999, Antirealism and falsity, *in* Gabbay and Wansing (1999).

Heyting, A.: 1974, Intuitionistic views on the nature of mathematics, *Synthese* **27**, 79–91.

Kuhn, T. S.: 1962, *The structure of scientfic revolutions*, Chicago University Press.

Lopez-Escobar, E. G. K.: 1972, Refutability and elementary number theory, *Indagationes Mathematicae* **34**, 362–74.

Marcus, R. B.: 1962, Interpreting quantification, *Inquiry* **5**, 252–259.

Martinez, C. et al. (eds): 1998, *Truth in Perspective: recent issues in logic, representation and ontology*, Ashgate.

Martino, E. and Usberti, G.: 1994, Temporal and atemporal truth in intuitionistic mathematics, *Topoi* **13**, 83–92.

McGuinness, B. and Olivieri, G. (eds): 1994, *The philosophy of Michael Dummett*, Kluwer Academic Publishers.

Miller, D.: 2006, *Out of Error: Further Essays on Critical Rationalism*, Ashgate.

Nelson, D.: 1949, Constructible Falsity, *The Journal of Symbolic Logic* **14**(1), 16–26.

Prawitz, D.: 1971, Ideas and results in proof theory, *in* J. E. Fenstad (ed.), *Proceedings of the Second Scandinavian Logic Symposium*, North Holland, Amsterdam, pp. 235–308.

Prawitz, D.: 1985, Remarks on some approaches to the concept of logical consequence, *Synthese* **62**, 153–171.

Prawitz, D.: 1987, Dummett on a theory of meaning and its impact on logic, *in* B. M. Taylor (ed.), *Michael Dummett: contributions to philosophy*, Martinus Nijhoff Publisher, Dordrecht.

Prawitz, D.: 1994, Meaning theory and antirealism, *in* McGuinness and Olivieri (1994).

Prawitz, D.: 1998a, Comments on the papers, *Theoria* **64**.

Prawitz, D.: 1998b, Truth and Objectivity from a verificationist point of view, *in* Dales and G. Olivieri (eds), *Truth in Mathematics*, Clarendon Press.

Prawitz, D.: 1998c, Truth from a constructive perspective, *in* Martinez et al. (1998).

Prawitz, D.: 2006, Meaning approached via proofs, *Synthese* **148**, 507–524.

Prawitz, D.: 2007, Pragmatist and verificationist theories of meaning, *in* Auxier and Hahn (2007).

Read, S.: 2000, Harmony and autonomy in classical logic, *Journal of Philosophical Logic* **29**, 123–154.

Restall, G.: 2005, Multiple Conclusions, *in* P. Hajek, L. Valdes-Villanueva and D. Westerstahl (eds), *Logic, Methodology and Philosophy of Science: Proceedings of the Twelfth International Congress*, Kings' College Publications, London, pp. 189–205.

Rumfitt, I.: 2007, Asserting and excluding: steps toward an anti-realist account of classical consequence, *in* Auxier and Hahn (2007), pp. 639–693.

Sandqvist, T.: 2009, Classical logic without bivalence, *Analysis* **69**(2), 211–218.

Schroeder-Heister, P.: 2004, On the notion of assumption in Logical Systems, *in* R. Bluhm and C. Nimtz (eds), *Selected Papers Contributed to the Sections of GAP5, Fifth International Congress of the Society for Analytical Philosophy, Bielefeld, 22-26 September 2003*, Mentis, Paderborn, pp. 27–48.

Schroeder-Heister, P.: 2006, Validity concepts in proof-theoretic semantics, *Synthese* **148**, 525–571.

Schroeder-Heister, P.: 2008, Proof-theoretic versus model-theoretic consequence, *in* M. Peliš (ed.), *The Logica Yearbook 2007*, Filosofia, Prague, pp. 187–200.

Schroeder-Heister, P.: 2009, Schluß und Umkehrschluß: ein Beitrag zur Definitionstheorie, *in* Gethmann (2009).

Shramko, Y.: 2005, Dual intuitionistic logic and a variety of negations: The logic of scientific research, *Studia Logica* **80**, 347–367.

Shramko, Y. and Wansing, H.: 2010, Truth-values, *in* E. N. Zalta (ed.), *The Standford Encyclopedia of Philosophy*.

Skura, T.: 1992, Refutation calculi for certain intermedialte propositional logics, *Notre Dame Journal of Formal Logic* **33**(4), 552–560.

Skura, T.: 1995, A Łucasiewicz-style refutation sysyem for the modal logic s4, *Journal of philosophical logic* **24**, 573–582.

Tennant, N.: 1987, *Anti-realism and Logic. Truth as eternal*, Oxford University Press.

Tennant, N.: 1999, Negation, absurdity and contrariety, *in* Gabbay and Wansing (1999).

Tranchini, L.: 2009, The meaning of negation. A proposal, *in* M. Peliš (ed.), *The Logica Yearbook 2008*.

Tranchini, L.: 2010, Refutation. A proof-theoretic account, *in* G. Turbanti and L. Bellotti (eds), *Pisa Colloquium*, ETS.

Urbas, I.: 1996, Dual-intuitionistic logic, *Notre Dame Journal of Formal Logic* **37**(3), 440–451.

Usberti, G.: 1995, *Significato e conoscenza*, Guerini Scientifica.

van Atten, M.: 2009, The hypothetical judgement in the history of intuitionistic logic, *in* C. Glymour, W. Wang and D. Westerstahl (eds), *Logic, Methodology, and Philosophy of Science XIII: Proceedings of the 2007 International Congress in Beijing*, King's College Publications, London.

van Dalen, D.: 1994, *Logic and Structure*, 3 edn, Springer.

Wansing, H.: 1999, Negation as falsity: a reply to Tennant, *in* Gabbay and Wansing (1999).

Wansing, H.: 2009, Negation in der konstruktiven Logik, *in* Gethmann (2009).