Structural and Enumerative Studies of Tropical Curves and Covers

Dissertation

der Mathematisch-Naturwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt von
Lou-Jean Leila Cobigo-Bihavan
aus Oldenburg

Tübingen
2025

Gedruckt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Eberhard Karls Universität Tübingen.

Tag der mündlichen Qualifikation: 28.04.2026

Dekan: Prof. Dr. Thilo Stehle

1. Berichterstatter/-in: Prof. Dr. Hannah Markwig

2. Berichterstatter/-in: Jun.-Prof. Dr. Daniele Agostini

Contents

Zusammenfassung

Die vorliegende Dissertation ist dem strukturellen und enumerativen Studium tropischer Kurven und Überlagerungen gewidmet. Die Analyse erfolgt anhand zweier spezifischer Forschungsfragen: Tropische spaltende Jacobische und tropische spin Hurwitzzahlen mit abgeschlossenen Zykeln.

In Teil 1 geht es um das Zusammenspiel von tropischen Kurven und tropischen abelschen Varietäten. Wir untersuchen strukturelle Aspekte tropischer spaltender Jacobischer von Kurven vom Geschlecht 2, und zwar sowohl auf globaler wie auch auf atomarer Ebene. Global erreichen wir ihre Charakterisierung in der Kategorie tropischer Kurven $\mathbb{T}\mathcal{C}$ (durch Überlagerungen) und in der Kategorie tropischer abelscher Varietäten $\mathbb{T}\mathcal{A}$ (als Quotient eines direktes Produkt von zwei elliptischen Kurven). Atomar identifizieren wir ihre Bausteine, ein Paar von Geschlecht 1 Kurven zusammen mit einer gewissen Untergruppe ihres direkten Produkts, und rekonstruieren daraus die Charakterisierungen in $\mathbb{T}\mathcal{A}$ und $\mathbb{T}\mathcal{C}$. Wir nutzen die atomare Perspektive, um unser Verständnis von spaltenden Jacobischen weiter zu kondensieren und gehen zu ihrer Betrachtung im Modulraum tropischer Kurven bzw. prinzipiell polarisierter tropischer abelscher Varietäten über. Hier untersuchen wir eine Variante des tropischen Schottky Problems für spaltende Jacobische bzw. dessen Umkehrung. Wann immer möglich, nutzen wir tropische Geometrie, um abstrakte Charakterisierungen durch konkrete Algorithmen zu untermauern.

In Teil 2 geht es um das Zusammenspiel von tropischer Geometrie mit anderen Disziplinen. Wir nutzen diese Interaktion (im Rahmen der enumerativen Geometrie) zur Untersuchung einer geometrisch motivierten Zahl aus, der Spin Hurwitzzahlmit abgeschlossenen Zykeln. Dazu führen wir eine tropische Zählung von verzweigten Überlagerungen ein, welche mit der ursprünglichen Zahl übereinstimmt, und verwenden schließlich Methoden der tropischen Geometrie, um strukturelle Eigenschaften dieser Zahl (Polynomialität und Wanddurchquerungsformeln) zu untersuchen.

Acknowledgements

Sincere thanks are due to my advisor, Hannah Markwig, without whom this text/research would not have been possible. I would like to thank her for the intellectual openness brought to numerous discussions, for taking any one of my questions seriously, for being a role model in how she teaches and thinks, and, finally, for being much more light-hearted than I am. Her thoughtful comments, questions and advice have let to significant improvement of this work.

I am also grateful for the works of Cavalieri, Johnson, Len, Markwig, Mikhalkin, Ranganathan, Röhrle, Schmitt, Ulirsch, Zakharov, Zharkov and many more and the inspiration they have provided. I thank Felix Röhrle for many interesting discussions and for his valuable feedback on earlier versions of Chapter 2 and 3. I thank Thomas Blomme whose original and canny method for computing floor multiplicities naturally led to a question about Jacobians of curves of genus 2. I thank Marvin Hahn for inspiring discussions.

I am indebted to Helmut F. für seine herrlich unkonventionelle Art and for teaching me to draw nectar out of the struggle, Erwin E. for a correspondence that has made me smile and Hilke S., whose strength and stance towards the future have always inspired me.

Introduction

This thesis is based on three projects. We provide a thematic overview here. Each project will contain a more technical and detailed introductory section. Results and future research will be discussed in more detail there, too.

0.1 Part I: Tropical split Jacobians.

In algebraic as well as in tropical geometry, a curve comes together with an abelian variety, its Jacobian.

This defines a relation between the category of *tropical abelian varieties (tav)*, $\mathbb{T}\mathcal{A}$, and the category of *tropical curves*, $\mathbb{T}\mathcal{C}$, a relation that is also compatible with covers of curves. Understanding the nature of this connection means knowing how phenomena in $\mathbb{T}\mathcal{C}$ are expressed in $\mathbb{T}\mathcal{A}$ or vice versa.

On the level of moduli spaces, this relation is formalized by the tropical Torelli map t_g^{tr}, which connects the *moduli space* M_g^{tr} *of tropical curves* to the *moduli space* A_g^{tr} *of principally polarized tav (pptav)*. In this context, the *tropical Schottky problem* asks for the characterization of the locus of Jacobian varieties inside A_g^{tr}. It is the tropical counterpart of the classical Schottky problem, with origins in the work of Abel, Jacobi, and Riemann and which, despite extensive study and progress, is not fully resolved.

Our research embeds in this framework through two projects that study a specific phenomenon, *tropical split Jacobians*.

0.1.1 Results from Project 1

Project 1 (Chapter 2) is a holistic study: Jacobians of genus 2 curves are two-dimensional tav and as such more complicated than their one-dimensional cousins, elliptic curves. *Split* Jacobians are a simplification. They take on different forms depending on whether we look at them in $\mathbb{T}\mathcal{A}$ or $\mathbb{T}\mathcal{C}$:

Theorem 0.1.1. (Split Jacobians and Covers: Theorem 2.6.2)

Let Γ be a tropical curve of genus 2. Then $\mathrm{Jac}(\Gamma)$ splits if and only if Γ covers an elliptic curve.

In Project 1 we set up a categorical framework and develop tools/techniques necessary to establish and further study this correspondence:

Theorem 0.1.2. (Split Jacobians and Optimal Covers)
Introducing the notion of *optimal covers* resolves issues of non-uniqueness and non-computability: An optimal cover fixes a "canonical" representative for a splitting of $\mathrm{Jac}(\Gamma)$ (Theorem 2.6.4) and produces a second optimal cover (Theorem 2.6.9) that can both be computed algorithmically (Subsection 2.6.3).

On the level of categories, Theorem 0.1.1 and 0.1.2 connect morphisms in $\mathbb{T}\mathcal{C}$ to isogenies in $\mathbb{T}\mathcal{A}$ via $\mathcal{F}$ such that the following global picture emerges: Split Jacobians appear either as a pair of optimal covers in $\mathbb{T}\mathcal{C}$ or as an isogeny ϕ satisfying $\mathrm{Jac}_d(\mathbb{T}E) \cong \ker(\phi) \cong \mathrm{Jac}_d(\mathbb{T}E')$ in $\mathbb{T}\mathcal{A}$.

Further results include:

- Theoretical:
    - Categorical tools: Sections 2.2.2 and 2.2.3 extend the existing theory of real tori with integral structure. Section 2.3.2 transfers relevant concepts to $\mathbb{T}\mathcal{A}$, introduces exact sequences and studies their behaviour under dualization.
    - Analysis of morphisms in $\mathbb{T}\mathcal{C}$ and $\mathbb{T}\mathcal{A}$: We analyze the push-forward φ_* and its dual φ^* for covers $\varphi : \Gamma \to \mathbb{T}E$ of elliptic curves (Sections 2.5.5 and 2.5.6). Invoking the universal property of the Jacobian ([69], Proposition 4.14), we then relate factorizations of φ to factorizations of φ_*.
    - Criteria for optimality: Section 2.5.7.
- Concrete:
    - Algorithmic point of view on Theorems 2.1.2 and 2.1.3 (Section 2.6.3).
    - Morphisms in $\mathbb{T}\mathcal{C}$ and $\mathbb{T}\mathcal{A}$: Sections 2.5.5 and 2.5.6.
    - Criteria for optimality: Section 2.5.7.

This projects builds on the Preprint [32]. Section 2.1.2 contains a more detailed discussion of results.

0.1.2 Results from Project 2

Project 2 (Chapter 3) takes a different approach. Instead of looking at the phenomenon as a whole, we analyze its building blocks, a pair of elliptic curves $\mathbb{T}E'$ and $\mathbb{T}E$ together with a specific finite subgroup G of their product, and how to reassemble them into a Jacobian.

Theorem 0.1.3. (Reconstruction Theorem: Theorem 3.5.17) Given *splitting data* $(\mathbb{T}E', \mathbb{T}E, G)$, there exists a curve Γ of genus 2 and a pair of optimal covers $(\varphi' : \Gamma \to \mathbb{T}E', \varphi : \Gamma \to \mathbb{T}E)$ such that $\mathrm{Jac}(\Gamma)$ splits via an isogeny ϕ with $\mathrm{Jac}_d(\mathbb{T}E') \cong \ker(\phi) \cong \mathrm{Jac}_d(\mathbb{T}E)$.

We use this atomic understanding to tackle a Schottky type problem for split Jacobians, specifically investigating the locus $\mathcal{Q}$ of split Jacobians and the locus $\mathbb{T}\mathscr{L}$ of curves with split Jacobians in the moduli space A_2^{tr} of principally polarized tropical abelian varieties and the moduli space M_2^{tr} of tropical curves of genus 2.

We have a coarse parametrization $\mathscr{P} \twoheadrightarrow \mathcal{Q} \subset A_2^{tr}$ for $\mathcal{Q}$, in the sense that multiple parameter values may correspond to the same (isomorphism class of) objects (Section 3.6). This parametrization can be leveraged to gain a deeper understanding of the geometry of $\mathcal{Q}$ by analyzing its intersection with the boundary of A_2^{tr} (Section 3.6.1). To resolve the redundancy in $\mathscr{P}$ and establish the connection to tropical curves, we develop and implement an algorithm that selects a representative in $\mathcal{Q}$ for each parameter value and computes the pointwise preimage under the tropical Torelli map (Algorithm 2 and Section 3.7). This leads to a combinatorial characterization of the preimage $\mathbb{T}\mathscr{L}$.

Theorem 0.1.4. (Combinatorial Structure of the Split-Jacobian Locus: Theorem 3.6.9) The locus of curves with d-split Jacobian, $\mathbb{T}\mathscr{L}_d$, admits a description in terms of certain cones and their images in M_2^{tr}.

Further results include:

- Theoretical:
  - Categorical tools: Sections 3.2.2 and 3.2.3 develop one of the main tools for the reconstruction procedure, a tropical analogue of Mumford's Criterion (Proposition 3.2.8) and introduce further concepts such as adjoints in $\mathbb{T}\mathscr{A}$.
  - Theoretical foundations for the Reconstruction Theorem: This includes determining conditions for $(\mathbb{T}E', \mathbb{T}E, G)$ to constitute splitting data (Section 3.5.1), proving the existence of a pptav J and an associated isogeny $\mathbb{T}E' \oplus \mathbb{T}E \to J$ (Section 3.5.2) together with a pair of optimal covers $(\varphi' : \Gamma \to \mathbb{T}E', \varphi : \Gamma \to \mathbb{T}E)$ (Section 3.5.4).
- Concrete:
  - Concrete aspects of the Reconstruction Theorem: The existence proofs are constructive. We provide explicit procedures to build the genus 2 curve with Jacobian J from splitting data (Algorithm 2 and its implementation (Section 3.7)) and to compute the pair of optimal covers (using methods from Section 2.6.2).
  - Analysis of a Schottky-type problem for split Jacobians (Section 3.6.1).

This project builds on the Preprint [33]. Section 3.1.4 provides a more detailed discussion of results.

Projects 1 and 2 are written so as to be as self-contained as possible. The overlap is kept to a minimum, though not avoided when necessary for readability or to emphasize the structural parallels that unite the two projects.

0.1.3 Context

This research connects to a large body of work on tavs and tropical covers, mainly in the context of Prym varieties (see e.g. [7], [16], [62], [6], [14], [29], [57], [69]). It parallels classical questions from algebraic geometry that go back to Jacobi, Legendre and others (see e.g. [53], [38], [38]).

0.2 Part II: Tropical spin Hurwitz numbers with completed cycles.

In algebraic as well as in tropical geometry, enumerative geometry forms a central subdiscipline.

There is a relation between the category of *algebraic curves*, $\mathcal{C}$, and the category of *tropical curves*, $\mathbb{T}\mathcal{C}$, which associates to an algebraic curve a tropical curve (e.g. via dual graphs endowed with metrics) and to a branched cover, a tropical cover. This relation allows for the translation of enumerative questions between the two settings.

At the level of moduli spaces, enumerative theories involving curves and covers are related via the tropicalization map trop, which connects the *moduli space* $\overline{M}_{g,n}$ *of* n*-pointed genus* g *curves* to the *moduli space* $M_{g,n}^{tr}$ *of tropical curves with* n *marked points*.

Our research embeds in this framework through the study of a specific enumerative invariant, *spin Hurwitz numbers with completed cycles*.

0.2.1 Results from Project 3

Spin Hurwitz theory is an enumerative theory of geometric objects, covers of curves with a theta characteristic. The number of such covers, subject to fixed ramification constraints, is called a spin Hurwitz number. This number admits a representation-theoretic expression in terms of characters of Sergeev-group. Based on this expression, a purely representation-theoretic quantity known as the spin Hurwitz number with completed cycles (HWC^{spin}) (Definition 5.5.1) is introduced.

In Definition 8.2.2 we introduce a tropical geometric count:

Definition 0.2.1 (Tropical spin Hurwitz number with multi-completed cycles)**.** The *tropical spin Hurwitz number with multi-completed cycles* is a weighted sum

$$\sum_{\pi:\Gamma\to\mathbb{TP}^1} \mathrm{mult}(\pi)$$

of tropical covers $\pi : \Gamma \to \mathbb{TP}^1$ of the tropical projective line $\mathbb{TP}^1$ meeting prescribed constraints, with weights $\mathrm{mult}(\pi)$ as defined in Section 8.2.

Using the Fock space formalism and a graphical method for computing vacuum expectations, we prove a correspondence theorem equating the classical and tropical spin Hurwitz numbers:

Theorem 0.2.2. (Spin Hurwitz Correspondence for completed cycles: Theorem 8.2.3) For fixed discrete data, the classical and the tropical spin double Hurwitz number with multi-completed cycles agree:

$$HWC^{spin} = HWC^{spin}_{trop}.$$

This re-geometrization transforms operator-theoretic vacuum expectations into explicit tropical covers of $\mathbb{TP}^1$, paving the way for a tropical analogue of the spin GW/H-correspondence.

The tropical setting can also be leveraged to examine spin Hurwitz numbers with $(r+1)$-completed cycles for polynomiality.

Theorem 0.2.3 (Polynomiality and wall-crossing: Propositions 8.3.8, 8.3.29 and Theorems 8.3.16, 8.3.39)**.** The tropical invariants are piecewise polynomial in the ramification data and satisfy explicit wall crossing formulae.

Further results include:

- Theoretical: We define a generalization of the spin-double Hurwitz numbers with $r+1$ completed cycles (Section 5.5) and give a character formula for them (Theorem 5.5.3) that mirrors the one for classical Hurwitz numbers. To the best of our knowledge, this minor generalization has not been explicitly recorded.

- Computational: We give a closed formula for the connected spin Hurwitz numbers with 1 completed cycles, in analogy to [66] (Theorem 2) for Hurwitz numbers. This formula is used to rewrite the spin completed cycles cut-and-join operators in Section 7.5.1, which allows for a geometric reinterpretation of these operators.

Section 3.8.1 provides a more detailed discussion of results.

0.2.2 Context

Classical theory: type A

The enumeration of covers of a fixed Riemann surface with specified ramification behavior is a classical problem. With origins in the work of Hurwitz in the 19th century, it is a *geometric* problem at first. Yet, this setting is not necessarily the most productive one. Translating the associated Hurwitz numbers into other areas of mathematics reveals new facets of the problem and has led to a growing body of results, ranging from structural insights to explicit computations ([28], [66], [22], [23], [50]).

Central settings (algebraic geometry, tropical geometry, representation theory, group theory and Fock space formalism) together with their interconnections are illustrated in Figure 1. Among these, tropical geometry enjoys a central position with direct connections to nearly all areas that support Hurwitz numbers.

Within each setting, Hurwitz numbers acquire different expressions. The original count HW of ramified covers appears in the group theoretic setting as the count HW_{group} of tuples of elements of the symmetric group, which encode the monodromy of the covers. In the representation-theoretic setting, on the other hand, HW corresponds to a sum of products $HW_{rep.}$ of so-called central characters of the symmetric group. This connection arises naturally, as group-theoretic phenomena are often examined with the help of representations of the respective group. $HW_{rep.}$ can now be conveniently calculated using operator formalism of bosonic and fermionic Fock space. These are two infinite-dimensional inner product spaces over $\mathbb{C}$ with an action of a certain matrix algebra on them. The Hurwitz number is realized as a *vacuum expectation*, $\langle v_\varnothing, Ov_\varnothing \rangle$, where O a specific operator and $v_\varnothing$ the vacuum vector. The precise way to move between all settings is described in detail in Chapter 4. For now, an overarching impression of the network of mathematical areas that support Hurwitz numbers will suffice.

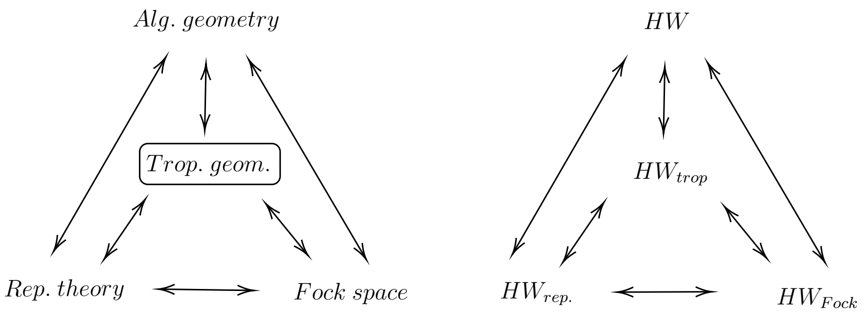


Figure 1: A network for type A theory: Areas supporting Hurwitz numbers (HW), on the left, and their realization within each setting, on the right. Double-headed arrows on the right indicate correspondences (equalities) between different realizations. Double-headed arrows on the left represent bidirectional bridges between the different settings.

Building on the representation-theoretic expression $HW_{rep.}$, a purely representation-theoretic number emerges: The *Hurwitz number with completed cycles (HWC)* (wavy arrow in Figure 2). Advantages include: Computability, connections to integrable systems, better behavior generalizing the piecewise polynomiality for double and quasi-polynomiality for single Hurwitz numbers discussed below.

Polynomial behavior and moduli spaces We introduce a dynamic viewpoint by considering the function that sends ramification data to the corresponding Hurwitz number with completed cycles. For *single* Hurwitz numbers this function is *quasi-polynomial* (a polynomial up to a combinatorial prefactor depending on the parts of the ramification partition). For *double* Hurwitz number it is only *piece-wise polynomial*, i.e. given by a collection of polynomials each describing the function on a subset of its domain.

This polynomial behavior is a geometric phenomenon, governed by the structure of certain moduli spaces:

Quasi-polynomiality for single Hurwitz numbers is a direct consequence of the ELSV formula, which expresses them as intersection numbers on the moduli space of stable curves with marked points $\overline{\mathcal{M}}_{g,n}$. Coefficients of these quasi-polynomials are themselves intersections numbers, meaning that single Hurwitz numbers encode information about the intersection theory of $\overline{\mathcal{M}}_{g,n}$.

Piecewise polynomiality, in turn, inspired Goulden, Jackson and Vakil to conjecture an ELSV-type formula for double Hurwitz numbers, whose anticipated shape is determined by finer knowledge of these polynomials (such as their degrees). In analogy with the single case, such a formula would not only explain this polynomiality, but also provide insights into the intersection theory of a conjectural compactification of $\mathrm{Pic}_{g,n}$ (the Picard variety of the universal curve over $\mathcal{M}_{g,n}$, see [43]).

In short, the presence of some kind of polynomial behavior suggests geometric structure in the background, likely given by a moduli space of some kind, and is therefore natural to seek and study.

Re-geometrization The representation-theoretic number HWC acquires a geometric meaning in two ways:

- The Gromov-Witten/Hurwitz (GW/H) correspondence ([66]): Gromov-Witten invariants (GWI) are intersection products on the moduli space $\mathcal{M}_{g,n}(\mathbb{P}^1, d)$ of degree d stable maps to $\mathbb{P}^1$, supposed to represent the number of covers of $\mathbb{P}^1$ satisfying specific geometric constraints. The (GW/H) correspondence identifies completed cycles Hurwitz numbers with certain GWI, thereby returning a geometric meaning to HWC.

- Their evaluation in Fock space formalism, which offers a natural translation in tropical language (see [28]): HWC become geometric by appearing as a count of tropical covers.

The result is a variant of Figure 1 for HWC:

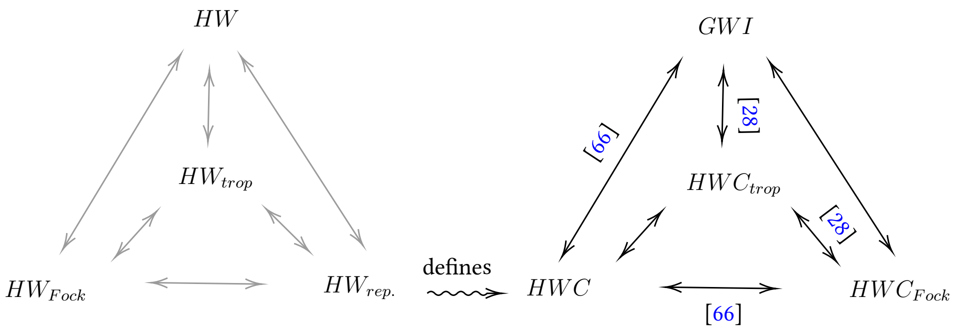


Figure 2: A network for type A theory: The network for Hurwitz numbers, on the left, is a blueprint for the completed-cycles variants on the right. The wavy arrow indicates that HWC is defined via the representation-theoretic expression of HW.

Spin theory: type B

Spin Hurwitz numbers enumerate branched covers of a fixed spin curve (a curve with theta characteristic) with specified ramification behavior and sign fixed by a lift of this theta characteristic to the source. They were introduced by Eskin-Okounkov-Pandharipande in 2008 for certain computations in the moduli space of differentials on a Riemann surface [37]. Recent work [41] establishes a spin GW/H correspondence, linking this theory to the spin Gromov-Witten theory of $\mathbb{P}^1$. These GWI, in turn, emerge from the ordinary GW theory of Kähler surfaces via localization by cosections, reducing it to the GW theory of an associated spin curve ([41]).

To date, the spin theory enjoys almost the same "global structure" as the non-spin theory (see Figure 3). This is the result of the cumulative work carried out in [54], [56], [45], [55], [40], which provides major components of Figure 3.

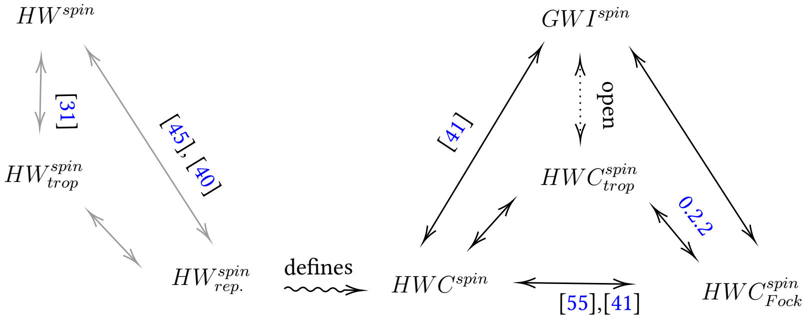


Figure 3: A network for type B theory from ordinary spin Hurwitz numbers, on the left, to completed cycles, on the right. Note: The absence of a Fock space expression for ordinary double spin Hurwitz numbers stems from a key difference from the classical theory. Double Hurwitz numbers (two special ramifications, 2-cycles elsewhere), agree with their 2-completed-cycle analogues . This explains the presence of the operator Formalism in Figure 1. This equality fails in the spin setting, as ordinary double spin Hurwitz numbers (two special ramifications, 3-cycles elsewhere) differ from their 3-completed-cycle analogues. The connection between GWI^{spin} and HWC^{spin}_{trop} (the dashed arrow) is yet to be established. This can be done by modeling [28] for type A theory and using the spin ingredients developed in this thesis and in [31].

We contribute by adding tropical counterparts: In previous work [31] we introduced a tropical count and established its equivalence to the algebraic one, using degeneration techniques (to complete the left part of Figure 3). The setting was exclusively geometric. Here, we turn to the right part of Diagram 3. The setting is representation-theoretic. The goal is

- To *re-geometrize* the spin invariants HWC^{spin}, completing the right part of Figure 3 and thereby returning the theory to a geometric setting.

- To analyze their structural properties (polynomial behavior, wall-crossing formulae).

This fits into the program initiated by [9], [28] and [23] and continued by [46] to tropicalize Hurwitz-type enumerative invariants and supports on-going research efforts in the spin-setting.

0.3 Preword

> « On ferait une bonne oeuvre, Hérodote, en procurant un abrégé de toute la matière [...].
>
> Et ceux aussi qui ont avancé suffisamment dans l'examen des ouvrages complets, il faut qu'ils gardent en mémoire le schéma, réduit aux éléments, du système entier ; car nous avons un besoin fréquent de la saisie de l'ensemble, non autant de celle des parties. Il faut donc aller continuellement vers les vues d'ensemble, mettre dans la mémoire cela seulement à partir de quoi sera possible la vue dominante jetée sur les choses, en même temps que se fera jour la connaissance précise et complète du particulier, du moment qu'auront été bien compris et bien retenus les schémas valables universellement. Car, même dans le cas de celui qui est parfaitement instruit, c'est cela qui permet, dans toute connaissance précise, de dominer le détail : le fait de pouvoir user rapidement des vues rassemblantes, en ramenant chaque chose à des éléments et à des termes simples. Car la concentration de la vue englobante qui enserre toutes choses sans faille, ne peut appartenir à qui n'est pas capable d'embrasser en lui-même, à l'aide de brèves formules, tout ce qui a été examiné avec précision dans sa particularité C'est pourquoi, une telle méthode étant utile à tous ceux qui sont familiarisés avec la science de la nature, moi qui recommande une activité incessante dans cette science, et qui trouve dans cette occupation ce qui procure le plus de calme à la vie, j'ai composé pour toi cet abrégé élémentaire de toutes mes opinions dont je parlais tout à l'heure.» (Épicure, *Lettre à Hérodote*)

This text is an attempt: I try to fit things in a system of thinking (category theory). To honor the interconnected nature of all topics considered, I deliberately include broad context and explain already established work. This is especially crucial regarding Hurwitz theory, where the essence lies in the interplay of different mathematical areas. An isolated tropical perspective would not do justice to the topic. I believe in explaining broadly and embedding new results within a wider framework. This shows: There is nothing truly new under the sun, rather old ideas in new settings. This work is a small part of this large cohesive mathematical narrative.

0.4 Guiding principles

We find Epicure's program of providing a "vue d'emsemble" in different forms throughout this thesis:

1. Literally, through the concept of *moduli spaces*.

2. Implicitly, through synthesis (e.g. placing questions into larger context, bringing various mathematical disciplines together and connecting abstract with concrete mathematics).

1. Moduli spaces A *moduli space* is an answer to a classification problem. It organizes a collection of mathematical objects by reducing them to their defining properties and by deciding which objects are to be identified. This reflects a sense of what is mathematically relevant. The result is a "vue d'emsemble" in the sense of Epicure, hiding details and placing objects into one setting. This perspective allows for the distinction between what is specific and what is general and how different objects interact. Approaches to study them are:

- *Via special phenomena/subclasses of objects:* In this thesis, we analyze M_2^{tr} and A_2^{tr} via tropical split Jacobians (Project 1 and 2).

- *Via enumerative geometry:* Enumerative invariants may often be expressed as intersection numbers on a suitable moduli space. Explicit formulas for these invariants inform about the intersection theory of the underlying space and, in turn, about the space itself. In this sense, moduli spaces of (tropical) Hurwitz covers are also present in the third project. More concretely, we conjecture via enumerative geometry that the numbers HW^{spin} (HW^{spin}_{trop}) are equal to the degree of a branch map from a moduli space of spin admissible covers to a symmetric product of $\mathbb{P}^1$, which encodes the location of the branch points of a spin cover. Furthermore, our study of the polynomiality behaviour of the numbers HWC^{spin} leads to the expectation that these numbers, analogously to the numbers HWC, can be expressed in terms of an intersection product on a moduli space.

Because of this central role played by moduli spaces in all projects, we discuss moduli spaces in detail, in particular in the category-theoretic framework, in the Preliminary section, Chapter 1, see Section 1.2.

2. Synthesis While the three projects deal with specific research problems, each project serves as a case study. A goal of this thesis is to is to present each project in its larger mathematical context.

Projects 1 and 2: The broader context is the connection between tavs and curves. A universal language, category theory, used in for example in Section 2.3 for tav, in Section 2.4 for curves, in Section 1.2 to introduce moduli spaces, or in Section 2.6 to carry over algebraic methods to the tropical setting, facilitates the task.

Project 3: The broader context is a network of mathematical disciplines hosting HWC^{spin}, visualized in Figure 3.15. Excerpts are: Tropical geometry links to algebraic geometry via degeneration ([31] proves a Correspondence Theorem), and to representation theory via Feynman graphs (Section 8.1).

The context unifying Parts I and II is the category of algebraic/tropical curves (see Figure 4), with two classes of morphisms: Optimal covers (used to understand split Jacobians) and tropical spin Hurwitz covers with completed cycles (which are to be counted).

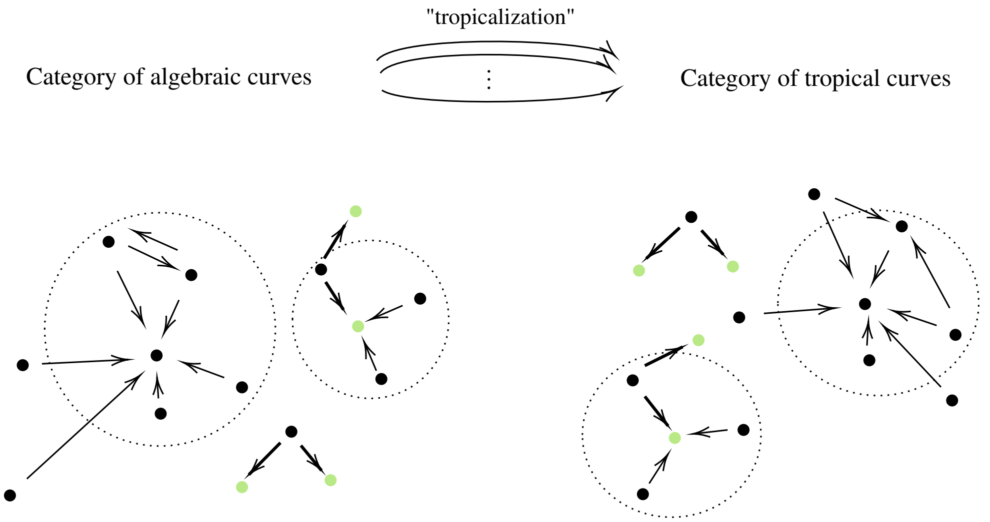


Figure 4: Vue d'emsemble: The category of algebraic/tropical curves with objects = algebraic/tropical Curves (green = genus 1) and morphisms = Hurwitz covers (fat arrows = optimal covers). Dashed circles enclose sets of morphisms that contributes to one Hurwitz number.

Tropical geometry is well-suited for realizing Point 2, due to an inherently connective character. We have already seen one expression of this character: It provides formalized bridges to various mathematical disciplines. This connecting nature also shows up in another form: As bridge between abstract and concrete mathematics. A recurring goal is to move as smoothly as possible between the two.

Indeed, tropical geometry offers a setting in which the two poles, abstract theory and concrete computations, often coexist closely. The tension between abstract and concrete is a reoccurring theme in this thesis, and we argue that it is a fruitful one. The reconstruction procedure in Section 3.5 exemplifies this, demonstrating how theoretical and explicit results stimulate each other.

Two perspectives on tropical geometry Tropical geometry is developed in formal analogy with classical algebraic geometry: Fundamental objects in algebraic geometry (curves, covers,...) will have a counterpart (tropical curves, tropical covers,...) that plays an equivalent role in the tropical theory. These counterparts are either imported from algebraic geometry (via "tropicalization") or defined intrinsically according to the same motivating principle as their algebraic analogues. This parallel development establishes a close connection between the two theories.

The present work splits according to these two approaches:

Intrinsic approach: Part I. Category theory offers a framework to formalize the intrinsic development of the tropical language. Objects in tropical geometry studied in this text fall in the category of rational polyhedral spaces and require methods from discrete mathematics and combinatorics. Objects in complex algebraic geometry build on solution sets of polynomial equations over $\mathbb{C}$. Methods emerging from analytic geometry have evolved into an abstract machinery that draws from various areas such as algebra, topology and more. Structural similarities between the two fields suggest that (in principle) many constructions in algebraic geometry can be defined "analogously" and made functional for the tropical setting. Category theory provides a structured framework to do so.

The present work contributes to this program (initiated by [69]) in the context of tropical abelian varieties and their relation to ramified covers. The starting point here is the concept of tropical split Jacobians, which has a formal counterpart in algebraic geometry but is intrinsically tropical. Through a tropical category-theoretic approach we are able to retain abstract concepts and strategies from the classical treatment, while taking advantage of the tropical setting to develop concrete algorithms to work with them.

Extrinsic approach: Part II. Enumerative problems in algebraic geometry provide a strong motivation that drives the development of the tropical language via the first approach (tropicalization). Degeneration techniques, for example, are commonly used to reduce their complexity. Tropical geometry encodes the "end product" in a combinatorial object, which is treated as the counterpart of the original object itself.

The occasion for further extending the tropical formalism in Part II is the classical count of spin Hurwitz numbers with completed cycles. We build on the work of [28] in the context of Gromov-Witten invariants and Hurwitz numbers who recognized the suitability of tropical geometry to encode expressions in the operator formalism of Fock space. The transition to spin Hurwitz numbers comes naturally. Previous work by [28] and [46] positioned tropical geometry within a broad network of mathematical areas connected by Hurwitz theory (see Figure 1). The present text sees this as opportunity to establish analogous connections in the spin case, thereby integrating itself into this ongoing research program.

0.5 Overview

Part I: Tropical split Jacobians. Chapter 2 establishes a categorical framework for tav and curves and derives a characterization of split Jacobians in $\mathbb{T}\mathcal{A}$ and $\mathbb{T}\mathcal{C}$.

Chapter 3 contains a procedure for constructing split Jacobians from their building blocks and a characterization in A_2^{tr} and M_2^{tr}.

Part II: Tropical spin Hurwitz numbers with completed cycles Part II is laid out according to the network of mathematical areas that host (spin) Hurwitz numbers, with a focus on creating a structured framework for further research:

We start with a review of the classical (type A) theory: Chapter 4 covers the algebraic, the representation theoretic, and the tropical setting and Chapter 6 the Fock space setting. This structure is mirrored in Chapters 5 and 7 for the spin (type B) theory. Apart from collecting known results, these chapters provide a unified framework that highlights the parallels between the type A and the type B theory, as well as the bridges between the different settings.

Building upon this structured foundation, Chapter 8 finds a tropical language and presents the main results of the third project.

Chapter 1

Preliminaries

This chapter introduces classical concepts around curves and moduli problems, providing the algebraic context in which this work is done. We structured this preliminary material so that it mirrors the framework used for the tropical setting in subsequent chapters. Those later chapters will contain technical introductory sections to the necessary tropical concepts. We close with an informal outlook towards their tropical analogues.

1.1 Category of algebraic curves

Objects: A *curve* C is a reduced, projective variety of dimension one, not necessarily connected, over $\mathbb{C}$, i.e. C is isomorphic a one dimensional subvariety of $\mathbb{P}^n$ (see [18], Section 1.5).

Assume $C \subset \mathbb{P}^n$ and let $I(C)$ be the ideal generated by all homogeneous polynomials vanishing on C. A point $p \in C$ is *smooth* (*singular*), if the Jacobian matrix $(\frac{\partial f_i}{\partial x_j}(p))_{i,j}$, where f_i are generators of $I(C)$, has rank at least $n - \operatorname{codim}_C(p) = n - 1$ (rank strictly smaller than $n - \operatorname{codim}_C(p) = n - 1$). If all points are smooth, we say that C is smooth, and C is singular otherwise.

Morphisms: Let C_i for $i = 1, 2$ be curves. A morphism of curves will be a regular map $f : C_1 \to C_2$ (i.e. locally of the form $f(x) = (g_0(x) : \dots : g_m(x)$, where each g_i is the quotient of homogeneous polynomials of the same degree in the homogeneous coordinate ring of C).

1.1.1 Smooth curves and Riemann surfaces

A fundamental invariant associated with a curve C is its genus (invariant under birational transformations). There are essentially two notions ([70], Definition 3.2):

- the *geometric genus* $p_g(C)$ given by the dimension of the space of global sections $H^0(C, \Omega^1_C)$ of the cotangent sheaf Ω^1_C, i.e. the $\mathbb{C}$-vector space of holomorphic 1-forms or differentials on C.

- the *arithmetic genus* $p_a(C) := 1 - \chi(\mathcal{O}_C)$, where $\mathcal{O}_C$ is the structure sheaf on C, i.e. the sheaf of regular functions on C, and χ its (algebraic not topological see [39]) Euler characteristic.

These agree when C is irreducible and smooth. We set $g(C) := p_a(C) = p_g(C)$. If C is reducible (equivalently disconnected since it is smooth), we set $g(C) := p_a(C) = 1 - k + \sum_{i=1}^{k} g(C_i)$, where $C = \bigcup_{i=1}^{k} C_i$ is its decomposition into irreducible components.

Irreducible and smooth curves form (with non-constant regular maps as morphism) a subcategory $\mathcal{C}$ whose analytic counterpart is the category of compact Riemann surfaces $\mathcal{RS}$.

Category of compact Riemann surfaces *Objects:* A *Riemann surface* X is a complex analytic manifold of dimension 1. Seen as a real manifold, a compact Riemann surface is a connected and compact orientable surface and hence homeomorphic to a connected sum of g tori (Theorem 2.4.3 [20]). The number $g(X) := g$ is the analytic counterpart of the genus and called the *(topological) genus* of X. *Morphisms:* Let X_i for $i = 1, 2$ be compact Riemann surfaces. A map of sets $f : X_1 \to X_2$ is a morphism, if f is holomorphic (see Definition 1.1.2).

Equivalence of categories Smooth curves and Riemann surfaces are equivalent notions. More precisely: There is a functor $\mathcal{F} : \mathcal{C} \to \mathcal{RS}$, which equips an irreducible smooth curve C of genus g with a unique complex structure turning C into a Riemann surface of (topological) genus g (see e.g. [20]).

Theorem 1.1.1 ([42], Chapter 1.3)**.** The functor $\mathcal{F}$ has a quasi-inverse, i.e. the categories of smooth and irreducible algebraic curves and compact Riemann surfaces are equivalent.

Theorem 1.1.1 justifies the use of either $\mathcal{C}$ or $\mathcal{RS}$ as a setting for new terminology or constructions. $\mathcal{RS}$ is the right setting for describing morphisms in more detail.

Maps between compact Riemann surfaces A combination of highly restrictive conditions, compactness of both target and base space together with holomorphicity, leads to maps with structured behavior.

Definition 1.1.2. Let C and D be compact Riemann surfaces. We call a map $f : C \to D$ *holomorphic* if and only if for every point $p \in C$ and every choice of charts ϕ_p around p and $\phi_{f(p)}$ around $f(p)$ the map $F := \phi_{f(p)} \circ f \circ {\phi_p}^{-1}$ is a holomorphic map between open sets of $\mathbb{C}$. We call F the *local expression* for f.

In fact, by carefully choosing our charts we can always achieve that (if f is non-constant) F is a power function $z \mapsto z^{k_p}$ where the positive integer k_p is uniquely determined. Here, carefully choosing means that we need to impose a restriction on the choice of charts ϕ_p and $\phi_{f(p)}$: We require that they are centered at $p, f(p)$, i.e. $\phi_p(p) = 0 = \phi_{f(p)}(f(p))$ (see [20], Theorem 4.2.1). In this case we call k_p the *ramification index* of f at p.

Generically, i.e. except at finitely (since C is compact) many points, $k_p = 1$. Points, where $k_p > 1$ are called *ramification points* of f and their images $f(p) \in D$ *branch points*. Except at branch points, each $q \in D$ has exactly d inverse images under f. This constant is called the *degree* of f. Figure 1.1 shows a local image of f around a generic point (on the left) and a branch point (on the right). The collection of ramification indices of preimages of branch points are called *ramification profile* and form a partition of the degree of f.

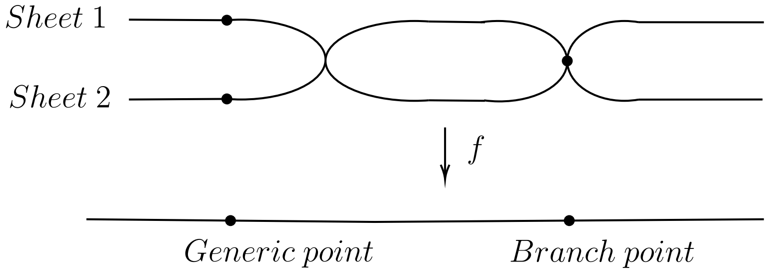


Figure 1.1: Real (local) picture of a degree 2 branched covering of $\mathbb{P}^1$ with ramification profiles (2) over the branch point. The generic point has ramification profile $(1, 1)$.

1.1.2 Singular curves: Nodal Curves

It will be useful to extend the study of curves to curves with mild singularities: A *node* is a singularity on the curve which is locally complex-analytically isomorphic to a neighborhood of the origin in the zero locus $\{xy = 0\} \subset \mathbb{C}^2$. A *nodal curve* is a curve such that every one of its points is either smooth or a node.

Example 1.1.3. Consider $C := \{[X : Y : Z] : f := ZY^2 + X^3 - ZX^2 = 0\} \subset \mathbb{P}^2$ ([70], Exercise 3.7). Then C has exactly one singularity $P := [0 : 0 : 1]$, which is the unique solution to:

$$\frac{\partial f}{\partial X} = 3X^2 - 2XZ = \frac{\partial f}{\partial Y} = 2YZ = \frac{\partial f}{\partial Z} = Y^2 - X^2 = 0.$$

We claim that P is a node, i.e. locally C looks like the union of two lines meeting transversally: Consider the chart $U_Z : \{Z \neq 0\} \subset \mathbb{P}^2$ with coordinates $(x, y) := \phi_Z([X : Y : Z]) = (X/Z, Y/Z)$. We have $\phi_Z(U_Z \cap C) = V(f(x, y, 1)) = V(y^2 + x^2(x - 1))$ and $\phi_Z(P) = (0, 0)$. Around the origin, $y^2 + x^2(x - 1) = 0$ is complex-analytically isomorphic to $0 = y^2 + x^2 \cdot (-1) = (y + x)(y - x)$, which is the union of the two lines $y = x$ and $y = -x$. More precisely, the map $(x, y) \mapsto (x\sqrt{1 - x}, y)$, where $\sqrt{1 - x}$ denotes the principal branch of the square root, is locally biholomorphic.

Both notions of genus can be extended to nodal curves: The geometric genus of a nodal curve C is given by the geometric genus of its normalization C^ν. The definition of arithmetic genus requires no further adjustments and relates to $p_a(C^\nu)$ as follows:

$$p_a(C) = p_a(C^\nu) + \#\{\text{ nodes of } C\}(\text{[70], Exercise 3.8}). \qquad (1.1)$$

These do not agree in general.

Example 1.1.4. Let C be the curve from Example 1.1.3 and $\nu : C^\nu \to C$ its normalization. We have $p_a(C) = 0 + 1 - 1 + 1 = 1$ and $p_g(C) = 0$.

As a slogan: $p_g(C)$ is the genus of the curve obtained from C after "unravelling" all the nodes and $p_a(C)$ is the genus of the curve obtained from C after "smoothing" all the nodes. Roughly, unravelling a node either creates a new connected component or decreases the number of holes. The summand $\#\{$ nodes of $C\}$ (Equation 1.1) compensates for the resulting decrease in genus and reveals a feature of $p_a(C)$: It stays constant in families of curves with possibly singular fibres (see Section 1.2.1).

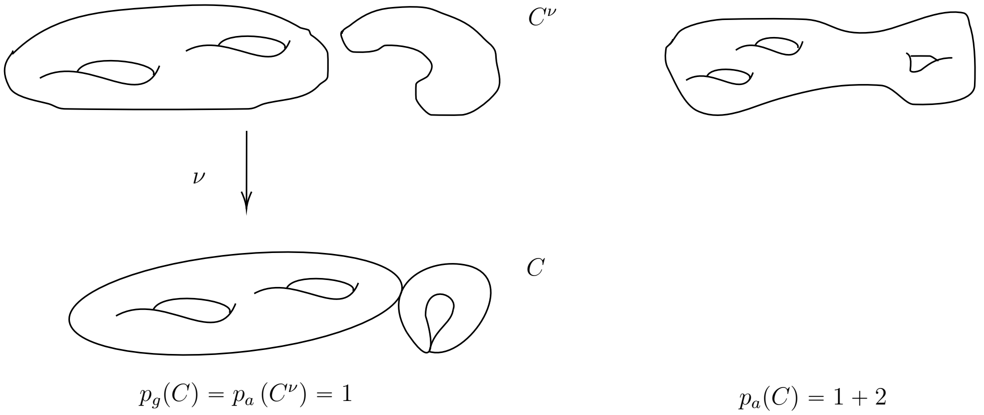


Figure 1.2: A nodal curve C, its normalization C^ν, on the left (obtained by "unravelling" all nodes), and the curve after "smoothing" all nodes, on the right.

Remark 1.1.5. We have an equivalence of data ([70], Fact 3.9)

$$(C \text{ nodal curve}, \{p_i\}_i \text{nodes}) \longleftrightarrow (C^\nu \text{smooth curve}, \{(p_i', p_i'')\}_i \text{marked points}),$$

given by associating to a curve C with normalization $\nu : C^\nu \to C$, the smooth curve C^ν together with the preimages $(p_i', p_i'') = \nu^{-1}(p_i)$ and, conversely, to a tuple $(C^\nu, \{(p_i', p_i'')\}_i)$ the unique curve obtained by identifying the points p_i' and p_i''. This means we can think of a nodal curve as a possibly disconnected smooth curve together with the data of marked points.

1.2 Moduli problems

Moduli problems related to the category of curves are a recurring theme of this text. They appear in Project 2 and, indirectly, in Project 3. We include a general introduction here, based on [70], [1], [47], [19] and [75]. Our main example is the moduli space of curves.

Classification is a fundamental task in mathematics, a *moduli problem* specifies this task and, in doing so, suggests what a "good" classification should be.

Definition 1.2.1. (Naive moduli problem)([47], Definition 2.8) Let $\mathcal{C}$ be a category with fiber products. Given a class of objects $\mathcal{P}$ in $\mathcal{C}$ and an equivalence relation $\sim$ on $\mathcal{P}$, find an object $M_{\mathcal{P}}$ in $\mathcal{C}$ whose points are in bijection with equivalence classes of objects in $\mathcal{P}$. The data $(\mathcal{P}, \sim)$ is referred to as a *naive moduli problem*.

There is often a natural notion of "proximity" in $\mathcal{P}$, that can be formalized by the notion of a *family* (see Example 1.2.9). The exact definition depends on the moduli problem. A good model is: Let $\mathcal{C} = \mathbf{Top}$ be the category of topological spaces. A family of $\mathcal{P}$-objects over the base (space) B is a space X together with a surjective morphism $\pi : X \to B$ whose fibers $\pi^{-1}(b)$ are objects in $\mathcal{P}$ ([19], Definition 1). Note: families over points correspond to objects in $\mathcal{P}$. Any notion of family that we will consider is a variant of this.

We extend a moduli problem given by $(\mathcal{P}, \sim)$ as in Definition 1.2.1 to include the data ([47], Definition 2.10):

- for each object B in $\mathcal{C}$, a pair $(\mathcal{P}_B, \sim_B)$ consisting of a *set of "families"* over B and an equivalence relation on $\mathcal{P}_B$,
- for each morphism $f : B' \to B$, a *pullback map* $f^* : \mathcal{P}_B \to \mathcal{P}_{B'}$,

such that the following compatibility conditions are satisfied:

1. $\sim_B$ specializes to $\sim$ on objects (to be made precise).
2. the pullback Id^* of $\mathrm{Id} : B \to B$ is the identity on $\mathcal{P}_B$.
3. $\sim_B$ is compatible with pullbacks, i.e.
   - for a morphism $f : B' \to B$ and families $X_1 \sim_B X_2$, we have $f^*(X_1) \sim_{B'} f^*(X_2)$.
   - for morphisms $g : B'' \to B'$,$f : B' \to B$ and $X \in \mathcal{P}_B$, $(f \circ g)^*(X) \sim_{B''} g^* \circ f^*(X)$.

We specialize to the setting $\mathcal{C} = \mathbf{Sch}_{\mathbb{C}}$ (the category of schemes over $\mathbb{C}$) and clarify Point 1: Families over $B := \mathrm{Spec}\mathbb{C}$ correspond to objects in $\mathcal{P}$. Hence, we require $(\mathcal{P}, \sim) = (\mathcal{P}_{\mathrm{Spec}\mathbb{C}}, \sim_{\mathrm{Spec}\mathbb{C}})$. This data can be conveniently packaged into a functor.

Definition 1.2.2. A *moduli functor/problem* for the setting above is a contravariant functor

$$\mathcal{F}_{\mathcal{P}} : \mathbf{Sch}_{\mathbb{C}} \to \mathbf{Set},$$

where

$$\mathcal{F}_{\mathcal{P}}(B) := \mathcal{P}_B/ \sim_B \text{ and } \mathcal{F}_{\mathcal{P}}(B' \xrightarrow{f} B) := f^*.$$

Definition 1.2.3. Let $\mathcal{F}_{\mathcal{P}}$ be a moduli problem (Definition 1.2.2). If $\mathcal{F}_{\mathcal{P}}$ is representable, i.e. if there exists a scheme $M_{\mathcal{P}}$ such that $\mathcal{F}_{\mathcal{P}} \cong \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{C}}}(\cdot, M_{\mathcal{P}})$ (as functors), $M_{\mathcal{P}}$ is called a *fine moduli space* for $\mathcal{F}_{\mathcal{P}}$.

Remark 1.2.4. A fine moduli space $M_{\mathscr{P}}$ for a problem $\mathscr{F}_{\mathscr{P}}$ satisfies all we could wish for: For every scheme B, we have a bijection between the set of families of $\mathscr{P}$-objects over B and the set of morphisms from B to $M_{\mathscr{P}}$ given by the component

$$T_B : \mathscr{F}_{\mathscr{P}}(B) = \{\text{families over } B\} \to \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{C}}}(B, M_{\mathscr{P}})$$

of the natural transformation $\mathscr{F}_{\mathscr{P}} \overset{T}{\cong} \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{C}}}(\cdot, M_{\mathscr{P}})$.

In particular, setting $B = \mathrm{Spec}\mathbb{C}$ yields a bijection between equivalence classes of objects in $\mathscr{P}$ and $\mathbb{C}$-points of $M_{\mathscr{P}}$.

Being representable is a strong condition and provides additional structure.

Definition 1.2.5. Let $M_{\mathscr{P}}$ be a fine moduli space for a problem $\mathscr{F}_{\mathscr{P}}$ (Definition 1.2.3). Denote by T a natural isomorphism $\mathscr{F}_{\mathscr{P}} \to \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{C}}}(\cdot, M_{\mathscr{P}})$. Then

$$U := T^{-1}(\mathrm{Id}_{M_{\mathscr{P}}}) \in \mathscr{F}_{\mathscr{P}}(M_{\mathscr{P}})$$

is called the *universal family.*

Remark 1.2.6. The adjective "universal" is used for U, since any family X over B can be obtained from U via pullback: Let $f := T_B(X)$ (see Remark 1.2.4) and consider the commutative diagram (with diagram-chase)

$$\begin{tikzcd}
\mathrm{Hom}(M_{\mathscr{P}}, M_{\mathscr{P}}) \arrow[r, "\circ f"] & \mathrm{Hom}(B, M_{\mathscr{P}}) \\
\mathscr{F}_{\mathscr{P}}(M_{\mathscr{P}}) \arrow[u, "T_{M_{\mathscr{P}}}"] \arrow[r, "f^*"] & \mathscr{F}_{\mathscr{P}}(B) \arrow[u, "T_B"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Id}_{M_{\mathscr{P}}} \arrow[d, mapsto] & f \arrow[l, mapsto] & & \\
U & X \arrow[u, mapsto] & U \arrow[r, mapsto] & f^*(U)
\end{tikzcd}$$

showing $X = f^*(U)$.

Not all moduli problems admit a fine moduli space, in fact many do not see Example (1.2.10). Relaxing (the requirements of) Definition 1.2.3 introduces the concept of a *coarse moduli space.*

Definition 1.2.7. Let $\mathscr{F}_{\mathscr{P}}$ be a moduli problem. A *coarse moduli space* for $\mathscr{F}_{\mathscr{P}}$ is a pair $(M_{\mathscr{P}}, \Phi)$, where $M_{\mathscr{P}}$ is a scheme and $\Phi : \mathscr{F}_{\mathscr{P}} \to \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{C}}}(\cdot, M_{\mathscr{P}})$ is a natural transformation, satisfying that

- for all pairs (M', Φ') as above, there exists a unique natural transformation Ψ such that
$$\begin{tikzcd}
\mathscr{F}_{\mathscr{P}} \arrow[r, "\Phi"] \arrow[dr, "\Phi'"'] & \mathrm{Hom}(\cdot, M_{\mathscr{P}}) \arrow[d, "\exists!\Psi"] \\
& \mathrm{Hom}(\cdot, M')
\end{tikzcd}$$
commutes.
- the map $\Phi_{\mathrm{Spec}\mathbb{C}}$ is a bijection.

Remark 1.2.8. A pair $(M_{\mathscr{P}}, \Phi_{\mathscr{P}})$ as in Definition 1.2.7 is (still) a moduli space: Condition 1 (Definition 1.2.7) spells that $\mathbb{C}$-points of $M_{\mathscr{P}}$ parameterize equivalence classes in $\mathscr{P}$

$$\mathscr{P}/\sim\,\cong \mathscr{F}_{\mathscr{P}}(\mathbb{C}) \xrightarrow{\Phi_{\mathrm{Spec}\mathbb{C}}} \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{C}}}(\mathrm{Spec}\mathbb{C}, M_{\mathscr{P}}) \cong M_{\mathscr{P}}(\mathbb{C}).$$

A pair $(M_{\mathscr{P}}, \Phi_{\mathscr{P}})$ is a (only) a "coarse" moduli space: The component maps Φ_B of Φ are not necessarily bijections. This mean that the geometry of $M_{\mathscr{P}}$ and the original moduli problem may be quite different.

We collect two examples of moduli problems in $\mathcal{C} = \mathbf{Sch}_{\mathbb{C}}$ from [70] we found enlightening.

Example 1.2.9. Let $\mathscr{P} := \{l : l \text{ line through the origin in } \mathbb{C}^2\}$ with equivalence relation $\sim$ being equality. A naive answer to Task/Definition 1.2.1 would be to set $M_{\mathscr{P}} = \mathbb{P}^1$ with bijective correspondence:

$$\left\langle \begin{pmatrix} X \\ Y \end{pmatrix} \right\rangle = l \mapsto [X : Y] =: [l].$$

However, we want $M_{\mathscr{P}}$ to encode additional structure: We have a natural notion of proximity. We say $l_1 = \left\langle \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \right\rangle$ is "close to" $l_2 = \left\langle \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} \right\rangle$ (where $X_i \neq 0$) if $\frac{Y_1}{X_1}$ is close to $\frac{Y_2}{X_2}$ with respect to the euclidean topology on $\mathbb{C}$ (see Figure 1.3).

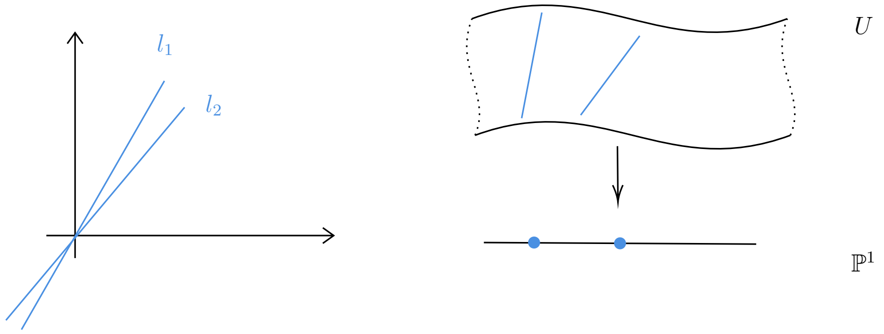


Figure 1.3: Intuitive notion of proximity in $\mathscr{P}^1$ (on the left). On the right, a sketch of the universal family over $\mathscr{P}^1$.

Families of lines. A family of lines through the origin (over a scheme)/(parameterized by) X is a line bundle $\pi : L \to X$ that is a subbundle of the trivial vector bundle $X \times \mathbb{C}^2 \to X$ (this means: we can find an injective morphism $L \xrightarrow{i} X \times \mathbb{C}^2$ of vector bundles whose cokernel exists (as a vector bundle) with equivalence relation on the set of families over X being the "existence of an isomorphism of line bundles that is compatible with both embeddings. More precisely, $(L \xrightarrow{i} X \times \mathbb{C}^2) \sim_X (L' \xrightarrow{i'} X \times \mathbb{C}^2)$, if there exists an

$$\mathcal{F}_{\mathcal{P}}(X) \ni \left[\begin{array}{c} L \\ \downarrow \pi \\ X \end{array}\right] \longleftrightarrow \left[\, X \longrightarrow \mathbb{P}^1 \,\right] \in \mathrm{Hom}\,(X,\mathbb{P}^1)$$

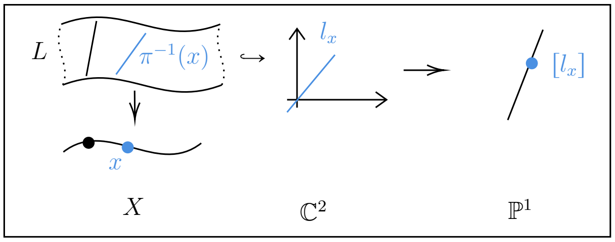


Figure 1.4: Visualization of T_X.

isomorphism $L \xrightarrow{\phi} L'$ of line bundles such that
$$\begin{tikzcd} L \arrow[r, "i"] \arrow[d, "\phi"] & X \times \mathbb{C}^2 \\ L' \arrow[ur, "i'"] & \end{tikzcd}$$
commutes. Note: This aligns with what we would want to call a family of $\mathcal{P}$-objects since the line bundles in question come together with fiberwise injective maps to $\mathbb{C}^2$ (given by i). The equivalence relation for families just means that after embedding the families are equal.

The extended moduli problem. Define

$$\mathcal{F}_{\mathcal{P}}(X) = \{(L \to X)\}/\sim_X \text{ and } \mathcal{F}_{\mathcal{P}}(X \xrightarrow{f} X') : \mathcal{F}_{\mathcal{P}}(X') \to \mathcal{F}_{\mathcal{P}}(X), (L \to X) \mapsto (f^*L \to X'),$$

where $(f^*L \to X')$ denotes the usual pullback. This data is compatible (i.e. it defines a functor $\mathcal{F}_{\mathcal{P}} : \mathbf{Sch}_{\mathbb{C}} \to \mathbf{Set}$) since

- the pullback takes subbundles to subbundles, i.e. $(f^*L \to X')$ is again (via f^*i) a subbundle of bundle $X' \times \mathbb{C}^2 \to X'$.
- $\mathcal{F}_{\mathcal{P}}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}_{\mathcal{P}}(X)}$ and the equivalence relations $\sim_X$ behave well with the pullback maps f^*.

A fine moduli space for $\mathcal{F}_{\mathcal{P}}$. There exists a natural isomorphism $\mathcal{F}_{\mathcal{P}} \overset{T}{\cong} \mathrm{Hom}(\cdot,\mathbb{P}^1)$, i.e. a collection of component maps

$$T_X : \mathcal{F}_{\mathcal{P}}(X) \xrightarrow{\sim} \mathrm{Hom}(X,\mathbb{P}^1)$$

satisfying some additional compatibility conditions. Here is an intuition for T_X's behavior on closed points (see also Figure 1.4): Let $L \xrightarrow{\pi} X \in \mathcal{F}_{\mathcal{P}}(X)$ and $l_x := i(\pi^{-1}(x)) \in \{x\} \times \mathbb{C}^2$ the line obtained after embedding the fiber of π above a point x of X. Then $L \xrightarrow{\pi} X$ gives rise to a morphism $X \to \mathbb{P}^1$ defined for closed points by $x \mapsto [l_x]$. To see why the construction of T_X must be restated in algebraic terms in order to associate to each family over a *scheme* a morphism of *schemes*, consider the following example:Let L be the trivial line bundle over $X := \text{Spec}(\mathbb{C}[x]) = \{[x-a], a \in \mathbb{C}\} \cup \{[0]\}$. On closed points, $\{[x-a], a \in \mathbb{C}\} \cong \mathbb{C}$, L together with its map to $X \times \mathbb{C}^2$ are given by

$$L = \mathbb{C} \times \mathbb{C} \xrightarrow{i} X \times \mathbb{C}^2, (a,t) \mapsto (a,(a,ta)).$$

Having constructed an assignment X to $\mathbb{P}^1$ on the level of closed points $a \mapsto [1:a]$, we need to address two issues: First, how to properly handle the point $[0]$ and then how to turn this into a genuine morphism of schemes. Regarding the first, the fiber $L_{[0]}$ of π over $[0]$ is the pullback of π along $\text{Spec}(\mathbb{C}(x)) \to \text{Spec}(\mathbb{C}[x]) = X$. This is given by $L_{[0]} = \text{Spec}(\mathbb{C}(x)[t])$, the affine line over $\text{Spec}(\mathbb{C}(x))$, with injective map to $\text{Spec}(\mathbb{C}(x)) \times \mathbb{A}^2 = \text{Spec}(\mathbb{C}(x))[u,v]$ induced by the map on rings $u \mapsto t$ and $v \mapsto tx$. We see that extending the assignement X to $\mathbb{P}^1$ to a full morphism of schemes is not immediate. For a formal construction we refer to [70], Example 2.4.

The universal family is given by the tautological bundle (see [70])

$$\{([l],c) \in \mathbb{P}^1 \times \mathbb{C}^2 : c \in l\} =: U \to \mathbb{P}^1, ([l],c) \mapsto [l].$$

Since $\mathbb{P}^1$ is a fine moduli space: Specifying the data of "families of lines" is equivalent to endowing $\mathbb{P}^1$ with a topology. It is the weakest topology induced by the collection of maps to $\mathbb{P}^1$ (that are in bijective correspondence with these families). It is the Zariski topology, when considering families of lines as above (corresponding to maps from schemes).

However, $\mathbb{P}^1$ also has a finer topology than the Zariski topology, the complex topology, which is induced by a metric. This would correspond to only looking at families of lines over complex spaces (i.e. line bundles over a complex spaces that are subbundles of the trivial bundle), thus endowing $\mathbb{P}^1$ with a metric that formalizes our notion of proximity.

Example 1.2.9 is as nice as it gets: The moduli functor is representable *and* the corresponding fine moduli space is compact. The situation is already very different for curves of genus 1.

Example 1.2.10. Consider the following classification problem: Let $\mathscr{P} := \{C : C \text{ smooth curve of genus } 1\}$ with relation $\sim$ given by $C_1 \sim C_2$ if and only if they are isomorphic. *A naive answer* to Task/Definition 1.2.1 exploits the following fact (see [70]): For C a smooth curve of genus 1 there exists $t \in \mathbb{C} \backslash \{0,1\}$ such that

$$C \cong E_t := \{[X:Y:Z] : Y^2Z + X(X-Z)(X-tZ) = 0\}.$$

The idea is to use $U := \mathbb{C}\backslash\{0,1\}$ as a first approximation to a parameter space. We have a surjective map

$$\phi : U \to \mathscr{P}/\sim, t \mapsto [E_t]$$

with $\phi^{-1}([E_t]) = \{t, \frac{1}{t}, 1-t, \frac{1}{1-t}, \frac{t-1}{t}, \frac{t}{t-1}\}$. In order to construct a scheme out of these ingredients, we describe the sets $\phi^{-1}([E_t])$ as orbits of a group action on U: Define $S_3 \times U \to$

U, $(\sigma,t) \mapsto \sigma\cdot t$ by $(12)\cdot t := \frac{1}{t}$ and $(23)\cdot t := 1-t$. Then $\phi^{-1}([E_t]) = S_3\cdot t$ and $U/S_3 \cong \mathscr{P}/\sim$ (as sets), where U/S_3 is the set of orbits of U under the action of S_3. Since S_3 can be viewed as an algebraic group (i.e. a group object in $\mathbf{Sch}_{\mathbb{C}}$, see [47], Definition 3.1 for more details) and the action of S_3 on U is algebraic (i.e. $S_3 \times U \to U$, is a morphism in $\mathbf{Sch}_{\mathbb{C}}$ that is compatible with the group structure (identity, inversion, multiplication), see [47], Definition 3.5 for more details), there is hope to turn the quotient U/S_3 into a scheme. It works in this case: The corresponding quotient in $\mathbf{Sch}_{\mathbb{C}}$ is given by the morphism

$$j : \mathbb{A}^1\backslash\{0,1\} \to \mathbb{A}^1, t \mapsto j(t) := 2^8\frac{(t^2-t+1)^3}{t^2(t-1)^2}$$

known as the *j-invariant*, where

- U is exactly the set of $\mathbb{C}$-points of $\mathbb{A}^1\backslash\{0,1\}$.
- preimages of closed points of $\mathbb{A}^1$ (i.e.those corresponding to $\mathbb{C}$) are exactly the orbits.

The data needed to specify the extended moduli problem/the moduli functor $\mathcal{F}_{\mathscr{P}}$ (i.e. a notion of family together with equivalence relations on the sets of families over a common base) will be defined in Section 1.2.1. At this point only the following remark: $M_1 := \mathbb{A}^1$ is a moduli space for $\mathcal{F}_{\mathscr{P}}$, but it is not a fine one.

A different issue and an informal discussion. Compactness is (among others) a desirable property. What should a *good* compactification of M_1 be? A compact space $\overline{M}_1$ such that $M_1 \subset \overline{M}_1$ open. Ideally, $\overline{M}_1$ itself should be a moduli space, a moduli space for smooth curves of genus 1 and for objects that are generalisations of them. Although $\mathbb{P}^1$ appears as the most natural candidate, the question remains whether it is a good compactification in the sense described above: We have "$j(t) = \infty$", if $t = 0, 1$. In these cases, the curves E_0 and E_1 are nodal cubics of arithmetic genus 1. This means we can view $\mathbb{P}^1$ as moduli space for curves of (arithmetic) genus 1, which are allowed to have nodes as singularities.

The "family" E_t parametrized by t satisfies $j(E_t) \underset{t\to\infty}{\to} \infty$. $\mathbb{P}^1$ provides a limit object for E_t at $t = \infty$. This shows that the compactification not only parametrizes a larger class of objects, but also answers the question: How do objects in a family degenerate?

The parameter spaces constructed in Examples 1.2.9 and 1.2.10 are not canonical, their definitions involve several arbitrary choices. However, having established that these are moduli spaces, the general theory guarantees that the corresponding Hom functors are unique up to natural isomorphism and, by Yoneda's Lemma, so are the respective schemes.

Remark 1.2.11. Example 1.2.10 represents an instance of a general procedure for constructing moduli spaces: Given a moduli functor $\mathcal{F}_{\mathscr{P}}$ (if possible) find a family of $\mathscr{P}$-objects $U \xrightarrow{\pi} S$ such that for each equivalence class $[p]$ of $\mathscr{P}$-objects there exists (at least one) point s of S with $\pi^{-1}(s)$ is a member of $[p]$ (in Example 1.2.10, $S = \mathbb{A}^1\backslash\{0,1\}$). If $U \xrightarrow{\pi} S$ has the local universal property ([47], Definition 2.28) and equivalence classes correspond exactly to orbits of the action of an algebraic group G, then any coarse moduli space for $\mathcal{F}_{\mathscr{P}}$ (if it exists) is an orbit space (see [47], Definition 3.23 and Proposition 3.35).

Obstructions to representability.

Being representable is a desirable but strong condition: Let M be in **Sch**$_{\mathbb{C}}$, then $\mathrm{Hom}(\cdot, M)$ is a sheaf in the Zariski topology (see e.g. [47], Exercise 2.18). This means: If a moduli functor $\mathscr{F}$ is representable, then $\mathscr{F}$ is a sheaf. In particular, compatible families glue uniquely (see [8], Section 2.1.1).

Example 1.2.12. [[70], p.49] Sketch example: The moduli functor $F_{\mathscr{P}}$ of curves of genus 1 (Example 1.2.10) is not representable. Let $\{U_1, U_2\}$ be the Zariski open cover of S, where S is a cycle of 4 rational curves as shown in Figure 1.5 (after Figure 25 of [70]). Consider the families $U_1 \times E \xrightarrow{p_1} U_1$ and $U_2 \times E \xrightarrow{p_1} U_2$, where E is a smooth curve of genus 1. Since the restriction of p_1 to the intersection $U_1 \cap U_2$, $p_{1|U_1 \cap U_2}$, agrees with $p_{2|U_1 \cap U_2}$, p_1 and p_2 are compatible.

If $F_{\mathscr{P}}$ is a sheaf, then p_1 and p_2 should glue to a unique family over S. Let $\sigma \in \mathrm{Aut}(E) \backslash \{\mathrm{Id}_E\}$ (this exists see [70]) and consider the family $C \xrightarrow{\pi} S$ obtained by gluing $U_1 \times E \xrightarrow{p_1} U_1$ and $U_2 \times E \xrightarrow{p_1} U_2$ along the two connected components U' and U'' of $U_1 \cap U_2$ via $(\mathrm{Id}, \mathrm{Id}) : U' \times E \to U' \times E$ and $(\mathrm{Id}, \sigma) : U' \times E \to U' \times E$. It can be shown that π is not isomorphic to the trivial family. Hence, there are at least to ways to glue p_1 and p_2.

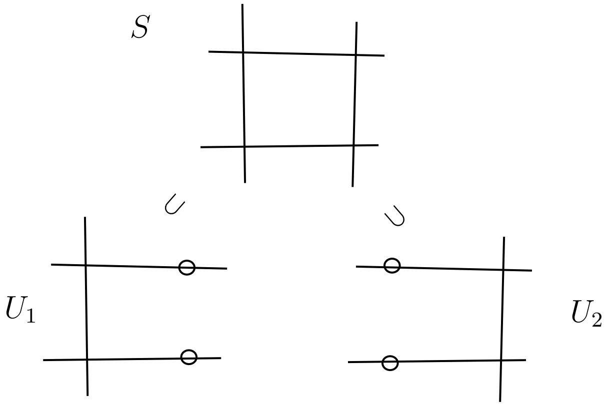


Figure 1.5: A sketch of the base S, reproduced from Figure 25 ([70]) for convenience of the reader.

Example 1.2.12 shows: The presence of non-trivial automorphism is (often) an obstruction to representability.

1.2.1 Moduli problems and the category of algebraic curves Main Example: Moduli space of (marked) curves

In the coming section, we shall dwell further on the problem of classifying curves of genus g. We start with:

The naive moduli problem. Example 1.2.10 and Example 1.2.12 serve as informal motivation for introducing the notion of a *stable (marked) curve*. These will be the objects we will want to classify up to isomorphism.

Definition 1.2.13. [[70], Definition 3.10 and Proposition 3.13] Let C be a connected curve with at most nodes as singularities and $\sigma = \{p_1, ..., p_n\}$ a set of distinct and smooth points of C. We call

- C *stable*, if $|\mathrm{Aut}(C) = \{\varphi : C \to C \mid \varphi \text{ is an automorphism}\}| < \infty$.
- (C, σ) *stable*, if $|\mathrm{Aut}(C, \sigma) = \{\varphi \in \mathrm{Aut}(C) : \varphi(p_i) = p_i\}| < \infty$.

A point $p_i \in \sigma$ is a *marked point*, the preimages of nodes and marked points under the normalization map $\nu : C^\nu \to C$ are *special*, and the tuple (C, σ) is an n*-marked/pointed curve.*

If C is smooth and $g(C) \geqslant 2$, then $|\mathrm{Aut}(C)| < \infty$. Hence, we can force the automorphism groups for nodal curves to be finite using the following marking rule: Let (C, σ) be an n-marked curve and C^ν its normalization. Then (C, σ) is stable if and only if every irreducible component $C_i \subset C^\nu$ has

- at least 3 special point, if $g(C_i) = 0$.
- at least 1 special point, if $g(C_i) = 1$.

Remark 1.2.14.

- We allow for nodal curves to have a chance to obtain a compact moduli space of curves of genus g.
- Automorphism groups are an obstruction to representability. We restrict to finite automorphism groups to ensure that the problem is bounded (see [70], Figure 10 for a sketch of an unbounded sequence nodal curves). The precise reason is that this condition ensures that the moduli stack of curves $\mathcal{M}_{g,n}$ is Deligne-Mumford (see e.g. [70], Section 5.1).

The extended moduli problem/the moduli functor Recall, a moduli functor is specified by the following data:

1. for each scheme S, a set of families over S.
2. for each set $\mathscr{P}_S$, an equivalence relation $\sim_S$.
3. for each morphism $S' \to S$, a pullback map $\mathscr{P}_S \to \mathscr{P}_{S'}$ of families.

1. Families of curves.

Definition 1.2.15. [[70], Definition 3.15] A *family* of n-marked (smooth/stable) genus g curves is a tuple $(\pi : C \to S, p_1, \ldots, p_n : S \to C)$, where

- π is a (smooth/flat), proper, surjective, finitely presented morphism of schemes such that the fibers $C_s := \pi^{-1}(s)$ over any geometric point $s \in S$ is a (smooth/stable) connected curve of arithmetic genus g.

- $p_1, \ldots, p_n$ are pairwise distinct sections of p_i with for any $s \in S$ $p_i(s)$ is a smooth point of C_s.

For S in $\mathbf{Sch}_{\mathbb{C}}$, let

- $\mathcal{P}_S^{(g,n)}$ be the set of smooth families with base S.

- $\overline{\mathcal{P}}_S^{(g,n)}$ be the set of stable families with base S.

Remark 1.2.16. Some explanatory words on Definition 1.2.15:

- Definition 1.2.15 only imposes requirements for *geometric* fibers, since we intend to work over algebraically closed fields.

- flat, proper, surjective, finitely presented is a collection of technical conditions for what constitutes a well-behaved family of curves: For example guaranteeing that various numerical invariants of fibers are constant such as the dimension, the arithmetic genus, the degree of π (see [76], 24.1).

- The morphisms $p_1, \ldots, p_n$ pick out exactly exactly n pairwise distinct smooth points $p_1(s), \ldots, p_n(s)$ in every fiber.

2. *Equivalence relation.* Two families with base S are *equivalent*

$$(\pi : C \to S, p_1, \ldots, p_n : S \to C) \sim_S (\pi' : C' \to S, p'_1, \ldots, p'_n : S \to C')$$

if there exists an isomorphism $\varphi : C \to C'$ such that for $i = 1, \ldots, n$ the diagram

$$\begin{tikzcd}
C \arrow[rr, "\varphi"] \arrow[dr, "\pi"] & & C' \arrow[dl, "\pi'"'] \\
& S \arrow[ul, bend left, "p_i"] \arrow[ur, bend right, "p'_i"'] &
\end{tikzcd}$$

commutes.

3. *Pullback maps.* Let $\pi : C \to S$ be a family of curves and $f : S' \to S$ any morphism of schemes. Consider their fiber product in $\mathbf{Sch}_{\mathbb{C}}$:

$$\begin{tikzcd}
C \times_S S' \arrow[r, dashed] \arrow[d, dashed, "\pi_{S'}"] & C \arrow[d, "\pi"] \\
S' \arrow[r, "f"] & S
\end{tikzcd} .$$

The map $\pi_{S'} : C_{S'} := C \times_S S' \to S'$ is again a family of curves (in the sense of Definition 1.2.15), since the collection of properties (proper, smooth/flat, finitely presented, surjective) is preserved under base change (see e.g. [76], Section 9.4).

For $f : S' \to S$, introduce the *pullback* of smooth/stable families

$$f^*(\pi : C \to S, p_1, ..., p_n : S \to C) := (\pi_{S'} : C_{S'} \to S', p_{1,S'}, ..., p_{n,S'} : S' \to C_{S'}),$$

where $p_{i,S'} := (p_i \circ f) \times \mathrm{Id}_{S'}$. Fact: Stability is also preserved under base change. We package data 1.-3. into moduli functors.

Definition 1.2.17. [[70], Definition 3.17] Let $g, n \geqslant 0$. Define the *moduli functor of smooth n-marked genus g curves* $\mathcal{M}_{g,n} : \mathbf{Sch}_{\mathbb{C}} \to \mathbf{Set}$ by

$$\mathcal{M}_{g,n}(S) = \mathscr{P}_S^{(g,n)} / \sim_S \text{ and } \mathcal{M}_{g,n}(S' \xrightarrow{f} S) := f^* : \mathcal{M}_{g,n}(S) \to \mathcal{M}_{g,n}(S')$$

and the "compactified moduli problem" given by the *moduli functor of stable n-marked genus g curves* $\overline{\mathcal{M}}_{g,n} : \mathbf{Sch}_{\mathbb{C}} \to \mathbf{Set}$:

$$\overline{\mathcal{M}}_{g,n}(S) = \overline{\mathscr{P}}_S^{(g,n)} / \sim_S \text{ and } \overline{\mathcal{M}}_{g,n}(S' \xrightarrow{f} S) := f^* : \overline{\mathcal{M}}_{g,n}(S) \to \overline{\mathcal{M}}_{g,n}(S').$$

If $n = 0$, write $\mathcal{M}_g := \mathcal{M}_{g,0}$, respectively $\overline{\mathcal{M}}_g := \overline{\mathcal{M}}_{g,0}$.

Solution 1: Schemes. The moduli problems $\mathcal{M}_{g,n}$ and its compactification $\overline{\mathcal{M}}_{g,n}$ (Definition 1.2.17) admit the following solution:

Theorem 1.2.18. (Deligne-Mumford-Knudsen, [70]) For $2g - 2 + n > 0$, there exists a variety $M_{g,n}$ and an irreducible, projective variety $\overline{M}_{g,n}$ containing $M_{g,n}$ as a nonempty, open and dense subvariety, both of dimension $3g - 3 + n$, that are coarse moduli spaces for $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$, respectively. Moreover, if $g = 0$, the situation is as good as it gets: $M_{0,n}$ and $\overline{M}_{0,n}$ are even fine moduli spaces.

Solution 2: Stacks. The primary reason for seeking an alternative solution to the classification problem is that the moduli functors, $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$, are not representable. We make the following trade: We renounce to the parameterizing object being in $\mathbf{Sch}_{\mathbb{C}}$. In return we get a more complicated object, a *Deligne-Mumford stack* (also denoted by $\overline{\mathcal{M}}_{g,n}$), but in which many properties of a fine moduli space are automatically built in. We now give a brief sketch of this new object, while keeping track of the following two properties we wish it to have:

S.1 A one-to-one correspondence between morphisms $S \to \overline{\mathcal{M}}_{g,n}$ and families over a scheme S.

S.2 Existence of a universal family.

Basic structure: The new object is a pair $(\overline{\mathcal{M}}_{g,n}, F : \overline{\mathcal{M}}_{g,n} \to \mathbf{Sch}_{\mathbb{C}})$, where $\overline{\mathcal{M}}_{g,n}$ is a category defined by

- *Objects:* Families of stable curves $(\pi : C \to S, p_1, ..., p_n : S \to C)$ (Definition 1.2.15),
- *Morphisms:* A morphism from $(\pi' : C' \to S')$ to $(\pi : C \to S)$ (leaving out sections to simplify notation) is a pair $(\hat{f}, f)$ that identifies π' as the pullback of π. More precisely:

$$\mathrm{Hom}(\pi' : C' \to S', \pi : C \to S) = \{(\hat{f}, f) \text{ such that }$$

$$\begin{tikzcd} C' \arrow[r, "\hat{f}"] \arrow[d, "\pi'"] & C \arrow[d, "\pi"] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

$$\text{commutes}\},$$

and $F : \overline{\mathcal{M}}_{g,n} \to \mathbf{Sch}_{\mathbb{C}}$ is a functor defined by

$$(\pi : C \to S) \mapsto S \text{ and } (\hat{f}, f) \mapsto f.$$

More structure: The *fiber over* S, $F^{-1}(S)$, which is the subcategory whose objects map to S (in this case: elements of $\overline{\mathcal{P}}_S^{(g,n)}$) and whose morphism map to Id_S (in this case: pairs $(\hat{f}, \mathrm{Id}_S)$ inducing an equivalence of families over S (compare to $\sim_S$)), is a groupoid, i.e. a category in which all morphisms are isomorphisms, and the pair $(\overline{\mathcal{M}}_{g,n}, F : \overline{\mathcal{M}}_{g,n} \to \mathbf{Sch}_{\mathbb{C}})$ is a *category fibered in groupoids over* $\mathbf{Sch}_{\mathbb{C}}$, i.e. a category $\mathcal{M}$ together with a functor $F : \mathcal{M} \to \mathbf{Sch}_{\mathbb{C}}$ such that "pullbacks exists up to canonical isomorphims" (see [1], Definition2.4.1). This means Property **S.1** is built into the pair $(\overline{\mathcal{M}}_{g,n}, F)$: A scheme S in $\mathbf{Sch}_{\mathbb{C}}$ can be considered a category fibered in groupoids over $\mathbf{Sch}_{\mathbb{C}}$ given by $(\mathbf{Sch}_S, F_S)$, where $\mathbf{Sch}_S$ is the category whose objects are schemes over S (i.e. schemes X together with a morphism $X \to S$) and whose morphisms are morphisms of schemes over S and $F_S : \mathbf{Sch}_S \to \mathbf{Sch}_{\mathbb{C}}$ just forgets the map to S. A *morphism* $S \xrightarrow{f} \overline{\mathcal{M}}_{g,n}$ is a functor $\mathbf{Sch}_S \to \overline{\mathcal{M}}_{g,n}$ such that

$$\begin{tikzcd} \mathbf{Sch}_S \arrow[rr, "f"] \arrow[dr, "F'_S"'] & & \overline{\mathcal{M}}_{g,n} \arrow[dl, "F"] \\ & S & \end{tikzcd}$$

commutes. The assignment

$$(S \xrightarrow{f} \overline{\mathcal{M}}_{g,n}) \mapsto f(S),$$

where S is viewed as an object of $\mathbf{Sch}_S$, yields the correspondence stated in Property **S.1**.

Even more structure: $\overline{\mathcal{M}}_{g,n}$ is not *just* a category fibered in groupoids, it is a *Deligne-Mumford stack.* Features include:

- A "sheaf-like behavior": Objects glue in a unique way (see e.g. [1], Definition 2.5.1 for a precise statement). Note, this addresses one obstruction to represenability, the presence of non-trivial automorphism (Example 1.2.12), essentially achieved by the notion of morphisms in the category $\overline{\mathcal{M}}_{g,n}$: Let S be a scheme. Instead of just having a set of equivalence classes of families over S (the moduli functor perspective), we now have a groupoid that additionally records how families are isomorphic.

- Existence of universal family: see Theorem 1.2.19 and see e.g. [1], Definition 3.1.27, for a general Definition, and Example 3.1.29.
- $\overline{\mathcal{M}}_{g,n}$ has an atlas: There is a (representable) surjective morphism (of categories fibered in groupoids) $U \to \overline{\mathcal{M}}_{g,n}$ from a smooth scheme U (interpreted as category fibered in groupoids), which allows us (in some sense) to study $\overline{\mathcal{M}}_{g,n}$ via U. This should be thought of in analogy to studying a scheme via an affine cover. In this context surjective means that the family $C \xrightarrow{\pi} U$ corresponding to $U \to \overline{\mathcal{M}}_{g,n}$ satisfies that every stable curve occurs as a fiber of π.

Many concepts that are defined for schemes and morphism of schemes (such as smoothness, irreducibility, dimension, different flavours of points) have an analogue in the theory of algebraic stacks and can often be studied by looking at the atlas U. Without precise Definitions here are features of $\overline{\mathcal{M}}_{g,n}$ parallel to and extending Theorem 1.2.18:

Theorem 1.2.19. ([70], Theorem 5.1)
For $2g - 2 + n > 0$, there exists (irreducible, proper and smooth as stacks) Deligne-Mumford stacks $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ of dimension $3g - 3 + n$ with $\overline{\mathcal{M}}_{g,n}$ containing $\mathcal{M}_{g,n}$ as a nonempty, open substack. The universal family is given by the morphism $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ that forgets a section.

Remark 1.2.20. It could be argued that categories fibered in groupoids are more fundamental than moduli functors, since there is a map sending a category fibered in groupoids to a moduli functor (i.e. a functor $\mathbf{Sch}_{\mathbb{C}} \to \mathbf{Set}$):

$$(\mathcal{M}, F : \mathcal{M} \to \mathbf{Sch}_{\mathbb{C}}) \mapsto (S \mapsto F^{-1}(S)/\text{Isom.}),$$

where $F^{-1}(S)$/Isom is the set of objects of $F^{-1}(S)$ up to morphisms in $F^{-1}(S)$. We can recover the moduli functor $\overline{\mathcal{M}}_{g,n} : \mathbf{Sch}_{\mathbb{C}} \to \mathbf{Set}$ (Definition 1.2.17) as the image of the pair $(\overline{\mathcal{M}}_{g,n}, F : \overline{\mathcal{M}}_{g,n} \to \mathbf{Sch}_{\mathbb{C}})$ and there is a morphism $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$, that induces a bijection on geometric points.This identifies the moduli space $\overline{M}_{g,n}$ as "coarse image" of the better behaved Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}$.

Towards a combinatorial description of $\overline{M}_{g,n}$.

Recall the equivalence of data from Remark 1.1.5:

$$(C, \{q_i\}_i \text{ nodes}, \{p_j\}_j \text{ marked pts.}) \longleftrightarrow (C^\nu, \{(q_i', q_i'')\}_i \cup \{p_j\}_j \text{ marked pts.}), \qquad (1.2)$$

where C is a connected nodal curve and C^ν is smooth, but possibly disconnected. Encode the right-hand side of (1.2) by means of a combinatorial object.

Definition 1.2.21 ([70])**.** Let $(C, p_1, ..., p_n)$ be a marked curve and C^ν its normalization. We construct the dual graph Γ_C in the following way:

- For each irreducible=connected component C_v^ν (alternatively for each irreducible component C_v of C) draw a vertex v (*dual to* C_v^ν) with weight $g(v)$ defined as the genus of C_v^ν.
- For each marked point p_i and node q with preimages $\{q', q''\}$ attach a half edge h, respectively a pair of half edges h', h'', to the vertex dual to the component that contains p_i, q' and q'', respectively.
- Connect two half edges h' and h'' to an edge, if they are dual to a pair of preimages $\{q', q''\}$ as above.

Stratification. $\overline{M}_{g,n}$ has a stratification according to dual graphs ([70], Proposition 4.14), i.e. it is the disjoint union $\bigcup_\Gamma M^\Gamma$ over all isomorphism classes Γ of stable graphs of genus g with n ends (Section 4.2.1), where $M^\Gamma \subset \overline{M}_{g,n}$ is the set of marked curves with dual graph Γ.

Recursive structure of the moduli problem. The equivalence of data (1.2) suggests: Lower dimensional moduli problems $\overline{\mathcal{M}}_{g',n'}$ (where $n' \leqslant n$ and $g' \leqslant g$) are building blocks of $\overline{\mathcal{M}}_{g,n}$ and stable graphs (i.e. dual graphs of stable curves) blueprints for their assembly. Here is a formalization (see [70], Proposition 4.15 and Proof): Let Γ be the dual graph of an n-marked stable curve of genus g and consider the moduli functor $\overline{\mathcal{M}}_\Gamma$ collecting building blocks according to Γ. On objects we have

$$S \mapsto \overline{\mathcal{M}}_\Gamma(S) := \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)},$$

where $n(v)$ is the number of half-edges adjacent to v. These building blocks are assembled to an n-marked curve of genus g according to a natural transformation $\hat{\zeta}_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$ whose component ζ_Γ^S "glues" a collection of families

$$(C_v \xrightarrow{\pi_v} S, (q_h : S \xrightarrow{q_h} C_v)_{h \in H(v)})_{v \in V(\Gamma)} \text{ with } H(v) := \{h : h \text{ half-edge adjacent to } v\}$$

along pairs of sections $\{q_{h'}, q_{h''}\}$ that correspond to edges of Γ (see [4] (Chapter X, Section 7) for a formal description of the gluing process). Since $\prod_{v \in V(\Gamma)} \overline{M}_{g(v),n(v)}$ is a coarse moduli space for $\overline{\mathcal{M}}_\Gamma$, we can complete the diagram (Definition 1.2.7)

$$\begin{tikzcd}
\overline{\mathcal{M}}_\Gamma \arrow[r] \arrow[d, "\hat{\zeta}_\Gamma"'] & \mathrm{Hom}(\cdot, \prod_{v \in V(\Gamma)} \overline{M}_{g(v),n(v)}) \arrow[d, dashed, "\exists!"] \\
\overline{\mathcal{M}}_{g,n} \arrow[r] & \mathrm{Hom}(\cdot, \overline{M}_{g,n})
\end{tikzcd}$$

and obtain a unique natural transformation whose restriction to $\mathbb{C}$-points gives rise to a finite morphism

$$\zeta_\Gamma : \overline{M}_\Gamma \to \overline{M}_{g,n}$$

with $\operatorname{im}(\zeta_\Gamma) = \overline{M}^\Gamma$, the closure of M^Γ in $\overline{M}_{g,n}$ (see [70], Proposition 4.15). The *gluing morphism* ζ_Γ identifies the set M^Γ ($\overline{M}^\Gamma$) as quotient of the product $\prod_{v\in V(\Gamma)} M_{g(v),n(v)}$ ($\prod_{v\in V(\Gamma)} \overline{M}_{g(v),n(v)}$) by $\operatorname{Aut}(\Gamma)$ and reveals much about the structure of $\overline{M}_{g,n}$ (see [70], Proposition 4.15 and Exercise 4.18):

- M^Γ are not just subsets. They are nonempty, irreducible, locally closed in $\overline{M}_{g,n}$ with
$$\dim(M^\Gamma) = \dim(\overline{M}^\Gamma) = \dim(\prod_{v\in V(\Gamma)} \overline{M}_{g(v),n(v)}) = \dim(\overline{M}_{g,n}) - \#E(\Gamma).$$
- The interior of $\overline{M}_{g,n}$ is equal to M^Γ, where Γ is a vertex of genus g with n ends.
- The boundary $\partial\overline{M}_{g,n} := \overline{M}_{g,n}\backslash M_{g,n}$ consists of subsets $\overline{M}^\Gamma$, where $\#E(\Gamma) = 1$, and $\overline{M}^\Gamma = \bigcup_{\Gamma'} M^{\Gamma'}$, where the union is over all Γ' such that Γ' is a specialization of Γ'.

In summary, we can see that the structure of the boundary follows combinatorial rules.

Moduli space of covers

Just as we have a parameter space for objects in the category of algebraic curves, there are also parameter spaces for its morphism:

- The moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(D,d)$: This space parametrizes degree d stable maps from genus g, n-pointed curves to a fixed target curve D ([66]).
- The moduli space $\overline{\mathcal{H}}_{g\to h,d}(\mu_1,\ldots,\mu_n)$, which is a compactification of the Hurwitz space: This space parametrizes admissible covers of degree d of a genus g curve with specified ramification behavior $(\mu_1,\ldots,\mu_n)$ over marked points $p_1,\ldots,p_r$, simple ramification over points $q_1,\ldots,q_s$ and no further ramification ([26]).

The first serves as setting for defining GW-invariants, while the second gives rise to Hurwitz numbers (as the degree of a map $\operatorname{br} : \overline{\mathcal{H}}_{g\to h,d}(\mu_1,\ldots,\mu_n) \to \overline{\mathcal{M}}_{g,r+s}$ called the branch map, which sends a cover to its base curve, marked at its branch points).

1.3 Towards tropical geometry

Tropical geometry is developed in formal analogy with classical algebraic geometry: Fundamental objects in algebraic geometry will have a "tropical counterpart" that plays an equivalent role in the tropical theory. These counterparts are either imported from algebraic geometry (via "tropicalization") or defined intrinsically according to the same motivating principle as their algebraic analogues. This parallel development establishes a close connection between the two theories.

We refrain from establishing the concepts we introduced in the algebraic realm here in the tropical realm and rather introduce them as needed in the two forthcoming parts of this dissertation. Here, we list the concepts in the form of a dictionary and provide forward references to their precise introductions:

1. Consider classical objects contained in the category of algebraic varieties. The ambient category for corresponding tropical objects is the category of rational polyhedral spaces. The correspondence for fundamental objects is as follows:

- variety ↦ tropical variety: A balanced polyhedral fan, roughly a collection of cones satisfying specific gluing conditions).
- in particular: stable marked curve ↦ stable tropical curve: The dual graph (Definition 1.2.21) equipped with a metric (introduced in Sections 2.4 and 4.2.1).
- branched cover ↦ tropical Hurwitz cover (introduced in Section 4.2.2).

Beyond formal analogy: Algebraic degeneration (abstract tropicalization) provides a structured way to map objects from the algebraic category into the polyhedral one.

2. The category of stacks has a tropical parallel in the category of stacky fans (roughly, collections of quotients of rational polyhedral cones satisfying specific gluing condition). We saw in Section 1.2.1 that $\overline{M}_{g,n}$ has a stratification according to dual graphs. Tropical geometry establishes the combinatorial part as a moduli space in its own right.

The moduli space $M^{\text{tr}}_{g,n}$, introduced in Section 3.4.3, should solve the moduli problem of parametrizing tropical curves of genus g with n marked points. In genus 0, $M^{\text{tr}}_{0,n}$ can be constructed as a tropical variety: It is a balanced polyhedral fan ([29]). For higher genus, graphs may have non-trivial automorphism groups. As in Section 1.2 this is an obstruction to being a tropical variety. A construction may now be performed within the more general category of stacky fans.

Beyond formal analogy: The combinatorial types of genus g tropical curves are precisely the dual graphs of stable curves in $\overline{\mathcal{M}}_{g,n}$, and $M^{\text{tr}}_{g,n}$ can be canonically identified with the boundary complex of the Deligne-Mumford compactification $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ ([30], Theorem 6.3.).

More recent work extends the category-theoretic framework from Section 1.2 to tropical moduli spaces: [21] introduce a *tropical* moduli functor for the naive moduli problem of classifying curves. This is a contravariant functor

$$\mathcal{M}^{\text{tr}}_{g,n} : \textbf{RPC} \to \textbf{Groupoid}$$

from the category **RPC** of rational polyhedral cones (rather than the category of schemes over $\mathbb{C}$). This setting provides a tropical analogue of Theorem 1.2.19:

Theorem 1.3.1 ([21], Theorems 1 and 2)**.** For $2g - 2 + n > 0$, the moduli functor $\mathcal{M}^{\text{tr}}_{g,n}$ is *representable by a cone stack*, i.e. by a geometric stack over the category of rational polyhedral cone complexes. The universal family is given by the tropical forgetful morphism $\mathcal{M}^{\text{tr}}_{g,n+1} \to \mathcal{M}^{\text{tr}}_{g,n}$.

Tropical curves will be discussed in detail in Sections 2.4 and 4.2, tropical moduli spaces in Part 3 and the process of tropicalization is carried out in more detail in [31] (for spin curves) and in Section 8.1 (for the completed cycles invariants).

Part I

Tropical split Jacobians of curves of genus 2

Index of Notation

$\mathbb{T}\mathcal{T}$ The category of real tori with integral structure. 22, 29

$\mathbb{T}\mathcal{A}$ The category of tropical abelian varieties. 22, 40, 82

$\mathbb{T}\mathcal{C}$ The category of tropical curves. 22, 89

Ab The category of abelian groups. 22, 35, 56, 98

Σ Depending on the context: A real torus with integral structure or pptav. 22, 29, 82

$\check{\Sigma}$ The dual of Sigma. 22, 29

ζ Polarization on Σ. 22, 38, 82

f_ζ Isogeny induced by ζ. 22, 38, 84

$f : \Sigma_1 \to \Sigma_2$ A morphism in $\mathbb{T}\mathcal{T}$. 22, 30

$\check{f} : \check{\Sigma}_2 \to \check{\Sigma}_1$ The dual of f. 22, 30, 83

$Ker(f)_0$ Kernel of f in $\mathbb{T}\mathcal{T}$ or $\mathbb{T}\mathcal{A}$. 22, 30

$\ker(f)$ Kernel of f in Ab. 22, 37

ζ_Γ Natural principal polarization on $\text{Jac}(\Gamma)$. 22, 46, 48, 71, 91

Φ_{P_0} The tropical Abel-Jacobi map with reference point P_0. 22, 46, 91

$\mathbb{T}E$ Tropical elliptic curve. 22, 49

Γ Tropical curve. 22, 78

J^{pp} The pptav from Construction 3.5.8. 22, 102, 105, 117

ζ^{pp} The principle polarization on J^{pp} from Construction 3.5.8. 22, 102, 107

Q^{pp} The quadractic form associated to J^{pp} with pp ζ^{pp}. 22, 108, 113

Chapter 2

A categorical framework and characterization in $\mathbb{T}\mathcal{A}$ and $\mathbb{T}\mathcal{C}$

2.1 Introduction

We explore the following connection between the category of *tropical abelian varieties (tav)*, $\mathbb{T}\mathcal{A}$, and the the category of *tropical curves*, $\mathbb{T}\mathcal{C}$: Let Γ, $\tilde{\Gamma}$ be tropical curves, $\varphi : \Gamma \to \tilde{\Gamma}$ a cover. Denote by $\mathrm{Jac}(\Gamma)$ and $\mathrm{Jac}(\tilde{\Gamma})$ the respective Jacobians and consider the assignment $\mathcal{F}$ sending $\Gamma \in \mathbb{T}\mathcal{C}$ to $\mathrm{Jac}(\Gamma) \in \mathbb{T}\mathcal{A}$, and sending φ to the push-forward $\varphi_* : \mathrm{Jac}(\Gamma) \to \mathrm{Jac}(\tilde{\Gamma})$.

As $\mathcal{F}$ provides a way to communicate between $\mathbb{T}\mathcal{C}$ and $\mathbb{T}\mathcal{A}$, it becomes essential to appreciate the nature of this communication: Given a phenomenon P in $\mathbb{T}\mathcal{C}$ (an object that satisfies a certain property, a collection of interrelated objects,...), one may wonder at *how* it is expressed as phenomenon in $\mathbb{T}\mathcal{A}$, or vice versa. The present works embeds in this framework by studying the phenomenon of *tropical split Jacobians*.

Tropical split Jacobians in $\mathbb{T}\mathcal{A}$ and in $\mathbb{T}\mathcal{C}$. Jacobians of genus 2 curves are two-dimensional tav and as such more complicated than their one-dimensional cousins. Whenever Γ, however, is a covering of an elliptic curve, it so happens that $\mathrm{Jac}(\Gamma)$ splits into simpler objects, the direct sum of elliptic curves (Theorem 2.1.1). This decrease in complexity comes with a multitude of advantages, but suffers from two major problems

- Indeterminacy: Non-uniqueness of the splitting.

- Indeterminability: Non-computability of the splitting.

Similar to algebraic geometry, optimal coverings (see Definition 2.4.3) offer a remedy for both: They resolve indeterminacy as they provide us with a *canonical* choice. They resolve indeterminability as they provide us with an algorithmic approach.

2.1.1 Context and Background

Attempts at unification of various constructions in the context of tropical curves and tavs (e.g. [62], [3]), using category theory as common language, have recently been made in [69] continuing the work of [57]. We follow up on this development and extend the theory in tropical geometry, see Sections 2.2, 2.3 and 2.5, to broaden its scope of action and application, and to provide a transparent and structured framework to its organic growth. There is a large body of work on tavs and tropical covers. Jacobians and Prym varieties are two classes of tavs that have been studied extensively, bringing forth connections between curves and tavs; connections that are fruitful at both levels, the level of objects and the level of moduli spaces (see e.g. [62], [6], [14], [29], [57], [58], [69]). Prym varieties provide a link to unramified covers (see [49]). Ramified tropical covers, on the other hand, have made a notable appearance in the context of various enumerative problems (see e.g. [22],[12]), as they open the way to a combinatorial approach.

The study of connections between ramified covers and tavs, initiated by [69], is new. Our work provides a new approach to study the interplay of ramified covers and tavs, here in the context of genus 2 curves covering elliptic curves and their Jacobians.

Inspiration from Algebraic Geometry

Once the tropical framework is established, we realize that classical questions like:

- When is the Jacobian of a curve C of genus 2 isogenous to a product of two elliptic curves, E and E'?
- Is such a decomposition unique and can it be made explicit?

that go back to Jacobi, Legendre and others are just as natural in the tropical world. Moreover, the kinship of both areas provides guidance for the tropical investigation; many constructions behave analogously. This lets us oftentimes import ideas from algebraic geometry. We give a short overview to make the parallels visible, for details we refer to ([53], [38]): The answer to the first question is positive if and only if there exists a finite cover $f : C \to E$. The answers to the remaining two questions are "in general it is not unique" and "making it explicit is not trivial". Whenever f is *optimal*, however, i.e. f does not factor through a non-trivial isogeny, the situation changes: There exists another elliptic curve E', an optimal covering $f' : C \to E'$ and an isogeny $\phi : \mathrm{Jac}(C) \to E \oplus E'$ whose kernel is isomorphic to the group of $\deg(f)$-torsion points of E. Explicit descriptions of f' and E' can already be found in the works of Jacobi and Legendre for degree $\leqslant 4$, Kuhn, Frey and Kani with a more modern approach obtain results for various cases of degree $\leqslant 11$ ([53],[38]).

We want to put forth tropical geometry as a setting that allows abstraction and concreteness to coexist: The classical treatment uses a powerful, rather abstract machinery that was developed over a long period of time. What makes a tropical investigation so interesting is that with tropical geometry we have a setting that makes both possible: We will work at the same level of abstraction, at the same time, make abstract concepts accessible by providing concrete algorithms to work with them.

2.1.2 Results

Our first result identifies a phenomenon in $\mathbb{T}\mathcal{C}$ that relates to split Jacobians as follows.

Theorem 2.1.1. (Theorem 2.6.2) Let Γ be a tropical curve of genus 2. Then $\mathrm{Jac}(\Gamma)$ splits if and only if Γ covers an elliptic curve.

This relation, however, is not a bijection in the sense that a cover fixes the splitting of $\mathrm{Jac}(\Gamma)$ and a splitting the cover. Our next results show that these failings can be remedied by introducing the concept of optimal covers (see Definition 2.4.3), which is motivated by a similar notion in algebraic geometry.

Theorem 2.1.2. (Theorem 2.6.4) If $\varphi : \Gamma \to \mathbb{T}E$ is an optimal cover of degree d, then $\mathbb{T}E' := \ker(\varphi_*)$ is a tropical elliptic curve and $\mathrm{Jac}(\Gamma)$ splits, i.e. there exists an isogeny $\phi : \mathbb{T}E' \bigoplus \mathbb{T}E \to \mathrm{Jac}(\Gamma)$ whose kernel is isomorphic to $\mathrm{Jac}_d(\mathbb{T}E)$, the group of d-torsion points of $\mathbb{T}E$.

Theorem 2.1.2 should be understood as follows: Optimal coverings allow us to fix a "canonical" representative for a splitting of $\mathrm{Jac}(\Gamma)$. The complement of $\mathbb{T}E$, $\mathbb{T}E'$, enjoys a similar relation with Γ:

Theorem 2.1.3. (Theorem 2.6.9) In the setting of Theorem 2.1.2: There exists another optimal cover $\varphi' : \Gamma \to \mathbb{T}E'$ of degree d such that φ' interacts "nicely" with φ.
The term "nicely" is made precise in Theorem 2.6.9. Informally, φ' gives rise to the same splitting ϕ whose kernel also satisfies $\ker(\phi) \cong \mathrm{Jac}_d(\mathbb{T}E')$.

Taken together, Theorem 2.1.2 and Theorem 2.1.3 describe the situation in $\mathbb{T}\mathcal{C}$ and $\mathbb{T}\mathcal{A}$, i.e. split Jacobians appear either as a pair of optimal covers in $\mathbb{T}\mathcal{C}$ or as an isogeny satisfying $\mathrm{Jac}_d(\mathbb{T}E) \cong \ker(\phi) \cong \mathrm{Jac}_d(\mathbb{T}E')$ in $\mathbb{T}\mathcal{A}$. Informal, but illustrative is Figure 2.1.

$In\ \mathbb{T}\mathcal{C}:$	$In\ \mathbb{T}\mathcal{A}:$
$\begin{tikzcd} & \Gamma \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ \mathbb{T}E & & \mathbb{T}E' \end{tikzcd}$	$\mathrm{Jac}(\Gamma) \simeq \mathbb{T}E' \oplus \mathbb{T}E$

Figure 2.1: Split Jacobians in $\mathbb{T}\mathcal{C}$ and in $\mathbb{T}\mathcal{A}$.

Breaking down the right-hand side of Figure 2.1 further, reveals the symmetry which the left-hand side already suggests. We have the diagram (see Diagram 2.23):

$$\begin{tikzcd}
\mathbb{T}E' \arrow[r, hook, "\iota_1"] \arrow[dd, "m_d"] \arrow[dr, "\varphi'_*"] & \mathbb{T}E' \bigoplus \mathbb{T}E \arrow[d, dashed, "\phi"] & \mathbb{T}E \arrow[l, hook', "\iota_1"'] \arrow[dd, "m_d"] \arrow[dl, "\varphi^*"'] \\
 & \operatorname{Jac}(\Gamma) \arrow[dl, "\varphi'_*"'] \arrow[d, dashed, "\tilde{\phi}"] \arrow[dr, "\varphi_*"] & \\
\mathbb{T}E' & \mathbb{T}E' \bigoplus \mathbb{T}E \arrow[l, "p_1"] \arrow[r, "p_2"] & \mathbb{T}E
\end{tikzcd} \tag{2.1}$$

where ι_i and p_i are the canonical injections, respectively projections, and m_d is the componentwise multiplication-by-d map.

We dedicate a separate analysis to φ_* and its dual φ^*: For covers of the form $\varphi : \Gamma \to \mathbb{T}E$, where Γ is of genus 2, see Subsection 2.5.5 and for covers without any restriction on the genus of Γ see Subsection 2.5.6. This analysis is detached from split Jacobians, but still focuses on covers of elliptic curves. One reason for this is that the target $\mathbb{T}E$ may be viewed as an object of $\mathbb{T}\mathcal{A}$ as well. This "double-identity" allows us to invoke the universal property of the Jacobian ([69], Proposition 4.14) and relate factorizations of φ to factorizations of φ_*.

An effort is made throughout to balance abstract and algorithmic techniques: Computational results concerning optimal covers, φ_*, and φ^* are discussed in Subsection 2.5.5 to 2.5.7. Concerning split Jacobians, Subsection 2.6.3 supports Theorems 2.1.2 and 2.1.3 with an algorithmic point of view.

The present work relies on technology developed in [52], [69], [57], [61] and many more for handling tavs efficiently, and which we develop further in Subsections 2.2.2, 2.2.3 and 2.3.2 to meet our needs.

2.1.3 Overview

Split Jacobians are studied in $\mathbb{T}\mathcal{A}$ and $\mathbb{T}\mathcal{C}$. This exposes them to two different theories. We treat these separately at first: See Sections 2.2 and 2.3 for the theory of tavs and Section 2.4 for the theory of curves. Section 2.5 brings them together and sets the stage for the remainder of Chapter 2.

We begin Section 2.2 with a reminder of real tori with integral structure. These form a category, denoted by $\mathbb{T}\mathcal{T}$, that $\mathbb{T}\mathcal{A}$ is built on. Still in $\mathbb{T}\mathcal{T}$, Subsections 2.2.2 and 2.2.3 extend the existing theory with further concepts relevant to split Jacobians. Tropical abelian varieties are discussed in Section 2.3, starting with preliminaries in Subsection 2.3.1. In 2.3.2 we transfer the notions developed in 2.2.2 and 2.2.3 to $\mathbb{T}\mathcal{A}$. Here, we also define exact sequences and study their behaviour under dualization. This concludes the part on tavs.

Section 2.4 is devoted to the category of tropical curves, recalling background on curves and covers in Subsection 2.4.1. We define optimal tropical covers in Subsection 2.4.2.

Section 2.5 establishes a connection between $\mathbb{T}\mathcal{C}$ and $\mathbb{T}\mathcal{A}$, one that we explore in 2.5.2 in the setting of curves of genus 2 covering curves of genus 1. Subsections 2.5.5 and 2.5.6 are a blend of computational and abstract results on φ_* and φ^*, merged together in Subsection 2.5.7 to obtain criteria for optimality.

Section 2.6, finally, is devoted to the phenomenon of tropical split Jacobians. Here we prove Theorem 2.1.1, 2.1.2 and 2.1.3, our main abstract results, which we accompany with examples and make algorithmically accessible in Subsection 2.6.3.

2.1.4 Future work

We investigate extensions to higher genus. Let Γ be a tropical curve of genus g. Then $\mathrm{Jac}(\Gamma)$ is a g-dimensional pptav.

- **Full splitting:** We say $\mathrm{Jac}(\Gamma)$ *fully splits*, if $\mathrm{Jac}(\Gamma)$ is isogenous to a g-fold product of elliptic curves.

  Question 2.1.4. How do fully split Jacobians appear in $\mathbb{T}\mathcal{C}$, respectively in $\mathbb{T}\mathcal{A}$?

  Investigating Question 2.1.4 would shed light on the behavior of the functor $\mathcal{F} : \mathbb{T}\mathcal{C} \to \mathbb{T}\mathcal{A}$, since we consider a wider class of phenomena. Building on the case of genus 2, we can still ask: Given a cover $\varphi : \Gamma \to \mathbb{T}E$ does $\mathrm{Jac}(\Gamma)$ split? Whether a single map is sufficient for such a strong condition is questionable: Let $\phi : \mathrm{Jac}(\Gamma) \to \mathbb{T}E_1 \oplus ... \oplus \mathbb{T}E_g$ be an isogeny. Then $\varphi_i := p_i \circ \phi \circ \Phi_{P_0}$, where Φ_{P_0} is the tropical Abel-Jacobi map with reference point P_0 and p_i is projection onto $\mathbb{T}E_i$, should give rise to a cover for $i = 1, ..., g$. An affirmative answer becomes less likely, as g increases. Whether all g maps are necessary, is questionable as well. Already in genus 2 we can get away with only one. A need of $g - 1$ covers is a suitable working hypothesis (perhaps with additional requirements concerning their interactions).

  Task 2.1.5. Develop criteria for $\mathrm{Jac}(\Gamma)$ to decompose into a product of elliptic curves.

  Just as for genus 2, we can perturb a splitting of $\mathrm{Jac}(\Gamma)$ by a g-fold product of isogenies of elliptic curves. To address non uniqueness we ask:

  Question 2.1.6. Suppose $\mathrm{Jac}(\Gamma)$ fully splits, can we single out a *canonical* representative for such a splitting? What does this mean for the covers φ_i's?

  Optimality seems to be a suitable notion in this context. The definition of an *optimal cover* carries over to higher genus without any further adjustment.

  Task 2.1.7. Characterize optimality in $\mathbb{T}\mathcal{A}$, using the kernel of the push-forward.

  Results from Task 2.1.7 are expected to clarify the relation between optimal covers and fully split Jacobians and hopefully offer a way to resolve non-uniqueness in higher genus. Further questions are: Suppose $\mathrm{Jac}(\Gamma)$ fully splits and φ_i (as defined above) is optimal for $i = 1, ..., g$. What can we say about their degree and their relation to each other?

- **Partial Splittings:** In addition to full splittings, several partial splittings involving Jacobians of curves of lower genus may be of interest. Determine necessary and sufficient conditions for

  1. Splitting off an elliptic curve, i.e. conditions for $\mathrm{Jac}(\Gamma)$ to be isogenous to $A \oplus \mathbb{T}E$, where A is a tav.
  2. Splitting into a product of lower dimensional Jacobians.
  3. Splitting off a (product of) lower dimensional Jacobians, i.e. conditions for $\mathrm{Jac}(\Gamma)$ to be isogenous to $A \oplus \mathrm{Jac}(\Gamma')$, where A is a tav and Γ' a curve of lower genus.

  We aim for a characterization in $\mathbb{T}\mathscr{A}$ (analyzing properties of the isogeny) and in $\mathbb{T}\mathscr{C}$ (analyzing the relation between curves involved in the splitting). Most approachable seems to be the first case, splitting off an elliptic curve. Assuming these conditions mimic the genus 2 case, that is they are related to the existence of covers between the respective curves, we could proceed as follows: Let $\varphi : \Gamma \to \mathbb{T}E'$ be a cover and consider

$$0 \to \ker(\varphi_*) \xrightarrow{i} \mathrm{Jac}(\Gamma) \to \text{coker}(i) \to 0$$

  in the category of groups. If $\ker(\varphi_*)$ is connected, $A := \ker(\varphi_*)$ and $\mathbb{TE} := \text{coker}(i)$ could be a candidate for a splitting of $\mathrm{Jac}(\Gamma)$ as in Point 1. A natural transition towards Point 2 is to investigate the case of genus 3, where A is itself the Jacobian of a curve of genus 2. This will yield first data on how more general splittings manifest in $\mathbb{T}\mathscr{C}$.

2.2 The category of real tori with integral structure

2.2.1 Preliminaries

We start with a description of a surrounding category $\mathbb{T}\mathscr{T}$, the category of *real tori with integral structure.*

Objects: More precisely, we start with a description of its objects (see [58], Section 2.3 and [69], Section 4): The data $\Sigma := (\Lambda, \Lambda', [\cdot,\cdot])$, where

- Λ and Λ' are finitely generated free abelian groups of the same rank,
- $[\cdot,\cdot] : \Lambda \times \Lambda' \to \mathbb{R}$ is a non-degenerate pairing,

contains all the information needed for building a *real torus with integral structure* and will therefore be identified as such. The torus is given by the quotient $\mathrm{Hom}(\Lambda, \mathbb{R})/\Lambda'$, where the pairing $[\cdot,\cdot]$ encodes the way in which Λ' is to be viewed as a lattice in $\mathrm{Hom}(\Lambda, \mathbb{R})$, namely via the embedding $\Lambda' \to \mathrm{Hom}(\Lambda, \mathbb{R}) : \lambda' \mapsto [\cdot, \lambda']$. Note that we also have an embedding $\Lambda \to \mathrm{Hom}(\Lambda', \mathbb{R}) : \lambda \mapsto [\lambda, \cdot]$ which turns Λ into a lattice in $\mathrm{Hom}(\Lambda', \mathbb{R})$. This allows for the construction of the *dual torus* $\check{\Sigma}$ which is realized as the quotient $\mathrm{Hom}(\Lambda', \mathbb{R})/\Lambda$

and identified by the data $\check{\Sigma} := (\Lambda', \Lambda, [\cdot,\cdot]^t)$, where $[\cdot,\cdot]^t$ denotes the transposed pairing. The *dimension* of Σ is the $\mathbb{R}$-vector space dimension of $\mathrm{Hom}(\Lambda, \mathbb{R})$ and equal to $\mathrm{rk}(\Lambda)$ (equivalently equal to $\mathrm{rk}(\Lambda')$).

Maps: Now that we have defined the objects of $\mathbb{T}\mathscr{T}$, we turn to structure-preserving maps. For $i = 1, 2$ let $\Sigma_i := (\Lambda_i, \Lambda_i', [\cdot,\cdot]_i)$ be real tori with integral structure. A pair of group homomorphisms $(f^\# : \Lambda_2 \to \Lambda_1, f_\# : \Lambda_1' \to \Lambda_2')$ is called a *morphism of integral tori* $f : \Sigma_1 \to \Sigma_2$ and denoted by $f := (f^\#, f_\#)$, if

$$[f^\#(\lambda_2), \lambda_1']_1 = [\lambda_2, f_\#(\lambda_1')]_2 \tag{2.2}$$

is satisfied for all $\lambda_1' \in \Lambda_1'$ and $\lambda_2 \in \Lambda_2$. From this data we construct a homomorphism of real tori as follows: Consider the $\mathbb{R}-$linear map $\mathrm{Hom}(f^\#) : \mathrm{Hom}(\Lambda_1, \mathbb{R}) \to \mathrm{Hom}(\Lambda_2, \mathbb{R})$ and note that the restriction of $\mathrm{Hom}(f^\#)$ to Λ_1' (viewed as lattice via the embedding given by $[\cdot,\cdot]_1$) is exactly $f_\#$. Hence, the compatibility condition (2.2) guarantees that $\mathrm{Hom}(f^\#)$ takes Λ_1' to Λ_2'. Passing to the quotients yields the requested morphism of tori.

Just as the data for one torus actually yields two tori, Σ and $\check{\Sigma}$, we obtain a second morphism from the data underlying f by considering the transposed pair, $(f_\#, f^\#) =: \check{f} : \check{\Sigma}_2 \to \check{\Sigma}_1$, called the *dual morphism* .

Since properties of the induced map of quotients are encoded as properties of the pair $(f^\#, f_\#)$ (see [69], Definition 4.8), we call f

- *surjective*, if $f^\#$ is injective.
- *finite*, if $f_\#$ is injective (equivalently if $[\Lambda_1 : f^\#(\Lambda_2)] < \infty$).
- *injective*, is f is finite and $f_\#(\Lambda_1')$ is saturated in Λ_2'.
- an *isogeny*, if f is surjective and finite.

The following categorical constructions have been introduced in e.g. [69] and [58].

Definition 2.2.1. To a morphism $f : \Sigma_1 \to \Sigma_2$ we associate the integral tori

1. $Ker(f)_0 := (\Lambda_1/\mathrm{Im}(f^\#)^{\mathrm{sat}}, \ker(f_\#), [\cdot,\cdot]_K)$, where $[\cdot,\cdot]_K$ is the pairing induced by $[\cdot,\cdot]_1$.
2. $\mathrm{Coker}(f) := (\ker(f^\#), \Lambda_2'/\mathrm{Im}(f_\#)^{\mathrm{sat}}, [\cdot,\cdot]_C)$, where $[\cdot,\cdot]_C$ is the pairing induced by $[\cdot,\cdot]_2$.
3. $\mathrm{Im}(f) := \mathrm{Ker}(q)_0$, where $q : \Sigma_2 \to \mathrm{Coker}(f)$ is the morphism induced by the natural maps on the lattices.

As the notation suggests, these are Kernels and Cokernels in the sense of category theory (see [69], Proposition 4.7).

Remark 2.2.2. Note that the torus built from the data $\mathrm{Ker}(f)_0$ is naturally identified with the connected component of the identity of the kernel of f (viewed as map on the quotients) (see [58], Section 2.3). The same applies to $\mathrm{Ker}(q)_0$ and the image of f, i.e. $\frac{\mathrm{Hom}(f^\#)(\mathrm{Hom}(\Lambda_1,\mathbb{R}))}{\mathrm{Hom}(f^\#)(\mathrm{Hom}(\Lambda_1,\mathbb{R}))\cap\Lambda_2'}$.

2.2.2 More categorical constructions

Here, we provide additional categorical constructions needed in Sections 2.5 and 2.6.

Lemma 2.2.3. Let Σ_2 be a real torus with integral structure and $\Sigma_1 \overset{i}{\hookrightarrow} \Sigma_2$ a subtorus, i.e. an injective morphism. The quotient group Σ_2/Σ_1 has an integral structure given by $(\ker(i^\#), \Lambda_2'/\operatorname{Im}(i_\#), [\cdot,\cdot]_Q)$, where $[\cdot,\cdot]_Q : \ker(i^\#) \times \Lambda_2'/\operatorname{Im}(i_\#) \to \mathbb{R}$ is the pairing induced by $[\cdot,\cdot]_2$. It is a torus of dimension $\operatorname{rk}(\Lambda_2) - \operatorname{rk}(\Lambda_1)$.

Remark 2.2.4. The integral structure of the quotient group Σ_2/Σ_1 presented here emerges from the proof of a later Lemma (Lemma 2.3.9), which adapts methods from the complex setting to the tropical one. Under the premise that $\mathbb{T}\mathscr{A}$ behaves like its algebraic counterpart, $(\ker(i^\#), \Lambda_2'/\operatorname{Im}(i_\#), [\cdot,\cdot]_Q)$ is therefore the lattice representation of Σ_2/Σ_1 one would expect. This integral structure can also be seen to agree with $\operatorname{Coker}(i)$, where $i : \Sigma_1 \hookrightarrow \Sigma_2$ is the inclusion, the definition expected from a tropical categorical point of view. This illustrates the proximity between the tropical and classical behaviour.

Proof. Note that $\Sigma^Q := (\ker(i^\#), \Lambda_2'/\operatorname{Im}(i_\#), [\cdot,\cdot]_Q)$ is an integral torus:

- $\ker(i^\#)$ and $\Lambda_2'/\operatorname{Im}(i_\#)$ are lattices: i being injective implies that $\operatorname{Im}(i_\#)$ is saturated. Hence, $\Lambda_2'/\operatorname{Im}(i_\#)$ is one as well as a direct summand of the lattice Λ_2.
- $\operatorname{rk}(\Lambda_2'/\operatorname{Im}(i_\#)) = \operatorname{rk}(\Lambda_2) - \operatorname{rk}(\Lambda_1) = \operatorname{rk}(\ker(i^\#))$.
- $[\cdot,\cdot]_Q$ is well-defined and non-degenerate since $[\cdot,\cdot]_2$ is.

We show that Σ^Q can be identified with the group theoretic quotient, i.e. that $\Sigma_2/\Sigma_1 \cong \Sigma^Q$ as groups: Let $V_2 := \operatorname{Hom}(\Lambda_2, \mathbb{R})/\Lambda_2'$ and $V_1 := \operatorname{Hom}(\Lambda_1, \mathbb{R})/\Lambda_1'$. We have

$$\Sigma_2/\Sigma_1 = V_2/i(V_1) \overset{\Phi_1}{\cong} \frac{V_2/\operatorname{Hom}(i^\#)(V_1)}{\Lambda_2'/\operatorname{Im}(i_\#)},$$

where Φ_1 is constructed from $V_2 \to \frac{V_2/\operatorname{Hom}(i^\#)(V_1)}{\Lambda_2'/\operatorname{Im}(i_\#)}, f \mapsto \bar{f}$ by applying two times the universal property of the quotient group. It is easy to see that Φ_1 is bijective. The morphism

$$V_2 \to \operatorname{Hom}(\ker(i^\#), \mathbb{R}), f \mapsto f_{|\ker(i^\#)}$$

satisfies $f \mapsto 0$ for $f \in \operatorname{Hom}(i^\#)(V_1)$ and hence, gives rise to a unique morphism

$$\Phi_2 : V_2/\operatorname{Hom}(i^\#)(V_1) \to \operatorname{Hom}(\ker(i^\#), \mathbb{R}).$$

Note that Φ_2 is $\mathbb{R}$-linear and injective: Suppose $f_{|\ker(i^\#)} = 0$. Since $i^\#$ is surjective we can complete

$$\begin{tikzcd}
\Lambda_1 \arrow[dr, dashed] & \Lambda_2 \arrow[l, "i^\#"'] \arrow[d, "f"] \\
 & \mathbb{R}
\end{tikzcd} \tag{2.3}$$

to a commutative diagram, hence $f \in \operatorname{Hom}(\ker(i^\#), \mathbb{R})$. For surjectivity, we take advantage of the additional structure of Φ_2 and conclude the proof with a dimension argument:

$$\begin{aligned}
&\dim_{\mathbb{R}}(\operatorname{Hom}(\ker(i^\#), \mathbb{R})) = \operatorname{rk}(\ker(i^\#)) = \operatorname{rk}(\Lambda_2) - \operatorname{rk}(\Lambda_1)\\
&\dim_{\mathbb{R}}(V_2/\operatorname{Hom}(i^\#)(V_1)) = \dim_{\mathbb{R}}(V_2) - \dim_{\mathbb{R}}(\operatorname{Hom}(i^\#)(V_1)) = \operatorname{rk}(\Lambda_2) - \operatorname{rk}(\Lambda_1).
\end{aligned}$$

☐

The quotient $\Sigma_2 \xrightarrow{\pi} \Sigma_2/\Sigma_1$, where $\pi = (\pi^\#, \pi_\#)$ is given by the inclusion, $\pi^\#$, and the quotient map, $\pi_\#$, satisfies a universal property that can be deduced from universal properties of the lattices it is built from.

Lemma 2.2.5. Let $f : \Sigma_2 \to \Sigma_3$ be a homomorphism of integral tori such that f vanishes on a subtorus $\Sigma_1 \xhookrightarrow{i} \Sigma_2$. Then there exists a unique homomorphism $g : \Sigma_2/\Sigma_1 \to \Sigma_3$ such that

$$\begin{tikzcd}
\Sigma_2 \arrow[r, "f"] \arrow[d, "\pi"] & \Sigma_3 \\
\Sigma_2/\Sigma_1 \arrow[ur, "g"'] &
\end{tikzcd} \tag{2.4}$$

commutes.

Proof. We can express $f \circ i = 0$ in terms of lattices as $i^\# \circ f^\# = 0$ and $f_\# \circ i_\# = 0$. By the universal property of the kernel $\ker(i^\#) \xhookrightarrow{\pi^\#} \Lambda_2$ and universal property of the quotient $\Lambda_2' \xrightarrow{\pi_\#} \Lambda_2'/\operatorname{Im}(i_\#)$ there exists unique group homomorphisms $g^\#$ and $g_\#$ such that

$$\begin{tikzcd}
\Lambda_2 & & \Lambda_3 \arrow[ll, "f^\#"'] \arrow[dll, "g^\#"] & & \Lambda_2' \arrow[rr, "f_\#"] \arrow[d, "\pi_\#"'] & & \Lambda_3' \\
\ker(i^\#) \arrow[u, "\pi^\#"] & & & & \Lambda_2'/\operatorname{Im}(i_\#) \arrow[urr, "g_\#"'] & &
\end{tikzcd} \tag{2.5}$$

commute. Then $g := (g^\#, g_\#)$ is a homomorphism of tori

$$[\lambda_3, g_\#(\bar{\lambda_2'})]_3 = [\lambda_3, f_\#(\lambda_2')]_3 = [f^\#(\lambda_3), \lambda_2']_2 = [g^\#(\lambda_3), \bar{\lambda_2'}]_Q$$

that satisfies $f = g \circ \pi$. ☐

Since free abelian groups are the building blocks of integral tori, it is not surprising that the notion of a product and a coproduct exists and strongly relies on the corresponding notion in the category of abelian groups.

Definition 2.2.6. Given integral tori Σ_1 and Σ_2, we define their *product* $\Sigma_1 \otimes \Sigma_2$ as follows:

- The underlying discrete data is $(\Lambda_1 \times \Lambda_2, \Lambda_1' \times \Lambda_2', [\cdot,\cdot]_1 + [\cdot,\cdot]_2)$, where $\Lambda_1 \times \Lambda_2$, respectively $\Lambda_1' \times \Lambda_2'$, denotes the direct product of groups and $[\cdot,\cdot]_1 + [\cdot,\cdot]_2$ is given by $((\lambda_1, \lambda_2), (\lambda_1', \lambda_2')) \mapsto [\lambda_1, \lambda_1']_1 + [\lambda_2, \lambda_2']_2$.
- The object $\Sigma_1 \otimes \Sigma_2$ is equipped with a pair of morphisms, $\pi_1 : \Sigma_1 \otimes \Sigma_2 \to \Sigma_1$ and $\pi_2 : \Sigma_1 \otimes \Sigma_2 \to \Sigma_2$, induced by the natural projection and inclusion maps between the lattices.

The resulting object is an integral torus, that is:

- The pairing $[\cdot,\cdot]_1 + [\cdot,\cdot]_2$ is non-degenerate.
- The projection maps π_1 and π_2 are morphism of tori.

The *coproduct* of Σ_1 and Σ_2 is defined analogously via the (co)product of the underlying lattices and will be denoted by $\Sigma_1 \oplus \Sigma_2$.

Remark 2.2.7. Although coproduct and product coincide (for finitely many factors), just as it is the case in the category of abelian groups (**Ab**), we maintain different notations to indicate which of the two universal property is being used.

Proof. The objects, $\Sigma_1 \otimes \Sigma_2$ and $\Sigma_1 \oplus \Sigma_2$, are both integral tori. We still need to verify, though, whether they deserve to be called product, respectively coproduct, in the sense of category theory. For the purpose of illustration we only examine the case $\Sigma_1 \otimes \Sigma_2$, arguments for the coproduct are completely analogous:
Let Σ_3 be a torus with morphisms f_1 and f_2 into each factor of $\Sigma_1 \otimes \Sigma_2$. We can complete the diagram

$$\begin{tikzcd}
& \Sigma_3 \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
\Sigma_1 & \Sigma_1 \otimes \Sigma_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \Sigma_2
\end{tikzcd}$$

to a commutative diagram by connecting the torus Σ_3 to $\Sigma_1 \otimes \Sigma_2$ through

$$f : \Sigma_3 \to \Sigma_1 \otimes \Sigma_2$$

defined by the pair $(f^{\#}, f_{\#})$, where

- $f_{\#}$ is the *unique* group homomorphism that makes

$$\begin{tikzcd}
& \Lambda_3' \arrow[dl, "f_{1\#}"'] \arrow[d, "f_{\#}"] \arrow[dr, "f_{2\#}"] & \\
\Lambda_1' & \Lambda_1' \times \Lambda_2' \arrow[l] \arrow[r] & \Lambda_2'
\end{tikzcd}$$

commute. Existence and uniqueness of $f_\#$ are an immediate consequence of the universal property of the direct products of groups.

- $f^\#$ is the *unique* group homomorphism that makes

$$\begin{tikzcd}
 & \Lambda_3 & \\
\Lambda_1 \arrow[ur, "f_1^\#"] \arrow[r] & \Lambda_1 \times \Lambda_2 \arrow[u, "f^\#"] & \Lambda_2 \arrow[l] \arrow[ul, "f_2^\#"']
\end{tikzcd}$$

  commute. Existence and uniqueness follow from viewing $\Lambda_1 \times \Lambda_2$ as coproduct (in $\mathbf{Ab}$) and applying the respective universal property.

Since f is clearly unique, we only need to check whether it is a morphism to conclude the argument. But this follows directly from the compatibility of the pair $(f^\#, f_\#)$: We have

$$\begin{aligned}
([\cdot,\cdot]_1 + [\cdot,\cdot]_2)((\lambda_1,\lambda_2), f_\#(\lambda_3')) &= [\lambda_1, f_{1\#}(\lambda_3')]_1 + [\lambda_2, f_{2\#}(\lambda_3')]_2 \\
&= [f_1^\#(\lambda_1), \lambda_3']_3 + [f_2^\#(\lambda_2), \lambda_3']_3 \\
&= [f_1^\#(\lambda_1) + f_2^\#(\lambda_2), \lambda_3']_3 \\
&= [f^\#(\lambda_1,\lambda_2), \lambda_3']_3
\end{aligned}$$

for all $\lambda_3' \in \Lambda_3'$ and $(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2$. Hence, $\Sigma_1 \otimes \Sigma_2$ has the universal property of the product. ☐

We conclude by defining *equalizers* and *coequalizers* in $\mathbb{T}\mathcal{T}$: For morphisms $\Sigma_1 \overset{f}{\underset{g}{\rightrightarrows}} \Sigma_2$, we consider the following pairs (L, ϕ_L) and (C, ϕ_C), where

- $L := (\Lambda_L, \Lambda_L', [\cdot,\cdot]_L)$ and $C := (\Lambda_C, \Lambda_C', [\cdot,\cdot]_{CE})$ are integral tori with

$$\begin{aligned}
&\Lambda_L := \Lambda_1/\mathrm{Im}(f^\# - g^\#)^{sat}, \Lambda_L' := \{\lambda_1' \in \Lambda_1' \ : \ f_\#(\lambda_1') = g_\#(\lambda_1')\} \\
&\Lambda_C := \{\lambda_2 \in \Lambda_2 \ : \ f^\#(\lambda_2) = g^\#(\lambda_2)\}, \Lambda_C' := \Lambda_2'/\mathrm{Im}(f_\# - g_\#)^{sat}
\end{aligned}$$

  and with pairings $[\cdot,\cdot]_L$ and $[\cdot,\cdot]_{CE}$ induced by $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$.

- $\phi_L : L \to \Sigma_1$ and $\phi_C : \Sigma_2 \to C$ are the morphisms induced by the natural maps on the lattices.

A word on well-definedness, exemplary for $[\cdot,\cdot]_L$: Let $\lambda_1, \lambda_2 \in \Lambda_1$ with $\lambda_1 - \lambda_2 \in \mathrm{Im}(f^\# - g^\#)$ and $\lambda' \in \Lambda_L'$. Using that $f - g$ is a morphism of tori we have:

$$[\underbrace{\lambda_1 - \lambda_2}_{=f^\# - g^\#(\lambda_3)}, \lambda']_1 = [\lambda_3, f_\# - g_\#(\lambda')]_2 = 0.$$

Lemma 2.2.8. The category of integral tori $\mathbb{T}\mathscr{T}$ has equalizers and coequalizers. It is finitely complete and cocomplete.

Proof. By the existence theorem for limits (colimits), $\mathbb{T}\mathscr{T}$ is finitely complete (finitely cocomplete), if it has binary products (binary coproducts) and binary equalizers (binary coequalizers). By Definition 2.2.6, $\mathbb{T}\mathscr{T}$ has binary product and coproducts, the existence of equalizers and coequalizers remains to be shown: Let $\Sigma_1 \underset{g}{\overset{f}{\rightrightarrows}} \Sigma_2$ be a parallel pair. We claim that their equalizer is given by (L, ϕ_L) as constructed above, i.e. that

1. f and g become equal when pulled back to L: $f \circ \phi_L = g \circ \phi_L$.
2. L is universal with this property.

We work on the level of lattices. Note that $(\Lambda'_L, \phi_{L\#})$ is the equalizer of the pair $(f_\#, g_\#)$ in the category of abelian groups, **Ab**, and $(\Lambda_L, \phi_L^\#)$ is related to the coequalizer $(\tilde{\Lambda}_L, \tilde{\phi}_L)$ of $(f^\#, g^\#)$ in **Ab** as follows: Λ_L is the torsion-free quotient group of $\tilde{\Lambda}_L$ and $\phi_L^\#$ the composition of $\tilde{\phi}_L$ with the natural projection $p : \tilde{\Lambda}_L \to \Lambda_L$. This yields

$$f_\# \circ \phi_{L\#} = g_\# \circ \phi_{L\#} \text{ and } \phi_L^\# \circ f^\# = \phi_L^\# \circ g^\#$$

proving point (1) of the claim. For point (2), let $(\tilde{L}, \phi_{\tilde{L}})$ be another pair satisfying (1). We need to construct a unique arrow $\tilde{L} \xrightarrow{\gamma} L$ with $\phi_{\tilde{L}} = \phi_L \circ \gamma$. Since $(\Lambda'_L, \phi_{L\#})$ is the equalizer of $(f_\#, g_\#)$, there exists a unique group homomorphism $\gamma_\#$ such that the left diagram of Figure 2.2 commutes.

$$\begin{tikzcd}
\Lambda'_{\tilde{L}} \arrow[r, "\phi_{\tilde{L}\#}"] \arrow[d, dashed, "\gamma_\#"'] & \Lambda'_1 \arrow[r, bend left, "f_\#"] \arrow[r, bend right, "g_\#"'] & \Lambda'_2 \\
\Lambda'_L \arrow[ru, "\phi_{L\#}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda_{\tilde{L}} & \Lambda_1 \arrow[l, "\phi_{\tilde{L}}^\#"'] \arrow[d, "\tilde{\phi}_L"] & \Lambda_2 \arrow[l, bend right, "g^\#"'] \arrow[l, bend left, "f^\#"] \\
\Lambda_L \arrow[u, dashed, "\gamma^\#"] & \tilde{\Lambda}_L \arrow[l, "p"] \arrow[lu, dashed, "\tilde{\gamma}^\#"] &
\end{tikzcd}$$

Figure 2.2: Applying the universal property of the equalizer (on the left) and the universal property of the quotient and of the coequalizer (on the right) in the category of abelian groups.

Similarly, we complete the righthand side of Figure 2.2 to a commutative diagram: The universal property of the coequalizer gives rise to a unique map $\tilde{\gamma}^\#$ whose kernel $\ker(\tilde{\gamma}^\#)$ contains the torsion subgroup of $\tilde{\Lambda}_L$ as $\Lambda_{\tilde{L}}$ is torsion-free. Applying the universal property of the quotient Λ_L then yields $\gamma^\#$ as desired. Moreover, $\gamma := (\gamma^\#, \gamma_\#)$ is a morphism of tori since

$$[\bar{\lambda}, \underbrace{\gamma_\#(\lambda')}_{=\phi_{\tilde{L}\#}(\lambda')}]_L = [\lambda, \phi_{\tilde{L}\#}(\lambda')]_1 = [\underbrace{\phi_{\tilde{L}}^\#(\lambda)}_{=\gamma^\#(\phi_L^\#(\lambda))}, \lambda']_{\tilde{L}} = [\gamma^\#(\underbrace{\phi_L^\#(\lambda)}_{=\bar{\lambda}}), \lambda']_{\tilde{L}}$$

holds for all $\lambda \in \Lambda_1$ and $\lambda' \in \Lambda'_{\tilde{L}}$ concluding point (2).
The statements for coequalizers, analogous to points (1) and (2), are verified by the pair (C, ϕ_C). The proof is similar and will therefore be omitted. $\square$

2.2.3 Factorization

We use factorizations to gain insight into the structure of a morphism. To that end, we consider factorizations with special properties.

Definition 2.2.9. Let $f : \Sigma_1 \to \Sigma_2$ be a surjective morphism of integral tori. We call a map $g_1 : \Sigma_1 \to \Sigma_3$ *minimal* for f, if f factors through g_1 (i.e. $f = g_2 \circ g_1$) and for all other factorizations $\tilde{g}_2 \circ \tilde{g}_1$ of f, where $\tilde{g}_1 : \Sigma_1 \to \tilde{\Sigma}_3$ and $\dim_{\mathbb{R}}(\tilde{\Sigma}_3) \leqslant \dim_{\mathbb{R}}(\Sigma_3)$, there exists a unique map h_1 such that the diagram on the left

$$\begin{tikzcd}
\Sigma_1 \arrow[r, "\tilde{g}_1"] \arrow[rr, bend left, "f"] \arrow[dr, "g_1"'] & \tilde{\Sigma}_3 \arrow[r, "\tilde{g}_2"] & \Sigma_2 \\
& \Sigma_3 \arrow[u, dashed, "\exists ! h_1"] \arrow[ur, "g_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma_1 \arrow[r, "\tilde{g}_1"] \arrow[rr, bend left, "f"] \arrow[dr, "g_1"'] & \tilde{\Sigma}_3 \arrow[r, "\tilde{g}_2"] \arrow[d, dashed, "\exists ! h_2"] & \Sigma_2 \\
& \Sigma_3 \arrow[ur, "g_2"'] &
\end{tikzcd} \tag{2.6}$$

commutes. In complete analogy, we call g_1 *maximal* for f, if f factors through g_1 and for all other factorizations of f with $\dim_{\mathbb{R}}(\tilde{\Sigma}_3) \geqslant \dim_{\mathbb{R}}(\Sigma_3)$ there exists a unique map h_2 such that the diagram on the right commutes.

Let us now bring these theoretical constructs to life by considering the *tropical Stein factorization* (see [10] for an analytic analogue): Let $f : \Sigma_1 \to \Sigma_2$ be surjective and consider

$$\begin{tikzcd}
\Sigma_1 \arrow[r, "\pi_f"] \arrow[rr, bend right, "f"'] & \Sigma_1/\operatorname{Ker}(f)_0 \arrow[r, "\phi_f"] & \Sigma_2
\end{tikzcd} \tag{2.7}$$

Since $\Sigma_1/\operatorname{Ker}(f)_0$ is a torus of dimension $\dim_{\mathbb{R}}(\Sigma_2)$ (Lemma 2.2.3), the projection map π_f is a morphism with $\ker(\pi_f) = \operatorname{Ker}(f)_0$ and ϕ_f an isogeny.

Lemma 2.2.10 (Tropical Stein Factorization)**.** Every surjective morphism of tori $f : \Sigma_1 \to \Sigma_2$ factors canonically through an isogeny and a morphism whose group-theoretic kernel is connected as in (2.7), such that π_f is minimal for f (see Definition 2.2.9).

Explicitly, (2.7) satisfies the following universal property: For all factorizations $g' \circ \pi'$ of f such that g' is an isogeny there exists a unique isogeny ϕ' such that

$$\begin{tikzcd}
\Sigma_1 \arrow[r, "\pi'"] \arrow[rr, bend left, "f"] \arrow[dr, "\pi_f"'] & \Sigma_3 \arrow[r, "g'"] & \Sigma_2 \\
& \Sigma_1/\operatorname{Ker}(f)_0 \arrow[u, dashed, "\exists ! \phi'"] \arrow[ur, "\phi_f"'] &
\end{tikzcd} \tag{2.8}$$

commutes.

Proof. For such a factorization $g' \circ \pi'$ of f we have

$$\ker(f) = \ker(g' \circ \pi') = \ker(\pi') + (\pi')^{-1}(\ker(g') \backslash \{0\}),$$

where $\pi'^{-1}(\ker(g') \backslash \{0\})$ is a finite set since g' is an isogeny and $\ker(\pi')$ a (not necessarily connected) group. This implies $\mathrm{Ker}(f)_0 \subset \ker(\pi')$ as $\mathrm{Ker}(f)_0$ is the connected component containing the identity. By the universal property of the quotient (Lemma 2.2.5) there exists a unique morphism of integral tori ϕ' such that (2.8) commutes, i.e. π' factors through π_f and ϕ_f through g'. Moreover, ϕ' is surjective (since π' is) with finite kernel (since $\ker(\phi') \subset \ker(\phi_f)$ holds), hence an isogeny. ☐

Remark 2.2.11. In [69], Röhrle and Zakharov prove that an isogeny canonically factors into a free isogeny and a dilation. This factorization enjoys a universal property similar to the tropical Stein factorization (see [69], Lemma 4.9). Note that both factorizations are complementary: The tropical Stein factorization decomposes *surjective* morphisms, when specialized to isogenies, however, it is trivial.

As a first application of the tropical Stein factorization we rediscover a fact whose analogue for complex tori is well-known (see [35], Proposition 4.6):

Lemma 2.2.12. Let $f : \Sigma \to \Sigma'$ be a morphism of integral tori. The number of connected components of $\ker(f)$ and $\ker(\check{f})$ agree.

Proof. The tropical Stein factorizations of the surjective morphisms $\Sigma \xrightarrow{f} \mathrm{Im}(f)$ and $\check{\Sigma}' \xrightarrow{\check{f}} \mathrm{Im}(\check{f})$ give rise to factorizations of f and $\check{f}$:

$$\Sigma \xrightarrow{\pi_f} \Sigma/\mathrm{Ker}(f)_0 \xrightarrow{\phi_f} \mathrm{Im}(f) \hookrightarrow \Sigma'$$
$$\check{\Sigma}' \xrightarrow{\pi_{\check{f}}} \Sigma'/\mathrm{Ker}(\check{f})_0 \xrightarrow{\phi_{\check{f}}} \mathrm{Im}(\check{f}) \hookrightarrow \Sigma.$$

Note that the number of connected components of each kernel, $c(\ker(f))$ and $c(\ker(\check{f}))$, is given by $|\ker(\phi_f)|$ and $|\ker(\phi_{\check{f}})|$) (i.e. by the geometric degree of the respective isogeny, ϕ_f and $\phi_{\check{f}}$ (see [69])). In particular, it is finite. We can dualize the first sequence to obtain another factorization of $\check{f}$

$$\check{\Sigma}' \twoheadrightarrow \widecheck{\mathrm{Im}}(f) \xrightarrow{\check{\phi}_f} (\widecheck{\Sigma/\mathrm{Ker}(f)_0}) \to \check{\Sigma},$$

where the last map factors through $\mathrm{Im}(\check{f})$ (simply because it is a factorization of $\check{f}$)

$$\check{\Sigma}' \twoheadrightarrow \widecheck{\mathrm{Im}}(f) \xrightarrow{\check{\phi}_f} (\widecheck{\Sigma/\mathrm{Ker}(f)_0}) \xrightarrow{g} \mathrm{Im}(\check{f}) \hookrightarrow \check{\Sigma}.$$

Since g is finite as (the restriction of the) dual of the surjective map π_f and surjective (artificially made so by restricting the target), it is an isogeny. With g being an isogeny, the composition $g \circ \check{\phi}_f$ is one as well. So minimality of the tropical Stein factorization (Lemma 2.2.10) forces $\phi_{\check{f}}$ to factor through $g \circ \check{\phi}_f$ and therefore $\ker(\phi_{\check{f}}) \supset \ker(\check{\phi}_f)$ holds. Using $\ker(\check{\phi}_f) \cong \ker(\phi_f)$ for isogenies, we can conclude that $c(\ker(f)) \leqslant c(\ker(\check{f}))$. Finally, "$\geqslant$" follows from $\check{\check{f}} = f$. $\square$

2.3 The category of tropical abelian varieties

2.3.1 Preliminaries

By enhancing an integral torus with the data of a polarization, we create a new object:

Definition 2.3.1. A *tropical abelian variety (tav)* $\Sigma = (\Lambda, \Lambda', [\cdot, \cdot])$ is a real torus with integral structure together with a *polarization*, i.e. a group homomorphism $\zeta : \Lambda' \to \Lambda$ such that the bilinear form $[\zeta(\cdot), \cdot] : \Lambda' \times \Lambda' \to \mathbb{R}$ is symmetric and positive definite. If ζ is a bijection, we call ζ a *principal polarization* (pp) and Σ a *principally polarized tropical abelian variety (pptav)*. In any case, the *dimension* of Σ is the $\mathbb{R}$-vector space dimension of $\mathrm{Hom}(\Lambda, \mathbb{R})$ and equal to $\mathrm{rk}(\Lambda)$ (equivalently equal to $\mathrm{rk}(\Lambda')$).

A polarization ζ defines an isogeny $f_\zeta := (\zeta, \zeta)$ between Σ and its dual $\check{\Sigma}$, which, endowed with the so-called *dual polarization* $\check{\zeta}$ (previously defined by Röhrle and Zakharov, see first version of [69]), is a tav as well. We can characterize its kernel, $\ker(f_\zeta)$, using the *type* of ζ, which is given by the invariant factors $(\alpha_1, ..., \alpha_n)$ (where $n := \mathrm{rk}(\Lambda)$) of its Smith normal form. We have:

$$\ker(f_\zeta) \cong \mathbb{Z}/\alpha_1\mathbb{Z} \times ... \times \mathbb{Z}/\alpha_n\mathbb{Z}.$$

If $(\lambda'_i)_{i=1}^n$ and $(\lambda_i)_{i=1}^n$ are lattice basis of Λ' and Λ, respectively, such that $\zeta(\lambda'_i) = \alpha_i \lambda_i$, then $\check{\zeta}$ is defined by $\lambda_i \mapsto \frac{\alpha_1 \alpha_n}{\alpha_i} \lambda'_i$.

Morphism of tav are not required to satisfy any additional conditions, i.e. the properties of morphisms of real tori with integral structures are, in a sense, sufficient: For example, they are, as is generally the case, a source of new objects (see Definition 2.3.4). To this end, let us first look at how polarizations can be transported along morphisms.

Definition 2.3.2. Let Σ_2 be a tav with polarization ζ_2, Σ_1 a real torus with integral structure, and $\phi : \Sigma_1 \to \Sigma_2$ an isogeny. Then $\phi^*\zeta_2 := \phi^\# \circ \zeta_2 \circ \phi_\#$ is a polarization on Σ_1 and called the *induced polarization* or alternatively *the pull-back* of ζ_2 by ϕ. Conversely, suppose Σ_1 carries a polarization ζ_1. We can define the *push-forward* of ζ_1 by ϕ as $\phi_*\zeta_1 := \check{\zeta}$, where $\zeta := \check{\phi}^*\check{\zeta}_1$. We say that an isogeny $\phi : \Sigma_1 \to \Sigma_2$ is *polarized* with respect to polarizations ζ_1 on Σ_1 and ζ_2 on Σ_2, if ζ_1 is the polarization induced by ϕ and ζ_2, in other words, if the diagram

$$\begin{tikzcd}\Sigma_1 \arrow[r, "\phi"] \arrow[d, "f_{\zeta_1}"] & \Sigma_2 \arrow[d, "f_{\zeta_2}"] \\ \check{\Sigma}_1 & \check{\Sigma}_2 \arrow[l, "\check{\phi}"]\end{tikzcd}$$

commutes.

Note that pulling back polarizations always turns ϕ into a polarized isogeny. The same is not true for the push-forward.

Remark 2.3.3.

- The pull-back of a polarization has been introduced in [58] and [69]. Another notion of push-forward can be extracted from Proposition 4.11 in [69] by using the identification

$$\mathrm{Hom}(\Lambda_1', \Lambda_1) \cong \mathrm{Hom}(\Lambda_1, \mathbb{Z}) \otimes \Lambda_1' \big(\cong \mathrm{H}_{1,1}(\Sigma_1), \big)$$

  where the $\mathrm{H}_{1,1}(\Sigma_1)\big)$ denotes the tropical homology group defined in [69]. Explicitly, $\zeta_1 \longmapsto \sum_{i=1}^{n} \mathrm{Hom}(\zeta_1(\lambda_i'), \mathbb{Z}) \otimes \lambda_i'$, where $(\lambda_i')_i$ is a lattice basis of Λ_1', and using the natural notion of push-forward in this setting, which is given by $f_* := \mathrm{Hom}(f^{\#}, \mathbb{Z}) \otimes f_{\#}$.

- Isogenies are special in that sense that they allow for bidirectional transport of polarizations. However, a surjective morphism is enough for the push-forward and a finite one for the pull-back.

Definition 2.3.4. ([57], Proposition 1.1.2) To a morphism $f : \Sigma_1 \to \Sigma_2$ we associate the following tropical abelian varieties:

1. $\mathrm{Ker}(f)_0$ (Definition 2.2.1) with polarization $\zeta_K := i^*\zeta_1$, where $i : \mathrm{Ker}(f)_0 \hookrightarrow \Sigma_1$ is the inclusion.
2. $\mathrm{Coker}(f)$ (Definition 2.2.1) with polarization $\zeta_C := q_*\zeta_2$, where $q : \Sigma_2 \to \mathrm{Coker}(f)$ is the quotient map.

Similarly, we can transfer the notions of (co-)products and (co-)equalizers to $\mathbb{T}\mathscr{A}$: To a pair of morphism $\Sigma_1 \overset{f}{\underset{g}{\rightrightarrows}} \Sigma_2$, we associate the following tropical abelian varieties:

1. (L, ϕ_L) (Lemma 2.2.8) with polarization $\zeta_L := \phi_L^*\zeta_1$, the *equalizer* of the pair (f, g).
2. $\mathrm{Coker}(f)$ (Lemma 2.2.8) with polarization $\zeta_C := \phi_{C*}\zeta_2$, the *coequalizer* of the pair (f, g).

Definition 2.3.5. Given tavs Σ_1 and Σ_2, we define their *product* $\Sigma_1 \otimes \Sigma_2$ as follows:

- The underlying real torus with integral structure is given by the product $\Sigma_1 \otimes \Sigma_2$ in $\mathbb{T}\mathscr{T}$ (see Definition 2.2.6).
- The polarization $\zeta_1 \times \zeta_2$ is defined component-wise.

The resulting object is a tav, that is the group homomorphism $\zeta_1 \times \zeta_2$ satisfies the condition described in Definition 2.3.1. The *coproduct* of Σ_1 and Σ_2 is defined analogously and will be denoted by $\Sigma_1 \oplus \Sigma_2$.

Putting all these pieces together we see that the category of tavs, $\mathbb{T}\mathscr{A}$, has a certain amount of structure. It makes it the perfect setting for several categorical constructions.

Lemma 2.3.6. The category of polarized tropical abelian varieties, $\mathbb{T}\mathscr{A}$, is additive, i.e.

1. it contains a zero object given by the trivial object: $(\{0\}, \{0\}, [\cdot, \cdot])$.
2. it has an abelian group structure on the Hom-Sets, which is given by the usual addition.
3. for any two objects, the product and coproduct are defined as in Definition 2.3.5.

Moreover, for every morphism f

- cokernel and kernel exist (Definition 2.3.4).
- such that if $\mathrm{Ker}(f)_0 = 0$, then f is the kernel of its cokernel.
- such that if $\mathrm{Coker}(f)_0 = 0$, then f is the cokernel of its kernel.

Hence, $\mathbb{T}\mathscr{A}$ is abelian.

This has already been remarked in [57]. The existence of binary coequalizers and equalizers further shows: $\mathbb{T}\mathscr{A}$ is finitely complete, respectively finitely cocomplete.

2.3.2 Exactness and Dualization

Although $\mathbb{T}\mathscr{A}$ has a notion of kernels, these are not suitable for defining exact sequences. We use group-theoretic kernels instead and define:

Definition 2.3.7. We call a sequence

$$0 \to \Sigma_1 \xrightarrow{f} \Sigma_2 \xrightarrow{g} \Sigma_3 \to 0, \tag{2.1}$$

of tropical abelian varieties *exact*, if f is injective, g is surjective, and $\mathrm{Im}(f) = \ker(g)$ holds. Note that in particular $\ker(g)$ is connected and itself a tropical abelian variety.

Warning: In contrast to the category of abelian groups not every morphism of tavs gives rise to a short exact sequence.

Lemma 2.3.8. Let

$$0 \to \Sigma_1 \xrightarrow{f} \Sigma_2 \xrightarrow{g} \Sigma_3 \to 0.$$

be a short exact sequence of tropical abelian varieties. Then the dual sequence

$$0 \to \check{\Sigma}_3 \xrightarrow{\check{g}} \check{\Sigma}_2 \xrightarrow{\check{f}} \check{\Sigma}_1 \to 0$$

is exact. In other words

$$\check{\ } : \mathbb{T}\mathcal{A} \to \mathbb{T}\mathcal{A}$$

is an exact functor.

Proof. Suppose $\Sigma_i = (\Lambda_i, \Lambda_i', [,]_i)$ and denote by V_i the universal cover $\mathrm{Hom}(\Lambda_i, \mathbb{R})$ of Σ_i for $i = 1, 2, 3$. We construct the commutative diagram in Figure 2.3 as follows:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda_1' \arrow[r] \arrow[d] & \Lambda_2' \arrow[r] \arrow[d] & \Lambda_3' \arrow[d] \arrow[dddll, rounded corners, "\partial"] & \\
0 \arrow[r] & V_1 \arrow[r, "F"] \arrow[d, "\pi_1"] & V_2 \arrow[r, "G"] \arrow[d, "\pi_2"] & V_3 \arrow[r] \arrow[d, "\pi_3"] & 0 \\
0 \arrow[r] & \Sigma_1 \arrow[r, "f"] \arrow[d] & \Sigma_2 \arrow[r, "g"] \arrow[d] & \Sigma_3 \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & 0 \arrow[r] & 0 &
\end{tikzcd}$$

Figure 2.3: Applying the snake lemma in the proof of Lemma 2.3.8.

Lift the composition $V_1 \xrightarrow{\pi_1} \Sigma_1 \xrightarrow{f} \Sigma_2$ to a map F into the universal covering V_2 of Σ_2 such that the respective square commutes and $F(0) = 0$. In fact F is given by $\mathrm{Hom}(f^{\#})$ and $\mathbb{R}$-linear. Hence, $\ker(F)$ is a vector space that satisfies $\ker(F) \subset \Lambda_1$. It follows $\ker(F) = \{0\}$. Proceed analogously for g to fill in the second row with an exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$.

Since the category of abelian groups is abelian, we can use the snake Lemma to get an exact sequence of kernels and cokernels (see Figure 2.3):

$$0 \to \Lambda_1' \xrightarrow{f_{\#}} \Lambda_2' \xrightarrow{g_{\#}} \Lambda_3' \to 0.$$

Applying the exact functor $\mathrm{Hom}(\cdot,\mathbb{R})$ yields an exact sequence between the universal covers of the dual varieties

$$0 \to \mathrm{Hom}(\Lambda_3', \mathbb{R}) \xrightarrow{\mathrm{Hom}(g_\#)} \mathrm{Hom}(\Lambda_2', \mathbb{R}) \xrightarrow{\mathrm{Hom}(f_\#)} \mathrm{Hom}(\Lambda_1', \mathbb{R}) \to 0. \tag{2.2}$$

The maps factor through the quotient. By pushing (2.2) down we get

$$\check{\Sigma}_3 \xrightarrow{\check{g}} \check{\Sigma}_2 \xrightarrow{\check{f}} \check{\Sigma}_1 \to 0.$$

We still need to check whether $\check{g}$ is injective, i.e. $g^\#(\Lambda_3)$ is saturated in Λ_2: Apply $\mathrm{Hom}(\cdot,\mathbb{R})$ to the second row of 2.3 to obtain

$$0 \to \Lambda_3 \to \Lambda_2 \to \Lambda_1 \to 0.$$

Since the Λ_i are free abelian groups, the sequence splits and Λ_3 is a direct summand of Λ_2. This finishes the proof. ☐

Quotients of tavs. We can always extend an exact sequence $0 \to \Sigma' \to \Sigma$ to a short exact sequence $0 \to \Sigma' \to \Sigma \to \Sigma'' \to 0$ in the category of *abelian groups*. Whether we can do so in the category of *tropical abelian varieties* is a priori not clear. We address this question in the following Lemmas, adapting methods from the complex world to the tropical setting.

Lemma 2.3.9. Let Σ be an abelian variety and suppose $\Sigma' \xrightarrow{i} \Sigma$ is an abelian subvariety. Then there exists an abelian variety Σ'' and a morphism $\Sigma \twoheadrightarrow \Sigma''$ whose kernel is Σ'.

Proof. Recall that we have a relatively good handle on when $\Sigma_1 \to \Sigma_2 \to 0$ extends (in the category of tori!) to the left. This is precisely the case when the kernel is connected. Seizing the opportunity, we make our lives easier by working with the dual instead: The map $\check{i} : \check{\Sigma} \to \check{\Sigma}'$ is surjective and gives rise to a short exact sequence in the category of *groups*

$$0 \to \ker(\check{i}) \to \check{\Sigma} \to \check{\Sigma}' \to 0. \tag{2.3}$$

We claim that (2.3) lives in the category of *tropical abelian varieties*, i.e. that $\ker(\check{i})$ is connected. Denote by $c(\ker(i))$ the number of connected components of $\ker(i)$. As i is injective, $\ker(i)$ is trivial and $1 = c(\ker(i)) = c(\ker(\check{i}))$ by Lemma 2.2.12. All that remains to be done is the translation back to our original setting: Apply the dualization functor to obtain a short exact sequence of tavs (Lemma 2.3.8)

$$0 \to \Sigma' \to \Sigma \to \widecheck{\ker}(\check{i}) \to 0$$

and set $\Sigma'' := \widecheck{\ker}(\check{i})$. ☐

Remark 2.3.10. From the preceding proof we can extract a concrete description of the quotient in terms of lattices. We have $\Sigma'' = (\ker(i^{\#}), \Lambda_2'/\mathrm{Im}(i_{\#}), [\cdot,\cdot]_K^t)$, where i denotes the inclusion map, $\Sigma' = (\Lambda_1', \Lambda_2', [\cdot,\cdot]')$, and $[\cdot,\cdot]_K$ is the pairing induced by the pairing $[\cdot,\cdot]$ on Σ, and recognize the quotient of integral tori from Lemma 2.2.3, but now automatically equipped with a polarization.

There is another kind of quotient that creates an object in $\mathbb{T}\mathcal{A}$, the group-theoretic quotient of a tav by a finite group.

Lemma 2.3.11. Let Σ be a tav and suppose $G \subset \Sigma$ is a finite subgroup. Then there exists a tav Σ_G and a free isogeny $\Sigma \twoheadrightarrow \Sigma_G$ whose kernel is G.

Proof. Let $\mathrm{Hom}(\Lambda, \mathbb{R}) \xrightarrow{\pi} \Sigma$ be the universal covering of Σ. Then $\pi^{-1}(G)$ is a lattice containing Λ' such that $[\pi^{-1}(G) : \Lambda'] < \infty$. The embedding $\pi^{-1}(G) \overset{i}{\hookrightarrow} \mathrm{Hom}(\Lambda, \mathbb{R})$ induces a non-degenerate pairing

$$[\cdot,\cdot]_G : \Lambda \times \pi^{-1}(G) \to \mathbb{R}, (\lambda, \lambda') \to [\lambda, \lambda']_G := i(\lambda')(\lambda)$$

which identifies the quotient $\Sigma_G := \Sigma/G$ as real torus with integral structure built from $(\Lambda, \pi^{-1}(G), [\cdot,\cdot]_G)$. Upgrading Σ_G to a tav requires the data of a polarization. To that extend, note that the quotient map $q : \Sigma \to \Sigma_G$ is an isogeny so its dual $\check{q}$ is one, too. Hence, we can use $\check{q}$ to define a polarization on $\check{\Sigma}_G$ as the induced polarization $\check{\zeta}_G := \check{q}^*\check{\zeta}$ and set $\zeta_G := \check{\check{\zeta}}_G$, i.e. $\zeta_G = q_*(\zeta)$. □

Remark 2.3.12. Note that q is not necessarily polarized with respect to ζ and ζ_G since $q^*(\zeta_G)$ and ζ do not agree in general. This will be discussed further in Project 2 (Chapter 3).

Factorization of morphisms of abelian varieties. Since morphisms of tavs are not required to satisfy any additional properties, we can transfer definitions and results of Subsection 2.2.3 to $\mathbb{T}\mathcal{A}$. In particular, we have that the tropical Stein factorization of a surjective morphism of tavs

$$\begin{tikzcd} \Sigma_1 \arrow[r, "\pi_f"] \arrow[rr, bend right, "f"'] & \Sigma_1/\operatorname{Ker}(f)_0 \arrow[r, "\phi_f"] & \Sigma_2 \end{tikzcd} \tag{2.9}$$

is legitimate in $\mathbb{T}\mathcal{A}$ as $\Sigma_1/\operatorname{Ker}(f)_0$ is a tav by Lemma 2.3.9.

2.4 The category of tropical curves

2.4.1 Preliminaries

In this section we describe the *category of tropical curves*, $\mathbb{T}\mathcal{C}$. We base our exposition on [3] and [6], drawing on the foundational work of Mikhalkin in [61] and of Mikhalkin and

Zharkov in [62]. Objects of $\mathbb{T}\mathcal{C}$ are *tropical curves*. These are, in analogy to the complex setting, (connected) topological spaces homeomorphic to a locally finite 1-dimensional simplicial complex and carry a tropical structure ([62], Definition 3.1). An equivalent way to package the information of a tropical curve is by means of a metric graph ([62], Section 3.3). This is the data we want to work with.

Definition 2.4.1. A tropical curve Γ is the geometric realization of a metric graph (G, l), i.e. a finite graph G with no legs/ends (i.e. no 1-valent vertices) together with a function $l : E(G) \to \mathbb{R}_{>0}$: It is the topological quotient

$$\Gamma := \dot{\bigcup_{e \in E(G)}} [0, l(e)]/ \sim$$

endowed with the path metric and with equivalence relation $\sim$ coming from the incidence relations of G. Any (G', l') that is obtained from (G, l) by adding or deleting vertices of valence 2 (and adapting the length function accordingly) is called a *model* for Γ and G its *combinatorial type*. The *genus* of Γ is the genus of a model and is given by the number $|E(G)| - |V(G)| + 1$, since we do *not* allow genus at vertices.

For us, the structure preserving maps of $\mathbb{T}\mathcal{C}$ will be *harmonic morphisms* (see e.g. [7] or [16], Section 2.1): Let Γ and $\tilde{\Gamma}$ be tropical curves. A continuous and surjective map $\varphi : \Gamma \to \tilde{\Gamma}$ is called a harmonic morphism if there exist models (G, l) and $(\tilde{G}, \tilde{l})$ such that

- $\varphi(V(G)) \subset V(\tilde{G})$ and $\varphi^{-1}(E(\tilde{G})) \subset E(G)$.
- φ is locally integer affine linear: On each edge $e \in E(G)$, φ restricts to an affine function with integer slope $d_e(\varphi)$ (possibly 0), called the *weight* or *expansion factor* of φ at e.
- φ is harmonic/balanced at every $P \in \Gamma$: For any $\tilde{v} \in T_{\varphi(P)}\tilde{\Gamma}$

  $$d_P(\varphi) := \sum_{v \in T_P\Gamma, v \mapsto \tilde{v}} d_v(\varphi)$$

  is independent of $\tilde{v}$, where $T_P\Gamma$ ($T_{\varphi(P)}\tilde{\Gamma}$) is the set of tangent directions emanating from P ($\varphi(P)$) and $d_v(\varphi)$ is the directional derivative of φ in the direction of v (i.e. $d_v(\varphi) := d_e(\varphi)$ for the edge e in direction of v).

In analogy to the complex setting, we refer to harmonic morphisms as *tropical covers*. A tropical cover φ is *finite*, if $d_e(\varphi) > 0$ for all edges e, and *non-finite* else. Its *degree* is given by the number $deg(\varphi) := \sum_{P \in \Gamma, P \mapsto \tilde{P}} d_P(\varphi)$, where $\tilde{P} \in \tilde{\Gamma}$ is an arbitrary point. We will work with tropical curves through choice of a model and by abuse of notation identify Γ with (G, l).

Remark 2.4.2. The ramification index of a point $P \in \Gamma$ is given by

$$R_P(\varphi) := 2d_P(\varphi) - 2 - \sum_{v \in T_P\Gamma} (d_v(\varphi) - 1).$$

One may then distinguish between *unramified covers*, i.e. covers that satisfy $R_P(\varphi) = 0$ for all $P \in \Gamma$, and *ramified* ones. The map φ in Figure 2.4 (b), for example, is ramified, while ϕ is not. We mostly consider ramified covers.

2.4.2 Optimal tropical covers

As in algebraic geometry (see e.g. [53]), there is a distinguished class of so-called *optimal covers* among the morphisms of $\mathbb{T}\mathcal{C}$. We introduce them here.

Definition 2.4.3. Let Γ and $\mathbb{T}E$ be curves of genus 2 and 1. We call a harmonic map $\varphi : \Gamma \to \mathbb{T}E$ *optimal* if it does not factor through a non-trivial cover, i.e. if there exists a curve $\mathbb{T}\tilde{E}$ and maps $\tilde{\varphi} : \Gamma \to \mathbb{T}\tilde{E}$, $\phi : \mathbb{T}\tilde{E} \to \mathbb{T}E$ such that

$$\begin{tikzcd}
\Gamma \arrow[r, "\tilde{\varphi}"] \arrow[dr, "\varphi"'] & \mathbb{T}\tilde{E} \arrow[d, "\phi"] \\
& \mathbb{T}E
\end{tikzcd}$$

commutes, then ϕ is an isomorphism (i.e. $deg(\phi) = 1$).

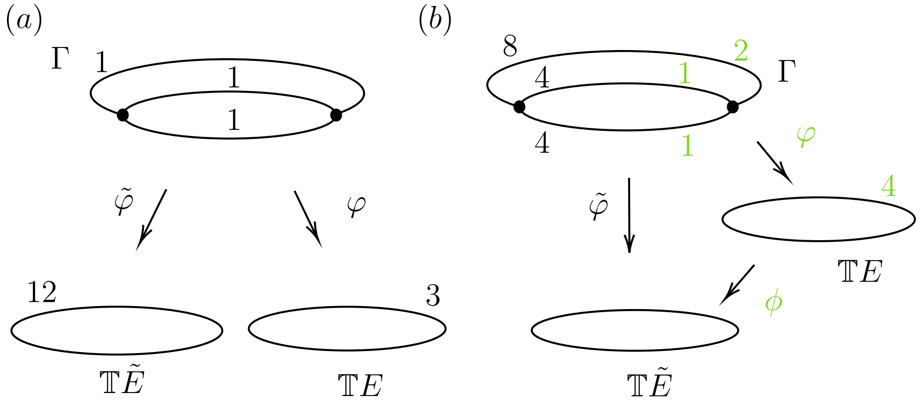


Figure 2.4: A curve of genus 2 covering two elliptic curves. The numbers in Figure 2.4 (a) are edge lengths and in Figure 2.4 (b) edge weights.

Example 2.4.4. Figure 2.4 (a) shows a cover of degree 2 on the right, which for degree reasons must be optimal. The cover on the left, however, is not as it factors, for example, through the first, giving rise to a cover of degree 4. This factorization is shown in Figure 2.4 (b), with numbers corresponding to edge weights, the green ones to those of φ and ϕ, and the blacks ones to those of $\tilde{\varphi}$.

2.5 Crossing Bridges

2.5.1 Preliminaries

We establish a functorial connection between $\mathbb{T}\mathcal{C}$ and $\mathbb{T}\mathcal{A}$. To an object Γ of $\mathbb{T}\mathcal{C}$ we assign an object of $\mathbb{T}\mathcal{A}$ through the following construction (see [6] or [58]):

Construction 2.5.1. Let (G, l) be an oriented model of Γ and $s, t : E(G) \to V(G)$ the source and target maps. Then (G, l) comes with two lattices that are related by a non-degenerate pairing:

- *The lattice of harmonic* 1*-forms*, $\Omega^1_G(\mathbb{Z})$: For each oriented edge e we introduce a formal symbol de called a *basic* 1-form on G and set $\Omega^1_G(\mathbb{Z})$ to be

  $$\{\omega := \sum_e \omega_e de : \omega_e \in \mathbb{Z}, \sum_{e:t(e)=V} \omega_e = \sum_{e:s(e)=V} \omega_e \, \forall \, V \in V(G)\}.$$

  It is the subgroup of the free group over $\{de : e \in E(G)\}$ consisting of harmonic $1-$forms on G.

- *The lattice of integral* 1*-cycles*, $\mathrm{H}_1(G, \mathbb{Z})$: It is the first simplicial homology group of G given by $\ker(\partial)$, where

  $$\partial : \mathrm{C}_1(G, \mathbb{Z}) \to \mathrm{C}_0(G, \mathbb{Z}), e \mapsto t(e) - s(e)$$

  is the boundary operator.

- *The integration pairing*, $\int_\cdot \cdot$: We can integrate a basic 1-form

  $$\int_e de' := \begin{cases} l(e), & \text{if } e = e' \\ 0, & \text{else} \end{cases}$$

  and extend linearly to obtain a perfect pairing

  $$\int_\cdot \cdot : \Omega^1_G(\mathbb{Z}) \times \mathrm{H}_1(G, \mathbb{Z}) \to \mathbb{R}, (\omega, c) \mapsto \int_c \omega.$$

These building blocks are independent of the choice of model (see also [6]). This means that lattices that arise from different models (that have compatible orientations) are related by isomorphisms, that leave the integration pairing invariant. We will write $\Omega^1_\Gamma(\mathbb{Z})$ and $\mathrm{H}_1(\Gamma, \mathbb{Z})$, instead, and complete Construction 2.5.1 by assigning a pptav to Γ.

Definition 2.5.2. The *Jacobian of* Γ is the pptav built from $(\Omega^1_\Gamma(\mathbb{Z}), \mathrm{H}_1(\Gamma, \mathbb{Z}), \int_\cdot \cdot)$ with principal polarization $\zeta_\Gamma : \mathrm{H}_1(\Gamma, \mathbb{Z}) \to \Omega^1_\Gamma(\mathbb{Z}), \sum a_e e \mapsto \sum a_e de$. It is related to Γ by the *tropical Abel-Jacobi map*:

$$\Phi_{P_0} : \Gamma \to \mathrm{Jac}(\Gamma), P \mapsto \int_{\gamma_P} \cdot,$$

where $P_0 \in V(\Gamma)$ is a fixed vertex and $\gamma_P \in \mathrm{C}_1(\Gamma, \mathbb{Z})$ is any path connecting P_0 to P in Γ.

Remark 2.5.3. Note that the construction of the tropical Jacobian works on exactly the same principle as the classical one over $\mathbb{C}$. In fact, the analogy goes one step further: We have an isomorphism $\mathrm{Jac}(\Gamma) \overset{\Phi_\Gamma}{\cong} \mathrm{Pic}^0(\Gamma)$ ([62], Theorem 6.2), where $\mathrm{Pic}^0(\Gamma) := \mathrm{Div}^0(\Gamma)/\,\mathrm{Prin}(\Gamma)$ is the group-theoretic quotient of $\mathrm{Div}^0(\Gamma)$, the group of divisors of degree 0, by $\mathrm{Prin}(\Gamma)$, the subgroup of principal divisors ([62], Section 4.2). This setting makes it easy to specify two homomorphsims that are induced by a cover $\varphi : \Gamma \to \tilde{\Gamma}$: The push-forward $\varphi_* : \mathrm{Pic}^0(\Gamma) \to \mathrm{Pic}^0(\tilde{\Gamma})$ and the pull-back $\varphi^* : \mathrm{Pic}^0(\tilde{\Gamma}) \to \mathrm{Pic}^0(\Gamma)$ of divisors defined by

$$\varphi_*\Big(\sum_i n_i P_i\Big) := \sum_i n_i \varphi(P_i) \text{ and } \varphi^*\Big(\sum_j \tilde{n}_j \tilde{P}_j\Big) := \sum_j \sum_{P_{ij} \mapsto \tilde{P}_j} \tilde{n}_j d_{P_{ij}}(\varphi) P_{ij}.$$

Now, Remark 2.5.3 suggests two natural candidates for playing the counterpart of a cover φ in $\mathbb{T}\mathscr{A}$. We start by describing these as morphisms of tavs and understanding their relationship: As the notation suggests, φ_* and φ^* are dual to each other.

Lemma 2.5.4. Let $\varphi : \Gamma \to \tilde{\Gamma}$ be a cover. We consider the morphism $\psi_* : \mathrm{Jac}(\Gamma) \to \mathrm{Jac}(\tilde{\Gamma})$ induced by the push-forward φ_* under the identification $\mathrm{Pic}^0(\Gamma) \to \mathrm{Jac}(\Gamma)$, respectively $\mathrm{Pic}^0(\tilde{\Gamma}) \to \mathrm{Jac}(\tilde{\Gamma})$, given by the tropical Abel-Jacobi map. Then the dual homomorphism $\check{\psi}_* : \widecheck{\mathrm{Jac}}(\tilde{\Gamma}) \to \widecheck{\mathrm{Jac}}(\Gamma)$ is induced by the pull-back on divisors $\varphi^* : \mathrm{Pic}^0(\tilde{\Gamma}) \to \mathrm{Pic}^0(\Gamma)$.

Lemma 2.5.4 is most certainly known, although a proof does not seem to appear in the literature. We include it here for the sake of completeness and as an illustration of Remark 2.5.3.

Proof. Let us fix reference points $Q_\Gamma \in \Gamma$ and $\varphi(Q_\Gamma) \in \tilde{\Gamma}$ and consider the map ψ_* that makes the following diagram commute:

$$\begin{tikzcd}
\mathrm{Pic}^0(\Gamma) \arrow[r, "\varphi_*"] \arrow[d, "\phi_\Gamma"] & \mathrm{Pic}^0(\tilde{\Gamma}) \arrow[d, "\phi_{\tilde{\Gamma}}"] \\
\mathrm{Jac}(\Gamma) \arrow[r, "\psi_*"] & \mathrm{Jac}(\tilde{\Gamma})
\end{tikzcd}.$$

In doing so, we may figure out the assignment rule of ψ_* by "diagram-chasing" elements: As the vertical arrows are the isomorphisms induced by the Abel-Jacobi map ([62], Theorem 6.2), i.e. defined by

$$\phi_\Gamma\Big(\sum_i n_i P_i\Big) = \sum_i n_i \int_{Q_\Gamma}^{P_i} (\cdot) \text{ and } \phi_{\tilde{\Gamma}}\Big(\sum_i n_i \tilde{P}_i\Big) = \sum_i n_i \int_{\varphi(Q_\Gamma)}^{\tilde{P}_i} (\cdot),$$

we get

$$\psi_* : \mathrm{Jac}(\Gamma) \to \mathrm{Jac}(\tilde{\Gamma}) : \sum_i n_i \int_{Q_\Gamma}^{P_i} (\cdot) \mapsto \sum_i n_i \int_{\varphi(Q_\Gamma)}^{\varphi(P_i)} (\cdot).$$

We claim that ψ_* is given by the pair $(\psi_*^\#, \psi_{*\#})$, where $\psi_*^\#$ is the pull-back of 1-forms and $\psi_{*\#}$ the push-forward of cycles, i.e. that for all $\sum_i n_i \int_{Q_\Gamma}^{P_i}(\cdot) \in \mathrm{Jac}(\Gamma)$

$$\mathrm{Hom}(\psi_*^\#, \mathbb{Z})\Big(\sum_i n_i \int_{Q_\Gamma}^{P_i}(\cdot)\Big) = \sum_i n_i \int_{Q_\Gamma}^{P_i} \psi_*^\#(\cdot) \text{ and } \sum_i n_i \int_{\varphi(Q_\Gamma)}^{\varphi(P_i)}(\cdot)$$

agree on $\Omega^1_{\tilde{\Gamma}}(\mathbb{Z})$. Inserting an arbitrary $\omega = \sum_k a_k d\tilde{e}_k \in \Omega^1_{\tilde{\Gamma}}(\mathbb{Z})$ and using linearity of the integral we see that we only need to verify the equality by summands, i.e. that for fixed tuple (k, i) we have

$$\int_{\varphi(Q_\Gamma)}^{\varphi(P)}(d\tilde{e}_k) = \sum_{e_j \mapsto \tilde{e}_k} \int_{Q_\Gamma}^{P} d_{e_j}(\varphi) de_j, \text{ where } P := P_i.$$

Choose a model of Γ ($\tilde{\Gamma}$) containing Q_Γ and P ($\varphi(Q_\Gamma)$ and $\varphi(P)$) as vertices and such that the set E ($\tilde{E}$) of edges, that cannot be subdivided further, satisfies the following conditions: No $e \in E$ ($\tilde{E}$) is a cycle, nor does $\varphi(e)$ contain any cycle. If $\gamma := \sum_j m_j e_j$ is a path (with $e_j \in E$) in Γ from Q_Γ to P, then $\tilde{\gamma} := \sum_j m_j \varphi(e_j)$ is a path connecting $\varphi(Q_\Gamma)$ to $\varphi(P)$ in $\tilde{\Gamma}$. Since integration is path-independent in the respective Jacobians, we use γ and $\tilde{\gamma}$ to compute the integrals above:

$$\int_{\varphi(Q_\Gamma)}^{\varphi(P)}(d\tilde{e}_k) = \Big(\sum_{e_j \mapsto \tilde{e}_k} m_j\Big) \cdot l(\tilde{e}_k) \text{ and } \sum_{e_j \mapsto \tilde{e}_k} \int_{Q_\Gamma}^{P} d_{e_j}(\varphi) de_j = \sum_{e_j \mapsto \tilde{e}_k} m_j \, d_{e_j}(\varphi) \, l(e_j).$$

Recalling how φ behaves with respect to edge lengths (i.e. $d_{e_j}(\varphi) l(e_j) = l(\tilde{e}_k)$ holds since by our choice of models we excluded winding) finally proves the claim.

The transposed pair $(\psi_{*\#}, \psi_*^\#,)$ is the dual of ψ_*. Under the natural isomorphisms $\Omega^1_\Gamma \overset{\zeta_\Gamma}{\cong} \mathrm{H}_1(\Gamma, \mathbb{Z})$ and $\Omega^1_{\tilde{\Gamma}} \cong \mathrm{H}_1(\tilde{\Gamma}, \mathbb{Z})$ we can interpret it as morphism between the Jacobians we denote by $\psi^* = (\psi^{*\#}, \psi^*_\#)$, where $\psi^{*\#}$ is the push-forward of 1-forms and $\psi^*_\#$ the pull-back of cycles. As above one shows that ψ^* is induced by the pull-back of divisors $\varphi^* : \mathrm{Pic}^0(\tilde{\Gamma}) \to \mathrm{Pic}^0(\Gamma)$, thereby completing the proof. $\square$

From now on the identification $\mathrm{Jac}(\Gamma) \cong \mathrm{Pic}^0(\Gamma)$ will be implicit and, justified by Lemma 2.5.4, we use the notation φ^* and φ_* for both settings.

2.5.2 Curves of genus 2 covering curves of genus 1.

We explore this connection in the setting of curves of genus 2 covering curves of genus 1: As Definition 2.4.3 suggests we will be interested in factorizing tropical coverings. A daunting task made easier by the following Lemma that allows us to transfer the factorization question from the category of tropical curves to the category of tropical abelian varieties.

Note that the tropical Abel-Jacobi map is an isomorphism (of rational polyhedral spaces) in the case of genus 1. It can be used to endow $\mathbb{T}E$ with the structure of a group. In analogy to the classical case we call $\mathbb{T}E$ an *elliptic curve* and depending on the context, regard $\mathbb{T}E$ as an object of $\mathbb{T}\mathcal{C}$ or, by abuse of notation, as an object of $\mathbb{T}\mathcal{A}$ (the identification of $\mathrm{Jac}(\mathbb{T}E)$ with $\mathbb{T}E$ will be used implicitly).

Lemma 2.5.5. Let $\varphi : \Gamma \to \mathbb{T}E$ and $\tilde{\varphi} : \Gamma \to \mathbb{T}\tilde{E}$ be harmonic maps and $\varphi_* : \mathrm{Jac}(\Gamma) \to \mathbb{T}E$ and $\tilde{\varphi}_* : \mathrm{Jac}(\Gamma) \to \mathbb{T}\tilde{E}$ the respective push-forwards. Then the following are equivalent:

- $\varphi = \phi \circ \tilde{\varphi}$ for an isogeny $\phi : \mathbb{T}\tilde{E} \to \mathbb{T}E$.
- $\varphi_* = \hat{\phi} \circ \tilde{\varphi}_*$ for an isogeny $\hat{\phi} : \mathbb{T}\tilde{E} \to \mathbb{T}E$.

Remark 2.5.6. The proof of Lemma 2.5.5 relies on a universal property of the Jacobian ([69], Proposition 4.14). Applied to our setting, the map $\mu : \mathrm{Jac}(\Gamma) \to \mathbb{T}E$ in [69] induced by $\varphi : \Gamma \to \mathbb{T}E$ corresponds to φ_*. To make the connection to [69] clearer we write μ for φ_* and $\tilde{\mu}$ for $\tilde{\varphi}_*$ in the proof of Lemma 2.5.5.

Proof of Lemma 2.5.5. The fact that factorization on the level of tropical covers corresponds to factorization on the level of Jacobians is essentially a consequence of Proposition 4.14. in [69].

To see this, note that φ and $\tilde{\varphi}$ are morphisms of rational polyhedral spaces whose targets are also integral tori. Hence, there exists unique homomorphisms μ and $\tilde{\mu}$ such that the following diagram commutes for all $P_0 \in \Gamma$:

$$\begin{tikzcd}
\mathbb{T}E \arrow[d, "t_{-\varphi(P_0)}"'] & \Gamma \arrow[l, "\varphi"'] \arrow[r, "\tilde{\varphi}"] \arrow[d, "\Phi_{P_0}"] & \mathbb{T}\tilde{E} \arrow[d, "t_{-\tilde{\varphi}(P_0)}"] \\
\mathbb{T}E & \mathrm{Jac}(\Gamma) \arrow[l, "\mu"] \arrow[r, "\tilde{\mu}"'] & \mathbb{T}\tilde{E}
\end{tikzcd}.$$

If $\varphi = \phi \circ \tilde{\varphi}$, choose $P_0 \in \tilde{\varphi}^{-1}(0)$ and observe that $t_{-\varphi(P_0)} = t_{-\tilde{\varphi}(P_0)} = id$. Then $\hat{\phi} := \phi$ satisfies $\mu = \hat{\phi} \circ \tilde{\mu}$. Conversely, suppose μ factors through an isogeny $\hat{\phi}$. Let $P_0 \in \varphi^{-1}(0)$ and obtain a factorisation of φ through an isogeny $\phi := \hat{\phi}$ and a harmonic map given by post-composing $\tilde{\varphi}$ with the translation $t_{-\tilde{\varphi}(P_0)}$:

$$\begin{tikzcd}
\Gamma \arrow[r, "\tilde{\varphi}"] \arrow[rrr, bend right, "\varphi"'] & \mathbb{T}\tilde{E} \arrow[r, "t_{-\tilde{\varphi}(P_0)}"] & \mathbb{T}\tilde{E} \arrow[r, "\hat{\phi}"] & \mathbb{T}E.
\end{tikzcd}$$

□

Lemma 2.5.5 defines our approach: Translating covering language into the language of pptav. This gives us the opportunity to address cover-related questions with a new set of tools and guides our next steps: We study

- isogenies of elliptic curves
- the push-forward and pull-back morphism

separately, and merge our results in Subsection 2.5.7 to develop criteria for verifying optimality.

2.5.3 Isogenies of Elliptic Curves.

We start by retrieving the analogy of an algebraic result in the tropical world: Recall that multiplication-by-a maps, where a is a complex number, yield isogenies between complex elliptic curves (in their complex analytic representation as 1-dimensional complex tori) and what is more, all possibilities are fully exhausted by this class of mappings. Essentially the same holds for 1-dimensional integral tori, that is tropical elliptic curves.

Lemma 2.5.7. Let ϕ be an isogeny between elliptic curves, $\mathbb{T}E$ and $\mathbb{T}\tilde{E}$, of length l and $\tilde{l}$. Under the identifications $\mathbb{T}E \cong \mathbb{R}/l\mathbb{Z}$ and $\mathbb{T}\tilde{E} \cong \mathbb{R}/\tilde{l}\mathbb{Z}$ we have: ϕ is induced by an $\mathbb{R}-$linear map $\tilde{\phi} : \mathbb{R} \to \mathbb{R}, z \mapsto a^{\#} \cdot z$, where $a^{\#} \in \mathbb{Z}$ and $\tilde{\phi}(\tilde{l}\mathbb{Z}) \subset l\mathbb{Z}$ holds.
Under the identifications $\mathbb{T}E \cong \mathrm{Jac}(\mathbb{T}E)$ and $\mathbb{T}\tilde{E} \cong \mathrm{Jac}(\mathbb{T}\tilde{E})$ we have: $\phi = (\phi^{\#}, \phi_{\#})$ is defined by a tuple $(a^{\#}, a_{\#}) \in \mathbb{Z}^2$ such that $\tilde{l}a^{\#} = la_{\#}$ holds and ϕ is an isomorphism precisely when $(a^{\#}, a_{\#}) = (\pm 1, \pm 1)$.

We see: On the quotient, ϕ corresponds to a multiplication-by-$a^{\#}$ map, where $a^{\#} \in \mathbb{Z}$. The tuple $(\phi^{\#}, \phi_{\#})$, of course, provides two multiplication maps, that characterize the winding and dilation behaviour of ϕ separately (see [69]).

Proof. We choose to work in the category of tropical abelian varieties. Thus, we will identify all elliptic curves involved with their respective Jacobians. An isogeny ϕ is then given by a pair of group homomorphisms

$$\begin{aligned} \phi^{\#} &: \Omega^1_{\mathbb{T}E} \to \Omega^1_{\mathbb{T}\tilde{E}} : \omega \mapsto a^{\#}\tilde{\omega} \\ \phi_{\#} &: \mathrm{H}_1(\mathbb{T}\tilde{E}, \mathbb{Z}) \to \mathrm{H}_1(\mathbb{T}E, \mathbb{Z}) : \tilde{B} \mapsto a_{\#}B \end{aligned}$$

where B ($\tilde{B}$) is a homology basis of $\mathbb{T}E$ ($\mathbb{T}\tilde{E}$) and ω ($\tilde{\omega}$) the associated basis of tropical 1-forms. The compatibility of $(\phi^{\#}, \phi_{\#})$ with the integration pairings on $\mathbb{T}E$ and $\mathbb{T}\tilde{E}$ poses additional requirements on the pair of integers $(a^{\#}, a_{\#})$:

$$\int_{\tilde{B}} \phi^{\#}(\omega) = \int_{\phi_{\#}(\tilde{B})} \omega.$$

Specifying the metric on $\mathbb{T}E$ and $\mathbb{T}\tilde{E}$ by the real numbers l and $\tilde{l}$ finally leads to relation, $\tilde{l}a^{\#} = la_{\#}$, as claimed. The actual isogeny ϕ is given by $z \mapsto a^{\#} \cdot z$. $\square$

Remark 2.5.8. In algebraic geometry elliptic curves (over algebraically closed fields) are characterized by their j-invariant. The appropriate tropical analogue is the cycle length (see [52], [61]). It is a geometric property that allows us to distinguish between non-isomorphic curves. Computations in Lemma 2.5.11, 2.5.18 and 2.5.15 should be seen in this light.

Continuing in the vein of Lemma 2.5.7, we obtain:

Lemma 2.5.9. For $i = 1, 2, 3$ let $\mathbb{T}E_i$ be an elliptic curve of length l_i. Let $\phi_1 : \mathbb{T}E_1 \to \mathbb{T}E_2$ and $\phi_2 : \mathbb{T}E_1 \to \mathbb{T}E_3$ be two isogenies given by tuples $(a_i^{\#}, a_{i\#}) \in \mathbb{Z}^2$, $i = 1, 2$ as in Lemma 2.5.7. Then ϕ_1 factors through ϕ_2 if and only the following holds:

$$a_3^{\#} := \frac{a_1^{\#}}{a_2^{\#}} \in \mathbb{Z} \text{ and } a_{3\#} := a_3^{\#} \cdot \frac{l_3}{l_2} \in \mathbb{Z}.$$

In this case, $(a_3^{\#}, a_{3,\#})$ defines an isogeny ϕ_3 that satisfies $\phi_1 = \phi_3 \circ \phi_2$.

Proof. Suppose ϕ_1 factors through ϕ_2 via an isogeny ϕ_3, i.e. ϕ_3 satisfies $\phi_{3\#} \circ \phi_{2\#} = \phi_{1\#}$ and $\phi_2^{\#} \circ \phi_3^{\#} = \phi_1^{\#}$. Expressed in terms of the defining tuples $(a_i^{\#}, a_{i\#})$ for $i = 1, 2, 3$ as in Lemma 2.5.7 we have, $a_{3\#} \cdot a_{2\#} = a_{1\#}$ and $a_2^{\#} \cdot a_3^{\#} = a_1^{\#}$, or equivalently, using $l_1 a_1^{\#} = l_2 a_{1\#}$ and $l_1 a_2^{\#} = l_3 a_{2\#}$:

$$a_3^{\#} = \tfrac{a_1^{\#}}{a_2^{\#}} \in \mathbb{Z} \text{ and } a_{3\#} = a_3^{\#} \cdot \tfrac{l_3}{l_2} \in \mathbb{Z}. \tag{2.10}$$

Conversely, suppose (2.10) holds. Then $(a_3^{\#}, a_{3,\#})$ gives rise to a well-defined morphism of tori ϕ_3, which satisfies $\phi_1 = \phi_3 \circ \phi_2$ by construction. ☐

2.5.4 Procedure to address a factorization problem.

Here, we describe a procedure to solve the factorization problem described below. How to perform specific computations mentioned in step 2 is covered in Subsection 2.5.5. Let Γ be of genus 2 and suppose Γ covers elliptic curves $\mathbb{T}E$ and $\mathbb{T}\tilde{E}$ via φ, $\tilde{\varphi}$ respectively (see Lemma 2.5.5):

When does the morphism φ_* factor through $\tilde{\varphi}_*$?

Step 1: Reduction to a lower dimensional factorization problem. This factorization problem can be simplified further: Let $\mu := \varphi_*$ and $\tilde{\mu} := \tilde{\varphi}_*$ and suppose $\ker(\tilde{\mu}) \subset \ker(\mu)$. Otherwise conclude immediately: μ does not factor through $\tilde{\mu}$. Let $\mu = \phi_\mu \circ \pi_\mu$ and $\tilde{\mu} = \phi_{\tilde{\mu}} \circ \pi_{\tilde{\mu}}$ be the Stein factorization of μ and $\tilde{\mu}$. Since

$$\mathrm{Ker}(\mu)_0 + \pi_\mu^{-1}(\ker(\phi_\mu)) = \ker(\mu) \supset \ker(\tilde{\mu}) = \mathrm{Ker}(\tilde{\mu})_0 + \pi_{\tilde{\mu}}^{-1}(\ker(\phi_{\tilde{\mu}}))$$

we have $\operatorname{Ker}(\tilde{\mu})_0 \subset \operatorname{Ker}(\mu)_0$. This yields a well-defined map pr such that the left part of

$$\begin{tikzcd}
\operatorname{Jac}(\Gamma) \arrow[r, "\pi_\mu"] \arrow[dr, "\pi_{\tilde{\mu}}"'] & \operatorname{Jac}(\Gamma)/\operatorname{Ker}(\mu)_0 \arrow[r, "\phi_\mu"] & \mathbb{T}E \\
 & \operatorname{Jac}(\Gamma)/\operatorname{Ker}(\tilde{\mu})_0 \arrow[u, "pr"] \arrow[r, "\phi_{\tilde{\mu}}"'] & \mathbb{T}\tilde{E} \arrow[u, dashed, "?"]
\end{tikzcd} \tag{2.11}$$

commutes. As $\operatorname{Ker}(\tilde{\mu})_0$ is a subgroup of finite index of $\operatorname{Ker}(\mu)_0$ since $\operatorname{Jac}(\Gamma)/\operatorname{Ker}(\tilde{\mu})_0$ and $\operatorname{Jac}(\Gamma)/\operatorname{Ker}(\mu)_0$ are both 1-dimensional tori, pr is an isogeny (and thus $\phi_\mu \circ pr$ as well). Moreover, $\ker(\phi_\mu \circ pr) \supset \ker(\phi_{\tilde{\mu}})$ holds. That is, the initial factorization problem reduces to the lower dimensional problem of factorizing $\phi_\mu \circ pr$ through $\phi_{\tilde{\mu}}$.

Step 2: Solve lower dimensional factorization problem. We use Lemma 2.5.9: Set $\mathbb{T}E_1 := \operatorname{Jac}(\Gamma)/\operatorname{Ker}(\tilde{\mu})_0$, $\mathbb{T}E_2 := \mathbb{T}E$, $\mathbb{T}E_3 := \mathbb{T}\tilde{E}$, $\phi_1 := \phi_\mu \circ pr$ and $\phi_2 := \phi_{\tilde{\mu}}$. If Γ is of type "theta", see Figure 2.6, compute l_1 and $(a_i^\#, a_{i\#})$ (for $i = 1, 2$) using Lemma 2.5.15. For the other maximal combinatorial type of a curve of genus 2, the dumbbell graph (see Figure 2.8), we have a formula for l_1 in Lemma 2.5.18. The pair $(a_i^\#, a_{i\#})$ is then computed as in the proof of Lemma 2.5.15. Then μ factors through $\tilde{\mu}$ if and only if the following holds:

- $\ker(\tilde{\mu}) \subset \ker(\mu)$,
- $a_3^\# := \frac{a_1^\#}{a_2^\#} \in \mathbb{Z}$ and $a_{3\#} := a_3^\# \cdot \frac{l_3}{l_2} \in \mathbb{Z}$.

In this case, $\hat{\phi} := \phi_3$ solves the initial factorization problem.

2.5.5 The push-forward

As before, let $\varphi : \Gamma \to \mathbb{T}E$ be a cover and $g(\Gamma) = 2$. Then φ_* is clearly surjective making its kernel the first object of interest. Note, however, that φ_* belongs to two different worlds, the category of tav ($\mathbb{T}\mathscr{A}$) and the category of abelian groups ($\mathbf{Ab}$). This means that its kernel will have two different identities as well.

We study both, first in detail for the case where Γ is of type "theta" as in Figure 2.6. For the case where the combinatorial type of Γ is the dumbbell-graph, we provide a collection of results in Subsection 2.5.5.

Type theta

Convention 2.5.10. This next part calls for consistency in naming that allows us to refer to individual parts of a cover $\varphi : \Gamma \to \mathbb{T}E$ explicitly. We introduce the labeling we want to use for the graphs underlying Γ and $\mathbb{T}E$ in Figure 2.5.

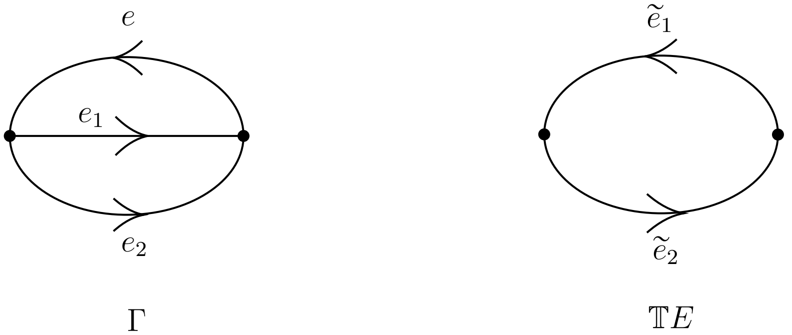


Figure 2.5: Labeling for the graphs underlying Γ and $\mathbb{T}E$.

Next, let us fix concrete homology basis (with edges oriented as in Figure 2.5):

- $(B_1, B_2) := (e + e_2, e_2 - e_1)$ of $\mathrm{H}_1(\Gamma, \mathbb{Z})$.
- $\tilde{B} := \tilde{e}_1 + \tilde{e}_2$ of $\mathrm{H}_1(\mathbb{T}E, \mathbb{Z})$.

We denote by (ω_1, ω_2) and $\tilde{\omega}$ the canonical basis of tropical 1-forms associated to (B_1, B_2), respectively $\tilde{B}$ (i.e. $\omega_i = \zeta_\Gamma(B_i)$ and $\tilde{\omega} = \zeta_{\mathbb{T}E}(\tilde{B})$). We want to stress that the following arguments apply to *any* choice of basis, the formulas below, however, will differ.

An arbitrary cover of degree d is captured by the following data:

- a triple of *winding numbers* $(n, n_1, n_2) \in \mathbb{N}_0^3$, where n_i (n) counts how many times the edge e_i (e) passes over the edge $\widetilde{e}_2$ ($\widetilde{e}_1$),
- a triple of *dilation factors* $(d_e(\varphi), d_{e_1}(\varphi), d_{e_2}(\varphi)) \in \mathbb{N}_0^3$,

that satisfy:

$$nd_e(\varphi) + (n_1 - 1)d_{e_1}(\varphi) + (n_2 - 1)d_{e_2}(\varphi) = d.$$
$$(n-1)d_e(\varphi) + n_1 d_{e_1}(\varphi) + n_2 d_{e_2}(\varphi) = d.$$

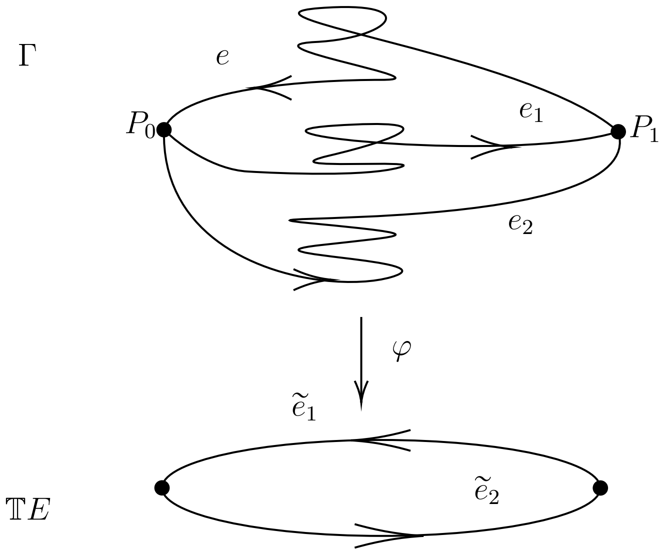


Figure 2.6: Schematic picture of a finite cover of an elliptic curve by a curve of genus 2 of type "theta".

Lemma 2.5.11. Let Γ be an abstract tropical curve of type "theta" and $\varphi : \Gamma \to \mathbb{T}E$ a tropical cover of degree d. In $\mathbb{T}\mathscr{A}$, the kernel of φ_* (i.e. $\mathrm{Ker}(\varphi_*)_0$) is an elliptic curve of length l_K given by the absolute value of the matrix product vMw^t, where

- $M := \begin{pmatrix} l(e)+l(e_2) & l(e_2) \\ l(e_2) & l(e_1)+l(e_2) \end{pmatrix}$ is the period matrix associated to the choice of basis from Convention 2.5.10.
- $v := (v_1, v_2)$ is any vector whose entries satisfy: $v_2 d_e(\varphi) + v_1 d_{e_1}(\varphi) = gcd(d_e(\varphi), d_{e_1}(\varphi))$.
- $w := (\frac{n_1-n_2}{gcd(n_2-n_1,n+n_2-1)}, \frac{n+n_2-1}{gcd(n_2-n_1,n+n_2-1)})$.

Proof. Recalling that φ_* is a surjective morphism from a 2-dimensional to a 1-dimensional torus, it is not surprising that (in $\mathbb{T}\mathscr{A}$) its kernel is a tav of dimension 1, in other words an elliptic curve (see Remark 2.5.12). More interesting is the statement about l_K, since the length is an isomorphy-invariant of elliptic curves. So, in order to capture of the geometry of $\mathrm{Ker}(\varphi_*)_0$, we need to look at it in more detail: Its lattice representation (see Definition 2.3.4)

$$(\Omega^1_\Gamma/\mathrm{Im}({\varphi_*}^{\#})^{\mathrm{sat}}, \ker(\varphi_{*\#}), [\cdot,\cdot]_K)$$

allows us to access l_K via integration on Γ, i.e. $l_K = |\int_B \omega|$, where B and ω are basis of the lattices defining $\mathrm{Ker}(\varphi_*)_0$. To obtain a concrete expression for l_K in terms of covering data, we compute B and ω explicitly.

We start with $\ker(\varphi_{*\#})$: The map $\varphi_{*\#} : \mathrm{H}_1(\Gamma, \mathbb{Z}) \to \mathrm{H}_1(\mathbb{T}E, \mathbb{Z})$, sends a cycle $\sum e_i$ to the cycle $\sum \varphi(e_i)$. This yields

$$\begin{aligned} \varphi_{*\#}(B_1) &= ((n_2 - 1) + (n - 1) + 1)\tilde{B} \text{ and} \\ \varphi_{*\#}(B_2) &= ((n_2 - 1) - (n_1 - 1))\tilde{B}, \end{aligned}$$

for our choice of basis (B_1, B_2) from Convention 2.5.10. A contracted cycle $B = w_1 B_1 + w_2 B_2 \in \mathrm{H}_1(\Gamma, \mathbb{Z})$ generates $\ker(\varphi_{*\#})$, if the pair of coefficients $(w_1, w_2) \in \mathbb{Z}^2$ in addition to satisfying

$$\begin{aligned} 0 &= \varphi_{*\#}(B) = w_1(n + n_2 - 1)\tilde{B} + w_2(n_2 - n_1)\tilde{B} \\ \Leftrightarrow 0 &= w_1(n + n_2 - 1) + w_2(n_2 - n_1), \end{aligned}$$

is relatively prime. This yields

$$w := (w_1, w_2) = \left(\frac{n_1 - n_2}{gcd(n_2 - n_1, n + n_2 - 1)}, \frac{n + n_2 - 1}{gcd(n_2 - n_1, n + n_2 - 1)} \right).$$

We continue with $\Omega^1_\Gamma / \mathrm{Im}({\varphi_*}^\#)^{\mathrm{sat}}$: The pullback of 1-forms ${\varphi_*}^\#$ is injective as φ_* is surjective. Hence, $\mathrm{Im}({\varphi_*}^\#)$ is generated by ${\varphi_*}^\#(\tilde{\omega})$ whose representation in terms of the basis (ω_1, ω_2) from Convention 2.5.10 is

$$\begin{aligned} {\varphi_*}^\#(\tilde{\omega} = d\tilde{e}_1 + d\tilde{e}_2) &= d_{e_1}(\varphi) de_1 + d_{e_2}(\varphi) de_2 + d_e(\varphi) de \\ &= d_e(\varphi)\omega_1 + (d_{e_2}(\varphi) - d_e(\varphi)) de_2 + d_{e_1}(\varphi) de_1 \\ &= d_e(\varphi)\omega_1 - d_{e_1}(\varphi)\omega_2. \end{aligned}$$

As a result, we get

$$\mathrm{Im}({\varphi_*}^\#)^{\mathrm{sat}} = \langle \omega_Q := \frac{d_e(\varphi)}{gcd(d_e(\varphi), d_{e_1}(\varphi))}\omega_1 - \frac{d_{e_1}(\varphi)}{gcd(d_e(\varphi), d_{e_1}(\varphi))}\omega_2 \rangle,$$

whereby it should be emphasized that $gcd(d_e(\varphi), d_{e_1}(\varphi)) \neq 0$ holds, even if φ is not finite: Suppose otherwise. Then $d_e(\varphi) = d_{e_1}(\varphi) = 0$ implies $d_{e_2}(\varphi) = 0$ due to balancing. So, φ cannot be surjective, a contradiction.

Note, the quotient $\Omega^1_\Gamma / \mathrm{Im}({\varphi_*}^\#)^{\mathrm{sat}}$ is naturally identified with a complement of $\mathrm{Im}({\varphi_*}^\#)^{\mathrm{sat}}$ in $\mathrm{H}_1(\Gamma, \mathbb{Z})$. To find one, we need to extend ω_Q to a basis of $\mathrm{H}_1(\Gamma, \mathbb{Z})$ by adding a 1-form $\omega := v_1\omega_1 + v_2\omega_2$. The requirement that the new pair (ω_Q, ω) can be obtained from (ω_1, ω_2) by an invertible coordinate change can be rephrased as condition imposed on the coefficients of ω:

$$v_2 d_e(\varphi) + v_1 d_{e_1}(\varphi) = gcd(d_e(\varphi), d_{e_1}(\varphi)).$$

Any such 1-form ω generates a complement and is therefore suitable for us (see Remark 2.5.13). This concludes our hunt for basis. The desired formula reads

$$l_K = |\int_B \omega| = |\int_{w_1 B_1 + w_2 B_2} (v_1\omega_1 + v_2\omega_2)|$$

and can be compactly written as matrix product $l_K = |vMw^t|$, where M is the period matrix associated to our choice of basis (see Convention 2.5.10). ☐

Remark 2.5.12. We interpret the 1-dimensional torus $\mathrm{Ker}(\varphi_*)_0 = (\langle\omega\rangle, \langle B\rangle, [\cdot,\cdot]_K)$, where $\omega \in \Omega^1_\Gamma$ and $B \in \mathrm{H}_1(\Gamma, \mathbb{Z})$ are lattice basis as in Lemma 2.5.11 such that $l_K := \int_B \omega > 0$, as Jacobian of a metric graph $\mathbb{T}E'$ with $l(\mathbb{T}E') = l_K$ via:

$$\Omega^1_{\mathbb{T}E'} = \langle\omega\rangle,\ \mathrm{H}_1(\mathbb{T}E', \mathbb{Z}) = \langle B\rangle,\ \text{and } \zeta_{\mathbb{T}E'},$$

the usual pp (Definition 2.5.2) and, by abuse of notation, denote $\mathrm{Jac}(\mathbb{T}E')$ again by $\mathbb{T}E'$ (see beginning of Subsection 2.5.2).

Remark 2.5.13. Note that the vector v from Lemma 2.5.11 is not uniquely determined: Solutions to

$$v_2 d_e(\varphi) + v_1 d_{e_1}(\varphi) = gcd(d_e(\varphi), d_{e_1}(\varphi))$$

differ by elements of $\langle(\frac{d_e(\varphi)}{gcd(d_e(\varphi), d_{e_1}(\varphi))}, \frac{-d_{e_1}(\varphi)}{gcd(d_e(\varphi), d_{e_1}(\varphi))})\rangle_{\mathbb{Z}}$, which corresponds to adding a multiple of the generator of $\mathrm{Im}({\varphi_*}^\#)^{\mathrm{sat}}$. The reason for this indeterminacy can be found in the proof of Lemma 2.5.11. It arises from the identification of the quotient $\Omega^1_\Gamma/\mathrm{Im}({\varphi_*}^\#)^{\mathrm{sat}}$ with a complement of $\mathrm{Im}({\varphi_*}^\#)^{\mathrm{sat}}$ and does not affect the matrix product vMw^t, since we have

$$\int_B {\varphi_*}^\#(\omega) = \int_{\varphi_{*\#}(B)} \omega = 0.$$

The transition to **Ab** is accompanied by a loss of structure (the kernel is no longer a tav). A trade-off that is necessary to fully understand it.

Lemma 2.5.14. The group-theoretic kernel of the push-forward map consists of translates of the 1-dimensional torus $\mathrm{Ker}(\varphi_*)_0$ by elements of $\varphi^* \mathrm{Jac}_d(\mathbb{T}E)$ (the image of the d-torsion subgroup $\mathrm{Jac}_d(\mathbb{T}E)$ of $\mathrm{Jac}(\mathbb{T}E)$ under φ^*). Moreover, the number of connected components is given by $|\ker(\gamma)|$, where $\gamma : \mathrm{Jac}(\Gamma)/\mathrm{Ker}(\varphi_*)_0 \to \mathrm{Jac}(\mathbb{T}E)$ is the natural map.

The beginning of the following proof is similar to the proof of Proposition 6.1. ([49]). As we work in a different setting, we present it here for the sake of completeness. Compare also to Proposition 4.7 in [69] for a different characterization of the connected components of $\ker(f)$ for general morphism f.

Proof. The kernel of φ_* is the image of $\mathrm{Hom}(\varphi_*^\#)^{-1}\,\mathrm{H}_1(\mathbb{T}E, \mathbb{Z})$ under the universal covering of $\mathrm{Jac}(\Gamma)$. Recalling that $\mathrm{Hom}(\varphi_*^\#)$ is $\mathbb{R}$-linear we can write the solution set $\mathrm{Hom}(\varphi_*^\#)^{-1}\,\mathrm{H}_1(\mathbb{T}E, \mathbb{Z})$ as translation of the kernel of $\mathrm{Hom}(\varphi_*^\#)$ by a set consisting of specific solutions to the equations

$$\mathrm{Hom}(\varphi_*^\#)(x_C) = C \text{ for } C \in \mathrm{H}_1(\mathbb{T}E, \mathbb{Z}).$$

We now compute such a specific solution. The isomorphisms $\mathrm{H}_1(\Gamma,\mathbb{Z}) \cong \Omega^1_\Gamma$ and $\mathrm{H}_1(\mathbb{T}E,\mathbb{Z}) \cong \Omega^1_{\mathbb{T}E}$ allow us to identify $\varphi_{*\#}$ with the push-forward of $1-$forms and consider the composition $\varphi_{*\#} \circ \varphi_*^{\#} : \Omega^1_{\mathbb{T}E} \to \Omega^1_{\mathbb{T}E}$. This is easily seen to be the multiplication-by-d map. Hence, we have

$$\mathrm{Hom}(\varphi_*^{\#}) \circ \mathrm{Hom}(\varphi_{*\#}) = \mathrm{Hom}(\varphi_{*\#} \circ \varphi_*^{\#}) = d \cdot id$$

for the Hom-duals and $\frac{1}{d}\,\mathrm{Hom}(\varphi_{*\#})(C)$ as solution of $\mathrm{Hom}(\varphi_*^{\#})(x_C) = C$.
We write

$$\mathrm{Hom}(\varphi_*^{\#})^{-1}\,\mathrm{H}_1(\mathbb{T}E,\mathbb{Z}) = \ker(\mathrm{Hom}(\varphi_*^{\#})) + \frac{1}{d}\,\mathrm{Hom}(\varphi_{*\#})\,\mathrm{H}_1(\mathbb{T}E,\mathbb{Z})$$

and descend to $\mathrm{Jac}(\Gamma)$ to obtain the desired description of $\ker(\varphi_*)$. Remembering that $\mathrm{Hom}(\varphi_{*\#})$ is the lift of φ^* to a map between the universal coverings (Lemma 2.5.4) and using $\mathrm{Jac}_d(\mathbb{T}E) = \frac{\mathrm{H}_1(\mathbb{T}E,\mathbb{Z})}{d\,\mathrm{H}_1(\mathbb{T}E,\mathbb{Z})}$, this looks like:

$$\begin{aligned}\ker(\varphi_*) &= \mathrm{Ker}(\varphi_*)_0 + \frac{\mathrm{Hom}(\varphi_{*\#})\,\mathrm{H}_1(\mathbb{T}E,\mathbb{Z})}{d \cdot \mathrm{Hom}(\varphi_{*\#})(\mathrm{H}_1(\mathbb{T}E,\mathbb{Z})) \cap \mathrm{H}_1(\Gamma,\mathbb{Z})} \\ &= \mathrm{Ker}(\varphi_*)_0 + \varphi^*\,\mathrm{Jac}_d(\mathbb{T}E).\end{aligned}$$

Note that we have just written $\ker(\varphi_*)$ as union of cosets of its subgroup $\mathrm{Ker}(\varphi_*)_0$. We did even more: We found a *finite* superset for a set of representatives, which is given by $\varphi^*\,\mathrm{Jac}_d(\mathbb{T}E)$. In other words, the number of connected components is finite and given by the index of $\mathrm{Ker}(\varphi_*)_0$ in $\ker(\varphi_*)$. We can consider the following commutative exact diagram in the category of abelian groups (see Figure 2.7)

and apply the snake Lemma to obtain an isomorphism $\frac{\ker(\varphi_*)}{\mathrm{Ker}(\varphi_*)_0} \xrightarrow{\partial} \ker(\gamma)$. In particular we see that the surjective map γ has a finite kernel, is therefore an isogeny, whose kernel counts the number of connected components as claimed. □

We give an explicit description of the isogeny γ from Lemma 2.5.14 and compute the number of connected components in terms of data from our cover.

Lemma 2.5.15. The tav $\mathrm{Jac}(\Gamma)/\mathrm{Ker}(\varphi_*)_0$ is an elliptic curve of length $\tilde{l} = |w^Q M v^{Q^t}|$, where

- M is the period matrix associated to the choice of basis from convention 2.5.10.
- $v^Q := (v_1^Q, v_2^Q)$ is any vector whose entries satisfy:
  $v_1^Q(n + n_2 - 1) - v_2^Q(n_1 - n_2) = gcd(n_2 - n_1, n + n_2 - 1)$.
- $w^Q := (-\frac{d_{e_1}(\varphi)}{gcd(d_e(\varphi), d_{e_1}(\varphi))}, \frac{d_e(\varphi)}{gcd(d_e(\varphi), d_{e_1}(\varphi))})$.

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ker(\gamma) \arrow[d] \arrow[dddll, rounded corners, "\partial"] & \\
 & \mathrm{Ker}(\varphi_*)_0 \arrow[r] \arrow[d] & \mathrm{Jac}(\Gamma) \arrow[r] \arrow[d, "id"] & \frac{\mathrm{Jac}(\Gamma)}{\mathrm{Ker}(\varphi_*)_0} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \ker(\varphi_*) \arrow[r] \arrow[d] & \mathrm{Jac}(\Gamma) \arrow[r] \arrow[d] & \mathrm{Jac}(\mathbb{T}E) \arrow[r] \arrow[d] & 0 \\
 & \frac{\ker(\varphi_*)}{\mathrm{Ker}(\varphi_*)_0} \arrow[r] & 0 \arrow[r] & 0 &
\end{tikzcd}$$

Figure 2.7: Applying the snake Lemma in the proof of Lemma 2.5.14.

Under the identifications $\mathrm{Jac}(\mathbb{T}E) \cong \mathbb{R}/l\mathbb{Z}$ and $\mathrm{Jac}(\Gamma)/\mathrm{Ker}(\varphi_*)_0 \cong \mathbb{R}/\tilde{l}\mathbb{Z}$ the isogeny γ from Lemma 2.5.14 takes the form:

$$\mathbb{R}/\tilde{l}\mathbb{Z} \to \mathbb{R}/l\mathbb{Z}, z \mapsto az,$$

where $a = gcd(d_e(\varphi), d_{e_1}(\varphi))$.

Proof. We derive a lattice representation $(\ker(i^{\#}), \Omega^1_\Gamma/\mathrm{Im}(i_{\#}), [\cdot,\cdot]^t_K)$ of the quotient variety $\mathrm{Jac}(\Gamma)/\mathrm{Ker}(\varphi_*)_0$ from Lemma 2.3.9 (or Remark 2.3.10) by choosing the natural inclusion $i = (i^{\#}, i_{\#})$ defined by the maps

$$i^{\#} : \Omega^1_\Gamma \twoheadrightarrow \Omega^1_\Gamma/\mathrm{Im}(\varphi_*{}^{\#})^{\mathrm{sat}} \text{ and } i_{\#} : \ker(\varphi_{*\#}) \hookrightarrow \mathrm{H}_1(\Gamma, \mathbb{Z})$$

as embedding of $\mathrm{Ker}(\varphi_*)_0$ into $\mathrm{Jac}(\Gamma)$. Just like in Lemma 2.5.11 we determine $\tilde{l}$ by expressing explicit basis ω_Q and B_Q of the lattices $\ker(i^{\#})$ and $\Omega^1_\Gamma/\mathrm{Im}(i_{\#})$ in terms of our chosen basis from Convention 2.5.10. The equality $\ker(i^{\#}) = \mathrm{Im}(\varphi_*{}^{\#})^{\mathrm{sat}}$ is immediate from the definition of i. Relying on the results of Lemma 2.5.11 we set

$$\omega_Q := \frac{d_e(\varphi)}{gcd(d_e(\varphi), d_{e_1}(\varphi))}\omega_2 - \frac{d_{e_1}(\varphi)}{gcd(d_e(\varphi), d_{e_1}(\varphi))}\omega_1.$$

Since $i_{\#}$ is just the inclusion, we have $\mathrm{Im}(i_{\#}) = \ker(\varphi_{*\#}) = \langle B \rangle$ (Lemma 2.5.11), where $B = w_1 B_1 + w_2 B_2 \in \mathrm{H}_1(\Gamma, \mathbb{Z})$ with coefficients

$$(w_1, w_2) = \left(\frac{n_1 - n_2}{gcd(n_2 - n_1, n + n_2 - 1)}, \frac{n + n_2 - 1}{gcd(n_2 - n_1, n + n_2 - 1)} \right).$$

Any cycle $B_Q := v_1^Q B_1 + v_2^Q B_2$ such that

$$v_1^Q(n + n_2 - 1) - v_2^Q(n_1 - n_2) = gcd(n_2 - n_1, n + n_2 - 1)$$

holds, generates a complement of $\mathrm{Im}(i_\#)$ in $\mathrm{H}_1(\Gamma, \mathbb{Z})$. We identify $\langle B_Q \rangle$ with the quotient $\Omega^1_\Gamma/\mathrm{Im}(i_\#)$. The length $\tilde{l}$ is then given by the absolute value of the integral

$$\int_{B_Q} \omega_Q = \int_{w_1^Q B_1 + w_2^Q B_2} (v_1^Q \omega_1 + v_2^Q \omega_2),$$

which written as matrix product reads $w^Q M v^{Q^t}$.

As a closer look at Figure 2.7 reveals, γ just sends an equivalence class $[x]$ to the image of a representative under the push-forward, i.e. $\gamma([x]) = \varphi_*(x)$. Hence, $\gamma^\# : \Omega^1_{\mathbb{T}E} \to \ker(i^\#)$, is defined by $\omega \mapsto \varphi_*^\#(\omega)$. We recognize this as multiplication by $a^\#$, where $a^\#$ satisfies $\varphi_*^\#(\omega) = a^\# \omega_Q$ and is therefore equal to $gcd(d_e(\varphi), d_{e_1}(\varphi))$. This together with the knowledge of $\tilde{l}$ and l is sufficient to give a concrete description of the isogeny γ under the identifications $\mathrm{Jac}(\mathbb{T}E) \cong \mathbb{R}/l\mathbb{Z}$ and $\mathrm{Jac}(\Gamma)/\mathrm{Ker}(\varphi_*)_0 \cong \mathbb{R}/\tilde{l}\mathbb{Z}$ (see Lemma 2.5.7):

$$\mathbb{R}/\tilde{l}\mathbb{Z} \to \mathbb{R}/l\mathbb{Z}, z \mapsto a^\# z.$$

□

Remark 2.5.16. A priory, all arguments, which lead us to $a^\#$, are based on a particular choice of basis (Convention 2.5.10). This should not be the case: The lengths $\tilde{l}$ and l are intrinsic characteristics of the curves, so should the description of γ be. And indeed, the balancing condition implies that $gcd(d_e(\varphi), d_{e_1}(\varphi)) = gcd(d_e(\varphi), d_{e_2}(\varphi))$, which in turn guarantees that interchanging the roles of e_1 and e_2 does not affect $a^\#$.

Using Lemma 2.5.7 we can also describe the full pair $\gamma = (\gamma^\#, \gamma_\#)$ since $\gamma_\#$ is the multiplication by $a_\# := \frac{\tilde{l} a^\#}{l}$.

Corollary 2.5.17. The number of connected components of $\ker(\varphi_*)$ is equal to $k+1$, where $k := max\{j \in \mathbb{N} : \frac{j}{a_\#} < 1\}$.

Proof. By Lemma 2.5.14 we know that the number of connected components is the order of $\ker(\gamma)$. We can use Lemma 2.5.15 and Remark 2.5.16 to write it down explicitly:

$$\ker(\gamma) = \{x \in \mathbb{R}/\tilde{l}\mathbb{Z} : \ a^\# x \in l\mathbb{Z}\} = \{x \in \mathbb{R}/\tilde{l}\mathbb{Z} : \frac{l \cdot a_\#}{\tilde{l}} x \in l\mathbb{Z}\}.$$

and conclude by specifying a set of representatives:

$$\{0, \frac{\tilde{l}}{a_\#}, \ldots, \frac{k \cdot \tilde{l}}{a_\#}\}, \text{ where } k := \max\{j \in \mathbb{N} : \frac{j}{a_\#} < 1\}.$$

□

Type dumbbell

The case where φ is finite is already addressed in Subsection 2.5.5. Therefore, working with a curve Γ of type "dumbbell" as in Figure 2.8 automatically leads us to the non-finite case since: Whenever Γ maps to a curve of genus 1, at least the edge connecting P_0 to P_1 is contracted (a sketch of all other possibilities is shown in Figure 2.8).

To formulate Lemma 2.5.11 and 2.5.15 in this case, we fix as before:

- the homology basis $(B_1, B_2) := (e_1, e_2)$ of $\mathrm{H}_1(\Gamma, \mathbb{Z})$ and $\tilde{B} := \tilde{e}$ of $\mathrm{H}_1(\mathbb{T}E, \mathbb{Z})$,
- the canonically associated basis (ω_1, ω_2) of $\Omega^1_\Gamma(\mathbb{Z})$ and $\tilde{\omega}$ of $\Omega^1_{\mathbb{T}E}(\mathbb{Z})$,

and characterize a cover of degree d by a tuple of *winding numbers* $(n_1, n_2) \in \mathbb{N}_0^2$ together with a tuple of *dilation factors* $(d_{e_1}(\varphi), d_{e_2}(\varphi)) \in \mathbb{N}_0^2$ that satisfy $n_1 d_{e_1}(\varphi) + n_2 d_{e_2}(\varphi) = d$.

Lemma 2.5.18. Let $\varphi : \Gamma \to \mathbb{T}E$ be a degree d cover as in Figure 2.8. In $\mathbb{T}\mathscr{A}$, the kernel of φ_* is an elliptic curve of length l_K given by the absolute value of the matrix product vMw^t, where

- $M := \begin{pmatrix} l(e_1) & 0 \\ 0 & l(e_2) \end{pmatrix}$ is the period matrix associated to the choice of basis.
- $v := (v_1, v_2)$ is any vector whose entries satisfy: $v_2 d_{e_1}(\varphi) - v_1 d_{e_2}(\varphi) = gcd(d_{e_1}(\varphi), d_{e_2}(\varphi))$.
- $w := (\frac{-n_2}{gcd(n_1,n_2)}, \frac{n_1}{gcd(n_1,n_2)})$.

The tav $\mathrm{Jac}(\Gamma)/\mathrm{Ker}(\varphi_*)_0$ is an elliptic curve of length $\tilde{l} = |w^Q M v^{Q^t}|$, where

- $v^Q := (v_1^Q, v_2^Q)$ is any vector whose entries satisfy:
  $-(n_2 v_1^Q + n_1 v_2^Q) = gcd(n_1, n_2)$.
- $w^Q := (\frac{d_{e_1}(\varphi)}{gcd(d_{e_1}(\varphi), d_{e_2}(\varphi))}, \frac{d_{e_2}(\varphi)}{gcd(d_{e_1}(\varphi), d_{e_2}(\varphi))})$.

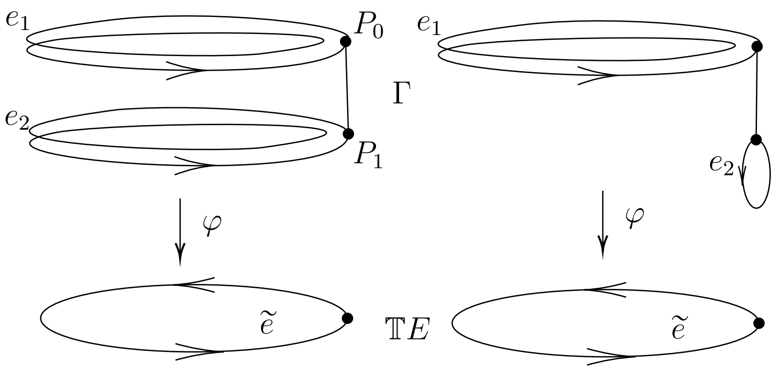


Figure 2.8: Schematic picture of two non-finite covers of an elliptic curve by a curve of genus 2 of "dumbell type".

2.5.6 The pull-back

Changing gears, we now work with divisors to study the dual of φ_*. Recall that from this perspective φ^* is simply the pull-back of divisors (see Remark 2.5.3). For background on divisors on tropical curves see e.g. [49], [2], [5] and references therein.

We are still interested in covers of the form $\varphi : \Gamma \to \mathbb{T}E$, but do not require Γ to have genus 2. Just as in Subsection 2.5.5, we approach φ^* by understanding its kernel $\ker(\varphi^*)$. We proceed as follows: The groundwork will be laid in Lemma 2.5.19, which we formulate for *any* harmonic map $\varphi : \Gamma \to \tilde{\Gamma}$. This means we find a superset of $\ker(\varphi^*)$ whose elements are easily described and allow for concrete manipulation. This will be used later in Construction 2.5.20.

Lemma 2.5.19. Let $\varphi : \Gamma \to \tilde{\Gamma}$ be a tropical cover of degree d. Then $\ker(\varphi^*) \subset \mathrm{Jac}_d(\tilde{\Gamma})$ and we have

$$D \in \ker(\varphi^*) \text{ if and only if all slopes of } \varphi^* f \text{ are divisible by } d,$$

where f is a rational function satisfying $d \cdot D = div(f)$.

Proof. We obtain a drastic simplification of φ^* by post-composing it with φ_*: As is easily verified, we have that $\varphi_*\varphi^*$ is given by the multiplication-by-d map on $\mathrm{Jac}(\tilde{\Gamma})$. This immediately gives us the desired superset for $\ker(\varphi^*)$ as the kernel of the composition, i.e. $\mathrm{Jac}_d(\tilde{\Gamma})$, and proves the first statement.
Since D has order d, there exists a piecewise linear function $f : \tilde{\Gamma} \to \mathbb{R}$ such that $d \cdot D = div(f)$. Suppose $\varphi^* D = div(g)$ for $g : \Gamma \to \mathbb{R}$. Then $div(d \cdot g) = \varphi^*(d \cdot D) = div(\varphi^* f)$ (recall: $div(\varphi^*(f)) = \varphi^*(div(f))$), shows that $\varphi^* f - d \cdot g$ is constant and $g = \frac{1}{d}\varphi^* f + c$ for $c \in \mathbb{R}$. Thus, g is only well-defined if $\frac{1}{d}\varphi^* f$ is, i.e. all slopes are divisible by d. ☐

As promised, applying Lemma 2.5.19 to the case where $\tilde{\Gamma}$ is of genus 1 provides a simple description of divisors that are potentially contained in $\ker(\varphi^*)$: By restricting the isomorphism $\mathbb{T}E \to \mathrm{Jac}(\mathbb{T}E)$ to $\mathbb{T}E[d]$ we see that we can write any $D \in \mathrm{Jac}_d(\mathbb{T}E)$ as $D_P := P - P_0$, where $P \in \mathbb{T}E[d]$ and $P_0 \in \mathbb{T}E$ denotes the identity (i.e. the origin in $\mathbb{T}E \cong \mathbb{R}/l\mathbb{Z}$).

Construction 2.5.20. For any point $P \in \mathbb{T}E = \mathbb{R}/l\mathbb{Z}$ let $\overline{PP_0}$ ($\overline{P_0P}$) be the oriented edge of length l_1 (l_2) connecting P with P_0 (P_0 with P). If $P \in l\mathbb{Q}/l\mathbb{Z}$, we have $ord(P) < \infty$. Hence, there exists a rational function f on $\mathbb{T}E$ whose set of zeroes and poles is $\{P, P_0\}$. We can translate the statement $div(f) = ord(P) \cdot D_P$ into a linear system

$$\begin{cases} s_1 + s_2 = ord_p(f) = ord(P) \\ -s_1 - s_2 = ord_{P_0}(f) = -ord(P), \\ l_1 s_1 - (l - l_1)s_2 = 0 \end{cases}$$

where

- l_1 encodes the position of p,
- (s_1, s_2) is the data specifying f, i.e. s_1 (s_2) denotes the slope of f on $\overline{PP_0}$ ($\overline{P_0P}$),

and solve for $s_1 = \frac{ord(P)(l-l_1)}{l}$ and $s_2 = \frac{ord(P)l_1}{l}$.

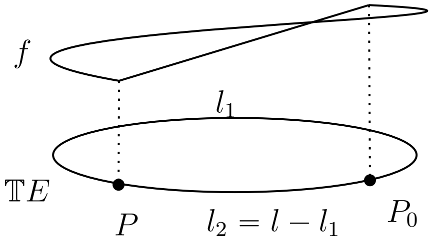


Figure 2.9: A rational function f on $\mathbb{T}E$ as described in Construction 2.5.20.

As the pullback of f along φ rescales slopes by local dilation factors, the divisor $\varphi^* D_P$ is principal if and only if every edge $e \in E(\Gamma)$ satisfies:

- If $e \mapsto e'$ for $e' \subset \overline{PP_0}$, then $d_e(\varphi)\frac{(l-l_1)}{l} \in \mathbb{Z}$ holds.
- If $e \mapsto e'$ for $e' \subset \overline{P_0P}$, then $d_e(\varphi)\frac{l_1}{l} \in \mathbb{Z}$ holds.

We package this information into a continuous function $q_\Gamma : [0, l] \to \mathbb{R}^{|E(\Gamma)|}$ that assigns to each $l_1 \in [0, l]$ the vector $q_\Gamma(l_1)$ with coordinates

$$q^e_\Gamma(l_1) := \begin{cases} d_e(\varphi)\frac{(l-l_1)}{l}, & \text{if } e \mapsto e' \subset \overline{PP_0}, \\ d_e(\varphi)\frac{l_1}{l}, & \text{if } e \mapsto e' \subset \overline{P_0P}, \end{cases} \text{ for } e \in E(\Gamma).$$

This allows $\ker(\varphi^*)$ to be described in a clean and concise manner.

Proposition 2.5.21. Let $\varphi : \Gamma \to \mathbb{T}E$ be a cover of degree d, where $g(\Gamma) = g$, and $q_\Gamma : [0, l] \to \mathbb{R}^{|E(\Gamma)|}$ the function from Construction 2.5.20. Then q_Γ decides whether a divisor lives in the kernel of the pullback map φ^*. We have:

$$\ker(\varphi^*) \cong q_\Gamma(\mathbb{T}E) \cap \mathbb{Z}^{|E(\Gamma)|},$$

where, by abuse of notation, we write $q_\Gamma(\mathbb{T}E)$ using that the domain of q_Γ parameterizes points of $\mathbb{T}E$. In particular, if $D_P \in \ker(\varphi^*)$, then $ord(D_P)$ divides d.

Proof. Note that $q^e_\Gamma(l_1) \in \mathbb{Z}$ is true if and only if $l_1 \in l\mathbb{Q}/l\mathbb{Z}$. Hence, the isomorphism is just a consequence of Construction 2.5.20. Suppose $D_P \in \ker(\varphi^*)$, i.e. $q_\Gamma(l_1) \in \mathbb{Z}^{|E(\Gamma)|}$ where l_1 is the length of $\overline{PP_0}$. Then l_1 is equal to $\frac{i}{ord(D_P)}l$ for an $i \in \{0, ..., ord(D_P) - 1\}$ and

$$q^e_\Gamma(l_1) = \begin{cases} d_e(\varphi)(1 - \frac{i}{ord(D_P)}), & \text{if } e \mapsto e' \subset \overline{PP_0}, \\ d_e(\varphi)\frac{i}{ord(D_P)}, & \text{if } e \mapsto e' \subset \overline{P_0P}, \end{cases} \in \mathbb{Z} \text{ for all } e \in E(\Gamma)$$

implies

$$ord(D_P) \,|\, d_e(\varphi) \text{ for all } e \in E(\Gamma).$$

Considering that $\sum_{e \mapsto e'} d_e(\varphi) = d$ holds for every edge e' of $\mathbb{T}E$, $ord(D_P)$ has to divide d as well. ☐

Remark 2.5.22. Note, that φ^* is injective whenever $q_\Gamma(\mathbb{T}E)$ does not contain any integer vector.

2.5.7 Criteria for Optimality.

Verifying optimality based on Definition 2.4.3 is tricky, in general. However, should the combinatorial type of Γ be the dumbbell-graph, then the structure of (not necessarily optimal) coverings of $\mathbb{T}E$ is particularly simple: They split into two "independent" isogenies. Lemma 2.5.23 gives a characterization of optimality in this case.

Lemma 2.5.23. Let $\varphi : \Gamma \to \mathbb{T}E$ be a degree d cover of dumbell type (Figure 2.8). Then φ factors through a non-trivial isogeny ϕ if and only if $gcd(d_\varphi(e_1), d_\varphi(e_2)) > 1$ or $gcd(n_1, n_2) > 1$ holds.

Proof. For the duration of this proof we will assume Γ to have the dumbbell-graph as combinatorial type and work in the notation of Lemma 2.5.18. We start with a structural observation: The map

$$\mathcal{F} : M := \{\varphi : \Gamma \to \mathbb{T}E\} \to \{(\varphi_1 : \mathbb{T}E_1 \to \mathbb{T}E, \varphi_2 : \mathbb{T}E_2 \to \mathbb{T}E)\}, \varphi \mapsto (\varphi_{|e_1}, \varphi_{|e_2})$$

is surjective with fibres $\mathcal{F}^{-1}(\varphi_1, \varphi_2) = \{\varphi_t : \Gamma_t \to \mathbb{T}E,\ t \in \mathbb{R}_{\geqslant 0}\}$, where Γ_t denotes the curve of type dumbbell obtained by connecting $\mathbb{T}E_i$ along an edge of length t. This procedure is not unique, but could be made so by forcing $\mathcal{F}$ to remember the positions of the vertices $P_0, P_1 \in \Gamma$. To avoid clutter, we do not mark points.
Now that we have identified the building blocks of a $\varphi \in M$, it becomes easy to construct a factorization of φ: Define isogenies

$$\phi : \mathbb{T}E' \to \mathbb{T}E \text{ and } \varphi'_i : \mathbb{T}E_i \to \mathbb{T}E',\ \text{for } i = 1, 2,$$

where $\mathbb{T}E'$ is the elliptic curve of length $l(e)$ with edge e and $\mathbb{T}E_i$ the one with edge e_i, by

$$n(\phi) := gcd(n_1, n_2), d_\phi(e) := gcd(d_\varphi(e_1), d_\varphi(e_2)), l(e) := \frac{n(\phi) l(\mathbb{T}E)}{d_\phi(e)},$$
$$n_i(\varphi'_i) := \frac{n_i(\varphi)}{n(\phi)}, d_{\varphi'_i}(e_i) := \frac{d_\varphi(e_i)}{d_\phi(e)}.$$

Set $\varphi' \in \mathcal{F}^{-1}(\varphi'_1, \varphi'_2)$ to be a cover whose source curve is Γ. We have $\varphi = \phi \circ \varphi'$ with $deg(\phi) = n(\phi) \cdot d_\phi(e)$, that is ϕ is non-trivial if and only if either $gcd(d_\varphi(e_1), d_\varphi(e_2)) > 1$ or

$gcd(n_1, n_2) > 1$ holds. We conclude the proof by observing that this factorization is maximal (in the sense that $deg(\phi)$ is maximal by construction), so that any other factorization of φ must factor through this one.

□

We could attempt a similar approach for the theta-graph and characterize the absence of optimality by the existence of a non-trivial solution to a system of equations analogous to, but more complex than, the one behind the previous proof. The following Proposition is the choice of a different path.

Proposition 2.5.24. A cover φ is tropically optimal if and only if $\ker(\varphi_*)$ is connected.

Proof. We know from Lemma 2.5.5 that factorizations of φ correspond to factorizations of the push-forward φ_*. Thus, consider

$$\begin{tikzcd}
\mathrm{Jac}(\Gamma) \arrow[r, "\tilde{\mu}"] \arrow[rr, bend right, "\varphi_*"'] & \mathbb{T}\tilde{E} \arrow[r, "\phi"] & \mathbb{T}E,
\end{tikzcd}$$

where $\tilde{\mu}$ is a morphism of tori and ϕ an isogeny. We make two observations:

1. Without loss of generality we can assume $\ker(\tilde{\mu})$ to be connected: Otherwise, decompose $\tilde{\mu}$ further as

$$\begin{tikzcd}
\mathrm{Jac}(\Gamma) \arrow[r, "\tilde{\tilde{\mu}}"] \arrow[rr, bend right, "\tilde{\mu}"'] \arrow[rrr, bend right=40, "\varphi_*"'] & \mathrm{Jac}(\Gamma)/\mathrm{Ker}(\tilde{\mu})_0 \arrow[r, "\tilde{\phi}"] & \mathbb{T}\tilde{E} \arrow[r, "\phi"] & \mathbb{T}E
\end{tikzcd}$$

   and replace $\tilde{\mu}$ by $\tilde{\tilde{\mu}}$ and ϕ by the isogeny $\tilde{\phi} \circ \phi$.

2. Assuming (1) (i.e. connectedness of the kernel) we know that $\ker(\tilde{\mu}) \subset \ker(\varphi_*)$ is a torus. More precisely, it is a subtorus of $\mathrm{Ker}(\varphi_*)_0$ of the same dimension. Since every 1-dimensional real torus is simple, we actually have an equality: $\ker(\tilde{\mu}) = \mathrm{Ker}(\varphi_*)_0$.

Combining observations (1) and (2) we see that any factorization of φ_* factors through the minimal map, $\pi_{\varphi_*} : \mathrm{Jac}(\Gamma) \to \mathrm{Jac}(\Gamma)/\mathrm{Ker}(\varphi_*)_0$ of the Stein factorization of φ_* (Lemma 2.2.10). Back to the statement of Proposition 2.5.24:
If $\ker(\varphi_*)$ is connected, then $\mathrm{Jac}(\Gamma)/\mathrm{Ker}(\varphi_*)_0 \cong \mathbb{T}E$ and $\pi_{\varphi_*} = \varphi_*$. In this case, we see that the minimal map is also maximal (Definition 2.2.9). Hence, φ must be optimal (again by Lemma 2.5.5).
Otherwise, the map $\mathrm{Jac}(\Gamma) \to \mathrm{Jac}(\Gamma)/\mathrm{Ker}(\varphi_*)_0$ gives rise to a non-trivial factorization of φ_* that contradicts optimality. This finishes the proof. □

Remark 2.5.25. The proof of Proposition 2.5.24 yields the following reformulation: A cover φ is tropically optimal if and only if the minimal factorization of φ_* is also maximal.

As a direct consequence we obtain:

Corollary 2.5.26. Any tropical cover $\varphi : \Gamma \to \mathbb{T}E$, where $g(\Gamma) = 2$, factors through an optimal cover.

This means: Optimal covers are not uncommon, but an integral part of every cover. While the first map, π_{φ_*}, in the Stein factorization of φ_* corresponds to the optimal cover described in Corollary 2.5.26, the second map, ϕ_{φ_*}, measures how far φ is from being optimal by counting the number of connected components of $\ker(\varphi_*)$.

2.6 From Covers to Curves: Tropical split Jacobians

We have seen: In genus 1, the curve and its Jacobian are (in some sense) indistinguishable and as 1-dimensional pptav easy to understand. As an increase in genus, however, entails an increase in complexity of the associated Jacobian, it seems natural to look for situations in which it is built of simpler objects.

Definition 2.6.1. Let Γ be a tropical curve of genus 2. We say that $\mathrm{Jac}(\Gamma)$ *splits*, if $\mathrm{Jac}(\Gamma)$ is isogeneous to the coproduct of two elliptic curve $\mathbb{T}E \oplus \mathbb{T}E'$. In this case we call $\mathbb{T}E'$ a *complement* of $\mathbb{T}E$ and vice versa.

This is a phenomenon in $\mathbb{T}\mathscr{A}$ that takes on the following form in $\mathbb{T}\mathscr{C}$.

Theorem 2.6.2. Let Γ be a tropical curve of genus 2. Then $\mathrm{Jac}(\Gamma)$ splits if and only if Γ covers an elliptic curve.

We will leave the proof of Theorem 2.6.2 to Subsection 2.6.5. At this point we only would like to suggest *why* this characterization might not be satisfactory: We can perturb a splitting of $\mathrm{Jac}(\Gamma)$ by isogenies of elliptic curves, leading to non-uniqueness of the complement or non-uniqueness of the cover, depending on the direction considered in Theorem 2.6.2.

Example 2.6.3. Suppose $\mathrm{Jac}(\Gamma)$ is isogeneous to $\mathbb{T}E \oplus \mathbb{T}E'$ via ϕ_1. Let $\mathbb{T}\tilde{E}$ be an elliptic curve of length $3 \cdot l_{\mathbb{T}E'}$ and $\phi_2 : \mathbb{T}E' \to \mathbb{T}\tilde{E}$ the corresponding isogeny of degree 3. Then $(id \oplus \phi_2) \circ \phi : \mathrm{Jac}(\Gamma) \to \mathbb{T}E \oplus \mathbb{T}\tilde{E}$ is (as composition of isogenies) an isogeny and $\mathbb{T}\tilde{E}$ another complement of $\mathbb{T}E$.

2.6.1 The canonical complement

Optimal covers improve the situation considerably and will set the scene for Subsections 2.6.1, 2.6.2 and 2.6.4. The exact setting is described in Theorem 2.6.4 and will be maintained throughout.

Theorem 2.6.4. If $\varphi : \Gamma \to \mathbb{T}E$ is an optimal cover of degree d, then $\mathbb{T}E' := \ker(\varphi_*)$ is a tropical elliptic curve and $\operatorname{Jac}(\Gamma)$ splits, i.e. there exists an isogeny $\phi : \mathbb{T}E' \bigoplus \mathbb{T}E \to \operatorname{Jac}(\Gamma)$ whose kernel is isomorphic to $\operatorname{Jac}_d(\mathbb{T}E)$.

Proof. We start by observing that optimal maps generate short exact sequences: Since φ is optimal, we know that $\mathbb{T}E' := \ker(\varphi_*)$ is a tropical abelian variety of dimension 1 (see Lemma 2.5.11) and even more, it is a tropical elliptic curve with principal polarization $\zeta_{\mathbb{T}E'}$ (see Remark 2.5.12). This means that we have an exact sequence of tropical abelian varieties

$$0 \to \mathbb{T}E' \xrightarrow{i} \operatorname{Jac}(\Gamma) \xrightarrow{\varphi_*} \operatorname{Jac}(\mathbb{T}E) \cong \mathbb{T}E \to 0,$$

where we used that for tropical elliptic curves the embedding $\mathbb{T}E \hookrightarrow \operatorname{Jac}(\mathbb{T}E)$, $P \mapsto P - Q_0$ for the choice of a point $Q_0 \in \mathbb{T}E$ is an isomorphism of rational polyhedral spaces ([69], Proposition 4.13). Picking Q_0 to be identity element, turns it into a morphism of tori. The dual sequence

$$0 \to \widecheck{\mathbb{T}E} \xrightarrow{\check{\varphi}_*} \widecheck{\operatorname{Jac}(\Gamma)} \xrightarrow{\check{i}} \widecheck{\mathbb{T}E'} \to 0$$

is also exact (see Lemma 2.3.8). Identifying $\widecheck{\mathbb{T}E'}$, $\widecheck{\operatorname{Jac}(\Gamma)}$ and $\widecheck{\mathbb{T}E}$ with their respective duals yields two exact sequences:

$$\begin{gathered} 0 \to \mathbb{T}E' \xrightarrow{i} \operatorname{Jac}(\Gamma) \xrightarrow{\varphi_*} \mathbb{T}E \to 0 \\ 0 \to \mathbb{T}E \xrightarrow{\varphi^*} \operatorname{Jac}(\Gamma) \xrightarrow{g} \mathbb{T}E' \to 0. \end{gathered} \tag{2.12}$$

Consider the coproduct $\mathbb{T}E' \bigoplus \mathbb{T}E$ in the category of tropical abelian varieties (see Definition 2.3.5) and let ϕ be the unique map making the diagram

$$\begin{tikzcd}
 & \operatorname{Jac}(\Gamma) & \\
\mathbb{T}E' \arrow[ur, "i"] \arrow[r] & \mathbb{T}E' \bigoplus \mathbb{T}E \arrow[u, dashed, "\phi"] & \mathbb{T}E \arrow[ul, "\varphi_*"'] \arrow[l]
\end{tikzcd}$$

commute. We claim that the morphism ϕ which is defined by $(P', P) \mapsto i(P') + \varphi^*(P)$ is an isogeny, i.e. surjective and finite. From the two sequences (2.12) we see that $\operatorname{Jac}(\Gamma)$ is the sum (in the usual sense, not the direct sum!) of two elliptic curves:

$$\operatorname{Jac}(\Gamma) \cong \operatorname{Jac}(\Gamma)/\ker(\varphi_*) + \ker(\varphi_*) \cong \operatorname{Im}(\varphi^*) + \operatorname{Im}(i).$$

This yields another exact sequence (now of abelian groups!):

$$0 \to \operatorname{Im}(\varphi^*) \cap \operatorname{Im}(i) \xrightarrow{f_1} \operatorname{Im}(\varphi^*) \bigoplus \operatorname{Im}(i) \xrightarrow{f_2} \operatorname{Jac}(\Gamma) \to 0, \tag{2.13}$$

where the map f_1 is given by $x \mapsto (x, -x)$ and f_2 is the usual addition. Recalling that i and φ^* are injective, identifies sequence (2.13) with the exact sequence associated to ϕ (as group homomorphism in the category of abelian groups). This proves surjectivity. Finally, since

$$\ker(\phi) \cong \operatorname{Im}(\varphi^*) \cap \operatorname{Im}(i) = \operatorname{Im}(\varphi^*) \cap \ker(\varphi_*) \cong \ker(\varphi_* \varphi^*) = \operatorname{Jac}_d(\mathbb{T}E)$$

is finite, ϕ is finite. $\square$

To avoid working exclusively in the abstract category of tav, we translate the proof into the explicit language of matrices.

Example 2.6.5. Let φ be the optimal cover from Example 2.4.4. Recall that the kernel of φ_* is an elliptic curve of length 1 with lattice representation given in terms of the base choice from Convention 2.5.10 (see also Lemma 2.5.11)

$$(\langle \omega := \omega_1 - \omega_2 \rangle, \langle B := B_2 \rangle, [\cdot, \cdot]_K).$$

Set $\mathbb{T}E' := \ker(\varphi_*)$. We already have a coordinate description of φ_*, let us now determine one for the natural inclusion $i : \mathbb{T}E' \hookrightarrow \operatorname{Jac}(\Gamma)$: As a map on the underlying tori, i is induced by

$$\operatorname{Hom}(i^\#) : \operatorname{Hom}(\langle \omega \rangle, \mathbb{R}) \to \operatorname{Hom}(\langle \omega_1, \omega_2 \rangle, \mathbb{R}),$$

where $i^\# : \langle \omega_1, \omega_2 \rangle \twoheadrightarrow \langle \omega_1, \omega_2 \rangle / \langle 2\omega_1 - \omega_2 \rangle \cong \langle \omega \rangle$ is the quotient map. For $f \in \operatorname{Hom}(\langle \omega \rangle, \mathbb{R})$ we have $f \circ i^\#(\omega_1) = f(\omega)$ and $f \circ i^\#(\omega_2) = 2f(\omega)$ (since $\omega_1 = \omega_1 - (2\omega_1 - \omega_2) = \omega$ in the quotient, analogously for ω_2), which makes the first exact sequence in (2.12) of the preceding proof explicit:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{T}E' \arrow[r, "i"] \arrow[d, "\cong"] & \operatorname{Jac}(\Gamma) \arrow[r, "\varphi_*"] \arrow[d, "\cong"] & \mathbb{T}E \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathbb{R}/\mathbb{Z} \arrow[r, "\begin{pmatrix} 1 \\ 2 \end{pmatrix}"] & \mathbb{R}^2 / \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbb{Z}^2 \arrow[r, "\begin{pmatrix} 2 & -1 \end{pmatrix}"] & \mathbb{R}/3\mathbb{Z} \arrow[r] & 0.
\end{tikzcd} \tag{2.14}$$

We proceed in the same way for the second and obtain

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{T}E \arrow[r, "\varphi^*"] \arrow[d, "\cong"] & \operatorname{Jac}(\Gamma) \arrow[r, "g"] \arrow[d, "\cong"] & \mathbb{T}E' \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathbb{R}/3\mathbb{Z} \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & \mathbb{R}^2 / \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbb{Z}^2 \arrow[r, "\begin{pmatrix} 0 & 1 \end{pmatrix}"] & \mathbb{R}/\mathbb{Z} \arrow[r] & 0.
\end{tikzcd} \tag{2.15}$$

By joining the left parts of the both sequences we obtain a coordinate description for the isogeny ϕ:

$$\begin{tikzcd}
\mathbb{T}E' \oplus \mathbb{T}E \arrow[r, "\phi"] \arrow[d, "\cong"] & \mathrm{Jac}(\Gamma) \arrow[d, "\cong"] \\
\mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/3\mathbb{Z} \arrow[r, "\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}"] & \mathbb{R}^2 / \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbb{Z}^2
\end{tikzcd} \tag{2.16}$$

whose kernel $\ker(\phi) = \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix} \} \cong \mathbb{R}/3\mathbb{Z}[2]$.

2.6.2 The complementary cover

The proof of Theorem 2.6.4 reveals a certain kind of symmetry: In terms of exact sequences $\mathbb{T}E'$ is related to $\mathrm{Jac}(\Gamma)$ in the same way as $\mathbb{T}E$ is. Hence, it is natural to ask, whether this symmetry does not hold at a more fundamental level, i.e. whether the sequences

$$\begin{gathered} 0 \to \mathbb{T}E' \xrightarrow{i} \mathrm{Jac}(\Gamma) \xrightarrow{\varphi_*} \mathbb{T}E \to 0 \\ \\ 0 \to \mathbb{T}E \xrightarrow{\varphi^*} \mathrm{Jac}(\Gamma) \xrightarrow{g} \mathbb{T}E' \to 0 \end{gathered} \tag{2.17}$$

are generated by an optimal map as well. To this end, consider:

Construction 2.6.6. Let Γ be of genus 2 with labeling as fixed in Figure 2.6 and Figure 2.8, depending on whether Γ is of type theta or of type dumbbell. Let us define $\varphi' : \Gamma \to \mathbb{T}E'$ as the composition

$$\Gamma \xrightarrow{\Phi_{P_0}} \mathrm{Jac}(\Gamma) \overset{f_{\zeta_\Gamma}}{\cong} \widecheck{\mathrm{Jac}(\Gamma)} \xrightarrow{\check{i}} \widecheck{\mathbb{T}E'} \overset{f_{\zeta_{\mathbb{T}E'}}}{\cong} \mathbb{T}E',$$

where Φ_{P_0} denotes the Abel-Jacobi map with base point $P_0 \in \Gamma$.

The following Lemma paves the way for a reinterpretation of the sequences (2.17) in terms of φ'.

Lemma 2.6.7. The map $\varphi' : \Gamma \to \mathbb{T}E'$ from Construction 2.6.6 is a tropical cover.

Proof of Lemma 2.6.7. We prove the statement in two steps. In step 1 we deal with $\mathbb{T}E'$ in its incarnation as the tav $\ker(\varphi_*)$ whose lattice representation is (see Definition 2.3.4)

$$(K, K', [\cdot,\cdot]_K) := (\Omega^1_\Gamma / \mathrm{Im}(\varphi_*{}^\#)^{\mathrm{sat}}, \ker(\varphi_{*\#}), [\cdot,\cdot]_K)$$

and show that φ' (now viewed as map to a tropical torus) is *tropical*, i.e. satisfies the following properties:

1. φ' is continuous and restricts to an affine function on each edge.

2. the differential $D\varphi'_P$ at $P \in \Gamma$ takes integral tangent vectors to integral tangent vectors.

3. φ' is harmonic.

For step 2, we turn to $\mathbb{T}E'$ as metric graph and recognize in φ' an actual morphism of graphs, a tropical cover.
Note that the first part of (1), φ' is continuous, is an immediate consequence of the continuity of the Abel-Jacobi map Φ_{P_0} (Theorem 4.1, [6]) since f_{ζ_Γ}, $f_{\zeta_{\mathbb{T}E'}}$ and $\check{i}$ are obviously continuous as morphisms. For the second part we utilize the local description of the lift $\widetilde{\Phi}_{P_0}$ of the Abel-Jacobi map to $\mathrm{Hom}(\Omega^1_\Gamma, \mathbb{R})$ given in [6] (Theorem 4.1) to determine a description of the lift $\tilde{\varphi}'$ of φ' to $\mathrm{Hom}(K, \mathbb{R})$. That is the map given by the upper path in the following diagram:

$$\begin{tikzcd}
\Gamma \arrow[r, "\widetilde{\Phi}_{P_0}"] \arrow[dr, "\Phi_{P_0}"] & \mathrm{Hom}(\Omega^1_\Gamma, \mathbb{R}) \arrow[r, "\mathrm{Hom}(\zeta_\Gamma)"] \arrow[d, "\pi_1"] & \mathrm{Hom}(\mathrm{H}_1(\Gamma, \mathbb{Z}), \mathbb{R}) \arrow[r, "\mathrm{Hom}(i_\#)"] \arrow[d, "\pi_2"] & \mathrm{Hom}(K', \mathbb{R}) \arrow[r, "\mathrm{Hom}(\zeta_{\mathbb{T}E'})"] \arrow[d, "\pi_3"] & \mathrm{Hom}(K, \mathbb{R}) \arrow[d, "\pi_4"] \\
 & \mathrm{Jac}(\Gamma) \arrow[r, "f_{\zeta_\Gamma}"] & \widecheck{\mathrm{Jac}}(\Gamma) \arrow[r, "\check{i}"] & \widecheck{\ker}(\varphi_*) \arrow[r, "f_{\zeta_{\mathbb{T}E'}}"] & \ker(\varphi_*)
\end{tikzcd}$$

where π_i for $i = 1, ..., 4$ are the canonical projections. Let $\hat{e}$ be an oriented edge with parametrization

$$[0, l(\hat{e})] \ni t \mapsto S + t \cdot v_{\hat{e}},$$

where S is the initial vertex and $v_{\hat{e}}$ a primitiv integral tangent vector at some interior point $P \in \hat{e}$. Then $\widetilde{\Phi}_{P_0}$ is given by the affine map

$$[0, l(\hat{e})] \ni t \mapsto \widetilde{\Phi}_{P_0}(S) + t \cdot (\frac{1}{l(\hat{e})} \int_{\hat{e}}) \in \mathrm{Hom}(\Omega^1_\Gamma, \mathbb{R})$$

and $\tilde{\varphi}'$ by

$$[0, l(\hat{e})] \ni t \mapsto \widetilde{\Phi}_{P_0}(S) \circ \zeta_\Gamma \circ i_\# \circ \zeta_{\mathbb{T}E'} + t \cdot (\frac{1}{l(\hat{e})} \int_{\hat{e}} \zeta_\Gamma \circ i_\# \circ \zeta_{\mathbb{T}E'}) \in \mathrm{Hom}(K, \mathbb{R}).$$

We (need to) show that (the image of) $(\frac{1}{l(\hat{e})} \int_{\hat{e}} \zeta_\Gamma \circ i_\# \circ \zeta_{\mathbb{T}E'})$ (under π_4) is an integral tangent vector of $\ker(\varphi_*)$, i.e. (that the lift is) an element of the dual lattice $\mathrm{Hom}(K, \mathbb{Z})$. Since $\zeta_{\mathbb{T}E'} : K' \to K$ is an isomorphism, it suffices to verify that

$$\frac{1}{l(\hat{e})} \int_{\hat{e}} \zeta_\Gamma \circ i_\#(\cdot) \in \mathrm{Hom}(K', \mathbb{Z})$$

holds. Recalling that $K' = \langle B \rangle_{\mathbb{Z}}$ with $B = w_1 B_1 + w_2 B_2$ and $w_1, w_2 \in \mathbb{Z}$ yields:

$$\frac{1}{l(\hat{e})} \int_{\hat{e}} \zeta_\Gamma \circ i_\#(B) = \frac{1}{l(\hat{e})} \int_{\hat{e}} (w_1(de + de_2) + w_2(de_2 - de_1)) \\ = \begin{cases} w_1, & \hat{e} = e \\ -w_2, & \hat{e} = e_1 \\ w_1 + w_2, & \hat{e} = e_2 \end{cases}, \tag{2.18}$$

whenever Γ is a theta-graph (Lemma 2.5.11 and notation therein), and

$$\frac{1}{l(\hat{e})} \int_{\hat{e}} \zeta_\Gamma \circ i_\#(B) = \frac{1}{l(\hat{e})} \int_{\hat{e}} (w_1(de_1) + w_2(de_2) \\ = \begin{cases} w_1, & \hat{e} = e_1 \\ w_2, & \hat{e} = e_2 \\ 0, & \hat{e} = e \end{cases}, \tag{2.19}$$

whenever Γ is a dumbbell-graph (Lemma 2.5.18 and notation therein). The maps $\frac{1}{l(\hat{e})} \int_{\hat{e}} \zeta_\Gamma \circ i_\#(\cdot)$ for $\hat{e} \in E(\Gamma)$ take integer values on a lattice base of K' and are therefore elements of $\mathrm{Hom}(K', \mathbb{Z})$. This proves (1) and (2).

Computations (2.18) and (2.19) also provide a coordinate description of the differential

$$D\varphi'_P : T_P(\Gamma) \to T_{\varphi'(P)}(\ker(\varphi_*))$$

of φ' at an interior point P of $\hat{e}$. Consider, for example, (2.18): A generator $v_{\hat{e}}$ of $T_P(\Gamma)$ is sent to $w_1, -w_2$ or $w_1 + w_2$, depending on whether $\hat{e}$ is (equal to) e, e_1 or e_2. With this one easily checks that φ' is balanced at the vertices of Γ, exemplarily for P_1 we have:

$$\sum_\lambda D\varphi'_{P_1}(\lambda) = w_1 + w_2 - (w_1 + w_2) = 0,$$

where the first sum is over all primitive integral tangent vectors λ at P_1, that is $\lambda = v_e, -v_{e_1}, -v_{e_2}$. The verification of the balancing condition (using (2.19)), when Γ is a dumbbell-graph, is analogous.

For step 2 we turn φ' into a morphism of metric graphs by viewing $\ker(\varphi'_*)$ as elliptic curve $\mathbb{T}E'$ with vertex set $V(\mathbb{T}E') := \{\varphi'(P_0), \varphi'(P_1)\}$. Since φ' is non-constant and harmonic at all vertices of Γ, it must be surjective (see [3], Remark 2.7) and therefore a tropical cover. ☐

Remark 2.6.8. Note that φ' is not necessarily finite (see Example 2.6.10).

Lemma 2.6.7 provides us with a second cover. From this perspective Theorem 2.6.4 reads as follows:

Theorem 2.6.9. In the setting of Theorem 2.6.4, there exists another cover $\varphi' : \Gamma \to \mathbb{T}E'$ such that

- φ' also generates the exact sequences (2.17), i.e. $i = \varphi'^*$ and $g = \varphi'_*$.
- φ' is optimal and of degree d.

Moreover, the isogeny $\phi : \mathbb{T}E' \bigoplus \mathbb{T}E \to \mathrm{Jac}(\Gamma)$ satisfies $\ker(\phi) \cong Jac_d(\mathbb{T}E')$ and is polarized with respect to f_{ζ_Γ} and $m_d \circ f_{\zeta_{\mathbb{T}E' \bigoplus \mathbb{T}E}}$ (see Definition 2.3.2), where $m_d : \mathbb{T}E' \bigoplus \mathbb{T}E \to \mathbb{T}E' \bigoplus \mathbb{T}E$ denotes the multiplication-by-d map and ζ_Γ, respectively $\zeta_{\mathbb{T}E' \bigoplus \mathbb{T}E}$, are the usual principal polarization on $\mathrm{Jac}(\Gamma)$ and $\mathbb{T}E' \bigoplus \mathbb{T}E$ (see Definition 2.3.5).

Proof. Let φ' be the cover from Construction 2.6.6. The notions of push-forward and pull-back morphisms that naturally come with it, make it worthwhile to re-examine the exact sequences (2.17). First, note that their labelling with φ^* and φ_* is not quite precise: We should actually replace $\mathbb{T}E$ with $\mathrm{Jac}(\mathbb{T}E)$, but have omitted the isomorphism $\mathbb{T}E \cong \mathrm{Jac}(\mathbb{T}E)$ to avoid cluttering notation. The same applies to the desired reinterpretation of i and g in terms φ'. We specify an embedding of $\mathbb{T}E'$ into its Jacobian via

$$j : \mathbb{T}E' \to \mathrm{Jac}(\mathbb{T}E'),\ P \mapsto P - 0_{\mathrm{Jac}(\mathbb{T}E')},$$

where $0_{\mathrm{Jac}(\mathbb{T}E')}$ is the identity element. Again, by abuse of notation we call g the map between the respective Jacobians to get a commutative diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "\Phi_{P_0}"] \arrow[d, "\varphi'"] & \mathrm{Jac}(\Gamma) \arrow[d, "g"] \\
\mathbb{T}E' \arrow[r, hook, "j"] & \mathrm{Jac}(\mathbb{T}E'),
\end{tikzcd} \tag{2.20}$$

which proves $g = \varphi'_*$. Turning towards Lemma 2.5.4 next, provides us with the right interpretation for i as pull-back of φ' and seeing that $\ker(\varphi'_*)$ is connected (exactness of the second sequence in (2.17)), ultimately shows that φ' is optimal (Lemma 2.5.24).
As tropical cover φ' satisfies

$$\varphi'_* \circ \varphi'^* = deg(\varphi')id.$$

Then $deg(\varphi')$ can be computed as $|\ker(\varphi'_* \circ \varphi'^*)|$. Using the reinterpretation of i and g in terms of φ' to interchange the roles of φ and φ' in the proof of Theorem 2.6.4, we obtain $\ker(\phi) \cong \ker(\varphi'_* \circ \varphi'^*)$ and thus $deg(\varphi') = d$ and $\ker(\phi) \cong \mathrm{Jac}_d(\mathbb{T}E')$, simultaneously.
We conclude by investigating the behaviour of ϕ with respect to the pp ζ_Γ and $\zeta_{\mathbb{T}E' \bigoplus \mathbb{T}E}$: Since ϕ is finite, it induces a polarization $\phi^*(\zeta_\Gamma)$ on $\mathbb{T}E' \bigoplus \mathbb{T}E$ that makes the diagram

$$\begin{tikzcd}
\mathrm{H}_1(E', \mathbb{Z}) \bigoplus \mathrm{H}_1(E, \mathbb{Z}) \arrow[r, "\phi_\#"] \arrow[d, "\phi^*(\zeta_\Gamma)"] & \mathrm{H}_1(\Gamma, \mathbb{Z}) \arrow[d, "\zeta_\Gamma"] \\
\Omega^1_{E'}(\mathbb{Z}) \bigoplus \Omega^1_E(\mathbb{Z}) & \Omega^1_\Gamma(\mathbb{Z}) \arrow[l, "\phi^\#"']
\end{tikzcd} \tag{2.21}$$

commute. Using that

- $\phi_{\#} = \varphi'_{*,\#} \oplus \varphi_{*,\#}$ is the composition of $(\varphi'_{*,\#}, \varphi_{*,\#})$ with the addition-of-components map,
- $\phi^{\#} = (\varphi'^{*,\#}, \varphi^{*,\#})$,
- $\varphi'_* \circ \varphi'^*$ and $\varphi^* \circ \varphi_*$ are multiplication-by-d maps on $\mathbb{T}E'$ and $\mathbb{T}E$,

together with exactness of (2.17), yields

$$\phi^*(\zeta_\Gamma)(\sum_{e'} a_{e'}e', \sum_e a_e e) = d \cdot (\sum_{e'} a_{e'}de', \sum_e a_e de)$$

as claimed. □

Example 2.6.10. Let us retrace Subsection 2.6.2 by continuing with Example 2.6.5. The natural principal polarization ζ_Γ provides us with an identification of $\mathrm{Jac}(\Gamma)$ with its dual. Due to our choice of basis for $\mathrm{H}_1(\Gamma, \mathbb{Z})$ and Ω^1_Γ (Convention 2.5.10), its representation matrix is simply I_2, analogously for $f_{\zeta_{\mathbb{T}E'}}$. Then φ' (as a tropical map, here we view its target as a torus and not as a curve) is given by the composition:

$$\begin{tikzcd}
\Gamma \arrow[r, hook, "\Phi_{P_0}"] & \mathrm{Jac}(\Gamma) \arrow[r, "f_{\zeta_\Gamma}"] \arrow[d, "\cong"] & \widecheck{\mathrm{Jac}}(\Gamma) \arrow[r, "\check{i}"] \arrow[d, "\cong"] & \widecheck{\mathbb{T}E'} \arrow[r, "f_{\zeta_{\mathbb{T}E'}}"] \arrow[d, "\cong"] & \mathbb{T}E' \arrow[d, "\cong."] \\
 & \mathbb{R}^2/\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbb{Z}^2 \arrow[r, "I_2"] & \mathbb{R}^2/\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbb{Z}^2 \arrow[r, "(0\ 1)"] & \mathbb{R}/\mathbb{Z} \arrow[r, "1"] & \mathbb{R}/\mathbb{Z}
\end{tikzcd} \tag{2.22}$$

We want to understand φ' as harmonic morphism of metric graphs next. By fixing P_0 as in Figure 2.10 (on the right), we obtain a representation of Γ inside its Jacobian (see Figure 2.10 on the left). Equipped with coordinates for the points of Γ we can simply compute $\varphi'(V(\Gamma))$ by

$$\varphi'(P_0) = 0 \text{ and } \varphi'(P_1) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \in \mathbb{R}/\mathbb{Z}$$

and set $V(\mathbb{T}E') := \{P := \varphi'(P_0)\}$. To determine the local descriptions of φ' we proceed as in the proof of Lemma 2.6.7. Since $B = B_2$, we have:

$$\varphi'_{|e} : [0,1] \ni t \mapsto P, \varphi'_{|e_1} : [0,1] \ni t \mapsto P - t \text{ and } \varphi'_{|e_2} : [0,1] \ni t \mapsto P + t.$$

These encode *how* Γ covers $\mathbb{T}E'$, so we see that φ'

- contracts e, i.e. that $d_e(\varphi') = 0$.
- maps e_i to e without any dilation (but reversing the orientation of e_1).

- is of degree 2.

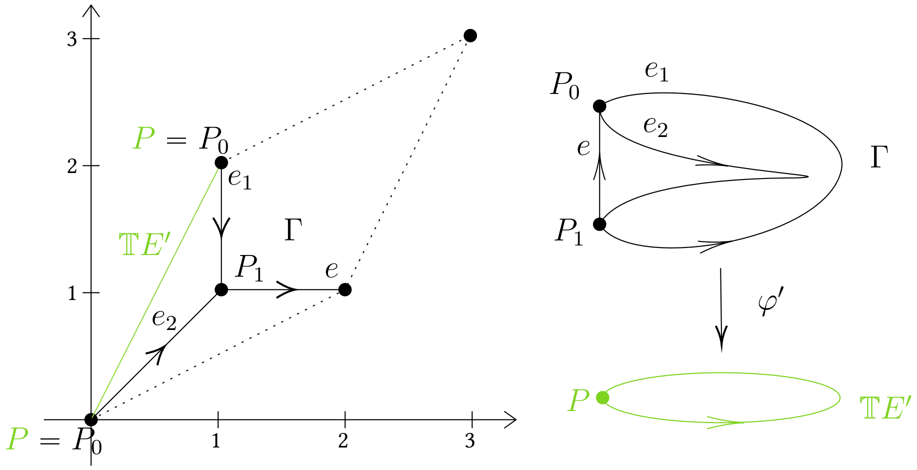


Figure 2.10: φ' as tropical cover on the left and as harmonic morphism of graphs on the right.

Recall from Example 2.6.5 that $\ker(\phi) = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}\}$. By projecting onto the first coordinate we see $\ker(\phi) \cong \mathbb{R}/\mathbb{Z}[2]$ and $\ker(\phi) \cong \mathbb{R}/3\mathbb{Z}[2]$ by projecting onto the second.

2.6.3 Algorithm for φ'

Identify a cover with the discrete data it is determined by, i.e. if Γ is of

- Type theta with: a triple of *winding numbers* $(n, n_1, n_2) \in \mathbb{N}_0^3$ and *dilation factors* $(d_e(\varphi), d_{e_1}(\varphi), d_{e_2}(\varphi)) \in \mathbb{N}_0^3$ such that

$$\begin{aligned} n d_e(\varphi) + (n_1 - 1) d_{e_1}(\varphi) + (n_2 - 1) d_{e_2}(\varphi) &= d, \\ (n-1) d_e(\varphi) + n_1 d_{e_1}(\varphi) + n_2 d_{e_2}(\varphi) &= d. \end{aligned}$$

- Type dumbbell with: a tuple of *winding numbers* $(n_1, n_2) \in \mathbb{N}_0^2$ and *dilation factors* $(d_{e_1}(\varphi), d_{e_2}(\varphi)) \in \mathbb{N}_0^2$ such that

$$n_1 d_{e_1}(\varphi) + n_2 d_{e_2}(\varphi) = d.$$

Given an optimal cover, we can compute the complementary cover:

Input: An optimal cover $\varphi : \Gamma \to \mathbb{T}E$.
Output: The complementary cover $\varphi' : \Gamma \to \mathbb{T}E'$.

1. Determine $\ker(\varphi_*)$ as described in Lemma 2.5.11, if Γ is of type theta, or Lemma 2.5.18, if Γ is of type dumbbell. Set $\mathbb{T}E' := \ker(\varphi_*)$.
2. Determine local dilation factors as described in Lemma 2.6.7 in computation (2.18), if Γ is of type theta, or computation (2.19), if Γ is of type dumbbell.
3. Determine winding numbers by using that φ' is locally integer affine linear: Each edge gives rise to an equation involving dilation factors, winding numbers, and the metric data of Γ and $\mathbb{T}E'$.

2.6.4 From Covers to Curves ultimately

Bringing together the two perspectives, that of φ and that of φ', finally reveals the symmetry hidden in the proof of Theorem 2.6.4: Given an optimal cover φ, we have constructed another optimal cover, φ', whose interaction with φ (through push-forward and pull-back of divisors) generates two so-called *complementary* exact sequences

$$0 \to \mathbb{T}E' \xrightarrow{\varphi'^*} \operatorname{Jac}(\Gamma) \xrightarrow{\varphi_*} \mathbb{T}E \to 0$$
$$0 \to \mathbb{T}E \xrightarrow{\varphi^*} \operatorname{Jac}(\Gamma) \xrightarrow{\varphi'_*} \mathbb{T}E' \to 0.$$

The pair of maps, (φ'^*, φ^*) and (φ'_*, φ_*), in turn give rise to isogenies, ϕ and $\tilde{\phi}$, by utilizing the coproduct, respectively the product, property of $\mathbb{T}E' \bigoplus \mathbb{T}E$ (recall that finite products coincide with finite coproducts in $\mathbb{T}\mathscr{A}$). We capture the pattern of their interactions in the diagram below:

$$\begin{tikzcd}
\mathbb{T}E' \arrow[r, hook, "\iota_1"] \arrow[dd, "m_d"] \arrow[dr, "\varphi'_*"] & \mathbb{T}E' \bigoplus \mathbb{T}E \arrow[d, dashed, "\phi"] & \mathbb{T}E \arrow[l, hook', "\iota_1"'] \arrow[dd, "m_d"] \arrow[dl, "\varphi^*"'] \\
 & \operatorname{Jac}(\Gamma) \arrow[dl, "\varphi'_*"'] \arrow[dr, "\varphi_*"] \arrow[d, dashed, "\tilde{\phi}"] & \\
\mathbb{T}E' & \mathbb{T}E' \bigoplus \mathbb{T}E \arrow[l, "p_1"] \arrow[r, "p_2"] & \mathbb{T}E
\end{tikzcd} \tag{2.23}$$

where ι_i and p_i are the canonical injections, respectively projections. The diagonals are formed by our exact sequences and a small diagram-chase shows that $\tilde{\phi} \circ \phi$ is the componentwise multiplication-by-d map.

Example 2.6.11. The following diagram summarizes the interaction of the two optimal covers φ and φ' from Example 2.6.10 (see Diagram 2.23):

$$\begin{tikzcd}
\mathbb{R}/\mathbb{Z} \arrow[r, hook, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[dd, "\cdot 2"'] \arrow[dr, "\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)"] & \mathbb{R}/\mathbb{Z}\bigoplus\mathbb{R}/3\mathbb{Z} \arrow[d, "\left(\begin{smallmatrix}1&1\\2&0\end{smallmatrix}\right)"] & \mathbb{R}/3\mathbb{Z} \arrow[l, hook', "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] \arrow[dl, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] \arrow[dd, "\cdot 2"] \\
 & \mathbb{R}^2/\begin{pmatrix}2&1\\1&2\end{pmatrix}\mathbb{Z}^2 \arrow[dl, "\left(\begin{smallmatrix}0&1\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}0&1\\2&-1\end{smallmatrix}\right)"] \arrow[dr, "\left(\begin{smallmatrix}2&-1\end{smallmatrix}\right)"] & \\
\mathbb{R}/\mathbb{Z} & \mathbb{R}/\mathbb{Z}\bigoplus\mathbb{R}/3\mathbb{Z} \arrow[l, "\left(\begin{smallmatrix}1&0\end{smallmatrix}\right)"] \arrow[r, "\left(\begin{smallmatrix}0&1\end{smallmatrix}\right)"'] & \mathbb{R}/3\mathbb{Z}
\end{tikzcd} \tag{2.24}$$

2.6.5 Proof of Theorem 2.6.2

We use methods and results from Subsection 2.6.1 and 2.6.2 to complete the proof of Theorem 2.6.2.

Proof of Theorem 2.6.2. Suppose $\mathrm{Jac}(\Gamma)$ splits, i.e. there exists an isogeny

$$\phi : \mathrm{Jac}(\Gamma) \to \mathbb{T}E \oplus \mathbb{T}E'.$$

We have to show that Γ covers an elliptic curve. To this end, define $\varphi : \Gamma \to \mathbb{T}E$ (in analogy to the classical case, see [36]) as the composition

$$\Gamma \xrightarrow{\Phi_{P_0}} \mathrm{Jac}(\Gamma) \xrightarrow{\phi} \mathbb{T}E \oplus \mathbb{T}E' \xrightarrow{p_1} \mathbb{T}E,$$

where p_1 is the projection onto the first factor. To realize that φ is a tropical cover we proceed exactly as described in the proof of Lemma 2.6.7. We only mention the points where adjustments have to be made. A lift $\tilde{\varphi}$ of φ is given by the upper path in the following diagram:

$$\begin{tikzcd}
\Gamma \arrow[r, "\widetilde{\Phi}_{P_0}"] \arrow[dr, "\Phi_{P_0}"] & \mathrm{Hom}(\Omega^1_\Gamma, \mathbb{R}) \arrow[r, "\mathrm{Hom}(\phi^\#)"] \arrow[d, "\pi_1"] & \mathrm{Hom}(\Omega^1_{\mathbb{T}E} \oplus \Omega^1_{\mathbb{T}E'}, \mathbb{R}) \arrow[r, "\mathrm{Hom}(p_1^\#)"] \arrow[d, "\pi_2"] & \mathrm{Hom}(\Omega^1_{\mathbb{T}E}, \mathbb{R}) \arrow[d, "\pi_3"] \\
 & \mathrm{Jac}(\Gamma) \arrow[r, "\phi"] & \mathbb{T}E \oplus \mathbb{T}E' \arrow[r, "p_1"] & \mathbb{T}E
\end{tikzcd} \tag{2.25}$$

where π_i for $i = 1, 2, 3$ are the canonical projections. To verify that

$$\frac{1}{l(\hat{e})} \int_{\hat{e}} \phi^\# \circ p_1^\#(\cdot) \in \mathrm{Hom}(\Omega^1_{\mathbb{T}E}, \mathbb{Z})$$

holds, let ω be a generator of $\Omega^1_{\mathbb{T}E}$ and let

$$\phi^{\#} \circ p_1^{\#}(\omega) = \phi^{\#}(\omega, 0) = \tilde{w}_1\omega_1 + \tilde{w}_2\omega_2$$

for some $\tilde{w}_1, \tilde{w}_2 \in \mathbb{Z}$ be basis representation. Finally, replace w_i in computations (2.18) and (2.19) by $\tilde{w}_i$ and note that this does not make any difference to the remaining arguments. For the other direction, suppose Γ covers an elliptic curve $\mathbb{T}E$. By Corollary 2.5.26 we know that any cover $\varphi : \Gamma \to \mathbb{T}E$ factors through an optimal cover, say φ'':

$$\begin{tikzcd}
\Gamma \arrow[r, "\varphi''"] \arrow[rr, bend right, "\varphi"'] & \mathbb{T}E'' \arrow[r, "\phi''"] & \mathbb{T}E.
\end{tikzcd}$$

Then, by Theorem 2.6.4 $\mathrm{Jac}(\Gamma)$ splits. $\square$

Chapter 3

Building blocks and characterization in A_2^{tr} and M_2^{tr}

3.1 Introduction

Embedded in the broader context of exploring connections between the category of tropical abelian varieties (tav), $\mathbb{T}\mathcal{A}$, and the category of tropical curves, $\mathbb{T}\mathcal{C}$, Project 1 (Chapter 2) ([32]) is a *holistic analysis*: Split Jacobians may appear either in the form of 2 dimensional abelian varieties that decompose into a product of two tropical elliptic curves, or as a pair of tropical optimal coverings.

Theorem 3.1.1. ([32], Theorem 61 and Theorem 66) Let Γ be a tropical curve of genus 2. For an optimal pair $(\mathbb{T}E, \varphi)$, that is a pair consisting of a tropical elliptic curve $\mathbb{T}E$ and a tropical optimal cover $\varphi : \Gamma \to \mathbb{T}E$, there exists another optimal pair $(\mathbb{T}E', \varphi')$ such that the tropical Jacobian of Γ, $\mathrm{Jac}(\Gamma)$, splits. Moreover, the splitting of $\mathrm{Jac}(\Gamma)$ is given by an isogeny

$$\phi : \mathbb{T}E' \bigoplus \mathbb{T}E \to \mathrm{Jac}(\Gamma),$$

whose kernel satisfies $\mathrm{Jac}_d(\mathbb{T}E') \cong \ker(\phi) \cong \mathrm{Jac}_d(\mathbb{T}E)$. In this case $\mathrm{Jac}(\Gamma)$ is said to be d-split and $(\mathbb{T}E, \mathbb{T}E', \ker(\phi))$ is referred to as splitting data.

More precisely, through push-forward and pull-back of divisors φ and φ' generate two exact sequences

$$0 \to \mathbb{T}E' \xrightarrow{\varphi'^*} \mathrm{Jac}(\Gamma) \xrightarrow{\varphi_*} \mathbb{T}E \to 0$$
$$0 \to \mathbb{T}E \xrightarrow{\varphi^*} \mathrm{Jac}(\Gamma) \xrightarrow{\varphi'_*} \mathbb{T}E' \to 0.$$

The pair of maps, (φ'^*, φ^*) and (φ'_*, φ_*), in turn give rise to isogenies, ϕ and $\tilde{\phi}$, by utilizing the coproduct, respectively the product, property of $\mathbb{T}E' \bigoplus \mathbb{T}E$ (recall that finite products coincide with finite coproducts in $\mathbb{T}\mathcal{A}$) and interact as follows:

$$\begin{tikzcd}
\mathbb{T}E' \arrow[r, hook, "\iota_1"] \arrow[dr, "{\varphi'}^*"] \arrow[dd, "m_d"] & \mathbb{T}E' \bigoplus \mathbb{T}E \arrow[d, dashed, "\phi"] & \mathbb{T}E \arrow[l, hook', "\iota_2"'] \arrow[dl, "\varphi^*"'] \arrow[dd, "m_d"] \\
 & \mathrm{Jac}(\Gamma) \arrow[dl, "{\varphi'}_*"'] \arrow[d, dashed, "\tilde{\phi}"] \arrow[dr, "\varphi_*"] & \\
\mathbb{T}E' & \mathbb{T}E' \bigoplus \mathbb{T}E \arrow[l, "p_1"] \arrow[r, "p_2"'] & \mathbb{T}E
\end{tikzcd} \tag{3.26}$$

where ι_i and p_i are the canonical injections, respectively projections. The diagonals are formed by our exact sequences and a small diagram-chase shows that $\tilde{\phi} \circ \phi$ is the componentwise multiplication-by-d map.

Project 2 (Chapter 3) opens with an *atomic perspective*.

Question 3.1.2. What are the basic components of split Jacobians and how are they assembled?

Let $\mathbb{T}E$ and $\mathbb{T}E'$ be tropical elliptic curves and G a finite subgroup of their product. Question 3.1.2 translates to:

Task 3.1.3. *Setting:* $\mathbb{T}\mathcal{A}$ and $\mathbb{T}\mathcal{C}$. Determine whether $(\mathbb{T}E, \mathbb{T}E', G)$ is splitting data and, if it is, construct a tropical curve Γ of genus 2 whose Jacobian splits accordingly (see Section 3.5 for a precise statement).

Task 3.1.3 paves the way for addressing issues of a more global nature in the moduli space A_2^{tr} of principally polarized tropical abelian varieties and the moduli space M_2^{tr} of tropical curves of genus 2.

Task 3.1.4. *Setting:* A_2^{tr} and M_2^{tr}. Analysis of the structure of the locus of split Jacobians and the locus of curves with split Jacobians.

3.1.1 Local: Task 3.1.3

The expected end product has been described in [32] (see Theorem 3.1.1). In consequence, we adopt the following approach:

Plan 3.1.5 (see Plan 3.5.1)**.** Given splitting data $(\mathbb{T}E, \mathbb{T}E', G)$, proceed as follows:

1. Determine a splitting $\phi : \mathbb{T}E \oplus \mathbb{T}E' \to J$ and generate a diagram D_ϕ modelled on Diagram 2.23,
2. Construct Γ,
3. Define covers φ and φ' of $\mathbb{T}E$ and $\mathbb{T}E'$.

3.1.2 Global: Task 3.1.4

A natural by-product of Plan 3.1.5 is a family of split Jacobians and a characterization of their pointwise preimage under the tropical Torelli map t_2^{tr}. The moduli spaces M_2^{tr} and A_2^{tr} offer the right setting for a change of perspective. We consider the following subsets:

- $\mathcal{Q} \subset A_2^{tr}$ the *locus of split Jacobians.*
- $\mathbb{T}\mathcal{L}_d \subset M_2^{tr}$ the *locus of curves with d-split Jacobians.*

We think of the first as a Schottky type problem for split Jacobians, whereas the second is a tropical analogue of the classical locus of curves with (d, d)-split Jacobians, which is known to be an irreducible 2 dimensional subvariety of $\mathcal{M}_2$ (e.g. [73] or [59]).

Concretely, we are interested in

- characterizing the intersection of $\mathcal{Q}$ with the boundary of A_2^{tr}.
- determining $\mathbb{T}\mathcal{L}_d$ has a "tropical" structure that in some sense reflects the algebraic one.

3.1.3 Context

The present work has its roots in classical algebraic geometry, as did the first project in this series, [32]. In [38] Frey and Kani describe how to construct curves of genus 2 covering two elliptic curves E and E' from an isomorphism

$$\alpha : E'[d] \to E[d]$$

whose graph G_α is isotropic with respect to the Weil pairing on $(E \times E')[d]$. The isotropy condition for G_α is necessary for one of the main tools in this context, Mumford's Criterion (Proposition 16.8 [63] or [65], p. 231), for which we develop a tropical analogue in Section 3.2.2: It guarantees that the candidate for the Jacobian, the quotient $(E' \times E)/G_\alpha$, can be equipped with a principal polarization (pp). Guided by their ideas, we consider an analogue starting point: We take the data contained in α to represent the buildings blocks of a classical split Jacobian and consider analogues of these building blocks for tropical split Jacobians. While the main steps in the construction of Γ can be found in the classical and tropical approach (see Plan 3.1.5), their implementation diverges. For more details on the classical side we refer to [38].

With two separate but related settings, Tasks 3.1.3 and 3.1.4 need to be tackled in different ways: Working in $\mathbb{T}\mathcal{C}$ and $\mathbb{T}\mathcal{A}$, we build on techniques developed in [16], [22],[12] for handling curves and covers and [52], [69], [57], [61], [32] and more for handling tav. Working in M_2^{tr} and A_2^{tr}, we rely on results of [3], [29], [14] and [17].

The philosophy is the one of [32], we want to use tropical geometry to make abstraction concrete and offer both, an abstract and a constructive approach. The perspective, on the other hand, is a different one: Abstraction appears in the form of moduli spaces, i.e. we leave the "local" point of view, a focus on objects, to notice that split Jacobians arise in families whose structure we tend to grasp.

3.1.4 Results

Here we describe our main contributions.

Theorem 3.1.6. (Theorem 3.5.17) Let $(\mathbb{T}E', \mathbb{T}E, G)$ be splitting data, i.e. G is the graph of an isomorphism α between the d-torsion points of $\mathbb{T}E'$ and $\mathbb{T}E$. Then there exists: A curve Γ of genus 2 and a pair of optimal covers $(\varphi' : \Gamma \to \mathbb{T}E', \varphi : \Gamma \to \mathbb{T}E)$, each of degree d, that induce a splitting

$$\phi : \mathbb{T}E' \bigoplus \mathbb{T}E \to \mathrm{Jac}(\Gamma)$$

with $\mathrm{Jac}_d(\mathbb{T}E') \cong \ker(\phi) \cong \mathrm{Jac}_d(\mathbb{T}E)$.

Remark 3.1.7. Depending on the combinatorial type of Γ, there exists either a unique curve or an infinite family of curves, and consequently, either a unique pair of maps or an infinite family of such pairs.

Theorem 3.1.6 shows: Plan 3.1.5 has two possible outcomes. These correspond to the two maximal combinatorial types of curves of genus 2, the theta and the dumbbell graph, and can be distinguished using Algorithm 2 (see Subsection 3.5.3 and Section 3.7 for its implementation).

This naturally translates into a question posed in terms of moduli spaces, asking for the intersection of the locus $\mathcal{Q}$ of split Jacobians with the boundary of A_2^{tr} (see Subsection 3.6.1). The results of which we use to obtain the following combinatorial characterization of $\mathbb{T}\mathcal{L}_d$.

Theorem 3.1.8. (Theorem 3.6.9)

The locus of curves with d-split Jacobian, $\mathbb{T}\mathcal{L}_d$, decomposes into $\varphi(d)$ subsets

$$\mathbb{T}\mathcal{L}_d = \bigcup_k L_k, \tag{3.1}$$

where $\varphi(d)$ denotes the Euler phi function. Each L_k corresponds to a 2 dimensional fan obtained by subdividing the positive quadrant: For each maximal cone $\tilde{\sigma} \subset L_k$ there exists a linear map $\phi_{\tilde{\sigma}} : \tilde{\sigma} \to M_2^{tr}$ that maps the open cone to a 2 dimensional cone in the theta part of M_2^{tr}. Restricting $\phi_{\tilde{\sigma}}$ to a ray r that is a face of two cones yields a linear map $\phi_{\tilde{\sigma}|r} \times id : r \times \mathbb{R}_{\geqslant 0} \to M_2^{tr}$ whose image in the dumbbell part of M_2^{tr} is $\phi_{\tilde{\sigma}}(r) \times r_B$, where r_B denotes the ray that corresponds to the bridge edge. Moreover, the linear maps are compatible in the sense that for two neighboring maximal cones $\tilde{\sigma}$ and $\tilde{\sigma}'$ the restriction of $\phi_{\tilde{\sigma}}$, respectively $\phi_{\tilde{\sigma}'}$, to a common face agree.

3.1.5 Overview

Chapter 3 consists of three parts. In part 1, Sections 3.2 to 3.4, we mainly cover background material on tavs, tropical curves and their Jacobians.

We start in Section 3.2 with a description of the category of tavs and review some categorical constructions such as kernels, products and coproducts that will be needed later.

The material we discuss is known, except for Subsections 3.2.2 and 3.2.3. Here we develop one of the main tools for part 2, a tropical analogue of Mumford's Criterion (Proposition 3.2.8) and define adjoints in $\mathbb{T}\mathcal{A}$, a notion we will find useful for concise notation. Section 3.3 introduces the category of tropical covers, combining [3] and [6], and includes a short recall on tropical optimal covers, which we introduced in [32].

In Section 3.4 we describe a connection between $\mathbb{T}\mathcal{C}$ and $\mathbb{T}\mathcal{A}$, which on the level of objects is given by $\Gamma \mapsto \mathrm{Jac}(\Gamma)$. Considering the reverse direction is tricky. The transition to moduli spaces (Section 3.4.3) offers a constructive approach since the objects parameterizing pptavs and curves are in some sense more concrete. Our exposition is based on [29] and [13]. We will be as as brief as possible, only aiming at a visualization of the objects involved. This concludes part 1.

The reconstruction procedure, which is the main focus of Project 3, is discussed in Section 3.5 with a computational supplement in Section 3.7. Here we prove Theorem 3.1.6, which may be retraced computationally using Algorithm 2.

In part 3, consisting of Section 3.6, we investigate part 2 from the perspective of moduli spaces, study a tropical analogue of the locus of curves with split Jacobians (Theorem 3.1.8) and a Schottky type problem in this context.

3.1.6 Future work

- **In genus 2:** The combinatorial structure associated to $\mathbb{T}\mathcal{L}_d$ (see Theorem 3.1.8) organizes $\mathcal{Q}$ according to Selling's reduction algorithm. Since we work with a set description of $\mathcal{Q}$ that is not a set of representatives, we expect a certain amount of redundancy in Description (3.1).

  Conjecture 3.1.9. The fans associated to L_1 and L_{d-1} have the same image in M_2^{tr}.

  The case $d = 3$ is verified in 3.6.12, examples for higher degree point in the same direction. More generally:

  Task 3.1.10. Describe the intersection behavior in M_2^{tr} of the fans associated to L_k for all k.

  Completing this task would give a more comprehensive understanding of the moduli space structure of $\mathbb{T}\mathcal{L}_d$.

- **Extension to higher genus:**

  1. Full splitting: Given g elliptic curve $\mathbb{T}E_1, ..., \mathbb{T}E_g$ and a finite subgroup G of their direct product:
     - When is $(\mathbb{T}E_1, ..., \mathbb{T}E_g, G)$ splitting data?
     - If it is, is it minimal data (i.e. both necessary and sufficient for constructing a curve of genus g)? Can we make this construction explicit?

Likewise, we define

- $\mathcal{Q} \subset A_g^{tr}$ as the *locus of split Jacobians*. It is a subset of the tropical Schottky locus $A_g^{\mathrm{cogr}} \subset A_g^{tr}$ of A_g^{tr}, where for genus > 3 this containment is strict.
- $\mathbb{T}\mathcal{L}_g \subset M_g^{tr}$ the *locus of curves with split Jacobians.*

Task 3.1.11. Describe the structure of $\mathcal{Q}$ and $\mathbb{T}\mathcal{L}_g$.

2. Let Γ be a curve of genus g. In parallel to Project 1 (Section 2.1), we can investigate the following intermediate situation from an atomic perspective
   (a) Splitting off an elliptic curve: $\mathrm{Jac}(\Gamma)$ is isogenous to $A \oplus \mathbb{T}E$, where A is a tav,
   (b) Splitting into a product: $\mathrm{Jac}(\Gamma)$ is isogenous to a product of lower dimensional Jacobians:
   (c) Splitting off a product: $\mathrm{Jac}(\Gamma)$ is isogenous to $A \oplus \prod_i \mathrm{Jac}(\Gamma_i)$, where the Γ_i's are curves of lower genus.

   We plan to reverse-engineer these splittings by constructing Jacobians from lower dimensional building blocks. More precisely, start with $\mathrm{Jac}(\Gamma_1)$, $\mathrm{Jac}(\Gamma_2)$ with $g = g_1 + g_2$ and a finite subgroup G and consider the quotient $J := \mathrm{Jac}(\Gamma_1) \oplus \mathrm{Jac}(\Gamma_1)/G$. Next: 1. Develop criteria for G such that J lies in the tropical Schottky locus of A_g^{tr} (in particular it is a pptav). 2 Determine the preimage of J under the tropical torelli map t_g^{tr}.

   As in Project 1, genus 3 is the most approachable next case.

3.2 Category of tropical abelian varieties

3.2.1 Preliminaries

A large part of the following work will take place in the category of tropical abelian varieties $\mathbb{T}\mathcal{A}$. Here we give a straight-to-the-point description. For a more complete introduction see Chapter 2, Sections 2.2 and 2.3. Let Σ be a real torus with integral structure, i.e. $\Sigma = (\Lambda, \Lambda', [\cdot,\cdot])$, where

- Λ and Λ' are finitely generated free abelian groups of the same rank,
- $[\cdot,\cdot] : \Lambda \times \Lambda' \to \mathbb{R}$ is a non-degenerate pairing,

with topological realization $\mathrm{Hom}(\Lambda, \mathbb{R})/\Lambda'$, where $\Lambda' \subset \mathrm{Hom}(\Lambda, \mathbb{R})$ via $[\cdot,\cdot]$, and $\check{\Sigma} := (\Lambda', \Lambda, [\cdot,\cdot]^t)$ its dual (see [32], Sections 2.1).

Definition 3.2.1. *Objects:* A *tropical abelian variety (tav)* $\Sigma = (\Lambda, \Lambda', [\cdot,\cdot])$ is a real torus with integral structure together with a *polarization*, i.e. a group homomorphism $\zeta : \Lambda' \to \Lambda$ such that the bilinear form $[\zeta(\cdot),\cdot] : \Lambda' \times \Lambda' \to \mathbb{R}$ is symmetric and positive definite. The *dimension* of Σ is the $\mathbb{R}$-vector space dimension of $\mathrm{Hom}(\Lambda, \mathbb{R})$ and equal to $\mathrm{rk}(\Lambda)$ (equivalently equal to $\mathrm{rk}(\Lambda'))$. Moreover, ζ is called a *principal polarization* and Σ a *principally polarized tropical abelian variety (pptav)*, whenever ζ is a bijection.

Morphisms: For $i = 1, 2$ let $\Sigma_i := (\Lambda_i, \Lambda'_i, [\cdot,\cdot]_i)$ be tavs. A *morphism of tavs* is a pair of group homomorphisms $f := (f^\# : \Lambda_2 \to \Lambda_1, f_\# : \Lambda'_1 \to \Lambda'_2)$ such that

$$[f^\#(\lambda_2), \lambda'_1]_1 = [\lambda_2, f_\#(\lambda'_1)]_2 \tag{3.27}$$

is satisfied for all $\lambda'_1 \in \Lambda'_1$ and $\lambda_2 \in \Lambda_2$.

Dual notions: The dualization functor $\check{\cdot} : \mathbb{T}\mathcal{A} \to \mathbb{T}\mathcal{A}$ sends a tav Σ to $\check{\Sigma}$ equipped with the so-called *dual polarization* $\check{\zeta}$ (see [32], Section 3.1, previously defined by Röhrle and Zakharov, see first version of [69]) and a morphism f, accordingly, to the *dual morphism*, $\check{f} : \check{\Sigma}_2 \to \check{\Sigma}_1$ obtained from the pair $(f_\#, f^\#)$.

Since properties of the induced map of quotients are encoded as properties of the pair $(f^\#, f_\#)$ (see [69], Definition 4.8), we call f

- *surjective*, if $f^\#$ is injective.
- *finite*, if $f_\#$ is injective (equivalently if $[\Lambda_1 : f^\#(\Lambda_2)] < \infty$).
- *injective*, is f is finite and $f_\#(\Lambda'_1)$ is saturated in Λ'_2.
- an *isogeny*, if it is surjective and finite.

The category of tavs is particularly nice, it is abelian (see [57]), finitely complete and finitely cocomplete ([32], Lemma 11). This means that we have notions of (co)kernels and (co)products (and the like) and they behave as we would expect from the category of abelian groups. Knowing this will be sufficient for our purposes. Therefore, we consider explicit constructions only to a limited extend and refer the reader to [32] for more details.

Definition 3.2.2. ([32], Definition 9) Given tavs Σ_1 and Σ_2, we define their *product* $\Sigma_1 \bigotimes \Sigma_2$ as follows:

- The underlying real torus with integral structure is $(\Lambda_1 \times \Lambda_2, \Lambda'_1 \times \Lambda'_2, [\cdot,\cdot]_1 + [\cdot,\cdot]_2)$, where $\Lambda_1 \times \Lambda_2$, respectively $\Lambda'_1 \times \Lambda'_2$, denotes the direct product of groups and $[\cdot,\cdot]_1 + [\cdot,\cdot]_2$ is given by $((\lambda_1, \lambda_2), (\lambda'_1, \lambda'_2)) \mapsto [\lambda_1, \lambda'_1]_1 + [\lambda_2, \lambda'_2]_2$.
- The polarization $\zeta_1 \times \zeta_2$ is defined component-wise.
- The object $\Sigma_1 \bigotimes \Sigma_2$ is equipped with a pair of morphisms, $\pi_1 : \Sigma_1 \bigotimes \Sigma_2 \to \Sigma_1$ and $\pi_2 : \Sigma_1 \bigotimes \Sigma_2 \to \Sigma_2$, induced by the natural projection and inclusion maps between the lattices.

The resulting object is a tav, that is:

- The pairing $[\cdot,\cdot]_1 + [\cdot,\cdot]_2$ is non-degenerate.

- The group homomorphism $\zeta_1 \times \zeta_2$ satisfies the condition described in Definition 3.2.1.
- The projection maps π_1 and π_2 are morphism of tori.

The *coproduct* of Σ_1 and Σ_2 is defined analogously and will be denoted by $\Sigma_1 \bigoplus \Sigma_2$.

Lemma 3.2.3. ([32], Lemma 26) Let Σ be a tav and suppose $G \subset \Sigma$ is a finite subgroup. Then there exists a tav Σ_G and a free isogeny $\Sigma \overset{q}{\twoheadrightarrow} \Sigma_G$ whose kernel is G.

We also include the following Lemma proved in Chapter 2.

Lemma 3.2.4. [[32], Lemma 23] Let

$$0 \to \Sigma_1 \xrightarrow{f} \Sigma_2 \xrightarrow{g} \Sigma_3 \to 0.$$

be a short exact sequence of tropical abelian varieties. Then the dual sequence

$$0 \to \check{\Sigma}_3 \xrightarrow{\check{g}} \check{\Sigma}_2 \xrightarrow{\check{f}} \check{\Sigma}_1 \to 0$$

is exact. In other words

$$\check{\cdot} : \mathbb{T}\mathcal{A} \to \mathbb{T}\mathcal{A}$$

is an exact functor.

3.2.2 Isogenies and Polarizations

Isogenies form a distinguished class of morphisms of tavs that are in some sense closely related to polarizations and interact well with them.

Polarizations induce isogenies

A polarization ζ defines an isogeny $f_\zeta := (\zeta, \zeta)$ between Σ and its dual $\check{\Sigma}$. We can characterize its kernel using the so-called *type* of ζ, which is given by the invariant factors $(\alpha_1, ..., \alpha_n)$ (where $n := \mathrm{rk}(\Lambda)$) of its Smith normal form. We have:

$$\ker(f_\zeta) \cong \mathbb{Z}/\alpha_1\mathbb{Z} \times ... \times \mathbb{Z}/\alpha_n\mathbb{Z}$$

and thus a "canonical" identification of Σ and $\check{\Sigma}$, whenever f_ζ is an isomorphism, or equivalently whenever ζ has type $(1, ..., 1)$ (i.e. ζ is a pp).

Bidirectional transport of polarizations via isogenies

Recall that isogenies ([32], Section 3.1) allow us to transfer polarizations from the target variety to the domain and vice versa. In other words, any integral torus that is isogenous to a tav can be turned into tav itself.

Definition 3.2.5. Let Σ_2 be a tav with polarization ζ_2, Σ_1 a real torus with integral structure, and $\phi : \Sigma_1 \to \Sigma_2$ an isogeny. Then $\phi^*\zeta_2 := \phi^\# \circ \zeta_2 \circ \phi_\#$ is a polarization on Σ_1 and called the *induced polarization* or alternatively *the pull-back* of ζ_2 by ϕ. Conversely, suppose Σ_1 carries a polarization ζ_1. We can define the *push-forward* of ζ_1 by ϕ as $\phi_*\zeta_1 := \check{\zeta}$, where $\zeta := \check{\phi}^*\check{\zeta}_1$. We say that an isogeny $\phi : \Sigma_1 \to \Sigma_2$ is *polarized* with respect to polarizations ζ_1 on Σ_1 and ζ_2 on Σ_2, if ζ_1 is the polarization induced by ϕ and ζ_2, in other words, if the diagram

$$\begin{tikzcd}
\Sigma_1 \arrow[r, "\phi"] \arrow[d, "f_{\zeta_1}"] & \Sigma_2 \arrow[d, "f_{\zeta_2}"] \\
\check{\Sigma}_1 & \check{\Sigma}_2 \arrow[l, "\check{\phi}"']
\end{tikzcd}$$

commutes.

Polarizing isogenies

Note that pulling back polarizations always turns ϕ into a polarized isogeny. The same is not true for the push-forward. Characterizing under which circumstances a polarization on the source is induced by a polarization on the target becomes relevant for Subsection 3.5.2 and is content of the subsequent Lemma.

Lemma 3.2.6. Let $\phi : \Sigma_1 \to \Sigma_2$ be an isogeny and $\zeta_1 : \Lambda_1' \to \Lambda_1$ a polarization on Σ_1. We denote by $(\gamma_1, ..., \gamma_n)$ the invariant factors of $i : \mathrm{Im}(\phi_\#) \to \Lambda_2'$ and by $\widehat{\alpha_1}$ the first invariant factor of $\zeta_1 : \Lambda_1' \to \mathrm{Im}(\phi^\#)$ (meaning the smallest with respect to divisibility). Then, there exists a polarization ζ_2 on Σ_2 such that $\phi^*\zeta_2 = \zeta_1$, if

1. $\mathrm{Im}(\phi^\#) \supset \mathrm{Im}(\zeta_1)$.
2. $\mathrm{Det}(i) := \prod_{j=1}^n \gamma_j$ is a divisor of $\widehat{\alpha_1}$.

While condition (1) in Lemma 3.2.6 is necessary for the existence of the polarization ζ_2, the divisibility condition (condition (2)) is only sufficient: It has been formulated so as to provide the reader with easily verifiable criterion. The sacrifice of necessity is deliberate.

Proof. Our goal is to complete

$$\begin{tikzcd}
\Lambda_1 & \Lambda_2 \arrow[l, "\phi^\#"'] \\
\Lambda_1' \arrow[u, "\zeta_1"] \arrow[r, "\phi_\#"] & \Lambda_2' \arrow[u, dashed, "\zeta_2"']
\end{tikzcd} \tag{3.28}$$

to a commutative diagram by connecting the lattices Λ_2' and Λ_2 by means of a polarization ζ_2 on Σ_2. We proceed in two steps:

(i) Invert Diagram 3.28 "as much as possible" and define a polarization $\tilde{\zeta_2}$ on a sublattice of Λ_2'.

(ii) Extend $\tilde{\zeta_2}$ to Λ_2' in an appropriate way.

Consider

$$\begin{tikzcd}
\mathrm{Im}(\phi^\#) \arrow[rr, bend left, "g_2"] & \Lambda_1 & \Lambda_2 \arrow[l, "\phi^\#"] \\
& \Lambda_1' \arrow[u, "\zeta_1"] \arrow[ul, bend left, "\zeta_1"] \arrow[r, "\phi_\#"] & \Lambda_2' \\
& & \mathrm{Im}(\phi_\#) \arrow[u, hook, "i"] \arrow[ul, bend left, "g_1"]
\end{tikzcd} \tag{3.29}$$

where g_1 and g_2 are the inverses of the isomorphisms obtained from $\phi^\#$ and $\phi_\#$ by restricting their respective codomains and define $\tilde{\zeta_2} : \mathrm{Im}(\phi_\#) \to \Lambda_2$ as the composition $g_2 \circ \zeta_1 \circ g_1$. By construction $\tilde{\zeta_2}$ satisfies $\phi^\# \circ \tilde{\zeta_2} \circ \phi_\# = \zeta_1$ and for all $\lambda \in \mathrm{Im}(\phi_\#)$ (i.e. $\lambda = \phi_\#(\lambda_1')$ for some $\lambda_1' \in \Lambda_1'$) we have:

$$\begin{aligned}
[\tilde{\zeta_2}(\lambda), \lambda]_2 &= [\tilde{\zeta_2}(\phi_\#(\lambda_1')), \phi_\#(\lambda_1')]_2 = [\phi^\# \circ \tilde{\zeta_2} \circ \phi_\#(\lambda_1')), \lambda_1']_1 \\
&= [\phi^\# \circ g_2 \circ \zeta_1 \circ g_1 \circ \phi_\#(\lambda_1')), \lambda_1']_1 = [\zeta_1(\lambda_1'), \lambda_1']_1.
\end{aligned}$$

Hence, $[\tilde{\zeta_2}(\cdot), \cdot]_2 : \mathrm{Im}(\phi_\#)_\mathbb{R} \times \mathrm{Im}(\phi_\#)_\mathbb{R} \to \mathbb{R}$ is symmetric and positive definite. This finishes step (i). For step (ii), let $n = \dim(\Sigma_1) = \dim(\Sigma_2)$ and recall that both $\mathrm{Im}(\phi_\#)$ and $\mathrm{Im}(\phi^\#)$ are full-dimensional sublattices since ϕ is an isogeny. We extend $\tilde{\zeta_2}$ injectively along i by looking at the diagram in Figure 3.1 (a) in coordinates (see Figure 3.1 (b)).

(a)

$$\begin{tikzcd}
\Lambda_2' \arrow[r, dashed] & \Lambda_2 \\
\mathrm{Im}(\phi_\#) \arrow[u, hook, "i"] \arrow[ur, "\tilde{\zeta_2}"'] &
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[r, dashed] & \mathbb{Z}^n \\
\mathbb{Z}^n \arrow[u, hook, "M(i)"] \arrow[ur, "M(\tilde{\zeta_2})"'] &
\end{tikzcd}$$

Figure 3.1: Transition to coordinates in the proof of Lemma 3.2.6.

On $\mathrm{Im}(\phi_\#)$ and Λ_2 these are chosen such that the transformation matrix of $\tilde{\zeta_2}$, $M(\tilde{\zeta_2})$, is in Smith normal form. We impose no restrictions on the choice for Λ_2'. Set

$$M(\zeta_2) := diag(\widehat{\alpha}_1, ..., \widehat{\alpha}_n) \cdot M(i)^{-1},$$

where $(\widehat{\alpha}_1, ..., \widehat{\alpha}_n)$ are the invariant factors of $\tilde{\zeta_2}$ and $M(i)^{-1}$ is the inverse of $M(i)$ over $\mathbb{Q}$. Since $M(i)^{-1}$ is the product of its adjoint and the scalar $\frac{1}{\mathrm{Det}(i)}$ and $\mathrm{Det}(i)$ is a divisor of $\widehat{\alpha_1}$, $M(\zeta_2)$ is an integer matrix. By forgetting coordinates, we obtain an extension, ζ_2, of $\tilde{\zeta_2}$ along i as desired. Note that ζ_2 is automatically injective as $\det(M(\zeta_2))$ is non zero.
We conclude the proof by verifying that ζ_2 induces a scalar product $\langle \cdot, \cdot \rangle := [\zeta_2(\cdot), \cdot]_2$ on ${\Lambda'_2}_{\mathbb{R}}$, equivalently that the matrix associated to $\langle \cdot, \cdot \rangle$ is symmetric and positive definite. In order to provide us with coordinate representations, C and C', of $\langle \cdot, \cdot \rangle$ and its restriction to $\mathrm{Im}(\phi_\#)_{\mathbb{R}} \times \mathrm{Im}(\phi_\#)_{\mathbb{R}}$, we fix lattice basis S, S' and S'' of Λ_2, Λ'_2 and $\mathrm{Im}(\phi_\#)$. Doing so, we observe that C and C' are related by the following equation:

$$C' =_{S''} {M^t}_{S'}(i) \cdot C \cdot_{S''} M_{S'}(i).$$

Since C' is symmetric, we have

$${}_{S''}{M^t}_{S'}(i) \cdot C^t \cdot_{S''} M_{S'}(i) = C'^t = C' =_{S''} {M^t}_{S'}(i) \cdot C \cdot_{S''} M_{S'}(i)$$

and conclude that $C^t = C$ using that ${}_{S''}M_{S'}(i)$ is invertible over $\mathbb{R}$. Choosing S' and S'' such that ${}_{S''}M_{S'}(i)$ is in Smith normal form with invariant factors $(\beta_1, ..., \beta_n)$, we can compute the leading principal minors $\det(C_k)$ of C in terms of the ones of C':

$$\det(C'_k) = \prod_{i=1}^{k} \beta_i^2 \cdot \det(C_k), k = 1, ..., n.$$

By Sylvester's criterion all of these must be positive, since C' is symmetric and positive definite. Hence, C is as well. ☐

Remark 3.2.7. In the setting of Lemma 3.2.6, note that whenever ζ_2 exists it is unique: The condition that Diagram 3.28 commutes requires any such ζ_2 to be an extension of $\tilde{\zeta_2}$, which is fixed (by ζ_1 and ϕ) on a sublattice of Λ'_2 of full rank. In this case we call ζ_2 the *inducing* polarization.
The divisibility criterion in Lemma 3.2.6 involves the invariant factors of the polarization ζ_1 on Σ_1, however with restricted target. These relate to the type $(\alpha_1, ..., \alpha_n)$ of ζ_1 in the following way ([67], Section 4):

$$\mathrm{lcm}(\,\widehat{\alpha}_{i+1}\widehat{\gamma}_{k-i} \,:\, 0 \leqslant i \leqslant k-1) \,|\, \alpha_k \,|\, \gcd(\,\widehat{\alpha}_{k-1+i}\widehat{\gamma}_{n-i+1} \,:\, 1 \leqslant i \leqslant n-k+1),$$

where $(\widehat{\gamma}_1, ..., \widehat{\gamma}_n)$ denote the invariant factors of the inclusion $\mathrm{Im}(\phi^\#) \hookrightarrow \Lambda_1$. If $n = 2$ this yields

$$\widehat{\alpha}_1\widehat{\gamma}_1 \,|\, \alpha_1 \,|\, \gcd(\widehat{\alpha}_1\widehat{\gamma}_2, \widehat{\alpha}_2\widehat{\gamma}_1) \text{ and } \mathrm{lcm}(\widehat{\alpha}_1\widehat{\gamma}_2, \widehat{\alpha}_2\widehat{\gamma}_1) \,|\, \alpha_2 \,|\, \widehat{\alpha}_2\widehat{\gamma}_2.$$

Clearly, the proof of Lemma 3.2.6 has an algorithmic flavor, which we now concretize. Algorithm 1 provides us with an if-and-only-if-criterion for the existence of ζ_2, which is accessible for algorithmic verification and turns into a step-by-step construction should ζ_2

Algorithm 1 Tropical Mumford's Criterion

```
Input: An isogeny $\phi$ and a polarization $\zeta_1$.
Output: A matrix $M$ that decides whether $\zeta_2$ exists.
  I. For $\mathbb{Z}$-basis $T'$ and $S$ of $\Lambda_1'$ and $\Lambda_2$, let $S''$ and $T''$ denote the basis on $\mathrm{Im}(\phi_\#)$ and
     $\mathrm{Im}(\phi^\#)$ induced by the injective maps $\phi_\#$ and $\phi^\#$. Set
     $A :=_{T'} M_{T''}(\zeta_1)$.
  II. Choose a $\mathbb{Z}$-basis $S'$ of $\Lambda_2'$ and let $B :=_{S''} M_{S'}(i)$.
  III. Compute $M := A \cdot B^{-1}$.
```

exist. The classical counterpart of Proposition 3.2.8 (see [59], Proposition 16.8 or [65], p.231) is of a different flavor: It places certain requirements on a pairing e^λ which is associated to the polarization λ and a matching Weil pairing. We consider this difference to be one of the main reasons why Plan 3.1.5 can be carried out explicitly in the tropical setting.

Proposition 3.2.8. Let $\phi : \Sigma_1 \to \Sigma_2$ be an isogeny and $\zeta_1 : \Lambda_1' \to \Lambda_1$ a polarization on Σ_1. There exists a polarization ζ_2 on Σ_2 such that $\phi^*\zeta_2 = \zeta_1$ if and only if the output M of Algorithm 1 is an integer matrix. In this case, M is a coordinate representation of ζ_2.

3.2.3 Adjoints

We will find the following to be a useful addition to our toolbox.

Definition 3.2.9. Let $f : \Sigma_1 \to \Sigma_2$ and $\tilde{f} : \Sigma_2 \to \Sigma_1$ be two morphism of integral tori. We say that $\tilde{f}$ is *adjoint* to f (alternatively that $(f, \tilde{f})$ is an *adjoint pair*), if their exists principal polarizations ζ_1 on Σ_1 and ζ_2 on Σ_2 such that the following diagram commutes:

$$\begin{tikzcd}
\Sigma_1 \arrow[d, "f_{\zeta_1}"] & \Sigma_2 \arrow[l, "\tilde{f}"'] \arrow[d, "f_{\zeta_2}."] \\
\check{\Sigma}_1 & \check{\Sigma}_2 \arrow[l, "\check{f}"]
\end{tikzcd} \tag{3.30}$$

Note that though the notion of adjoint is not unique in the strict sense of the word, it is "essentially" so (meaning unique up to conjugation by isomorphisms).

Lemma 3.2.10. (Properties of adjoint morphisms)
Let $(f, \tilde{f})$ be an adjoint pair. For $i = 1, 2$ denote by $\langle \cdot, \cdot \rangle_i$ the scalar product on $\Lambda'_{i\mathbb{R}}$ induced by ζ_i. We have:

1. $\langle \lambda_1', \tilde{f}_\#(\lambda_2') \rangle_1 = \langle f_\#(\lambda_1'), \lambda_2' \rangle_2$ for all $\lambda_1' \in \Lambda_1'$ and $\lambda_2' \in \Lambda_2'$.

2. $\mathrm{Ker}(\tilde{f})_0 \cong \widecheck{\mathrm{Coker}}(f)$.

Moreover, if f is an isogeny, then so is $\tilde{f}$ and the composition $\tilde{f} \circ f$ is given by $(f^*\zeta_2 \circ \zeta_1^{-1}, \zeta_1^{-1} \circ f^*\zeta_2)$.

Proof. Points (1),(2) and the formula for $\tilde{f} \circ f$ are easily derived by manipulating definitions. Since the dualization functor, $\check{\cdot}$, takes isogenies to isogenies, we see that f is an isogeny if and only if $\tilde{f}$ is one. $\square$

3.3 Category of tropical curves

3.3.1 Preliminaries

The second category we work in is the *category of tropical curves*, $\mathbb{T}\mathcal{C}$. Our exposition is based on [3] and [6], relying on the foundational work of Mikhalkin in [61] and of Mikhalkin and Zharkov in [62].

Definition 3.3.1. *Objects*: A metric graph (G, l) is a finite graph G with no legs/ends (i.e. no 1-valent vertices) and a function $l : E(G) \to \mathbb{R}_{>0}$. The geometric realization Γ of (G, l) is called a *tropical curve* of genus $g(\Gamma) := b_1(G)$ with *model* (G, l) and *combinatorial type* G, where $b_1(G)$ denotes the first Betti number of G.

Morphisms: A continuous and surjective map $\varphi : \Gamma \to \tilde{\Gamma}$ between two tropical curves Γ and $\tilde{\Gamma}$ is called a *tropical cover* (see e.g. [7] or [16], Section 2.1) if there exists models (G, l) and $(\tilde{G}, \tilde{l})$ such that

- $\varphi(V(G)) \subset V(\tilde{G})$ and $\varphi^{-1}(E(\tilde{G})) \subset E(G)$.
- φ is locally integer affine linear: On each edge $e \in E(G)$, φ restricts to an affine function with integer slope $d_e(\varphi)$ (possibly 0), called the *weight* or *expansion factor* of φ at e.
- φ is harmonic/balanced at every $P \in \Gamma$: For any $\tilde{v} \in T_{\varphi(P)}\tilde{\Gamma}$
$$d_P(\varphi) := \sum_{v \in T_P\Gamma, v \mapsto \tilde{v}} d_v(\varphi)$$
is independent of $\tilde{v}$, where $T_P\Gamma$, respectively $T_{\varphi(P)}\tilde{\Gamma}$, is the set of tangent directions emanating from P, respectively $\varphi(P)$, and $d_v(\varphi)$ is the directional derivative of φ in the direction of v (i.e. $d_v(\varphi) := d_e(\varphi)$ for the edge e in direction of v).

We say that φ is *finite*, if $d_e(\varphi) > 0$ for all edges e, and *non-finite* else. In any case, $deg(\varphi) := \sum_{P \in \Gamma, P \mapsto \tilde{P}} d_P(\varphi)$, where $\tilde{P} \in \tilde{\Gamma}$ is an arbitrary point, provides a well-defined notion of *degree*, a tropical analogue of the corresponding notion in algebraic geometry.

We will work with tropical curves through choice of a model and by abuse of notation identify Γ with (G, l).

3.3.2 Setup

In Sections 3.5 and 3.6 we only work in the genus 1 and 2 part of $\mathbb{T}\mathcal{C}$. We recall a class of morphism introduced in 2 that is relevant in this context:

Definition 3.3.2. Let Γ and $\mathbb{T}E$ be tropical curves of genus 2 and 1. We call a cover $\varphi : \Gamma \to \mathbb{T}E$ *optimal* if it does not factor through a non-trivial cover, i.e. if there exists a curve $\mathbb{T}\tilde{E}$ and maps $\tilde{\varphi} : \Gamma \to \mathbb{T}\tilde{E}$, $\phi : \mathbb{T}\tilde{E} \to \mathbb{T}E$ such that

$$\begin{tikzcd}
\Gamma \arrow[r, "\tilde{\varphi}"] \arrow[dr, "\varphi"'] & \mathbb{T}\tilde{E} \arrow[d, "\phi"] \\
& \mathbb{T}E
\end{tikzcd}$$

commutes, then ϕ is an isomorphism (i.e. $deg(\phi) = 1$).

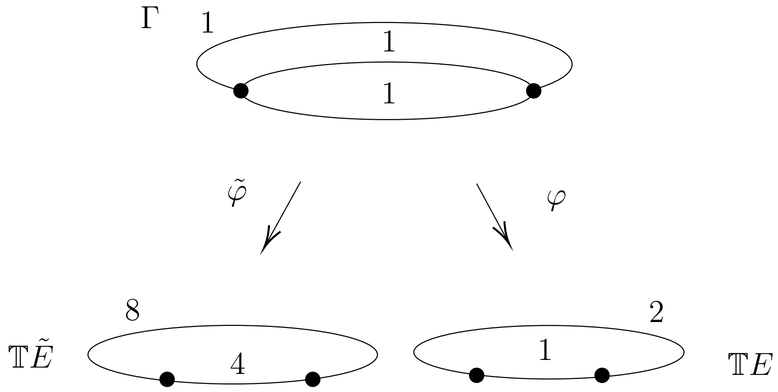


Figure 3.2: A curve of genus 2 covering two elliptic curves. The numbers are edge lengths.

Example 3.3.3. Figure 3.2 shows a cover of degree 2 on the right, which must be optimal. The cover on the left, however, is not as it factors, for example, through the first, giving rise to a cover of degree 4.

3.4 Bridging $\mathbb{T}\mathcal{C}$ and $\mathbb{T}\mathcal{A}$

3.4.1 Preliminaries

The categories $\mathbb{T}\mathcal{C}$ and $\mathbb{T}\mathcal{A}$ are connected:

- To each object Γ we assign an object $\mathrm{Jac}(\Gamma)$ in $\mathbb{T}\mathcal{A}$,
- To each morphism $\varphi : \Gamma_1 \to \Gamma_2$ we associate a morphism φ_* in $\mathbb{T}\mathcal{A}$,

where $\mathrm{Jac}(\Gamma)$ denotes the Jacobian of Γ and $\varphi_* : \mathrm{Jac}(\Gamma_1) \to \mathrm{Jac}(\Gamma_2)$ the push-forward of φ (Section 2.5.5). This connection, especially in the case of curves of genus 2 covering curves of genus 1, was discussed in detail in Chapter 2. We briefly recall some constructions relevant in this context.

Construction 3.4.1. (see [6] or [58]) Let (G, l) be an oriented model of Γ and $s, t : E(G) \to V(G)$ the source and target maps. Then (G, l) comes with two lattices that are related by a non-degenerate pairing:

- *The lattice of harmonic* 1*-forms*, $\Omega^1_G(\mathbb{Z})$: For each oriented edge e we introduce a formal symbol de called a *basic* 1-form on G and set $\Omega^1_G(\mathbb{Z})$ to be

  $$\{\omega := \sum_e \omega_e de : \omega_e \in \mathbb{Z}, \sum_{e:t(e)=V} \omega_e = \sum_{e:s(e)=V} \omega_e \,\forall\, V \in V(G)\}.$$

  It is the subgroup of the free group over $\{de : e \in E(G)\}$ consisting of harmonic $1-$forms on G.

- *The lattice of integral* 1*-cycles*, $\mathrm{H}_1(G, \mathbb{Z})$: It is the first simplicial homology group of G given by $\ker(\partial)$, where

  $$\partial : \mathrm{C}_1(G, \mathbb{Z}) \to \mathrm{C}_0(G, \mathbb{Z}), e \mapsto t(e) - s(e)$$

  is the boundary operator.

- *The integration pairing*, $\int_\cdot \cdot$: We can integrate a basic 1-form

  $$\int_e de' := \begin{cases} l(e), & \text{if } e = e' \\ 0, & \text{else} \end{cases}$$

  and extend linearly to obtain a perfect pairing

  $$\int_\cdot \cdot : \Omega^1_G(\mathbb{Z}) \times \mathrm{H}_1(G, \mathbb{Z}) \to \mathbb{R}, (\omega, c) \mapsto \int_c \omega.$$

These building blocks are independent of the choice of model (see also [6]). This means that lattices that arise from different models (that have compatible orientations) are related by isomorphisms, that leave the integration pairing invariant. We will write $\Omega^1_\Gamma(\mathbb{Z})$ and $\mathrm{H}_1(\Gamma, \mathbb{Z})$, instead, and complete Construction 3.4.1 by assigning a pptav to Γ.

Definition 3.4.2. The *Jacobian of* Γ is the pptav built from $(\Omega^1_\Gamma(\mathbb{Z}), \mathrm{H}_1(\Gamma, \mathbb{Z}), \int_\cdot \cdot)$ with principal polarization $\zeta_\Gamma : \mathrm{H}_1(\Gamma, \mathbb{Z}) \to \Omega^1_\Gamma(\mathbb{Z}), \sum a_e e \mapsto \sum a_e de$. It is related to Γ by the *tropical Abel-Jacobi map*:

$$\Phi_{P_0} : \Gamma \to \mathrm{Jac}(\Gamma), P \mapsto \int_{\gamma_P} \cdot,$$

where $P_0 \in V(\Gamma)$ is a fixed vertex and $\gamma_P \in \mathrm{C}_1(\Gamma, \mathbb{Z})$ is any path connecting P_0 to P in Γ.

As in classical algebraic geometry, we can identify $\mathrm{Jac}(\Gamma)$ with $\mathrm{Pic}^0(\Gamma) := \mathrm{Div}^0(\Gamma)/\mathrm{Prin}(\Gamma)$, where $\mathrm{Div}^0(\Gamma)$ is the group of divisors of degree 0 and $\mathrm{Prin}(\Gamma)$, the subgroup of principal divisors ([62], Section 4.2). Then φ_* is the morphism induced by the push-forward of divisors under this identification, whose dual φ^* is induced by the pull-back ([32], Lemma 35).

3.4.2 Setup

Specializing to the case of curves of genus 2 covering curves of genus 1, we are interested in a particular class of objects in $\mathbb{T}\mathcal{A}$, so-called *tropical split Jacobians*.

Definition 3.4.3. ([32], Definition 58) Let Γ be a tropical curve of genus 2. We say that $\mathrm{Jac}(\Gamma)$ *splits*, if $\mathrm{Jac}(\Gamma)$ is isogeneous to the coproduct of two elliptic curve $\mathbb{T}E \oplus \mathbb{T}E'$. In this case we call $\mathbb{T}E'$ a *complement* of $\mathbb{T}E$ and vice versa.

Convention 3.4.4. We fix a labelling for the combinatorial types we are going to work with (see Figure 3.3)

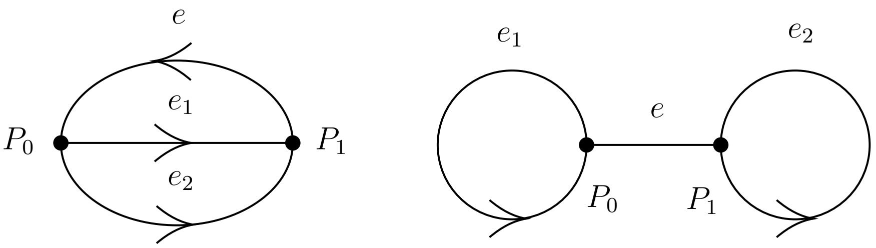


Figure 3.3: A curve of type theta on the left and of type dumbbell on the right.

and for explicit computations basis

- $(B_1, B_2) := (e+e_2, e_2-e_1)$ of $\mathrm{H}_1(\Gamma, \mathbb{Z})$, if Γ is of type theta, and (B_1, B_2) $(B_1, B_2) := (e_1, e_2)$, if Γ is of type dumbbell.
- in both cases the canonically associated basis (ω_1, ω_2) of $\Omega^1_\Gamma(\mathbb{Z})$ (see [32], Section 5).

3.4.3 From the perspective of moduli spaces

The connection between $\mathbb{T}\mathcal{A}$ and $\mathbb{T}\mathcal{C}$ focuses on individual objects. Transferred to the level of moduli spaces, it gives rise to the *tropical Torelli map*

$$t_2^{tr} : M_2^{tr} \to A_2^{tr}, \Gamma \mapsto \mathrm{Jac}(\Gamma)$$

where M_2^{tr} and A_2^{tr} denote the moduli space of tropical curves of genus 2 and the moduli space of pptav of dimension 2, respectively ([17], [14]). For these moduli spaces to be well-behaved, we need slightly more general notions of tropical curves, their Jacobians and correspondingly of pptav. For example, to guarantee that M_2^{tr} is both complete (i.e. closed under specialization) and relates well to its algebraic counterpart $\overline{m}_2$ (via taking dual graphs) (see [29], Remark 2.4), we need to generalize Definition 3.3.1 and allow our curves to carry genus at their vertices: A tropical curve Γ will be a metric graph (G, l) as in Definition 3.3.1 together with a genus function $g : V(G) \to \mathbb{Z}_{\geqslant 0}$ that satisfies a certain stability condition (see [29], Definition 2.1). To include this new piece of data, we define (G, g) to be the *combinatorial type* of Γ and the number $g(G) + \sum_{v \in V(G)} g(v)$ to be its genus. The definition of the Jacobian extends to this case and results in a slightly generalized notion of pptav (by relaxing the positive definite-condition to positive semidefinite and renouncing to the non-degeneracy of $\int_{\cdot} \cdot$). We refer to [29] (Section 4 and 6) or [13] (Section 4 and 5) for more details as it will be of no further relevance to us.

Now, the tropical Torelli-map t_2^{tr} is a morphism in the category of *stacky fans*, a category that is suitable for the construction of tropical moduli spaces. Naturally, A_2^{tr} and M_2^{tr} are both objects therein.

A pptav Σ is a real integral torus that carries a pp ζ, alternatively a pair $(\frac{\mathrm{Hom}(\Lambda,\mathbb{R})}{\Lambda'}, Q)$, where Q is the quadratic form associated to the symmetric bilinear form $[\zeta(\cdot), \cdot]$. After suitable choice of basis one can find a representative of the isomorphism class of a 2-dimesnional pptav Σ of the form $(\mathbb{R}^2/\mathbb{Z}^2, Q)$, where Q is an element of $\tilde{S}^2_{\geqslant 0}$, the space of positive semidefinite 2×2 matrices with rational nullspace. Note that in A_2^{tr} the notion of isomorphism is *stronger* than in $\mathbb{T}\mathscr{A}$: An isomorphism of pptavs (in A_2^{tr}) is (in $\mathbb{T}\mathscr{A}$) an isomorphism $f : \Sigma_1 \to \Sigma_2$ of integral tori such that f is polarized with respect to the pp ζ_1 on Σ_1 and ζ_2 on Σ_2 (see Definition 3.2.5 and [69], Definition 4.10).

Moreover, Σ_1 and Σ_2 are isomorphic if and only if for their selected representatives, $(\mathbb{R}^2/\mathbb{Z}^2, Q_1)$ and $(\mathbb{R}^2/\mathbb{Z}^2, Q_2)$, there exists $X \in GL(\mathbb{Z})$ with $Q_2 = X^t Q_1 X$. This suggests the quotient $\tilde{S}^2_{\geqslant 0}/GL(\mathbb{Z})$, where the action of $GL(\mathbb{Z})$ on $\tilde{S}^2_{\geqslant 0}$ is given by $GL(\mathbb{Z}) \times \tilde{S}^2_{\geqslant 0} \ni (X, Q) \mapsto X \cdot Q := X^T Q X$, as a candidate for A_2^{tr}. Indeed, there is a point-wise bijection between the two, which leads us to work with this interpretation of A_2^{tr} in Subsection 3.5.3. More precisely, A_2^{tr} is a stacky fan with cells

$$\begin{aligned}
C(D_1) &:= \bar{\sigma}_{D_1}/\operatorname{Stab}(\sigma_{D_1}), \text{ where } \bar{\sigma}_{D_1} := \left\langle \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle_{\mathbb{R}\geqslant 0} \\
C(D_2) &:= \bar{\sigma}_{D_2}/\operatorname{Stab}(\sigma_{D_2}), \text{ where } \bar{\sigma}_{D_2} := \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle_{\mathbb{R}\geqslant 0}, \\
C(D_3) &:= \bar{\sigma}_{D_3}/\operatorname{Stab}(\sigma_{D_3}), \text{ where } \bar{\sigma}_{D_3} := \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle_{\mathbb{R}\geqslant 0}, \\
C(D_4) &:= \bar{\sigma}_{D_4}/\operatorname{Stab}(\sigma_{D_4}) := \{0\},
\end{aligned}$$

where $\operatorname{Stab}(\sigma_{D_i}) \leqslant GL(\mathbb{Z})$ is the setwise stabilizer of σ_{D_i}. These are glued together according to the equivalence relation

$$Q_1 \sim Q_2 \Leftrightarrow X^t Q_1 X = Q_2 \text{ for some } X \in GL(\mathbb{Z}).$$

As pointed out in [29] (Section 4), gluing $C(D_i)$ for $i = 2, 3, 4$ does not change $C(D_1)$, so that A_2^{tr} is homeomorphic to $C(D_1)$ (see Figure 3.4).

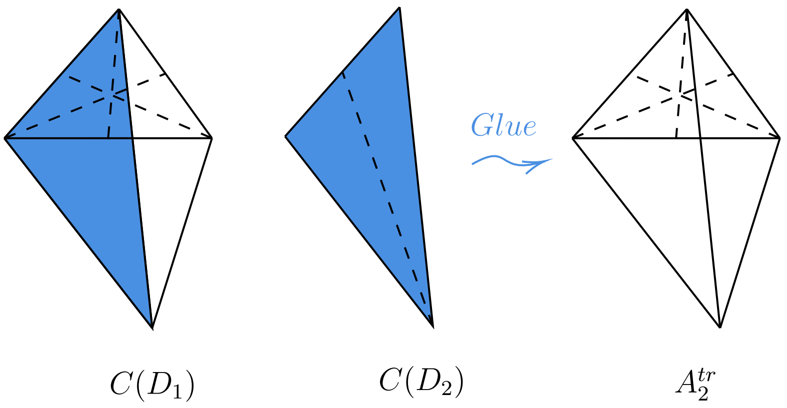


Figure 3.4: Visualization of A_2^{tr} as in [29] with dashed lines indicating symmetry.

The stacky fan M_2^{tr} (see [29], Definition and Theorem 3.4) consists of cells

$$C(G,g) := \mathbb{R}^{|E(G)|}_{\mathbb{R}_{\geqslant 0}}/\mathrm{Aut}(G,\omega)$$

for each combinatorial type (G,g), where $\mathrm{Aut}(G,g)$ is the automorphism group, i.e. the set of graph automorphisms that preserve g, and gluing determining equivalence relation: the relation of specialization (see Figure 3.5). In this setting, the tropical Torelli map sends Γ to its tropical period matrix ([13], Definition 5.1).

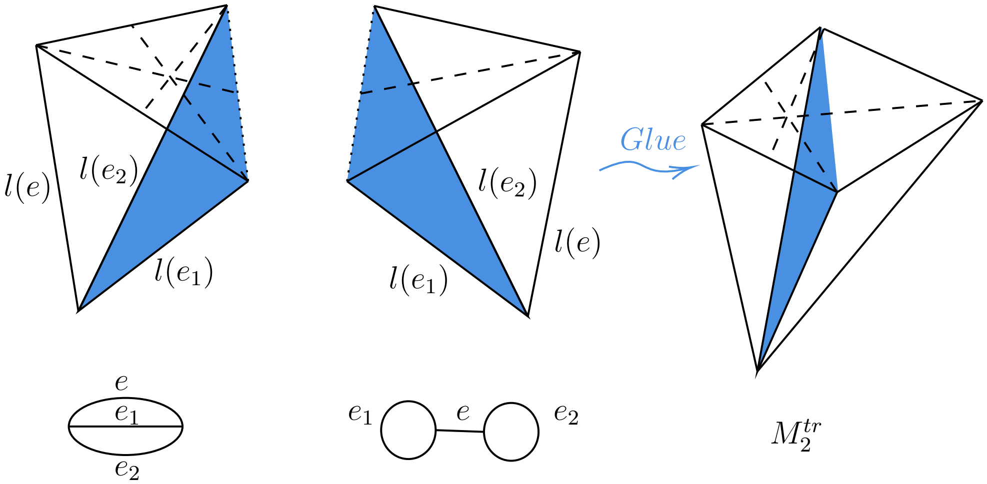


Figure 3.5: Visualization of M_2^{tr} as in [29] with dashed lines indicating symmetries.

3.5 Constructing Curves of genus 2 covering Curves of genus 1

Let Γ be a curve of genus 2. Following Chapter 2 ([32]) we can compute a splitting of $\mathrm{Jac}(\Gamma)$ whenever Γ covers an elliptic curve. Such a splitting is given by an isogeny $\phi : \mathbb{T}E' \bigoplus \mathbb{T}E \rightarrow$

$\mathrm{Jac}(\Gamma)$ from which we only want to record the *splitting data*, i.e. $(\mathbb{T}E, \mathbb{T}E', \ker(\phi))$. In this section we reverse the procedure and ask, given $\mathbb{T}E$ and $\mathbb{T}E'$, together with a finite subgroup G of their direct product:

- When is $(\mathbb{T}E, \mathbb{T}E', G)$ actual splitting data?
- If it is, is it minimal data, i.e. both necessary and sufficient, for constructing a curve of genus 2 covering $\mathbb{T}E$ and $\mathbb{T}E'$?

3.5.1 Planning the reconstruction process

Results of [32] (see Section 3.1) determine our strategy.

Plan 3.5.1. Given splitting data $(\mathbb{T}E, \mathbb{T}E', G)$, proceed as follows:

1. Determine a splitting $\phi : \mathbb{T}E \oplus \mathbb{T}E' \to J$ and generate a diagram D_ϕ modelled on Diagram 2.23.
2. Construct Γ.
3. Define covers φ and φ'.

The properties of splitting data

We address the feasibility of Plan 3.5.1 first and ask whether splitting data captures the essence of Diagram 2.23, i.e. the data that is both, necessary and sufficient, for its reconstruction.

A closer look reveals: The lower triangle (corresponding to the product property $\mathbb{T}E' \bigoplus \mathbb{T}E$) arises from the upper one (the one corresponding to the coproduct property $\mathbb{T}E' \bigoplus \mathbb{T}E$) as image through the dualization functor, $\check{\cdot} : \mathbb{T}\mathscr{A} \to \mathbb{T}\mathscr{A}$ (see Definition 2.3.8), and by subsequent identification of the duals with the original spaces. The data required for its reconstruction is therefore: An isogeny ϕ and canonical isomorphisms to the duals. We mimic this procedure to construct a diagram D_ϕ associated to ϕ.

Construction 3.5.2. Let J be a pptav and $\phi : \mathbb{T}E' \bigoplus \mathbb{T}E \to J$ an isogeny with principal polarizations $\zeta_{\mathbb{T}E' \bigoplus \mathbb{T}E}$ on $\mathbb{T}E' \bigoplus \mathbb{T}E$ (Definition 3.2.2) and ζ on J, both providing isomorphisms to the dual varieties. Start building D_ϕ from ϕ by drawing arrows that correspond to the canonical injections and projections, ι_i and p_i. These (drawn as solid arrows in Diagram 3.31) form its skeleton. Next, complete the upper part to a commutative triangle and set $f_i := \phi \circ \iota_i$ for $i = 1, 2$. For the base triangle, let $\tilde{\phi}$ be the adjoint of ϕ (with respect to the polarizations $\zeta_{\mathbb{T}E' \bigoplus \mathbb{T}E}$ and ζ as in Definition 3.2.9) and define the maps g_i by $p_i \circ \tilde{\phi}$ to finish the construction.

$$\begin{tikzcd}
\mathbb{T}E' \arrow[r, hook, "\iota_1"] \arrow[dr, dashed, "f_1"] & \mathbb{T}E' \bigoplus \mathbb{T}E \arrow[d, "\phi"] & \mathbb{T}E \arrow[l, hook', "\iota_1"'] \arrow[dl, dashed, "f_2"'] \\
& J \arrow[dl, dashed, "g_1"'] \arrow[d, dashed, "\tilde{\phi}"] \arrow[dr, dashed, "g_2"] & \\
\mathbb{T}E' & \mathbb{T}E' \bigoplus \mathbb{T}E \arrow[l, "p_1"] \arrow[r, "p_2"'] & \mathbb{T}E
\end{tikzcd} \tag{3.31}$$

Lemma 3.5.3. Let $\phi : \mathbb{T}E' \bigoplus \mathbb{T}E \to J$ be an isogeny to a pptav J. The diagram D_ϕ as in (3.31) has exact diagonals if and only if there exists an integer d such that the following holds:

- $\ker(\phi) \cong \{(\lambda, \alpha(\lambda)) : \lambda \in \mathbb{T}E'[d]\}$, where $\alpha : \mathbb{T}E'[d] \to \mathbb{T}E[d]$ is an isomorphism.
- $\phi^* \zeta = (m_d, m_d)$, where ζ is the principal polarization on J and m_d the multiplication-by-d map.

We will use the following characterization of subgroups of the direct product of two groups.

Lemma 3.5.4. (Goursat's Lemma, [44]) Let G_1, G_2 be groups and $H \leqslant G_1 \oplus G_2$ a subgroup such that the projections $p_1 : G_1 \oplus G_2 \to G_1$ and $p_2 : G_1 \oplus G_2 \to G_2$ remain surjective when restricted to H. If N_i is the kernel of the restriction of p_i to H, then the image of H in the direct product $G_1/N_2 \bigoplus G_2/N_1$ is the graph of an isomorphism

$$\alpha : G_1/N_2 \to G_2/N_1.$$

Proof of Lemma 3.5.3. Let $\phi : \mathbb{T}E' \bigoplus \mathbb{T}E \to J$ be an isogeny between pptavs and $\tilde{\phi}$ its adjoint with respect to the polarizations $\zeta_{\mathbb{T}E' \bigoplus \mathbb{T}E}$ and ζ (Definition 3.2.9). For D_ϕ to have exact diagonals means that for $i, j \in \{1, 2\}$

1. f_i is injective.
2. g_i is surjective.
3. $\ker(g_i) = \mathrm{Im}(f_j)$ holds whenever $i \neq j$.

Recall from Construction 3.5.2 that for $i \neq j$, g_j is given by $\check{f}_i$ (up to pre- and post-composition with isomorphisms). Then f_i being injective implies that $\check{f}_i$ is surjective and exactness of both diagonals is already equivalent to (1) and (3).

We express these in terms of requirements on ϕ that match the statement of Lemma 3.5.3. As

$$\ker(f_i) = \ker(\phi \circ \iota_i) = \ker(\phi) \cap \mathrm{Im}(\iota_i),$$

we have:

$$(1) \Leftrightarrow \ker(\phi) \cap \mathbb{T}E' \bigoplus \{0\} = \{0\} \text{ and } \ker(\phi) \cap \{0\} \bigoplus \mathbb{T}E = \{0\}.$$

For (3), we address $" \supset "$ first and observe:

$$\begin{aligned} g_2 \circ f_1 = 0 \text{ and } g_1 \circ f_2 = 0 \Leftrightarrow \quad & \tilde{\phi} \circ \phi(\mathbb{T}E' \bigoplus \{0\}) \subset \mathbb{T}E' \bigoplus \{0\} \text{ and} \\ & \tilde{\phi} \circ \phi(\{0\} \bigoplus \mathbb{T}E) \subset \{0\} \bigoplus \mathbb{T}E \\ \Leftrightarrow \quad & \phi^*\zeta = (m_{\alpha_1}, m_{\alpha_2}) \text{ for some } \alpha_1, \alpha_2 \in \mathbb{N}, \end{aligned}$$

where the last equivalence holds because of the following arguments: From Lemma 3.2.10 we know that the composition of ϕ with its adjoint, $\tilde{\phi}$, is given by

$$\tilde{\phi} \circ \phi = (\phi^*\zeta \circ \zeta^{-1}_{\mathbb{T}E' \bigoplus \mathbb{T}E}, \zeta^{-1}_{\mathbb{T}E' \bigoplus \mathbb{T}E} \circ \phi^*\zeta)$$

So $\tilde{\phi} \circ \phi$ respects the product $\mathbb{T}E' \bigoplus \mathbb{T}E$ (i.e. $\tilde{\phi} \circ \phi$ is a product of morphisms between the individual factors) if and only if its lift $\mathrm{Hom}(\phi^*\zeta \circ \zeta^{-1}_{\mathbb{T}E' \bigoplus \mathbb{T}E})$ does so. This in turn means that $\phi^*\zeta$ itself must be of the form (h_1, h_2) for group homomorphisms h_1 and h_2, given that $\zeta^{-1}_{\mathbb{T}E' \bigoplus \mathbb{T}E}$ is and that the Hom-functor reflects the property of being a product. Finally, recalling that all lattices involved are 1-dimensional, yields $h_i = m_{\alpha_i}$, whereby, after suitable choice of basis, we can assume that $\alpha_i \in \mathbb{N}$.

Taken together, (1) and the $" \supset "-$part of (3) determine the shape of $\ker(\phi)$ to some extent: From $\tilde{\phi} \circ \phi = (m_{\alpha_1}, m_{\alpha_2})$ we see $\ker(\tilde{\phi} \circ \phi) = \mathbb{T}E'[\alpha_1] \bigoplus \mathbb{T}E[\alpha_2]$ and $\ker(\phi)$ is a subgroup that projects (via p_1 and p_2) onto subgroups, say $\mathbb{T}E'[a_1]$, respectively $\mathbb{T}E[a_2]$, of each factor.

Subgroups of direct product that project surjectively onto each factor are characterized by Goursat's lemma (Lemma 3.5.4): If N_i is the kernel of the restriction of p_i to $\ker(\phi)$, the image of $\ker(\phi)$ in the direct product $\mathbb{T}E'[a_1]/N_2 \bigoplus \mathbb{T}E[a_2]/N_1$ is the graph of an isomorphism

$$\alpha : \mathbb{T}E'[a_1]/N_2 \to \mathbb{T}E[a_2]/N_1.$$

We remark that (1) forces both

$$N_2 = (\mathbb{T}E' \bigoplus \{0\}) \cap \ker(\phi) \text{ and } N_1 = (\{0\} \bigoplus \mathbb{T}E) \cap \ker(\phi)$$

to be trivial and thus $d := a_1 = a_2$ as α is an isomorphism.

At this point we should notice that we have almost arrived at the statement of Lemma 3.5.3. The only thing left is to take advantage of the $" \subset "-$part of (3) and claim that, under the previous conditions, $" \subset "$ holds if and only if $d = \alpha_1 = \alpha_2$ is also true.

For the "only-if"-direction, suppose (without loss of generality) that $d \neq \alpha_2$. This means that $p_{2|\ker(\phi)}$ does not project surjectively onto $\mathbb{T}E[\alpha_2]$, i.e. that there exists a $\lambda_2 \in \mathbb{T}E[\alpha_2] \backslash \mathbb{T}E[d]$. Then $\tilde{\phi} \circ \phi(0, \lambda_2) = (0, 0)$ shows $\phi(0, \lambda_2) \in \ker(g_2)$, but $\phi(0, \lambda_2) \notin \mathrm{Im}(f_1)$.

Otherwise $\phi(0, \lambda_2) = \phi(\lambda_1, 0)$ would imply $(-\lambda_1, \lambda_2) \in \ker(\phi)$, a contradiction to $\lambda_2 \in \mathbb{T}E[\alpha_2] \backslash \mathbb{T}E[d]$. For the "if"-direction, consider a point $\lambda \in \ker(g_2)$, i.e. $\tilde{\phi}(\lambda) \in \mathbb{T}E' \bigoplus \{0\}$. Since ϕ is surjective, we can write λ as $\phi(\mu_1, \mu_2)$ for some $(\mu_1, \mu_2) \in \mathbb{T}E' \bigoplus \mathbb{T}E$. Applying $\tilde{\phi}$, yields $\mu_2 \in \mathbb{T}E[\alpha_2]$. Then, $\mathbb{T}E[\alpha_2] = \mathbb{T}E[d]$ guarantees that we can find a point $(\tilde{\mu}_1, \mu_2) \in \ker(\phi)$ and obtain

$$\lambda = \lambda + 0 = \phi(\mu_1, \mu_2) - \phi(\tilde{\mu}_1, \mu_2) = \phi(\mu_1 - \tilde{\mu}_1, 0) \in \mathrm{Im}(f_1)$$

as desired. This finishes the proof. □

Notation 3.5.5. We now start with Plan 3.5.1: Take two elliptic curves $\mathbb{T}E'$, $\mathbb{T}E$ and a finite subgroup G as in Lemma 3.5.3, i.e. G is the graph of an isomorphism α between the d-torsion points of $\mathbb{T}E'$ and $\mathbb{T}E$, where $d \in \mathbb{N}$.

We will continue to use this notation for the remaining sections.

3.5.2 Step 1 of Plan 3.5.1

A first natural choice for J and ϕ, suggested by the classical treatment in [38], is given by

$$(\mathbb{T}E' \bigoplus \mathbb{T}E)_G \text{ and } q : \mathbb{T}E' \bigoplus \mathbb{T}E \to (\mathbb{T}E' \bigoplus \mathbb{T}E)_G,$$

where $(\mathbb{T}E' \bigoplus \mathbb{T}E)_G$ is the quotient of $\mathbb{T}E' \bigoplus \mathbb{T}E$ by G, and q the quotient map (Lemma 3.2.3).

Key observation

Using the "Cover-to-Splitting" direction discussed in [32] (Theorem 3.1.1) to probe the "classical" choice shows: q fails to meet the requirements of Lemma 3.5.3. This, understood well enough, will eventually show what role q and $(\mathbb{T}E' \bigoplus \mathbb{T}E)_G$ have to play.

In the setting of Theorem 3.1.1 of [32] we find q to be part of the following commutative triangle

$$\begin{tikzcd}
\mathbb{T}E' \bigoplus \mathbb{T}E \arrow[r, "\phi"] \arrow[d, "q"] & \mathrm{Jac}(\Gamma) \\
(\mathbb{T}E' \bigoplus \mathbb{T}E)_G \arrow[ur, "\bar{\phi}"'] &
\end{tikzcd} \tag{3.32}$$

where $G := \ker(\phi)$ and $\bar{\phi}$ is defined by the universal property of the quotient in **Ab**.

Breaking down (3.32) further, to the level of lattices, shows $\bar{\phi}$ is a morphism of tori:

$$\begin{tikzcd}
\Omega^1_{\mathbb{T}E'}(\mathbb{Z}) \bigoplus \Omega^1_{\mathbb{T}E}(\mathbb{Z}) & \Omega^1_{\mathbb{T}E'}(\mathbb{Z}) \bigoplus \Omega^1_{\mathbb{T}E}(\mathbb{Z}) \arrow[l, "q^{\#}"'] & \Omega^1_{\Gamma}(\mathbb{Z}) \arrow[l, "\bar{\phi}^{\#}"'] \arrow[ll, bend left, "\phi^{\#}"] \\
\mathrm{H}_1(\mathbb{T}E', \mathbb{Z}) \bigoplus \mathrm{H}_1(\mathbb{T}E, \mathbb{Z}) \arrow[r, hook, "q_{\#}"] \arrow[rr, bend right, "\phi_{\#}"'] & \pi^{-1}(G) \arrow[r, "\bar{\phi}_{\#}"] & \mathrm{H}_1(\Gamma, \mathbb{Z})
\end{tikzcd} \tag{3.33}$$

where

- $\phi = (\phi^{\#}, \phi_{\#}) = ((\varphi'_*, \varphi_*), ({\varphi'}^* \oplus \varphi^*))$.
- $q = (q^{\#}, q_{\#}) = (Id, \int + \int(\cdot))$
- $\bar{\phi} = (\bar{\phi}^{\#}, \bar{\phi}_{\#}) = (\phi^{\#}, \int(\cdot)^{-1} \circ \mathrm{Hom}(\phi^{\#})_{|\pi^{-1}(G)})$

  with

$$\int(\cdot) : \mathrm{H}_1(\Gamma, \mathbb{Z}) \to \mathrm{Hom}(\Omega^1_\Gamma, \mathbb{R}), B \to \int_B \cdot$$
$$\int + \int(\cdot) : \mathrm{H}_1(E', \mathbb{Z}) \bigoplus \mathrm{H}_1(E, \mathbb{Z}) \to \pi^{-1}(G), (B', B) \to (\int_{B'} + \int_B)(\cdot, \cdot)$$

the inclusions induced by the natural pairings on $\mathrm{Jac}(\Gamma)$, respectively on $\mathbb{T}E' \bigoplus \mathbb{T}E$, π the universal covering of $\mathbb{T}E' \oplus \mathbb{T}E$, and $\int(\cdot)^{-1}$ denotes the isomorphism $\mathrm{Im}(\int(\cdot)) \cong \mathrm{H}_1(\Gamma, \mathbb{Z})$ given by $\int_{B_i} \cdot \mapsto B_i$ for a lattice Basis (B_1, B_2) of $\mathrm{H}_1(\Gamma, \mathbb{Z})$ and the corresponding lattice Basis $(\int_{B_1}, \int_{B_2})$ of $\mathrm{Im}(\int(\cdot))$. One verifies that $\bar{\phi}_{\#}$ is a group isomorphism, (3.33) commutes, and $\bar{\phi}$ is compatible with the pairings, using $\pi^{-1}(G) = \mathrm{Hom}(\phi^{\#})^{-1}(\mathrm{H}_1(\Gamma, \mathbb{Z}))$ (since $\ker(\phi) = G$) and using that $\mathrm{Hom}(\phi^{\#})$ is a vector space isomorphism (it is a surjective map between spaces of equal dimension) that agrees with $\phi_{\#}$ (lifted to a map between the universal covers).

Now both $\mathbb{T}E' \bigoplus \mathbb{T}E$ and $\mathrm{Jac}(\Gamma)$ carry principal polarizations that provide a link between the upper and lower path. We have:

$$\begin{tikzcd}
\Omega^1_{\mathbb{T}E'}(\mathbb{Z}) \bigoplus \Omega^1_{\mathbb{T}E}(\mathbb{Z}) & \Omega^1_{\mathbb{T}E'}(\mathbb{Z}) \bigoplus \Omega^1_{\mathbb{T}E}(\mathbb{Z}) \arrow[l, "q^{\#}"'] & \Omega^1_{\Gamma}(\mathbb{Z}) \arrow[l, "\bar{\phi}^{\#}"'] \\
\mathrm{H}_1(\mathbb{T}E', \mathbb{Z}) \bigoplus \mathrm{H}_1(\mathbb{T}E, \mathbb{Z}) \arrow[u, "m_d \circ \zeta_{\mathbb{T}E' \oplus \mathbb{T}E}"] \arrow[r, hook, "q_{\#}"] & \pi^{-1}(G) \arrow[u, "\bar{\phi}^* \zeta_\Gamma"] \arrow[r, "\bar{\phi}_{\#}"] & \mathrm{H}_1(\Gamma, \mathbb{Z}). \arrow[u, "\zeta_\Gamma"]
\end{tikzcd}$$

Figure 3.6: The interaction of the natural pp on $\mathbb{T}E' \bigoplus \mathbb{T}E$ and $\mathrm{Jac}(\Gamma)$, providing a blueprint for ϕ.

This brings us to 3 important observations:

1. By default (meaning with default polarization (see [32], Lemma 26) $(\mathbb{T}E' \bigoplus \mathbb{T}E)_G$ is *not* a pptav.

2. The map $\bar{\phi}$ is *not* an isomorphism of tav, though it is, of course, an isomorphism of abelian groups. This factors ϕ into a free isogeny q and a dilation (i.e. an injective isogeny) $\bar{\phi}$ since $q^{\#}$ and $\bar{\phi}_{\#}$ are isomorphisms (compare to [69], Lemma 4.9).

3. The polarization $\bar{\phi}^* \zeta_\Gamma$ is *not* principal.

Example 3.5.6. We examine triangle (3.32) for the splitting generated by the optimal cover from Figure 3.2 with our usual choice of coordinates (for details on the computations see [32], Section 6). Then

$$\begin{tikzcd}
\mathbb{R}/\mathbb{Z} \bigoplus \mathbb{R}/3\mathbb{Z} \arrow[r, "\left(\begin{smallmatrix} 1 & 1 \\ 2 & 0 \end{smallmatrix}\right)"] \arrow[d, "I_2"] & \mathbb{R}^2 / \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbb{Z}^2 \\
\mathbb{R}^2 / \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix} \rangle_{\mathbb{Z}} \arrow[ur, "\left(\begin{smallmatrix} 1 & 1 \\ 2 & 0 \end{smallmatrix}\right)"'] &
\end{tikzcd} \tag{3.34}$$

breaks down into

$$\begin{tikzcd}
\mathbb{Z}^2 & \mathbb{Z}^2 \arrow[l, "M(\phi^{\#}) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}"'] \arrow[dl, "M(\bar{\phi}^{\#}) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}"] & \mathbb{Z}^2 \arrow[r, "M(\phi_{\#}) = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}"] \arrow[d, "M(q_{\#}) = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}"'] & \mathbb{Z}^2 \\
\mathbb{Z}^2 \arrow[u, "M(q^{\#}) = I_2"] & & \mathbb{Z}^2 \arrow[ur, "M(\bar{\phi}_{\#}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}"'] &
\end{tikzcd} \tag{3.35}$$

This yields coordinate representations for the polarizations $\bar{\phi}^*\zeta_\Gamma$ and ζ_G

$$M(\bar{\phi}^*\zeta_\Gamma) = M(\phi^\#) \cdot M(\zeta_\Gamma) \cdot M(\bar{\phi}_\#) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \cdot I_2 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$
$$M(\zeta_G) = M(\check{\zeta}_G) = M(q_\#) \cdot M(\check{\zeta}_{\mathbb{T}E' \oplus \mathbb{T}E}) \cdot M(q^\#) = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

that reflect our observations from above:

1. $(\mathbb{T}E' \oplus \mathbb{T}E)_G$ with default polarization ζ_G is *not* a pptav since ζ_G is of type $(1, 2)$.
2. $\bar{\phi}$ is only a dilation as $\phi^\#$ is not an isomorphism.
3. $\bar{\phi}^*\zeta_\Gamma$ is *not* principal.

The previous discussion demonstrates:

- The failure of q to meet the requirements of Lemma 3.5.3 can be traced back to observations (2) and (3).
- The failure of $(\mathbb{T}E' \oplus \mathbb{T}E)_G$ to provide an adequate basis for supporting a principle polarization that interacts "well" with q (see Lemma 3.5.7 for a precise statement) can be traced back to observation (1). It is confirmed by Lemma 3.5.7.

The previous discussion also demonstrates: These issues can be fixed by using Figure 3.6 as a blueprint for ϕ. We treat q as building block and construct ϕ step-wise according to Figure 3.6: The following Lemma takes care of the second vertical arrow.

Lemma 3.5.7. Let $J := (\mathbb{T}E' \oplus \mathbb{T}E)_G$ and $q : \mathbb{T}E' \bigoplus \mathbb{T}E \to J$ be the quotient map. Then there exists a unique polarization ζ on J with $q^*\zeta = m_d \circ \zeta_{\mathbb{T}E' \bigoplus \mathbb{T}E}$. Moreover, ζ is not principal.

Proof. We show existence and uniqueness of ζ using Lemma 3.2.6 and suggest referring to Diagram 3.33 for an overview of the lattices involved. Since $q^\#$ is an isomorphism, condition (1), that is $\mathrm{Im}(q^\#) \supset \mathrm{Im}(m_d \circ \zeta_{\mathbb{T}E' \bigoplus \mathbb{T}E})$, is automatically satisfied. For the divisibility condition, condition (2), we determine the invariant factors of

$$m_d \circ \zeta_{\mathbb{T}E' \bigoplus \mathbb{T}E} : \mathrm{H}_1(\mathbb{T}E', \mathbb{Z}) \bigoplus \mathrm{H}_1(\mathbb{T}E, \mathbb{Z}) \to \mathrm{Im}(q^\#) \text{ and } i : \mathrm{Im}(q_\#) \hookrightarrow \pi^{-1}(G).$$

The former is just the component-wise multiplication-by-d-map whose first and second invariant factor is d. For the latter, we first establish a coordinate representation of $\pi^{-1}(G)$ and $\mathrm{Im}(q_\#)$ inside $\mathrm{Hom}(\Omega^1_{\mathbb{T}E'}, \mathbb{R}) \oplus \mathrm{Hom}(\Omega^1_{\mathbb{T}E}, \mathbb{R})$ with respect to the dual basis $(\omega, 0)^*, (0, \tilde{\omega})^*$. Doing so we identify $\mathrm{Im}(q_\#)$ with $\Lambda_1' := \langle \begin{pmatrix} l_{\mathbb{T}E'} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ l_{\mathbb{T}E} \end{pmatrix} \rangle_{\mathbb{Z}}$, which corresponds to choosing $((B, 0), (0, \tilde{B}))$ as basis on $\mathrm{H}_1(\mathbb{T}E', \mathbb{Z}) \bigoplus \mathrm{H}_1(\mathbb{T}E, \mathbb{Z})$ and embedding it by means of $\int + \int(\cdot)$. For $\pi^{-1}(G)$, recall that G is the graph of an isomorphism α between cyclic groups of order

d (Lemma 3.5.3). Under the identification of $\mathbb{T}E$ with $\mathbb{R}/(l_{\mathbb{T}E}\mathbb{Z})$, $\frac{l_{\mathbb{T}E}}{d}$ is a generator of $\mathbb{T}E[d]$ whose preimage under α, in turn, generates $\mathbb{T}E'[d] \cong \mathbb{R}/(l_{\mathbb{T}E'}\mathbb{Z})[d]$, i.e. is of the form $\frac{k \cdot l_{\mathbb{T}E'}}{d}$ for some $k \in \mathbb{N}$ with $1 \leqslant k \leqslant d-1$ and $\gcd(k,d) = 1$. Then, $G = \langle \begin{pmatrix} \frac{k \cdot l_{\mathbb{T}E'}}{d} \\ \frac{l_{\mathbb{T}E}}{d} \end{pmatrix} \rangle_{\mathbb{Z}}$ and its lift $\pi^{-1}(G)$ to the universal cover is the 2-dimensional lattice $G + \Lambda'_1$ generated by $\begin{pmatrix} l_{\mathbb{T}E'} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{k \cdot l_{\mathbb{T}E'}}{d} \\ \frac{l_{\mathbb{T}E}}{d} \end{pmatrix}$. With respect to these we can specify the representation matrix of i since $\begin{pmatrix} 0 \\ l_{\mathbb{T}E} \end{pmatrix} = d \cdot \begin{pmatrix} \frac{k \cdot l_{\mathbb{T}E'}}{d} \\ \frac{l_{\mathbb{T}E}}{d} \end{pmatrix} - k \cdot \begin{pmatrix} l_{\mathbb{T}E'} \\ 0 \end{pmatrix}$:

$$M(i) = \begin{pmatrix} 1 & -k \\ 0 & d \end{pmatrix}$$

and compute its type (γ_1, γ_2) by

$$\gamma_1 = \gcd(1, k, d) = 1 \text{ and } \gamma_2 = \det(M(i)) = d.$$

We see, that $\operatorname{Det}(i) = \gamma_1 \cdot \gamma_2 = 1 \cdot d$ is a divisor of the first invariant factor of $m_d \circ \zeta_{\mathbb{T}E' \bigoplus \mathbb{T}E}$. Hence, by Lemma 3.2.6, ζ exists and is unique. Moreover, ζ cannot be principal: Otherwise $q^{\#} \circ \zeta \circ q_{\#}$ would have the same type as $q_{\#}$, $(1, d)$ as previously computed (since $q^{\#} \circ \zeta$ is an isomorphism), but $q^{\#} \circ \zeta \circ q_{\#} = m_d \circ \zeta_{\mathbb{T}E' \bigoplus \mathbb{T}E}$ has type (d, d), a contradiction. ☐

Determine the splitting ϕ

We now construct ϕ. For convenience, let $J = (\mathbb{T}E' \bigoplus \mathbb{T}E)_G$, as before, and relabel the target of ϕ (see step 1 of Plan 3.5.1) from J to J^{pp}, instead.

Construction 3.5.8. Let $(\mathbb{T}E', \mathbb{T}E, G)$ be splitting data and ζ be the unique polarization from Lemma 3.5.7. For lattice basis (ω_1, ω_2) of $\Omega^1_{\mathbb{T}E'}(\mathbb{Z}) \bigoplus \Omega^1_{\mathbb{T}E}(\mathbb{Z})$ and (B_1, B_2) of $\pi^{-1}(G)$ define:

- $\zeta^{pp} : \pi^{-1}(G) \to \Omega^1_{\mathbb{T}E'}(\mathbb{Z}) \bigoplus \Omega^1_{\mathbb{T}E}(\mathbb{Z})$ by $B_i \mapsto \omega_i$ for $i = 1, 2$.
- $\bar{\phi} := (\bar{\phi}^{\#}, Id)$, where $\bar{\phi}^{\#}(\omega_i) := \zeta(B_i)$.
- $J^{pp} := (\Omega^1_{\mathbb{T}E'}(\mathbb{Z}) \bigoplus \Omega^1_{\mathbb{T}E}(\mathbb{Z}), \pi^{-1}(G), [\cdot,\cdot]^{pp})$, where
  $[\omega, B]^{pp} := [\bar{\phi}^{\#}(\omega), B]_G$ for all $B \in \pi^{-1}(G)$ and $\omega \in \Omega^1_{\mathbb{T}E'}(\mathbb{Z}) \bigoplus \Omega^1_{\mathbb{T}E}(\mathbb{Z})$.

Proposition 3.5.9. Let J^{pp}, ζ^{pp} and $\bar{\phi}$ be defined as in Construction 3.5.8. Then, J^{pp} with polarization ζ^{pp} is a pptav and $\phi := \bar{\phi} \circ q$ a morphism of tavs satisfying the conditions described in Lemma 3.5.3.

Proof. Proposition 3.5.9 naturally decomposes into two parts, which we address separately:

1. J^{pp} is a pptav.
2. ϕ satisfies the conditions of Lemma 3.5.3.

Because $[\cdot,\cdot]_G$ is a non-degenerate pairing and $\bar{\phi}^{\#}$ is injective and linear, $[\cdot,\cdot]^{pp}$, which is equal to the composition $[\cdot,\cdot]_G \circ (\bar{\phi}^{\#}, Id)$, equips the quotient with the structure of an integral torus. To see that ζ^{pp} even grants J^{pp} the status of a pptav, it is only the "polarized-part" that needs checking (principal it will be, since ζ^{pp} is a group isomorphism). Thus, let us verify that ζ^{pp} induces a scalar product on $\pi^{-1}(G) \otimes \mathbb{R}$: For $\lambda_1', \lambda_2' \in \pi^{-1}(G) \otimes \mathbb{R}$ write $\lambda_1' = aB_1 + bB_2$ in terms of the basis (B_1, B_2). Unpacking definitions we obtain

$$\begin{aligned}[\zeta^{pp}(\lambda_1'), \lambda_2']^{pp} =& [a\omega_1 + b\omega_2, \lambda_2']^{pp} = [\bar{\phi}^{\#}(a\omega_1 + b\omega_2), \lambda_2']_G \\ =& [a\zeta(B_1) + b\zeta(B_1), \lambda_2']_G = [\zeta(\lambda_1'), \lambda_2']_G\end{aligned}$$

and recognize, in the last step, the bilinear form induced by ζ on J (with the quotient pairing). Since ζ is a polarization, this is indeed a scalar product. For part (2), note that by construction

$$[\bar{\phi}^{\#}(\lambda_1), \lambda_2']_G = [\lambda_1, \bar{\phi}_{\#}(\lambda_2')]^{pp}$$

holds for all $\lambda_2' \in \pi^{-1}(G)$ and $\lambda_1 \in \Omega^1_{\mathbb{T}E'}(\mathbb{Z}) \bigoplus \Omega^1_{\mathbb{T}E}(\mathbb{Z})$. Hence, $\bar{\phi}$ is a morphism, even an isogeny, and ϕ as composition of isogenies, is one as well. Target and domain varieties are principally polarized by part (1). Its kernel agrees with $\ker(q)$, since $\bar{\phi}$ is a group isomorphism, and is, therefore, the graph of an isomorphis α as required by Lemma 3.5.3 (condition 1). Condition 2, finally, holds by construction:

$$\phi^*\zeta^{pp} = (\bar{\phi} \circ q)^*\zeta^{pp} = q^*(\bar{\phi}^*\zeta^{pp}) = q^*(\zeta) = m_d \circ \zeta_{\mathbb{T}E' \bigoplus \mathbb{T}E}.$$

□

Corollary 3.5.10. Let ϕ be the map from Proposition 3.5.9 and D_ϕ the diagram associated to ϕ as in Construction 3.5.2. Then D_ϕ has exact diagonals.

Proof. This follows from Lemma 3.5.3. □

Example 3.5.11. Let $\mathbb{T}E$ and $\mathbb{T}E'$ be two elliptic curves with $l_{\mathbb{T}E} = 3$ and $l_{\mathbb{T}E'} = 1$ and $\alpha : \mathbb{T}E'[2] \to \mathbb{T}E[2]$ the unique isomorphism. We choose $\omega_1 := (\omega, 0), \omega_2 := (0, \tilde{\omega})$ as basis of $\Omega^1_{\mathbb{T}E'}(\mathbb{Z}) \bigoplus \Omega^1_{\mathbb{T}E}(\mathbb{Z})$ and let the two generators (of $\pi^{-1}(G)$) play the role of B_1 and B_2:

$$\pi^{-1}(G) = \langle (\int_{P'}^{Q'}, \int_{P}^{Q}), (\omega, 0)^* \rangle_{\mathbb{Z}} \subset \mathrm{Hom}(\Omega^1_{\mathbb{T}E'}, \mathbb{R}) \oplus \mathrm{Hom}(\Omega^1_{\mathbb{T}E}, \mathbb{R}),$$

where $Q' \in \mathbb{T}E'$ and $Q \in \mathbb{T}E$ are the only non-trivial points of order two and $P' \in \mathbb{T}E'$ and $P \in \mathbb{T}E$ are fixed reference points (that yield isomorphisms to the respective Jacobians). Under the identification $\mathbb{T}E' \cong \mathbb{R}/\mathbb{Z}$ and $\mathbb{T}E \cong \mathbb{R}/3\mathbb{Z}$ we have

$$G = \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix} \} \subset \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/3\mathbb{Z} \text{ and } \pi^{-1}(G) = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix} \rangle_{\mathbb{Z}}.$$

We continue working in coordinates and compute ζ (see Lemma 3.5.7), more precisely ${}_{(B_1,B_2)}M_{(\omega_1,\omega_2)}(\zeta)$, via Algorithm 1:

$$
{}_{(B_1,B_2)}M_{(\omega_1,\omega_2)}(\zeta) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} .
$$

Setting ζ^{pp}, $\bar{\phi}$ and J^{pp} as described above Proposition 3.5.9 yields

$$
{}_{(B_1,B_2)}M_{(B_1,B_2)}(\bar{\phi}_{\#}) = I_2 \text{ and } {}_{(\omega_1,\omega_2)}M_{(\omega_1,\omega_2)}(\bar{\phi}^{\#}) = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} ,
$$

such that all maps interact as expected (see Diagram 3.6):

$$
\begin{tikzcd}
\mathbb{Z}^2 & \mathbb{Z}^2 \arrow[l, "I_2"'] & \mathbb{Z}^2 \arrow[l, dashed, "\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}"'] \\
\mathbb{Z}^2 \arrow[u, "\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}"'] & \mathbb{Z}^2 \arrow[u, dashed, "\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}"] \arrow[r, "I_2"'] & \mathbb{Z}^2. \arrow[u, "I_2"]
\end{tikzcd}
\tag{3.36}
$$

With coordinate representations at hand, we check that ζ^{pp} is a polarization simply by computing its Gram matrix

$$
M([\zeta^{pp}(\cdot), \cdot]^{pp}) = M(\zeta)^t M([\cdot, \cdot]_G) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}
$$

and obtain

$$
\phi : \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/3\mathbb{Z} \to \mathbb{R}^2 / \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbb{Z}^2, x \mapsto \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} x,
$$

an isogeny of pptav whose associated diagram D_ϕ has exact diagonals:

$$\begin{tikzcd}
\mathbb{R}/\mathbb{Z} \arrow[r, hook, "\binom{0}{1}"] \arrow[dd, "\cdot 2"'] \arrow[dr, "\binom{2}{1}"] & \mathbb{R}/\mathbb{Z} \bigoplus \mathbb{R}/3\mathbb{Z} \arrow[d, "\left(\begin{smallmatrix} 2 & 0 \\ 1 & 1 \end{smallmatrix}\right)"] & \mathbb{R}/3\mathbb{Z} \arrow[l, hook', "\binom{0}{1}"'] \arrow[dl, "\binom{1}{0}"'] \arrow[dd, "\cdot 2"] \\
 & \mathbb{R}^2 / \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbb{Z}^2 \arrow[dl, "(1\ 0)"'] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ -1 & 2 \end{smallmatrix}\right)"] \arrow[dr, "(-1\ 2)"] & \\
\mathbb{R}/\mathbb{Z} & \mathbb{R}/\mathbb{Z} \bigoplus \mathbb{R}/3\mathbb{Z} \arrow[l, "(1\ 0)"] \arrow[r, "(0\ 1)"'] & \mathbb{R}/3\mathbb{Z}
\end{tikzcd}$$

Figure 3.7: The diagram associated to the isogeny ϕ from Example 3.5.11.

Note that in Example 3.5.11 we used the output data from Example 62 ([32]) to generate D_ϕ, but ended up in a slightly different place. Remark 3.5.12 clarifies this.

Remark 3.5.12. Construction 3.5.8 should produce a pptav J^{pp} and a dilation $\bar{\phi}$ that extend

$$\begin{tikzcd}
\Omega^1_{\mathbb{T}E'}(\mathbb{Z}) \bigoplus \Omega^1_{\mathbb{T}E}(\mathbb{Z}) & \Omega^1_{\mathbb{T}E'}(\mathbb{Z}) \bigoplus \Omega^1_{\mathbb{T}E}(\mathbb{Z}) \arrow[l, "q^{\#}"'] \\
\mathrm{H}_1(\mathbb{T}E', \mathbb{Z}) \bigoplus \mathrm{H}_1(\mathbb{T}E, \mathbb{Z}) \arrow[u, "m_d \circ \zeta_{\mathbb{T}E' \oplus \mathbb{T}E}"] \arrow[r, hook, "q_{\#}"] & \pi^{-1}(G) \arrow[u, "\zeta"']
\end{tikzcd} \tag{3.37}$$

to a commutative Diagram as in Figure 3.6. The specific pp ζ^{pp} and the isomorphism $\bar{\phi}_{\#}$ (see Construction 3.5.8) are just a matter of choice. We argue that these are natural in anticipation of Step 2 of Plan 3.5.1 (see also Definition 3.4.2): Given an arbitrary $\bar{\phi}_{\#} \in \mathrm{Aut}(\pi^{-1}(G))$, Construction 3.5.8 has to be adapted as follows. Set

$$\zeta^{\tilde{pp}}(\bar{\phi}_{\#}(B_i)) := \omega_i \text{ and } [\omega, B]^{\tilde{pp}} := [\bar{\phi}^{\#}(\omega), \bar{\phi}_{\#}^{-1}(B)]_G,$$

where $\omega \in \Omega^1_{\mathbb{T}E'}(\mathbb{Z}) \bigoplus \Omega^1_{\mathbb{T}E}(\mathbb{Z})$ and $B \in \pi^{-1}(G)$. Then $J^{\tilde{pp}}$ and J^{pp} differ by an isomorphism $f := (id, \bar{\phi}_{\#}) : J^{pp} \to J^{\tilde{pp}}$ of pptavs that respects the pp, i.e. f satisfies $f^*\zeta^{\tilde{pp}} = \zeta^{pp}$.

3.5.3 Step 2 of Plan 3.5.1

Specifically, we ask whether J^{pp} from Construction 3.5.8 is the Jacobian of a curve of genus 2. Widening our perspective, we recognize here the incarnation of a classical problem, *the tropical Schottky problem.* It is useful to embed step 2 in this more general framework and to rephrase it as follows:

- Does J^{pp} lie in the *tropical Schottky locus*, i.e. in the image of the *tropical Torelli map* $t_2^{tr} : M_2^{tr} \to A_2^{tr}$?

- If yes, can we compute the preimage(s) of J^{pp} under t_2^{tr} explicitly?

When $g \leqslant 3$ the tropical Torelli map is surjective, hence feasibility (the first question) will not be an issue. It is the second point that is of real interest, especially since t_2^{tr} is, in contrast to the classical case, *not* injective (see [13], p. 24,25).

A concrete approach.

Here, we take a practical approach to step 2 and ignore (as far as possible) the structural framework it is embedded in. We also provide a computational tool for constructing Γ and use it in Section 3.6 for gaining experience of how Γ depends on the choice of splitting data (see Lemma 3.6.2 and Lemma 3.6.3).

Interim Setup. Recall from Subsection 3.4.3: The points of A_2^{tr} are in bijection with the points of the quotient $\tilde{S}^2_{\geqslant 0}/GL(\mathbb{Z})$, where $\tilde{S}^2_{\geqslant 0}$ denotes the space of positive semidefinite matrices with rational nullspace and the action of $GL(\mathbb{Z})$ on $\tilde{S}^2_{\geqslant 0}$ is given by $GL(\mathbb{Z}) \times \tilde{S}^2_{\geqslant 0} \ni (X, Q) \mapsto X \cdot Q := X^TQX$ ([29], Section 4.3).

An essential ingredient in this context is a result by Selling. Paraphrased for our purposes it reads:

Proposition 3.5.13 (Selling)**.** Any positive definite 2×2 matrix Q is $GL(\mathbb{Z})$-equivalent to a matrix in

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \rangle_{\mathbb{R}\geqslant 0},$$

which can be computed explicitly using *Selling's Reduction Algorithm* (see [77], Section 2.3.3, and [71]). If Q satisfies $q_{12} < 0$, then transformations of type (1) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and (2) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are already sufficient.

Proposition 3.5.14. Let J^{pp} be the pptav from Construction 3.5.8. Then one of two cases can occur:

1. J^{pp} is the Jacobian of a unique tropical curve C whose combinatorial type is the theta-graph.

2. There exists a family of tropical curves $\mathcal{C}$ over $\mathbb{R}_{\geqslant 0}$ such that J^{pp} is the Jacobian of each fiber.

Using Algorithm 2 we can identify which one does and give an explicit description of the tropical curve(s) for each specific case.

Proof. In order to access the methods developed in the context of the tropical Schottky problem ([13]), we first have to reformulate our input (J^{pp} with principal polarization ζ^{pp}) in adequate language: The equivalence class of (J^{pp}, ζ^{pp}) corresponds to $\bar{Q}^{pp} \in \tilde{S}^2_{\geqslant 0}/GL(\mathbb{Z})$, where Q^{pp} is a coordinate representation of the scalar product $[\zeta^{pp}(\cdot), \cdot]^{pp}$. We have to compute Q^{pp} next: We know from the proof of Proposition 3.5.9 that $[\zeta^{pp}(\cdot), \cdot]^{pp}$ agrees with $[\zeta(\cdot), \cdot]_G$. Thus, upon fixing $\mathbb{Z}$-basis

- $S' := (\begin{pmatrix} l_{\mathbb{T}E'} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{k \cdot l_{\mathbb{T}E'}}{d} \\ \frac{l_{\mathbb{T}E}}{d} \end{pmatrix})$ of $\pi^{-1}(G)$,
- $S := ((\omega, 0), (0, \tilde{\omega}))$ on $\Omega^1_{\mathbb{T}E'} \bigoplus \Omega^1_{\mathbb{T}E}$ with dual basis S^*,

we see that $Q^{pp} := M_{(S', S')}([\zeta^{pp}(\cdot), \cdot]^{pp})$ factors as ${}_{S'}M_S(\zeta)^t \cdot M_{(S,S')}([\zeta(\cdot), \cdot]_G)$. Since the quotient pairing ([32], Lemma 26) is defined by

$$[\cdot, \cdot]_G : \Omega^1_{\mathbb{T}E'} \bigoplus \Omega^1_{\mathbb{T}E} \times \pi^{-1}(G) \to \mathbb{R}, (\lambda, \lambda') \to j(\lambda')(\lambda),$$

where j denotes the embedding $\pi^{-1}(G) \hookrightarrow \mathrm{Hom}(\Omega^1_{\mathbb{T}E'}, \mathbb{R}) \bigoplus \mathrm{Hom}(\Omega^1_{\mathbb{T}E}, \mathbb{R})$, we have:

$$M_{(S,S')}([\zeta(\cdot), \cdot]_G) =_{S'} M_{S*}(j) = \begin{pmatrix} l_{\mathbb{T}E'} & \frac{k \cdot l_{\mathbb{T}E'}}{d} \\ 0 & \frac{l_{\mathbb{T}E}}{d} \end{pmatrix}.$$

To obtain a coordinate representation for ζ (the polarization inducing $m_d \circ \zeta_{\mathbb{T}E' \bigoplus \mathbb{T}E}$) next, we use Algorithm 1, just to note that most of the work has already been carried out in the proof of Lemma 3.5.7: With the notation therein, fix $T' := ((B, 0), (0, \tilde{B}))$ as basis for $\mathrm{H}_1(\mathbb{T}E', \mathbb{Z}) \bigoplus \mathrm{H}_1(\mathbb{T}E, \mathbb{Z})$ and take the basis S and S' as above. Then ${}_{S'}M_S(\zeta)$ is given by the matrix product

$${}_{T'}M_S(m_d \circ \zeta_{\mathbb{T}E' \bigoplus \mathbb{T}E})_{S''}M_{S'}^{-1}(i) = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} d & k \\ 0 & 1 \end{pmatrix}$$

and Q^{pp} finally by

$$M_{(S',S')}([\zeta^{pp}(\cdot), \cdot]^{pp}) = \begin{pmatrix} d \cdot l_{\mathbb{T}E'} & k \cdot l_{\mathbb{T}E'} \\ k \cdot l_{\mathbb{T}E'} & \frac{k^2 l_{\mathbb{T}E'} + l_{\mathbb{T}E}}{d} \end{pmatrix}.$$

The idea is to recognize in Q^{pp} the period matrix of genus 2 graph Γ and recover Γ by just reading off edge lengths from its matrix entries. However, the non-injectivity of t_2^{tr} requires a more systematic approach: In the spirit of [13], we therefore compute a representative Q_Γ for each relevant combinatorial type Γ and check whether Q^{pp} is $GL(\mathbb{Z})$-equivalent to one of these. From Section 3.4, with the corresponding base choice, we get

$$Q^1_\Gamma = \begin{pmatrix} l(e) + l(e_2) & l(e_2) \\ l(e_2) & l(e_1) + l(e_2) \end{pmatrix} \text{ and } Q^2_\Gamma = \begin{pmatrix} l(e_1) & 0 \\ 0 & l(e_2) \end{pmatrix}$$

for the theta-graph and the dumbbell-graph with corresponding closed cones

$$\sigma_\Gamma^1 = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rangle_{\mathbb{R}_{\geqslant 0}} \text{ and } \sigma_\Gamma^2 = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle_{\mathbb{R}_{\geqslant 0}}.$$

Since σ_Γ^2 is just a face of σ_Γ^1, it suffices to consider σ_Γ^1, which we will, for the sake of convenience, replace by

$$\sigma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \sigma_\Gamma^1 = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \rangle_{\mathbb{R}_{\geqslant 0}},$$

accordingly Q^{pp} by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot Q^{pp}$, but retain our old notation Q^{pp} in this case. Next, we perform Selling's Reduction Algorithm (Proposition 3.5.13) to reduce Q^{pp} to a form lying in σ. Non-uniqueness poses a problem for determining edge lengths in a consistent way (see Remark 3.5.16 and 3.5.15). We therefore extend Selling's Reduction Algorithm (as described in [77], Section 2.3.3) by applying transformations $X \in \mathrm{Stab}(\sigma)$ to get a unique representative $\tilde{Q}^{pp}$ in

$$F := \langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \rangle_{\mathbb{R}_{\geqslant 0}}.$$

Since $F \subset \sigma$ we can write $\tilde{Q}^{pp}$ as linear combination of the extreme rays of σ:

$$\tilde{Q}^{pp} = l_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + l_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + l_3 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

If $\tilde{Q}^{pp}$ lies in the interior of σ, we conclude that Q^{pp} is the period matrix of the theta-graph whose metric is fully determined by the coefficients l_i for $i = 1, 2, 3$, i.e. we have

$$l(e) = l_1, l(e_1) = l_2, l(e_2) = l_3.$$

Suppose, however, that $\tilde{Q}^{pp}$ lies in a face of σ, then it can only lie in the interior of a 2-dimensional face since $\det(Q^{pp}) \neq 0$. Thus Q^{pp} is equivalent to a form in σ_Γ^2. In other words, it is the period matrix of a family of curves $\mathcal{C} \to \mathbb{R}_{\geqslant 0}$ with

- general fibre C_t whose combinatorial type is the dumbbell-graph with $l(e_1), l(e_2)$ determined by the two non-zero coefficients and bridge edge of length t.
- special fibre C_0 whose combinatorial type is the common specialization of the theta- and the dumbbell-graph obtained by contracting one edge.

□

An algorithmic perspective on Proposition 3.5.14: We implemented this algorithm in SINGULAR and provide instructions for use that are of interest to us here, as they clarify the question of length assignement in the case that Q lives in the boundary. For more details on Algorithm 2 see Section 3.7.

Algorithm 2 :

Input: A pptav J^{pp} as in Proposition 3.5.14.
Output: A tropical curve C with $\mathrm{Jac}(C) = J^{pp}$.

1. Perform Selling's reduction algorithm to obtain a representative Q of Q^{pp} in σ. (Output option: the Selling Parameters of Q and a list of integers that records the number of transformations of type (1) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and (2) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ yielding Q (in reverse order of their application, ending with type (1))).
2. For all $X \in \mathrm{Stab}(\sigma) \cong S_3$ compute $X \bullet Q$. If $X \bullet Q \in F$, set $Q := X \bullet Q$.
3. Let L be a list. Write Q as linear combination of the extremal rays of σ. Add the coefficients to L.
4. Return L.

Remark 3.5.15. Length output interpretation for Algorithm 2:

1. If all entries of L are non-zero, then Q^{pp} is the period matrix of the theta-graph and the entries of L correspond to the edge lengths $l(e), l(e_1), l(e_2)$ (in this order).
2. Else, Q^{pp} is the period matrix of a family of curves whose combinatorial type is the dumbbell graph. Using the labeling from Figure 3.3 we interpret the entries of L as follows:
    - If $l_3 = 0$ (i.e. $Q \in \sigma_\Gamma^2$), set $l(e_i) = l_i$.
    - If $l_2 = 0$ (i.e. $Q \notin \sigma_\Gamma^2$), consider $\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \bullet Q$ and set $l(e_1) = l_1$ and $l(e_2) = l_3$.
    - If $l_1 = 0$ (i.e. $Q \notin \sigma_\Gamma^2$), consider $\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \bullet Q$ and set $l(e_1) = l_3$ and $l(e_2) = l_2$.

From the perspective of moduli spaces.

As we move away from a sole focus on the algorithmic part of the problem, we bring the structural framework back into the picture: From the perspective of moduli spaces (Section 3.4.3) Proposition 3.5.14 asks for a special fibre, the preimage of Q^{pp} under t_2^{tr}. Since $\{\bar{\sigma}_{D_1}, \bar{\sigma}_{D_2}, \bar{\sigma}_{D_3}, \bar{\sigma}_{D_4}\}$ is a complete set of representatives for $Gl(\mathbb{Z})$-equivalence classes of secondary cones (see [29], Example 4.10), we (only) know that $Q^{pp} \in X \cdot \bar{\sigma}_{D_1}$ for an $X \in Gl(\mathbb{Z})$. We first find *an* X with $X^t Q^{pp} X \in \bar{\sigma}_{D_1}$ using Selling's Reduction Algorithm. The cone $\bar{\sigma}_{D_1}$, however, still has symmetries that arise from the action of its stabilzer $\mathrm{Stab}(\sigma_{D_1})$. So we pick a fundamental domain, the cone F in the proof of Proposition 3.5.14

(see blue-shaded area in Figure 3.8), and compute the *unique* representative $\tilde{Q}^{pp}$ of Q^{pp} in F. If $l_i \neq 0$ holds for $i = 1, 2, 3$ (see Proposition 3.5.14 and its proof), in other words, if Q^{pp} is a *generic* point (see [14], Definition 2.1.1) of A_2^{tr}, we have

$$(t_2^{tr})^{-1}(Q^{pp}) = \{C\}.$$

This reflects the fact that t_2^{tr} is of tropical degree 1. Else

$$(t_2^{tr})^{-1}(Q^{pp}) = \{\mathcal{C}\},$$

where $\mathcal{C}$ is the family of tropical curves from Proposition 3.5.14. Figure 3.8 offers a visualization of both situations.

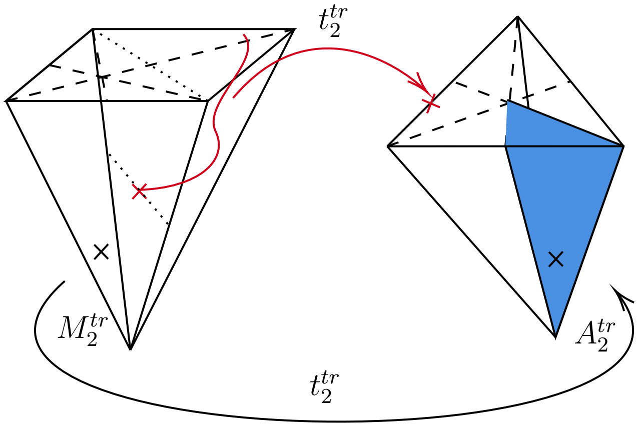


Figure 3.8: Propostion 3.5.14 for case (1) in black and case (2) in red.

Remark/Reference 3.5.16. The reconstruction is based on the approach of [13], where they construct an *arbitrary* curve with a given tropical Jacobian. Not only arbitrary in the sense, that a positive definite quadratic form Q can be the period matrix associated to curves with different combinatorial types (the natural non-injectivity of t_2^{tr}). But also arbitrary when specifying the metric for a fixed combinatorial type (G, g). There is not necessarily a unique $Gl(\mathbb{Z})$ representative of Q in the corresponding cone $\sigma_{(G,g)}$, since we can still operate on Q with $X \in \mathrm{Stab}(\sigma_{(G,g)})$. Depending on our choice of representative, we get different metrics on (G, g). The reason for this is as follows: The action of $\mathrm{Stab}(\sigma_{(G,g)})$ on elements of $\sigma_{(G,g)}$ really corresponds to permuting edges of G. To determine edge lengths, however, we fix a cycle base and an order on $E(G)$, write Q as linear combination of the corresponding extreme rays and match the resulting coefficients to the edges according to our chosen order. In order to do justice to the geometric meaning behind the $\mathrm{Stab}(\sigma_{(G,g)})$-action, we remove the element of arbitrariness and only use the representative of Q in F to determine the metric.

3.5.4 Step 3 of Plan 3.5.1

Fix $\Gamma := C$ or $\Gamma := C_0, \Gamma_t := C_t$ for $t \in \mathbb{R}_{>0}$, $P_0 \in \Gamma$ as in Figure 3.3, depending on whether case (1) or (2) occurs in Proposition 3.5.14, and define maps

$$\varphi := g_2 \circ \Phi_{P_0} \text{ and } \varphi' := g_1 \circ \Phi_{P_0}$$

adding a subscript t (φ_t, φ'_t) for case (2). At this point, all that remains is to assemble the building blocks from steps 1 and 2 of Plan 3.5.1.

Theorem 3.5.17. Let $(\mathbb{T}E', \mathbb{T}E, G)$ be splitting data, i.e. G is the graph of an isomorphism α between the d-torsion points of $\mathbb{T}E'$ and $\mathbb{T}E$. Then there exists

- a (family of) curve(s) of genus 2 Γ (Γ_t for $t \geqslant 0$),
- a (family of) pair(s) of optimal covers $(\varphi' : \Gamma \to \mathbb{T}E', \varphi : \Gamma \to \mathbb{T}E)$ ((φ_t, φ'_t) for $t \geqslant 0$) of degree d,

that induce a splitting

$$\phi : \mathbb{T}E' \bigoplus \mathbb{T}E \to \mathrm{Jac}(\Gamma)$$

with $\mathrm{Jac}_d(\mathbb{T}E') \cong \ker(\phi) \cong \mathrm{Jac}_d(\mathbb{T}E)$.

Proof. Let ϕ, Γ, φ' and φ be as in steps 1-3, omitting the subscript t for clarity. All that remains is to show that the maps φ and φ' are optimal covers of degree d with $g_1 = \varphi'_*$, $g_2 = \varphi_*$, $f_1 = \varphi^*$ and $f_2 = {\varphi'}^*$. Compare the definition of φ and φ' with the complementary cover ([32], Section 6.2) and note that they are constructed according to the same pattern. We therefore refrain from proving the covering-property and refer to Lemma 64 in [32], instead. With two covers at our disposal we embark to reinterpret Diagram 3.31 in terms of φ and φ': We specify embeddings

$$j_1 : \mathbb{T}E' \to \mathrm{Jac}(\mathbb{T}E'), P' \mapsto P' - 0_{\mathrm{Jac}(\mathbb{T}E')}$$
$$j_2 : \mathbb{T}E \to \mathrm{Jac}(\mathbb{T}E), P \mapsto P - 0_{\mathrm{Jac}(\mathbb{T}E)}$$

that are compatible with Φ_{P_0}. Under these identifications (i.e. $\mathbb{T}E' \cong \mathrm{Jac}(\mathbb{T}E')$ and $\mathbb{T}E' \cong \mathrm{Jac}(\mathbb{T}E')$ via j_1 and j_2) we see that

- g_1 (g_2) corresponds to the push-forward φ'_* (φ_*) since

$$\begin{tikzcd} \Gamma \arrow[r, "\Phi_{P_0}"] \arrow[d, "\varphi'"] & \mathrm{Jac}(\Gamma) \arrow[d, "g_1"] & \Gamma \arrow[r, "\Phi_{P_0}"] \arrow[d, "\varphi"] & \mathrm{Jac}(\Gamma) \arrow[d, "g_2"] \\ \mathbb{T}E' \arrow[r, hook, "j_1"] & \mathrm{Jac}(\mathbb{T}E') & \mathbb{T}E \arrow[r, hook, "j_2"] & \mathrm{Jac}(\mathbb{T}E), \end{tikzcd} \tag{3.38}$$

commute.

- f_1 (f_2) corresponds to the pull-back φ'^* (φ^*) ([32], Lemma 35)

which, together with the exactness of the diagonals in 3.31, implies optimality of φ and φ'. We can now compute $deg(\varphi)$ and $deg(\varphi')$ as in the proof of Theorem 66 ([32]) For φ we have

$$\begin{aligned} deg(\varphi)id &= \varphi_* \circ \varphi^* = (g_2 \circ j_2) \circ (j_2^{-1} \circ f_2) \\ &= p_2 \circ \tilde{\phi} \circ \phi \circ \iota_2, = p_2 \circ (m_d, m_d) \circ \iota_2 = m_d \end{aligned}$$

and thus $deg(\varphi) = d$. The analogous computation for φ' shows $deg(\varphi') = d$ as well. ☐

We let Example 3.5.11 rest for reconstruction step 2, as it was already conceivable from the "Cover-to-Splitting" direction (see Figure 3.2 and Example 67 ([32]), that J^{pp} is the Jacobian of the theta-graph whose edges have all length 1. We pick it up again to illustrate step 3 of Plan 3.5.1.

Example 3.5.18. We define

$$\begin{tikzcd} \varphi' : \Gamma \arrow[r, hook, "\Phi_{P_0}"] & J^{pp} = \mathrm{Jac}(\Gamma) \arrow[r, "g_1"] \arrow[d, "\cong"] & \mathrm{Jac}(\mathbb{T}E') \arrow[r, "j_1"] \arrow[d, "\cong"] & \mathbb{T}E' \\ & \mathbb{R}^2/\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbb{Z}^2 \arrow[r, "(1\ 0)"] & \mathbb{R}/\mathbb{Z} & \end{tikzcd} \tag{3.39}$$

and

$$\begin{tikzcd} \varphi : \Gamma \arrow[r, hook, "\Phi_{P_0}"] & J^{pp} = \mathrm{Jac}(\Gamma) \arrow[r, "g_2"] \arrow[d, "\cong"] & \mathrm{Jac}(\mathbb{T}E) \arrow[r, "j_2"] \arrow[d, "\cong"] & \mathbb{T}E \\ & \mathbb{R}^2/\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbb{Z}^2 \arrow[r, "(-1\ 2)"] & \mathbb{R}/3\mathbb{Z} & \end{tikzcd} \tag{3.40}$$

where coordinate representations are taken from Figure 3.7. In order to understand φ and φ' as morphisms of graphs we work locally as [32] (see Example 67 for details). We can easily read off parametrizations of e, e_1 and e_2 in J^{pp} from Figure 10 ([32]). By acting on these with $M(g_i)$ we finally obtain

$$\begin{aligned} &\varphi'_{|e}(t) = t + 0_{\mathrm{Jac}(\mathbb{T}E')}, \varphi'_{|e_1}(t) = 0_{\mathrm{Jac}(\mathbb{T}E')}, \varphi'_{|e_2}(t) = t + 0_{\mathrm{Jac}(\mathbb{T}E')}, \\ &\varphi_{|e}(t) = -t + 0_{\mathrm{Jac}(\mathbb{T}E)}, \varphi_{|e_1}(t) = -2t + 0_{\mathrm{Jac}(\mathbb{T}E)}, \varphi_{|e_2}(t) = t + 0_{\mathrm{Jac}(\mathbb{T}E)}, \end{aligned}$$

where $t \in [0, 1]$.

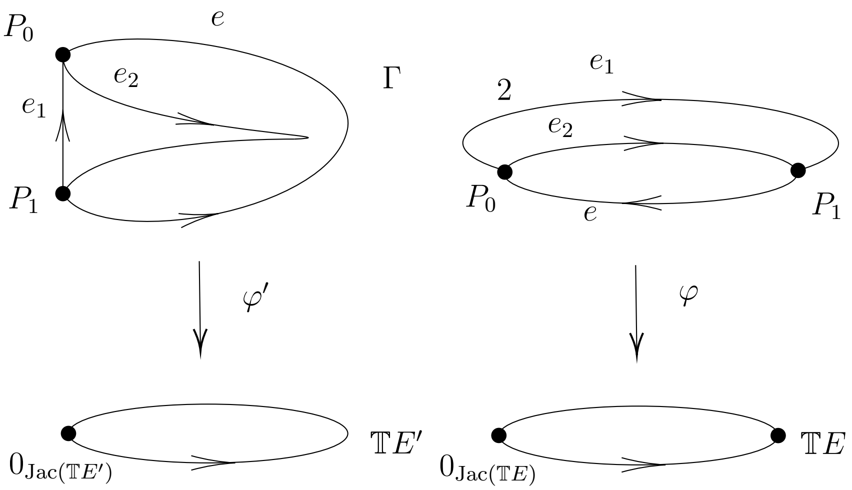


Figure 3.9: φ' and φ as morphisms of graphs.

Remark 3.5.19. We can now answer the two questions at the beginning Section 3.5: $(\mathbb{T}E', \mathbb{T}E, G)$ is splitting data if and only if it satisfies the conditions described in Lemma 3.5.3. Moreover, Theorem 3.5.17 shows, it is both necessary and sufficient.

3.6 Moduli space perspective

We investigate a Schottky-type problem and study the fibres of t_2^{tr} for the case of split Jacobians. In this context, we are interested in the following subsets:

1. $\mathcal{Q} \subset A_2^{tr}$ the *locus of split Jacobians.*
2. $\mathbb{T}\mathcal{L}_d \subset M_2^{tr}$ the *locus of curves with d-split Jacobians.*

The second is a tropical analogue of the locus of genus 2 curves with (d,d)-split Jacobian studied for example in [73], [59]) or [74]. The notation from the previous Section is maintained.

3.6.1 A Schottky-type Problem

As a result of our reconstruction procedure in Section 3.5, we have a representative for each isomorphism-class of pptavs that gives rise to a split Jacobian, i.e. we have a description for $\mathcal{Q}$:

$$\mathcal{Q} = \{Q^{pp}(d, k, l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \mid d \in \mathbb{N}_{>1}, k \in \mathbb{Z}_d^*, l_{\mathbb{T}E'}, l_{\mathbb{T}E} \in \mathbb{R}_{>0}\}.$$

With this family of pptavs coming from geometry at hand, it is natural to ask for $\mathcal{Q} \cap \partial A_2^{tr}$. For each specific case, Proposition 3.5.14 tells us whether Q^{pp} is a generic point, but it does so only *a posteriori* meaning after running through Algorithm 2. Much more satisfying would be an *a priori* characterization, one that is inspired by the following observation

$$\begin{aligned} Q^{pp} \in \sigma \text{ is not a generic point } \Leftrightarrow & \, l_{\mathbb{T}E} = kl_{\mathbb{T}E'}(d-k) \text{ for } k, d \in \mathbb{N}, \\ & k < d, \gcd(k,d) = 1. \end{aligned}$$

This can be seen immediately by writing Q^{pp} as linear combination of the extreme rays of σ and noting that only one of the coefficients can vanish. More generally:

Remark 3.6.1. We have used the description of σ in terms of generators. As a system of inequalities it is given by

$$\{ \begin{pmatrix} q_{11} \\ q_{12} \\ q_{22} \end{pmatrix} \in \mathbb{R}^3 \,|\, q_{12} \leqslant 0, q_{11} + q_{12} \geqslant 0, q_{22} + q_{12} \geqslant 0 \},$$

where we identify a 2×2 symmetric matrix with entries q_{ij} with a vector in $\mathbb{R}^3$. Then, $Q \in \sigma$ lies in a 2-dimensional face if and only if one of the inequalities is not strict.

We exploit Remark 3.6.1 and Algorithm 2 to give a concrete description of the intersection with the boundary for two subfamilies of $\mathcal{Q}$.

Lemma 3.6.2. Consider the same setting as in Proposition 3.5.14 and let $k = 1$. Then case (2) holds if and only if there exists $\alpha \in \mathbb{N}_{>0}$ with

$$\alpha \cdot l_{\mathbb{T}E} = (d - \alpha) \cdot l_{\mathbb{T}E'}.$$

Proof. The proof is based on two simple observations:

1. A reduced representative Q of Q^{pp} inside σ can be obtained only using the transformation $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.
2. Q lies in a 2-dimensional face if and only if $q_{22} + q_{12} = 0$ holds.

We start with (1): Consider the sequence

$$a_n := -q_{12}^{pp} - (n+1)q_{22}^{pp}, \; n \in \mathbb{N}$$

and note that $(a_n)_{n \in \mathbb{N}}$ is strictly decreasing. Let $\tilde{\alpha} := \min\{n \in \mathbb{N} | a_n \leqslant 0\}$. We claim that $Q := \begin{pmatrix} 1 & 0 \\ \tilde{\alpha} & 1 \end{pmatrix} \cdot Q^{pp} \in \sigma$, i.e. that Q satisfies the inequalities from Remark 3.6.1. A computation shows

$$\begin{aligned} q_{12} &= q_{12}^{pp} + \tilde{\alpha} q_{22}^{pp}, \\ q_{11} + q_{12} &= q_{11}^{pp} + (2\tilde{\alpha} + 1) q_{12}^{pp} + \tilde{\alpha}(\tilde{\alpha} + 1) q_{22}^{pp}, \\ q_{22} + q_{12} &= q_{12}^{pp} + (\tilde{\alpha} + 1) q_{22}^{pp}, \end{aligned}$$

i.e.

$$\begin{aligned} q_{12} &= -a_{\tilde{\alpha}-1} < 0 \\ q_{11} + q_{12} &= q_{11}^{pp} + (\tilde{\alpha}+1)q_{12}^{pp} - \tilde{\alpha}a_{\tilde{\alpha}} \geqslant q_{11}^{pp} + (\tilde{\alpha}+1)q_{12}^{pp} \\ q_{22} + q_{12} &= -a_{\tilde{\alpha}} \geqslant 0 \end{aligned}$$

as an immediate consequence of the definition of $\tilde{\alpha}$ and the fact that $(a_n)_{n\in\mathbb{N}}$ is strictly decreasing. Recalling that $q_{11}^{pp} = dl_{\mathbb{T}E'}$ and $q_{12}^{pp} = -kl_{\mathbb{T}E'}$, we obtain

$$q_{11}^{pp} + (\tilde{\alpha}+1)q_{12}^{pp} \geqslant 0$$

since $d \geqslant \tilde{\alpha}+1$ (again by minimality of $\tilde{\alpha}$) and conclude (1). Note that point (2) is just a special case of Remark 3.6.1 when combined with the following observation:

$$\begin{aligned} q_{11} + q_{12} = 0 &\Leftrightarrow q_{11}^{pp} + (\tilde{\alpha}+1)q_{12}^{pp} = 0 \text{ and } \tilde{\alpha}a_{\tilde{\alpha}} = 0 \\ &\Leftrightarrow d = \tilde{\alpha}+1 \text{ and } \tilde{\alpha}(l_{\mathbb{T}E'} - (\tilde{\alpha}+1)\frac{l_{\mathbb{T}E'} + l_{\mathbb{T}E}}{d}) = 0 \\ &\Leftrightarrow d = \tilde{\alpha}+1 \text{ and } (l_{\mathbb{T}E'} - (\tilde{\alpha}+1)\frac{l_{\mathbb{T}E'} + l_{\mathbb{T}E}}{d}) = 0, \end{aligned}$$

where the last equivalence holds since $d > 1$ (and thus $\tilde{\alpha} \neq 0$), so we have a contradiction. As intermediate result (from (1) and (2)) we get: Case (2) in Proposition 3.5.14) holds if and only if $q_{22}+q_{12}$, i.e. $a_{\tilde{\alpha}} = l_{\mathbb{T}E'} - (\tilde{\alpha}+1)\frac{l_{\mathbb{T}E'}+l_{\mathbb{T}E}}{d}$ vanishes. This shows the "only if"-direction (by setting $\alpha := \tilde{\alpha}+1$ of the statement of Lemma 3.6.2. For the "if"-direction suppose there exists such an $\alpha \in \mathbb{N}_{>0}$, then $a_{\alpha-1} = 0$ and $\alpha - 1 = \tilde{\alpha}$ since $(a_n)_{n\in\mathbb{N}}$ is strictly decreasing. This means $Q \in \sigma$ by point (1), in particular Q is in a 2-dimensional face by (2). ☐

A symmetrical result is content of the following Lemma.

Lemma 3.6.3. Consider the same setting as in Proposition 3.5.14 and let $d \geqslant 3$, $k = d-1$. Then case (2) holds if and only if there exists $\beta \in \mathbb{N}_{>0}$ with

$$\beta \cdot l_{\mathbb{T}E} = (d-\beta) \cdot l_{\mathbb{T}E'}.$$

The proof is based on the same methods, but their implementation is tedious and more complicated. We leave it at a rough sketch.

Sketch of Proof. If $Q^{pp} \in \sigma$, then case (2) holds if and only if $l_{\mathbb{T}E} = (d-1) \cdot l_{\mathbb{T}E'}$ (see Observation above Remark 3.6.1. If not, replace the two observations from the proof of Lemma 3.6.2 by:

1. We can find a reduced representative Q of Q^{pp} inside σ of the form

$$\begin{pmatrix} 1 & \tilde{\beta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \boldsymbol{\cdot} Q^{pp}.$$

2. Q lies in a 2-dimensional face if and only if $q_{11} + q_{12} = 0$ holds.

Consider a sequence $(b_n)_{n\in\mathbb{N}}$ analogous to $(a_n)_{n\in\mathbb{N}}$ and set

$$\tilde{\beta} := \min\{n \in \mathbb{N} | b_n \leqslant 0\}.$$

Then check that the entries of Q satisfy the inequalities from Remark 3.6.1 and argue that equality can hold at most once, namely for $q_{11} + q_{12}$. A final substitution provides us with the claimed equation. ☐

Example 3.6.4. Consider two elliptic curves $\mathbb{T}E$ and $\mathbb{T}E'$ with $l_{\mathbb{T}E} = 5$ and $l_{\mathbb{T}E'} = 3$. We are looking for subgroups G of the direct product $\mathbb{T}E' \oplus \mathbb{T}E$ that arise from isomorphisms between their d-torsions. These should be as required by Lemma 3.6.2 and 3.6.3 of the form (here in coordinates, i.e. as map $\mathbb{R}/3\mathbb{Z}[d] \to \mathbb{R}/5\mathbb{Z}[d]$):

$$\frac{3}{d} \mapsto \frac{5}{d} \text{ (as in Lemma 3.6.2) and } \frac{(d-1)3}{d} \mapsto \frac{5}{d} \text{ (as in Lemma 3.6.3)}$$

and yield for each d a pptav J^{pp} (Constrution 3.5.8) that is the Jacobian of a family of curves whose combinatorial type is the dumbbell-graph. According to the previous discussion consider

$$a \cdot 5 = (d - a) \cdot 3,$$

which is satisfied, for example, by the tuple $(d = 16, a = 6)$ or $(d = 24, a = 9)$. Running through Algorithm 2 we find that the special fiber of the corresponding family of genus 2 curves is defined by $(l(e_1) = \frac{1}{2}, l(e_2) = 30)$, for the tuple $(d = 16, a = 6)$, and $(l(e_1) = \frac{1}{3}, l(e_2) = 45)$, for the tuple $(d = 24, a = 9)$. The general fibers are as in the proof of Proposition 3.5.14.

Example 3.6.4 illustrates: Geometrically, the subfamilies of pptavs from Lemma 3.6.2 and 3.6.3 correspond to fixing a special class of isomorphisms between the d-torsion points of $\mathbb{T}E'$ and $\mathbb{T}E$. The reason for explicit results in these cases is simply that the (type of) reduced representative in σ that is produced by Algorithm 2 is fixed. For a general $Q^{pp} \in \mathcal{Q}$ we only know that there exists a sequence $(\alpha_1, \beta_1, ..., \alpha_n, \beta_n)$ of positive integers such that

$$\begin{pmatrix} 1 & \beta_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha_n & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & \beta_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha_1 & 1 \end{pmatrix} \cdot Q^{pp} \in \sigma.$$

If we want to extend our previous approach, the question arises whether there exist other subfamilies of $\mathcal{Q}$ that restrict the possibilities for $(\alpha_1, \beta_1, ..., \alpha_n, \beta_n)$ (e.g. that fix n). In other words whether $(\alpha_1, \beta_1, ..., \alpha_n, \beta_n)$ has geometric meaning.

Example 3.6.5. In the vein of previous results, we aim at classes of isomorphisms whose graphs are the subgroups G (Lemma 3.5.3). But more conservatively, work with fixed $d = 5$ and $k = 2$ first. The result is sobering: For $l_{\mathbb{T}E} = 2$ and $l_{\mathbb{T}E'} = 3$, for example, we have type $(2, 0)$, and for $l_{\mathbb{T}E} = 10$ and $l_{\mathbb{T}E'} = 100$, type $(3, 1)$.

Hence, from now on we will fix $(\alpha_1, \beta_1, ..., \alpha_n, \beta_n)$ as additional, purely algebraic datum and split the initial question about the intersection $\mathcal{Q} \cap \partial A_2^{tr}$:

1. Given G (i.e. given k, d), can we find $\mathbb{T}E$ and $\mathbb{T}E'$ such that J^{pp} as constructed in 3.5.8 is equal to $\mathrm{Jac}(\mathcal{C})$ (see Proposition 3.5.14, case (2))?
2. Given $\mathbb{T}E$ and $\mathbb{T}E'$, can we find G such that $J^{pp} = \mathrm{Jac}(C_t)$ for each member C_t of $\mathcal{C}$?

Question (1). The following Lemma can be viewed as a generalization of Lemma 3.6.2 and 3.6.3, assuming that the representative is fixed (!).

Lemma 3.6.6. Let $1 \leqslant k \leqslant d$ be coprime and $(\alpha_1, \beta_1, ..., \alpha_n, \beta_n) \in \mathbb{N}\backslash\{0\}^{2n-1} \cup \mathbb{N}$. Then there exists elliptic curves $\mathbb{T}E$ and $\mathbb{T}E'$ with $J^{pp} = \mathrm{Jac}(\mathcal{C})$ as in case (2) of Proposition 3.5.14, if one of the following two linear equations has a solution $(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \in \mathbb{R}_{>0}$:

$$c_1^j(\alpha_i, \beta_i, k, d) l_{\mathbb{T}E'} + c_2^j(\alpha_i, \beta_i) l_{\mathbb{T}E} = 0, \ j = 1, 2,$$

where the coefficients c_1^j and c_2^j for $j = 1, 2$ are fixed by G (i.e. by k, d) and $(\alpha_1, \beta_1, ..., \alpha_n, \beta_n)$.

Proof. Let $Q := \begin{pmatrix} 1 & \beta_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha_n & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & \beta_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha_1 & 1 \end{pmatrix} \cdot Q^{pp}$. We claim that $Q \in \partial\sigma$, whenever one of the following holds:

$$q_{11} + q_{12} = 0 \text{ or } q_{22} + q_{12} = 0. \tag{3.41}$$

Suppose $q_{11} + q_{12} = 0$. We have to show that the entries of Q satisfy the remaining inequalities from Remark 3.6.1. Note that $q_{11} > 0$ since q_{11}^{pp} is and, given that transformations of type $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ keep one of the diagonal entries of Q^{pp} fixed, the other does not change sign as well.

Exemplary for $\tilde{Q} := \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \cdot Q^{pp}$: We have $\tilde{q}_{11} = q_{11}^{pp} + \alpha q_{12}^{pp}$ and $\tilde{q}_{22} = q_{22}^{pp}$. Then $0 < \tilde{q}_{11}\tilde{q}_{22} - (\tilde{q}_{12})^2$ ($\tilde{Q}$ is positive definite) implies that $\tilde{q}_{11}$ and $\tilde{q}_{22}$ have the same sign.

This shows $q_{12} = -q_{11} < 0$ and $q_{22} + q_{12} = q_{22} - q_{11} > 0$ follows from $0 < q_{11}q_{22} - (q_{12})^2 = q_{11}q_{22} - (q_{11})^2$. Proceeding analogously for the case $q_{22} + q_{12} = 0$ proves the claim.

To obtain equations of the form given in Lemma 3.6.6, consider, equivalently to Equations (3.41), their d-multiple. It then follows from the structure of Q^{pp} that these are linear in $l_{\mathbb{T}E'}$ and $l_{\mathbb{T}E}$, i.e. with no constant term: By acting with $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ on Q^{pp} we simply swap the entries of Q^{pp}, which are linear functions in $l_{\mathbb{T}E'}$ and $l_{\mathbb{T}E}$, for linear combinations of them. ☐

Remark 3.6.7.

- The existence of a solution to the equations in Lemma 3.6.6 is equivalent to c_1^j and c_2^j having different sign. A closer look at both reveals that $c_2^j > 0$. The sign of c_1^j, however, depends on G and $(\alpha_1, \beta_1, ..., \alpha_n, \beta_n)$.
- The proof shows that Equations (3.41) cannot be satisfied simultaneously.

Question (2). Unlike Question (1), answering Question (2) involves finding a special solution to a quadratic equation. By considering

$$q_{11} + q_{12} = 0 \text{ or } q_{22} + q_{12} = 0$$

as equations in d and k we get:

$$c_1^j(\alpha_i, \beta_i) l_{\mathbb{T}E'} \cdot k^2 + c_2^j(\alpha_i, \beta_i) l_{\mathbb{T}E'} \cdot kd + c_3^j(\alpha_i, \beta_i) l_{\mathbb{T}E'} \cdot d^2 + c_4^j(\alpha_i, \beta_i) l_{\mathbb{T}E} = 0,$$

where $c_i^j \in \mathbb{Z}$ with $c_1^j, c_3^j, c_4^j > 0$ and $c_2^j < 0$ for $j = 1, 2$ are fixed by $(\alpha_1, \beta_1, ..., \alpha_n, \beta_n)$. Then G exists, if one of the above has a solution (d, k) with $d \in \mathbb{N}$ and $k \in \mathbb{Z}_d^*$.

3.6.2 Locus of curves with split Jacobian

Subsections 3.5.3 and 3.6.1 give us a way to understand and organize the locus of curves with d-split Jacobians that follows naturally from previous considerations.

Construction 3.6.8. The locus $\mathbb{T}\mathcal{L}_d := (t_2^{tr})^{-1}(\mathcal{Q}_d)$ decomposes into

$$\mathbb{T}\mathcal{L}_d = \bigcup_{k \in \mathbb{Z}_d^*} L_k,$$

where $L_k := (t_2^{tr})^{-1}(\mathcal{Q}_{d,k})$ and $\mathcal{Q}_d \subset \mathcal{Q}$ and $\mathcal{Q}_{d,k} \subset \mathcal{Q}$ are the subsets obtained by fixing d, respectively d and k. Now, $\mathcal{Q}_{d,k}$ is the disjoint union of $\mathcal{Q}_{d,k} \cap \partial\mathcal{A}_2^{tr}$ and $\mathcal{Q}_{d,k} \backslash (\mathcal{Q}_{d,k} \cap \partial\mathcal{A}_2^{tr})$.

By Lemma 3.6.6 we know that $\mathcal{Q}_{d,k} \cap \partial\mathcal{A}_2^{tr}$ is in 1:1 correspondence with a union of half rays:

$$\bigcup_{(\alpha_i, \beta_i)} \{(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \in \mathbb{R}^2_{>0} \,|\, c_1(\alpha_i, \beta_i, k, d) l_{\mathbb{T}E'} + c_2(\alpha_i, \beta_i, k, d) l_{\mathbb{T}E} = 0\},$$

where (α_i, β_i) are so that $c_1(\alpha_i, \beta_i, k, d) > 0$ (Remark 3.6.7). In this case we say (α_i, β_i) is *feasible* (for d and k).

We define the fan Δ_k associated to L_k as the collection of cones obtained by subdividing $\mathbb{R}^2_{>0}$ as above and taking the Euclidean closure.

Theorem 3.6.9. The locus of curves with d-split Jacobian decomposes into $\varphi(d)$ subsets

$$\mathbb{T}\mathcal{L}_d = \bigcup_{k \in \mathbb{Z}_d^*} L_k,$$

where $\varphi(d)$ denotes the Euler phi function. The fan Δ_k (see Construction 3.6.8) has maximal cones $\sigma_{(\alpha_i, \beta_i)}$ indexed by feasible types and relates to L_k via a family of linear maps $\{\phi_{\sigma_{(\alpha_i,\beta_i)}} : \sigma_{(\alpha_i, \beta_i)} \to M_2^{tr}\}$ such that

- $\phi_{\tilde{\sigma}}$ maps the interior of $\tilde{\sigma} := \sigma_{(\alpha_i,\beta_i)}$ to a 2 dimensional cone in the theta part of M_2^{tr}.
- restricting $\phi_{\tilde{\sigma}}$ to a ray r that is a face of two maximal cones yields a linear map $\phi_{\tilde{\sigma}|r} \times id : r \times \mathbb{R}_{\geqslant 0} \to M_2^{tr}$ whose image in the dumbbell part of M_2^{tr} is $\phi_{\tilde{\sigma}}(r) \times r_B$, where r_B denotes the ray that corresponds to the bridge edge.
- the maps in $\{\phi_{\sigma_{(\alpha_i,\beta_i)}}\}$ are compatible in the sense that for two neighboring maximal cones $\tilde{\sigma}$ and $\tilde{\sigma}'$ the restriction of $\phi_{\tilde{\sigma}}$, respectively $\phi_{\tilde{\sigma}'}$, to a common face agree.

Proof. We address the structure of Δ_k first and then describe its realization in M_2^{tr}.

1. Maximal cones are indexed by feasible types:

A point $(l_{\mathbb{T}E'}, l_{\mathbb{T}E})$ in the interior of a maximal cone $\tilde{\sigma} \in \Delta_k$ corresponds to $Q^{pp}(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \in (\mathcal{Q}_{d,k} \backslash (\mathcal{Q}_{d,k} \cap \partial \mathcal{A}_2^{tr}))$, i.e. we have

$$q_{12} < 0, -q_{12} - q_{11} < 0, -q_{12} - q_{22} < 0, \tag{3.42}$$

where $Q := (q_{ij}) := (\alpha_i, \beta_i) \bullet Q^{pp}(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \in \sigma$ denotes the representative (in σ) obtained by Selling's reduction algorithm (see Subsection 3.5.3). Since inequalities (3.42) depend continuously on $(l_{\mathbb{T}E'}, l_{\mathbb{T}E})$, points in a small neighborhood of $(l_{\mathbb{T}E'}, l_{\mathbb{T}E})$ yield pp of type (α_i, β_i) as well. This is true as long as inequalities (3.42) are preserved.

If they are not, then $Q(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \in \partial A_2^{tr}$ and hence $Q^{pp}(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \in \partial A_2^{tr}$. More precisely: If $-q_{12} - q_{11} = 0$ or $-q_{12} - q_{22} = 0$ holds, then $Q^{pp}(l_{\mathbb{T}E'}, l_{\mathbb{T}E})$ has type (α_i, β_i) as well and $(l_{\mathbb{T}E'}, l_{\mathbb{T}E})$ lies on the ray that corresponds to (α_i, β_i). If $q_{12} = 0$, then $(l_{\mathbb{T}E'}, l_{\mathbb{T}E})$ is a point on the ray defined by $(\tilde{\alpha}_i, \tilde{\beta}_i)$, where $(\tilde{\alpha}_i, \tilde{\beta}_i)$ differs from (α_i, β_i) by an α or by a β transformation: $q_{12} = 0$ does not occur when performing Selling's reduction algorithm, but $Q \in \sigma$. We compare inequalities (3.42) before and after an α, respectively β transformation:

$$\begin{aligned} q_{12}^{after} &= -q_{23}^{before}, q_{13}^{after} = q_{13}^{before} + 2q_{23}^{before}, q_{23}^{after} = 2q_{23}^{before} + q_{12}^{before} \\ q_{12}^{after} &= -q_{13}^{before}, q_{13}^{after} = q_{12}^{before} + 2q_{13}^{before}, q_{23}^{after} = q_{23}^{before} + 2q_{13}^{before} \end{aligned}$$

We see $q_{12}^{after} = 0$ means, either $q_{23}^{before} = 0$, or $q_{13}^{before} = 0$. Selling's Algorithm has to terminate earlier.

2. Construction of $\phi_{\tilde{\sigma}}$: For each maximal cone $\tilde{\sigma}$ with corresponding type (α_i, β_i) write $Q(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) := (\alpha_i, \beta_i) \bullet Q^{pp}(l_{\mathbb{T}E'}, l_{\mathbb{T}E})$ as linear combination of the extreme rays of σ

$$l_1(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + l_2(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + l_3(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

and set $\phi_{\tilde{\sigma}}(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) := (l_1, l_2, l_3)$. Note that $\phi_{\tilde{\sigma}} : \tilde{\sigma} \to M_2^{tr}$ is linear since

$$l_1 = q_{12} + q_{11}, l_2 = q_{12} + q_{22}, l_3 = -q_{12}.$$

3. Compatibility: Combining Points 1 and 2, we see

$$\phi_{\tilde{\sigma}} = \begin{pmatrix} q_{12}^{after} + q_{11}^{after} \\ q_{12}^{after} + q_{22}^{after} \\ -q_{12}^{after} \end{pmatrix} \text{ and } \phi_{\tilde{\sigma}'} = \begin{pmatrix} q_{12}^{before} + q_{11}^{before} \\ q_{12}^{before} + q_{22}^{before} \\ -q_{12}^{before} \end{pmatrix}$$

for cones $\tilde{\sigma}$ and $\tilde{\sigma}'$ having a common face r. As r is defined by $q_{12}^{before} + q_{11}^{before} = 0$ or $q_{12}^{before} + q_{22}^{before} = 0$, $\phi_{\tilde{\sigma}|r}$ and $\phi_{\tilde{\sigma}'|r}$ agree up to permutation.

4. The interior of $\tilde{\sigma}$: $\phi_{\tilde{\sigma}}(\tilde{\sigma})$ is a cone that maps the interior of $\tilde{\sigma}$ to the theta part of M_2^{tr}. This is an immediate consequence of Point 2. We show that $\phi_{\tilde{\sigma}}(\tilde{\sigma})$ is 2 dimensional: First, suppose $\mathbb{R}_{\geqslant 0} \cdot l_{\mathbb{T}E'}$ is not a face of $\tilde{\sigma}$ and let γ be a path between the two extreme rays of $\tilde{\sigma}$ with constant $l_{\mathbb{T}E}$ coordinate (see Figure 3.10). Given $(l_{\mathbb{T}E'}, l_{\mathbb{T}E}), (\tilde{l}_{\mathbb{T}E'}, \tilde{l}_{\mathbb{T}E}) \in \tilde{\sigma}$, observe that $Q^{pp}(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \sim Q^{pp}(\tilde{l}_{\mathbb{T}E'}, \tilde{l}_{\mathbb{T}E})$ in A_2^{tr} implies $\tilde{l}_{\mathbb{T}E'} \cdot \tilde{l}_{\mathbb{T}E} = l_{\mathbb{T}E'} \cdot l_{\mathbb{T}E}$ (by taking determinants). Then $\phi_{\tilde{\sigma}|\gamma}$ is injective and for a shift $\gamma_\epsilon := \gamma + (0, \epsilon)$, where $\epsilon > 0$ is big enough, we have $\phi_{\tilde{\sigma}}(\mathrm{Im}(\gamma)) \cap \phi_{\tilde{\sigma}}(\mathrm{Im}(\gamma_\epsilon)) = \varnothing$ by the same argument. This proves that $\phi_{\tilde{\sigma}}(\tilde{\sigma})$ is 2 dimensional. Now, suppose $\mathbb{R}_{\geqslant 0} \cdot l_{\mathbb{T}E'}$ is a face of $\tilde{\sigma}$ and consider a path with constant $l_{\mathbb{T}E'}$ coordinate instead.

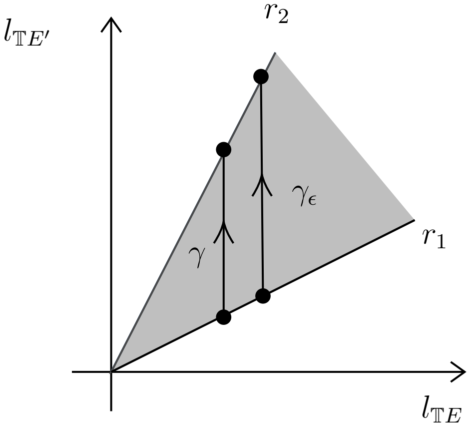


Figure 3.10: Paths γ and γ_ϵ in the proof of Theorem 3.6.9

5. The boundary of $\tilde{\sigma}$: If $(l_{\mathbb{T}E'}, l_{\mathbb{T}E})$ lies on a common face r of two maximal cones, then $l_{\mathbb{T}E'}$ and $l_{\mathbb{T}E}$ are both non-zero and give rise to a pp that lives in the boundary of $\mathcal{A}_2^{tr}$. Then, by Proposition 3.5.14, its preimage under t_2^{tr} consists of a family of tropical curves of type dumbbell whose cycle lengths are fixed by the non-zero entries of $\phi_{\tilde{\sigma}}(l_{\mathbb{T}E'}, l_{\mathbb{T}E})$. This induces a map

$$\phi_{\tilde{\sigma}|r} \times id : r \times \mathbb{R}_{\geqslant 0} \to M_2^{tr},$$

whose image in the dumbbell part of M_2^{tr} is $\phi_{\tilde{\sigma}}(r) \times r_B$, where r_B denotes the ray that corresponds to the bridge edge.

□

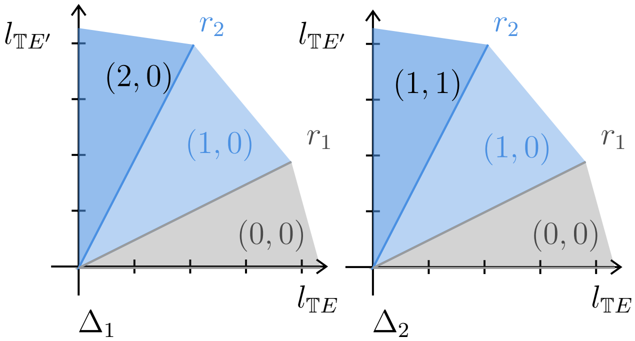


Figure 3.11: The fan Δ_1 on the left and the fan Δ_2 on the right with cones labeled by their type (α, β).

Remark 3.6.10.

- The maps $\phi_{\sigma_{(\alpha_i,\beta_i)}}$ can be viewed as local inverses of the Torelli map.
- By taking the Euclidean closure in $\mathbb{R}^2$ of the cones induced by subdividing $\mathbb{R}^2_{>0}$ according to feasible types, we get points that do not fit into the setting of Section 3.5, namely points whose $l_{\mathbb{T}E}$- or $l_{\mathbb{T}E'}$-coordinate is 0. Nevertheless, these give insight into the behavior of the reconstruction procedure in the limit $l(\mathbb{T}E) \to 0$ or $l(\mathbb{T}E') \to 0$ for the inputs $(\mathbb{T}E, \mathbb{T}E', G)$. The corresponding genus 2 curve tends towards a curve whose combinatorial type is a cycle with a vertex of genus 1.

Remark 3.6.11. With Lemma 3.6.2 and 3.6.3 we have an explicit description of the fans Δ_1 and Δ_{d-1} associated to L_1 and L_{d-1}: For Δ_1 we have a collection of d cones obtained by subdividing $\mathbb{R}^2_{\geqslant 0}$ as follows:

$$\bigcup_{\alpha=1}^{d-1} \{(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \in \mathbb{R}^2_{>0} \,|\, \alpha l_{\mathbb{T}E} + (\alpha - d) l_{\mathbb{T}E'} = 0\}.$$

For Δ_{d-1} the subdivision is induced by

$$\bigcup_{\beta=1}^{d-1} \{(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \in \mathbb{R}^2_{>0} \,|\, \beta l_{\mathbb{T}E} + (\beta - d) l_{\mathbb{T}E'} = 0\}$$

and generates d cones.

Example 3.6.12. Let $d = 3$, then $\mathbb{T}\mathcal{L}_3 = L_1 \cup L_2$. By Remark 3.6.11 we have an explicit description for both: Δ_1 is induced by

$$\{(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \in \mathbb{R}^2_{>0} \,|\, l_{\mathbb{T}E} - 2l_{\mathbb{T}E'} = 0\} \cup \{(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \in \mathbb{R}^2_{>0} \,|\, 2l_{\mathbb{T}E} - l_{\mathbb{T}E'} = 0\}$$

and the fan Δ_{3-1} by:

$$\{(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \in \mathbb{R}^2_{>0} \,|\, l_{\mathbb{T}E} - 2l_{\mathbb{T}E'} = 0\} \cup \{(l_{\mathbb{T}E'}, l_{\mathbb{T}E}) \in \mathbb{R}^2_{>0} \,|\, 2l_{\mathbb{T}E} - l_{\mathbb{T}E'} = 0\}.$$

See Figure 3.11. Following the proof of Theorem 3.6.9 we compute the maps associated to the maximal cones of Δ_1:

$$\phi_{(0,0)} = \begin{pmatrix} 2l_{\mathbb{T}E'} \\ \frac{l_{\mathbb{T}E}}{3} - \frac{2l_{\mathbb{T}E'}}{3} \\ l_{\mathbb{T}E'} \end{pmatrix} \phi_{(1,0)} = \begin{pmatrix} \frac{2l_{\mathbb{T}E}}{3} + \frac{2l_{\mathbb{T}E'}}{3} \\ \frac{-l_{\mathbb{T}E'}}{3} + \frac{2l_{\mathbb{T}E}}{3} \\ \frac{-l_{\mathbb{T}E}}{3} + \frac{2l_{\mathbb{T}E'}}{3} \end{pmatrix} \phi_{(2,0)} = \begin{pmatrix} 2l_{\mathbb{T}E} \\ l_{\mathbb{T}E} \\ \frac{l_{\mathbb{T}E'}}{3} - \frac{2l_{\mathbb{T}E}}{3} \end{pmatrix} \tag{3.43}$$

Compatibility is verified exemplarily for $\phi_{(0,0)}$ and $\phi_{(1,0)}$: r_1 is given by the equation $2l' = l$. Substituting in (3.43) yields

$$\phi_{(0,0)} = \begin{pmatrix} 2l' \\ 0 \\ l' \end{pmatrix} \text{ and } \phi_{(1,0)} = \begin{pmatrix} 2l' \\ l' \\ 0 \end{pmatrix}.$$

We conclude with a visualization in Figure 3.12. Repeating the same procedure for Δ_2 we get

$$\phi_{(0,0)} = \begin{pmatrix} l_{\mathbb{T}E'} \\ \frac{l_{\mathbb{T}E}}{3} - \frac{2l_{\mathbb{T}E'}}{3} \\ 2l_{\mathbb{T}E'} \end{pmatrix} \phi_{(1,0)} = \begin{pmatrix} \frac{-l_{\mathbb{T}E'}}{3} + \frac{2l_{\mathbb{T}E}}{3} \\ \frac{2l_{\mathbb{T}E}}{3} + \frac{2l_{\mathbb{T}E'}}{3} \\ \frac{-l_{\mathbb{T}E}}{3} + \frac{2l_{\mathbb{T}E'}}{3} \end{pmatrix} \phi_{(1,1)} = \begin{pmatrix} l_{\mathbb{T}E} \\ 2l_{\mathbb{T}E} \\ \frac{l_{\mathbb{T}E'}}{3} - \frac{2l_{\mathbb{T}E}}{3} \end{pmatrix}$$

and see that the image of Δ_1 and Δ_2 in M_2^{tr} agree.

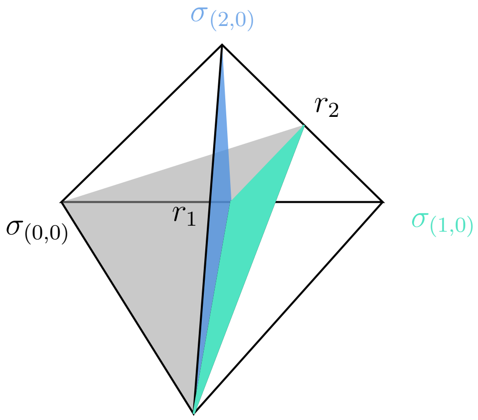


Figure 3.12: The image of Δ_1 in M_2^{tr}.

3.7 Appendix

This section follows Subsection 3.5.3 and the notation therein. We review Selling's reduction algorithm in greater detail (see [77], Section 2.3.3 and [71]) and provide an overview of some SINGULAR procedures, which may be used in the context Split Jacobians.

A 2-dimensional positive definite quadratic form q with associated matrix $Q := (q_{ij})$ is represented by its *Selling parameters*:

$$q_{12},\ q_{13} := -q_{11} - q_{12},\ q_{23} := -q_{11} - q_{12}.$$

Selling's reduction algorithm reduces q to a form whose Selling parameters are all non-positive: Without loss of generality suppose $q_{12} < 0$, else consider $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot Q$ and do:

- Compute the Selling Parameters of Q.
- If $q_{13} > 0$, set $Q := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot Q$.
- If $q_{23} > 0$, set $Q := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot Q$.

Repeat until $q_{13} \leqslant 0$ and $q_{23} \leqslant 0$.

Procedures available in the context of split Jacobians are:

Procedure: `SELLING`.
Input: A symmetric and positive definite 2×2 matrix Q with $q_{12} < 0$.
Output: Selling parameters of a represantative $\tilde{Q}$ of Q in σ and a list of integers that records the number of transformations of type (1) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and (2) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ needed to obtain $\tilde{Q}$.

Procedure: FD_VERTETER.
Input: A symmetric and positive definite 2×2 matrix Q with non-positive Selling parameters.
Output: A represenative of Q in the fundamental domain F (see Subsection 3.5.3).

Procedure: LENGTH_OUTPUT
Input: A matrix $Q \in F$.
Output: A curve Γ whose Jacobian is defined by Q.

Now Steps 1 and 2 of Plan 3.5.1 may be retraced computationally: Let $(\mathbb{T}E, \mathbb{T}E', G)$ be splitting data, where $\mathbb{T}E$ and $\mathbb{T}E'$ are elliptic curves of length 1, respectively 3, and G is the graph of

$$f : \mathbb{R}/3\mathbb{Z}[18] \to \mathbb{R}/\mathbb{Z}[18], \qquad \frac{7 \cdot 3}{18} \longmapsto \frac{1}{18}.$$

Step 1 of Plan 3.5.1: The splitting data $(\mathbb{T}E, \mathbb{T}E', G)$ is recorded as

$$(d, k, l_{\mathbb{T}E'}, l_{\mathbb{T}E}) = (18, 7, 3, 1).$$

We compute the pptav J^{pp} by running SETMATRIX, which returns:

```
> matrix Q=SETMATRIX(18,7,3,1);
> Q;
Q[1,1]=54
Q[1,2]=-21
Q[2,1]=-21
Q[2,2]=74/9
```

i.e. $Q^{pp} = \begin{pmatrix} 54 & -21 \\ -21 & \frac{74}{9} \end{pmatrix}$ (see Subsection 3.5.3).

Step 2 of 3.5.1: Running SELLING and RED_VERTRETER returns

```
> list L=SELLING(Q);L;
 [1]:
-1/3
 [2]:
-11/9
 [3]:
-5/3
 [4]:
1
 [5]:
2
```

and

```
> matrix Qn=RED_VERTRETER(L);
> Qn;
Qn[1,1]=26/9
Qn[1,2]=-5/3
Qn[2,1]=-5/3
Qn[2,2]=2
```

where Qn is a representative of Q^{pp} in σ whose Selling parameters are $\frac{-1}{3}, \frac{-11}{9}, \frac{-5}{3}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ is a sequence of transformations that relates Q^{pp} to Qn. The unique representative inside F is obtained by invoking

```
> matrix Qnn=FD_VERTRETER(Qn);Qnn;
Qnn[1,1]=14/9
Qnn[1,2]=-1/3
Qnn[2,1]=-1/3
Qnn[2,2]=2
```

and

```
> list Ln=LENGTH_OUTPUT(Qnn);Ln;
[1]:
Jacobian of tropical curve of type T:
[2]:
[1]:
11/9
[2]:
5/3
[3]:
1/3
```

returns the length data $L = \frac{11}{9}, \frac{5}{3}, \frac{1}{3}$ (see Algorithm 2) together with the combinatorial type of the curve Γ whose Jacobian is J^{pp}.

Part II

Spin Hurwitz numbers with completed cycles

3.8 Introduction

The second part of this thesis moves from an intrinsic focus on tropical geometry to its relation to other fields. The setting is a large network of mathematical disciplines, whose interaction we study through the lens of enumerative geometry, Hurwitz numbers in particular.

Classical theory: type A Hurwitz numbers enumerate covers of a fixed Riemann surface with specified ramification behavior. While their origin is geometric, translating these numbers into other areas of mathematics has led to a growing body of results, ranging from structural insights to explicit computations ([28], [66], [22], [23], [50]). Figure 3.13 sketches the network:

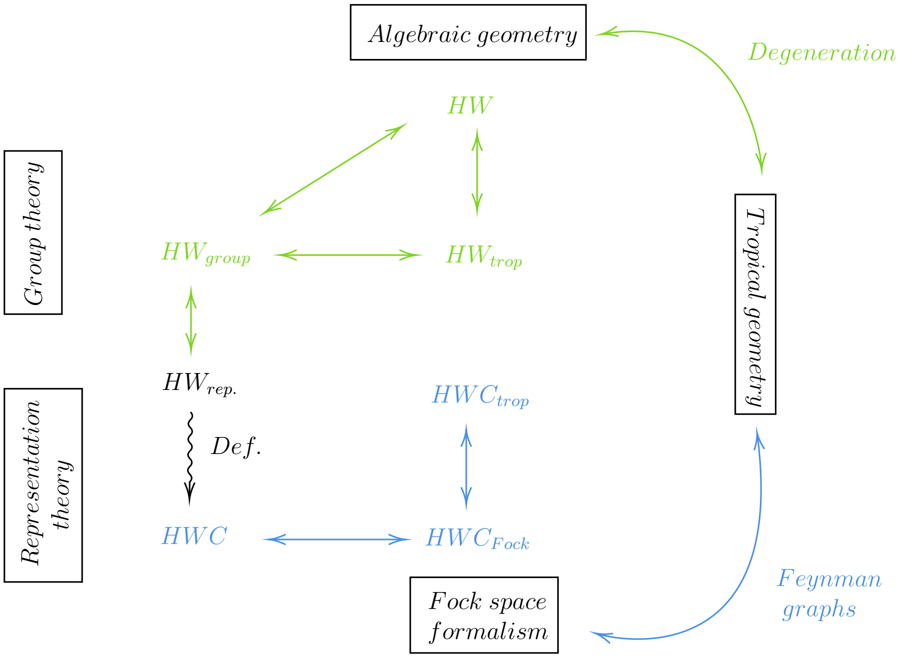


Figure 3.13: Areas supporting Hurwitz numbers (HW) and Hurwitz numbers with completed cycles (HWC). The double-headed arrows indicate a correspondence (equality) between the respective invariants, the wavy arrow the transition to completed cycles via the substitution rule $f_\mu \mapsto p_\mu$.

In green: The geometric count (HW) is translated into a group theoretic (HW_{group}) one by encoding the monodromy of ramified covers as tuples of elements of the symmetric group. The number of such tuples becomes the coefficient of a product of elements in the class algebra of the symmetric group and yields a representation-theoretic formula ($HW_{rep.}$) for Hurwitz numbers given by products of so-called central characters f_μ.

In blue: For target $\mathbb{P}^1$, *Hurwitz numbers with completed cycles (HWC)* are defined purely representation-theoretically, by the substitution rule $f_\mu \mapsto p_\mu$ applied to the expression $HW_{rep.}$, where the p_μ are the shifted symmetric power sums. This comes with several advantages: First, HWC can be expressed in the operator formalism of fermionic/bosonic Fock space (HW_{Fock}) as *vacuum expectation* of a specific operator (Definition 6.4.1). This makes an efficient computation possible. Second, this expression identifies the generating function of HWC as a tau-function of the KP hierarchy and links Hurwitz theory to integrable systems. Third, HWC enjoy much better properties than classical HW such as piece-wise polynomiality.

Re-geometrization: This purely representation-theoretic number acquires a geometric meaning in two ways:

- The Gromov-Witten/Hurwitz (GW/H) correspondence (see [66]), which identifies certain Gromov–Witten invariants on $\mathcal{M}_{g,n}(\mathbb{P}^1, d)$ with completed-cycles Hurwitz numbers.

- Their evaluation in Fock space formalism, which offers a natural translation in tropical language (see [28]), where HWC appear as a count of tropical covers.

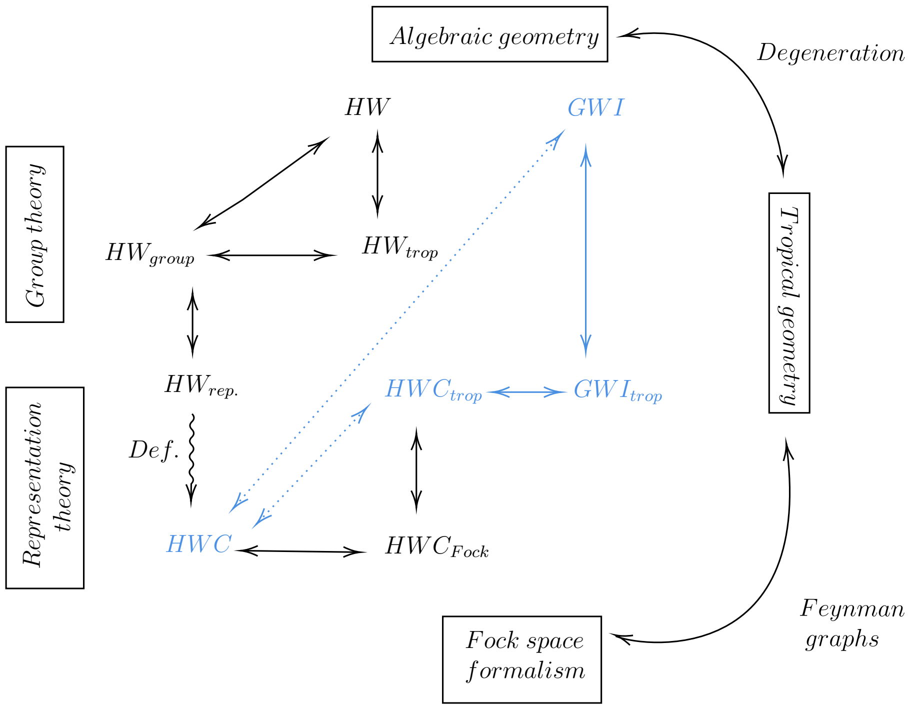


Figure 3.14: The two ways in which HWC acquires geometric meaning: 1. via the GW/H correspondence (the upper dashed arrow). 2. by "tropicalizing" HWC_{Fock}.

We see: Tropical geometry is central, as it enjoys direct connections to nearly all settings that support Hurwitz numbers.

Spin theory: type B Spin Hurwitz numbers enumerate branched covers of a fixed spin curve (a curve with theta characteristic) with specified ramification behavior and sign fixed by a lift of this theta characteristic to the source. They were introduced by Eskin-Okounkov-Pandharipande in 2008 for certain computations in the moduli space of differentials on a Riemann surface [37]. As of now this theory enjoys almost the same global structure (Figure 3.15) as Hurwitz theory:

- (upper green triangle, Geometric setting): Starting from [37], [54] and [56] develop degeneration techniques, which [31] subsequently uses to introduce a tropical count and prove its equivalence to the classical one.

- (lower blue triangle, Representation-theoretic and Fock space setting): Following [45] representation-theoretic formulation of spin Hurwitz numbers, [40] introduced their completed cycles analogues, specifically spin Hurwitz numbers with $(r+1)$-completed cycles (building on [37] and [55]). In this work, we use Fock space formalism from [40] to re-geometrize completed cycles spin Hurwitz numbers, and study piecewise polynomiality, their polynomiality domains (walls) and wall-crossing formulas. These have been studied previously in the operator formalism of [40]. We provide a combinatorial point of view.

This fits within the lineage of [28] and [23] studies on double Hurwitz numbers and GWI, alongside [46] studies on monotone-Hurwitz numbers, and completes Diagram 3.15 for the spin variants, building on prior work in [31].

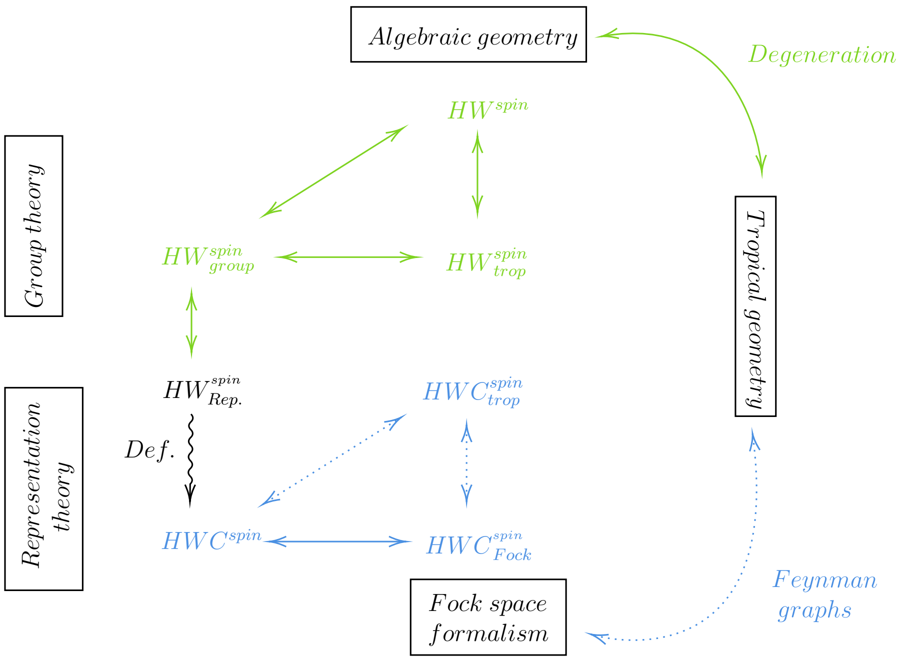


Figure 3.15: Areas supporting spin Hurwitz numbers (HW^{spin}) and spin Hurwitz numbers with completed cycles (HWC^{spin}). Solid double-headed arrows indicate established correspondences, while dashed ones represent those yet to be established.

3.8.1 Results

Our first result provides a tropical count (Definition 8.2.2) and a correspondence theorem that re-geometrizes the purely representation-theoretic invariant.

Theorem 3.8.1. [Theorem 8.2.3] Fix discrete data: Positive integers $d, g > 0$, even integers $r_1, ..., r_b > 0$ and odd partitions μ, ν of $d \in \mathbb{N}$ such that $r_1 + ... + r_b = 2g + l(\mu) + l(\nu) - 2$ holds. The classical and the tropical spin double Hurwitz number with multi-completed cycles agree:

$$\mathbb{T}h_g^{spin}(\mu, \overline{(r_1+1)}, ..., \overline{(r_b+1)}, \nu) = h_g^{spin}(\mu, \overline{(r_1+1)}, ..., \overline{(r_b+1)}, \nu)$$
$$\mathbb{T}h_g^{spin,\bullet}(\mu, \overline{(r_1+1)}, ..., \overline{(r_b+1)}, \nu) = h_g^{spin,\bullet}(\mu, \overline{(r_1+1)}, ..., \overline{(r_b+1)}, \nu).$$

If $r := r_1 = ... = r_b$, we simply write $\mathbb{T}h_g^{spin,r}(\mu,\nu)/\mathbb{T}h_g^{spin,r,\bullet}(\mu,\nu)$.

The discrete data governing both invariants now has a geometric interpretation: The tropical invariant enumerates covers (with suitable multiplicity) of the tropical projective line $\mathbb{TP}^1$ of degree d by genus g curves with odd edge weights and vertex valencies fixed by the integers r_i.

Theorem 3.8.1 also paves the way for investigating structural properties of the map:

$$\mathbb{T}h_g^{spin,r} : \mathcal{H} \to \mathbb{Q}, x \mapsto \mathbb{T}h_g^{spin,r}(x^-, x^+), \tag{3.2}$$

where $\mathscr{H} := \{x \in \mathbb{Z}^n : \sum_{i=1}^n x_i = 0 \text{ and } x_i \text{ odd}\}$, using combinatorial methods.

We obtain another proof that (3.2) is piecewise polynomial and new wall crossing formulae. The wall crossing formula of [40] is a coefficient of a product of vacuum expectation of B-Okounkov-Pandharipande operators and expressions derived from the generating series for multi-completed cycles ([40], Definition 6.15). Our formula expresses wall crossing in terms of completed cycles spin Hurwitz numbers for smaller counting problems, a structure that mirrors that of formulae obtained for related problems through tropical methods ([28],[23],[46]). These numbers are inherently tied to the covers we enumerate as they reflect building blocks of covers on either side of the wall (see Section 8.3).

Theorem 3.8.2. [Proposition 8.3.8 and Theorem 8.3.16, Proposition 8.3.29 and Theorem 8.3.39] The function $\mathbb{T}h_g^{spin,r}$ is piecewise polynomial and satisfies explicit wall crossing formulae.

Strategy Powerful techniques for proving theorems of this type were introduced by [23] in the context of double Hurwitz numbers. We adapt these methods to the spin setting, further evidence of their broad applicability.

We first consider the case where the source is of genus 0. Just like for double Hurwitz numbers, this setting is entirely combinatorial. The vertex multiplicities (local spin Hurwitz numbers, we determine in Section 8.2) are constant. Thus, the genus 0 case reduces to enumerating graphs of fixed type, whose multiplicities depend only on the structure of each graph (see Definition 8.2.2).

Extending to higher genus adds

- Combinatorial complexity: Cycles lead to auxiliary variables for edge weights, in addition to variables for end weights.
- Algebraic complexity: The multiplicity of each covers requires an algebraic input in the form of local spin Hurwitz numbers.

This requires a change of approach. Following [23], we introduce a new counting unit, the x-graph, that encompasses all covers that share the same underlying undirected graph. Its multiplicity is now independent of all auxiliary variables, since it is computed by integrating over them. The challenge: In our spin setting this means integrating over odd lattice points of bounded polytopes in a hyperplane arrangement. In order to use the general theory that deals with lattice points in polytopes (Erhart theory), we work with rational hyperplane arrangements. Care is needed since summing a polynomial over lattice points of a rational polytope usually yields a quasi-polynomial (see Lemma 8.3.31 and 8.3.33). Finally, we recover Theorem 3.8.2 for $g > 0$.

3.8.2 Overview

We review the classical (type A) theory in Chapters 4 and 6, the spin (type B) theory in Chapters 5 and 7, and find a tropical language in Chapter 8.

Chapter 4 explains the transition from Hurwitz numbers to completed cycles and locates these new invariants in a broader framework (the wavy arrow connecting the green with the blue part of Diagram 3.15). We start by reviewing Hurwitz numbers in the algebraic, respectively tropical setting, before moving on to the representation-theoretic setting, which leads to the definition of Hurwitz numbers with completed cycles. Chapter 6 provides a brief but sufficiently general introduction to Fock spaces and an overview of the method for translating Hurwitz-type enumerative problems into operator formalism. A solid understanding proves useful for later chapters.

Chapter 4 and 6 serve as a preparation for Chapters 5 and 7, where the same method is applied to spin Hurwitz numbers. By establishing this structured framework here, we have a blueprint for Chapters 5 and 7.

Chapters 5 and 7 are organized so as to make parallels visible to the type A theory visible: In 5 we review the algebraic, tropical and representation-theoretic setting, which leads to completed cycles spin Hurwitz numbers (HWC^{spin}). Here we generalize the definition to arbitrary completed cycles and give a character formula for them, providing a slight generalization of [40] that mirrors the classical approach. In 7, we review the Fock space formalism of type B and HWC^{spin}_{Fock}, which prepares for re-geometrization of completed cycles in Chapter 8. This re-geomtrization builds on well-known methods for computing certain vacuum expectations of products of bosonic operators via Feynman diagram and follows the program carried out for the non-spin case in [28]. We enhance these diagrams to tropical covers of $\mathbb{TP}^1$ and introduce tropical completed cycles numbers, a count with geometric meaning that agrees with the representation-theoretic one. Finally, we analyze properties such as piecewise polynomiality and wall-crossing behavior, building on the work in [22] and [23].

3.8.3 Future work

Natural extension of this work are:

- Further study the polynomial behavior established in [40] from a tropical perspective.
    - Polynomiality of one-part spin Hurwitz numbers: [40] give a concrete formula in terms of Stirling numbers of the second kind (Proposition 7.11). It would certainly be interesting to have a geometric/graph-theoretic explanation for the appearance of Stirling numbers in this context.
    - Quasi-polynomiality of spin single Hurwitz numbers: Already in genus 0, the tropical approach poses difficulties which are due to the structure of the Riemann-Hurwitz formula. For double Hurwitz numbers terms involving the degree d of a cover cancel. For single Hurwitz numbers, however, they do not: As a consequence, the number of branch points increases with d. This complicates the classification of all possible combinatorial types, a challenge inherent to the tropical approach that does not arise when working with operator formalism.

- Complete Diagram 3.15: Investigate Gromov-Witten theory of spin curves tropically and obtain a spin analogue of the Gromov-Witten/Hurwitz correspondence in the tropical setting. The type A analogue found in [28]) serves as a guide.

Further directions:

- Investigate the relationship between Hurwitz and spin Hurwitz numbers: Determine cases where both agree (up to a sign). Is there a rule that describes how the even and odd part of a spin Hurwitz number partitions the Hurwitz number? Does this partition reflect (perhaps asymptotically for $d \to \infty$) the count of odd and even theta characteristics on a curve of genus g, $2^{g-1}(2^g+1)/2^{g-1}(2^g-1)$?
- Tropical moduli spaces and intersection theory: Tropical moduli spaces and their relation to classical ones have been extensively studied (see e.g. [26], [27], [25]). As proposed in [68], these relations can be leveraged to connect tropical and algebraic invariants not only on an individual basis, but systematically at the level of moduli spaces.
    - Flavor 1: Construct a moduli space for (tropical) spin Hurwitz covers and define (tropical) spin Hurwitz numbers as degree of a (tropical) branch map. A potential approach is to combine the (tropicalization of) the space of admissible covers [26] with the moduli space of (tropical) spin curves [18]). We know that the (algebraic) moduli space of spin curves admits a map to the moduli space of curves of degree 2^{2g}. Similarly, one could relate the space of (tropical) admissible covers and the (tropical) admissible spin covers by forgetting the respective spin structures. This connection could be used to explore the relationship between Hurwitz and spin Hurwitz numbers via pullback/pushforward along the forgetful morphism.
    - Flavor 2: Towards a tropical ELSV/ELSV-type formula for single/double Hurwitz numbers. With the ongoing development of intersection theory on tropical spaces, an opportunity arises to lift the definition of Hurwitz numbers/spin Hurwitz numbers to the level of the moduli space of CURVES (not maps) and potentially address the conjecture of Goulden/Vakil from a tropical perspective.
- New counts: (Tropical) Split Hurwitz numbers, a subcount that only considers Hurwitz coverings $\varphi : \Gamma \to B$ for which the Jacobian $\mathrm{Jac}(\Gamma)$ splits. A related question: What percentage of Hurwitz cover satisfies this splitting property? More generally, one can count maps according to properties that are tractable in families. This translates to asking: How frequently does a given property occur in moduli space of covers?

Chapter 4

Type A theory

This chapter surveys the theory of type A theory, i.e. the classical theory around Hurwitz numbers. The first part covers their definition in the algebraic and in the tropical setting. Degeneration techniques provide the most direct connection between the two.

The second part reviews Hurwitz numbers in the representation-theoretic setting, which leads to the definition of Hurwitz numbers with completed cycles. The material presented in this chapter is standard. Our exposition follows Okounkov's and Pandharipande's foundational paper [66], supplemented by [20] and [9].

4.1 Algebraic setting

Fix

- a connected compact Riemann surface (equivalently: an irreducible and smooth curve) D of genus h.
- a collection of points $q^1, ..., q^n \in D$.
- positive integers d and g.
- a collection $\mu_1, ..., \mu_n$ of partitions of d.

Define a *(connected/potentially disconnected) Hurwitz cover* of type $(g \to h, d, \mu_1, ..., \mu_n)$ to be a holomorphic map $f : C \to D$ such that

- C is a compact and (connected/potentially disconnected) Riemann surface of genus g,
- f is of degree d and
- ramified at q_i as prescribed by the partition μ_i.

Introduce an equivalence relation on the set of Hurwitz covers: Let $f : C \to D$ and $\tilde{f} : \tilde{C} \to D$ be of type $(g \to h, d, \mu_1, ..., \mu_n)$. Then

$$f \text{ is equivalent to } \tilde{f} :\Leftrightarrow \exists \text{ isomorphism } \phi : C \to \tilde{C} \text{ such that } f = g \circ \phi.$$

There are only finitely many equivalence classes $[f]$ and the cardinality $|\mathrm{Aut}(\tilde{f})|$ of the automorphism group of $\tilde{f} \in [f]$ is finite and independent of the choice of representative.

Definition 4.1.1. The *connected/disconnected Hurwitz number* $H_{g \xrightarrow{d} h}(\mu_1, ..., \mu_n)/H^\bullet_{g \xrightarrow{d} h}(\mu_1, ..., \mu_n)$ is the weighted sum

$$H_{g \xrightarrow{d} h}(\mu_1, ..., \mu_n)/H^\bullet_{g \xrightarrow{d} h}(\mu_1, ..., \mu_n) := \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|},$$

of equivalence classes $[f]$ of (connected/potentially disconnected) Hurwitz covers of type $(g \to h, d, \mu_1, ..., \mu_n)$.

Remark 4.1.2. A necessary condition for the existence of Hurwitz covers of type $(g \to h, d, \mu_1, ..., \mu_n)$ is given by the Riemann Hurwitz formula:

$$2 - 2g = d(2 - 2h) + \sum_{i=1}^{n} (d - l(\mu_i)).$$

Optionally, one of the decorations g, h or d will be omitted and understood as determined by the Riemann Hurwitz formula.

Section 4.4 requires a generalization of Definition 4.1.1 for target curve $\mathbb{P}^1$.

Definition 4.1.3. [[66], Section 0.3.2]

Extend Definition 4.1.1 for target curve $\mathbb{P}^1$ to encompass the following cases:

- Let $H^\bullet_{d:=0}(\varnothing, ..., \varnothing) := 1$.
- If $|\lambda_i| > d$ for some i, let $H^\bullet_d(\lambda_1, ..., \lambda_n) := 0$.
- If $|\lambda_i| \leqslant d$, let $\tilde{\lambda}_1$ be the partition of size d obtained by adding parts of size 1 to λ_i. Then: $H^\bullet_d(\lambda_1, ..., \lambda_n) := \prod_{i=1}^n \binom{m_1(\tilde{\lambda}_i)}{m_1(\lambda_i)} H^\bullet_{g \xrightarrow{d} 0}(\tilde{\lambda}_1, ..., \tilde{\lambda}_n)$, where $m_1(\lambda)$ counts the number of 1's occurring in λ and g is determined by the Riemann Hurwitz formula.

4.1.1 Degeneration formula

Within (the framework of) algebraic geometry, degeneration techniques offer an approach to the Hurwitz problem and reveal its recursive structure.

Theorem 4.1.4 ([20], Theorem 7.5.1)**.** The following degeneration formulas hold for all Hurwitz numbers:

1. Reducing the genus of a higher genus base curve

$$H^{\bullet}_{g \xrightarrow{d} h}(\mu_1, \ldots, \mu_s) = \sum_{\mu} \prod_{i=1}^{l} \mu^i |\mathrm{Aut}(\mu)| \cdot H^{\bullet}_{g-l(\mu) \xrightarrow{d} h-1}(\mu_1, \ldots, \mu_s, \mu, \mu)$$

2. Base curve of genus zero: reducing branch points

$$H^{\bullet}_{g \xrightarrow{d} 0}(\mu_1, \ldots, \mu_s, \nu_1, \ldots, \nu_t) = \sum_{\mu} \prod_{i=1}^{l} \mu^i |\mathrm{Aut}(\mu)| \cdot H^{\bullet}_{g_1 \xrightarrow{d} 0}(\mu_1, \ldots, \mu_s, \mu) H^{\bullet}_{g_2 \xrightarrow{d} 0}(\mu, \nu_1, \ldots, \nu_t).$$

   Here, g_1 and g_2 are determined by the Riemann-Hurwitz formula and satisfy the condition $g_1 + g_2 + l(\mu) - 1 = g$, where $l(\mu)$ denotes the length of the partition $\mu = (\mu^i)_i$.

Iterating these degeneration steps yields:

Lemma 4.1.5 ([20], Theorem 7.5.3.)**.** All disconnected Hurwitz numbers can be expressed in terms of Hurwitz numbers for a base of genus 0 with only 3 ramification points, $H^{\bullet}_{g \xrightarrow{d} 0}(\lambda_1, \lambda_2, \lambda_3)$.

This reduces the Hurwitz problem to a combinatorial one requiring only triple Hurwitz numbers as algebraic input, the foundation of the tropical approach. While the barrier for specific computations is lower with smaller algebraic input, it still prevents general ones: Specific numbers can be computed via the character formula, but there are no closed formulas for all the classes of triple Hurwitz numbers that can appear in a degeneration.

4.2 Tropical setting

4.2.1 Weighted tropical curves and morphisms

We generalize Section 2.4 to include weighted tropical curves with ends.

Definition 4.2.1. *Curves:* A (vertex-weighted) abstract tropical curve Γ is a triple (G, l, g) consisting of

1. a metric graph (G, l), i.e. a finite graph G with $E^{\infty}(G) \subset E(G)$ the subset of *ends* (i.e. edges adjacent to a single vertex), and a function $l : E(G) \to \mathbb{R}_{\geqslant 0} \cup \infty$ such that $l(e) = \infty$ if and only if $e \in E^{\infty}(G)$.

2. a weighting $g : V(G) \to \mathbb{Z}_{\geqslant 0}$ on the vertices.

The pair (G, g) is its *combinatorial type*.

For a vertex v, the value $g(v)$ is called the *genus of* v. The *genus of* Γ is defined as

$$g(\Gamma) = b_1(G) + \sum_{v \in V(G)} g(v),$$

where $b_1(G)$ denotes the first Betti number of the graph.

As in Section 2.4, any (G', l') that is obtained from (G, l) by adding or deleting vertices of valence 2 (and adapting the length function accordingly) is called a *model* for Γ. For a standalone tropical curve, we identify Γ with its *essential model* (G, l), which is chosen so that $V(G)$ is the set of points with $g(v) > 0$ or $\mathrm{val}(v) \neq 2$. The curve Γ is called

- *explicit*, if $g(\Gamma) = b_1(\Gamma)$.
- *stable*, if for all $v \in V(\Gamma)$ we have

  $$2g(v) - 2 + \mathrm{val}(v) > 0.$$

  This means that each genus 0 vertex has to be at least trivalent and 2-valent vertices have to be of higher genus.

Morphisms: In analogy to the unweighted case (Section 2.4.1), we call a map $\pi : \Gamma \to B$ a *harmonic morphism/tropical cover* if π is a harmonic morphism/tropical cover between the metric graphs ([3], Section 2), i.e. π is surjective and continuous and there exist models (G, l) for Γ and $(\tilde{G}, \tilde{l})$ for B such that

i $\pi(V(G)) \subset V(\tilde{G})$ and $\pi^{-1}(E(\tilde{G})) \subset E(G)$,

ii locally integer affine linear, i.e. π restricts on each edge e to an affine linear function $\pi_{|e} : [0, l(e)] \to \mathbb{R} : t \mapsto a + \omega(e)t$ whose slope is a positive integer. We call $\omega(e)$ the weight $\omega(e)$.

iii harmonic/balanced: Let $v \in \Gamma$ and e' adjacent to $\pi(v)$. We require the integer $d_v := \sum_{e \mapsto e'} \omega(e)$ to be independent of the choice of e'. In this case call d_v the *local degree* of π at v.

When working with covers, we identify Γ and B with models that fit these conditions.

Remark 4.2.2.

- Let Γ and B be metric graphs. A graph homomorphism $\pi : G \to \tilde{G}$ (on the underlying graphs) that satisfies Condition i can be promoted to a morphism of metric graphs (i.e. a graph homomorphism satisfying Condition i and ii) by a suitable assignment of edge weights: For all contracted edges e, we have $\omega(e) = 0$. For a non-contracted edge e of finite length, $\omega(e)$ is uniquely determined by the equation $l(\pi(e)) = \omega(e)l(e)$. Since $\omega(e) > 0$, this relation implies that

  - $l(e) = \infty$ if and only if $l(\pi(e)) = \infty$ (i.e. non-contracted ends are mapped to ends).
  - end weights are not uniquely determined by the metrics.

  If the balancing condition is satisfied on $E(\Gamma)\backslash E^{\infty}(\Gamma)$, then π can further be promoted to harmonic morphism by choosing end weights such that the balancing condition is satisfied. If not, π remains a morphism of metric graphs for any choices of end weights, but fails to be harmonic.

- For a harmonic morphism, we will often drop the distance labelings of B and Γ and consider only their combinatorial type. This is due to the fact that for a non-contracted edge e in Γ its length is uniquely determined be $\omega(e)$ and the length of $\pi(e)$.
- Note the change of notation from Section 2.4: We use $\omega(e)$ instead of $d_e(\pi)$. This notation hides the fact that the $\omega(e)$ come from a cover, but is more common in the context of Hurwitz theory.

4.2.2 Tropical Hurwitz covers

We introduce a specific class of tropical covers that emerge from the classical enumeration via degeneration.

Definition 4.2.3. A tropical cover π is a *tropical Hurwitz cover* if at every point $v \in \Gamma$ the local *Riemann Hurwitz condition* is satisfied, i.e. we have

$$r_{\pi,v} := d_v(2 - 2g(h(v))) - \sum(\omega(e) - 1) - (2 - 2g(v)) \geqslant 0, \tag{4.1}$$

where the sum is over all edges adjacent to v and the genus function g is extended on Γ by mapping points that are not vertices to 0.

The cover π is an *isomorphism* if there exists a tropical Hurwitz cover π' such that $\pi \circ \pi' = \pi' \circ \pi = \mathrm{id}$. An isomorphism $\pi : \Gamma \to \Gamma$ from Γ to itself is called an *automorphism* of Γ. Concretely, an *automorphism* of Γ is an automorphism of the underlying graph that preserves edge lengths and vertex genera.

Let $\pi \colon \Gamma \to B$ be a tropical Hurwitz cover. In analogy to the complex setting, we introduce the following concepts

- Degree: The *degree* of π is given by the integer $\deg(\pi) := \sum_{p' \in \pi^{-1}(p)} d_{p'}$, where $p \in B$ is an arbitrary point. The balancing condition ensures that $\deg(\pi)$ is independent of the choice of p.
- Ramification: A subset E of Γ, where $\pi(E)$ is a point, is called a *ramification component* if E is a connected component of $\pi^{-1}(\pi(E))$, such that
  - E contains an edge $e \in E(\Gamma)\backslash E^{\infty}(\Gamma)$,

- E contains a vertex $v \in V(\Gamma)$ with $r_{\pi,v} > 0$ or
- E contains a point that lives on an end $e \in E^{\infty}(\Gamma)$ with $\omega(e) > 1$.

Let $\mu = (\mu^1, \ldots, \mu^n)$ a partition. We say an end e has *ramification profile* μ with respect to π if $\pi^{-1}(e) = \{e_1, ..., e_n\}$ a set of ends of Γ with weights $\omega(e_i) = \mu^i$.

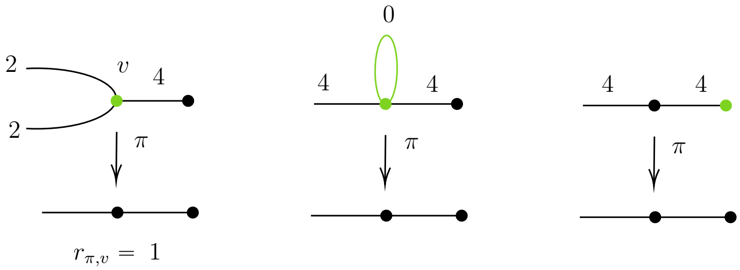


Figure 4.1: Local picture of a tropical cover. Ramification components E are in green. The black numbers are edge weights.

4.2.3 Tropical Hurwitz numbers

Fix

- a connected explicit tropical curve B of genus h with n ends.
- positive integers d and g.
- a collection $\mu_1, ..., \mu_n$ of partitions of d.

Define a *(connected/potentially disconnected) tropical Hurwitz cover* of type $(g \to h, d, \mu_1, ..., \mu_n)$ to be a tropical Hurwitz cover $\pi\colon \Gamma \to B$ of degree d such that

- Γ is a (connected/potentially disconnected) tropical curve of genus g,
- π is unramified over $B \backslash E^{\infty}(B)$ and
- there exists a labeling of the n ends of B, such that the end labeled i has ramification profile μ_i.

Introduce an equivalence relation on the set of tropical Hurwitz covers: Let $\pi\colon \Gamma \to B$ and $\tilde{\pi}\colon \tilde{\Gamma} \to B$ be of type $(g \to h, d, \mu_1, ..., \mu_n)$. Then

$$\pi \text{ is equivalent to } \tilde{\pi} :\Leftrightarrow \exists \text{ isomorphism } \phi : \Gamma \to \tilde{\Gamma} \text{ such that } \pi = \tilde{\pi} \circ \phi.$$

There are only finitely many equivalence classes $[\pi]$ and the cardinality $|\mathrm{Aut}(\pi)|$ of the automorphism group of $\tilde{\pi} \in [\pi]$ is finite and independent of the choice of representative. The *multiplicity* of $[\pi]$ is:

$$\mathrm{mult}([\pi]) := \frac{1}{|\mathrm{Aut}(\pi)|} \prod_{e \in E(\Gamma) \setminus E^{\infty}(\Gamma))} \omega(e) \prod_{v \in V(\Gamma)} |\mathrm{Aut}(\mu_v)| \cdot H_{g(v) \to g(\pi(v))}(\mu_v),$$

where

- μ_v is the *local ramification profile at* $v \in V(\Gamma)$: Let $e_1, ..., e_l \in E(B)$ be adjacent to $v' = \pi(v)$ and $e_i^1, ..., e_i^{l_i} \in E(\Gamma)$ the collection of edges adjacent to v that map to e_i, $i = 1, ..., l$. Define $\mu_v = (\mu_1, \ldots, \mu_l)$ to be the collection partitions
$$\mu_i = (\omega(e_i^1), \ldots, \omega(e_i^{l_i})) \vdash d_v.$$
- $|\mathrm{Aut}(\mu_v)| = \prod_{i=1}^{l} |\mathrm{Aut}(\mu^i)|$.
- $H_{g(v) \to g(\pi(v))}(\mu_v)$ is the *local Hurwitz number.*

As usual, we identify equivalent covers and write $\mathrm{mult}(\pi)$ for $\mathrm{mult}([\pi])$.

Definition 4.2.4. The *connected/disconnected tropical Hurwitz number* $\mathbb{T}H_{g \xrightarrow{d} h}(\mu_1, ..., \mu_n)/\mathbb{T}H^{\bullet}_{g \xrightarrow{d} h}(\mu_1, ..., \mu_n)$ is the weighted sum

$$\mathbb{T}H_{g \xrightarrow{d} h}(\mu_1, ..., \mu_n)/\mathbb{T}H^{\bullet}_{g \xrightarrow{d} h}(\mu_1, ..., \mu_n) := \sum_{[\pi]} \mathrm{mult}(\pi),$$

of equivalence classes $[\pi]$ of (connected/potentially disconnected) tropical Hurwitz covers of type $(g \to h, d, \mu_1, ..., \mu_n)$.

4.2.4 Correspondence

The following correspondence theorem clarifies the relation between classical and tropical Hurwitz numbers.

Theorem 4.2.5 ([22],[9])**.** Let g, h non-negative integers, d a positive integer and $\mu_1, ..., \mu_n$ partitions of d. Then

$$H_{g \to h}(\mu_1, ..., \mu_n) = \mathbb{T}H_{g \to h}(\mu_1, ..., \mu_n).$$

Remark 4.2.6.

- Theorem 4.2.5 shows that the tropical count is independent from the base curve, just as the classical is.
- [9] proves Theorem 4.2.5 for tropical covers with *explicit* tropical base curves B. The same argument can be used to omit this condition.

4.3 Representation-theoretic setting

Via *monodromy representations*, Hurwitz numbers can be translated to a multiplication problem in the class algebra $\mathcal{Z}(d)$ of the symmetric group S_d, leading to a representation-theoretic expression, *the character formula.*

4.3.1 Monodromy representation

A *(connected/possibly disconnected) monodromy representation* of type $(g \to h, d, \mu_1, ..., \mu_n)$ is a tuple of permutations $(\sigma_1, ..., \sigma_n, a_1, b_1, ..., a_h, b_h) \in {S_d}^{n+2h}$ satisfying

- σ_i has cycle type μ_i,
- $\prod_{i=1}^{n} \sigma_i \prod_{i=1}^{h} [a_i, b_i] = \text{id}$, where $[a_i, b_i] := a_i b_i {a_i}^{-1} {b_i}^{-1}$,
- for connected: $\langle \sigma_1, ..., \sigma_n, a_1, b_1, ..., a_h, b_h \rangle$ acts transitively on $\{1, ..., d\}$.

There is a correspondence between Hurwitz covers and monodromy representations: A Hurwitz cover of type $(g \to h, d, \mu_1, ..., \mu_n)$ defines a homomorphism $\rho : \pi_1(D \backslash B, c_o) \to S_d$, where

$$\pi_1(D \backslash B, c_o) = \langle \alpha_1, \beta_1, ..., \alpha_h, \beta_h, r_1, ..., r_n \mid \prod_{i=1}^{n} r_i \prod_{i=1}^{h} [\alpha_i, \beta_i] = \text{id} \rangle$$

is the fundamental group of the target surface D with base point c_0, punctured at the branch locus B with standard generating set $\{\alpha_1, \beta_1, ..., \alpha_h, \beta_h, r_1, ..., r_n\}$ consisting of loops r_i around the punctures of D and handle generators α_i, β_i. The monodromy representation associated to f is the list $(\rho(r_1), ..., \rho(r_n), \rho(\alpha_1), \rho(\beta_1), ..., \rho(\alpha_h), \rho(\beta_h))$, which uniquely defines ρ. The image $\rho(r_i)$, for example, is a permutation with cycle type given by the ramification profile of f.

Theorem 4.3.1 ([20], Theorem 7.3.1 and Theorem 7.3.2)**.** The connected/disconnected Hurwitz number $H_{g \xrightarrow{d} h}(\mu_1, ..., \mu_n) / H^\bullet_{g \xrightarrow{d} h}(\mu_1, ..., \mu_n)$ (Definition 4.1.1) is equal to $\frac{|M|}{d!}$, where M is the set of (connected/possibly disconnected) monodromy representations of type $(g \to h, d, \mu_1, ..., \mu_n)$.

4.3.2 Character formula

The *class algebra of* S_d is the center $\mathcal{Z}(d)$ of its group algebra $\mathbb{C}[S_d]$. The algebra $\mathcal{Z}(d)$ has two natural basis (as a vector space)

- The conjugacy class basis: $\{C_\mu | \mu \vdash d\}$, where $C_\mu := \sum_\sigma \sigma \in \mathcal{Z}(d)$ is the sum of permutations of cycle type μ.

- The semisimple basis: $\{e_\lambda | \lambda \vdash d\}$ indexed by the irreducible representations λ of S_d (also identified with partitions of d).

The change-of-base matrix is fully determined by the character table of S_d ([20], Theorem 8.4.2).

Theorem 4.3.1 can be easily encoded in terms of the conjugacy class basis.

Theorem 4.3.2 ([20], Proposition 9.2.3)**.** The disconnected Hurwitz number $H^\bullet_{g\xrightarrow{d}0}(\mu_1, ..., \mu_n)$ (Definition 4.1.1) is equal to $\frac{1}{d!}[C_{(1,...,1)}]C_{\mu_n}...C_{\mu_1}$.

By changing basis we obtain the *character formula.*

Theorem 4.3.3 ([20], Theorem 9.3.1)**.** The disconnected Hurwitz number $H^\bullet_{g\xrightarrow{d}0}(\mu_1, ..., \mu_n)$ (Definition 4.1.1) is equal to

$$\sum_{|\lambda|=d} (\frac{\dim(\lambda)}{d!})^2 \prod_{i=1}^{n} \frac{|C_{\mu_i}|\chi_\lambda(\mu_i)}{\dim(\lambda)},$$

where $\chi_\lambda(\mu_i)$ is the character of the irreducible representation λ evaluated at any permutation of cycle type μ_i and $|C_{\mu_i}|$ is the number of elements of cycle type μ_i.

4.3.3 Towards completed cycles

Notation: Let $\mathscr{P}$ be the set of all partitions. Define $f_\mu : \mathscr{P} \to \mathbb{C}$ by

- $f_\mu \equiv 1$, if $\mu = \varnothing$.
- $f_\mu(\lambda) := \binom{|\lambda|}{|\mu|} \frac{|C_\mu|\chi_\lambda(\mu)}{\dim(\lambda)}$ else. (If $|\mu| > |\lambda|$, $f_\mu \equiv 0$. If $|\mu| < |\lambda|$, then $\chi_\lambda(\mu)$ makes sense since $S_{|\lambda|}$ includes all permutations of cycle type μ.)

Rewrite the formula in Theorem 4.3.3

$$\sum_{|\lambda|=d} (\frac{\dim(\lambda)}{d!})^2 \prod_{i=1}^{n} f_{\mu_i}(\lambda). \tag{4.2}$$

Broader context: The algebra $\mathbb{C}^{\mathscr{P}}$ of $\mathbb{C}$ valued functions on $\mathscr{P}$ has a distinguished subalgebra: The algebra Λ^* of shifted symmetric functions defined as the image of the injective linear map ([66], Section 0.4)

$$\bigoplus_{d=0}^{\infty} \mathcal{Z}(d) \xrightarrow{\phi} \mathbb{C}^{\mathscr{P}}, C_\mu \mapsto f_\mu.$$

(Comment: Although Λ^* is only canonically isomorphic to a subalgebra of $\mathbb{C}^{\mathscr{P}}$, presenting it as the image of ϕ is better suited for our purposes.)

As an algebra, Λ^* is freely generated by the *shifted symmetric power sums* $p_k \in \mathbb{C}^{\mathscr{P}}$ ([66], (0.14)), i.e. $\Lambda^* = \mathbb{C}[p_1, p_2, ...]$.

As a vector space, Λ^* admits two key basis for our applications: $\{f_\mu | \mu \in \mathscr{P}\}$ and $\{p_\mu | \mu \in \mathscr{P}\}$, where $p_\mu := \prod_i p_{\mu_i}$.

This provides another vector space basis for $\bigoplus_{d=0}^{\infty} \mathcal{Z}(d)$ via ϕ: The counterpart of the conjugacy class basis is given by $\{\bar{\mu} | \mu \in \mathscr{P}\}$, where $\bar{\mu} := \frac{1}{\prod_i \mu_i} \phi^{-1}(p_\mu)$ is called the *completed cycle conjugacy class*. These will be used in the next section to define Hurwitz numbers with multi-completed cycles.

4.4 Completed cycles Hurwitz numbers

For the completed cycle theory we restrict to target genus $h = 0$. The index h will be dropped, with Hurwitz numbers labeled exclusively by either d or g, depending on the context. Completed cycles are linear combinations of ordinary conjugacy classes (see 4.3.3). We define Hurwitz number with multi-completed cycles, accordingly.

Definition 4.4.1. Fix positive integers $d, g, k_1, ..., k_n$ and partitions μ, ν of $d \in \mathbb{N}$ such that $k_1 + ... + k_n = 2g + l(\mu) + l(\nu) - 2$ holds. The *disconnected Hurwitz number with multi-completed cycles* $h_d^\bullet(\mu, \overline{(k_1+1)}, ..., \overline{(k_n+1)}, \nu)$ is defined the linear combination of ordinary extended disconnected Hurwitz numbers (Definition 4.1.3)

$$h_d^\bullet(\mu, \overline{(k_1+1)}, ..., \overline{(k_n+1)}, \nu) := \sum_{\lambda_1, ..., \lambda_n} \rho_{(k_1+1),\lambda_1} \cdot ... \cdot \rho_{(k_n+1),\lambda_n} \cdot H_d^\bullet(\mu, \lambda_1, ..., \lambda_n, \nu),$$

where the summation is over all partitions λ_i of integers less than or equal to $k_i + 1$ and the coefficients $\rho_{(k_i+1),\lambda_i}$ come from the expansion $\overline{(k_i+1)} = (k_i+1) + \sum_{\lambda_1} \rho_{(k_i+1),\lambda_i} \lambda_i$ of the completed cycles in ordinary conjugacy classes given in [66], (0.21).

Remark 4.4.2. Hurwitz numbers with completed cycles have a geometric interpretation as a count of stable maps from a (not necessarily connected) source curve to $\mathbb{P}^1$ ([72], Proposition 2.7.) Introduce *connected Hurwitz numbers with multi-completed cycles* $h_d(\mu, \overline{(k_1+1)}, ..., \overline{(k_n+1)}, \nu)$ by imposing a connectedness condition on the source.

By replacing f_μ in Equation (4.2) by $p_\mu / \prod_i \mu_i$ we obtain the *character formula for Hurwitz number with multi-completed cycles.*

Theorem 4.4.3. [[66], Section 3.1.1]

The degree d disconnected double Hurwitz number with multi-completed cycles (Definition 4.4.1) is given by

$$h_d^\bullet(\mu, \overline{(k_1+1)}, ..., \overline{(k_1+1)}, \nu) = \frac{\prod_{i=1}^n k_i!}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \sum_\lambda \chi_\mu^\lambda \chi_\nu^\lambda \cdot \prod_{i=1}^n \frac{p_{k_i+1}(\lambda)}{(k_i+1)!},$$

where $\mathfrak{z}(\mu) := |\mu|!/|C_\mu| (= |\mathrm{Aut}(\mu)| \prod_{i=1} \mu_i)$ and $\chi_\mu^\lambda := \chi_\lambda(\mu)$ is the character of the irreducible representation λ evaluated at any permutation of cycle type μ.

A generating function for multi-completed cycles: We have seen in Section 4.1 that Hurwitz numbers exhibit a recursive structure. A standard technique to deal with such sequences is to embed them in a single function: Introduce

$$F_{\mu,\nu}(z_1, ..., z_n) := \sum_{k_1,...,k_n=-2}^{\infty} h_d^{\bullet}(\mu, \overline{(k_1+1)}, ..., \overline{(k_n+1)}, \nu) \prod_{i=1}^{n} z_i^{k_i+1},$$

and its expression in terms of characters

$$F_{\mu,\nu}(z_1, ..., z_n) = \frac{\prod_{i=1}^{n} k_i!}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \sum_{\lambda \vdash d} \chi_\mu^\lambda \chi_\nu^\lambda \cdot \prod_{i=1}^{n} e(\lambda, z_i),$$

where we used Theorem 4.4.3 together with

$$p_k(\lambda) = k![z^k]e(\lambda, z),\ e(\lambda, z) = \sum_{i=1}^{\infty} e^{\lambda_i - i + \frac{1}{2}}$$

and conventions

$$p_{k+1} := \begin{cases} 0, & \text{if } k = -1. \\ -1, & \text{if } k = -2. \end{cases}$$

Chapter 5

Type B theory

This chapter reviews the theory of type B theory around spin Hurwitz numbers.

5.1 Algebraic setting

5.1.1 Spin curves

Definition 5.1.1. Let C be a compact Riemann surface/smooth curve. A *spin structure or theta characteristic* on C is either of the following equivalent objects:

- A divisor L, such that $2L \sim K_C$, where K_C is the canonical divisor on C.
- A line bundle $L \to C$ such that $L^{\otimes 2} \sim \omega_C$, where ω_C is the canonical bundle.

We call a tuple (S, L) a *spin curve*. Moreover, we define the *parity* $p(L)$ of L as

$$p(L) = h^0(C, L) \operatorname{mod} 2,$$

where $h^0(C, L)$ is the dimension of the vector space $H^0(C, L)$. If $p(L) = 0$, we say L has even parity, while for $p(L) = 1$, we say L has odd parity.

Example 5.1.2. The curve $\mathbb{P}^1$ admits only one isomorphism class of spin structures given by the divisor $L = -[\mathrm{pt}]$. Since $h^0(\mathbb{P}^1, L) = 0$, the parity of L is even.

5.1.2 Spin Hurwitz numbers

Fix

- a spin curve (D, L) of genus h with parity $p := h^0(L) \mod 2$.
- a collection of points $q^1, ..., q^n \in D$.
- positive integers d and g.

- a collection $\mu_1, ..., \mu_n$ of *odd* partitions of d (i.e., partitions whose parts are all odd).

A *(connected/potentially disconnected) spin Hurwitz cover* of type $(g \to (h, p), d, \mu_1, ..., \mu_n)$ is a map $f : (C, N_{f,L}) \to (D, L)$, where

- $f : C \to D$ is (connected/potentially disconnected) Hurwitz cover of type $(g \to h, d, \mu_1, ..., \mu_n)$, and

- $N_{f,L}$ the theta characteristic defined by $N_{f,L} := f^*(L) \otimes \mathcal{O}(\frac{1}{2}R_f)$ with $\mathcal{R}_f$ being the *branch divisor* of f. (Note that $\frac{1}{2}R_f$ is well-defined precisely because of the oddness of the μ_i's.)

The *parity* $p(f)$ of f (as a spin Hurwitz cover) is the parity of the pull-back theta characteristic $N_{f,L}$. Any Hurwitz cover can be enhanced to a spin Hurwitz cover once a spin structure on the base is fixed, provided that the μ_i are odd.

Definition 5.1.3. The *connected/disconnected spin Hurwitz number* $H^p_{g\xrightarrow{d}h}(\mu_1, ..., \mu_n)/H^{\bullet,p}_{g\xrightarrow{d}h}(\mu_1, ..., \mu_n)$ is the weighted sum

$$H^p_{g\xrightarrow{d}h}(\mu_1, ..., \mu_n)/H^{\bullet,p}_{g\xrightarrow{d}h}(\mu_1, ..., \mu_n) := \sum_{[f]} \frac{(-1)^{p(f)}}{|\mathrm{Aut}(f)|}, \tag{5.1}$$

of equivalence classes $[f]$ of (connected/potentially disconnected) Hurwitz covers of type $(g \to h, d, \mu_1, ..., \mu_n)$.
Moreover, we define the sub-counts

$$H^{(p,+)}_{g\to h}/H^{\bullet,(p,+)}_{g\to h} = \sum_{p(f)\ \text{even}} \frac{1}{|\mathrm{Aut}(f)|} \quad \text{and} \quad H^{(p,-)}_{g\to h}/H^{\bullet,(p,-)}_{g\to h} = \sum_{p(f)\ \text{odd}} \frac{1}{|\mathrm{Aut}(f)|}$$

that yield enumerations of covers with fixed parity.

Definition 5.1.4. Extend Definition 5.1.3 for target $\mathbb{P}^1$ to encompass the following cases:

- Let $H^\bullet_{d=0}(\emptyset, ..., \emptyset) := 1$.

- If $|\mu_i| > d$ for some i, let $H^\bullet_d(\mu_1, ..., \mu_n) := 0$.

- If $|\mu_i| \leqslant d$, let $\tilde{\mu}_i$ be the partition of size d obtained by adding $1'$s to μ_i. Then: $H^\bullet_d(\mu_1, ..., \mu_n) := \prod_{i=1}^n \binom{m_1(\tilde{\mu}_i)}{m_1(\mu_i)} H^\bullet_{g\xrightarrow{d}0}(\tilde{\mu}_1, ..., \tilde{\mu}_n)$, where $m_1(\mu)$ counts the number of 1's occurring in μ and g is determined by the Riemann Hurwitz formula.

5.1.3 Degeneration formula

Lee and Parker extend the degeneration method from Section 4.1 to maps between spin curves, revealing that spin Hurwitz numbers satisfy recursive relations similar to ordinary Hurwitz numbers (see [56] for details or [31] for a sketch proof).

Theorem 5.1.5 ([56], Theorem 1.1, [45], Theorem 4.5)**.** The following degeneration formulas hold for all spin Hurwitz numbers:

1. If h_1, h_2 with $h_1 + h_2 = h$ and p_1, p_2 parities with $p = p_1 + p_2 \bmod 2$ and any $0 \leqslant n_0 \leqslant n$ then
$$H^{\bullet,p}_{g\to h}(\mu_1, \ldots, \mu_n) = \sum_{\mu}\prod_{i} \mu^i \cdot \mathrm{Aut}(\mu) \cdot H^{\bullet,p_1}_{g_1\to h_1}(\mu_1, \ldots, \mu_{n_0}, \mu) H^{\bullet,p_2}_{g_2\to h_2}(\mu, \mu_{n_0+1}, \ldots, \mu_n),$$
where g_1 and g_2 are determined by the Riemann Hurwitz formula.
2. If $h \geqslant 2$ or $(h, p) = (1, 0)$ then
$$H^{\bullet,p}_{g\to h}(\mu_1, \ldots, \mu_n) = \sum_{\mu}\prod_{i} \mu^i \cdot \mathrm{Aut}(\mu) \cdot H^{\bullet,p}_{\tilde{g}\to h-1}(\mu, \mu, \mu_1, \ldots, \mu_n),$$
where $\tilde{g}$ is determined by the Riemann Hurwitz formula.

The sums are taken over all odd partitions $\mu = (\mu^i)_i$ of d.

5.2 Tropical setting

5.2.1 Tropical spin curves

We extend Section 4.2.1 to include tropical curves with spin structures. For more details, we refer to [81, 18].

Let Γ a graph, with edge set E and vertex set V. Then, we consider the $\mathbb{F}_2$ vector spaces $\mathcal{E}$ and $\mathcal{V}$ spanned by E and V respectively. Every element of $\mathcal{E}$ is canonically identified with a subset of E. We consider the linear map

$$\partial \colon \mathcal{E} \to \mathcal{V}$$

where for an edge e with vertices v and w, we have $\partial(e) = v + w$. We call $C_\Gamma = \ker(\partial)$ the *cycle space* of Γ. Moreover, for $P \in C_\Gamma$, we denote by Γ/P the graph obtained by contracting all edges in P.

Definition 5.2.1. We define a *tropical spin curve* to be a triple (Γ, P, s), such that

- Γ is an abstract tropical curve,

- $P \in \mathcal{C}_\Gamma$ is a collection of cycles in Γ,
- $s\colon \Gamma/P \to \mathbb{Z}/2\mathbb{Z}$ is the *sign (or parity) function*, such that $s(v) = 0$ if $g(v) = 0$.

The parity of a tropical spin curve is defined as

$$s(\Gamma) = \sum_{v \in V(G/P)} s(v).$$

5.2.2 Tropical spin Hurwitz covers

We enhance the covers from Section 4.2.2 with spin structures, obtaining a class of covers that emerges naturally from the degeneration of their classical (spin) counterparts.

Definition 5.2.2. A *tropical spin cover* is a tuple (π, s, s_B), such that

- $\pi\colon \Gamma \to B$ is a tropical Hurwitz cover,
- all edge weights are odd,
- $(\Gamma, [0], s)$ is a tropical spin curve, i.e. we choose the empty cycle for P,
- $(B, [0], s_B)$ is a tropical spin curve as well.

We define the parity of (π, s) to be

$$p(\pi, s) = \sum_{v \in V(\Gamma)} s(v) \in \mathbb{Z}/2\mathbb{Z}.$$

5.2.3 Tropical spin Hurwitz numbers

Fix

- a tropical spin curve an $(B, [0], s_B)$, where B is an abstract tropical curve of genus h with n ends and parity $p := s_B(B)$.
- positive integers d and g.
- a collection $\mu_1, \dots, \mu_n$ of *odd* partitions of d.

A *(connected/potentially disconnected) tropical spin Hurwitz cover* of type $(g \to (h, p), d, \mu_1, \dots, \mu_n)$ is a tropical spin Hurwitz cover $\pi : (\Gamma, s) \to (B, s_B)$, where $\pi : \Gamma \to B$ is (connected/potentially disconnected) tropical Hurwitz cover of type $(g \to h, d, \mu_1, \dots, \mu_n)$ and $s_B(B) = p$. Introduce an equivalence relation on the set of tropical spin Hurwitz covers: Let $(\pi, s) : (\Gamma, s) \to (B, s_B)$ and $(\tilde{\pi}, \tilde{s}) : (\tilde{\Gamma}, \tilde{s}) \to (B, s_B)$ be of type $(g \to (h, p), d, \mu_1, \dots, \mu_n)$. Then

$$\begin{aligned} \pi \text{ is equivalent to } \tilde{\pi} :\Leftrightarrow \exists \text{ isomorphism } \phi : \Gamma \to \tilde{\Gamma} \text{ such that } \pi = \tilde{\pi} \circ \phi \\ \text{and } s(v) = \tilde{s}(\phi(v)) \text{ for all } v \in V(\Gamma). \end{aligned}$$

As usual, we want to identify equivalent spin Hurwitz covers and introduce the *multiplicity of* (π, s, s_B) as record of the number of ways in which (π, s, s_B) may be promoted to a degenerate spin Hurwitz cover of nodal curves with same local parities. Using the same notation as in Section 4.2.3 we define:

Definition 5.2.3. Let B a tropical curve and (π, s, s_B) a tropical spin cover of B, that is unramified over $B\backslash E^{\infty}(B)$. We define its multiplicity

$$\operatorname{mult}(\pi, s, s_B) = (-1)^{\sum_{v\in V(\Gamma)} s(v)} \frac{1}{|\operatorname{Aut}(\pi)|} \prod_{e\in E(\Gamma)\backslash E^{\infty}(\Gamma)} \omega(e) \prod_{v\in V(\Gamma)} |\operatorname{Aut}(\mu_v)| \cdot H^{s_B(v'),s(v)}_{g(v)\to g(v')}(\mu_v), \tag{5.2}$$

where we use v' to denote $\pi(v)$ and $H^{s_B(v'),s(v)}_{g(v)\to g(v')}(\mu_v)$ as in Definition 5.1.3.

Definition 5.2.4. The *connected/disconnected tropical spin Hurwitz number* $\mathbb{T}H^{p}_{g\xrightarrow{d}h}(\mu_1, ..., \mu_n)/\mathbb{T}H^{\bullet,p}_{g\xrightarrow{d}h}(\mu_1, ..., \mu_n)$ is the weighted sum

$$\mathbb{T}H^{p}_{g\xrightarrow{d}h}(\mu_1, ..., \mu_n)/\mathbb{T}H^{\bullet,p}_{g\xrightarrow{d}h}(\mu_1, ..., \mu_n) := \sum_{(\pi,s,s_B)} \operatorname{mult}(\pi), \tag{5.3}$$

of equivalence classes of (connected/potentially disconnected) tropical spin Hurwitz covers of type $(g \to (h, p), d, \mu_1, ..., \mu_n)$.

Remark 5.2.5.

- We note that in the definition of tropical spin Hurwitz numbers, we do not impose an explicit compatibility condition between the sign functions s on Γ and s_B on B. This contrasts [31], where the parity function $s : V(\Gamma) \to \mathbb{Z}/2\mathbb{Z}$ on Γ is required to be *admissible*. That is a function s, such that for all $v \in V(\Gamma)$ there exists a spin Hurwitz Cover f (with connected domain curve) contributing to the local Hurwitz number associated to v with $p(f) = s(v)$. Here, compatibility issues are encoded in the multiplicity: If the parity function is not admissible, then the cover contributes 0 to the count.
- Moreover, as in the classical case, we do not include B in the notation of the tropical Hurwitz numbers. Again, the tropical spin Hurwitz number is independent of the choice of the genus g curve B, however it does depend on the parity of the spin structure on B. This is clarified in Theorem 5.3.1.
- A collection of examples and closed formulae for special cases is available in [31].

5.3 Correspondence

The following correspondence theorem clarifies the relation between classical and tropical spin Hurwitz numbers.

Theorem 5.3.1 ([31]). Let g, h non-negative integers, d a positive integer, $\mu_1, \ldots, \mu_n$ partitions of d. Moreover, let $p \in \mathbb{Z}/2\mathbb{Z}$, then we have

$$H^{\bullet,p}_{g\to h}(\mu_1,\ldots,\mu_n) = \mathbb{T}H^{\bullet,p}_{g\to h}(\mu_1,\ldots,\mu_n).$$

5.4 In the language of representation theory

Spin Hurwitz theory and Hurwitz theory share the same backbone structure. To emphasize these parallels, we present a representation-theoretic expression as a substitution rule: Replace

- the symmetric group S_d by the *Sergeev group* $\mathfrak{h}_d$.
- irreducible representation by irreducible *spin representations*.
- the algebra $\mathbb{C}[S_d]$ by the *superalgebra* $\mathcal{H}_d$.
- the center $\mathcal{Z}(d)$ of $\mathbb{C}[S_d]$ by the *supercenter* $\mathcal{Z}_d$ of $\mathcal{H}_d$.

Recall: We presented a representation-theoretic expression for Hurwitz numbers as result of translating ramified coverings into monodromy representations. Monodromy representations also appear in [37] as initial connection between the geometric and the representation-theoretic framework for spin Hurwitz numbers. Section 5.4 briefly references these to motivate the substitution $S_d \mapsto \mathfrak{h}_d$, that sets the representation theoretic machinery in motion (Section 5.4).

Monodromy representation

The enumeration of Hurwitz covers in (5.1) takes parities into account. Their monodromy representations, however, do not retain this data. Still: [37] uses the maps in (5.1) as building blocks new covers whose monodromy define morphisms $\rho : \pi_1(D\backslash B, c_o) \to B(d) \subset S_{2d}$ (as in Section 4.3.1), where $B(d)$ is the hyperoctahedral group.

The group $B(d)$ admits a central extension $\mathfrak{h}_d$

$$0 \to \mathbb{Z}_2 \to \mathfrak{h}_d \to B(d) \to 0,$$

called the Sergeev group. The parity of the Hurwitz cover is encoded through the existence/non-existence of a "canonical lift" of ρ to $\mathfrak{h}_d$.

We give an alternative definition of $\mathfrak{h}_d$ (not as central extension), which is better suited for our purposes:

Definition 5.4.1. Let d a positive integer. The *Sergeev group* $\mathfrak{h}_d$ is the semi-direct product $\mathfrak{h}_d = S_d \ltimes \mathrm{Cl}_d$, where

- $\mathrm{Cl}_d := \langle \epsilon, \xi_1, \ldots, \xi_d | \xi_i \xi_j = \xi^2 = \epsilon^2 = 1, \epsilon \xi_j \xi_i \text{ for } i \neq j \rangle$ is the *Clifford group* and
- $\sigma \in S_d$ acts on Cl_d by $\epsilon \mapsto \epsilon$ and $x_i \mapsto x_{\sigma(i)}$.

While the unsigned count of morphisms $\rho : \pi_1(D \backslash B, c_o) \to G$ that send the loop generator r_i to a fixed conjugacy class of a finite group G requires considering all irreducible representations of G (see Theorem 4.3.3 for $G = S_d$), letting $G = \mathfrak{h}_d$ and including signs amounts to considering only representations of $\mathfrak{h}_d$ where ϵ acts as -1. These representations are called *spin representations (of $B(d)$)* ([45], Section 1.5).

Character formula

Irreducible spin representations correspond to irreducible modules of the *twisted group algebra of* $\mathfrak{h}_d$ ([45], Section 1.5)

$$\mathcal{H}_d = \mathbb{C}[\mathfrak{h}_d]/1 + \epsilon.$$

These representations are studied in the framework of superalgebras, revealing further parallels to the classical theory.

Superalgebra structure. A *superalgebra* is a $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$ with $A_i A_j \subset A_{i+j}$ for $i, j \in \mathbb{Z}_2$.

The morphism

$$\deg : \mathfrak{h}_d \to \mathbb{Z}_2, \deg(\xi_i) := 1 \text{ and } \deg(\epsilon) = \deg(\sigma) := 0, \sigma \in S_d$$

defines a $\mathbb{Z}_2$–grading on $\mathfrak{h}_d$, which turns $\mathbb{C}[\mathfrak{h}_d]$ into a superalgebra with direct sum decomposition $\mathbb{C}_0[\mathfrak{h}_d] \oplus \mathbb{C}_1[\mathfrak{h}_d]$, where $\mathbb{C}_i[\mathfrak{h}_d] := \mathbb{C}[\{g \in \mathfrak{h}_d | \deg(g) = i\}]$. The quotient algebra $\mathcal{H}_d$ inherits this superalgebra structure.

Classification of irreducible supermodules Irreducible spin representations arise from irreducible supermodules of $\mathcal{H}_d$ ([45], Proposition 1.13). Mirroring the classical setting (Section 4.3.1), these are classified by *strict* partitions of d ([45], Propositions 1.16).

The supercenter $\mathcal{Z}_d$ Let $\mathcal{Z}_d$ denote the *supercenter* of $\mathcal{H}_d$ ([45], Section 1.4). Since $\mathcal{H}_d$ is semisimple ([15], by Lemma 3.6), $\mathcal{Z}_d$ is equal to the even part of the ordinary (algebra) center of $\mathcal{H}_d$.

The algebra $\mathcal{Z}_d$ has two natural vector space basis indexed by *odd* and *strict* partitions of d.

- The conjugacy class basis: $\{c_\mu | \mu \in \mathrm{OP}(d)\}$, where c_μ is the image of the sum $\sum_x x \in \mathbb{C}[\mathfrak{h}_d]$ over all x in the conjugacy class of a permutation $\sigma \in S_d \subset \mathfrak{h}_d$.

- The semisimple basis: $\{e^B_\lambda | \lambda \vdash d\}$ of orthogonal idempotents indexed by irreducible supermodules V^λ of $\mathcal{H}_d$ for $\lambda \in \mathrm{SP}(d)$ (see [45], Remark 1.11 using that $\mathcal{H}_d$ is semisimple).

For $\lambda \in \mathrm{SP}(d)$, let $\zeta_\lambda : \mathfrak{h}_d \to \mathbb{C}$ be the (usual) character of the spin representation associated to the supermodule V^λ by forgetting the grading and write $\zeta_\lambda(\mu)$ for its value at $\sigma \in S_d \subset \mathfrak{h}_d$ of cycle type $\mu \in \mathrm{OP}(d)$.

Theorem 5.4.2 ([45], Theorem 1.10)**.** The disconnected spin Hurwitz number $H^{\bullet,0}_{g\to 0}(\mu_1, \ldots, \mu_n)$ (Definition 5.1.3) is equal to

$$2^{\left(\sum_i (\ell(\mu_i)-d)-2d\right)} \sum_{\lambda \in \mathrm{SP}(d)} \left(\frac{\dim(V^\lambda)}{2^{\delta(d)/2} d!}\right)^2 \prod_{i=1}^{n} \frac{|c_{\mu_i}| \zeta_\lambda(\mu_i)}{\dim(V^\lambda)},$$

where $\delta(d) = 1$ if d is odd and $\delta(d) = 0$ if d is even and $|c_{\mu_i}|$ is the number of elements in the conjugacy class of a permutation $\sigma \in S_d \subset \mathfrak{h}_d$ of cycle type μ_i.

Towards spin completed cycles *Notation:* Let SP be the set of all strict partitions. Define *the central character* $f_\mu : \mathrm{SP} \to \mathbb{C}$ by

- $f_\mu \equiv 1$, if $\mu = \varnothing$.
- $f_\mu(\lambda) := \binom{m_1(\mu)+|\lambda|-|\mu|}{|\lambda|-|\mu|} \frac{|c_\mu| \zeta_\lambda(\mu)}{\dim(V^\lambda)}$, where $m_1(\mu)$ is the number of parts of μ equal to 1, else.

Rewrite the formula in Theorem 5.4.2

$$2^{\left(\sum_i (\ell(\mu_i)-d)-2d\right)} \sum_{\lambda \in \mathrm{SP}(d)} \left(\frac{\dim(V^\lambda)}{2^{\delta(d)/2} d!}\right)^2 \prod_{i=1}^{n} f_{\mu_i}(\lambda). \tag{5.4}$$

Broader context: The algebra $\mathbb{C}^{\mathrm{SP}}$ of $\mathbb{C}$ valued functions on SP has a distinguished subalgebra: The algebra Λ^B of supersymmetric functions is defined as the image of the injective linear map ([40], Proposition 3.14)

$$\bigoplus_{d=0}^{\infty} \mathcal{Z}_d \xrightarrow{\phi} \mathbb{Q}^{\mathrm{SP}}, c_\mu \mapsto f_\mu.$$

(Comment: Usually Λ^B is defined as a subalgebra of the algebra of symmetric functions, which itself can be viewed as subalgebra of $\mathbb{C}^{\mathcal{P}}$. Since the central characters are only defined for strict partitions, we work with $\mathbb{C}^{\mathrm{SP}}$ instead using that symmetric functions are uniquely determined by value on strict partitions (see [48], Proposition 6.1.))

As an algebra, Λ^B is freely generated by the usual *symmetric power sums* $p_k \in \mathbb{C}^{\mathcal{P}}$ with k odd ([48]), i.e. $\Lambda^B = \mathbb{C}[p_1, p_3, \ldots]$.

As a vector space, Λ^B admits two key basis for our applications: $\{f_\mu | \mu \in \mathrm{OP}\}$ and $\{p_\mu | \mu \in \mathrm{OP}\}$, where $p_\mu := \prod_i p_{\mu_i}$.

This provides another vector space basis for $\bigoplus_{d=0}^{\infty} \mathcal{Z}_d$ via ϕ: The counterpart of the conjugacy class basis is given by $\{\bar{\mu} \mid \mu \in \mathrm{OP}\}$, where $\bar{\mu} := \frac{1}{\prod_i \mu_i} \phi^{-1}(p_\mu)$ is called the *spin completed cycle conjugacy class.* These will be used in the next section to define spin Hurwitz numbers with multi-completed cycles.

5.5 Completed cycles spin Hurwitz numbers

The completed cycle theory is restricted to target $\mathbb{P}^1$. The index h will be dropped, with spin Hurwitz numbers labeled exclusively by either d or g, depending on the context.

Definition 5.5.1. Fix positive integers $d, g > 0$, even integers $r_1, \ldots, r_b > 0$ and odd partitions μ, ν of $d \in \mathbb{N}$ such that $r_1 + \ldots + r_b = 2g + l(\mu) + l(\nu) - 2$ holds. The *disconnected spin double Hurwitz number with multi-completed cycles* $h_g^{spin,\bullet,0}(\mu, \overline{(r_1+1)}, \ldots, \overline{(r_b+1)}, \nu)$ is defined as the linear combination of ordinary spin Hurwitz numbers

$$\frac{|\mathrm{Aut}(\mu)||\mathrm{Aut}(\nu)|}{b!} \sum_{\lambda_1, \ldots, \lambda_b} \rho_{(r_1+1),\lambda_1} \cdot \ldots \cdot \rho_{(r_b+1),\lambda_b} \cdot H_{\tilde{g}}^{\bullet,0}(\mu, \lambda_1, \ldots, \lambda_b, \nu), \tag{5.5}$$

where the summation is over all odd partitions λ_i of integers smaller or equal to $r_i + 1$, $\tilde{g}$ is defined by the Riemann Hurwitz formula (for $(d, h = 0, \tilde{\lambda}_i)$ with $\tilde{\lambda}_i$ the partition of size d obtained by adding $1'$s to λ_i), and the coefficients $\rho_{(r_i+1),\lambda_i}$ come from the expansion $\overline{(r_i+1)} = (r_i+1) + \sum_{\lambda_i} \rho_{(r_i+1),\lambda_i} \lambda_i$ of the completed cycles in ordinary conjugacy classes (see [41], Lemma 2.11).

Remark 5.5.2.

- To make sense of the right hand side of (5.5), refer to Definition 5.1.4 for extended spin Hurwitz numbers.
- Definition 5.5.1 specializes to Definition 4.5 in [40] by setting $r_i + 1 = r + 1$ for $i = 1, \ldots, b$.
- Introduce *connected spin double Hurwitz number with multi-completed cycles* $h_d^{spin,0}(\mu, \overline{(k_1+1)}, \ldots, \overline{(k_n+1)}, \nu)$ by imposing a connectedness condition on the source.

We derive a character formula for spin Hurwitz number with multi-completed cycle.

Theorem 5.5.3. The degree d disconnected spin double Hurwitz number with multi-completed cycles (Definition 5.5.1) is given by

$$h_g^{spin,\bullet,0}(\mu, \overline{(r_1+1)}, \ldots, \overline{(r_b+1)}, \nu) = \frac{2^{1-g}}{b! \prod_i \mu_i \prod_j \nu_j} \sum_{\lambda \in \mathrm{SP}(d)} \frac{\zeta_\mu^\lambda \zeta_\nu^\lambda}{2^{\delta(\lambda)+l(\mu)+l(\nu)}} \cdot \prod_{i=1}^{b} \frac{\mathbf{p}_{r_i+1}(\lambda)}{(r_i+1)},$$

where $\zeta_\mu^\lambda := \zeta_\lambda(\mu)$ (ζ_μ^λ) denotes the usual character of the irreducible representation corresponding to λ at μ (ν) ([40], Definition 3.11).

The case, $r_i + 1 = r + 1$ for all i, is stated in [40], but without explicit proof. For completeness and to present a general formula that mirrors Theorem 4.4.3 for type A, we include a proof here.

Proof. The linear expansion on the right hand side of Equation (5.5) corresponds to replacing the central characters $f_{(r_i+1)}$ by the usual power sum functions $p_{(r_i+1)}/(r_i + 1)$. The rest follows from a simple computation, using

- the relation between central and the usual characters (see Section 5.4 or [40], Definition 3.11) to obtain:

$$\frac{\dim(V^\lambda) f_\mu^\lambda}{2^{\delta(\lambda)/2} d!} = \frac{2^d \zeta_\mu^\lambda}{2^{\delta(\lambda)/2+l(\mu)} \mathfrak{z}(\mu)},$$

  where $\mathfrak{z}(\mu) = |\mathrm{Aut}(\mu)| \prod_{i=1} \mu_i$ and $f_\mu^\lambda := f_\mu(\lambda)$.

- the Riemann-Hurwitz formula for the factor 2^{1-g}.

□

Chapter 6

Type A Fock space formalism

6.1 Introduction

The *theory of Fock spaces* is a standalone theory that first emerged from physics literature and has since then been applied across various branches of mathematics. This chapter presents the Fock space formalism through the lens of its application to enumerative theories such as Hurwitz- and Gromov-Witten theory. From this perspective, it offers a framework for handling generating functions of Hurwitz numbers efficiently as it allows the class algebras of all symmetric groups (collected in $\oplus_{d \in \mathbb{N}} \mathcal{Z}(d)$) to be treated simultaneously. Recall that $\oplus_{d \in \mathbb{N}} \mathcal{Z}(d)$ has two basis that are naturally indexed by partitions: The conjugacy class basis $\{C_\mu\}_\mu$ and the semi-simple basis $\{e_\lambda\}_\lambda$ corresponding to the irreducible representations λ of S_d. Accordingly, there are two types of Fock spaces, fermionic and bosonic Fock space. One constructs these as vector spaces over $\mathbb{C}$:

- *Fermionic Fock space* with basis corresponding to the irreducible representations.
- *Bosonic Fock space* (which can be identified with the algebra of symmetric or shifted symmetric functions (see Remark 6.5.3) with basis given by the power sum functions $\{p_\mu\}$ corresponding to (completed cycles) conjugacy classes.

Translating between the two settings is formalized by the so-called *boson-fermion correspondence*, which provides an isomorphism between bosonic and fermionic Fock space as representations of a Heisenberg algebra.

The general framework is established Sections 6.2 and 6.3, introducing the two types of Fock spaces together with the Lie algebras that act on them. Section 6.5 relates these two settings. The material is a synthesis of [11] and [51]. We make no proofs but attempt to convey ideas whenever helpful to understand a certain construction. Additional background material and helpful ideas are drawn from [50] and [19]. As the literature uses different notations and conventions, this compilation is an attempt to present the material in a unified and, to the best of our ability, consistent setting.

Tailored towards applications in Hurwitz theory are Sections 6.4 and 6.5.1. Following the foundational work of Okounkov and Pandharipande in [66], we introduce *vacuum expectations* as values assigned to operators acting on Fock space and define key operators on the fermionic side. An insight from the general framework is that these operators arise naturally from the action of a certain matrix algebra and that their vacuum expectations can be viewed as a specific element of the corresponding infinite dimensional matrix. Within this framework, the Hurwitz generating function is encoded as the vacuum expectation of a specific operator. The boson-fermion correspondence (Section 6.5) then allows for a translation from fermionic to bosonic Fock space and, subsequently for the re-geometrization of the original enumerative problem via tropical geometry (see [28]).

6.2 Relevant Lie algebras

Vector spaces with bilinear forms give rise to Lie algebras: If the bilinear form is symmetric, the result is a *Clifford algebra*.

Definition 6.2.1 ([40], Definition 3.20)**.** Let V be a vector space over $\mathbb{K}$ equipped with a quadratic form q. The *Clifford algebra* $\mathcal{C}\ell(V,q)$ of (V,q) is the unital free algebra on V (i.e. the tensor algebra of V with unit 1) modulo the relations

$$v^2 = q(v) \cdot 1$$

for all $v \in V$. The commutator Lie bracket turns $\mathcal{C}\ell(V,q)$ into a Lie algebra.

Remark 6.2.2. If $\mathrm{Char}(\mathbb{K}) \neq 2$, then a quadratic form is the same as a symmetric bilinear form $\langle \cdot, \cdot \rangle$. We will write $\mathcal{C}\ell(V, \langle \cdot,\cdot \rangle)$ in this case. Using $\langle u, v\rangle := \frac{1}{2}(q(u+v) - q(u) - q(v))$, one rewrites the relations $v^2 = q(v) \cdot 1$ as $uv + vu = 2\langle u, v\rangle \cdot 1$, where $uv + vu$ is called the *anti commutator* of v and u.

If the bilinear form is skew-symmetric, the result is a *Heisenberg algebra*.

Definition 6.2.3. Let V be a vector space over $\mathbb{K}$ and $\omega : V \times V \to \mathbb{K}$ a skew-symmetric bilinear form. The *Heisenberg algebra* $\mathcal{H}eis(V,\omega)$ of (V,ω) is the Lie algebra with underlying vector space $V \oplus \mathbb{K}$ and Lie bracket

$$[(v,k),(v',k')] := (0, \omega(v,v')), \tag{6.1}$$

where $v, v' \in V$ and $k, k' \in \mathbb{K}$.

Definition 6.2.4. A *representation of a Lie algebra/projective representation of a Lie algebra* g on a $\mathbb{C}$-vector space V is a Lie algebra homomorphism ρ from g to the algebra $\mathrm{gl}(V)$ of endomorphisms of V/the quotient $\mathrm{gl}(V)/\mathbb{C} \cdot \mathrm{Id}$. Moreover, ρ is said to be *irreducible* if V does not contain any non-trivial ρ-invariant subspaces.

The following *abstract* Lie algebras will play a central role:

- The Heisenberg algebra $\mathcal{H}^A := \mathcal{Heis}(V, \omega)$, generated by the complex vector space V with basis of formal symbols $\{a_n, n \in \mathbb{Z}\}$ and skew-symmetric bilinear form $\omega(a_n, a_m) := n\delta_{n,-m}$, where $\delta_{n,-m}$ denotes the Kronecker delta. We identify the underlying vector space $V \oplus \mathbb{C}$ (via $(0, 1) \mapsto \gamma$) with the $\mathbb{C}$-span of $\{\gamma, a_n, n \in \mathbb{Z}\}$. The resulting commutator relations (6.1) are: $[a_n, a_m] = n\delta_{n,-m}\gamma$ and $[\gamma, a_n] = 0$.
- The Clifford algebra $\mathcal{C\ell}^A := \mathcal{C\ell}(V, \langle \cdot, \cdot \rangle)$, generated by the complex vector space V with basis of formal symbols $\{\psi_k, \psi_k^*, k \in \mathbb{Z} + \frac{1}{2}\}$ and symmetric bilinear form given by: $\langle \psi_k, \psi_l^* \rangle = \frac{\delta_{k,l}}{2}$ and $\langle \psi_k^*, \psi_l^* \rangle = \langle \psi_k, \psi_l \rangle = 0$.

Operators will come from a collection of the following *concrete* Lie algebras: We denote by

$$\mathrm{gl}(\infty) := \bigoplus_{i,j \in \mathbb{Z} + \frac{1}{2}} E_{i,j},$$

the algebra of infinite-dimensional matrices, and by

$$\mathcal{A}_\infty := \{ \sum_{i,j \in \mathbb{Z} + \frac{1}{2}} a_{ij} E_{i,j} \,|\, a_{ij} = 0 \text{ for } |j - i| >> 0 \},$$

the larger algebra of all matrices supported in a bounded strip with Lie Bracket given by the commutator.

6.3 Vector spaces and representations

6.3.1 Bosonic Fock space

The *bosonic Fock space* B is the space of polynomials in infinitely many variables $\mathbb{C}[x_1, x_2, ...]$. We have a distinguished vector, the constant polynomial 1, called the *vacuum vector*. The terminology is of physical origin and will become clear in Subsection 6.3.2 (together with the Boson-Fermion correspondence). There is a natural (vector space) basis $\{x_\mu := \prod x_{\mu_i}\}_\mu$ indexed by partitions.

Representations on bosonic Fock space

For $\mu \in \mathbb{R}$ the map $\rho_\mu : \mathcal{H}^A \to \mathrm{gl}(B)$ given by

$$\rho_\mu(a_n) = \frac{\partial}{\partial x_n},\ \rho_\mu(a_{-n}) = n x_n,\ \rho_\mu(a_0) = \mu \text{ and } \rho_\mu(c) = 1,$$

where $n \in \mathbb{N}$ and $\mathrm{gl}(B)$ denotes the endomorphism Lie algebra, defines an irreducible representation of $\mathcal{H}$ on B (Definition 6.2.4):

The map $\rho_\mu(a_n)$ is clearly linear and compatibility with the Lie brackets of $\mathcal{H}^A$ and $\mathrm{gl}(B)$ is readily verified. For irreducibility, note that the polynomials

$$\rho_\mu(c)(1), \rho_\mu(a_{-1})^{k_1} \circ \ldots \circ \rho_\mu(a_{-n})^{k_n}(1) = \prod_{i=1}^{n} i^{k_i} x_i^{k_i}, k_i \in \mathbb{N} \tag{6.2}$$

form a basis of B. Since any ρ_μ-invariant subspace $\{0\} \neq W \subset B$ contains the vacuum (by repeatedly applying a_n to a $0 \neq w \in W$), we have $W = B$.

B may be equipped with an inner product $\langle \cdot, \cdot \rangle$ such that the vacuum has norm 1, i.e $\langle 1, 1 \rangle = 1$, and such that $\rho_\mu(a_n)$ is adjoint to $\rho_\mu(a_{-n})$. Then (6.2) forms an orthogonal basis with respect to $\langle \cdot, \cdot \rangle$ and we have

$$\begin{aligned}
&\langle \rho_\mu(a_{-1})^{k_1} \circ \ldots \circ \rho_\mu(a_{-n})^{k_n}(1), \rho_\mu(a_{-1})^{k_1} \circ \ldots \circ \rho_\mu(a_{-n})^{k_n}(1) \rangle \\
&= \langle 1, \rho_\mu(a_n)^{k_n} \circ \ldots \circ \rho_\mu(a_1)^{k_1} \rho_\mu(a_{-1})^{k_1} \circ \ldots \circ \rho_\mu(a_{-n})^{k_n}(1) \rangle \\
&= \langle 1, \frac{\partial}{\partial^{k_n} x_n} \circ \ldots \circ \frac{\partial}{\partial^{k_1} x_1} (\prod_{i=1}^{n} i^{k_i} x_i^{k_i}) \rangle = \prod_{i=1}^{n} k_i!.
\end{aligned}$$

Existence and uniqueness of $\langle \cdot, \cdot \rangle$ together with the orthogonality can be verified with a similar computation.

6.3.2 Fermionic Fock space — the infinite wedge

The *fermionic Fock space*, alternatively *the (half) infinite wedge*, is an inner product space $(\bigwedge^{\frac{\infty}{2}} V, \langle \cdot, \cdot \rangle)$ that is constructed as follows: Let V be the vector space with $\mathbb{C}$-basis $\{\underline{s}\}_{s \in \mathbb{Z}+\frac{1}{2}}$. Then $\bigwedge^{\frac{\infty}{2}} := \bigoplus_S \mathbb{C} v_S$ is the $\mathbb{C}$ vector space with basis $\{v_S := \underline{s_1} \wedge \underline{s_2} \wedge \ldots\}_S$ indexed by sequences $S := (s_i)_{i \in \mathbb{N}} \in (\mathbb{Z} + \frac{1}{2})^{\mathbb{N}}$ such that

- S is decreasing,
- only finitely many positive slots of $\mathbb{Z} + \frac{1}{2}$ are occupied, i.e. $S_+ := S \backslash (\mathbb{Z}_{\geqslant 0} - \frac{1}{2})$, is finite,
- only finitely many negative slots of $\mathbb{Z} + \frac{1}{2}$ are vacant, i.e. $S_- := (\mathbb{Z}_{\geqslant 0} - \frac{1}{2}) \backslash S$ is finite,

and inner product $\langle \cdot, \cdot \rangle : \bigwedge^{\frac{\infty}{2}} V \times \bigwedge^{\frac{\infty}{2}} V \to \mathbb{C}$ defined by $\langle v_{S_1}, v_{S_2} \rangle := \delta_{S_1, S_2}$.

Remark 6.3.1.

- The symbol $\wedge$ is purely formal, but is reminiscent of the exterior algebra of V. In this sense, $\wedge$ is multilinear and alternating and we call vectors v_S *semi-infinite wedges*.
- This construction is best understood against its physical background (see [51], Section 4.2): The infinite wedge is a mathematical model for Dirac's positron theory and build according to the following assumptions

- (Negative energy states): The wave equation of an electron admits solutions with negative energy.
- (Pauli's exclusion principle): Two electrons cannot occupy the same energy state
- (Dirac sea): A perfect vacuum is a region where all positive energy states are unoccupied and all negative energy states are occupied.

The vector space V records energy states of a single electron. These can be positive or negative. The infinite wedge records $\bigwedge^{\frac{\infty}{2}} V$ the energy states of a distribution of electrons in a region. The perfect vacuum is modeled by the *vacuum vector*

$$v_{\varnothing} := \frac{-1}{2} \wedge \frac{-3}{2} \wedge \frac{-5}{2} \wedge \ldots.$$

It plays a distinguished role in the mathematical theory, often serving as a reference state for evaluating operators (Definition 6.4.1).

By definition, any vector v_S differs from the vacuum only in a finite number of places: An occupied energy state is thought of as an electron of that energy and a vacant negative energy state as a positron. Example: The vector $v := \frac{3}{2} \wedge \frac{-3}{2} \wedge \frac{-5}{2} \wedge \frac{-7}{2} \wedge \ldots$ corresponds to an electron of energy $\frac{3}{2}$ and a positron of energy $\frac{1}{2}$.

Charge decomposition. To each v_S we can associate an integer $m := |S_+| - |S_-|$, called its *charge*, and obtain a corresponding decomposition $\bigwedge^{\frac{\infty}{2}} V = \oplus_{m \in \mathbb{Z}} \bigwedge_m^{\frac{\infty}{2}} V$, where $\bigwedge_m^{\frac{\infty}{2}} V$ is the subspace generated by semi-infinite wedges of charge m. Each $\bigwedge_m^{\frac{\infty}{2}} V$ has a basis indexed by partitions. Exemplary for $\bigwedge_0^{\frac{\infty}{2}} V$: We have a bijection between the set of basis vectors v_S of charge 0 and partitions given by $v_S \mapsto \lambda := (\lambda_k := s_k + k - \frac{1}{2})_{k \in \mathbb{N}}$. We argue that the sequence $\lambda \in \mathbb{Z}^{\mathbb{N}}$ is indeed a partition: The vacuum vector is sent to $(\lambda_k)_k = 0$, which we identify with the empty partition $\varnothing$. Suppose $v_S \neq v_{\varnothing}$. We have $0 < s_1 = S_v + (|S_+| - 1) + \frac{1}{2}$, where S_v is the number of vacant slots between $-1/2$ and s_1. So $\lambda_1 \in \mathbb{N}$. Since $-(\lambda_k - \lambda_{k+1})$ is equal to the number of vacant slots between s_k and s_{k+1}, $(\lambda_k)_k$ is decreasing and 0 for k large enough. Indeed, choose K such that $s_k - s_{k+1} = 1$ for all $K \leqslant k$. Then

$$\lambda_k = \lambda_1 - \underbrace{\#(\text{vacant slots between } s_1 \text{ and } s_k)}_{= S_v + |S_-|} = |S_+| - |S_-| = 0, k \geqslant K$$

since v_S has charge 0. Hence, λ is a finite sequence with $\lambda_k \in \mathbb{N}$, i.e. a partition. We will use this bijection and write v_λ instead of v_S.

Representations on fermionic Fock space

We understand how the Lie algebras in Subsection 6.2 operate on $\bigwedge^{\frac{\infty}{2}} V$.

Construction outline: We start by constructing a representation r_0 of $\mathrm{gl}(\infty)$ and extend r_0 to a representation r_1 of $\mathcal{A}_\infty$:

$$\begin{tikzcd}
\mathrm{gl}(\infty) \arrow[r, hook] \arrow[d, "r_0"] & \mathcal{A}_\infty \arrow[d, "r_1"] \\
\mathrm{gl}(\bigwedge^{\frac{\infty}{2}} V) \arrow[r, two heads] & \mathrm{gl}(\bigwedge^{\frac{\infty}{2}} V)/\mathbb{C} \cdot \mathrm{Id}.
\end{tikzcd}$$

As r_1 will only be a *projective representation* (Definition 6.2.4), we have to pass to a bigger Lie Algebra $\hat{\mathcal{A}}_\infty$, which is a *central extension* of $\mathcal{A}_\infty$, to get an ordinary representation $\hat{r}_1$ on $\bigwedge^{\frac{\infty}{2}} V$:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \cdot c \arrow[r, hook] \arrow[d] & \hat{\mathcal{A}}_\infty \arrow[r, two heads] \arrow[d, "\hat{r}_1"] & \mathcal{A}_\infty \arrow[r] \arrow[d, "r_1"] & 0 \\
0 \arrow[r] & \mathbb{C} \cdot \mathrm{Id} \arrow[r, hook] & \mathrm{gl}(\bigwedge^{\frac{\infty}{2}} V) \arrow[r, two heads] & \mathrm{gl}(\bigwedge^{\frac{\infty}{2}} V)/\mathbb{C} \cdot \mathrm{Id} \arrow[r] & 0
\end{tikzcd},$$

where we will set $\hat{\mathcal{A}}_\infty := \mathcal{A}_\infty \oplus \mathbb{C} \cdot c$ with $c \in Z(\hat{\mathcal{A}}_\infty)$.

1. Construction of r_0. The representation r_0 is a generalization of the following: The n-dimensional matrix Lie algebra $\mathrm{Mat}(n, \mathbb{C})$ has a representation on any n-dimensional vector space $W \cong \mathbb{C}^n$ given by $E_{i,j} \cdot e_k = \delta_{jk} e_i$. Via

$$A \cdot (w_1 \wedge \ldots \wedge w_k) := \sum_{i=1}^{k} w_1 \wedge \ldots \wedge (A \cdot w_i) \wedge \ldots \wedge w_k$$

we also get a representation of $\mathrm{Mat}(n, \mathbb{C})$ on the Grassmann algebra $\bigwedge W$.

Accordingly, we identify V with $\mathbb{C}^\infty$ via $\underline{s} \mapsto e_s$, where e_s denotes the standard basis vector with index $s \in \mathbb{Z} + \frac{1}{2}$ and let $E_{i,j} \in \mathrm{gl}(\infty)$ act on V via $E_{i,j} \cdot \underline{s} = \delta_{js} \underline{i}$. Then

$$r_0(A)(v_S) := A\underline{s}_1 \wedge \underline{s}_2 \wedge \ldots + \underline{s}_1 \wedge A\underline{s}_2 \wedge \ldots + \ldots,$$

where $A \in \mathrm{gl}(\infty)$ and $v_S \in \bigwedge^{\frac{\infty}{2}} V$, is linear and well-defined, since

$$r_0(E_{i,j})(v_S) = \begin{cases} 0 & j \notin S, \\ \underline{s}_1 \wedge \ldots \wedge \underline{i} \wedge \ldots, & \text{else}, \end{cases}$$

is and every such A is a finite linear combination of the matrices $E_{i,j}$. One verifies compatibility with the Lie brackets to conclude that r_0 is a representation.

2. Construction of r_1. To extend r_0 to $\mathcal{A}_\infty$ we have to deal with finite linear combinations of matrices with entries only on the $\mathbb{Z} \ni c$-shifted diagonal:

$$A_c := \sum_{i,j \in \mathbb{Z} + \frac{1}{2}} a_{ij} E_{i,i+c},$$

The r_0-definition works for $c \neq 0$, since again

$$A_c \underline{s_1} \wedge \underline{s_2} \wedge \ldots + \underline{s_1} \wedge A_c \underline{s_2} \wedge \ldots + \ldots,$$

is a finite: We have $A_c\underline{s} = a_i \cdot \underline{i}$, if $s = i + c$, and 0 else. Moreover, there exists an index k such that $\forall i \leqslant k$ we have $i, i + c \in S$. The corresponding summand vanishes as $\wedge$ is alternating. For $c = 0$, however, we have $A_0\underline{s} = a_s \cdot \underline{s}$ and

$$A_0 v_S = (a_{s_1} + a_{s_2} + \dots)v_S$$

may diverge. As any v_S has only finitely many positive terms, it suffices to introduce a correction term for the action of $E_{i,i}$ for $i < 0$ and define: $\sigma(E_{i,j}) := \begin{cases} r_0(E_{i,j}) - \mathrm{Id}, & i = j < 0 \\ r_0(E_{i,j}), & \text{else.} \end{cases}$

This gives rise to a well-defined map $\sigma : \mathcal{A}_\infty \to \mathrm{gl}(\bigwedge^{\frac{\infty}{2}} V)$ whose behavior with respect to the Lie brackets is given by

$$[\sigma(A_1), \sigma(A_2)] = \sigma([A_1, A_2]) + a(A_1, A_2) \cdot \mathrm{Id},$$

where $a : \mathcal{A}_\infty \times \mathcal{A}_\infty \to \mathbb{C}$ is the bilinear map defined by

$$a(E_{i,j}, E_{k,l}) = \begin{cases} 1 & i = l < 0 \text{ and } j = k > 0, \\ -1 & i = l > 0 \text{ and } j = k < 0, \\ 0 & \text{else.} \end{cases} \tag{6.3}$$

(see [11], Lemma 1.3).

Passing to the quotient yields a Lie algebra homomorphism r_1

$$\mathcal{A}_\infty \xrightarrow{\sigma} \mathrm{gl}(\bigwedge^{\frac{\infty}{2}} V) \twoheadrightarrow \mathrm{gl}(\bigwedge^{\frac{\infty}{2}} V)/\mathbb{C} \cdot \mathrm{Id}$$

and thus a projective representation of $\mathcal{A}_\infty$.

3. Construction of $\hat{r}_1$. An ordinary representation is obtained as follows: Introduce a *central extension* $\hat{\mathcal{A}}_\infty$ of $\mathcal{A}_\infty$ by a 1-dimensional center; this is the Lie algebra $\hat{\mathcal{A}}_\infty := \mathcal{A}_\infty \oplus \mathbb{C} \cdot c$ with Lie bracket/commutator relations $[A_1, A_2]^\wedge := A_1A_2 - A_2A_1 + a(A_1, A_2) \cdot c$ for $A_i \in \mathcal{A}_\infty$ and $c \in Z(\hat{\mathcal{A}}_\infty)$. Define $\hat{r}_1 : \hat{\mathcal{A}}_\infty \to \mathrm{gl}(\bigwedge^{\frac{\infty}{2}} V)$ by

$$\hat{r}_1(A + \lambda \cdot c) := r_1(A) + \lambda \cdot \mathrm{Id}.$$

The map $\hat{r}_1$ is a representation of $\hat{\mathcal{A}}_\infty$ that *preserves charges*, i.e. it restricts to representations on the charge spaces: $\hat{r}_1^m : \hat{\mathcal{A}}_\infty \to \mathrm{gl}(\bigwedge_m^{\frac{\infty}{2}} V)$ for $m \in \mathbb{Z}$.

6.4 Operators and vacuum expectations

This section aims at two results: Application 1 and 2 (6.4.4 and 6.5.1) that encode Hurwitz numbers as the action of certain operators on bosonic and fermionic Fock space. For that purpose, we start by fixing terminology in Section 6.4.1, continue by defining the key operators to finally arrive at an expression for Hurwitz numbers as vacuum expectation on fermionic Fock space in Section 6.4.4. To derive a similar expression on bosonic Fock space, we use Boson-Fermion correpondence (Section 6.5)

6.4.1 Terminology

The fermionic Fock space is an inner product space with infinite dimensional matrices operating on it. This motivates the following terminology.

Definition 6.4.1. Let A be an operator on $\bigwedge^{\frac{\infty}{2}} V$. The expression $\langle v_S, A(v_{S'})\rangle$ for basis vectors v_S and $v_{S'}$ is called a *matrix element* of A. The diagonal matrix element $\langle A\rangle := \langle v_\varnothing, A(v_\varnothing)\rangle$ corresponding to the vacuum vector is the *vacuum expectation*.

Remark 6.4.2.

- Consider the inner product space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{\mathbb{R}^n})$, where $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ denotes the standard scalar product. Terminology 6.4.1 is to be understood in analogy to: Let $A \in \mathrm{Mat}(n,\mathbb{R})$ and $e_1, ..., e_n$, the standard basis vectors. Then $\langle e_s, Ae_{s'}\rangle$ is equal to the matrix entry $e_s^T Ae_{s'} = A_{ss'}$.
- Since the boson-fermion correspondence (Section 6.5) allows us to transport the representation of $\mathcal{A}_\infty$ on $\bigwedge^{\frac{\infty}{2}} V$ to a representation on bosonic Fock space, we may transfer Definition 6.4.1 to this setting and call a product of the form, $\langle 1, A(1)\rangle$ for an operator A, *vacuum expectation of* A.

6.4.2 Charged fermionic operators

Consider the following family of operators: For $k \in \mathbb{Z} + \frac{1}{2}$, let

$$\psi_k : \bigwedge^{\frac{\infty}{2}} V \to \bigwedge^{\frac{\infty}{2}} V, v_S \mapsto \underline{k} \wedge v_S, \text{ i.e. } \psi_k(v_S) = \begin{cases} (-1)^{|\{s\in S: s>k\}|} v_{S\cup\{k\}}, & \text{if } k \notin S, \\ 0, & \text{else,} \end{cases}$$

with adjoint (with respect to the inner product $\langle\cdot,\cdot\rangle$) ψ_k^* given by

$$\psi_k^*(v_S) = \begin{cases} (-1)^{|\{s\in S: s>k\}|} v_{S\setminus\{k\}}, & \text{if } k \in S, \\ 0, & \text{else,} \end{cases}.$$

Accordingly, the operators $\{\psi_k, \psi_{-k}^*\}_{k>0}$ are called *creation* and $\{\psi_k, \psi_{-k}^*\}_{k<0}$ *annihilation* operators. Note that the basis $\{v_S\}_S$ can be obtained through the action of the creation operators on $v_\varnothing$.

Concrete realization of $\mathcal{Cl}^A$*:* The subalgebra $\mathcal{C}$ of $\mathrm{gl}(\bigwedge^{\frac{\infty}{2}} V)$ with basis $\{\psi_k, \psi_k^*, k \in \mathbb{Z}+\frac{1}{2}\}$ is a Clifford algebra: These operators satisfy the anti-commutation relations ([66], Section 2.1.2)

$$\psi_k \circ \psi_l^* + \psi_k^* \circ \psi_l = \delta_{kl} \cdot \mathrm{Id} \text{ and } \psi_k \circ \psi_l + \psi_l \circ \psi_k = \psi_k^* \circ \psi_l^* + \psi_l^* \circ \psi_k^* = 0.$$

Hence, we find $\mathcal{C}$ to be isomorphic to $\mathcal{Cl}^A$ (Section 6.2).

Regarding to the framework of Section 6.3, we have $\hat{r}_1(E_{i,j}) =: \psi_i \circ \psi_j^* :$, where

$$: \psi_i \circ \psi_j^* ::= \begin{cases} \psi_i \circ \psi_j^*, & \text{if } j > 0 \\ -\psi_j^* \circ \psi_i, & \text{if } j < 0 \end{cases}$$

denotes the *normally ordered product*: Recall that $\hat{r}_1(E_{i,j}) = \sigma(E_{i,j})$ and use that for $j < 0$ and $i \neq j$, we have $: \psi_i \circ \psi_j^* := \psi_i \circ \psi_j^*$.

6.4.3 Bosonic operators

We can realize the Heisenberg algebra $\mathcal{H}^A$ from Section 6.2 as subalgebra of the central extension $(\hat{\mathcal{A}}_\infty, [\cdot,\cdot]^\wedge)$: Let $\Lambda_n := \sum_{i \in \mathbb{Z}+\frac{1}{2}} E_{i,i+n} \in \hat{\mathcal{A}}_\infty$, for $n \in \mathbb{Z}$, be the matrix with 1 on the n-th diagonal and denote by $\mathcal{H}$ the algebra with basis $\{\Lambda_n, n \in \mathbb{Z}\} \cup \{c\}$. Then

$$[c, \Lambda_n]^\wedge = 0 \text{ and } [\Lambda_n, \Lambda_m]^\wedge = [\Lambda_n, \Lambda_m] + a(\Lambda_n, \Lambda_m) \cdot c = a(\Lambda_n, \Lambda_m),\ n, m \in \mathbb{Z}$$

since the Λ_n generate a commutative subalgebra of $(\mathcal{A}_\infty, [\cdot,\cdot])$ and c is a central element. Using (6.3) together with bilinearity one computes

$$[\Lambda_n, \Lambda_m]^\wedge = \sum_{i,j} a(E_{i,i+n}, E_{j,j+m}) \cdot c = n \cdot \delta_{-n,m} \cdot c$$

and finally obtains $\mathcal{H} \cong \mathcal{H}^A$ via $\Lambda_n \mapsto a_n$ and $c \mapsto \gamma$ (Subsection 6.2).

The operators $\alpha_n := \hat{r}_1(\Lambda_n) \in \mathrm{gl}(\bigwedge^{\frac{\infty}{2}} V)$ are called *bosonic operators*. We examine their action on the basis $\{v_S := \underline{s_1} \wedge \underline{s_2} \wedge \dots\}_S$ (see Subsection 6.3.2):

$$\alpha_n(v_S) = \sum_{i : i+n = s_j \in S} \underline{s_1} \wedge \dots \wedge \underline{s_{j-1}} \wedge \underline{i} \wedge \underline{s_{j+1}} \wedge \dots, \tag{6.4}$$

where the i-th summand, $\underline{s_1} \wedge \dots \wedge \underline{s_{j-1}} \wedge \underline{i} \wedge \underline{s_{j+1}} \wedge \dots$, vanishes, if $i \in S$, and is equal to $\pm v_{S'}$ with $S' = S \backslash \{i+n\} \cup \{i\}$, else. The sign is given by $(-1)^{|\{s \in S : s > i\}|}$, where $|\{s \in S : s > i\}|$ is the number swaps necessary to move i to the right position. In particular, we see: bosonic operators *preserve charges.*

Much of the significance of bosonic operators in the context of Hurwitz theory, more broadly in the context of character theory of the symmetric group, comes from reformulating the action of bosonic operators (6.4) in terms of partitions. Restrict α_n to the subspace/sector $\bigwedge_0^{\frac{\infty}{2}} V$ of charge 0 and recall that v_S, as well as v_S', can be indexed by partitions (Subsection 6.3.2). Then, (6.4) can be seen to match the Murnaghan-Nakayama recursion, a recursive algorithm for computing characters the symmetric group. An operator-theoretic formulation is as follows.

Lemma 6.4.3 ([50], Theorem (Murnaghan-Nakayama)). Let μ be a partition of d, then

$$\prod_{i=1}^{l(\mu)} \alpha_{-\mu_i}(v_\varnothing) = \sum_{\lambda \vdash d} \chi_\mu^\lambda v_\lambda,$$

where $\chi^\lambda_\mu := \chi^\lambda(\sigma)$ is the character of the representation corresponding to λ evaluated at a permutation σ of cycle type μ.

Example 6.4.4. Let μ and ν be partitions of d. Using Lemma 6.4.3 and the orthogonality of the vectors v_λ we compute:

$$\langle \prod_{i=1}^{l(\nu)} \alpha_{-\nu_i}(v_\varnothing), \prod_{i=1}^{l(\mu)} \alpha_{-\mu_i}(v_\varnothing) \rangle = \langle \sum_{\lambda \vdash d} \chi^\lambda_\nu v_\lambda, \sum_{\lambda' \vdash d} \chi^{\lambda'}_\mu v_{\lambda'} \rangle = \sum_{\lambda \vdash d} \chi^\lambda_\nu \chi^\lambda_\mu.$$

6.4.4 Application 1: Hurwitz numbers as vacuum expectation on fermionic Fock space

We want to express Hurwitz numbers with multi-completed cycles as vacuum expectation on fermionic Fock space. With Lemma 6.4.3 we have operators that take care of the "character-part" ($\sum_\lambda \chi^\lambda_\mu \chi^\lambda_\nu$) of

$$h^\bullet_d(\mu, \overline{(k_1+1)}, \dots, \overline{(k_1+1)}, \nu) = \frac{\prod_{i=1}^n k_i!}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \sum_\lambda \chi^\lambda_\mu \chi^\lambda_\nu \cdot \prod_{i=1}^n \frac{p_{k_i+1}(\lambda)}{(k_i+1)!}.$$

Indeed, $\langle \prod_{i=1}^{l(\nu)} \alpha_{\nu_i} \prod_{i=1}^{l(\mu)} \alpha_{-\mu_i} \rangle$ is equal to $\sum_{\lambda \vdash d} \chi^\lambda_\nu \chi^\lambda_\mu$ (see Example 6.4.4). For each λ we still need a coefficient $\prod_{i=1}^n \frac{p_{k_i+1}(\lambda)}{(k_i+1)!}$, i.e. we want to insert operators $\mathscr{P}_{k+1}$ (between $\prod_{i=1}^{l(\nu)} \alpha_{\nu_i}$ and $\prod_{i=1}^{l(\mu)} \alpha_{-\mu_i}$) that send v_λ to $p_{k+1}(\lambda) \cdot v_\lambda$. Rather than defining these operators individually, we construct a generating function $\mathscr{E}_0(z)$ that relates to $\mathscr{P}_k$ as $e(\lambda, z)$ relates to p_k (i.e. via $\mathscr{P}_k = k![z^k]\mathscr{E}_0(z)$) and derive a vacuum expectation for the entire generating function for multi-completed cycles instead: $F_{\mu,\nu}(z_1, \dots, z_n) =$

$$\sum_{k_1,\dots,k_n=-2}^{\infty} h^\bullet_d(\mu, \overline{(k_1+1)}, \dots, \overline{(k_n+1)}, \nu) \prod_{i=1}^n z_i^{k_i+1} = \frac{\prod_{i=1}^n k_i!}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \cdot \sum_{\lambda \vdash d} \chi^\lambda_\mu \chi^\lambda_\nu \cdot \prod_{i=1}^n e(\lambda, z_i). \tag{6.5}$$

Here, $\mathscr{E}_0(z)$ is an operator on $\bigwedge_0^{\frac{\infty}{2}} V[[z]]$ and $\langle \cdot, \cdot \rangle$ extends to $\bigwedge_0^{\frac{\infty}{2}} V[[z]]$ via $\langle z^i, z^j \rangle = z^{i+j}$. According to (6.5), we want $\mathscr{E}_0(z)$ to have eigenvalues $e(\lambda, z)$ with corresponding eigenvectors v_λ.

***Ansatz*:** Define $E_0 := \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{zk} E_{kk} \in \hat{\mathscr{A}}_\infty[[z]]$ and consider the r_0 action of E_0 on a basis vector $v_\varnothing \neq v_S \in \bigwedge_0^{\frac{\infty}{2}} V$:

$$E_0 v_S = \underbrace{(\sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{zk} E_{kk} \underline{s_1})}_{=e^{zs_1}\underline{s_1}} \wedge \underline{s_2} \wedge \dots + \underline{s_1} \wedge \underbrace{(\sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{zk} E_{kk} \underline{s_2})}_{=e^{zs_2}\underline{s_2}} \wedge \underline{s_2} \wedge \dots + \dots$$

$$= (\sum_{s_i \in S} e^{zs_i}) v_S = (\sum_{i=0}^{\infty} e^{z(\lambda_i - i + \frac{1}{2})}) v_\lambda = e(\lambda, z) v_\lambda,$$

where $v_S = v_\lambda$ for the partition $\lambda := (\lambda_i := s_i + i - \frac{1}{2})_{i\in\mathbb{N}}$ (see Section 6.3.2). Recall from Section 6.3.2 that the r_0 action is not necessarily well-defined, i.e. we need to work with $\hat{r}_1$. One computes

$$\hat{r}_1(E_0)(v_S) = e(\lambda, z)v_\lambda - \underbrace{(\sum_{0>k\in\mathbb{Z}+\frac{1}{2}} e^{zk})}_{=\frac{1}{e^{z/2}-e^{-z/2}}} v_\lambda$$

and adjusts for the second summand by defining

$$E_0 := \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{zk}E_{kk} + \frac{1}{e^{z/2}-e^{-z/2}} \cdot \mathrm{Id} \text{ and}$$

$$\mathcal{E}_0(z) := \hat{r}_1(E_0) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{zk} : \psi_k\psi_k^* : + \frac{1}{e^{z/2}-e^{-z/2}} \cdot \mathrm{Id}.$$

Proposition 6.4.5 ([66], Proposition 3.1)**.** The generating series $F_{\mu,\nu}(z_1, \ldots, z_n)$ can be expressed on fermionic Fock space:

$$F_{\mu,\nu}(z_1, \ldots, z_n) = \frac{\prod_{i=1}^n k_i!}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \cdot \langle \prod_{i=1}^{l(\mu)} \alpha_{\mu_i} \prod_{i=1}^{n} \mathcal{E}_0(z_i) \prod_{i=1}^{l(\nu)} \alpha_{-\nu_i} \rangle$$

We extract operators $\mathcal{P}_k$ to arrive at an expression for individual Hurwitz numbers.

Definition 6.4.6. For $k > 0$ define $\mathcal{P}_k := k![z^k]\mathcal{E}_0(z) = \sum_{j\in\mathbb{Z}+\frac{1}{2}} j^k : \psi_j\psi_j^* :$.

By taking the coefficient of $z_1^{k_1+1} \cdot \ldots \cdot z_n^{k_n+1}$ in Proposition 6.4.5, we find:

Corollary 6.4.7. The Hurwitz number $h_d^\bullet(\mu, \overline{(k_1+1)}, \ldots, \overline{(k_1+1)}, \nu)$ is given by: $\frac{\prod_{i=1}^n k_i!}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \cdot \langle \prod_{i=1}^{l(\mu)} \alpha_{\mu_i} \prod_{i=1}^{n} \mathcal{P}_{k_i+1} \prod_{i=1}^{l(\nu)} \alpha_{-\nu_i} \rangle$.

This formalism was primarily developed for Gromov-Witten invariants. For Hurwitz numbers the additional factor of $\prod_{i=1}^n k_i!$ might be taken care of by setting $\tilde{\mathcal{P}}_k := [z^k]\mathcal{E}_0(z)$, instead. Then, $\tilde{\mathcal{P}}_k$ has eigenvalues $\frac{p_k(\lambda)}{k!}$.

6.5 Type A boson-fermion correspondence

Following [51],we connect the representation of $\mathcal{H}^A$ on bosonic Fock space B (Section 6.3.1) to the one on fermionic Fock space (Section 6.3.2).

Theorem 6.5.1. [[51], Section 5.1] The representations $\hat{r}_1^m$ and ρ_m (see Subsection 6.3.1) of the Heisenberg algebra $\mathcal{H}^A$ are equivalent, i.e. there exists a vector space isomorphism $f_m : \bigwedge_m^{\frac{\infty}{2}} V \to B$ such that

$$\begin{tikzcd} \bigwedge_m^{\frac{\infty}{2}} V \arrow[r, "\hat{r}_1^m(A)"] \arrow[d, "f_m"] & \bigwedge_m^{\frac{\infty}{2}} V \arrow[d, "f_m"] \\ B \arrow[r, "\rho_m(A)"] & B \end{tikzcd}$$

commutes for all operators $A \in \mathcal{H}$. Moreover, f_m satisfies $v_\varnothing^m \mapsto 1$, where $v_\varnothing^m := \underline{m - \frac{1}{2}} \wedge \underline{m - \frac{3}{2}} \wedge \ldots$ denotes the charge m vacuum vector.

Theorem 6.5.1 provides a dictionary (f_m) to translate between $\hat{r}_1^m(\mathcal{H})$ and $\mathrm{Im}(\rho_m)$, where

$$\alpha_n \leftrightarrow \begin{cases} \frac{\partial}{\partial x_n}, & \text{for } n > 0, \\ n \cdot x_n, & \text{for } n < 0, \\ m, & \text{for } n = 0. \end{cases} \tag{6.6}$$

Bypassing ρ_m and $\hat{r}_1^m$, we can still use the isomorphism f_m to transport operators acting on $\bigwedge_m^{\frac{\infty}{2}} V$ to operators acting on B and vice versa: Let $A \in \mathrm{gl}(\bigwedge_m^{\frac{\infty}{2}} V)$ or $A \in \mathrm{gl}(B)$, then conjugation with f_m, respectively f_m^{-1}, yields $f_m \circ A \circ f_m^{-1} \in \mathrm{gl}(B)$ or $f_m^{-1} \circ A \circ f_m \in \mathrm{gl}(\bigwedge_m^{\frac{\infty}{2}} V)$.

We want to extend this procedure to $\bigwedge^{\frac{\infty}{2}} V$ while preserving the compatibility described in (6.6). For each $m \in \mathbb{Z}$ introduce a copy $B_m := z^m \mathbb{C}[x_1, x_2, \ldots]$ of B and define

$$\bigwedge^{\frac{\infty}{2}} V = \bigoplus_{m \in \mathbb{Z}} \bigwedge_m^{\frac{\infty}{2}} V \xrightarrow{f := \oplus_{m \in \mathbb{Z}} \tilde{f}_m} \bigoplus_{m \in \mathbb{Z}} B_m = \mathbb{C}[z, z^{-1}, x_1, x_2, \ldots],$$

where $\tilde{f}_m$ is the composition of f_m with the (vector space) isomorphism $B \to B_m, p \mapsto z^m p$. The map f is compatible with Theorem 6.5.1, in the sense that f respects charge m vacuum vectors ($\tilde{f}_m(v_\varnothing^m) := z^m$) and (6.6):

$$\tilde{f}_m \circ \alpha_n \circ \tilde{f}_m^{-1} \leftrightarrow \begin{cases} \frac{\partial}{\partial x_n}, & \text{for } n > 0, \\ n \cdot x_n, & \text{for } n < 0, \\ m, & \text{for } n = 0. \end{cases} \tag{6.7}$$

Theorem 6.5.2 ([51], Theorem 6.1 and Section 5.1)**.** The map $f := \oplus_{m \in \mathbb{Z}} \tilde{f}_m$ is an isomorphism of vector spaces. On a basis, $\tilde{f}_0(v_\lambda) = s_\lambda(\frac{x_1}{1}, \frac{x_2}{2}, \ldots)$, where s_λ is the Schur polynomial indexed by λ, and $\tilde{f}_0^{-1}(x_\mu) = \sum_\lambda \chi^\lambda(\mu) v_\lambda$.

Remark 6.5.3. We can identify B with the ring of symmetric functions Λ via $x_i \mapsto \tilde{p}_i$, where $\tilde{p}_i$ is the usual i-th power-sum symmetric function in infinitely many variables.

That is we find the basis $\{x_\mu\}_\mu$ to correspond to the power-sum basis $\{\tilde{p}_\mu\}_\mu$ (and thus to the conjugacy class basis of $\bigoplus_{d=0}^{\infty} \mathcal{Z}(d)$). Recall that Λ has another prominent basis $\{s_\lambda\}_\lambda$ indexed by irreducible representations λ, namely the Schur polynomials that (via Theorem 6.5.2) correspond to the basis $\{v_\lambda\}_\lambda$ of (the charge 0 sector of) fermionic Fock space. In Λ, both basis are not "equal": While power-sum functions are easier to define, Schur polynomials have the advantage of forming an orthonormal basis with respect to the inner product on Λ.

Fock space formalism removes this imbalance by creating two separate settings. In each, one of the two bases will be the most natural and (rather than cumbersome definitions) it is their relationship (i.e. base change) that will be recorded by f_0.

Remark 6.5.4. [28] considers a rescaled version $\tilde{\rho}_m$ of the representation ρ_m given by

$$\tilde{\rho}_m(a_n) = n\frac{\partial}{\partial x_n},\ \tilde{\rho}_m(a_{-n}) = x_n,\ \tilde{\rho}_m(a_0) = m \text{ and } \tilde{\rho}_m(c) = 1,$$

which is equivalent to $\hat{r}_1^m$ via the isomorphism $g \circ f_m : \bigwedge_m^{\frac{\infty}{2}} V \to B$, where $g : B \to B$ is defined by $x_n \mapsto \frac{x_n}{n}$ and $1 \mapsto 1$.

6.5.1 Application 2: Hurwitz numbers as vacuum expectation on bosonic Fock space

Using the boson-fermion correspondence, we can transport $\mathcal{E}_0(z)$ to B. Conjugation with f (in the sense of Section 6.5) yields the following:

Lemma 6.5.5 (Lemma 1.4.8, [64])**.** The bosonic description of $\mathcal{E}_0(z)$ is given by:

$$\mathcal{E}_0^B(z) := \frac{1}{\varsigma(z)} \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{i_1+\ldots+i_l=0} \frac{\varsigma(i_1 z)\cdot \ldots \cdot \varsigma(i_l z)}{i_1 \cdot \ldots \cdot i_l} : a_{i_1} \cdot \ldots \cdot a_{i_l} :,$$

where $\varsigma(z) := e^{z/2} - e^{-z/2}$.

The bosonic counterparts $\mathcal{F}_k := k![z^k]\mathcal{E}_0^B(z)$ of $\mathcal{P}_k$'s are called *cut-and-join operators.* Together with Lemma we obtain a bosonic version of Corollary 6.4.7:

Lemma 6.5.6. The Hurwitz number $h_d^\bullet(\mu, \overline{(k_1+1)}, \ldots, \overline{(k_1+1)}, \nu)$ is given by: $\frac{\prod_{i=1}^n k_i!}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \cdot \langle \prod_{i=1}^{l(\mu)} a_{\mu_i} \prod_{i=1}^{n} \mathcal{F}_{k_i+1} \prod_{i=1}^{l(\nu)} a_{-\nu_i} \rangle$.

Chapter 7

Type B Fock space formalism

7.1 Introduction

This chapter serves as an introduction to the type B theory and follows the structural blueprint established in Chapter 6.

The general framework is developed in Sections 7.2 and 7.3, introducing the two types of Fock spaces and the Lie algebras that act on them. Section 7.5 relates these two settings via the type B boson-fermion correspondence.

Tailored towards applications in spin Hurwitz theory, Sections 7.4 and 7.5.1 derive a fermionic and a bosonic expression for spin Hurwitz numbers, ultimately enabling the re-geometrization of the original enumerative problem in Section 8.1. The material is taken from [40],[80], supplemented with original work in Section 7.5.1.

7.2 Relevant Lie algebra

The following *abstract* Lie algebras will play a central role in the type B theory:

- The Heisenberg algebra $\mathscr{H}^{\mathrm{B}}$ generated by the complex vector space with basis of formal symbols $\{a_n^{\mathrm{B}}, n \in \mathbb{Z} \text{ odd }\} \cup \{\gamma\}$ and with skew-symmetric bilinear form $\omega(a_n^{\mathrm{B}}, a_m^{\mathrm{B}}) := \frac{m}{2}\delta_{n,-m}$ und $\omega(\gamma, a_n^{\mathrm{B}}) := 0$. The commutator relations are: $[a_n^{\mathrm{B}}, a_m^{\mathrm{B}}] = \frac{m}{2}\delta_{n,-m}\gamma$ and $[\gamma, a_n^{\mathrm{B}}] = 0$.
- The Clifford algebra $\mathscr{C}\ell^{\mathrm{B}}$ generated by the complex vector space with basis of formal symbols $\{\varphi_n, n \in \mathbb{Z}\}$ and with symmetric bilinear form $\langle \varphi_k \varphi_l \rangle = \frac{(-1)^k}{2}\delta_{k,-l}$.

Concrete Lie algebra: The role of the matrix algebra $\mathscr{A}_\infty$ in type A theory is assumed by one of its subalgebras.

Definition 7.2.1. Let $\mathscr{B}_\infty$ be the subalgebra of $\mathscr{A}_\infty$ (Section 6.2) consisting of those matrices that satisfy $\iota(A) = -A$, where ι is the involution $\iota : \mathscr{A}_\infty \to \mathscr{A}_\infty, E_{i,j} \mapsto (-1)^{i+j}E_{-j,-i}$. The matrices $B_{j,k} := E_{-j,k} - (-1)^{k+j}E_{-k,j}$ for $j > k$ form a basis of $\mathscr{B}_\infty$.

7.3 Vector spaces and representations

7.3.1 Bosonic Fock space of type B

The *bosonic Fock space* of type B is the space of polynomials $\mathbb{C}[x_1, x_3, x_5, ...]$ with indeterminates indexed by odd natural numbers. We have a distinguished vector, the constant polynomial 1, called the *vacuum vector*. There is a natural (vector space) basis $\{x_\mu := \prod x_{\mu_i}\}_{\mu \in \mathrm{OP}}$ indexed by odd partitions.

Represention on bosonic Fock space of Type B

The map $\rho^{\mathrm{B}} : \mathcal{H}^{\mathrm{B}} \to \mathrm{gl}(\mathbb{C}[x_1, x_3, x_5 ...])$ given by

$$\rho^{\mathrm{B}}(a_n^{\mathrm{B}}) = n \cdot \frac{\partial}{\partial x_n},\ \rho_\mu^{\mathrm{B}}(a_{-n}^{\mathrm{B}}) = \frac{x_n}{2} \text{ and } \rho^{\mathrm{B}}(c^{\mathrm{B}}) = 1,$$

where $n \in \mathbb{N}$ is odd and 1 denotes the identity operator in $\mathrm{gl}(\mathbb{C}[x_1, x_3, x_5 ...])$, defines an irreducible representation of $\mathcal{H}^{\mathrm{B}}$ on $\mathbb{C}[x_1, x_3, x_5 ...]$ (see [40], Section 3.3).

$\mathbb{C}[x_1, x_3, x_5 ...]$ may be equipped with an inner product $\langle \cdot, \cdot \rangle$ such that the vacuum has norm 1, i.e $\langle 1, 1 \rangle = 1$, and $\rho^{\mathrm{B}}(a_n^{\mathrm{B}})$ is adjoint to $\rho^{\mathrm{B}}(a_{-n}^{\mathrm{B}})$.

7.3.2 Fermionic Fock space of type B

Since the action of charged fermions generates fermionic Fock space of type A, the Clifford algebra $\mathcal{C}\ell^A$ is considered the more fundamental object. The B-type theory starts from the Clifford algebra $\mathcal{C}\ell^B$.

Neutral fermions

Explicitly, we can realize $\mathcal{C}\ell^B$ as subalgebra of $\mathcal{C}\ell^A$: Call the following linear combinations of charged fermions

$$\varphi_n := \frac{1}{\sqrt{2}}(\psi_{n-\frac{1}{2}} + (-1)^n \psi^*_{-n-\frac{1}{2}}),\ n \in \mathbb{Z},$$

neutral fermions, the algebra they generate the *algebra neutral fermions*, and verify the anticommutation relations of $\mathcal{C}\ell^B$:

$$\varphi_n \varphi_m + \varphi_m \varphi_n = (-1)^n \delta_{n,-m}. \tag{7.1}$$

This shows: The algebra neutral fermions is isomorphic to $\mathcal{C}\ell^B$. Following the example of the type A theory, define the fermionic Fock space of type B ($\mathcal{F}^B$) to be the vector space generated by the action of $\mathcal{C}\ell^B$ on the vacuum $v_\varnothing \in \bigwedge^{\frac{\infty}{2}} V$ with inner product $\langle \cdot, \cdot \rangle$

inherited from $\bigwedge^{\frac{\infty}{2}} V$. We have

$$\varphi_n(v_\varnothing) = \begin{cases} 0 & \text{if } n < 0, \\ \frac{1}{\sqrt{2}}(\underline{n-\frac{1}{2}} \wedge \underline{\frac{-1}{2}} \wedge \underline{\frac{-3}{2}} \wedge \ldots) & \text{if } n > 0, \\ \frac{1}{\sqrt{2}}(\underline{\frac{-3}{2}} \wedge \ldots) & \text{if } n = 0. \end{cases}$$

Using the anticommutation relations (7.1) and the fact that neutral fermion with negative index annihilate the vacuum, one can arrange for any element $\varphi_{m_1} \circ \ldots \circ \varphi_{m_r}(v_\varnothing)$, where $m_j \in \mathbb{Z}$, to be of the form $\varphi_{n_1} \circ \ldots \circ \varphi_{n_l}(v_\varnothing)$, where the n_j are strictly ordered all non-negative integers. The set

$$\{\varphi_{n_1} \circ \ldots \circ \varphi_{n_l}(v_\varnothing) : n_1 > n_2 > \ldots > n_l \geqslant 0 \text{ and } l > 0\} \cup \{v_\varnothing\},$$

is a basis of $\mathcal{F}^B$. The decomposition $\mathcal{C}\ell^B = \mathcal{C}\ell_0^B \oplus \mathcal{C}\ell_1^B$, where $\mathcal{C}\ell_0^B$ and $\mathcal{C}\ell_1^B$ are spanned by all products $\varphi_{n_1} \circ \ldots \circ \varphi_{n_l}$ of even, respectively odd length, induces a vector space decomposition $\mathcal{F}^B = \mathcal{F}_0^B \oplus \mathcal{F}_1^B$. The space $\mathcal{F}_0^B$ is the type B analogue of the charge 0 space $\bigwedge_0^{\frac{\infty}{2}} V$. It has an orthonormal basis $\{v_\lambda\}_\lambda$ indexed by strict partitions (see [55], Section 3.2.1): Given $\lambda \in \mathrm{SP}$, we define

$$v_\lambda := \begin{cases} \varphi_{\lambda_1} \cdots \varphi_{\lambda_{l(\lambda)}}(v_\varnothing), & \text{if } l(\lambda) \text{ is even.} \\ \sqrt{2}\varphi_{\lambda_1} \cdots \varphi_{\lambda_{l(\lambda)}}\varphi_0(v_\varnothing), & \text{if } l(\lambda) \text{ is odd.} \end{cases}$$

Representation of $\mathcal{B}_\infty$ on $\mathcal{F}_0^{\mathrm{B}}$

In contrast to the type A theory, normal ordering is not needed to get a well-defined action of $\mathcal{B}_\infty$ on $\mathcal{F}_0^B$.

Lemma 7.3.1. The assignements

$$B_{j,k} \mapsto (-1)^j : \varphi_j \varphi_k : \text{ and } B_{j,k} \mapsto (-1)^j \varphi_j \varphi_k$$

yield two irreducible representations of $\mathcal{B}_\infty$, a projective and an ordinary one,

$$r_1^B : \mathcal{B}_\infty \to \mathrm{gl}(\mathcal{F}_0^B)/\mathbb{C}\mathrm{Id} \text{ and } r_0^B : \mathcal{B}_\infty \to \mathrm{gl}(\mathcal{F}_0^B).$$

Denote by $\hat{r}_1^B : \hat{\mathcal{B}}_\infty \to \mathrm{gl}(\mathcal{F}_0^B)$ the ordinary representation obtained from r_1^B (The construction of the central extension $\hat{\mathcal{B}}_\infty$ is analogous to the one of $\hat{\mathcal{A}}_\infty$ in Section 6.3.2).

7.4 Application 1: Spin Hurwitz numbers as vacuum expectation on fermionic Fock space of type B

7.4.1 Bosonic operators of type B

Definition 7.4.1. The *bosonic operators* α_n^B *(of type B)* are defined by:

$$\alpha_n^B := \frac{-1}{2} \sum_{k \in \mathbb{Z}} (-1)^k : \varphi_k \varphi_{-k-n} :, \text{ where } n \in \mathbb{Z}.$$

Remark 7.4.2.

- Bosonic operators vanish for n even: Using $:\varphi_i\varphi_j: := -:\varphi_j\varphi_i:$ and $:\varphi_i\varphi_i: := 0$ for $i, j \in \mathbb{Z}$ ([80], Section 1.4.6), we obtain

$$\alpha_n^B = -\frac{1}{2}\sum_{k\in\mathbb{Z}}(-1)^k :\varphi_k\varphi_{-k-n} := -\frac{1}{2}\sum_{k>-k-n}(-1)^k :\varphi_k\varphi_{-k-n}:$$
$$+ \frac{-1}{2}\sum_{k<-k-n}(-1)^k \underbrace{:\varphi_k\varphi_{-k-n}:}_{=-:\varphi_{-k-n}\varphi_k:}$$
$$\overset{(l:=-k-n)}{=} \frac{-1}{2}\sum_{k>-k-n}(-1)^k :\varphi_k\varphi_{-k-n}: + \frac{(-1)^{-n}}{2}\sum_{l>-l-n}(-1)^{-l} :\varphi_l\varphi_{-l-n}:$$
$$= \begin{cases} -\sum_{k>-k-n}(-1)^k :\varphi_k\varphi_{-k-n}:, & \text{if } n \text{ is odd,}\\ 0, & \text{if } n \text{ is even.}\end{cases}$$

- For $n\mathbb{Z}$ odd, the preceding computation shows:

$$\alpha_n^B = -\frac{1}{2}\sum_{k\in\mathbb{Z}}(-1)^k :\varphi_k\varphi_{-k-n} := -\sum_{k>-k-n}(-1)^k :\varphi_k\varphi_{-k-n} := -\sum_{k>-k-n}(-1)^k\varphi_k\varphi_{-k-n}.$$

  Note, the normal ordering in Definition 7.4.1 is not necessary since $:\varphi_k\varphi_{-k-n}: \neq \varphi_k\varphi_{-k-n}$ if and only if $k = k + n$, i.e. $n = 0$.

- The adjoint $(\alpha^B)_n^*$ is given by

$$(\alpha^B)_n^* = -\frac{1}{2}\sum_{k\in\mathbb{Z}}(-1)^k \underbrace{(:\varphi_k\varphi_{-k-n}:)^*}_{=:\varphi^*_{-k-n}\varphi^*_k:} = -\frac{1}{2}\sum_{k\in\mathbb{Z}}(-1)^k \underbrace{(-1):\varphi_{k+n}\varphi_{-k}:}_{=:\varphi_{-k}\varphi_{k+n}:} = \alpha^B{}_{-n},$$

  where $\varphi_k^* = \begin{cases} -\varphi_{-k}, & \text{if } k \text{ is odd}\\ \varphi_{-k}, & \text{if } k \text{ is even}\end{cases}$ follows from recalling that φ_k is defined in terms of charged fermions.

Lemma 7.4.3 ([40], Lemma 3.31)**.** The operators $\{\alpha_n^B\}_{n\in\mathbb{Z}\text{ odd}}$ satisfy

$$[\alpha_n^B, \alpha_m^B] = \frac{m}{2}\delta_{n,-m}.$$

Together with the identity they generate an algebra isomorphic to the Heisenberg algebra $\mathcal{H}^B$.

Remark 7.4.4. For $n \in \mathbb{Z}$ odd, define $\Lambda_n^B := \sum_{-k-n<k} B_{k,-k-n} \in \hat{\mathcal{B}}_\infty$. Then $\{\Lambda_n^B\}_{n\in\mathbb{Z}\text{ odd}} \cup \{c^B\}$, realizes $\mathcal{H}^B$ as a subalgebra of the central extension induced by the representation r_1 (compare to Section 6.4.3): $(\hat{\mathcal{B}}_\infty := \mathcal{B}_\infty \oplus \mathbb{C}\cdot c^B, [\cdot,\cdot]^\wedge)$. Since we did not properly define the Lie bracket $[\cdot,\cdot]^\wedge$, verifying the commutation relations in Lemma 7.4.3 is best done in $\mathrm{gl}(\mathcal{F}_0^B)$.

We have a type B analogue of the Murnaghan-Nakayama rule.

Lemma 7.4.5 ([40], Corollary 3.33)**.** For $\mu \in \mathrm{OP}$ a partition, we have:

$$\prod_{i=1}^{l(\mu)} \alpha^B_{-\mu_i}(v_\varnothing) = \sum_{\lambda \in \mathrm{SP}} \frac{\zeta^\lambda_\mu}{2^{\delta(\lambda)/2+l(\mu)}} v_\lambda,$$

where $\delta(\lambda) := \begin{cases} 1, & \text{if } l(\lambda) \text{ is odd.} \\ 0, & \text{if } l(\lambda) \text{ is even.} \end{cases}$

7.4.2 The vacuum expectation formula

With the type B analogue of the Murnaghan-Nakayama rule (Lemma 7.4.5) we can realize the "character-part" of the character formula for spin Hurwitz numbers (Theorem 5.5.3) as vacuum expectation: Let d be a positive integer and $\mu, \nu \in \mathrm{OP}(d)$. Using the orthogonality of the vectors v_λ and Remark 7.4.2:

$$\langle \prod_{i=1}^{l(\nu)} \alpha^B_{\nu_i} \prod_{i=1}^{l(\mu)} \alpha^B_{-\mu_i} \rangle = \langle \prod_{i=1}^{l(\nu)} \alpha^B_{-\nu_i}(v_\varnothing), \prod_{i=1}^{l(\mu)} \alpha^B_{-\mu_i}(v_\varnothing) \rangle = \sum_{\lambda \in \mathrm{SP}} \frac{\zeta^\lambda_\nu \zeta^\lambda_\mu}{2^{\delta(\lambda)+l(\nu)+l(\mu)}}.$$

The coefficients $\prod_{i=1}^{b} \frac{p_{r_i+1}(\lambda)}{(r_i+1)}$ will come from the action of operators $\mathscr{P}^{\mathrm{B}}_{r+1}$ with eigenvalues $p_{r_i+1}(\lambda)$ for $\lambda \in \mathrm{SP}$.

Definition 7.4.6 ([40], Definition 3.34 and Lemma 3.35)**.** The operators

$$\mathscr{P}^{\mathrm{B}}_{r+1} := \frac{1}{2} \sum_{k \in \mathbb{Z}} (-1)^k k^{r+1} : \varphi_k \varphi_{-k} := \sum_{k>0} (-1)^k k^{r+1}, \varphi_k \varphi_{-k}$$

where $r \in \mathbb{N}$ even, have eigenvalues $p_{r+1}(\lambda)$ and corresponding eigenvectors v_λ.

Let $k > 0$. Then

$$(-1)^k \varphi_k \varphi_{-k}(v_\lambda) = \begin{cases} v_\lambda, & \text{if } k = \lambda_i \\ 0, & \text{else} \end{cases}$$

is easily verified. One obtains:

$$\mathscr{P}^{\mathrm{B}}_{r+1}(v_\lambda) = \sum_{k>0} (-1)^k k^{r+1} \varphi_k \varphi_{-k}(v_\lambda) = (\sum_i {\lambda_i}^{r+1}) v_\lambda = p_{r+1}(\lambda)(v_\lambda).$$

As an immediate consequence, we have the following (see [40] for the case $r_i + 1 = r + 1$ for all i):

Proposition 7.4.7. The spin Hurwitz number with completed cycle can be expressed on fermionic Fock space:

$$h_g^{spin,\bullet,0}(\mu, \overline{(r_1+1)}, ..., \overline{(r_b+1)}, \nu) = \frac{2^{1-g}}{b! \prod_{i=1} \mu_i \prod_{j=1} \nu_j \prod_{i=1}^{b} (r_i+1)} \cdot \langle \prod_{i=1}^{l(\mu)} \alpha^{\mathrm{B}}_{\nu_i} \prod_{i=1}^{b} \mathcal{P}^{\mathrm{B}}_{r_i+1} \prod_{i=1}^{l(\nu)} \alpha^{\mathrm{B}}_{-\mu_i} \rangle$$

Remark 7.4.8. In Section 7.5.1 we use results from [40] that are expressed in terms of generating functions: Introduce the type B analogue of $\mathcal{E}_0(z)$ ([40], Definition 5.5)

$$\mathcal{E}_0^{\mathrm{B}}(z) = \sum_{k \in \mathbb{Z}} \frac{1}{2} \sum_{k \in \mathbb{Z}} e^{kz} (-1)^k : \varphi_k \varphi_{-k} : \in \hat{\mathcal{B}}_\infty[[z]]$$

with $\mathcal{P}^{\mathrm{B}}_{r+1} = (r+1)![z^{r+1}]\hat{\mathcal{E}}_0(z)$ for even r. Generally speaking: Working with generating series simplifies (among other things) the determination of the bosonic expression of $\mathcal{P}^{\mathrm{B}}_{r+1}$ (i.e. the transport of the action of $\mathcal{P}^{\mathrm{B}}_{r+1}$ to an action on $\mathbb{C}[x_1, x_3, ...]$).

7.5 Type B boson-fermion correspondence

Bosonic and fermionic Fock space of type B are connected in a similar way to how the corresponding objects are connected in the type A theory.

Theorem 7.5.1. [[40], Theorem 3.32] The representations $\hat{r}_1^{\mathrm{B}}$ and ρ^{B} (see Sections 7.3.2 and 7.3.1) of the Heisenberg algebra $\mathcal{H}^{\mathrm{B}}$ are equivalent. This equivalence is realized by the vector space isomorphism $f^{\mathrm{B}} : \mathcal{F}_0^{\mathrm{B}} \to \mathbb{C}[x]$, which is given by $v_\lambda \mapsto 2^{-l(\lambda)} Q_\lambda(\frac{1}{2}x)$, where $x := (x_1, x_3, ...)$ and Q_λ is the Schur Q-function associated to $\lambda \in \mathrm{SP}$ ([40], Section 3.2).

7.5.1 Application 2: Spin Hurwitz numbers as vacuum expectation on bosonic Fock space and related computations

The boson-fermion correspondence of type B provides a way to transport operators acting on $\mathcal{F}_0^{\mathrm{B}}$ to operators $\mathbb{C}[x]$. We exploit Theorem 7.5.1 for the operator $\mathcal{E}_0^{\mathrm{B}}(z)$.

Lemma 7.5.2 ([40], Lemma 5.7)**.** The bosonic description of $\mathcal{E}_0^{\mathrm{B}}(z)$ is given by:

$$\coth(\frac{z}{2}) \sum_{n=1}^{\infty} \sum_{k_1+\cdots+k_n=0, k_i \text{ odd}} \frac{2^{n-2}}{n!} : \prod_{i=1}^{n} \zeta(k_i z) \frac{a^B_{k_i}}{k_i} :,$$

where we write $a^B_{k_i}$ for $\rho^{\mathrm{B}}(a^B_{k_i}) \in \mathrm{gl}(\mathbb{C}[x_1, x_3, ...])$, $\zeta(z) := 2\sinh(\frac{z}{2})$, and $: a^B_{k_i} :$ for the *normal product* of the $a^B_{k_i}$'s, which means that the indices from left to right are increasing.

As usual, call the coefficients $\mathcal{F}^{\mathrm{B}}_{r+1} := r![z^{r+1}]\mathcal{E}^{\mathrm{B}}_0(z) =$

$$r![z^{r+1}]\left(\coth(\frac{z}{2})\sum_{n=1}^{\infty}\sum_{k_1+\dots+k_n=0, k_i \text{ odd}}\frac{2^{n-2}}{n!}:\prod_{i=1}^{n}\zeta(k_i z)\frac{a^B_{k_i}}{k_i}:\right) \tag{7.2}$$

spin completed cycles cut-and-join operators.

Lemma 7.5.3. The spin Hurwitz number $h_g^{spin,\bullet,0}(\mu, \overline{(r_1+1)}, \dots, \overline{(r_b+1)}, \nu)$ is given by:

$$\frac{2^{1-g}}{b!\prod_i \mu_i \prod_j \nu_j}\cdot\langle\prod_{i=1}^{l(\mu)} a^B_{\mu_i}\prod_{i=1}^{n}\mathcal{F}^{\mathrm{B}}_{r_i+1}\prod_{i=1}^{l(\nu)} a^B_{-\nu_i}\rangle.$$

Note that multiplying by $r!$ instead of $(r+1)!$ eliminates the factor $\frac{1}{\prod_{i=1}^{b}(r_i+1)}$ in Proposition 7.4.7.

Towards a graph-theoretic interpretation. The action of the cut-and-join operators $\mathcal{F}^{\mathrm{B}}_{r+1}$ has a geometric interpretation in terms of algebraic degeneration (see Section 8.1) that is not visible yet. Indeed, $\mathcal{F}^{\mathrm{B}}_{r+1}$ is the coefficient of a certain power of z in a gigantic sum that mixes products of functions in z together with bosonic operators. Rewriting Equation (7.2) as in Lemma 7.5.6 is a first step. For that purpose introduce the following notation:

$$\zeta(z) := 2\sinh(\frac{z}{2}),\ \mathcal{S}(z) = \frac{\zeta(z)}{z},\ \rho(z) = \frac{1}{2}\cosh(\frac{z}{2})$$

and

$$\begin{aligned} S\mathbb{Z}^{r+2-2g} := \{x \in \mathbb{Z}^{r+2-2g} : x_1 \leqslant \dots \leqslant x_l < 0 < x_{l+1} \leqslant \dots \leqslant x_{r+2-2g}, \\ \sum_{i=1}^{r+2-2g} x_i = 0 \text{ and } x_i \text{ odd}\} \end{aligned}$$

for the set of ordered integer vectors of length $r+2-2g$ with odd entries that sum up to zero. We denote by x^+ (x^-) the partitions formed by the positive (negative) part of an element $x \in S\mathbb{Z}^{r+2-2g}$ and write $h_g^{spin,0,r}(x^+, x^-)$ ($h_g^{spin,\bullet,0,r}(x^+, x^-)$) for $h_g^{spin,0}(\mu, \overline{(r+1)}, \nu)$ ($h_g^{spin,\bullet,0}(x^+, \overline{(r+1)}, x^-)$), where $x^+ = \mu$ and $x^- = \nu$.

Lemma 7.5.6 simplifies $\mathcal{F}^{\mathrm{B}}_{r+1}$ to a weighted sum of *words* in the bosonic operators $a^B_{k_i}$. The next Lemma provides an interpretation for the coefficients appearing in this sum.

Lemma 7.5.4. The connected spin Hurwitz numbers with 1 completed cycles are given by:

$$h_g^{spin,0,r}(x^+, x^-) = 2^{1-g} r![z^{2g}]\rho(z)\frac{\prod_i \mathcal{S}(x_i^- z)\prod_j \mathcal{S}(x_j^+ z)}{\mathcal{S}(z)}.$$

Proof. An explicit computation of the vacuum expectation (see [40], Proposition 6.1)

$$h_g^{spin,\bullet,0,r}(x^+,x^-)=\frac{r!\cdot 2^{1-g}}{\prod_i x_i^+\prod_j x_j^-}[z^{r+1}]\langle\prod_i \alpha^B_{x_i^+}\mathcal{E}_0^B(z)\prod_j \alpha^B_{-x_j^-}\rangle, \tag{7.3}$$

is carried out by using the commutation relations derived from the properties of the B-Okounkov-Pandharipande operators ([40], Lemma 5.8):

$$[\alpha^B_m,\mathcal{E}^B_n(z)]=\left(\frac{1}{2}+\left(\frac{-1}{2}\right)^{m+1}\right)\zeta(mz)\mathcal{E}^B_{n+m}(z) \tag{7.4}$$

$$[\mathcal{E}^B_m(z),\alpha^B_n]=\left(\frac{-1}{2}+\left(\frac{-1}{2}\right)^{n}\right)\zeta(nz)\mathcal{E}^B_{n+m}(z) \tag{7.5}$$

$$[\alpha^B_m,\alpha^B_n]=\frac{m}{2}\delta_{-n,m} \tag{7.6}$$

to reduce (7.3) to a linear combination of vacuum expectations of the form $\langle\mathcal{E}^B_m(z)\rangle=\delta_m\frac{\rho(z)}{\zeta(z)}$. More precisely: 1. Start by moving $\mathcal{E}^B_0(z)$ to the left by substituting $\alpha^B_m\mathcal{E}^B_0(z)$ by

$$\mathcal{E}^B_0(z)\alpha^B_m+\underbrace{[\alpha^B_m,\mathcal{E}^B_0(z)]}_{(\ldots)\mathcal{E}^B_m(z)}.$$

This produces two terms: One in which $\mathcal{E}^B_0(z)$ absorbs α^B_m increasing its subscript by m and another in which $\mathcal{E}^B_0(z)$ is shifted one step further to the left. Repeat until (7.3) reduces to a linear combination of terms of the form $\langle\mathcal{E}^B_k(z)\prod_{i>0}\alpha^B_i\prod_{j<0}\alpha^B_j\rangle$.

2. Since α^B-operators with positive subscript annihilate the vacuum, we move them to the right using $\alpha^B_i\alpha^B_j=\alpha^B_j\alpha^B_i+[\alpha^B_i,\alpha^B_j]$. This *always* yields a term in which the operator with a positive subscript is further to the right, eventually resulting in 0 after iteration. *And*, if $i+j$ happens to be equal to 0, a second term is created in which both cancel, producing a scalar.

The surviving terms are of the form $c\cdot\langle\mathcal{E}^B_k(z)\prod_{j<0}\alpha^B_j\rangle$, where c is a scalar and the subscripts sum up to 0.

3. Finally, move $\mathcal{E}^B_k(z)$ to the right using relations (7.4). From (7.4) also note: Non-zero contributions to (7.3) come from $\mathcal{E}^B_0(z)$ absorbing α^B-operators whose subscripts sum up to 0 such that the remaining operators can pairwise cancel.

However, *non-trivially* mixed-terms (i.e. obtained by pairwise cancellation *and* absorption) contribute to the disconnected number only (see [66], Section 3.2). It then follows that the Formula in Lemma 7.5.4, the connected part of (7.3), corresponds to $\mathcal{E}^B_0(z)$ absorbing the α^B-operator in each step.

□

A sanity check. Completed cycles spin Hurwitz numbers can be compute in a number of ways, e.g. by using Lemma 7.5.4 or by extracting the connected part of Definition 5.5.1. We verify that both agree.

Example 7.5.5. Consider $h_1^{spin,0,2}((3),(3))$. We have

$$\rho(z)\frac{(\mathcal{S}(3z))^2}{\mathcal{S}(z)} = \frac{1}{2} + \frac{5}{12}z^2 + \ldots \text{ and } h_g^{spin,0,2}((3),(3)) = 2^{1-1}\cdot 2![z^2](\rho(z)\frac{(\mathcal{S}(3z))^2}{\mathcal{S}(z)}) = \frac{5}{6},$$

by Lemma 7.5.4. On the other hand, $h_g^{spin,\bullet,0,2}((3),(3))$ is a linear combination of ordinary (extended) spin Hurwitz numbers (Definition 5.1.4) whose connected part is

$$\rho_{(3),(3)}\binom{0}{0}H^0_{\tilde{g}=1}((3),(3),(3)) + (\rho_{(3),(1,1)}\binom{2+1}{1} + \rho_{(3),(1)}\binom{1+2}{2})H^0_{\tilde{g}=0}((3),(1,1,1),(3))$$

with coefficients determined by the expansion of $(\bar{3})$ in ordinary cycles ([41], Lemma 2.11)

$$\begin{aligned}(\bar{3}) &= \rho_{(3),(3)}(3) + \rho_{(3),(1,1,1)}(1,1,1) + \rho_{(3),(1,1)}(1,1) + \rho_{(3),(1)}(1) + \rho_{(3),\varnothing}\varnothing \\ &= (3) + (1,1) + \frac{1}{6}(1) - \frac{1}{720}\varnothing.\end{aligned}$$

Excluding the empty partition (since we are interested in the connected number) and using [54] finally yields:

$$h_1^{spin,0,2}((3),(3)) = \frac{-1}{3} + (1\cdot 3 + \frac{1}{6}\cdot 3)\frac{1}{3} = \frac{5}{6}.$$

Lemma 7.5.6. The operators $\mathcal{F}^{\mathrm{B}}_{r+1}$ satisfy

$$\mathcal{F}^{\mathrm{B}}_{r+1} = \sum_{g\geqslant 0}\sum_{x\in S\mathbb{Z}^{r+2-2g}} \frac{2^{l(x)-1+g}}{|\mathrm{Aut}(x^+)||\mathrm{Aut}(x^-)|} h_g^{spin,0,r}(x^+,x^-)\prod_i a^B_{x_i^-}\prod_j a^B_{x_j^+}.$$

Proof. Recall, the right hand side of (7.2) is a sum of words in the $a^B_{k_i}$ that contribute to the coefficient of z^{r+1}. Rewriting it as power series in z we see: The coefficient of z^{r+1} is formed by a choice of summand in the series expansion of $\coth(z)$ together with a word in $a^B_{k_i}$. Note that each individual $a^B_{k_i}$ contributes with a power of z coming from a summand of $\zeta(k_i z)$. The series expansions

$$\begin{aligned}\coth(z) &= \frac{1}{z} + \frac{z}{3} + \mathcal{O}(z^3)/\text{h.o.t. in odd powers of } z \\ \zeta(z) &= z + \mathcal{O}(z^3)/\text{h.o.t. in odd powers of } z\end{aligned}$$

form the basis for the following observation.

1. The longest word in the $a^B_{k_i}$ has length $r+2$: Choose the l.o.t. in each series.
2. There are words of length $r+1$ given by choosing the $z-$term in each series.
3. Any further deviation from the choices above give rise to a decrease of the word length by 2 since $\coth(z)$ and $\zeta(z)$ are odd in z.

Let g remember the difference between word length and power of z, i.e.

$$(r+2)-2g = (\text{word length}).$$

Expression (7.2) becomes

$$r!\sum_{g\geqslant 0}\sum_{x\in S\mathbb{Z}^{r+2-2g}}\frac{2^{l(x)-2}}{|\mathrm{Aut}(x^+)||\mathrm{Aut}(x^-)|}[z^{r+1}]\left(\coth(z)\frac{\prod_i \zeta(x_i z)}{\prod_i x_i}\right)\prod_i a^B_{x_i^-}\prod_j a^B_{x_j^+},$$

where the factor $|\mathrm{Aut}(x^+)||\mathrm{Aut}(x^-)|$ appears by ordering the positive, respectively negative part of x. Substituting $\coth(z)=\frac{\cosh(z)}{\sinh(z)}$ and $\mathcal{S}(z)=\frac{\zeta(z)}{z}$ we obtain

$$r!\sum_{g\geqslant 0}\sum_{x\in S\mathbb{Z}^{r+2-2g}}\frac{2^{l(x)-1}}{|\mathrm{Aut}(x^+)||\mathrm{Aut}(x^-)|}[z^{r+1}]\left(\cosh(\frac{z}{2})z^{l(x)-1}\frac{\prod_i \mathcal{S}(x_i z)}{\mathcal{S}(z)}\right)\prod_i a^B_{x_i^-}\prod_j a^B_{x_j^+} =$$
$$\sum_{g\geqslant 0}\sum_{x\in S\mathbb{Z}^{r+2-2g}}\frac{2^{l(x)-1}}{|\mathrm{Aut}(x^+)||\mathrm{Aut}(x^-)|}[z^{2g}]\left(r!\cosh(\frac{z}{2})\frac{\prod_i \mathcal{S}(x_i z)}{\mathcal{S}(z)}\right)\prod_i a^B_{x_i^-}\prod_j a^B_{x_j^+}$$

Using $\rho(z)=\frac{1}{2}\cosh(\frac{z}{2})$ together with Lemma 7.5.4 yields the result. □

Chapter 8

Tropical type B theory

This is the point at which the discussion of the type B theory comes full circle. The tropical theory connects to the classical one via algebraic degeneration (Chapter 5). The operator formalism reveals a second connection: In the type A theory, vacuum expectations of the form $\langle a^B_{i_1} \ldots a^B_{i_n} \rangle$, are known to be computable graphically, as a weighted sum of Feynman diagrams (see [28]).

We extend this approach to the type B theory and identify the graphs in question as tropical covers of $\mathbb{TP}^1$. This allows us to convert the representation-theoretic count HWC^{spin} into a geometric count HWC^{spin}_{trop}, of tropical spin Hurwitz covers with completed cycles (see Figure 3.15). We then use the tropical setting to analyze properties of these numbers, such as polynomiality and wall-crossing behavior, already of interest in type A theory, and thereby recover type B results of [40].

8.1 Re-geometrization: Boson-Tropical Correspondence

Recall the expression of completed spin Hurwitz numbers as vacuum expectation on bosonic Fock space:

$$h_g^{spin,\bullet,0}(\mu, \overline{(r_1+1)}, \ldots, \overline{(r_b+1)}, \nu) = \frac{2^{1-g}}{b! \prod_i \mu_i \prod_j \nu_j} \cdot \langle \prod_{i=1}^{l(\mu)} a^B_{\mu_i} \prod_{i=1}^{n} \mathcal{F}^{\mathrm{B}}_{r_i+1} \prod_{i=1}^{l(\nu)} a^B_{-\nu_i} \rangle. \quad (8.1)$$

What follows is based on two observations:

- The vacuum expectation decomposes into a sum of vacuum expectations of the form $\langle m_0 \ldots m_{b+1} \rangle$, where m_i denotes a normally ordered product of bosonic operators, and m_0 (m_{b+1}) contains only factors a^B_k with $k > 0$ ($k < 0$).

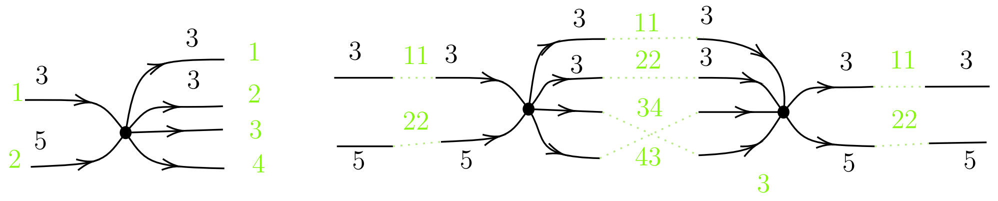
- The only non-zero contribution comes from polynomials P (in the operators a^B_k) of the form $P = m_0 \ldots m_{b+1}$, where the m_i fit together nicely (see Proposition 8.1.2).

Construction 8.1.1. [compare to [28], Definition 5.4.1]
To a normally ordered product of bosonic operators $m := (: \prod_{i=1}^{l} a_{k_i}^B :)$ we associated a vertex v with an edge germ of weight k_i for each $a_{k_i}^B$ whose orientation is determined by the sign of k_i. If $k_i < 0$, the edge germ points towards the left, and if $k_i > 0$, towards the right. Each edge germ is provided with a marking (to keep track later on of all the ways different edge germs can be glued together). This is called the *Feynman fragment* associated to m (see Figure 8.1).

To a polynomial $P = m_0 ... m_{b+1}$ as above we associate a collection of ordered Feynman fragments $(v_1, ..., v_b)$ (one for each m_i with $i \neq 0, b+1$) together with a set of left and a set of right pointing ends corresponding to the monomials m_0 and m_{b+1}.

Graphical condition. A collection of Feynman fragments can be merged to a *Feynman graph*, if edge germs can be paired together to form an edge. More precisely, if for each right pointing edge germ, there exists a left pointing edge germ belonging to a fragment of equal weight further to the right. The resulting graph will have edges labeled by tuples of germ markings (see Figure 8.1). This "graphical condition" that allows Feynman fragments to be merged to Feynman graphs characterizes precisely the condition of P making a non-zero contribution.

To keep track of this merging process, we mark all edge germs.



$m = a_{-5}^B a_{-3}^B (a_1^B)^2 (a_3^B)^2$

$\langle P := (a_5^B a_3^B)(a_{-5}^B a_{-3}^B (a_1^B)^2 (a_3^B)^2)((a_{-1}^B)^2 (a_{-3}^B)^2 a_5^B a_3^B)(a_{-5}^B a_{-3}^B)\rangle$

Figure 8.1: On the left, a normally ordered product and its Feynman fragment. On the right, a vacuum expectation and Feynman graph contributing to it. The black numbers are edge weights, the green ones markings.

Proposition 8.1.2 (Graphical condition, Wick's theorem of type B)**.** Let P be a polynomial appearing in the linear expansion of (8.1). The vacuum expectation $\langle P \rangle$ is non-zero if and only if the collection of Feynman fragments associated to P can be connected to a graph. Moreover, $\langle P \rangle$ is equal to the weighted sum of all Feynman graphs associated to P, where each graph contributes with the product over all edge weights (including ends) each counted with a factor of $\frac{1}{2}$.

Proof. The proof is analogous to the proof of the type A version of Wick's theorem (see [28], Proposition 5.4.3). The only difference consists in the factors of $\frac{1}{2}$ that come from the commutation relations of the type B bosonic operators (see Lemma 7.4.3). ☐

Remark 8.1.3. The connected part $\langle P\rangle^\circ$ of the vacuum expectation $\langle P\rangle$ is defined by subtracting all possible products of vacuum expectations that are obtained by factorizing P ([66], Section 3.3.1). A Feynman graph that contributes to such a product of vacuum expectations is disconnected, with connected components corresponding to the individual factors. Thus, we only need to consider connected Feynmnan graphs when computing connected invariants.

We can immediately transfer this "graphical tracking-technique" for the calculation of vacuum expectations to the case of spin Hurwitz numbers.

Proposition 8.1.4. The spin Hurwitz number $h_g^{spin,0}(\mu, \overline{(r+1)}^b, \nu)$ is equal to the weighted count of *connected* Feynman graphs associated to monomials P, that appear in the linear expansion of (8.1). Each such graph contributes with the product of its edge weights (without a factor of $\frac{1}{2}$) times a factor of $\frac{h_g^{spin,0,r}(x_i^+,x_i^-)}{|\mathrm{Aut}(x^+)||\mathrm{Aut}(x^-)|}$ for every vertex v_i, where x_i^- (x_i^+) denotes the partition of weights of ingoing (outgoing edges).

Proof. Linear expansion of the right-hand side of Equation (8.1) leaves us with a weighted sum of vacuum expectations of monomials $P = m_0 \dots m_{b+1}$ as in Proposition 8.1.2. Each of the b cut-and-join operator $\mathcal{F}_{r+1}^{\mathrm{B}}$ donates a normally ordered product of bosonic operators m_i and a coefficient of the form $2^{r+1-g_i}\frac{h_g^{spin,0,r}(x_i^+,x_i^-)}{|\mathrm{Aut}(x^+)||\mathrm{Aut}(x^-)|}$. Hence, P appears with coefficient $\prod_{i=1}^b 2^{r+1-g_i}\frac{h_g^{spin,0,r}(x_i^+,x_i^-)}{|\mathrm{Aut}(x^+)||\mathrm{Aut}(x^-)|}$ in the expansion of (8.1). Let G be a Feynman graph associated to P. To facilitate later computations we attach the number $g(v_i) := g_i$ to the vertex v_i corresponding to m_i. Taking into consideration Proposition 8.1.2 we have to count a graph G with weight

$$\underbrace{\prod_{e\in E(G)} \frac{\omega(e)}{2}}_{\text{commutation relation}} \underbrace{\prod_{v_i\in V(G)} 2^{r+1-g(v_i)} \frac{h_g^{spin,0,r}(x_i^+,x_i^-)}{|\mathrm{Aut}(x^+)||\mathrm{Aut}(x^-)|}}_{\text{operator contribution}}.$$

We claim that factors of 2 cancel with the prefactor 2^{1-g} in (8.1). A look into the proof of Lemma 7.5.6 tells us that $r+1-g(v_i)$ is equal to $\mathrm{val}(v_i)-1+g(v_i)$. We get:

$$\frac{1}{2^{|E(G)|}}\prod_{v_i\in V(G)} 2^{r+1-g(v_i)} = \frac{1}{2^{|E(G)|}}\prod_{v\in V(G)} 2^{\mathrm{val}(v)-1+g(v)} = \frac{1}{2^{|E(G)|}} 2^{2|E(G)|-|V(G)|+\sum_{v\in V(G)} g(v)}.$$

Recall, that the first Betti number of G, $b_1(G)$, satisfies the following identity:
$|V(G)|-|E(G)| = 1-b_1(G)$. With $|V(G)| = b$ and

$$\begin{aligned}|E(G)| &= \frac{1}{2}\Big(\sum_{i=1}^b l(x_i^-,x_i^+) - l(\mu) - l(\nu)\Big) = \frac{1}{2}\Big(b(r+2) - 2\sum_{i=1}^b g(v_i) - l(\mu) - l(\nu)\Big)\\ &= g-1+b-2\sum_{i=1}^b g(v_i),\end{aligned}$$

we obtain

$$1 - b_1(G) = |V(G)| - |E(G)| = g - \sum_{i=1}^{b} g(v_i)$$

and hence $|E(G)| - |V(G)| + \sum_{i=1}^{b} g(v) = g - 1$. This proves the claim. $\square$

Remark 8.1.5. Note, Proposition 8.1.4 generalizes immediately to spin Hurwitz numbers $h_g^{spin,0}(\mu, \overline{(r_1+1)}, ..., \overline{(r_b+1)}, \nu)$ and to the disconnected count $h_g^{spin,\bullet,0}(\mu, \overline{(r_1+1)}, ..., \overline{(r_b+1)}, \nu)$ by removing the connectedness condition.

Example 8.1.6. Figure 8.2 shows all Feynman graphs that count towards (connected) $h_{g=2}^{spin,0}((3), \overline{(3)}, \overline{(3)}, (3))$. Note that the second graph occurs 6 times with markings $\{i_1, i_2, i_3\} = \{j_1, j_2, j_3\} = \{1, 2, 3\}$.

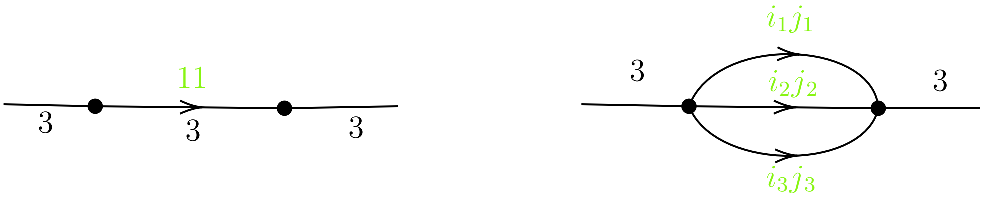


Figure 8.2: Feynman graphs contributing to $h_{g=2}^{spin,0}((3), \overline{(3)}, \overline{(3)}, (3))$. The first graph has vertices of genus 1, while the second has vertices of genus 0. To avoid overloading the drawing, these have been omitted.

8.2 Tropical spin completed cycles Hurwitz numbers

Uprading Feynman graphs to tropical Covers. Let $\mathbb{TP}^1$ be the tropical line with b vertices and G a connected Feynman graph associated to a polynomial $P = m_0...m_{b+1}$ that appears in the expansion of (8.1). Upgrade G to a tropical cover $\pi : \Gamma := (G, l, g) \to \mathbb{TP}^1$, where

- the metric l is determined by the metric on $\mathbb{TP}^1$.
- the genus function is defined as in the proof of Proposition 8.1.4 and we forget the marking on the edges of G.

Note that Γ is balanced since edge weights at every vertex are given by a vector $(x^-, x^+) \in \mathcal{S}\mathbb{Z}^{r+2-2g(v)}$ and has by Proposition 8.1.4 genus $g(\Gamma) = b_1(G) + \sum_{v \in V(G)} g(v) = g$. We call π a *tropical spin completed cycles Hurwitz cover.*

Example 8.2.1. The left hand side of Figure 8.3 shows all tropical spin completed cycles Hurwitz covers that contribute to the connected count $h_{g=2}^{spin,0}((3), \overline{(3)}, \overline{(3)}, (3))$, obtained by updgrading the Feynman diagrams from Example 8.1.6. Linearly expanding the local completed cycle Hurwitz numbers (Definition 5.5.1) at each vertex, yields the ordinary spin Hurwitz covers on the right side of Figure 8.3 (coefficients of the linear expansion have been omitted). Following [28], we add a fat end for each completed 3-cycle to differentiate between the two.

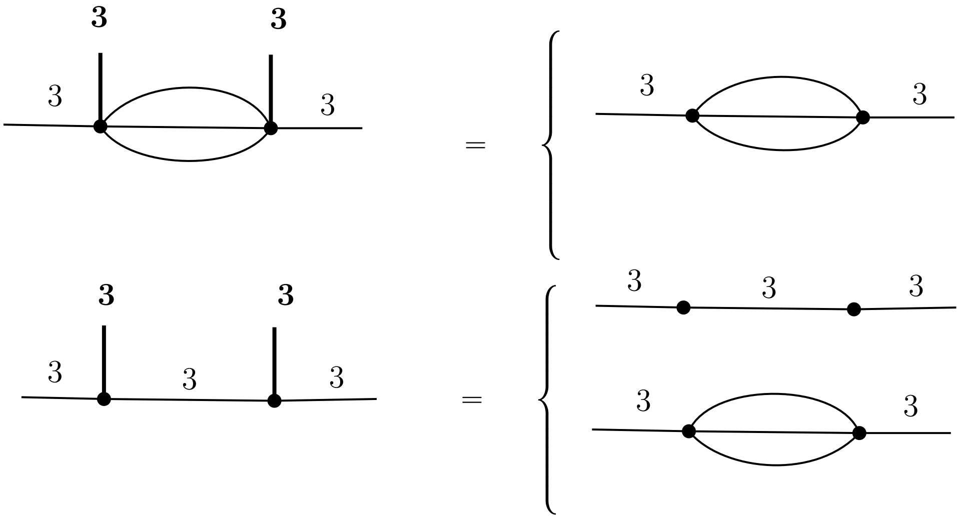


Figure 8.3: Tropical spin completed cycles Hurwitz covers with a fat end for each completed 3-cycle to differentiate them from the ordinary covers on the right (see Example 8.2.1)

We give a tropical analogue of the numbers in Definition 5.5.1.

Definition 8.2.2. Fix positive integers $d, g > 0$, even integers $r_1, ..., r_b > 0$ and odd partitions μ, ν of $d \in \mathbb{N}$ such that $r_1 + ... + r_b = 2g + l(\mu) + l(\nu) - 2$ holds.

Let $\mathbb{TP}^1$ be the tropical line with vertices $p_1, ..., p_b$. The *connected/disconnected tropical spin double Hurwitz number with multi-completed cycles* $\mathbb{T}h_g(\mu, \overline{(r_1 + 1)}, ..., \overline{(r_b + 1)}, \nu)/\mathbb{T}h_g^{spin,\bullet}(\mu, \overline{(r_1 + 1)}, ..., \overline{(r_b + 1)}, \nu)$ is given by

$$\sum_{\pi:\Gamma\to\mathbb{TP}^1} \mathrm{mult}(\pi)$$

where the sum is over all covers $\pi : \Gamma \to \mathbb{TP}^1$ such that

- Γ is connected/disconnected tropical curve of genus g with odd edge weights only,
- the unbounded left (resp. right) pointing ends of Γ have weights as prescribed by μ (resp. ν),
- over each p_i there is a unique vertex v_i of Γ whose valence is equal to $2 + r_i - 2g(v_i)$,

and

$$\mathrm{mult}(\pi) = \frac{1}{|\mathrm{Aut}(\pi)|} \prod_{e\in E(\Gamma)} \omega(e) \prod_{i=1}^{b} h_g^{spin,0,r_i}(x_i^+, x_i^-),$$

where x_i^- (x_i^+) denotes the partition of weights of ingoing (outgoing edges) at $v_i \in V(\Gamma)$ and, as usual, $\mathrm{Aut}(\pi) := \{\pi' : \Gamma' \to \mathbb{TP}^1 | \exists \phi \in \mathrm{Aut}(\Gamma) \pi = \pi' \circ \phi\}$ is the automorphism group of π.

If $r := r_1 = ... = r_b$, we simply write $\mathbb{T}h_g^{spin,r}(\mu,\nu)/\mathbb{T}h_g^{spin,r,\bullet}(\mu,\nu)$.

The isomorphism class of π contains exactly $\frac{1}{|\mathrm{Aut}(\pi)|}\prod_{v\in V(\Gamma)}(|\mathrm{Aut}(x_v^+)||\mathrm{Aut}(x_v^-)|)$ Feynman graphs (a factor of $\prod_{v\in V(\Gamma)}(|\mathrm{Aut}(x_v^+)||\mathrm{Aut}(x_v^-)|)$ for the number of ways to label edge germs and a factor $\frac{1}{|\mathrm{Aut}(\pi)|}$ to account for overcounting).

Theorem 8.2.3. For the discrete data as above, the classical and tropical counts agree:

$$\mathbb{T}h_g(\mu, \overline{(r_1+1)}, ..., \overline{(r_b+1)}, \nu) = h_g(\mu, \overline{(r_1+1)}, ..., \overline{(r_b+1)}, \nu)$$
$$\mathbb{T}h_g^{spin,\bullet}(\mu, \overline{(r_1+1)}, ..., \overline{(r_b+1)}, \nu) = h_g^{spin,\bullet}(\mu, \overline{(r_1+1)}, ..., \overline{(r_b+1)}, \nu).$$

Genus 0 vertex multiplicities.

Lemma 8.2.4. The genus 0 vertex multiplicities are equal to $h_g^{spin,0,r}(x^+,x^-) = r!$.

Proof. We use Lemma 7.5.4. Recall that the constant term of the quotient of two series in z is the quotient of the constant terms. With $\mathcal{S}(xz) = 1 + \frac{x^2}{2^2 4!}z^2 + \frac{x^4}{2^4 5!}z^4 + ...$ and $\frac{1}{\mathcal{S}(z)} = 1 - \frac{z^2}{24} + \frac{7z^4}{5760} + ..\ \rho(z) = \frac{1}{2}(1 + \frac{z^2}{2^2 2!} + \frac{z^4}{2^4 4!} + ...)$ Since $h_g^{spin,0,r}(x^+,x^-)$ we obtain $h_g^{spin,0,r}(x^+,x^-) = 2r!\frac{1}{2}$. □

8.3 Geometric properties: Polynomiality and Wall-crossing

Thus far, we have analyzed counting problems with fixed data, treating each one in isolation. More structure is to be uncovered by a change of perspective.

Definition 8.3.1. For $k, l \geqslant 0$ consider the subspace $\mathcal{H}(k,l) \subset \mathbb{N}^l \times \mathbb{N}^k$ that consists of pairs of odd partitions (μ,ν) of the same number with fixed lengths $l(\mu) = l$ and $l(\nu) = k$, i.e. we have $\mathcal{H}(l,k) := \{(\mu,\nu) : \sum_{i=1}^l \mu_i = \sum_{j=1}^k \nu_j\}$. Any choice of a positive integer g and an even integer r yields a map

$$\mathbb{T}h_g^{spin,r} : \mathcal{H}(k,l) \to \mathbb{Q}, (\mu,\nu) \mapsto \mathbb{T}h_g^{spin,r}(\mu,\nu).$$

and an assignment $(\mu,\nu) \mapsto \mathbb{T}S(\mu,\nu)$, where $\mathbb{T}S(\mu,\nu)$ is the *set of relevant covers* for the number $\mathbb{T}h_g^{spin,r}(\mu,\nu)$ (see Definition 8.2.2).
(The set $\mathbb{T}S(\mu,\nu)$ depends on g and r, but we suppress this in the notation as both should be clear from the context.)

Remark 8.3.2. We note that

- $\mathbb{T}h_g^{spin,r}$ is the trivial map if the data (k,l,g,r) does not satisfy the Riemann-Hurwitz condition. As of now, we implicitly assume that it is met.
- preimages of 0 and ∞ are marked, which is essential to guarantee piecewise polynomiality of the function $\mathbb{T}h_g^{spin,r}$. This means that $\mathrm{Aut}(\Gamma)$ is independent of μ and ν for any Γ that contributes to $\mathbb{T}S(\mu,\nu)$. Moreover, if $g=0$, then $\mathrm{Aut}(\Gamma)$ is trivial.

The transition from considering spin Hurwitz numbers for a fixed pair of partitions (μ,ν) to viewing them as function on the space $\mathcal{H}(k,l)$ has uncovered structure that was previously hidden by the static nature of the counting problem: the function $\mathbb{T}h_g^{spin,r}$ is piecewise polynomial, i.e. it subdivides the space $\mathcal{H}(k,l)$ into loci of polynomiality $c \subset \mathcal{H}(k,l)$ we call *chambers* separated by hyperplanes $W_{I,J}$ called *walls*.

8.3.1 Genus 0

Among all genera, the genus 0 case proves most accessible from a combinatorial standpoint (compare to Section 8.3.2): By generalizing the proofs of [22] (Section 6), we recover similar polynomiality results and wall-crossing formulas *only* using polynomiality of spin-completed cycle Hurwitz numbers with three branch points as algebraic input. This follows from Lemma 8.3.3 ([22], Lemma 6.4), which holds for rational tropical curves in general (not only for tri-valent trees), and its Corollary.

Lemma 8.3.3 ([22], Lemma 6.4)**.** Let Γ be a rational tropical curve. If we choose the weight μ_i for the in-end labeled i and ν_j for the out-end labeled j, then the weights $\omega(e)$ of all inner edges e are uniquely determined (but might be negative). The weight $\omega(e)$ equals

$$\omega(e) = \sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j,$$

where $I \subset \{1,...,l\}$ and $J \subset \{1,...,k\}$ are the subsets of in- and out-ends belonging to the connected component of $\Gamma \backslash \{e\}$ from which e points away.

Corollary 8.3.4. Let Γ be a rational tropical curve.
The parity of the edge weights of Γ is entirely determined by the graph structure, i.e. it is independent of any specific choice of odd end weights.
The property "$\omega(e)$ odd $\forall e \in \mathrm{E}(\Gamma)$" does not depend on any specific choice (μ,ν) of odd end weights.

Proof. By Lemma 8.3.3 we have

$$\omega(e) = \sum_{i \in I_e} \mu_i - \sum_{j \in J_e} \nu_j$$

for $e \in \mathrm{E}(\Gamma)$ and subsets $I_e \subset \{1,...,l\}$ and $J_e \subset \{1,...,k\}$ of in- and out-ends. Write $\mu_i = 2k_1 + 1$ and $\nu_j = 2\hat{k}_j + 1$, then

$$\omega(e) = 2(\sum_{i \in I_e} k_i - \sum_{j \in J_e} \hat{k}_j) + |I_e| - |J_e|$$

is odd if and only if $|I_e| - |J_e|$ is odd. ☐

We combine the above with the fact that a graph will stay relevant to the count as long as edge weights stay positive.

Lemma 8.3.5. The assignment $(\mu, \nu) \mapsto \mathbb{T}S(\mu, \nu)$ naturally gives rise to a hyperplane arrangement $\mathcal{W}(l, k)$ in $\mathcal{H}(k, l)$ consisting of hyperplanes of the form $\mathcal{W}_{I,J} = \{(\mu, \nu) : \sum_{i \in I} \mu_i = \sum_{j \in J} \nu_i\}$, where $I \subset \{1, ..., l\}$ and $J \subset \{1, ..., k\}$ are non-empty subsets. The connected components of the complement $\mathcal{H}(k, l) \backslash \mathcal{W}(l, k)$ describe precisely the locus where the sets $\mathbb{T}S(\mu, \nu)$ remain constant under variation of (μ, ν), i.e. the set of underlying directed graphs with variables as edge weights does not change.

Proof. Fix a point $(\mu', \nu') \in \mathcal{H}(k, l)$ and let $\pi : \Gamma \to \mathbb{TP}^1 \in \mathbb{T}S(\mu', \nu')$ be a cover. Lemma 8.3.3 gives edge weights of Γ in terms of (μ', ν'). Corollary 8.3.4 shows that we may vary (μ', ν') without losing any of the directed graphs that correspond to a feasible cover π as long as all edge weights stay positive (see Remark 8.3.6 for correspondence between covers of $\mathbb{TP}^1$ and directed graphs). Hence, the inequalities

$$\omega(e)(\mu, \nu) > 0 \text{ for all } e \in E(\Gamma) \text{ and } \Gamma \in \mathbb{T}S(\mu', \nu')$$

describe the maximal subset of $\mathcal{H}(k, l)$ such that $\mathbb{T}S(\mu', \nu')$ remains unchanged. ☐

Remark 8.3.6. Note that by Lemma 8.3.5 it makes sense to associate a set of relevant covers $\mathbb{T}S$ to a chamber c, i.e. a connected component of the complement $\mathcal{H}(k, l) \backslash \mathcal{W}(l, k)$, instead of viewing all sets $\mathbb{T}S(\mu, \nu)$ for $(\mu, \nu) \in c$ separately. Depending on the situation at hand we identify an element of $\mathbb{T}S$ with

1. a family of tropical covers π over the discrete base $c \subset \mathcal{H}(k, l)$ whose total space is a cover of $\mathbb{TP}^1$ by a tree Γ with polynomial edge weights that satisfies the balancing condition. More precisely, a fibre over (μ, ν) is the cover in $\mathbb{T}S(\mu, \nu)$ given by evaluating edge weights at (μ, ν).

2. a directed tree Γ with polynomial edge weights that satisfies the balancing condition, together with an ordering $<$ on the vertices $V(\Gamma)$ that is compatible with the edge directions, i.e. source v_1 and target vertex v_2 satisfy $v_1 < v_2$.

The first emphasizes the dynamic nature of the situation, while the second offers an elegant way of analyzing wall-crossing (see following paragraph).

Remark 8.3.7. The hyperplane arrangement $\mathcal{W}(l, k)$ is natural from another point of view: The connected components of the complement $\mathcal{H}(k, l) \backslash \mathcal{W}(l, k)$ describe precisely the locus, where connected and disconnected counts agree. A disconnected cover of degree d can be expressed as a union of covers of smaller degrees, say d_1 and d_2 with $d = d_1 + d_2$. Therefore, such a cover exists only if the underlying discrete data (d, μ, ν) admits a decomposition into smaller counting problems- that is, if there exists subsets $I \subset \{1, ..., l\}$ and $J \subset \{1, ..., k\}$ with $d_1 = \sum_{i \in I} \mu_i = \sum_{j \in J} \nu_i$, equivalently $d_2 = \sum_{i \in I^c} \mu_i = \sum_{j \in J^c} \nu_i$.

Proposition 8.3.8. [[40], Theorem 6.4] The function $\mathbb{T}h_0^{spin,r}$ is a homogeneous polynomial of degree $\frac{k+l-2}{r} - 1$ within each chamber $c \subset \mathcal{H}(k,l)$ cut out by the hyperplanes $\mathcal{W}_{I,J}$ from Lemma 8.3.5.

Proof. Restrict to a chamber c and use Lemma 8.3.5 and Remark 8.3.6 to count families $\pi : \Gamma \to \mathbb{TP}^1 \in \mathbb{T}S$ of tropical covers with $(\mu,\nu)-$dependent weight.

Since local vertex multiplicities are constants (Lemma 8.2.4) and edge weights are homogeneous polynomials of degree 1 in (μ,ν), $\mathrm{mult}(\Gamma)$ is a homogeneous polynomial of degree $|E(\Gamma)| = |V(\Gamma)| - 1 = \frac{l+k-2}{r} - 1$ for any $\Gamma \in \mathbb{T}S$. As $\mathbb{T}S$ is finite, $\mathbb{T}h_0^{spin,r}$ is a finite sum of homogeneous polynomials of equal degree, and therefore homogenous itself. This concludes the proof. □

Example 8.3.9. Consider the counting problem

$$\mathbb{T}h_0^{spin,2} : \mathcal{H}(3,3) \to \mathbb{Q}, (\mu,\nu) \mapsto \mathbb{T}h_0^{spin,2}(\mu,\nu). \tag{8.2}$$

Figure 8.4 shows a directed tree Γ corresponding to a family of covers π (see Remark 8.3.6) with $\pi \in \mathbb{T}S$ for any chamber c and whose contribution to (8.2) is given by:

$$\mathrm{mult}(\Gamma) = (2!)^2(\mu_1 + \mu_2 + \mu_3).$$

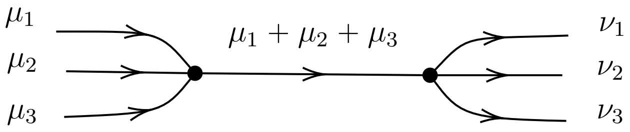


Figure 8.4: A directed tree that contributes to $\mathbb{T}h_0^{spin,2}(\mu,\nu)$.

Wall crossing formulae. A natural question to ask is what happens to $\mathbb{T}h_0^{spin,r}$ as we cross a wall. This information is encoded in so-called wall crossing formulae.

Definition 8.3.10. Let c_1 and c_2 be two neighboring chambers separated by a wall $\mathcal{W}_{I,J}$, i.e. c_1 and c_2 are defined by the same inequalities except for the one corresponding to $\mathcal{W}_{I,J}$. Let c_1 be given by $\sum_{i\in I}\mu_i - \sum_{j\in J}\nu_i > 0$ and c_2 by $\sum_{i\in I}\mu_i - \sum_{j\in J}\nu_i < 0$. The polynomial $WC^r_{0,I,J} = \mathbb{T}h_0^{spin,r}{}_{|c_1} - \mathbb{T}h_0^{spin,r}{}_{|c_2}$ is called a wall crossing formula.

Remark 8.3.11. Note: The restriction of $\mathbb{T}h_0^{spin,r}$ to a single chamber, together with all wall crossing formulae, is enough to reconstruct the full function $\mathbb{T}h_0^{spin,r}$. This encoding is efficient since wall crossing formulae depend only on smaller counting problems (see Theorem 8.3.16).

Let us denote the set of relevant covers in the chamber c_1 (c_2) by $\mathbb{T}S_1$ ($\mathbb{T}S_2$) and by $\mathbb{T}S_1^{\omega}$ ($\mathbb{T}S_2^{-\omega}$) the subset of all covers that have an edge e of weight $\omega := \omega(\mu,\nu) := \sum_{i\in I}\mu_i - \sum_{j\in J}\nu_i$ ($-\omega$). Note that by Lemma 8.3.3 this edge is unique. We define an equivalence relation on $\mathbb{T}S_1^{\omega}$, that "makes the structure of the set $\mathbb{T}S_1^{\omega}$ visible" by identifying common building blocks of its members:

$$(\Gamma, <) \sim_1 (\tilde{\Gamma}, \tilde{<}) \text{ if and only if the connected components of } \Gamma \backslash \{e\} \text{ and } \tilde{\Gamma} \backslash \{\tilde{e}\}$$
$$\text{together with their inherited orders agree.}$$

The same construction yields an equivalence relation $\sim_2$ on $\mathbb{T}S_2^{-\omega}$.

Example 8.3.12. Consider the counting problem

$$\mathbb{T}h_0^{spin,2} : \mathcal{H}(4,4) \to \mathbb{Q}, (\mu, \nu) \mapsto \mathbb{T}h_0^{spin,2}(\mu, \nu)$$

that asks for 4-valent directed trees with 3 vertices and ends of weight (μ, ν). Let c_1 and c_2 be chambers separated by the wall $W = \{\omega := \nu_4 - \mu_3 + \mu_4 = 0\}$. Figure 8.5 shows two equivalent covers in $\mathbb{T}S_2^{-\omega}$ and their common building blocks.

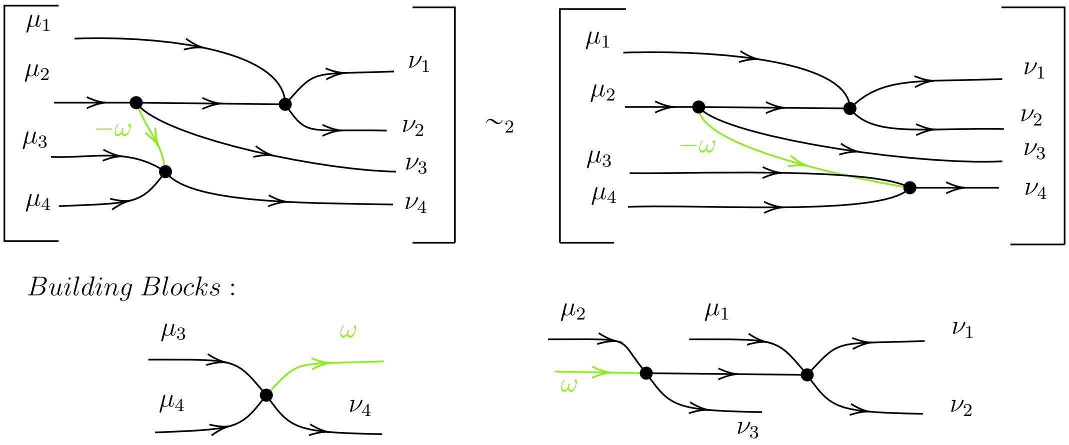


Figure 8.5: An example for the equivalence relation $\sim_2$.

Remark 8.3.13. It is worth looking at the following from a "cross-chamber-perspective": We know that edge weights of a balanced graph are poynomials in the end weights, talking about their sign does not make sense. However, should we have two graphs in different chambers that have edges e and $\tilde{e}$ such that $|\omega(e)| = |\omega(\tilde{e})|$ holds, we can. If $sgn(\omega(e)) = sgn(\omega(\tilde{e}))$ is the case, we say e and $\tilde{e}$ point in the same direction, and in opposite direction otherwise. When viewed as a cover, this "orientation-reversal" corresponds to switching the order in which the vertices adjacent to e are mapped to $\mathbb{TP}^1$. Indeed, this is a just consequence of the balancing condition.

Proposition 8.3.14. We have a bijection $\phi : \mathbb{T}S_1^{\omega}/\sim_1 \to \mathbb{T}S_2^{-\omega}/\sim_2$ that behaves sign-reversing with respect to the multiplicity, i.e. $\operatorname{mult}(\phi([\Gamma]_1)) = -\operatorname{mult}([\Gamma]_1)$ holds, and satisfies $|[\Gamma]_1| + |\phi([\Gamma]_1)| = \binom{a}{a_1, a_2}$, where $a = \frac{k+l-2}{r}$, $a_1 = \frac{|I|+|J|-1}{r}$ and $a_2 = a - a_1$.

Proof. We construct ϕ as follows: Given $(\Gamma, <) \in \mathbb{T}S_1^{\omega}$ with edge of weight ω denoted by e, consider $\Gamma \backslash \{e\} = \Gamma_1 \cup \Gamma_2$. We claim that there exists $(\tilde{\Gamma}, \tilde{<}) \in \mathbb{T}S_2^{-\omega}$ such that $\tilde{\Gamma} \backslash \{\tilde{e}\} = \Gamma_1 \cup \Gamma_2$, where $\tilde{e}$ is the edge of weight $-\omega$, and define $\tilde{\phi} : \mathbb{T}S_1^{\omega} \to \mathbb{T}S_2^{-\omega} / \sim_2$ by $\tilde{\phi}(\Gamma) := [\tilde{\Gamma}]_2$. By construction $\tilde{\phi}$ factors through the quotient (i.e. well-defined on equivalence classes) and yields an injective map $\phi : \mathbb{T}S_1^{\omega} / \sim_1 \to \mathbb{T}S_2^{-\omega} / \sim_2$. To see that ϕ is surjective as well requires an argument similar to the one needed to prove the existence of $\tilde{\Gamma}$. We address the existence first.

Note, we already know $\tilde{\Gamma}$ as a graph: it consists of the two components Γ_1 and Γ_2 connected by an edge of weight $|\omega|$. It remains to be shown that $\tilde{\Gamma}$ can be turned into a cover, i.e. that we find $\tilde{<}$ such that $(\tilde{\Gamma}, \tilde{<}) \in \mathbb{T}S_2^{-\omega}$ holds. The inequality $\omega < 0$ in c_2 forces $\tilde{e}$ to have polynomial weight $-\omega$ since relevant covers have positive edge weights when evaluated at partitions. Recall that by Remark 8.3.13 we interpret this as "switching the direction" of e in Γ. Why is this always possible? We can only produce a sink or a source, if one of the vertices adjacent to e, say v, has e as unique in-going or out-going edge. Without loss of generality suppose v has in-going edges of weight $\omega_1, \ldots, \omega_n$ and outgoing edge e such that $\sum_{i=1}^{n} \omega_i = \omega$ (as polynomials). Reversing the orientation of e creates a sink. We argued previously that all inequalities defining c_1 and c_2 are the same except for the one given by the wall. Hence, $\omega_i > 0$ holds in both chambers and ω can never be zero, a contradiction.

We can now define $\tilde{<}$ on $V(\tilde{\Gamma}) = V(\Gamma)$ by modifying $<$ as follows: interchange/switch the positions of the vertices adjacent to e in $<$. This proves the claim. By the same type of argument (starting from $[\Gamma]_2 \in \mathbb{T}S_2^{-\omega} / \sim_2$ instead of $[\Gamma]_1 \in \mathbb{T}S_1^{\omega} / \sim_1$) we see that ϕ is surjectiv as well.

The previous construction shows that image and preimage of ϕ are made of the same building blocks:

1. a directed weighted tree Γ_1 with $a_1 := |V(\Gamma_1)| = \frac{|I|+|J|-1}{r}$ (by Riemann-Hurwitz) ordered vertices from which precisely one is not balanced and end weights given by μ_I and ν_J.

2. a directed weighted tree Γ_2 with $a_2 := |V(\Gamma_2)| = \frac{|I^c|+|J^c|-1}{r}$ ordered vertices from which precisely one is not balanced and end weights given by μ_{I^c} and ν_{J^c}.

In order to obtain a graph in $[\Gamma]_1$ respectively $\phi([\Gamma]_1)$, connect the two unbalanced vertices by an edge of weight $|\omega|$ and pick an ordering $<$ that refines the ordering on $V(\Gamma_1)$ and $V(\Gamma_2)$. If $v_1 < v_2$ holds, we get a graph in $[\Gamma]_1$, and one in $\phi([\Gamma]_1)$ otherwise. We see that $|[\Gamma]_1| + |\phi([\Gamma]_1)|$ is given by the number of ordering on $a := a_1 + a_2 = \frac{k+l-2}{r}$ vertices that refines two given orderings on a_1 and a_2 vertices, which is $\binom{a}{a_1, a_2}$: such an order corresponds to a vector v of length $r_1 + r_2$. We can position the r_1 vertices freely in v with the only condition being that we respect the order dictated by Γ_1. The positions of the remaining r_2 vertices are then fixed.

To finish let us address how ϕ behaves with respect to the multiplicity. First, note that this is well-defined (on equivalence classes) since all members of an equivalence class share the same underlying graph. We have just seen that $\phi([\Gamma]_1)$ for $[\Gamma]_1 \in \mathbb{T}S_1^{\omega} / \sim_1$ is given by the equivalence class of covers obtained by switching the direction of e in Γ. Thus, each member is counted with the same multiplicity up to a change of sign: Recall that $\operatorname{mult}(\Gamma)$ is a product over edge weights and local completed cycles spin Hurwitz numbers. The only terms affected by the wall-crossing are:

1. The edge weight ω. It changes by a minus sign.
2. The local spin completed cycles Hurwitz numbers attached to the vertices adjacent to e. These change by moving ramification from 0 to ∞ or backwards, which has no effect since these are even in the entries of the partition corresponding to the respective ramification profiles.

□

Example 8.3.15. Refer to Example 8.3.12 and consider $\Gamma \in \mathbb{T}S_1^{\omega}$. Then Figure 8.6 shows $[\Gamma]_1$, $\phi([\Gamma]_1)$ and verifies $|[\Gamma]_1| + |\phi([\Gamma]_1)| = 1 + 2 = \binom{3}{1,2}$.

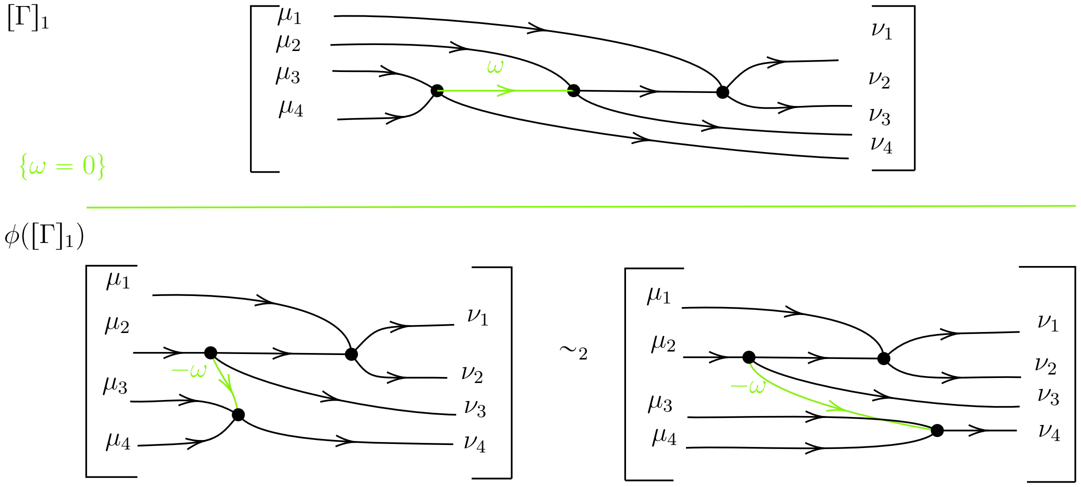


Figure 8.6: The map ϕ from Proposition 8.3.14.

Theorem 8.3.16. For the wall the $\mathcal{W}_{I,J}$ with adjacent chambers c_1 and c_2 the wall crossing formula is given by

$$WC^r_{0,I,J} = \binom{a}{a_1, a_2} \mathbb{T}h_0^{spin,r}(\mu_I, (\nu_J, \omega)) \mathbb{T}h_0^{spin,r}((\mu_{I^c}, \omega), \nu_{J^c})\omega.$$

The whole function $\mathbb{T}h_0^{spin,r}$ can be recovered from its value inside one chamber and all its wall-crossing formulas. Theorem 8.3.16 states that this encoding is efficient, since wall-crossing formulas are determined by spin Hurwitz numbers (with completed cycles) with less input data.

Proof. Let $(\Gamma, <) \in \mathbb{T}S_1$. We differentiate two cases, either $(\Gamma, <) \in \mathbb{T}S_1^{\omega}$ or not. If not, then $(\Gamma, <) \in \mathbb{T}S_2$ as well and their contribution cancel in $WC^r_{0,I,J}$ since they are counted with the same multiplicity on both sides of the wall. This yields

$$WC^r_{0,I,J} = \mathbb{T}h_0^{spin,r}{}_{|c_1} - \mathbb{T}h_0^{spin,r}{}_{|c_2} = \sum_{\Gamma \in \mathbb{T}S_1^{\omega}} \operatorname{mult}(\Gamma) - \sum_{\Gamma' \in \mathbb{T}S_2^{-\omega}} \operatorname{mult}(\Gamma').$$

Since multiplicities are well-defined on equivalence classes, we get by Proposition 8.3.14:

$$\begin{aligned} WC^r_{0,I,J} &= \sum_{[\Gamma]_1 \in \mathbb{T}S_1^{\omega}/\sim_1} |[\Gamma]_1| \cdot \operatorname{mult}([\Gamma]_1) - \sum_{[\Gamma']_2 \in \mathbb{T}S_2^{\omega}/\sim_2} |[\Gamma']_2| \cdot \operatorname{mult}([\Gamma']_2) \\ &= \sum_{[\Gamma]_1 \in \mathbb{T}S_1^{\omega}/\sim_1} (|[\Gamma]_1| + |\phi([\Gamma]_1)|) \cdot \operatorname{mult}([\Gamma]_1) = \sum_{[\Gamma]_1 \in \mathbb{T}S_1^{\omega}/\sim_1} \binom{a}{a_1, a_2} \cdot \operatorname{mult}([\Gamma]_1). \end{aligned}$$

By definition $\mathbb{T}S_1^{\omega}/\sim_1$ is in bijection with the set of tuples of graphs (Γ_1, Γ_2) as described in the proof of Proposition 8.3.14. We can turn Γ_1/Γ_2 into a cover that counts towards the spin Hurwitz number $\mathbb{T}h_0^{spin,r}(\mu_I, (\nu_J, \omega))/\mathbb{T}h_0^{spin,r}((\mu_{I^c}, \omega), \nu_{J^c})$ by attaching an outward/inward directed end of weight ω to the unique unbalanced vertex of Γ_1/Γ_2. This modification yields a bijection to the set of tuples of covers that contribute to $\mathbb{T}h_0^{spin,r}(\mu_I, (\nu_J, \omega))$ and $\mathbb{T}h_0^{spin,r}((\mu_{I^c}, \omega), \omega), \nu_{J^c})$. Thus

$$\begin{aligned} WC^r_{0,I,J} &= \binom{a}{a_1, a_2} \omega \sum_{(\Gamma_1, \Gamma_2)} \operatorname{mult}(\Gamma_1) \operatorname{mult}(\Gamma_2) \\ &= \binom{a}{a_1, a_2} \mathbb{T}h_0^{spin,r}(\mu_I, (\nu_J, \omega)) \mathbb{T}h_0^{spin,r}((\mu_{I^c}, \omega), \nu_{J^c}) \omega \end{aligned}$$

proves the claim.

□

8.3.2 Higher genus

Following [23] we do not fix the lengths of ramification profiles over 0 and ∞ separately: We investigate properties of the map

$$\mathbb{T}h_g^{spin,r} : \mathcal{H} \to \mathbb{Q}, x \mapsto \mathbb{T}h_g^{spin,r}(x^-, x^+),$$

where $\mathcal{H} := \{x \in \mathbb{Z}^n : \sum_{i=1}^n x_i = 0 \text{ and } x_i \text{ odd}\}$.

Remark 8.3.17. The domain of the map $\mathbb{T}h_g^{spin,r}$ is discrete. We can extend $\mathbb{T}h_g^{spin,r}$ to the space $\mathcal{H}_{\mathbb{R}} := \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0\}$ of real valued vectors as follows: For $x \in \mathcal{H}$, $\mathbb{T}h_g^{spin,r}(x)$ is a weighted sum of graphs. As for the genus 0 case, the set of these graphs remains unchanged "near x" (see Proposition 8.3.29 for a precise statement). By equipping these graphs with real valued end and edge weights, we can extend $\mathbb{T}h_g^{spin,r}$ locally to a real-valued function. However, for $x \in \mathcal{H}_\mathbb{R} \backslash \mathcal{H}$, $\mathbb{T}h_g^{spin,r}(x)$ is not a completed cycles spin Hurwitz number and graphs are not tropical completed cycle spin Hurwitz covers. These graphs arise from discrete counting problems but, with real weights, the parity notion is lost.

Added complexity. Lemma 8.3.3 is what makes the genus 0 case clean. Together with Corollary 8.3.4 it allows us to associate to each chamber c a set $\mathbb{T}S$ whose elements are families of tropical covers over c - an association that fails in higher genus. The reason is as follows: For each independent cycle we get a free parameter, that is we have a certain amount of choice when equipping the graph with a balanced weighting. Consequently, the subset of $\mathcal{H}$ where all edge weights remain positive now depends on these extra parameters. The chambers of polynomiality (see Lemma 8.3.27), however, do not.

To handle this added complexity, we broaden our notion of "smallest counting unit" from a directed weighted graph with vertex ordering to an x-*graph*.

Definition 8.3.18. Given a completed cycle spin Hurwitz cover $\pi : \Gamma \to \mathbb{TP}^1$ arising from a counting problem $\mathbb{T}h_g^{spin,r}(x)$ fixed by $g > 0$ and $x = (x_1, \ldots, x_n) \in \mathcal{H}$ with $x_i \neq 0$, we build a graph with variables as edge weights: The x-*graph* $\Gamma(x)$ associated to π is obtained by

- compactifying Γ, i.e. adding a 1-valent vertex to each end (thereby capping all infinite rays).
- forgetting edge directions and vertex ordering.

An x-graph is equipped with a fixed orientation, the *reference orientation*, that allows us to endow $\Gamma(x)$ with a *reference weighting*: Let $(B_1, \ldots, B_{b_1(\Gamma)})$ be a canonical homology basis of $\Gamma(x)$ and e_i a fixed edge in B_i. Assign to e_i a parameter j_i, its weight. The remaining edge weights are fixed by the balancing condition, i.e. can be expressed uniquely in terms of x and $(j_1, \ldots, j_{b_1(\Gamma)})$. Note, $\Gamma(x)$ is geometrically meaningful for odd weight and end vectors!

To view $\mathbb{T}h_g^{spin,r}$ as a sum over x-graphs, we must be able to reconstruct from $\Gamma(x)$ all covers whose x-graph is $\Gamma(x)$. This reconstruction problem reduces to finding all compatible pairs $((j_1, \ldots, j_{b_1(\Gamma)}), <)$, where $(j_1, \ldots, j_{b_1(\Gamma)})$ is a weight vector and $<$ a vertex ordering. The challenge is that we do not a priori know how these pieces of data interact (The relationship between weight vectors and vertex orderings, as well as the relationship between different weight vectors, is unclear.): To resolve the situation, we construct a suitable "container" space, that naturally organizes this data.

Construction 8.3.19. As $\Gamma := \Gamma(x)$ is a simplicial object, we can consider the associated chain complex

$$0 \to C_1^\Gamma \xrightarrow{\partial} C_0^\Gamma \to 0,$$

where

- C_0^Γ is the free abelian group of $0-$chains with coefficients in $\mathbb{R}$.
- C_1^Γ is the free abelian group of $1-$chains with coefficients in $\mathbb{R}$.
- $\partial : C_1^\Gamma \to C_0^\Gamma$ is the boundary operator.

Recall that the groups C_0^Γ and C_1^Γ both have natural basis given by the set of vertices $V(\Gamma)$ (interior and 1-valent vertices), of edges $E(\Gamma)$ (interior edges and ends) respectively. Hence, we may identify the coefficient-vector of a $1-$chain c (represented with respect to the natural basis of C_1^Γ) with an assignment of $\mathbb{R}$-valued weights to the edges of Γ ($\mathbb{R}$-weighting on Γ). The $\mathbb{R}$-weighting satisfies the balancing condition at every interior vertex if and only if the image of c under ∂ lies in the subgroup U generated by all $1-$valent vertices with coefficient vector $x \in \mathcal{H}_\mathbb{R}$, i.e.

$$\partial(c) = \sum_i x_i v_i, \text{ where } x_i = 0, \text{ if } \operatorname{val}(v_i) > 1, \text{ and } \sum_i x_i = 0.$$

The spaces $\partial^{-1}(\sum_i x_i v_i)$ will therefore parameterize all possible ways to endow Γ with a balanced $\mathbb{R}$-weighting. As they also emerge naturally from Γ, they qualify as the container spaces we are looking for.

Definition 8.3.20. To an x-graph Γ we attach the *bundle of container spaces* $\mathcal{C}_\Gamma \xrightarrow{\pi_\Gamma} \mathcal{H}_\mathbb{R}$, that records all balanced $\mathbb{R}$-weightings as end weights of Γ vary. We define $\mathcal{C}_\Gamma \to \mathcal{H}_\mathbb{R}$ to be the union of the sets $\{x\} \times \partial^{-1}(x)$ for all $x \in \mathcal{H}_\mathbb{R}$, where x is identified with an element of U. The fibers $\partial^{-1}(x)$ are translates of the $b_1(\Gamma)-$dimensional vector space $\ker(\partial) = \mathrm{H}_1(\Gamma, \mathbb{R})$, and $\mathcal{C}_\Gamma$ therefore a subset of the trivial bundle $\mathcal{H}_\mathbb{R} \times C_1^\Gamma$.

The following Lemma provides a justification for why Definition 8.3.20 is suitable for our combinatorial problem.

Lemma 8.3.21. Let Γ be an x-graph of genus g. The property "$\omega(e)$ odd $\forall e \in \mathrm{E}(\Gamma)$" does not depend on any specific choice of end weigths, $x \in \mathcal{H}$, or odd edge weights, $(j_1, ..., j_{b_1(\Gamma)})$, (Definition 8.3.18).

Proof. Cut Γ at the edges labelled by $(j_1, ..., j_{b_1(\Gamma)})$ and use Corollary 8.3.4. ☐

Note that although any point ω in a fiber of $\mathcal{C}_\Gamma$ will give rise to a balanced $\mathbb{R}$-weighting, it does not automatically correspond to a tropical cover: The orientation ω induces on Γ might not allow for a total order on the vertices, thus ω is not *adequate*. This singles out the induced orientation as *the* property to keep track of when it comes to deciding whether a weight vector ω is adequate. In fact "inducing the same orientation" is an equivalence relation on the fibre and partitions ${\pi_\Gamma}^{-1}(x)$ in a natural way.

Definition 8.3.22 ([24]). Let $\mathcal{C}_\Gamma \xrightarrow{\pi_\Gamma} \mathcal{H}_\mathbb{R}$ be the bundle of container spaces for a given x-graph Γ and view ${\pi_\Gamma}^{-1}(x)$ as a subspace of $\mathbb{R}^{|E(\Gamma)|}$ (identifying C_1^Γ with $\mathbb{R}^{|E(\Gamma)|}$ via the isomorphism that takes the basis consisting of edges to the standard basis). The set of coordinate hyperplanes $\{e_i = 0\}$, where e_i denotes the coordinate of the i-th standard basis vector associated to an *internal edge*, restricts to a hyperplane arrangement $\mathcal{A}(x)$ inside each fiber ${\pi_\Gamma}^{-1}(x)$. The partition induced by $\mathcal{A}(x)$ is exactly the partition corresponding to the equivalence relation from above, i.e. where the chambers of $\mathcal{A}(x)$ are equivalence classes indexed by fixed orientations on Γ.

Example 8.3.23. We illustrate Construction 8.3.19, Definitions 8.3.20 and 8.3.22 using the x-graph Γ shown in Figure 8.7. As this example will be revisited, we take the time to embed it in a broader context here: Let the data $g = 1, n = 6, r = 2$ define the following counting problem

$$\mathbb{T}h_1^{spin,2,+} : \mathcal{H} \to \mathbb{Q}, (x_1, ..., x_6) \mapsto \mathbb{T}h_1^{spin,2,+}(x^-, x^+). \tag{8.3}$$

Then (8.3) asks for graphs of genus 1 with 6 ends, 3 inner vertices of valency 4 or 2 depending on the genus at that vertex. Note that Γ is such a graph.

One computes $\ker(\partial) = \langle e_1 + e_3 - e_2 \rangle \subset \mathbb{R}^9$. Hence, π_Γ is contained in the trivial bundle $\mathcal{H}_\mathbb{R} \times \mathbb{R}^9$ via the embedding i (see Figure 8.7 for a sketch in the quotient $\mathbb{R}^9/\mathbb{R}^6 \cong \mathbb{R}^3$ corresponding to $\langle e_1, e_2, e_3 \rangle$).

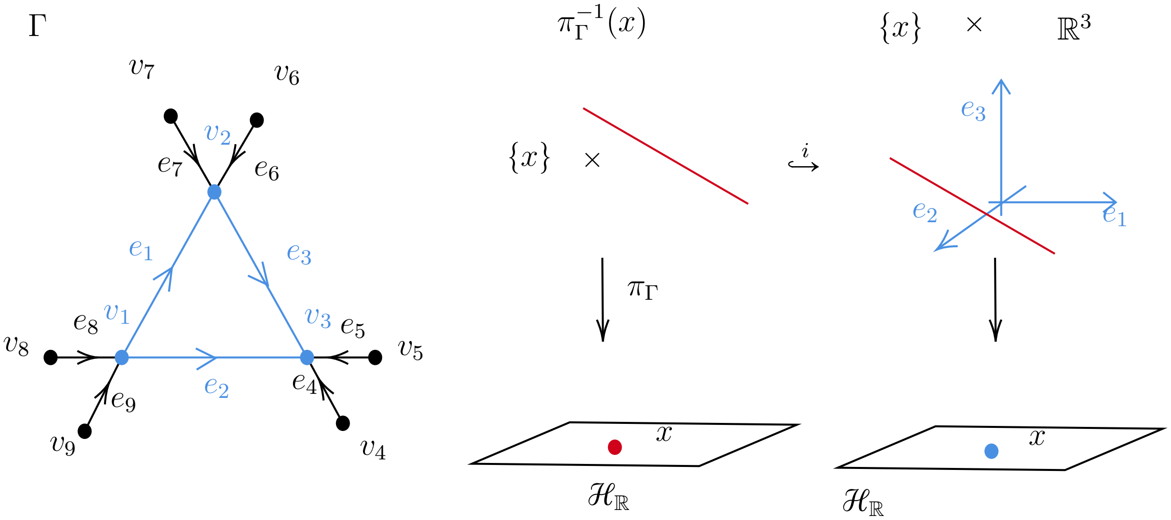


Figure 8.7: On the left the x-graph Γ, on the right a sketch of π_Γ.

Inside each fibre ${\pi_\Gamma}^{-1}(x) \cong \mathbb{R}$, we have the hyperplane arrangement $\mathcal{A}(x)$ as sketched in Figure 8.9: Here, $\mathcal{A}(x)$ consists of three walls, $j = 0, j = x_2 + x_3$ and $j = -x_4 - x_5$, corresponding to interior edges, which together cut out two bounded chambers.

Building on Example 8.3.23, we generalize the observations concerning the "shape" of $\mathcal{A}(x)$:

Remark 8.3.24. The arrangement $\mathcal{A}(x)$ exhibits structural properties that are independent of x, e.g. it comes with a minimal number of non-transversalities that are forced by the structure of the graph: Each vertex induces a non-transversality given by the intersection of the hyperplanes corresponding to its adjacent edges. These cannot be resolved by variations of x, which can only create additional non-transversalities. This means that a change in topology of $\mathcal{A}(x)$ is always accompanied by a vanishing chamber.

Lemma 8.3.25 ([24], Lemma 2.12)**.** Let c be a chamber of $\mathcal{A}(x)$. The corresponding orientation allows for the set of internal vertices to be totally ordered if and only if c is bounded.

$\mathbb{T}h_g^{spin,r}$ **as sum over** x**-graphs.** Let $\tilde{x} \in \mathcal{H}$. Lemma 8.3.25 identifies the *locus of adequate weight vectors* in ${\pi_\Gamma}^{-1}(\tilde{x})$ as the union of bounded chambers of $\mathcal{A}(\tilde{x})$. The contribution of Γ to $\mathbb{T}h_g^{spin,r}(\tilde{x})$ is obtained by summing the contributions from all covers π whose underlying $x-$graph is Γ:

$$\mathrm{mult}(\Gamma) := \sum_{\pi} \mathrm{mult}(\pi).$$

Then, $\mathrm{mult}(\Gamma)$ is given by the weighted integral of the polynomial

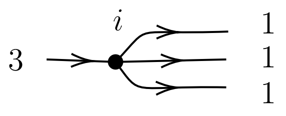
$$P_\Gamma(\underline{e}) := \prod_{i=1}^{|E^{\mathrm{in}}(\Gamma)|} e_i \prod_{i=1}^{|V\mathrm{in}(\Gamma)|} h_i(\underline{e}),\ \underline{e} := (e_1, ..., e_{|E(\Gamma)|}) \in \mathbb{R}^{|E^{\mathrm{in}}(\Gamma)|} \oplus \mathbb{R}^n,$$

over the *odd* lattice points of the bounded chambers c of $\mathcal{A}(\tilde{x})$:

$$\mathrm{mult}(\Gamma) = \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{c \text{ bounded}} \mathrm{sgn}(P_\Gamma(\underline{e}))m(c) \sum_{\underline{e} \in c \cap (2\mathbb{Z}+1)^{|E(\Gamma)|}} P_\Gamma(\underline{e}),$$

where

- $E^{\mathrm{in}}(\Gamma)$ ($V^{\mathrm{in}}(\Gamma)$) is the set of internal edges (vertices).
- $h_i(\underline{e})$ is the local completed cycle spin Hurwitz number at vertex i (Figure 8.8).
- $\mathrm{sgn}(P_\Gamma(\underline{e}))$ is notation for the sign of $\prod_{i=1}^{|E^{\mathrm{in}}(\Gamma)|} e_i$ on c (This sign is constant on c (see Definition 8.3.22). The sign of $h_i(\underline{e})$ is even constant on $\mathcal{H}_\mathbb{R}$ as it is an even polynomial in the edge weights (see Lemma 7.5.4)).
- $\mathrm{Aut}(\Gamma)$ is the automorphism group of the x-graph Γ (disregarding edge weights and orientation).



$$h_i(\underline{e}) = h_0^{0,2}((3),(1,1,1))$$

Figure 8.8: The local completed cycle Hurwitz number at vertex i. The non-zero entries of $\underline{e}$ are $3, 1, 1, 1$.

Finally, we update the counting unit from π to Γ:

$$\mathbb{T}h_g^{spin,r}(\tilde{x}) = \sum_{\Gamma \in \mathbb{T}S_x} \operatorname{mult}(\Gamma),$$

where $\mathbb{T}S_x$ denotes the set of $x-$graphs relevant for $\mathbb{T}h_g^{spin,r}(\tilde{x})$ and extend $\mathbb{T}h_g^{spin,r}$ to $\mathcal{H}_\mathbb{R}$ in the obvious way (see Proposition 8.3.29).

Remark 8.3.26. The decomposition ${\pi_\Gamma}^{-1}(\tilde{x}) = V \oplus \{\tilde{x}\} \subset \mathbb{R}^{|E^{\text{in}}(\Gamma)|} \oplus \mathbb{R}^n$ allows us to view ${\pi_\Gamma}^{-1}(\tilde{x})$ and $\mathcal{A}(\tilde{x})$ as subsets of $\mathbb{R}^{b_1(\Gamma)}$ via the isomorphism $\mathbb{R}^{|E^{\text{in}}(\Gamma)|} \cong \mathbb{R}^{b_1(\Gamma)}$ induced by the reference weighting (Definition 8.3.18). Accordingly, we write

$$\operatorname{mult}(\Gamma)(\tilde{x}) = \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{c(\tilde{x}) \text{ bounded}} m(c(x)) \sum_{\underline{j} \in c(\tilde{x}) \cap (2\mathbb{Z}+1)^{b_1(\Gamma)}} \operatorname{sgn}(P_\Gamma(\tilde{x}, \underline{j}) P_\Gamma(\tilde{x}, \underline{j})$$

and observe that the new counting unit Γ behaves similarly to the genus 0 case: It is independent of the extra parameters $\underline{j}$. However, unlike the genus 0 case, the set $\mathbb{T}S_x$ is now even independent of specific chambers: Whether Γ contributes to $\mathbb{T}h_g^{spin,r}(\tilde{x})$ is encoded in $\mathcal{A}(\tilde{x})$.

Lemma 8.3.27. To each $\Gamma \in \mathbb{T}S_x$ we associate an assignment $\phi_\Gamma : \mathcal{H}_\mathbb{R} \to \mathbb{Q} : x \mapsto b(\mathcal{A}(x))$ that counts the number of bounded chambers $b(\mathcal{A}(x))$ in $\mathcal{A}(x)$. Individually, each ϕ_Γ is locally constant except along certain hyperplanes. Collectively, they naturally gives rise to a hyperplane arrangement $\mathcal{W}$ in $\mathcal{H}_\mathbb{R}$ consisting of hyperplanes of the form $\mathcal{W}_I = \{x : \sum_{i \in I} x_i = 0\}$, where $I \subsetneq \{1, \ldots, n\}$ is a non-empty subset. The connected components of the complement $\mathcal{H}_\mathbb{R} \backslash \mathcal{W}$ describe precisely the locus where all functions ϕ_Γ are constant.

The proof of Lemma 8.3.27 requires the following technical ingredient:

Lemma 8.3.28 (Variant of Farkas' Lemma [60], Chapter 6)**.** Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then either $Ax \leqslant b$ has a solution $x \in \mathbb{R}^n$ or $A^t y = 0$ has a solution $y \geqslant 0$ with $b^t y < 0$.

Proof. From Remark 8.3.24 we know that the number $b(\mathcal{A}(x))$ changes whenever the system of inequalities defining a bounded chamber c in $\mathcal{A}(x)$

$$A \cdot \omega \leqslant b, \text{ where we write } b_i := b(x)_i = \sum_{i \in I_i} x_i \text{ for } I_i \subset \{1, \ldots, n\} \tag{8.4}$$

(renaming $x_i := -x_i$ if necessary) and A is a matrix with integer entries, becomes unsolvable. Consider x' and x'' such that system (8.4) is solvable for x' and not for x''. By Farkas' Lemma there exists a vector $y \geqslant 0$ that satisfies $y^t A = 0$ such that $b(x')^t y \geqslant 0$, but $b(x'')^t y < 0$. This can only happen, if a component of $b(x)$ changes sign, i.e. there exists x''' with $0 = b(x''')_i = \sum_{i \in I_i} x'''_i$, thus $x''' \in \mathcal{W}_{I_i} =: \mathcal{W}_I$. Conversely, suppose $x''' \in \mathcal{W}_I$. We construct an $x-$graph Γ such that ϕ_Γ drops at x''':

- If $|I|$ is odd: Choose $x-$graphs Γ_1 and Γ_2 that contribute to the counts $\mathbb{T}h^{spin,r}_{g_1}(x_I, -\sum_{i\in I} x_i)$ and $\mathbb{T}h^{spin,r}_{g-g_1}(\sum_{i\in I} x_i, x_{I^c})$, respectively, and glue them along their end of weight $\sum_{i\in I} x_i$.
- If $|I|$ is even: Choose $x-$graphs Γ_1 and Γ_2 that contribute to $\mathbb{T}h^{spin,r}_{g_1}(x_I, -\sum_{i\in I} x_i - y, -y)$ and $\mathbb{T}h^{spin,r}_{g-g_1-1}(\sum_{i\in I} x_i + y, y, x_{I^c})$ (note: since $|I|$ is even, $g > 0$) and identify the ends of weight $\sum_{i\in I} x_i + y$ and y, thus forming a new cycle in the glued graph.

It follows immediately that for x sufficiently generic (i.e. $x \in \mathcal{H}_\mathbb{R}\backslash\mathcal{W}$) ϕ_Γ is constant in a neighborhood of x (since the topology of $\mathcal{A}(x)$ is preserved) and that the locus where all functions ϕ_Γ are constant corresponds exactly to the connected components of the complement $\mathcal{H}_\mathbb{R}\backslash\mathcal{W}$. ☐

Proposition 8.3.29. Let $\Gamma \in \mathbb{T}S_x$ contribute to $\mathbb{T}h^{spin,r}_g$. For $\tilde{x} \in \mathcal{H} \cap (\mathcal{H}_\mathbb{R}\backslash\mathcal{W})$, $\mathrm{mult}(\Gamma)$ is a polynomial in the $\tilde{x}_i$ of degree $2g(\Gamma) + |V^{\mathrm{in}}(\Gamma)| - 1 = 2g + b - 1$.

Proposition 8.3.29 falls naturally within the scope of Erhart theory. We present a proof here while deferring the proofs of certain required Erhart-theoretic results until later.

Proof. A bounded chamber $c(\tilde{x})$ in $\mathcal{A}(\tilde{x}) \subset \mathbb{R}^{b_1(\Gamma)}$ induces a family $\mathbb{Z}^n \to \mathbb{R}^{b_1(\Gamma)}, x \mapsto c(x)$ of integer polytopes such that: The topology is fixed across the family (i.e. the set of facets including their *outward* normal vector is fixed up to translation and scaling), as long as x varies within the connected component c_H of $\mathcal{H}_\mathbb{R}\backslash\mathcal{W}$ that contains $\tilde{x}$ (Lemma 8.3.27).More precisely, $c(x)$ is the convex hull of a *fixed* set $V(c(x))$ of x-dependent vertices, whose coordinates are affine integer linear functions in x.

Applying the affine map $\phi : \mathbb{R}^{b_1(\Gamma)} \to \mathbb{R}^{b_1(\Gamma)}, j \mapsto \frac{1}{2}\mathrm{Id}j - \frac{1}{2}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$ yields another family of polytopes $\mathbb{Z}^n \cap c_H \to \mathbb{R}^{b_1(\Gamma)}, x \mapsto \tilde{c}(x) := \phi(c(x))$ with vertex set $\{\phi(v(x))|v(x) \in V(c(x))\}$, where vertex coordinates are now affine linear functions in x with coefficients in $\frac{1}{2}\mathbb{Z}$.

Then the contribution

$$\sum_{j\in c(x)\cap(2\mathbb{Z}+1)^{b_1(\Gamma)}} \mathrm{sgn}(P_\Gamma(x,j))P_\Gamma(x,j) \tag{8.5}$$

of $c(x)$ to $\mathrm{mult}(\Gamma)$ may be rewritten in terms of $\tilde{c}(x)$ as

$$\sum_{k\in\tilde{c}(x)\cap\mathbb{Z}^{b_1(\Gamma)}} \mathrm{sgn}(\tilde{P}_\Gamma(x,k))\tilde{P}_\Gamma(x,k) = \sum_\alpha c_\alpha(x) \sum_{k\in\tilde{c}(x)\cap\mathbb{Z}^{b_1(\Gamma)}} k^\alpha, \text{ where}$$

$\tilde{P}_\Gamma(x,k) := P_\Gamma(x, 2k+1) := \sum_{\alpha\in\mathbb{N}^{b_1(\Gamma)}} c_\alpha(x)k^\alpha$ is its decomposition as polynomial in k with coefficients $c_\alpha(x) \in \mathbb{Q}[x]$. By Lemma 8.3.31 (for the family $\tilde{c}(x)$ and polynomial weight k^α)

$$F : \mathbb{Z}^n \cap c_H \to \mathbb{R}, x \mapsto \sum_{k\in\tilde{c}(x)\cap\mathbb{Z}^{b_1(\Gamma)}} k^\alpha$$

is a quasi-polynomial of degree $\dim(\tilde{c}(x)) + |\alpha| = b_1(\Gamma) + |\alpha|$ and period "at most" $(2, \ldots, 2)^T \in \mathbb{Z}^n$, i.e. there exists at most 2^n polynomials F_i such that

$$F(x) = \begin{cases} F_1(x), x_i \equiv 1 \bmod 2 \text{ for } i = 1, \ldots, n. \\ F_2(x), x_1 \equiv 0 \bmod 2, x_i \equiv 1 \bmod 2 \text{ for } i = 2, \ldots, n. \\ \vdots \\ F_{2^n}, x_i \equiv 0 \bmod 2 \text{ for } i = 1, \ldots, n. \end{cases}$$

Restricting to $\mathcal{H} \cap c_H$, one obtains polynomials

$$\mathcal{H} \cap c_H \ni x \mapsto \sum_{k \in \tilde{c}(x) \cap \mathbb{Z}^{b_1(\Gamma)}} k^\alpha \text{ and } \mathcal{H} \cap c_H \ni x \mapsto \sum_\alpha c_\alpha(x) \sum_{k \in \tilde{c}(x) \cap \mathbb{Z}^{b_1(\Gamma)}} k^\alpha$$

of degree $b_1(\Gamma) + |\alpha|$ and $b_1(\Gamma) + \deg(P_\Gamma(x, j))$. Combining the Euler characteristic formula for Γ with the fact that the local completed-cycles spin Hurwitz number $h_i(x, j)$ at vertix v_i has degree $2g(v_i)$ ([40], Theorem 6.4), shows

$$\deg(P_\Gamma(x, j)) = |E^{\text{in}}(\Gamma)| + \sum_{i=1}^{|V^{in}(\Gamma)|} \deg(h_i(x, j)) = |V^{in}(\Gamma)| + b_1(\Gamma) - 1 + \sum_{i=1}^{|V^{in}(\Gamma)|} 2g(v_i) \text{ and}$$
$$b_1(\Gamma) + \deg(P_\Gamma(x, j)) = 2g(\Gamma) + |V^{in}(\Gamma)| - 1.$$

It follows that $\mathcal{H} \cap c_H \ni x \mapsto \text{mult}(\Gamma)(x)$ is a polynomial of degree $2g(\Gamma) + |V^{in}(\Gamma)| - 1 = 2g + b - 1$, since Γ contribute to $\mathbb{T}h_g^{spin,r}$. ☐

Example 8.3.30. We return to Example 8.3.23: Let Γ be the x-graph from Figure 8.7 and consider the connected component c_H of $\mathcal{H}_\mathbb{R} \backslash \mathcal{W}$ defined by: $x_1, x_3, x_3 \geqslant 0$, $x_4, x_5, x_6 \leqslant 0$, $x_2 + x_3 \leqslant -x_4 - x_5$. Figure 8.9 illustrates for $\tilde{x} = (5, 3, 3, -5, -5, -1) \in c_H$ the hyperplane arrangement $\mathcal{A}(\tilde{x})$ with its two bounded chambers, $c_1(\tilde{x})$ and $c_2(\tilde{x})$, and the corresponding induced orientations.

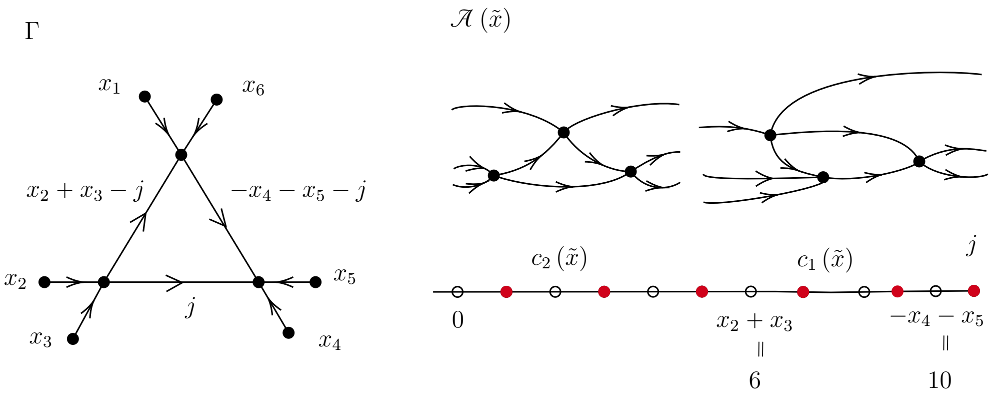


Figure 8.9: The x-graph from Example 8.3.30 with reference weighting and hyperplane arrangement $\mathcal{A}(\tilde{x})$ for $\tilde{x} \in c_H$.

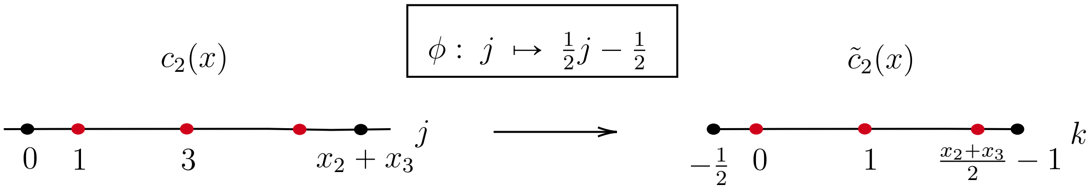


Figure 8.10: Sketch of affine coordinate change ϕ. The red points correspond to the lattice points that contribute to the count.

We follow the strategy outlined in the proof of Proposition 8.3.29 to determine Γ's contribution to $\mathbb{T}h_1^{spin,2,+}$. As $\operatorname{mult}(\Gamma)$ splits additively across the chambers, $c_1(\tilde{x})$ and $c_2(\tilde{x})$, we can handle each chamber individually. Below, we detail the computation for $c_2(\tilde{x})$, $c_1(\tilde{x})$ is treated similarly. We can express the contribution of $c_H \cap (2\mathbb{Z}+1) \ni x \mapsto c_2(x)$ to $\operatorname{mult}(\Gamma)$ as

$$m(c_2(x)) \sum_{j\in c_2(x)\cap(2\mathbb{Z}+1)} \operatorname{sgn}(P_\Gamma(x,j))P_\Gamma(x,j) = \sum_{0\leqslant j\leqslant x_2+x_3 \text{ odd}} P_\Gamma(x,j), \text{where}$$

$$m(c_2(x)) = 1 = \operatorname{sgn}(P_\Gamma(x,j)) \text{ and } P_\Gamma(x,j) = (x_2+x_3-j)(-x_4-x_5-j)j(2!)^3$$

(2! accounts for the local vertex multiplicities), in terms of the family $\tilde{c}_2(x)$ obtained via the affine coordinate change "$j = 2k+1$" (see Figure 8.10):

$$\sum_{k\in\tilde{c}_2(x)\cap\mathbb{Z}} \tilde{P}_\Gamma(x,k) = \sum_{k=0}^{\lfloor\frac{x_2+x_3}{2}-\frac{1}{2}\rfloor} (2!)^3(x_2+x_3-2k-1)(-x_4-x_5-2k-1)(2k+1).$$

Since all x_i are odd, the upper summation bound $\lfloor\frac{x_2+x_3}{2}\rfloor$ is equal to $\frac{x_2+x_3}{2}-1$, a polynomial (as opposed to a quasi-polynomial) expression in the x_i. This shows

$$\begin{aligned}\sum_{k\in\tilde{c}_2(x)\cap\mathbb{Z}} \tilde{P}_\Gamma(x,k) &= \sum_{k=0}^{\frac{x_2+x_3}{2}-1} (2!)^3(x_2+x_3-2k-1)(-x_4-x_5-2k-1)(2k+1) \\ &= \frac{-8}{24}(x_2+x_3)(x_2^2+2x_2x_3+x_3^2+2)(2x_4+x_2+x_3+2x_5),\end{aligned}$$

which is indeed a polynomials of degree $2g+3-1=4$.

Lemma 8.3.31. Let $\mathbb{Z}^{n'} \to \mathbb{R}^n, x \mapsto P(x)$ be a family of n-dimensional rational polytopes and $v_i(x) = G_i x + h_i$ with $G_i \in (\frac{1}{2}\mathbb{Z})^{n'\times n}$ and $h_i \in (\frac{1}{2}\mathbb{Z})^n, i = 1,\ldots,l,$ such that, for all x in a region $c \subset \mathbb{Q}^{n'}$, the vertex set $V(P(\underline{x}))$ of $P(\underline{x})$ is given by:

$$V(P(x)) = \{v_1(x),\ldots,v_l(x)\}.$$

Then, for any $f \in \mathbb{Q}[j_1,\ldots,j_n]$

$$\mathbb{Z}^{n'} \cap c \to \mathbb{R}, x \mapsto \sum_{j\in P(x)\cap\mathbb{Z}^n} f(j)$$

is a quasi-polynomial in x of degree $n + \deg(f)$ whose period $(Q_1, ..., Q_{n'})$ satisfies $Q_i \in \{1, 2\} \forall i$.

It is known that for a family of polytopes as in Lemma 8.3.31, the lattice-point enumerator ($f = 1$) is a quasi-polynomial of period at most 2. To obtain the corresponding result for the weighted enumeration we use the methodology of [34], converting the weighted sum into a lattice point count in a higher dimensional polytope.

Proof. Without loss of generality, assume $P(x) \subset \mathbb{R}^n_{\geqslant 0}$ (if not: see Remark 8.3.32) and $f(j) = j_1^{\alpha_1} \cdot ... \cdot j_n^{\alpha_n}$. The statement of Lemma 8.3.31 for general $f' \in \mathbb{Q}[j_1, ..., j_n]$ follows by decomposing f' into a sum of monomials.

Step 1: Lift to the unweighted case. Convert the V-description of $P(x)$ on c into an H-description and write $P(x)$ in the form $P(x) = \{y | A(x)y = b(x), y \geqslant 0\}$ with $A(x) \in \mathbb{Z}[x]^{s \times \tilde{n}}, b(x) \in \mathbb{Z}[x]^s$ by introducing additional variables (i.e. $n \leqslant \tilde{n}$).

Next, consider the weight-lifting polytope $P^*(x)$ for $P(x)$ and weight f constructed as in [34] (Corollary 2.5 and 2.6):

$$P^* = \left\{ \begin{pmatrix} y \\ y' \end{pmatrix} \mid \begin{pmatrix} \multicolumn{2}{c}{A(x)} & 0 \\ \underbrace{D}_{n} & \underbrace{0}_{\tilde{n}-n} & C \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} b(x) \\ -e \end{pmatrix}, y, y' \geqslant 0 \right\} \subset \mathbb{R}^{\tilde{n}+m}, \text{ where}$$

$$e = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix}, C = \begin{pmatrix} C_1 & 0 & 0 & \dots & 0 \\ 0 & C_2 & 0 & \dots & 0 \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & \dots & \dots & \dots & C_n \end{pmatrix} \text{ with } C_i = \begin{pmatrix} I_{\alpha_i} & I_{\alpha_i} \end{pmatrix} \text{ and}$$

$$D = \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & d_n \end{pmatrix} \text{ with } d_i = \left.\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\right\} \alpha_i,$$

and note: For $y \in P(x)$ with $y_i > 1$, the set $\left\{ y' | \left(\begin{smallmatrix} D & 0 \end{smallmatrix} \right) y + e = Cy', y' \geqslant 0 \right\}$ is equal to the product $(y_1 - 1)C(\alpha_1) \times ... \times (y_n - 1)C(\alpha_n)$ of dilated hypercubes $(y_i - 1)C(\alpha_i)$ of size α_i associated to y.

For fixed $x \in \mathbb{Z}^{n'}$, Theorem 1.1 ([34]) gives:

$$\sum_{y \in P(x) \cap \mathbb{Z}^{\tilde{n}}} f(y) = \sum_{\left(\begin{smallmatrix} y \\ y' \end{smallmatrix}\right) \in P^*(x) \cap \mathbb{Z}^{\tilde{n}+m}} 1. \tag{8.6}$$

Step 2: Proof of the unweighted case. Without loss of generality, suppose that the vertices of $P(x)$ have coordinates strictly larger than 1. If not, translate P by $v \in \mathbb{Z}^{\tilde{n}}$, $P^*(x)$ by $(v, 0)^T$ and consider

$$\sum_{y \in \left(P(x)+v\right) \cap \mathbb{Z}^{\tilde{n}}} f(y - v) = \sum_{\left(\begin{smallmatrix} y \\ y' \end{smallmatrix}\right) \in \left(P^*(x) + \left(\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right)\right) \cap \mathbb{Z}^{\tilde{n}+m}} 1. \tag{8.7}$$

If (8.7) gives rise to a quasiploynomial in x of period at most 2, so does (8.6).

By Lemma 8.3.33, the vertices of $P^*(x)$ are of the form $v^*(x) = (y(x), y'(x))$, where

- $y(x) := (y_1(x), ..., y_{\tilde{n}}(x))$ is a vertex of $P(x)$.
- $y'(x)$ is a vertex of the product of hypercubes $\prod_{i=1}^{n}(y_i(x) - 1)C(\alpha_i)$ associated to $y(x)$.

In particular, $v^*(x)$ has only coordinates $y_i(x)$, $y_i(x) - 1$ or 0 for $y(x)$ a vertex of $P(x)$.

This means: The weight-lifting process (Step 1) gives rise to an $|\alpha| + n$-dimensional family $c \ni x \mapsto P^*(x)$ with the property that for every $x \in c$, each $v^*(x) \in V(P^*(x))$ is an affine linear function in x with coefficients in $\frac{1}{2}\mathbb{Z}$ (since this holds for the vertices of $P(x)$ by assumption). We can apply Proposition 1 ([79]) to conclude that

$$\mathbb{Z}^{n'} \cap c \to \mathbb{R}, x \mapsto \sum_{j \in P^*(x) \cap \mathbb{Z}^{\tilde{n}+m}} 1$$

is a quasi-polynomial in x of degree $n+|\alpha| = n+\deg(f)$ whose period $(Q_1, ..., Q_{n'})$ satisfies $Q_i \in \{1, 2\} \forall i$.

□

Remark 8.3.32. If $P(x) \subset \mathbb{R}^n_{\geqslant 0}$ does not hold, then writing $P(x)$ in the form $A(x)y = b(x), y \geqslant 0$ necessitates introducing for each original variable j_i in P two new variables j_i^+ and j_i^- (representing positive and negative parts, respectively). Under this transformation: A monomial j^α becomes a *sum* of monomials in the variables j_i^+ and j_i^- (e.g. $j_1 j_2$ turns into $j_1^+ j_2^+ + j_1^- j_2^- - j_1^+ j_2^- - j_1^- j_2^+$). This requires the following adjustments in the proof of Lemma 8.3.31:

- Prove Step 1 for a monomial f in $2n$ variables (replace index n by $2n$ from Step 1 onward).
- Deduce the statement of Lemma 8.3.31 for polytopes $\tilde{P}(x)$ in the form $A(x)y = b(x), y \geqslant 0$ and general polynomial weight f.
- Recover Lemma 8.3.31 for the original $P(x)$.

Lemma 8.3.33. Let

- $P \subset \mathbb{R}^n$ be a rational polytope in the form $\{x | Ax = b, x \geqslant 0\}$, where $A \in \mathbb{Z}^{s \times n}$ and $b \in \mathbb{Z}^s$, whose vertices have coordinates strictly greater than 1.
- $P^* \subset \mathbb{R}^{n+m}$ be the weight-lifting polytope for the weight function $\omega(\underline{x}) := x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$ constructed in [34], Corollary 2.5 and 2.6.

Then, the vertex set $V(P^*)$ satisfies

$$V(P^*) \subset \{(x, x')|x \in V(P) \text{and } x' \in V(\prod_{i=1}^{n}(x^i - 1)C(\alpha_i))\}, \text{ where}$$

$(x^1 - 1)C(\alpha_1) \times ... \times (x^n - 1)C(\alpha_n)$ denotes the product of dilated hypercubes $(x^i - 1)C(\alpha_i)$ of size α_i associated to x.

Proof. Without loss of generality, assume $\alpha_i > 0$ for $i = 1, ..., n$. Otherwise, permute the coordinates of P such that the first $\tilde{n}$ coordinates correspond to $\alpha_i > 0$, and construct P^* as in the proof of Lemma 8.3.31. The following then applies, noting that the product of hypercubes is only taken up to $\tilde{n}$.

Let $M := \{(x, x')|x \in V(P) \text{ and } x' \in V(\prod_{i=1}^{n}(x^i - 1)C(\alpha_i))\}$. We show $P^* = \text{conv}(M)$.
Observations:

1. A vertex of the dilated hypercube $(x^i - 1)C(\alpha_i)$ has coordinates in $\{0, x^i - 1\}$.
2. A vertex of the product $\prod_{i=1}^{n}(x^i - 1)C(\alpha_i)$ is a product of vertices from each of the factors.

Consider a point $(\tilde{x}, \tilde{x}') \in P^*$. Then, $\tilde{x} \in P$ and $\tilde{x}' \in \prod_{i=1}^{n}(\tilde{x}^i - 1)C(\alpha_i)$, where $\tilde{x}' \in \prod_{i=1}^{n}(\tilde{x}^i - 1)C(\alpha_i)$ is the product of positively dilated hypercubes associated to $\tilde{x}$ (Recall: $\tilde{x}^i - 1 > 1$ holds by assumption on the vertex coordinates of P). Hence, we can write

- $\tilde{x} = \sum_k \lambda_k x_k$ for some $\lambda_k \geqslant 0$ with $\sum_k \lambda_k = 1$ and $x_k \in V(P)$.
- $\tilde{x}' = \sum_j \lambda'_j y_j$ for some $\lambda'_j \geqslant 0$ with $\sum_j \lambda'_j = 1$ and $y_j \in V(\prod_{i=1}^{n}(\tilde{x}^i - 1)C(\alpha_i))$.

Combining Observations 1 and 2, we use the coefficients λ_k to express each vertex y_j as a convex combination of vertices y_k^j of the product $\prod_{i=1}^{n}(x_k^i - 1)C(\alpha_i)$ associated to $x_k \in V(P)$: $y_j = \sum_k \lambda_k y_k^j$. Here, the superscript j does not denote a component, but indicates that the choice of vertex y_k^j of $\prod_{i=1}^{n}(x_k^i - 1)C(\alpha_i)$ depends on y_j. Then, $\tilde{x}' = \sum_j \lambda'_j y_j = \sum_j \sum_k \lambda'_j \lambda_k y_k^j$ yields

$$(\tilde{x}, \tilde{x}') = \sum_j \sum_k \lambda'_j \lambda_k (x_k, y_k^j), \text{ where } (x_k, y_k^j) \in M,$$

as desired. □

The following example was computed using

`polymake`

Example 8.3.34. Consider a rectangle $P \subset \mathbb{R}^2$ with vertices $(\frac{7}{2}, 2), (\frac{7}{2}, \frac{5}{2}), (2, \frac{5}{2}), (2, 2)$. Write P in the form $\{x|Ax = b, x \geqslant 0\}$:

```
$P = new Polytope<Rational>( EQUATIONS => [[-11,2, 0,2,0,0,0,0,0],
[-9,0, 2,0,2,0,0,0,0],[-2,1,0,0,0,-1,0,0,0],[-2,0,1,0,0,0,-1,0,0],
[-2,0,0,1,0,0,0,-1,0],[-2,0,0,0,1,0,0,0,-1]], INEQUALITIES =>
[[0, 1, 0,0,0,0,0,0,0],[0, 0, 1,0,0,0,0,0,0],[0, 0, 0, 1,0,0,0,0,0],
[0,0,0,0,1,0,0,0,0],[0,0,0,0,0,1,0,0,0],[0,0,0,0,0,0,1,0,0],
[0,0,0,0,0,0,0,1,0],[0,0,0,0,0,0,0,0,1]] );

polytope > print $P->VERTICES;
polymake: used package ppl
The Parma Polyhedra Library ([[wiki:external_software#PPL]]):
A C++ library for convex polyhedra
and other numerical abstractions.
http://www.cs.unipr.it/ppl/

1 7/2 2 2 5/2 3/2 0 0 1/2
1 7/2 5/2 2 2 3/2 1/2 0 0
1 2 5/2 7/2 2 0 1/2 3/2 0
1 2 2 7/2 5/2 0 0 3/2 1/2
```

Let P^* be the weight-lifting polytope for the weight function $\omega(\underline{x}) := x_1^1 x_2^2$:

```
polytope > $P = new Polytope<Rational>( EQUATIONS =>
[[-11,2, 0,2,0,0,0,0,0,0,0,0,0,0,0],[-9,0, 2,0,2,0,0,0,0,0,0,0,0,0,0],
[-2,1,0,0,0,-1,0,0,0,0,0,0,0,0,0],[-2,0,1,0,0,0,-1,0,0,0,0,0,0,0,0],
[-2,0,0,1,0,0,0,-1,0,0,0,0,0,0,0],[-2,0,0,0,1,0,0,0,-1,0,0,0,0,0,0],
[-1,1,0,0,0,0,0,0,0,-1,-1,0,0,0,0],[-1,0,1,0,0,0,0,0,0,0,0,-1,0,-1,0],
[-1,0,1,0,0,0,0,0,0,0,0,0,-1,0,-1]],INEQUALITIES =>
[[0, 1, 0,0,0,0,0,0,0,0,0,0,0,0,0],[0,0, 1, 0,0,0,0,0,0,0,0,0,0,0,0],
[0,0,0,1,0,0,0,0,0,0,0,0,0,0,0],[0,0,0,0,1,0,0,0,0,0,0,0,0,0,0],
[0,0,0,0,0,1,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,1,0,0,0,0,0,0,0,0],
[0,0,0,0,0,0,0,1,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,1,0,0,0,0,0,0],
[0,0,0,0,0,0,0,0,0,1,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0,1,0,0,0,0],
[0,0,0,0,0,0,0,0,0,0,0,1,0,0,0],[0,0,0,0,0,0,0,0,0,0,0,0,1,0,0],
[0,0,0,0,0,0,0,0,0,0,0,0,0,1,0],[0,0,0,0,0,0,0,0,0,0,0,0,0,0,1]]);
```

Then, P^* has exactly $|V(P)| \cdot 2^1 \cdot 2^2 = 32$ vertices:

```
polytope > print $P->VERTICES;
1 7/2 2 2 5/2 3/2 0 0 1/2 5/2 0 1 0 0 1
1 7/2 2 2 5/2 3/2 0 0 1/2 0 5/2 1 0 0 1
1 7/2 5/2 2 2 3/2 1/2 0 0 5/2 0 3/2 3/2 0 0
```

```
1 7/2 2 2 5/2 3/2 0 0 1/2 0 5/2 0 1 1 0
1 7/2 2 2 5/2 3/2 0 0 1/2 5/2 0 0 1 1 0
1 7/2 5/2 2 2 3/2 1/2 0 0 5/2 0 0 3/2 3/2 0
1 7/2 5/2 2 2 3/2 1/2 0 0 0 5/2 0 3/2 3/2 0
1 7/2 5/2 2 2 3/2 1/2 0 0 0 5/2 3/2 0 0 3/2
1 7/2 5/2 2 2 3/2 1/2 0 0 5/2 0 3/2 0 0 3/2
1 7/2 5/2 2 2 3/2 1/2 0 0 0 5/2 0 0 3/2 3/2
1 7/2 5/2 2 2 3/2 1/2 0 0 5/2 0 0 0 3/2 3/2
1 7/2 5/2 2 2 3/2 1/2 0 0 0 5/2 3/2 3/2 0 0
1 7/2 2 2 5/2 3/2 0 0 1/2 0 5/2 1 1 0 0
1 7/2 2 2 5/2 3/2 0 0 1/2 5/2 0 0 0 1 1
1 7/2 2 2 5/2 3/2 0 0 1/2 0 5/2 0 0 1 1
1 7/2 2 2 5/2 3/2 0 0 1/2 5/2 0 1 1 0 0
1 2 2 7/2 5/2 0 0 3/2 1/2 1 0 1 1 0 0
1 2 2 7/2 5/2 0 0 3/2 1/2 0 1 0 0 1 1
1 2 2 7/2 5/2 0 0 3/2 1/2 1 0 0 0 1 1
1 2 2 7/2 5/2 0 0 3/2 1/2 0 1 1 1 0 0
1 2 5/2 7/2 2 0 1/2 3/2 0 0 1 3/2 3/2 0 0
1 2 5/2 7/2 2 0 1/2 3/2 0 1 0 0 0 3/2 3/2
1 2 5/2 7/2 2 0 1/2 3/2 0 0 1 0 0 3/2 3/2
1 2 5/2 7/2 2 0 1/2 3/2 0 1 0 3/2 0 0 3/2
1 2 2 7/2 5/2 0 0 3/2 1/2 1 0 1 0 0 1
1 2 2 7/2 5/2 0 0 3/2 1/2 0 1 1 0 0 1
1 2 5/2 7/2 2 0 1/2 3/2 0 1 0 3/2 3/2 0 0
1 2 2 7/2 5/2 0 0 3/2 1/2 0 1 0 1 1 0
1 2 2 7/2 5/2 0 0 3/2 1/2 1 0 0 1 1 0
1 2 5/2 7/2 2 0 1/2 3/2 0 1 0 0 3/2 3/2 0
1 2 5/2 7/2 2 0 1/2 3/2 0 0 1 0 3/2 3/2 0
1 2 5/2 7/2 2 0 1/2 3/2 0 0 1 3/2 0 0 3/2

polytope > print $P->N_VERTICES;
32
```

Wall crossing formulae. Let C_1 and C_2 be two chambers separated by a wall $\mathscr{W}_I$ and assume that $\sum_{i\in I} x_i < 0$ holds in C_1, but not in C_2. Once again, we seek a more intrinsic expression for

$$WC^r_{g,I} : C_2 \to \mathbb{Q},\ x \mapsto \mathbb{T}h_g^{spin,r}{}_{|C_2}(x) - \mathbb{T}h_g^{spin,r}{}_{|C_1}(x), \tag{8.8}$$

in terms of Hurwitz numbers. Following the strategy described in [23], we proceed in two steps.

Step 1: Explicit Wall-Crossing formula. As in the genus 0 case, the analysis of $WC^r_{g,I}$ reduces to studying the wall-crossing behavior of individual x−graphs. By Remark 8.3.24 and Proposition 8.3.29, it is enough to consider graphs Γ for which ϕ_Γ jumps when crossing $\mathscr{W}_I$, otherwise the contribution to $\mathbb{T}h_g^{spin,r}{}_{|C_2}(x)$ cancels with the one to $\mathbb{T}h_g^{spin,r}{}_{|C_1}(x)$. Doing so we note: The condition "ϕ_Γ drops for $x \in W_I$" means that $\mathcal{A}_\Gamma(x_2)$ for $x_2 \in C_2$ contains bounded chambers that do not exist in $\mathcal{A}_\Gamma(x_1)$ for $x_1 \in C_1$ or, equivalently, the inequality $\sum_{i \in I} x_i < 0$ (that holds in C_1) excludes certain orientations adequate in C_2 and vice versa.

We observed in genus 0 that we could relate these by cutting and reversing the orientation of the *unique* edge of weight $\sum_{i \in I} x_i$. The situation is similar for higher genus, except that we usually have to cut and reorient *more than one* edge to obtain a particular orientation in $\mathcal{A}_\Gamma(x_1)$ from one in $\mathcal{A}_\Gamma(x_2)$. The *set of $I-$cuts* comprises such cuts.

Definition 8.3.35 ([24], Definition 6.4). For a bounded chamber c in $\mathcal{A}_\Gamma$, we denote by Γ_c the corresponding directed graph and associate to each subset of ends indexed by $I \subset \{1, ..., n\}$ a poset $Cuts_{\Gamma_c}(I) \subset E(\Gamma_c)$, where $E \in Cuts_{\Gamma_c}(I)$ holds, if $E = \varnothing$ or

1. Cutting Γ_c along E distributes the ends of Γ on precisely two connected components, one containing all ends in I (*the initial component*), the other I^c (*the final component*). In particular, $\Gamma_c \backslash E$ is disconnected.
2. The directed graph Γ_c / E^c is acyclic with initial vertex dual to the initial component and final vertex dual to the final component.

Each $E \in Cuts_{\Gamma_c}(I)$ is ranked by $\mathrm{rk}(E) := \nu(\Gamma_c \backslash E) - 1$, where $\nu(\Gamma_c \backslash E)$ denotes the number of components of $\Gamma_c \backslash E$.

Despite the added combinatorial complexity, we encounter a familiar phenomenon: The complements of $I-$cuts (i.e. the connected components of $\Gamma_c \backslash E$ for $E \in Cuts_{\Gamma_c}(I)$) form the common building blocks of directed graphs Γ_c that occur on both sides of W_I (compare to Proposition 8.3.14). This allows us to express $WC^r_{g,I}(x)$ in terms of these building blocks:

Theorem 8.3.36. [Explicit Wall crossing formula] For the wall W_I the wall crossing formula is given by:

$$WC^r_{g,I}(x_2) = \sum_\Gamma \sum_{c \in \mathcal{BC}(\mathcal{A}_\Gamma(x_2))} \left(\sum_{E \in Cuts_{\Gamma_c}(I)} (-1)^{rk(E)-1} \binom{a}{a_0, a_1, ..., a_N, a_{N+1}} \right) \left(\sum_{j \in c \cap (2\mathbb{Z}+1)^{b_1(\Gamma)}} P_\Gamma(x_2, j) \right),$$

where

1. $\mathcal{BC}(\mathcal{A}_\Gamma(x_2))$ is the set of bounded chambers of $\mathcal{A}_\Gamma(x_2)$.
2. $N = \mathrm{rk}(E) - 1$ is the number of inner connected components of $\Gamma_c \backslash E$.
3. a_0 and a_{N+1} ($a_1, ..., a_N$) denote the number of vertices in the initial, respectively final, (inner) commponent(s) of $\Gamma_c \backslash E$.

Theorem 8.3.36 is explicit. One can read off the wall crossing behavior of individual x-graphs from it since:

- $\binom{a}{a_0,a_1,\ldots,a_N,a_{N+1}}$ is the number of orderings on $a := |V(\Gamma)|$ vertices that refines $N+2$ given orderings on $a_0, a_1, \ldots, a_N$ and a_{N+1} vertices (see Proposition 8.3.14 in genus 0 for reference).

- $(-1)^{rk(E)-1}$ plays a twofold role depending on whether cutting and regluing Γ_c along reoriented edges of E yields an adequate orientation on Γ or not: If it does, then $(-1)^{rk(E)-1}$ provides the right prefactor, and ensures, if otherwise, that this orientation does not contribute (see [24], Lemma 6.12, for a precise statement).

For the proof of Theorem 8.3.36, we essentially refer to [23]. However, our slightly different setup requires us to extend relevant machinery from integral to rational hyperplane arrangements (an arrangement is integral, if each hyperplane is defined by an equation with integral coordinates and the vertices of the arrangement have integral coordinates as well). Consequently, this proof is only a sketch. It is not self-contained, as it borrows notation from the original paper.

Proof Sketch 8.3.37 (Theorem 8.3.36)**.** Denote by $WC[\Gamma]$ the wall-crossing contribution of an x-graphs Γ. The affine map

$$\phi : \mathbb{R}^{b_1(\Gamma)} \to \mathbb{R}^{b_1(\Gamma)}, j \mapsto \frac{1}{2}\mathrm{Id} j - \frac{1}{2}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

transforms the family of integer hyperplane arrangements $\mathbb{Z}^n \to \mathbb{R}^{b_1(\Gamma)}, x \mapsto \mathcal{A}_\Gamma(x)$ into a family $\mathbb{Z}^n \to \mathbb{R}^{b_1(\Gamma)}, x \mapsto \tilde{\mathcal{A}}_\Gamma(x) := \phi(\mathcal{A}_\Gamma(x))$ whose bounded chambers are polytopes with vertex coordinates being affine linear functions in x with coefficients in $\frac{1}{2}\mathbb{Z}$ (see Proof of Proposition 8.3.29). Just like in [23], consider the Gauss-Manin connection $\nabla_{\Gamma,12}$ of the vector bundle arising from $x \mapsto \tilde{\mathcal{A}}_\Gamma(x)$ and its adjoint $\nabla^*_{\Gamma,12}$ ([23], Definition 5.1.). By Proposition 8.3.29 the contribution of a bounded chamber $\tilde{c}$ of $\tilde{\mathcal{A}}_\Gamma(x_1)$ (where $x_1 \in C_1 \cap \mathcal{H}$) is a polynomial F on C_1. The value $F(x_2)$ at $x_2 \in C_2$ can be obtained by summing $\tilde{P}_\Gamma(x_2, k)$ over a signed sum of bounded chambers of $\tilde{\mathcal{A}}_\Gamma(x_2)$. This signed sum is encoded in the Gauss-Manin connection (see [78]). Analogously to [23] (Equation (9)) we have:

$$\begin{aligned} WC[\Gamma] &= \sum_{\hat{c} \in \tilde{\mathcal{A}}_\Gamma(x_2))} WC[\Gamma, \hat{c}] \, (\underbrace{\sum_{k \in \hat{c} \cap \mathbb{Z}^{b_1(\Gamma)}} \tilde{P}_\Gamma(x_2, k))}_{=: P^c_\Gamma(x_2)), \phi(c) = \hat{c}}, \text{ where} \\ WC[\Gamma, \hat{c}] &:= \mathrm{sgn}(\hat{c})(m(\hat{c}) - \sum_{\tilde{c} \in \mathcal{BC}(\tilde{\mathcal{A}}_\Gamma(x_1))} m(\tilde{c}) \langle \tilde{c}, \nabla^*_{\Gamma,12}(\hat{c})), \\ \mathrm{sgn}(\hat{c}) &:= \mathrm{sgn}(\tilde{P}_\Gamma(x_2, k)), \text{ and } m(\tilde{c}) := m(\phi^{-1}(\tilde{c})), \end{aligned}$$

and therefore:

$$WC[\Gamma] = \sum_{c\in\mathcal{A}_\Gamma(x_2))} WC[\Gamma,c]P^c_\Gamma(x_2)),\ WC[\Gamma,c] := WC[\Gamma,\hat{c}]$$

since ϕ defines a bijection between the set $\mathcal{BC}(\mathcal{A}_\Gamma(x))$ of bounded chambers in $\mathcal{A}_\Gamma(x))$ and the set $\mathcal{BC}(\tilde{\mathcal{A}}_\Gamma(x))$ of bounded chambers in $\tilde{\mathcal{A}}_\Gamma(x))$.

With this setup Theorem 8.3.36 follows from the discussion in Section 6 and 7 of [23].

Step 2: Clean Wall-Crossing formula. Leveraging our combinatorial understanding of wall crossing, we have decomposed the algebraic expression $WC^r_{g,I}(x)$ according to building blocks of individual x-graphs and arrived at an explicit combinatorial formula that directly reflects their wall-crossing behavior (Step 1). To revert to an algebraic formulation, we consider particularly nice cuts (Definition 8.3.38) that organize contributions from the inner components of $\Gamma_c\backslash E$.

Definition 8.3.38 ([24], Definition 8.1 **).** A cut $T \in Cuts_{\Gamma_c}(I) \subset E(\Gamma_c)$, where every edge is adjacent to either the connected component with ends x_I or the one with ends x_{I^c}, is called a *thin cut*. We say T divides Γ_c into three *parts*, the initial and the final component, and a (possibly disconnected) middle part.

Thin cuts induce a partition of the set of $I-$Cuts

$$Cuts_{\Gamma_c}(I) = \bigcup_T P(T),\ \text{where}\ P(T) := \{E \in Cuts_{\Gamma_c}(I)\ :\ T \subset E\},$$

according to which we split the sum in Theorem 8.3.36 (see Lemma 8.2 [24] as written in [46], Remark 4.14)

$$\sum_{E\in Cuts_{\Gamma_c}(I)} (-1)^{\mathrm{rk}(E)-1}\binom{a}{a_0,a_1,\ldots,a_N,a_{N+1}} = \sum_T (-1)^t\binom{a}{a_0,t,a_{N+1}},$$

where $t := a_1 + \ldots + a_N$. Next, consider the three parts of $\Gamma_c\backslash T$ as $x-$graphs (for smaller counting problems) with adequate orientation inherited from Γ_c. Denote their underlying x-graphs by $\Gamma_1, \Gamma_2, \Gamma_3$ and factor $\frac{1}{|\mathrm{Aut}(\Gamma)|}P_\Gamma$ accordingly:

$$\frac{1}{|\mathrm{Aut}(\Gamma)|}P_\Gamma = \frac{\prod y_i \prod z_i}{y!z!}\prod_{i=1}^{3}\frac{1}{|\mathrm{Aut}(\Gamma_i)|}P_{\Gamma_i},\ \text{where}$$

- y and z are the partitions of edge weights connecting Γ_1 to Γ_2 and Γ_2 to Γ_3.
- $\frac{1}{y!z!}$ comes from forgetting the labels on the edges corresponding to $y-$ and $z-$ends after gluing.

Observing that Γ_1, Γ_2 and Γ_3 contribute to $\mathbb{T}h^{spin,r}_{g(\Gamma_1)}(x_I,-y), \mathbb{T}h^{spin,r,\bullet}_{g(\Gamma_2)}(y,-z)$ and $\mathbb{T}h^{spin,r}_{g(\Gamma_3)}(z,x_{I^c})$, finally paves the way for Theorem 8.3.39.

Theorem 8.3.39. [Clean Wall Crossing formula] For the wall W_I the wall-crossing formula is given by:

$$WC^r_{g,I}(x) = \sum_{(y,z,g_i,a_i)} \binom{a}{a_1, a_2, a_3} \frac{\prod y_i \prod z_i}{y!z!} \mathbb{T}h^{spin,r}_{g_1}(x_I, -y)\mathbb{T}h^{spin,r,\bullet}_{g_2}(y, -z)\mathbb{T}h^{spin,r}_{g_3}(z, x_{I^c}),$$

where

- a is the number of ramification of $(r + 1)-$completed cycle type of the original counting problem $\mathbb{T}h^{spin,r}_g(x)$.
- y, z are partitions such that $\sum y_i = \sum z_i = \sum_{i \in I} x_i$ and define together with integers g_i, a_i smaller counting problems.

Proof. See [24], Section 8. ☐

Remark 8.3.40. Note that a summand for the discrete data $(y, z, g_1, g_2, g_3, a_1, a_2, a_3)$ contributes to $WC^r_{g,I}(x)$ if and only if the respective Riemann-Hurwitz conditions are satisfied.

Bibliography

[1] Jarod Alper. Stacks and moduli, February 2020. URL `https://sites.math.washington.edu/~jarod/`.

[2] Omid Amini and Matthew Baker. Linear series on metrized complexes of algebraic curves. *Math. Ann.*, 362(1-2):55–106, 2015. ISSN 0025-5831,1432-1807. doi: 10.1007/s00208-014-1093-8. URL `https://doi.org/10.1007/s00208-014-1093-8`.

[3] Omid Amini, Matthew Baker, Erwan Brugallé, and Joseph Rabinoff. Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta. *Res. Math. Sci.*, 2:Art. 7, 67, 2015. ISSN 2522-0144,2197-9847. doi: 10.1186/s40687-014-0019-0. URL `https://doi.org/10.1186/s40687-014-0019-0`.

[4] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths. *Geometry of algebraic curves. Volume II. With a contribution by Joseph Daniel Harris*, volume 268 of *Grundlehren Math. Wiss.* Berlin: Springer, 2011. ISBN 978-3-540-42688-2; 978-3-540-69392-5. doi: 10.1007/978-3-540-69392-5.

[5] Matthew Baker. Specialization of linear systems from curves to graphs. *Algebra Number Theory*, 2(6):613–653, 2008. ISSN 1937-0652,1944-7833. doi: 10.2140/ant.2008.2.613. URL `https://doi.org/10.2140/ant.2008.2.613`. With an appendix by Brian Conrad.

[6] Matthew Baker and Xander Faber. Metric properties of the tropical Abel-Jacobi map. *J. Algebraic Combin.*, 33(3):349–381, 2011. ISSN 0925-9899,1572-9192. doi: 10.1007/s10801-010-0247-3.

[7] Matthew Baker and Serguei Norine. Harmonic morphisms and hyperelliptic graphs. *Int. Math. Res. Not. IMRN*, (15):2914–2955, 2009. ISSN 1073-7928,1687-0247. doi: 10.1093/imrn/rnp037.

[8] Dori Bejleri. Moduli spaces in algebraic geometry, 2020. URL `https://math.umd.edu/~dbejleri/`.

[9] Benoît Bertrand, Erwan Brugallé, and Grigory Mikhalkin. Tropical open Hurwitz numbers. *Rend. Semin. Mat. Univ. Padova*, 125:157–171, 2011. ISSN 0041-8994. doi: 10.4171/RSMUP/125-10. URL `https://doi.org/10.4171/RSMUP/125-10`.

[10] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*, volume 302 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2004. ISBN 3-540-20488-1. doi: 10.1007/978-3-662-06307-1. URL `https://doi.org/10.1007/978-3-662-06307-1`.

[11] Spencer Bloch and Andrei Okounkov. The character of the infinite wedge representation. *Adv. Math.*, 149(1):1–60, 2000. ISSN 0001-8708. doi: 10.1006/aima.1999.1845.

[12] Janko Böhm, Kathrin Bringmann, Arne Buchholz, and Hannah Markwig. Tropical mirror symmetry for elliptic curves. *J. Reine Angew. Math.*, 732:211–246, 2017. ISSN 0075-4102,1435-5345. doi: 10.1515/crelle-2014-0143. URL `https://doi.org/10.1515/crelle-2014-0143`.

[13] Barbara Bolognese, Madeline Brandt, and Lynn Chua. From curves to tropical Jacobians and back. In *Combinatorial algebraic geometry*, volume 80 of *Fields Inst. Commun.*, pages 21–45. Fields Inst. Res. Math. Sci., Toronto, ON, 2017. ISBN 978-1-4939-7485-6; 978-1-4939-7486-3.

[14] Silvia Brannetti, Margarida Melo, and Filippo Viviani. On the tropical Torelli map. *Adv. Math.*, 226(3):2546–2586, 2011. ISSN 0001-8708,1090-2082. doi: 10.1016/j.aim.2010.09.011.

[15] Jonathan Brundan and Alexander Kleshchev. Projective representations of symmetric groups via Sergeev duality. *Math. Z.*, 239(1):27–68, 2002. ISSN 0025-5874. doi: 10.1007/s002090100282.

[16] Lucia Caporaso. Gonality of algebraic curves and graphs. In *Algebraic and complex geometry*, volume 71 of *Springer Proc. Math. Stat.*, pages 77–108. Springer, Cham, 2014. ISBN 978-3-319-05403-2. doi: 10.1007/978-3-319-05404-9_4.

[17] Lucia Caporaso and Filippo Viviani. Torelli theorem for graphs and tropical curves. *Duke Math. J.*, 153(1):129–171, 2010. ISSN 0012-7094,1547-7398. doi: 10.1215/00127094-2010-022. URL `https://doi.org/10.1215/00127094-2010-022`.

[18] Lucia Caporaso, Margarida Melo, and Marco Pacini. Tropicalizing the moduli space of spin curves. *Selecta Mathematica*, 26(1):16, 2020.

[19] Renzo Cavalieri. Moduli spaces of pointed rational curves, 2016. URL `https://sites.google.com/view/renzocavalieri/`.

[20] Renzo Cavalieri and Eric Miles. *Riemann Surfaces and Algebraic Curves, A First Course in Hurwitz Theory*. University Lecture Series.

[21] Renzo Cavalieri, Melody Chan, Martin Ulirsch, and Jonathan Wise. A moduli stack of tropical curves. *Forum Math. Sigma*, 8. ISSN 2050-5094.

[22] Renzo Cavalieri, Paul Johnson, and Hannah Markwig. Tropical Hurwitz numbers. *J. Algebr. Comb.*, 32(2):241–265, 2010. ISSN 0925-9899. doi: 10.1007/s10801-009-0213-0.

[23] Renzo Cavalieri, Paul Johnson, and Hannah Markwig. Chamber structure for double Hurwitz numbers. In *Proceedings of the 22nd annual international conference on formal power series and algebraic combinatorics, FPSAC 2010, San Francisco, USA, August 2–6, 2010*, pages 227–238. Nancy: The Association. Discrete Mathematics & Theoretical Computer Science (DMTCS), 2010. URL `www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/view/dmAN0109`.

[24] Renzo Cavalieri, Paul Johnson, and Hannah Markwig. Wall crossings for double Hurwitz numbers. *Adv. Math.*, 228(4):1894–1937, 2011. ISSN 0001-8708,1090-2082. doi: 10.1016/j.aim.2011.06.021.

[25] Renzo Cavalieri, Simon Hampe, Hannah Markwig, and Dhruv Ranganathan. Moduli spaces of rational weighted stable curves and tropical geometry. *Forum Math. Sigma*, 4: 35, 2016. ISSN 2050-5094. doi: 10.1017/fms.2016.7. Id/No e9.

[26] Renzo Cavalieri, Hannah Markwig, and Dhruv Ranganathan. Tropicalizing the space of admissible covers. *Math. Ann.*, 364(3-4):1275–1313, 2016. ISSN 0025-5831. doi: 10.1007/s00208-015-1250-8. URL `https://doi.org/10.1007/s00208-015-1250-8`.

[27] Renzo Cavalieri, Hannah Markwig, and Dhruv Ranganathan. Tropical compactification and the Gromov-Witten theory of $\mathbb{P}^1$. *Sel. Math., New Ser.*, 23(2):1027–1060, 2017. ISSN 1022-1824. doi: 10.1007/s00029-016-0265-7.

[28] Renzo Cavalieri, Paul Johnson, Hannah Markwig, and Dhruv Ranganathan. A graphical interface for the Gromov-Witten theory of curves. In *Algebraic geometry: Salt Lake City 2015. 2015 summer research institute in algebraic geometry, University of Utah, Salt Lake City, UT, USA, July 13–31, 2015. Proceedings. Part 2*, pages 139–167. Providence, RI: American Mathematical Society (AMS); Cambridge, MA: Clay Mathematics Institute, 2018. ISBN 978-1-4704-3578-3; 978-1-4704-4667-3; 978-1-4704-4680-2. doi: 10.1090/pspum/097.2/01702. URL `eprints.whiterose.ac.uk/133417/1/GWGUI.pdf`.

[29] Melody Chan. Combinatorics of the tropical Torelli map. *Algebra Number Theory*, 6(6): 1133–1169, 2012. ISSN 1937-0652,1944-7833. doi: 10.2140/ant.2012.6.1133.

[30] Melody Chan. Lectures on tropical curves and their moduli spaces. In *Moduli of curves. CIMAT Guanajuato, Mexico 2016. Lecture notes of a CIMPA-ICTP school, Guanajuato, Mexico, February 22 – March 4, 2016*, pages 1–26. Cham: Springer, 2017. ISBN 978-3-319-59485-9; 978-3-319-59486-6. doi: 10.1007/978-3-319-59486-6_1.

[31] Lou-Jean Leila Cobigo. Tropical spin hurwitz numbers, 2022. available on the webpage https://www.math.uni-tuebingen.de/de/forschung/kombinatorische-algebraische-geometrie/personen/hannah-markwig/arbeitsgruppe/loujean-cobigo.

[32] Lou-Jean Leila Cobigo. Tropical Split Jacobians of genus 2 and optimal covers. Preprint, arXiv:2410.13459 (2024), 2024. URL https://arxiv.org/abs/2410.13459.

[33] Lou-Jean Leila Cobigo. Tropical split jacobians of curves of genus 2 ii, 2025. URL https://arxiv.org/abs/2502.05624.

[34] Jesús A. De Loera, Laura Escobar, Nathan Kaplan, and Chengyang Wang. Sums of Weighted Lattice Points of Polytopes. Preprint, arXiv:2402.11328 [math.CO] (2024), 2024. URL https://arxiv.org/abs/2402.11328.

[35] Olivier Debarre. *Tores et variétés abéliennes complexes*, volume 6 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris; EDP Sciences, Les Ulis, 1999. ISBN 2-86883-427-2.

[36] Kevin D. Doerksen. *On the arithmetic of genus 2 curves with (4,4)-split Jacobians*. PhD thesis, Simon Fraser University, 2011.

[37] Alex Eskin, Andrei Okounkov, and Rahul Pandharipande. The theta characteristic of a branched covering. *Adv. Math.*, 217(3):873–888, 2008. ISSN 0001-8708. doi: 10.1016/j.aim.2006.08.001.

[38] Gerhard Frey and Ernst Kani. Curves of genus 2 covering elliptic curves and an arithmetical application. In *Arithmetic algebraic geometry (Texel, 1989)*, volume 89 of *Progr. Math.*, pages 153–176. Birkhäuser Boston, Boston, MA, 1991. ISBN 0-8176-3513-0. doi: 10.1007/978-1-4612-0457-2_7.

[39] Andreas Gathmann. Lecture notes in algebraic geometry, 2002/2003. URL https://math.rptu.de/ags/agag/personen/leitung/gathmann.

[40] Alessandro Giacchetto, Reinier Kramer, and Danilo Lewański. A new spin on Hurwitz theory and ELSV via theta characteristics. Preprint, arXiv:2104.05697 [math-ph] (2021), 2021. URL https://arxiv.org/abs/2104.05697.

[41] Alessandro Giacchetto, Reinier Kramer, Danilo Lewański, and Adrien Sauvaget. The spin Gromov–Witten/Hurwitz correspondence for $\mathbb{P}^1$. *J. Eur. Math. Soc.*, January 2025. ISSN 1435-9863. doi: 10.4171/jems/1588. URL http://dx.doi.org/10.4171/JEMS/1588.

[42] Ernesto Girrondo and Gabino Gonzalez-Diez. *Introduction to Compact Riemann Surfaces and Dessins d'Enfants.* Cambridge University Press, 2012.

[43] I. P. Goulden, D. M. Jackson, and R. Vakil. Towards the geometry of double Hurwitz numbers. *Adv. Math.*, 198(1):43–92, 2005. ISSN 0001-8708. doi: 10.1016/j.aim.2005.01.008.

[44] E. Goursat. Sur les substitutions orthogonales et les divisions régulières de l'espace. *Ann. Sci. Éc. Norm. Supér. (3)*, 6:9–102, 1889. ISSN 0012-9593. doi: 10.24033/asens.317.

[45] Sam Gunningham. Spin hurwitz numbers and topological quantum field theory. *Geom. & Topol.*, 20(4):1859–1907, 2016.

[46] Marvin Anas Hahn and Danilo Lewański. Wall-crossing and recursion formulae for tropical Jucys covers. *Trans. Am. Math. Soc.*, 373(7):4685–4711, 2020. ISSN 0002-9947. doi: 10.1090/tran/8006.

[47] Victoria Hoskins. Moduli problems and geometric invariant theory, 2015/2016.

[48] V. N. Ivanov. Gaussian limit for projective characters of large symmetric groups. *J. Math. Sci., New York*, 121(3):2330–2344, 2001. ISSN 1072-3374. doi: 10.1023/B:JOTH.0000024615.07311.fe.

[49] David Jensen and Yoav Len. Tropicalization of theta characteristics, double covers, and Prym varieties. *Selecta Math. (N.S.)*, 24(2):1391–1410, 2018. ISSN 1022-1824,1420-9020. doi: 10.1007/s00029-017-0379-6. URL `https://doi.org/10.1007/s00029-017-0379-6`.

[50] Paul Johnson. Double Hurwitz numbers via the infinite wedge. *Trans. Am. Math. Soc.*, 367(9):6415–6440, 2015. ISSN 0002-9947. doi: 10.1090/S0002-9947-2015-06238-2.

[51] Victor G. Kac, Ashok K. Raina, and Natasha Rozhkovskaya. *Bombay lectures on highest weight representations of infinite dimensional Lie algebras*, volume 29 of *Adv. Ser. Math. Phys.* Hackensack, NJ: World Scientific, 2nd ed. edition, 2013. ISBN 978-981-4522-18-2; 978-981-4522-19-9; 978-981-4522-21-2. doi: 10.1142/8882.

[52] Eric Katz, Hannah Markwig, and Thomas Markwig. The j-invariant of a plane tropical cubic. *J. Algebra*, 320(10):3832–3848, 2008. ISSN 0021-8693,1090-266X. doi: 10.1016/j.jalgebra.2008.08.018. URL `https://doi.org/10.1016/j.jalgebra.2008.08.018`.

[53] Robert M. Kuhn. Curves of genus 2 with split Jacobian. *Trans. Amer. Math. Soc.*, 307(1):41–49, 1988. ISSN 0002-9947,1088-6850. doi: 10.2307/2000749.

[54] Junho Lee. Degree-three spin Hurwitz numbers. *Pac. J. Math.*, 263(2):399–417, 2013. ISSN 1945-5844. doi: 10.2140/pjm.2013.263.399.

[55] Junho Lee. A square root of Hurwitz numbers. *Manuscr. Math.*, 162(1-2):99–113, 2020. ISSN 0025-2611. doi: 10.1007/s00229-019-01113-0.

[56] Junho Lee and Thomas H. Parker. Spin Hurwitz numbers and the Gromov-Witten invariants of Kähler surfaces. *Comm. Anal. Geom.*, 21(5):1015–1060, 2013. ISSN 1019-8385. doi: 10.4310/CAG.2013.v21.n5.a6. URL `https://doi.org/10.4310/CAG.2013.v21.n5.a6`.

[57] Yoav Len and Martin Ulirsch. Skeletons of Prym varieties and Brill-Noether theory. *Algebra Number Theory*, 15(3):785–820, 2021. ISSN 1937-0652,1944-7833. doi: 10.2140/ant.2021.15.785.

[58] Yoav Len and Dmitry Zakharov. Kirchhoff's theorem for Prym varieties. *Forum Math. Sigma*, 10:Paper No. e11, 54, 2022. ISSN 2050-5094. doi: 10.1017/fms.2021.75. URL `https://doi.org/10.1017/fms.2021.75`.

[59] Kay Magaard, Tanush Shaska, and Helmut Völklein. Genus 2 curves that admit a degree 5 map to an elliptic curve. *Forum Math.*, 21(3):547–566, 2009. ISSN 0933-7741. doi: 10.1515/FORUM.2009.027.

[60] Jiří Matoušek and Bernd Gärtner. *Understanding and using linear programming*. Universitext. Berlin: Springer, 2007. ISBN 3-540-30697-8. doi: 10.1007/978-3-540-30717-4.

[61] Grigory Mikhalkin. Tropical geometry and its applications. In *International Congress of Mathematicians. Vol. II*, pages 827–852. Eur. Math. Soc., Zürich, 2006. ISBN 978-3-03719-022-7.

[62] Grigory Mikhalkin and Ilia Zharkov. Tropical curves, their Jacobians and theta functions. In *Curves and abelian varieties*, volume 465 of *Contemp. Math.*, pages 203–230. Amer. Math. Soc., Providence, RI, 2008. ISBN 978-0-8218-4334-5. doi: 10.1090/conm/465/09104.

[63] J. S. Milne. Abelian varieties. Arithmetic geometry, Pap. Conf., Storrs/Conn. 1984, 103-150 (1986)., 1986.

[64] Ellena Moskovsky. *Enumerative geometry, topological recursion, and the semi-infinite wedge.* PhD thesis, Monash University, 2022.

[65] David Mumford. *Abelian varieties*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics.* Tata Institute of Fundamental Research, Bombay; by Oxford University Press, London, 1970.

[66] A. Okounkov and R. Pandharipande. Gromov-Witten theory, Hurwitz theory, and completed cycles. *Ann. Math. (2)*, 163(2):517–560, 2006. ISSN 0003-486X. doi: 10.4007/annals.2006.163.517.

[67] João F. Queiró and Eduardo M. Sá. Singular values and invariant factors of matrix sums and products. *Linear Algebra Appl.*, 225:43–56, 1995. ISSN 0024-3795,1873-1856. doi: 10.1016/0024-3795(93)00317-S. URL https://doi.org/10.1016/0024-3795(93)00317-S.

[68] Dhruv Ranganathan. *Skeletons, degenerations, & Gromov–Witten theory.* PhD thesis, Yale University, 2016.

[69] Felix Röhrle and Dmitry Zakharov. The tropical n-gonal construction. *Algebr. Comb.*, 8(2):319–378, 2025. ISSN 2589-5486. doi: 10.5802/alco.410.

[70] Johannes Schmitt. The moduli space of curves, September 2020.

[71] E. Selling. On binary and ternary quadratic forms. *J. Reine Angew. Math.*, 77:143–229, 1873. ISSN 0075-4102. URL https://eudml.org/doc/148235.

[72] S. Shadrin, L. Spitz, and D. Zvonkine. On double Hurwitz numbers with completed cycles. *J. Lond. Math. Soc., II. Ser.*, 86(2):407–432, 2012. ISSN 0024-6107. doi: 10.1112/jlms/jds010.

[73] Elira Shaska and Tanush Shaska. Machine learning for moduli space of genus two curves and an application to isogeny-based cryptography, February 2025. ISSN 1572-9192. URL http://dx.doi.org/10.1007/s10801-025-01393-8.

[74] T. Shaska. Genus 2 fields with degree 3 elliptic subfields. *Forum Math.*, 16(2):263–280, 2004. ISSN 0933-7741. doi: 10.1515/form.2004.013.

[75] R. Vakil. The moduli space of curves and Gromov-Witten theory. In *Enumerative invariants in algebraic geometry and string theory. Lectures given at the C.I.M.E. summer school, Cetraro, Italy, June 6–11, 2005*, pages 143–198. Berlin: Springer, 2008. ISBN 978-3-540-79813-2. doi: 10.1007/978-3-540-79814-9_4.

[76] R. Vakil. The Rising Sea, foundations of algebraic geometry, 2017.

[77] F. Vallentin. *Sphere coverings, Lattices, and Tilings (in low dimensions).* PhD thesis, Technische Universität München, 2003.

[78] A. N. Varchenko. Combinatorics and topology of the situation of affine hyperplanes in real space. *Funct. Anal. Appl.*, 21(1-3):9–19, 1987. ISSN 0016-2663. doi: 10.1007/BF01077981.

[79] Sven Verdoolaege, Rachid Seghir, Kristof Beyls, Vincent Loechner, and Maurice Bruynooghe. Counting integer points in parametric polytopes using Barvinok's rational functions. *Algorithmica*, 48(1):37–66, 2007. ISSN 0178-4617. doi: 10.1007/s00453-006-1231-0. URL lirias.kuleuven.be/handle/123456789/124371.

[80] M. Wheeler. Free fermions in classical and quantum integrable models. Preprint, arXiv:1110.6703 [math-ph] (2011), 2011. URL `https://arxiv.org/abs/1110.6703`.

[81] Ilia Zharkov. Tropical theta characteristics. 527:165–168, 2010. doi: 10.1090/conm/527/10403. URL `https://doi.org/10.1090/conm/527/10403`.