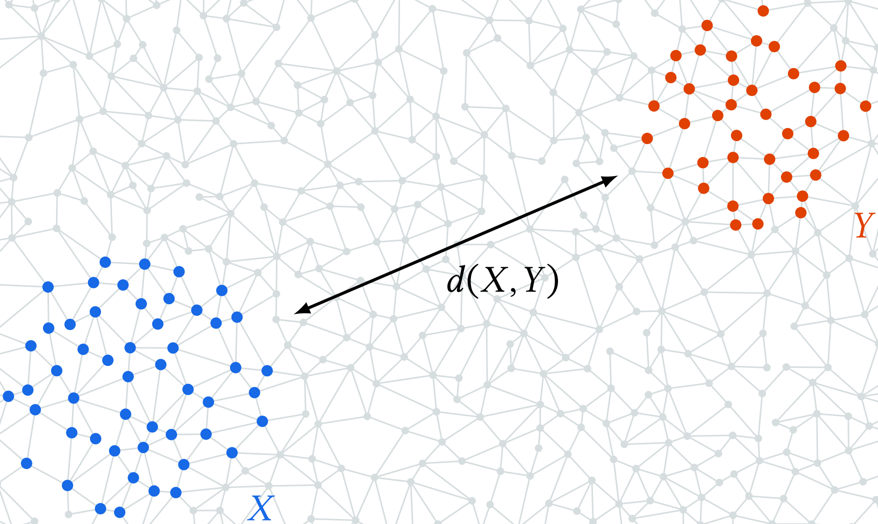
Locality in Quantum Lattice Systems

Tom Wessel



Locality in Quantum Lattice Systems

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Tom Wessel
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Abstract

This thesis analyses several locality properties of extended quantum lattice systems, considering both spins and fermions. We begin by assuming locality of the Hamiltonian and then discuss how this translates into locality of the time evolution and the thermal states.

Starting from a local Hamiltonian, it is well known that the Heisenberg time evolution satisfies Lieb-Robinson bounds, which quantify its locality. We strengthen these bounds for long-range interacting fermions and for commuting interactions. Furthermore, we establish the first Lieb-Robinson bounds for Hamiltonians whose local terms can grow linearly away from the origin. We also prove stability of Lieb-Robinson bounds with respect to local perturbations in the Hamiltonian.

For the thermal states we focus on three types of locality known as *decay of correlations*, the *local perturbations perturb locally* (LPPL) principle and *local indistinguishability*. As a key tool we use spectral filters, which were introduced by Hastings in the 2000s for various applications. It enable us to use Lieb-Robinson bounds to obtain information about locality properties of states. In particular, we use *quantum belief propagation* to prove an equivalence of decay of correlations, LPPL and local indistinguishability for Gibbs states at any positive temperature. Using the improved Lieb-Robinson bounds mentioned before, we obtain improvements for decay of correlations in the ground state, the locality of the spectral flow and LPPL for long-range interacting systems and commuting Hamiltonians. Furthermore, we adjust the filter functions to achieve exponential decay for certain applications that were previously limited to stretched exponential decay.

Spectral filters are also used to rigorously justify response theory, which usually requires a gapped system to begin with. To relax this assumption, we introduce various notions of *local gaps*, discuss their relations and use one to justify response theory and Kubo's formula in the absence of a global gap.

In another result we use local indistinguishability as an input to prove bulk-edge correspondence at finite temperature, in the sense that the bulk magnetisation is equal to the edge current. This result has mostly been studied for non-interacting models or at zero temperature before.

Finally, we present a class of weakly interacting spin systems with an on-site gap and prove that the ground state satisfies a *strong* form of LPPL. Moreover, we show that these systems satisfy decay of correlations and local indistinguishability uniformly in temperature.

Zusammenfassung

Diese Arbeit analysiert mehrere Lokalitätseigenschaften von ausgedehnten Quanten-Gittersystemen, wobei wir Spin Systeme und Fermionen betrachten. Wir gehen zunächst von der Lokalität des Hamiltonian aus und analysieren dann, wie sich diese auf die Lokalität der Zeitentwicklung und der thermischen Zustände auswirkt.

Für lokale Hamiltonians ist die Lokalität der Heisenberg-Zeitentwicklung bekannt und wird mittels Lieb-Robinson Schranken quantifiziert. Wir verbessern diese Schranken für Fermionen mit langreichweitigen Wechselwirkungen und für Systeme mit kommutierenden Wechselwirkungen. Darüber hinaus zeigen wir die ersten Lieb-Robinson Schranken für Hamiltonians, deren lokale Terme im Abstand zum Ursprung linear anwachsen. Wir beweisen auch die Stabilität der Lieb-Robinson Schranken in Bezug auf lokale Störungen im Hamiltonian.

Für die thermischen Zustände konzentrieren wir uns auf drei Arten von Lokalität, die als *Korrelationsabfall* (decay of correlations), *lokale Störungen stören lokal* (local perturbations perturb locally, LPPL) und *lokale Ununterscheidbarkeit* (local indistinguishability) bekannt sind. Als wichtigstes Werkzeug verwenden wir spektrale Filter, die von Hastings in den 2000er Jahren für verschiedene Anwendungen eingeführt wurden. Sie ermöglichen es, Lieb-Robinson Schranken zu verwenden, um Informationen über die Lokalitätseigenschaften von Zuständen zu erhalten. Insbesondere verwenden wir *Quanten Glaubensausbreitung* (quantum belief propagation), um eine Äquivalenz zwischen dem Korrelationsabfall, LPPL und lokaler Ununterscheidbarkeit für Gibbs-Zustände bei jeder positiven Temperatur zu beweisen. Mit den zuvor erwähnten verbesserten Lieb-Robinson Schranken erhalten wir auch Verbesserungen für den Korrelationsabfall im Grundzustand, die Lokalität des Spektralflusses und LPPL für langreichweitige Wechselwirkungen und kommutierende Hamiltonians. Außerdem verändern wir die Filterfunktionen, um einen exponentiellen Abfall für bestimmte Anwendungen zu bekommen, die zuvor nur mit gestrecktem exponentiellen Abfall bekannt waren.

Spektrale Filter werden auch verwendet, um die lineare Antworttheorie mathematisch zu rechtfertigen, was meist ein System mit spektraler Lücke erfordert. Um diese Annahme zu lockern, führen wir verschiedene Begriffe der *lokalen Lücke* ein, diskutieren deren Relationen und nutzen eine Variante, um die lineare Antworttheorie und Kubos Formel in Abwesenheit einer globalen Lücke zu rechtfertigen.

In einem weiteren Ergebnis verwenden wir die lokale Ununterscheidbarkeit als Annahme, um Volumen-Rand-Korrespondenz bei positiver Temperatur zu zeigen. Genauer gesagt zeigen wir, dass die Volumen-Magnetisierung gleich dem Randstrom ist.

Dieses Ergebnis wurde bisher hauptsächlich für nicht wechselwirkende Modelle oder im Grundzustand untersucht.

Schließlich stellen wir eine Klasse von schwach wechselwirkenden Spin Systemen vor, deren Einzelsysteme an jedem Gitterpunkt ohne die Wechselwirkung eine spektrale Lücke aufweisen, und beweisen, dass der Grundzustand eine *starke* Form von LPPL erfüllt. Darüber hinaus zeigen wir, dass diese Systeme bei positiver Temperatur Korrelationsabfall und lokale Ununterscheidbarkeit erfüllen, und zwar mit Konstanten die unabhängig von der Temperatur sind.

Acknowledgements

First and foremost, I am deeply grateful to my supervisor, Stefan Teufel, for our many discussions, for encouraging me throughout the past few years and for sharing his extensive knowledge of mathematics and physics.

I would also like to express my thanks to my second supervisor, Marius Lemm, for his invaluable support and for initiating collaborative projects that enriched my research.

This work would not have been possible without the many collaborators and colleagues who engaged in stimulating discussions and shared their insights. I would particularly like to mention Ángela Capel and Massimo Moscolari, with whom I had many discussions, and who enabled me to participate in a variety of scientific activities. Special thanks go to Sven Bachmann, who hosted me in Vancouver for three months and shared his ideas, which was both professionally and personally enriching. Thank also go Joscha Henheik and Marius Wesle for our discussions about quantum lattice systems.

Many thanks also go to my office mates, for our daily discussions and friendship, and to the entire Mathematical Physics group in Tübingen, including everyone I met during my time there. The cooking group in particular made lunchtimes much more enjoyable, healthy and flavourful. I would also like to thank everyone in the community, especially within our joint research units, for their warmth and welcoming attitude.

I would also like to acknowledge the Deutsche Forschungsgemeinschaft for their financial support, which enabled me to attend numerous conferences and expand my scientific knowledge.

Lastly, I am deeply thankful to my family for their love, encouragement, and patience throughout this journey.

Contents

List of publications

P1 **Local stability of ground states in locally gapped and weakly interacting quantum spin systems**.
Joscha Henheik, Stefan Teufel, and Tom Wessel.
Published in Letters in Mathematical Physics **112**, 9 (2022).

P2 **On adiabatic theory for extended fermionic lattice systems**.
Joscha Henheik and Tom Wessel.
Published in Journal of Mathematical Physics **63**, 121101 (2022).

P3 **Equality of magnetization and edge current for interacting lattice fermions at positive temperature**.
Jonas Lampart, Massimo Moscolari, Stefan Teufel, and Tom Wessel.
Published in Mathematical Physics, Analysis and Geometry **27**, 24 (2024).

P4 **From decay of correlations to locality and stability of the Gibbs state**.
Ángela Capel, Massimo Moscolari, Stefan Teufel, and Tom Wessel.
Published in Communications in Mathematical Physics **406**, 43 (2025).

P5 **Enhanced Lieb-Robinson bounds for commuting long-range interactions**.
Marius Lemm and Tom Wessel.
Published in Journal of Mathematical Physics **66**, 091901 (2025).

P6 **Response theory for locally gapped systems**.
Joscha Henheik and Tom Wessel.
Preprint available at arXiv:2410.10809 (2024).

P7 **Lieb-Robinson bounds, automorphic equivalence and LPPL for long-range interacting fermions**.
Stefan Teufel and Tom Wessel.
Preprint available at arXiv:2507.03319 (2025).

P8 **Uniform-in-temperature locality estimates for weakly interacting quantum systems**.
Arka Adhikari, Joscha Henheik, Marius Lemm, and Tom Wessel.
Preprint available at arXiv:2508.15907 (2025).

P9 **Gaussian filters in quantum lattice systems: Applications to spectral flow, local perturbations, clustering, and the quantum Hall effect**.
Sven Bachmann, Zhiqian (Simon) Du, Martin Fraas, and Tom Wessel.
Preprint available at arXiv:2508.15913 (2025).

P10 **Dynamics generated by spatially growing derivations on quasi-local algebras**.
Stefan Teufel, Marius Wesle, and Tom Wessel.
Preprint available at arXiv:2511.02941 (2025).

Personal contribution

The authors are listed in alphabetical order in all the publications. Most of the ideas underlying the proofs were developed through multiple discussions. Furthermore, the presentation of the results was typically refined several times to enhance their readability and applicability. As this process included all authors, it is difficult to attribute all contributions in detail.

Publication P1 started as a project towards the end of my master's thesis. As far as I can remember, Joscha Henheik suggested looking at the original work by Yarotsky [219]. I then did the initial estimates and formulated a result like theorem P1-3 that already allowed for arbitrary perturbations at the boundary of Λ. Together, we proved corollary P1-5 and improved the proof and presentation.

Publication P2 is mostly a review of existing results and was written by Joscha Henheik and me in equal parts. It provides a good starting point to understand the expansion underlying the generalized super adiabatic theorems in a unified notation and was the foundation for publication P6.

Publication P3 started while Massimo Moscolari was still in Tübingen. Inspired by a similar result for non-interaction fermions in the continuum by Massimo Moscolari, Stefan Teufel and a co-author [65], we together discussed what we expect to hold based on a local indistinguishability assumption in the bulk. I then did most of the technical estimates for theorems P3-I to P3-III. The proof of theorem P3-IV was mostly contributed by Jonas Lampart.

Publication P4 dropped out from a reading group in Tübingen. The general idea for quantum belief propagation (QBP) and results proving local indistinguishability using it were already in the literature. In a joint effort we understood these results, filled the gaps and composed a rigorous proof. Additionally, I realized how to use QBP to handle non-finite-range interactions and remove the scaling with $|\partial \Lambda'|$, which had been there in previous results. Moreover, I came up with the stability of Lieb-Robinson bounds against local perturbations, lemma P4-42.

Publication P5 was initiated by Marius Lemm presenting the strong Lieb-Robinson bounds for commuting finite-range interactions in Tübingen. Together, we came up with an initial proof for long-range interactions and simplified it to obtain the included proof, which even holds for decay $\alpha > 0$ on finite lattices, theorem P5-6. I then added the proofs for the exemplary applications and the sharpness result, which was requested by a referee.

Publication P6 was initiated in multiple discussions with Joscha Henheik and Stefan Teufel, about response theory within gapped parts of a system and proper notions for such local gaps. The locality preserving response theory and the necessary technical estimates from appendix P6-A were mostly provided by me. In collaboration with Joscha Henheik we adjusted the proofs of response theory to work with the local dynamical gap condition. Many of the characterisations of local gaps were developed in multiple discussions. Joscha Henheik proved the implications between them, and figured out the explicit examples.

Publication P7 is based on similar previous results [160, 84]. It is mostly written by me, with regular feedback from Stefan Teufel.

Publication P8 was written by all authors quite collaboratively. The two important ingredients are an analytic bound from [217] and the cluster expansion and swapping trick, which was contributed by Arka Adhikari. We together developed the proof of decay of correlations to the point where it is now. The modifications to prove local indistinguishability and LPPL where mostly done by myself.

Publication P9 started while I was visiting Sven Bachmann in Vancouver. He originally proved the exact automorphic equivalence [23]. I contributed most of sections P9-3 and P9-4, based on ideas from [120], regularly discussing with Sven Bachmann. Simon Du, a PhD student of Martin Fraas, contributed the proofs for section P9-5, on which they started working independently of Sven Bachmann and me. Sven Bachmann combined everything and wrote most of section P9-6, based on [14].

Publication P10 started when Marius Wesle and Stefan Teufel brought up the problem in the context of automorphic equivalence along paths of gapped Hamiltonians with varying magnetic field. After some discussions with Marius Wesle, I realized that we can prove Lieb-Robinson bounds for strictly local observables and short times. Marius Wesle then came up with a generalization for polynomially decaying observables and a formal proof to show existence of the automorphism for all times. Together, we improved the result to the current form that also provides exponential light cones.

Overview

1 Introduction

Locality is an important feature of quantum many-body systems. Both from a physical perspective and as a property which makes rigorous analysis possible. In this thesis we study locality properties of quantum many-body systems on extended lattices. These systems arise as simplifications of continuous theories for example by a tight-binding approximation. More precisely, we consider the two classes of lattice spin systems and fermionic lattice systems. The former are widely studied in quantum information theory and mathematical physics. We understand them as a fixed lattice of particles, where each particle has internal degrees of freedom, describing for example the intrinsic spins of atoms at the lattice sites. The interaction of the system is given by a coupling of these internal degrees of freedom. Such systems can arise in condensed matter theory as crystalline systems, serve as toy models for magnetism and quantum computers or quantum memory, and are experimentally implemented using optical lattices. Lattice fermions, on the other hand, model interacting particles that can jump between lattice sites, which might have some extra internal degrees of freedom. Importantly, by the Pauli exclusion principle, every lattice site (or internal degree of freedom) can only be occupied by one particle. These systems serve as models for electrons in a homogeneous structure like metals, semiconductors and insulators. A similar physical interpretation as a lattice gas also exists for quantum spin systems, where one interprets the internal degree of freedom as counting the number of particles present at the site.

Quantum lattice systems are particularly important for the understanding of the (fractional) quantum Hall effect, adiabatic and response theory for extended quantum systems, (topological) phases of matter, quantum phase transitions, thermalization, state preparation and quantum computers.

As we are interested in locality properties, we consider systems where the Hamiltonian is given by a sum of local terms (SLT) operator

$$H_\Lambda = \sum_{Z \subset \Lambda} \Phi(Z),$$

where each $\Phi(Z)$ acts only on $Z \subset \Lambda$ and Λ (for now) is a finite lattice. Moreover, we require that $\|\Phi(Z)\|$ decays in the size of Z. The main objective of this thesis is to understand how the locality of the Hamiltonian transfers to locality of the time evolution and the thermal states, and which consequences this locality has for physically relevant questions.

Note that, while we mostly consider finite lattices, all definitions and statements are to be understood to hold uniformly in the lattice size, for example for all finite lattices $\Lambda \Subset \mathbb{Z}^D$. In many cases it is then possible to take the thermodynamic limit.

Lieb-Robinson bounds

One important result concerning lattice systems is the locality of the Heisenberg time evolution $\tau_t^\Lambda(A) = \mathrm{e}^{\mathrm{i}H_\Lambda t} A \, \mathrm{e}^{-\mathrm{i}H_\Lambda t}$ generated by H_Λ. Its locality is characterized in terms of Lieb-Robinson bounds, which provide upper bounds on the commutator

$$\|[\tau_t^\Lambda(A), B]\|$$

that decay in the distance between the supports of the observables A and B. As a first consequence, they allow to rigorously define the dynamics on infinite lattices Γ as the limit

$$\tau_t^\Gamma(A) := \lim_{\Lambda \nearrow \Gamma} \tau_t^\Lambda(A)$$

of the dynamics on finite lattices $\Lambda \Subset \Gamma$ [see e.g. 174], where the limit can be taken along any increasing and exhausting sequence $(\Lambda_n) \subset \Gamma$.

Lieb-Robinson bounds were first proven by Lieb and Robinson [154] for translation invariant finite-range interaction. Since then, there have been many results for spin systems [118, 56, 174] and recently an interest in better bounds [89, 160, 61, 84, 149, 210] for *long-range* systems where $\|\Phi(Z)\|$ decays polynomially in $\operatorname{diam}(Z)$. Similar results are known also for fermionic lattice systems [173, 56]. Unfortunately, some of the results, in particular for long-range interactions, do not hold for fermions or have other restrictions.

In publication P7 we prove better bounds for long-range fermionic systems (which also hold for spin systems). Moreover, in publication P5 we prove enhanced bounds for commuting interactions, focussing on long-range commuting interactions. In particular, we show that commuting interactions obey a linear light cone for a broader range of parameters. In publication P10 we provide the first Lieb-Robinson bounds for interactions whose local terms are allowed to grow linearly away from a fixed lattice site, which in particular allows concluding global existence and uniqueness of the infinite volume dynamics. As a technical requirement for some results, we also prove stability of Lieb-Robinson bounds with respect to localized perturbations added to the Hamiltonian.

While Lieb-Robinson bounds are interesting in its own because they bound information and particle propagation, they are also one of the most important ingredients in the further analysis of lattice systems in general and their locality in particular.

Locality measures for the states

Before we continue discussing the implications of Lieb-Robinson bounds, let us briefly introduce the locality properties for thermal states that we are using within this thesis. There are many ways to quantify locality of states or quantum systems more generally like decay of the covariance (also known as clustering), decay of conditional entropies or (conditional) mutual information, local topological quantum order, and local indistinguishability. We focus on properties of ground and Gibbs states and in particular on decay of correlations, local indistinguishability and the local perturbations perturb locally (LPPL) principle.

Decay of correlations, means that the truncated correlation function or covariance

$$\mathrm{Cov}_{\rho}(A, B) := \mathrm{tr}(\rho\, A\, B) - \mathrm{tr}(\rho\, A)\ \mathrm{tr}(\rho\, B)$$

decays in the distance between the supports of the observables A and B.

Local indistinguishability is the property that expectation values of local observables B in the ground or Gibbs state ρ_{Λ} at inverse temperature β of the Hamiltonian restricted to different lattices $\Lambda' \subset \Lambda$ almost agree. More precisely,

$$\left|\mathrm{tr}\big(\rho_{\Lambda}\, B\big) - \mathrm{tr}\big(\rho_{\Lambda'}\, B\big)\right|$$

decays in the distance between the support of B and $\Lambda \setminus \Lambda'$.

LPPL quantifies how much expectation values of local observables B change, if the Hamiltonian is perturbed by some V supported far away from B. More precisely, a system satisfies LPPL if

$$\left|\mathrm{tr}\big(\rho[H_{\Lambda}]\, B\big) - \mathrm{tr}\big(\rho[H_{\Lambda} + V]\, B\big)\right|$$

decays in the distance between the supports of B and V.

Automorphic equivalence, response theory and LPPL

As one important application, Lieb-Robinson bounds allow quantifying locality of certain functions of the spectrum using so-called *spectral filters*. For gapped Hamiltonians, these are used to prove locality of the spectral flow and automorphic equivalence [120, 23, 167, 31], which underlie the analysis and definition of gapped phases of matter, symmetry protected phases of matter, the adiabatic theorem and the rigorous justification of response theory [17, 165, 203, 32], the quantum Hall effect [13, 216, 204], and the local perturbation perturb locally (LPPL) principle [23, 74].

In publication P7, we use the improved Lieb-Robinson bounds to proof locality of the spectral flow and thus automorphic equivalence and LPPL for long-range interactions. And with the enhanced Lieb-Robinson bounds for commuting Hamiltonians we also obtain qualitatively better decay of LPPL for these systems in publication P5.[1] Moreover, in publication P9 we consider a modified spectral filter to prove exponential decay of an almost spectral flow for exponentially decaying interactions, which for example allows concluding LPPL with exponential decay.

Using the standard filter, we rigorously justify response theory in publication P6 under the assumption of a local gap in a certain sense that goes beyond the usual

[1] We actually use the results from [214], which avoid the use of the spectral flow, to obtain the LPPL bound in publication P5.

definition of a bulk gap and also works on finite systems. As the notion of a local gap that we use is quite specific, we also discuss other possible definitions of local gaps and their relations.

Decay of correlations for gapped ground states

Another spectral filter allows proving decay of correlations for gapped ground states [113, 118, 171, 14]. Using the improved Lieb-Robinson bounds, we obtain qualitatively better decay of correlations in publication P5 for gapped ground states of commuting Hamiltonians, where the correlation length is independent of the gap. Moreover, in publication P9 we use a modified spectral filter to obtain exponential decay also for ground state sectors with a ground state energy splitting in finite systems, while previously used spectral filters only provide superpolynomial decay in finite volumes in this case. Using a complex analysis proof, [214] prove a similar result.

Quantum belief propagation

Yet another concept based on spectral filters is the so-called *quantum belief propagation* [115, 134, 80, 109]. It allows writing a differential equation for the Gibbs states of a path of Hamiltonians $s \mapsto H(s)$ similarly to the spectral flow for gapped ground states. In its original formulation by Hastings [115], it is actually a differential equation for the non-normalized exponential $\mathrm{e}^{-\beta H(s)}$. It found several applications mostly in the quantum information theory for finite-range interactions in various contexts [134, 135, 45, 131, 109, 8, 145, 194, 136, 62].

In publication P4, we give a rigorous proof of quantum belief propagation and write down the differential equation for the Gibbs state. Moreover, we use it to prove an equivalence between decay of correlations, LPPL and local indistinguishability for Gibbs states at any positive temperature. All these results are formulated for interactions with arbitrary decay, as long as they satisfy a Lieb-Robinson bound, which includes exponentially and polynomially decaying interactions. Building on previous results on decay of correlations for certain systems in one dimension or at high temperature, we thus prove local indistinguishability and LPPL for these systems.

Bulk-edge correspondence

As mentioned above, an important problem in the analysis of quantum lattice systems is the quantum Hall effect. One aspect of it is the bulk-edge correspondence, relating the bulk magnetization with edge currents. So far this is mostly studied for non-interacting systems [7, 155, 143, 65]. Stronger results on bulk-edge correspondence for the transport coefficients, i.e. relating the Hall conductivity and the edge conductance, exist for non-interacting gapped systems [197, 82, 65] and weakly interacting systems [100, 159] at zero temperature. Moreover, for non-interacting fermions in the continuum with a gapped ground state, [65] proves that the derivatives of the bulk magnetization and edge current with respect to the chemical potential converge to the Hall conductivity and the edge current, respectively, in the zero temperature limit.

In publication P3, we prove equality of the magnetization and the edge current for interacting fermions on a square lattice, assuming that the system is homogeneous in the bulk and that the Gibbs state satisfies local indistinguishability. We also prove differentiability of these quantities with respect to the chemical potential in analogy to the above result. As an intermediate result, we obtain Bloch's theorem for two-dimensional square lattices, stating that persistent currents vanish in the bulk. See [215, 21] for related results.

Locally weakly interacting quantum spin systems with an on-site gap

We also consider an exemplary class of spin system, for which we can prove very strong locality properties. The Hamiltonian of these systems is given by a sum of gapped on-site terms with a unique gapped ground state and a weak finite-range interaction that in particular does not close the gap. For these systems, we reinterpret a result by Yarotsky [219] in publication P1, to prove that the ground state satisfies local indistinguishability and a very strong form of LPPL, with a bound that is independent of the strength of the perturbation.

Moreover, assuming that the interaction is form-bounded with respect to the on-site Hamiltonian in a strong sense (see (3.42)), we prove in publication P8 that the Gibbs state also satisfies decay of correlations and local indistinguishability uniformly in temperature. Decay of correlations uniformly at low temperature for special (classes of) quantum spin systems [205, 161, 40, 66, 67, 94] and also fermions [112, 118, 73, 94, 102] has been studied before. However, to our knowledge, proving local indistinguishability uniformly in temperature has not been approached in the literature so far, and we provide a robust proof for uniform-in-temperature decay of correlations and local indistinguishability.

Structure of this thesis

After this general introduction, we introduce the mathematical objects used to describe quantum lattice systems in chapter 2. In particular, we define the notions of decay of correlations, LPPL and local indistinguishability in section 2.6. Afterwards, we summarize the important results and give some ideas of the proofs that are obtained in the included publications in chapter 3. There, we also provide more connections to the literature. Reprints of the publications are included afterwards. When we reference statements from the included publications, their numbers are prefixed with the number of the publication. For example, theorem 34 from publication P4 is referenced as theorem P4-34 in the main text.

2 Mathematical setup

This chapter introduces the mathematical concepts used to describe quantum lattice systems and their properties. We point out the differences between the setup in the publications and compare to related notions used in the literature. This lays the fundament to the discussion of the results in chapter 3 and shall allow the reader to follow the publications and their proofs.

2.1 Lattice geometry

We first introduce the lattices considered throughout this thesis. The prototypical example we consider is the lattice $\Gamma = \mathbb{Z}^D$ with ℓ^1-distance or finite subsets thereof.

In principle, multiple generalizations are possible and give similar results, but need to be treated slightly differently. For example, one could consider a set of points $\Gamma \subset \mathbb{R}^D$ together with the induced metric [84], or any countable metric space (Γ, d) [174]. These generalizations have in common that one can straightforwardly define an infinite system and deduce properties by first considering finite subsets $\Lambda \Subset \Gamma$. The dimension of Λ is then defined as the dimension of Γ.

On the other hand, we would like to include statements about torus and cylinder geometries, where the finite systems cannot just be defined as restrictions of the infinite lattices with the restricted metric. Instead, we mostly use *surface-regular* graphs and prove statements, which are uniform in the graph after fixing the dimension and growth rate. Throughout the thesis we use the terms *lattice* and *graph* interchangeably.

Definition 2.1 (Surface-regular graph)**.** A graph (Γ, E) with graph distance d such that there exist constants $D \in \mathbb{N}$ and $\mathscr{C}_\mathrm{A} \geq 1$ such that

$$\left|S_y(R)\right| := \left|\{\, x \in \Gamma \,|\, d(y,x) = R \,\}\right| \leq \mathscr{C}_\mathrm{A}\, R^{D-1} \quad \text{for all } y \in \Gamma \text{ and } R \geq 1 \tag{2.1}$$

is called $(\mathscr{C}_\mathrm{A}, D)$-*surface-regular*. The set of all D-dimensional surface-regular graphs with growth constant $\mathscr{C}_\mathrm{A}$ is denoted $\mathscr{G}(D, \mathscr{C}_\mathrm{A})$. ◇

Noticeably, any subset $\Gamma \subset \mathbb{Z}^D$ with ℓ^1-distance is a D-surface-regular graph and so are the boxes $\Lambda_L = \mathbb{Z}^D \cap [-L, L]$ with periodic boundary condition in some (or all) directions.

A slightly more general notion is that of a *D-regular* graphs, which is similar but just requires a bound on the balls

$$\left|B_y(R)\right| := \left|\{ x \in \Gamma \,|\, d(y,x) \leq R \}\right| \leq \mathscr{C}_{\mathrm{vol}}\,(1+R)^D \quad \text{for all } y \in \Gamma \text{ and } R \geq 0. \tag{2.2}$$

Clearly, every D-surface-regular graph is also D-regular. However, especially for long-range interactions, the bounds one can obtain are often better for surface-regular graphs. Hence, for simplicity, we only consider D-surface-regular graphs from here on.[2]

For any graph Γ and $Z \subset \Gamma$, we denote the cardinality $|Z|$. Finite subsets are denoted with "$\Subset$" and the set of finite subsets is denoted

$$\mathscr{P}_0(\Gamma) = \{ Z \Subset \Gamma \}.$$

Moreover, we denote the diameter

$$\operatorname{diam}(Z) = \sup_{x,y \in Z} d(x,y),$$

and for the distance between a point and a set or between two sets, we also write $d(\cdot,\cdot)$.

For a set $X \subset \Lambda$, we denote the *fattening*

$$X_r := \{ y \in \Lambda \,|\, d(y,X) \leq r \}, \tag{2.3}$$

and it will be clear from the context on which lattice the fattening is considered. Likewise, the context should determine, when we use the same notation to enumerate sets $Z_1, \ldots, Z_n \subset \Lambda$.

2.2 Operator algebra

To describe an interacting quantum system on a lattice, one usually uses the operator algebraic description. We will consider spin systems and fermions, and it turns out that their algebraic descriptions are quite similar. The basic idea is to consider the bounded operators on the Hilbert space associated to the lattice Λ as an algebra $\mathscr{A}_\Lambda$ and states as linear functionals on this algebra. These functionals can still be identified with density matrices for finite lattices, see section 2.5. For subsets $X \subset \Lambda$ one then identifies $\mathscr{A}_X \subset \mathscr{A}_\Lambda$ as a subalgebra. More details on this identification and properties thereof are discussed in the next sections, when we introduce the operator algebras for spin systems and fermions separately. Importantly, many results for spins or fermions can immediately be translated to the other. However, there are some points, where one needs to be very careful. See section P7-6 for one example.

[2] In publications P6 and P9, we actually use regular graphs. And in publications P1 to P4 and P8 we restrict to (subsets of) $\mathbb{Z}^D$. Also note, that the included publications might use d or ν instead of D to denote the dimension.

2.2.1 Operator algebra for lattice spin systems

Under a quantum spin lattice system, we understand a system where a q-level quantum system sits at each lattice site. Physically, this can be implemented by an atom on every site and modelling the internal spin degrees of these atoms. We consider spin systems in publications P1, P4 to P6, P8 and P9, but only the methods and results in publications P1 and P8 particularly rely on this.

Mathematically, we associate to each site $x \in \Gamma$ a finite-dimensional Hilbert space $\mathcal{H}_x = \mathbb{C}^q$ and the corresponding space of linear operators $\mathcal{A}_x = \mathcal{B}(\mathcal{H}_x)$. Then, for each finite $\Lambda \subset \Gamma$, we define the joint Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$ and the operator algebra $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$. Due to the product structure of $\mathcal{H}_\Lambda$, also $\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{A}_x$. Moreover, for any $X \subset \Lambda$, we can consider $\mathcal{A}_X \subset \mathcal{A}_\Lambda$ as a subalgebra by identifying any $A \in \mathcal{A}_X$ with $A \otimes \mathbb{1}_{\Lambda\setminus X} \in \mathcal{A}_\Lambda$. From now on, this identification is always implicitly understood, and we define the support $\operatorname{supp} A$ as the smallest set such that $A \in \mathcal{A}_{\operatorname{supp} A}$.

This identification allows to define the algebra of local observables on Γ

$$\mathcal{A}_\Gamma^{\mathrm{loc}} = \bigcup_{\Lambda \Subset \Gamma} \mathcal{A}_\Lambda,$$

which contains observables with finite but arbitrarily large support. Its completion with respect to the operator norm is the C^*-algebra of *quasi-local* observables

$$\mathcal{A}_\Gamma = \overline{\mathcal{A}_\Gamma^{\mathrm{loc}}}^{\|\cdot\|}.$$

The product structure of the operator algebra also implies that observables with disjoint support commute. Given two disjoint sets $X, Y \subset \Gamma$ and operators $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ one has $[A, B] = 0$.

2.2.2 Operator algebra for fermions on a lattice

Another kind of system we are interested in are fermions on a lattice, which can be described by an operator algebra with very similar properties. The physical interpretation is clearly different and models fermionic particles that can move on the lattice. These systems are explicitly considered in publications P2, P3, P7 and P10, but the results from publications P4 to P6 and P9 mostly hold for fermionic lattice systems as well.

Mathematically, a single particle on a lattice Γ is described by a wave function in $\mathcal{H}^\Gamma = \ell^2(\Gamma, \mathbb{C}^q) \simeq \ell^2(\Gamma \times \{1, \dots, q\}, \mathbb{C})$, where $q \in \mathbb{N}$ is the number of internal degrees of freedom. For fermions, the N-particle Hilbert spaces is given by the antisymmetric tensor product $\mathcal{H}_N^\Gamma = \bigwedge_{k=1}^N \mathcal{H}^\Gamma$, and we define $\mathcal{H}_0^\Gamma = \mathbb{C}$. Then, let $\mathcal{F}^\Gamma = \bigotimes_{N=0}^\infty \mathcal{H}_N^\Gamma$ be the Fock space. The many-body description of the system is given by the CAR algebra $\mathcal{A}_\Gamma = \mathrm{CAR}(\mathcal{H}^\Gamma) \subset \mathcal{B}(\mathcal{F}^\Gamma)$, which is the C^*-algebra with norm given by

$\|\cdot\| = \|\cdot\|_{\mathcal{B}(\mathcal{F}^\Gamma)}$ generated by the identity $\mathbb{1}$ and the creation and annihilation operators $a^*_{x,i}$ and $a_{y,j}$ satisfying the canonical anti-commutation relations (CAR)

$$\{a_{x,i}, a_{y,j}\} = \{a^*_{x,i}, a^*_{y,j}\} = 0 \qquad \text{and} \qquad \{a_{x,i}, a^*_{y,j}\} = \delta_{x,y}\, \delta_{i,j}\, \mathbb{1} \tag{2.4}$$

for all $x, y \in \Gamma$ and $i, j \in \{1, \dots, q\}$, where $\{A, B\} = AB + BA$ denotes the anticommutator. Moreover, for $\Lambda \subset \Gamma$, we define $\mathcal{A}_\Lambda \subset \mathcal{A}_\Gamma$ as the subalgebra generated by the creation and annihilation operators localized in Λ.

Notably, one considers $\mathcal{A}_\Gamma$ instead of the bounded operators on Fock space $\mathcal{B}(\mathcal{F}^\Gamma)$, because states, which we introduce in section 2.5, on $\mathcal{A}_\Gamma$ can describe systems with an infinite number of particles, while states in the Fock space $\mathcal{F}^\Gamma$ have an arbitrary large but finite number of particles.

In comparison to the setting for spin systems, we do not need to take a completion, because $\mathcal{A}_\Gamma \subset \mathcal{B}(\mathcal{F}^\Gamma)$ and the construction of a C^*-algebra already includes taking the closure. But analogously to the construction for spin systems, if the lattice Λ is finite then $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{F}^\Lambda) = \mathcal{B}\big(\bigotimes_{N=0}^{q|\Lambda|} \mathcal{H}_N^\Gamma\big)$ by the Pauli exclusion principle. Via the creation and annihilation operators we can abstractly identify $\mathcal{A}_{\Lambda'} \subset \mathcal{A}_\Lambda$ as a subalgebra for any $\Lambda' \subset \Lambda$. Even without knowing $\mathcal{A}_\Gamma$ from above, one can then define the local algebra

$$\mathcal{A}_\Gamma^{\mathrm{loc}} = \bigcup_{\Lambda \Subset \Gamma} \mathcal{A}_\Lambda,$$

for infinite lattices Γ. Moreover,

$$\mathcal{A}_\Gamma = \overline{\mathcal{A}_\Gamma^{\mathrm{loc}}}^{\|\cdot\|},$$

where the right-hand side can either be understood as the completion – like for spin systems – or the closure within the bounded operators on Fock space $\mathcal{B}(\mathcal{F}^\Gamma)$.

In comparison to the algebra for spin systems, two observables with disjoint support need not commute. Indeed, for $x \neq y$ the creation operators a_x and a_y anti-commute. This motivates the definition of the even operator algebra $\mathcal{A}_\Gamma^+$, which is the subalgebra generated by identity and even monomials in the creation and annihilation operators, and its subalgebras $\mathcal{A}_\Lambda^+ = \mathcal{A}_\Lambda \cap \mathcal{A}_\Gamma^+$ for subsets $\Lambda \subset \Gamma$. Then, two operators $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y^+$ with disjoint supports $X, Y \subset \Gamma$ indeed commute $[A, B] = 0$. Moreover, the reverse is also true: if two operators with disjoint supports commute, at least one must be even [173, Proposition 2.1].

For any finite $Z \subset \Gamma$ we define the number operator[3]

$$\mathcal{N}_Z = \sum_{x \in Z} \sum_{j=1}^{q} a^*_{x,j}\, a_{x,j}$$

and the subalgebra $\mathcal{A}_Z^{\mathcal{N}} \subset \mathcal{A}_Z^+$ of operators commuting with $\mathcal{N}_Z$.

[3] In publication P2 it is just denoted N_Z.

2.3 Interactions

All extensive operators on the lattice, like the Hamiltonian, are give by so-called *interactions*, which are maps

$$\Phi\colon \{Z \Subset \Gamma\} \to \mathcal{A}_\Gamma, \\ Z \mapsto \Phi(Z) \in \mathcal{A}_Z.$$

With any interaction Φ, we associate a *sum of local terms* (SLT) operator for each $\Lambda \Subset \Gamma$

$$A_\Lambda = \sum_{Z \subset \Lambda} \Phi(Z).$$

The name *interaction* evolved historically and is a little misleading, as interactions not only model the interaction between particles or lattice sites, but also external potentials and the kinetic energy. More generally, interactions are used to define all SLT operators and the Hamiltonian is only one example.

Whenever each term of an interaction is self-adjoint, even, or particle preserving,[4] we give the interaction and the associated SLT operator the same name. Moreover, an interaction is commuting, if all terms mutually commute, $[\Phi(Z_1), \Phi(Z_2)] = 0$ for all $Z_1, Z_2 \Subset \Gamma$.

The properties of the interaction can be utilised to quantify locality of the corresponding SLT operator. The simplest situation is when Φ is a *finite-range* interaction. That is, there exists an $R \in \mathbb{N}$, such that $\Phi(Z) = 0$ unless $\operatorname{diam}(Z) \leq R$. The smallest such R is the range of Φ and in the special case of $R = 0$ we say that the interaction is *on-site*, since it only acts on individual lattice sites. We say that an interaction is *k-body*, if $\Phi(Z) = 0$ unless $|Z| \leq k$.

More generally, we want to quantify the decay of $\|\Phi(Z)\|$ in the size of Z. A common approach is to introduce so-called *interaction norms*, and there are many variants in the literature. The most basic versions require decay in the cardinality $|Z|$. For example,

$$\sup_{z \in \Gamma} \sum_{Z \Subset \Gamma : z \in Z} \mathrm{e}^{|Z|-1} \|\Phi(Z)\| < \infty \tag{2.5}$$

is enough to define the thermodynamic limit of the Heisenberg evolution [195, theorem 7.6.2]. We usually require better decay, which also takes into account the diameter of Z. Therefore, consider a decaying function $F\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ and let

$$\|\Phi\|_F^{\mathrm{tb}} = \sup_{x,y \in \Gamma} \sum_{\substack{Z \Subset \Gamma: \\ x,y \in Z}} \frac{\|\Phi(Z)\|}{F\big(d(x,y)\big)}. \tag{2.6}$$

[4] The notions of being even or particle preserving are clearly only relevant for fermions.

This norm is a good choice when proving Lieb-Robinson bounds and a natural generalization of two-body interactions, for which this norm exactly quantifies the decay $\|\Phi(\{x,y\})\| \le \|\Phi\|_F^{\mathrm{tb}}\, F\big(d(x,y)\big)$.

However, the above interaction norm makes it difficult to control terms involving the cardinality $|Z|$ and requires additional assumptions on F to make $\sum_{Z\Subset\Gamma:z\in\Gamma}\Phi(Z)$ summable. It is often simpler to work with the slightly stronger interaction norm

$$\|\Phi\|_F = \sup_{z\in\Gamma}\sum_{\substack{Z\Subset\Gamma:\\ z\in Z}}\frac{\|\Phi(Z)\|}{F\big(\mathrm{diam}(Z)\big)}. \tag{2.7}$$

This version can also be generalized to quantify the localization of an SLT operator around a set $\Omega\subset\Gamma$, for which one defines

$$\|\Phi\|_{F;\Omega} = \sup_{z\in\Gamma}\sum_{\substack{Z\Subset\Gamma:\\ z\in Z}}\frac{\|\Phi(Z)\|}{F\big(\mathrm{diam}(Z)\big)\,F\big(d(z,\Omega)\big)}. \tag{2.8}$$

Such interactions are called *Ω-localized* and this form of interaction norm ensures that the commutator of an Ω-localized SLT operator with another SLT operator is still Ω-localized, as one would expect. See lemmata P6-A.5 and P6-2.1 for details.

It is left, to quantify the decay function F, used in the various interaction norms. We say that an interaction is *exponentially decaying* if $F(r)\le \mathrm{e}^{-br}$ for some $b>0$ and *stretched exponentially decaying* if $F(r)\le \chi_{b,p}(r) = \mathrm{e}^{-br^p}$ for some $b>0$ and $p\in(0,1)$. In both cases, we call such interactions *short-range*. Moreover, we say that an interaction is *polynomially decaying* or *long-range*, if $F(r)\le F_\alpha(r) = (1+r)^{-\alpha}$ for some $\alpha>0$.

In particular for long-range two-body interactions we want to point out the difference between the interaction norms (2.6) and (2.7) which is important to compare different results obtained with either of those. Therefore, consider a two-body interaction Φ with $\|\Phi(\{x,y\})\|\le C\,F_{\alpha_{\mathrm{tb}}}\big(d(x,y)\big)$ for all $x,\ y\in\Gamma$ and some $C>0$ and $\alpha_{\mathrm{tb}}>D$. As explained above, it satisfies $\|\Phi\|^{\mathrm{tb}}_{F_{\alpha_{\mathrm{tb}}}}\le C$. However, for the interaction norm (2.7) we only obtain

$$\|\Phi\|_{F_\alpha} \le \sup_{x\in\Gamma}\sum_{y\in\Gamma}\frac{C\,F_{\alpha_{\mathrm{tb}}}\big(d(x,y)\big)}{F_\alpha\big(d(x,y)\big)} = C\sup_{x\in\Gamma}\sum_{y\in\Gamma}F_{\alpha_{\mathrm{tb}}-\alpha}\big(d(x,y)\big)<\infty,$$

for all $\alpha<\alpha_{\mathrm{tb}}-D$ on D-surface-regular graphs.

As mentioned above, one often needs to bound sums over terms $|Z|\,\|\Phi(Z)\|$. Using the interaction norm (2.7), this can be done by utilizing the decay in $\mathrm{diam}(Z)$. Alternatively, we can introduce another variant of the interaction norms (2.6) and (2.7)

$$\|\Phi\|^{\mathrm{tb}}_{F,n} = \sup_{x,y\in\Gamma}\sum_{\substack{Z\Subset\Gamma:\\ x,y\in Z}}\frac{|Z|^n\,\|\Phi(Z)\|}{F\big(d(x,y)\big)} \qquad\text{and}\qquad \|\Phi\|_{F,n} = \sup_{z\in\Gamma}\sum_{\substack{Z\Subset\Gamma:\\ z\in Z}}\frac{|Z|^n\,\|\Phi(Z)\|}{F\big(\mathrm{diam}(Z)\big)} \tag{2.9}$$

by adding a factor $|Z|^n$ for some $n \in \mathbb{N}_0$ to the sum. This is particularly helpful for (2.6), but also allows obtaining better results for long-range interactions with the norm (2.7) as it only adds restrictions on the cardinality $|Z|$ but not the decay in $\operatorname{diam}(Z)$. In particular, for k-body interactions, it merely changes the interaction norm by a factor k^n.

Whenever we consider time-dependent systems, we also need *time-dependent interactions* on an interval $I \subset \mathbb{R}$, which are functions $\Phi(\cdot,\cdot)$, where $\Phi(\cdot,t)$ is an interaction for each $t \in I$ and $\Phi(Z,\cdot)\colon I \to \mathcal{A}_Z$ is norm continuous. We also abbreviate $\Phi(t) = \Phi(\cdot,t)$, which allows specifying the interaction norm $\|\Phi(t)\|_F = \|\Phi(\cdot,t)\|_F$. We simply write

$$\|\Phi\|_F = \sup_{t\in I} \|\Phi(t)\|_F$$

for time-dependent interactions, as many bounds then hold for time-independent and time-dependent interactions in the same way.

Note, that a condition asking for an interaction norm to be finite in only a good condition if the underlying lattice is infinite. Instead, our results will often be formulated on finite lattices, where we prove results which are uniform in the size of the lattice Λ and all interactions Φ on Λ that have the same interaction norm $C = \|\Phi\|_F$.

We use the notation introduced here for the main part of this thesis. It also introduces all interaction norms used in the included publications. However, as most publications only use one of these norms, they often use an abbreviated form and one has to check the exact definition in the respective mathematical setups.

2.4 Dynamics

Given a Hilbert space $\mathcal{H}$, an interval $I \subset \mathbb{R}$ and a continuous Hamiltonian $H\colon I \to \mathcal{B}(\mathcal{H})$, the time-evolution of a state $\psi_0 \in \mathcal{H}$ is given as the unique solution of the Schrödinger equation

$$\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\psi(t) = H(t)\,\psi(t) \qquad \text{with} \qquad \psi(t_0) = \psi_0.$$

Moreover, these solutions can be characterised by the propagator $U(t,s)$, which is a two-parameter family of unitaries satisfying $\psi(t) = U(t,s)\,\psi(s)$ for $s,\ t \in I$. From the Schrödinger equation, it turns out that the propagator satisfies

$$\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,U(t,s) = H(t)\,U(t,s) \qquad \text{and} \qquad U(s,s) = \mathbb{1}.$$

Instead of using the Schrödinger picture, quantum lattice systems are usually described in the Heisenberg picture, where instead of evolving the state, one evolves the observables. By the Schrödinger equation, for every observable $A \in \mathcal{B}(\mathcal{H})$ it holds that

$$\langle \psi(t), A\,\psi(t)\rangle = \langle \psi(s), U(t,s)^*\,A\,U(t,s)\,\psi(s)\rangle,$$

and we thus define the Heisenberg time evolution $\tau_{t,s}(A) = U(t,s)^* A U(t,s)$. It is alternatively characterised as the unique solution of the Heisenberg equation

$$-\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\tau_{t,s}(A) = \tau_{t,s}\big([H(t), A]\big) \qquad \text{with} \qquad \tau_{s,s}(A) = A, \tag{2.10}$$

for all $A \in \mathscr{A}_\Lambda$ and $s, t \in I$. In this convention, the Heisenberg time evolution satisfies $\tau_{r,s}\,\tau_{t,r} = \tau_{t,s}$ for all $t, r, s \in I$. For finite systems, where states are given by density matrices, see section 2.5, one can also define the evolution $\hat{\tau}_{t,s}$, which solves the von Neumann equation

$$\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\hat{\tau}_{t,s}(\rho) = [H(t), \hat{\tau}_{t,s}(\rho)] \qquad \text{with} \qquad \hat{\tau}_{t,t} = \mathrm{id}$$

and satisfies $\hat{\tau}_{t,s} = \tau_{s,t}$. In our convention, $\hat{\tau}_{t,s}$ satisfies the usual cocycle property $\hat{\tau}_{t,r}\,\hat{\tau}_{r,s} = \hat{\tau}_{t,s}$. One can alternatively swap the indices in both evolutions so that $\tau_{t,s}$ satisfies the usual cocycle property, which is the convention used in publication P10.

One advantage of the Heisenberg picture is that one can make sense of the dynamics on infinite volume systems. Therefore, consider an infinite lattice Γ and a (time-dependent) interaction Φ on Γ. By the above construction, for each $\Lambda \Subset \Gamma$ the Hamiltonian $t \mapsto H_\Lambda(t) \in \mathscr{A}_\Lambda$ gives rise to the Heisenberg time evolution $\tau^\Lambda_{t,s}$ as the unique solution of the Heisenberg equation (2.10) with Hamiltonian $H_\Lambda(t)$. One can then define the dynamics on $\mathscr{A}_\Gamma$ as the limit $\tau^\Gamma_{t,s}(A) = \lim_{\Lambda \nearrow \Gamma} \tau^\Lambda_{t,s}(A)$, given locality of the Hamiltonian, for example as in (2.5) or in the sense of a Lieb-Robinson bound [154, 174].

2.5 Thermal states

Besides the evolution, the states of a quantum system play a key role. We are mostly interested in the ground and Gibbs states of the system under consideration. For finite lattices Λ, a general state is given by a density matrix, that is a positive operator $\rho \in \mathscr{B}(\mathscr{H})$ with $\mathrm{tr}(\rho) = 1$.

In the algebraic description states are positive linear functionals ω over $\mathscr{A}_\Lambda$ satisfying $\omega(\mathbb{1}) = 1$. Indeed, on finite lattices, there is a one-to-one correspondence between these notions. Given a density matrix ρ, the functional $\omega\colon A \mapsto \mathrm{tr}(\rho A)$ is a state in the algebraic sense and for every such functional there exists a density matrix. Within this thesis, we mostly use density matrices or projections, as we mostly consider finite systems. We only switch to functionals when infinite systems are considered, where the notion of density matrices is not available.

2.5.1 Ground states

One state of particular interest is the ground state of a system, which describes the system at zero-temperature $T = 0$ and minimizes the energy $\mathrm{tr}(\rho H)$ of the Hamiltonian $H \in \mathscr{A}_\Lambda$ of the system. We often denote the ground state projection P as the spectral projection onto the ground state energy $E_0 = \min \sigma(H)$, where $\sigma(H)$ denotes the spectrum of H. Then, a ground state is a state satisfying $\rho = P \rho P$. Most of the time, we are interested in gapped ground states (or more precisely ground states of gapped Hamiltonians), for which $\sigma(H) \setminus \{E_0\} \geq E_0 + g$ for some gap $g > 0$. While all finite systems are technically gapped (with the size of the gap vanishing with increasing system size), we always mean a gap uniform in system size or explicitly specify the size of the gap g.

Another common feature of finite systems is that the ground state energy splits up into several eigenvalues and only in the thermodynamic limit converges to a degenerate ground state. For such cases, we also denote with P the projection onto the ground state sector $\sigma_* \subset \sigma(H)$. This notion is made precise when we use it.

Whenever, we want to denote the ground state of a particular Hamiltonian H we write $\rho[H]$ and if the interaction is clear from the context, we write $\rho_\Lambda := \rho[H_\Lambda]$.

2.5.2 Gibbs states

The other type of state we are interested in are Gibbs states, which are thermal states at positive temperature $T > 0$. They are usually labelled by the inverse temperature $\beta = 1/T$ and defined as

$$\rho(\beta) = \frac{\mathrm{e}^{-\beta H}}{Z(\beta)} \qquad \text{with} \qquad Z(\beta) = \mathrm{tr}(\mathrm{e}^{-\beta H})$$

the partition function. As for ground states, we write $\rho(\beta)[H]$ to specify the Hamiltonian and $\rho_\Lambda^\beta = \rho_\Lambda(\beta)$ if the interaction is clear from the context. The analogues of Gibbs states in the algebraic setting on infinite volumes are KMS states [47, section 5.3], which we briefly consider in theorem 3.17.

2.6 Locality measures

There are various levels on which a quantum system can exhibit locality, the simplest of which is the locality of the Hamiltonian that we understand in terms of the interaction norms introduced in section 2.3. The locality of the Hamiltonian directly implies locality of the dynamics in the sense of Lieb-Robinson bounds, which we discuss in detail in section 3.1.

More interesting is the locality of the (thermal) states, for which there are many different measures available. In the following sections we describe the ones which are used in this thesis.

2.6.1 Decay of correlations

For any state ρ on $\mathscr{A}_\Lambda$, we define the *covariance*

$$\mathrm{Cov}_\rho(A,B) := \mathrm{tr}(\rho\, A\, B) - \mathrm{tr}(\rho\, A)\ \mathrm{tr}(\rho\, B)$$

for any $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$ and $X, Y \subset \Lambda$. One way to quantify correlations in the state ρ, is to understand how the covariance decays with $d(X,Y)$.

Definition 2.2 (Decay of correlations)**.** Let Λ be a finite graph and ρ a state on $\mathscr{A}_\Lambda$. We say that ρ satisfies *decay of correlations* with respect to the positive functions ζ_{Cov}, f_{Cov} if and only if

$$|\mathrm{Cov}_\rho(A,B)| \le \|A\|\,\|B\|\, f_{\mathrm{Cov}}\big(|X|,|Y|\big)\, \zeta_{\mathrm{Cov}}\big(d(X,Y)\big)$$

for all $X, Y \subset \Lambda$ and $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$. ◇

While decay of correlations is a property of a state, we can ask whether a ground or Gibbs state satisfies decay of correlations, which translates decay of correlations to a property of the Hamiltonian. Moreover, we say that an interaction Φ satisfies *uniform decay of correlations* (at inverse temperature[5] β) on Λ if and only if all ground or Gibbs states $\rho(\beta)[H_{\Lambda'}]$ satisfy decay of correlations with respect to the same functions for every $\Lambda' \subset \Lambda$. This is also known as *uniform clustering* [45, 132].

2.6.2 Local perturbations perturb locally

The next concept not only quantifies locality, but also stability of a system. The question is how much a local perturbation in the Hamiltonian changes expectation values of the ground or Gibbs state far away from the perturbation. We refer to it as the *local perturbations perturb locally* (LPPL) principle, a term used for ground states for example in [23, 74] before.

Definition 2.3 (Local perturbations perturb locally (LPPL))**.** Let Λ be a finite graph and $H \in \mathscr{A}_\Lambda$ be self-adjoint. We say that H satisfies *LPPL* (at inverse temperature β) with respect to the positive functions f_{LPPL}, g_{LPPL}, ζ_{LPPL} and $n \in \mathbb{N}$, if and only if

$$\big|\mathrm{Tr}\big(\rho(\beta)[H_\Lambda]\, B\big) - \mathrm{Tr}\big(\rho(\beta)[H_\Lambda + V]\, B\big)\big| \le \|B\|\, |X|^n\, f_{\mathrm{LPPL}}\big(|Y|\big)\, g_{\mathrm{LPPL}}\big(\|V\|\big)\, \zeta_{\mathrm{LPPL}}\big(d(X,Y)\big)$$

for all $X, Y \subset \Lambda$, $V \in \mathscr{A}_X$ self-adjoint and $B \in \mathscr{A}_Y$. ◇

In principle, we could allow other scalings in $|X|$, for example exponential. But as this is not relevant in the following, we keep this simpler form.

We say that an interaction Φ satisfies *uniform LPPL* (at inverse temperature β) on Λ if and only if $H_{\Lambda'}$ satisfies LPPL with respect to the same functions and n for every $\Lambda' \subset \Lambda$.

[5] For the definitions of the locality properties, we identify the Gibbs state at inverse temperature $\beta = +\infty$ with the ground state. And if it is clear from the context later on, we do not explicitly state, whether H_Λ satisfies uniform decay of correlations for the ground or Gibbs state.

2.6.3 Local indistinguishability

The final locality measure, which we consider, asks whether expectation values of local operators in thermal state on Λ can be well approximated by expectation values in the thermal state on $\Lambda' \subset \Lambda$, as long as they are localized far away from $\Lambda \setminus \Lambda'$. We call this property *local indistinguishability*, and this term is also used in [45, 36]. For Gibbs states, it was also called *locality of temperature* in [139].

Definition 2.4 (Local indistinguishability)**.** Let Λ be a finite graph and Φ be an interaction. We say that Φ satisfies *local indistinguishability (everywhere[6])* (at inverse temperature β) on Λ with respect to the positive functions f_{LI} and ζ_{LI} if and only if

$$\left|\mathrm{Tr}\big(\rho(\beta)[H_\Lambda]\, B\big) - \mathrm{Tr}\big(\rho(\beta)[H_{\Lambda'}]\, B\big)\right| \leq \|B\|\, f_{\mathrm{LI}}\big(|Y|\big)\, \zeta_{\mathrm{LI}}\big(d(Y, \Lambda \setminus \Lambda')\big)$$

for all $Y \subset \Lambda' \subset \Lambda$ and $B \in \mathcal{A}_Y$. ◇

At the first glance, local indistinguishability looks similar to *local topological quantum order* (LTQO), which is a widely used name various related properties of ground states. The key difference is that LTQO is also a statement about different ground states on the same volume (see e.g. [162, section 4.1, 175, section 2.2.2]) for Hamiltonians with a degenerate ground state space, while local indistinguishability compares expectation values of the ground or Gibbs states on two different volumes. In the case of degenerate ground states, we might think of the whole (possibly normalized) ground state projection for local indistinguishability.

One simple consequence of local indistinguishability is that the ground or Gibbs states (as functionals) have a thermodynamic limit. Moreover, at least for finite-range interactions, local indistinguishability immediately implies decay of correlations as we discuss at the end of section 3.6.

[6] In publication P3 we also use the notion of *local indistinguishability in the bulk*, which we explain in section 3.7.

3 Results

In this chapter we summarize the results obtained in the included publications. As many results build on Lieb-Robinson bounds and we also provide multiple improvements, we begin with discussing Lieb-Robinson bounds in section 3.1 and in particular the ones obtained in publications P4, P5, P7 and P10. Afterwards, we briefly describe what spectral filters are and how Lieb-Robinson bounds make them useful for the analysis of extended quantum lattice systems in section 3.2. Then we discuss the spectral flow, which is one instance of a spectral filter, and LPPL in section 3.3 focusing on the results obtained in publications P5, P7 and P9. Afterwards, we discuss decay of correlations for gapped ground states in section 3.4 and in particular enhanced decay for commuting Hamiltonians obtained in publication P5 and a modified filter function constructed in publication P9. In section 3.5 we come back to the spectral filter used to construct the spectral flow and discuss response theory and adiabatic theory. There, we also discuss several notions of local gaps and how one can use them to prove response theory for locally gapped systems. In section 3.6 we discuss the main result of publication P4, an equivalence of decay of correlations, LPPL and local indistinguishability, which is obtained via another spectral filter named quantum belief propagation. Afterwards, in section 3.7 we discuss an instance of bulk-edge correspondence for interacting fermions at positive temperature, which was obtained in publication P3. Eventually, in section 3.8, we discuss the results of publications P1 and P8, which prove very strong locality in terms of decay of correlations, LPPL and local indistinguishability for the ground and Gibbs state of a class of weakly interacting spin systems with an on-site gap.

3.1 Lieb-Robinson bounds

One of the main tools to analyse locality in lattice systems are so called Lieb-Robinson bounds, which were first proven for translation invariant systems by Lieb and Robinson [154]. They characterize the locality of the Heisenberg time evolution and are a very important tool for the analysis of quantum lattice systems. One of the very general forms of Lieb-Robinson bounds is given in [174], and variants for other flavours of interaction norms also exist [118, 173]. In particular, for exponentially decaying interactions Φ in the sense that $\|\Phi\|_F < \infty$ for $F(r) = \mathrm{e}^{-br}\,(1+r)^{-(D+1)}$ and some $b > 0$, one has

$$\left\|[\tau_t^\Gamma(A), B]\right\| \leq C\, \min\left\{|X|, |Y|\right\} \|A\|\, \|B\|\, \mathrm{e}^{b\,(vt-d(X,Y))}, \tag{3.1}$$

for all $X, Y \subset \Gamma$, $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$, where $v \sim \|\Phi\|_F/b$. One crucial observation is that this commutator is small, whenever $r = d(X,Y) \geq v\,t$, and decays exponentially outside the cone $r = v\,t$. In this sense, information propagation is *ballistic* up to exponential errors. One also says, that the Lieb-Robinson bounds has a *linear light cone*[7] in analogy to the light cone in special relativity, although there are small tails leaking outside the cone.

In comparison, following the same proofs for long-range interactions, one obtains bounds scaling like $\mathrm{e}^{c\,vt}\,F_{\alpha-D}(r)$, see e.g. [118], which gives an exponential light cone. Recall that $F_\alpha(r) = (1+r)^\alpha$.

Only recently, Kuwahara and Saito [149] proved Lieb-Robinson bounds obeying a linear light cone for spin systems with long-range interactions Φ, if $\|\Phi\|^{\mathrm{tb}}_{F_{\alpha_{\mathrm{tb}}}} < \infty$ for some $\alpha_{\mathrm{tb}} \geq 2D+1$. More precisely, these interactions satisfy a Lieb-Robinson bound with polynomial decay in $d(X,Y) - vt$. Afterwards, Tran et al. [210] extended this to an algebraic light cone in the range $2D \leq \alpha_{\mathrm{tb}} \leq 2D+1$. For more details on light cones, we refer to publications P5 and P7.

While the first bounds by Lieb and Robinson appeared already in 1972, they regained attention after Hastings [113] realized in 2004 that they can also be used to infer static properties about the systems, like decay of correlations for gapped ground states. Therefore, one uses spectral filters, which we explain in section 3.2 and then use in sections 3.3, 3.5 and 3.6.

We now summarize the results on Lieb-Robinson bounds for certain classes of systems, which we obtained in the publications.

3.1.1 Publication P5: Enhanced Lieb-Robinson bounds for commuting interactions

In publication P5 we prove improved Lieb-Robinson bounds for commuting, time-independent interactions. Commuting interactions have been studied from the quantum information theory perspective extensively. All stabilizer Hamiltonians, which are used as quantum error correcting codes, are commuting, examples are the toric code [138] and the CSS codes [57, 200]. Moreover, they are often used to study open quantum systems, because their Davies generator is local and hence accessible to rigorous analysis. Examples of long-range commuting interactions are Euclidean implementations of low-density parity check (LDPC) codes on expander graphs, which are *good* quantum codes. See section P5-1.3 for more details on these codes and example P5-1 for an explicit long-range toric-code.

For commuting, finite-range interactions, it is known [186] that the time evolution

[7] In contrast to relativity, the Lieb-Robinson velocity depends on the interaction and is not universal, which lead some people refer to the cones as *sound cones* [139, 78]. But *light cone* seems to be the more common and widely accepted term in the field.

has a fixed light cone. More precisely, assume that Φ is of range $R > 0$. Then

$$\tau_t^{H_\Lambda}(A) := \mathrm{e}^{\mathrm{i}tH_\Lambda} A\, \mathrm{e}^{-\mathrm{i}tH_\Lambda} = \mathrm{e}^{\mathrm{i}tH_{\Lambda,\cap X}} A\, \mathrm{e}^{-\mathrm{i}tH_{\Lambda,\cap X}} \in \mathcal{A}_{X_R}, \tag{3.2}$$

where X_R is the fattening from (2.3) and $H_{\Lambda,\cap X} := \sum_{Z\subset\Lambda: Z\cap X\neq\emptyset} \Phi(Z)$ includes all terms that intersect X such that $H_\Lambda = H_{\Lambda\setminus X} + H_{\Lambda,\cap X}$. The equalities hold by commutativity and are explained below. For more general interactions, we obtain the following result.

Theorem 3.1 (Lieb-Robinson bound for commuting interactions (summary))**.** *Let Φ be a time-independent interaction on a finite graph Λ and $F\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ a decaying function. Then, the Heisenberg dynamics associated to Φ satisfies the Lieb-Robinson bound*

$$\|[\tau_t^{H_\Lambda}(A), B]\| \leq 4\, \|\Phi\|_F^{\mathrm{tb}}\, \|A\|\, \|B\|\, |t| \sum_{x\in X}\sum_{y\in Y} F\big(d(x,y)\big) \tag{3.3}$$

for all $X, Y \subset \Lambda$ and operators $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$. Moreover, this bound is sharp.

The remaining sums in (3.3) *can always be bounded by $|X|\,|Y|\, F\big(d(X,Y)\big)$. If Λ is a D-surface-regular lattice, one also has the bounds*

$$C\, \min\{|X|, |Y|\}\, F_{\alpha-D}\big(d(X,Y)\big) \qquad \text{if} \qquad F(r) = F_\alpha(r) := (1+r)^{-\alpha} \tag{3.4}$$

and

$$C_{b,b'}\, \min\{|X|, |Y|\}\, \chi_{b',p}\big(d(X,Y)\big) \qquad \text{if} \qquad F(r) = \chi_{b,p}(r) := \mathrm{e}^{-b\, r^p} \tag{3.5}$$

for $0 < b' < b$ and $0 < p \leq 1$.

The general result (3.3) is obtained in corollary P5-10 based on the approximation of the Heisenberg dynamics, theorem P5-9. The sharpness is discussed in section P5-3.5.1 and the individual statements for finite-range, short-range and long-range interactions are given in theorems P5-3 to P5-6.

The proof is significantly simpler than the proof of Lieb-Robinson bounds for general (non-commuting) interactions. Since all interaction terms commute, we can factor $\mathrm{e}^{\mathrm{i}tH_\Lambda} = \mathrm{e}^{\mathrm{i}tH_{\Lambda\setminus X}}\, \mathrm{e}^{\mathrm{i}tH_{\Lambda,\cap X}}$ where $H_{\Lambda,\cap X} = \sum_{Z\subset\Lambda} \Phi_{\cap X}(Z) = \sum_{Z\subset\Lambda} \mathbb{1}_{X\cap Z\neq\emptyset}\, \Phi(Z)$ contains all the terms of $\Phi(Z)$ where Z intersects the support of X. Then $\tau_t^{H_\Lambda}(A) = \tau_{t,\cap X}^{H_\Lambda}(A)$ for all $A \in \mathcal{A}_X$, where the latter evolution is generated by $\Phi_{\cap X}$. This already proves the finite-range Lieb-Robinson bound (3.2). Within the commutator, only the terms $\Phi_{\cap X}(Z)$ with $Z \cap Y \neq \emptyset$ contribute, so that only terms directly connecting X and Y are relevant. Summing these up, one obtains (3.3).

This proof also explains why the Lieb-Robinson bounds for commuting interactions are so much stronger than their counterparts for general interactions. Compared to the general bound from (3.1), theorem 3.8 gives linear instead of exponential dependence on time. And also the bound for long-range interactions is a qualitative improvement: For commuting long-range interactions (3.3) together with the trivial bound for the sums gives a linear light cone for $\alpha \geq 1$ in any dimension, compared to the threshold $\alpha \geq 2D+1$ for general interactions [149]; see also figure P5-1.

Some exemplary applications of these LRBs are given in sections 3.3.2 and 3.4.1.

3.1.2 Publication P7: Lieb-Robinson bounds for long-range interactions

In publication P7, we improve the Lieb-Robinson bounds from Matsuta et al. [160] and Else et al. [84] for long-range interactions. Our main motivation is to prove locality of the quasi-local inverse of the Liouvillian and automorphic equivalence for long-range interacting fermions, see section 3.3.1. Therefore, we want Lieb-Robinson bounds that hold for very general, time-dependent interactions, that appear as the time evolution of another interaction.

Note that the long-range Lieb-Robinson bounds [149, 210] with linear light cones, which we mentioned before, do not hold for fermions, as we comment at the end of this section. Moreover, their proofs are very long and technical. Instead, we prove the following Lieb-Robinson bound with algebraic light cones for fermionic systems, based on the simpler proofs from [160, 84].

Theorem 3.2. *Let $D \in \mathbb{N}_+$, $\mathscr{C}_\mathrm{A} > 0$ and $\alpha > D$. Then there exists a constant $C > 0$ such that for all lattices $\Lambda \in \mathscr{G}(D, \mathscr{C}_\mathrm{A})$, all $\sigma \in \big((D+1)/(\alpha+1), 1\big)$, all intervals $I \subset \mathbb{R}$, time-dependent interactions Φ on I, $X, Y \subset \Lambda$, $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$ with A or B even, and $s, t \in I$ the following holds*

$$\big\|\big[\tau_{t,s}(A), B\big]\big\| \leq 2\,\|A\|\,\|B\| \min\{|X|, |Y|\} \\ \times \Big(\mathrm{e}^{v\,|t-s|-r^{1-\sigma}} + C_\sigma\,(r+1)^{-\sigma\alpha}\,v\,|t-s|\,\big(1 + (v|t-s|)^{D/(1-\sigma)}\big)\Big).$$

Here $v = \max\{2\mathrm{e}\,\|F_\alpha\|_\Lambda\,\|\Phi\|_{F_\alpha}, \|\Phi\|_{F_\alpha,1}\}$, $r = d(X,Y)$ and $C_\sigma = C\left(\sigma - \frac{D+1}{\alpha+1}\right)^{-2}\frac{1}{1-\sigma}\,\Gamma\!\left(\frac{D}{1-\sigma}\right)$ with Γ the gamma function.

In comparison to the previous result [84], we obtain the factor $\min\{|X|, |Y|\}$ instead of a factor $|X|^n$ with $n = \lceil \sigma D/(\sigma\alpha - D)\rceil + 2$ and explicitly write down the constants v and C_σ.

The idea of the proof is to iteratively use the interaction picture to split the interaction into different length scales. More precisely, one uses (see lemma P7-2)

$$\big\|\big[\tau_{t,s,<R}(A), B\big]\big\| \leq \big\|\big[\tau_{t,s,<R'}(A), B\big]\big\| + 2\,\|B\| \sum_{\substack{Z\subset\Lambda:\\ R'\leq\mathrm{diam}(Z)<R}} \int_{\min\{s,t\}}^{\max\{s,t\}} \big\|\big[\tau_{t,\theta,<R'}(A), \Phi(Z,\theta)\big]\big\|\,\mathrm{d}\theta, \quad (3.6)$$

where $\tau_{t,s,<R}(A)$ is the Heisenberg dynamic generated by all interaction terms $\Phi(Z)$ with $\mathrm{diam}(Z) < R$. Since the bound does not depend on R, one can take R large enough so that $\tau_{t,s,<R}(A)$ is the dynamics on the whole finite Λ. One then uses a Lieb-Robinson bound with exponential decay $\mathrm{e}^{v|t-s|-r/R'}$ for interactions of range R' to bound the two commutators in (3.6) and chooses $R' \sim r^\sigma = d(X,Y)^\sigma$. Importantly, the main improvement is to use the finite-range Lieb-Robinson bound with $|X|$ for

the first term and with $|Z|$ for the second term. The $|Z|$ later gets absorbed by the interaction norm and the bound of the sum includes a factor $|X|$. In this way, we obtain a bound that scales with $|X|$ instead of $|X|^2$ as in [160] after applying (3.6) once. The final statement of theorem 3.2 is obtained after applying (3.6) repetitively to the integrand of the bound (3.6) and gives the improvement over [84] as explained below the statement. See section P7-3 for more details.

Unfortunately, this procedure does not provide Lieb-Robinson bounds with linear light cones for two reasons: To obtain the second term on the right-hand side of (3.6) in the proof of lemma P7-2, we use the trivial estimate for the outer commutator in

$$\left[\tau_{\theta,s,<R'}\,\tau_{s,\theta}(B), \left[\tau_{t,s,<R'}(A), \tau_{\theta,s,<R'}\big(H_{\geq R'}(\theta)\big)\right]\right].$$

This then requires us to choose $R' \sim r^{\sigma}$ to obtain some decay in r. And choosing $R' \sim r^{\sigma}$, the bound for the first term in (3.6) does not have a linear light cone. To obtain linear light cones, the works [149, 210] instead prove bounds for the interaction picture dynamic $\tau_{\theta,s,<R'}\,\tau_{s,\theta}$ and then provide stronger bounds for the nested commutator, to obtain better light cones. However, their proofs only work for spin systems, since they use the *conditional expectation trick*, which allows them to first prove Lieb-Robinson bounds for observables A and B supported on single sites and then extend them to observables with arbitrary support. The details and problems for fermions are explained in section P7-6. Lieb-Robinson bounds with linear light cones are interesting from both a physical perspective, as ballistic transport of information is believed even for long-range interactions, and a technical perspective, as Lieb-Robinson bounds underlie many results.

Lieb-Robinson bounds with linear light cones for polynomially decaying interactions would for example enhance the results obtained in [32] and enable us to prove automorphic equivalence for Hamiltonians with varying magnetic fields, see section 3.1.4.

Moreover, [149, 210] also rely on a quite involved iteration. Hence, the concise proof in publication P7, which is more transparent, has value in its own. And it allows proving automorphic equivalence (see section 3.3.1 and [31]) and generalized superadiabatic theorems [32] for polynomially decaying interactions.

3.1.3 Publication P4: Lieb-Robinson bounds for perturbed dynamics

In publication P4 we prove a stability result for Lieb-Robinson bounds. The motivation for this type of stability mostly lies in a clean LPPL result, where one prefers to impose conditions only on the unperturbed Hamiltonian H. One is then left to prove that the perturbed Hamiltonian $H+V$ also satisfies a Lieb-Robinson bound. The original proof in publication P4 is written for time-independent Hamiltonians, but since the proof is exactly the same, we give the statement for time-dependent Hamiltonians here.

Proposition 3.3 (Stability of Lieb-Robinson bounds, see lemma P4-42)**.** *Let Λ be a finite lattice, $I \subset \mathbb{R}$ an interval and $t \mapsto H(t) \in \mathscr{A}_\Lambda$ be a continuous and point-wise self-adjoint Hamiltonian, and assume that H satisfies a Lieb-Robinson bound such that for some $\zeta_{\mathrm{LR}} : \{X \subset \Lambda\} \times \{Y \subset \Lambda\} \times [0,\infty) \to [0,\infty)$ it holds that*

$$\left\|[\tau^H_{t,s}(A), B]\right\| \le \|A\| \, \|B\| \, \zeta_{\mathrm{LR}}\big(X, Y, |t-s|\big)$$

for all $X, Y \subset \Lambda$, $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$ and $s,t \in I$.

Moreover, let $Z \subset \Lambda$, $t \mapsto V(t) \in \mathscr{A}_Z$ be continuous and point-wise self-adjoint and denote with $\tau^{H+V}_{t,s}$ the dynamics associated to $H+V$. It satisfies the Lieb-Robinson bound

$$\left\|[\tau^{H+V}_{t,s}(A), B]\right\| \le \|A\| \, \|B\| \left(\zeta_{\mathrm{LR}}\big(X, Y, |t-s|\big) + 2 \sup_{r\in I} \|V(r)\| \min_{W\in\{X,Y\}} \left| \int_s^t \zeta_{\mathrm{LR}}\big(Z, W, |r-s|\big) \, \mathrm{d}r \right| \right)$$

for all $X, Y \subset \Lambda$, $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$ and $s,t \in I$.

The statement immediately generalizes to infinite lattices, and can be generalized to SLT perturbations $V = \sum_{Z' \subset Z} V_{Z'}$ with a bound in interaction norm.

We use this result in publication P4, so that we only need to assume decay of correlations and a Lieb-Robinson bound for the Hamiltonian H, to conclude LPPL for the Gibbs state, see theorem 3.13. Moreover, we use it in publication P5, to obtain better Lieb-Robinson bounds for the perturbed Hamiltonian $H + \lambda V$, where H is given by a commuting interaction but V is not necessarily commuting with H, see theorem 3.7.

For the proof, one just compares the two evolutions in the standard way using the fundamental theorem of calculus and then applies Lieb-Robinson bounds. A detailed proof for time-independent interactions is given in section P4-10.2.1 and the same arguments also hold for time-dependent Hamiltonians.

This is one of the simplest ways to obtain stability of Lieb-Robinson bounds. As noted above, one can generalize this to SLT perturbations, basically using a statement like [174, Theorem 3.4]. In theorem P5-15, we show that one can prove Lieb-Robinson bounds for dynamics generated by $\lambda H^0 + H^1$ with a velocity independent of λ, if H^0 is given by a commuting and finite-range SLT operator and H^1 is sufficiently local to satisfy Lieb-Robinson bounds, which is another kind of stability. With similar ideas, Toniolo and Bose [208] discuss stability in the following sense: Assuming there exists a region in space where the Lieb-Robinson bound shows very slow transport (usually subballistic, as for commuting interactions) and the observables are located on different sides of this region, then the slow-down of propagation in the region in-between is enough to obtain improved Lieb-Robinson bounds. The same authors also recover a similar result to ours in [207].

3.1.4 Publication P10: Lieb-Robinson bounds and existence for dynamics with spatially growing generators

In contrast to all Lieb-Robinson bounds we have discussed so far, we consider dynamics with unbounded generators in publication P10. More precisely, we allow the local terms to grow linearly in space. To formulate this assumption properly, we consider so called *zero-chains* instead of interactions. A time-dependent zero-chain on the interval $I \subset \mathbb{R}$ and (possibly infinite) D-regular lattice Γ is a map

$$\Phi \colon I \times \Gamma \to \mathcal{A}_\Gamma^+, \ (t, x) \mapsto \Phi_x(t),$$

such that each term $\Phi_x(t)$ is self-adjoint, $t \mapsto \Phi_x(t)$ is norm-continuous and

$$\mathcal{L}_{\Phi(t)} A := \sum_{x \in \Gamma} [\Phi_x(t), A]$$

converges unconditionally for all $A \in \mathcal{A}_\Gamma^{\mathrm{loc}}$. As we consider possibly infinite lattices here, the Hamiltonian is not a well-defined object, but the Liouvillian $\mathcal{L}_{\Phi(t)}$ is. Additionally, we require that each term $\Phi_x(t)$ is localized around x. To quantify this localization, we introduce the norms

$$\|A\|_{F,x} := \|A\| + \sup_{r \geq 0} \frac{\|(1 - \mathbb{E}_{B_x(r)})A\|}{F(r)},$$

for $x \in \Gamma$ and bounded functions F. Then, we assume that there is $x_0 \in \Gamma$ and $C_\Phi > 0$ such that Φ satisfies

$$\sup_{t \in I} \|\Phi_x(t)\|_{\mathrm{e}^{-b\cdot},x} \leq C_\Phi \left(1 + d(x, x_0)\right) \quad \text{for all} \quad x \in \Gamma.$$

The precise results in publication P10, only require decay with any function G instead of $\mathrm{e}^{-b\cdot}$ and additionally assume Lieb-Robinson bounds with linear light cones for zero-chains Ψ that satisfy a uniform bound $\sup_{t \in I} \sup_{x \in \Gamma} \|\Psi_x(t)\|_{G,x} \leq C_\Psi$, see assumption P10-2. For fermions, such Lieb-Robinson bounds are currently only known for exponentially decaying interactions. We also expect them to hold for polynomially decaying interactions, similarly to the result for spin systems [149], but their proof does not immediately work for fermions as explained in section 3.1.2. Hence, we restrict to the results for short-range interactions here.

Under these assumptions we prove existence of the dynamics generated by Φ and a Lieb-Robinson bound with exponential light cones. This notably enlarges the class of systems, known to have well-defined dynamics on infinite lattices.

Theorem 3.4 (Simplified version of theorem P10-8). *The time-dependent zero-chain Φ generates a unique cocycle of automorphisms $(\alpha_{s,t})_{s,t \in I}$ on $\mathcal{A}_\Gamma$. Moreover, for each $v > 0$,* *there are* $C_\nu, \gamma_\nu > 0$ *that do not depend on* Φ, *such that for all* $s, t \in I$ *and* $A \in \mathscr{A}_\Gamma$ *satisfying* $\|A\|_{F_\nu, x_0} < \infty$ *we have the bound*

$$\|\alpha_{s,t}\, A\|_{F_\nu, x_0} \leq C_\nu\, \mathrm{e}^{\gamma_\nu\, C_\Phi\, |t-s|}\, \|A\|_{F_\nu, x_0}, \tag{3.7}$$

where $F_\nu(r) = (1+r)^{-\nu}$.

Note, that $\|A\|_{F_\nu, x} < \infty$ for some $x \in \Gamma$, implies $\|A\|_{F_\nu, y} < \infty$ for all $Y \in \Gamma$. Hence, we prove that polynomially decaying quasi-local observables A are mapped to quasi-local observables $\alpha_{s,t}(A)$ with the same decay. Moreover, we can phrase (3.7) in terms of usual commutator Lieb-Robinson bounds. Therefore, let $A \in \mathscr{A}_{B_k(x_0)}$ and $B \in \mathscr{A}_Y^+$ for any $Y \subset \Gamma$ with $r := d\big(B_k(x_0), Y\big) > 0$. Then,

$$\|[\alpha_{s,t}\, A, B]\| \leq 2\, \|A\|\, \|B\|\, C_\nu\, \mathrm{e}^{\gamma_\nu\, C_\Phi\, |t-s| - \nu \ln(1 + r/(1+k))},$$

which is an exponential light cone, since the bound is small whenever

$$r \gg (1+k)\, \mathrm{e}^{C_\Phi\, \gamma_\nu\, \nu^{-1}\, |t-s|}.$$

As mentioned above, theorem 3.4 enlarges the class of interactions known to have a well-defined infinite volume dynamics. Our motivation mostly comes from automorphic equivalence and the definition of gapped phases. In particular, once Lieb-Robinson bounds with linear light cones for polynomially decaying interactions for fermions are known, our result proves automorphic equivalence for gapped phases of matter with varying homogeneous magnetic fields. While the Hamiltonians of these systems are usually given by a bounded interactions Φ_x^b, the derivative $\partial_b\, \Phi_x^b$ with respect to the magnetic field b grows linearly in x. Previously available results about the spectral flow[8] generated by $\mathscr{I}_{H^b}(\partial_b\, H^b)$ hence cannot be applied, while our conditional result gives a simple and physically motivated proof of existence and locality of the spectral flow once Lieb-Robinson bounds with linear light cones for polynomially decaying interactions are known.[9]

Our result has some analogy with solutions to first order ODEs on $\mathbb{R}^D$, which exist as long as the velocity field is locally Lipschitz continuous and has a linear upper bound. Then, integral curves do not reach infinity in finite time and the distance to the starting point grows at most exponentially.

The proof, however, is not related to this analogy. Instead, we construct a sequence of cocycles $\alpha_{s,t}^{(k)}$ by restricting Φ to act non-trivially only on balls $B_{x_0}(k)$. These restricted zero-chains $\Phi^{(k)}$ are bounded in space

$$\sup_{t \in I}\, \sup_{x \in \Gamma} \|\Phi_x^{(k)}(t)\|_{\mathrm{e}^{-b\,\cdot}, x} \leq C_\Phi\, (1 + k/2)$$

[8] The spectral flow is used to connect ground states along paths of gapped Hamiltonians by local dressings. It is discussed in detail in section 3.3.

[9] For exponentially localized Hamiltonians, the quasi-local inverse Liouvillian decays [174, Corollary 6.6] like $\mathrm{e}^{-x/\ln(x)^2}$, so Lieb-Robinson bounds with linear light cones for stretched exponential decay are enough.

and hence satisfy a Lieb-Robinson bound with a linear light cone and velocity $v_{\mathrm{LR}}^{(k)} \sim k$. Then, we consider $A \in \mathcal{A}_{B_{x_0}(k/8)}$. The evolution $\alpha_{s,t}^{(k)}$ spreads the support of A, but for a short time $|t-s| < \tau$, we can well approximate $\alpha_{s,t}^{(k)}(A)$ within an enlarged ball of radius $k/8 + v_{\mathrm{LR}}^{(k)}\,\tau \leq k/4$, after choosing τ small enough such that $v_{\mathrm{LR}}^{(k)}\,\tau \leq k/8$. Note, that this choice of τ only depends on the constants and is uniform in k. As $\alpha_{s,t}^{(k)}$ and $\alpha_{s,t}$ have almost the same generator within $B_{x_0}(k/4)$, we understand that $\alpha_{s,t}^{(k)}(A)$ is a good approximation of $\alpha_{s,t}(A)$ for small times. Indeed, $\alpha_{s,t}(A) = \lim_{k\to\infty} \alpha_{s,t}^{(k)}(A)$ for small times and we also prove the bound $\|\alpha_{s,t}\, A\|_{F_v,x_0} \leq \gamma_v\, \|A\|_{F_v,x_0}$ for $|t-s| \leq \tau$ by this argument. We then basically iterate this bound to obtain the statement in theorem 3.4 for all times.

3.2 Spectral filter functions

In this section, we explain the general idea behind spectral filters. They can be used to prove decay of correlations of gapped ground states, see section 3.4 and [118, 171]. Moreover, they are used to construct the spectral flow, which underlies automorphic equivalence of gapped ground states, see section 3.3 and [120, 23], some LPPL results for gapped ground states, see the same section and [23], and adiabatic theorems and response theory, see section 3.5 and [17, 203]. Another filter is used in quantum belief propagation to write a differential equation for Gibbs states, see section 3.6 and [115].

Let Φ be a self-adjoint interaction and H the corresponding SLT operator and assume that $\tau_t^H = \mathrm{e}^{\mathrm{i}Ht}\, V\, \mathrm{e}^{-\mathrm{i}Ht}$ satisfies a Lieb-Robinson bound. Furthermore, let $V \in \mathcal{A}_X$ be self-adjoint. We decompose H in its spectral decomposition $H = \sum_j E_j\, P_j$ and use the Fourier transform[10] to write

$$\mathcal{Q}(V) := \int_{\mathbb{R}} f(t)\, \mathrm{e}^{\mathrm{i}Ht}\, V\, \mathrm{e}^{-\mathrm{i}Ht}\; \mathrm{d}t = \sqrt{2\pi}\, \sum_{j,k} \hat{f}(E_k - E_j)\, P_j\, V\, P_k. \tag{3.8}$$

The filter function $f\colon \mathbb{R} \to \mathbb{R}$ usually has good decay, so that one can infer locality of (3.8) as we explain next. Its Fourier transform $\hat{f}$ on the other hand allows us to write certain function of the spectrum.

To use localization, one approximates $\tau_t^H(V)$ with an operator in $\mathcal{A}_{X_r}$, where X_r is the fattening defined in (2.3). For this approximation one often uses a conditional expectation $\mathbb{E}_{X_r} : \mathcal{A}_\Lambda \to \mathcal{A}_{X_r}$, which for spin systems is just the partial trace. From an explicit construction of the conditional expectation, one then obtains a bound

$$\big\|\big(\mathrm{id} - \mathbb{E}_{X_r}\big)\, \tau_t^H(V)\big\| \leq \sup_{\substack{B \in \mathcal{A}_{\Lambda\setminus X_r} \\ \|B\|=1}} \|[\tau_t^H(V), B]\|. \tag{3.9}$$

[10] Here and in most publications we use the convention that $\hat{f}(\omega) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t$ for the Fourier transform. Only in publication P4, we omit the prefactor $1/\sqrt{2\pi}$ in the Fourier transform.

Hence, $\mathbb{E}_{X_r}(\tau_t^H(V))$ is a good approximation as long as t is small compared to r, given Lieb-Robinson bounds for τ_t^H. And, if $|f(t)|$ decays in $|t|$, one can split the integral at $T > 0$, which will eventually depend on r, to bound

$$\begin{aligned}\left\|\left(\mathrm{id} - \mathbb{E}_{X_r}\right) \mathcal{Q}(V)\right\| &= \left\|\left(\mathrm{id} - \mathbb{E}_{X_r}\right) \int_{\mathbb{R}} f(t)\, \tau_t^H(V)\ \mathrm{d}t\right\| \\ &\leq \int_{|t|\leq T} \sup_{\substack{B\in\mathscr{A}_{\Lambda\setminus X_r}\\ \|B\|=1}} \|[\tau_t^H(V), B]\|\ \mathrm{d}t + 2\,\|V\| \int_{|t|>T} |f(t)|\ \mathrm{d}t.\end{aligned}$$

Hence, also $\mathcal{Q}(V)$ can be approximated within X_r.

The conditional expectation can be explicitly constructed for spin systems and fermions and some properties are given in lemmata P7-11 and P7-19, based on [174] and [173], respectively. For fermions, the bound (3.9) only holds for $V \in \mathscr{A}_X^+$, which poses some problems, as explained in section P7-6. Alternatively to the direct bound (3.9), we can also approximate $\tau_t^H(V)$ with $\tau_t^{H_{X_r}}(V)$. Here, since we restrict the evolution to the terms supported in X_r, also $\tau_t^{H_{X_r}}(V) \in \mathscr{A}_{X_r}$. Comparing the two evolutions using the fundamental theorem of calculus, decay of the interaction Φ and Lieb-Robinson bounds for $\tau_t^{H_{X_r}}$, we again see that this approximation is good as long as t is small compared to r. See [174, Theorem 3.4 (ii)] or the proof of theorem P5-9 for details. This approximation also works for non-even V in fermionic systems. It is advantageous to compute the approximation numerically, as one only to do computations on the smaller algebra $\mathscr{A}_{X_r}$. However, it is a less good approximation, since it does not include effects from interactions with $\mathscr{A}_{\Lambda\setminus X_r}$.

In the next sections, we discuss some particular applications of spectral filters.

3.3 Spectral flow and automorphic equivalence

We now explain the spectral flow, which was developed in [120, 23, 173] to connect the ground state sectors along a differentiable path of Hamiltonians. It serves as a characterisation for gapped phases of matter [23], including topological phases characterized by various indices [179, 178, 12, 14, 15]. The underlying spectral filter is also important for rigorous justification of response theory and the adiabatic theorem for many-body lattice systems [17, 165, 203, 124, 32], which we discuss in section 3.5.

In the following, we consider Hamiltonians $H(s)$ with a uniformly gapped part of the spectrum, the locality of $H(s)$ and $\dot{H}(s)$ will be specified later. More precisely, we assume the following.

Assumption 3.5. For a differentiable family of Hamiltonians $s \mapsto H(s)$ with $s \in I$ and I a closed interval, we assume that for all $s \in I$ the spectrum $\sigma(H(s))$ has a gapped part $\sigma_*(s) \subset \sigma(H(s))$ satisfying the following: There exist $g > 0$ and continuous functions $f_\pm : I \to \mathbb{R}$ such that

(i) $f_\pm(s) \in \mathbb{R} \setminus \sigma(H(s))$,
(ii) $\big[f_-(s), f_+(s)\big] \cap \sigma(H(s)) = \sigma_*(s)$ and
(iii) $\operatorname{dist}\big(\sigma_*(s), \sigma(H(s)) \setminus \sigma_*(s)\big) \geq g$

for all $s \in I$. ◇

The prime example is the case where $\sigma_*(s)$ is the ground state sector with possible eigenvalue splitting and where each $H(s)$ has a uniform gap above the ground state.

Note that assumption 3.5 implies differentiability of $s \mapsto P(s)$, using standard arguments involving the Cauchy formula for the resolvent. And taking a derivative of $P(s)^2 = P(s)$ one finds

$$\mathrm{i}\,\dot{P}(s) = \big[\mathrm{i}\,[\dot{P}(s), P(s)], P(s)\big] = [G_{\mathrm{Kato}}(s), P(s)], \tag{3.10}$$

with the self-adjoint Kato-generator $G_{\mathrm{Kato}}(s) := \mathrm{i}\,[\dot{P}(s), P(s)]$. Hence, the solution of

$$\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}s}\,U_{\mathrm{Kato}}(s) = G_{\mathrm{Kato}}(s)\,U_{\mathrm{Kato}}(s) \quad \text{with} \quad U_{\mathrm{Kato}}(0) = \mathbb{1}$$

is unitary and satisfies

$$P(s) = U_{\mathrm{Kato}}(s)\,P(0)\,U_{\mathrm{Kato}}(s)^*. \tag{3.11}$$

However, the Kato-generator $G_{\mathrm{Kato}}(s)$ does not come from a local interaction in general and thus cannot be used to infer locality results.

To circumvent this, we notice that (3.11) only fixes, how the unitary acts on $P(0)$, and can be satisfied by many other unitaries as well. But fixing the generator G_{Kato} also fixes how U_{Kato} acts on all other states. Indeed, looking back at (3.10), we can add anything to the generator that commutes with $P(s)$, and still satisfy (3.11). The alternative we use is commonly referred to as the *Hastings generator* $G(s) = -\mathcal{I}_{H(s),g}(\dot{H}(s))$, where $\mathcal{I}_{H,g}$ is the *quasi-local inverse of the Liouvillian*, which is the spectral filter

$$\mathcal{I}_{H,g} \colon \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda, \quad \mathcal{I}_{H,g}(A) = \int_{\mathbb{R}} \mathcal{W}_g(t)\,\mathrm{e}^{\mathrm{i}Ht}\,A\,\mathrm{e}^{-\mathrm{i}Ht}\,\mathrm{d}t, \tag{3.12}$$

with some filter function $\mathcal{W}_g \in L^1(\mathbb{R})$ with decay such that

$$\sup_{|t|>1} |t|^n\,|\mathcal{W}_g(t)| < \infty \quad \text{for all } n \in \mathbb{N}$$

and with Fourier transform $\hat{\mathcal{W}}_g \in C^\infty(\mathbb{R})$ satisfying

$$\hat{\mathcal{W}}_g(\omega) = \frac{-\mathrm{i}}{\sqrt{2\pi}\,\omega} \quad \text{for all } |\omega| \geq g \qquad \text{and} \qquad \hat{\mathcal{W}}_g(0) = 0.$$

The quasi-local inverse of the Liouvillian $\mathcal{I}_{H,g}$ is quasi-local since the decay of $\mathcal{W}_g$ allows good locality estimates as outlined in section 3.2, and it inverts the Liouvillian on off-diagonal (w.r.t. to the projection P onto the gapped sector σ_*) operators. More precisely, for any $A \in \mathcal{A}_\Lambda$ such that $A = P\,A\,P^\perp + P^\perp\,A\,P$, where $P^\perp = \mathbb{1} - P$, it holds that

$$-\mathrm{i}\,[H, \mathcal{I}_{H,g}(A)] = A. \tag{3.13}$$

One can also rewrite (3.12) as

$$\mathcal{I}_{H,g}(A) = \int_{\mathbb{R}} \mathrm{d}t\, w_g(t) \int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{i}Hs}\, A\, \mathrm{e}^{-\mathrm{i}Hs}, \tag{3.14}$$

where $w_g \in L^1(\mathbb{R})$ has similar decay, satisfies $\int_{\mathbb{R}} w_g = 1$, and its Fourier transform $\hat{w}_g$ is continuously differentiable, symmetric, and compactly supported in $\operatorname{supp} \hat{w}_g \subset [-g, g]$. The functions $\mathcal{W}_g$ and w_g are related by $\mathcal{W}_g(t) = -\int_{-\infty}^t \mathrm{d}s\, w_g(s) + \mathbb{1}_{[0,\infty)}(t)$, and we use both variants (3.12) and (3.14) in the following.

The existence of the functions $\mathcal{W}_g$ and w_g is a problem in itself, and was solved in [23, 175].[11] Moreover, one cannot achieve the same with a function $\mathcal{W}_g$ that decays exponentially [114, 126, section V] but almost exponential decay (up to logarithmic corrections) is possible [23, 174, section VI.E]. In section 3.3.3 we discuss how one can construct an *almost* inverse Liouvillian with better decay, which is beneficial for some applications.

Let us now come back to the spectral flow connecting the gapped projections $P(s)$. Since $\dot{P}$ is off-diagonal, one concludes on finite lattices that

$$\mathrm{i}\,\dot{P} = \big[H, \mathcal{I}_{H,g}(\dot{P})\big] = \mathcal{I}_{H,g}\big([H, \dot{P}]\big) = -\mathcal{I}_{H,g}\big([\dot{H}, P]\big) = \big[-\mathcal{I}_{H,g}(\dot{H}), P\big],$$

where we use (3.13) in the first step and the explicit integral form of $\mathcal{I}_{H,g}$ and $[H, P] = 0$ in the others. Hence, $G(s) = -\mathcal{I}_{H(s),g}(\dot{H}(s))$, which we call the Hastings generator because Hastings pioneered the use of filter functions in this context [110, 120, 114, 169], generates a unitary $U(s)$ which also satisfies (3.11). In this sense, the projections $P(s)$ are obtained from $P(0)$ by local dressings. In the case that $P(s)$ are the ground state projections, this property is usually referred to as *automorphic equivalence* and serves many purposes as mentioned before.

One interesting application of automorphic equivalence, is the LPPL principle for perturbations which do not close the gap. They were first observed by Bachmann et al. [23] in the context of short-range Hamiltonians. There, the authors consider Hamiltonians $H(s) = H + s\,W$, were H is given by a short-range interaction and $W \in \mathcal{A}_X$. Then, one assumes that $s \mapsto H(s)$ satisfies assumption 3.5. By the outlined locality

[11] A function $\mathcal{W}_g$ with the mentioned properties was constructed in [23]. [174] provides an extensive overview on the topic and in particular discusses how to obtain the better decay $\mathrm{e}^{-x/\ln(x)^2}$. Monaco and Teufel [165] modified the function to additionally satisfy $\hat{\mathcal{W}}_g(\omega) = 0$ for $|\omega| < \delta$, which is beneficial for the analysis of the transport within the gapped sector as long as $\operatorname{diam}(\sigma_*(s)) < \delta < \gamma$ since it implies $P\,\mathcal{I}_{H,g}(A)\,P = 0$ for all $A \in \mathcal{A}_\Lambda$.

of the quasi-local inverse Liouvillian, the generator of the spectral flow $-\mathcal{I}_{H,g}(W)$ is localized around X, and the intertwining unitaries $U(s)$ can be approximated by strictly local unitaries $U_R(s) \in \mathcal{A}_{X_R}$. A simple argument then gives

$$\begin{aligned} \left|\operatorname{tr}\big(P(s)\,B\big) - \operatorname{tr}\big(P(0)\,B\big)\right| &= \left|\operatorname{tr}\big(P(0)\,U(s)^*\,[B, U(s) - U_R(s)]\big)\right| \\ &\leq 2\ \operatorname{rank} P(0)\,\|B\|\,\|U(s) - U_R(s)\|, \end{aligned} \tag{3.15}$$

for all $B \in \mathcal{A}_Y$ and $R \leq d(X, Y)$, which ensures that B and $U_R(s)$ commute. For short-range interactions, one then obtains stretched exponential decay in R by the locality of the spectral flow [174, section VI.E].

In the following sections, we describe the extensions and improvements we contribute in publications P5, P7 and P9.

3.3.1 Publication P7: Spectral flow and automorphic equivalence for long-range interactions

In publication P7, we prove that the inverse Liouvillian for long-range interactions maps long-range interactions to long-range interactions, see proposition P7-8. This in particular allows to prove automorphic equivalence for long-range interactions.

Theorem 3.6 (Automorphic equivalence for long-range interactions, see theorem P7-9). *Let $D \in \mathbb{N}$, $\mathscr{C}_\mathrm{A} > 0$, $I = [0,1]$, $n \in \mathbb{N}$, $\beta > 0$, $\alpha > (n+1)\,D + 1 + \beta$, $g > \delta \geq 0$, and $v_* \geq 1$. Then there exists a constant $C > 0$ such that for all finite lattices $\Lambda \in \mathcal{G}(D, \mathscr{C}_\mathrm{A})$ the following holds:*

Let Φ be a time-dependent interaction on I such that $v = \max\{2\mathrm{e}\,\|F_\alpha\|_\Lambda\,\|\Phi\|_{F_\alpha}, \|\Phi\|_{F_\alpha,1}\} \leq v_*$, and let H_0 be a time-independent on-site Hamiltonian. Furthermore, assume that the corresponding operator family H satisfies assumption 3.5. Then the projections $P(s)$ associated to $\sigma_*(s)$ (see assumption 3.5) are unitarily equivalent

$$P(s) = U(s)\,P(0)\,U(s)^*, \tag{3.16}$$

where the unitaries have a local generator $G = -\mathcal{I}_{H,g}(\dot{H})$, which is given by an interaction Ψ, satisfying $\|\Psi(t)\|_{\beta,n} \leq C\,\|\dot{\Phi}(t)\|_{\beta,n+1}$.

For the proof we follow the usual strategy, which is outlined in section 3.3, combined with the improved Lieb-Robinson bounds from theorem 3.2. The full proof is given in appendix P7-D. A similar result, but for larger α or with $\|\Psi(t)\|_{\beta,n} \leq C\,\|\dot{\Phi}(t)\|_{\beta,n+m}$ for potentially large m, can also be obtained using the earlier Lieb-Robinson bounds by Matsuta et al. [160] and Else et al. [84], a detailed analysis and comparison is given after theorem P7-9.

A direct consequence of theorem 3.6 is automorphic equivalence for interactions that decay faster than any polynomial. This was believed to hold true before, but not actually proven to our knowledge. An analogous statement in the thermodynamic limit, i.e. for infinite graphs, was recently obtained in [31], by using our Lieb-Robinson bounds from theorem 3.2. Moreover, the same authors also used this Lieb-Robinson bound to prove response theory directly in the thermodynamic limit [32].

As discussed above, one important application of automorphic equivalence is the LPPL principle. For long-range interactions, we obtain the following statement.

Theorem 3.7 (LPPL for long-range interactions, see theorem P7-10). *Let $D \in \mathbb{N}$, $\mathscr{C}_{\mathrm{A}} > 0$, $n \in \mathbb{N}$, $\alpha > D$, $g > \delta \geq 0$, $v_* \geq 1$ and $\varepsilon \in (0, \alpha - D)$. Then there exists a constant $C > 0$ such that for all lattices $\Lambda \in \mathscr{G}(D, \mathscr{C}_{\mathrm{A}})$ the following holds:*

Let Φ be a time-dependent interaction on I such that $v = \max\{2\mathrm{e}\,\|F_\alpha\|_\Lambda\,\|\Phi\|_{F_\alpha}, \|\Phi\|_{F_\alpha,1}\} \leq v_*$, $\dot{\Phi}(s) \in \mathscr{A}_X^+$ for some $X \subset \Lambda$ and, and let H_0 be a time-independent on-site Hamiltonian. Furthermore, assume that the corresponding operator family H satisfies assumption P7-7. Then, for all $A \in \mathscr{A}_Y$ with $Y \subset \Lambda$ and $Y \cap X = \emptyset$ it holds that

$$\big|\mathrm{tr}\big(P(s)\,A\big) - \mathrm{tr}\big(P(0)\,A\big)\big| \leq C\,s\;\mathrm{rank}\big(P(0)\big)\,|Y|\,\|A\|\,\sup_{t\in I}\|\dot{H}(t)\|\,\big(d(X,Y)+1\big)^{-\alpha+\varepsilon} \tag{3.17}$$

and

$$\big|\mathrm{tr}\big(P(s)\,A\big) - \mathrm{tr}\big(P(0)\,A\big)\big| \leq C\,s\;\mathrm{rank}\big(P(0)\big)\,|Y|\,\|A\|\,\|\dot{\Phi}\|_{0,1}\,\big(d(X,Y)+1\big)^{-\alpha+D+\varepsilon}. \tag{3.18}$$

In the long-range setting, we do not follow the proof outlined in (3.15), which uses automorphic equivalence. Instead, we adjust the proof of theorem 3.6 and obtain a statement with loosened assumptions, compared to a result which could be obtained from theorem 3.6 together with (3.15). The full proof is given in appendix P7-D.

A similar statement on LPPL was recently obtained by Wang and Hazzard [214] using a complex analysis method, which avoids the use of the spectral flow. In this way, they obtain a result like (3.17), with better decay but only for two-body interactions. More precisely, they use a Lieb-Robinson bound [89] for two-body interactions satisfying $\|\Phi\|^{\mathrm{tb}}_{F_{\alpha_{\mathrm{tb}}}} \leq C$ with $\alpha_{\mathrm{tb}} > 2D$ as input, to obtain (3.17) but with α replaced by α_{tb}. As explained in section 2.3, these interactions are included in the assumptions of theorem 3.7 as $\|\Phi\|_{F_\alpha} \leq \|\Phi\|^{\mathrm{tb}}_{F_{\alpha_{\mathrm{tb}}}}$ with $\alpha = \alpha_{\mathrm{tb}} - D$, but their result keeps the better decay with α_{tb}.

3.3.2 Publication P5: LPPL for commuting interactions

In publication P5, we use the enhanced Lieb-Robinson bounds for commuting interactions, see (3.3), to obtain LPPL for systems where the unperturbed Hamiltonian is commuting. More precisely, we consider $H(s) = H + s\,W$ and assume that H is given by a commuting interaction. As we do not want to require that W commutes with H, we use the stability of the Lieb-Robinson bounds from section 3.1.3 to obtain a good Lieb-Robinson bound also for $H + s\,W$. While one can use the spectral flow to prove LPPL also for commuting interactions, in publication P5, we instead use the complex analysis method from [214], which we already mentioned above.

Theorem 3.8 (LPPL for gapped ground states of commuting Hamiltonians)**.** *Let Φ be a commuting interaction on a finite lattice Λ and let $V \in \mathscr{A}_X$ with $X \in \Lambda$ be some perturbation. Moreover, assume that $H + sV$ has a unique ground state $P(s)$ and a gap of size at least $g > 0$ above the ground state for all $s \in [0,1]$.*

For polynomial decay $F_\alpha(r) := (1+r)^{-\alpha}$ with $\alpha > 0$ and for all $Y \subset \Lambda$ and $B \in \mathscr{A}_Y$, it holds that

$$\begin{aligned}&\Big|\mathrm{Tr}\big(P(0)\,B\big) - \mathrm{Tr}\big(P(1)\,B\big)\Big| \\ &\qquad \leq 32\,\|\Phi\|^{\mathrm{tb}}_{F_\alpha}\,\|B\|\,\big(\|V\| + \|V\|^2\big)\,|X|\,\frac{g+2}{g^3}\begin{cases}|Y|\,F_\alpha(r) & \text{for all } \alpha > 0 \text{ and}\\ C\,F_{\alpha-D}(r) & \text{for all } \alpha > D,\end{cases}\end{aligned} \tag{3.19}$$

where $r = d(X,Y)$ and C is the constant from the long-range commuting Lieb-Robinson bound (3.4).

For stretched-exponential decay $\chi_{b,p}(r) := \mathrm{e}^{-br^p}$ with $b > b' > 0$ and $p \in (0,1]$ and for all $Y \subset \Lambda$ and $B \in \mathscr{A}_Y$, it holds that

$$\Big|\mathrm{Tr}\big(P(0)\,B\big) - \mathrm{Tr}\big(P(1)\,B\big)\Big| \leq 8\,C_{b,b'}\,\|\Phi\|^{\mathrm{tb}}_{\chi_{b,p}}\,\|B\|\,\big(\|V\| + \|V\|^2\big)\,\min\big\{|X|,|Y|\big\}\,\chi_{b',p}(r)\,\frac{g+2}{g^3},$$

where $r = d(X,Y)$, and $C_{b,b'}$ is the constant from the short-range commuting Lieb-Robinson bound (3.5).

Interestingly, for exponentially decaying interactions, we obtain exponential decay in the distance and the exponent does not depend on the gap. In comparison, the original result using the spectral flow [23] and the complex analysis technique [214] can only obtain stretched exponential scaling for non-commuting Hamiltonians and the exponent scales linearly in the gap g. Using a different filter-function, we can also obtain exponential scaling for non-commuting interactions, see section 3.3.3, but without explicitly knowing the scaling with the gap g.

3.3.3 Publication P9: Gaussian filters for better spatial decay

In publication P9 we use Gaussian filter functions to improve the decay in certain applications. We recall that the exact quasi-local inverse Liouvillian defined in (3.12) uses a filter function that cannot have exponential decay. However, we can define an *almost* inverse Liouvillian, which uses a Gaussian filter function and hence decays faster than exponential Clearly, the Fourier transformation is also Gaussian and hence not compactly supported. This compact support of w_g was the reason why the inverse Liouvillian (3.14) is exact. Notably, Hastings and Wen [120] used a similar filter function for the original "quasi-adiabatic continuation" due to the lack of a compactly supported w_g with the properties described in section 3.3.

More precisely, for any[12] $\beta > 0$ we let

$$\phi_\beta(t) := \frac{\beta}{\sqrt{\pi}}\, \mathrm{e}^{-\beta^2 t^2}.$$

It is such that $\int \phi_\beta = 1$ and the Fourier transform is

$$\hat{\phi}_\beta(\omega) := \frac{1}{\sqrt{2\pi}} \int \mathrm{d}t\, \phi_\beta(t)\, \mathrm{e}^{-\mathrm{i}t\omega} = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{\omega^2}{4\beta^2}} \tag{3.20}$$

for which $\int \hat{\phi}_\beta = \beta\sqrt{\frac{2}{\pi}}$. Then, for any Hamiltonian H and observable A we define the *almost inverse Liouvillian*[13]

$$\mathcal{I}_{H,\beta}(A) := \int_{-\infty}^{\infty} \mathrm{d}t\, \phi_\beta(t) \int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{i}Hs}\, A\, \mathrm{e}^{-\mathrm{i}Hs}. \tag{3.21}$$

Following the argument in section 3.2, one can see from the decay of ϕ_β and the Lieb-Robinson bounds that $\mathcal{I}_{H,\beta}(A)$ is exponentially localized for exponentially localized H. The exact rate of localization depends on β and the decay of H.

We then prove, that $\mathcal{I}_{H(s),\beta}(\dot{H}(s))$ generates an almost spectral flow under suitable assumptions on the Hamiltonian. The missing minus sign compared to the previous sections comes from the fact that we use the notation from publication P9, where the state ω_s is a functional on the algebra and the spectral flow $\alpha_{s,0}$ acts on observables. This allows to compare the almost spectral flow to the exact spectral flow on local observables more naturally.

Theorem 3.9 (Almost spectral flow, theorem P9-12)**.** *Let $D \in \mathbb{N}$, $\mathscr{C}_\mathrm{A} > 0$, $b' > b > 0$, $C_\mathrm{int} > 0$, $g > 0$. Then there exist constants C and $c > 0$ such that the following holds. For all finite $\Lambda \in \mathscr{G}(D, \mathscr{C}_\mathrm{A})$ and smooth Hamiltonians H that satisfy assumption 3.5 with gap g and are given by interactions Φ such that $\|\Phi\|_{\chi_{b'},1} < C_\mathrm{int}$ and $\|\dot{\Phi}\|_{\chi_{b'},1} < C_\mathrm{int}$, the flow $\alpha^\beta_{0,s}$ generated by $\mathcal{I}_{H(s),\beta}(\dot{H}(s))$ is an almost spectral flow in the sense that*

$$\left|\omega_s(A) - \omega_0 \circ \alpha^\beta_{0,s}(A)\right| \le C\, |X|^2\, \|A\|\, \mathrm{e}^{-c\beta^{-2}}, \tag{3.22}$$

for all $X \subset \Lambda$ and $A \in \mathcal{A}_X$ and $\beta \in \left(0, \min\left\{1, \sqrt{2b}\, v\right\}\right)$, where $v \sim \|\Phi\|_{\chi_b,1}$ is the Lieb-Robinson velocity from lemma P9-2.

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[12] For consistency with the publication P9, we use β as a parameter in the filter function. We emphasize that this is *not* an inverse temperature.

[13] In accordance to the notation in publication P9, we use $\mathcal{I}_{H,\beta}$ to denote the *almost* inverse Liouvillian in this section. It is not to be confused with the *exact* inverse Liouvillian as introduced in (3.12). The exact inverse Liouvillian is denoted $\mathcal{I}_H$ in publication P8, without the extra index to specify the gap.

We refer to publication P9 for the exact statement of the Lieb-Robinson bound for exponentially decaying interactions which we use. It was proven in [156, theorem 7.3.3] based on a previous result for time-independent Hamiltonians in [22, theorem A.1].

Since the almost spectral flow does not have the nice spectral properties from the exact spectral flow, we do not have a differential equation for the state as in (3.10). Instead, for the proof we use the fact that $\omega_s(A) = \omega_0 \circ \alpha_{s,0}(A)$ is given by the exact spectral flow and then compare $\alpha_{s,0}(A)$ with $\alpha^{\beta}_{s,0}(A)$. In this comparison, however, we cannot utilize locality of the exact spectral flow, as it is not exponential. Instead, we use triangle inequality to compare both automorphisms with the modified automorphism $\alpha^{\beta,X}_{s,0}(A)$, which is generated by

$$\sum_{Z\subset\Lambda} \mathscr{I}_{H(t),\beta_{X,Z}}(\dot{\Phi}(t,Z)), \tag{3.23}$$

where $\beta_{X,Z}$ is chosen such that $\beta_{X,Z} = \beta$ for Z in a surrounding of X and $\beta_{X,Z} \to 0$ as $d(X,Z) \to \infty$. In this way, we can control the spectral properties well enough to estimate $|\omega_0 \circ (\alpha_{s,0} - \alpha^{\beta,X}_{s,0})(A)|$, while $\alpha^{\beta,X}_{s,0}$ and $\alpha^{\beta}_{s,0}$ agree around X and are localized enough to estimate $\|(\alpha^{\beta,X}_{s,0} - \alpha^{\beta}_{s,0})(A)\|$.

Theorem 3.10 (LPPL with Gaussian filter function, theorem P9-13)**.** *Let $D \in \mathbb{N}$, $\mathscr{C}_{\mathrm{A}} > 0$, $b' > 0$, $C_{\mathrm{int}} > 0$, $g > 0$. Then there exist constants C and $c > 0$ such that the following holds. For all $\Lambda^{\mathrm{pert}} \subset \Lambda \in \mathscr{G}(D,\mathscr{C}_{\mathrm{A}})$ finite, smooth Hamiltonians H that satisfy assumption 3.5 with gap g and are given by interactions Φ such that $\|\Phi\|_{\chi_{b'},1} < C_{\mathrm{int}}$, $\|\dot{\Phi}\|_{\chi_{b'},1} < C_{\mathrm{int}}$, and $\dot{\Phi}(Z) = 0$ unless $Z \subset \Lambda^{\mathrm{pert}}$,*

$$\big|\omega_s(A) - \omega_0(A)\big| \le C\,|X|^2\,\|A\|\,\mathrm{e}^{-cd(X,\Lambda^{\mathrm{pert}})}, \tag{3.24}$$

for all $s \in [0,1]$, $X \subset \Lambda$ and $A \in \mathscr{A}_X$.

As for the long-range result, we do not obtain the LPPL statement from theorem 3.9. Instead, we follow large parts of the proof. In particular, we do not need to consider the modified automorphism $\alpha^{\beta,X}$, as the restriction that $\dot{\Phi}$ vanishes around X provides enough locality.

Note that the gap assumption is crucial and means that one can only consider Hamiltonians $H(s) = H + sV$, when V does not close the gap. Theorem 3.10 improves the statement from (3.15), which was originally obtained in [23], to exponential decay. A similar statement was obtained in [214] using an approach that circumvents the inverse Liouvillian.

3.4 Decay of correlations for gapped ground states

Another well known application of spectral filter functions in quantum lattice systems is the proof of rapid decay of correlations for gapped ground states [113, 118, 171]. These results prove exponential decay of correlations for exponentially decaying Hamiltonians without ground state energy splitting. They have been extended for ground state sectors with energy splitting by [14].

Let us sketch the proof used in [14] before we discuss our contributions. Therefore, consider a Hamiltonian H with a ground state sector $\sigma_* \subset \sigma(H)$, i.e. $\sigma_* < \sigma(H) \setminus \sigma_*$, such that $\mathrm{dist}(\sigma_*, \sigma(H) \setminus \sigma_*) > g$ and $\mathrm{diam}(\sigma_*) < \delta$ for some $0 < \delta < g$. Then, we consider a spectral filter $\mathscr{J}$ as defined in (3.8) where $\hat{f}$ is a smeared out step function such that $\hat{f}(\omega) = 0$ for $\omega < \delta$ and $\hat{f}(\omega) = 1$ for $\omega > g$. Denoting the spectral projection onto σ_* as P and using (3.8) one finds

$$\mathscr{J}(A)\,P = 0 \qquad \text{and} \qquad P\,\mathscr{J}(A) = P\,\mathscr{J}(A)\,P^\perp = P\,A\,P^\perp. \tag{3.25}$$

This immediately implies for any ground state ρ, i.e. $\rho = P\,\rho\,P$, $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$ that

$$\begin{aligned} \mathrm{tr}\big(\rho\,[\mathscr{J}(A), B]\big) &= \mathrm{tr}\big(\rho\,P\,\mathscr{J}(A)\,B\big) - \mathrm{tr}\big(\rho\,B\,\mathscr{J}(A)\,P\big) \\ &= \mathrm{tr}(\rho\,P\,A\,P^\perp\,B) \\ &= \mathrm{tr}(\rho\,A\,B) - \mathrm{tr}(\rho\,A\,P\,B) \end{aligned}$$

and the left-hand side decays in $d(X, Y)$ by Lieb-Robinson bounds for H and decay of f. For non-degenerate ground states $\rho = |\Omega\rangle\langle\Omega| = P$, the right-hand side equals the usual covariance $\mathrm{Cov}_\rho(A, B)$.

Similarly to the situation with the spectral flow, the inverse Fourier transform can not decay exponentially and [14] only obtains superpolynomial decay in $d(X, Y)$ for exponentially localized Hamiltonians. The proof from [118, 171] is slightly different and provides exponential decay of correlations for ground state sectors without energy splitting.

3.4.1 Publication P5: Enhanced decay of correlations for commuting interactions

As for LPPL, the enhanced Lieb-Robinson bounds for commuting interactions lead to an improved decay of correlations for gapped ground states of commuting Hamiltonians. The original results from [118] already prove decay of correlations for general long-range interactions. Therefore, they use the Lieb-Robinson bounds with exponential light-cones that we mentioned in section 3.1. Recent improvements use newer Lieb-Robinson bounds and an alternative proof via complex analytic arguments [214].

Following the original proof, we obtain the following decay for commuting interactions, where we also restrict to systems without energy splitting of the ground state sector.

Theorem 3.11 (Decay of correlations for gapped ground states of commuting Hamiltonians, theorem P5-11)**.** *Let Φ be a commuting interaction on a finite graph Λ and* *assume that the spectrum of the associated Hamiltonian H has a gap $g > 0$ above the ground state sector $\sigma_* = \{E_0\}$ as above. Let P be the projection onto the ground state sector and ρ_0 be any ground state, i.e. $\rho_0 = P\,\rho_0\,P$.*

For polynomial decay $F_\alpha(r) := (1+r)^{-\alpha}$ with $\alpha > 0$ and for all disjoint X and $Y \subset \Lambda$ and $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$, it holds that

$$\begin{aligned}&\left|\mathrm{Tr}\big(\rho_0\,A\,B\big) - \tfrac{1}{2}\Big(\mathrm{Tr}\big(\rho_0\,A\,P\,B\big) + \mathrm{Tr}\big(\rho_0\,B\,P\,A\big)\Big)\right| \\ &\qquad \leq 8\,\|A\|\,\|B\|\,|X|\,|Y|\,\|\rho_0\|_1 \left(\sqrt{\frac{\alpha}{\pi}}\,\frac{\|\Phi\|^{\mathrm{tb}}_{F_\alpha}}{g} + 1\right) \ln(1+r)\,F_\alpha(r),\end{aligned} \tag{3.26}$$

where $r = d(X,Y)$.

For stretched-exponential decay $\chi_{b,p}(r) := \mathrm{e}^{-br^p}$ with $b > \tilde{b} > 0$ and $p \in (0,1]$ and for all disjoint X and $Y \subset \Lambda$ and $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$, it holds that

$$\begin{aligned}&\left|\mathrm{Tr}\big(\rho_0\,A\,B\big) - \tfrac{1}{2}\Big(\mathrm{Tr}\big(\rho_0\,A\,P\,B\big) + \mathrm{Tr}\big(\rho_0\,B\,P\,A\big)\Big)\right| \\ &\qquad \leq \|A\|\,\|B\|\,\min\big\{|X|,|Y|\big\}\,\|\rho_0\|_1 \left(\frac{C\,\|\Phi\|^{\mathrm{tb}}_{\chi_{b,p}}}{g} + 1\right) \chi_{\tilde{b},p}(r),\end{aligned} \tag{3.27}$$

where $r = d(X,Y)$ and C is an explicit constant given in (P5-29) *that only depends on b and $\tilde{b}$.*

Compared to the results for general (non-commuting) interactions [118, 214], the decay exponent in (3.26) does not depend on the gap. Moreover, the correlation length does not scale with the gap either. This is best understood for short-range interactions, for which the previous results [118, 171, 214] prove bounds with decay $\mathrm{e}^{-d(X,Y)/\xi}$ and a correlation length $\xi \sim 1/g$ that decays for small gaps g. For commuting interactions we find that the correlation length $\xi \sim 1$ is independent of the gap, see (3.27).

3.4.2 Publication P9: Exponential clustering with Gaussian filter functions

Similarly to how we used a Gaussian filter to obtain better locality properties for the spectral flow, one can also improve the filter used to prove decay of correlations for gapped ground states. Therefore, instead of $\hat{f}$ discussed above (3.25), we consider the convolution of a step function with discontinuity at $g/2$ and a Gaussian. This function only approximately vanishes for $\omega < \delta$ and only approximately equals 1 for $\omega > g$. Hence, the equalities in (3.25) also hold only up to small errors. However, the relevant part of the inverse Fourier transform decays like a Gaussian as well and a careful analysis shows that one obtains exponential decay of correlations for exponentially decaying Hamiltonians, even with splitting of the ground state energy. This proof is closer to the original results [118, 171]. For simplicity, we restrict to the situation, where $\delta < \gamma/4$ and then obtain the following statement. A similar result was also obtained in [214] using a completely different proof.

Theorem 3.12 (Exponential decay of correlations, theorem P9-17)**.** *Let $D \in \mathbb{N}, \mathscr{C}_{\mathrm{A}} > 0$, $b' > 0$, $C_{\mathrm{int}} > 0$, $\gamma > 0$ and $0 < \delta < \gamma/4$. Then there exist constants $C, c > 0$, such that the following holds:*

For all $\Lambda \in \mathscr{G}(D, \mathscr{C}_{\mathrm{A}})$ finite and Hamiltonians H that have a gapped ground state sector $\sigma_* \subset \sigma(H)$ such that $\operatorname{dist}(\sigma_*, \sigma(H) \setminus \sigma_*) \leq g$ and $\operatorname{diam}(\sigma_*) < \delta$ and are given by interactions Φ such that $\|\Phi\|_{\chi_{b'},1} < C_{\mathrm{int}}$ the following holds:

For any normalized state $\Omega \in \operatorname{Ran}(P)$, where P is the spectral projection onto σ_*

$$|\langle \Omega, A\, B\, \Omega \rangle - \langle \Omega, A\, P\, B\, \Omega \rangle| \leq C\, \|P\|_1 \|A\|\, \|B\|\, \mathrm{e}^{-cd(X,Y)}$$

for all disjoint $X, Y \subset \Lambda$ and $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$.

Using the almost spectral flow and this clustering result, we also improve the convergence of the charge transport to quantized values in the results about the Hall effect from [14]. Details are given in section P9-6.

3.5 Response and adiabatic theory

One application of the quasi-local inverse Liouvillian is to rigorously justify adiabatic and response theory in extended lattice systems [17, 165, 203]. Unlike most other results discussed in this thesis, we here analyse a dynamical question. More precisely, we consider Hamiltonians

$$H^{\varepsilon}(t) = H_0(t) + \varepsilon\, V(t), \tag{3.28}$$

and are interested in the adiabatic time evolution of ground states[14] under this Hamiltonian. As before H_0 and V are SLT operators. One main motivation is a rigorous justification of Kubo's formula and linear response theory [142], which was formulated as one of the “Fifteen Problems in Mathematical Physics” by Simon [198]. Let us explain the latter, before we discuss the more general results. Therefore, assume that $H_0(t) = H_0$ is time-independent and has a gapped ground state ρ_0. We are then interested in the change

$$\operatorname{tr}(\rho_\varepsilon\, A) - \operatorname{tr}(\rho_0\, A) = \varepsilon\, \sigma_A + o(\varepsilon),$$

where ρ_ε is the state one obtains after adiabatically, i.e. slowly, switching on the perturbation $\varepsilon\, V$. While the above difference looks similar to what we compared for LPPL or local indistinguishability, the major difference is that ρ_ε is the result of a time evolution with a Hamiltonian $H_0 + \varepsilon\, f(\eta\, t)\, V$, where $f(t)$ is a switching function and η a small

[14] While the results [17, 165, 203] also hold for other spectral projections of gapped sectors, we restrict to ground states here.

parameter quantifying the duration of the switching process. The resulting state ρ_ε is usually not an equilibrium state of H^ε, but almost invariant under its time evolution.

As the perturbation is turned on over a long time η^{-1}, the question goes beyond standard perturbation theory and requires an adiabatic type theorem. However, the standard adiabatic theorem is also not enough for two main reasons: First, an adiabatic theorem requires H^ε to be gapped and then asserts that the ground state of H^ε is close to ρ_ε. However, we only want to require $H_0(t)$ to be gapped and allow $\varepsilon\, V(t)$ to close the gap. Second, results from standard adiabatic theorems are formulated as norm bounds, which do not hold for extended systems due to the orthogonality catastrophe.

Let us now give a more formal statement of the *generalized super-adiabatic theorems* one aims to obtain. Therefore, consider $H^\varepsilon(t)$ as in (3.28). We assume that $H_0(t)$ is given by a short-range interaction and uniformly gapped, i.e. it satisfies assumption 3.5. The perturbation $V(t)$ can be a sum of two terms, one is another short-range interaction, and the other is a *Lipschitz potential.* A Lipschitz potentials is an external potential

$$V = \sum_{z\in\Lambda} v(z)\, \mathcal{N}_{\{z\}},$$

where $v\colon \Lambda \to \mathbb{R}$ has a bounded Lipschitz constant. Lipschitz potentials can for example be used to describe an electric field across the lattice, where one would choose $v(z) = z_1$ for an electric field in 1-direction. And although these are sums of on-site terms, they do not necessarily have a bounded interaction norm, as $v(z)$ can grow to infinity in the thermodynamic limit. Crucially, we do not assume that $H^\varepsilon(t)$ is gapped, and in the example of a constant electric field the gap eventually closes for large Λ, see figure P2-1.

Then there exists operators $S^{\varepsilon,\eta}(t)$, such that $\Pi^{\varepsilon,\eta}(t) = \mathrm{e}^{\mathrm{i}S^{\varepsilon,\eta}(t)}\, \rho_0(t)\, \mathrm{e}^{-\mathrm{i}S^{\varepsilon,\eta}(t)}$ is almost invariant under the adiabatic time evolution generated by $\frac{1}{\eta}\, H^\varepsilon(t)$ in the following sense:[15] Let $\rho^{\varepsilon,\eta,t_0}(t)$ be the solution of the von Neumann equation

$$\mathrm{i}\,\eta\, \frac{\mathrm{d}}{\mathrm{d}t}\, \rho^{\varepsilon,\eta,t_0}(t) = \big[H_\varepsilon(t), \rho^{\varepsilon,\eta,t_0}(t)\big] \qquad \text{with} \qquad \rho^{\varepsilon,\eta,t_0}(t_0) = \Pi^{\varepsilon,\eta}(t_0),$$

then

$$\Big|\mathrm{tr}\big(\rho^{\varepsilon,\eta,t_0}(t)\, A\big) - \mathrm{tr}\big(\Pi^{\varepsilon,\eta}(t)\, A\big)\Big| \le C_n\, \frac{\varepsilon^{n+1} + \eta^{n+1}}{\eta^{d+1}}\, \big(1 + |t - t_0|^{d+1}\big)\, \|A\|\, |X|^2. \tag{3.29}$$

The generator of the dressing $S^{\varepsilon,\eta}(t)$ is a polynomial in ε and η with smallest degree 1 and thus

$$\big|\mathrm{tr}(\rho_0(t)\, A) - \mathrm{tr}(\Pi^{\varepsilon,\eta}(t)\, A)\big| \le C\,(\varepsilon + \eta)\, \|A\|\, |X|^2.$$

[15] For better readability, we chose to write the evolution on the state here. This is similar to the notation we use in publication P6. In publication P2 and parts of the thesis, we instead apply the Heisenberg time evolutions to the operators. In the trace, this has the same effect, but the latter has an analogue in the thermodynamic limit, where states are only given as functionals. See also section 2.5.

Moreover, the coefficients and thus $\Pi^{\varepsilon,\eta}(t)$ only depend on H_0 and V and their derivatives at time t. And if all these derivatives vanish at some time t_0, then $\Pi^{\varepsilon,\eta}(t_0) = \Pi^{\varepsilon,0}(t_0)$ and $\Pi^{\varepsilon,0}(t_0)$ is almost stationary under the time evolution generated by the time-independent Hamiltonian $H_\varepsilon(t_0)$ in the sense that

$$\begin{aligned} &\left|\operatorname{tr}\!\left(\mathrm{e}^{-\mathrm{i}tH_\varepsilon(t_0)}\, \Pi^{\varepsilon,0}(t_0)\, \mathrm{e}^{\mathrm{i}tH_\varepsilon(t_0)}\, A\right) - \operatorname{tr}\!\left(\Pi^{\varepsilon,0}(t_0)\, A\right)\right| \\ &\qquad \leq C_n\, \varepsilon^{n+1} \left(1 + |t|^{d+1}\right) \|A\|\, |X|^2 \end{aligned} \tag{3.30}$$

with a bound as in (3.29) without any η. Therefore, $\Pi^{\varepsilon,0}(t_0)$ is called *non-equilibrium almost stationary state* (NEASS). In similar results, they are also called resonances or metastable states [220, 221]. If, additionally, the perturbation $V(t_0)$ vanishes at this t_0, which means $\varepsilon = 0$, then $\Pi^{\varepsilon,\eta}(t_0) = \Pi^{\varepsilon,0}(t_0) = \Pi^{0,0}(t_0) = \rho_0(t_0)$ equals the instantaneous ground state $\rho_0(t_0)$ of $H_0(t_0)$.

These properties together let us immediately conclude the above statement on response theory. Therefore, suppose that f is smooth and $f(t) = 0$ for $t \leq -1$ and $f(t) = 1$ for $t \geq 0$ and consider the switching Hamiltonian $H^\varepsilon(t) = H_0 + \varepsilon\, f(t)\, V$. Then, $\Pi^{\varepsilon,\eta}(t) = \rho_0$ equals the ground state of H_0 for $t < -1$ and $\Pi^{\varepsilon,\eta}(t) = \Pi^{\varepsilon,0}(t)$ is a NEASS for $H_0 + \varepsilon V$ for $t > 0$. Moreover, the time evolution generated by $t \mapsto H^\varepsilon(t)$ drives ρ_0 into a state close to $\Pi^{\varepsilon,0}(t)$ in the sense of the quantitative bound (3.29).

The *super-adiabatic theorem* in this language, is the statement for gapped Hamiltonians $H_0(t)$, i.e. for $\varepsilon = 0$. It says that the instantaneous ground state $\rho_0(t_0)$ stays close to the instantaneous ground state at later times when evolved with the time evolution generated by $t \mapsto \frac{1}{\eta} H_0(t)$. It was proven for extended (but finite) lattice spin systems by Bachmann et al. [17], based on the quasi-local inverse of the Liouvillian, which we discussed in section 3.3, and extended to lattice fermions by Monaco and Teufel [165]. Based on these results, Teufel [203] proved the *generalized* super-adiabatic theorem, which extends the previous results to $\varepsilon > 0$. This in particular includes the case where the perturbation $\varepsilon V(t)$ closes the spectral gap. These results have then been generalized to the infinite volume in various ways [123, 122, 32].

3.5.1 Publication P2: Review of generalized super-adiabatic theorems for extended fermionic systems

In publication P2, we give an introduction into generalized super-adiabatic theorems and review four recent results and their proofs. The review discusses the result for extended finite systems from Teufel [203] and its thermodynamic limit from Henheik and Teufel [123]. These results assume that the Hamiltonians H_0 is uniformly gapped for all $\Lambda \subset \mathbb{Z}^D$. Afterwards, we discuss the results from Henheik and Teufel [122], which only assumes a gap in the thermodynamic limit. It first proves a generalized super-adiabatic theorem in the thermodynamic limit and then uses an assumption on the convergence of the ground states to relate it back to finite volumes. The more recent result by Becker et al. [32], which is not discussed in publication P2, removes any assumption on convergence of the ground states and directly proves the result in the thermodynamic limit without boundary.

Assuming only a gap in the thermodynamic limit leaves the possibility for gap closing edge states in the finite volumes. In this sense, [122] is assuming a gap only in the bulk of the system. This motivated us to investigate response theory under the assumption of a local gap, which we discuss in the next section.

3.5.2 Publication P6: Response theory with local gaps

We have seen already that the perturbed Hamiltonian $H^\varepsilon(t)$ does not need to be gapped and that response theory also works under the possible presence of gap closing edge states [122]. In publication P6, we relate various notions of local gaps and use them to justify response theory on systems where H_0 only has a local gap.

The results discussed so far relied on the exact inverse Liouvillian, which we discussed in section 3.3. Its defining property that it inverts the Liouvillian on off-diagonal operators, (3.13), can be rephrased as the condition that

$$[\mathscr{L}_{H_0}\, \mathscr{I}_{H_0,g}\, A - A, \rho_0] = 0 \qquad \text{for all} \qquad A \in \mathscr{A}_\Lambda,$$

where ρ_0 is the ground state of H_0. Using the trace norm and its dual with respect to the Hilbert Schmidt inner product, this is equivalent to asking

$$\operatorname{tr}\big(\rho_0\, [\mathscr{L}_{H_0}\, \mathscr{I}_{H_0,g}\, A - A, B]\big) = 0 \qquad \text{for all} \qquad A,\ B \in \mathscr{A}_\Lambda. \tag{3.31}$$

Remember at this point that $\mathscr{I}_{H_0,g}$ was defined using the gap g of the Hamiltonian. And it indeed turns out, that (3.31) is equivalent to H_0 being gapped in the usual sense, see proposition P6-3.1. In analogy to this, we say that H_0 is *locally dynamically gapped* of size g in a region $\Lambda^{\mathrm{gap}} \subset \Lambda$ if (3.31) is small for X and Y well inside the gapped region Λ^{gap} in the sense that

$$\begin{aligned}\big|\operatorname{tr}\big(\rho_0\, [\mathscr{L}_{H_0}\, \mathscr{I}_{H_0,g}\, A - A, B]\big)\big| \le C\, \|A\|\, \|B\| \,\big(\operatorname{diam}(X) + \operatorname{diam}(Y)\big)^{\ell} \\ \times \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}}) + d(Y, \Lambda \setminus \Lambda^{\mathrm{gap}})\big).\end{aligned}$$

Under this assumption, we can still follow the construction of the NEASS $\Pi^{\varepsilon,\eta}(t)$. However, as we do not assume a global spectral gap, we cannot use the spectral flow for the parallel transport of the instantaneous ground states, and hence need to restrict to time-independent unperturbed Hamiltonians H_0. We still obtain response theory for the switching Hamiltonian $H^\varepsilon(t) = H_0 + \varepsilon\, f(t)\, V$ with $V \in \Lambda^{\mathrm{pert}}$. Starting in the ground state, the system evolves into a state close to the NEASS $\Pi^{\varepsilon,0}(t)$, which is almost stationary, similarly to (3.29) and (3.30). In both statements the bounds are modified such that they only provide good estimates as long as $d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}}) \gg |\log(\varepsilon)|$, i.e. if the perturbation V is far away from the non-gapped region.

As one can imagine, the local dynamical gap assumption we use to justify response theory, is very much designed for this proof. Hence, we also discuss a variety of other local gap conditions and the relations between them, see section P6-3. We find two classes of local gap conditions, which we call extrinsic and intrinsic.

For extrinsic local gap conditions, we assume that the Hamiltonian of interest can be written as $H = H_* + J$, where H_* is globally gapped and J is localized in $\Lambda \setminus \Lambda^{\mathrm{gap}}$. It is then left to quantify in which sense the ground states ρ and ρ_* of H and H_*, respectively, are related. For example, one can assume that they have similar expectation values within Λ^{gap}, similarly to a strong LPPL principle, which we prove for weakly interacting spin systems in section 3.8.1. Unfortunately, this notion is not strong enough to conclude a local dynamical gap of ρ. Instead, one needs to assume that ρ_* and ρ are close to each other within Λ^{gap} in a stronger topology, and we discuss multiple possibilities in section P6-3.2.

Intrinsic local gap conditions are conditions that do not need a reference to a globally gapped Hamiltonian H_*. As such, the local dynamical gap assumption is an intrinsic condition. The other intrinsic gap conditions we discuss are generalizations of the algebraic condition for a global gap. Indeed, a unique ground state ρ of the Hamiltonian H on a finite lattice Λ is gapped with gap $g > 0$, if

$$\mathrm{i}\,\langle A^* \mathscr{L}_H[\![A]\!]\rangle_\rho \geq g\left(\langle A^* A\rangle_\rho - \left|\langle A\rangle_\rho\right|^2\right), \tag{3.32}$$

for all observables $A \in \mathscr{A}$, where $\mathscr{L}_H[\![\cdot]\!] := -\mathrm{i}\,[H, \cdot]$ denotes the Liouvillian. We then define two intrinsic local gap conditions by requiring (3.32) to hold up to a multiplicative or additive errors on the right-hand side that vanishes in $d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})$. For the version with the multiplicative error, we basically replace g with

$$g\left(1 - C\ \mathrm{diam}(X)\,\chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big)\right)$$

which has the interpretation that the gap vanishes once X is near the boundary but approaches g in the bulk. We then prove that this form of *gap decay* implies the local dynamical gap for pure product states.

3.6 Publication P4: Quantum belief propagation

In publication P4, we provide a rigorous analysis of *quantum belief propagation*, an idea originally introduced by Hastings [115] that provides an analogue of the spectral flow for Gibbs states. With it, we prove the equivalence of uniform decay of correlations, LPPL and local indistinguishability, see figure 3.1. This result holds in any dimension and only requires locality of the Hamiltonian in terms of Lieb-Robinson bounds. Quantum belief propagation was investigated and used before, see e.g. [134, 135, 80, 45, 131, 109, 8, 145, 194, 137], with applications often restricted to finite-range interactions. Our contribution is a rigorous analysis of quantum belief propagation and

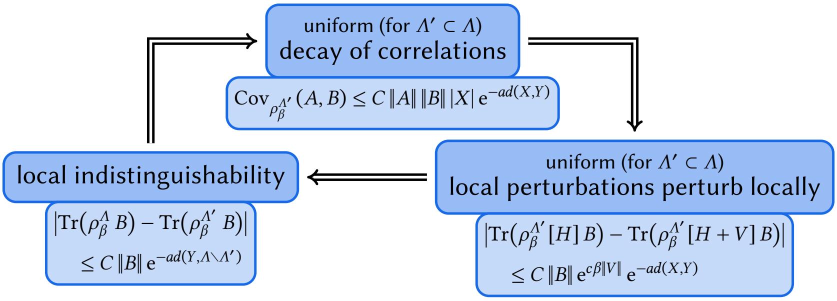

Figure 3.1. The diagram shows the main implications from publication P4. In particular, we show an equivalence of the three concepts decay of correlations, LPPL and local indistinguishability. Note, that the formulas are mainly illustrative for the concepts and in particular the constants change, see remark P4-33. A crucial ingredient in all the implications is quantum belief propagation coupled with Lieb-Robinson bounds. The main entry point into this circle is decay of correlations, which is known to hold for various one-dimensional systems [9, 36, 185, 136] (see also theorem P4-6) and above a threshold temperature [139].

the circle of equivalences for general interactions satisfying a Lieb-Robinson bound. Indeed, for non-finite-range interactions, we need to use quantum belief propagation for each of the implications shown in figure 3.1.

While we think that the equivalence is interesting in its own, the main entry point into this circle for applications is decay of correlations. It is known to hold for various one-dimensional systems [9, 36, 185, 136] (see also theorem P4-6) and in higher dimensions above a threshold temperature [139, 98]. Specific bounds on LPPL and local indistinguishability under these assumptions are discussed in section P4-3.

Next, we discuss the differential equation and spectral filter behind quantum belief propagation. Afterwards, we briefly explain each of the three implications from figure 3.1.

Quantum belief propagation

To derive the differential equation for the Gibbs state, consider a time-dependent Hamiltonian $H(s)$ and use Duhamel's formula to write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{e}^{-\beta H(s)} &= -\beta\int_0^1 \mathrm{e}^{-\beta\tau H(s)}\,\dot{H}(s)\,\mathrm{e}^{-\beta(1-\tau)H(s)}\,\mathrm{d}\tau\\ &= -\frac{\beta}{2}\Big\{\mathrm{e}^{-\beta H(s)}, \sum_{j,k}\hat{f}_\beta(E_k-E_j)\,P_j\,\dot{H}(s)\,P_k\Big\},\end{aligned}$$

after inserting the spectral decomposition $H(s) = \sum_j E_j P_j$ and some algebra, see section P4-4 for more details. The function $\hat{f}_\beta$ is given as $\hat{f}_\beta(\omega) = \tanh(\beta\,\omega/2)/(\beta\,\omega/2)$ and its inverse Fourier transform $f_\beta \in L^1(\mathbb{R})$ is positive, satisfies $\int_{\mathbb{R}} f_\beta = 1$ and decays exponentially for large $|t|$. Hence, writing the spectral filter

$$\Phi_\beta^{H(s)}(V) = \int_{\mathbb{R}} f_\beta(t)\,\tau_t^{H(s)}(V)\,\mathrm{d}t,$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{e}^{-\beta H(s)} = -\frac{\beta}{2}\Big\{\mathrm{e}^{-\beta H(s)}, \Phi_\beta^{H(s)}(\dot{H}(s))\Big\},$$

and together with the chain rule also a differential equation for the Gibbs state

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\rho_\beta(s) = -\frac{\beta}{2}\Big\{\rho_\beta(s), \Phi_\beta^{H(s)}\big(\dot{H}(s) - \langle\dot{H}(s)\rangle_{\rho_\beta(s)}\big)\Big\}, \tag{3.33}$$

where $\langle V\rangle_{\rho_\beta(s)} := \mathrm{Tr}\big(\rho_\beta(s)\,V\big)$.

From these equations, we obtain operators $\eta(s)$ and $\tilde{\eta}(s)$, such that

$$\mathrm{e}^{-\beta H(s)} = \eta(s)\,\mathrm{e}^{-\beta H(0)}\,\eta(s)^* \quad \text{and} \quad \rho_\beta(s) = \tilde{\eta}(s)\,\rho_\beta(0)\,\tilde{\eta}(s)^*.$$

Compared to the spectral flow (3.11), these operators cannot be unitary in general, but we can still use locality of the spectral filter to approximate them locally, if $H(s)$ and $\dot{H}(s)$ are local.

From now on, we only consider $H(s) = H + s\,V$ with $\dot{H}(s) = V$, as we can connect any two Hamiltonians along a linear path.[16] If $V \in \mathcal{A}_X$, then $\tilde{\eta}(s)$ can be approximated by some $\tilde{\eta}_r(s) \in \mathcal{A}_{X_r}$, and

$$\|\tilde{\eta}(s) - \tilde{\eta}_r(s)\| \le \beta\,s\,\|V\|\,\mathrm{e}^{\beta s\|V\|}\,\zeta_{\mathrm{QBP}}(X, r)$$

decays in r. Here, ζ_{QBP} is an explicitly given decay function that depends on the inverse temperature β and the Lieb-Robinson bound of τ^H. Its precise definition is given in (P4-11) and bounds for short- and long-range interactions can be found in lemmata P4-20 and P4-46.

LPPL from decay of correlations

Our first result is that decay of correlations implies LPPL.

[16] If one wants to connect the ground states of two Hamiltonians H_1 and H_2 with the spectral flow as described in section 3.3, one needs to find a uniformly gapped path of Hamiltonians. Such a path might not be the direct connection $H(s) = (1-s)\,H_1 + s\,H_2$. Here, there is no such assumption and it is enough to analyse this direct connection, where $V = H_2 - H_1$.

Theorem 3.13 (LPPL from decay of correlations, see theorem P4-22)**.** *Let $\Lambda \Subset \mathbb{Z}^\nu$, $H \in \mathcal{A}_\Lambda$ be a local self-adjoint Hamiltonian, $X \subset \Lambda$, and $V \in \mathcal{A}_X$ self-adjoint. For $s \in [0,1]$, consider the path of Hamiltonians $H(s) := H + s\,V$ with Gibbs states $\rho_\beta(s)$. Then, for all $Y \subset \Lambda$, $B \in \mathcal{A}_Y$ and $r \geq 0$, we have*

$$\left|\operatorname{Tr}\big(\rho_\beta(0)\,B\big) - \operatorname{Tr}\big(\rho_\beta(1)\,B\big)\right| \leq \mathrm{e}^{2\beta\|V\|}\,\|B\|\,\big(\operatorname{Cov}_{\rho_\beta(0)}(X_r;Y) + 4\,\beta\,\|V\|\,\zeta_{\mathrm{QBP}}(X,r)\big). \quad (3.34)$$

To prove LPPL, we would like to follow the argument in (3.15) for the ground state. However, as $\tilde{\eta}(s)$ is not unitary, we cannot just insert $\mathbb{1} = U(s)^*\,U(s)$ as we did there. Instead, to prove theorem 3.13, we abbreviate $\rho = \rho_\beta(0)$, $\tilde{\eta} = \tilde{\eta}(1)$ and $\tilde{\eta}_r = \tilde{\eta}_r(1)$ and obtain

$$\begin{aligned}
&\left|\operatorname{Tr}\big(\rho_\beta(1)\,B\big) - \operatorname{Tr}\big(\rho_\beta(0)\,B\big)\right| \\
&\quad = \left|\operatorname{Tr}\big((\tilde{\eta} - \tilde{\eta}_r)\,\rho\,\tilde{\eta}^*\,B\big) + \operatorname{Tr}\big(\tilde{\eta}_r\,\rho\,(\tilde{\eta}^* - \tilde{\eta}_r^*)\,B\big) + \operatorname{Tr}\big(\rho\,\tilde{\eta}_r^*\,\tilde{\eta}_r\,B\big) - \operatorname{Tr}\big(\rho\,B\big)\right| \\
&\quad \leq 2\,\|\tilde{\eta} - \tilde{\eta}_r\|\,\big(\|\tilde{\eta}\| + \|\tilde{\eta}_r\|\big)\,\|B\| + \left|\operatorname{Cov}_\rho(\tilde{\eta}_r^*\,\tilde{\eta}_r, B)\right|.
\end{aligned}$$

And so, besides the decay of $\|\tilde{\eta} - \tilde{\eta}_r\|$, we also need decay of the covariance to prove LPPL for the Gibbs state with quantum belief propagation.

Theorem 3.13 only requires the Gibbs state $\rho_\beta(0)$ to satisfy decay of correlations, but it scales exponentially in $\beta\,\|V\|$. One can alternatively first use the fundamental theorem of calculus and then integrate (3.33), to obtain a similar result which scales linearly in $\beta\,\|V\|$ assuming that all $\rho_\beta(s)$ satisfy decay of correlations. The precise statement can be found in theorem P4-25.

Local indistinguishability from LPPL

For the next implication in the circle, we prove local indistinguishability from LPPL.

Theorem 3.14 (Local indistinguishability from uniform LPPL, see theorem P4-29)**.** *Let $\Lambda \Subset \mathbb{Z}^\nu$, F a decay function, and Ψ be an interaction such that $\|\Psi\|_F < \infty$. Assume that all Gibbs states $\rho_\beta^{\Lambda'}$ with $\Lambda' \subset \Lambda$ satisfy LPPL with at most polynomial growth in the size of the support of the perturbation and fast enough decay. Then, there exist $C(\cdot,\cdot)$ and ζ, explicitly given in theorem P4-29, which depend on the specific LPPL statement and decay of the interaction, such that*

$$\left|\operatorname{Tr}\big(\rho_\beta^{\Lambda}\,B\big) - \operatorname{Tr}\big(\rho_\beta^{\Lambda'}\,B\big)\right| \leq C\big(|Y|, \|\Psi\|_F\big)\,\zeta\big(d(Y, \Lambda \setminus \Lambda')\big)$$

for all $Y \subset \Lambda'$ and $B \in \mathcal{A}_Y$.

The result is similar to the result of Brandão and Kastoryano [45], but improves it in three aspects: Their result only holds for finite-range interactions and scales with $\partial\Lambda'$, which ours does not. As a more subtle difference, our result also holds under the assumption that only Gibbs states on contractible sets $\Lambda' \subset \Lambda$ satisfy decay of correlations and thus LPPL. This might be relevant for topological insulators.

For the proof, we enumerate $\Lambda \setminus \Lambda' = \{x_j\}$ and then remove one lattice site after another. If the interaction Ψ is finite range with range R, we can just apply LPPL with $V = \sum_{Z\subset\Lambda: x_j\in Z} \Psi(Z) \in \mathcal{A}_{B_{x_j}(R)}$ to remove the site x_j. By the lattice geometry, there can be at most $|Y|\,\mathscr{C}_\Lambda\, r^{D-1}$ points x_j with $d(Y, x_j) = r$ and if the decay from LPPL is good enough, we are able to bound this sum uniformly in Λ. The more interesting case is, when Ψ is not finite range. Then we cannot remove all interactions with x_j at once, because their support can overlap with Y, in which case we do not obtain decay from LPPL. Instead, we consider two contributions

$$V_1 = \sum_{\substack{Z\subset\Lambda:\\ x_j\in Z, \operatorname{diam}(Z)\le R}} \Psi(Z) \qquad \text{and} \qquad V_2 = \sum_{\substack{Z\subset\Lambda:\\ x_j\in Z, \operatorname{diam}(Z)> R}} \Psi(Z).$$

Then, we first remove V_1 using LPPL as before. For V_2, we observe that $\|V_2\| \le \|\Psi\|_F\, F(R)$ and remove this small term using quantum belief propagation directly by (P4-9).

Decay of correlations from local indistinguishability

To conclude the circle and thus the equivalence of the three locality measures, we also prove decay of correlations from local indistinguishability.

Theorem 3.15. *Let $\Lambda \Subset \mathbb{Z}^\nu$, F be a decay function, and Ψ be an interaction such that $\|\Psi\|_F < \infty$ and assume that Ψ satisfies local indistinguishability at inverse temperature β with respect to f_{LI} and ζ_{LI} in the sense of Definition 2.4. Then, for all disjoint $X, Y \subset \Lambda$, and $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$*

$$\operatorname{Cov}_{\rho_\beta^\Lambda}(A, B) \le \|A\|\,\|B\|\, \max\bigl\{|X|, f_{\mathrm{LI}}\bigl(|X| + |Y|\bigr)\bigr\}\, \zeta\bigl(d(X, Y)\bigr),$$

where

$$\zeta(r) := 3 \inf_{0\le\ell<r/2} \Bigl(\zeta_{\mathrm{LI}}(\ell) + (\mathrm{e}^{2\beta\|\Psi\|_F} - 1)\,(2\ell+1)^\nu\, F(r - 2\ell)\Bigr).$$

Again, the idea of the proof is straightforward for finite-range interactions. Assume that ρ_β^Λ satisfies local indistinguishability and let $X, Y \subset \Lambda$. Then, let $\ell = \lfloor (d(X, Y) - R)/2 \rfloor$ so that $d(X_\ell, Y_\ell) \ge R$. We now apply local indistinguishability for $\Lambda' = X_\ell \cup Y_\ell \subset \Lambda$ to obtain

$$\operatorname{tr}(\rho_\beta^\Lambda\, A\, B) \approx \operatorname{tr}(\rho_\beta^{\Lambda'}\, A\, B) = \operatorname{tr}(\rho_\beta^{X_\ell}\, A)\, \operatorname{tr}(\rho_\beta^{Y_\ell}\, B) \approx \operatorname{tr}(\rho_\beta^\Lambda\, A)\, \operatorname{tr}(\rho_\beta^\Lambda\, B),$$

for all $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$. For the central equality we use that $\rho_\beta^{\Lambda'}$ factors for finite-range interactions with range R, as then $H_{\Lambda'} = H_{X_\ell} + H_{Y_\ell}$ and these two Hamiltonians commute. If the interaction is not finite-range, we instead need to use quantum belief propagation, to remove the additional interactions between X_ℓ and Y_ℓ first, before we can factorize the trace.

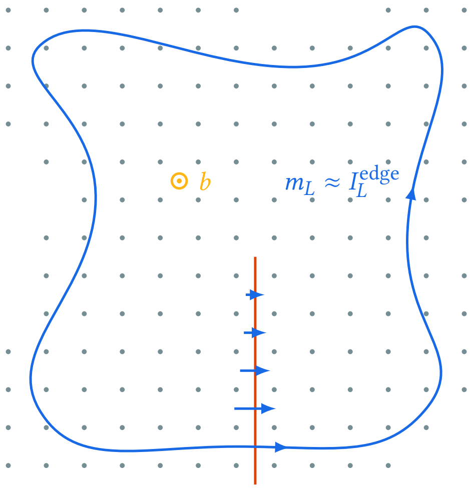

Figure 3.2. Pictorial representation of the main results from publication P3: For locally interacting fermions on a two-dimensional lattice with perpendicular magnetic field b, satisfying local indistinguishability at positive temperature, the edge current I_L^{edge}, which is the bond current through the vertical line, is localized near the boundary and approximately equals the magnetization m_L (theorem P3-I). The latter is a bulk quantity, i.e. it converges in the thermodynamic limit $L \to \infty$, does not depend on the details near the boundary (theorem P3-II), and can be computed directly from a system without boundary (theorem P3-IV). The independence from the boundary terms allows for the rough edges in the picture.

3.7 Publication P3: Bulk-edge correspondence

In publication P3, we consider interacting fermions on finite square lattices $\Lambda_L := \big([-L, L] \times [0, 2L]\big) \cap \mathbb{Z}^2$ subject to an external magnetic field. Under certain assumptions, we then prove a bulk-edge correspondence in the sense that the edge current is determined by the bulk magnetization. The main idea is depicted in figure 3.2, and we discuss the precise assumptions and results in the following.

More precisely, we consider Hamiltonians

$$\begin{aligned} H_L(b) &= \sum_{x,y\in\Lambda_L} a_x^* \, T_b^{\text{bulk}}(x,y)\, a_y + \sum_{X\subset\Lambda_L} \Phi^{\text{bulk}}(X) \\ &\quad + \sum_{x,y\in\Lambda_L} a_x^* \, T_b^{\text{edge}}(x,y)\, a_y + \sum_{X\subset\Lambda_L} \Phi^{\text{edge}}(X) \\ &=: \sum_{x,y\in\Lambda_L} a_x^* \, T_b(x,y)\, a_y + \sum_{X\subset\Lambda_L} \Phi(X), \end{aligned} \tag{3.35}$$

which are explicitly split in the kinetic part, which includes the external magnetic field, and an interaction. We eventually take a thermodynamic limit $L \to \infty$, and hence split each term in a bulk part, which is defined on $\mathbb{Z}^2$ and assumed to be invariant under magnetic translations, and an edge part, which is localized in a strip of fixed width at the lower edge[17] $\mathbb{Z}\times\{0, 1, \dots, E-1\}$. All terms are assumed to be finite range. Moreover, the kinetic terms

$$T_b^{\mathrm{bulk/edge}}(x, y) := \mathrm{e}^{\mathrm{i}\frac{x_2+y_2}{2} b(x_1-y_1)}\, T^{\mathrm{bulk/edge}}(x, y) \tag{3.36}$$

are given by a Peierls phase, which models a constant magnetic field perpendicular to the lattice, and a hopping amplitude $T^{\mathrm{bulk/edge}}$. The bulk hopping amplitudes are invariant under translation, $T^{\mathrm{bulk}}(x-z, y-z) = T^{\mathrm{bulk}}(x, y)$ so that the full kinetic term $T_b^{\mathrm{bulk}}(x, y)$ is invariant under magnetic translations. The additional interactions $\Phi^{\mathrm{bulk/edge}}$ are uniformly bounded and termwise self-adjoint. Moreover, we assume that the operators $\sum_{X\subset\Lambda_L} \Phi^{\mathrm{bulk/edge}}(X)$ commute with all local number operators $\mathcal{N}_{\{z\}}$, which is in particular satisfied for density-density interactions. Like the kinetic term, the bulk interaction Φ^{bulk} is assumed to be invariant under magnetic translations, see section P3-3.1.1 for details.

A class of examples is given by the discrete magnetic Laplacian for the kinetic term and an additional density-density interaction. More specifically, the Hofstadter-Hubbard model

$$H_L^{\mathrm{HH}}(b) = \sum_{\substack{x,y\in\Lambda_L:\\ |x-y|=1}} \mathrm{e}^{\mathrm{i}\frac{x_2+y_2}{2} b(x_1-y_1)} \sum_{j\in\{1,2\}} a_{x,j}^*\, a_{y,j} + \sum_{x\in\Lambda_L} a_{x,1}^*\, a_{x,1}\, a_{x,2}^*\, a_{x,2}$$

for spin $1/2$, i.e. $q = 2$, is one example. Near the edge, one could for example add an additional external field or remove some sites by subtracting all hoppings that connect them.

For these systems, we consider the Gibbs state at inverse temperature $\beta > 0$, chemical potential $\mu \in \mathbb{R}$ and magnetic field $b \in \mathbb{R}$, defined as

$$\rho_L(\beta, \mu, b) := \frac{\mathrm{e}^{-\beta(H_L(b)-\mu\,\mathcal{N}_L)}}{Z_L(\beta, \mu, b)},$$

where $Z_L(\beta, \mu, b) := \mathrm{tr}\big(\mathrm{e}^{-\beta(H_L(b)-\mu\,\mathcal{N}_L)}\big)$ is the partition function. The magnetization is defined as the derivative of the grand canonical pressure $p_L(\beta, \mu, b) := -|\Lambda_L|^{-1}\,\beta^{-1}\ln\big(Z_L(\beta, \mu, b)\big)$ with respect to the magnetic field b, namely

$$m_L(\beta, \mu, b) := \frac{\partial}{\partial b}\, p_L(\beta, \mu, b). \tag{3.37}$$

[17] In the included reprint of publication P3, the width of this strip at the lower edge is denoted D instead of E. We change this here, to avoid confusion with the dimension $D = 2$ of the lattice.

The bond current on Λ_L through the dual edge $e_{1,z}$ perpendicular to the edge between sites $z = (m, n)$ and $(m + 1, n)$ is defined as basically[18]

$$J_1^z(b) := \mathrm{i} \sum_{\substack{x,y \in \Lambda_L: \\ \overline{xy} \text{ intersects } e_{1,z}}} \operatorname{sgn}(x_1 - y_1)\, a_x^* \, T_b(x, y)\, a_y.$$

The index "1" refers to direction 1, and an analogous definition also holds in direction 2. These bond currents are chosen such that the sum over all dual edges gives the full current defined as the commutator with the position operator $X_{1,L} = \sum_{z\in\Lambda_L} z_1 \mathcal{N}_z$

$$J_{1,L}(b) := \mathrm{i}\left[X_{1,L}, H_L(b)\right] = \sum_{m=-L}^{L-1} \sum_{n=0}^{2L} J_1^{(m,n)}(b), \tag{3.38}$$

which vanishes due to current conservation. This is explained in section P3-3.1.2 in more detail. Note that only the kinetic terms contribute to the commutator and the local current operators, since we assume that the interaction Φ commutes with all local number operators. For $d \in \{1, \dots, L\}$, we define the edge current

$$I_L^{d\,\text{edge}}(\beta, \mu, b) := \sum_{n=0}^{d-1} \operatorname{tr}\!\left(\rho_L(\beta, \mu, b)\, J_1^{(0,n)}(b)\right) \tag{3.39}$$

as the bond current over a fiducial line perpendicular to the boundary.

Besides the structure of the Hamiltonian, our main assumption is that $\rho_L(\beta, \mu, b)$ is homogeneous in the bulk. More precisely, we assume that $\rho_L(\beta, \mu, b)$ satisfies local indistinguishability *everywhere* as introduced in definition 2.4 with ζ-decay, which scales like $\zeta(r) \leq C\,(r+1)^{-n}$ with $n \geq 3$. In definition P3-1, we also introduce the notion of *local indistinguishability in the bulk*, for which we replace the distance $d(Y, \Lambda_L \setminus \Lambda')$ in definition 2.4 by $d(Y, \mathbb{Z}^2 \setminus \Lambda')$. The precise statements in publication P3 distinguish between local indistinguishability *everywhere* and *in the bulk*, since we usually expect better decay in the bulk than close to the lower edge due to possible edge states. Local indistinguishability *everywhere* is mostly needed for technical reasons and to take the thermodynamic limit, for which it is important to know that $\operatorname{tr}\!\left(\rho_L(\beta, \mu, b)\, J_1^{(0,n)}(b)\right)$ converges as $L \to \infty$.

Under this assumption, we can prove a relation between magnetization and edge current in the finite volumes Λ_L.

[18] For simplicity of the presentation we ignore the following two issues in this summary: First, if a bond intersects two neighbouring dual edges, it needs to be counted with a factor $1/2$ to each of the dual edges, see (P3-8). Only then, equality (3.38) holds. Secondly, as the sum is only over x, $y \in \Lambda_L$, the definition depends on L, but is constant for L large enough, i.e. whenever $L > |m| + R$ and $2L > n + R$.

Theorem 3.16 (Simplified version of theorem P3-I)**.** *Under the above assumptions, the edge current is given by the magnetization in the sense that*

$$\left|m_L(\beta,\mu,b) - I_L^{\text{edge}}(\beta,\mu,b)\right| \le C\, L^{-(n-1)/(n+1)} \qquad \textit{for all } L \ge E + R,$$

and it is indeed localized near the edge

$$\left|I_L^{d\,\text{edge}}(\beta,\mu,b) - I_L^{\text{edge}}(\beta,\mu,b)\right| \le C\,(d-R-E)^{n-1} \qquad \textit{for all } d \in [R+E, L]. \tag{3.40}$$

Under the mentioned assumptions, the thermodynamic limits exist and this property also carries over to the infinite volume $\lim_{L\to\infty} \Lambda_L = \mathbb{Z}_+^2$. More precisely, by local indistinguishability we can define the infinite volume state on the local algebra by $\rho_\infty(\beta,\mu,b)(A) = \lim_{L\to\infty} \operatorname{tr}\big(\rho_L(\beta,\mu,b)\,A\big)$ for all $A \in \mathscr{A}_{\mathbb{Z}^2}^{\text{loc}}$, and the bond current through the fiducial line as

$$I^{d\,\text{edge}}(\beta,\mu,b) := \rho_\infty(\beta,\mu,b)\Bigg(\sum_{n=0}^{d} J_1^{(0,n)}(b)\Bigg).$$

Then, see theorem P3-II, the thermodynamic limit of the magnetization $m(\beta,\mu,b) := \lim_{L\to\infty} m_L(\beta,\mu,b)$ and the total edge current $I^{\text{edge}}(\beta,\mu,b) := \lim_{d\to\infty} I^{d\,\text{edge}}(\beta,\mu,b)$ both exist and are equal. Moreover, also the total edge current is localized in the sense of (3.40).

While this establishes an equality between the magnetization and edge current, we so far only considered the system with a boundary. Instead, we would like to see an equality between an edge quantity in the system with an edge and a bulk quantity in the system on $\mathbb{Z}^2$. To this end, let $H_L^{\text{bulk}}(b)$ be the bulk Hamiltonian given by the bulk contributions on boxes $B_L := [-L,L]^2 \cap \mathbb{Z}^2$, see (P3-17) for an explicit definition, and let

$$\tau_t^{\text{bulk}}(A) = \lim_{L\to\infty} \mathrm{e}^{\mathrm{i}t(H_L^{\text{bulk}}(b)-\mu\mathcal{N}_{B_L})}\, A\, \mathrm{e}^{-\mathrm{i}t(H_L^{\text{bulk}}(b)-\mu\mathcal{N}_{B_L})}$$

be the corresponding infinite volume dynamics, which exist due to Lieb-Robinson bounds. Then, we prove that the magnetization of any KMS state of the system on $\mathbb{Z}^2$ equals the magnetization $m(\beta,\mu,b)$ of the edge system.

Theorem 3.17 (Simplified statement of theorem P3-IV)**.** *Let ω be any $(\tau^{\text{bulk}},\beta)$-KMS state. Under the above assumptions, the per volume pressure of ω equals the limit* $\lim_{L\to\infty} p_L(\beta,\mu,b)$*. Moreover, $b \mapsto p(\beta,\mu,b)$ is differentiable and the derivative agrees with the magnetization $m(\beta,\mu,b)$ obtained from the edge system.*

In the non-interacting setting [65], one can also show the better known bulk-edge correspondence for the transport coefficients in the zero temperature limit when assuming a gapped ground state. In this setting, the derivative of the magnetization and edge current with respect to the chemical potential converge to the Hall conductivity and edge conductance, respectively. As a first step in this direction, we establish the differentiability of the magnetization and edge current also for interacting systems in theorem P3-III.

The proofs mostly rely on local indistinguishability and current conservation. As the Gibbs state is stationary, the total number of particles in any box $Z \subset \Lambda_L$ is not changed by the time evolution, and so the in- and out-going currents through the boundary of Z exactly cancel. See section P3-3.1.2 for a formal proof. We can now use this insight for the left half of Λ_L to conclude that the total current across any vertical line through the whole Λ_L vanishes. And by local indistinguishability, the local currents in the bulk are almost the same and hence almost vanish. This decay of the currents through dual edges in the bulk is known as Bloch's theorem in the literature [39, 215, 12, 21] and plays an important role in the further analysis. The formal statement in our setting is given in proposition P3-5.

Since the local currents vanish in the bulk, there can only be a non-vanishing current in 1-direction at the lower and upper edge of Λ_L, and the localization of the edge current near the boundary follows. One is then left to explicitly calculate the magnetization, see (P3-25), and use current conservation together with Bloch's theorem cleverly to obtain theorem 3.16.

To obtain theorem 3.17, we adjust the arguments for translation invariant systems from [11] to our setting where the bulk interaction is invariant under magnetic translations. We first show that the per volume pressures of $H_L(b)$ and $H_L^{\mathrm{bulk}}(b)$ agree up to terms vanishing as $L \to \infty$. As in [11], we then observe that the per volume pressure of any KMS state agrees with this limit. The differentiability then follows by the earlier statements on differentiability of the quantities on Λ_L and uniform convergence.

3.8 Stability of locally gapped and weakly interacting spin systems

We now introduce a class of spin systems, for which decay of correlations, local indistinguishability and LPPL can be proven at any temperature. These systems are called *weakly interacting spin systems* and their Hamiltonian is of the form $H_\Lambda = H_\Lambda^0 + H_\Lambda^{\mathrm{int}}$. The first part $H_\Lambda = \sum_{x\in\Lambda} h_x$ is a sum of on-site terms $h_x \in \mathscr{A}_x$ that have a non-degenerate ground state with ground state energy 0 and a uniformly lower-bounded spectral gap. More precisely, there exists $g > 0$ such that for each x, there exists ψ_x such that $h_x\,\psi_x = 0$ and $h_x|_{\mathscr{H}_x \ominus \psi_x} \geq g$. The additional interaction $H_\Lambda^{\mathrm{int}} = \sum_{x\in\Lambda} v_x$ is a sum of finite-range terms $v_x \in B_x(R)$ with small uniformly bounded norm $\|\!|v|\!\| := \sup_{x\in\Lambda} \|v_x\|$.

For small enough H_Λ^{int}, these systems are gapped [218]. But the structure described above is much richer. In particular, note that the ground state of H_Λ^0 is a product state and that the ground state of H_Λ is a local dressing of the ground state of H_Λ^0, which can be seen by the spectral flow we discussed in section 3.3. Moreover, as H_Λ^0 is non-interacting, the system can be viewed as a weak quantum perturbation of a classical system.

The main example we have in mind is a lattice with a two-level system at each site, where H_Λ^0 is an external magnetic field in z-direction, e.g. $h_x = \sigma_z^x$, where $\sigma_\#^x$ is the #-Pauli matrix at site x, and H_Λ^{int} is an arbitrary, small, nearest-neighbour coupling.

3.8.1 Publication P1: Stability of the ground state

As mentioned above, Hamiltonian is gapped [218], and hence, the ground state satisfies decay of correlations by the results discussed in section 3.4. Moreover, Yarotsky [219] proves uniqueness of the infinite-volume ground state, based on a result similar to local indistinguishability or LTQO for these systems. The latter is stated in theorem P1-7.

In publication P1 we use this result to prove a very strong form of LPPL.

Theorem 3.18 (Strong LPPL, see theorem P1-3)**.** *Let $R \in \mathbb{N}$ and $g > 0$. There exist constants $c, c_1, c_2 > 0$, such that for any $\Lambda \Subset (\mathbb{Z}^\nu)$ and any weakly interacting spin system $H = H^0 + H^{\mathrm{int}}$ on Λ with on-site gap at least g, interaction range R, and interaction strength $\|\!|v|\!\| \leq c$ the following holds:*

Let $X \subset \Lambda$ be non-empty and $W \in \mathcal{A}_X$ be self-adjoint. Set $H^W = H + W$. Then for any ground state ρ of H, any ground state ρ_W of H^W, and all $A \in \mathcal{A}_Y$ with $Y \subset \Lambda$ it holds that

$$\left|\mathrm{tr}\big((\rho_W - \rho)\, A\big)\right| \leq \mathrm{e}^{c_1|Y|}\, \|A\|\, \mathrm{e}^{-c_2 d(Y,X)}. \tag{3.41}$$

We call this a *strong LPPL* because the bound does not depend on $\|W\|$. The heuristic explanation for this is that a perturbation can at most fix the ground state in X and selects a boundary condition in this way. Due to the small interaction, the perturbation might influence the state in the surrounding of X, but the very strong on-site gap quickly has a stronger effect and the bulk of the system does not see this boundary condition.

The full statement in publication P1 also allows for unbounded operators H^0 and perturbations W which are relatively bounded with respect to H. We restrict to the simpler case here for a cleaner presentation. Moreover, publication P1 we also lift this result, to systems, which only have the structure of a weakly interacting spin system locally within a region that contains Y, see corollary P1-5.

3.8.2 Publication P8: Stability of the Gibbs state at any temperature

In publication P8 we analyse the Gibbs states of weakly interacting spin systems and prove decay of correlations and local indistinguishability uniformly in temperature.

For high enough temperature, decay of correlations and local indistinguishability for weakly interacting spin systems follow from previous results [139, 98]. Moreover, *one-dimensional* short-range translation invariant systems are known to satisfy decay of correlations [9, 36, 185] and the restriction to translation invariant and short-range interactions has recently been overcome [136]. However, all these results obtain bounds that diverge as $\beta \to \infty$ and hence do not give uniform decay of correlations at low temperatures. As such, they do not exclude phase transitions as $\beta \to \infty$.

One way to exclude phase transitions as $\beta \to \infty$ is to consider quantum perturbations of stable classical phases, as we do here. This approach was used to obtain decay of correlations as $\beta \to \infty$ for special models [205, 161] and certain translation-invariant Hamiltonians [205, 40, 66, 67, 94] before. While these results can handle non-degenerate ground states, the proofs are quite involved. Our approach instead leads to rather simple self-contained proofs and not only allows to prove decay of correlations and local indistinguishability uniformly in temperature, which has not been considered in the literature before.

In contrast to the result for the ground state described in section 3.8.1, we need stronger assumptions on[19] H_Λ^{int}. More precisely, we assume that the local terms v_x are relatively form bounded with respect to the non-interacting Hamiltonian in the following strong form: There exists an $a \in (0,1)$ such that

$$|\langle \psi, v_x \psi \rangle| \le \frac{a}{|B_x(R)|} |\langle \psi, H^0_{B_x(R)} \psi \rangle| \qquad \text{for all } \psi \in \mathscr{H}_{B_x(R)}, \tag{3.42}$$

which in particular implies that

$$|\langle \psi, H_\Lambda^{\mathrm{int}} \psi \rangle| \le a\, |\langle \psi, H_\Lambda^0 \psi \rangle| \qquad \text{for all } \psi \in \mathscr{H}_\Lambda.$$

Heuristically, this assumption can be understood as follows. While the ground state properties are mostly determined by the interaction range and the gap above the ground state, the Gibbs state depends on the whole spectrum of H. And the bound (3.42) ensures that the many-body density of states of the perturbed Hamiltonian H behaves similarly to the many-body density of states of the non-interacting H^0. In particular, it ensures that the part of the spectrum above the ground state cannot collapse into a single energy by addition of H^{int}, something that is not prevented by known results for bounded interactions [52].

To state the results, we need to introduce the notion of *R-connected* sets $X \subset \Lambda$, which are such that for every two points $x, y \in X$ there exists a sequence of points $z_1 = x, z_2, \dots, z_m \in X, z_{m+1} = y$ such that $d(z_i, z_{i+1}) \le 2R$ for all $i \in \{1, \dots, m\}$. See also definitions P8-2.1 and P8-3.6.

Theorem 3.19 (Decay of correlations, see theorem P8-2.2)**.** *Let $D, q, R \in \mathbb{N}$ and $C_{\mathrm{int}} > 0$. Then there exist $a \in (0,1)$ and $C_1, C_2, C_3 > 0$ such that the following holds. Consider*

[19] In publication P8, we denote the interaction with $V_\Lambda = H_\Lambda^{\mathrm{int}}$.

the lattice $\Lambda \Subset \mathbb{Z}^D$ *and a Hamiltonian* $H_\Lambda^0 + H_\Lambda^{\text{int}}$ *as defined above with* $|||h|||, |||v||| \le C_{\text{int}}$, v_x *of range* $R \in \mathbb{N}$, *and* v_x *relatively a-bounded w.r.t.* H_Λ^0 *in the sense* (3.42). *Then the Gibbs state* ρ_Λ^β *at any inverse temperature* $\beta \in (0, \infty)$ *satisfies*

$$
\begin{aligned}
&|\operatorname{tr}(A\, B\, \rho_\Lambda^\beta) - \operatorname{tr}(A\, \rho_\Lambda^\beta) \operatorname{tr}(B\, \rho_\Lambda^\beta)| \\
&\qquad \le C_1\, \|A\|\, \|B\|\, \mathrm{e}^{C_2\, (|X|+|Y|)}\, \mathrm{e}^{-C_3\, d(X,Y)}
\end{aligned}
\tag{3.43}
$$

for all R-connected sets $X, Y \subset \Lambda$ *and observables* $A \in \mathcal{A}_X$ *and* $B \in \mathcal{A}_Y$.

On a very high level, our proof is based on an inclusion-exclusion principle from [217], which allows expanding the exponential $\mathrm{e}^{-\beta H_\Lambda}$. We then use a cluster expansion and the swapping trick from [3], to cancel many terms in the covariance. The remaining terms involve clusters that connect the supports X and Y and hence decay by estimates from [217].

While the results are only formulated for R-connected sets in publication P8, the proofs can be extended to sets with finite R-connected components. In this case, we expect the constant C_1 to increase faster than exponential in the number of R-connected components. This still allows concluding local indistinguishability and LPPL using quantum belief propagation, which we discussed in section 3.6.[20] However, the resulting bounds naturally have a prefactor that diverges as $\beta \to \infty$.

Instead, our proof strategy from publication P8 can be adjusted to directly prove local indistinguishability uniformly in temperature.

Theorem 3.20 (Local indistinguishability, see theorem P8-2.3)**.** *Let* $D, q, R \in \mathbb{N}$ *and* $C_{\text{int}} > 0$. *Then there exist* $a \in (0,1)$ *and* $C_1, C_2, \xi_{\text{LI}} > 0$ *such that the following holds. Consider the lattice* $\Lambda \Subset \mathbb{Z}^D$ *and a Hamiltonian* $H_\Lambda^0 + H_\Lambda^{\text{int}}$ *as defined above with* $|||h|||$, $|||v||| \le C_{\text{int}}$, v_x *of range* $R \in \mathbb{N}$, *and* v_x *relatively a-bounded w.r.t.* H_Λ^0 *in the sense* (3.42). *Moreover, let* $\Lambda' \subset \Lambda$ *and denote with* $H_{\Lambda'}$ *the Hamiltonian restricted to* Λ'. *Then the Gibbs states* ρ_Λ^β *and* $\rho_{\Lambda'}^\beta$ *at any inverse temperature* $\beta \in (0, \infty)$ *satisfy*

$$
|\operatorname{tr}(B\, \rho_\Lambda^\beta) - \operatorname{tr}(B\, \rho_{\Lambda'}^\beta)| \le C_1\, \|B\|\, \mathrm{e}^{C_2\, |Y|}\, \mathrm{e}^{-C_3\, d(Y, \Lambda \setminus \Lambda')}
\tag{3.44}
$$

for all R-connected sets $Y \subset \Lambda$ *and observables* $B \in \mathcal{A}_Y$.

In a similar way, the proof can also be adjusted to obtain LPPL. However, without further assumptions, we obtain a prefactor which diverges exponentially as $\beta \to \infty$ as we have too little knowledge about the perturbed system. A precise statement is given in theorem P8-4.1.

[20] The results from publication P4, which are discussed in section 3.6, do not apply directly if one only knows decay of correlations with exponential scaling in the supports of the observables. This restriction is necessary to handle short-range or long-rang interactions, and to preserve the exponential decay in the implications. Here, we only consider finite-range interactions and quantum belief propagation can be used to conclude local indistinguishability and LPPL at the cost of loosing exponential decay.

Publications

Publication P1

Local stability of ground states in locally gapped and weakly interacting quantum spin systems

Joscha Henheik [*] Stefan Teufel [†] Tom Wessel [†]

2022

Abstract

Based on a result by Yarotsky (J. Stat. Phys. 118, 2005), we prove that localized but otherwise arbitrary perturbations of weakly interacting quantum spin systems with uniformly gapped on-site terms change the ground state of such a system only locally, even if they close the spectral gap. We call this a *strong version* of the *local perturbations perturb locally* (LPPL) principle which is known to hold for much more general gapped systems, but only for perturbations that do not close the spectral gap of the Hamiltonian. We also extend this strong LPPL-principle to Hamiltonians that have the appropriate structure of gapped on-site terms and weak interactions only locally in some region of space.

While our results are technically corollaries to a theorem of Yarotsky, we expect that the paradigm of systems with a locally gapped ground state that is completely insensitive to the form of the Hamiltonian elsewhere extends to other situations and has important physical consequences.

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[*] Institute of Science and Technology Austria (IST Austria),
Am Campus 1, 3400 Klosterneuburg, Austria.
[†] Mathematisches Institut, Eberhard Karls Universität Tübingen,
Auf der Morgenstelle 10, 72076 Tübingen, Germany.

Contents

1 Introduction

We consider weakly interacting quantum spin systems on finite subsets Λ of the lattice $\mathbb{Z}^\nu$, $\nu \in \mathbb{N}$, described by a self-adjoint Hamiltonian

$$H = H_0 + H_{\text{int}}, \tag{1}$$

which is composed of a non-interacting part H_0 and an interacting part H_{int}. The non-interacting Hamiltonian H_0 is a sum of non-negative on-site Hamiltonians h_x, $x \in \Lambda$. Each h_x is assumed to have a non-degenerate ground state with ground state energy 0 and spectral gap of size at least g above the ground state. The interaction Hamiltonian H_{int} is a sum of interaction terms Φ_x of finite range R and of small uniformly bounded norm $\|\Phi_x\|$. We show that for such Hamiltonians a strong version of the *local perturbations perturb locally* (LPPL) principle holds: For any self-adjoint perturbation P, supported in a region $X \subset \Lambda$, any ground state ρ_P of the perturbed Hamiltonian $H + P$ agrees with the ground state ρ of the unperturbed Hamiltonian H when tested against observables A supported in a region $Y \subset \Lambda$ up to an error that is exponentially small in the distance $\text{dist}(Y, X)$. More precisely, theorem 3 states that there are positive constants $c, c_1, c_2 > 0$ depending only on R and g, but not on Λ, A, H or P, such that whenever $\|\Phi_x\| \leq c$ for all $x \in \Lambda$, it holds that

$$\left|\text{tr}\big((\rho_P - \rho)\, A\big)\right| \leq \mathrm{e}^{c_1 |Y|}\, \|A\|\, \mathrm{e}^{-c_2\, \text{dist}(Y,X)}. \tag{2}$$

Note that the uniformity of the error estimate with respect to the system size $|\Lambda|$ is one key aspect which makes this estimate non-trivial. Note also, that the bound on $\|\Phi_x\|$ implies that H has a gap above its unique ground state ρ as we discuss below. However, for our result we neither require nor actually have any uniform lower bound on the gap above the possibly non-unique ground state ρ_P of the perturbed Hamiltonian $H + P$.

As a corollary of our main theorem, we show that a bound of the form (2) also holds for systems that have the appropriate structure of gapped on-site terms and weak interactions only locally in some region of space. In particular, this shows that the notion of a locally gapped ground state, which is completely insensitive to the form of the Hamiltonian elsewhere, is perfectly valid in this setup.

The LPPL-principle was coined by Bachmann, Michalakis, Nachtergaele, and Sims in [23], where a similar estimate with subexponential decay was proven. While their result covers much more general interacting quantum spin systems, it requires the gap above the ground state to remain open also for the perturbed Hamiltonian $H+P$. More precisely, it relies on connecting $H(0) := H$ with $H(1) := H+P$ by a continuous path $[0,1] \ni t \mapsto H(t)$ in the space of Hamiltonians, such that the gap above the ground state of $H(t)$ remains open uniformly along the whole path. Then the locality of the quasi-adiabatic evolution introduced by Hastings and Wen in [120] can be used to prove the result. Their subexponential bound was improved to exponential precision for finite-range interactions by De Roeck and Schütz in [74]. See also [174, 175] for recent developments.

While we prove the strong version of the LPPL-principle only for weakly interacting spin systems, we expect it to hold somewhat more generally. For example, we expect it to hold for fermions on the lattice with weak finite range interactions, a physical setup where the strong LPPL-principle would have important consequences. It would imply that a gapped ground state for such a system with periodic boundary conditions remains unchanged in the bulk when introducing open boundary conditions that may close the global gap due to the emergence of edge states. And as a consequence, it would also explain why the adiabatic response to external fields in the bulk of such systems is not affected by edge states that close the gap, see [17, 165, 203, 123, 122] for related results. However, it is known that the strong LPPL-principle cannot hold in general, but requires further conditions on the unperturbed ground state sector such as local topological quantum order (LTQO) [162, 176].

Shortly before resubmitting the final version of this article, Bachmann et al. published a preprint containing a closely related result. In [16] they prove an LPPL-bound as in (2), but with subexponential decay, assuming LTQO for a unique frustration-free gapped ground state of the unperturbed Hamiltonian which has no long-range entanglement.

Our result is a corollary of a result by Yarotsky [219] (see theorem 7 below), which provides a bound on the difference of so-called finite volume ground states in quantum spin systems described by Hamiltonians of the form (1). His aim and main result in that work was to show the uniqueness of the ground state of such systems in the thermodynamic limit. In a different work Yarotsky [218] has shown that Hamiltonians of the form (1) with $\|\Phi\| < c$ indeed have a unique ground state separated by a gap $\tilde{g} > 0$ from the rest of the spectrum, with $\tilde{g}$ independent of Λ (see [73, 93, 116] for similar results). Closely related to the stability of the gap is the stability of phase diagrams at low temperatures, see [40, 66, 67].

2 Main results

Consider the lattice $\mathbb{Z}^\nu$ for fixed $\nu \in \mathbb{N}$ equipped with the ℓ^1-metric $d\colon \mathbb{Z}^\nu \times \mathbb{Z}^\nu \to \mathbb{N}_0$ and define $\mathcal{P}_0(\mathbb{Z}^\nu) = \{ \Lambda \subset \mathbb{Z}^\nu \,|\, |\Lambda| < \infty \}$, where $|\Lambda|$ denotes the cardinality of Λ. With each site $x \in \mathbb{Z}^\nu$ one associates a (possibly infinite dimensional) Hilbert space $\mathcal{H}_x$. For $\Lambda \in \mathcal{P}_0(\mathbb{Z}^\nu)$ set $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$ and denote the algebra of bounded linear operators on $\mathcal{H}_\Lambda$ by $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$. Due to the tensor product structure, we have $\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x)$. Hence, for $\Lambda' \subset \Lambda \in \mathcal{P}_0(\mathbb{Z}^\nu)$, any $A \in \mathcal{A}_{\Lambda'}$ can be viewed as an element of $\mathcal{A}_\Lambda$ by identifying A with $A \otimes \mathbb{1}_{\Lambda\setminus\Lambda'} \in \mathcal{A}_\Lambda$, where $\mathbb{1}_{\Lambda\setminus\Lambda'}$ denotes the identity in $\mathcal{A}_{\Lambda\setminus\Lambda'}$. Note that

$$[A, B] = 0 \quad \text{for all} \quad A \in \mathcal{A}_\Lambda,\ B \in \mathcal{A}_{\Lambda'} \quad \text{with} \quad \Lambda \cap \Lambda' = \emptyset.$$

Similarly, we will also denote by K the closure of $\mathbb{1}_{\Lambda\setminus\Lambda'} \otimes K$ on $\mathcal{H}_{\Lambda\setminus\Lambda'} \otimes D(K)$ for any self-adjoint operator K on $\mathcal{H}_{\Lambda'}$. Here and in the following, $D(K)$ denotes the domain of the operator K.

Our main result will be formulated for a Hamiltonian

$$H = H_0 + H_{\text{int}} \in \mathcal{A}_\Lambda$$

that is composed of a non-interacting part H_0 and an interacting part H_{int}. The non-interacting part H_0 is assumed to be of the form

$$H_0 = \sum_{x\in\Lambda} h_x,$$

where each h_x is a non-negative self-adjoint (possibly unbounded) operator on $\mathcal{H}_x$ with a unique gapped ground state $\psi_x \in D(h_x)$ satisfying

$$h_x \psi_x = 0 \quad \text{and} \quad h_x\big|_{D(h_x)\ominus\psi_x} \geq g, \tag{3}$$

for some fixed $g > 0$. The latter means that $\langle \varphi_x, (h_x - g\mathbb{1}_x)\varphi_x\rangle \geq 0$ for all $\varphi_x \in D(h_x)$ with $\langle \psi_x, \varphi_x\rangle = 0$. In other words, all Hamiltonians h_x have a spectral gap of size at least g above the bottom of their spectrum. The interacting part is of the form

$$H_{\text{int}} = \sum_{x\in\Lambda} \Phi_x,$$

with $\Phi_x \in \mathcal{A}_{b_x(R)}$ self-adjoint for each $x \in \Lambda$ and some fixed $R \in \mathbb{N}$. Here $b_x(R) := \{ y \in \Lambda \,|\, d(x, y) \leq R \}$ denotes the ℓ^1-ball with radius R centered at $x \in \Lambda$. We set

$$|||\Phi||| := \sup_{x\in\Lambda} \|\Phi_x\|.$$

Definition 1 (Weakly interacting spin system)**.** For any $\Lambda \in \mathcal{P}_0(\mathbb{Z}^\nu)$ we call a Hamiltonian $H = H_0 + H_{\text{int}}$ on $\mathcal{H}_\Lambda$ with H_0 and H_{int} satisfying the above conditions a *weakly interacting spin system on* Λ with on-site gap g, interaction range R and interaction strength $|||\Phi|||$. ◇

We use the following definition for ground states and briefly explain how it is connected to the standard definition in appendix A.

Definition 2. Let $\Lambda \in \mathscr{P}_0(\mathbb{Z}^\nu)$ and K be a self-adjoint and bounded below operator on $\mathscr{H}_\Lambda$. We say that $[K, A]$ is a bounded operator $B \in \mathscr{A}_\Lambda$, whenever A leaves $D(K)$ invariant and $[K, A] = B$ on $D(K)$.

A state $\rho \in \mathscr{A}_\Lambda$, i.e. a positive semi-definite bounded operator with trace equal to one, is called a ground state of K, if

$$\mathrm{tr}(A^* [K, A] \rho) \geq 0 \quad \text{for all } A \in \mathscr{A}_\Lambda \text{ such that } [K, A] \text{ is bounded.} \qquad \diamond$$

Our first main result is the following.

Theorem 3 (The strong LPPL-principle)**.** *Let $R \in \mathbb{N}$ and $g > 0$. There exist constants $c, c_1, c_2 > 0$, such that for any $\Lambda \in \mathscr{P}_0(\mathbb{Z}^\nu)$ and any weakly interacting spin system $H = H_0 + H_{\mathrm{int}}$ on Λ with on-site gap at least g, interaction range R, and interaction strength $|||\Phi||| \leq c$ the following holds:*

Let $X \subset \Lambda$ be non-empty and P be a symmetric operator on $\mathscr{H}_X$ such that P is relatively bounded with respect to H with H-bound less than one. Set $H_P = H + P$. Then for any ground state ρ of H, any ground state ρ_P of H_P, and all $A \in \mathscr{A}_Y$ with $Y \subset \Lambda$ it holds that

$$\left|\mathrm{tr}\big((\rho_P - \rho)\, A\big)\right| \leq \mathrm{e}^{c_1 |Y|}\, \|A\|\, \mathrm{e}^{-c_2\, \mathrm{dist}(Y,X)}. \tag{4}$$

Under the assumptions of the theorem, Yarotsky has proven in [218] that H has a unique ground state ρ, whenever $c > 0$ is small enough.[1] In the following we will assume that this is the case.

For X at the edge of Λ, the perturbation P can be employed to realize all kinds of boundary conditions, e.g. if $\Lambda = \{-M, \dots, M\}^\nu$ is a box, periodic boundary conditions can be modeled by some P connecting opposite sites in Λ. Therefore, if X is at the edge, one can take the thermodynamic limit $\Lambda \nearrow \mathbb{Z}^\nu$ in (4) and conclude that there exists a unique ground state ρ, i.e. a normalized positive functional, on the C^*-algebra of quasi-local observables $\mathscr{A} = \overline{\mathscr{A}_{\mathrm{loc}}}^{\|\cdot\|}$, independent of the imposed boundary conditions for the finite systems. This uniqueness of ground states for the infinite system was the main result of [219] and has been shown by Yarotsky based on theorem 7, which we quote below.

As mentioned in the introduction, we expect a similar strong LPPL-principle to hold also for fermionic lattice systems with weak finite range interactions. As discussed in [123, 122], this would have important consequences for linear response and adiabatic theorems for systems with a gap only in the bulk.

[1] Note that the systems for which Yarotsky proves existence and uniqueness of the ground state in [218] differ slightly from our definition of weakly interacting spin systems in the treatment of interaction terms near the boundary of the domain. To obtain the same result with our definition, one extends the Hamiltonian H to $\Omega \supset \Lambda$ as in the proof of theorem 3, applies the result from [218] and restricts the resulting ground state to Λ by taking the partial trace.

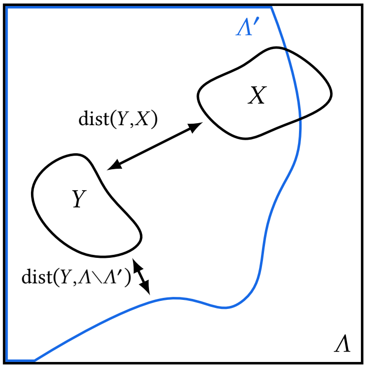

Figure 1. Depicted is the setting from corollary 5. The system H defined on Λ is assumed to be weakly interacting and to have an on-site gap in $\Lambda' \subset \Lambda$. For any perturbation P acting on $X \subset \Lambda$, ground states of H and $H + P$ agree in regions Y away from X and $\Lambda \setminus \Lambda'$.

Our second main result is a local version of theorem 3, where we assume the on-site gap and the weak interaction only locally.

Definition 4 (Locally weakly interacting spin system)**.** For any $\Lambda \in \mathscr{P}_0(\mathbb{Z}^\nu)$ and $\Lambda' \subset \Lambda$ we say that a self-adjoint operator H on $\mathscr{H}_\Lambda$ is *weakly interacting in the region* Λ' with on-site gap g, range R and strength s, if and only if there exists a weakly interacting spin system $\tilde{H} = \tilde{H}_0 + \tilde{H}_{\mathrm{int}}$ on Λ with on-site gap g, range R and strength $\|\Phi\| = s$ such that $H - \tilde{H} = \mathbb{1}_{\mathscr{H}_{\Lambda'}} \otimes Q$ with Q a possibly unbounded symmetric operator on $\mathscr{H}_{\Lambda\setminus\Lambda'}$ such that Q is infinitesimally $\tilde{H}$-bounded. ◇

Corollary 5 (The strong LPPL-principle for local gaps)**.** *Let $R \in \mathbb{N}$, $g > 0$, and $c, c_1, c_2 > 0$ be the constants from theorem 3. Then for any $\Lambda \in \mathscr{P}_0(\mathbb{Z}^\nu)$, $\Lambda' \subset \Lambda$, and any self-adjoint operator H on $\mathscr{H}_\Lambda$ which is weakly interacting in the region Λ' with on-site gap at least g, range R and strength $s \leq c$ the following holds:*

Let $X \subset \Lambda$ be non-empty and P be a symmetric operator on $\mathscr{H}_X$ such that P is relatively bounded with respect to H with H-bound less than one. Set $H_P = H + P$ (see figure 1). Then for any ground state ρ of H, any ground state ρ_P of H_P, and all $A \in \mathscr{A}_Y$ with $Y \subset \Lambda'$ it holds that

$$\left|\operatorname{tr}\big((\rho_P - \rho)\, A\big)\right| \leq 2\, \mathrm{e}^{c_1 |Y|}\, \|A\|\, \mathrm{e}^{-c_2 \min\{\operatorname{dist}(Y,X), \operatorname{dist}(Y,\Lambda\setminus\Lambda')\}}.$$

Proof. Let $\tilde{H}$ and Q be as in definition 4. Then $\tilde{H}$, $\tilde{H} + Q$ and $\tilde{H} + Q + P$ are self-adjoint. For the latter this follows, because also $Q + P$ is relatively bounded with respect to $\tilde{H}$ with $\tilde{H}$-bound less than one. This is not obvious, but the proof is a straightforward calculation that we skip.

Let $\tilde{\rho}$ be a ground state of $\tilde{H}$, see the comment after theorem 3 for existence. Then the triangle inequality and two applications of theorem 3 yield

$$\begin{aligned}\left|\operatorname{tr}\big((\rho_P - \rho)\, A\big)\right| &\leq \left|\operatorname{tr}\big((\rho_P - \tilde{\rho})\, A\big)\right| + \left|\operatorname{tr}\big((\rho - \tilde{\rho})\, A\big)\right| \\ &\leq \mathrm{e}^{c_1|Y|}\, \|A\| \left(\mathrm{e}^{-c_2 \operatorname{dist}(Y, X \cup (\Lambda\setminus\Lambda'))} + \mathrm{e}^{-c_2 \operatorname{dist}(Y,\Lambda\setminus\Lambda')}\right).\end{aligned}$$ □

3 Proof

The proof of theorem 3 is essentially a reinterpretation of a result by Yarotsky [219]. Since we only deal with finite volumes, we modify Yarotsky's notion of *finite volume ground states* to *ground states in the bulk*. To make the arguments as transparent as possible, we will add superscripts to Hamiltonians and states indicating on which subset of $\mathbb{Z}^\nu$ they are defined. These superscripts are also used to distinguish different operators and states. From now on let

$$\mathscr{D}_\Lambda := \{ A \in \mathscr{A}_\Lambda \,|\, [H_0^\Lambda, A] \text{ is bounded} \}$$

and note, that also $\{ A \in \mathscr{A}_\Lambda \,|\, [H_0^\Lambda + K, A] \text{ is bounded} \} = \mathscr{D}_\Lambda$ for all bounded operators $K \in \mathscr{A}_\Lambda$.

Definition 6 (Ground states in the bulk)**.** Let $R \in \mathbb{N}$, $\Lambda_* \subset \Lambda \in \mathscr{P}_0(\mathbb{Z}^\nu)$ and $H^{\Lambda_*} = H_0^{\Lambda_*} + H_{\text{int}}^{\Lambda_*} \in \mathscr{A}_{\Lambda_*}$ be a weakly interacting spin system on Λ_* with range R. Then we call

$$\Lambda_*^\circ := \{ x \in \Lambda_* \,|\, \operatorname{dist}(x, \mathbb{Z}^\nu \setminus \Lambda_*) > 2R \}$$

the *bulk* of the Hamiltonian H^{Λ_*} and any state $\rho^\Lambda \in \mathscr{A}_\Lambda$ satisfying

$$\operatorname{tr}\big(\rho^\Lambda A^* [H^{\Lambda_*}, A]\big) \geq 0 \quad \text{for all} \quad A \in \mathscr{D}_{\Lambda_*^\circ}$$

a *ground state in the bulk of* H^{Λ_*}. ◇

Our proof is based on the following theorem due to Yarotsky [219].

Theorem 7 ([219, Theorem 2])**.** *Let $R \in \mathbb{N}$ and $g > 0$. There exist constants $c, c_1, c_2 > 0$ such that for any $\Lambda_* \in \mathscr{P}_0(\mathbb{Z}^\nu)$, and any weakly interacting spin system $H^{\Lambda_*} = H_0^{\Lambda_*} + H_{\text{int}}^{\Lambda_*}$ on Λ_* with on-site gap at least g, range R and interaction strength $|||\Phi||| \leq c$ the following holds:*

Let $\Lambda \in \mathscr{P}_0(\mathbb{Z}^\nu)$ be such that $\Lambda_* \subset \Lambda$. Then for any two ground states ρ_1^Λ and $\rho_2^\Lambda \in \mathscr{A}_\Lambda$ in the bulk of H^{Λ_*} in the sense of definition 6, $Y \subset \Lambda_*$, and $A \in \mathscr{A}_Y$ it holds that

$$\big|\operatorname{tr}\big((\rho_1^\Lambda - \rho_2^\Lambda)\, A\big)\big| \leq e^{c_1 |Y|} \, \|A\| \, e^{-c_2 \operatorname{dist}(Y, \mathbb{Z}^\nu \setminus \Lambda_*^\circ)}.$$

Note that the set denoted by Λ in [219, Theorem 2] corresponds to our set Λ_*. Note, moreover, that any ground state ρ^Λ in the bulk of H^{Λ_*} trivially defines a finite-volume ground state $A \mapsto \operatorname{tr}(\rho^\Lambda (A \otimes \mathbb{1}_{\Lambda \setminus \Lambda_*}))$ of H^{Λ_*} in the sense of [219, Definition 2]. Allowing an arbitrary on-site gap $g > 0$ instead of $g = 1$, as in [219], is achieved by simple scaling.

Lemma 8. *Let $R \in \mathbb{N}$, $\Lambda_* \subset \Lambda \in \mathscr{P}_0(\mathbb{Z}^\nu)$ and $H^\Lambda = H_0^\Lambda + H_{\text{int}}^\Lambda \in \mathscr{A}_\Lambda$ be a weakly interacting spin system. Then the canonical restriction of H^Λ to Λ_* defined by*

$$H^\Lambda|_{\Lambda_*} = H_0^\Lambda|_{\Lambda_*} + H_{\text{int}}^\Lambda|_{\Lambda_*} := \sum_{x \in \Lambda_*} h_x + \sum_{\substack{x \in \Lambda_* : \\ \operatorname{dist}(x, \Lambda \setminus \Lambda_*) > R}} \Phi_x$$

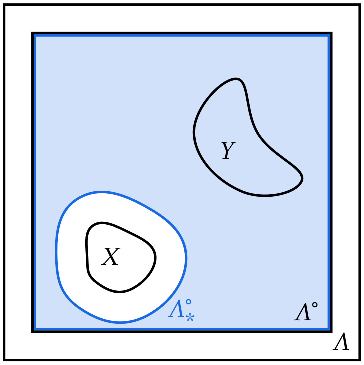

Figure 2. Depicted is the setting from the proof of proposition 9. The subset $X \subset \Lambda$ is the region where the perturbation P acts, and we choose $\Lambda_* = \Lambda \setminus X$. The shaded region Λ_*° is the bulk of H^{Λ_*}. $Y \subset \Lambda_*^\circ$ is the support of the observable A. This indicates why (5) holds.

is a weakly interacting spin system on Λ_* with the same on-site gap, range and strength and has the following property: For any symmetric operator Q on $\mathscr{H}_{\Lambda\setminus\Lambda_*^\circ}$ such that Q is relatively bounded with respect to H^Λ with H^Λ-bound less than one, any ground state of $H^\Lambda + Q$ is also a ground state in the bulk of $H^\Lambda|_{\Lambda_*}$.

Proof. It is clear that $H^\Lambda|_{\Lambda_*}$ is a weakly interacting spin system on Λ_*. A simple calculation shows, that Q is also relatively bounded with respect to $H_0^{\Lambda\setminus\Lambda_*^\circ} = \sum_{x\in\Lambda\setminus\Lambda_*^\circ} h_x$ with $H_0^{\Lambda\setminus\Lambda_*^\circ}$-bound less than one. Hence, $K := (H^\Lambda - H^\Lambda|_{\Lambda_*} + Q)$ is a self-adjoint operator on $\mathscr{H}_{\Lambda\setminus\Lambda_*^\circ}$. Moreover, any $A \in \mathscr{A}_{\Lambda_*^\circ}$ leaves invariant the domain of $\mathbb{1}_{\Lambda_*^\circ} \otimes K$ and satisfies

$$\left[\mathbb{1}_{\Lambda_*^\circ} \otimes K, A \otimes \mathbb{1}_{\Lambda\setminus\Lambda_*^\circ}\right] = 0.$$

Similarly, for all $A \in \mathscr{D}_{\Lambda_*^\circ}$, $[H^\Lambda + Q, A]$ is bounded and satisfies

$$[H^\Lambda + Q, A] = [H^\Lambda|_{\Lambda_*}, A].$$

Therefore, any ground state of $H^\Lambda + Q$ is also a ground state in the bulk of $H^\Lambda|_{\Lambda_*}$. □

Before we prove theorem 3, let us give an intermediate result, which follows rather directly from theorem 7 and lemma 8.

Proposition 9. *Let $R \in \mathbb{N}$ and $g > 0$. There exist constants $c, c_1, c_2 > 0$ such that for any $\Lambda \in \mathscr{P}_0(\mathbb{Z}^\nu)$ and any weakly interacting spin system $H^\Lambda = H_0^\Lambda + H_{\text{int}}^\Lambda$ on Λ with on-site gap at least g, interaction range R, and interaction strength $|\!|\!|\Phi|\!|\!| \leq c$ the following holds:*

Let $X \subset \Lambda$ be non-empty and P be a symmetric operator on $\mathscr{H}_X$ such that P is relatively bounded with respect to H^Λ with H^Λ-bound less than one. Set $H_P^\Lambda = H^\Lambda + P$. Then for any ground state ρ^Λ of H^Λ, any ground state ρ_P^Λ of H_P^Λ, and all $A \in \mathscr{A}_Y$ with $Y \subset \Lambda$ it holds that

$$\left|\operatorname{tr}\left((\rho_P^\Lambda - \rho^\Lambda)\, A\right)\right| \leq \mathrm{e}^{c_1|Y|}\, \|A\|\, \mathrm{e}^{-c_2 \min\{\operatorname{dist}(Y,\mathbb{Z}^\nu\setminus\Lambda^\circ),\, \operatorname{dist}(Y,X)-2R\}}.$$

Proof. Assume w.l.o.g. that $Y \subset \Lambda^\circ$. Otherwise, the statement in proposition 9 is trivially satisfied after a possible adjustment of c_1.

Let $\Lambda_* = \Lambda \setminus X$, and let $H^\Lambda|_{\Lambda_*}$ be the canonical restriction of H^Λ to Λ_* as defined in lemma 8. Then $\Lambda_*^\circ \cap X = \emptyset$. We can assume w.l.o.g. that $\mathrm{dist}(X, Y) > 2R$ since otherwise the statement in proposition 9 is trivially satisfied after a possible adjustment of c_1. Then also $Y \subset \Lambda_*^\circ$ (compare figure 2). By application of lemma 8 with $Q = P$ and $Q = 0$ we find that both, ρ_P^Λ and ρ^Λ, are ground states in the bulk of $H^\Lambda|_{\Lambda_*}$. Hence, theorem 7 implies that

$$\left|\mathrm{tr}\big((\rho_P^\Lambda - \rho^\Lambda)\, A\big)\right| \le \mathrm{e}^{c_1|Y|}\, \|A\|\, \mathrm{e}^{-c_2\, \mathrm{dist}(Y, \mathbb{Z}^\nu \setminus \Lambda_*^\circ)}.$$

From

$$\mathbb{Z}^\nu \setminus \Lambda_*^\circ = (\mathbb{Z}^\nu \setminus \Lambda^\circ) \cup \{\, x \in \mathbb{Z}^\nu \,|\, \mathrm{dist}(x, X) \le 2R \,\}$$

we immediately conclude that

$$\mathrm{dist}(Y, \mathbb{Z}^\nu \setminus \Lambda_*^\circ) = \min\{\mathrm{dist}(Y, \mathbb{Z}^\nu \setminus \Lambda^\circ),\ \mathrm{dist}(Y, X) - 2R\}, \tag{5}$$

which yields the claim. □

We now extend this result to obtain theorem 3.

Proof of theorem 3. In the following, we add superscripts Λ to the Hamiltonians and states from the statement of theorem 3.

Let $\Omega \in \mathscr{P}_0(\mathbb{Z}^\nu)$ be such that $\Lambda \subset \Omega$. For each $x \in \Omega \setminus \Lambda$ let $h_x \in \mathscr{A}_{\{x\}}$ be a self-adjoint operator with gap at least g and non-degenerate ground state ψ_x satisfying (3). Then $\rho^{\Omega\setminus\Lambda} = \bigotimes_{x \in \Omega\setminus\Lambda} |\psi_x\rangle\langle\psi_x|$ is the ground state of

$$H_0^{\Omega\setminus\Lambda} := \sum_{x \in \Omega\setminus\Lambda} h_x.$$

Moreover, $\rho^\Omega := \rho^\Lambda \otimes \rho^{\Omega\setminus\Lambda}$ is a ground state of $H^\Omega := H^\Lambda + H_0^{\Omega\setminus\Lambda}$ which is a weakly interacting spin system on Ω with on-site gap at least g, range R, and interaction strength $|||\Phi|||$. And also $\rho_P^\Omega := \rho_P^\Lambda \otimes \rho^{\Omega\setminus\Lambda}$ is a ground state of $H_P^\Omega := H_P^\Lambda + H_0^{\Omega\setminus\Lambda} = H^\Omega + P$.

According to proposition 9 we have

$$\left|\mathrm{tr}\big((\rho_P^\Omega - \rho^\Omega)\, A\big)\right| \le \mathrm{e}^{c_1|Y|}\, \|A\|\, \mathrm{e}^{-c_2 \min\{\mathrm{dist}(Y, \mathbb{Z}^\nu \setminus \Omega^\circ),\, \mathrm{dist}(Y, X) - 2R\}}$$

for all $A \in \mathscr{A}_Y$ and $Y \subset \Omega$. By requiring $Y \subset \Lambda$ we obtain

$$\left|\mathrm{tr}\big((\rho_P^\Lambda - \rho^\Lambda)\, A\big)\right| = \left|\mathrm{tr}\big((\rho_P^\Omega - \rho^\Omega)\, A\big)\right| \le \mathrm{e}^{c_1|Y|}\, \|A\|\, \mathrm{e}^{-c_2 \min\{\mathrm{dist}(\Lambda, \mathbb{Z}^\nu \setminus \Omega^\circ),\, \mathrm{dist}(Y, X) - 2R\}}.$$

Since this bound is independent of Ω, we can choose Ω sufficiently large such that $\mathrm{dist}(\Lambda, \mathbb{Z}^\nu \setminus \Omega^\circ) > \mathrm{dist}(Y, X) - 2R$. Absorbing e^{2c_2R} in c_1 yields the claim. □

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Conflicts of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

A Characterization of ground states

In the following lemma we show that every ground state in the usual sense, i.e. every minimizer of the energy functional, is also a ground state according to definition 2. While definition 2 is often used as a characterization of ground states in the context of extended quantum lattice systems, we could not find any reference in the literature, which covers the statement of the following lemma also for unbounded operators.

Lemma 10. *Let $\Lambda \in \mathscr{P}_0(\mathbb{Z}^\nu)$ and K be a self-adjoint and bounded below operator on $\mathscr{H}_\Lambda$. A state $\rho \in \mathscr{A}_\Lambda$ is a ground state in the usual sense, i.e.*

$$\operatorname{tr}(K\,\rho) \le \operatorname{tr}(K\,\tilde\rho) \quad \textit{for all states } \tilde\rho \in \mathscr{A}_\Lambda, \tag{6}$$

if and only if $\operatorname{ran}(\rho) \subset D(K)$ *and*

$$\operatorname{tr}(A^*\,[K,A]\,\rho) \ge 0 \quad \textit{for all } A \in \mathscr{A}_\Lambda \textit{ such that } [K,A] \textit{ is bounded.} \tag{7}$$

Here we adopt the convention that the trace of an operator that is not trace class is $+\infty$.

For bounded K this implies that for any state $\rho \in \mathscr{A}_\Lambda$ the conditions (6) and (7) are equivalent. And for unbounded K, any ground state in the usual sense is a ground state according to our definition 2. It could be that equivalence extends to unbounded operators, i.e. that (7) implies $\operatorname{ran}(\rho) \subset D(K)$. However, we could not find a proof for this.

Proof of lemma 10. Let $E_0 := \inf \sigma(K)$ and let (ϕ_n) be a Weyl sequence for E_0, i.e. $\phi_n \in D(K)$, $\|\phi_n\| = 1$ and $\|(K - E_0)\,\phi_n\| \le 1/n$ for all $n \in \mathbb{N}$.

Assume that ρ satisfies (6). Since $\operatorname{tr}(K\,|\phi_n\rangle\langle\phi_n|) \le E_0 + 1/n$, it follows that $\operatorname{tr}(K\rho) = E_0$. Hence, E_0 is an eigenvalue of K and the range of ρ is contained in the ground state eigenspace. Let $A \in \mathscr{A}_\Lambda$ such that $[K,A]$ is bounded. Then the operator $A^*\,(K - E_0)\,A$ is non-negative and

$$\operatorname{tr}\big(A^*\,[K,A]\,\rho\big) = \operatorname{tr}\big(\rho^{\frac12}\,A^*\,[K,A]\,\rho^{\frac12}\big) = \operatorname{tr}\big(\rho^{\frac12}\,A^*\,(K - E_0)\,A\,\rho^{\frac12}\big) \ge 0$$

follows.

Now assume that ρ is a ground state in the sense of definition 2, denote by $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ a spectral decomposition of ρ with ψ_i normalized. Since $\mathrm{ran}(\rho) \subset D(K)$, also $\psi_i \in D(K)$. The operator $A_{n,j} := |\phi_n\rangle\langle\psi_j|$ then has a bounded commutator with K and inequality (7) yields

$$0 \leq \mathrm{tr}\big(A_{n,j}^* \, [K, A_{n,j}] \, \rho\big) = \lambda_j \, \big\langle \phi_n, [K, A_{n,j}] \, \psi_j \big\rangle \leq \lambda_j \left(E_0 + \tfrac{1}{n} - \langle \psi_j, K\psi_j \rangle\right).$$

Thus, $\langle \psi_j, K \, \psi_j \rangle \leq E_0$ for all j. Hence, $\mathrm{tr}(K \, \rho) = E_0$ and ρ is indeed a ground state of K. □

Publication P2

On adiabatic theory for extended fermionic lattice systems

Joscha Henheik [*] Tom Wessel [†]

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Abstract

We review recent results on adiabatic theory for ground states of extended gapped fermionic lattice systems under several different assumptions. More precisely, we present generalized super-adiabatic theorems for extended but finite and infinite systems, assuming either a *uniform gap* or a *gap in the bulk* above the unperturbed ground state. The goal of this note is to provide an overview of these adiabatic theorems and briefly outline the main ideas and techniques required in their proofs.

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[*] Institute of Science and Technology Austria (IST Austria),
Am Campus 1, 3400 Klosterneuburg, Austria
[†] Fachbereich Mathematik, Eberhard-Karls-Universität Tübingen,
Auf der Morgenstelle 10, 72076 Tübingen, Germany

Contents

1 Introduction

In this article, we review four recent results on adiabatic theory for ground states of extended finite and infinite fermionic lattice systems at zero temperature [203, 123, 122]. These results are *generalized super-adiabatic theorems* (see Section 1.2) and concern Hamiltonians of the form

$$H^{\varepsilon} = H_0 + \varepsilon V,$$

where the unperturbed Hamiltonian H_0 is a sum-of-local-terms (SLT) operator describing short-range interacting fermions and is assumed to have a spectral gap above its ground state. This gap might be closed by the (small) perturbation εV, which is given by a short-range Hamiltonian, a Lipschitz potential, or a sum of both. Consequently, the results presented in this article are adiabatic theorems for resonances of H^ε (cf. [2, 83]).

The most important corollary and main motivation for proving such theorems in the context of extended fermionic lattice systems is the rigorous justification of linear response theory [203, 124] and the Kubo formula [142] for (topological) insulators [157], such as quantum Hall systems [206], where the prototypical relevant perturbation is a linear external potential modeling a constant electric field closing the gap of H_0 for every $\varepsilon \neq 0$ (see Figure 1 on page 95).

In the remainder of this introduction, we first briefly discuss the connection between linear response and adiabatic theory in Section 1.1 (see also [203, 124]). Furthermore, we point out the key ingredients and developments, which allowed to prove the four adiabatic theorems reviewed in this paper. Afterwards, in Section 1.2, we explain the notion of generalized super-adiabatic theorems and thereby introduce *(super-adiabatic) non-equilibrium almost-stationary states* (NEASSs) [203] as the above mentioned resonances of H^ε. A first brief but somewhat precise statement and overview of the results is given in Section 1.3.

1.1 Linear response and adiabatic theory

The formalism of linear response theory [142] has been widely used in physics to calculate the response of a system in thermal equilibrium to external perturbations. Put briefly, linear response theory provides an answer to the following question: What is the response of a system described by a Hamiltonian H_0, that is initially in an equilibrium state ρ_0, to a small static perturbation εV? Or, in somewhat more mathematical terms: What is the change[1]

$$\rho_\varepsilon(A) - \rho_0(A) = \varepsilon\, \sigma_A + o(\varepsilon)$$

of the expectation value of an observable A induced by the perturbation εV to leading order in its strength $0 < \varepsilon \ll 1$? Here, ρ_ε denotes the state of the system after the perturbation has been (adiabatically) turned on and σ_A denotes the linear response coefficient.

The answer to this fundamental question of linear response clearly hinges on the problem of determining ρ_ε. Although in few particular situations one expects ρ_ε to remain an equilibrium state for the perturbed Hamiltonian $H^\varepsilon = H_0 + \varepsilon V$, the original linear response theory [142] was developed for situations where the system is driven out of equilibrium, i.e. ρ_ε being a resonance state. As prominently formulated by Simon [198] in his "Fifteen problems in mathematical physics" from 1984, the latter non-equilibrium situation causes the main challenges in a rigorous mathematical treatment. However, in either case, the linear response coefficient σ_A is customarily

[1] To be consistent with the rest of the paper, we view states as linear functionals on the algebra of observables (see Section 2).

expected to be given by the celebrated *Kubo formula* [142], and rigorously justifying it was formulated as one of the problems by Simon [198]. For a more detailed recent review on the (mathematical) problem of proving Kubo's formula and its relevance in the context of quantum Hall systems, we refer to [124].

In a nutshell, the problem of justifying linear response theory and proving Kubo's formula is thus to verify that a system, initially in an equilibrium state ρ_0, is adiabatically driven by a small perturbation εV into a non-equilibrium state $\rho_\varepsilon \approx \rho_0$. Since the perturbation acts over a very long (macroscopic) time, this problem clearly supersedes standard perturbation theory: The change of the state being small is *not* a trivial consequence of the smallness of the perturbation εV. Instead, verifying that the two states, ρ_ε and ρ_0, are close to each other requires an adiabatic type theorem.

However, even in our rather simple setting (zero temperature, assuming that ρ_0 is the gapped ground state of H_0 describing an extended fermionic lattice system, the perturbation εV might close the gap), the problem of justifying the linear response formalism also goes beyond standard adiabatic theory. In fact, the applicability of the standard adiabatic theorem of quantum mechanics is rather restrictive for the following three reasons:

(i) The standard adiabatic theorem requires the perturbation εV to *not close the spectral gap*. In that scenario, it asserts that ρ_ε is (close to) the gapped ground state of $H^\varepsilon = H_0 + \varepsilon V$ and as such a (nearly) equilibrium state.

(ii) Even if we neglect the first issue, the usual adiabatic theorem estimates the difference between ρ_ε and the ground state of the perturbed Hamiltonian H^ε in *operator norm*, leaving the translation to local differences in expectation values as an additional and potentially non-trivial step.

(iii) In general, extended systems are plagued by the *orthogonality catastrophe*: Whenever for single-particle states $\psi, \tilde{\psi}$ we have $\|\psi - \tilde{\psi}\| \sim \varepsilon$, the non-interacting many-particle states $\otimes_{x\in\Lambda}\psi_x$ and $\otimes_{x\in\Lambda}\tilde{\psi}_x$ satisfy $\|\otimes_{x\in\Lambda}\psi_x - \otimes_{x\in\Lambda}\tilde{\psi}_x\| \sim \varepsilon|\Lambda|$, i.e. the norm-estimate deteriorates when $|\Lambda| \to \infty$. This means that the approximation error in the standard adiabatic theorem grows with the systems size, and it is thus not applicable for macroscopic systems.

A major breakthrough in overcoming these obstacles has recently been achieved by Bachmann, De Roeck and Fraas [17] (see also their introductory lecture notes [18]). They proved the first adiabatic theorem for extended (but finite) lattice systems with short-range interactions, thereby solving the second and third problem in the list above. More precisely, their result concerns differences in expectation values and provides error estimates, which are uniform in the system size.

For these lattice systems with short-range interactions, well known *Lieb-Robinson bounds* [154, 173, 174] ensure a finite speed of correlation and prevent build-up of long-range entanglement. Having Lieb-Robinson bounds at hand allowed Bachmann et al. [23] to prove that the generator of the *spectral flow*, introduced by Hastings and Wen [120], is an SLT operator and thus preserves good locality properties. The general spectral flow technique can then be used to prove automorphic equivalence of two gapped ground states ρ_0 and ρ_1 of Hamiltonians $H(0)$ and $H(1)$, respectively: Given a smooth path $s \mapsto H(s)$ of (uniformly) gapped SLT Hamiltonians, their ground states are automorphically equivalent (equal up to a conjugation by unitaries) with the generator of the automorphism being an SLT operator [23]. This automorphic equivalence allowed Bachmann et al. [17] to prove a *super-adiabatic theorem* (see Section 1.2 for an explanation of this notion) for such systems, however, still requiring the spectral gap not only for H_0 but also for H^ε, i.e. the gap must remain open.[2]

The four theorems presented in this article also solved the last remaining problem given under item (i) in the above list, i.e. they allow the perturbation εV to close the spectral gap of H_0. The main idea for establishing this generalization is that a spatially local gap should suffice for an adiabatic theorem to hold. This underlies the space-time adiabatic perturbation theory originally developed for non-interacting fermions by Panati, Spohn and Teufel [182, 180], where one utilizes a gap that exists locally in space (and time) but does not exist globally. It also underlies the recent results by De Roeck, Elgart and Fraas [70], where an adiabatic theorem holds even if the "spectral gap" is filled with eigenvalues, whose corresponding eigenvectors are spatially localized, leaving a gap (with smaller size) locally open. Finally, this is also the idea behind the Theorems III and especially IV presented below, where one still has an adiabatic type theorem although the gap closes at the boundary of the lattices.

Combining the ideas from the space-time adiabatic perturbation theory with the methods invented in [17], the first of the four theorems presented in this article was proven by Teufel [203]. It concerns extended but finite systems and requires a spectral gap for H_0, uniformly in the system size (see Assumption ($\mathrm{GAP}_{\mathrm{unif}}$)). The precise statement is formulated in Theorem I below. In order to extend this result from finite lattices to an infinite system, Henheik and Teufel [123] adapted ideas from Nachtergaele, Sims and Young [174] on controlling the thermodynamic limit of automorphisms with SLT generators. This result is formulated in Theorem II below.

So far, all mentioned results were obtained under the assumption of a (uniform) spectral gap for the finite systems (which also implies a gap for the infinite system). However, the recent result on automorphic equivalence with a *gap only in the bulk* (via the GNS construction) by Moon and Ogata [167], opened the door for a new class of adiabatic theorems, where the unperturbed Hamiltonian H_0 is no longer required to have a uniform spectral gap. Instead, Theorem III, originally proven by Henheik and Teufel [122], is a result for the infinite volume states and requires a gap in the bulk. This technically means a gap for the infinite system (cf. Assumption ($\mathrm{GAP}_{\mathrm{bulk}}$)) but can be understood as requiring a local gap in the interior of the finite lattices (cf. Remark 4).

[2] A slight generalization of their result can be found in [165], where the authors used an alternative gauge with a time-dependent vector potential for a quantum Hall model.

Moreover, by employing strong locality estimates from [174, 167] and assuming fast convergence of ground states, Theorem III can be traced back to extended but finite systems which only have a gap in the bulk. This was also proven in [122] and is formulated in Theorem IV below.

1.2 Non-equilibrium almost-stationary states

For the results presented in this paper, we consider time-dependent families

$$H^{\varepsilon}(t) = H_0(t) + \varepsilon V(t), \quad t \in I \subset \mathbb{R}, \tag{1}$$

of many-body Hamiltonians for lattice fermions in $\Gamma \subset \mathbb{Z}^d$ with short-range interactions. Here, Γ will either be a finite box Λ or the whole of $\mathbb{Z}^d$. For each $t \in I$, we denote by $\rho_0(t)$ the instantaneous ground state of $H_0(t)$ on the (quasi-local) algebra of observables $\mathcal{A}_\Gamma$. For simplicity of the presentation, we shall assume that the ground state is unique.[3] Moreover, we assume that the ground state is separated by a gap from the rest of the spectrum (see Assumptions ($\mathrm{GAP}_{\mathrm{unif}}$) and ($\mathrm{GAP}_{\mathrm{bulk}}$) in Section 3 for the precise formulation). The perturbation $V(t)$ can be a Hamiltonian with short-range interactions or a possibly unbounded external Lipschitz potential or a sum of both (see Section 2 and the Assumptions (INT_1)–(INT_4) in Section 3).

As mentioned above, the main results presented in this article are so-called *generalized super-adiabatic theorems* for $\rho_0(t)$, which we briefly explain in the following. For $\varepsilon = 0$, the results are *'standard' super-adiabatic theorems* and establish the existence of super-adiabatic states $\rho_0^\eta(t)$ on $\mathcal{A}_\Gamma$ close to $\rho_0(t)$, i.e.

$$\left|\rho_0^\eta(t)(A) - \rho_0(t)(A)\right| = \mathcal{O}(\eta),$$

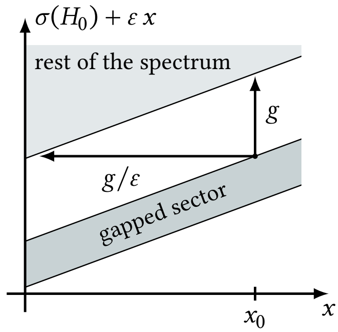
such that the adiabatic time-evolution $\mathfrak{U}_{t,t_0}^\eta$ on $\mathcal{A}_\Gamma$ generated by $\frac{1}{\eta}H_0(\cdot)$ intertwines the super-adiabatic states to all orders in the adiabatic parameter $\eta > 0$, i.e.

$$\left|\rho_0^\eta(t_0)(\mathfrak{U}_{t,t_0}^\eta[\![A]\!]) - \rho_0^\eta(t)(A)\right| = \mathcal{O}(\eta^\infty) \tag{2}$$

for all A in a dense subspace $\mathcal{D} \subset \mathcal{A}_\Gamma$. Throughout the entire paper, we shall study our system in the *Heisenberg picture*, meaning that the observable A evolves in time, not the state $\rho_0^\eta(t_0)$ (see also Proposition 3). Note that the comparison state $\rho_0^\eta(t)$ does *not* involve any time evolution but simply depends on the Hamiltonian at time t (see Definition 1 for details). Here and in the following, we write the arguments of (densely defined) linear operators on $\mathcal{A}_\Gamma$ inside the brackets $[\![\cdot]\!]$ for better readability.

[3] We refer to the original papers [203, 123, 122] for the most general assumptions. However, note that the results from [122], corresponding to our Theorems III and IV, are only formulated for a unique ground state, although the underlying result on automorphic equivalence of gapped phases [167] can easily be generalized to any gapped pure state (see [167, Remark 1.4]). In general, allowing for a degenerate ground state (or even a gapped spectral patch) requires understanding an enhanced modification of the spectral flow.

Figure 1. Let H_0 be a Hamiltonian with a gapped sector and a gap g. Perturbing with a Lipschitz potential $v(x) = \varepsilon x$, the gap gets closed (for large enough lattices). But, as indicated in the figure, a local gap persists and an electron at location x_0 would either need to overcome the gap (vertical arrow) or tunnel along the distance g/ε (horizontal arrow) in order to make a transition from the gapped sector. [203, 124]



For $\varepsilon > 0$, the scope of the adiabatic theorem (2) extends considerably since the perturbation $\varepsilon V(t)$ might close the spectral gap and turn the ground state $\rho_0(t)$ of $H_0(t)$ into an instantaneous resonance state $\Pi^\varepsilon(t)$ for $H^\varepsilon(t)$. These states have a life-time of order $\mathcal{O}(\varepsilon^{-\infty})$ for the dynamics $s \mapsto \mathrm{e}^{\mathrm{i}s\mathcal{L}_{H^\varepsilon(t)}}$ with $\mathcal{L}_{H^\varepsilon(t)}[\![\cdot]\!] := [H^\varepsilon(t), \cdot]$ (formally) denoting the derivation associated to $H^\varepsilon(t)$. That is, for all $n \in \mathbb{N}$ and fixed t, it holds that

$$\left|\Pi^\varepsilon(t)(\mathrm{e}^{\mathrm{i}s\mathcal{L}_{H^\varepsilon(t)}}[\![A]\!]) - \Pi^\varepsilon(t)(A)\right| = \mathcal{O}\Big(\varepsilon^n \left(1 + |s|^{d+1}\right)\Big),$$

which is why they were called *non-equilibrium almost-stationary states* (NEASSs) in this context by Teufel [203]. The generalized super-adiabatic theorems then establish the existence of a *super-adiabatic NEASS* $\Pi^{\varepsilon,\eta}(t)$ on $\mathcal{A}_\Gamma$ close to $\Pi^\varepsilon(t)$ such that the adiabatic time-evolution $\mathfrak{U}^{\varepsilon,\eta}_{t,t_0}$ generated by $\frac{1}{\eta}H^\varepsilon(\cdot)$ approximately intertwines the super-adiabatic NEASSs in the following sense: for any $n > d$ and for all $A \in \mathcal{D} \subset \mathcal{A}_\Gamma$, we have

$$\left|\Pi^{\varepsilon,\eta}(t_0)(\mathfrak{U}^{\varepsilon,\eta}_{t,t_0}[\![A]\!]) - \Pi^{\varepsilon,\eta}(t)(A)\right| = \mathcal{O}\left(\eta^{n-d} + \frac{\varepsilon^{n+1}}{\eta^{d+1}}\right) \tag{3}$$

uniformly for t in compact sets, which we call a *generalized super-adiabatic theorem.*

In our setting of gapped Hamiltonians H_0 describing insulating materials, there is indeed a clear and simple physical picture suggesting the existence of NEASSs for H^ε as observed in [203, 124] (see Figure 1). For simplicity, assume that H_0 is a periodic one-body operator in one spatial dimension and that the Fermi energy μ (chemical potential) lies in a gap of size g. For the perturbation, we consider the potential of a small constant electric field ε. In the initial state ρ_0, before the perturbation is turned on, all one-body states with energy smaller than μ are occupied. After the voltage has been applied, the energy of an electron located at position x_0 gets substantially shifted by εx_0, but is only subject to small force of order ε. As indicated in Figure 1, in order to make a transition, such an electron must either overcome the gap of size g or tunnel a macroscopic distance of order g/ε. Thus, although ρ_0 is neither close to the ground state nor any other equilibrium state of the perturbed Hamiltonian $H^\varepsilon = H_0 + \varepsilon V$, it is still almost stationary for H^{ε}. This heuristic picture remains valid if short-range interactions between the electrons are taken into account.

While for $\varepsilon = 0$ the generalized super-adiabatic theorem (3) reduces to the standard one (2), for $0 < \varepsilon \ll 1$ the right-hand side of (3) is small if and only if also η is small, but not too small compared to ε, i.e. $\varepsilon^{\frac{n+1}{d+1}} \ll \eta \ll 1$ for some $n \in \mathbb{N}$. Physically, this simply means that the adiabatic approximation breaks down when the adiabatic switching occurs at times that exceed the lifetime of the NEASS, an effect that has been observed in adiabatic theory for resonances before, see, e.g., [2, 83]. It can also be heuristically understood from the tunneling picture given in Figure 1.

Moreover, in view of the linear response problem discussed in Section 1.1, let us only mention here that a statement like (3), in fact, yields a solution to this problem after expanding the state $\Pi^{\varepsilon,\eta}(t)$ in powers of ε, where the linear term (eventually stemming from the first order operator A_1 given in (18)) does, in fact, constitute Kubo's formula. See [203, 124, 123] for details.

1.3 Brief statement of the results

We shall establish the existence of super-adiabatic NEASSs in four generally quite different situations, the main differences are also summarized in Table 1:

(I) On finite systems $\Lambda_k \Subset \mathbb{Z}^d$ with suitable boundary conditions, assuming that the unperturbed Hamiltonians $H_0^{\Lambda_k}(t)$ have a gapped ground state uniformly in Λ_k, there exists NEASSs on $\mathcal{A}_{\Lambda_k}$ such that the constants in (3) are independent of Λ_k. See Theorem I and [203].

(II) Additionally assuming convergence of the Hamiltonians (they have a thermodynamic limit, cf. Definition 2) and ground states, there also exists a super-adiabatic NEASS on $\mathcal{A}_{\mathbb{Z}^d}$ after taking the thermodynamic limit $\Lambda_k \nearrow \mathbb{Z}^d$. See Theorem II and [123].

(III) For the infinite system $\mathbb{Z}^d$, assuming that the unperturbed Hamiltonian H_0 has a unique gapped ground state (via the GNS construction), there exists a NEASS on $\mathcal{A}_{\mathbb{Z}^d}$, while a (uniform) spectral gap for finite sub-systems is not required. See Theorem III and [122].

(IV) Additionally assuming a quantitative control on the convergence of the finite volume Hamiltonians $H^{\Lambda_k}(t)$ (they have a rapid thermodynamic limit, cf. Definition 5) and the *unperturbed* ground states in the thermodynamic limit, there also exist NEASSs on $\mathcal{A}_{\Lambda_k}$ (again with a uniform constant) up to an error vanishing faster than any inverse polynomial in the distance to the boundary. See Theorem IV and [122].

	Finite volume	Infinite volume
Uniform gap	Theorem I; see [203]	Theorem II; see [123]
Gap in the bulk	Theorem IV; see [122]	Theorem III; see [122]

Table 1. Overview of the adiabatic theorems and the original papers.

A typical example of a physically relevant class of Hamiltonians [165, 203, 124], to which the above generalized super-adiabatic theorems apply, is given by

$$\begin{aligned} H_0^{\Lambda_k} = &\sum_{x,y\in\Lambda_k} a_x^* \, T(x-y)\, a_y \; + \sum_{x\in\Lambda_k} a_x^* \, \phi(x)\, a_x \\ &+ \sum_{x,y\in\Lambda_k} a_x^* a_x \, W\big(d^{\Lambda_k}(x,y)\big)\, a_y^* a_y \; - \; \mu\, N_{\Lambda_k}, \end{aligned} \tag{4}$$

modeling Chern or topological insulators. In agreement with the precise locality assumptions (INT_1)–(INT_4) in Section 3, we suppose that the kinetic term $T\colon \mathbb{Z}^d \to \mathscr{L}(\mathbb{C}^r)$ is an exponentially decaying function with $T(-x) = T(x)^*$, the potential term $\phi\colon \mathbb{Z}^d \to \mathscr{L}(\mathbb{C}^r)$ is a bounded function taking values in the self-adjoint matrices, and the two-body interaction $W\colon [0,\infty) \to \mathscr{L}(\mathbb{C}^r)$ is exponentially decaying and also takes values in the self-adjoint matrices. Note, that $x-y$ in the kinetic term refers to the difference modulo the imposed boundary condition on Λ_k. In the first line of (4), a_x is the column vector of the annihilation operators $a_{x,i}$ (i labels internal degrees of freedom such as spin) and a_x^* the row vector of the creation operators $a_{x,i}^*$ (see Section 2). And with a slight abuse of notation in the second line of (4), we wrote $a_x^* a_x$ for the *row* vector with entries $a_{x,i}^* a_{x,i}$ and $a_y^* a_y$ for the *column* vector with entries $a_{y,i}^* a_{y,i}$.

It is well known that non-interacting Hamiltonians H_0, i.e. with $W \equiv 0$, of the type (4) on a *torus* (periodic boundary condition) have a *uniform spectral gap* (see Assumption ($\mathrm{GAP}_{\mathrm{unif}}$)) whenever the chemical potential μ multiplying the number operator lies in a gap of the spectrum of the corresponding one-body operator on the infinite domain. It was recently shown [116, 73], that the spectral gap remains open when perturbing by sufficiently small short-range interactions $W \neq 0$. On the other hand, the Hamiltonian H_0 on a *cube* with open boundary condition has, in general, no longer a spectral gap because of the appearance of edge states. However, away from the boundary, a *gap in the bulk* (see Assumption ($\mathrm{GAP}_{\mathrm{bulk}}$)) is still present. While also *uniqueness* of the ground state is expected to hold for such models, to our knowledge it has been shown only for certain types of quantum spin systems, cf. [219, 93, 175, 176, P1]. For further details, we refer to the original papers [203, 123, 122]. Finally, it is an interesting program to extend Table 1 by further rows representing different notions of a spectral gap for H_0, e.g. a *local gap* as in [P1] or even only a *mobility gap* (see [70] for a first result in this direction).

After a brief introduction to the relevant mathematical framework in Section 2, we formulate the four main theorems in Section 3. Ideas of their proofs are provided in Section 4.

2 Mathematical Framework

In this section, we briefly introduce the (standard) mathematical framework used in the formulation of the adiabatic theorems. More explanations and details are provided in [203, 123, 122].

2.1 Algebra of observables

We consider fermions with r spin or other internal degrees of freedom on the lattice $\mathbb{Z}^d$. Let $\{X \Subset \mathbb{Z}^d\} := \{X \subset \mathbb{Z}^d \,|\, |X| < \infty\}$ denote the set of finite subsets of $\mathbb{Z}^d$, where $|X|$ is the number of elements in X. For each $X \Subset \mathbb{Z}^d$ let $\mathfrak{F}_X$ be the fermionic Fock space built up from the one-body space $\ell^2(X, \mathbb{C}^r)$. The C^*-algebra of bounded operators $\mathcal{A}_X := \mathcal{L}(\mathfrak{F}_X)$ is generated by the identity element $\mathbb{1}_{\mathcal{A}_X}$ and the creation and annihilation operators $a^*_{x,i}$, $a_{x,i}$ for $x \in X$ and $1 \le i \le r$, which satisfy the canonical anti-commutation relations (CAR). Whenever $X \subset X'$, then $\mathcal{A}_X$ is naturally embedded as a subalgebra of $\mathcal{A}_{X'}$. For infinite systems, the *algebra of local observables* is defined as the inductive limit

$$\mathcal{A}_{\mathrm{loc}} := \bigcup_{X \Subset \mathbb{Z}^d} \mathcal{A}_X, \qquad \text{and its completion} \qquad \mathcal{A}_{\mathbb{Z}^d} := \overline{\mathcal{A}_{\mathrm{loc}}}^{\|\cdot\|}$$

with respect to the operator norm $\|\cdot\|$ is a C^*-algebra, called the *quasi-local algebra*. The even elements $\mathcal{A}^+_{\mathbb{Z}^d} \subset \mathcal{A}_{\mathbb{Z}^d}$ form a C^*-subalgebra. Also, note that for any $X \Subset \mathbb{Z}^d$ the set of elements $\mathcal{A}^N_X$ commuting with the number operator $N_X := \sum_{x \in X} a^*_x\, a_x := \sum_{x \in X} \sum_{i=1}^r a^*_{x,i}\, a_{x,i}$ forms a subalgebra of the even subalgebra, i.e. $\mathcal{A}^N_X \subset \mathcal{A}^+_X \subset \mathcal{A}_X$. As only even observables will be relevant to our considerations, we will drop the superscript $^+$ from now on and redefine $\mathcal{A}_{\mathbb{Z}^d} := \mathcal{A}^+_{\mathbb{Z}^d}$.

Since a very similar construction is common for quantum spin systems (see, e.g., [174]), all the results immediately translate to this setting.

2.2 Interactions and operator families

We shall consider sequences of Hamiltonians defined on centered boxes $\Lambda_k := \{-k, \dots, +k\}^d$ of size $2k$ with metric $d^{\Lambda_k}(\cdot,\cdot)$. This *metric* may differ from the standard ℓ^1-distance $d(\cdot,\cdot)$ on $\mathbb{Z}^d$ restricted to Λ_k if one considers discrete tube or torus geometries, but satisfies the bulk-compatibility condition

$$\forall k \in \mathbb{N}\ \forall x, y \in \Lambda_k \,:\, d^{\Lambda_k}(x,y) \le d(x,y) \ \text{ and } \ d^{\Lambda_k}(x,y) = d(x,y) \text{ whenever } d(x,y) \le k.$$

An *interaction on a domain* Λ_k is a map

$$\Phi^{\Lambda_k}\colon \{X \subset \Lambda_k\} \to \mathcal{A}^N_{\Lambda_k},\ X \mapsto \Phi^{\Lambda_k}(X) \in \mathcal{A}^N_X$$

with values in the self-adjoint operators. Note that the maps Φ^{Λ_k} can be extended to maps on the whole $\{X \Subset \mathbb{Z}^d\}$ or restricted to a smaller Λ_l, trivially. In order to describe fermionic systems on the lattice $\mathbb{Z}^d$ in the thermodynamic limit, one considers sequences $\Phi = \left(\Phi^{\Lambda_k}\right)_{k\in\mathbb{N}}$ of interactions on domains Λ_k and calls the whole sequence an *interaction.*

An *infinite volume interaction* is a map

$$\Psi\colon \{X \Subset \mathbb{Z}^d\} \to \mathcal{A}^N_{\text{loc}},\ X \mapsto \Psi(X) \in \mathcal{A}^N_X,$$

again with values in the self-adjoint operators. Such an infinite volume interaction defines a general interaction $\Psi = \left(\Psi^{\Lambda_k}\right)_{k\in\mathbb{N}}$ by restriction, i.e. by setting $\Psi^{\Lambda_k} := \Psi|_{\{X\subset\Lambda_k\}}$.[4] With any interaction Φ, one associates an *operator family*, which is a sequence $A = (A^{\Lambda_k})_{k\in\mathbb{N}}$ of self-adjoint operators

$$A^{\Lambda_k} := A^{\Lambda_k}(\Phi) := \sum_{X\subset\Lambda_k} \Phi^{\Lambda_k}(X) \in \mathcal{A}^N_{\Lambda_k}.$$

For any $a > 0$ and $n \in \mathbb{N}_0$, we define the norm

$$\|\Phi\|_{a,n} := \sup_{k\in\mathbb{N}} \sup_{x,y\in\mathbb{Z}^d} \sum_{\substack{X\subset\Lambda_k:\\ x,y\in X}} d^{\Lambda_k}\text{-diam}(X)^n\, \mathrm{e}^{a\cdot d^{\Lambda_k}(x,y)}\, \|\Phi^{\Lambda_k}(X)\| \tag{5}$$

on the space of interactions.[5] Note that these norms depend on the sequence of metrics d^{Λ_k} on the cubes Λ_k, i.e. on the boundary conditions.

Similar constructions for interactions and interaction norms are long known. More commonly, the norms are independent of the particular lattice Λ_k and the interaction $(\Phi^{\Lambda_k})_{k\in\mathbb{N}}$ is given by restrictions of a single infinite volume interaction. Moreover, in earlier works [191, 195] the authors did not require additional decay properties, which were only added later (see, e.g., [199, 118, 174]). The use of interactions and corresponding norms, which are *not* simply restrictions of an infinite volume interaction, originates in [165] to incorporate non-trivial boundary conditions. In order to control commutators with Lipschitz potentials (see Section 2.3), the dependence on the diameter d^{Λ_k}-diam(X) was added in [203]. Finally, to ensure the existence of the thermodynamic limit, it is necessary to require the bulk-compatibility condition [123,

[4] We will use the convention that Φ denotes general interactions and Ψ infinite volume interactions.

[5] One should be aware that the norm definition (5) is slightly modified compared to the original works [203, 123, 122] for the sake of simplicity in presentation. For more general and precise statements of the theorems we refer the reader to the original works.

122]. Yet another variant of defining interaction norms is to replace $\operatorname{dist}(x, y)$ with $\operatorname{diam}(X)$ in (5) (see, e.g., [118, 16]).

In order to quantify the difference of interactions in the bulk (see Section 3.2), we also introduce for any interaction Φ^{Λ_l} on the domain Λ_l and any $\Lambda_M \subset \Lambda_l$ the quantity

$$\|\Phi^{\Lambda_l}\|_{a,n,\Lambda_M} := \sup_{x,y\in\Lambda_M} \sum_{\substack{X\subset\Lambda_M: \\ x,y\in X}} \operatorname{diam}(X)^n \, \mathrm{e}^{a\cdot d(x,y)} \, \|\Phi^{\Lambda_l}(X)\|,$$

where d and diam now refer to the ℓ^1-distance on $\mathbb{Z}^d$.

Let $\mathcal{B}_{a,n}$ be the *Banach space of interactions* with finite $\|\cdot\|_{a,n}$-norm and define the space of *exponentially localized interactions* as the intersection $\mathcal{B}_{a,\infty} := \bigcap_{n\in\mathbb{N}_0} \mathcal{B}_{a,n}$. In the literature, the vector spaces of operator families, which can be written in terms of such interactions, are denoted by $\mathcal{L}_{a,n}$ and $\mathcal{L}_{a,\infty}$. Moreover, we will be a bit sloppy in the following terminology and call the elements A^{Λ_k} of an operator sequence A *sum-of-local-terms* (SLT) operators, whenever its interaction Φ_A has a finite interaction norm similar to (5), but with the exponential replaced by a function growing faster than any polynomial. This will allow us to formulate the results and the ideas of the proofs without too many details. For the precise conditions see, e.g., [123, Section 2.2].

Now, let $I \subset \mathbb{R}$ be an open interval. We say that a map $\Phi \colon I \to \mathcal{B}_{a,n}$ is *smooth and bounded* whenever it is (i) term- and point-wise smooth in $t \in I$, i.e. $t \mapsto \Phi^{\Lambda_k}(t, X)$ are C^∞-functions for all $k \in \mathbb{N}$ and $X \subset \Lambda_k$, and (ii) $\sup_{t\in I}\|\frac{\mathrm{d}^i}{\mathrm{d}t^i}\,\Phi(t)\|_{a,n} < \infty$ for all $i \in \mathbb{N}_0$. The corresponding spaces of smooth and bounded time-dependent interactions and operator families are denoted by $\mathcal{B}_{I,a,n}$ and $\mathcal{L}_{I,a,n}$ and are equipped with the norm $\|\Phi\|_{I,a,n} := \sup_{t\in I}\|\Phi(t)\|_{a,n}$. We say that $\Phi \colon I \to \mathcal{B}_{a,\infty}$ is smooth and bounded, if $\Phi \colon I \to \mathcal{B}_{a,n}$ is smooth and bounded for all $n \in \mathbb{N}_0$, and we write $\mathcal{B}_{I,a,\infty}$ and $\mathcal{L}_{I,a,\infty}$ for the corresponding spaces of *time-dependent exponentially localized interactions* and *operator families* respectively.

For (time-dependent) *infinite volume interactions* Ψ, we add a superscript $^\circ$ to the norms and to the normed spaces defined above, emphasizing in particular the use of open boundary conditions, i.e. $d^{\Lambda_k} \equiv d$. Note that the compatibility condition for the metrics d^{Λ_k} implies that $\|\Psi\|_{a,n} \leq \|\Psi\|^\circ_{a,n}$.

2.3 Lipschitz potentials

For the perturbation we will allow external potentials $v = (v^{\Lambda_k} \colon \Lambda_k \to \mathbb{R})_{k\in\mathbb{N}}$ that satisfy the Lipschitz condition

$$\mathcal{C}_v := \sup_{k\in\mathbb{N}} \sup_{\substack{x,y\in\Lambda_k: \\ x\neq y}} \frac{\left|v^{\Lambda_k}(x) - v^{\Lambda_k}(y)\right|}{d^{\Lambda_k}(x,y)} < \infty, \tag{6}$$

and call them for short *Lipschitz potentials.*[6] With a Lipschitz potential v we associate the corresponding operator-sequence $V_v = (V_v^{\Lambda_k})_{k\in\mathbb{N}}$ defined by

$$V_v^{\Lambda_k} := \sum_{x\in\Lambda_k} v^{\Lambda_k}(x)\, a_x^*\, a_x$$

and denote the space of Lipschitz potentials by $\mathscr{V}$. We emphasize that, since $\sup_{k\in\mathbb{N}} \sup_{x\in\Lambda_k} |v^{\Lambda_k}(x)|$ might be infinite, V_v is in general no SLT operator. However, this is still more restrictive than general onsite potentials, because it only varies slowly in space. Moreover, we say that the map $v\colon I \to \mathscr{V}$ is smooth and bounded whenever (i) $v^{\Lambda_k}(x,\cdot)$ are C^∞-functions for all $k \in \mathbb{N}$ and $x \in \Lambda_k$, and (ii) satisfies $\sup_{t\in I} C_{\frac{\mathrm{d}^i}{\mathrm{d}t^i} v(t)} < \infty$ for all $i \in \mathbb{N}_0$. The space of smooth and bounded *time-dependent Lipschitz potentials* is denoted by $\mathscr{V}_I$.

As above, we also introduce *infinite volume Lipschitz potentials* $v_\infty\colon \mathbb{Z}^d \to \mathbb{R}$, which, again by restriction and invoking the compatibility condition for the metrics d^{Λ_k}, can be viewed as a Lipschitz potential with $d^{\Lambda_k} \equiv d$ in (6). And analogously to Section 2.2, for (time-dependent) *infinite volume Lipschitz potentials*, we add a superscript $^\circ$ to the constant from (6) and to the spaces, emphasizing the use of open boundary conditions. Note that the compatibility condition for the metrics d^{Λ_k} implies that $\mathscr{C}_v \geq \mathscr{C}_v^\circ$.

3 Adiabatic theorems for gapped quantum systems

As mentioned in the introduction, we shall distinguish two generally quite different settings regarding the presence of a spectral gap of the unperturbed Hamiltonian H_0 grouped as Theorem I and Theorem II in Section 3.1 as well as Theorem III and Theorem IV in Section 3.2. First, in Section 3.1, we will work under the assumption that there exists a sequence of subsystems $(\Lambda_k)_{k\in\mathbb{N}}$ equipped with an appropriate metric (reflecting, e.g., periodic boundary conditions), ensuring that $H_0^{\Lambda_k}$ have a *uniform gap* above their ground state, which is made precise in Assumption ($\mathrm{GAP}_{\mathrm{unif}}$) below. Then, in Section 3.2, however, we drop this assumption and solely assume that H_0 has a *gap in the bulk*, meaning that the GNS Hamiltonian, describing the system in the thermodynamic limit, has a spectral gap above its ground state eigenvalue zero (see Assumption ($\mathrm{GAP}_{\mathrm{bulk}}$)). Note that the second group of results is more general than the first group with regard to the gap condition, since a uniform gap for finite systems guarantees a spectral gap for the GNS Hamiltonian describing the infinite system (see Proposition 5.4 in [20]). Therefore, the second row in Table 1 somewhat improves the results in the first row since finding a suitable geometry for which one already has a spectral gap for finite systems is no longer necessary.

[6] Teufel [203] instead allowed slightly more general *slowly-varying potentials*. And while the phrase captures the idea very well, the technical definition is less transparent and slightly complicates the presentation of the proofs. Hence, we here, as in [123, 122], restrict to the subclass of Lipschitz potentials.

In the precise formulation of the adiabatic theorems, we shall frequently use the abbreviating phrase that a state $\Pi^{\varepsilon,\eta}(t)$ *is a super-adiabatic NEASS* (see Section 1.2), which we generally define as follows, reminiscent of [203, 123, 122].

Definition 1. (Super-adiabatic non-equilibrium almost-stationary states)
We assume to be in the following general setting, which is made precise in concrete situations: For (small) $\varepsilon > 0$, define the time-dependent Hamiltonian

$$H^\varepsilon(t) = H_0(t) + \varepsilon V(t), \quad t \in I, \quad \text{on} \quad \Gamma \subset \mathbb{Z}^d$$

and let $\rho_0(t)$ be (close to)[7] the ground state of $H_0(t)$. Moreover, denote the Heisenberg time-evolution on the algebra of (quasi-local) observables $\mathcal{A}_\Gamma$ generated by $\frac{1}{\eta}H^\varepsilon(t)$ as $\mathfrak{U}^{\varepsilon,\eta}_{t,t_0}$, where $t, t_0 \in I$ for some open interval $I \subset \mathbb{R}$ and $\eta > 0$ is a (small) adiabatic parameter.

Then, we say that a state $\Pi^{\varepsilon,\eta}(t)$ on $\mathcal{A}_\Gamma$ *is a super-adiabatic non-equilibrium almost-stationary state* for the state $\rho_0(t)$ and the time-evolution $\mathfrak{U}^{\varepsilon,\eta}_{t,t_0}$ on $\mathcal{A}_\Gamma$ if it satisfies the following properties:

1. $\Pi^{\varepsilon,\eta}$ almost **intertwines the time evolution**: For any $n \in \mathbb{N}$, there exists a constant C_n such that for any $t, t_0 \in I$ and for all $X \Subset \Gamma$ and $A \in \mathcal{A}_X \subset \mathcal{A}_\Gamma$ we have

$$\left|\Pi^{\varepsilon,\eta}(t_0)(\mathfrak{U}^{\varepsilon,\eta}_{t,t_0}[\![A]\!]) - \Pi^{\varepsilon,\eta}(t)(A)\right| \leq C_n \frac{\varepsilon^{n+1} + \eta^{n+1}}{\eta^{d+1}} \left(1 + |t - t_0|^{d+1}\right) \|A\| \, |X|^2. \quad (7)$$

2. $\Pi^{\varepsilon,\eta}$ is **local in time**: $\Pi^{\varepsilon,\eta}(t)$ only depends on H_0 and V and their time derivatives at time t.

3. $\Pi^{\varepsilon,\eta}$ is **stationary** whenever the Hamiltonian is stationary: If for some fixed $t \in I$ all time-derivatives of H_0 and V vanish at time t, then $\Pi^{\varepsilon,\eta}(t)$ equals the NEASS[8] $\Pi^\varepsilon(t)$ for the instantaneous ground state $\rho_0(t)$ and the time-evolution $s \mapsto \mathrm{e}^{\mathrm{i}s\mathcal{L}_{H^\varepsilon(t)}}$ generated by the time-independent Hamiltonian $H^\varepsilon(t)$.

4. $\Pi^{\varepsilon,\eta}$ equals the (approximate) ground state ρ_0 of H_0 whenever the perturbation vanishes and the Hamiltonian is stationary: If for some $t \in I$ all time-derivatives of H_0 and V vanish at time t and $V(t) = 0$, then $\Pi^{\varepsilon,\eta}(t) = \Pi^{\varepsilon,0}(t) = \rho_0(t)$. ◇

We could have written bound (7) in a more general form as indicated by (3). For example, we could allow $(1+|t-t_0|^{d+1})$ to be replaced by a constant $C_K < \infty$, depending only on a compact subset $K \subset I$ of times, or, similarly, $|X|^2$ to be replaced by a constant

[7] See the comment below Assumption ($\mathrm{S}_{\mathrm{bulk}}$) on page 110 for a precise definition.

[8] It follows from the construction sketched in Section 4.1.1 that $\Pi^\varepsilon(t) = \Pi^{\varepsilon,0}(t)$. Moreover, $\Pi^\varepsilon(t)$ is almost stationary with a bound as in (7) where the fraction is replaced by ε^{n+1}.

$C_X < \infty$, depending only on the support $X \Subset \mathbb{Z}^d$ of the observable A. Also, the power of η in the denominator could be allowed to be more general, e.g. some constant $C_d < \infty$ instead of $d+1$. However, the concrete form of (7) indeed matches the precise bounds of the results in Section 3.

3.1 Systems with a uniform gap

Throughout this section, we assume that H_0 has a uniformly gapped unique ground state in the following sense.

($\mathbf{GAP_{unif}}$) Assumptions on the ground state of H_0**.** Let $\Phi_{H_0} = \big(\Phi_{H_0}^{\Lambda_k}\big)_{k\in\mathbb{N}}$ be an interaction. There exists $L \in \mathbb{N}$ such that for all $t \in I$, $k \geq L$ and corresponding Λ_k the operator $H_0^{\Lambda_k}(t)$ has a simple gapped ground state eigenvalue $E_0^{\Lambda_k}(t) = \inf\sigma(H_0^{\Lambda_k}(t))$, i.e. there exists $g > 0$ such that $\operatorname{dist}\big(E_0^{\Lambda_k}(t), \sigma(H_0^{\Lambda_k}(t)) \setminus \{E_0^{\Lambda_k}(t)\}\big) \geq g$, for all $t \in I$, $k \geq L$. We denote the spectral projection of $H_0^{\Lambda_k}(t)$ corresponding to $E_0^{\Lambda_k}(t)$ by $P_0^{\Lambda_k}(t)$ and write $\rho_0^{\Lambda_k}(t)(\cdot) := \operatorname{tr}\big(P_0^{\Lambda_k}(t)\,\cdot\,\big)$ for the canonically associated state on $\mathcal{A}_{\Lambda_k}$. ◇

A physically relevant class of Hamiltonians satisfying this assumption (possibly up to the uniqueness, which we require for simplicity of the presentation) was given in (4) in Section 1.3. In the following, we shall present adiabatic theorems for extended but finite systems (Theorem I) as well as for infinite systems (Theorem II) under Assumption ($\mathrm{GAP_{unif}}$).

3.1.1 Extended but finite systems

The basic assumption on the Hamiltonian says that it is composed of exponentially localized interactions and/or a Lipschitz potential.

($\mathbf{INT_1}$) Assumptions on the interactions. Let H_0, H_1 be the Hamiltonians of two time-dependent exponentially localized interactions, i.e. $\Phi_{H_0}, \Phi_{H_1} \in \mathcal{B}_{I,a,\infty}$ for some $a > 0$, and $v \in \mathcal{V}_I$ be a time-dependent Lipschitz potential. ◇

The following results due to Teufel [203] marks the starting point for generalized super-adiabatic theorems for extended fermionic lattice systems.

Theorem I. (Adiabatic theorem for finite systems with a uniform gap [203, Theorem 5.1])
Under Assumptions ($\mathrm{GAP_{unif}}$) *and* ($\mathrm{INT_1}$), *there exists a sequence of near-identity*[9] *automorphisms* $\beta^{\varepsilon,\eta,\Lambda_k}(t) = \mathrm{e}^{\mathrm{i}\varepsilon\mathcal{L}^{\Lambda_k}_{S^{\varepsilon,\eta}(t)}}$ *with SLT generators* $S^{\varepsilon,\eta}$ *for any* $\varepsilon, \eta \in (0,1]$ *and* $t \in I$ *such that the states*

$$\Pi^{\varepsilon,\eta,\Lambda_k}(t) := \rho_0^{\Lambda_k}(t) \circ \beta^{\varepsilon,\eta,\Lambda_k}(t) \tag{8}$$

[9] Indeed, $\sup_{k\in\mathbb{N}}\big\|A - \beta^{\varepsilon,\eta,\Lambda_k}(t)[\![A]\!]\big\| \leq (\varepsilon+\eta)\,C\,\|A\|\,|X|\,|t|$ for $A \in \mathcal{A}_X$ and t in a bounded interval by [174, Theorem 3.4(i)].

are super-adiabatic NEASSs for the Heisenberg time-evolution $\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}$ *on* $\mathcal{A}_{\Lambda_k}$ *generated by* $\frac{1}{\eta}H^{\varepsilon,\Lambda_k}(\cdot)$ *with*

$$\frac{1}{\eta}H^{\varepsilon,\Lambda_k}(t) := \frac{1}{\eta}\left(H_0^{\Lambda_k}(t) + \varepsilon\left(V_v^{\Lambda_k}(t) + H_1^{\Lambda_k}(t)\right)\right)$$

uniformly in $k \geq L$*. That is, for every* $n \in \mathbb{N}$*, there exists a constant* C_n*, such that for any* $A \in \mathcal{A}_X$*,* $\varepsilon, \eta \in (0,1]$ *and all* $t, t_0 \in I$ *it holds that*

$$\sup_{k\geq L}\left|\Pi^{\varepsilon,\eta,\Lambda_k}(t_0)(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]) - \Pi^{\varepsilon,\eta,\Lambda_k}(t)(A)\right| \leq C_n\,\frac{\varepsilon^{n+1}+\eta^{n+1}}{\eta^{d+1}}\left(1+|t-t_0|^{d+1}\right)\|A\|\,|X|^2.$$

The proof of this result fundamentally builds on space-time adiabatic perturbation theory [182, 180] and technical estimates originally derived in [17]. The latter show that the operations necessary for the construction of the generator of the near-identity automorphism in the definition of the NEASS in (8) (almost) preserve exponential localization required for the Hamiltonian (see Section 4). As already mentioned in the introduction, although the adiabatic theorem in [17] is at first sight quite similar to the one above, it requires the perturbation to *not* close the spectral gap of the Hamiltonian H_0 and is thus not generalized in the sense explained in Section 1.2.

3.1.2 Infinite systems

The next result is obtained from Theorem I by taking $\Lambda_k \nearrow \mathbb{Z}^d$. This requires the interactions and the Lipschitz potential composing the Hamiltonian (1) to *have a thermodynamic limit* [123] in the following sense.

Definition 2. (Thermodynamic limit of interactions and potentials)

(a) An exponentially localized time-dependent interaction $\Phi \in \mathcal{B}_{I,a,\infty}$ is said to *have a thermodynamic limit* (have a TDL) if there exists an infinite volume interaction $\Psi \in \mathcal{B}^{\circ}_{I,a,\infty}$ such that

$$\forall\, n \in \mathbb{N},\ i \in \mathbb{N}_0,\ M \in \mathbb{N}:\ \lim_{k\to\infty}\sup_{t\in I}\left\|\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\Psi - \Phi^{\Lambda_k}\right)(t)\right\|_{a,n,\Lambda_M} = 0,$$

and we write $\Phi \xrightarrow{\text{td}} \Psi$ in this case.

An operator family is said to *have a TDL* if and only if the corresponding interaction does.

For more general (non-exponentially localized) SLT operators, the definition is completely analogous.

(b) A Lipschitz potential $v \in \mathscr{V}_I$ is said to *have a TDL* if there exists an infinite volume Lipschitz potential $v_\infty \in \mathscr{V}_I^\circ$ such that

$$\forall M \in \mathbb{N} \quad \exists K \geq M \quad \forall k \geq K,\ t \in I\ :\ v^{\Lambda_k}(t,\cdot)|_{\Lambda_M} = v_\infty(t,\cdot)|_{\Lambda_M}.$$

Again, we write $v \xrightarrow{\text{td}} v_\infty$ in this case. ◇

Note that, whenever $\Phi = \Psi$ for some infinite-volume interaction Ψ, or $v = v_\infty$ for some infinite volume Lipschitz potential v_∞, both Φ and v trivially have a TDL.

The following proposition is a standard consequence of Lieb-Robinson bounds and shows that the property of having a TDL for interactions and Lipschitz potentials guarantees the existence of the thermodynamic limit for the corresponding evolution operators [174, 56]. We remark that it remains true under less restrictive assumptions on the localization quality of the interaction (see, e.g., Proposition 2.2 in [123]).

Proposition 3. (Thermodynamic limit of evolution operators)
Let $K_0 \in \mathscr{L}_{I,a,\infty}$ and $w \in \mathscr{V}_I$ both have a thermodynamic limit, i.e. $\Phi_{K_0} \xrightarrow{\text{td}} \Psi_{K_0}$ and $w \xrightarrow{\text{td}} w_\infty$ for some $\Psi_{K_0} \in \mathscr{B}^\circ_{I,a,\infty}$ and $w_\infty \in \mathscr{V}_I^\circ$. Set $K = K_0 + V_w$ and let $U^{\eta,\Lambda_k}(t,t_0)$ denote the evolution family generated by $K^{\Lambda_k}(t)$ in scaled time with $\eta > 0$, i.e. the solution to the Schrödinger equation

$$\mathrm{i}\,\eta\,\frac{\mathrm{d}}{\mathrm{d}t}\,U^{\eta,\Lambda_k}(t,t_0) = K^{\Lambda_k}(t)\,U^{\eta,\Lambda_k}(t,t_0)$$

with $U^{\eta,\Lambda_k}(t_0,t_0) = \mathrm{id}$. Then, there exists a co-cycle of automorphisms $\mathfrak{U}^\eta_{t,t_0} : \mathscr{A}_{\mathbb{Z}^d} \to \mathscr{A}_{\mathbb{Z}^d}$ such that for all $A \in \mathscr{A}_{\mathrm{loc}}$,

$$\mathfrak{U}^\eta_{t,t_0}[\![A]\!] = \lim_{k\to\infty} \mathfrak{U}^{\eta,\Lambda_k}_{t,t_0}[\![A]\!] := \lim_{k\to\infty} U^{\eta,\Lambda_k}(t,t_0)^*\,A\,U^{\eta,\Lambda_k}(t,t_0).$$

The co-cycle $\mathfrak{U}^\eta_{t,t_0}$ only depends on Ψ_{K_0} and w_∞ and is generated by the time-dependent (closed) derivation $(\mathscr{L}_{K(t)}, D(\mathscr{L}_{K(t)}))$ associated with $K(t)$.

As mentioned above, since the following Theorem II is deduced from Theorem I by taking $\Lambda_k \nearrow \mathbb{Z}^d$, we will need to assume the existence of a thermodynamic limit for the building blocks of the Hamiltonian (1).

(INT$_2$) Assumptions on the interactions. For $\Psi_{H_0}, \Psi_{H_1} \in \mathscr{B}^\circ_{I,a,\infty}$ for some $a > 0$ and $v_\infty \in \mathscr{V}_I^\circ$ there exist $\Phi_{H_0}, \Phi_{H_1} \in \mathscr{B}_{I,a,\infty}$ and $v \in \mathscr{V}_I$ with appropriate boundary conditions (encoded in the definition of the norms defining the spaces $\mathscr{L}$ and the Lipschitz condition) all having a TDL with the respective object as the limit, i.e. $\Phi_{H_0} \xrightarrow{\text{td}} \Psi_{H_0}$, $\Phi_{H_1} \xrightarrow{\text{td}} \Psi_{H_1}$ and $v \xrightarrow{\text{td}} v_\infty$. ◇

We also assume the convergence of ground states, by means of the Banach-Alaoglu Theorem (the unit sphere in $\mathscr{A}^*_{\mathbb{Z}^d}$ is weak*-compact), essentially only in order to avoid the extraction of a subsequence.

(S_{unif}) Assumptions on the convergence of states. Assume that for every $t \in I$ the sequence $\big(\rho_0^{\Lambda_k}(t)\big)_{k\in\mathbb{N}}$ of ground states (naturally extended to the whole of $\mathcal{A}_{\mathbb{Z}^d}$) converges in the weak*-topology to a state $\rho_0(t)$ on $\mathcal{A}_{\mathbb{Z}^d}$, which we call the *gapped limit ground state at $t \in I$.* ◇

We can now formulate the second generalized super-adiabatic theorem concerning infinite systems with a uniform gap [123].

Theorem II. (Adiabatic theorem for infinite systems with a uniform gap [123, Theorems 3.2 and 3.5])
Under the Assumptions (GAP_{unif}), (INT_2) *and* (S_{unif})*, there exists a near-identity automorphism* $\beta^{\varepsilon,\eta}(t) = \mathrm{e}^{\mathrm{i}\varepsilon \mathcal{L}_{S^{\varepsilon,\eta}(t)}}$ *with SLT generators* $S^{\varepsilon,\eta}$ *for any* $\varepsilon, \eta \in (0,1]$ *and* $t \in I$*, such that the state*

$$\Pi^{\varepsilon,\eta}(t) := \rho_0(t) \circ \beta^{\varepsilon,\eta}(t)$$

is a super-adiabatic NEASS for the Heisenberg time-evolution on $\mathcal{A}_{\mathbb{Z}^d}$ *generated by* $\frac{1}{\eta}\Psi_{H^\varepsilon(\cdot)}$ *with*

$$\Psi_{H^\varepsilon} := \Psi_{H_0} + \varepsilon\,(V_{v_\infty} + \Psi_{H_1}).$$

The crucial point in the proof of Theorem II in [123] is to show that the property of having a TDL is designed in such a way that it is preserved under all necessary operations for the construction of the NEASS (see Section 4). Therefore, also the near-identity automorphism from (8) converges as $\Lambda_k \nearrow \mathbb{Z}^d$ by means of Proposition 3.

3.2 Systems with a gap in the bulk

In this section, we drop Assumption (GAP_{unif}) of a uniform gap for finite systems, but merely work under the condition of a *gap in the bulk*, which is formulated via the Gelfand-Naimark-Segal (GNS) construction in Assumption (GAP_{bulk}) below: Let $\Psi_{H_0} \in \mathcal{B}^{\circ}_{a,0}$ be an infinite volume interaction and $\mathcal{L}_{H_0}$ denote the induced derivation on (a dense subset of) $\mathcal{A}_{\mathbb{Z}^d}$. A state ω on $\mathcal{A}_{\mathbb{Z}^d}$ is called an $\mathcal{L}_{H_0}$-ground state, if and only if $\omega(A^*\mathcal{L}_{H_0}(A)) \geq 0$ for all $A \in D(\mathcal{L}_{H_0})$. Let ω be an $\mathcal{L}_{H_0}$-ground state and $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the corresponding GNS triple ($\mathcal{H}_\omega$ a Hilbert space, $\pi_\omega\colon \mathcal{A} \to \mathcal{L}(\mathcal{H}_\omega)$ a representation and $\Omega_\omega \in \mathcal{H}_\omega$ a cyclic vector). Then, there exists a unique densely defined, self-adjoint positive operator $H_{0,\omega} \geq 0$ on $\mathcal{H}_\omega$ satisfying

$$\pi_\omega(\mathrm{e}^{\mathrm{i}t\mathcal{L}_{H_0}}[\![A]\!]) = \mathrm{e}^{\mathrm{i}tH_{0,\omega}}\,\pi_\omega(A)\,\mathrm{e}^{-\mathrm{i}tH_{0,\omega}} \quad \text{and} \quad \mathrm{e}^{-\mathrm{i}tH_{0,\omega}}\,\Omega_\omega = \Omega_\omega \tag{9}$$

for all $A \in \mathcal{A}$ and $t \in \mathbb{R}$. We call this $H_{0,\omega}$ the *bulk Hamiltonian* (or *GNS Hamiltonian*) associated with Ψ_{H_0} and ω. See [47] for the general theory.

We assume that Ψ_{H_0} has a unique gapped ground state in the following sense (cf. [167, 122]):

(GAP_{bulk}) Assumptions on the ground state of Ψ_{H_0}.

(i) **Uniqueness.** For each $t \in I$, there exists a unique $\mathcal{L}_{H_0(t)}$-ground state $\rho_0(t)$.

(ii) **Gap.** There exists $g > 0$ such that $\sigma\big(H_{0,\rho_0(t)}(t)\big) \setminus \{0\} \subset [g,\infty)$ for all $t \in I$.

(iii) **Regularity.** For any strictly positive $f \in \mathcal{S}(\mathbb{R})$ (Schwarz functions), define $\mathcal{D}_f$ as the set of observables $A \in \mathcal{A}_{\mathbb{Z}^d}$ for which $\|A\|_f := \|A\| + \sup_{k\in\mathbb{N}}\big(\|(1-\mathbb{E}_{\Lambda_k})[\![A]\!]\|/f(k)\big) < \infty$, where $\mathbb{E}_{\Lambda_k}[\![\cdot]\!]$ denotes the conditional expectation (see [123, Appendix C]). Then, for any $A \in \mathcal{D}_f$, $t \mapsto \rho_0(t)(A)$ is differentiable and there exists a constant C_f such that

$$\sup_{t\in I}|\dot{\rho}_0(t)(A)| \leq C_f\,\|A\|_f. \qquad \diamond$$

The smoothness of expectation values of (almost) exponentially localized observables as under item (iii) is a rather technical condition and a consequence of a uniform gap as in Assumption ($\mathrm{GAP_{unif}}$) (see Remark 4.15 in [167] and Lemma 6.0.1 in [166]). Although uniqueness of the ground state in item (i), which we required throughout the paper, is expected to hold for the physically relevant type of Hamiltonian (4), it has been shown, to our present knowledge, only in very specific quantum spin systems. These include (a) weak perturbations of non-interacting gapped frustration-free systems [219, P1], and (b) short-range interacting frustration-free models fulfilling *local topological quantum order* (LTQO) [176, 16].

Remark 4. As mentioned in the beginning of Section 3, item (ii) holds, in particular, if one has a uniform gap for finite systems as spelled out in Assumption ($\mathrm{GAP_{unif}}$), since it cannot close abruptly in the thermodynamic limit for the GNS Hamiltonian (see Proposition 5.4 in [20]). However, we observe that a considerably *weaker* sufficient condition for having a gap for the GNS Hamiltonian as in Assumption ($\mathrm{GAP_{bulk}}$) (ii) is to have a *gap in the bulk for the finite systems* Λ_k in the following sense: There exists $g > 0$ such that for all $k \in \mathbb{N}$ there exists some $l = l(k) \in \mathbb{N}$ with $l(k) \to \infty$ as $k \to \infty$, and we have

$$\rho_0^{\Lambda_k}(t)\big(A^*\mathcal{L}^{\Lambda_k}_{H_0(t)}[\![A]\!]\big) \geq g\left(\rho_0^{\Lambda_k}(t)(A^*A) - \left|\rho_0^{\Lambda_k}(t)(A)\right|^2\right) \tag{10}$$

for all $A \in \mathcal{A}_{\Lambda_l}$ and all $t \in I$, where $\rho_0^{\Lambda_k}(t)$ denotes a suitable ground state of $H_0^{\Lambda_k}(t)$. Indeed, assuming that $\rho_0^{\Lambda_k}(t) \rightharpoonup \rho_0(t)$ for every $t \in I$,[10] this simply follows after taking the limit $k \to \infty$ on both sides of (10) and realizing that, as $k \to \infty$, the set of admissible observables $A \in \mathcal{A}_{\Lambda_{l(k)}}$ exhausts $\mathcal{A}_{\mathrm{loc}}$, which is dense in $\mathcal{A}_{\mathbb{Z}^d}$ by definition. The resulting inequality immediately yields the desired spectral gap for the GNS Hamiltonian (cf. [47, Proposition 5.3.19] and [175, Section 7]). $\diamond$

In the following, we shall present adiabatic theorems for infinite systems (Theorem III) as well as for extended but finite systems (Theorem IV) under Assumption ($\mathrm{GAP_{bulk}}$).

[10] Note that the sequence $(\rho_0^{\Lambda_k}(t))_{k\in\mathbb{N}}$ is compact for every fixed $t \in I$ (Banach-Alaoglu Theorem). Moreover, it is shown in Proposition 5.3.25 in [47] that every limit point of a sequence of ground states associated to a converging sequence of derivations $\mathcal{L}^{\Lambda_k}_{H_0(t)} \to \mathcal{L}_{H_0(t)}$ is a ground state of the limiting derivation.

3.2.1 Infinite systems

Analogously to Section 3.1, the basic assumptions on the Hamiltonian say that it is composed of exponentially localized interactions and/or a Lipschitz potential. In addition, the Hamiltonian H_0 satisfies a technical regularity assumption in t, for which we recall that $I \subset \mathbb{R}$ denotes an open time interval.

(INT_3) Assumptions on the interactions.

(i) Let $\Psi_{H_0}, \Psi_{H_1} \in \mathcal{B}^{\circ}_{I,a,\infty}$ be time-dependent infinite volume interactions and $v_\infty \in \mathcal{V}^{\circ}_{I}$ a time-dependent infinite volume Lipschitz potential.

(ii) Assume that the map $I \to \mathcal{B}^{\circ}_{a,\infty}, t \mapsto \Psi_{H_0(t)}$ is continuously differentiable.[11] ◇

We can now formulate the third generalized super-adiabatic theorem concerning infinite systems with a gap in the bulk [122].

Theorem III. (Adiabatic theorem for infinite systems with a gap in the bulk [122, Theorem 3.4])
Under Assumptions ($\mathrm{GAP}_{\mathrm{bulk}}$) *and* ($\mathrm{INT}_3$)*, there exists a near-identity automorphism* $\beta^{\varepsilon,\eta}(t) = \mathrm{e}^{\mathrm{i}\varepsilon \mathcal{L}_{S^{\varepsilon,\eta}(t)}}$ *on* $\mathcal{A}_{\mathbb{Z}^d}$ *with SLT generators* $S^{\varepsilon,\eta}$ *for any* $\varepsilon,\eta \in (0,1]$ *and* $t \in I$ *such that the state*

$$\Pi^{\varepsilon,\eta}(t) := \rho_0(t) \circ \beta^{\varepsilon,\eta}(t)$$

is a super-adiabatic NEASS for $\rho_0(t)$ *and the Heisenberg time-evolution on* $\mathcal{A}_{\mathbb{Z}^d}$ *generated by* $\frac{1}{\eta}\Psi_{H^\varepsilon(\cdot)}$ *with*

$$\Psi_{H^\varepsilon} := \Psi_{H_0} + \varepsilon\,(V_{v_\infty} + \Psi_{H_1}).$$

The key role of the spectral gap condition is that it allows to construct an inverse of the Liouvillian $\mathcal{L}_{H_0(t)}$, appearing in the construction of the NEASS, which maps SLT operators to SLT operators with slightly deteriorated locality properties. Hence, the inverse of $\mathcal{L}_{H_0(t)}$ is called the *quasi-local inverse of the Liouvillian.*[12] Assuming a gap only in the bulk, as done in ($\mathrm{GAP}_{\mathrm{bulk}}$), means that the action of the Liouvillian can only be *inverted in the bulk* (see Section 4).

3.2.2 Extended but finite systems

Contrary to the results in Section 3.1, the adiabatic theorem describing an infinite system with a gap in the bulk did *not* require any notion of having a TDL in its formulation. Instead, in order to derive a finite-volume analogue from Theorem III (with

[11] Note that this technical assumption does not follow from $\Psi_{H_0} \in \mathcal{B}^{\circ}_{I,a,\infty}$, as the spaces of smooth and bounded interactions are defined via term-wise and point-wise time derivatives (cf. Section 2.2).
[12] This particular phrase was used in [203]. Others call it *local inverse* [17, 165] or just *inverse* [124, 123, 174]. In [122], it was called *SLT inverse*. To avoid confusion, we want to reserve the SLT prefix for operators, i.e. SLT operator or SLT generator, but not for maps between SLT operators.

qualitative additional error terms, see Theorem IV below), we need to introduce the stronger notion of *having a rapid thermodynamic limit* for the exponentially localized interactions and the Lipschitz potential. We refer to [122] for a detailed discussion of this property.

Definition 5. (Rapid thermodynamic limit of interactions and potentials)

(a) An exponentially localized time-dependent interaction $\Phi \in \mathcal{B}_{I,a,\infty}$ is said to *have a rapid thermodynamic limit with exponent* $\gamma \in (0,1)$ (have a RTDL$_\gamma$) if there exists an infinite volume interaction $\Psi \in \mathcal{B}^{\circ}_{I,a,\infty}$ such that

$$\forall n \in \mathbb{N},\ i \in \mathbb{N}_0\ \exists \lambda, C > 0\ \forall M \in \mathbb{N}\ \forall k \geq M + \lambda M^\gamma\ : \\ \sup_{t \in I} \left\| \frac{\mathrm{d}^i}{\mathrm{d}t^i} \big(\Psi - \Phi^{\Lambda_k}\big)(t) \right\|_{a,n,\Lambda_M} \leq C\,\mathrm{e}^{-aM^\gamma}, \tag{11}$$

and we write $\Phi \xrightarrow{\text{rtd}} \Psi$ in this case.

A family of operators is said to *have a RTDL* if and only if the corresponding interaction does.

For more general (non-exponentially localized) SLT operators, the definition is completely analogous.

(b) A Lipschitz potential $v \in \mathcal{V}_I$ is said to *have a RTDL$_\gamma$* if it is eventually independent of k, i.e. if there exists an infinite volume Lipschitz potential $v_\infty \in \mathcal{V}_I^{\circ}$ such that

$$\exists \lambda > 0\ \forall M \in \mathbb{N}\ \forall k \geq M + \lambda M^\gamma,\ t \in I\ :\ v_\infty(t,\cdot)|_{\Lambda_M} = v^{\Lambda_k}(t,\cdot)|_{\Lambda_M}.$$

Again, we write $v \xrightarrow{\text{rtd}} v_\infty$ in this case. ◇

In a nutshell, having a RTDL$_\gamma$ means that the interaction (or the Lipschitz potential) essentially agrees with a corresponding infinite volume object, up to terms located on a thin shell with relative size of order $k^{\gamma-1}$ right at the boundary of Λ_k. Note that, whenever $\Phi = \Psi$ for some infinite-volume interaction Ψ, or $v = v_\infty$ for some infinite volume Lipschitz potential v_∞, both Φ and v trivially have a RTDL$_\gamma$ (with any exponent $\gamma \in (0,1)$).

The following Theorem IV is deduced from Theorem III by comparing the time evolution $\mathfrak{U}^{\varepsilon,\eta}_{t,t_0}$ and the near identity automorphism $\beta^{\varepsilon,\eta}$ in the definition of the NEASS on the infinite system $\mathbb{Z}^d$ with the same objects for large (but finite) systems Λ_k. Therefore, we will need to assume the existence of a rapid thermodynamic limit for the building blocks of the Hamiltonian (1).

(INT$_4$) Assumptions on the interactions. The interactions $\Phi_{H_0}, \Phi_{H_1} \in \mathcal{B}_{I,a,\infty}$ and the Lipschitz potential $v \in \mathcal{V}_I$ all have a RTDL, i.e. $\Phi_{H_0} \xrightarrow{\text{rtd}} \Psi_{H_0}$, $\Phi_{H_1} \xrightarrow{\text{rtd}} \Psi_{H_1}$ and $v \xrightarrow{\text{rtd}} v_\infty$. The limiting objects Ψ_{H_0}, Ψ_{H_1} and v_∞ satisfy Assumption (INT$_3$). ◇

In Theorem IV we shall consider finite volume states $\rho_0^{\Lambda_k}(t)$, which are close to the infinite volume ground state $\rho_0(t)$ away from the boundary in following sense.

(S_{bulk}) Assumption on the convergence of states. The sequence $\big(\rho_0^{\Lambda_k}(t)\big)_{k\in\mathbb{N}}$ of states on $\mathcal{A}_{\Lambda_k}$ converges rapidly to $\rho_0(t)$ in the bulk: there exist $C \in \mathbb{R}$, $m \in \mathbb{N}$ and $h \in \mathcal{S}$ such that for any finite $X \subset \mathbb{Z}^d$, $A \in \mathcal{A}_X$ and $\Lambda_k \supset X$

$$\sup_{t\in I}\left|\rho_0(t)(A) - \rho_0^{\Lambda_k}(t)(A)\right| \leq C\,\|A\|\,\mathrm{diam}(X)^m\, h\big(\mathrm{dist}(X, \mathbb{Z}^d \setminus \Lambda_k)\big). \qquad \diamond$$

While the sequence $\rho_0^{\Lambda_k}(t) \equiv \rho_0(t)|_{\mathcal{A}_{\Lambda_k}}$ of simple restrictions satisfies Assumption (S_{bulk}) trivially, the adiabatic theorem ensures the existence of a super-adiabatic NEASS constructed for *any* such sequence.[13] Most interesting for physical application would be a sequence of ground states $\rho_0^{\Lambda_k}(t)$ of the finite volume Hamiltonians $H_0^{\Lambda_k}(t)$. While the above assumption is expected to hold for any sequence of finite volume ground states for Hamiltonians modeling Chern or topological insulators like in (4), the only result we are aware of indeed *proving* such a statement is again (see the discussion below Assumption ($\mathrm{GAP}_{\mathrm{bulk}}$)) for weakly interacting spin systems [219]. In spirit, assuming (S_{bulk}) is very similar to supposing that the system satisfies *local topological quantum order* (LTQO) [162, 176] or a strong *local perturbations perturb locally* (LPPL) principle for perturbations acting at the boundary of the system [P1, 16].

We can now formulate the fourth and last generalized super-adiabatic theorem concerning finite systems with a gap in the bulk [122].

Theorem IV. (Adiabatic theorem for finite systems with gap in the bulk [122, Theorem 4.1])
Under the Assumptions ($\mathrm{GAP}_{\mathrm{bulk}}$), ($\mathrm{INT}_4$) *and* ($S_{\mathrm{bulk}}$), *there exists a sequence of near-identity automorphisms* $\beta^{\varepsilon,\eta,\Lambda_k}(t) = \mathrm{e}^{\mathrm{i}\varepsilon\mathcal{L}^{\Lambda_k}_{S^{\varepsilon,\eta}(t)}}$ *with SLT generators* $S^{\varepsilon,\eta}$ *for any* $\varepsilon, \eta \in (0,1]$ *and* $t \in I$, *such that the states*

$$\Pi^{\varepsilon,\eta,\Lambda_k}(t) := \rho_0^{\Lambda_k}(t) \circ \beta^{\varepsilon,\eta,\Lambda_k}(t)$$

are super-adiabatic NEASSs for the Heisenberg time-evolution $\mathfrak{U}^{\varepsilon,\eta,\Lambda_k}_{t,t_0}$ *on* $\mathcal{A}_{\Lambda_k}$ *generated by* $\frac{1}{\eta}\, H^{\varepsilon,\Lambda_k}(\cdot)$ *with*

$$\frac{1}{\eta}\, H^{\varepsilon,\Lambda_k}(t) := \frac{1}{\eta}\Big(H_0^{\Lambda_k}(t) + \varepsilon\left(V_v^{\Lambda_k}(t) + H_1^{\Lambda_k}(t)\right)\Big),$$

up to an error vanishing faster than any inverse polynomial in the distance to the boundary. That is, for any $n \in \mathbb{N}$ *there exists a constant* C_n *and for any compact* $K \subset I$ *and*

[13] This is why we wrote *'(close to) a ground state'* in Definition 1.

$m \in \mathbb{N}$ *there exists a constant* $\tilde{C}_{n,m,K}$ *such that for all* $k \in \mathbb{N}$*, all* $X \subset \Lambda_k$*, all* $A \in \mathscr{A}_X$ *and all* $t, t_0 \in K$

$$\begin{aligned} \left| \Pi^{\varepsilon,\eta,\Lambda_k}(t_0)(\mathfrak{U}^{\varepsilon,\eta,\Lambda_k}_{t,t_0}[\![A]\!]) - \Pi^{\varepsilon,\eta,\Lambda_k}(t)(A) \right| & \\ \leq C_n \, \frac{\varepsilon^{n+1} + \eta^{n+1}}{\eta^{d+1}} \left(1 + |t - t_0|^{d+1}\right) \|A\| \, |X|^2 & \qquad (12) \\ + \; \tilde{C}_{n,m,K} \left(1 + \eta \, \operatorname{dist}(X, \mathbb{Z}^d \setminus \Lambda_k)\right)^{-m} \|A\| \operatorname{diam}(X)^{2d}. & \end{aligned}$$

The above theorem asserts that by assuming ($\mathrm{GAP}_{\mathrm{bulk}}$), one obtains similar adiabatic bounds also for states of finite systems (without a spectral gap!) which are close to the infinite volume ground state in the bulk as formulated in Assumption ($\mathrm{S}_{\mathrm{bulk}}$). Since adiabaticity potentially breaks at the boundaries of the finite systems, non-adiabatic effects arising close to the boundary may propagate into the bulk. Therefore, an additional error term appears, but it decays faster than any polynomial in the size of the finite system for any fixed η. The actual form of the additional error term in the last line of (12) coming out of the proof in [122, Section 5] is slightly better but more complicated, which is why we refrain from stating it here.

The main points in the proof of Theorem IV, which we discuss in Section 4, are to show that (i) the property of having a RTDL_γ is preserved under all necessary operations for the construction of the NEASS (similarly as for Theorem II) and (ii) having a RTDL_γ for an interaction provides an explicit rate of convergence for the associated evolution family as in Proposition 3.

4 Idea of the proofs

The goal of the present section is to convey the main ideas relevant for proving the individual theorems from Section 3, where we already glimpsed the key steps required in their proofs. For many technical details we refer the reader to the original works [203, 123, 122].

4.1 Systems with a uniform gap

The fundamental conceptual idea behind the proof for all four variants of the generalized super-adiabatic theorems is a perturbative scheme, which was called *space-time adiabatic perturbation theory* in [182, 180]. The basic structure of this computation is most easily presented for finite systems, where no further technical difficulties arise since all appearing operators are in fact matrices and thus bounded. However, it is still necessary to show that all estimates are uniform in the size of the system Λ_k.

4.1.1 Extended but finite systems: Proof of Theorem I

The form in which we presented Theorem I differs slightly from the original result [203, Theorem 5.1]. The original statement concerns a sequence $\Pi_n^{\varepsilon,\eta,\Lambda_k}(t) := \rho_0^{\Lambda_k}(t) \circ \beta_n^{\varepsilon,\eta,\Lambda_k}(t)$ of states on Λ_k (indexed by $n \in \mathbb{N}$), where

$$\beta_n^{\varepsilon,\eta,\Lambda_k}(t)[\![A]\!] := \mathrm{e}^{-\mathrm{i}\varepsilon \mathscr{L}^{\Lambda_k}_{S_n^{\varepsilon,\eta}(t)}}[\![A]\!] \quad \text{and} \quad \varepsilon \mathscr{L}^{\Lambda_k}_{S_n^{\varepsilon,\eta}(t)} := \sum_{j=1}^{n} \varepsilon^j \mathscr{L}^{\Lambda_k}_{A_j^{\varepsilon,\eta}(t)}.$$

From this, Theorem I (and similarly all other three theorems) follows by a simple *resummation* of the $\varepsilon^j \mathscr{L}^{\Lambda_k}_{A_j^{\varepsilon,\eta}(t)}$, which will be discussed in Section 4.3 below.

The main idea of the proof is to choose each operator $A_j^{\varepsilon,\eta,\Lambda_k}(t)$, $j = 1, \dots, n$, in such a way that the j^{th}-order term in the perturbative scheme vanishes. For the n-dependent result (i.e. prior to resummation), we apply the fundamental theorem of calculus to get

$$\Pi_n^{\varepsilon,\eta,\Lambda_k}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\big) - \Pi_n^{\varepsilon,\eta,\Lambda_k}(t)(A) = -\int_{t_0}^{t} \mathrm{d}s \, \frac{\mathrm{d}}{\mathrm{d}s}\, \rho_0^{\Lambda_k}(s)\Big(\beta_n^{\varepsilon,\eta,\Lambda_k}(s) \circ \mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\Big) \tag{13}$$

and then aim to bound the integrand. Calculating the derivative by using the chain rule and Duhamel's formula leaves us with

$$\frac{\mathrm{d}}{\mathrm{d}s}\, \rho_0^{\Lambda_k}(s)\Big(\beta_n^{\varepsilon,\eta,\Lambda_k}(s) \circ \mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\Big) = -\frac{\mathrm{i}}{\eta}\, \rho_0^{\Lambda_k}(s)\Big(\Big[Q_n^{\varepsilon,\eta,\Lambda_k}(s), \beta_n^{\varepsilon,\eta,\Lambda_k}(s) \circ \mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\Big]\Big), \tag{14}$$

where $Q_n^{\varepsilon,\eta,\Lambda_k}(s)$ is a shorthand notation for

$$\begin{aligned} &\eta\, \mathscr{I}_s^{\Lambda_k}\big(\dot{H}_0^{\Lambda_k}(s)\big) + \eta \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{-\mathrm{i}\lambda\varepsilon S_n^{\varepsilon,\eta,\Lambda_k}(s)}\, \varepsilon \dot{S}_n^{\varepsilon,\eta,\Lambda_k}(s)\, \mathrm{e}^{\mathrm{i}\lambda\varepsilon S_n^{\varepsilon,\eta,\Lambda_k}(s)} \\ &\qquad + \mathrm{e}^{-\mathrm{i}\varepsilon S_n^{\varepsilon,\eta,\Lambda_k}(s)}\, \big(H_0^{\Lambda_k}(s) + \varepsilon V^{\Lambda_k}(s)\big)\, \mathrm{e}^{\mathrm{i}\varepsilon S_n^{\varepsilon,\eta,\Lambda_k}(s)} \\ &=: H_0^{\Lambda_k}(s) + \sum_{j=1}^{n} \varepsilon^j R_j^{\varepsilon,\eta,\Lambda_k}(s) + \varepsilon^{n+1} R_{n+1}^{\varepsilon,\eta,\Lambda_k}(s), \end{aligned} \tag{15}$$

and $V^{\Lambda_k} = V_v^{\Lambda_k} + H_1^{\Lambda_k}$. Here, $\mathscr{I}_s^{\Lambda_k}\big(\dot{H}_0^{\Lambda_k}(s)\big)$ is the SLT generator of the parallel transport within the vector-bundle $\Xi_{0,I}^{\Lambda_k}$ over I defined by $t \mapsto \rho_0^{\Lambda_k}(t)$.[14] This parallel transport is also known as the *spectral flow*, which plays a fundamental role in proving automorphic equivalence of gapped ground state phases (see e.g. [23, 17]). Moreover, the

[14] Since we assumed uniqueness of the ground state $\rho_0^{\Lambda_k}(t)$, the vector-bundle $\Xi_{0,I}^{\Lambda_k}$ is one-dimensional. If this were not the case, one had to include further terms generating the internal dynamic in $\Xi_{0,I}^{\Lambda_k}$ (see [203, 123]).

operator $\mathscr{I}_s^{\Lambda_k} : \mathscr{A}_{\Lambda_k} \to \mathscr{A}_{\Lambda_k}$ is called the *quasi-local inverse of the Liouvillian*[12] $\mathscr{L}_{H_0(s)}^{\Lambda_k}$, since it satisfies [17, 203]

$$\rho_0^{\Lambda_k}(s)\Big(\big[\mathscr{L}_{H_0(s)}^{\Lambda_k} \circ \mathscr{I}_s^{\Lambda_k}[\![B_1]\!] - \mathrm{i}\, B_1, B_2\big]\Big) = 0 \quad \text{for all} \quad B_1, B_2 \in \mathscr{A}_{\Lambda_k}, \quad s \in I, \tag{16}$$

and also preserves good localization of its argument (in particular, it maps SLT operators to SLT operators). This combined property of $\mathscr{I}_s^{\Lambda_k}$ heavily relies on the ground state $\rho_0^{\Lambda_k}(s)$ being gapped [17, 203, 174] and will be of fundamental importance in the following.

In the last line of (15), we expanded in powers of ε and η in the sense that $R_j^{\varepsilon,\eta,\Lambda_k}(s)$, for $j \le n$, are polynomials in η/ε of order (at most) j with ε- and η-independent SLT operators as coefficients. A more detailed step-by-step calculation can be found in the proof of Proposition 5.1 in [203]. Let us here only report the general structure

$$R_j^{\varepsilon,\eta,\Lambda_k}(s) = -\mathrm{i}\, \mathscr{L}_{H_0(s)}^{\Lambda_k}\big(A_j^{\varepsilon,\eta,\Lambda_k}(s)\big) + \tilde{R}_j^{\varepsilon,\eta,\Lambda_k}(s), \tag{17}$$

where the first remainder term is given by

$$\tilde{R}_1^{\varepsilon,\eta,\Lambda_k}(s) = \frac{\eta}{\varepsilon}\, \mathscr{I}_s^{\Lambda_k}\big(\dot{H}_0^{\Lambda_k}(s)\big) - V^{\Lambda_k}(s)$$

and all other $\tilde{R}_j^{\varepsilon,\eta,\Lambda_k}(s)$ are composed of iterated commutators of the operators $A_i^{\varepsilon,\eta,\Lambda_k}(s)$ and $\dot{A}_i^{\varepsilon,\eta,\Lambda_k}(s)$, for $i < j \le n$, with $H_0^{\Lambda_k}(s)$ and $V^{\Lambda_k}(s)$. In contrast to general onsite potentials, the commutator of a Lipschitz potential with an SLT operator is an SLT operator itself [203, Lemma 2.1]. For the commutator of SLT operators, this is easy to see.

We now consider individual terms from (15) when plugged into (14). The zero-order term vanishes, because $\rho_0^{\Lambda_k}(s)$ is the ground state of $H_0^{\Lambda_k}(s)$. By application of (16) we can iteratively choose

$$A_j^{\varepsilon,\eta,\Lambda_k}(s) = -\mathscr{I}_s^{\Lambda_k}\big(\tilde{R}_j^{\varepsilon,\eta,\Lambda_k}(s)\big) \tag{18}$$

such that (14) vanishes up to

$$-\mathrm{i}\, \frac{\varepsilon^{n+1}}{\eta}\, \rho_0^{\Lambda_k}(s)\Big(\Big[R_{n+1}^{\varepsilon,\eta,\Lambda_k}(s), \beta_n^{\varepsilon,\eta,\Lambda_k}(s) \circ \mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\Big]\Big). \tag{19}$$

Moreover, all the operations involved in calculating the $A_j^{\varepsilon,\eta,\Lambda_k}(s)$, i.e. taking commutators and applying the quasi-local inverse of the Liouvillian preserve the locality properties of the operators as shown in the appendices of [203, 165], which are heavily based on [17]. Hence, also all $A_j^{\varepsilon,\eta,\Lambda_k}$ are SLT operators.

It turns out that also $R_{n+1}^{\varepsilon,\eta,\Lambda_k}$ is a polynomial in η/ε of order at most $n+1$ and its coefficients, as we just explained, are SLT operators [203, Proof of Proposition 6.1].

Thus, the absolute value of (13) is bounded by

$$\begin{aligned}
&\frac{\varepsilon^{n+1}}{\eta}\left|\int_{t_0}^{t} \mathrm{d}s\, \rho_0^{\Lambda_k}(s)\Big(\Big[R_{n+1}^{\varepsilon,\eta,\Lambda_k}(s), \beta_n^{\varepsilon,\eta,\Lambda_k}(s)\circ \mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\Big]\Big)\right| \\
&\quad\le \frac{\varepsilon^{n+1}}{\eta}\,|t-t_0| \sup_{s\in[t_0,t]}\left\|\Big[\big(\beta_n^{\varepsilon,\eta,\Lambda_k}\big)^{-1}(s)\big(R_{n+1}^{\varepsilon,\eta,\Lambda_k}(s)\big), \mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\Big]\right\| \\
&\quad\le C_n\,\frac{\varepsilon^{n+1}}{\eta}\,|t-t_0|\left(1+\Big(\frac{\eta}{\varepsilon}\Big)^{n+1}\right)\left(1+\Big(\frac{|t-t_0|}{\eta}\Big)^{d}\right)\|A\|\,|X|^2 \qquad (20)\\
&\quad\le C_n\,\frac{\varepsilon^{n+1}+\eta^{n+1}}{\eta^{d+1}}\,|t-t_0|\,\big(1+|t-t_0|^d\big)\,\|A\|\,|X|^2,
\end{aligned}$$

where we essentially used a generalized Lieb-Robinson bound [203, Lemma B.5] to estimate the commutator. Note that the $(1+\big(|t-t_0|/\eta\big)^d)$-factor comes from the Lieb-Robinson bound and the adiabatic $1/\eta$-scaling of the time evolution $\mathfrak{U}_{t,s}$. The $(1+(\eta/\varepsilon)^{n+1})$-factor comes from bounding the interaction norm of $R_{n+1}^{\varepsilon,\eta,\Lambda_k}(s)$ by separating the polynomial dependence on η/ε such that C_n is independent of Λ_k, ε and η. We have thus shown that the NEASS almost intertwines the time evolution, i.e. item 1 of Definition 1.

We are left with discussing the remaining three characterizing properties of the NEASS given in Definition 1: By construction, all $A_j^{\varepsilon,\eta,\Lambda_k}(t)$ depend only on $H_0^{\Lambda_k}(t)$ and $V^{\Lambda_k}(t)$ and their j^{th} derivatives at time t. This shows that the NEASS is local in time, i.e. item 2. Moreover, if all time derivatives of H_0 and V vanish for some $t\in I$, then all non-constant (i.e. in front of some positive power of η/ε) coefficients in $R_j^{\varepsilon,\eta,\Lambda_k}$ vanish and $\Pi_n^{\varepsilon,\eta,\Lambda_k}(t)=\Pi_n^{\varepsilon,0,\Lambda_k}(t)$. This shows that the NEASS is stationary whenever the Hamiltonian is stationary, i.e. item 3. If, for some $t\in I$, $\dot H_0^{\Lambda_k}(t)$ and $V^{\Lambda_k}(t)$ vanish, then $\tilde R_1^{\varepsilon,\eta,\Lambda_k}$ and thus $A_1^{\varepsilon,\eta,\Lambda_k}$ vanish. If additionally all derivatives of $H_0^{\Lambda_k}$ and V^{Λ_k} at t vanish, also $\tilde R_j^{\Lambda_k}(t)$ and thus $A_j^{\varepsilon,\eta,\Lambda_k}(t)$ vanish. Hence, $\beta_n^{\varepsilon,\eta,\Lambda_k}(t)=\mathbb{1}_{\Lambda_k}$ and the NEASS equals the ground state, i.e. item 4 holds.

The above listed general arguments immediately translate to the other three theorems.

4.1.2 Infinite systems: Proof of Theorem II

Without any further assumptions, the sequence Hamiltonian $H^{\varepsilon,\Lambda_k}$ and its constituents $H_0^{\Lambda_k}$ and V^{Λ_k} could have nothing in common for different lattice sizes k (they might even describe different physical systems), so taking the limit $\Lambda_k\nearrow\mathbb{Z}^d$ might not be well-defined. In order to avoid this somewhat meaningless situation, we assumed that the building blocks of the Hamiltonian *have a TDL* (see Definition 2 and Assumption (INT$_2$)) and also the sequence of ground states $\big(\rho_0^{\Lambda_k}(t)\big)_{k\in\mathbb{N}}$ converges (Assumption (S_{unif})). Since the property of having a TDL guarantees the existence of the thermodynamic limit for the corresponding evolution operators (see Proposition 3 and [174]), it remains to show that the operator sequences $\big(A_j^{\varepsilon,\eta,\Lambda_k}(t)\big)_{k\in\mathbb{N}}$, $j = 1,\dots,n$, constructed in Section 4.1.1 have a TDL. More precisely, one needs to show that taking time-derivatives, sums of commutators with the building blocks of H^ε (and $\dot{H}_0$), and the inverse of the Liouvillian (see (17)) leaves the property of having a TDL for SLT operators invariant, which is in fact the main point of the proof in [123]. It is then straightforward to show that compositions of states and automorphisms, all having a thermodynamic limit, converge as $\Lambda_k \nearrow \mathbb{Z}^d$. Since the constant C_n from (20) is uniformly bounded in k, the (sketch of a) proof of Theorem II is complete.

4.2 Systems with a gap in the bulk

For systems having a spectral gap only in the bulk (i.e. for the GNS Hamiltonian), the characteristic (16) of $\mathscr{I}_s^{\Lambda_k}$, that it essentially inverts the Liouvillian $\mathscr{L}_{H_0(s)}^{\Lambda_k}$ (and still maps SLT operators to SLT operators), is now only fulfilled for certain B_1 and B_2 in a dense domain $\mathscr{D} \subset \mathscr{A}_{\mathbb{Z}^d}$ after taking the limit $\Lambda_k \nearrow \mathbb{Z}^d$ (see [122, Proposition 3.3]). Presuming that the limit actually exists, this point is the main challenge in proving an adiabatic theorem under the less restrictive gap Assumption (GAP$_{\text{bulk}}$).

4.2.1 Infinite systems: Proof of Theorem III

As just explained, the main difficulty in proving Theorem III is that (16) only holds if $H_0^{\Lambda_k}$ is gapped. On top of that, we cannot handle the limit $\Lambda_k \nearrow \mathbb{Z}^d$ of the $\tilde{R}_j^{\varepsilon,\eta,\Lambda_k}$ directly nor could they be used in the infinite volume version of (16) because it only holds for $B_1, B_2 \in \mathscr{D} \subset \mathscr{A}_{\mathbb{Z}^d}$. However, the rest of the construction from Section 4.1.1 is still valid, but the lower order terms in (13) have a non-vanishing contribution in finite domains. We thus repeat this construction but take coefficients $A_j^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(t)$, which are built up from $H_0^{\Lambda_k}(t)$ but restricting the perturbations $\dot{H}_0(t)$ and $V(t)$ to Λ_l with $l < k$. In this way, one can take the limit $\Lambda_k \nearrow \mathbb{Z}^d$ in (16) with $B_1 = \lim_{k\to\infty} \tilde{R}_j^{\varepsilon,\eta,\Lambda_k,\Lambda_l} \in \mathscr{A}_{\mathbb{Z}^d}$ (see (22) and (23) and the comment thereafter for technical obstructions in taking the limit). Using this notational convention, we introduce the states

$$\Pi_n^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(t) = \rho_0(t) \circ \beta^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(t),$$

where $\rho_0(t)$ is the infinite volume ground state, and compare them to the actual objects in infinite volume while estimating

$$\begin{aligned}
&\left|\Pi_n^{\varepsilon,\eta}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta}[\![A]\!]\big) - \Pi_n^{\varepsilon,\eta}(t)(A)\right| \\
&\quad\leq \left|\Pi_n^{\varepsilon,\eta}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta}[\![A]\!]\big) - \Pi_n^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k,\Lambda_l}[\![A]\!]\big)\right| \\
&\qquad+ \left|\Pi_n^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k,\Lambda_l}[\![A]\!]\big) - \Pi_n^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(t)(A)\right| \\
&\qquad+ \left|\Pi_n^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(t)(A) - \Pi_n(t)(A)\right|
\end{aligned} \tag{21}$$

by means of the triangle inequality. Since all the interactions (and the Lipschitz potential) have a TDL, one can prove [122, Section 5.1(b)] that the first and last summand in (21) can be made arbitrarily small for $k, l \in \mathbb{N}$ large enough, and we can thus focus on the second summand. However, since (16) only holds in the limit $\Lambda_k \nearrow \mathbb{Z}^d$ and also $\rho_0(t)$ is not necessarily a ground state of $H_0^{\Lambda_k}(t)$, the lower order terms in the analogues of (14) and (15) do not vanish for finite k and l. Instead, only

$$\lim_{k\to\infty} \rho_0(s)\Big(\big[H_0^{\Lambda_k}(s), \beta_n^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(s) \circ \mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k,\Lambda_l}[\![A]\!]\big]\Big) = 0 \tag{22}$$

and

$$\lim_{k\to\infty} \rho_0(s)\Big(\big[R_j^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(s), \beta_n^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(s) \circ \mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k,\Lambda_l}[\![A]\!]\big]\Big) = 0 \tag{23}$$

for all $l \in \mathbb{N}$ and uniformly for s and t in compacts. These statements require a careful analysis of deteriorating localization properties along the expansion as well as convergence estimates in norms measuring the quality of localization (cf. the norm $\|\cdot\|_f$ introduced in Assumption (GAP$_{\text{bulk}}$) (iii)), such that the limits really converge to the infinite volume version of (16) with B_1 and B_2 in a dense domain $\mathscr{D} \subset \mathscr{A}_{\mathbb{Z}^d}$. For further details, we refer to Proposition 3.2 and the statements in Appendix B of [122], which are adaptions of technical estimates that were originally established for the proof of automorphic equivalence with a gap only in the bulk [167]. Now, combining (22) and (23) with the estimates on the first and third summand in (21), we conclude that all the lower order terms vanish in the limit $k \to \infty$ followed by $l \to \infty$, which finishes our sketch of the proof of Theorem III.

4.2.2 Extended but finite systems: Proof of Theorem IV

Let us briefly explain the strategy to prove Theorem IV. In order to show (12), we first estimate

$$\begin{aligned}\Big|\Pi_n^{\varepsilon,\eta,\Lambda_k}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\big) - \Pi_n^{\varepsilon,\eta,\Lambda_k}(t)(A)\Big| \\ \leq \Big|\Pi_n^{\varepsilon,\eta,\Lambda_k}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\big) - \Pi_n^{\varepsilon,\eta}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta}[\![A]\!]\big)\Big| \\ + \Big|\Pi_n^{\varepsilon,\eta}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta}[\![A]\!]\big) - \Pi_n^{\varepsilon,\eta}(t)(A)\Big| \\ + \Big|\Pi_n^{\varepsilon,\eta,\Lambda_k}(t)(A) - \Pi_n^{\varepsilon,\eta}(t)(A)\Big|\end{aligned} \tag{24}$$

and treat the three summands separately. The second summand corresponds to the infinite system and can be estimated by means of Theorem III, such that it accounts for the first contribution on the RHS of (12). We are left with bounding the remaining two summands in (24). These contribute the additional error term on the RHS of (12). To estimate them, we need explicit control on the speed of convergence (it must be faster than any inverse polynomial) for the states (see Assumption (S_{bulk})) and automorphisms $\beta_n^{\varepsilon,\eta,\Lambda_k}$ and $\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}$. For the time evolution $\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}$, the rapid convergence to $\mathfrak{U}_{t,t_0}^{\varepsilon,\eta}$ is ensured by supposing that the building blocks of H^ε have a RTDL (see Definition 5 and Assumption (INT_4)). This was carried out in [122, Appendix B], building on estimates from [174, Section 3]. We remark that the adiabatic $1/\eta$-scaling of the time evolution is responsible for the factor η appearing in the additional error term in (12). In order to show that also $\beta_n^{\varepsilon,\eta,\Lambda_k} \to \beta_n^{\varepsilon,\eta}$ sufficiently fast, we need to show that all $A_j^{\varepsilon,\eta}$ have a RTDL, i.e. the operations involved in constructing the generator of $\beta_n^{\varepsilon,\eta}$ leave the property of having a RTDL (essentially) invariant (see [122, Appendix C]). This finishes the sketch of the proof of Theorem IV and we refer to [122, Section 5.2] for further details.

4.3 Resummation of the NEASS

As mentioned in the beginning of Section 4.1.1, the statements formulated in Section 3 require a resummation, which we explain in the following. First, note that the generator $\varepsilon S_n^{\varepsilon,\eta}$ of $\beta_n^{\varepsilon,\eta}$ constructed above can be rewritten as $\varepsilon S_n^{\varepsilon,\eta} = \sum_{j=1}^n \sum_{i=0}^j \varepsilon^i \eta^{j-1} A_{j,i}$, where the coefficients $A_{j,i}$ are time-dependent SLT operators and independent of ε and η. Now, it is easy to show (see [123, Lemma E.1]) that there exists a sequence $\delta_j \to 0$ and constants C_n such that the resummed generator

$$\varepsilon S^{\varepsilon,\eta} = \sum_{j=1}^{\infty} \chi_{[0,1]}(\varepsilon/\delta_j)\, \chi_{[0,1]}(\eta/\delta_j) \sum_{i=0}^{j} \varepsilon^i \eta^{j-i} A_{j,i} \tag{25}$$

satisfies

$$\big\|\varepsilon S^{\varepsilon,\eta} - \varepsilon S_n^{\varepsilon,\eta}\big\|_{\text{SLT}} \leq C_n\,(\varepsilon^n + \eta^n),$$

where $\|\cdot\|_{\mathrm{SLT}}$ denotes an interaction norm similar to (5). Resummations of this type are standard, e.g., in microlocal analysis [158] and the above estimate immediately leads to the bounds (cf. [123, Lemmata E.3, E.4])

$$\sup_{t\in I}\left|\Pi^{\varepsilon,\eta,\Lambda_k}(t)(A) - \Pi_n^{\varepsilon,\eta,\Lambda_k}(t)(A)\right| \le C_n' \left(\varepsilon^n + \eta^n\right) \|A\| \, |X|^2 \tag{26}$$

and

$$\begin{aligned} &\left|\Pi^{\varepsilon,\eta,\Lambda_k}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\big) - \Pi_n^{\varepsilon,\eta,\Lambda_k}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\big)\right| \\ &\qquad \le C_n'' \, \frac{\varepsilon^n + \eta^n}{\eta^{d+1}} \left(1 + |t - t_0|\right)^{d+1} \|A\| \, |X|^2, \end{aligned} \tag{27}$$

uniformly in the size of the system Λ_k. In the context of Theorem II and Theorem III, corresponding estimates hold in infinite volume, i.e. without the subscript Λ_k.

Next, since the sum in (25) is finite for every fixed $\varepsilon > 0$, also the resummed generator $S^{\varepsilon,\eta,\Lambda_k}$ has a TDL as soon as $S_n^{\varepsilon,\eta,\Lambda_k}$ has a TDL. Therefore, the states $\Pi^{\varepsilon,\eta,\Lambda_k}$ constructed using the $S^{\varepsilon,\eta,\Lambda_k}$ instead of the $S_n^{\varepsilon,\eta,\Lambda_k}$ have a well-defined thermodynamic limit $\Pi^{\varepsilon,\eta}$ (see [123, Lemma E.2]) and since the bounds (26) and (27) are independent of Λ_k, they also hold for the respective objects in the thermodynamic limit. Hence, the results formulated in Section 3 can be concluded by combining the n-dependent statements discussed earlier in this section with the bounds (26) and (27) (or their infinite volume correspondents).

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Equality of magnetization and edge current for interacting lattice fermions at positive temperature

Jonas Lampart [*] Massimo Moscolari [†] Stefan Teufel [‡]

Tom Wessel [‡]

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Abstract

We prove that the magnetization is equal to the edge current in the thermodynamic limit for a large class of models of lattice fermions with finite-range interactions satisfying local indistinguishability of the Gibbs state, a condition known to hold for sufficiently high temperatures. Our result implies that edge currents in such systems are determined by bulk properties and are therefore stable against large perturbations near the boundaries. Moreover, the equality persists also after taking the derivative with respect to the chemical potential. We show that this form of bulk-edge correspondence is essentially a consequence of homogeneity in the bulk and locality of the Gibbs state. An important intermediate result is a new version of Bloch's theorem for two-dimensional systems, stating that persistent currents vanish in the bulk.

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[*] CNRS & LICB (UMR 6303), Université de Bourgogne,
9 Av. A. Savary, 21078 Dijon Cedex, France

[†] Dipartimento di Matematica, Politecnico di Milano,
Piazza Leonardo da Vinci 32, 20133 Milano, Italy

[‡] Fachbereich Mathematik, Universität Tübingen,
Auf der Morgenstelle 10, 72076 Tübingen, Germany

Contents

1 Introduction

We show that extended fermion systems subject to homogeneous magnetic fields exhibit a form of bulk-edge correspondence in the thermodynamic limit, namely exact equality of magnetization and edge current, at positive temperatures. Roughly speaking, our assumptions are finite-range interactions, homogeneity in the bulk of the Hamiltonian, and local indistinguishability of the Gibbs state. The first two are explicit assumptions on the Hamiltonian, the last is known to hold for sufficiently high temperatures [139] and expected to hold much more generally.

In [65] a similar result was established for non-interacting fermion systems. There it is also shown, how this result relates to the better known bulk-edge correspondence of the transport coefficients: Under the assumption of a gapped ground state and in the zero temperature limit, the derivative of the magnetization with respect to the chemical potential converges to the Hall conductivity and the derivative of the edge current with respect to the chemical potential converges to the edge conductance. In this paper we establish the differentiability of the magnetization with respect to the chemical potential also for interacting systems and thus also the equality of the corresponding derivatives.

Let us now be more specific. We consider a system of interacting fermions modelled by a sequence of finite-range Hamiltonians $(H_L(b))_L$ defined on boxes $\Lambda_L = \{-L,\dots,L\}\times\{0,\dots,2L\}$ and dependent on a homogeneous magnetic field b perpendicular to the plane. We think of Λ_L as a subset of the upper half plane of $\mathbb{Z}^2$ and consider a strip $\{-L,\dots,L\}\times\{0,\dots,D-1\}$ of fixed width D as the edge region and its complement as the bulk. In the bulk we assume translation invariance of the Hamiltonian with respect to magnetic translations.

For inverse temperature $\beta > 0$, chemical potential $\mu \in \mathbb{R}$, and magnetic field $b \in \mathbb{R}$ the Gibbs, or thermal, state is defined as

$$\rho_L(\beta,\mu,b) := \frac{\mathrm{e}^{-\beta(H_L(b)-\mu\mathcal{N}_L)}}{\mathcal{Z}_L(\beta,\mu,b)}, \tag{1}$$

where $\mathcal{N}_L$ is the number operator and $\mathcal{Z}_L(\beta,\mu,b) := \mathrm{tr}\big(\mathrm{e}^{-\beta(H_L(b)-\mu\mathcal{N}_L)}\big)$ is the partition function. In the absence of interactions, $\rho_L(\beta,\mu,b)$ is naturally a local object, namely it has an integral kernel in which it is possible to identify a bulk and an edge region, see e.g. [65]. However, in the interacting setting, the locality of $\rho_L(\beta,\mu,b)$ is a delicate issue, which has been investigated e.g. in [67, 66, 219, 139, 98], see also the more recent [P4]. In the present work locality of the Gibbs state in the form of *local indistinguishability* is one of the crucial assumptions: Let $X \subset \Lambda' \subset \Lambda_L$, then we assume that the expectation value of an observable $A \in \mathcal{A}_X$ can be approximated by the Gibbs state of the Hamiltonian restricted to Λ',

$$\mathrm{tr}\big(\rho_L(\beta,\mu,b)\,A\big) \approx \mathrm{tr}\big(\rho_{\Lambda'}(\beta,\mu,b)\,A\big),$$

up to terms that vanish in the distance of X to the boundary $\partial\Lambda'$ of Λ'. A subtle point here is that the definition of $\partial\Lambda'$ depends on whether we consider Λ' as a subset of $\mathbb{Z}^2$ or as a subset Λ_L. In the first case $\partial\Lambda'$ could include parts of the physical edge $\{-L,\dots,L\}\times\{0\}$ and local indistinguishability is only demanded for X in the bulk of the system. In the second case local indistinguishability is also required for X located at the edge of the system. For this reason we speak of *local indistinguishability in the bulk* for the former case and *local indistinguishability everywhere* for the latter. Note that in our setting a sufficient condition implying local indistinguishability everywhere is a sufficiently high temperature [139]. However, for systems with short-range interactions, one may generally expect local indistinguishability to hold away from critical points, i.e. whenever the system has a unique thermal state in the thermodynamic limit. Such a state has decaying correlations [199, chapter 4], which implies local indistinguishability, at least if the decay is sufficiently fast [P4].

The magnetization is defined as the derivative of the grand canonical pressure $p_L(\beta,\mu,b) := -|\Lambda_L|^{-1}\,\beta^{-1}\ln\big(\mathcal{Z}_L(\beta,\mu,b)\big)$ with respect to the magnetic field b, namely

$$m_L(\beta,\mu,b) := \frac{\partial}{\partial b}\,p_L(\beta,\mu,b). \tag{2}$$

Our main result states that whenever the family of finite volume Gibbs states satisfies *local indistinguishability in the bulk* then the magnetization approximately equals the bond current $I_L^{\text{edge}}(\beta, \mu, b)$ through an orthogonal line of length L at the lower edge of the sample, see Figure 1 and equation (9),

$$\left| m_L(\beta, \mu, b) - I_L^{\text{edge}}(\beta, \mu, b) \right| = \mathcal{O}(L^{-1}). \tag{3}$$

Moreover, this current is very well localized near the edge and thus called *edge current*. Both statements are contained in Theorem I and depicted in Figure 1. Furthermore, if the finite volume Gibbs state satisfies *local indistinguishability everywhere*, then we show that the thermodynamic limit $m(\beta, \mu, b) := \lim_{L\to\infty} m_L(\beta, \mu, b)$ exists and obtain an equality between the edge current $I^{\text{edge}}(\beta, \mu, b)$ and the magnetization $m(\beta, \mu, b)$ in the infinite volume system, see Theorem II. And while the orbital magnetization $m_L(\beta, \mu, b)$ and the edge current $I_L^{\text{edge}}(\beta, \mu, b)$ of the finite systems in principle depend on the bulk and edge part of the system, we show that the limits $m(\beta, \mu, b)$ and $I^{\text{edge}}(\beta, \mu, b)$ are independent of the specific shape of the interactions at the edge of the system.

However, since we obtain the infinite volume magnetization $m(\beta, \mu, b)$ from a limit of finite systems with edges converging to a system on the upper half plane with an infinite edge, one might ask whether $m(\beta, \mu, b)$ can be considered a pure bulk quantity. To answer this question in the affirmative, we show in Theorem IV that the magnetization obtained from any KMS state at (β, μ) for the translation invariant bulk Hamiltonian defined on the entire plane coincides with $m(\beta, \mu, b)$.

Finally, in Theorem III we establish the differentiability of $m(\beta, \mu, b)$ with respect to μ. By comparison to the non-interacting setting, one would expect that, in the presence of a spectral gap and with weak interactions, the zero temperature limit of $\partial_\mu m(\beta, \mu, b)$ converges to the quantized Hall conductivity. While this result is not present in the literature and out of the scope of the paper, we show here a preliminary regularity result of $m(\beta, \mu, b)$ with respect to μ.

A crucial ingredient to our proofs is a new version of Bloch's Theorem for two-dimensional systems. We show that local indistinguishability together with current conservation implies that currents decay quickly with the distance to the edge, or, put differently, that in equilibrium currents can only flow near the edge of a sample. See [215, 21] for recent related results and the discussion below.

Let us end the introduction with a few more comments on the literature. As already mentioned, analogous mathematical results relating magnetization in the bulk with edge currents for non-interacting systems were obtained in [65], with predecessors e.g. in [7, 155, 143]. Notice that the equality between edge current and magnetization in two-dimensional systems can be also interpreted as a quantum mechanical, microscopic version of Ampère's law, as it is sometimes addressed in the physics literature, see for instance [177], where the effect of a time-dependent magnetic field on the magnetization of localized states is analyzed in a discrete, non-interacting setting.

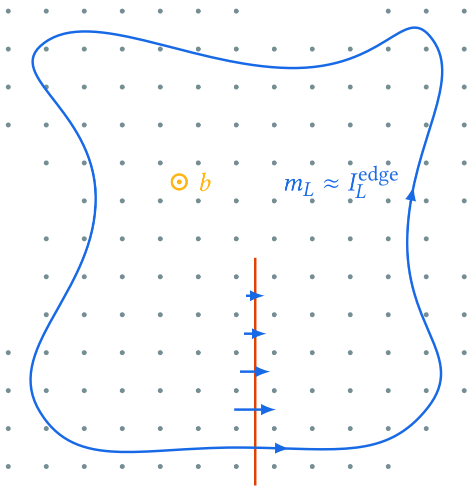

Figure 1. Pictorial representation of our main results: For locally interacting fermions on a two-dimensional lattice with perpendicular magnetic field b, satisfying local indistinguishability (see details in section 2.1) at positive temperature, the edge current I_L^{edge}, which is the bond current through the vertical line, is localized near the boundary and approximately equals the magnetization m_L (Theorem I). The latter is a bulk quantity, i.e. it converges in the thermodynamic limit $L \to \infty$ and does not depend on the details near the boundary (Theorem II). This independence allows for the rough edges in the picture.

The existence and properties of edge states of magnetic Schrödinger operators were studied e.g. in [91, 68]. The mathematical literature on bulk-edge correspondence for transport coefficients is vast but concerns almost exclusively non-interacting systems at zero temperature and with a gap in the bulk, e.g. [197, 82]. In [95, 92, 96] the authors derive, starting from the assumption of an incompressible bulk, effective actions for the bulk and the edge system. While they do not start from a many-body fermion model as we do, they are able to derive much more far-reaching consequences for quantum Hall systems from a seemingly innocuous assumption about the bulk. In microscopic models of interacting fermions the bulk-edge correspondence of transport coefficients was established at zero-temperature for weakly interacting gapped systems in [100, 159].

2 Mathematical framework and main results

2.1 The Hamiltonian

Let $\mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty)$ and $\mathbb{Z}^2_+ = \mathbb{Z} \times \mathbb{Z}_+$, both equipped with the 1-metric $\operatorname{dist}(x, y) := |x_1 - y_1| + |x_2 - y_2|$. For any finite subset $X \Subset \mathbb{Z}^2$ let $\mathfrak{h}_X := \ell^2(X, \mathbb{C}^s)$ be the one-body space and $\mathfrak{F}_X := \mathfrak{F}^-(\mathfrak{h}_X)$ the corresponding fermionic Fock space. By $\mathscr{A}_X$ we denote the algebra of all bounded operators in $\mathscr{L}(\mathfrak{F}_X)$ that commute with the number operator $\mathscr{N}_X := \sum_{x\in X} a_x^* \, a_x := \sum_{x\in X} \sum_{j=1}^{s} a_{x,j}^* \, a_{x,j}$ and by $\mathscr{A}_{\mathrm{loc}} := \bigcup_{X\Subset\mathbb{Z}^2} \mathscr{A}_X$ the algebra of all local observables that preserve particle number. Its closure

$$\mathscr{A} := \overline{\mathscr{A}_{\mathrm{loc}}}^{\|\cdot\|}$$

is a C^*-algebra and called the quasi-local algebra.

We consider sequences $(H_L(b))_{L\in\mathbb{N}}$ of Hamiltonians defined on boxes $\Lambda_L := \big([-L, L] \times [0, 2L]\big) \cap \mathbb{Z}^2$ that are of the form

$$\begin{aligned} H_L(b) = & \sum_{x,y\in\Lambda_L} a_x^* \, T_b^{\mathrm{bulk}}(x, y) \, a_y + \sum_{X\subset\Lambda_L} \Phi^{\mathrm{bulk}}(X) \\ & + \sum_{x,y\in\Lambda_L} a_x^* \, T_b^{\mathrm{edge}}(x, y) \, a_y + \sum_{X\subset\Lambda_L} \Phi^{\mathrm{edge}}(X) \\ =: & \sum_{x,y\in\Lambda_L} a_x^* \, T_b(x, y) \, a_y + \sum_{X\subset\Lambda_L} \Phi(X). \end{aligned} \tag{4}$$

We sometimes need to restrict this Hamiltonian to other finite sets $\Lambda' \subset \Lambda_L$. In this case we write $H_L(b)\big|_{\Lambda'}$ which means that the sums in (4) only run over $x, y \in \Lambda'$ and $X \subset \Lambda'$, respectively.

The Hamiltonian is split into a “bulk” part, which is invariant under magnetic translations and an “edge” part, which lives on the lower edge. Each consists of two contributions: The kinetic terms

$$T_b^{\text{bulk/edge}}(x,y) := \mathrm{e}^{\mathrm{i}\frac{x_2+y_2}{2}b(x_1-y_1)}\, T^{\text{bulk/edge}}(x,y) \tag{5}$$

are a Peierls phase times a hopping amplitude $T^{\text{bulk/edge}}\colon \mathbb{Z}^2\times\mathbb{Z}^2 \to \mathscr{L}(\mathbb{C}^s)$, which is uniformly bounded $\sup_{x,y\in\mathbb{Z}^2}\|T^{\text{bulk/edge}}(x,y)\| \le C$ and satisfies $T^{\text{bulk/edge}}(x,y) = T^{\text{bulk/edge}}(y,x)^*$. The interactions

$$\Phi^{\text{bulk/edge}}\colon \{X \Subset \mathbb{Z}^2\} \to \mathscr{A}_{\text{loc}}, \quad X \mapsto \Phi^{\text{bulk/edge}}(X) \in \mathscr{A}_X$$

are self-adjoint, the terms are uniformly bounded, $\sup_{X\subset\mathbb{Z}^2}\|\Phi^{\text{bulk/edge}}(X)\| \le C$, and the corresponding operators $\sum_{X\subset\Lambda_L}\Phi^{\text{bulk/edge}}(X)$ are assumed to commute with all local number operators $\mathcal{N}_{\{z\}}$ for $z\in\Lambda_L$. The last condition is satisfied, e.g., for density-density interactions or external potentials.

Furthermore, all terms are assumed to be of finite range $R\in\mathbb{N}$, i.e. $T^{\text{bulk/edge}}(x,y) = 0$ if $\operatorname{dist}(x,y) > R$ and $\Phi^{\text{bulk/edge}}(X) = 0$ if $\operatorname{diam}(X) > R$. As mentioned above, the bulk contributions are assumed to be invariant under magnetic translations, i.e. $T^{\text{bulk}}(x-z, y-z) = T^{\text{bulk}}(x,y)$ only depends on the difference $x-y$ and Φ^{bulk} satisfies (20). And the edge contributions are supported on a strip of fixed width D along the lower edge, i.e. $T^{\text{edge}}(x,y) = 0$ unless $x,y\in\mathbb{Z}\times\{0,1,\dots,D-1\}$ and $\Phi^{\text{edge}}(X) = 0$ unless $X\subset\mathbb{Z}\times\{0,1,\dots,D-1\}$. Without loss of generality we choose $D\ge R$, since the presence of the boundary already modifies the Hamiltonian in $\mathbb{Z}\times\{0,1,\dots,R-1\}$.

A canonical example of a magnetic Hamiltonian with interactions is the Hofstadter-Hubbard model, i.e. the second quantization of the discrete magnetic Laplacian together with an on-site density-density interaction. More precisely, for the Hofstadter-Hubbard model we have $\mathfrak{h}_{\{x\}} = \mathbb{C}^2$,

$$T^{\text{bulk}}(x,y) = \mathrm{id}_{\mathbb{C}^2}\cdot\delta_{|x-y|=1}, \quad \text{and} \quad \Phi^{\text{bulk}}(X) = a_{x,1}^*\, a_{x,1}\, a_{x,2}^*\, a_{x,2}\cdot\delta_{X=\{x\}},$$

which leads to

$$H_L^{\text{HH}}(b) = \sum_{\substack{x,y\in\Lambda_L:\\ |x-y|=1}} \mathrm{e}^{\mathrm{i}\frac{x_2+y_2}{2}b(x_1-y_1)} \sum_{j\in\{1,2\}} a_{x,j}^*\, a_{y,j} + \sum_{x\in\Lambda_L} a_{x,1}^*\, a_{x,1}\, a_{x,2}^*\, a_{x,2}.$$

Near the edge, one could, for example, add an external potential $\Phi^{\text{edge}}(X) = \phi(x)\,\mathcal{N}_{\{x\}}\,\delta_{X=\{x\}}$ or effectively remove individual sites by subtracting all hoppings connected to them.

For a finite subset $\Lambda \Subset \mathbb{Z}_+^2$, a Hamiltonian $H\in\mathscr{A}_\Lambda$, inverse temperature $\beta > 0$, chemical potential $\mu\in\mathbb{R}$, and magnetic field $b\in\mathbb{R}$ we denote the grand canonical partition function by

$$\mathcal{Z}_\Lambda[H](\beta,\mu) := \operatorname{tr}\big(\mathrm{e}^{-\beta(H-\mu\,\mathcal{N}_\Lambda)}\big),$$

and the Gibbs state by

$$\rho_\Lambda[H](\beta,\mu) := \frac{\mathrm{e}^{-\beta(H-\mu\mathcal{N}_\Lambda)}}{\mathcal{Z}_\Lambda[H](\beta,\mu)}.$$

When we consider truncated Hamiltonians in the proofs, we drop the index and informally write $\rho\big[H_L(b)|_{\Lambda'}\big](\beta,\mu) \equiv \rho_{\Lambda'}\big[H_L(b)|_{\Lambda'}\big](\beta,\mu)$. On the boxes Λ_L we abbreviate

$$\begin{aligned}\mathcal{N}_L &:= \mathcal{N}_{\Lambda_L},\\ \mathcal{Z}_L(\beta,\mu,b) &:= \mathcal{Z}_{\Lambda_L}\big[H_L(b)\big](\beta,\mu)\end{aligned}$$

and

$$\rho_L(\beta,\mu,b) := \rho_{\Lambda_L}\big[H_L(b)\big](\beta,\mu).$$

The key hypothesis for our results is that the Gibbs state is locally determined by the local terms in the Hamiltonian. This property is often called *local indistinguishability* and made precise in the following definition.

Definition 1 (Local indistinguishability of the Gibbs state)**.** Let $\zeta\colon \mathbb{N}_0 \to \mathbb{R}_+$ be non-increasing with $\lim_{n\to\infty}\zeta(n) = 0$ and $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ non-decreasing with $g\big((2R+1)^2 + 1\big) = 1$. The family of Hamiltonians $(H_L(b))_{L\in\mathbb{N}}$ is said to satisfy *local indistinguishability of the Gibbs state at* (β,μ,b) with ζ-decay,

(a) *in the bulk* if and only if for all $L \in \mathbb{N}$, $X \subset \Lambda' \subset \Lambda_L$ and $A \in \mathcal{L}(\mathcal{F}_X) \subset \mathcal{A}$

$$\begin{aligned}&\big|\mathrm{tr}\big(\rho_L(\beta,\mu,b)\,A\big) - \mathrm{tr}\big(\rho_{\Lambda'}\big[H_L(b)|_{\Lambda'}\big](\beta,\mu)\,A\big)\big|\\ &\quad\le \|A\|\,g\big(|X|\big)\,\zeta\big(\mathrm{dist}(X,\mathbb{Z}^2\setminus\Lambda')\big),\end{aligned} \tag{6}$$

(b) *everywhere* if and only if for all $L \in \mathbb{N}$, $X \subset \Lambda' \subset \Lambda_L$ and $A \in \mathcal{L}(\mathcal{F}_X) \subset \mathcal{A}$

$$\begin{aligned}&\big|\mathrm{tr}\big(\rho_L(\beta,\mu,b)\,A\big) - \mathrm{tr}\big(\rho_{\Lambda'}\big[H_L(b)|_{\Lambda'}\big](\beta,\mu)\,A\big)\big|\\ &\quad\le \|A\|\,g\big(|X|\big)\,\zeta\big(\mathrm{dist}(X,\Lambda_L\setminus\Lambda')\big).\end{aligned} \tag{7}$$

◇

Note the difference between $\mathbb{Z}^2\setminus\Lambda'$ and $\Lambda_L\setminus\Lambda'$ in the distance in (6) and (7), if Λ' includes parts at the boundary of Λ_L, see Figure 2. In particular $\mathrm{dist}(X,\mathbb{Z}^2\setminus\Lambda') = \min\{\mathrm{dist}(X,\Lambda_L\setminus\Lambda'),\mathrm{dist}(X,\mathbb{Z}^2\setminus\Lambda_L)\}$, so indistinguishability everywhere implies the property in the bulk. This distinction is useful, because we expect a better decay in the bulk and a worse decay at the boundary due to the presence of edge states. For some of the statements we however need local indistinguishability also near the boundary and might accept a slower decay. In particular, local indistinguishability directly implies decay of correlations, see Lemma 13, and we do not expect that to hold with good decay near the boundary due to edge states, see e.g. [159].

For most of our results, we will require local indistinguishability with decay at least $\zeta \in \ell^1$, but any better decay will yield better results, in particular concerning localization near the boundary. For example, local indistinguishability everywhere with

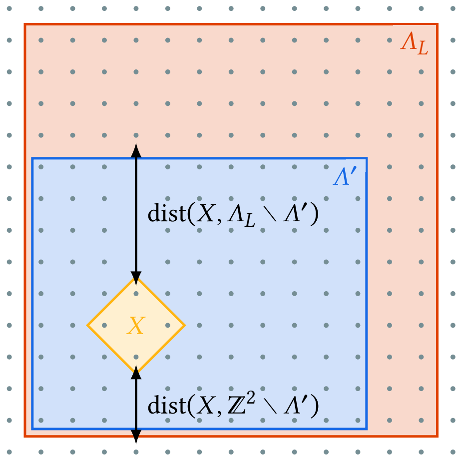

Figure 2. Sketch of the two distances used in Definition 1. If the Hamiltonian satisfies local indistinguishability in the bulk, the bound (6) decays in the distance to the boundary of Λ' (in $\mathbb{Z}^2$). While local indistinguishability everywhere also gives a good estimate if X is close to the boundary of Λ' as long as the boundaries of Λ' and Λ_L coincide in that region.

exponential decay function ζ is known to hold for sufficiently high temperature in systems with finite-range interactions [139, Corollary 2 and 5]. The decay rate and constants depend on β, but can be chosen uniformly for small β. As is shown in [P4], decay of correlations at some positive temperature implies local indistinguishability at the same temperature, and the converse also holds, see Lemma 13.

The normalization in Definition 1 is chosen such that all later bounds, where we always restrict to sets $|X| \leq (2R+1)^2 + 1$ so that $g\big(|X|\big) \leq 1$, do not depend on g. We need to allow for larger X only to define the thermodynamic limit $\rho_\infty(\beta, \mu, b)$.

2.2 The edge current and the magnetization

Denote by $B_L^x(\ell) := \{\, y \in \Lambda_L \,|\, \mathrm{dist}(x,y) \leq \ell \,\}$ the ball around x in Λ_L with radius ℓ. The set $B_L^x(R)$ contains all points which have non-vanishing interaction with x. Then the current operator $J_L(b)$ has components ($k = 1, 2$)

$$\begin{aligned} J_{k,L}(b) := \mathrm{i}\big[X_{k,L}, H_L(b)\big] &= \mathrm{i}\Bigg[\sum_{z\in\Lambda_L} z_k\, a_z^*\, a_z,\ \sum_{x,y\in\Lambda_L} a_x^*\, T_b(x,y)\, a_y\Bigg] \\ &= \mathrm{i} \sum_{x\in\Lambda_L} \sum_{y\in B_L^x(R)} (x_k - y_k)\, a_x^*\, T_b(x,y)\, a_y. \end{aligned}$$

We now rewrite this sum as a sum of currents through edges of the dual lattice. For that, denote by $e_{k,z} \subset \mathbb{R}^2$ the dual edge which intersects the edge between lattice points z and $z + \hat{\mathrm{e}}_k$ and by $\overline{e_{k,z}}$ the edge together with the attached vertices (see Figure 3). Here, $\hat{\mathrm{e}}_k$ denotes the unit vector in k direction, e.g. $\hat{\mathrm{e}}_1 = (1,0)$. Moreover, denote by $\overline{xy} \subset \mathbb{R}^2$

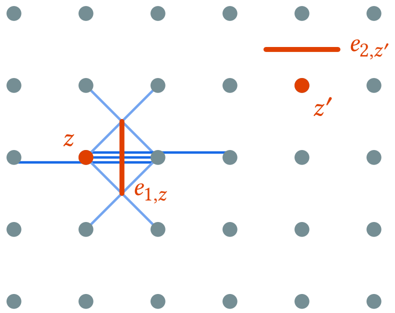

Figure 3. The figure shows a small section of Λ_L and two points z and $z' \in \Lambda_L$ with their dual edges $e_{1,z}$ and $e_{2,z'}$, respectively. For z, also all lines $\overline{xy}$ which contribute to $J^z_{1,L}(b)$ for $R = 2$ are drawn. The light blue lines only intersect the endpoints of $\overline{e_{1,z}}$ and thus come with a prefactor $1/2$ in (8).

the line connecting x and y. We define the current through the dual edge $e_{k,z}$ as

$$J^z_{k,L}(b) := \frac{\mathrm{i}}{2}\Bigg(\sum_{\substack{x,y\in\Lambda_L:\\ \overline{xy}\,\cap\, e_{k,z}\neq\emptyset}} \operatorname{sgn}(x_k - y_k)\, a^*_x\, T_b(x,y)\, a_y + \sum_{\substack{x,y\in\Lambda_L:\\ \overline{xy}\,\cap\, \overline{e_{k,z}}\neq\emptyset}} \operatorname{sgn}(x_k - y_k)\, a^*_x\, T_b(x,y)\, a_y \Bigg). \quad (8)$$

Thus, each hopping term $a^*_x T_b(x,y)\, a_y$ is included once in $J^z_{k,L}(b)$ if $\overline{xy}$ intersects the dual edge $e_{k,z}$, or half if $\overline{xy}$ intersects only the endpoints $\overline{e_{k,z}} \setminus e_{k,z}$ of the dual edge. In the latter case it appears for twice as many different z. Since $\overline{xy}$ intersects $|x_k - y_k|$ vertical lines we can rewrite

$$J_{1,L}(b) = \sum_{m=-L}^{L-1} \sum_{n=0}^{2L} J^{(m,n)}_{1,L}(b) \quad \text{and} \quad J_{2,L}(b) = \sum_{m=-L}^{L} \sum_{n=0}^{2L-1} J^{(m,n)}_{2,L}(b)$$

by summing over all edges. Note, that the L-dependence of $J^z_{k,L}(b)$ only stems from missing hopping terms near the boundary, and we define

$$J^z_k(b) := J^z_{k,L}(b)$$

for all $L > |z_1| + R$ and $2L > z_2 + R$ consistently.

Moreover, for $d \in \{1,\dots,L\}$, we define the edge current as

$$I^{d\,\mathrm{edge}}_L(\beta,\mu,b) := \sum_{n=0}^{d-1} \operatorname{tr}\big(\rho_L(\beta,\mu,b)\, J^{(0,n)}_{1,L}(b)\big) \quad (9)$$

and introduce the shorthand $I^{\mathrm{edge}}_L = I^{L\,\mathrm{edge}}_L$ for the current along the lower half of the system Λ_L. By current conservation it equals the currents through lines connecting the center of Λ_L with the midpoints of the other boundaries of Λ_L, see proof of Proposition 9, we only choose this edge because it persists in the thermodynamic limit $L \to \infty$ in our geometry.

It remains to recall the definition of (orbital) magnetization. For inverse temperature $\beta > 0$, chemical potential $\mu \in \mathbb{R}$, and magnetic field $b \in \mathbb{R}$, the grand canonical pressure is given by

$$p_L(\beta,\mu,b) \;:= -(2L+1)^{-2}\beta^{-1}\ln\big(\mathscr{Z}_L(\beta,\mu,b)\big), \tag{10}$$

and the magnetization by

$$m_L(\beta,\mu,b) \;:= \frac{\partial}{\partial b}\, p_L(\beta,\mu,b).$$

2.3 Main results

Our first main result deals with the magnetization and the edge current at finite volume. For this type of result only local indistinguishability of the Gibbs state in the bulk is needed.

Theorem I. *Let $\zeta^{\text{bulk}} \in \ell^1$ and $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonians of the form* (4)*. Then, there exists a null sequence θ and a constant $C > 0$ such that the following holds: If $(H_L(b))_{L\in\mathbb{N}}$ satisfies local indistinguishability of the Gibbs state in the bulk at (β,μ,b) with ζ^{bulk}-decay in the sense of Definition 1, then*

$$\big|m_L(\beta,\mu,b) - I_L^{\text{edge}}(\beta,\mu,b)\big| \le \theta(L) \qquad \textit{for all } L \ge D+R. \tag{11}$$

Moreover, the edge current is localized near the edge in the sense that for all $L \ge d \ge R+D$

$$\big|I_L^{d\,\text{edge}}(\beta,\mu,b) - I_L^{\text{edge}}(\beta,\mu,b)\big| \le C \sum_{n=d-R-D}^{\infty} \zeta^{\text{bulk}}(n). \tag{12}$$

Remark 2. The sequence θ is given as a function of ζ^{bulk} by (30). If $\zeta^{\text{bulk}}(r) \le C'\,(r+1)^{-(n+1)}$ with $n > 1$, then $\theta(L) \le C''\,L^{-n/(n+2)}$ and (12) is bounded by $C'''\,(d-R-D)^{-n}$. While (12) scales basically like ζ^{bulk}, the best possible decay in (11) is $\theta(L) \sim 1/L$, which results from the fact that the fraction of the total area occupied by the boundary scales like $1/L$ in two dimensions. However, unless both the magnetization and the edge current vanish, no better decay can be expected even in the non-interacting case, cf. [65]. ◇

Then, if we further assume local indistinguishability of the Gibbs state everywhere we can also analyze the thermodynamic limit of (12). First, notice that if the family of Hamiltonians $(H_L(b))_{L\in\mathbb{N}}$ satisfies local indistinguishability with ζ^{edge}-decay everywhere, then for all finite $X \subset \mathbb{Z}_+^2$ and all observables $A \in \mathscr{A}_X$ the expectation values $\operatorname{tr}\big(\rho_L(\beta,\mu,b)\,A\big)$, which are defined if $\Lambda_L \supset X$, form a Cauchy sequence in L, and we define $\rho_\infty(\beta,\mu,b)(A)$ to be its limit. Thus, there exists a unique thermodynamic limit state $\rho_\infty(\beta,\mu,b)$ on $\mathscr{A}_{\text{loc}}$, and we can define the edge current in this state by

$$I^{d\,\text{edge}}(\beta,\mu,b) \;:= \rho_\infty(\beta,\mu,b)\bigg(\sum_{n=0}^{d} J_1^{(0,n)}(b)\bigg). \tag{13}$$

Theorem II. *Let $\zeta^{\mathrm{bulk}} \in \ell^1$ and ζ^{edge} tend to zero. Let $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonians of the form* (4) *satisfying local indistinguishability of the Gibbs state at (β, μ, b) with ζ^{bulk}-decay in the bulk and ζ^{edge}-decay everywhere, in the sense of Definition 1. Then the thermodynamic limit*

$$m(\beta, \mu, b) := \lim_{L\to\infty} m_L(\beta, \mu, b) \tag{14}$$

exists, and the total edge current

$$I^{\mathrm{edge}}(\beta, \mu, b) := \lim_{d\to\infty} I^{d\,\mathrm{edge}}(\beta, \mu, b), \tag{15}$$

satisfies

$$m(\beta, \mu, b) = I^{\mathrm{edge}}(\beta, \mu, b).$$

The edge current is localized near the edge in the sense that there is $C > 0$ so that for all $d \geq D + R$

$$\left|I^{d\,\mathrm{edge}}(\beta, \mu, b) - I^{\mathrm{edge}}(\beta, \mu, b)\right| \leq C \sum_{n=d-R-D}^{\infty} \zeta^{\mathrm{bulk}}(n).$$

Moreover, $m(\beta, \mu, b)$ and $I^{\mathrm{edge}}(\beta, \mu, b)$ are independent of the specific edge contributions T_b^{edge} and Φ^{edge}.

The precise statement on the independence from the edge Hamiltonian is given in Proposition 12, where we also derive an explicit bound on the difference of the finite-volume edge currents, see (33).

Note, that local indistinguishability everywhere with ζ^{edge}-decay implies local indistinguishability in the bulk with decay at most $\zeta^{\mathrm{bulk}} \leq \zeta^{\mathrm{edge}}$. But we assume the decay in the bulk separately in order to take into account the possibility that the localization properties of the edge current might be better than its speed of convergence in the thermodynamic limit, cf. Proposition 10, which is the situation in the non-interacting setting [65, 168].

Remark 3. We present Theorems I and II with an edge interaction only at the lower boundary for simplicity. The proofs allow for a more general setting where an interaction can be added on all four sides. Moreover, then the magnetization $m(\beta, \mu, b)$ and the edge current $I^{\mathrm{edge}}(\beta, \mu, b)$ are independent of these boundary perturbations. More precisely, we have the following: For each $L \in \mathbb{N}$ let $\Phi_L^{\mathrm{boundary}}$ be a finite-range interaction supported on $Q_L := \Lambda_L \setminus [-L+D, L-D] \times [0, 2L-D]$, a strip of width D around the remaining boundaries of Λ_L, commuting with all local number operators $\mathcal{N}_{\{z\}}$ and with interaction terms bounded uniformly in L, and let T_b^{boundary} be a hopping as in (5) with T^{boundary} supported on Q_L. Let $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonians of the form (4) and let

$$\tilde{H}_L(b) := H_L(b) + \sum_{x,y\in\Lambda_L} a_x^* \, T_b^{\mathrm{boundary}}(x, y) \, a_y + \sum_{X\subset\Lambda_L} \Phi_L^{\mathrm{boundary}}(X).$$

Then the respective statements of Theorem I and II also hold for the family of Hamiltonians $(\tilde{H}_L(b))_{L\in\mathbb{N}}$. Moreover, if both $\tilde{H}_L(b)$ and $H_L(b)$ satisfy local indistinguishability, then $\tilde{m}(\beta,\mu,b) = m(\beta,\mu,b)$ and $\tilde{I}^{\mathrm{edge}}(\beta,\mu,b) = I^{\mathrm{edge}}(\beta,\mu,b)$ are independent of the boundary terms. ◇

Supposing that the assumptions of Theorem II hold uniformly in some open interval of chemical potentials around μ, we obtain differentiability of $m(\beta,\mu,b)$ with respect to μ and thus, as explained in the introduction, a further step in the direction of proving the equality of transport coefficients.

Theorem III. *Let $n \mapsto n^2\,\zeta^{\mathrm{edge}}(n) \in \ell^1$. Let $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonians of the form* (4) *satisfying local indistinguishability of the Gibbs state everywhere with ζ^{edge}-decay at (β,μ,b), for μ in some open interval. Then $m(\beta,\mu,b)$ and $I^{\mathrm{edge}}(\beta,\mu,b)$ defined in Theorem II are differentiable, and thus*

$$\partial_\mu\, m(\beta,\mu,b) = \partial_\mu\, I^{\mathrm{edge}}(\beta,\mu,b).$$

Remark 4. As we already know from Theorem II that $m(\beta,\mu,b) = I^{\mathrm{edge}}(\beta,\mu,b)$, Theorem III will follow from differentiability of $I^{\mathrm{edge}}(\beta,\mu,b)$, see Proposition 17. Additionally, we prove a quantitative bound for the difference of the two quantities in finite volume in Propositions 18, and localization near the boundary in Proposition 17. ◇

An important ingredient in the proof of Theorem I is the vanishing of the equilibrium current in the bulk. This result is known in the literature as Bloch's theorem (see e.g. [215, 21] and references therein) and has an importance on its own. In our setting with open boundary conditions, it is just a consequence of current conservation coupled with the local indistinguishability of the Gibbs state. This allows for better decay rates than in the setting with periodic boundary conditions.

Proposition 5 (Bloch's Theorem)**.** *Let ζ^{bulk} be a null sequence and $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonian of the form* (4)*. There exists $C_{\mathrm{B}} > 0$ such that the following holds: If $(H_L(b))_{L\in\mathbb{N}}$ satisfies local indistinguishability of the Gibbs state in the bulk at (β,μ,b) with ζ^{bulk}-decay in the sense of Definition 1, then*

$$\left|\mathrm{tr}\big(\rho_L(\beta,\mu,b)\, J^z_{k,L}(b)\big)\right| \le C_{\mathrm{B}}\,\zeta^{\mathrm{bulk}}\Big(\big[\mathrm{dist}(z,\mathbb{Z}^2\setminus\Lambda_L) - D - R\big]_+\Big). \tag{16}$$

We have established the equality of the edge current and the magnetization and proven that the edge current is an edge quantity in the sense, that it is localized near the edge. Finally, we argue that the magnetization is a bulk quantity by showing that it can be obtained directly in the infinite volume without any edge. Denote by $B_L := [-L,L]^2 \cap \mathbb{Z}^2$ the boxes centered on the origin and let

$$H^{\mathrm{bulk}}_L(b) = \sum_{x,y\in B_L} a^*_x\, T^{\mathrm{bulk}}_b(x,y)\, a_y + \sum_{X\subset B_L} \Phi^{\mathrm{bulk}}(X) \tag{17}$$

be the bulk Hamiltonian and

$$\tau_t^{\mathrm{bulk}}(A) = \lim_{L\to\infty} \mathrm{e}^{\mathrm{i}t(H_L^{\mathrm{bulk}}(b)-\mu\mathcal{N}_{B_L})}\, A\, \mathrm{e}^{-\mathrm{i}t(H_L^{\mathrm{bulk}}(b)-\mu\mathcal{N}_{B_L})} \tag{18}$$

be the infinite-volume dynamics generated by the bulk Hamiltonian (adjusted by the chemical potential). Following the arguments from [11] we will first show that the pressure of the infinite volume limits of the edge and bulk system agree. Under a somewhat stronger hypothesis and using ideas similar to Theorem III, the magnetization $m(\beta,\mu,b)$ can then also be obtained directly in the infinite-volume system.

To this end, note that the pressure for general states ρ on $\mathscr{A}_{B_L}$ is defined as

$$P(\rho) := \mathrm{tr}\Big(\rho\left(H_L^{\mathrm{bulk}}(b) - \mu\,\mathcal{N}_{B_L}\right)\Big) - \beta^{-1}\,S(\rho) \quad \text{with} \quad S(\rho) := -\,\mathrm{tr}(\rho\ln\rho), \tag{19}$$

which agrees with $-\beta^{-1}\ln\mathscr{Z}_{B_L}\big[H_L^{\mathrm{bulk}}(b)\big](\beta,\mu)$ for the Gibbs state of $H_L^{\mathrm{bulk}}(b)$, compare (10). The following theorem states that the pressure per unit volume, $\lim_{L\to\infty} P(\omega|_{B_L})/(2L+1)^2$, where $\omega|_{B_L}$ is the restriction of a bulk equilibrium state ω to $\mathscr{A}_{B_L}$, equals the thermodynamic limit of the pressure in the system with an edge.

Theorem IV. *Let $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonians of the form* (4). *For any $\beta>0$, $\mu, b\in\mathbb{R}$ the thermodynamic limit*

$$p(\beta,\mu,b) := \lim_{L\to\infty} p_L(\beta,\mu,b)$$

of the pressure exists and is independent of the boundary terms. Moreover, for any $(\tau^{\mathrm{bulk}},\beta)$-KMS state ω the pressure per volume of ω equals $p(\beta,\mu,b)$,

$$p(\beta,\mu,b) = \lim_{L\to\infty} \frac{P\big(\omega|_{B_L}\big)}{(2L+1)^2}.$$

Additionally, given $\zeta^{\mathrm{bulk}}\in\ell^1$ and ζ^{edge} tending to zero, assume that $\big(H_L(b)\big)_{L\in\mathbb{N}}$ satisfies local indistinguishability in the bulk with ζ^{bulk}-decay and everywhere with ζ^{edge}-decay at (β,μ,b), for b in some open interval. Then $b\mapsto p(\beta,\mu,b)$ is differentiable and its derivative agrees with the magnetization $m(\beta,\mu,b)$ defined in (14),

$$\partial_b\, p(\beta,\mu,b) = m(\beta,\mu,b).$$

3 Proofs

3.1 Bloch's theorem

This section is devoted to the proof of Bloch's theorem, namely Proposition 5. The proof is based on the bulk homogeneity of the system and on the continuity equation for the current provided by Lemma 6, together with the local indistinguishability assumption for the Gibbs state. The homogeneity in the bulk of the system is encoded in the invariance under magnetic translation, which we briefly recall in the next section.

3.1.1 Magnetic translations

On the one-particle Hilbert space $\mathfrak{h} = \ell^2(\mathbb{Z}^2)$ of the full lattice $\mathbb{Z}^2$, for $y \in \mathbb{Z}^2$ and $b > 0$ the magnetic translation $U_y(b)$ is defined by its action on $\psi \in \mathfrak{h}$ as

$$\big(U_y(b)\,\psi\big)(x) = \mathrm{e}^{\mathrm{i}by_2x_1}\,\psi(x-y) \quad \text{with adjoint} \quad \big(U_y^*(b)\,\psi\big)(x) = \mathrm{e}^{-\mathrm{i}by_2(x_1+y_1)}\,\psi(x+y).$$

Then, in second quantization, for which we use the same symbol, we obtain

$$U_y^*(b)\,a_x^*\,U_y(b) = \mathrm{e}^{-\mathrm{i}y_2bx_1}\,a_{x-y}^*.$$

With this definition, the kinetic part of the bulk Hamiltonian (4) is invariant under magnetic translations since

$$\begin{aligned} U_z^*(b)\,a_x^*\,T_b^{\mathrm{bulk}}(x,y)\,a_y\,U_z(b) &= U_z^*(b)\,a_x^*\,U_z(b)\,\mathrm{e}^{\mathrm{i}b\frac{x_2+y_2}{2}(x_1-y_1)}\,T^{\mathrm{bulk}}(x,y)\,U_z^*(b)\,a_y\,U_z(b) \\ &= a_{x-z}^*\,\mathrm{e}^{\mathrm{i}b\frac{x_2+y_2}{2}(x_1-y_1)-\mathrm{i}b\frac{2z_2x_1}{2}+\mathrm{i}b\frac{2z_2y_1}{2}}\,T^{\mathrm{bulk}}(x-z,y-z)\,a_{y-z} \\ &= a_{x-z}^*\,T_b^{\mathrm{bulk}}(x-z,y-z)\,a_{y-z}. \end{aligned}$$

Moreover, we assume that the bulk interaction Φ^{bulk} is invariant under magnetic translations, namely

$$U_z^*(b)\,\Phi^{\mathrm{bulk}}(X)\,U_z(b) = \Phi^{\mathrm{bulk}}(X-z). \tag{20}$$

Hence, the complete bulk part of the Hamiltonian (4) is invariant under magnetic translations, and

$$U_z^*(b)\,H_L(b)\big|_X\,U_z(b) = H_L(b)\big|_{X-z}$$

for $X \subset \Lambda_L$ and $z \in \mathbb{Z}^2$ such that $\mathrm{dist}(X, \mathbb{Z}^2 \setminus \Lambda_L)$ and $\mathrm{dist}(X-z, \mathbb{Z}^2 \setminus \Lambda_L) > D$. This property carries over to the local Gibbs state in the sense, that

$$U_z^*(b)\,\rho\big[H_L(b)\big|_X\big](\beta,\mu)\,U_z(b) = \rho\big[H_L(b)\big|_{X-z}\big](\beta,\mu).$$

Moreover, the above calculation also shows that $U_z^*(b)\,J_k^y(b)\,U_z(b) = J_k^{y-z}(b)$ for all $y_2, z_2 \geq R + D$, due to the simple structure of $J_k^y(b)$. Then, it follows that

$$\mathrm{tr}\big(\rho\big[H_L(b)\big|_X\big](\beta,\mu)\,J_1^y(b)\big) = \mathrm{tr}\big(\rho\big[H_L(b)\big|_{X-z}\big](\beta,\mu)\,J_1^{y-z}(b)\big) \tag{21}$$

if X, y and z fulfill all mentioned conditions.

3.1.2 The continuity equation

We prove the continuity equation for the local currents defined in (8) and the resulting current conservation for stationary states. These two facts play a key role in the proof of Bloch's theorem.

For this purpose, let us define the dual edge boundary of a set $Z \subset \Lambda_L$ as

$$\partial_{\Lambda_L} Z := \left\{ (k,z) \in \{1,2\} \times \begin{pmatrix} \{-L,\dots,L-1\} \\ \times\{0,\dots,2L-1\} \end{pmatrix} \middle| \begin{matrix} z \in Z \text{ and } z + \hat{e}_k \in \Lambda_L \setminus Z \quad \text{or} \\ z \in \Lambda_L \setminus Z \text{ and } z + \hat{e}_k \in Z \end{matrix} \right\}.$$

This is exactly the set of labels (k,z), such that the union of the dual edges $\overline{e_{k,z}}$ is the boundary of the set $\bigcup_{z\in Z} z + [-1/2, 1/2]^2$ in $\bigcup_{z\in\Lambda_L} z + [-1/2,1/2]^2$.

Lemma 6 (Continuity equation). *For any $z \in \Lambda_L$, the currents defined in* (8) *satisfy the continuity equation*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{e}^{\mathrm{i}H_L(b)\,t}\, \mathcal{N}_{\{z\}}\, \mathrm{e}^{-\mathrm{i}H_L(b)\,t}\Big|_{t=0} &= \operatorname{div}_z J^z_{k,L}(b) \\ &:= J^z_{1,L}(b) - J^{z-\hat{e}_1}_{1,L}(b) + J^z_{2,L}(b) - J^{z-\hat{e}_2}_{2,L}(b). \end{aligned} \tag{22}$$

Proof. Calculating the derivative, we find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{e}^{\mathrm{i}H_L(b)\,t}\, \mathcal{N}_{\{z\}}\, \mathrm{e}^{-\mathrm{i}H_L(b)\,t}\Big|_{t=0} &= \mathrm{i}\,[H_L(b), \mathcal{N}_{\{z\}}] \\ &= \mathrm{i} \sum_{x,y\in\Lambda_L} \left[a^*_x\, T_b(x,y)\, a_y, \mathcal{N}_{\{z\}}\right] \\ &= \mathrm{i} \sum_{x\in\Lambda_L\setminus\{z\}} a^*_x\, T_b(x,z)\, a_z - \mathrm{i} \sum_{y\in\Lambda_L\setminus\{z\}} a^*_z\, T_b(z,y)\, a_y \\ &= \mathrm{i} \sum_{x\in\Lambda_L\setminus\{z\}} a^*_x\, T_b(x,z)\, a_z - a^*_z\, T_b(z,x)\, a_x, \end{aligned}$$

where we used that $\left[\sum_{Z\subset\Lambda_L} \Phi^{\mathrm{bulk}}(Z), \mathcal{N}_{\{z\}}\right] = \left[\sum_{Z\subset\Lambda_L} \Phi^{\mathrm{edge}}(Z), \mathcal{N}_{\{z\}}\right] = 0$ by assumption.

It is left to rewrite the sum in terms of $J^z_{k,L}(b)$. Each $\overline{xz}$ in the sum will cross the rectangle around z formed by the four dual edges $\overline{e_{1,z}}$, $\overline{e_{1,z-\hat{e}_1}}$, $\overline{e_{2,z}}$ and $\overline{e_{2,z-\hat{e}_2}}$ of the dual lattice at one point. If this point lies within a dual edge $e_{k,q}$, the term in the sum will contribute to $J^q_{k,L}(b)$ with weight 1. Otherwise, $\overline{xz}$ crosses the rectangle at a corner and the contribution is attributed evenly to the two adjacent dual edges with weight 1/2. The sum on the right-hand side of (22) still contains some more terms coming from elements $x, y \in \Lambda_L \setminus \{q\}$ such that the line $\overline{xy}$ intersects two of the four dual edges. As can be easily checked, the corresponding contributions come with different signs and cancel each other. For example, the term $a^*_x\, T_b(x,y)\, a_y$ for $x = z + \hat{e}_2$ and $y = z + \hat{e}_1$ appears with negative sign in $J^z_{1,L}(b)$, because $x_1 - y_1 < 0$, and with positive sign in $J^z_{2,L}(b)$, because $x_2 - y_2 > 0$. □

As a simple consequence, in a stationary state it follows that the net current into any set $Z \subset \Lambda_L$ is zero. This is an important ingredient for the following proof.

Corollary 7 (Current conservation). *For any $Z \subset \Lambda_L$ and stationary state*[1] *ρ, current conservation holds*

$$\sum_{(k,z)\in\partial_{\Lambda_L} Z} (-1)^{\delta_{z\in Z}}\ \mathrm{tr}\big(\rho\, J^z_{k,L}(b)\big) = 0. \tag{23}$$

Here $\delta_{z\in Z} = 1$ if $z \in Z$ and 0 otherwise, takes the role of the normal vector in the continuous analogue.

Proof. Taking the expectation value of (22) and summing over $z \in Z$ yields

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\ \mathrm{tr}\Big(\mathrm{e}^{-\mathrm{i}H_L(b)\,t}\,\rho\,\mathrm{e}^{\mathrm{i}H_L(b)\,t}\,\mathcal{N}_Z\Big)\Big|_{t=0} = \sum_{\substack{k\in\{1,2\}\\ z\in Z}} \mathrm{tr}\big(\rho\, J^z_{k,L}(b)\big) - \mathrm{tr}\big(\rho\, J^{z-\hat{\mathrm{e}}_k}_{k,L}(b)\big),$$

due to stationarity of ρ and cyclicity of the trace. In the sum, the positive term for $z \in Z$ is cancelled by the negative one for $z + \hat{\mathrm{e}}_k \in Z$ and only (23) remains. □

3.1.3 Proof of Proposition 5

Proposition 5 (Bloch's Theorem). *Let ζ^{bulk} be a null sequence and $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonian of the form* (4). *There exists $C_{\mathrm{B}} > 0$ such that the following holds: If $(H_L(b))_{L\in\mathbb{N}}$ satisfies local indistinguishability of the Gibbs state in the bulk at (β,μ,b) with ζ^{bulk}-decay in the sense of Definition 1, then*

$$\big|\mathrm{tr}\big(\rho_L(\beta,\mu,b)\, J^z_{k,L}(b)\big)\big| \le C_{\mathrm{B}}\, \zeta^{\mathrm{bulk}}\Big(\big[\mathrm{dist}(z,\mathbb{Z}^2\setminus\Lambda_L) - D - R\big]_+\Big). \tag{16}$$

Proof. We do the proof for $k = 1$, i.e. currents in x_1-direction, since the case $k = 2$ is analogous. Let $d > D + R$. By current conservation for the rectangle $\Lambda_L \cap \{x_1 \ge 0\}$, whose boundary in Λ_L is simply the vertical line at $x_1 = 0$ (see edge set (a) in Figure 5 for $m = 0$), we find

$$\begin{aligned}
0 &= \sum_{n=0}^{2L} \mathrm{tr}\big(\rho_L(\beta,\mu,b)\, J^{(0,n)}_{1,L}(b)\big)\\
&= \sum_{n=0}^{d-1} \mathrm{tr}\big(\rho_L(\beta,\mu,b)\, J^{(0,n)}_{1,L}(b)\big) + \mathrm{tr}\big(\rho_L(\beta,\mu,b)\, J^{(0,2L-n)}_{1,L}(b)\big)\\
&\quad + \sum_{n=d}^{2L-d} \mathrm{tr}\big(\rho_L(\beta,\mu,b)\, J^{(0,n)}_{1}(b)\big) - \mathrm{tr}\big(\rho\big[H_L(b)|_{B^{(0,n)}(\ell)}\big](\beta,\mu)\, J^{(0,n)}_{1}(b)\big)\\
&\quad + \sum_{n=d}^{2L-d} \mathrm{tr}\big(\rho\big[H_L(b)|_{B^{(0,n)}(\ell)}\big](\beta,\mu)\, J^{(0,n)}_{1}(b)\big)
\end{aligned}$$

[1] The statement actually holds for all bounded operators $\rho \in \mathcal{A}_{\Lambda_L}$ but is naturally interesting for states or similar objects (see section 3.4).

$$=: A_1 + A_2 + A_3,$$

where $B^x(\ell) := \{y \in \mathbb{Z}^2 \,|\, \mathrm{dist}(x,y) \leq \ell\}$ is the ball in $\mathbb{Z}^2$, $R < \ell < d - D$ and we replaced $J^x_{1,L}(b) = J^x_1$ in the last two sums in view of the previous remark that the L-dependence of $J^z_{k,L}(b)$ only stems from missing hopping terms near the boundary. Due to their structure, $J^{(m,n)}_{1,L}(b)$ are bounded operators with norm bound $\|J^{(m,n)}_{1,L}(b)\| < C_J$ uniform for all m and n. Hence, also $\big|\mathrm{tr}\big(\rho_L(\beta,\mu,b)\, J^{(m,n)}_{1,L}(b)\big)\big| < C_J$ and $|A_1|$ is bounded by $2\,d\,C_J$. For the second sum A_2 notice that $J^{(m,n)}_{1,L}(b) \in \mathscr{A}_{B^{(m,n)}(R)}$. Thus, by using the local indistinguishability of the Gibbs state, see Definition 1, we have

$$|A_2| \leq \sum_{n=d}^{2L-d} C_J\, \zeta^{\mathrm{bulk}}(\ell - R) = (2(L-d)+1)\, C_J\, \zeta^{\mathrm{bulk}}(\ell - R).$$

And by the translation invariance of the Gibbs state, namely (21), we get

$$(2\,(L-d)+1)\ \mathrm{tr}\Big(\rho\big[H_L(b)|_{B^{(0,L)}(\ell)}\big](\beta,\mu)\, J^{(0,L)}_1(b)\Big) = A_3 = -(A_1 + A_2),$$

which together with the previous bound on A_1 and A_2, implies

$$\Big|\mathrm{tr}\Big(\rho\big[H_L(b)|_{B^{(0,L)}(\ell)}\big](\beta,\mu)\, J^{(0,L)}_1(b)\Big)\Big| \leq \frac{C_J\, d}{L-d} + C_J\, \zeta^{\mathrm{bulk}}(\ell - R).$$

As the left-hand side is actually independent of L, it is bounded by the infimum $C_J\, \zeta^{\mathrm{bulk}}(\ell - R)$ of the right-hand side.

We can now prove the same for every $J^{(m,n)}_{1,L}(b)$ with $z = (m,n)$ in a finite box by using (21) together with

$$\rho\big[H_L(b)|_{B^{(m,n)}(\ell)}\big](\beta,\mu)\, J^{(m,n)}_1(b) = \rho\big[H_{L'}(b)|_{B^{(m,n)}(\ell)}\big](\beta,\mu)\, J^{(m,n)}_1(b)$$

if $\mathrm{dist}\big((m,n), \mathbb{Z}^2 \setminus \Lambda_L\big) > R + D$. Indeed, we have

$$\begin{aligned}
&\Big|\mathrm{tr}\big(\rho_L(\beta,\mu,b)\, J^{(m,n)}_{1,L}(b)\big)\Big| \\
&\quad \leq \Big|\mathrm{tr}\big(\rho_L(\beta,\mu,b)\, J^{(m,n)}_1(b)\big) - \mathrm{tr}\big(\rho\big[H_L(b)|_{B^{(m,n)}(\ell)}\big](\beta,\mu)\, J^{(m,n)}_1(b)\big)\Big| \\
&\qquad + \Big|\mathrm{tr}\big(\rho\big[H_L(b)|_{B^{(0,L)}(\ell)}\big](\beta,\mu)\, J^{(0,L)}_1(b)\big)\Big| \\
&\quad \leq 2\, C_J\, \zeta^{\mathrm{bulk}}(\ell - R)
\end{aligned}$$

by using local indistinguishability and the bound for $J^{(0,L)}_{1,L}(b)$. We can now choose $\ell = \mathrm{dist}\big((m,n), \mathbb{Z}^2 \setminus \Lambda_L\big) - D$, which proves the statements for $\mathrm{dist}\big((m,n), \mathbb{Z}^2 \setminus \Lambda_L\big) > R + D$. And since we argued above that the LHS of (16) is in any case bounded for all $z \in \Lambda_L$, the full statement follows with $C_{\mathrm{B}} = C_J\ \max\{\zeta^{\mathrm{bulk}}(0)^{-1}, 2\}$. □

3.2 Proof of Theorem I

We split the proof of Theorem I in two parts: we briefly discuss the localization of the edge current first and then the equality between the magnetization and the current.

3.2.1 Localization of the current

The localization of the edge current near the edge is a straightforward consequence of Bloch's Theorem.

Proposition 8. *Let* $\zeta^{\text{bulk}} \in \ell^1$, $(H_L(b))_{L\in\mathbb{N}}$ *be a family of Hamiltonians of the form* (4) *and* C_{B} *be the constant from Bloch's Theorem. If* $(H_L(b))_{L\in\mathbb{N}}$ *satisfies local indistinguishability of the Gibbs state in the bulk at* (β,μ,b) *with* ζ^{bulk}*-decay in the sense of Definition 1, then for all* $L \geq d \geq D+R$,

$$\left|I_L^{d\,\text{edge}}(\beta,\mu,b) - I_L^{\text{edge}}(\beta,\mu,b)\right| \leq C_{\text{B}} \sum_{n=d-D-R}^{\infty} \zeta^{\text{bulk}}(n).$$

Proof. For the proof we just apply Bloch's Theorem to obtain

$$\left|I_L^{d\,\text{edge}} - I_L^{\text{edge}}\right| \leq \sum_{n=d}^{L-1} \left|\operatorname{tr}\Big(\rho_L(\beta,\mu,b)\, J_1^{(0,n)}(b)\Big)\right| \leq C_{\text{B}} \sum_{n=d}^{\infty} \zeta^{\text{bulk}}(n-D-R). \qquad \square$$

3.2.2 Magnetization in finite systems

Let us compute the magnetic derivative of the Hamiltonian. Notice that for every fixed L the Hamiltonian $H_L(b)$ is a smooth function of b in the operator norm topology. Taking into account (5), we find that

$$\begin{aligned} H_L'(b) \;:=\; \frac{\partial}{\partial b} H_L(b) &= \sum_{x\in\Lambda_L} \sum_{y\in B_L^x(R)} \tfrac{\mathrm{i}}{2}\,(x_2+y_2)\,(x_1-y_1)\, a_x^*\, T_b(x,y)\, a_y \\ &= \sum_{m=-L}^{L-1} \sum_{n=0}^{2L} n\, J_{1,L}^{(m,n)}(b). \end{aligned} \tag{24}$$

Where $J_{1,L}^{(m,n)}(b)$ has been defined in (8). To see the last equality, we compare the coefficients in front of $a_x^*\, T_b(x,y)\, a_y$. Without loss of generality we only consider $x_1 < y_1$ and $x_2 \leq y_2$. By (8), the coefficient on the right-hand side is

$$-\mathrm{i} \sum_{m=x_1}^{y_1-1} \tfrac{1}{2} \Bigg(\sum_{\substack{n\in\mathbb{N}:\\ \overline{xy}\,\cap\, e_{1,(m,n)}\neq\emptyset}} n \;+ \sum_{\substack{n\in\mathbb{N}:\\ \overline{xy}\,\cap\, \overline{e_{1,(m,n)}}\neq\emptyset}} n \Bigg).$$

By point symmetry around the center $(x+y)/2$ of $\overline{xy}$, whenever $\overline{xy} \cap \overline{e_{1,(m,n)}} \neq \emptyset$ for $m = x_1 + k$ and $n \in \mathbb{N}$, then also $\overline{xy} \cap \overline{e_{1,(m',n')}} \neq \emptyset$ for $m' = y_1 - k - 1$ and $n' = y_2 - (n - x_2)$. The same holds for the edges without the endpoints. Thus, the coefficient equals

$$-\mathrm{i}\,\frac{1}{2}\sum_{k=0}^{y_1-x_1-1}\frac{1}{2}\Bigg(\sum_{\substack{n\in\mathbb{N}:\\ \overline{xy}\,\cap\, e_{1,(x_1+k,n)}\neq\emptyset}} n+\big(y_2-(n-x_2)\big)+\sum_{\substack{n\in\mathbb{N}:\\ \overline{xy}\,\cap\, \overline{e_{1,(x_1+k,n)}}\neq\emptyset}} n+\big(y_2-(n-x_2)\big)\Bigg)=\frac{\mathrm{i}}{2}\,(x_1-y_1)\,(y_2+x_2).$$

This is exactly the coefficient on the left-hand side of (24).

By using Duhamel's formula and (24), we can explicitly compute the magnetization as follows

$$\begin{aligned} m_L(\beta,\mu,b) &= -(2L+1)^{-2}\,\beta^{-1}\,\frac{\partial}{\partial b}\ln\Big(\mathrm{tr}\big(\mathrm{e}^{-\beta(H_L(b)-\mu\,\mathcal{N}_L)}\big)\Big)\\ &= -\frac{1}{(2L+1)^2\,\beta\,\mathcal{Z}_L(\beta,\mu,b)}\,\mathrm{tr}\Big(\frac{\partial}{\partial b}\,\mathrm{e}^{-\beta(H_L(b)-\mu\,\mathcal{N}_L)}\Big)\\ &= \frac{1}{(2L+1)^2\,\mathcal{Z}_L(\beta,\mu,b)}\,\mathrm{tr}\Big(\int_0^1 \mathrm{d}s\,\mathrm{e}^{-s\beta(H_L(b)-\mu\,\mathcal{N}_L)}\,H_L'(b)\,\mathrm{e}^{-(1-s)\beta(H_L(b)-\mu\,\mathcal{N}_L)}\Big)\\ &= \frac{1}{(2L+1)^2}\,\mathrm{tr}\big(\rho_L(\beta,\mu,b)\,H_L'(b)\big)\\ &= \frac{1}{(2L+1)^2}\sum_{m=-L}^{L-1}\sum_{n=0}^{2L} n\,\mathrm{tr}\big(\rho_L(\beta,\mu,b)\,J_{1,L}^{(m,n)}(b)\big). \end{aligned} \tag{25}$$

Proposition 9. *Let $\zeta^{\mathrm{bulk}} \in \ell^1$ and $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonians of the form* (4) *satisfying local indistinguishability of the Gibbs state in the bulk at (β,μ,b) with ζ^{bulk}-decay in the sense of Definition 1. Then*

$$\Big|m_L(\beta,\mu,b) - I_L^{d\,\mathrm{edge}}(\beta,\mu,b)\Big| \le C_{\mathrm{B}}\left(\frac{4\,d^2}{L}+\sum_{n=d-R-D}^{\infty}\zeta^{\mathrm{bulk}}(n)\right)$$

for all $d > D + R$ with C_{B} from Bloch's Theorem.

Proof. We decompose the sum from (25) into five regions (see Figure 4)

$$\begin{aligned} &\frac{1}{(2L+1)^2}\sum_{m=-L}^{L-1}\sum_{n=0}^{2L} n\,\mathrm{tr}\big(\rho_L(\beta,\mu,b)\,J_{1,L}^{(m,n)}(b)\big)\\ &\quad= A^{\mathrm{bulk}} + A^{\mathrm{left}} + A^{\mathrm{right}} + A^{\mathrm{bottom}} + A^{\mathrm{top}}. \end{aligned} \tag{26}$$

We will show that A^{bulk}, A^{left} and A^{right} are small and that A^{top} and A^{bottom} resemble the edge current. Abbreviating $j_{k,L}^{(m,n)} := \mathrm{tr}\big(\rho_L(\beta,\mu,b)\,J_{k,L}^{(m,n)}(b)\big)$, the individual

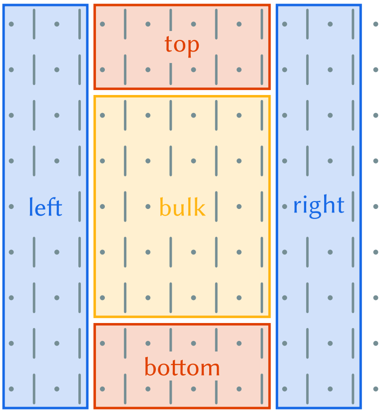

Figure 4. Depicted are Λ_L for $L = 4$ (dots) and the corresponding dual edges $e_{1,(m,n)}$ (lines) as defined above (8). They were used to define the current $J_{1,L}^{(m,n)}(b)$ in (8). Since there are less vertical dual edges than vertices in Λ_L, the right most vertices have no corresponding dual edge. The coloured boxes group the dual edges into the five groups of the decomposition (26) for $d = 2$.

contributions are

$$A^{\text{bulk}} := \frac{1}{(2L+1)^2} \sum_{m=-L+d}^{L-d-1} \sum_{n=d}^{2L-d} n\, j_{1,L}^{(m,n)},$$

$$A^{\text{left}} := \frac{1}{(2L+1)^2} \sum_{m=-L}^{-L+d-1} \sum_{n=0}^{2L} n\, j_{1,L}^{(m,n)},$$

$$A^{\text{right}} := \frac{1}{(2L+1)^2} \sum_{m=L-d}^{L-1} \sum_{n=0}^{2L} n\, j_{1,L}^{(m,n)},$$

$$A^{\text{bottom}} := \frac{1}{(2L+1)^2} \sum_{m=-L+d}^{L-d-1} \sum_{n=0}^{d-1} n\, j_{1,L}^{(m,n)}, \quad \text{and}$$

$$A^{\text{top}} := \frac{1}{(2L+1)^2} \sum_{m=-L+d}^{L-d-1} \sum_{n=2L-d+1}^{2L} n\, j_{1,L}^{(m,n)},$$

where A^{bulk} is the bulk part, A^{left} and A^{right} are the sum over the left and right edge regions, A^{bottom} and A^{top} are the sum over the upper and lower edge regions. Note that $d > R + D$.

By Bloch's Theorem, $|j_{1,L}^{z}| \leq \zeta(\operatorname{dist}(z, \mathbb{Z}^2 \setminus \Lambda_L))$ with $\zeta(r) := C_{\mathrm{B}}\, \zeta^{\text{bulk}}(r - D - R)$. Thus, we can bound the inner part A^{bulk} by summing over shells $\{z \in \Lambda_L \mid \operatorname{dist}(z, \mathbb{Z}^2 \setminus \Lambda_L) = r\}$

$$\left|A^{\text{bulk}}\right| \leq \frac{1}{2L+1} \sum_{m=-L+d}^{L-d-1} \sum_{n=d}^{2L-d} \left|j_{1,L}^{(m,n)}\right| \leq \frac{L}{2L+1} \sum_{r=d}^{L} \zeta(r). \tag{27}$$

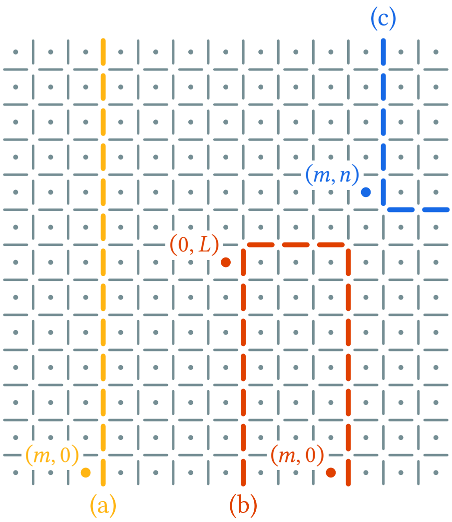

Figure 5. The picture shows Λ_L for $L = 6$ and all dual edges. A few connected sets of dual edges are highlighted. Together with the boundary all these sets are rectangles, and we use current conservation through the edges in the proofs. More precisely,

(a) $\sum_{n=0}^{2L} j_{1,L}^{(m,n)} = 0$,

(b) $\sum_{n=0}^{L} \left(-j_{1,L}^{(0,n)} + j_{1,L}^{(m,n)}\right) + \sum_{m'=1}^{m} j_{2,L}^{(m',L)} = 0$,

(c) $\sum_{n'=n}^{2L} j_{1,L}^{(m,n')} + \sum_{m'=m+1}^{L} j_{2,L}^{(m',n-1)} = 0$.

Then, let us consider the right and left edge regions, A^{left} and A^{right}. We write only the case of A^{right}, since the other one is analogous. First, using a discrete version of partial integration, we find

$$\sum_{m=L-d}^{L-1} \sum_{n=0}^{2L} n\, j_{1,L}^{(m,n)} = \sum_{m=L-d}^{L-1} \sum_{n=1}^{2L} \sum_{n'=n}^{2L} j_{1,L}^{(m,n')}$$

and by current conservation (see edge set (c) in Figure 5) and $\left|j_{2,L}^{(m',n)}\right| < C_{\mathrm{B}}$,

$$\left| \sum_{n'=n}^{2L} j_{1,L}^{(m,n')} \right| = \left| \sum_{m'=m+1}^{L} j_{2,L}^{(m',n-1)} \right| \leq (L-m)\, C_{\mathrm{B}}.$$

Hence,

$$\left|A^{\text{right}}\right| = \left| \frac{1}{(2L+1)^2} \sum_{m=L-d}^{L-1} \sum_{n=0}^{2L} n\, j_{1,L}^{(m,n)} \right| \leq \frac{d^2\, C_{\mathrm{B}}}{2L+1}.$$

Let us now consider the contributions coming from the lower and upper edge regions. The contribution A^{bottom} is small due to the prefactor $n \approx 0$:

$$\left| \frac{1}{(2L+1)^2} \sum_{m=-L+d}^{L-d-1} \sum_{n=0}^{d-1} n\, j_{1,L}^{(m,n)} \right| \leq \frac{1}{(2L+1)^2}\, 2(L-d)\, d\, d\, C_{\mathrm{B}} \leq \frac{d^2\, C_{\mathrm{B}}}{2L+1}.$$

We have now proven that all contributions up to A^{top} are small. In A^{top} we replace n with $2L$ making again an error of

$$\left|\frac{1}{(2L+1)^2}\sum_{m=-L+d}^{L-d-1}\sum_{n=2L-d+1}^{2L}(n-2L)\,j_{1,L}^{(m,n)}\right|\leq\frac{d^2\,C_{\mathrm{B}}}{2L+1}.$$

It now seems, that the magnetization only stems from the top part. That, however, is only due to our choice of the gauge. Indeed, the top and bottom contributions would equal in a gauge with Peierls phase $\mathrm{e}^{\mathrm{i}\frac{x_2+y_2-2L}{2}b(x_1-y_1)}$ in (5) – which corresponds to exactly our Hamiltonian but on boxes $[-L,L]^2$ centered around the origin.

However, also with our natural choice of gauge, we can rewrite the magnetization in terms of the currents near the bottom edge using current conservation (see edge set (a) in Figure 5) and vanishing of the currents in the bulk:

$$\begin{aligned}&\left|\frac{1}{(2L+1)^2}\sum_{m=-L+d}^{L-d-1}\sum_{n=2L-d+1}^{2L}2L\,j_{1,L}^{(m,n)}-\frac{1}{(2L+1)^2}\sum_{m=-L+d}^{L-d-1}\sum_{n=0}^{d-1}2L\left(-j_{1,L}^{(m,n)}\right)\right|\\&\leq\frac{1}{2\,(2L+1)}\sum_{m=-L+d}^{L-d-1}\sum_{n=d}^{2L-d}\left|j_{1,L}^{(m,n)}\right|.\end{aligned}$$

The last step here follows exactly from the calculation (27) for the inner part.

So far, we have proven that

$$\left|m_L(\beta,\mu,b)-\frac{-2L}{(2L+1)^2}\sum_{m=-L+d}^{L-d-1}\sum_{n=0}^{d-1}j_{1,L}^{(m,n)}\right|\leq\frac{4\,d^2\,C_{\mathrm{B}}}{2L+1}+\sum_{n=d}^{\infty}\zeta(n). \tag{28}$$

It remains to show that all contributions equal the one at $m=0$. Using current conservation once more (now for edge set (b) in Figure 5), for any $m>0$

$$\left|\sum_{n=0}^{d-1}\left(j_{1,L}^{(m,n)}-j_{1,L}^{(0,n)}\right)\right|\leq\sum_{n=d}^{L}\left(\left|j_{1,L}^{(m,n)}\right|+\left|j_{1,L}^{(0,n)}\right|\right)+\sum_{m'=1}^{m}\left|j_{2,L}^{(m',L)}\right|, \tag{29}$$

and similarly for $m<0$. Thus, by using (29) and the analogue estimate for $m>0$, we can estimate the error that we make by replacing $j_{1,L}^{(m,n)}$ with $j_{1,L}^{(0,n)}$ in (28). The error has three terms coming from the estimate (29) and each of them can be bounded using again the strategy from (27):

$$\frac{2L}{(2L+1)^2}\sum_{m=-L+d}^{L-d-1}\sum_{n=d}^{L}\left|j_{1,L}^{(m,n)}\right| \le \sum_{n=d}^{\infty}\zeta(n),$$

$$\frac{2L}{(2L+1)^2}\sum_{m=-L+d}^{L-d-1}\sum_{n=d}^{L}\left|j_{1,L}^{(0,n)}\right| \le \sum_{n=d}^{\infty}\zeta(n),$$

$$\frac{2L}{(2L+1)^2}\sum_{m=-L+d}^{L-d-1}\sum_{m'=1}^{m}\left|j_{1,L}^{(m',L)}\right| \le \sum_{m'=1}^{L-d-1}\zeta(L-m') \le \sum_{n=d}^{\infty}\zeta(n).$$

As a final step, we bound

$$\left|\left(\tfrac{2L\,2(L-d)}{(2L+1)^2}-1\right)\sum_{n=0}^{d-1} j_{1,L}^{(0,n)}\right| \le \frac{4\,d^2\,C_{\mathrm{B}}}{2L+1}$$

Hence, altogether we have proven that

$$\left|m_L(\beta,\mu,b) - I_L^{d\ \mathrm{edge}}(\beta,\mu,b)\right| \le C_{\mathrm{B}}\left(\frac{8\,d^2}{2L+1} + \sum_{n=d-R-D}^{\infty}\zeta^{\mathrm{bulk}}(n)\right). \qquad \square$$

We conclude this section with the proof of Theorem I.

Proof of Theorem I. Combining Propositions 8 and 9, gives

$$\left|m_L(\beta,\mu,b) - I_L^{\mathrm{edge}}(\beta,\mu,b)\right| \le C_{\mathrm{B}}\left(\frac{4\,d^2}{L} + 2\sum_{n=d-R-D}^{\infty}\zeta^{\mathrm{bulk}}(n)\right)$$

for all $d \ge R + D$. Taking the minimum over d gives (11) with

$$\theta(L) = 2\,C_{\mathrm{B}} \min_{R+D\le d\le L}\left(\frac{2\,d^2}{L} + \sum_{n=d-R-D}^{\infty}\zeta^{\mathrm{bulk}}(n)\right), \tag{30}$$

which tends to zero for $L \to \infty$ (choose, e.g., $d = L^{1/4}$). The bound (12) follows directly from Proposition 8. $\square$

3.3 Proof of Theorem II

Theorem II is basically the thermodynamic limit version of Theorem I plus some additional remarks. We split the proof into three parts: in the first part we show the localization of the edge current in the thermodynamic limit, namely Proposition 10, in the second part we prove the equality with the thermodynamic limit of the magnetization, that is Proposition 11, and in the last part we show the independence of the edge interaction, see Proposition 12.

Let us start with an important remark. In section 2.3 we already discussed that local indistinguishability everywhere with ζ^{edge}-decay implies existence of a unique thermodynamic limit state $\rho_\infty(\beta,\mu,b)$. More precisely, for finite $X \subset \mathbb{Z}^2_+$ and $A \in \mathcal{A}_X$ the convergence is given by

$$\left|\rho_\infty(\beta,\mu,b)(A) - \operatorname{tr}\big(\rho_L(\beta,\mu,b)\,A\big)\right| \le \|A\|\, g\big(|X|\big)\, \zeta^{\text{edge}}\big(\operatorname{dist}(X, \mathbb{Z}^2_+ \setminus \Lambda_L)\big) \qquad (31)$$

due to local indistinguishability.

3.3.1 Localization of the current in the thermodynamic limit

We first note that Bloch's Theorem and the localization of the edge current carry over to the thermodynamic limit.

Proposition 10. *Let $\zeta^{\text{bulk}} \in \ell^1$, ζ^{edge} a null sequence and $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonians of the form* (4) *satisfying local indistinguishability of the Gibbs state at (β,μ,b) with ζ^{bulk}-decay in the bulk and ζ^{edge}-decay everywhere, in the sense of Definition 1. Then the unique thermodynamic limit state $\rho_\infty(\beta,\mu,b)$ on $\mathcal{A}_{\text{loc}}$ satisfies Bloch's theorem, that is, for all $z \in \mathbb{Z}^2_+$*

$$\left|\rho_\infty(\beta,\mu,b)\big(J^z_1(b)\big)\right| \le 2\, C_{\text{B}}\, \zeta^{\text{bulk}}\Big(\big[\operatorname{dist}(z, \mathbb{Z}^2 \setminus \mathbb{Z}^2_+) - D - R\big]_+\Big),$$

with C_{B} from Bloch's Theorem. Hence, also the infinite edge current, defined in (15)*, is localized near the boundary in the sense that*

$$\left|I^{d\,\text{edge}}(\beta,\mu,b) - I^{\text{edge}}(\beta,\mu,b)\right| \le 2\, C_{\text{B}} \sum_{n=d-D-R}^{\infty} \zeta^{\text{bulk}}(n)$$

for all $d \ge D + R$. Moreover, for all $L \ge R$ and $d \le L$

$$\left|I^{d\,\text{edge}}_L(\beta,\mu,b) - I^{d\,\text{edge}}(\beta,\mu,b)\right| \le d\, \zeta^{\text{edge}}(L - R).$$

Proof. Combining (31) with Bloch's Theorem we find for all $z \in \mathbb{Z}^2_+$ and L such that $z \in \Lambda_{L-R}$ (remember that then $J^z_1(b) = J^z_{1,L}(b)$) that

$$\begin{aligned}
&\left|\rho_\infty(\beta,\mu,b)\big(J^z_1(b)\big)\right| \\
&\quad \le \left|\rho_\infty(\beta,\mu,b)\big(J^z_{1,L}(b)\big) - \operatorname{tr}\big(\rho_L(\beta,\mu,b)\, J^z_{1,L}(b)\big)\right| + \left|\operatorname{tr}\big(\rho_L(\beta,\mu,b)\, J^z_{1,L}(b)\big)\right| \\
&\quad \le \zeta^{\text{edge}}\big(\operatorname{dist}(z, \mathbb{Z}^2_+ \setminus \Lambda_L) - R\big) + C_{\text{B}}\, \zeta^{\text{bulk}}\Big(\big[\operatorname{dist}(z, \mathbb{Z}^2 \setminus \Lambda_L) - D - R\big]_+\Big),
\end{aligned}$$

where the first quantity converges to zero, and $\operatorname{dist}(z, \mathbb{Z}^2 \setminus \Lambda_L) = \operatorname{dist}(z, \mathbb{Z}^2 \setminus \mathbb{Z}^2_+)$ for L large enough. The localization of the current now follows exactly as in Proposition 8. For the convergence in L, we estimate

$$\begin{aligned}\left|I_L^{d\,\text{edge}}(\beta,\mu,b) - I^{d\,\text{edge}}(\beta,\mu,b)\right| &\le \sum_{n=0}^{d-1} \left|\operatorname{tr}\Big(\rho_L(\beta,\mu,b)\, J_1^{(0,n)}(b)\Big) - \rho_\infty(\beta,\mu,b)\Big(J_1^{(0,n)}(b)\Big)\right| \\ &\le d\,\zeta^{\text{edge}}(L-R). \qquad \square\end{aligned}$$

3.3.2 Magnetization in the thermodynamic limit

Proposition 11. *Let $\zeta^{\text{bulk}} \in \ell^1$ and ζ^{edge} tend to zero. If $(H_L(b))_{L\in\mathbb{N}}$ is a family of Hamiltonians of the form* (4) *satisfying local indistinguishability of the Gibbs state at (β,μ,b) with ζ^{bulk}-decay in the bulk and ζ^{edge}-decay everywhere, in the sense of Definition 1, then*

$$\lim_{L\to\infty} m_L(\beta,\mu,b) = I^{\text{edge}}(\beta,\mu,b).$$

Proof. We combine Propositions 9 and 10 to obtain

$$\left|m_L(\beta,\mu,b) - I^{\text{edge}}(\beta,\mu,b)\right| \le C_{\mathrm{B}}\left(\frac{4\,d^2}{L} + \sum_{n=d-R-D}^{\infty} \zeta^{\text{bulk}}(n)\right) + d\,\zeta^{\text{edge}}(L-R), \tag{32}$$

which will tend to zero as $L\to\infty$ for an appropriate choice of d, depending on L. $\square$

3.3.3 Independence of the specific edge Hamiltonian T_b^{edge} and Φ^{edge}

In order to complete the proof of Theorem II it remains to show that the thermodynamic limit of the magnetization and the current does not depend on the specific edge contributions T_b^{edge} and Φ^{edge}. We prove this by showing that the edge currents of two finite systems with different edge Hamiltonians are asymptotically equivalent and then taking the thermodynamic limit.

Taking into account the differentiability result with respect to μ of Theorem III, this also implies the insensitivity to boundary perturbations of the μ-derivatives in the thermodynamic limit.

Proposition 12. *Let $\zeta^{\text{bulk}} \in \ell^1$, ζ^{edge} tend to zero, and let $(H_L(b))_{L\in\mathbb{N}}$ and $(\tilde{H}_L(b))_{L\in\mathbb{N}}$ be two families of Hamiltonians of the form* (4) *that only differ in the definition of the edge contributions T_b^{edge}, Φ^{edge} and $\tilde{T}_b^{\text{edge}}$, $\tilde{\Phi}^{\text{edge}}$. Assume that both satisfy local indistinguishability of the Gibbs state at (β,μ,b) with ζ^{bulk}-decay in the bulk and ζ^{edge}-decay everywhere in the sense of Definition 1. Then, denoting the quantities of $\tilde{H}_L(b)$ with an additional tilde,*

$$I^{\text{edge}}(\beta,\mu,b) = \tilde{I}^{\text{edge}}(\beta,\mu,b).$$

This result shows, that a local perturbation near the edge might change, where exactly the edge current flows, but it does not change the total net current near the boundary. In this sense, the edge current $I^{\text{edge}}(\beta,\mu,b)$ is dictated by the bulk.

Proof. The idea of the proof is to use current conservation and Bloch's Theorem to show that the current along the lower and upper edge are the same up to a sign (for both interactions). Using local indistinguishability we can then prove that the currents along the upper edge are almost the same for both edge interactions. Thus, also the currents along the lower edge almost agree.

As in section 3.2.2, we abbreviate $j_{k,L}^{(m,n)} := \operatorname{tr}\big(\rho_L(\beta,\mu,b)\, J_{k,L}^{(m,n)}(b)\big)$. And for better readability, we drop the arguments (β,μ,b) in the following. The current flowing in the upper edge of the box can be written as

$$I_L^{d\,\text{up. edge}} := \sum_{n=0}^{d-1} j_{1,L}^{(0,2L-n)}.$$

By current conservation (see edge set (a) in Figure 5) and Bloch's Theorem we find

$$\left|I_L^{d\,\text{edge}} + I_L^{d\,\text{up. edge}}\right| \le \sum_{n=d}^{2L-d} \left|j_{1,L}^{(0,n)}\right| \le 2\,C_{\text{B}} \sum_{n=d-R-D}^{\infty} \zeta^{\text{bulk}}(n)$$

for all $L \ge d \ge D+R$. Then, using local indistinguishability for $\Lambda' = B_L^{(0,2L-n)}(2L-n-D)$, with $0<n<d-1$, which is chosen such that T_b^{edge} and Φ^{edge} vanish on Λ', we find

$$\left|j_{1,L}^{(0,2L-n)} - \operatorname{tr}\Big(\rho\big[H_L\big|_{\Lambda'}\big]\, J_{1,L}^{(0,2L-n)}\Big)\right| \le \zeta^{\text{edge}}(2L-n-R-D).$$

And the same also holds for the corresponding properties of $\tilde{H}_L$ denoted by an additional tilde. Due to the choice of Λ', $H_L\big|_{\Lambda'} = \tilde{H}_L\big|_{\Lambda'}$, and we also have $J_{1,L}^{(0,2L-n)} = \tilde{J}_{1,L}^{(0,2L-n)}$ for $2L-n > D+R$. Hence,

$$\begin{aligned}
\left|I_L^{d\,\text{edge}} - \tilde{I}_L^{d\,\text{edge}}\right| &\le \left|I_L^{d\,\text{up. edge}} - \tilde{I}_L^{d\,\text{up. edge}}\right| + \left|I_L^{d\,\text{edge}} + I_L^{d\,\text{up. edge}}\right| + \left|\tilde{I}_L^{d\,\text{up. edge}} + \tilde{I}_L^{d\,\text{edge}}\right| \\
&\le \sum_{n=0}^{d-1}\left|j_{1,L}^{(0,2L-n)} - \tilde{j}_{1,L}^{(0,2L-n)}\right| + 4\,C_{\text{B}} \sum_{n=d-R-D}^{\infty} \zeta^{\text{bulk}}(n) \\
&\le 2\,d\,\zeta^{\text{edge}}(2L+1-d-D-R) + 4\,C_{\text{B}} \sum_{n=d-R-D}^{\infty} \zeta^{\text{bulk}}(n). \qquad (33)
\end{aligned}$$

The statement now follows from Proposition 10. □

3.4 Proof of Theorem III

In this section, we discuss the μ-derivative of the edge current. Defining

$$\mathscr{F}_L(\beta,\mu,b) := \partial_\mu\, \rho_L(\beta,\mu,b) = \beta\left(\mathcal{N}_L - \langle \mathcal{N}_L\rangle_{\rho_L(\beta,\mu,b)}\right)\rho_L(\beta,\mu,b), \tag{34}$$

we find

$$\partial_\mu \operatorname{tr}\big(\rho_L(\beta,\mu,b)\, J^z_{1,L}(b)\big) = \operatorname{tr}\big(\mathscr{F}_L(\beta,\mu,b)\, J^z_{1,L}(b)\big) \tag{35}$$

because $J^z_{1,L}(b)$ does not depend on μ.

To prove Theorem III we use the same strategy of the proofs of Theorems I and II, whose main ingredient is Bloch's Theorem. Therefore, we start by proving a similar statement to Bloch's Theorem for the "state" $\mathscr{F}_L(\beta,\mu,b)$. A key point in the proof of Bloch's Theorem is the use of local indistinguishability for the Gibbs state. However, since it is not clear whether $\mathscr{F}_L(\beta,\mu,b)$ satisfies local indistinguishability, we need to adapt the strategy a bit.

Let us start by showing that $\mathscr{F}_L(\beta,\mu,b)$ has a thermodynamic limit. As a starting point, note that local indistinguishability everywhere implies decay of correlations, a property needed in the following.

Lemma 13. *Let $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonians of the form* (4) *satisfying local indistinguishability of the Gibbs state everywhere at (β,μ,b) with ζ^{edge}-decay in the sense of Definition 1.*

Then the Gibbs state satisfies decay of correlations, that is for all $X,Y\subset\Lambda_L$, with $\operatorname{dist}(X,Y)>R$, $A\in\mathscr{A}_X$ and $B\in\mathscr{A}_Y$ the covariance

$$\operatorname{Cov}_{\rho_L(\beta,\mu,b)}(A,B) := \operatorname{tr}\big(\rho_L(\beta,\mu,b)\,A\,B\big) - \operatorname{tr}\big(\rho_L(\beta,\mu,b)\,A\big)\operatorname{tr}\big(\rho_L(\beta,\mu,b)\,B\big)$$

is bounded by

$$\left|\operatorname{Cov}_{\rho_L(\beta,\mu,b)}(A,B)\right| \le 3\,g\big(|X|+|Y|\big)\,\|A\|\,\|B\|\,\zeta^{\mathrm{edge}}\left(\left\lfloor\frac{\operatorname{dist}(X,Y)-R-1}{2}\right\rfloor\right).$$

Proof. Let $\ell = \big\lfloor\big(\operatorname{dist}(X,Y)-R-1\big)/2\big\rfloor$ and $\Lambda' = X_\ell\cup Y_\ell$, with the ℓ-neighbourhoods $X_\ell := \big\{z\in\Lambda_L\,\big|\,\operatorname{dist}(z,X)\le\ell\big\}$ and Y_ℓ. For better readability we again drop the arguments (β,μ,b). Then, for all $Q\in\mathscr{A}_{X\cup Y}$

$$\Big|\operatorname{tr}\big(\rho_L\,Q\big) - \operatorname{tr}\big(\rho\big[H_L|_{\Lambda'}\big]\,Q\big)\Big| \le g\big(|X|+|Y|\big)\,\|Q\|\,\zeta^{\mathrm{edge}}(\ell).$$

Moreover, $\rho\big[H_L|_{\Lambda'}\big] = \rho\big[H_L|_{X_\ell}\big]\otimes\rho\big[H_L|_{Y_\ell}\big]$, since $\operatorname{dist}(X_\ell,Y_\ell)>R$, and thus $\operatorname{tr}\big(\rho\big[H_L|_{\Lambda'}\big]\,A\,B\big) = \operatorname{tr}\big(\rho\big[H_L|_{\Lambda'}\big]\,A\big)\operatorname{tr}\big(\rho\big[H_L|_{\Lambda'}\big]\,B\big)$. The statement then follows from the triangle inequality. □

This allows us to prove convergence in the thermodynamic limit of the expectation value in the state $\mathscr{F}_L(\beta,\mu,b)$ of observables that may be supported near the boundary. We denote by

$$B^x_+(\ell) := \big\{y\in\mathbb{Z}^2_+ \mid \operatorname{dist}(x,y)\le\ell\big\}$$

the ball in $\mathbb{Z}_+^2$, similar to how we denoted with $B_L^x(\ell)$ and $B^x(\ell)$ the balls in Λ_L and $\mathbb{Z}^2$, respectively.

Lemma 14. *Let $n \mapsto n^2\,\zeta^{\mathrm{edge}}(n) \in \ell^1$ and $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonians of the form* (4)*. There exists a non-increasing $\xi \in \ell^1$, explicitly given in* (38)*, such that the following holds: If $(H_L(b))_{L\in\mathbb{N}}$ satisfies local indistinguishability of the Gibbs state everywhere at (β,μ,b) with ζ^{edge}-decay in the sense of Definition 1, then, for all $x \in \mathbb{Z}_+^2$, $A \in \mathcal{A}_{B_+^x(R)}$ and $L' < L$ such that $B_+^x(R) \subset \Lambda_{L'}$, it holds that*

$$\Big|\mathrm{tr}\big(\mathscr{F}_L(\beta,\mu,b)\,A\big) - \mathrm{tr}\big(\mathscr{F}_{L'}(\beta,\mu,b)\,A\big)\Big| \le \beta\,\|A\|\,\xi\big(\mathrm{dist}(x,\Lambda_L\setminus\Lambda_{L'}) - R\big). \tag{36}$$

Proof. For the proof let $X = B_+^x(R)$, $\ell = \lfloor \mathrm{dist}(X,\Lambda_L\setminus\Lambda_{L'})/3\rfloor$ and $Z = B_+^x(R+2\ell) \subset \Lambda_{L'}$. The main idea of the proof is to write the number operators as sums of single-site operators. Then we can use local indistinguishability (for the single-site operators supported in Z) and decay of correlations (for the single-site operators supported outside Z) of the Gibbs state to conclude the result. For better readability we drop the arguments (β,μ,b). Then

$$\begin{aligned}
&\Big(\mathrm{tr}\big(\mathscr{F}_L\,A\big) - \mathrm{tr}\big(\mathscr{F}_{L'}\,A\big)\Big)/\beta \\
&\quad= \sum_{z\in Z} \mathrm{tr}\big(\rho_L\,A\,\mathcal{N}_z\big) - \mathrm{tr}\big(\rho_{L'}\,A\,\mathcal{N}_z\big) \\
&\qquad - \sum_{z\in Z} \mathrm{tr}\big(\rho_L\,A\big)\,\mathrm{tr}\big(\rho_L\,\mathcal{N}_z\big) - \mathrm{tr}\big(\rho_L\,A\big)\,\mathrm{tr}\big(\rho_{L'}\,\mathcal{N}_z\big) \\
&\qquad + \sum_{z\in Z} \mathrm{tr}\big(\rho_{L'}\,A\big)\,\mathrm{tr}\big(\rho_{L'}\,\mathcal{N}_z\big) - \mathrm{tr}\big(\rho_L\,A\big)\,\mathrm{tr}\big(\rho_{L'}\,\mathcal{N}_z\big) \\
&\qquad + \sum_{z\in\Lambda_L\setminus Z} \mathrm{tr}\big(\rho_L\,A\,\mathcal{N}_z\big) - \mathrm{tr}\big(\rho_L\,A\big)\,\mathrm{tr}\big(\rho_L\,\mathcal{N}_z\big) \\
&\qquad - \sum_{z\in\Lambda_{L'}\setminus Z} \mathrm{tr}\big(\rho_{L'}\,A\,\mathcal{N}_z\big) - \mathrm{tr}\big(\rho_{L'}\,A\big)\,\mathrm{tr}\big(\rho_{L'}\,\mathcal{N}_z\big).
\end{aligned} \tag{37}$$

Since A has bounded support, the first three sums can now be bounded using local indistinguishability of the Gibbs state because $\rho_{L'} = \rho\big[H_L|_{\Lambda_{L'}}\big]$, while the last two can be bounded using decay of correlations in the form of Lemma 13. Using $\|\mathcal{N}_z\| = 1$ and $g(|X|+1) \le 1$, we find

$$\Big|\mathrm{tr}\big(\mathscr{F}_L\,A\big) - \mathrm{tr}\big(\mathscr{F}_{L'}\,A\big)\Big| \Big/ \big(\beta\,\|A\|\big)$$

$$\begin{aligned}&\leq 3 \sum_{z \in Z} \zeta^{\mathrm{edge}}\big(\operatorname{dist}(X \cup \{z\}, \Lambda_L \setminus \Lambda_{L'})\big)\\&\quad + \sum_{z \in \Lambda_L \setminus Z} 3\, \zeta^{\mathrm{edge}}\Big(\Big\lfloor \frac{\operatorname{dist}(X,z) - R - 1}{2} \Big\rfloor\Big)\\&\quad + \sum_{z \in \Lambda_{L'} \setminus Z} 3\, \zeta^{\mathrm{edge}}\Big(\Big\lfloor \frac{\operatorname{dist}(X,z) - R - 1}{2} \Big\rfloor\Big).\end{aligned}$$

Summing over shells as in (27), the sum over $z \in Z$ is bounded by

$$\sum_{z \in Z} \zeta^{\mathrm{edge}}\big(\operatorname{dist}(X \cup \{z\}, \Lambda_L \setminus \Lambda_{L'})\big) \leq 4\,(R + 2\ell) \sum_{n=\ell}^{2\ell} \zeta^{\mathrm{edge}}(n) + (2R+1)^2\, \zeta^{\mathrm{edge}}(2\ell),$$

and each of the other sums can be bounded by

$$12 \sum_{n=2\ell+1}^{\infty} (n + R)\, \zeta^{\mathrm{edge}}\big(\lfloor (n - R - 1)/2 \rfloor\big) \leq 12 \sum_{m=\ell}^{\infty} (2m + 2 + R)\, \zeta^{\mathrm{edge}}\big(m - \lfloor (R-1)/2 \rfloor\big).$$

Defining

$$\xi(r) = 3 \Big(12 \sum_{m=\ell}^{\infty} (2m + 2 + R)\, \zeta^{\mathrm{edge}}\big(m - \lfloor (R-1)/2 \rfloor\big) + (2R+1)^2\, \zeta^{\mathrm{edge}}(2\ell) \Big)\Big|_{\ell = \lfloor r/3 \rfloor}, \quad (38)$$

which is in ℓ^1 since $n \mapsto n^2\, \zeta^{\mathrm{edge}}(n) \in \ell^1$, yields the claim. □

We will use this Lemma later to prove differentiability of I^{edge}. But first, we adjust the proof to show that also the μ-derivative of the local bond currents decay in the bulk, i.e. a Bloch's Theorem for $\mathscr{F}_L(\beta, \mu, b)$. However, the two simplest approaches to adjust the proof of Bloch's Theorem do not work. At first, one might want to use local indistinguishability of the Gibbs state and view $\mathscr{N}_L - \langle \mathscr{N}_L \rangle$ as part of the operator in (7). This fails, since $\mathscr{N}_L - \langle \mathscr{N}_L \rangle$ is supported over all Λ_L and not bounded uniformly in L. Alternatively, one could try to use local indistinguishability of the "state" $\mathscr{F}_L(\beta, \mu, b)$ and follow the proof of the Bloch's Theorem afterwards. And while Lemma 14 already looks similar to local indistinguishability, it can only be used to compare $\partial_\mu\, \rho_L(\beta, \mu, b)$ with $\mathscr{F}\big[H_L(b)|_{\Lambda'}\big](\beta, \mu) := \partial_\mu\, \rho\big[H_L(b)|_{\Lambda'}\big](\beta, \mu)$ for $\Lambda' = \Lambda_{L'}$. But within the proof of Bloch's Theorem we also need to compare to more general sets $\Lambda' \subset \Lambda_L$, in particular to sets that include an edge of Λ_L (not only the lower one).

One might hope to adapt the proof of Lemma 14 to prove local indistinguishability. However, that needs decay of correlations in $\mathscr{F}\big[H_L(b)|_{\Lambda'}\big](\beta, \mu)$ which we could not prove. It would follow from local indistinguishability everywhere of $\rho\big[H_L(b)|_{\Lambda'}\big](\beta, \mu)$, which might be a viable assumption since the Hamiltonian is translation invariant. To avoid these more general assumptions, we take a different approach for which we introduce

$$\mathscr{F}^Z\big[H_L(b)\big|_{\Lambda'}\big](\beta,\mu) := \beta\left(\mathscr{N}_Z - \langle\mathscr{N}_Z\rangle_{\rho[H_L(b)|_{\Lambda'}](\beta,\mu)}\right)\rho\big[H_L(b)\big|_{\Lambda'}\big](\beta,\mu) \tag{39}$$

for $Z \subset \Lambda' \subset \Lambda_L$. These "states" can be handled easily since the problematic sum is not present. We prove the following statement which is similar to local indistinguishability in the state $\mathscr{F}_L(\beta,\mu,b)$.

Lemma 15. *Let $n \mapsto n^2\,\zeta^{\text{edge}}(n) \in \ell^1$, $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonians of the form* (4) *and $\xi \in \ell^1$ as in Lemma 14. If $(H_L(b))_{L\in\mathbb{N}}$ satisfies local indistinguishability of the Gibbs state everywhere at (β,μ,b) with ζ^{edge}-decay in the sense of Definition 1, then, for all $x \in \Lambda_L$, $\ell \in \mathbb{N}_0$, $Z = B^x_L(R+2\ell)$, $\Lambda' = B^x_L(R+3\ell)$ and $A \in \mathscr{A}_{B^x_L(R)}$*

$$\Big|\mathrm{tr}\big(\mathscr{F}_L(\beta,\mu,b)\,A\big) - \mathrm{tr}\big(\mathscr{F}^Z\big[H_L(b)\big|_{\Lambda'}\big](\beta,\mu)\,A\big)\Big| \le \beta\,\|A\|\,\xi(3\ell).$$

Proof. The proof is exactly the same as the proof of Lemma 14 but with $\rho_{L'}$ replaced by $\rho\big[H_L\big|_{\Lambda'}\big]$ and without the last sum in (37). Hence, we only need decay of correlations in ρ_L, which is provided by Lemma 13. □

Lemma 15 allows us to prove an analogous statement to Bloch's Theorem for the "state" $\mathscr{F}_L(\beta,\mu,b)$, namely that $\big|\mathrm{tr}\big(\mathscr{F}_L(\beta,\mu,b)\,J^{(m,n)}_{k,L}(b)\big)\big|$ decays rapidly inside the bulk. In particular, we also prove boundedness of the μ-derivative of local bond currents, which is not clear a priori because $\mathscr{N}_L - \langle\mathscr{N}_L\rangle$ is not uniformly bounded in L.

Proposition 16. *Let $n \mapsto n^2\,\zeta^{\text{edge}}(n) \in \ell^1$, $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonians of the form* (4) *and $\xi \in \ell^1$ as in Lemma 14. There exists $C_{\mathscr{F}} > 0$ such that the following holds: If $(H_L(b))_{L\in\mathbb{N}}$ satisfies local indistinguishability of the Gibbs state everywhere at (β,μ,b) with ζ^{edge}-decay in the sense of Definition 1, then*

$$\big|\mathrm{tr}\big(\mathscr{F}_L(\beta,\mu,b)\,J^z_{k,L}(b)\big)\big| \le C_{\mathscr{F}}\,\beta\,\xi\Big(\big[\mathrm{dist}(z,\mathbb{Z}^2\setminus\Lambda_L) - D - R\big]_+\Big).$$

Proof. For better readability, we drop the arguments (β,μ,b) in the proof. And as in the proof of Bloch's Theorem in section 3.1.3 we only do the proof for $k=1$, i.e. currents in x_1-direction.

We first prove uniform boundedness of the left-hand side. Recall that $J^z_{k,L} \in \mathscr{A}_{B^z_L(R)}$ and $\|J^z_{k,L}\| \le C_J$. For $\ell = 0$, Lemma 15 with $B^z_L(R) = Z = \Lambda'$ yields

$$\begin{aligned}\big|\mathrm{tr}\big(\mathscr{F}_L\,J^z_{k,L}\big)\big| &\le \Big|\mathrm{tr}\big(\mathscr{F}_L\,J^z_{k,L}\big) - \mathrm{tr}\big(\mathscr{F}^Z\big[H_L\big|_Z\big]\,J^z_{k,L}\big)\Big| \\ &\quad + \big|\mathrm{tr}\big(\mathscr{F}^Z\big[H_L\big|_Z\big]\,J^z_{k,L}\big)\big| \\ &\le \beta\,C_J\,\xi(0) + \beta\,(2R+1)^2\,C_J \\ &=: \beta\,C_0.\end{aligned}$$

To prove decay in the bulk, note that $\mathscr{F}_L$ is stationary due to stationarity of the Gibbs state and since H_L commutes with the full number operator $\mathscr{N}_L$ appearing in the definition of $\mathscr{F}_L$. Hence, we have current conservation in the "state" $\mathscr{F}_L$, see Corollary 7, which we apply for the rectangle $\Lambda_L \cap \{x_1 \geq 0\}$. Then we choose $d > D + R$ and $0 < \ell < (d - D - R)/3$ to obtain

$$\begin{aligned} 0 &= \sum_{n=0}^{2L} \mathrm{tr}\big(\mathscr{F}_L\, J_{1,L}^{(0,n)}\big) \\ &= \sum_{n=0}^{d-1} \mathrm{tr}\big(\mathscr{F}_L\, J_{1,L}^{(0,n)}\big) + \mathrm{tr}\big(\mathscr{F}_L\, J_{1,L}^{(0,2L-n)}\big) \\ &\quad + \sum_{n=d}^{2L-d} \mathrm{tr}\big(\mathscr{F}_L\, J_1^{(0,n)}\big) - \mathrm{tr}\big(\mathscr{F}^{B^{(0,n)}(R+2\ell)}\big[H_L|_{B^{(0,n)}(R+3\ell)}\big]\, J_1^{(0,n)}\big) \\ &\quad + \sum_{n=d}^{2L-d} \mathrm{tr}\big(\mathscr{F}^{B^{(0,n)}(R+2\ell)}\big[H_L|_{B^{(0,n)}(R+3\ell)}\big]\, J_1^{(0,n)}\big), \end{aligned}$$

which is the same decomposition as in the proof of Bloch's Theorem. The first sum is bounded by $2\, d\, \beta\, C_0$. With Lemma 15, the second sum is bounded by

$$\big(2(L-d)+1\big)\, \beta\, C_J\, \xi(3\ell).$$

All terms in the third sum equal the one at $(0, L)$ due to translation invariance of the Hamiltonian. Thus,

$$\Big|\mathrm{tr}\big(\mathscr{F}^{B^{(0,L)}(R+2\ell)}\big[H_L|_{B^{(0,L)}(R+3\ell)}\big]\, J_1^{(0,L)}\big)\Big| \leq \frac{C_0\, \beta\, d}{L-d} + \beta\, C_J\, \xi(3\ell) \to \beta\, C_J\, \xi(3\ell)$$

for $L \to \infty$.

We can now relate this result back to any point $x \in \Lambda_L$. Choosing $d = \mathrm{dist}(x, \mathbb{Z}^2 \setminus \Lambda_L)$, $\ell = \lfloor (d - D - R)/3 \rfloor$ we find

$$\begin{aligned} \Big|\mathrm{tr}\big(\mathscr{F}_L\, J_1^x\big)\Big| &\leq \Big|\mathrm{tr}\big(\mathscr{F}_L\, J_1^x\big) - \mathrm{tr}\big(\mathscr{F}^{B^x(R+2\ell)}\big[H_L|_{B^x(R+3\ell)}\big]\, J_1^x\big)\Big| \\ &\quad + \Big|\mathrm{tr}\big(\mathscr{F}^{B^{(0,L)}(R+2\ell)}\big[H_L|_{B^{(0,L)}(R+3\ell)}\big]\, J_1^{(0,L)}\big)\Big| \\ &\leq 2\, \beta\, C_J\, \xi(3\ell), \end{aligned}$$

which proves the claim with $C_{\mathscr{F}} = \max\big\{C_0\, \xi(0)^{-1}, 2\, C_J\big\}$, using $\xi(k) = \xi\big(3\, \lfloor k/3 \rfloor\big)$ due to the explicit form (38). □

We can now prove differentiability of I^{edge}.

Proposition 17. *Let $n \mapsto n^2\,\zeta^{\mathrm{edge}}(n) \in \ell^1$, $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonians of the form* (4). *If $(H_L(b))_{L\in\mathbb{N}}$ satisfies local indistinguishability of the Gibbs state everywhere with ζ^{edge}-decay at (β,μ,b) for all μ in an open interval M, then $I^{\mathrm{edge}}(\beta,\mu,b)$, $I^{d\,\mathrm{edge}}(\beta,\mu,b)$ are differentiable functions of $\mu \in M$. Moreover, the derivative of $I^{\mathrm{edge}}(\beta,\mu,b)$ is localized near the boundary, with the decay estimate*

$$\left|\partial_\mu I^{\mathrm{edge}}(\beta,\mu,b) - \partial_\mu I^{d\,\mathrm{edge}}(\beta,\mu,b)\right| \le \beta\, C_{\mathscr{F}} \sum_{n=d-R-D}^{\infty} \xi(n), \tag{40}$$

where $C_{\mathscr{F}}$, ξ are as in Proposition 16.

Proof. In finite volume, it is clear that $I_L^{d\,\mathrm{edge}}$ is differentiable in μ. We will thus use local indistinguishability, respectively Proposition 16, to take the limit $L \to \infty$ and then $d \to \infty$ uniformly in μ.

We abbreviate $j_{1,L}^{(0,n)}(\mu) = \operatorname{tr}\big(\rho_L(\beta,\mu,b)\, J_{1,L}^{(0,n)}(b)\big)$ and $j_1^{(0,n)}(\mu) = \rho_\infty(\beta,\mu,b)\big(J_1^{(0,n)}(b)\big)$ as before (the first will only be used for $n \le L$ so that $J_{1,L}^{(0,n)}(b) = J_1^{(0,n)}(b)$). First, by Lemma 14,

$$\partial_\mu\, j_{1,L}^{(0,n)}(\mu) = \operatorname{tr}\Big(\mathscr{F}_L(\beta,\mu,b)\, J_1^{(0,n)}(b)\Big)$$

is a Cauchy sequence in L. Denoting its limit by $c_1^{(0,n)}(\mu)$, we have

$$\left|c_1^{(0,n)}(\mu) - \partial_\mu\, j_{1,L}^{(0,n)}(\mu)\right| \le \beta\, C_J\, \xi(L-R).$$

By local indistinguishability we know that $j_{1,L}^{(0,n)}(\mu)$ converges to its limit $j_1^{(0,n)}(\mu)$, and this convergence is uniform since ζ^{edge} is independent of μ. Hence, by completeness of $C^1(M)$, $j_1^{(0,n)}(\mu)$ is differentiable in μ and $\partial_\mu\, j_1^{(0,n)}(\mu) = c_1^{(0,n)}(\mu)$. Thus, $I^{d\,\mathrm{edge}}(\beta,\mu,b)$ is differentiable and satisfies

$$\left|\partial_\mu I^{d\,\mathrm{edge}}(\beta,\mu,b) - \partial_\mu I_L^{d\,\mathrm{edge}}(\beta,\mu,b)\right| \le \beta\, d\, C_J\, \xi(L-R).$$

To take $d \to \infty$, observe that by Proposition 16

$$\begin{aligned}|\partial_\mu\, j_1^{(0,n)}(\mu)| &\le \left|\partial_\mu\, j_1^{(0,n)}(\mu) - \partial_\mu\, j_{1,L}^{(0,n)}(\mu)\right| + \left|\partial_\mu\, j_{1,L}^{(0,n)}(\mu)\right| \\ &\le \beta\, C_J\, \xi(L-R) + C_{\mathscr{F}}\, \beta\, \xi(n-D-R),\end{aligned}$$

which converges to $\beta\, C_{\mathscr{F}}\, \xi(n-D-R)$ as $L \to \infty$. Summation over n shows that $I^{\mathrm{edge}}(\beta,\mu,b)$ is differentiable and satisfies (40). □

Note that Proposition 17 together with the equality $m(\beta,\mu,b) = I^{\mathrm{edge}}(\beta,\mu,b)$ from Theorem II proves Theorem III. Additionally, we provide a bound on the difference of $\partial_\mu\, m_L(\beta,\mu,b)$ and $\partial_\mu I_L^{\mathrm{edge}}(\beta,\mu,b)$ in finite volume, which is analogous to the bound from Theorem I.

Proposition 18. *Let $n \mapsto n^2 \zeta^{\mathrm{edge}}(n) \in \ell^1$, $(H_L(b))_{L\in\mathbb{N}}$ be a family of Hamiltonians of the form* (4). *There exists a null sequence η so that the following holds: If $(H_L(b))_{L\in\mathbb{N}}$ satisfies local indistinguishability of the Gibbs state everywhere with ζ^{edge}-decay at (β, μ, b) for μ in some open interval, then*

$$\left|\partial_\mu m_L(\beta,\mu,b) - \partial_\mu I_L^{\mathrm{edge}}(\beta,\mu,b)\right| \le \beta\,\eta(L).$$

Proof. Differentiating (25) by using (35), we obtain

$$\partial_\mu m_L(\beta,\mu,b) = \frac{1}{(2L+1)^2} \sum_{m=-L}^{L-1} \sum_{n=0}^{2L} n \operatorname{tr}\bigl(\mathscr{F}_L(\beta,\mu,b)\, J_{1,L}^{(m,n)}(b)\bigr).$$

Following the proof of Proposition 9, where we only used Bloch's Theorem and current conservation in $\rho_L(\beta,\mu,b)$, whose analogues here are Proposition 16 and current conservation in $\mathscr{F}_L(\beta,\mu,b)$ (the latter holds, because $\mathscr{F}_L(\beta,\mu,b)$ is stationary), we obtain

$$\left|\partial_\mu m_L(\beta,\mu,b) - \partial_\mu I_L^{d\,\mathrm{edge}}(\beta,\mu,b)\right| \le \beta\, C_{\mathscr{F}} \left(\frac{4\,d^2}{L} + \sum_{n=d-R-D}^{\infty} \xi(n)\right).$$

Combining this with Proposition 16 to compare $\partial_\mu I_L^{d\,\mathrm{edge}}(\beta,\mu,b)$ and $\partial_\mu I_L^{\mathrm{edge}}(\beta,\mu,b)$, proves the claim with

$$\eta(L) = 2\,C_{\mathscr{F}} \min_{D+R\le d\le L} \left(\frac{2\,d^2}{L} + \sum_{n=d-R-D}^{\infty} \xi(n)\right). \qquad \square$$

3.5 Proof of Theorem IV

We first prove that the limit of the finite volume pressures $p_L(\mu,\beta,b)$ defined in (10) exists and is independent of the edge Hamiltonian. Therefore, let

$$\rho_L^{\mathrm{bulk}}(\beta,\mu,b) = \rho_{B_L}\bigl[H_L^{\mathrm{bulk}}(b)\bigr](\beta,\mu), \quad \text{and} \quad \mathscr{Z}_L^{\mathrm{bulk}} = \mathscr{Z}_{B_L}\bigl[H_L^{\mathrm{bulk}}(b)\bigr](\beta,\mu) \tag{41}$$

be the Gibbs state and partition function of the bulk Hamiltonian on the centered boxes B_L. For the statement we introduce

$$C_H^{\mathrm{bulk/edge}} := \sup_{x\in\mathbb{Z}^2} \left(2 \sum_{y\in\mathbb{Z}^2} \bigl\|a_x^*\, T_b^{\mathrm{bulk/edge}}(x,y)\, a_y\bigr\| + \sum_{\substack{X\subset\mathbb{Z}^2:\\ x\in X}} \bigl\|\Phi^{\mathrm{bulk/edge}}(X)\bigr\|\right) + \mu, \tag{42}$$

which bounds the norm of all hoppings and interactions which include a particular site.

Proposition 19. *Let* $\big(H_L(b)\big)_{L\in\mathbb{N}}$ *be a family of Hamiltonians of the form* (4) *and let* $H_L^{\mathrm{bulk}}(b)$ *be the corresponding bulk Hamiltonian defined in* (17). *Then*

$$\left| p_L(\beta,\mu,b) - \frac{P\big(\rho_L^{\mathrm{bulk}}(\beta,\mu,b)\big)}{(2L+1)^2} \right| \le \frac{C_H^{\mathrm{edge}}\, D}{2L+1} \tag{43}$$

for all $\beta > 0$, $\mu, b \in \mathbb{R}$. *Moreover, the thermodynamic limit* $p(\beta,\mu,b) := \lim_{L\to\infty} p_L(\beta,\mu,b)$ *of the pressure exists and*

$$\left| p(\beta,\mu,b) - \frac{P\big(\rho_L^{\mathrm{bulk}}(\beta,\mu,b)\big)}{(2L+1)^2} \right| \le \frac{4\,R\,C_H^{\mathrm{bulk}}}{2L+1}. \tag{44}$$

The proof is based on [11, section 9.2], where the convergence for translation invariant interactions is discussed. Instead, here we have a bulk part which is invariant under magnetic translations and an additional edge contribution.

Proof. Within the proof we fix β, μ and denote $P(H) = \beta^{-1}\ \ln \mathrm{tr}\big(\mathrm{e}^{-\beta H}\big)$ for self-adjoint operators H such that $(2L+1)^2\, p_L(\beta,\mu,b) = P\big(\rho_L(\beta,\mu,b)\big) = P\big(H_L(b) - \mu\,\mathcal{N}_L\big)$, i.e. we write the pressure of the Gibbs state by just specifying the exponent.

We begin with the important observation, that the pressure is continuous and bounded in the Hamiltonian, i.e. $\big|P(H_1) - P(H_2)\big| \le \|H_1 - H_2\|$ and $\big|P(H_1)\big| \le \|H_1\|$, for all self-adjoint H_1 and H_2. To see this, consider self-adjoint A_1 and A_2 and $A(\lambda) := \lambda\, A_1 + (1-\lambda)\, A_2$. Then

$$\begin{aligned} \big|\ln \mathrm{tr}(\mathrm{e}^{-A_1}) - \ln \mathrm{tr}(\mathrm{e}^{-A_2})\big| &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \mathrm{tr}\big(\mathrm{e}^{-A(\lambda)}\big)\,\mathrm{d}\lambda \right| \\ &= \left| \int_0^1 \frac{\mathrm{tr}\big((A_1 - A_2)\,\mathrm{e}^{-A(\lambda)}\big)}{\mathrm{tr}\big(\mathrm{e}^{-A(\lambda)}\big)}\,\mathrm{d}\lambda \right| \le \|A_1 - A_2\|, \end{aligned}$$

where we used that $\mathrm{e}^{-A(\lambda)}\big/\,\mathrm{tr}\big(\mathrm{e}^{-A(\lambda)}\big)$ is a normalized state in the last step. The result for P follows immediately because the factors of β cancel.

We first show that the pressure in finite volume is almost independent of the edge terms. Therefore, let $W := \sum_{x,y\in\Lambda_L} a_x^*\, T_b^{\mathrm{edge}}(x,y)\, a_y + \sum_{X\subset\Lambda_L} \Phi^{\mathrm{edge}}(X)$ be the edge contribution to the Hamiltonian $H_L(b)$ such that $H_L(b) - W$ is the Hamiltonian from (4) without any additional edge terms. Then,

$$\big|P\big(H_L - \mu\,\mathcal{N}_L\big) - P\big(H_L(b) - W - \mu\,\mathcal{N}_L\big)\big| \le \|W\| \le C_H^{\mathrm{edge}}\, D\,(2L+1).$$

Thus, the per volume pressure $p_L(\beta,\mu,b)$ is independent of the edge terms up to an error $C_H^{\mathrm{edge}}\, D\big/(2L+1) \to 0$ as $L \to \infty$ and we only consider the Hamiltonian without edge terms in the following.

To shorten notation in the following we denote $H^{\mathrm{bulk}}_{\Lambda} := H^{\mathrm{bulk}}_{L}\big|_{\Lambda}$ for $\Lambda \subset B_L$. As discussed in section 3.1.1, $H^{\mathrm{bulk}}_{\Lambda+x} = U_{-x}(b)\, H^{\mathrm{bulk}}_{\Lambda}\, U^{*}_{-x}(b)$ and thus the partition function and the pressure of the respective states agree, $P(H^{\mathrm{bulk}}_{\Lambda+x}) = P(H^{\mathrm{bulk}}_{\Lambda})$. This in particular proves that $P\big(H_L(b) - \mu\, \mathcal{N}_{\Lambda_L}\big) = P\big(H^{\mathrm{bulk}}_{L}(b) - \mu\, \mathcal{N}_{B_L}\big)$, i.e. the pressure of the system on Λ_L without edge contribution exactly agrees with that on B_L. Together with the above estimate, (43) follows.

We now prove convergence of $p^{\mathrm{bulk}}_L := P\big(\rho^{\mathrm{bulk}}_L(\beta,\mu,b)\big)\big/(2L+1)^2$ as $L \to \infty$. For $L' < L$ one can fit $n = \big\lfloor \frac{2L+1}{2L'+1} \big\rfloor^2$ disjoint boxes $B_{L'} + x_j$ in B_L. By the estimate on the pressure, we find

$$\begin{aligned}\Big|P\big(H^{\mathrm{bulk}}_{B_L} - \mu\, \mathcal{N}_{B_L}\big) - P\big(H^{\mathrm{bulk}}_{\bigcup_j B_{L'}+x_j} - \mu\, \mathcal{N}_{\bigcup_j B_{L'}+x_j}\big)\Big| &\le C^{\mathrm{bulk}}_H \left(|B_L| - n\,|B_{L'}|\right)\\ &\le 2\, C^{\mathrm{bulk}}_H\, (2L'+1)\,(2L+1).\end{aligned}$$

In the second step we used $\big|\lfloor q \rfloor^2 - q^2\big| \le 2\,q$ for $q > 0$. In the next step we remove the hoppings and interactions between the individual boxes

$$\begin{aligned}&\Big|P\big(H^{\mathrm{bulk}}_{\bigcup_j B_{L'}+x_j} - \mu\, \mathcal{N}_{\bigcup_j B_{L'}+x_j}\big) - P\Big(\sum_{j=1}^{n} H^{\mathrm{bulk}}_{B_{L'}+x_j} - \mu\, \mathcal{N}_{B_{L'}+x_j}\Big)\Big|\\ &\qquad\le 4\, C^{\mathrm{bulk}}_H\, R\, (2L'+1)\, n\\ &\qquad\le 4\, C^{\mathrm{bulk}}_H\, R\, (2L+1)^2\, (2L'+1)^{-1}.\end{aligned}$$

Then, we observe that the trace of the non-interacting parts factors

$$\begin{aligned}P\Big(\sum_{j=1}^{n} H^{\mathrm{bulk}}_{B_{L'}+x_j} - \mu\, \mathcal{N}_{B_{L'}+x_j}\Big) &= \beta^{-1}\, \ln \prod_{j=1}^{n} \mathrm{tr}\big(\beta\, H^{\mathrm{bulk}}_{B_{L'}+x_j} - \beta\,\mu\, \mathcal{N}_{B_{L'}+x_j}\big)\\ &= \sum_{j=1}^{n} P\Big(H^{\mathrm{bulk}}_{B_{L'}+x_j} - \mu\, \mathcal{N}_{B_{L'}+x_j}\Big)\\ &= n\, P\big(H^{\mathrm{bulk}}_{B_{L'}} - \mu\, \mathcal{N}_{B_{L'}}\big),\end{aligned}$$

where we used that the pressures of the individual boxes all agree. As a last step we bound

$$\left|\left(n - \frac{|B_L|}{|B_{L'}|}\right) P\big(H^{\mathrm{bulk}}_{B_{L'}} - \mu\, \mathcal{N}_{B_{L'}}\big)\right| \le 2\, C^{\mathrm{bulk}}_H\, (2L'+1)\,(2L+1).$$

Using triangle inequality and dividing everything by $|B_L|$, we obtain an estimate for the per volume pressures

$$\left|p^{\mathrm{bulk}}_L - p^{\mathrm{bulk}}_{L'}\right| \le 4\, C^{\mathrm{bulk}}_H \left(\frac{2L'+1}{2L+1} + \frac{R}{2L'+1}\right). \tag{45}$$

Equation (45) shows that $\{p_L^{\mathrm{bulk}}\}_{L\in\mathbb{N}}$ is a Cauchy sequence and thus it is convergent. Together with (43) also $p_L(\beta,\mu,b)$ converges to the same limit $p(\beta,\mu,b)$. The convergence in (44) follows from (45) after taking the limit $L\to\infty$. □

Next we show that the limit of the pressures of the finite volume boxes agrees with the pressure of any infinite volume KMS state of the system without an edge.

Proposition 20. *Let* $\left(H_L^{\mathrm{bulk}}(b)\right)_{L\in\mathbb{N}}$ *be a family of Hamiltonians of the form* (17) *satisfying the assumptions from section 2.1, and let* τ^{bulk} *be the corresponding dynamics defined in* (18)*. For every* $(\tau^{\mathrm{bulk}},\beta)$*-KMS state* ω *the pressure per volume of the restriction of* ω *to* B_L *defined by* (19)*, satisfies*

$$\left|\frac{P\big(\rho_L^{\mathrm{bulk}}(\beta,\mu,b)\big)}{(2L+1)^2}-\frac{P\big(\omega|_{B_L}\big)}{(2L+1)^2}\right|\leq\frac{8\,C_H^{\mathrm{bulk}}\,R}{2L+1}.$$

Proof. We follow the ideas of [11, Proposition 12.1]. Let ω be a KMS state, denote its restriction $\omega_L:=\omega|_{B_L}$ and abbreviate $\rho_L^{\mathrm{bulk}}=\rho_L^{\mathrm{bulk}}(\beta,\mu,b)$. The difference

$$\beta\,P(\omega_L)-\beta\,P(\rho_L^{\mathrm{bulk}})=\mathrm{tr}(\omega_L\ln\omega_L)-\mathrm{tr}(\omega_L\ln\rho_L^{\mathrm{bulk}})$$

equals the relative entropy $S(\omega_L|\rho_L^{\mathrm{bulk}})\geq 0$. Since the relative entropy is monotone under restrictions (see [47, Theorem 6.2.33]), we have

$$S(\omega_L|\rho_L^{\mathrm{bulk}})\leq S(\omega|\rho)$$

for any extension ρ of ρ_L^{bulk} from $\mathscr{A}_{B_L}$ to $\mathscr{A}$. By [11, Theorem 7.5], ω satisfies the Gibbs condition and a natural choice for this extension is given by the perturbation of ω where all interactions between B_L and the rest of the system are deleted (compare [11, Corollary 7.8]). To be precise, let

$$W_L=\sum_{\substack{x,y\in\mathbb{Z}^2:\\ \{x,y\}\cap B_L\neq\emptyset,\\ \{x,y\}\cap\mathbb{Z}^2\setminus B_L\neq\emptyset}}a_x^*\,T^{\mathrm{bulk}}(x,y)\,a_y+\sum_{\substack{X\subset\mathbb{Z}^2:\\ X\cap B_L\neq\emptyset,\\ X\cap\mathbb{Z}^2\setminus B_L\neq\emptyset}}\Phi^{\mathrm{bulk}}(X)$$

be the surface interaction, which is an element of $\mathscr{A}$ with norm bounded by $4\,C_H^{\mathrm{bulk}}\,R\,(2L+1)$ since all interactions are of finite range R. The state corresponding to subtraction of W_L from the Hamiltonian can be expressed in the GNS representation $(\mathfrak{h}_\omega,\pi_\omega,\Omega)$ for ω by

$$\rho(A)=\Big\langle \mathrm{e}^{-\beta(H_\omega-\pi_\omega(W_L))/2}\,\mathrm{e}^{\beta H_\omega/2}\,\Omega,\pi_\omega(A)\,\mathrm{e}^{-\beta(H_\omega-\pi_\omega(W_L))/2}\,\mathrm{e}^{\beta H_\omega/2}\,\Omega\Big\rangle\Big/\mathscr{Z}^{W_L},$$

where H_ω is the generator of the dynamics induced by τ^{bulk} in $\mathfrak{h}_\omega$ and $\mathscr{Z}^{W_L}$ the normalizing factor (see [47, Theorem 5.4.4] and note that $\mathrm{e}^{\beta H_\omega/2}$ acts trivially on the cyclic vector Ω since ω is invariant). With this, we have (cf. [47, below Definition 6.2.29])

$$S(\omega|\rho)\leq-\omega(\beta\,W_L)+\rho(\beta\,W_L)\leq 2\,\beta\,\|W_L\|,$$

and thus

$$0 \le \frac{P(\rho_L^{\mathrm{bulk}}) + P(\omega_L)}{(2L+1)^2} \le \frac{8\, C_H^{\mathrm{bulk}}\, R}{2L+1}. \qquad \square$$

Now we are able to prove Theorem IV.

Proof of Theorem IV. The convergence of $p_L(\beta,\mu,b)$ and independence of boundary terms follows from Proposition 19. Equality of the pressure with the per volume pressure of any $(\tau^{\mathrm{bulk}},\beta)$-KMS state follows from Propositions 19 and 20.

Now assume that $\big(H_L(b)\big)_{L\in\mathbb{N}}$ satisfies local indistinguishability uniformly in b. By Theorem II, $m_L(\beta,\mu,b) = \partial_b\, p_L(\beta,\mu,b)$ converges to $m(\beta,\mu,b)$ as $L\to\infty$, and in view of the estimate (32) this convergence is uniform in b. Then, the convergence of the primitives $p_L(\beta,\mu,b)$, which converge pointwise by Proposition 19, must also be uniform and $p(\beta,\mu,b)$ is differentiable with derivative $m(\beta,\mu,b)$. □

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Publication P4

From decay of correlations to locality and stability of the Gibbs state

Ángela Capel [*][†] Massimo Moscolari [‡] Stefan Teufel [*]
Tom Wessel [*]

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Abstract

We show that whenever the Gibbs state of a quantum spin system satisfies decay of correlations, then it is stable, in the sense that local perturbations affect the Gibbs state only locally, and it satisfies local indistinguishability, i.e. it exhibits local insensitivity to system size. These implications hold in any dimension, require only locality of the Hamiltonian, and are based on Lieb-Robinson bounds and on a detailed analysis of the locality properties of the quantum belief propagation for Gibbs states.

To demonstrate the versatility of our approach, we explicitly apply our results to several physically relevant models in which the decay of correlations is either known to hold or is proved by us. These include Gibbs states of one-dimensional spin chains with polynomially decaying interactions at any temperature, and high-temperature Gibbs states of quantum spin systems with finite-range interactions in any dimension. We also prove exponential decay of correlations above a threshold temperature for Gibbs states of one-dimensional finite spin chains with translation-invariant and exponentially decaying interactions, and then apply our general results.

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[*] Fachbereich Mathematik, Universität Tübingen,
Auf der Morgenstelle 10, 72076 Tübingen, Germany
[†] Department of Applied Mathematics and Theoretical Physics, University of Cambridge
Wilberforce Road, Cambridge CB3 0WA, United Kingdom
[‡] Dipartimento di Matematica, Politecnico di Milano,
Piazza Leonardo da Vinci 32, 20133 Milano, Italy

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1 Introduction

One of the characteristic features of quantum many-body systems is the locality of their interactions. In the last decades, this property has been largely exploited in the characterization of their thermal (Gibbs) states and their ground states. In this work we are interested in Gibbs states and, in particular, in their locality and in their stability against local perturbations. Understanding how the locality of Hamiltonians of many-body systems translates to the locality of the Gibbs states is of crucial importance for the preparation and simulation of quantum states [201, 46, 60].

There are several ways to describe how local a quantum state is. Here we focus on three main concepts: *local indistinguishability*, the principle that *local perturbations perturb locally (LPPL)* and the well-known *decay of correlations*. The main purpose of this work is to show how these three locality properties are related to each other and to show that they are all equivalent under certain conditions. Let us first describe these concepts informally.

Consider an interacting quantum spin system defined on $\Lambda \Subset \mathbb{Z}^\nu$, $\nu \in \mathbb{N}$, with a Hamiltonian H_Λ that is a sum of local terms. Let ρ_β^Λ denote the Gibbs state of H_Λ at inverse temperature $\beta > 0$,

$$\rho_\beta^\Lambda := \mathrm{e}^{-\beta H_\Lambda} \big/ \operatorname{Tr}_\Lambda\big(\mathrm{e}^{-\beta H_\Lambda}\big).$$

For every $\Lambda' \subset \Lambda$, we define the local Gibbs state $\rho_\beta^{\Lambda'}$ to be the Gibbs state of the Hamiltonian consisting only of terms supported on Λ'. Then, let A be an observable supported in an inner region $X \subset \Lambda' \subset \Lambda$: local indistinguishability of the Gibbs state (see Definition 3) is a quantitative version of the statement that

$$\operatorname{Tr}_\Lambda\big(\rho_\beta^\Lambda A\big) \approx \operatorname{Tr}_{\Lambda'}\big(\rho_\beta^{\Lambda'} A\big).$$

The concept of local indistinguishability [45, 36] goes also under the name of locality of temperature [139]. Roughly speaking, local indistinguishability says that local observables cannot distinguish between the Gibbs state of the full Hamiltonian and the Gibbs state of its truncated version. On the other hand, the concept of LPPL, originally introduced in the context of gapped ground states [23, 16, 74, P1], is concerned with localized perturbations of a quantum system. Let V be a generic perturbation supported on a region $Y \subset \Lambda$ and let $\tilde{H}_\Lambda := H_\Lambda + V$. We denote by $\tilde{\rho}_\beta^\Lambda$ the Gibbs state associated to $\tilde{H}_\Lambda$. Let A be an observable supported on a set X *far* from Y, then LPPL (see Definition 2) is a quantitative version of the statement that

$$\mathrm{Tr}_\Lambda\left(\rho_\beta^\Lambda\, A\right) \approx \mathrm{Tr}_\Lambda\left(\tilde{\rho}_\beta^\Lambda\, A\right),$$

which means that the expectation values of local observables supported far from the perturbation are not influenced by the presence of the perturbation. The notion of LPPL thus concerns the Gibbs states of two different Hamiltonians, the perturbed and unperturbed one.

A condition similar to local indistinguishability named *Local Topological Quantum Order (LTQO)* has been previously considered in the literature [52, 51, 162] in the context of ground states of topologically ordered Hamiltonians. For frustration-free Hamiltonians, LTQO implies stability of the spectral gap. In a sense, it states that the effect of boundary conditions is exponentially suppressed in the bulk. And while generally difficult to prove, LTQO is known to hold for one-dimensional matrix product states with normal tensors [63] and for some projected entangled pair states with commuting parent Hamiltonians [196] in two dimensions.

Another standard way to measure the locality of a quantum state ρ is in terms of the decay of correlations in such a state. This amounts to study the behaviour of the covariance [139, 45, 9, 98, 185, 36]

$$\mathrm{Cov}_\rho(X;Y) := \sup_{\substack{A\in\mathcal{A}_X:\ \|A\|=1,\\ B\in\mathcal{A}_Y:\ \|B\|=1}} \left|\mathrm{Tr}_\Lambda(\rho\, A\, B) - \mathrm{Tr}_\Lambda(\rho\, A)\ \mathrm{Tr}_\Lambda(\rho\, B)\right|$$

with respect to the distance $d(X,Y)$ between the support of the observables.

Our main result shows that decay of correlations of the Gibbs state (see Definition 1) implies the stability of the Gibbs state against local perturbation, namely LPPL. The main ingredient in our proof is the so-called quantum belief propagation (QBP) introduced by Hastings [115], and with a recent increase of attention due to its various applications, see e.g. [134, 135, 80, 45, 131, 109, 8, 145, 194]. We show that quantum belief propagation, together with the well-known Lieb-Robinson bounds [154, 173, 174], allows exploiting the local structure of the Hamiltonian in order to prove LPPL.

Then, by using QBP again, we show that uniform decay of correlations implies uniform stability to local perturbations (*uniform LPPL*), which in turn implies local indistinguishability of the Gibbs state. Uniformity here means that a property does not only hold for a given Gibbs state or Hamiltonian, but also for the Gibbs states of the restrictions of the Hamiltonian to smaller domains.

Finally, to close the circle, we also show that local indistinguishability implies uniform decay of correlations. This is well-known in the case of finite-range Hamiltonians, but the proof for arbitrary local Hamiltonians requires more care and is again based on QBP.

Combining these three implications, our main result is that *local indistinguishability*, *uniform LPPL* and *uniform decay of correlations* are actually three equivalent ways to describe the locality properties of Gibbs states. See also Figure 1.

Main Result (informal). *Let H be the Hamiltonian of a finite interacting quantum spin system and $\beta > 0$. Then the following three properties are equivalent:*

(a) *The Gibbs state at inverse temperature β satisfies uniform decay of correlations (Definition 1).*

(b) *The Gibbs state at inverse temperature β satisfies uniform LPPL, i.e. the expectation values of observables are stable against local perturbations of the Hamiltonian (Definition 2).*

(c) *The Gibbs state at inverse temperature β satisfies local indistinguishability (Definition 3).*

In the main text, the three implications (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are split into the three Theorems 22, 29 and 31, respectively, and refined results for one-dimensional spin chains can be found in Theorems 36, 37 and 38.

We emphasize that we present a rigorous framework that allows relating LPPL and local indistinguishability to decay of correlations for very general interactions that have a finite interaction norm of the form (1). In this way, we are able to treat finite-range, short-range, and long-range interactions in a unified way. To guide the reader through our results, we always show, along with the general statements, the implications for short-range interactions as an immediate and demonstrative example. The core message of our work is that, whenever there are results on the decay of correlations for the Gibbs state, one can immediately obtain quantitative versions of LPPL and local indistinguishability by applying our theorems.

To illustrate this idea, in Section 3 we collect some physically relevant applications of our general results and prove LPPL and local indistinguishability for systems for which these properties have not been known before. First, we focus on one-dimensional quantum spin chains: assuming translation invariance, (uniform) exponential decay of correlations is known to hold at every temperature if the interactions are also finite-range [9, 36], and above a critical temperature in the short-range case [185] for the infinite-chain. We first extend the latter result to finite chains, and then, by applying our framework, we show LPPL and local indistinguishability. Quite recently it has been shown [136] that decay of correlations for short-range interactions holds at any positive temperature if one relaxes the decay to stretched exponential. With that input, we directly obtain LPPL and local indistinguishability with a stretched exponential decay rate for such systems. Furthermore, we also apply our results to spin chains with long-range interactions, where polynomial decay of correlations has recently been shown [136] at any positive temperature. Finally, in any dimension, for quantum spin systems with finite-range interactions at high enough temperature, it is known that Gibbs states satisfy (uniform) exponential decay of correlation, as proven in [139]. Thus, as a byproduct of our results, we also recover the known results [139] of uniform LPPL and local indistinguishability, both with exponential decay. Compared to the proof in [139] our methods have the advantage of relying only on decay of the covariance of the unperturbed Gibbs state, rather than on decay of the generalized covariance of the perturbed Gibbs states, an object about which much less is known.

On the side, we use QBP to recover a known result saying that the Gibbs state is stable (in trace norm) against small (in norm) perturbations. Moreover, we show that local expectation values are stable against perturbations by sums of small local terms in the underlying Hamiltonian, assuming some algebraic decay of correlations. A similar result was recently indicated in [187]. Furthermore, we note that in the recent work [212], the stability against sum of local terms perturbations has been analysed in connection to quantum simulations and quantum advantage. Related and in some respects stronger results for small perturbations of classical systems at low temperature were obtained in [66, 67].

1.1 Organization of the paper

In Section 2 we introduce the mathematical framework and provide a precise definition of decay of correlations, LPPL and local indistinguishability. After that, in Section 3 we conclude the introductory sections by showing several applications of our main results to Gibbs states of one-dimensional quantum spin chains and to high-temperature Gibbs states in any dimensions. Section 4 is devoted to the analysis of quantum belief propagation, which is the tool at the core of our proofs. The proofs of the results of Section 4 are given in Section 10. Section 5 contains the theorems, which allow concluding (uniform) LPPL from (uniform) decay of correlations. Then, in Section 6 and Section 7, we provide the equivalence shown in Figure 1. In Section 8 we show stability of local expectation values against perturbations in the underlying Hamiltonian. Finally, Section 9 is dedicated to the analysis of one-dimensional spin chains.

2 Mathematical setup and important concepts

Consider the regular lattice $\mathbb{Z}^\nu$, for fixed $\nu \in \mathbb{N}$, equipped with the ℓ^1-metric $d\colon \mathbb{Z}^\nu \times \mathbb{Z}^\nu \to \mathbb{N}$. We denote arbitrary subsets as $\Lambda \subset \mathbb{Z}^\nu$ (including equality) and finite subsets by $\Lambda' \Subset \Lambda$ (again including equality if Λ is finite). The cardinality of a set $\Lambda \Subset \mathbb{Z}^\nu$ is denoted by $|\Lambda|$. Given any two subsets $X, Y \subset \mathbb{Z}^\nu$ we denote by $d(X, Y)$ their distance with respect to the metric d. Likewise, we denote by $\operatorname{diam}(X) := \sup_{x,y\in X} d(x, y)$ the diameter of X. For a set $X \subset \Lambda$ and $r > 0$, we denote the r-neighbourhood

$$X_r := \left\{ x \in \Lambda \,\middle|\, d(x, X) \leq r \right\},$$

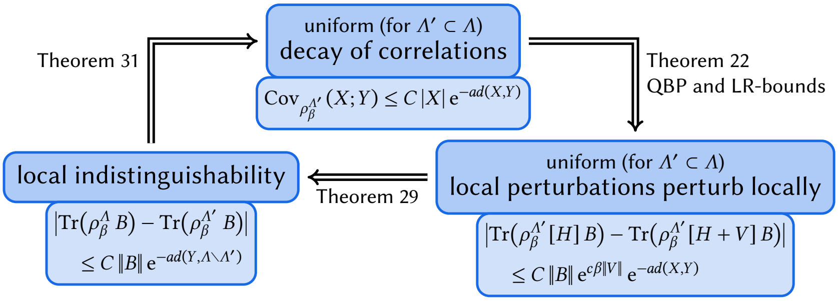

Figure 1. The diagram shows the main implications discussed in this work for short-range interactions. In particular, we show "equivalence" of the three concepts in the picture. Note, that the formulas are mainly illustrative for the concepts and in particular the constants change, see Remark 33. A crucial ingredient in all the implications is quantum belief propagation (QBP) coupled with Lieb-Robinson bounds. For precise statements we refer to the Theorems. In certain physical dimensions and temperature regimes, exponential decay of correlations is known to hold by earlier results, for which all three properties are thus satisfied.

where the base set Λ will be clear from the context.

With every site $x \in \mathbb{Z}^\nu$ we associate a finite-dimensional local Hilbert space $\mathcal{H}_x \equiv \mathbb{C}^D$ with the corresponding space of linear operators denoted by $\mathcal{A}_x := \mathcal{B}(\mathbb{C}^D)$. For each $\Lambda \Subset \mathbb{Z}^\nu$ we define the Hilbert space $\mathcal{H}_\Lambda := \bigotimes_{x\in\Lambda} \mathcal{H}_x$, and denote the algebra of bounded linear operators on $\mathcal{H}_\Lambda$ by $\mathcal{A}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda)$. Due to the tensor product structure, we have $\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x)$. Hence, for $X \subset \Lambda \Subset \mathbb{Z}^\nu$, any $A \in \mathcal{A}_X$ can be viewed as an element of $\mathcal{A}_\Lambda$ by identifying A with $A \otimes \mathbb{1}_{\Lambda\setminus X} \in \mathcal{A}_\Lambda$, where $\mathbb{1}_{\Lambda\setminus X}$ denotes the identity in $\mathcal{A}_{\Lambda\setminus X}$. This identification is always understood implicitly and for $B \in \mathcal{A}_\Lambda$ we denote by $\mathrm{supp}(B)$ the smallest $Y \subset \Lambda$ such that $B \in \mathcal{A}_Y$. For every $\Lambda \subset \mathbb{Z}^\nu$, let

$$\mathcal{A}_\Lambda := \overline{\bigcup_{\Lambda' \Subset \Lambda} \mathcal{A}_{\Lambda'}}^{\|\cdot\|}$$

be the algebra of quasi-local observables, where completion is with respect to the operator norm and is only relevant if Λ is not finite.

An *interaction* on $\Lambda \subset \mathbb{Z}^\nu$ is a function

$$\Psi : \{X \Subset \Lambda\} \to \mathcal{A}_{\mathbb{Z}^\nu}, \quad X \mapsto \Psi(X) \in \mathcal{A}_X \quad \text{with} \quad \Psi(X) = \Psi(X)^*.$$

For each $\Lambda' \Subset \Lambda$, the corresponding local Hamiltonian is then defined as

$$H_{\Lambda'} := \sum_{X \subset \Lambda'} \Psi(X).$$

There are several types of interactions we will consider in the following. The first are *finite-range interactions*, for which there exists $R > 0$ and $J > 0$ such that $\Psi(X) = 0$ whenever $\operatorname{diam}(X) > R$ and $\|\Psi(X)\| \leq J$ for all $X \subset \Lambda$.

The other are general decaying interactions, for which we define the interaction norm

$$\|\Psi\|_F := \sup_{z\in\Lambda} \sum_{\substack{Z\Subset\Lambda:\\ z\in Z}} \frac{|Z|\,\|\Psi(Z)\|}{F\big(\operatorname{diam}(Z)\big)}, \tag{1}$$

for some decaying $F\colon [0,\infty) \to (0,\infty)$ with $F(0) \leq 1$ and require $\|\Psi\|_F < \infty$. The extra $|Z|$ is included in the norm to obtain simpler explicit bounds. Along all the general statements, we will provide exemplary results for so-called *short-range* or *exponentially decaying* interactions with $F(r) = \exp(-b\,r)$, for some $b > 0$. Other regularly used classes are *stretched exponentially decaying* interactions with $F(r) = \exp(-b\,r^p)$ for some $p \in (0,1)$ and so-called *long-range* or *polynomially decaying* interactions with $F(r) = (r+1)^{-\alpha}$ for some $\alpha > 0$. Relevant applications to all those types of interactions can be found in Section 3.

Note that on finite Λ, every interaction can be seen as a finite-range interaction with range $R \leq \operatorname{diam}(\Lambda)$. However, the range R will enter our bounds explicitly. Hence, even on finite lattices, it makes sense to consider general decaying interactions, for which only the interaction norm (1) enters the bounds.

We also emphasize that the various constants, which appear in all definitions and results below, do not depend on Λ. Thus, all results are uniform in $|\Lambda|$, and one can take the thermodynamic limit.

For any Hilbert space $\mathscr{H}$ and self-adjoint operator $H \in \mathscr{B}(\mathscr{H})$ we denote the Gibbs state at inverse temperature $\beta \in (0,\infty)$ by

$$\rho_\beta^{\mathscr{H}}[H] := \frac{\mathrm{e}^{-\beta H}}{\operatorname{Tr}(\mathrm{e}^{-\beta H})},$$

where Tr denotes the (unnormalized) trace over $\mathscr{H}$. For a lattice system on $\Lambda \subset \mathbb{Z}^\nu$ and $\Lambda' \Subset \Lambda$ we abbreviate

$$\rho_\beta^{\Lambda'}[K] := \rho_\beta^{\mathscr{H}_{\Lambda'}}[K] \quad \text{for} \quad K \in \mathscr{A}_{\Lambda'} \qquad \text{and} \qquad \rho_\beta^{\Lambda'} := \rho_\beta^{\Lambda'}[H_{\Lambda'}].$$

Additionally, whenever we consider a path $H_{\Lambda'}(s) = H_{\Lambda'} + s\,V \in \mathscr{A}_{\Lambda'}$, we denote

$$\rho_\beta^{\Lambda'}(s) := \rho_\beta^{\Lambda'}[H_{\Lambda'}(s)].$$

Let us now introduce the three concepts for which we want to prove equivalence as described in the introduction and depicted in Figure 1.

2.1 Decay of correlations

Let $\Lambda \Subset \mathbb{Z}^\nu$ and ρ be a state on $\mathscr{A}_\Lambda$. Then the *covariance* of two operators $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$, localized in $X, Y \subset \Lambda$ is defined as

$$\mathrm{Cov}_\rho(A, B) := \mathrm{Tr}\big(\rho\, A\, B\big) - \mathrm{Tr}\big(\rho\, A\big)\, \mathrm{Tr}\big(\rho\, B\big). \tag{2}$$

To remove the explicit dependence on the operators, we define

$$\mathrm{Cov}_\rho(X; Y) := \sup_{\substack{A \in \mathscr{A}_X:\, \|A\|=1,\\ B \in \mathscr{A}_Y:\, \|B\|=1}} \big|\mathrm{Cov}_\rho(A, B)\big|.$$

One of the main concepts, we will use later is *decay of correlations*, sometimes also called *clustering (of correlations)*.

Definition 1 ((Uniform) decay of correlations)**.** Let $\Lambda \Subset \mathbb{Z}^\nu$ and ρ be a state on $\mathscr{A}_\Lambda$. We say that ρ satisfies *decay of correlations* with respect to the continuous functions ζ_{Cov}, $f_{\mathrm{Cov}} : [0, \infty) \to [0, \infty)$ and $n \geq 0$ if and only if

$$\mathrm{Cov}_\rho(X; Y) \leq |X|^n\, f_{\mathrm{Cov}}\big(|Y|\big)\, \zeta_{\mathrm{Cov}}\big(d(X, Y)\big)$$

for all $X, Y \subset \Lambda$.

With a little abuse of notation, we say that an interaction Ψ satisfies *uniform decay of correlations* (at inverse temperature β) on Λ if and only if the Gibbs states $\rho_\beta^{\Lambda'}[H_{\Lambda'}]$ satisfy decay of correlations with respect to the same functions and n for every $\Lambda' \subset \Lambda$. ◇

One could equivalently take the minimum with the same bound where X and Y are exchanged. But for simplicity in the presentation, we will always write the bound in this way without writing the minimum explicitly. We choose this rather general definition with arbitrary functions as a compromise between understandable proofs and validity for different results on decay of correlations coming from previous literature [9, 139, 136]. A possible extension is to consider a bound where the growth is only in the size of the boundaries ∂X and ∂Y of the sets. This setting will be discussed in detail for one-dimensional spin chains, where $|\partial X| = 2$ for all intervals X.

We say that ρ satisfies exponential decay of correlations, if $\zeta_{\mathrm{Cov}}(r) \leq C_{\mathrm{Cov}}\, \mathrm{e}^{-c_{\mathrm{Cov}} r}$ for some $C_{\mathrm{Cov}}, c_{\mathrm{Cov}} > 0$.

2.2 Local perturbations perturb locally

The second concept measures stability of a system against local perturbations, in the sense that expectation values of local observables supported far away from the perturbation change only very little. We refer to this stability as *local perturbations perturb locally* (LPPL), a term used for the local stability of ground states in previous works [23, 74, P1, 16].

Definition 2 ((Uniform) local perturbations perturb locally (LPPL)). Let $\Lambda \Subset \mathbb{Z}^\nu$ and $H \in \mathscr{A}_\Lambda$ be self-adjoint. We say that H satisfies *LPPL* (at inverse temperature β) with respect to the continuous functions $f_{\mathrm{LPPL}}, g_{\mathrm{LPPL}}, \zeta_{\mathrm{LPPL}} \colon [0,\infty) \to [0,\infty)$ and $n \geq 0$, if and only if

$$\left|\mathrm{Tr}\big(\rho_\beta^\Lambda[H]\, B\big) - \mathrm{Tr}\big(\rho_\beta^\Lambda[H+V]\, B\big)\right| \leq \|B\|\, |X|^n\, f_{\mathrm{LPPL}}\big(|Y|\big)\, g_{\mathrm{LPPL}}\big(\|V\|\big)\, \zeta_{\mathrm{LPPL}}\big(d(X,Y)\big)$$

for all $X, Y \subset \Lambda$, $V \in \mathscr{A}_X$ self-adjoint and $B \in \mathscr{A}_Y$.

We say that an interaction Ψ satisfies *uniform LPPL* (at inverse temperature β) on Λ if and only if $H_{\Lambda'}$ satisfies LPPL with respect to the same functions and n for every $\Lambda' \subset \Lambda$. ◇

2.3 Local indistinguishability

The final property we are interested in is locality of the Gibbs state, in the sense that expectation values of local operators can be well approximated by the expectation values in the Gibbs state on a smaller set $\Lambda' \subset \Lambda$ as long as they are localized far apart from $\Lambda \setminus \Lambda'$. This property goes under the name *local indistinguishability* [45, 36]. We note that it was also called *locality of temperature* in [139].

Definition 3 (Local indistinguishability). Let $\Lambda \Subset \mathbb{Z}^\nu$ and Ψ be an interaction. We say that Ψ satisfies *local indistinguishability* (at inverse temperature β) on Λ with respect to the continuous functions f_{LI} and $\zeta_{\mathrm{LI}} \colon [0,\infty) \to [0,\infty)$ if and only if

$$\left|\mathrm{Tr}\big(\rho_\beta^\Lambda[H_\Lambda]\, B\big) - \mathrm{Tr}\big(\rho_\beta^{\Lambda'}[H_{\Lambda'}]\, B\big)\right| \leq \|B\|\, f_{\mathrm{LI}}\big(|Y|\big)\, \zeta_{\mathrm{LI}}\big(d(Y, \Lambda \setminus \Lambda')\big)$$

for all $Y \subset \Lambda' \subset \Lambda$ and $B \in \mathscr{A}_Y$. ◇

A similar property for ground states is also of great interest and often called local topological quantum order (LTQO) for historical reasons. See [175, section 2.2.2] for a discussion of LTQO. To emphasize the above flavour, it sometimes gets more descriptive names like "indistinguishability radius" [175]. In most works, LTQO is taken as an assumption, see e.g. [162, 16]. It is only known to be satisfied in very simple systems, see [175] for an overview and [63, 196, P1, 16].

Remark 4. Note that decay of correlations is a property of a state, LPPL is a property of a Hamiltonian, and local indistinguishability is a property of an interaction. Clearly in all three concepts the most relevant information is the decay function: ζ_{Cov} encodes the rate of decay of correlations in a state between different regions in space, ζ_{LPPL} controls the rate at which the influence of a perturbation decays in the distance to the region where it is supported, and ζ_{LI} encodes the rate at which the influence of the boundary on a Gibbs state decays into the bulk of a system. ◇

Remark 5. Let us briefly comment on how our results extend to infinite volume systems and how they are related and might be useful for related questions in infinite volume. Since all our estimates are uniform in the system size $|\Lambda|$, they extend to the KMS state for the infinite volume system that is obtained as the weak*-limit of the finite volume Gibbs states ρ_β^Λ for $\Lambda \nearrow \mathbb{Z}^\nu$. The existence of this limit is guaranteed, for example, by the local indistinguishability property. In particular, our circle of equivalences implies that uniform decay of correlations or uniform LPPL at a certain inverse temperature β are also sufficient conditions for the existence of this limit. Note, however, that even if one had a property like local indistinguishability uniformly for different boundary conditions, uniqueness of the KMS state is not expected to our knowledge [49, chapter 6].

Related questions concerning the stability of KMS states in infinite volume that have been discussed intensively in the literature are return and approach to equilibrium (see e.g. [190, 128] and references therein). Roughly speaking, *return to equilibrium* is the property that a KMS state of a locally perturbed system returns under the unperturbed dynamics to the associated KMS state of the unperturbed system in the limit $t \to \infty$ and in the weak*-topology [190]. The idea is that local changes in the KMS state disperse or propagate to spatial infinity under the unperturbed evolution, which is usually assumed to be asymptotically abelian. Note that this problem is somewhat independent of LPPL, which claims that a local perturbation leads to a local change in the Gibbs or KMS state. It is conceivable that a local perturbation changes the KMS state only locally, but that this change then persists under the unperturbed dynamics. On the other hand, it is also possible that a system does not satisfy LPPL, but still exhibits return to equilibrium. This is because LPPL implies that the perturbed KMS state is normal with respect to the unperturbed one, while this is not a necessary condition for return to equilibrium [190].

The problem of *approach to equilibrium* in its general formulation proposed in [128] is completely open. The question is whether general initial states approach in the long time average and in the weak* sense a superposition of KMS states. In [128] this problem was studied for initial states that are themselves KMS states for a different system. While to our understanding this problem is also independent of the LPPL property, our Theorem 34 establishes a form of Lipschitz continuity of the Gibbs state as a function of the defining interaction, which might be a useful ingredient when studying approach to equilibrium. ◇

3 Applications of the general results

Before formally stating and proving our main results, in this section we first present, also as a motivation, a collection of applications to various classes of spin systems. In all cases, we start from quantitative bounds on the decay of correlations, which have either been shown before or will be shown later.

We focus first on one-dimensional quantum spin systems, for which we consider the following cases separately: translation-invariant short-range interactions at high enough temperature, short-range interactions at any positive temperature, and long-range interactions at any positive temperature.

For all of them, decay of correlations (with different decay rates) is known to hold [9, 36, 185, 136] or is proved in this paper (see Theorem 6). We then turn our attention to finite-range quantum spin systems of arbitrary dimension, for which (uniform) exponential decay of correlations above a threshold temperature is known to be satisfied [139].

3.1 One-dimensional translation-invariant short-range systems

In this section, we restrict our attention to translation-invariant short-range spin systems. For one-dimensional translation-invariant systems, Araki [9] proved that the infinite chain satisfies exponential decay of correlations at every positive temperature, and this was subsequently extended to short-range interactions in [185] above a threshold temperature. As a consequence of this, the analogous result for finite-range finite chains was recently shown in [36]. In the current manuscript, we extend this to short-range interactions above a threshold temperature. A precise definition of translation-invariant interactions is given in Section 9, basically it means that $\Psi(X)$ and $\Psi(X+n)$ are the same operator on different parts $\mathcal{A}_X \simeq \mathcal{A}_{X+n}$ of the lattice.

Theorem 6. *Let $b > 0$ and Ψ be a translation-invariant interaction with $\|\Psi\|_{\exp(-b\,\cdot)} < \infty$ and denote $\beta^* := b/(2\,\|\Psi\|_1)$. Then, for all $\beta \in (0, \beta^*)$, the Gibbs state satisfies decay of correlations in the sense that there exist $C_{\mathrm{Cov}}, c_{\mathrm{Cov}} > 0$ such that for every finite interval $I \subset \mathbb{Z}$ and subintervals $X, Y \subset I$, it holds that*

$$\mathrm{Cov}_{\rho_\beta^I}(X;Y) \le C_{\mathrm{Cov}}\, \mathrm{e}^{-c_{\mathrm{Cov}} d(X,Y)}.$$

From Theorems 36 and 37, we conclude the following statement.

Corollary 7. *Let b and $C_{\mathrm{int}} > 0$, and Ψ be a translation-invariant interaction with $\|\Psi\|_{\exp(-b\,\cdot)} < \infty$. Denote $\beta^* := b/(2\,\|\Psi\|_1)$. Then, for all $\beta \in (0, \beta^*)$, there exist constants $C_{\mathrm{LPPL}}, c_{\mathrm{LPPL}}, C_{\mathrm{LI}}, c_{\mathrm{LI}} > 0$ such that the following statements hold for all intervals $I \Subset \mathbb{Z}$.*

(a) *The Gibbs state satisfies LPPL in the sense that*

$$\left|\mathrm{Tr}\big(\rho_\beta^I[H_I]\,B\big) - \mathrm{Tr}\big(\rho_\beta^I[H_I + V]\,B\big)\right| \le C_{\mathrm{LPPL}}\, \mathrm{e}^{3\beta\|V\|}\left(1 + \|V\|\right)\|B\|\, \mathrm{e}^{-c_{\mathrm{LPPL}}\, d(X,Y)}$$

for all subintervals $X, Y_1, Y_2 \subset I$, such that $Y_1 < X < Y_2$, $Y = Y_1 \cup Y_2$, all self-adjoint perturbations $V \in \mathcal{A}_X$ and all observables $B = B_1 \otimes B_2$ with $B_1 \in \mathcal{A}_{Y_1}$ and $B_2 \in \mathcal{A}_{Y_2}$.

(b) *The interaction satisfies local indistinguishability in the sense that*

$$\left|\mathrm{Tr}\big(\rho_\beta^I\, B\big) - \mathrm{Tr}\big(\rho_\beta^{I\setminus X}\, B\big)\right| \le C_{\mathrm{LI}}\, \|B\|\, \mathrm{e}^{-c_{\mathrm{LI}} d(Y,X)}$$

for all subintervals $X, Y_1, Y_2 \subset I$, such that $Y_1 < X < Y_2$, $Y = Y_1 \cup Y_2$, and all observables $B = B_1 \otimes B_2$ with $B_1 \in \mathcal{A}_{Y_1}$ and $B_2 \in \mathcal{A}_{Y_2}$.

The proof of this result is a straightforward application of Theorems 36 and 37 under the conditions of Theorem 6. We refer to Section 9 for a detailed exposition of all these results.

3.2 One-dimensional short-range systems

In this and the next section we restrict to so called k-local interactions, which satisfy $\Psi(Z) = 0$ if $|Z| > k$. Additionally we will – as before – require a decay of the terms $\Psi(Z)$ in $\operatorname{diam}(Z)$ by specifying an interaction norm. And again, we only consider one-dimensional systems $\Lambda \Subset \mathbb{Z}$. For such systems, Kimura and Kuwahara [136] recently proved decay of correlations for short-range interactions.

Theorem 8 ([136, Theorem 1]). *Let b, C_{int} and $k > 0$. Then, for any $\beta > 0$ there exist constants C_{Cov}, $c_{\mathrm{Cov}} > 0$ such that all k-local interactions Ψ with $\|\Psi\|_{\exp(-b\,\cdot)} < C_{\mathrm{int}}$ satisfy decay of correlations such that*

$$\operatorname{Cov}_{\rho_\beta^\Lambda}(X;Y) \leq C_{\mathrm{Cov}}\, \mathrm{e}^{-c_{\mathrm{Cov}}\sqrt{d(X,Y)}}$$

for all $\Lambda \Subset \mathbb{Z}$, and intervals $X, Y \subset \Lambda$.

From Theorems 22 and 29 we obtain the following statement. A short proof is given in Appendix A.1.

Corollary 9. *Let b, C_{int} and $k > 0$. Then, for any $\beta > 0$ there exist constants C_{LPPL}, c_{LPPL}, $C_{\mathrm{LI}} > 0$ such that the following statements hold for all k-local interactions Ψ with $\|\Psi\|_{\exp(-b\,\cdot)} < C_{\mathrm{int}}$.*

(a) *The Gibbs state satisfies LPPL in the sense that*

$$\left|\operatorname{Tr}\!\left(\rho_\beta^\Lambda[H_\Lambda]\, B\right) - \operatorname{Tr}\!\left(\rho_\beta^\Lambda[H_\Lambda + V]\, B\right)\right| \leq C_{\mathrm{LPPL}}\, \mathrm{e}^{3\beta\|V\|}\left(1 + \|V\|\right)\|B\|\, \mathrm{e}^{-c_{\mathrm{LPPL}}\sqrt{d(X,Y)}} \tag{3}$$

for all $\Lambda \Subset \mathbb{Z}$, intervals $X, Y \subset \Lambda$, $V \in \mathcal{A}_X$ self-adjoint and $B \in \mathcal{A}_Y$.

(b) *The interaction satisfies local indistinguishability in the sense that*

$$\left|\operatorname{Tr}\!\left(\rho_\beta^\Lambda\, B\right) - \operatorname{Tr}\!\left(\rho_\beta^{\Lambda'}\, B\right)\right| \leq C_{\mathrm{LI}}\, \|B\| \left(1 + \sqrt{d(Y, \Lambda \setminus \Lambda')}\right) \mathrm{e}^{-c_{\mathrm{LI}}\sqrt{d(Y,\Lambda\setminus\Lambda')}}$$

for all $\Lambda' \subset \Lambda \Subset \mathbb{Z}$, intervals $Y \subset \Lambda'$ and $B \in \mathcal{A}_Y$, with $c_{\mathrm{LI}} = b\, c_{\mathrm{LPPL}} / \sqrt{b^2 + c_{\mathrm{LPPL}}^2}$.

Kimura and Kuwahara [136] prove a more general statement, which includes stretched exponentially decaying interactions. For such interactions, one obtains a similar result by using different Lieb-Robinson bounds to calculate ζ_{QBP}. Moreover, they prove decay of correlations for long-range interactions, which we discuss in the next section.

3.3 One-dimensional long-range systems

Let us consider k-local interactions (see Section 3.2) with polynomial decay in the sense that

$$\|\Psi\|_{F_\alpha} := \sup_{x,y\in\Lambda} \sum_{\substack{Z\Subset\Lambda:\\ x,y\in Z}} \frac{\|\Psi(Z)\|}{F_\alpha\big(d(x,y)\big)} < \infty, \tag{4}$$

with $F_\alpha(r) := (r+1)^{-\alpha}$. For such interactions, Kimura and Kuwahara [136] recently obtained a bound on decay of correlations in one-dimensional systems.

Theorem 10 ([136, Theorem 1]). *Let $\alpha > 2$, $\alpha_{\mathrm{Cov}} < \alpha - 2$, C_{int} and $k > 0$. Then, for any $\beta > 0$ there exist constants $C_{\mathrm{Cov}}, c_{\mathrm{Cov}} > 0$ such that all k-local interactions Ψ with $\|\Psi\|_{F_\alpha} < C_{\mathrm{int}}$ satisfy decay of correlations such that*

$$\mathrm{Cov}_{\rho_\beta^\Lambda}(X;Y) \le C\, F_{\alpha_{\mathrm{Cov}}}\big(d(X,Y)\big)$$

for all $\Lambda \Subset \mathbb{Z}$ and intervals $X, Y \subset \Lambda$.

From Theorems 22 and 29 we obtain the following statement. A short proof is given in Appendix A.2.

Corollary 11. *Let $C_{\mathrm{int}} > 0$, k, $\beta > 0$. The following statements hold.*

(a) *For every $\alpha > 2$, $\alpha_{\mathrm{LPPL}} < \alpha - 2$ there exist a constant $C_{\mathrm{LPPL}} > 0$ such that for all k-local interactions Ψ on $\mathbb{Z}$ with $\|\Psi\|_{F_\alpha} < C_{\mathrm{int}}$*

$$\big|\mathrm{Tr}\big(\rho_\beta^\Lambda[H_\Lambda]\,B\big) - \mathrm{Tr}\big(\rho_\beta^\Lambda[H_\Lambda + V]\,B\big)\big| \le C_{\mathrm{LPPL}}\, \mathrm{e}^{3\beta\|V\|}\,\big(1+\|V\|\big)\,\|B\|\,F_{\alpha_{\mathrm{LPPL}}}\big(d(X,Y)\big) \tag{5}$$

for all $\Lambda \Subset \mathbb{Z}$, intervals $X, Y \subset \Lambda$, $V \in \mathcal{A}_X$ self-adjoint and $B \in \mathcal{A}_Y$.

(b) *For every $\alpha > 3$, $\alpha_{\mathrm{LI}} < \alpha - 3$ there exist a constant $C_{\mathrm{LI}} > 0$ such that for all k-local interactions Ψ on $\mathbb{Z}$ with $\|\Psi\|_{F_\alpha} < C_{\mathrm{int}}$*

$$\big|\mathrm{Tr}\big(\rho_\beta^\Lambda\,B\big) - \mathrm{Tr}\big(\rho_\beta^{\Lambda'}\,B\big)\big| \le C_{\mathrm{LI}}\,\|B\|\,F_{\alpha_{\mathrm{LI}}}\big(d(Y,\Lambda\setminus\Lambda')\big)$$

for all $\Lambda' \subset \Lambda \Subset \mathbb{Z}$, intervals $Y \subset \Lambda$ and $B \in \mathcal{A}_Y$.

3.4 ν-dimensional short-range systems at high temperature

At high enough temperature, Gibbs states of finite-range Hamiltonians in any dimension satisfy uniform exponential decay of correlations [139] (see Remark 14). This behaviour is expected also for systems that have short-range interactions.

Conjecture 12. *Let $C_{\text{int}}, b > 0$ and $\nu \in \mathbb{N}$, then there exists $\beta^*, C_{\text{Cov}}, c_{\text{Cov}} > 0$ and $n \in \mathbb{N}$ such that the following holds: Let Ψ be an interaction on $\mathbb{Z}^\nu$ such that $\|\Psi\|_{\exp(-b\,\cdot)} < C_{\text{int}}$. Then, for every $\beta < \beta^*$ and $\Lambda \Subset \mathbb{Z}^\nu$, the Gibbs state ρ_β^Λ satisfies decay of correlations in the sense of Definition 1 with constants such that*

$$\operatorname{Cov}_{\rho_\beta^\Lambda}(X;Y) \leq C_{\text{Cov}}\, |X|^n\, |Y|^n\, \mathrm{e}^{-c_{\text{Cov}} d(X,Y)}$$

for all $X, Y \subset \Lambda$.

If the Conjecture 12 is satisfied, then our general Theorems 22 and 29 imply that the Gibbs states of such systems are also stable against local perturbations and satisfy local indistinguishability. A short proof of how the claimed exponential decay rates are obtained from the general formulas is given in Appendix A.3.

Corollary 13. *Assume that Conjecture 12 holds true. Let $C_{\text{int}}, b > 0$ and $\nu \in \mathbb{N}$, and let β^* and n as in Conjecture 12. Then there exist constants $C, c > 0$ such that the following holds for all interactions Ψ on $\mathbb{Z}^\nu$ satisfying $\|\Psi\|_{\exp(-b\,\cdot)} < C_{\text{int}}$:*

(a) *For all $\Lambda \Subset \mathbb{Z}^\nu$, $X, Y \subset \Lambda$, $V \in \mathcal{A}_X$ self-adjoint, $B \in \mathcal{A}_Y$ and $\beta < \beta^*$*

$$\left|\operatorname{Tr}\big(\rho_\beta^\Lambda[H_\Lambda]\, B\big) - \operatorname{Tr}\big(\rho_\beta^\Lambda[H_\Lambda + V]\, B\big)\right| \leq C\, \mathrm{e}^{3\beta\|V\|}\, \big(1 + \|V\|\big)\, \|B\|\, |X|^n\, |Y|^n\, \mathrm{e}^{-cd(X,Y)}.$$

(b) *For all $Y \subset \Lambda' \subset \Lambda \Subset \mathbb{Z}^\nu$, $B \in \mathcal{A}_Y$ and $\beta < \beta^*$*

$$\left|\operatorname{Tr}\big(\rho_\beta^\Lambda\, B\big) - \operatorname{Tr}\big(\rho_\beta^{\Lambda'}\, B\big)\right| \leq C\, \beta\, \|B\|\, |Y|^{n+1}\, \mathrm{e}^{-cd(Y,\Lambda\setminus\Lambda')}.$$

Remark 14. A few remarks are in order:

- Conjecture 12 is known to be true for finite-range interactions, as it has been shown in [139], which is also one of the very few previous results on LPPL and local indistinguishability for finite-range interactions. Indeed, by using the explicit decay of correlations obtained in [139] together with our Theorem 22 and Theorem 29 we recover the result of [139], namely estimates analogous to those in 13 (a) and 13 (b) but with scaling $|\partial Y|^2$ instead of the $|X|^n\,|Y|^n$ and $|Y|^{n+1}$, respectively.[1] However, we emphasize that the proof in [139] requires not only the control on the decay of correlations but also on the decay of the generalized covariance for all the Gibbs states of the perturbed Hamiltonian $H(s)$. In this respect, our result makes the locality and stability results in principle more accessible.

[1] First, note that ∂Y in [139] is the boundary in the interaction graph, i.e. it scales with the range R. To obtain the claimed scaling, we slightly modify their proof: After [139, eq. (47)], we apply the same bound, but with $x_0 = 1$, to upper bound [139, eq. (47)] by $2\,|\partial Y|\,\mathrm{e}^{-L/\xi(\beta)}/(1 - \mathrm{e}^{-1/\xi(\beta)})$ for all $L > L_0 := \xi(\beta)\,\ln\big(|\partial Y|/\big(1 - \mathrm{e}^{-1/\xi(\beta)}\big)\big)$, where $\xi(\beta)$ as in [139, eq. (10)]. Multiplying this bound by $|\partial Y|/(1 - \mathrm{e}^{-1/\xi(\beta)}) = \mathrm{e}^{L_0/\xi(\beta)} > 1$ makes it valid for all $L \geq 0$ due to the trivial bound $|\operatorname{Cov}_\rho(X;Y)| \leq 2$.

- In dimension $\nu = 1$, Conjecture 12 is true for translation-invariant interactions. Moreover, the critical temperature β^* vanishes when reducing to translation-invariant finite-range interactions in one dimension due to [185]. This setting is addressed in detail in Section 9.

- Theorem 3.2 in [98] implies Conjecture 12. However, after finishing this manuscript, it was pointed out to us [35] that the proof presented in [98] only proves a scaling $\exp(|X|+|Y|)$ [97] instead of the claimed $|X|\,|Y|$. The exponential prefactor, which is not considered to be optimal, is insufficient to apply our results. ◇

4 Quantum belief propagation

This section is devoted to the introduction of the *quantum belief propagation* (QBP), the main tool for our analysis. The concept was originally introduced by Hastings [115]. More recently, the concept of QBP attracted more attention and has been used for different applications, see for example the works [134, 135, 80, 45, 131, 109, 8, 145, 187, 194]. See also the review [5] and Remark 21 for a more detailed discussion on the literature.

Here, we offer a rigorous exploration of quantum belief propagation, expanding upon earlier findings. In particular, we show how to extend the technique from the exponential of a Hamiltonian to its Gibbs state. A step which was used in some of the mentioned works without justification. Moreover, we also include short- and long-range interactions instead of finite-range interactions only. To enhance the text's clarity, we have chosen to postpone the proofs to Section 10.

The first theorem does not need any locality properties of the Hamiltonians. Therefore, we state the theorem for general self-adjoint operators on a finite-dimensional Hilbert space. Notice that in [80] the authors extend the QBP equation for the perturbed exponential, namely equation (6), to KMS states of general W^*-dynamical systems.

Proposition 15 (Quantum belief propagation, general statement). *Let $\mathscr{H}$ be a finite-dimensional Hilbert space and H and V self-adjoint operators on $\mathscr{H}$. We consider the path of Hamiltonians $H(s) := H + sV$. Then, the following holds true.*

(a) *The exponentials $\mathrm{e}^{-\beta H(s)}$ satisfy the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{e}^{-\beta H(s)} = -\frac{\beta}{2}\left\{\mathrm{e}^{-\beta H(s)}, \Phi_\beta^{H(s)}(V)\right\}, \tag{6}$$

where $\Phi_\beta^{H(s)}(V)$ is defined by

$$\Phi_\beta^{H(s)}(W) := \int_{-\infty}^{\infty} \mathrm{d}t\, f_\beta(t)\,\mathrm{e}^{-\mathrm{i}tH(s)}\,W\,\mathrm{e}^{\mathrm{i}tH(s)}, \tag{7}$$

for all self-adjoint operators W on $\mathscr{H}$ with f_β an exponentially decaying L^1-function, which is explicitly given in (26). *Clearly* $\|\Phi_\beta^{H(s)}(W)\| \leq \|W\|$.

Moreover, there exists a path of operators $s \mapsto \eta(s)$, *given in* (28), *such that* $e^{-\beta H(s)} = \eta(s)\, e^{-\beta H(0)}\, \eta(s)^*$ *and* $\|\eta(s)\| \leq e^{\frac{\beta}{2} s\|V\|}$.

(b) *Equation* (6) *implies that the path of perturbed Gibbs state* $s \mapsto \rho_\beta(s) := \rho_\beta^{\mathscr{H}}[H(s)]$ *satisfies the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}s}\, \rho_\beta(s) = -\frac{\beta}{2} \Big\{\rho_\beta(s), \Phi_\beta^{H(s)}\big(V - \langle V\rangle_{\rho_\beta(s)}\big)\Big\}. \tag{8}$$

Moreover, there exists a path of operators $s \mapsto \tilde{\eta}(s)$ *such that* $\rho_\beta(s) = \tilde{\eta}(s)\, \rho_\beta(0)\, \tilde{\eta}(s)^*$, $\|\tilde{\eta}(s)\| \leq e^{\beta s\|V\|}$ *and*

$$\|\rho_\beta(0) - \rho_\beta(s)\|_1 \leq e^{2\beta s\|V\|} - 1. \tag{9}$$

Proposition 15 not only gives two useful differential equations, but also proves continuity of the Gibbs state in the Hamiltonian and stability of the Gibbs state against small perturbations, see equation (9). To make the result more explicit, we often use the bound

$$e^{xy} - 1 \leq x\,(e^y - 1) \qquad \text{for all } x \in [0,1],\ y \geq 0, \tag{10}$$

which for example allows to bound $e^{2\beta s\|V\|} - 1 \leq s\big(e^{2\beta\|V\|} - 1\big)$.

Note that the existence of an operator $\kappa(s)$ such that $e^{-\beta H(s)} = \kappa(s)\, e^{-\beta H(0)}\, \kappa(s)^*$ is not surprising, and the same holds for the normalized state. Indeed, many choices are possible since we only require one state to be mapped to another. One can also choose $\kappa(s) = e^{-\beta H(s)/2}\, e^{\beta H(0)/2}$, see [49, Corollary 5.4.2] together with the discussion following it and [5, section III.B]. This is enough to obtain continuity of the Gibbs state, i.e. $\|\rho_\beta(0) - \rho_\beta(s)\| \to 0$ for $s \to 0$ but without the explicit bound in (9), see [49, Theorem 5.4.4(3)].

The main advantage of the specific operators $\eta(s)$ and $\tilde{\eta}(s)$ provided by quantum belief propagation is their bounded norm and the locality, which will be stated in Proposition 18. For the $\kappa(s)$ given above, boundedness and locality can be proven above a threshold temperature for finite-range Hamiltonian, see [5, eq. (36), (37)]. But for smaller temperatures, there are nearest neighbour Hamiltonians [41, Main Theorem], where these proofs fail and where we expect that the norm of $\kappa(s)$ diverges in the thermodynamic limit.

Remark 16. A very similar concept appears in the context of Hamiltonians with gapped ground states [23]. There

$$-\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}s}\, P(s) = -\big[P(s), \Phi_\gamma^{H(s)}(V)\big],$$

where $P(s)$ is the projection onto the ground state of $H(s)$ and $\Phi_Y^{H(s)}(V)$ is defined as in (7) but with a different function f_Y. One then solves the Schrödinger equation and obtains the unitary propagator $U(s)$ such that

$$P(s) = U(s)\,P(0)\,U(s)^*,$$

which one can also approximate. It satisfies $\|U(s)\| = \|U(s)^*\| = 1$ and $U(s)^* = U(s)^{-1}$, which we do not have for the $\eta(s)$. For the ground state projection, the uniform (in s) gap above the ground state is crucial, which is not the case for the Gibbs state. However, the estimates we obtain for the Gibbs state are not useful to analyse the $\beta \to \infty$ regime. ◇

Proposition 15 becomes even more interesting when we apply it to the lattice setting described in Section 2, because locality of the Hamiltonian H_Λ and localization of V will result in quasi-locality of $\Phi_\beta^{H_\Lambda(s)}$. The locality of the Hamiltonian is often measured in terms of Lieb-Robinson bounds [154, 173, 174]. These bounds measure how fast (the support of) an operator spreads under the Heisenberg time evolution.

Definition 17 (Lieb-Robinson bound)**.** Let $\Lambda \Subset \mathbb{Z}^\nu$. We say that a Hamiltonian $H \in \mathcal{A}_\Lambda$ satisfies a Lieb-Robinson bound with decay $\zeta_{\mathrm{LR}} : \{X \subset \Lambda\}\times\{Y \subset \Lambda\}\times[0,\infty) \to [0,\infty)$ if

$$\big\|\big[\mathrm{e}^{-\mathrm{i}tH}\,A\,\mathrm{e}^{\mathrm{i}tH}, B\big]\big\| \le \|A\|\,\|B\|\,\zeta_{\mathrm{LR}}\big(X, Y, |t|\big)$$

for all $X, Y \subset \Lambda$, $A \in \mathcal{A}_X$,$B \in \mathcal{A}_Y$ and $t \in \mathbb{R}$. ◇

For short-range interactions, the Lieb-Robinson bound decay can be proven to be

$$\zeta_{\mathrm{LR}}\big(X, Y, |t|\big) = 2\,\min\big\{|X|, |Y|\big\}\,\mathrm{e}^{b(v|t|-d(X,Y))}$$

with the Lieb-Robinson velocity $v = 2\,\|\Psi\|_{\exp(-b\,\cdot)}/b$, see Proposition 44 for a precise statement. Such Lieb-Robinson bounds are known for spin systems [174, Theorem 3.1] and lattice fermions [173, Theorem 3.1(i)]. Similarly, for long-range interactions, Lieb-Robinson bounds with polynomial decay in the distance $d(X,Y)$ are known [84], see Proposition 45. We keep this very general assumption on the Lieb-Robinson bounds to render our results applicable for a large class of interactions. Moreover, it allows us to obtain improved results for one-dimensional systems when we restrict X to be an interval.

Proposition 18 (Quantum belief propagation on a lattice)**.** *Let $\Lambda \Subset \mathbb{Z}^\nu$, $H \in \mathcal{A}_\Lambda$ self-adjoint, $X \subset \Lambda$, and $V \in \mathcal{A}_X$ self-adjoint. For $s \in [0,1]$, consider the path of Hamiltonians $H(s) := H + s\,V$. Moreover, assume that all Hamiltonians $H(s)$ satisfy a Lieb-Robinson bound with ζ_{LR} decay uniformly in s. Let $\zeta_{\mathrm{QBP}} : \{\, X \Subset \Lambda \,\} \times [0,\infty) \to [0,\infty)$ with*

$$\zeta_{\mathrm{QBP}}(X,r) := \min\Big\{2, \inf_{T\ge 0}\|\zeta_{\mathrm{LR}}(X, \Lambda\setminus X_r, \cdot)\|_{L^\infty([-T,T])} + 4\,\mathrm{e}^{-\frac{\pi}{\beta}T}\Big\}, \tag{11}$$

which only depends on ζ_{LR} and the inverse temperature β. Then the following holds:

(a) *For any $W \in \mathscr{A}_X$, the operators $\Phi_\beta^{H(s)}(W)$ defined in* (7) *can be approximated by local operators $\Phi_{\beta,r}^{H(s)}(W) \in \mathscr{A}_{X_r}$ supported on the r-neighbourhood X_r, such that $\|\Phi_{\beta,r}^{H(s)}(W)\| \le \|\Phi_\beta^{H(s)}(W)\|$ and*

$$\|\Phi_\beta^{H(s)}(W) - \Phi_{\beta,r}^{H(s)}(W)\| \le \|W\| \, \zeta_{\mathrm{QBP}}(X,r).$$

(b) *The operators $\eta(s)$ defined in Proposition 15 (a) can be approximated by local operators $\eta_r(s) \in \mathscr{A}_{X_r}$ supported on X_r, such that $\|\eta_r(s)\| \le \|\eta(s)\|$ and*

$$\|\eta(s) - \eta_r(s)\| \le \frac{\beta}{2} \, s \, \|V\| \, \mathrm{e}^{\frac{\beta}{2} s \|V\|} \, \zeta_{\mathrm{QBP}}(X,r).$$

(c) *The operators $\tilde{\eta}(s)$ defined in Proposition 15 (b) can be approximated by operators $\tilde{\eta}_r(s) \in \mathscr{A}_{X_r}$ supported on X_r, such that $\|\tilde{\eta}_r(s)\| \le \|\tilde{\eta}(s)\|$ and*

$$\|\tilde{\eta}(s) - \tilde{\eta}_r(s)\| \le \beta \, s \, \|V\| \, \mathrm{e}^{\beta s \|V\|} \, \zeta_{\mathrm{QBP}}(X,r). \tag{12}$$

When using the approximations given by Proposition 18, it is important to note that both maps $\sigma \mapsto \tilde{\eta}(s) \, \sigma \, \tilde{\eta}^*(s)$ and $\sigma \mapsto \tilde{\eta}_r(s) \, \sigma \, \tilde{\eta}_r^*(s)$ are completely positive, but in general not trace preserving. Only for the Gibbs state $\rho_\beta(s)$, clearly, $\mathrm{Tr}\big(\tilde{\eta}(s) \, \rho_\beta(0) \, \tilde{\eta}^*(s)\big) = \mathrm{Tr}\big(\rho_\beta(s)\big) = 1$, and by (12) one can conclude $\mathrm{Tr}\big(\tilde{\eta}_r(s) \, \rho_\beta(0) \, \tilde{\eta}_r^*(s)\big) \approx 1$.

For the applications discussed in this work, it is enough to have some local approximation of $\Phi_\beta^{H(s)}(W)$, which we construct in Section 10.2.2 using a conditional expectation, which is basically a partial trace for spin systems. For numerical implementations, it might be useful to instead use $\Phi_\beta^{H_{X_r}}(W)$, i.e. (7) with the Hamiltonian truncated to X_r, as an approximation. Qualitatively, the result would be as in Proposition 18 by using a bound like [174, Theorem 3.4 (ii)] instead of (32) in the proof.

Remark 19. In Proposition 18 we assume a uniform Lieb-Robinson bound for all Hamiltonians $H(s) = H + sV$. This simplifies the statement and allows for better bounds when we later prove local indistinguishability. The assumption can be dropped by the stability result for the Lieb-Robinson bound we provide in Lemma 42 at the price of an extra factor $\|V\|$ in ζ_{QBP}, see Lemma 20 (b). ◇

Lemma 20 (ζ_{QBP} for short-range interactions)**.** *Let $b > 0$ and Ψ be an interaction on $\Lambda \Subset \mathbb{Z}^\nu$ satisfying $\|\Psi\|_{\exp(-b\,\cdot)} < \infty$, $X \subset \Lambda$ and $V \in \mathscr{A}_X$ self-adjoint. Then for all Hamiltonians $H(s) = H + sV$ the following holds:*

(a) *If also $\|\Psi + V\|_{\exp(-b\,\cdot)} < \infty$, then*

$$\zeta_{\mathrm{QBP}}(X,r) \le 6 \, |X| \, \mathrm{e}^{-\frac{b}{1+a\beta} r},$$

where $a = \frac{2}{\pi} \, \max\big\{\|\Psi\|_{\exp(-b\,\cdot)}, \|\Psi + V\|_{\exp(-b\,\cdot)}\big\}$.

(b) *In general, it holds that*

$$\zeta_{\mathrm{QBP}}(X,r) \le C_{\mathrm{QBP}}\left(1+\|V\|\right)|X|\,\mathrm{e}^{-\frac{b}{1+a\beta}r}, \tag{13}$$

where $C_{\mathrm{QBP}} = 6\,\max\{1, 2/(b\,v_b)\}$ *and* $a = \frac{2}{\pi}\|\Psi\|_{\exp(-b\,\cdot)}$.

Moreover, if $X = B_z(R) := \{x \in \Lambda \,|\, d(x,z) \le R\}$ *is a ball (for some* $z \in \Lambda$ *and* $R \ge 1$*) we can replace* $|X|$ *in both bounds with* $\frac{1+b}{b}|\partial X|$*, where* $\partial X = B_z(R) \setminus B_z(R-1)$*. In particular, for intervals* $X \subset \mathbb{Z}$ *in dimension* $\nu = 1$*, we can replace* $|X|$ *by* $2\,\frac{1+b}{b}$.

We also provide similar estimates for long-range interactions in Lemma 46, the proofs are given in Section 10.3.

Remark 21. The original proof of the quantum belief propagation is due to Hastings [115] and was only concerned with a differential equation for the perturbed exponential $\mathrm{e}^{-\beta H(s)}$, namely the first part of Proposition 15 (a) and Proposition 18 (a) and 18 (b). However, note that the equation obtained in [115] was different than the one we show here, which is instead the one that is commonly used nowadays and, as far as we know, appeared first in [134, 135]. The other works that used QBP [134, 135, 80, 45, 131, 109, 8, 145] were mainly focused on a differential equation for the perturbed exponential, while our main focus is the use of QBP directly on the Gibbs state, namely the differential equation (8) and its locality properties, for which we provide a thorough discussion in Section 10. A similar approach was used for a slightly different application and finite-range interactions in [194]. ◇

5 LPPL from decay of correlations

Using quantum belief propagation, we now show that Gibbs states are stable against local perturbations in the Hamiltonian whenever they satisfy decay of correlations.

Theorem 22 (LPPL from correlations in the unperturbed state). *Let* $\Lambda \Subset \mathbb{Z}^\nu$*,* $H \in \mathcal{A}_\Lambda$ *self-adjoint,* $X \subset \Lambda$*, and* $V \in \mathcal{A}_X$ *self-adjoint. For* $s \in [0,1]$*, consider the path of Hamiltonians* $H(s) := H + sV$ *with Gibbs states* $\rho_\beta(s)$*. Moreover, assume that all Hamiltonians* $H(s)$ *satisfy a Lieb-Robinson bound with* ζ_{LR}*-decay uniformly in s, and let* ζ_{QBP} *be the function from Proposition 18. Then, for all* $Y \subset \Lambda$*,* $B \in \mathcal{A}_Y$ *and* $r \ge 0$*, we have*

$$\left|\mathrm{Tr}\big(\rho_\beta(0)\,B\big) - \mathrm{Tr}\big(\rho_\beta(1)\,B\big)\right| \le \mathrm{e}^{2\beta\|V\|}\,\|B\|\left(\mathrm{Cov}_{\rho_\beta(0)}(X_r;Y) + 4\,\beta\,\|V\|\,\zeta_{\mathrm{QBP}}(X,r)\right). \tag{14}$$

Remark 23 (Simplified short-range version). Let Ψ be a short-range interaction and assume that the Gibbs state $\rho_\beta(0)$ satisfies exponential decay of correlations with respect to $\zeta_{\mathrm{Cov}}(r) = C_{\mathrm{Cov}}\,\mathrm{e}^{-c_{\mathrm{Cov}}r}$, $f_{\mathrm{Cov}}(|Y|) = |Y|^m$ and $n \ge 0$ in the sense of Definition 1.

Then Theorem 22 implies that for every $c < c_0 := c_{\mathrm{Cov}}\, c_{\mathrm{QBP}}/(c_{\mathrm{Cov}} + c_{\mathrm{QBP}})$, where c_{QBP} is the decay exponent in (13), there exists a constant $C > 0$ such that for all $X, Y \subset \Lambda$, $V \in \mathscr{A}_X$ self-adjoint and $B \in \mathscr{A}_Y$

$$\left|\mathrm{Tr}\big(\rho_\beta(0)\, B\big) - \mathrm{Tr}\big(\rho_\beta(1)\, B\big)\right| \le C\, \mathrm{e}^{3\beta\|V\|}\, \big(1 + \|V\|\big)\, \|B\|\, |X|^{\max\{1,n\}}\, |Y|^m\, \mathrm{e}^{-cd(X,Y)}. \qquad \diamond$$

Notice that we only need to know decay of correlations in the unperturbed state $\rho_\beta(0)$ in order to control the bound (14). This allows to use it in conjunction with decay of correlations in translation-invariant systems in one dimension [9, 185]. Although the bound (14) holds for all temperatures, it still diverges for $\beta \to \infty$ and is thus not useful in the zero temperature limit.

Proof. For the proof, we drop the subscript β. Under the assumptions of the proposition, let $\tilde{\eta}(s)$ be the operators from Proposition 15 (b) such that $\rho(s) = \tilde{\eta}(s)\, \rho(0)\, \tilde{\eta}(s)^*$, and $\tilde{\eta}_r(s)$ be their local approximations from Proposition 18 (c) with $r < d(X, Y)$ such that $[B, \tilde{\eta}_r] = 0$. We abbreviate $\rho := \rho(0)$, $\tilde{\eta} := \tilde{\eta}(s)$, and $\tilde{\eta}_r := \tilde{\eta}_r(s)$. Then,

$$\begin{aligned}
&\mathrm{Tr}\big(\rho(s)\, B\big) - \mathrm{Tr}\big(\rho\, B\big) \\
&\quad = \mathrm{Tr}\big(\tilde{\eta}\, \rho\, \tilde{\eta}^*\, B\big) - \mathrm{Tr}\big(\rho\, B\big) \\
&\quad = \mathrm{Tr}\big((\tilde{\eta} - \tilde{\eta}_r)\, \rho\, \tilde{\eta}^*\, B\big) + \mathrm{Tr}\big(\tilde{\eta}_r\, \rho\, (\tilde{\eta}^* - \tilde{\eta}_r^*)\, B\big) + \mathrm{Tr}\big(\rho\, \tilde{\eta}_r^*\, \tilde{\eta}_r\, B\big) - \mathrm{Tr}\big(\rho\, B\big).
\end{aligned}$$

The first two terms are bounded by $\|\tilde{\eta} - \tilde{\eta}_r\|\, \big(\|\tilde{\eta}\| + \|\tilde{\eta}_r\|\big)\, \|B\|$. Thus, for $B = \mathbb{1}$, we obtain $\left|\mathrm{Tr}(\rho\, \tilde{\eta}_r^*\, \tilde{\eta}_r) - 1\right| \le \|\tilde{\eta} - \tilde{\eta}_r\|\, \big(\|\tilde{\eta}\| + \|\tilde{\eta}_r\|\big)$. Hence, we can replace $\mathrm{Tr}\big(\rho\, B\big)$ by $\mathrm{Tr}(\rho\, \tilde{\eta}_r^*\, \tilde{\eta}_r)\ \mathrm{Tr}\big(\rho\, B\big)$ to recover the covariance and obtain

$$\begin{aligned}
&\left|\mathrm{Tr}\big(\rho(s)\, B\big) - \mathrm{Tr}\big(\rho\, B\big)\right| \\
&\quad \le 2\, \|\tilde{\eta} - \tilde{\eta}_r\|\, \big(\|\tilde{\eta}\| + \|\tilde{\eta}_r\|\big)\, \|B\| + \left|\mathrm{Cov}_\rho(\tilde{\eta}_r^*\, \tilde{\eta}_r, B)\right| \\
&\quad \le 4\, \|B\|\, \beta\, s\, \|V\|\, \mathrm{e}^{2\beta s\|V\|}\, \zeta_{\mathrm{QBP}}(X, r) + \|B\|\, \mathrm{e}^{2\beta s\|V\|}\ \mathrm{Cov}_\rho(X_r; Y). \qquad \square
\end{aligned}$$

Remark 24. The same calculations can also be done in the ground state setting (see Remark 16). There one does not need to define a $\tilde{U}(s)$ because $U(s)$ is already the mapping of the ground state projection. Moreover, one does not need to assume decay of correlations because $\mathrm{Cov}_P(U^*\, U, B) = 0$ anyway since $U^*\, U = \mathbb{1}$. And the scaling in $\|V\|$ is also better. However, as pointed out above, the gap is necessary and thus LPPL of this form only holds if V does not close the spectral gap, which is in general only true for small $\|V\|$. $\diamond$

Furthermore, if we know how to control the correlations for all Gibbs states along the path, we obtain a better scaling in the norm of the perturbation, as is shown in the next proposition.

Theorem 25 (LPPL from correlations along the path)**.** *Let $\Lambda \Subset \mathbb{Z}^\nu$, $H \in \mathscr{A}_\Lambda$ self-adjoint, $X \subset \Lambda$, and $V \in \mathscr{A}_X$ self-adjoint. For $s \in [0, 1]$, consider the path of Hamiltonians*

$H(s) := H + sV$ with Gibbs states $\rho_\beta(s)$. Moreover, assume that all Hamiltonians $H(s)$ satisfy a Lieb-Robinson bound with ζ_{LR}-decay uniformly in s and let ζ_{QBP} be the function from Proposition 18. Then, for all $Y \subset \Lambda$, $B \in \mathcal{A}_Y$ and all $r \geq 0$, we have

$$\left|\mathrm{Tr}\big(\rho_\beta(0)\,B\big) - \mathrm{Tr}\big(\rho_\beta(1)\,B\big)\right| \leq \beta\,\|V\|\,\|B\| \left(\sup_{s\in[0,1]} \mathrm{Cov}_{\rho_\beta(s)}(X_r; Y) + 2\,\zeta_{\mathrm{QBP}}(X,r)\right).$$

Remark 26 (Simplified short-range version). Let Ψ and $\Psi + V$ be short-range interactions and assume that all Gibbs states $\rho_\beta(s)$ satisfy exponential decay of correlations with respect to $\zeta_{\mathrm{Cov}}(r) = C_{\mathrm{Cov}}\,\mathrm{e}^{-c_{\mathrm{Cov}} r}$, $f_{\mathrm{Cov}}(|Y|) = |Y|^m$ and $n \geq 0$ in the sense of Definition 1. Then Theorem 25 implies that for every $c < c_0$, with c_0 as in Remark 23, there exist a constant $C > 0$ such that for all $Y \subset \Lambda$ and $B \in \mathcal{A}_Y$

$$\left|\mathrm{Tr}\big(\rho_\beta(0)\,B\big) - \mathrm{Tr}\big(\rho_\beta(1)\,B\big)\right| \leq C\,\beta\,\|V\|\,\|B\|\,|X|^{\max\{1,n\}}\,|Y|^m\,\mathrm{e}^{-cd(X,Y)}. \qquad \diamond$$

Proof. We again drop the subscript $_\beta$. Integrating (8), we obtain

$$\begin{aligned}
&\mathrm{Tr}\big(\rho(1)\,B\big) - \mathrm{Tr}\big(\rho(0)\,B\big) \\
&\quad = -\frac{\beta}{2}\int_0^1 \mathrm{Tr}\Big(\Big\{\rho(s), \Phi_\beta^{H(s)}(V) - \mathrm{Tr}\big(\rho(s)\,\Phi_\beta^{H(s)}(V)\big)\Big\}\,B\Big)\,\mathrm{d}s \\
&\quad = -\frac{\beta}{2}\int_0^1 \mathrm{Cov}_{\rho(s)}\big(\Phi_\beta^{H(s)}(V), B\big) + \mathrm{Cov}_{\rho(s)}\big(B, \Phi_\beta^{H(s)}(V)\big)\,\mathrm{d}s.
\end{aligned}$$

Approximating $\Phi_\beta^{H(s)}(V)$ in X_r using Proposition 18 (a) and $\|\Phi_{\beta,r}^{H(s)}(V)\| \leq \|V\|$, gives

$$\left|\mathrm{Tr}\big(\rho(1)\,B\big) - \mathrm{Tr}\big(\rho(0)\,B\big)\right| \leq \beta\,\|V\|\,\|B\| \left(\int_0^1 \mathrm{Cov}_{\rho(s)}(X_r; Y)\,\mathrm{d}s + 2\,\zeta_{\mathrm{QBP}}(X,r)\right).$$

Bounding the integral with the supremum concludes the proof. □

A result similar to Theorem 25 was obtained by Kliesch et al. [139]. They also start with a differential equation for ρ_β. But later they need to use not only decay of correlations for all s, but decay of the generalized covariance for all s, which, they can only prove in finite-range systems at high temperature. Instead, Theorem 25 can also be applied if one only has information about decay of correlations. And in Theorem 22 it is enough to know decay of correlations in the unperturbed state $\rho_\beta(0)$. See in particular the applications we give in Section 3. In Section 9, we will discuss consequences of the results for translation-invariant one-dimensional spin chains, where decay of correlations is known for finite-range interactions at any temperature [9, 36].

Remark 27. Since the proofs of Theorems 22 and 25 are mainly based on quantum belief propagation and Lieb-Robinson bounds, which hold true also for fermionic systems, analogous propositions can be stated and proved in a fermionic setting with minor modifications. Similarly, the following Theorems 29 and 31 hold for fermionic systems as well. In this paper we focus on quantum spin systems for simplicity. ◇

6 Local indistinguishability from uniform LPPL

In this section we prove that uniform LPPL implies local indistinguishability if also the interaction decays fast enough. The main idea is to remove $\Lambda \setminus \Lambda'$ from Λ point by point, for which we have to assume LPPL at all intermediate steps. The idea is inspired by Brandão and Kastoryano [45], who removed point by point a boundary region $\partial\Lambda' \subset \Lambda$ to decouple the system in Λ' from the rest. With our improved method we can improve the scaling with Λ' and extend the result from finite- to short- and long-range interactions. The first result will later be used to remove single vertices.

Lemma 28. *Let $\Lambda \Subset \mathbb{Z}^\nu$, F be a decay function, and Ψ be an interaction such that $\|\Psi\|_F < \infty$. Moreover, assume that ρ_β^Λ satisfies LPPL with respect to f_{LPPL}, g_{LPPL}, ζ_{LPPL}, and $n \geq 0$ as in Definition 2. Then, for any $X \subset \Lambda$ it holds that*

$$\left|\mathrm{Tr}\big(\rho_\beta^\Lambda B\big) - \mathrm{Tr}\big(\rho_\beta^{\Lambda\setminus X} B\big)\right| \leq \|B\|\,|X|^n\, f_{\mathrm{LPPL}}\big(|Y|\big)\, g\big(|X|\,\|\Psi\|_F\big)\, \zeta\big(d(Y,X)\big)$$

for all $Y \subset \Lambda \setminus X$ and $B \in \mathcal{A}_Y$, where

$$g(v) := \max\big\{g_{\mathrm{LPPL}}(v), \big(\mathrm{e}^{2\beta v}-1\big)\big\} \quad \textit{and} \quad \zeta(r) := \min_{0\leq R\leq r} (2R+1)^{n\nu}\,\zeta_{\mathrm{LPPL}}(r-R) + F(R).$$

Proof. In the proof we will compare $\rho_\Lambda := \rho_\beta^\Lambda[H_\Lambda]$ and $\rho_{\Lambda\setminus X} := \rho_\beta^\Lambda[H_{\Lambda\setminus X}]$ such that we can always use the same trace. By the different normalizations in the Gibbs state, it holds that

$$\mathrm{Tr}_\Lambda\big(\rho_{\Lambda\setminus X} B\big) = \mathrm{Tr}_\Lambda\big(\rho_\beta^\Lambda[H_{\Lambda\setminus X}]\, B\big) = \mathrm{Tr}_{\Lambda\setminus X}\big(\rho_\beta^{\Lambda\setminus X}[H_{\Lambda\setminus X}]\, B\big) = \mathrm{Tr}_{\Lambda\setminus X}\big(\rho_\beta^{\Lambda\setminus X} B\big)$$

for all $B \in \mathcal{A}_{\Lambda\setminus X}$ and the result follows.

We split the difference $H_\Lambda - H_{\Lambda\setminus X} = V_1 + V_2$ into a finite-range part and the rest,

$$V_1 := \sum_{\substack{Z\subset\Lambda:\\ X\cap Z\neq\emptyset,\\ \mathrm{diam}(Z)\leq R}} \Psi(Z) \qquad \text{and} \qquad V_2 := \sum_{\substack{Z\subset\Lambda:\\ X\cap Z\neq\emptyset,\\ \mathrm{diam}(Z)> R}} \Psi(Z)$$

for some $R \geq 0$, and use $\tilde\rho := \rho_\beta^\Lambda[H_\Lambda - V_1]$ as an intermediate step. The perturbation V_1 is supported in X_R and bounded $\|V_1\| \leq |X|\,\|\Psi\|_F$. Hence, by LPPL we find

$$\left|\mathrm{Tr}(\rho_\Lambda B) - \mathrm{Tr}(\tilde\rho\, B)\right| \leq \|B\|\,|X_R|^n\, f_{\mathrm{LPPL}}\big(|Y|\big)\, g_{\mathrm{LPPL}}\big(|X|\,\|\Psi\|_F\big)\, \zeta_{\mathrm{LPPL}}\big(d(Y,X)-R\big)$$

for all $B \in \mathcal{A}_Y$. The remaining perturbation V_2 is small,

$$\|V_2\| \leq \sum_{\substack{Z\subset\Lambda:\\ X\cap Z\neq\emptyset,\\ \mathrm{diam}(Z)> R}} \|\Psi(Z)\| \leq |X|\,\|\Psi\|_F\, F(R),$$

and with the bound (9) from Proposition 15 (b) and (10) it holds that

$$\left|\mathrm{Tr}(\tilde\rho\, B) - \mathrm{Tr}(\rho_{\Lambda\setminus X}\, B)\right| \le \|B\| \left(\mathrm{e}^{2\beta\|V_2\|} - 1\right) \le \|B\|\, F(R) \left(\mathrm{e}^{2\beta|X|\|\Psi\|_F} - 1\right).$$

The result follows by triangle inequality. □

The bound provided in Lemma 28 on its own is not very good because it scales at least exponentially in $|X|$. This might be enough in one-dimensional systems where one needs to remove only a constant number of sites to decouple two halves of a system. But in general it is more advantageous to remove X site by site. Therefore, we assume that there exists a sequence $(\Lambda_i)_{i=0}^N$ such that $\Lambda_0 = \Lambda$ and $\Lambda_N = \Lambda'$ along which each $\rho_\beta^{\Lambda_i}$ satisfies LPPL.

Theorem 29. *Let $\Lambda \Subset \mathbb{Z}^\nu$, F a decay function, and Ψ be an interaction such that $\|\Psi\|_F < \infty$. Let $\Lambda' \subset \Lambda$ and assume that there exists a sequence $(x_i)_{i=1}^N \subset \Lambda\setminus\Lambda'$ defining $\Lambda_i := \Lambda\setminus\{x_j \,|\, j = 1,\dots,i\}$ such that $\Lambda_N = \Lambda'$ and $\rho_\beta^{\Lambda_i}$ satisfies LPPL with respect to f_{LPPL}, g_{LPPL}, ζ_{LPPL}, and $n \ge 0$ as in Definition 2 for every $i = 0,\dots,N-1$.*

Then,

$$\left|\mathrm{Tr}\big(\rho_\beta^{\Lambda}\, B\big) - \mathrm{Tr}\big(\rho_\beta^{\Lambda'}\, B\big)\right| \le \|B\|\, f_{\mathrm{LPPL}}\big(|Y|\big)\, g\big(\|\Psi\|_F\big) \sum_{i=1}^{N} \zeta\big(d(Y,x_i)\big) \tag{15}$$

for all $Y \subset \Lambda'$ and $B \in \mathcal{A}_Y$, where g and ζ as in Lemma 28.

Moreover, if ζ decays fast enough, i.e. such that $\tilde\zeta(0)$ defined below converges, one can bound

$$\sum_{i=1}^{N} \zeta\big(d(Y,x_i)\big) \le |Y|\, \tilde\zeta\big(d(Y,\Lambda\setminus\Lambda')\big) \quad \textit{where} \quad \tilde\zeta(r) := 2^\nu \sum_{q=r}^{\infty} q^{\nu-1}\, \zeta(q). \tag{16}$$

Remark 30 (Simplified short-range version). Let Ψ be a short-range interaction on $\Lambda \Subset \mathbb{Z}^\nu$ such that $\|\Psi\|_{\exp(-b\,\cdot)} < \infty$ and assume that all Gibbs states $\rho_\beta^{\Lambda'}$ with $\Lambda' \subset \Lambda$ satisfy LPPL with respect to $\zeta_{\mathrm{LPPL}}(r) = C_{\mathrm{LPPL}}\, \mathrm{e}^{-c_{\mathrm{LPPL}} r}$, $f_{\mathrm{LPPL}}(|Y|) = |Y|^m$, $g_{\mathrm{LPPL}}(v) = \mathrm{e}^{3\beta v}$ and $n \ge 0$ in the sense of Definition 2. Then, for every $c < b\, c_{\mathrm{LPPL}}/(b + c_{\mathrm{LPPL}})$ there exists $C > 0$ such that for all $Y \subset \Lambda' \subset \Lambda$ and $B \in \mathcal{A}_Y$

$$\left|\mathrm{Tr}\big(\rho_\beta^{\Lambda}\, B\big) - \mathrm{Tr}\big(\rho_\beta^{\Lambda'}\, B\big)\right| \le C\, \|B\|\, |Y|^{m+1}\, \mathrm{e}^{3\beta\|\Psi\|_{\exp(-b\,\cdot)}}\, \mathrm{e}^{-c d(Y,\Lambda\setminus\Lambda')}. \qquad \diamond$$

Proof. Equation (15) follows from Lemma 28 and triangle inequality. To prove (16) we split $\Lambda\setminus\Lambda'$ into shells

$$S_q := \left\{ z \in \Lambda\setminus\Lambda' \,\middle|\, d(Y,z) = q \right\}$$

which clearly satisfy $\left|S_q\right| \le |Y|\, 2^\nu\, q^{\nu-1}$. Then, $\{x_i \,|\, i = 1,\dots,N\} = \bigcup_{q=d(Y,\Lambda\setminus\Lambda')}^{\infty} S_q$ and (16) follows. □

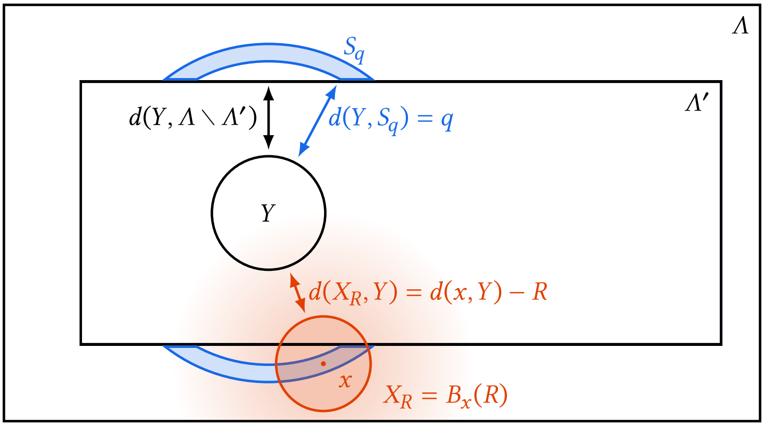

Figure 2. Depicted is the main idea for the proof of local indistinguishability from uniform LPPL. The idea is to remove all points $x \in \Lambda \setminus \Lambda'$ one by one. Therefore, we first apply LPPL to the sum of all interaction terms connecting x with its R-neighbourhood $B_x(R)$. For short-range interactions, the remaining interaction terms including x are exponentially small in R and can be removed using QBP. Furthermore, the points x are grouped into shells $S_q \ni x$ according to their distance $q := d(x, Y)$ to Y. We then choose the parameter R depending on q, so that the error for operators $B \in \mathscr{A}_Y$ introduced by removing all points in S_q decays exponentially in q. This allows to sum the error terms introduced by removing all shells with q and still obtain exponential decay in the distance $d(Y, \Lambda \setminus \Lambda')$.

To recover the result from Brandão and Kastoryano [45], note that they restrict to finite-range interactions and therefore only need to remove enough points along the boundary of Λ' to decouple the system in Λ' from the rest. Then they use the trivial bound $|\Lambda\setminus\Lambda'|\,\zeta\big(d(Y,\Lambda\setminus\Lambda')\big)$ for the sum in (15) and thus obtain a linear scaling in $|\partial\Lambda'|$. Our improvement is to observe (16) such that the statement is independent of $|\Lambda\setminus\Lambda'|$ and applicable to short-range interactions.

More specific results can be found in Section 9 for one-dimensional spin chains.

7 Uniform decay of correlations from local indistinguishability

In this section we briefly discuss how to close the circle of implications in Figure 1, i.e. how to conclude decay of correlations from local indistinguishability. This is a simple and well known consequence for finite-range Hamiltonians. We present the statement here for short- and long-range Hamiltonians. We will also discuss in detail the dependence on the support of the observables.

Theorem 31. *Let $\Lambda \Subset \mathbb{Z}^\nu$, F a decay function, and Ψ be an interaction such that $\|\Psi\|_F < \infty$ and assume that Ψ satisfies local indistinguishability at inverse temperature β with respect to f_{LI} and ζ_{LI} in the sense of Definition 3. Then, for all disjoint $X, Y \subset \Lambda$,*

$$\mathrm{Cov}_{\rho_\beta^\Lambda}(X;Y) \le \max\big\{|X|, f_{\mathrm{LI}}\big(|X|+|Y|\big)\big\}\,\zeta\big(d(X,Y)\big),$$

where

$$\zeta(r) := 3 \inf_{0\le\ell<r/2}\Big(\zeta_{\mathrm{LI}}(\ell) + (\mathrm{e}^{2\beta\|\Psi\|_F}-1)\,(2\ell+1)^\nu\,F(r-2\ell)\Big).$$

Remark 32 (Simplified short-range version). Let $\Lambda \Subset \mathbb{Z}^\nu$ and Ψ be a short-range interaction such that $\|\Psi\|_{\exp(-b\,\cdot)} < \infty$ and assume that it satisfies local indistinguishability at inverse temperature β with respect to $f_{\mathrm{LI}}\big(|Y|\big) = |Y|^m$ and $\zeta_{\mathrm{LI}}(r) = C_{\mathrm{LI}}\,\mathrm{e}^{-c_{\mathrm{LI}}r}$. Then, for every $c < b\,c_{\mathrm{LI}}/(2b+c_{\mathrm{LI}})$, there exists $C>0$ such that for all $X, Y \subset \Lambda$,

$$\mathrm{Cov}_{\rho_\beta^\Lambda}(X;Y) \le C\,\big(|X|+|Y|\big)^{\max\{1,m\}}\,\mathrm{e}^{-cd(X,Y)}. \qquad \diamond$$

Proof. Let $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$ with unit norm. Then choose $0 \le \ell < d(X,Y)/2$ and $\Lambda' = X_\ell \cup Y_\ell$. We first use local indistinguishability to approximate

$$\Big|\mathrm{Cov}_{\rho_\beta^\Lambda}(A,B) - \mathrm{Cov}_{\rho_\beta^{\Lambda'}}(A,B)\Big| \le 3\, f_{\mathrm{LI}}\big(|X|+|Y|\big)\,\zeta_{\mathrm{LI}}\big(\ell\big). \tag{17}$$

In the case of finite-range interactions with range R, one could choose ℓ so that $d(X_\ell,Y_\ell) > R$ for which $\rho_\beta^{\Lambda'} = \rho_\beta^{X_\ell} \otimes \rho_\beta^{Y_\ell}$ and $\mathrm{Cov}_{\rho_\beta^{\Lambda'}}(A,B) = 0$. For short-range

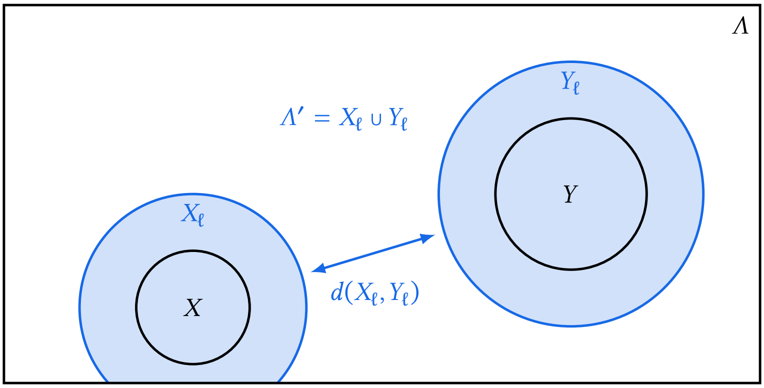

Figure 3. Depicted is the situation from the proof of Theorem 31. By local indistinguishability, the covariance of the Gibbs state on Λ and $\Lambda' = X_\ell \cup Y_\ell$ are similar. The remaining distance $d(X_\ell, Y_\ell)$ must be chosen so large that the remaining interactions coupling both regions are small. In the case of finite-range interactions, the distance must be chosen larger than the interaction range, so that the regions completely decouple.

interactions, however, we have to remove the interactions coupling regions X_ℓ and Y_ℓ first. Therefore, enumerate $X_\ell = \{x_1, x_2, \dots, x_N\}$ and let

$$V_i := \sum_{\substack{Z \subset \Lambda' : \\ x_i \in Z,\ x_1,\dots,x_{i-1} \notin Z, \\ Z \cap Y_\ell \neq \emptyset}} \Psi(Z), \quad \text{which satisfies} \quad \|V_i\| \leq \sum_{\substack{Z \subset \mathbb{Z}^\nu : \\ x_i \in Z, \\ \operatorname{diam}(Z) \geq d(X_\ell, Y_\ell)}} \|\Psi(Z)\| \leq \|\Psi\|_F\, F\big(d(X_\ell, Y_\ell)\big).$$

Moreover, $H_{\Lambda'} - \sum_{i=1}^N V_i = H_{X_\ell} + H_{Y_\ell}$ and by Proposition 15 (b) and (10)

$$\left|\operatorname{Cov}_{\rho_\beta^{\Lambda'}[H]}(A, B) - \operatorname{Cov}_{\rho_\beta^{\Lambda'}[H - V_i]}(A, B)\right| \leq 3\, F\big(d(X_\ell, Y_\ell)\big)\left(\mathrm{e}^{2\beta\|\Psi\|_F} - 1\right),$$

for all Hamiltonians $H \in \mathscr{A}_{\Lambda'}$ and all $i = 1, \dots, N$ as long as $d(X_\ell, Y_\ell) \geq 0$. Choosing $H = H_{\Lambda'} - \sum_{j=1}^{i-1} V_j$, by triangle inequality and vanishing of $\operatorname{Cov}_{\rho_\beta^{\Lambda}[H_{X_\ell} + H_{Y_\ell}]}(A, B)$ it follows that

$$\left|\operatorname{Cov}_{\rho_\beta^{\Lambda'}}(A, B)\right| \leq 3\, |X|\, (2\ell + 1)^\nu\, F\big(d(X_\ell, Y_\ell)\big)\left(\mathrm{e}^{2\beta\|\Psi\|_F} - 1\right). \tag{18}$$

Combining (17) and (18) concludes the proof. □

This statement closes the circle depicted in Figure 1.

Remark 33. Notice that the constants get worse in each step, when going around the circle shown in Figure 1. Indeed, let $b > 0$ and Ψ be an interaction such that $\|\Psi\|_{\exp(-b\,\cdot)} < \infty$. Moreover, let $\Lambda \Subset \mathbb{Z}^\nu$ and assume that ρ_β^Λ satisfies uniform exponential decay of correlations in the sense that there are constants C, c and n such that

$$\mathrm{Cov}_{\rho_\beta^{\Lambda'}}(X;Y) \le C\left(|X|+|Y|\right)^n \mathrm{e}^{-cd(X,Y)} \qquad \text{for all } X,Y \subset \Lambda' \subset \Lambda.$$

Then there exist constants $\tilde{C} > C$ and $\tilde{c} < c$, which in particular depend on C, c, β and $\|\Psi\|_{\exp(-b\,\cdot)}$, such that after going once through the statements about LPPL and local indistinguishability as indicated in Figure 1 one obtains

$$\mathrm{Cov}_{\rho_\beta^{\Lambda'}}(X;Y) \le \tilde{C}\left(|X|+|Y|\right)^{n+1} \mathrm{e}^{-\tilde{c}d(X,Y)} \qquad \text{for all } X,Y \subset \Lambda' \subset \Lambda. \qquad \diamond$$

8 Stability against small SLT perturbations from decay of correlations

This section is devoted to a result which is not part of the implications depicted in Figure 1. In Proposition 15 we already observed stability of the Gibbs state against small (in norm) perturbations. Using the idea from Theorem 25, we can extend this to stability against perturbations which are small in an interaction norm. We call these *sum-of-local-terms (SLT)* perturbations. Their norm grows like $|\Lambda|$. Hence, if we aim to find a bound uniform in the system size, they are not small in norm. In contrast to the bound in (9), which is in trace norm, we compare local expectation values of the Gibbs states, which is the natural topology for extended systems.

The idea to this observation comes from [187] and we give a rigorous proof here.

Theorem 34 (Stability against small SLT perturbations)**.** *Let $C_{\mathrm{int}} > 0$, $n \in \mathbb{N}_0$, $\beta_0 > 0$ and*

(a) *$\alpha_{\mathrm{Cov}} > \nu$, $b > 0$ and $F(r) := \mathrm{e}^{-br}$ or*

(b) *$\alpha, \alpha_{\mathrm{Cov}} > (n+1)\,\nu$ and $F(r) = F_\alpha(r) := (1+r)^{-\alpha}$.*

Then there exists a constant $C > 0$ such that the following holds: Let $\Lambda \Subset \mathbb{Z}^\nu$, $\beta \in (0,\beta_0)$, Ψ_H and Ψ_V be interactions such that $\|\Psi_H\|_F$, $\|\Psi_V\|_F < C_{\mathrm{int}}$ and denote the Gibbs state of $\Psi_H + s\,\Psi_V$ by $\rho_\beta^\Lambda(s)$. Assume that all Gibbs states satisfy decay of correlations with respect to $\zeta_{\mathrm{Cov}} \le (1+\cdot)^{-\alpha_{\mathrm{Cov}}}$, f_{Cov} and n as in Definition 1, then for all $Y \subset \Lambda$ and $B \in \mathscr{A}_Y$ we have

$$\left|\mathrm{Tr}\big(\rho_\beta(0)\,B\big) - \mathrm{Tr}\big(\rho_\beta(1)\,B\big)\right| \le C\,\beta\,\|\Psi_V\|_F\,|Y|\left(1 + f_{\mathrm{Cov}}\big(|Y|\big)\right)\|B\|.$$

Remark 35. Concerning temperature dependence, we observe that $\rho^{\Lambda}_{\beta+\Delta\beta}[H] = \rho^{\Lambda}_{\beta}\big[H + \frac{\Delta\beta}{\beta}H\big]$. Hence, Theorem 34 implies that, assuming all Gibbs states for inverse temperatures in $[\beta, \beta + \Delta\beta]$ satisfy decay of correlations (w.r.t. the functions above), local expectation values change slowly in β, namely

$$\big|\mathrm{Tr}\big(\rho^{\Lambda}_{\beta}\, B\big) - \mathrm{Tr}\big(\rho^{\Lambda}_{\beta+\Delta\beta}\, B\big)\big| \le C\, \Delta\beta\, \|\Phi_H\|_F\, |Y| \Big(1 + f_{\mathrm{Cov}}\big(|Y|\big)\Big)\, \|B\|,$$

uniformly in Λ with the constant from above. For only this statement, the proof could be simplified since $\Phi^{H(s)}_{\beta}(H) = H$ in (6) and (8). ◇

Proof. As in the proof of Theorem 25, we drop the subscript $_\beta$ and have

$$\begin{aligned}&\big|\mathrm{Tr}\big(\rho(1)\, B\big) - \mathrm{Tr}\big(\rho(0)\, B\big)\big|\\ &\quad\le \frac{\beta}{2} \sup_{s\in[0,1]} \Big(\big|\mathrm{Cov}_{\rho(s)}\big(\Phi^{H(s)}_{\beta}(V), B\big)\big| + \big|\mathrm{Cov}_{\rho(s)}\big(B, \Phi^{H(s)}_{\beta}(V)\big)\big|\Big),\end{aligned}$$

where $V = \sum_{Z\subset\Lambda} \Psi_V(Z)$. We now bound the first term, as the second is bounded analogously,

$$\begin{aligned}&\big|\mathrm{Cov}_{\rho(s)}\big(\Phi^{H(s)}_{\beta}(V), B\big)\big|\\ &\quad\le \sum_{k=0}^{\infty} \sum_{\substack{Z\subset\Lambda:\\ d(Z,Y)=k}} \Big|\mathrm{Cov}_{\rho(s)}\Big(\Phi^{H(s)}_{\beta}\big(\Psi_V(Z)\big), B\Big)\Big|\\ &\quad\le 2\,\|B\| \sum_{y\in Y} \sum_{\substack{Z\subset\Lambda:\\ y\in Z}} \|\Psi_V(Z)\|\\ &\qquad+ \|B\| \sum_{k=1}^{\infty} \sum_{\substack{Z\subset\Lambda:\\ d(Z,Y)=k}} \|\Psi_V(Z)\| \min_{0\le r}\Big(|Z_r|^n\, f_{\mathrm{Cov}}\big(|Y|\big)\, \zeta_{\mathrm{Cov}}\big(d(Z_r, Y)\big) + 2\,\zeta_{\mathrm{QBP}}(Z, r)\Big),\end{aligned}$$

where we used the approximation $\Phi^{H(s)}_{\beta,r}\big(\Psi_V(Z)\big)$ from Proposition 18 (a), $\mathrm{Cov}_{\rho(s)}(A, B) \le 2\,\|A\|\,\|B\|$, and the assumption on decay of correlations. The first summand is bounded by $2\,|Y|\,\|\Psi_V\|_F\,\|B\|$. For the second summand, we first replace $d(Z_r, Y) = \max\{d(Z, Y) - r, 0\}$. Then notice that for every $Z \subset \Lambda$ with $d(Z, Y) = k$, there exist $y \in Y$ and $z \in Z$ with $d(z, y) = k$. Moreover, under the assumptions (a) and (b), $\zeta_{\mathrm{QBP}}(Z, r) = |Z|\,\zeta_{\mathrm{QBP}}(r)$ scales linearly in $|Z|$ for some decaying $\zeta_{\mathrm{QBP}}(r)$, by Lemmata 20 and 46, respectively. Hence, by overcounting each of the terms with $k \ge 1$, the second summand is bounded by

$$\sum_{y\in Y}\sum_{\substack{z\in\Lambda:\\ d(z,y)=k}}\sum_{\substack{Z\subset\Lambda:\\ z\in Z}}\|\Psi_V(Z)\|\min_{0\le r\le k}\Big(|Z|^n\,(1+2r)^{\nu n}\,f_{\mathrm{Cov}}\big(|Y|\big)\,\zeta_{\mathrm{Cov}}\big(k-r\big)+2\,|Z|\,\zeta_{\mathrm{QBP}}(r)\Big)$$

$$\le |Y|\,2^\nu\,k^{\nu-1}\,C_1\min_{0\le r\le k}\Big((1+2r)^{\nu n}\,f_{\mathrm{Cov}}\big(|Y|\big)\,\zeta_{\mathrm{Cov}}\big(k-r\big)+2\,|Z|\,\zeta_{\mathrm{QBP}}(r)\Big),$$

where

$$C_1\ :=\sup_{z\in Z}\sum_{\substack{Z\subset\Lambda:\\ z\in Z}}|Z|^{\max\{1,n\}}\,\|\Psi_V(Z)\|<C_1'\,\|\Psi_V\|_F$$

and we used $\big|\{\,z\in\Lambda\,|\,d(z,y)=k\,\}\big|\le 2^\nu k^{\nu-1}$ and $|Z_r|\le|Z|\,(1+2r)^\nu$. Defining

$$C\ :=2+2^\nu\,C_1'\sum_{k=1}^{\infty}k^{\nu-1}\min_{0\le r\le k}\Big(\,(1+2r)^{\nu n}\,\zeta_{\mathrm{Cov}}\big(k-r\big)+2\,\zeta_{\mathrm{QBP}}(r)\Big),$$

the bound given in the statement follows when $C<\infty$.

For the case of short-range interactions with assumption (a), $\zeta_{\mathrm{QBP}}(r)\le 6\,\mathrm{e}^{-c_{\mathrm{QBP}}\,r}$ for some $c_{\mathrm{QBP}}>0$, which can be chosen uniformly in $\beta\in(0,\beta_0)$ and $\|\Psi_H+s\,\Psi_V\|_F\le 2\,C_{\mathrm{int}}$ by Lemma 20 (a). To bound C_1, we use $|Z|\le(\mathrm{diam}(Z)+1)^\nu$ to find $C_1'\le\max_{r\in\mathbb{N}}(r+1)^{n\nu}\,\mathrm{e}^{-br}<\infty$, which only depends on b and n. Finally, C is bounded if $\zeta_{\mathrm{Cov}}(k)\le(1+k)^{-q}$ with $q>\nu$, by choosing $r=k^{(q-\nu)/(2\nu n)}/2$.

For long-range interactions, $C_1'\le 1$ and for every $\alpha_{\mathrm{QBP}}<\alpha$ and β_0 there exist C_{QBP} such that $\zeta_{\mathrm{QBP}}(r)\le C_{\mathrm{QBP}}\,F_{\alpha_{\mathrm{QBP}}}(r)$ by Lemma 46 (a). Thus, C can be bounded after choosing $r=k/2$ under the assumptions specified in (b). □

9 Results for one-dimensional short-range systems

We now restrict our attention to translation-invariant one-dimensional spin chains.

For every $x\in\mathbb{Z}$, $n\in\mathbb{N}$ consider $Y=[x,x+n]\subset\mathbb{Z}$ and for every $A\in\mathcal{A}_{\{x\}}$ define $t_n(A)\in\mathcal{A}_{\{x+n\}}$ by $t_n(A)=A\otimes\mathbb{1}_{Y\setminus\{x+n\}}$, where we made use of the canonical identification of $\mathcal{A}_{Y\setminus\{x+n\}}$ as a subalgebra of $\mathcal{A}_Y$. Then, let $I\Subset\mathbb{Z}$ and consider the set $I+n=\{\,x\in\mathbb{Z}\,|\,x-n\in I\,\}$. Let $Y\subset\mathbb{Z}$ such that $I,I+n\subset Y$. Define the algebra $*$-isomorphism $t_n^I\colon\mathcal{A}_I\to\mathcal{A}_{I+n}$ by $t_n^I(\otimes_{i\in I}A_i)=\otimes_{i\in I}t_n(A_i)\otimes\mathbb{1}_{Y\setminus(I+n)}$. The $*$-isomorphisms t_n^I induce a $*$-algebra automorphism τ_n of the algebra of quasi-local observables $\mathcal{A}_{\mathbb{Z}}$. Analogously, one can define τ_n for $-n\in\mathbb{N}$. The family $\{\tau_n\}_{n\in\mathbb{Z}}$ is called the family of lattice translations. Translation-invariant interactions are interactions that satisfy the additional constraint that for all $X\Subset\mathbb{Z}$ and $n\in\mathbb{Z}$

$$\tau_n\,\Psi(X)=\Psi(X+n).$$

In this section we assume $\nu = 1$ and consider only local translation-invariant interactions on $\mathbb{Z}$. For every finite interval $I = [x, y] \Subset \mathbb{Z}$ and every inverse temperature $\beta > 0$, we additionally denote the Gibbs state as a functional $\varphi_\beta^{[x,y]} : \mathscr{A}_{[x,y]} \to \mathbb{C}$ on the operator algebra, defined by

$$\varphi_\beta^{[x,y]}(Q) := \mathrm{Tr}\big(\rho_\beta^{[x,y]} Q\big) \quad \text{for all} \quad Q \in \mathscr{A}_{[x,y]}$$

and extended to a state on $\mathscr{A}_\mathbb{Z}$, e.g. by using the Hahn-Banach Theorem as in [48, Proposition 2.3.24]. We denote by φ_β the unique KMS state over the infinite chain at inverse temperature $\beta > 0$ [10]. Following the discussion in e.g. [49, Proposition 6.2.15], for every increasing and absorbing sequence $I_n \nearrow \mathbb{Z}$, the sequence of states $\varphi_\beta^{I_n}$ is weak*-convergent to φ_β, in particular

$$\varphi_\beta(Q) = \lim_{n\to\infty} \varphi_\beta^{I_n}(Q) \quad \text{for all} \quad Q \in \mathscr{A}_\mathbb{Z}.$$

In the infinite-chain regime, we define the covariance to measure correlations as an extension of equation (2) by

$$\mathrm{Cov}_{\rho_\beta}(A, B) := \varphi_\beta(A\,B) - \varphi_\beta(A)\,\varphi_\beta(B),$$

for all $A, B \in \mathscr{A}_\mathbb{Z}$.

Araki [9] proved that a translation-invariant, finite-range interaction satisfies *exponential decay of correlations*, and this was recently extended to short-range interactions in [185]. That is, there exist constants $\mathscr{K}, \alpha > 0$ such that for every $x \in \mathbb{Z}$, $k \in \mathbb{N}$, $A \in \mathscr{A}_{(-\infty,x]}$ and $B \in \mathscr{A}_{[x+k,\infty)}$,

$$\big|\varphi_\beta(A\,B) - \varphi_\beta(A)\,\varphi_\beta(B)\big| \le \mathscr{K}\, \mathrm{e}^{-\alpha\, k}\, \|A\|\, \|B\|,$$

whenever $\beta < \beta^*$, where the precise form of β^* is given in Theorem 6. For finite-range interactions, the result was extended to the finite-chain regime in [36], where it was proven that any condition of uniform decay of correlations in the infinite-chain can be transferred to the finite-chain, and vice versa. It is natural to ask whether this is also correct in the presence of exponentially decaying interactions. And one of the main results of this section is a positive answer to this question. The precise formulation of this theorem has already appeared in Section 3.1, but we restate it here for convenience with a new notation. To simplify notation, we will drop the union sign $XY := X \cup Y$ from unions of finite intervals $X, Y \subset \mathbb{Z}$, and we will write hereafter XY whenever $X < Y$ and YX for $Y < X$. And since we only deal with short-range interactions, we abbreviate

$$\|\Psi\|_b := \|\Psi\|_{\exp(-b\,\cdot)}, \qquad b \ge 0.$$

We restate Theorem 6 with this notation. Its proof is deferred to Section 9.1.

Theorem 6. *Let $b > 0$ and Ψ be a translation-invariant interaction such that $\|\Psi\|_b < \infty$ and denote $\beta^* := b/(2\,\|\Psi\|_0)$. Then, for all $\beta \in (0, \beta^*)$, the Gibbs state satisfies decay of correlations in the sense that there exist $C_{\mathrm{Cov}}, c_{\mathrm{Cov}} > 0$ such that for every finite interval $I \subset \mathbb{Z}$ and subintervals $X, Y \subset I$, it holds that*

$$\mathrm{Cov}_{\rho_\beta^I}(X;Y) \leq C_{\mathrm{Cov}}\, \mathrm{e}^{-c_{\mathrm{Cov}} d(X,Y)}.$$

Note that, for finite-range interactions, we have $\beta^* = \infty$, recovering the results of [9, 36]. Additionally, in [36], it was shown that having exponential decay of correlations for finite-range interactions is equivalent to local indistinguishability. This is extended below to exponentially decaying interactions as well. Following the lines of the equivalence of notions of locality and decay of correlations presented in Figure 1, we first show that exponential decay of correlations implies LPPL, and subsequently prove that the latter implies local indistinguishability. This is reflected in Figure 4. The use of quantum belief propagation and Lieb-Robinson bounds is again pivotal to derive these results.

In contrast to the results at high temperature, Theorem 6 only provides decay of correlations between two operators each supported on an interval. As a consequence, we will only prove LPPL for perturbations V supported in an interval $X \subset I$ and observables $B = B_1 \otimes B_2$ with B_1 and B_2 supported on intervals Y_1 and Y_2, respectively, where $Y_1 < X < Y_2$ as in Figure 5. Thus, we use slightly different definition for the three concepts decay of correlations, LPPL, and local indistinguishability and adjust some of the arguments. One could extend all to arbitrary operators in $\mathcal{A}_{Y_1 \cup Y_2}$ by using the Schmidt decomposition and allowing for an exponential growth in $|Y_1 \cup Y_2|$.

Theorem 36. *For every b, C_{int}, β, C_{Cov} and $c_{\mathrm{Cov}} > 0$, there exist constants C_{LPPL} and $c_{\mathrm{LPPL}} > 0$ such that the following holds: Let Ψ be an interaction on $\mathbb{Z}$ satisfying $\|\Psi\|_b \leq C_{\mathrm{int}}$, $I \Subset \mathbb{Z}$ be a finite interval, and assume that the corresponding Gibbs state ρ_β^I satisfies decay of correlations in the sense given in Theorem 6 with constants C_{Cov} and c_{Cov}. Then the Gibbs state satisfies LPPL in the sense that for all subintervals $X, Y_1, Y_2 \subset I$, such that $Y_1 < X < Y_2$, $Y = Y_1 \cup Y_2$, all self-adjoint perturbations $V \in \mathcal{A}_X$ and all observables $B = B_1 \otimes B_2$ with $B_1 \in \mathcal{A}_{Y_1}$ and $B_2 \in \mathcal{A}_{Y_2}$, it holds true that*

$$\left|\mathrm{Tr}\big(\rho_\beta^I[H_I]\,B\big) - \mathrm{Tr}\big(\rho_\beta^I[H_I + V]\,B\big)\right| \leq C_{\mathrm{LPPL}}\,\|B\|\,\mathrm{e}^{3\beta\|V\|}\,\big(1 + \|V\|\big)\,\mathrm{e}^{-c_{\mathrm{LPPL}} d(X,Y)}.$$

If also $\|\Psi + V\|_b \leq C_{\mathrm{int}}$, one can drop the factor $\big(1 + \|V\|\big)$.

The geometry described in the statement is depicted in Figure 5. In Theorem 36 included is the case where Y_i is empty and $B_i = \mathbb{1}$ which corresponds to Y being a single interval. Together with Theorem 6, Theorem 36 shows that LPPL holds for local, translation-invariant interactions Ψ satisfying $\|\Psi\|_b < \infty$ at inverse temperatures $\beta < \beta^* := b/(2\,\|\Psi\|_0)$.

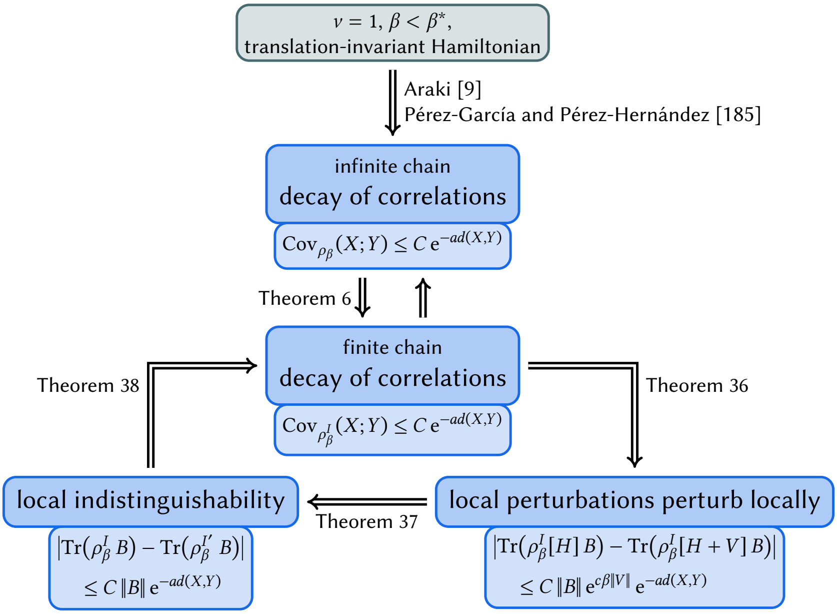

Figure 4. The diagram shows the main implications for one-dimensional (translation-invariant) spin chains, which are discussed in this section. Here, $I \subset \mathbb{Z}$ is a finite interval, $X \subset I$ a subinterval and $Y \subset I$ a union of two intervals. In particular, we show "equivalence" of the four concepts in the picture. Note that the constants are not the same, and we refer to the Theorems for precise statements. A crucial ingredient in all the implications is quantum belief propagation (QBP) coupled with Lieb-Robinson bounds. For finite-range or exponentially decaying interactions, exponential decay of correlations is known to hold by earlier results for the infinite-chain regime at every positive or high enough temperature, respectively, for which all four properties are thus satisfied.

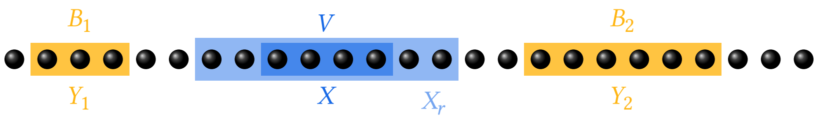

Figure 5. Representation of an interval I with subintervals $X, Y_1, Y_2 \subset I$. An example of a perturbation $V \in \mathscr{A}_X$ such that the distance between X_r and $Y = Y_1 \cup Y_2$ is at least r. Here, $r = \lfloor d(X,Y)/2 \rfloor = 2$.

Proof. Let us denote $r := \lfloor d(X,Y)/2 \rfloor$ and abbreviate $\rho := \rho^I_\beta[H_I]$ and $\tilde\rho := \rho^I_\beta[H_I + V]$. From the last line in the proof of Theorem 22 we obtain

$$\left|\mathrm{Tr}\big(\rho\, B\big) - \mathrm{Tr}\big(\tilde\rho\, B\big)\right| \leq \mathrm{e}^{2\beta\|V\|}\Big(\sup_{\substack{W\in\mathcal{A}_{X_r}:\\ \|W\|=1}} \left|\mathrm{Cov}_\rho(W,B)\right| + 4\,\beta\,\|V\|\,\|B\|\,\zeta_{\mathrm{QBP}}(X,r)\Big), \tag{19}$$

in this setting, where we write the covariance with a supremum only in the first argument, to use the product structure of $B = B_1 \otimes B_2 = B_1\, B_2$ to obtain

$$\mathrm{Cov}_\rho(W, B_1\, B_2) = \mathrm{Cov}_\rho(W\, B_1, B_2) + \mathrm{Tr}(\rho\, B_2)\ \mathrm{Cov}_\rho(W, B_1) - \mathrm{Tr}(\rho\, W)\ \mathrm{Cov}_\rho(B_1, B_2).$$

By Theorem 6, there exist constants C_{Cov} and c_{Cov} only depending on $\|\Psi\|_b$ and β such that

$$\left|\mathrm{Cov}_\rho(W, B_1\, B_2)\right| \leq 3\,C_{\mathrm{Cov}}\,\|W\|\,\|B_1\|\,\|B_2\|\,\mathrm{e}^{-c_{\mathrm{Cov}} d(X_r,Y)}.$$

And since $B = B_1 \otimes B_2$ is a tensor product, $\|B_1\|\,\|B_2\| = \|B\|$, such that

$$\sup_{W\in\mathcal{A}_{X_r}:\,\|W\|=1} \left|\mathrm{Cov}_\rho(W,B)\right| \leq 3\,C_{\mathrm{Cov}}\,\|B\|\,\mathrm{e}^{-\alpha d(X,Y)/2}. \tag{20}$$

For the decay function ζ_{QBP} we use Lemma 20 for intervals X. Thus, for every b, C_{int}, β, C_{Cov} and c_{Cov}, there exists C_{QBP} and c_{QBP} such that

$$\zeta_{\mathrm{QBP}}(X,r) \leq C_{\mathrm{QBP}}\,\big(1 + \|V\|\big)^\gamma\,\mathrm{e}^{-c_{\mathrm{QBP}} r}, \tag{21}$$

with $\gamma = 0$ if $\|\Psi + V\|_b \leq C_{\mathrm{int}}$ and $\gamma = 1$ otherwise. Replacing now (20) and (21) into (19), we obtain

$$\begin{aligned}
&\left|\mathrm{Tr}\big(\rho\, B\big) - \mathrm{Tr}\big(\tilde\rho\, B\big)\right| \\
&\qquad \leq \mathrm{e}^{2\beta\|V\|}\,\|B\|\,\Big(3\,C_{\mathrm{Cov}}\,\mathrm{e}^{-c_{\mathrm{Cov}} d(X,Y)/2} + 4\,\beta\,\|V\|\,C_{\mathrm{QBP}}\,\mathrm{e}^{-c_{\mathrm{QBP}}\left(d(X,Y)/2-1\right)}\Big) \\
&\qquad \leq C_{\mathrm{LPPL}}\,\mathrm{e}^{3\beta\|V\|}\,\|B\|\,\mathrm{e}^{-c_{\mathrm{LPPL}} d(X,Y)},
\end{aligned}$$

where $c_{\mathrm{LPPL}} := \min\{c_{\mathrm{Cov}}, c_{\mathrm{QBP}}\}/2$ and $C_{\mathrm{LPPL}} := 3\,C_{\mathrm{Cov}} + 4\,C_{\mathrm{QBP}}\,\mathrm{e}^{c_{\mathrm{QBP}}}$, and we have used that $x \leq \mathrm{e}^x$. □

Next, we show that this slightly restricted version of LPPL also implies local indistinguishability in one-dimensional spin chains. Together with Theorems 6 and 36, Theorem 37 indeed gives that local indistinguishability holds for local, translation-invariant interactions Ψ satisfying $\|\Psi\|_b < \infty$ at inverse temperatures $\beta < \beta^* := b/(2\,\|\Psi\|_0)$.

Theorem 37. *For every b, β, C_{LPPL} and $c_{\mathrm{LPPL}} > 0$, there exist constants C_{LI} and $c_{\mathrm{LI}} > 0$ such that the following holds: Let Ψ be an interaction on $\mathbb{Z}$ satisfying $\|\Psi\|_b < \infty$, $I \Subset \mathbb{Z}$ be a finite interval, and assume that the corresponding Gibbs state ρ^I_β satisfies LPPL in* *the sense given in Theorem 36 with constants C_{LPPL} and c_{LPPL}. Then the Gibbs state satisfies local indistinguishability in the sense that for all subintervals $X, Y_1, Y_2 \subset I$, so that $Y_1 < X < Y_2$, $Y = Y_1 \cup Y_2$, and all observables $B = B_1 \otimes B_2$ with $B_1 \in \mathcal{A}_{Y_1}$ and $B_2 \in \mathcal{A}_{Y_2}$, it holds true that*

$$\left|\mathrm{Tr}\big(\rho_\beta^I\, B\big) - \mathrm{Tr}\big(\rho_\beta^{I\setminus X}\, B\big)\right| \le C_{\mathrm{LI}}\, \|B\|\, \mathrm{e}^{-c_{\mathrm{LI}} d(Y,X)}.$$

Proof. Let us first explain the conceptual difference to the proof of local indistinguishability in Section 6: Since we do not have uniform decay of correlations, we cannot remove X site by site. And looking back at Lemma 28 we should not remove X in one step, because that leads to an exponential scaling in $|X|$. Instead, here we only remove the interactions between X and $I \setminus X$, which leads to an exponential scaling in $|\partial X| = 2$.

If $X \cap Y \neq \emptyset$ the statement is trivial, otherwise we have

$$\mathrm{Tr}\big(\rho_\beta^{I\setminus X} B\big) = \mathrm{Tr}\big(\rho_\beta^{I\setminus X} \otimes \rho_\beta^X\, B\big) = \mathrm{Tr}\big(\rho_\beta^I [H_{I\setminus X} + H_X]\, B\big)$$

for all $B \in \mathcal{A}_Y$. Thus, we can apply Theorem 36 with the perturbation given by $V := H_I - H_{I\setminus X} - H_X$, i.e. all interaction terms whose support intersects X and $I \setminus X$. Following the idea of Lemma 28, we split V into

$$V_0 := \sum_{\substack{Z\subset I:\\ (I\setminus X)\cap Z\neq\emptyset\\ X\cap Z\neq\emptyset\\ \mathrm{diam}(Z)\le R}} \Psi(Z) \qquad \text{and} \qquad V' := \sum_{\substack{Z\subset I:\\ (I\setminus X)\cap Z\neq\emptyset\\ X\cap Z\neq\emptyset\\ \mathrm{diam}(Z)> R}} \Psi(Z),$$

for some $R \in \mathbb{N}$ to be chosen later. Since each Z in the sum of V_0 needs to contain a site in $\{\, x \in X \,|\, d(x, I \setminus X) \le R \,\}$ we have $\|V_0\| \le |\partial X|\, R\, \|\Psi\|_0$ and $V_0 \in \mathcal{A}_{X_R}$. Thus, by Theorem 36 we find

$$\begin{aligned}\left|\mathrm{Tr}\big(\rho_\beta^I\, B\big) - \mathrm{Tr}\big(\rho_\beta^I [H_I - V_0]\, B\big)\right| &\le C_{\mathrm{LPPL}}\, \|B\|\, \mathrm{e}^{6\beta\|\Psi\|_0 R}\, \mathrm{e}^{-c_{\mathrm{LPPL}}(d(X,Y)-R)}\\ &\le C_{\mathrm{LPPL}}\, \|B\|\, \mathrm{e}^{-c_{\mathrm{LPPL}} d(X,Y)/2},\end{aligned}$$

by choosing $\alpha := c_{\mathrm{LPPL}}\, (6\beta\|\Psi\|_0 + c_{\mathrm{LPPL}})^{-1}/2$ and $R := \lfloor \alpha\, d(X,Y) \rfloor$. Then, denoting $q_n := \{\, x \in X \,|\, d(x, I \setminus X) = n \,\}$ and $Q_n := \bigcup_{i=1}^{n-1} q_i$ we split $V' = \sum_{n=1}^{\infty} V_n$ with

$$V_n := \sum_{\substack{Z\subset I\setminus Q_n:\\ Z\cap q_n\neq\emptyset\\ Z\cap(I\setminus X)\neq\emptyset\\ \mathrm{diam}(Z)>R}} \Psi(Z),$$

where the sum is actually finite. Then, $\|V_n\| \le 2\, \|\Psi\|_b\, \mathrm{e}^{-b\max\{R+1,n\}}$. Using Proposition 15 (b) and equation (10), we thus find

$$\left|\mathrm{Tr}\big(\rho_\beta^I [H_I - \textstyle\sum_{i=0}^{n-1} V_i]\, B\big) - \mathrm{Tr}\big(\rho_\beta^I [H_I - \textstyle\sum_{i=0}^{n} V_i]\, B\big)\right| \le \|B\|\, \mathrm{e}^{-b\max\{R+1,n\}} \left(\mathrm{e}^{4\beta\|\Psi\|_b} - 1\right).$$

Hence, by triangle inequality, $\sum_{n=R+1}^{\infty} \mathrm{e}^{-bn} < \mathrm{e}^{-bR}/b$, $\sup_{R>0} R\,\mathrm{e}^{-bR/2} \leq 2/(b\,\mathrm{e})$ and $2/\mathrm{e} + 1 < 2$ we obtain

$$\left|\mathrm{Tr}\big(\rho_\beta^I [H_I - V_0]\, B\big) - \mathrm{Tr}\big(\rho_\beta^I [H_{I\setminus X} + H_X]\, B\big)\right| \leq 2\, b^{-1}\, \|B\|\, \mathrm{e}^{-bR/2} \left(\mathrm{e}^{4\beta\|\Psi\|_b} - 1\right).$$

Again by triangle inequality,

$$\begin{aligned}\left|\mathrm{Tr}\big(\rho_\beta^I\, B\big) - \mathrm{Tr}\big(\rho_\beta^{I\setminus X}\, B\big)\right| &\leq C_{\mathrm{LPPL}}\, \|B\|\, \mathrm{e}^{-c_{\mathrm{LPPL}} d(X,Y)/2} + 2\, b^{-1}\, \|B\|\, \mathrm{e}^{-bR/2} \left(\mathrm{e}^{4\beta\|\Psi\|_b} - 1\right) \\ &\leq C_{\mathrm{LI}}\, \|B\|\, \mathrm{e}^{-c_{\mathrm{LI}}\, d(Y,X)},\end{aligned}$$

with $C_{\mathrm{LI}} := C_{\mathrm{LPPL}} + 2\, b^{-1} \left(\mathrm{e}^{4\beta\|\Psi\|_b} - 1\right) \mathrm{e}^{-b/2}$ and $c_{\mathrm{LI}} := \min\{c_{\mathrm{LPPL}}/2, \alpha\, b/2\}$. □

To conclude the circle in Figure 4, we would need to show that local indistinguishability implies decay of correlations. This is completely analogous to Theorem 31 for any dimension at high enough temperature.

Theorem 38. *For every b, β, C_{LI} and $c_{\mathrm{LI}} > 0$, there exist constants C_{Cov} and $c_{\mathrm{Cov}} > 0$ such that the following holds: Let Ψ be an interaction on $\mathbb{Z}$ satisfying $\|\Psi\|_b < \infty$, $I \Subset \mathbb{Z}$ be a finite interval, and assume that the corresponding Gibbs state ρ_β^I satisfies local indistinguishability in the sense given in Theorem 37 with constants C_{LI} and c_{LI}. Then, the Gibbs state satisfies decay of correlations in the sense that for all disjoint intervals $X, Y \subset I$, and all $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$, it holds true that*

$$\mathrm{Cov}_{\rho_\beta^I}(X; Y) \leq C_{\mathrm{Cov}}\, \mathrm{e}^{-c_{\mathrm{Cov}} d(X,Y)}. \tag{22}$$

Proof. Without loss of generality we assume $X < Y$ and extend them to the boundary of I such that $I = X\chi Y$ where also $\chi \subset I$ is an interval and X, Y and χ are pairwise disjoint. This puts us in a situation, where we can proceed almost as in Theorem 31, but with the restricted version of local indistinguishability where we can only remove one interval.

Take $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$, let $r < d(X,Y)/2$ to be chosen later and consider $I' := X_r \cup Y_r$. We first use local indistinguishability to cut out a part between X and Y

$$\left|\mathrm{Cov}_{\rho_\beta^I}(A, B) - \mathrm{Cov}_{\rho_\beta^{I'}}(A, B)\right| \leq 3\, C_{\mathrm{LI}}\, \|A\|\, \|B\|\, \mathrm{e}^{-c_{\mathrm{LI}} r}.$$

Then we remove the remaining interactions coupling the regions X_r and Y_r as in the proof of Theorem 31 with a slight modification: We choose the enumeration $X_r = \{x_1, x_2, \dots, x_N\}$ decreasing such that $\|V_i\| \leq \|\Phi\|_b\, \mathrm{e}^{-b\left(d(X_r,Y_r)+(i-1)\right)}$, then we can resum as in the proof of Theorem 37 to obtain a result independent of $|X|$:

$$\left|\mathrm{Cov}_{\rho_\beta^{I'}}(A, B)\right| \leq 3\, \mathrm{e}^b\, b^{-1}\, \|A\|\, \|B\|\, \mathrm{e}^{-bd(X_r,Y_r)} \left(\mathrm{e}^{2\beta\|\Psi\|_b} - 1\right).$$

Choosing $\alpha := b\,(c_{\mathrm{LI}} + 2b)^{-1}$ and $r := \lfloor \alpha\, d(X,Y) \rfloor$ gives the bound in (22) with $c_{\mathrm{Cov}} := b\, c_{\mathrm{LI}}\, (c_{\mathrm{LI}} + 2b)^{-1}$ and $C_{\mathrm{Cov}} := 3\, C_{\mathrm{LI}}\, \mathrm{e}^{c_{\mathrm{LI}}} + 3\, \mathrm{e}^b\, b^{-1} \left(\mathrm{e}^{2\beta\|\Psi\|_b} - 1\right)$. □

There are various reasons for studying the case of one-dimensional spin chains separately. On one hand, all results from this section present the obvious advantage with respect to those from Section 3.4 in the range of β for which they hold, since the β^* in this case reduces to ∞ for super-exponentially decaying interactions, as opposed to the case of high dimensions. However, they have the drawback that one needs to assume translation invariance for this to be true. This is a direct consequence of the regimes where correlations are known to decay exponentially fast in one- and higher-dimensions, respectively. Other cases in which correlations are known to decay with slower rates for one-dimensional systems, such as those of short-range and long-range interactions at every positive temperature, are discussed in Sections 3.2 and 3.3, respectively.

On the other hand, the study of decay of correlations, with different measures than that given by the covariance, in one-dimensional spin chains with translation-invariant, finite-range interactions, has been incredibly fruitful in the past few years. Given a finite interval $I \subset \mathbb{Z}$, with $X, Y \subset I$, $X \cap Y \neq \emptyset$, a state ρ on I, and denoting $\rho_Z := \operatorname{tr}_{I \setminus Z}(\rho)$ for $Z \subset I$, some other quantities of relevance in this context are for example the *mutual information*, given by $I_\rho(X, Y) := \operatorname{Tr}\big(\rho_{XY} (\log \rho_{XY} - \log \rho_X \otimes \rho_Y)\big)$, and the *mixing condition*, given by $\left\| \rho_{XY}\, \rho_X^{-1} \otimes \rho_Y^{-1} - \mathbb{1}_{XY} \right\|$. It is not difficult to show, see [36, Section 3.1], that, for any state ρ on I such that ρ_{XY} is full-rank, the following holds:

$$\tfrac{1}{2} \operatorname{Cov}_\rho(X; Y)^2 \leq I_\rho(X, Y) \leq \left\| \rho_{XY}\, \rho_X^{-1} \otimes \rho_Y^{-1} - \mathbb{1}_{XY} \right\|.$$

Thus, a ζ-decay with $d(X, Y)$ for the mixing condition implies the same for the mutual information and a $\sqrt{2\zeta}$-decay for the covariance. Interestingly, in one-dimensional spin chains with translation-invariant, finite-range interactions, a converse is proven for the Gibbs state $\rho = \rho_\beta^I$, and all the latter conditions are shown to have equivalent decays. We expect that a similar result can be derived in the short-range regime. See [37] for an analogous result in high dimensions, at high-enough temperature.

9.1 Exponential decay of correlations for short-range interactions

This subsection is devoted to the proof of Theorem 6. The procedure we will follow is very similar to that of [36, Theorem 6.2]. First, note that I can be written as $I = Z_1 X Z_2 Y Z_3$ for certain intervals $Z_1, Z_2, Z_3 \subset I$. Without loss of generality, let us assume for this proof that both Z_1 and Z_3 are empty so that we only have to prove the result for the case in which $I = XZY$. If not, we enlarge X and Y, which will only allow for more observables A and B and yield the same bound.

For the rest of the section, we fix $b > 0$, a translation-invariant interaction Ψ satisfying $\|\Psi\|_b < \infty$ and $\beta^* := b/(2\,\|\Psi\|_0)$.

We need to rephrase some results from [185], where the authors use a different interaction norm[2]

$$\|\Psi\|_{\lambda} := \sum_{n=0}^{\infty} \mathrm{e}^{\lambda n} \sup_{z\in\mathbb{Z}} \sum_{\substack{Z\Subset\mathbb{Z}:\\ z\in Z,\\ \mathrm{diam}(Z)\geq n}} \|\Psi(Z)\|.$$

It can be upper bounded with our norm for every $\lambda < b$, i.e.

$$\|\Psi\|_{\lambda} \leq \sum_{n=0}^{\infty} \mathrm{e}^{\lambda n} \|\Psi\|_b \, \mathrm{e}^{-bn} \leq \frac{1}{1-\mathrm{e}^{\lambda-b}} \|\Psi\|_b,$$

such that $\|\Psi\|_b < \infty$ implies $\|\Psi\|_{\lambda} < \infty$ for all $\lambda < b$. The statements we use from [185] all hold for $\beta < \beta^*_{\lambda} := \lambda/(2\,\|\Psi\|_0)$ if $\|\Psi\|_{\lambda} < \infty$. With the above observation, they thus also hold for all $\beta < \beta^*$.

Next, for $\beta > 0$, $a \in \mathbb{Z}$, and $p, q \in \mathbb{N}_0$, we define the expansional

$$E^{\beta}_{a,p,q} = \mathrm{e}^{-\beta H_{[a-p,a+1+q]}} \, \mathrm{e}^{\beta H_{[a-p,a]}} \, \mathrm{e}^{\beta H_{[a+1,a+1+q]}}.$$

Then we extract the following Lemma from [185, Corollary 3.3 and section 4.1].

Lemma 39. *Let $\beta < \beta^*$. Then there exist constants $\mathscr{G} > 1$ and $\delta > 0$ such that for all $a \in \mathbb{Z}$, and $p, q \geq 0$,*

$$\|E^{\beta}_{a,p,q}\|, \|(E^{\beta}_{a,p,q})^{-1}\| \leq \mathscr{G},$$

and for all $q' \geq q$,

$$\|E^{\beta}_{a,p,q} - E^{\beta}_{a,p,q'}\| \leq \mathscr{G}\mathrm{e}^{-\delta q}.$$

For $V, W \Subset \mathbb{Z}$, we introduce the slightly more general expansionals

$$E^{\beta}_{V,W} := \mathrm{e}^{-\beta H_{VW}} \, \mathrm{e}^{\beta H_V} \, \mathrm{e}^{\beta H_W}.$$

As a consequence of the previous lemma, we can provide the following bounds for these expansionals.

Lemma 40. *Let $\beta < \beta^*$. Then there exist constants $\mathscr{G} > 1$ and $\delta > 0$ such that for all disjoint intervals $V, W \Subset \mathbb{Z}$,*

$$\|E^{\beta}_{V,W}\|, \|(E^{\beta}_{V,W})^{-1}\| \leq \mathscr{G},$$

and if we append intervals $\tilde{V}$ and $\tilde{W} \Subset \mathbb{Z}$ to V and W, respectively, it holds that

$$\|E^{\beta}_{V,W} - E^{\beta}_{\tilde{V}V,W\tilde{W}}\| \leq \mathscr{G}\,\mathrm{e}^{-\delta q},$$

as long as $|V|, |W| \geq q$.

[2] While the symbol is the same, this is obviously a different norm than the one used for long-range interactions in Section 3.3.

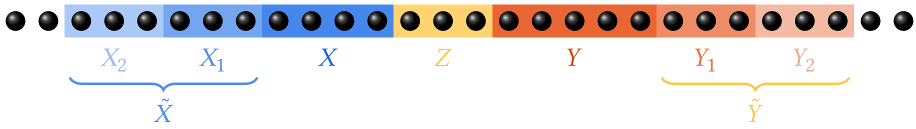

Figure 6. Splitting of an enlarged interval $\tilde{I}$ of an original interval I with $X, Y \subset I$ and $d(X,Y) > 0$, to which we have appended regions $\tilde{X}$ and $\tilde{Y}$ on the left and the right side, respectively, which are subsequently divided in two subregions each.

Now, let us enlarge the finite interval $I = XZY \subset \mathbb{Z}$ to $\tilde{I} = \tilde{X}XZY\tilde{Y}$ as in Figure 6. The appended systems $\tilde{X}$ and $\tilde{Y}$ are split into two subsystems each, denoted $\tilde{X} = X_2 X_1$ and $\tilde{Y} = Y_1 Y_2$. Let us further assume, without loss of generality, that $|X|, |Y| \geq |Z|$, otherwise we can again redefine X, Y and Z by moving the left third of Z into X and the right third of Z into Y, reducing $|Z|$ to $1/3$ of its original value. Finally, we also choose $|X_1|, |X_2|, |Y_1|, |Y_2| > |Z|$.

This brings us in position of proving exponential decay of correlations in this setting for which we now also fix $\beta < \beta^*$ and the constants $\mathscr{G}$ and δ from Lemma 40. We define

$$E^{-\beta/2}_{\tilde{X},XZY,\tilde{Y}} := \mathrm{e}^{\frac{\beta}{2}H_{\tilde{X}XZY\tilde{Y}}}\,\mathrm{e}^{-\frac{\beta}{2}H_{\tilde{X}}-\frac{\beta}{2}H_{XZY}-\frac{\beta}{2}H_{\tilde{Y}}},$$

which can be easily shown to coincide with

$$E^{-\beta/2}_{\tilde{X}XZY,\tilde{Y}}\,E^{-\beta/2}_{\tilde{X},XZY}.$$

Then, by Lemma 40

$$\begin{aligned}
&\left\|E^{-\beta/2}_{\tilde{X},XZY,\tilde{Y}} - E^{-\beta/2}_{X_1,X}\,E^{-\beta/2}_{Y,Y_1}\right\| \\
&\quad= \left\|E^{-\beta/2}_{\tilde{X}XZY,\tilde{Y}}\,E^{-\beta/2}_{\tilde{X},XZY} - E^{-\beta/2}_{X_1,X}\,E^{-\beta/2}_{Y,Y_1}\right\| \\
&\quad\leq \left\|E^{-\beta/2}_{\tilde{X}XZY,\tilde{Y}}\,E^{-\beta/2}_{\tilde{X},XZY} - E^{-\beta/2}_{Y,Y_1}\,E^{-\beta/2}_{\tilde{X},XZY}\right\| + \left\|E^{-\beta/2}_{Y,Y_1}\,E^{-\beta/2}_{\tilde{X},XZY} - E^{-\beta/2}_{X_1,X}\,E^{-\beta/2}_{Y,Y_1}\right\| \\
&\quad\leq \mathscr{G}\left\|E^{-\beta/2}_{\tilde{X}XZY,\tilde{Y}} - E^{-\beta/2}_{Y,Y_1}\right\| + \mathscr{G}\left\|E^{-\beta/2}_{\tilde{X},XZY} - E^{-\beta/2}_{X_1,X}\right\| \\
&\quad\leq 2\,\mathscr{G}\,\mathrm{e}^{-\delta d(X,Y)}.
\end{aligned}$$

Next, following the calculations of [36, Section 6], and recalling that $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$, it is not difficult to realize that

$$\varphi^I_\beta(A\,B) = \frac{\varphi^{\tilde{I}}_\beta\Big(\big(E^{-\beta/2}_{\tilde{X},XZY,\tilde{Y}}\big)\,A\,B\,\big(E^{-\beta/2}_{\tilde{X},XZY,\tilde{Y}}\big)^*\Big)}{\varphi^{\tilde{I}}_\beta\Big(\big(E^{-\beta/2}_{\tilde{X},XZY,\tilde{Y}}\big)\,\big(E^{-\beta/2}_{\tilde{X},XZY,\tilde{Y}}\big)^*\Big)}.$$

In a slight abuse of notation, let us consider the map

$$\begin{array}{ccc} \mathscr{A}_I & \to & \mathscr{A}_I \\ Q & \mapsto & \mathrm{Tr}_I(\rho_\beta^I Q) \end{array}$$

for the unnormalized partial trace in I

$$\mathrm{Tr}_I : \begin{array}{ccc} \mathscr{A}_J & \to & \mathscr{A}_J \\ R \otimes S & \mapsto & \mathrm{Tr}(R)(\mathbb{1}_I \otimes S) \end{array}$$

for any $I \subset J$, and any $R \in \mathscr{A}_I$, $S \in \mathscr{A}_{J\setminus I}$. Then, it is contractive as a consequence of the Russo-Dye theorem, and using this as well as Lemma 40, we see that

$$\mathscr{G}^{-4} \leq \varphi_\beta^{\tilde{I}}\Big(\big(E^{-\beta/2}_{\tilde{X},XZY,\tilde{Y}}\big)\,\big(E^{-\beta/2}_{\tilde{X},XZY,\tilde{Y}}\big)^*\Big)$$

and

$$\varphi_\beta^{\tilde{I}}\Big(\big(E^{-\beta/2}_{X_1,X}\big)\,\big(E^{-\beta/2}_{X_1,X}\big)^*\,\big(E^{-\beta/2}_{Y,Y_1}\big)\,\big(E^{-\beta/2}_{Y,Y_1}\big)^*\Big) \leq \mathscr{G}^4.$$

This is fundamental for proving that $\varphi_\beta^I(A\,B)$ can be approximated by

$$\frac{\varphi_\beta^{\tilde{I}}\Big(\big(E^{-\beta/2}_{X_1,X}\big)\,A\,\big(E^{-\beta/2}_{X_1,X}\big)^*\,\big(E^{-\beta/2}_{Y,Y_1}\big)\,B\,\big(E^{-\beta/2}_{Y,Y_1}\big)^*\Big)}{\varphi_\beta^{\tilde{I}}\Big(\big(E^{-\beta/2}_{X_1,X}\big)\,\big(E^{-\beta/2}_{X_1,X}\big)^*\,\big(E^{-\beta/2}_{Y,Y_1}\big)\,\big(E^{-\beta/2}_{Y,Y_1}\big)^*\Big)}$$

up to an error that decays exponentially with the distance between X and Y. Indeed, using the following standard inequality for scalars $a, a', b, b' \in \mathbb{C}$

$$\left|\frac{a}{b} - \frac{a'}{b'}\right| \leq \frac{1}{|b|}\,|a - a'| + \frac{|a'|}{|b|\,|b'|}\,|b - b'|, \tag{23}$$

we can prove

$$\begin{aligned} &\left|\varphi_\beta^I(A\,B) - \frac{\varphi_\beta^{\tilde{I}}\Big(\big(E^{-\beta/2}_{X_1,X}\big)\,A\,\big(E^{-\beta/2}_{X_1,X}\big)^*\,\big(E^{-\beta/2}_{Y,Y_1}\big)\,B\,\big(E^{-\beta/2}_{Y,Y_1}\big)^*\Big)}{\varphi_\beta^{\tilde{I}}\Big(\big(E^{-\beta/2}_{X_1,X}\big)\,\big(E^{-\beta/2}_{X_1,X}\big)^*\,\big(E^{-\beta/2}_{Y,Y_1}\big)\,\big(E^{-\beta/2}_{Y,Y_1}\big)^*\Big)}\right| \\ &\quad\leq \mathscr{G}^4\,\Big\|\big(E^{-\beta/2}_{\tilde{X},XZY,\tilde{Y}}\big)\,A\,B\,\big(E^{-\beta/2}_{\tilde{X},XZY,\tilde{Y}}\big)^* - \big(E^{-\beta/2}_{X_1,X}\big)\,A\,\big(E^{-\beta/2}_{X_1,X}\big)^*\,\big(E^{-\beta/2}_{Y,Y_1}\big)\,B\,\big(E^{-\beta/2}_{Y,Y_1}\big)^*\Big\| \\ &\qquad + \mathscr{G}^{12}\,\|A\|\,\|B\|\,\Big\|\big(E^{-\beta/2}_{\tilde{X},XZY,\tilde{Y}}\big)\,\big(E^{-\beta/2}_{\tilde{X},XZY,\tilde{Y}}\big)^* - \big(E^{-\beta/2}_{X_1,X}\big)\,\big(E^{-\beta/2}_{X_1,X}\big)^*\,\big(E^{-\beta/2}_{Y,Y_1}\big)\,\big(E^{-\beta/2}_{Y,Y_1}\big)^*\Big\| \\ &\quad\leq \tilde{\mathscr{G}}\,\|A\|\,\|B\|\,\mathrm{e}^{-\delta d(X,Y)}, \end{aligned} \tag{24}$$

for $\tilde{\mathscr{G}} > 1$ a constant. Since we have approximated $\varphi_\beta^I(A\,B)$ by an expression that is independent of X_2 and Y_2, we can take a limit with $|X_2|, |Y_2| \to \infty$. Thus, $\varphi_\beta^I(A\,B)$ can be also approximated by

$$\frac{\varphi_\beta\Big(\big(E_{X_1,X}^{-\beta/2}\big)\,A\,\big(E_{X_1,X}^{-\beta/2}\big)^*\,\big(E_{Y,Y_1}^{-\beta/2}\big)\,B\,\big(E_{Y,Y_1}^{-\beta/2}\big)^*\Big)}{\varphi_\beta\Big(\big(E_{X_1,X}^{-\beta/2}\big)\,\big(E_{X_1,X}^{-\beta/2}\big)^*\,\big(E_{Y,Y_1}^{-\beta/2}\big)\,\big(E_{Y,Y_1}^{-\beta/2}\big)^*\Big)}$$

keeping the same error as in the finite chain case. Now we can apply decay of correlations for the infinite chain as proven in [185].

Theorem 41 ([185, Theorem 4.4]). *Let $\beta < \beta^*$. Then there exist constants $\mathscr{G} > 1$ and $\delta > 0$ such that for all $a, b \in \mathbb{Z}$ with $a < b$, $A \in \mathscr{A}_{(-\infty,a]}$ and $B \in \mathscr{A}_{[b,\infty)}$ it holds that*

$$\big|\varphi_\beta(A\,B) - \varphi_\beta(A)\,\varphi_\beta(B)\big| \leq \|A\|\,\|B\|\,\mathscr{G}\,\mathrm{e}^{-\delta|b-a|}.$$

Thus, together with (23) and similar inequalities to the ones above for the infinite chain, we can find $\mathscr{K} > 1$ and $\gamma > 0$ such that

$$\left\| \frac{\varphi_\beta\Big(\big(E_{X_1,X}^{-\beta/2}\big)\,A\,\big(E_{X_1,X}^{-\beta/2}\big)^*\,\big(E_{Y,Y_1}^{-\beta/2}\big)\,B\,\big(E_{Y,Y_1}^{-\beta/2}\big)^*\Big)}{\varphi_\beta\Big(\big(E_{X_1,X}^{-\beta/2}\big)\,\big(E_{X_1,X}^{-\beta/2}\big)^*\,\big(E_{Y,Y_1}^{-\beta/2}\big)\,\big(E_{Y,Y_1}^{-\beta/2}\big)^*\Big)} - \frac{\varphi_\beta\Big(\big(E_{X_1,X}^{-\beta/2}\big)\,A\,\big(E_{X_1,X}^{-\beta/2}\big)^*\Big)}{\varphi_\beta\Big(\big(E_{X_1,X}^{-\beta/2}\big)\,\big(E_{X_1,X}^{-\beta/2}\big)^*\Big)}\,\frac{\varphi_\beta\Big(\big(E_{Y,Y_1}^{-\beta/2}\big)\,B\,\big(E_{Y,Y_1}^{-\beta/2}\big)^*\Big)}{\varphi_\beta\Big(\big(E_{Y,Y_1}^{-\beta/2}\big)\,\big(E_{Y,Y_1}^{-\beta/2}\big)^*\Big)} \right\| \leq \mathscr{K}\,\|A\|\,\|B\|\,\mathrm{e}^{-\gamma|Z|}, \tag{25}$$

so that together with (24) the second summand in the norm is an approximation of $\varphi_\beta^I(A\,B)$. By choosing $A = \mathbb{1}$ we also obtain

$$\left| \varphi_\beta^I(A) - \frac{\varphi_\beta\Big(\big(E_{X_1,X}^{-\beta/2}\big)\,A\,\big(E_{X_1,X}^{-\beta/2}\big)^*\Big)}{\varphi_\beta\Big(\big(E_{X_1,X}^{-\beta/2}\big)\,\big(E_{X_1,X}^{-\beta/2}\big)^*\Big)} \right| \leq \tilde{\mathscr{K}}\,\|A\|\,\|B\|\,\mathrm{e}^{-\tilde{\gamma}|Z|}$$

from (24) and (25), and an analogous statement also for the expectation value of B with sets Y, Y_1. Combining these three approximations, we conclude that there exist $C_{\mathrm{Cov}} > 1$ and $c_{\mathrm{Cov}} > 0$ such that

$$\big|\varphi_\beta^I(A\,B) - \varphi_\beta^I(A)\,\varphi_\beta^I(B)\big| \leq C_{\mathrm{Cov}}\,\|A\|\,\|B\|\,\mathrm{e}^{-c_{\mathrm{Cov}}d(X,Y)},$$

thus concluding the result.

10 Properties of quantum belief propagation

In this section we review the proof of the quantum belief propagation for the Gibbs state. The central point is to analyse how the Gibbs state of a certain Hamiltonian is modified when the system is perturbed.

10.1 Differential equations

We begin with the discussion of Proposition 15 where no underlying lattice structure is necessary. Therefore, let $\mathscr{H}$ be a finite-dimensional Hilbert space and H and V be two self-adjoint operators on $\mathscr{H}$. Furthermore, let $H(s) := H + sV$.

10.1.1 Differential equation for the exponential $\mathrm{e}^{-\beta H(s)}$

The basic idea is to write a differential equation for $\mathrm{e}^{-\beta H(s)}$. Hastings [115] proves a version where

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{e}^{-\beta H(s)} = A\,\mathrm{e}^{-\beta H(s)} + \mathrm{e}^{-\beta H(s)}\,A^*$$

for some quasi-local (in the case of an underlying lattice structure) A, while later works use an anti-commutator form

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{e}^{-\beta H(s)} = -\frac{\beta}{2}\left\{\mathrm{e}^{-\beta H(s)}, \Phi_\beta^{H(s)}(V)\right\}$$

which we prove in the following.

We mostly follow the proof from Hastings [115] but more explicitly work in the energy eigenbasis of $H(s)$. Let $\{\psi_a(s)\}_a$ be the eigenbasis of $H(s)$ such that $H(s) = \sum_a E_a(s)\,|\psi_a(s)\rangle\langle\psi_a(s)|$. Then we can also write V in this basis as

$$V = \sum_{a,b} V_{a,b}(s)\,|\psi_a(s)\rangle\langle\psi_b(s)|.$$

Using Duhamel's formula we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{e}^{-\beta H(s)} &= -\beta\int_0^1 \mathrm{e}^{-\beta\tau H(s)}\,V\,\mathrm{e}^{-\beta(1-\tau)H(s)}\,\mathrm{d}\tau\\
&= -\beta\sum_{a,b}\int_0^1 \mathrm{e}^{-\beta\tau H(s)}\,V_{a,b}(s)\,|\psi_a(s)\rangle\langle\psi_b(s)|\,\mathrm{e}^{\beta\tau H(s)}\,\mathrm{d}\tau\,\mathrm{e}^{-\beta H(s)}\\
&= -\beta\sum_{a,b}V_{a,b}(s)\int_0^1 \mathrm{e}^{\beta\tau\Delta E_{a,b}(s)}\,\mathrm{d}\tau\,|\psi_a(s)\rangle\langle\psi_b(s)|\,\mathrm{e}^{-\beta H(s)}\\
&= -\beta\sum_{a,b}V_{a,b}(s)\left(1+\mathrm{e}^{\beta\Delta E_{a,b}(s)}\right)^{-1}\int_0^1 \mathrm{e}^{\beta\tau\Delta E_{a,b}(s)}\,\mathrm{d}\tau\,\{\mathrm{e}^{-\beta H(s)}, |\psi_a(s)\rangle\langle\psi_b(s)|\}\\
&= -\frac{\beta}{2}\left\{\mathrm{e}^{-\beta H(s)}, \Phi_\beta^{H(s)}(V)\right\},
\end{aligned}$$

with $\Delta E_{a,b}(s) := E_b(s) - E_a(s)$ and

$$\Phi_\beta^{H(s)}(V) := \sum_{a,b}\hat{f}_\beta\big(\Delta E_{a,b}(s)\big)\,V_{a,b}(s)\,|\psi_a(s)\rangle\langle\psi_b(s)| = \int_{-\infty}^{\infty} f_\beta(t)\,\mathrm{e}^{-\mathrm{i}tH(s)}\,V\,\mathrm{e}^{\mathrm{i}tH(s)}\,\mathrm{d}t,$$

where

$$\hat{f}_\beta(\omega) := 2\left(1+\mathrm{e}^{\beta\omega}\right)^{-1} \int_0^1 \mathrm{e}^{\beta\tau\omega}\,\mathrm{d}\tau = \begin{cases} \dfrac{2}{\beta\omega}\,\dfrac{\mathrm{e}^{\beta\omega}-1}{\mathrm{e}^{\beta\omega}+1} & \omega \neq 0 \\ 1 & \omega = 0 \end{cases} = \frac{\tanh\frac{\beta\omega}{2}}{\frac{\beta\omega}{2}}$$

and $f_\beta(t)$ is its inverse Fourier transform given by, see [8, SM Sec. 5.1] and [80, appendix C],

$$f_\beta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\,\hat{f}_\beta(\omega)\,\mathrm{d}\omega = \frac{2}{\beta\pi}\,\log\!\left(\frac{\mathrm{e}^{\pi|t|/\beta}+1}{\mathrm{e}^{\pi|t|/\beta}-1}\right). \tag{26}$$

The inverse Fourier transform satisfies $\|f_\beta\|_{L^1(\mathbb{R})} = 1$ and, since $\ln(x) \leq x-1$,

$$f_\beta(t) \leq \frac{4}{\beta\pi}\,\frac{1}{\mathrm{e}^{\pi|t|/\beta}-1}, \tag{27}$$

which decays exponentially in $|t|$.

We can now give the general solution of this equation. Therefore, let

$$\begin{aligned} \eta(s) &:= \mathcal{T}\exp\!\left(-\tfrac{\beta}{2}\int_0^s \Phi_\beta^{H(\sigma)}(V)\,\mathrm{d}\sigma\right) \\ &:= \sum_{n=0}^{\infty}\left(-\tfrac{\beta}{2}\right)^n \int_0^s \mathrm{d}\sigma_1 \int_0^{\sigma_1}\mathrm{d}\sigma_2 \cdots \int_0^{\sigma_{n-1}} \mathrm{d}\sigma_n\, \Phi_\beta^{H(\sigma_1)}(V)\,\Phi_\beta^{H(\sigma_2)}(V)\cdots\Phi_\beta^{H(\sigma_n)}(V), \end{aligned} \tag{28}$$

with $\mathcal{T}$ being the time-ordering operator. Then its adjoint is

$$\begin{aligned} \eta(s)^* &:= \overline{\mathcal{T}}\exp\!\left(-\tfrac{\beta}{2}\int_0^s \Phi_\beta^{H(\sigma)}(V)\,\mathrm{d}\sigma\right) \\ &:= \sum_{n=0}^{\infty}\left(-\tfrac{\beta}{2}\right)^n \int_0^s \mathrm{d}\sigma_1 \int_0^{\sigma_1}\mathrm{d}\sigma_2 \cdots \int_0^{\sigma_{n-1}} \mathrm{d}\sigma_n\, \Phi_\beta^{H(\sigma_n)}(V)\,\Phi_\beta^{H(\sigma_{n-1})}(V)\cdots\Phi_\beta^{H(\sigma_1)}(V), \end{aligned}$$

with $\overline{\mathcal{T}}$ being the reverse-time-ordering operator. By definition, these satisfy

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\eta(s) = -\tfrac{\beta}{2}\,\Phi_\beta^{H(s)}(V)\,\eta(s) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}s}\,\eta(s)^* = -\tfrac{\beta}{2}\,\eta(s)^*\,\Phi_\beta^{H(s)}(V).$$

Hence,

$$\mathrm{e}^{-\beta H(s)} = \eta(s)\,\mathrm{e}^{-\beta H(0)}\,\eta(s)^*.$$

Moreover, we have

$$\|\eta(s)\| \leq \mathrm{e}^{\frac{\beta}{2}s\|V\|} \qquad \text{and} \qquad \|\eta(s) - \mathbb{1}\| \leq \mathrm{e}^{\frac{\beta}{2}s\|V\|} - 1.$$

Later, in Section 10.2, we discuss how to approximate $\Phi_\beta^{H(s)}(V)$ and $\eta(s)$ by strictly local operators if one has an underlying lattice structure.

10.1.2 Differential equation for the Gibbs state $\rho_\beta(s)$

In the previous section, we discussed an evolution for the exponential $\mathrm{e}^{-\beta H(s)}$. Due to the missing normalization, this cannot be directly applied to the Gibbs state $\rho_\beta(s) = \mathrm{e}^{-\beta H(s)}/\operatorname{Tr}(\mathrm{e}^{-\beta H(s)})$, which was overlooked by some previous works, e.g. [45]. Instead, we will discuss how to deal with the full Gibbs state $\rho_\beta(s)$ in the following.

By Leibniz rule, equation (6) allows writing the differential equation for the normalized Gibbs state $\rho_\beta(s)$ as

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\rho_\beta(s) = -\frac{\beta}{2}\Big\{\rho_\beta(s), \Phi_\beta^{H(s)}(V)\Big\} + \beta\,\rho_\beta(s)\operatorname{Tr}\Big(\rho_\beta(s)\,\Phi_\beta^{H(s)}(V)\Big). \tag{29}$$

It appears, with one missing $\rho_\beta(s)$, already in [134, equation (32)] (see also [135, Lemma 17)]) and, tested against local observables, in [194, appendix C]. Note that we have $\operatorname{Tr}\big(\rho_\beta(s)\,\Phi_\beta^{H(s)}(V)\big) = \operatorname{Tr}\big(\rho_\beta(s)\,V\big) =: \langle V\rangle_{\rho_\beta(s)}$ due to cyclicity of the trace. Hence, we can further simplify (29) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\rho_\beta(s) = -\frac{\beta}{2}\Big\{\rho_\beta(s), \Phi_\beta^{H(s)}\big(V - \langle V\rangle_{\rho_\beta(s)}\big)\Big\}.$$

This equation is not linear in $\rho_\beta(s)$ anymore, but presuming we know $\rho_\beta(s)$, it still gives a nice evolution

$$\rho_\beta(s) = \tilde\eta(s)\,\rho_\beta(0)\,\tilde\eta(s)^*,$$

with

$$\tilde\eta(s) := \mathcal{T}\exp\bigg(-\frac{\beta}{2}\int_0^s \Phi_\beta^{H(\sigma)}\big(V - \langle V\rangle_{\rho_\beta(\sigma)}\big)\,\mathrm{d}\sigma\bigg).$$

This $\tilde\eta(s)$ is very similar to $\eta(s)$ from (28) and thus has similar properties. In particular,

$$\|\tilde\eta(s)\| \le \mathrm{e}^{\frac{\beta}{2}s\sup_{\sigma\le s}\|V-\langle V\rangle_{\rho_\beta(\sigma)}\|} \le \mathrm{e}^{\beta s\|V\|} \qquad \text{and} \qquad \|\tilde\eta(s) - \mathbb{1}\| \le \mathrm{e}^{\beta s\|V\|} - 1.$$

As a simple consequence,

$$\begin{aligned}
\big\|\rho_\beta(0) - \rho_\beta(s)\big\|_1 &= \sup_{\|A\|=1}\Big|\operatorname{Tr}\Big(\big(\rho_\beta(0) - \rho_\beta(s)\big)\,A\Big)\Big| \\
&= \sup_{\|A\|=1}\big\|\rho_\beta(0)\big\|_1\,\big\|A - \tilde\eta^*(s)\,A\,\tilde\eta(s)\big\| \\
&\le \sup_{\|A\|=1}\Big(\big\|A\,\big(\mathbb{1} - \tilde\eta(s)\big)\big\| + \big\|\big(\mathbb{1} - \tilde\eta^*(s)\big)\,A\,\tilde\eta(s)\big\|\Big) \\
&\le \mathrm{e}^{2\beta\|V\|} - 1 \\
&\le 2\beta\,\|V\|\,\mathrm{e}^{2\beta\|V\|}.
\end{aligned}$$

The formula for $\tilde{\eta}(s)$ given above is particularly useful to obtain the same locality results as for $\eta(s)$, see the next section. However, since $\langle V\rangle_{\rho_\beta(\sigma)}$ is just a number (with an implicit unit $\mathbb{1}$ in the definition of $\tilde{\eta}(s)$), we can also factor

$$\tilde{\eta}(s) = \exp\left(-\frac{\beta}{2}\int_0^s \langle V\rangle_{\rho_\beta(\sigma)}\,\mathrm{d}\sigma\right)\eta(s).$$

Observing

$$\tilde{\eta}(s)\,\rho_\beta(0)\,\tilde{\eta}^*(s) = \rho_\beta(s) = \frac{\mathrm{Tr}\big(\mathrm{e}^{-\beta H(0)}\big)}{\mathrm{Tr}\big(\mathrm{e}^{-\beta H(s)}\big)}\,\eta(s)\,\rho_\beta(0)\,\eta^*(s),$$

we can thus conclude

$$\frac{\mathrm{Tr}\big(\mathrm{e}^{-\beta H(0)}\big)}{\mathrm{Tr}\big(\mathrm{e}^{-\beta H(s)}\big)} = \exp\left(-\beta\int_0^s \langle V\rangle_{\rho_\beta(\sigma)}\,\mathrm{d}\sigma\right).$$

This might help to understand the difference between the two differential equations (6) and (8) for $\mathrm{e}^{-\beta H(s)}$ and $\rho_\beta(s)$, respectively. In particular, $\eta(s)\,\rho_\beta(0)\,\eta^*(s)$ differs from $\rho_\beta(s)$ by exactly this factor. However, for explicit computations of $\rho_\beta(s)$ the original approach of [115] to compute $\eta(s)\,\mathrm{e}^{-\beta H(0)}\,\eta^*(s)$ and normalize might be more practical.

10.2 Locality properties

In this section we discuss, how to obtain Proposition 18 from Proposition 15, which we discussed in the previous section. Therefore, we now restrict to the Hilbert spaces $\mathscr{H}_\Lambda$ with underlying lattice structure as discussed in Section 2.

We now fix the interaction Ψ of the Hamiltonian H_Λ and the perturbation $V \in \mathscr{A}_X$ with $X \subset \Lambda$. We will only write the proof of Proposition 18 (b). Part 18 (c) then follows by choosing $\tilde{V} := V - \langle V\rangle_{\rho_\beta(s)}$ and noting that we can write $\langle V\rangle_{\rho_\beta(s)} = \langle V\rangle_{\rho_\beta(s)}\,\mathbb{1}_X \in \mathscr{A}_X$ such that also $\tilde{V} \in \mathscr{A}_X$.

10.2.1 Lieb-Robinson bound for the perturbed Hamiltonian $H_\Lambda + s\,V$

To prove locality of the generator $\Phi_\beta^{H(s)}(V)$ and the exponential $\eta(s)$ in the next sections, we need Lieb-Robinson bounds for the Hamiltonian $H(s) = H + V$. Hence, we need to extend the Lieb-Robinson bounds for $H(0)$ to $s > 0$.

Lemma 42 (Lieb-Robinson bound for perturbed Hamiltonians)**.** *Let $\Lambda \subset \mathbb{Z}^\nu$ and $H \in \mathscr{A}_\Lambda$ self-adjoint, and assume that H satisfies a Lieb-Robinson bound with decay ζ_{LR}^H as in Definition 17. Moreover, let $V \in \mathscr{A}_X$ self-adjoint, then $H + V$ satisfies the Lieb-Robinson bound*

$$\begin{aligned}&\big\|\big[\mathrm{e}^{-\mathrm{i}t(H+V)}\,A\,\mathrm{e}^{\mathrm{i}t(H+V)}, B\big]\big\| \\ &\qquad \le \|A\|\,\|B\|\left(\zeta_{\mathrm{LR}}^H(Y,Z,|t|) + 2\,\|V\|\min_{W\in\{Z,Y\}}\int_0^{|t|}\zeta_{\mathrm{LR}}^H(X,W,s)\,\mathrm{d}s\right)\end{aligned} \tag{30}$$

for all $A \in \mathscr{A}_Y$, $B \in \mathscr{A}_Z$ *and* $t \in \mathbb{R}$.

We will apply the above result in the cases $A = V$ or $A = \tilde{V}$ for which $X = Y$ and the minimum is attained at $W = Z$. Moreover, since Lieb-Robinson bounds are increasing in s, the integral in (30) can be bounded by $|t|\,\zeta^H_{\mathrm{LR}}(X, W, |t|)$ so that (30) is bounded by

$$\|A\|\,\|B\|\,\big(1 + 2\,\|V\|\,|t|\big)\,\zeta^H_{\mathrm{LR}}\big(X, Y, |t|\big)$$

in this case.

Proof. We abbreviate $\tau_t(A) := \mathrm{e}^{-\mathrm{i}tH}\,A\,\mathrm{e}^{\mathrm{i}tH}$ and $\tilde{\tau}_t(A) := \mathrm{e}^{-\mathrm{i}t(H+V)}\,A\,\mathrm{e}^{\mathrm{i}t(H+V)}$. We will later prove

$$\|\tilde{\tau}_t(A) - \tau_t(A)\| \le \|V\|\,\|A\| \int_0^{|t|} \zeta^H_{\mathrm{LR}}(X, Y, s)\,\mathrm{d}s, \tag{31}$$

which allows to bound

$$\begin{aligned} \big\|\big[\tilde{\tau}_t(A), B\big]\big\| &\le \big\|\big[\tau_t(A), B\big]\big\| + \big\|\big[\tilde{\tau}_t(A) - \tau_t(A), B\big]\big\| \\ &\le \|A\|\,\|B\|\,\Big(\zeta^H_{\mathrm{LR}}(Y, Z, |t|) + 2\,\|V\| \int_0^t \zeta^H_{\mathrm{LR}}(X, Y, |s|)\,\mathrm{d}s\Big). \end{aligned}$$

The minimum can be concluded using $\big\|\big[\tilde{\tau}_t(A), B\big]\big\| = \big\|\big[A, \tilde{\tau}_{-t}(B)\big]\big\|$ and the above argument with the roles of A and B exchanged.

To prove (31) we follow very closely the proof of [174, Theorem 3.4 (i)]. By fundamental theorem of calculus,

$$\tilde{\tau}_t(A) - \tau_t(A) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\Big(\tilde{\tau}_s \circ \tau_{t-s}(A)\Big)\,\mathrm{d}s = \mathrm{i} \int_0^t \tilde{\tau}_s\Big(\big[V, \tau_{t-s}(A)\big]\Big)\,\mathrm{d}s,$$

which gives (31) by using the Lieb-Robinson bound for $\big\|\big[V, \tau_s(A)\big]\big\|$. □

In the special case of a short-range Hamiltonian H satisfying the Lieb-Robinson bound given in Proposition 44, we can actually carry out the integration in (30) and obtain the following result.

Corollary 43. *Let* Ψ *be a short-range interaction with* $\|\Psi\|_{\exp(-b\,\cdot)} < \infty$ *for some* $b > 0$. *Then, for all* $X, Y \subset \Lambda \Subset \mathbb{Z}^\nu$, $V \in \mathscr{A}_X$ *self-adjoint,* $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$ *and* $t \in \mathbb{R}$

$$\big\|\big[\mathrm{e}^{-\mathrm{i}t(H+V)}\,A\,\mathrm{e}^{\mathrm{i}t(H+V)}, B\big]\big\| \le 2\,\|A\|\,\|B\|\left(1 + \frac{2\,\|V\|}{b\,v_b}\right)\mathrm{e}^{b v_b |t|} \sum_{x \in X} \mathrm{e}^{-b d(x,Y)},$$

with v_b *as in Proposition 44.*

Lemma 42 justifies to assume that $H(s)$ satisfies a Lieb-Robinson bound with decay ζ_{LR} uniformly in s. In particular, the perturbation V only changes the constant in the Lieb-Robinson bound, but not the Lieb-Robinson velocity. This Lieb-Robinson bound allows to approximate the Heisenberg time evolution $\mathrm{e}^{-\mathrm{i}tH(s)}\,W\,\mathrm{e}^{\mathrm{i}tH(s)}$ of any $W \in \mathscr{A}_X$ with a local operator in $\mathscr{A}_{X_r}$.

10.2.2 Locality of the "generator" $\Phi_\beta^{H(s)}(V)$

Given the Lieb-Robinson bound, there exists a conditional expectation [174, Lemma 4.1] $\mathbb{E}_{X_r} : \mathscr{A}_\Lambda \to \mathscr{A}_{X_r}$, which for us is just the normalized partial trace since we are in finite dimensions, such that

$$\left\|(\mathbb{1} - \mathbb{E}_{X_r})\left(\mathrm{e}^{-\mathrm{i}tH(s)}\, W\, \mathrm{e}^{\mathrm{i}tH(s)}\right)\right\| \leq \|W\|\, \zeta_{\mathrm{LR}}(X, \Lambda \setminus X_r, t) \quad \text{for all } W \in \mathscr{A}_X. \tag{32}$$

It allows decomposing $\Phi_\beta^{H(s)}(V) = \Delta_r(s) + \overline{\Delta}_r(s)$ into two parts

$$\begin{aligned} \Delta_r(s) &:= \int_{-\infty}^{\infty} \mathrm{d}t\, f_\beta(t)\, \mathbb{E}_{X_r}\left(\mathrm{e}^{-\mathrm{i}tH(s)}\, V\, \mathrm{e}^{\mathrm{i}tH(s)}\right) \quad \text{and} \\ \overline{\Delta}_r(s) &:= \int_{-\infty}^{\infty} \mathrm{d}t\, f_\beta(t)\, (\mathbb{1} - \mathbb{E}_{X_r})\left(\mathrm{e}^{-\mathrm{i}tH(s)}\, V\, \mathrm{e}^{\mathrm{i}tH(s)}\right). \end{aligned}$$

By definition $\Delta_r(s) \in \mathscr{A}_{X_r}$ is strictly local and bounded by $\|\Delta_r(s)\| \leq \|V\|$. Using the Lieb-Robinson bound and the conditional expectation we can bound the remainder by

$$\begin{aligned} \|\overline{\Delta}_r(s)\| &\leq \|V\| \int_{|t|\leq T} f_\beta(t)\, \zeta_{\mathrm{LR}}(X, \Lambda \setminus X_r, t)\, \mathrm{d}t + 2\,\|V\| \int_{|t|\geq T} f_\beta(t)\, \mathrm{d}t \\ &\leq \|V\| \left(\|f_\beta\|_{L^1(\mathbb{R})}\, \|\zeta_{\mathrm{LR}}(X, \Lambda \setminus X_r, \cdot)\|_{L^\infty([-T,T])} + \frac{32}{\pi^2}\, \mathrm{e}^{-\frac{\pi}{\beta}T} \right). \end{aligned}$$

Here, we bounded the second integral assuming $T \geq \frac{\beta}{\pi} \ln 2$, by using (27) and substituting $u = \frac{\pi}{\beta} t - \ln 2$

$$\int_{|t|\geq T} f_\beta(t)\, \mathrm{d}t \leq \frac{8}{\pi^2} \int_{\frac{\pi}{\beta}T - \ln 2}^{\infty} \frac{1}{2\,\mathrm{e}^u - 1}\, \mathrm{d}u \leq \frac{16}{\pi^2}\, \mathrm{e}^{-\frac{\pi}{\beta}T}.$$

Additionally, we have the trivial bound $\|\overline{\Delta}_r(s)\| \leq 2\,\|V\|\, \|f_\beta\|_{L^1(\mathbb{R})} = 2\,\|V\|$, such that one can now optimize the bound for a given ζ_{LR}. More precisely, we obtain

$$\|\overline{\Delta}_r(s)\| \leq \|V\|\, \zeta_{\mathrm{QBP}}(X, r)$$

with

$$\zeta_{\mathrm{QBP}}(X, r) := \min\left\{2, \inf_{T \geq \frac{\beta}{\pi} \ln 2} \|\zeta_{\mathrm{LR}}(X, \Lambda \setminus X_r, \cdot)\|_{L^\infty([-T,T])} + 4\,\mathrm{e}^{-\frac{\pi}{\beta}T}\right\},$$

where we bounded $32/\pi^2 < 4$. Then, $4\,\mathrm{e}^{-\frac{\pi}{\beta}T} > 2$ for $T < \frac{\beta}{\pi} \ln 2$, which together with the trivial bound allows us to take the infimum over all $T \geq 0$ instead of $T \geq \frac{\beta}{\pi} \ln 2$.

We bound ζ_{QBP} explicitly for short- and long-range interactions in Section 10.3.

10.2.3 Locality of the exponential $\eta(s)$

The local approximation of $\Phi_\beta^{H(s)}(V)$ from the previous section also allows us to approximate $\eta(s)$ by a local version

$$\eta_r(s) := \mathscr{T} \exp\left(-\frac{\beta}{2} \int_0^s \Delta_r(\sigma)\, \mathrm{d}\sigma\right). \tag{33}$$

It is easy to show by induction that for all operators $A_1, \dots, A_n$ and $B_1, \dots, B_n$ it holds that

$$A_1 A_2 \cdots A_n = B_1 B_2 \cdots B_n + \sum_{j=1}^{n} A_1 \cdots A_{j-1} (A_j - B_j) B_{j+1} \cdots B_n.$$

Hence,

$$\|A_1 A_2 \cdots A_n - B_1 B_2 \cdots B_n\| \le n \left(\sup_{j\in\{1,n\}} \max\{\|A_j\|, \|B_j\|\} \right)^{n-1} \sup_{j\in\{1,n\}} \|A_j - B_j\|. \tag{34}$$

Using (34) for $A_j := \Phi_\beta^{H(\sigma_j)}(V)$ and $B_j := \Delta_r(\sigma_j)$ together with (28) we find

$$\begin{aligned}\|\eta(s) - \eta_r(s)\| &\le \sum_{n=0}^{\infty} \left(\frac{\beta}{2}\right)^n \frac{s^n}{n!}\, n\, \|V\|^{n-1}\, \|V\|\, \zeta_{\mathrm{QBP}}(X, r) \\ &\le \frac{\beta}{2}\, s\, \|V\|\, \mathrm{e}^{\frac{\beta}{2} s \|V\|}\, \zeta_{\mathrm{QBP}}(X, r).\end{aligned}$$

And by the definition (33) we have the same bound $\|\eta_r(s)\| \le \mathrm{e}^{\frac{\beta}{2} s\|V\|}$ as for $\eta(s)$.

10.3 Specific decay ζ_{QBP} for short- and long-range interactions

We now obtain the specific ζ_{QBP} stated in Section 4 for short- and long-range interactions. Moreover, we will explain how to optimize the bound on $\zeta_{\mathrm{QBP}}(X, r)$ for balls X, which we use for intervals in the one-dimensional setting. In all cases, we start with a discussion of available Lieb-Robinson bounds.

10.3.1 Short-range interactions

For short-range interactions, Lieb-Robinson bounds have been proven in several different forms, for example in [154, 118, 174, 156]. A more complete discussion about the history is given in [174]. For our specific interaction norm, we adopt the proof of [156, Theorem 7.3.1] which is based on the proof of [174, Theorem 3.1].

Proposition 44 (Lieb-Robinson bound). *Let $\Lambda \Subset \mathbb{Z}^\nu$ and Ψ be a short-range interaction on Λ with $\|\Psi\|_{\exp(-b\,\cdot)} < \infty$ for some $b > 0$. Then, for all $X, Y \subset \Lambda$, operators $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$, and $t \in \mathbb{R}$*

$$\left\|\left[\mathrm{e}^{-\mathrm{i}tH_\Lambda}\, A\, \mathrm{e}^{\mathrm{i}tH_\Lambda}, B\right]\right\| \leq 2\,\|A\|\,\|B\|\,\mathrm{e}^{bv_b|t|} \sum_{x\in X} \mathrm{e}^{-bd(x,Y)},$$

where $v_b := 2\,\|\Psi\|_{\exp(-b\,\cdot)}/b$ is the Lieb-Robinson velocity.

Proof. The interaction norm used in [156] does not have the extra factor $|Z|$. In the proof of the Lieb-Robinson bound, they need to bound a factor $|Z|$, for which they use a fraction of the decay in $\mathrm{diam}(Z)$ and $\sup_{Z\Subset\Lambda}|Z|\,\mathrm{e}^{-c\,\mathrm{diam}(Z)} < \infty$, instead, we use the extra $|Z|$ in the interaction norm.

More specifically, instead of [156, eq. (7.3.5)], we start with the induction hypothesis

$$a_k(X,Y) \leq \|\Psi\|^k_{\exp(-b\,\cdot)}\, g_b(X,Y), \qquad \text{where} \qquad g_b(X,Y) := \sum_{x\in X} \mathrm{e}^{-bd(x,Y)},$$

where $a_k(X,Y)$ is defined in [156, p. 313]. The proof for $a_1(X,Y)$ is the same, while for the induction step, we find

$$\begin{aligned} a_{k+1}(X,Y) &\leq \|\Psi\|^k_{\exp(-b\,\cdot)} \sum_{x\in X} \sum_{\substack{Z\subset\Lambda:\\ x\in Z}} \frac{\|\Psi(Z)\|}{\mathrm{e}^{-b\,\mathrm{diam}(Z)}} \sum_{z\in Z} \mathrm{e}^{-b\,\mathrm{diam}(Z)}\,\mathrm{e}^{-bd(z,Y)} \\ &\leq \|\Psi\|^k_{\exp(-b\,\cdot)} \sum_{x\in X} \mathrm{e}^{-bd(x,Y)} \sum_{\substack{Z\subset\Lambda:\\ x\in Z}} \frac{\|\Psi(Z)\|}{\mathrm{e}^{-b\,\mathrm{diam}(Z)}}\,|Z| \\ &\leq \|\Psi\|^{k+1}_{\exp(-b\,\cdot)}\, g_b(X,Y). \end{aligned}$$

With this small modification, the proof in [156] yields the claim. □

We are now ready to obtain ζ_{QBP} for short-range interactions.

Proof of Lemma 20. Note that $\|\Psi + s\,V\|_{\exp(-b\,\cdot)} \leq \max\big\{\|\Psi\|_{\exp(-b\,\cdot)}, \|\Psi + V\|_{\exp(-b\,\cdot)}\big\}$ by convexity. Hence, to prove Lemma 20 (a) we can choose the Lieb-Robinson velocity $v_b = 2\,\max\big\{\|\Psi\|_{\exp(-b\,\cdot)}, \|\Psi + V\|_{\exp(-b\,\cdot)}\big\}/b$. We begin with the proof for general sets and bound

$$\sum_{x\in X} \mathrm{e}^{-bd(x,Y)} \leq |X|\,\mathrm{e}^{-bd(X,Y)}, \tag{35}$$

which yields $\zeta_{\mathrm{LR}}(X, \Lambda\setminus X_r, t) \leq C_{\mathrm{LR}}\,|X|\,\mathrm{e}^{b(v_b|t|-r)}$ with $C_{\mathrm{LR}} = 2$. This ζ_{LR} attains its supremum for $t \in [-T,T]$ at $t = T$. Hence, the infimum in (11) is attained at

$$T = \frac{b\,r - c}{\frac{\pi}{\beta} + b\,v_b} \qquad \text{with} \qquad c := \ln\!\Big(C_{\mathrm{LR}}\,|X|\,b\,v_b\,\beta/(4\,\pi)\Big).$$

To obtain a simpler result, we just choose $T := br/(\frac{\pi}{\beta} + b\, v_b)$ to get

$$\zeta_{\mathrm{QBP}}(X,r) \le (C_{\mathrm{LR}} + 4)\, |X|\, \mathrm{e}^{-\frac{b}{1+bv_b\beta/\pi} r}, \tag{36}$$

where $C_{\mathrm{LR}} + 4 = 6$.

To obtain (13), we start with the Lieb-Robinson bound for the perturbed Hamiltonian from Corollary 43, which means to replace C_{LR} by $2\left(1 + 2/(b\, v_b)\, \|V\|\right)$. In (36) we then bound $2\left(1 + 2/(b\, v_b)\, \|V\|\right) + 4 \le 6\, \max\{1, 2/(b\, v_b)\}\left(1 + \|V\|\right)$.

If $X = B_z(R)$ is a ball, we can improve the bound in (35) by summing over shells $S_k = \{\, x \in X \,|\, d(z,x) = k \,\}$ which all satisfy $|S_k| \le |\partial X|$ such that

$$\sum_{x\in X} \mathrm{e}^{-bd(x,\Lambda\setminus X_r)} \le \sum_{k=0}^{R} |S_k|\, \mathrm{e}^{-b(r+R-k)} \le |\partial X| \left(\mathrm{e}^{-br} + \int_r^\infty \mathrm{e}^{-bq}\, \mathrm{d}q \right) \le \tfrac{1+b}{b}\, |\partial X|\, \mathrm{e}^{-br}.$$

Hence, we just replace $|X|$ with $\frac{1+b}{b}\, |\partial X|$ in ζ_{QBP} to obtain the improved result for balls. With the same proof it is also clear that this replacement works for every interval $X \subset \mathbb{Z}$, while strictly speaking, only intervals with an odd number of sites are balls. □

10.3.2 Long-range interactions

Also for long-range interactions various Lieb-Robinson bounds are available. Some of them only apply for a restricted set of times t or only for two-body interactions. In particular, there was a focus on proving so-called linear light cones in the last years. The bounds have the property that for each $\varepsilon > 0$ there exists $v > 0$ such that $\zeta_{\mathrm{LR}}\big(X, Y, d(X,Y)/v\big) < \varepsilon$ in the limit $d(X,Y) \to \infty$, see e.g. [149, 210] and the discussion of light-cones in [84]. For our applications, it turns out that we need a good decay in $d(X,Y)$ and a bound which holds for all t, while the growth in t is not too important. This is due to the fact, that we only need to use the Lieb-Robinson bound until T in (11) and can then use the exponential decay in T in the second term to our advantage.

One bound, which provides a good decay and allows for general interaction, is proven in [84], which is based on [160]. We give a slightly improved version here. From now on, we will fix $F_\alpha(r) := (1+r)^\alpha$.

Proposition 45. *Let $\alpha > \nu$ and $\sigma \in \big((\nu+1)/(\alpha+1), 1\big)$. Then there exist constants C and $c > 0$ such that for all $\Lambda \Subset \mathbb{Z}^\nu$ and interactions Ψ on Λ with $\|\Psi\|_{F_\alpha} < \infty$ the following Lieb-Robinson bound hold: For all $X, Y \subset \Lambda$, $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$*

$$\left\| \left[\mathrm{e}^{-\mathrm{i}tH_\Lambda} A\, \mathrm{e}^{\mathrm{i}tH_\Lambda}, B \right] \right\| \le C\, \|A\|\, \|B\| \sum_{x\in X} \left(\mathrm{e}^{v|t| - r^{1-\sigma}} + v\,|t| \left(1 + (v\,|t|)^{\nu/(1-\sigma)}\right) F_{\sigma\alpha}(r) \right) \tag{37}$$

where $r = d(x,Y)$ and $v := c\, \|\Psi\|_{F_\alpha}$.

Proof. First note, that the assumptions in [84, eq. (3), (5)] are satisfied if $\alpha > \nu$ and $\|\Psi\|_{F_\alpha} < \infty$. Then [84, Theorem 1] proves the statement for a single point $X = \{x\}$. From [209, Lemma 4] one obtains the statement for $|X| > 1$ with the sum over $x \in X$.

Strictly speaking, the bound in [84] is only provided for ℓ^2-distance on $\mathbb{Z}^\nu$, which agrees with our setting only for $\nu = 1$. However, the proof can be modified to obtain the same result also with ℓ^1-distance and directly with a linear scaling in $|X|$. Details will be given in a future work, where we also consider fermions, for which [209, Lemma 4] is not applicable. □

Before we estimate ζ_{QBP}, let us further upper bound (37) to obtain the simpler

$$\zeta_{\mathrm{LR}}(X,Y,t) \leq C \sum_{x\in X} \left(\mathrm{e}^{vt - d(x,Y)^{1-\sigma}} + \left(1 + (v\,t)^{1+\nu/(1-\sigma)}\right) F_{\sigma\alpha}\big(d(x,Y)\big)\right) \tag{38}$$

after including a factor 2 in C. Then, for the dynamics of $H_\Lambda + V$ with $V \in \mathcal{A}_X$ we obtain

$$\zeta_{\mathrm{LR}}(X,Y,t) \leq C\left(1 + \|V\|\right) \sum_{x\in X} \left(\mathrm{e}^{vt - d(x,Y)^{1-\sigma}} + \left(1 + (v\,t)^{2+\nu/(1-\sigma)}\right) F_{\sigma\alpha}\big(d(x,Y)\big)\right) \tag{39}$$

using Lemma 42, again after adjusting C.

Lemma 46 (ζ_{QBP} for long-range interactions)**.** *Let $\alpha > \nu$, $\alpha_{\mathrm{QBP}} < \alpha$, $\beta_0 > 0$ and $C_{\mathrm{int}} > 0$. Then there exists a constant $C_{\mathrm{QBP}} > 0$ such that the following holds:*

Let Ψ be an interaction on $\Lambda \Subset \mathbb{Z}^\nu$, $X \subset \Lambda$, $V \in \mathcal{A}_X$ self-adjoint and $\beta \in (0, \beta_0)$. Then for all Hamiltonians $H(s) = H + s\,V$ the following holds:

(a) *If $\|\Psi\|_{F_\alpha}$, $\|\Psi + V\|_{F_\alpha} < C_{\mathrm{int}}$, then*

$$\zeta_{\mathrm{QBP}}(X, r) \leq C_{\mathrm{QBP}} |X|\, F_{\alpha_{\mathrm{QBP}}}(r).$$

(b) *If only $\|\Psi\|_{F_\alpha} < C_{\mathrm{int}}$, then*

$$\zeta_{\mathrm{QBP}}(X, r) \leq C_{\mathrm{QBP}} \left(1 + \|V\|\right) |X|\, F_{\alpha_{\mathrm{QBP}}}(r).$$

Moreover, if $X = \{x \in \Lambda \,|\, d(x,z) \leq R\}$ is a ball (for some $z \in \Lambda$ and $R > 0$) and $\alpha_{\mathrm{QBP}} < \alpha - 1$ we can replace $|X|$ in both bounds with $|\partial X|$. This replacement in particular also works for all intervals $X \subset \mathbb{Z}$, where $|\partial X| = 2$.

Proof. For the proof, we will assume the $\zeta_{\mathrm{LR}}(X,Y,t)$ from (39) but without the $\left(1+\|V\|\right)$. It clearly upper bounds (38) and we can later add the factor $\left(1 + \|V\|\right)$ as in the proof of Lemma 20. This way, we only need to bound (11) once.

As in the proof of Lemma 20, we start with the result for general $|X|$, where we replace the sum with $|X|$ and r with $d(X,Y)$. Then,

$$\begin{aligned}\|\zeta_{\mathrm{LR}}(X,\Lambda\setminus X_r,\cdot)\|_{L^\infty([-T,T])}\\ &\le C\,|X|\left(\mathrm{e}^{vT-r^{1-\sigma}}+\left(1+(v\,T)^{2+v/(1-\sigma)}\right)F_{\sigma\alpha}(r)\right)\\ &\le C\,|X|\left(\mathrm{e}^{r^{p(1-\sigma)}-r^{1-\sigma}}+F_{\sigma\alpha-2p(1-\sigma)-vp}(r)\right)\end{aligned}$$

after choosing $p\in(0,1)$, $T=r^{p(1-\sigma)}/v$ and adjusting the constant by a factor 2 after using $1+r^x\le 2\,(1+r)^x$. We now choose σ and p such that $\alpha_{\mathrm{QBP}}=\sigma\alpha-2p(1-\sigma)-vp$, which is possible, because we assumed $\alpha_{\mathrm{QBP}}<\alpha$ and can choose σ arbitrarily close to 1 and p arbitrarily close to 0, which will only change the constant C according to Proposition 45. For every choice of σ and p, we can also upper bound $\mathrm{e}^{r^{p(1-\sigma)}-r^{1-\sigma}}\le C\,F_{\alpha_{\mathrm{QBP}}}(r)$ for some $C>0$. Thus, we obtain

$$\|\zeta_{\mathrm{LR}}(X,\Lambda\setminus X_r,\cdot)\|_{L^\infty([-T,T])}\le C\,|X|F_{\alpha_{\mathrm{QBP}}(r)},$$

where T is chosen as above and the constant C only depends on α, α_{QBP} and v as in the Lieb-Robinson bound. For the second summand in (11) we bound $4\,\mathrm{e}^{-\frac{\pi}{\beta}T}=4\,\mathrm{e}^{-\frac{\pi}{\beta v}r^{p(1-\sigma)}}\le C\,F_{\alpha_{\mathrm{QBP}}}(r)$ for some $C>0$, where this C additionally depends on β_0 and C_{int}. Combining the two, proves Lemma 46 (a). And as said above, adding the additional factor $(1+\|V\|)$ also proves Lemma 46 (b).

For balls $X=B_z(R)$ we use the same strategy as in the proof of Lemma 20. Here, we bound

$$\sum_{x\in X}F_\alpha\big(d(x,\Lambda\setminus X_r)\big)\le\sum_{k=0}^{R}|S_k|\,F_\alpha(r+R-k)\le|\partial X|\sum_{k=r}^{\infty}F_\alpha(k)\le\tfrac{\alpha}{\alpha-1}\,|\partial X|\,F_{\alpha-1}(r),$$

using

$$\sum_{k=r}^{\infty}F_\alpha(k)\le F_\alpha(r)+\int_r^\infty F_\alpha(x)\,\mathrm{d}x\le\tfrac{\alpha}{\alpha-1}\,F_{\alpha-1}(r)\quad\text{for}\quad\alpha>1 \tag{40}$$

for the second summand in (38) and (39). For the first summand one obtains a polynomial correction in r, see e.g. [84, Lemma 3] which will be absorbed by the exponential decay in the end. Hence, after adjusting the constants, the result also holds with $|\partial X|$ instead of $|X|$ for $\alpha_{\mathrm{QBP}}<\alpha-1$. As in Lemma 20, the replacement also works for all intervals $X\subset\mathbb{Z}$. □

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A Details on Section 3

In this section we collect short proofs for the statements in Section 3.

A.1 One-dimensional short-range systems

We begin with the proof for short-range interactions.

Proof of Corollary 9. LPPL follows from Theorem 22 by choosing $r = d(X,Y)/2$, using Theorem 8 to bound the covariance and using the specific form of ζ_{QBP} from Lemma 20 (b) for intervals. For more details, one can follow the proof of Theorem 36.

For local indistinguishability we first observe, that (3) also holds without the factor $\big(1 + \|V\|\big)$, if $\|\Psi + V\|_{\exp(-b\,\cdot)} < C_{\mathrm{int}}$, see Lemma 20 (a). This is the form of LPPL that we need in the proof of Lemma 28. Notice moreover, that we indeed provide uniform LPPL for all $\Lambda \Subset \mathbb{Z}$, with respect to $f_{\mathrm{LPPL}} = C_{\mathrm{LPPL}}$, $g_{\mathrm{LPPL}}(v) = \mathrm{e}^{3\beta v}$, $\zeta_{\mathrm{LPPL}}(r) = \mathrm{e}^{-c_{\mathrm{LPPL}}\sqrt{r}}$ and $n = 0$ (in a restricted sense, where X and Y are intervals). Hence, we can apply Theorem 29 by removing the sites from $\Lambda \setminus \Lambda'$. Therefore, we calculate, see Lemma 28, $g(v) = \mathrm{e}^{3\beta v}$ and bound

$$\zeta(r) \leq \big(\mathrm{e}^{-c_{\mathrm{LPPL}}\sqrt{(1-\alpha)\,r}} + \mathrm{e}^{-b\sqrt{\alpha\,r}}\big) \leq 2\,\mathrm{e}^{-c_{\mathrm{LI}}\sqrt{r}},$$

where we bounded $\mathrm{e}^{-bx} \leq \mathrm{e}^{-b\sqrt{x}}$ for $x \geq 1$ and chose $R = \alpha\,r$, $\alpha = c_{\mathrm{LPPL}}^2/(b^2 + c_{\mathrm{LPPL}}^2)$ and $c_{\mathrm{LI}} = b\,\sqrt{\alpha}$. It remains to bound the sum in (15), which in this case is

$$\sum_{x\in\Lambda\setminus\Lambda'} \zeta\big(d(Y,x_i)\big) \leq 2\sum_{k=r}^{\infty}\zeta(k) \leq 4\,\mathrm{e}^{-c_{\mathrm{LI}}\sqrt{r}} + 4\int_r^{\infty}\mathrm{e}^{-c_{\mathrm{LI}}\sqrt{q}}\,\mathrm{d}q \leq \frac{8}{c_{\mathrm{LI}}^2}\,\big(c_{\mathrm{LI}}\sqrt{r} + 1 + c_{\mathrm{LI}}^2\big)\,\mathrm{e}^{-c_{\mathrm{LI}}\sqrt{r}},$$

where we abbreviated $r = d(Y, \Lambda \setminus \Lambda')$. Absorbing all the constants in C_{LI} proves the statement. □

Remark 47. With a refined proof as in Section 9, one could also obtain LPPL for observables $B = B_1\,B_2$ with $B_1 \in \mathcal{A}_{Y_1}$, $B_2 \in \mathcal{A}_{Y_2}$ and Y_1, $Y_2 \subset \Lambda$ intervals such that $Y_1 < X < Y_2$. ◇

A.2 One-dimensional long-range systems

We proceed with the proof for long-range interactions.

Proof of Corollary 11. The interaction norm used in [136], which we give in (4) and which is used in the formulation of Theorem 10 does not agree with the interaction norm $\|\cdot\|_{F_\alpha}$, which we use in the rest of the paper. Clearly

$$\|\!|\Psi|\!\|_{F_\alpha} = \sup_{x,y\in\Lambda} \sum_{\substack{Z\Subset\Lambda:\\ x,y\in Z}} \frac{\|\Psi(Z)\|}{F_\alpha\big(d(x,y)\big)} \leq \sup_{x\in\Lambda} \sum_{\substack{Z\Subset\Lambda:\\ x\in Z}} \frac{\|\Psi(Z)\|}{F_\alpha\big(\operatorname{diam}(Z)\big)} = \|\Psi\|_{F_\alpha},$$

so their result also applies for our norm. But it will be advantageous in terms of possible α to bound the $\|\Psi\|_{F_{\alpha'}}$ norm with the $\|\!|\Psi|\!\|_{F_\alpha}$ for some α' and α and formulate the assumptions using $\|\!|\Psi|\!\|_{F_\alpha}$. Therefore, observe the following: For each $Z \Subset \mathbb{Z}$ one can find $x, y \in \mathbb{Z}$ such that $\operatorname{diam}(Z) = d(x,y) \leq d(x,z) + d(y,z)$ for every $z \in Z$ by triangle inequality. Thus, for every $z \in Z$ one can find $w \in Z$ (which is x or y) such that $\operatorname{diam}(Z) \leq 2\,d(z,w)$. This allows to obtain the following bound for k-body interactions

$$\begin{aligned}\|\Psi\|_{F_{\alpha'}} &= \sup_{z\in\mathbb{Z}} \sum_{\substack{Z\Subset\mathbb{Z}:\\ z\in Z}} \frac{|Z|\,\|\Psi(Z)\|}{F_{\alpha'}\big(\operatorname{diam}(Z)\big)}\\ &\leq 2^{\alpha'}\,k\,\sup_{z\in\mathbb{Z}} \sum_{w\in\mathbb{Z}} F_{\alpha-\alpha'}\big(d(z,w)\big) \sum_{\substack{Z\Subset\mathbb{Z}:\\ z,w\in Z}} \frac{\|\Psi(Z)\|}{F_\alpha\big(d(z,w)\big)}\\ &\leq C_{\alpha,\alpha'}\,\|\!|\Psi|\!\|_{F_\alpha}\end{aligned}$$

with

$$C_{\alpha,\alpha'} = 2^{\alpha'}\,k\,\sup_{z\in\mathbb{Z}} \sum_{w\in\mathbb{Z}} F_{\alpha-\alpha'}\big(d(z,w)\big) < \infty \qquad \text{if } \alpha > \alpha' + 1.$$

With the estimates on the interaction norms, for every $\alpha > 2$ and $\alpha_{\mathrm{QBP}} < \alpha' - 1 < \alpha - 2$ we have

$$\zeta_{\mathrm{QBP}}(X,r) \leq C_{\mathrm{QBP}}\,\big(1 + \|V\|\big)\,|X|\,F_{\alpha_{\mathrm{QBP}}}(r)$$

from Lemma 46 (b) for intervals $X \subset \Lambda$. We now choose $\alpha_{\mathrm{QBP}} = \alpha_{\mathrm{Cov}} = \alpha_{\mathrm{LPPL}}$ and apply Theorem 22 with $r = d(X,Y)/2$. Using $x < \mathrm{e}^x$ to absorb the $\beta\,\|V\|$ in the exponential, we obtain the statement on LPPL.

For local indistinguishability we again observe, that (5) also holds without the factor $\big(1 + \|V\|\big)$, if $\|\Psi + V\|_{F_\alpha} < C_{\mathrm{int}}$, see Lemma 46 (a). As in the proof of Corollary 9, we can apply Theorem 29, since we provide uniform LPPL for all $\Lambda \Subset \mathbb{Z}$, with respect to $f_{\mathrm{LPPL}} = C_{\mathrm{LPPL}}$, $g_{\mathrm{LPPL}}(v) = \mathrm{e}^{3\beta v}$, $\zeta_{\mathrm{LPPL}}(r) = F_{\alpha_{\mathrm{LPPL}}}(r)$ and $n = 0$ (in a restricted sense, where X and Y are intervals). After choosing $R = d(X,Y)/2$, we obtain, see Lemma 28, $g(v) = \mathrm{e}^{3\beta v}$ and bound $\zeta(r) \leq 2^{\alpha_{\mathrm{LPPL}}} F_{\alpha_{\mathrm{LPPL}}}(r)$. For Theorem 29 it is left to bound

$$\sum_{x \in \Lambda \setminus \Lambda'} F_{\alpha_{\mathrm{LPPL}}}\big(d(Y,x)\big) \leq 2 \sum_{k=d(Y,\Lambda \setminus \Lambda')}^{\infty} F_{\alpha_{\mathrm{LPPL}}}(k) \leq \frac{\alpha_{\mathrm{LPPL}}}{\alpha_{\mathrm{LPPL}}-1} F_{\alpha_{\mathrm{LPPL}}-1}\big(d(Y, \Lambda \setminus \Lambda')\big),$$

where we used (40). Putting everything together and absorbing the constants in C_{LI} proves the claim. □

A.3 ν-dimensional short-range systems at high temperature

Proof of Corollary 13. LPPL follows from Theorem 22 by choosing $r = d(X,Y)$, using Conjecture 12 to bound the covariance and using ζ_{QBP} from Lemma 20 (b). See also Remark 23.

To obtain local indistinguishability, we first use Theorem 25 together with ζ_{QBP} from Lemma 20 (a), to obtain a better LPPL for the perturbations considered in the proof of Lemma 28, similar to what is given in Remark 26. This way, we obtain a linear scaling $\beta \|V\|$ in the LPPL bound with a constant that can be chosen uniformly for $\beta < \beta^*$. To keep this scaling also for local indistinguishability, we bound $\mathrm{e}^{2\beta v} - 1 \leq \beta \frac{1}{\beta^*} (\mathrm{e}^{2\beta^* v} - 1)$ to obtain $g\big(\|\Psi\|_{\exp(-b\,\cdot)}\big) \leq C\,\beta\,\|\Psi\|_{\exp(-b\,\cdot)}$ for some C, which only depends on β^* and C_{int}, in Theorem 29. Since ζ_{LPPL} and F decay exponentially, $\tilde{\zeta}(r)$ in Theorem 29 converges and decays exponentially. Thus, the result follows. □

Publication P5

Enhanced Lieb-Robinson bounds for commuting long-range interactions

Marius Lemm [*]   Tom Wessel [*]

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Abstract

Recent works have revealed the intricate effect of long-range interactions on information transport in quantum many-body systems: In D spatial dimensions, interactions decaying as a power-law $r^{-\alpha}$ with $\alpha > 2D+1$ exhibit a Lieb-Robinson bound (LRB) with a linear light cone and the threshold $2D+1$ is sharp in general. Here, we observe that mutually commuting, long-range interactions satisfy an enhanced LRB of the form $t\,r^{-\alpha}$ for any $\alpha > 0$, and this scaling is sharp. In particular, the linear light cone occurs at $\alpha = 1$ in any dimension. Part of our motivation stems from quantum error-correcting codes. As applications, we derive enhanced bounds on ground state correlations and an enhanced local perturbations perturb locally (LPPL) principle for which we adapt a recent subharmonicity argument of Wang-Hazzard. Similar enhancements hold for commuting interactions with stretched exponential decay.

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[*] Department of Mathematics, University of Tübingen,
72076 Tübingen, Germany

Contents

1 Introduction

The Lieb-Robinson bound (LRB) is a central theorem in quantum many-body physics. It underpins the proofs of fundamental structural properties such as the decay of correlations of ground states of Hamiltonians with a spectral gap [118, 171], the existence of the thermodynamic limit as a strongly continuous automorphism on the quasi-local operator algebra [49, 170], and the mathematical definition of a topological quantum phase [120, 23]. At the same time, the LRB is a cornerstone of the growing area of quantum information theory: on the one hand, it was a decisive tool for landmark results such as the area law for the entanglement entropy [111] and on the other hand it provides bounds on dynamical entanglement generation [53], quantum messaging and quantum state transfer [85], and efficient quantum simulability of many-body dynamics [106].

The LRB is concerned with quantum lattice Hamiltonians. These are defined by fixing a finite graph Λ and defining the Hilbert space

$$\mathscr{H}_\Lambda := \bigotimes_{x\in\Lambda} \mathbb{C}^q$$

with the standard inner product. On this Hilbert space, one considers a distinguished self-adjoint linear operator, the Hamiltonian H, which is taken as a sum of terms that act "locally", i.e.

$$H_\Lambda := \sum_{Z\subset\Lambda} \Psi(Z), \tag{1}$$

where each summand is a bounded, linear, self-adjoint operator of the form $\Psi(Z) \equiv \Psi(Z)\otimes \mathbb{1}_{\Lambda\setminus Z}$. The precise mathematical setup in terms of quasi-local algebras is given in section 2.

Instead of studying solutions to the Schrödinger equation directly, it is advantageous to study its dual, the quantum dynamics on operators, known as *Heisenberg dynamics*, that is defined by unitary conjugation with the solution operator of the Schrödinger equation, i.e.

$$\tau_t^\Lambda(A) = \mathrm{e}^{\mathrm{i}tH_\Lambda}\, A\, \mathrm{e}^{-\mathrm{i}tH_\Lambda}. \tag{2}$$

For a set $X \subset \Lambda$ and $\ell > 0$, we denote its ℓ-neighbourhood by

$$X_\ell := \left\{ x \in \Lambda \,\middle|\, d(x, X) \le \ell \right\}. \tag{3}$$

To put our results in context, we first review the two kinds of LRBs that are well-established for the case where the interaction Ψ is of finite-range, i.e. there exists a range $R > 0$ such that $\Psi(Z) = 0$ if $\mathrm{diam}(Z) > R$, or exponentially decaying, i.e. $\|\Psi(Z)\|$ decays exponentially in $\mathrm{diam}(Z)$. For such rapidly decaying interactions, one has the following two types of LRBs [154, 118, 171, 174].

(a) **Commutator version of the LRB.** There exist constants $C, b, v > 0$ such that for all finite $X, Y \subset \Lambda$ and bounded operators $A_X \equiv A_X \otimes \mathbb{1}_{\Lambda\setminus X}$ and $B_Y \equiv B_Y \otimes \mathbb{1}_{\Lambda\setminus Y}$, it holds that

$$\left\|[\tau_t^\Lambda(A_X), B_Y]\right\| \le C \min\left\{|X|, |Y|\right\} \|A_X\| \, \|B_Y\| \, \mathrm{e}^{b\,(vt-d(X,Y))}. \tag{4}$$

(b) **Operator localization version of the LRB.** There exist constants $C, b, v > 0$ such that for all finite $X \subset \Lambda$ and bounded operators $A_X \equiv A_X \otimes \mathbb{1}_{\Lambda\setminus X}$, it holds that

$$\left\|\tau_t^\Lambda(A_X) - \mathbb{E}_{X_r}\left(\tau_t^\Lambda(A_X)\right)\right\| \le C\,|X|\,\|A_X\|\,\mathrm{e}^{b(vt-r)}. \tag{5}$$

Here, $\mathbb{E}_{X_r}$ is the conditional expectation on X_r, i.e. the partial trace over the Hilbert space associated to $\Lambda \setminus X_r$; see (12).

The LRBs are a mathematically precise way to capture that quantum information propagates at most with a speed v in these systems. The speed v and the constants C and b might depend on the details of the interaction but are uniformly bounded in the system size $|\Lambda|$. Indeed, since the right-hand sides are small for $v\,t \ll d(X,Y)$ and $v\,t \ll r$, respectively, the LRBs establish the existence of an effective finite propagation speed for many-body Hamiltonians of the form (1). The fact that the bound is effective until $vt \sim r$ is also called a "linear light cone" (with slope v) or a "ballistic bound" and we use both of these phrases interchangeably.

1.1 Enhanced Lieb-Robinson bounds

Given the important role played by LRBs, it is unsurprising that significant effort has been invested to extend its validity to other classes of many-body systems. Of particular interest in the past ten years has been the extension of LRBs to so-called *long-range* interactions, whose operator norm decays as a power law. In this introduction, we focus for simplicity on two-body interactions, i.e. $\Psi(Z) = 0$ for $|Z| > 2$, and we refer to theorems 5 and 6 for the more general statements. For two-body interactions, the power-law decay assumption can be simply phrased as the operator norm bound

$$\|\Psi(\{x,y\})\| \leq C\, d(x,y)^{-\alpha} \tag{6}$$

and $\|\Psi(\{x\})\| \leq C$.

After intensive research efforts and the introduction of a variety of new techniques, the sharp form of the LRBs for long-range interactions (6) were nailed down [79, 188, 81, 84, 149, 209, 211, 210, 192]. Also, propagation bounds for Bose-Hubbard type Hamiltonians with unbounded long-range hopping terms have recently been studied [87, 86, 150, 151, 213, 152].

Our goal is to study the effect of the additional assumption that the interactions mutually commute, i.e.

$$\left[\Psi(X), \Psi(Y)\right] = 0 \qquad \text{for all } X, Y \subset \Lambda. \tag{7}$$

The commutativity (7) is a significant restriction, but it is physically very relevant in the context of quantum error correcting codes as we explain in section 1.3. Commutativity (7) allows us to prove significantly enhanced LRBs compared to general long-range interactions, which satisfy only (6) but not necessarily (7). We now summarize the differences between previous bounds; see also figure 1. Detailed statements are given in theorems 5 and 6. The parameter D captures the dimension of the graph; think of $\mathbb{Z}^D$ and see definition 2.

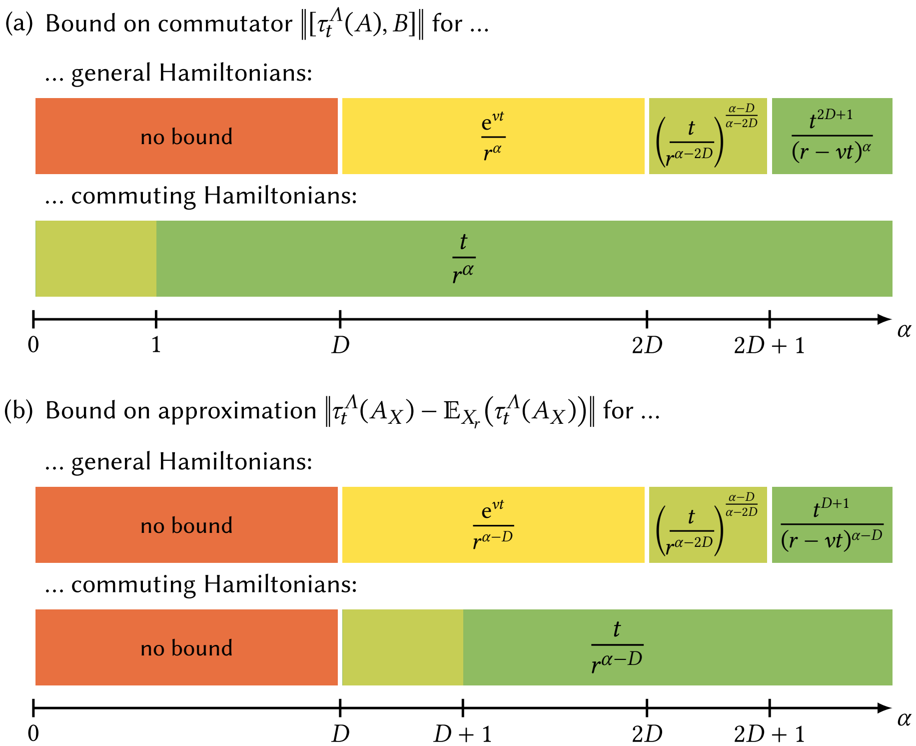

Figure 1. Comparison of the previously shown (sharp) Lieb-Robinson bounds for general long-range interactions and the enhanced Lieb-Robinson bounds proven here for long-range *commuting* interactions. The general bounds mentioned above are proven in [210] for $\alpha \in (2D, 2D+1)$ and [149] for $\alpha > 2D+1$. The bounds displayed for $\alpha \in (D, 2D)$ hold for all $\alpha > D$, the commutator version was proven in [118] and the operator localization version in [147, section S.III.A]. The enhanced bounds for commuting Hamiltonians are stated in theorems 5 and 6.

In figure 1, we compare the results for general long-range interactions [118, 171, 79, 188, 81, 84, 149, 209, 211, 210, 192] to the ones we prove here for mutually commuting long-range interactions. The LRBs for commuting long-range interactions are enhanced compared to the LRB for general long-range interactions in two ways.

(i) *The range of α where an LRB exists at all is increased.* For general interactions, there are no LRBs known at all for $\alpha \leq D$. By contrast, for commuting long-range interactions, the commutator version of the LRB holds for any $\alpha > 0$ and the localization version holds for any $\alpha > D$.

(ii) *The LRBs are stronger, meaning they are effective in larger spacetime regions.* The central first question is whether the light cone is indeed linear, i.e. of the form $vt \sim r$. A LRB corresponding to a *linear light cone* says that information transport is at most *ballistic*, i.e. it can spread at most a distance proportional to t in time t. We use the term linear light cone and ballistic LRB interchangeably, as both are common in the literature. In the finite-range case, both LRBs (4) and (5) are ballistic because they become effective at distance vt, which is linear in t. For general long-range interactions, both versions of the LRB become ballistic for $\alpha > 2D + 1$. For commuting interactions, the commutator version of the LRB is ballistic already at $\alpha = 1$ and the localization version is ballistic at $\alpha = D + 1$. Moreover, the LRBs for commuting interactions even become subballistic, i.e. the critical distance scales sublinearly in time, above these thresholds.

The LRBs stated for general (non-commuting) long-range interactions in figure 1(a) are proven to be sharp in the following sense: bounds which assume better scaling of t versus r can be violated by explicit choices of time-dependent Hamiltonians; see for example [84, 209, 211]. Similarly, the bounds for commuting interactions are also sharp, as we discuss in section 3.5.

The proofs we give are quite short and more direct than the usual proofs of LRBs, thanks to the commutativity. We first prove a bound similar to the operator localization version of the LRB, theorem 9, which bounds $\|\tau_t^{\Lambda}(A_X) - \tau_t^{\Lambda'}(A_X)\|$ where the second evolution is with the Hamiltonian restricted to $\Lambda' = \Lambda \setminus Y$ or $\Lambda' = X_r$ to obtain the commutator or operator localization version, respectively. This is different to the usual approach for non-commuting interactions, where one commonly first proves the commutator version and then derives the operator localization version. In both approaches, one usually looses a power D in the decay for the operator localization compared to the commutator version. Only for $\alpha \in (2D, 2D + 1)$, the proof of the general Lieb-Robinson bound, see figure 1, directly works with the operator localization, which always implies the commutator version with the same decay. Thus, the spatial decay in this regime is $r^{\alpha - D}$ for both LRBs and might be improved for the commutator version.

We remark in passing that a different direction in which the general bounds on long-range interactions can be improved is for non-interacting particles, which display a linear light cone at $\alpha > D + 1$ [209].

1.2 Further results

As mentioned in the introduction, LRBs and operator localization bounds have a plethora of applications – both structural ones in mathematical physics and practical ones in quantum information theory, e.g. on the speed of entanglement generation [53], quantum messaging and quantum state transfer [85]. Here, we focus on two applications which use LRBs to constrain the structure of ground states of Hamiltonians with a uniform spectral gap, namely decay of correlations and the principle that "local perturbations perturb locally" (LPPL).

1.2.1 Enhanced correlation bounds for gapped ground states

Modifying the proof of Hastings and Koma [118], we show that a ground state ρ_0, which is gapped from the rest of the spectrum by a gap g, satisfies *decay of correlations* in the sense that

$$\left|\mathrm{Tr}\big(\rho_0\, A_X\, B_Y\big) - \mathrm{Tr}\big(\rho_0\, A_X\big)\ \mathrm{Tr}\big(\rho_0\, B_Y\big)\right| \lesssim \|A_X\|\,\|B_Y\|\,|X|\,|Y|\,g^{-1}\,d(X,Y)^{-\alpha}$$

for small g and any $\alpha > 0$. The precise statement is given in theorem 11. In this bound, the inverse spectral gap g^{-1} plays a similar role as time in figure 1. In particular, the analogue of a ballistic bound is that the relevant localization length scales like g^{-1}, which is the case for $\alpha \geq 1$. This is to be compared with the best known bound $r^{-\alpha g/v}$ for general long-range interactions with $\alpha \in (D, 2D]$ [214]. First, no bound for $\alpha \leq D$ is known in the general case. And second, writing $r^{-\alpha g/v} = \mathrm{e}^{-(\alpha g/v)\ln r}$, one sees that the localization length is exponentially large in g^{-1}. Again, we see a fundamental, qualitative improvement under the additional assumption of commuting interactions.

1.2.2 Enhanced LPPL for gapped ground states

We also obtain a similar improvement for the *local perturbations perturb locally* (LPPL) principle by adapting the complex-analytic subharmonicity argument from Wang and Hazzard [214]: Let H be a long-range Hamiltonian with decay α as before and V_X a perturbation localized in $X \subset \Lambda$. Moreover, assume that for all $\lambda \in [0, 1]$, the ground state ρ_λ of $H + \lambda\, V_X$ is gapped with gap at least g. Then,

$$\left|\mathrm{Tr}\big(\rho_0\, B_Y\big) - \mathrm{Tr}\big(\rho_1\, B_Y\big)\right| \lesssim \|B_Y\|\,\big(\|V_X\| + \|V_X\|^2\big)\,|X|\,|Y|\,g^{-3}\,d(X,Y)^{-\alpha}.$$

The precise statement is given in theorem 13. This bound becomes ballistic for $\alpha = 3$ in the same sense as before. However, the best known bound for general interactions scales exactly the same as the bound for decay of correlations [214] and thus the localization length is again exponentially large in g^{-1}. A comparison of the different regimes and scalings is provided in table 1.

type	interaction decay	commutator LRB	decay of correlations LPPL
long-range	$r^{-\alpha}$, $\alpha > D$	$e^{v\|t\|}\, r^{-\alpha}$ [118]	$r^{-\alpha g/v}$ [214]
	$r^{-\alpha}$, $\alpha > 2D$	$t^{2D+1}\,(r - vt)^{-\alpha}$ [149][a]	$r^{-\alpha}$ [209, 214]
commuting long-range	$r^{-\alpha}$, $\alpha > 0$	$\|t\|\, r^{-\alpha}$	$r^{-\alpha}$
short-range	e^{-br}	$e^{b'\,(v\|t\|-r)}$ [118]	$e^{-g/v\,r}$ [118, 214]
commuting short-range	e^{-br^p}	$\|t\|\, e^{-b'r^p}$	$e^{-b'r^p}$

[a]This bound is valid only for $\alpha > 2D + 1$. For $\alpha \in (2D, 2D + 1)$ one has $t^{\frac{\alpha-D}{\alpha-2D}}\, r^{\alpha-D}$ from [210], see figure 1(a). Both focus on the shape of the light cone, while the proof of decay of correlations in [214] relies on a bound with better decay in r proven in [89].

Table 1. Comparisons between previous results and our improvements for commuting interactions. These are rough summaries, and we refer to the theorems for the precise statements. In particular, we drop logarithmic corrections in r and, for decay of correlations and LPPL, we focus on the scaling as $g \to 0$, and we drop polynomial prefactors depending on the gap g.

1.2.3 A stability result

Additionally, in section 4.3, we prove a result on *stability of LRBs* for more general Hamiltonians which are only assumed to have a commuting part, and we prove a LRB that is independent of the strength of the commuting part. Stability-type results for LRBs have been proved in other settings before, e.g. in [174, 211, 149, 208].

This finishes our presentation of the main results. We discuss future directions in section 6.

1.3 Motivation from quantum error-correcting codes

The results described above show that there is a fundamental, qualitative difference between the quantum many-body dynamics produced by general long-range interactions and commuting long-range interactions.

In this paragraph, we briefly review why Hamiltonians with commuting non-local interactions have recently received a lot of interest in the quantum information and quantum computing communities in the context of quantum error-correcting codes. The goal of a quantum error-correcting code is typically to robustly store quantum information, often by harnessing topological order; early examples are the toric code [138] and CSS stabilizer codes [57, 200]. Most quantum error-correcting codes are based on *commuting* interactions. Non-local *and* commuting interactions are of particular interest lately because they arise in Euclidean constructions of efficient quantum codes. The reason is as follows: Recall that a $[[n, k, d]]$ quantum code is a quantum code on system size n that can store k logical qubits with code distance d. The goal is to have k and d as large as possible, so that the code is both efficient and robust. Then, the Bravyi-Poulin-Terhal (BPT) theorem [54] says that any $[[n, k, d]]$ quantum code with short-range interactions in D Euclidean dimensions must satisfy the bound $k\,d^{2/(D-1)} \lesssim n$. An extension to general graphs was given in [29]. A quantum code is called good if it satisfies $k,\ d\ \sim\ n$ (which is optimal) and the BPT theorem says that local commuting interactions in D Euclidean dimensions cannot produce good quantum codes. Then, in 2021, the 30-year-old open problem of constructing good quantum codes was resolved by quantum low-density parity check (qLDPC) codes on expander graphs and related constructions [181, 55, 153]. Implementing these codes as Hamiltonians in the Euclidean setting, naturally produces Hamiltonians with non-local, commuting interactions [75, 30, 125, 184, 28] (as expected in view of the BPT theorem [54]). Beside their potential for near-term fault-tolerant quantum computation, quantum codes are also emerging as a rich source of new models for condensed-matter physics, e.g., they lead to a proof of the no low-energy trivial state (NLTS) conjecture of Freedman and Hastings [90, 117] and provide a rich class of gapped topological quantum phases [72].

For illustration purposes of the kind of quantum code that is within our scope, we discuss a simple example of a long-range toric code below. Of course, Euclidean implementations of good qLDPC codes coming from expander graphs would look more complicated. Indeed, the pioneering investigations for Euclidean implementations of efficient qLDPC codes and related error-correcting codes [75, 30, 125, 184, 28] have not fully settled on a specific Hamiltonian and non-locality in these works is not necessarily the same as power-law decay. In some cases it should rather be understood as few-body and not necessarily decaying in distance. Nonetheless, these proposals provide strong motivation for us to study quantum dynamics of Hamiltonians with commuting long-range interactions like the following one.

Example 1 (Long-range toric code). A non-trivial example of a Hamiltonian satisfying our assumptions for any $\alpha > 0$ is given by a long-range version of Kitaev's famous toric code model [138]. The basic model consists of spins sitting at the edges of the lattice $(\mathbb{Z}/L\mathbb{Z})^2$. We denote the set of vertices V, edges E and faces F of $(\mathbb{Z}/L\mathbb{Z})^2$. Then the Hilbert space is given by $\mathscr{H} := \bigotimes_{e\in E} \mathscr{H}_e := \bigotimes_{e\in E} \mathbb{C}^2$. To each vertex $s \in V$ one associates a *star* operator $A_s := -\mathbb{1} + \prod_{e\in E:\, v\in e} \sigma_e^X$, and to each face $p \in F$ one associates a *plaquette* operator $B_p := -\mathbb{1} + \prod_{e\in\partial(p)} \sigma_e^Z$, where $\sigma_e^\#$ is the Pauli # matrix acting on $\mathscr{H}_e$ and $\partial(p)$ denotes the four edges touching p. The Hamiltonian is then given as a sum

$$H := -\sum_{s\in V} A_s - \sum_{p\in F} B_p,$$

which turns out to be commuting and gapped [138] and has ground state energy 0. Moreover, it is frustration free, i.e. its ground state projection P also minimizes all individual terms $A_s\, P = 0$ and $B_p\, P = 0$. Adding a term of the form

$$H_{\text{long-range}} := \sum_{s_1,s_2\in V} f(s_1,s_2)\, A_{s_1}\, A_{s_2} + \sum_{p_1,p_2\in F} g(p_1,p_2)\, B_{p_1}\, B_{p_2},$$

with $0 \le f(s_1,s_2) \le C\, d(s_1,s_2)^{-\alpha}$ and $0 \le g(p_1,p_2) \le C\, d(p_1,p_2)^{-\alpha}$ which is positive and satisfies $H_{\text{long-range}}\, P = 0$ can only increase the gap. The perturbed Hamiltonian $H + H_{\text{long-range}}$ is still mutually commuting, gapped, has ground state P and is long-range. This example can be generalized in various ways, e.g. to higher dimensions. One can also add higher order products $(-1)^k A_{s_1} \cdots A_{s_k}$ and $(-1)^k B_{p_1} \cdots B_{p_k}$ or allow for small negative f and g and still obtain a spectral gap by adopting the BHM strategy [52, 175]. Having a spectral gap is not needed for the Lieb-Robinson bounds. It is only relevant for the applications to gapped ground states that we discussed in section 1.2. ◇

2 Mathematical setup

In this work we want to consider spin systems on D-regular graphs. The graph is seen as a metric space (Γ, d), where d is the graph distance. For any $Z \subset \Gamma$ we denote its cardinality by $|Z|$ and its diameter by $\operatorname{diam}(Z) := \sup_{x,y\in Z} d(x,y)$. Given two sets $X, Y \subset \Gamma$ we denote by $d(X,Y)$ their distance with respect to the metric d. We use $\Lambda \Subset \Gamma$ to denote that Λ is a finite subset of Γ. The ball with radius r around $x \in \Gamma$ is denoted $B_x^\Gamma(r) := \{ z \in \Gamma \,|\, d(x,z) \le r \}$.

Definition 2 (Surface-regular graph [192])**.** We call the graph (κ, D)-surface-regular, if

$$\left|B_x^\Gamma(r) \setminus B_x^\Gamma(r-1)\right| \le \kappa\, r^{D-1} \qquad \text{for all} \qquad x \in \Gamma \quad \text{and} \quad r \in \mathbb{N}. \qquad \diamond$$

The prototypical example of a (κ, D)-surface-regular lattice is $\Gamma = \mathbb{Z}^D$. Any D-surface-regular graph is also D-regular [192], in the sense that

$$\left|B_x^\Gamma(r)\right| \le (1+\kappa)\, r^D \qquad \text{for all} \qquad x \in \Gamma \quad \text{and} \quad r \in \mathbb{N}_{>0}.$$

This notion of D-regular graphs is more common in the literature. However, all lattices we are interested in are D-surface-regular, and we can prove better bounds with this stronger assumption. An explicit construction of a counter example can be found in [202].

Often however, we restrict to *finite* (κ, D)-surface-regular lattices Λ. This might seem trivial, but it is important to note that all our results and constants will be independent of the specific graph Λ and only depend on κ and D. In certain cases, this will allow us to take the limit $\Lambda_n \to \Gamma$ where all Λ_n are finite (κ, D)-regular lattices and Γ is an infinite one.

With every site $x \in \Gamma$ one associates a finite-dimensional local Hilbert space $\mathscr{H}_x := \mathbb{C}^q$ with the corresponding space of linear operators denoted by $\mathscr{A}_x := \mathscr{B}(\mathbb{C}^q)$. And for every finite $\Lambda \Subset \Gamma$ we define the Hilbert space $\mathscr{H}_\Lambda := \bigotimes_{x\in\Lambda} \mathscr{H}_x$, and denote the algebra of bounded linear operators on $\mathscr{H}_\Lambda$ by $\mathscr{A}_\Lambda := \mathscr{B}(\mathscr{H}_\Lambda)$. Due to the tensor product structure, we have $\mathscr{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathscr{A}_x$. Hence, for $X \subset \Lambda \Subset \Gamma$, any $A \in \mathscr{A}_X$ can be viewed as an element of $\mathscr{A}_\Lambda$ by identifying A with $A \otimes \mathbb{1}_{\Lambda\setminus X} \in \mathscr{A}_\Lambda$, where $\mathbb{1}_{\Lambda\setminus X}$ denotes the identity in $\mathscr{A}_{\Lambda\setminus X}$. This identification is always understood implicitly and for $B \in \mathscr{A}_\Lambda$ we denote by $\operatorname{supp}(B)$ the smallest $Y \subset \Lambda$ such that $B \in \mathscr{A}_Y$. Moreover, the algebra of local observables on Γ is given by

$$\mathscr{A}_\Gamma^{\mathrm{loc}} := \bigcup_{\Lambda \Subset \Gamma} \mathscr{A}_\Lambda, \qquad \text{and its completion} \qquad \mathscr{A}_\Gamma := \overline{\mathscr{A}_\Gamma^{\mathrm{loc}}}^{\|\cdot\|}$$

with respect to the operator norm is the algebra of *quasi-local* observables, which is only relevant if Γ is infinite.

An *interaction* on Γ is a function

$$\Psi \colon \{Z \Subset \Gamma\} \to \mathscr{A}_\Gamma, \quad Z \mapsto \Psi(Z) \in \mathscr{A}_Z \quad \text{with} \quad \Psi(Z) = \Psi(Z)^*. \tag{8}$$

For each $\Lambda \Subset \Gamma$, the corresponding local Hamiltonian is then defined as

$$H_\Lambda := \sum_{Z \subset \Lambda} \Psi(Z).$$

And the Heisenberg time-evolution of an operator $A \in \mathscr{A}_\Lambda$ is denoted by

$$\tau_t^\Lambda(A) := \mathrm{e}^{\mathrm{i}tH_\Lambda} A \,\mathrm{e}^{-\mathrm{i}tH_\Lambda}. \tag{9}$$

For infinite Γ, the Hamiltonian is not an object of the algebra $\mathscr{A}_\Gamma$ and the dynamics cannot be defined as in (9). Instead, one can define the time evolution on finite subgraphs $\Lambda \Subset \Gamma$ as above. The dynamics on Γ can then be defined as the limit

$$\tau_t^\Gamma(A) := \lim_{\Lambda \nearrow \Gamma} \tau_t^\Lambda(A) \tag{10}$$

if it exists for all $A \in \mathscr{A}_\Gamma^{\mathrm{loc}}$. It might then be extended to a cycle of automorphism on $\mathscr{A}_\Gamma$. This limit is understood in the following way: Let $\{\Lambda_n\}_{n\in\mathbb{N}} \subset \{\Lambda \Subset \Gamma\}$ be an increasing and exhaustive sequences, i.e. $\Lambda_n \subset \Lambda_{n+1}$ for all n and for each $Z \Subset \Gamma$ there is $n \in \mathbb{N}$ such that $Z \subset \Lambda_n$. If the limit $\lim_{n\to\infty} \tau_t^{\Lambda_n}(A)$ exists and is independent of the chosen sequence, it equals the above limit. We comment on the existence after introducing interaction norms.

To measure locality of an interaction so-called interaction norms are widely used in the literature. Therefore, let $F\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ be a decaying function. We define (for interactions on Γ)

$$\|\Psi\|_F := \sup_{x,y\in\Gamma} \sum_{\substack{Z\Subset\Gamma:\\ x,y\in Z}} \frac{\|\Psi(Z)\|}{F\big(d(x,y)\big)}, \tag{11}$$

and say that Ψ is F-local if $\|\Psi\|_F < \infty$. We choose to use this norm, because it allows for very general interactions. For two-body interactions with decay $\|\Psi(\{x,y\})\| \leq C\,F\big(d(x,y)\big)$, one directly computes $\|\Psi\|_F \leq C$.

Our bounds will depend on a suitable interaction norm $\|\Psi\|_F$. Beyond this, however, the constants in all our results will be independent of the particular lattice Γ. In this way, if one defines an interaction on an infinite graph Γ such that $\|\Psi\|_F^{(\Gamma)} < \infty$, one obtains bounds that are uniform in $\Lambda \Subset \Gamma$, by using $\|\Psi\|_F^{(\Lambda)} \leq \|\Psi\|_F^{(\Gamma)}$. Here, we add an extra index to highlight on which graph (11) is used. This in particular holds for the prototypical example $\Gamma = \mathbb{Z}^D$, where one requires

$$\|\Psi\|_F^{(\mathbb{Z}^D)} := \sup_{x,y\in\mathbb{Z}^D} \sum_{\substack{Z\Subset\mathbb{Z}^D:\\ x,y\in Z}} \frac{\|\Psi(Z)\|}{F\big(d(x,y)\big)}$$

and uniform obtains bounds on arbitrary subsets $\Lambda \Subset \mathbb{Z}^D$. But with the more general framework, one can for example also consider rectangles with torus geometry, or more general graphs like the honeycomb lattice.

With the notion of interaction norms at hand, we can come back to the thermodynamic limit. For interactions with decay $F(r) \leq (1+r)^{-\alpha}$ with $\alpha > D$ on a D-surface-regular graph, existence of the thermodynamic limit is known, see e.g. [174, Theorem 3.5].[1] For slower decaying interactions, one does not expect a thermodynamic limit to exist in general. In fact, one can expect that the interaction at least need to be summable in the sense

$$\sup_{z\in\Gamma} \sum_{\substack{Z\Subset\Gamma: \\ z\in Z}} \|\Psi(Z)\| < \infty,$$

to take a thermodynamic limit. See for example [195, Theorem 7.6.2] for the existence of the thermodynamic limit without explicitly requiring decay in the diameter $\mathrm{diam}(Z)$ of the supports.

To discuss the operator localization version of the LRB, we introduce the conditional expectation $\mathbb{E}_Y : \mathscr{A}_\Lambda \to \mathscr{A}_Y$, which has many crucial properties, which are for example discussed in [174, Lemma 4.1]. For us, it is only important to know that $\|\mathbb{E}_Y\| \leq 1$ and that its restriction to $\mathscr{A}_Y$ is the identity. For finite Λ it is just given by the partial trace on the complement [53]

$$\mathbb{E}_Y(A) = \frac{1}{\mathrm{Tr}_{\Lambda\setminus Y}(\mathbb{1}_{\Lambda\setminus Y})} \mathrm{Tr}_{\Lambda\setminus Y}(A), \tag{12}$$

and a generalization to infinite volume systems, which we will use in some of the statements, are defined in [174, Lemma 4.2]. Extensions to fermionic systems [173] also exist, and the LRBs we discuss here similarly apply for fermions.

3 Lieb-Robinson bounds for commuting interactions

We now restrict to mutually commuting interactions satisfying $[\Psi(X), \Psi(Y)] = 0$ for all X and Y. We split the discussion according to the decay of the interactions.

3.1 Finite-range interactions

The simplest type of interactions are finite-range interactions, i.e. interactions for which there exists R such that $\Psi(Z) = 0$ if $\mathrm{diam}(Z) > R$. For commuting, finite-range interactions, the Heisenberg time evolution remains localized for all times in the following sense.

Theorem 3 (Lieb-Robinson bound for commuting, finite range Hamiltonians)**.** *Let Ψ be a commuting interaction on a graph Γ. Assume that Ψ is of range R, i.e. $\Psi(Z) = 0$ if* $\mathrm{diam}(Z) > R$*, and uniformly bounded, i.e. there exists a constant C_Ψ such that $\|\Psi(Z)\| < C_\Psi$ for all $Z \Subset \Gamma$.*

[1] To apply their results, one needs to realize that F_α satisfies the convolution condition [174, equation 3.9] on a D-surface-regular graph Γ if $\alpha > D$, by the same arguments as given in [174, section 8.1.1] for $\Gamma = \mathbb{Z}^D$. Alternatively, one can just use (18) as outlined in the main text.

Then the corresponding Heisenberg time-evolution satisfies

$$\tau_t^{\Gamma}(A) = \tau_t^{X_R}(A) \in \mathcal{A}_{X_R} \qquad \textit{for all} \qquad X \Subset \Gamma \quad \textit{and} \quad A \in \mathcal{A}_X,$$

where $X_R := \{ z \in \Gamma \,|\, d(z, X) \leq R \}$ *is the* R *neighbourhood of* X*. Thus, it also satisfies the following Lieb-Robinson bound: For all* $X, Y \Subset \Gamma$ *and operators* $A \in \mathcal{A}_X$*,* $B \in \mathcal{A}_Y$*,*

$$\left\| [\tau_t^{\Gamma}(A), B] \right\| = 0 \qquad \textit{if} \qquad d(X, Y) > R.$$

This behaviour of commuting finite-range interactions was observed before, for example in [186]. We note that this behaviour dramatically differs from the general LRB for non-commuting, finite-range Hamiltonians. Our goal will be to show that this extreme locality persists to some degree for long-range commuting interactions.

3.2 Short-range interactions

Another class of widely used interactions are so-called short-range or (stretched) exponentially-decaying interactions, which have a bounded interaction norm with decay function

$$F_{b,p}(r) := \mathrm{e}^{-b\, r^p}$$

for some $b > 0$ and $p \in (0, 1]$.

Theorem 4. *Let* $0 < b' < b$*,* $p \in (0, 1]$*,* $\kappa > 0$ *and* $D \in \mathbb{N}$*, then there exists a constant* $C_{b,b'} > 0$ *such that the following holds: Let* Ψ *be a commuting interaction on a* (κ, D)*-regular graph* Γ *satisfying* $\|\Psi\|_{F_{b,p}} < \infty$*. Then the corresponding Heisenberg time-evolution satisfies the following Lieb-Robinson bounds:*

(i) *For all disjoint* $X, Y \Subset \Gamma$ *and operators* $A \in \mathcal{A}_X$*,* $B \in \mathcal{A}_Y$*,*

$$\left\| [\tau_t^{\Gamma}(A), B] \right\| \leq C_{b,b'} \, \|\Psi\|_{F_{b,p}} \, \|A\| \, \|B\| \, \min\big\{|X|, |Y|\big\} \, |t| \, F_{b',p}\big(d(X, Y)\big). \tag{13}$$

(ii) *For all* $X \Subset \Gamma$*, operators* $A \in \mathcal{A}_X$*, and* $r \geq 0$*,*

$$\left\| \tau_t^{\Gamma}(A) - \mathbb{E}_{X_r}\big(\tau_t^{\Gamma}(A)\big) \right\| \leq C_{b,b'} \, \|\Psi\|_{F_{b,p}} \, \|A\| \, |X| \, |t| \, F_{b',p}(r).$$

A more general result for short-range interactions, assuming only that the commutator of interaction terms is small (but not necessarily zero), was proven by Haah et al. [106]. In the limit of vanishing commutators, their bound gives the same LRB. The result stated here, gives a logarithmic light cone, which Toniolo and Bose [208] call "slow dynamics". They then prove stability of the LRBs to local perturbations similar to (30) but for extended perturbations given by an interaction.

3.3 Long-range interactions

Recently, many efforts were made to prove LRBs for long-range interactions, which are described by the decay function

$$F_\alpha(r) := (1 + r)^{-\alpha}.$$

While this is quite involved for general (non-commuting) Hamiltonians, one can very easily obtain LRBs for commuting Hamiltonians. Moreover, it is often necessary to distinguish different regimes in α. In particular, on a D-surface-regular graph, one cannot expect a thermodynamic limit of the dynamics for $0 < \alpha \leq D$, which is the reason for us to provide two distinct statements. The first works also for infinite graphs and allows for two different bounds, one scales better in the support of the two observables, the other scales better in the distance between them.

Theorem 5. *Let $\kappa > 0$, $D \in \mathbb{N}$ and $\alpha > D$, then there exists a constant $C > 0$ such that the following holds: Let Ψ be a commuting interaction on a (κ, D)-surface-regular graph Γ satisfying $\|\Psi\|_{F_\alpha} < \infty$. Then the corresponding Heisenberg time-evolution satisfies the following Lieb-Robinson bounds:*

(i) *For all disjoint $X, Y \Subset \Gamma$ and operators $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$,*

$$\left\|[\tau_t^\Gamma(A), B]\right\| \leq C\, \|\Psi\|_{F_\alpha}\, \|A\|\, \|B\|\, \min\{|X|, |Y|\}\, |t|\, F_{\alpha-D}\big(d(X,Y)\big), \tag{14}$$

and

$$\left\|[\tau_t^\Gamma(A), B]\right\| \leq 4\, \|\Psi\|_{F_\alpha}\, \|A\|\, \|B\|\, |X|\, |Y|\, |t|\, F_\alpha\big(d(X,Y)\big). \tag{15}$$

(ii) *For all $X \Subset \Gamma$, operators $A \in \mathscr{A}_X$, and $r \geq 0$,*

$$\left\|\tau_t^\Gamma(A) - \mathbb{E}_{X_r}\big(\tau_t^\Gamma(A)\big)\right\| \leq C\, \|\Psi\|_{F_\alpha}\, \|A\|\, |X|\, |t|\, F_{\alpha-D}(r). \tag{16}$$

This gives a linear light cone (a ballistic LRB) in the sense of local approximations for $\alpha > D+1$ which is an improvement over the tight $\alpha > 2D+1$ found for non-commuting two-body interactions [210].

But even for $\alpha > 0$, we can obtain a bound as in (15) on arbitrarily large, but finite graphs.

Theorem 6. *Let $\alpha > 0$ and Λ be a finite graph. Let Ψ be a commuting interaction on Λ satisfying $\|\Psi\|_{F_\alpha} < \infty$. Then the corresponding Heisenberg time-evolution satisfies the Lieb-Robinson bound given in* (15)*.*

This gives some kind of linear light cone for commuting long-range interactions with $\alpha > 1$, which are not even summable in dimensions $D > 1$. In this parameter range, the Heisenberg dynamics do not converge to a thermodynamic limit in general. Hence, a bound like (14) cannot hold, as it implies convergence of the Heisenberg dynamics in the thermodynamic limit. Thus, we cannot discuss LRBs for the limiting dynamics on infinite graphs.

Remark 7. The Lieb-Robinson bounds in theorem 5 are sharp. For example, to see that (14) is sharp, let $X = \{x\}$, $Y = \{y\}$ and choose the interaction with only non-vanishing term $\Psi(\{x,y\}) = F_\alpha\big(d(x,y)\big)\,U$, where $U = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}(\mathbb{1}_x+\sigma_x^Z)(\mathbb{1}_y-\sigma_y^X)}$ is a standard CNOT gate between sites x and y. The CNOT gate U satisfies $\mathrm{e}^{\mathrm{i}tU} = \cos(t)\,\mathbb{1}+\mathrm{i}\sin(t)\,U$ and $\|U\| = 1$. Moreover, let $A = \sigma_x^X$ be a spin flip and $B = \sigma_y^Z$ measure the Z-component of the qubit at site y. Now, let ψ be any pure state on $\Lambda\setminus\{x,y\}$. Then

$$[\tau_t^\Lambda(A), B]\,|{\downarrow\downarrow}\rangle_{xy}\otimes\psi = -2\,\mathrm{i}\sin\big(t\,F_\alpha\big(d(x,y)\big)\big)\,|{\uparrow\uparrow}\rangle_{xy}\otimes\psi,$$

and thus

$$\big\|[\tau_t^\Lambda(A), B]\big\| \geq 2\,\big|\sin\big(t\,F_\alpha\big(d(x,y)\big)\big)\big|,$$

which for small $t\,F_\alpha\big(d(x,y)\big)$ has the same scaling as (15). In section 3.5, we present a generalization of this example that extends sharpness of (15) to the case of general supports X, Y and we also show sharpness of the other bounds from theorem 5. ◇

3.4 Proof of the Lieb-Robinson bound for commuting Hamiltonians

The proof of all the stated LRBs boils down to a very simple observation, which was already made before, e.g. in [186]. It drastically simplifies the Heisenberg evolution of local operators by only considering the interaction terms, which overlap with the support of the operator.

Lemma 8. *Let Ψ be a commuting interaction on a finite graph Λ. For all $X\subset\Lambda$ and operators $A\in\mathcal{A}_X$, the corresponding Heisenberg time-evolution satisfies*

$$\tau_t^\Lambda(A) = \tau_{t,\cap X}^\Lambda(A),$$

where $\tau_{t,\cap X}^\Lambda$ is generated by the interaction

$$\Psi_{\cap X}(Z) := \begin{cases}\Psi(Z) & \text{if } X\cap Z\neq\emptyset \text{ and}\\ 0 & \text{else.}\end{cases} \tag{17}$$

Proof. Due to commutativity of the interaction, the exponential satisfies

$$\mathrm{e}^{\mathrm{i}tH_\Lambda} = \mathrm{e}^{\mathrm{i}tH_{\Lambda\setminus X}}\,\mathrm{e}^{\mathrm{i}t(H_\Lambda-H_{\Lambda\setminus X})}.$$

Using this for both exponentials in (9) and realizing that $\big[H_{\Lambda\setminus X}, A\big] = 0$ due to their disjoint support, one has

$$\tau_t^\Lambda(A) = \mathrm{e}^{-\mathrm{i}t(H_\Lambda-H_{\Lambda\setminus X})}\,A\,\mathrm{e}^{\mathrm{i}t(H_\Lambda-H_{\Lambda\setminus X})} = \tau_{t,\cap X}^\Lambda(A).$$ □

From here, the statement about finite-range Hamiltonians on finite graphs in theorem 3 is immediately clear. The extension to infinite graphs follows from the next theorem, and we comment on the details at the end of this section.

Lemma 8 also allows to approximate the time-evolution locally.

Theorem 9 (Local approximation of Heisenberg evolution)**.** *Let Ψ be a commuting interaction on a finite graph Λ. For all $X \subset \Lambda' \subset \Lambda$ and $A \in \mathscr{A}_X$ it holds that*

$$\left\|\tau_t^{\Lambda}(A) - \tau_t^{\Lambda'}(A)\right\| \leq 2\,\|A\|\,|t| \sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\emptyset,\\ Z\cap\Lambda\setminus\Lambda'\neq\emptyset}} \left\|\Psi(Z)\right\|.$$

Moreover, if $\|\Psi\|_F < \infty$ for some decaying function $F\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, then

$$\left\|\tau_t^{\Lambda}(A) - \tau_t^{\Lambda'}(A)\right\| \leq 2\,\|A\|\,\|\Psi\|_F\,|t| \sum_{x\in X} \sum_{y\in\Lambda\setminus\Lambda'} F\big(d(x,y)\big). \tag{18}$$

Proof. By lemma 8 we have

$$\begin{aligned}
\tau_t^{\Lambda}(A) - \tau_t^{\Lambda'}(A) &= \tau_{t,\cap X}^{\Lambda}(A) - \tau_{t,\cap X}^{\Lambda'}(A)\\
&= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\, \tau_{s,\cap X}^{\Lambda} \circ \tau_{t-s,\cap X}^{\Lambda'}(A)\,\mathrm{d}s\\
&= \sum_{\substack{Z\subset\Lambda:\\ Z\cap\Lambda\setminus\Lambda'\neq\emptyset}} \int_0^t \tau_{s,\cap X}^{\Lambda}\left(\left[\Psi_{\cap X}(Z), \tau_{t-s,\cap X}^{\Lambda'}(A)\right]\right)\mathrm{d}s,
\end{aligned}$$

and thus the statement after bounding

$$\left\|\int_0^t \tau_{s,\cap X}^{\Lambda}\left(\left[\Psi_{\cap X}(Z), \tau_{t-s,\cap X}^{\Lambda'}(A)\right]\right)\mathrm{d}s\right\| \leq 2\,|t|\,\|\Psi_{\cap X}(Z)\|\,\|A\|.$$

For the second statement, we additionally bound

$$\sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\emptyset,\\ Z\cap\Lambda\setminus\Lambda'\neq\emptyset}} \left\|\Psi(Z)\right\| \leq \sum_{x\in X}\sum_{y\in\Lambda\setminus\Lambda'} F\big(d(x,y)\big) \sum_{\substack{Z\subset\Lambda:\\ x,y\in Z}} \frac{\left\|\Psi(Z)\right\|}{F\big(d(x,y)\big)} \leq \|\Psi\|_F \sum_{x\in X}\sum_{y\in\Lambda\setminus\Lambda'} F\big(d(x,y)\big)$$

by overcounting. □

As a direct consequence, we also obtain the two versions of the LRB.

Corollary 10 (Lieb-Robinson bound for commuting Hamiltonians)**.** *Let Ψ be a commuting interaction on a finite graph Λ. Let $F\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ be a decaying function and assume that Ψ is F local in the sense that $\|\Psi\|_F < \infty$. Then the corresponding Heisenberg time-evolution satisfies the following Lieb-Robinson bounds:*

(i) *For all disjoint $X, Y \subset \Lambda$ and operators $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$,*

$$\left\|[\tau_t^\Lambda(A), B]\right\| \leq 4\,\|\Psi\|_F\,\|A\|\,\|B\|\,|t| \sum_{x\in X}\sum_{y\in Y} F\big(d(x,y)\big).$$

(ii) *For all $X \subset \Lambda$, operators $A \in \mathcal{A}_X$, and $r \geq 0$,*

$$\left\|\tau_t^\Lambda(A) - \mathbb{E}_{X_r}\big(\tau_t^\Lambda(A)\big)\right\| \leq 4\,\|\Psi\|_F\,\|A\|\,|t| \sum_{x\in X}\sum_{y\in\Lambda\setminus X_r} F\big(d(x,y)\big).$$

Proof. Both statements directly follow from theorem 9. For (i) choose $\Lambda' = \Lambda\setminus Y$. Then $\tau_t^{\Lambda\setminus Y}(A) \in \mathcal{A}_{\Lambda\setminus Y}$ and $B \in \mathcal{A}_Y$ have disjoint support and thus commute. Hence,

$$\left\|[\tau_t^\Lambda(A), B]\right\| \leq \left\|[\tau_t^\Lambda(A) - \tau_t^{\Lambda\setminus Y}(A), B]\right\| \leq 2\left\|\tau_t^\Lambda(A) - \tau_t^{\Lambda\setminus Y}(A)\right\|\|B\|.$$

For (ii) choose $\Lambda' = X_r$. Then using $\mathbb{E}_{X_r}(\tau_t^{X_r}(A)) = \tau_t^{X_r}(A)$ and $\|\mathbb{E}_{X_r}\| = 1$ we obtain

$$\begin{aligned}\left\|\tau_t^\Lambda(A) - \mathbb{E}_{X_r}\big(\tau_t^\Lambda(A)\big)\right\| &\leq \left\|\tau_t^\Lambda(A) - \tau_t^{X_r}(A)\right\| + \left\|\mathbb{E}_{X_r}\big(\tau_t^{X_r}(A)\big) - \mathbb{E}_{X_r}\big(\tau_t^\Lambda(A)\big)\right\| \\ &\leq 2\left\|\tau_t^\Lambda(A) - \tau_t^{X_r}(A)\right\|.\end{aligned}$$

□

Note, that the local approximation is different between theorem 9 for $\Lambda' = X_r$ and corollary 10(ii). The first compares the evolution of A to the time evolution generated by the local Hamiltonian on X_r, while the latter gives an approximation of $\tau_t^\Lambda(A)$ in X_r by the conditional expectation. The former implies the latter.

We can now prove theorems 4 to 6 on finite graphs. The extension to infinite graphs will be discussed afterwards. Notice that

$$\sum_{x\in X}\sum_{y\in Y} F\big(d(x,y)\big) \leq |X|\,|Y|\,F\big(d(X,Y)\big) \tag{19}$$

for any F. Together with corollary 10, this proves theorem 6 and (15) from theorem 5(i). Alternatively, one can bound

$$\sum_{x\in X}\sum_{y\in Y} F\big(d(x,y)\big) \leq \min\big\{|X|, |Y|\big\} \sup_{x\in\Lambda} \sum_{\substack{y\in\Lambda:\\ d(x,y)\geq d(X,Y)}} F\big(d(x,y)\big)$$

if the right-hand side is bounded. Indeed, for the decay function $F_\alpha(r) := (1+r)^{-\alpha}$, a simple integration shows that for all $\kappa > 0$, $D \in \mathbb{N}$, $\alpha > D$, there exists a constant $C/4 > 0$ such that

$$\sup_{x\in\Lambda} \sum_{\substack{y\in\Lambda:\\ d(x,y)\geq R}} F_\alpha\big(d(x,y)\big) \leq \frac{C}{4}\,F_{\alpha-D}(R) \tag{20}$$

for all (κ, D)-surface-regular graphs Λ. This proves the remaining bounds from theorem 5.

Similarly, for theorem 4, we observe that for all $0 < b' < b$, $p \in (0,1]$, $\kappa > 0$ and $D \in \mathbb{N}$, there exists a constant $C > 0$ such that

$$\sup_{x\in\Lambda} \sum_{\substack{y\in\Lambda:\\ d(x,y)\geq R}} F\big(d(x,y)\big) \leq C\,\mathrm{e}^{-b'R^p} \tag{21}$$

for all (κ, D)-regular graphs.

It is left to obtain the results also on infinite graphs Γ. As discussed in section 2, we first define the Heisenberg dynamics on $\mathscr{A}_\Gamma^{\mathrm{loc}}$ as the limit

$$\tau_t^\Gamma(A) := \lim_{\Lambda\nearrow\Gamma} \tau_t^\Lambda(A),$$

which is well-defined if $\|\Psi\|_{F_\alpha} < \infty$ for some $\alpha > D$. To obtain the LRBs on Γ one uses triangle inequality in theorem 9 to bound

$$\big\|\tau_t^\Gamma(A) - \tau_t^{X_r}(A)\big\| \leq \big\|\tau_t^\Gamma(A) - \tau_t^\Lambda(A)\big\| + \big\|\tau_t^\Lambda(A) - \tau_t^{X_r}(A)\big\|,$$

where the first term vanishes in the limit $\Lambda \nearrow \Gamma$ and the second gives the previously proven LRBs which did not depend on Λ.

3.5 Sharpness of the Lieb-Robinson bound

The bounds in corollary 10 are sharp and we now construct an example 'protocol' which attains the bound. This easily implies sharpness of theorem 5 as well, as we explain afterwards. The main difference to protocols for non-commuting interactions as in [149, 209] is that due to commutativity only terms in the Hamiltonian directly connecting the supports X and Y have an influence.

3.5.1 Sharpness of Corollary 10

Let Λ be any finite graph and consider two-dimensional on-site Hilbert spaces $\mathscr{H}_x = \mathbb{C}^2$, and denote with $\sigma_x^\#$ the Pauli # matrix on $\mathscr{H}_x$. We fix $X, Y \subset \Lambda$. Then, for some $C > 0$ let

$$\Psi(Z) = \begin{cases} F_\alpha\big(d(x,y)\big)\,C\,U_{x,y} & \text{if } Z = \{x,y\} \text{ for some } x \in X \text{ and } y \in Y \\ 0 & \text{else,} \end{cases}$$

where

$$U_{x,y} = \sigma_x^Z\sigma_y^Z = |{\uparrow\uparrow}\rangle\langle{\uparrow\uparrow}|_{xy} - |{\uparrow\downarrow}\rangle\langle{\uparrow\downarrow}|_{xy} - |{\downarrow\uparrow}\rangle\langle{\downarrow\uparrow}|_{xy} + |{\downarrow\downarrow}\rangle\langle{\downarrow\downarrow}|_{xy}$$

has norm $\|U_{x,y}\| = 1$, such that $\|\Psi\|_{F_\alpha} = C$. Clearly,

$$\mathrm{e}^{\mathrm{i}tU_{x,y}} = \mathrm{e}^{+\mathrm{i}t}\,|{\uparrow\uparrow}\rangle\langle{\uparrow\uparrow}|_{xy} + \mathrm{e}^{-\mathrm{i}t}\,|{\uparrow\downarrow}\rangle\langle{\uparrow\downarrow}|_{xy} + \mathrm{e}^{-\mathrm{i}t}\,|{\downarrow\uparrow}\rangle\langle{\downarrow\uparrow}|_{xy} + \mathrm{e}^{+\mathrm{i}t}\,|{\downarrow\downarrow}\rangle\langle{\downarrow\downarrow}|_{xy},$$

so that the dynamics generated by $U_{x,y}$ add a phase $\mathrm{e}^{\pm\mathrm{i}t}$ in the computational basis.

In the following, we abbreviate $|a\rangle|b\rangle|c\rangle = |a\rangle_X \otimes |b\rangle_Y \otimes |c\rangle_{\Lambda\setminus(X\cup Y)}$, $|{\Uparrow}\rangle = |{\uparrow\cdots\uparrow}\rangle$ and $|{\Downarrow}\rangle = |{\downarrow\cdots\downarrow}\rangle$. Let ψ be any state on $\Lambda \setminus (X \cup Y)$. Since $U_{x,y}$ mutually commute, we have

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}tH_\Lambda}\,|{\Uparrow}\rangle|{\Uparrow}\rangle|\psi\rangle &= \prod_{x\in X}\prod_{x\in X} \mathrm{e}^{\mathrm{i}tCF_\alpha(d(x,y))U_{x,y}}\,|{\Uparrow}\rangle|{\Uparrow}\rangle|\psi\rangle \\
&= \mathrm{e}^{\mathrm{i}tC\sum_{x\in X}\sum_{y\in Y}F_\alpha(d(x,y))}\,|{\Uparrow}\rangle|{\Uparrow}\rangle|\psi\rangle \\
&=:\ \mathrm{e}^{\mathrm{i}tc}\,|{\Uparrow}\rangle|{\Uparrow}\rangle|\psi\rangle
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}tH_\Lambda}|{\Uparrow}\rangle|{\Downarrow}\rangle|\psi\rangle &= \mathrm{e}^{-\mathrm{i}tc}\,|{\Uparrow}\rangle|{\Downarrow}\rangle|\psi\rangle, \\
\mathrm{e}^{\mathrm{i}tH_\Lambda}|{\Downarrow}\rangle|{\Uparrow}\rangle|\psi\rangle &= \mathrm{e}^{-\mathrm{i}tc}\,|{\Downarrow}\rangle|{\Uparrow}\rangle|\psi\rangle \quad \text{and} \\
\mathrm{e}^{\mathrm{i}tH_\Lambda}|{\Downarrow}\rangle|{\Downarrow}\rangle|\psi\rangle &= \mathrm{e}^{\mathrm{i}tc}\,|{\Downarrow}\rangle|{\Downarrow}\rangle|\psi\rangle,
\end{aligned}$$

where $c = C\sum_{x\in X}\sum_{y\in Y}F_\alpha\big(d(x,y)\big)$. Similarly to the protocol in remark 7, we choose $A = \prod_{x\in X}\sigma_x^X$ such that $A\,|{\Downarrow}\rangle|\phi\rangle|\psi\rangle = |{\Uparrow}\rangle|\phi\rangle|\psi\rangle$ for all states ϕ on Y. Moreover, we choose $B = |{\Uparrow}\rangle\langle{\Downarrow}|_Y + |{\Downarrow}\rangle\langle{\Uparrow}|_Y$, which acts as identity on $|{\Uparrow}\rangle + |{\Downarrow}\rangle$. Hence,

$$\begin{aligned}
\big[\tau_t(A), B\big]|{\Downarrow}\rangle\big(|{\Downarrow}\rangle + |{\Uparrow}\rangle\big)|\psi\rangle &= (1-B)\,\mathrm{e}^{\mathrm{i}tH}\,A\,\mathrm{e}^{-\mathrm{i}tH}\,|{\Downarrow}\rangle\big(|{\Downarrow}\rangle + |{\Uparrow}\rangle\big)|\psi\rangle \\
&= (1-B)\,|{\Uparrow}\rangle\big(\mathrm{e}^{-2\mathrm{i}tc}\,|{\Downarrow}\rangle + \mathrm{e}^{2\mathrm{i}tc}\,|{\Uparrow}\rangle\big)|\psi\rangle \\
&= |{\Uparrow}\rangle\Big(\big(\mathrm{e}^{-2\mathrm{i}tc} - \mathrm{e}^{2\mathrm{i}tc}\big)\,|{\Downarrow}\rangle + \big(\mathrm{e}^{2\mathrm{i}tc} - \mathrm{e}^{-2\mathrm{i}tc}\big)\,|{\Uparrow}\rangle\Big)|\psi\rangle,
\end{aligned}$$

and thus

$$\big\|\big[\tau_t(A), B\big]\big\| \geq \big|\mathrm{e}^{2\mathrm{i}tc} - \mathrm{e}^{-2\mathrm{i}tc}\big| = 2\,\sin\Big(2\,t\,\|\Psi\|_{F_\alpha}\sum_{x\in X}\sum_{y\in Y}F_\alpha\big(d(x,y)\big)\Big).$$

Using that $\sin(x) \approx x$ for small x, we see that the LRB obtained in corollary 10(i) is sharp whenever its right-hand side is indeed small.

As discussed before, the operator localization version of the LRB implies the commutator version of the LRB. More precisely, corollary 10(ii) implies

$$\big\|[\tau_t^\Lambda(A), B]\big\| \leq 8\,\|\Psi\|_F\,\|A\|\,\|B\|\,|t|\sum_{x\in X}\sum_{y\in\Lambda\setminus X_r}F\big(d(x,y)\big)$$

for all $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_{\Lambda\setminus X_r}$. Hence, the above argument implies that corollary 10(ii) is also optimal up to a factor 2.

3.5.2 Sharpness of Theorem 5

To understand that the scaling in the long-range LRBs given in (14) and (16) is optimal, it is enough to understand that the bound

$$\sum_{x\in X}\sum_{y\in Y} F_\alpha\big(d(x,y)\big) \le |X|\,\frac{C}{4}\,F_{\alpha-D}\big(d(X,Y)\big) \quad \text{for all disjoint } X, Y \subset \Gamma,$$

which was used to obtain the results from corollary 10, is also optimal. And indeed, for $X = \{0\}$ and $Y = \{\, y \in \Gamma \,|\, d(x,y) \ge R \,\}$ on $\Gamma = \mathbb{Z}^D$, it is straightforward to check that

$$\sum_{y\in Y} F_\alpha\big(d(0,y)\big) = \sum_{n=R}^{\infty} \sum_{\substack{y\in\Gamma:\\ d(0,y)=n}} F_\alpha(n) \ge \sum_{n=R}^{\infty} |Q_n|\, F_\alpha(n) = \frac{1}{(D-1)!} \sum_{n=R}^{\infty} F_{\alpha-(D-1)}(n),$$

where $Q_n = \big\{ z \in \mathbb{R}^{D-1} \,\big|\, \sum_{k=1}^{D-1} z_k \le n+1, \forall k \,:\, z_k \ge 0 \big\}$. Moreover, this sequence only converges for $\alpha > D$ and then can be lower bounded by

$$\sum_{n=R}^{\infty} F_{\alpha-(D-1)}(n) \ge \int_R^\infty \frac{1}{(r+1)^{\alpha-(D-1)}}\,\mathrm{d}r = \frac{1}{\alpha-D}\,F_{\alpha-D}(R).$$

Hence, the obtained decay in theorem 5 is optimal.

4 Exemplary applications

In recent years, LRBs have been used for a vast range of applications. Here, we improve some of the results from the literature for commuting interactions. In particular, we discuss decay of correlations and the 'local perturbations perturb locally' principle in gapped ground states of long-range Hamiltonians in sections 4.1 and 4.2, respectively. In section 4.3, we show that the LRB for a Hamiltonian, which is a sum of a finite-range, commuting Hamiltonian and a general Hamiltonian, does not depend on the strength of the commuting part.

4.1 Decay of correlations

Hastings and Koma [118] proved that local interactions and a uniform spectral gap imply decay of correlations in the ground state in the thermodynamic limit. Their argument includes long-range interactions with $\alpha > D$. However, they use a LRB that follows from the usual proof and is outperformed by more recent approaches, e.g. [160, 210], even for non-commuting Hamiltonians.

For commuting Hamiltonians, the improved LRBs also result in improved decay of correlations as the following theorem shows. The necessary adjustments to the original proof are discussed in section 5.1.

Theorem 11 (Spectral gap implies decay of correlations). *Let Ψ be a commuting interaction on a finite graph Λ and assume that the spectrum of H has a gap $g > 0$ above the ground state energy E_0 in the sense that*

$$E_0 \in \sigma(H), \quad E_0 < \sigma(H) \setminus \{E_0\} \quad \text{and} \quad d\big(\{E_0\}, \sigma(H) \setminus \{E_0\}\big) \geq g.$$

Let P_0 be the projection onto the ground state sector and ρ_0 be any ground state, i.e. $\rho_0 = P_0\, \rho_0\, P_0$.

If Ψ is polynomially decaying with $\|\Psi\|_{F_\alpha} < \infty$ for some $\alpha > 0$ where $F_\alpha(r) := (1+r)^{-\alpha}$, then, for all disjoint X and $Y \subset \Lambda$ and $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$, it holds that

$$\begin{aligned} &\Big|\mathrm{Tr}\big(\rho_0\, A\, B\big) - \tfrac{1}{2}\Big(\mathrm{Tr}\big(\rho_0\, A\, P_0\, B\big) + \mathrm{Tr}\big(\rho_0\, B\, P_0\, A\big)\Big)\Big| \\ &\qquad \leq 8\, \|A\|\, \|B\|\, |X|\, |Y|\, \|\rho_0\|_1 \left(\sqrt{\frac{\alpha}{\pi}}\, \frac{\|\Psi\|_{F_\alpha}}{g} + 1\right) \ln(1+r)\, F_\alpha(r), \end{aligned} \tag{22}$$

where $r = d(X,Y)$.

If Ψ is a short-range interaction satisfying $\|\Psi\|_{F_{b,p}} < \infty$ for some $b > \tilde{b} > 0$ and $p \in (0,1]$ where $F_{b,p}(r) := \mathrm{e}^{-br^p}$, then, for all disjoint X and $Y \subset \Lambda$ and $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$, it holds that

$$\begin{aligned} &\Big|\mathrm{Tr}\big(\rho_0\, A\, B\big) - \tfrac{1}{2}\Big(\mathrm{Tr}\big(\rho_0\, A\, P_0\, B\big) + \mathrm{Tr}\big(\rho_0\, B\, P_0\, A\big)\Big)\Big| \\ &\qquad \leq \|A\|\, \|B\|\, \min\big\{|X|, |Y|\big\}\, \|\rho_0\|_1 \left(\frac{C\, \|\Psi\|_{b,p}}{g} + 1\right) F_{\tilde{b},p}(r), \end{aligned} \tag{23}$$

where $r = d(X,Y)$ and C is an explicit constant given in (29) and only depends on b and $\tilde{b}$.

The results concerning decay of correlations from [118] have been improved for general (non-commuting) interactions in [214] using analogous methods to those used in [214] to prove LPPL, which we will discuss in the next section. In the case of commuting interactions, the improved method qualitatively yields the same results, and we decided to use the previous method by [118] to emphasize the broad applicability of improved LRBs for commuting Hamiltonians.

Not that the first lines in (22) and (23) simplify to the more common

$$\big|\mathrm{Tr}(\rho_0\, A\, B) - \mathrm{Tr}(\rho_0\, A)\ \mathrm{Tr}(\rho_0\, B)\big|$$

in the case $\rho_0 = P_0$.

We only give a finite volume version of the statement in [118] here. For $\alpha > D$ (or short-range interactions) there exists a thermodynamic, see (10). Then, a statement similar to theorem 11 also holds in the thermodynamic limit, where it is enough that a gapped spectral patch converges to a unique ground state energy as in [118, (2.12)].

Note, that the decay exponent in (22) does not depend on the gap and equals the decay of the interaction. Thus, it is better than the trivial bound $2\,\|A\|\,\|B\|\,\|\rho_0\|_1$ for $r \gtrsim g^{-1/\alpha}$. Hence, this result is a qualitative improvement over the original one in [118] and also the more recent [214]. In both references, the decay exponent in the bound scales like $\alpha\, g$ for small g, meaning that the bound becomes non-trivial only for $\ln r \gtrsim g^{-1}$.

Moreover, this qualitative improvement is also apparent in the result for short-range interactions: The previous results in [118, 214] both have a correlation length $\xi \sim 1/g$ for small g, i.e. the bounds they obtain scale with $\mathrm{e}^{-d(X,Y)/\xi}$. In [101] an improved scaling $\xi \sim 1/\sqrt{g}$ was obtained for frustration-free Hamiltonians. In contrast, here we prove that the correlation length is independent of the gap, $\xi \sim 1$ if the underlying interaction is commuting.

Remark 12. For $\alpha > D$, one can also use (14) and obtain the same statement with the bound from (22) replaced by

$$8\,C\,\|A\|\,\|B\|\,\min\{|X|,|Y|\}\,\|\rho_0\|_1\,\|\Psi\|_{F_\alpha}\left(\sqrt{\frac{\alpha}{\pi}}\,\frac{\|\Psi\|_{F_\alpha}}{g}+1\right)\ln(1+r)\,F_{\alpha-D}(r),$$

where C is the constant from (14). ◇

4.2 Local perturbations perturb locally principle

Wang and Hazzard [214] recently proved a version of the local perturbations perturb locally (LPPL) principle for long-range systems with a new technique. They avoid the spectral flow [120], which was used by previous results for short-range Hamiltonians [23]. The general idea is that the ground state of a gapped system only changes locally around a small perturbation, in the sense that expectations values of local observables away from the perturbation do not change.

With the improved LRBs we can also improve these results for gapped commuting Hamiltonians.

Theorem 13 (LPPL for gapped ground states)**.** *Let Ψ be a commuting interaction on a finite graph Λ and let $V \in \mathcal{A}_X$ with $X \in \Lambda$ be some perturbation. Moreover, assume that $H + \lambda V$ has a unique ground state ρ_λ and a gap of size at least $g > 0$ above the ground state for all $\lambda \in [0, 1]$.*

If Ψ is polynomially decaying with $\|\Psi\|_{F_\alpha} < \infty$ for some $\alpha > 0$ where $F_\alpha(r) := (1+r)^{-\alpha}$, then, for all $Y \subset \Lambda$ and $B \in \mathcal{A}_Y$, it holds that

$$\left|\mathrm{Tr}\big(\rho_0\, B\big) - \mathrm{Tr}\big(\rho_1\, B\big)\right| \le 32\,\|\Psi\|_{F_\alpha}\,\|B\|\,\big(\|V\| + \|V\|^2\big)\,|X|\,|Y|\,F_\alpha(r)\,\frac{g+2}{g^3}, \tag{24}$$

where $r = d(X,Y)$.

If Ψ is a short-range interaction satisfying $\|\Psi\|_{F_{b,p}} < \infty$ for some $b > b' > 0$ and $p \in (0,1]$ where $F_{b,p}(r) := \mathrm{e}^{-br^p}$, then, for all $Y \subset \Lambda$ and $B \in \mathcal{A}_Y$, it holds that

$$\left|\mathrm{Tr}\left(\rho_0\, B\right) - \mathrm{Tr}\left(\rho_1\, B\right)\right| \leq 8\, C_{b,b'}\, \|\Psi\|_{b,p}\, \|B\| \left(\|V\| + \|V\|^2\right) \min\{|X|,|Y|\}\, F_{b',p}(r)\, \frac{g+2}{g^3},$$

where $r = d(X,Y)$, and $C_{b,b'}$ is the constant from (13).

As in the result on decay of correlations, the decay exponent in (24) does not depend on the gap and equals the decay of the interaction. It is better than the trivial bound $2\,\|B\|$ for $r \gtrsim g^{-3/\alpha}$. Thus, it is a qualitative improvement compared to the bound in [214] for general interactions, where the exponent scales like $\alpha\, g$ for small g, which make the bound better than the trivial one for $\ln r \gtrsim g^{-1}$. The same is also apparent in the short-range setting, where we obtain a decay $\mathrm{e}^{-b' d(X,Y)^p}$ for any $b' < b$ uniformly in $g > 0$ for commuting interactions, while [214] only proves a stretched-exponential decay with $b' \sim b\, g$ for small g for general interactions.

To obtain the statement, we need improved LRBs also for the perturbed Hamiltonian $H + V$. And since the statement is trivial, if also the perturbation V commutes with all Hamiltonian terms, we cannot just rely on the results from section 3.3. Instead, we use a previous result on LRBs for perturbed Hamiltonians [P4, Lemma 33], details are discussed in section 5.2.

Remark 14. For $\alpha > D$, one can also use (14) and obtain the same statement with the bound from (24) replaced by

$$32\, C\, \|\Psi\|_{F_\alpha}\, \|B\| \left(\|V\| + \|V\|^2\right) \min\{|X|,|Y|\}\, F_{\alpha-D}(r)\, \frac{g+2}{g^3},$$

where C is the constant from (14). ◇

4.3 General Hamiltonians with a commuting part

In contrast to the rest of the paper, we want to investigate non-commuting Hamiltonians which only have a commuting part in this section. More specifically, we assume to be given two interactions Φ and Ψ, where Ψ is commuting as before, Φ might not be commuting and in particular Ψ and Φ are not required to commute. For simplicity, we assume that the commuting interaction Ψ is of finite range $R > 0$, meaning that $\Psi(Z) = 0$ unless $\mathrm{diam}(Z) > R$. The arguments work similarly for short- or long-range interactions, as long as the decay of the commuting part Ψ is better than the one of the non-commuting part Φ. Moreover, we allow Φ to be time-dependent. We write $\Phi(t)$ for the interaction at time t, $\Phi(Z,t) := \Phi(t)(Z)$ the interaction term at $Z \subset \Lambda$ at time t, and

$$\|\Phi\|_F := \sup_{t\in I} \|\Phi(t)\|_F$$

for the norm of a time-dependent interaction Φ. The time interval $I \subset \mathbb{R}$ of interest will be clear from the context.

In this setting we obtain the following LRB for the evolution of the full system.

Theorem 15. *Let Λ be a finite graph and $F\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ a decaying function. Assume that we have a general Lieb-Robinson bound given in terms of a function ζ, such that for any interaction Φ with $\|\Phi\|_F < \infty$, it holds that for all $X, Y \subset \Lambda$, $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$*

$$\big\|[\tau_{t,s}(A), B]\big\| \leq \zeta\Big(\|\Phi\|_F, \|A\|, \|B\|, |X|, |Y|, d(A,B), |t-s|\Big).$$

Furthermore, let Ψ be a commuting interaction of range $R > 0$ and let Φ be a general, time-dependent interaction on Λ satisfying $\|\Phi\|_{F(\cdot+2R)} < \infty$. Then the dynamics $\tau_{t,s}$ for the sum $\Phi + \Psi$ satisfy the following modified Lieb-Robinson bound: For all $X, Y \subset \Lambda$, $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$ it holds that

$$\big\|[\tau_{t,s}(A), B]\big\| \leq \zeta\Big(c^2 \|\Phi\|_{F(\cdot+2R)}, \|A\|, \|B\|, c\,|X|, c\,|Y|, d(A,B) - 2R, |t-s|\Big),$$

with $c = (1+\kappa)\, R^D$.

In particular, the Lieb-Robinson velocity and the bound do not depend on the norm of the commuting part Ψ. As long as the evolution exists it could even be unbounded if one works in a setting with infinite dimensional local Hilbert spaces.

As an example we provide the following result for exponentially decaying interactions, where we use the LRB from [174].

Corollary 16. *Let Λ be a finite graph, Ψ a commuting interaction of range $R > 0$ and Φ an exponentially decaying interaction with $\|\Phi\|_{F(\cdot+2R)} < \infty$ where $F(r) := \mathrm{e}^{-br}/(1+r)^{D+1}$. Then the full dynamics $\tau_{t,s}$ generated by the interaction $\Psi+\Phi$ satisfies the following Lieb-Robinson bound: For all $X, Y \subset \Lambda$, $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$, it holds that*

$$\big\|[\tau_{t,s}(A), B]\big\| \leq C\,\|A\|\,\|B\|\,\min\big\{|X|, |Y|\big\}\,\mathrm{e}^{b\,(v_b|t-s| - d(X,Y))},$$

where $C = 2\,\|F\|\,C_F^{-1}\,\mathrm{e}^{2bR}$ and $v_b = 2\,C_F\,(1+\kappa)^2\,R^{2D}\,\|\Phi\|_{F(\cdot+2R)}/b$, and $\|F\|$ and C_F are constants depending on F that are defined in [174, equations (3.8) and (3.9)].

Again, we highlight that the velocity does not depend on the strength of Ψ, but only on its range.

As an easy example, consider the XXZ spin chain on the Hilbert space $\bigotimes_{x=1}^{L} \mathbb{C}^{2S+1}$

$$H_{XXZ} := -\sum_{x=1}^{L-1} \big(S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2 + \Delta S_x^3 S_{x+1}^3\big)$$

with anisotropy parameter $\Delta > 1$ and each $\vec{S}_x$ an irreducible spin-S representation of $\mathfrak{su}(2)$. Applying the above result to the commuting interactions $\Psi(\{x, x+1\}) := \Delta S_x^3\, S_{x+1}^3$, we obtain a LRB that is independent of Δ.

5 Proofs of applications

5.1 Proof of Theorem 11

Since we only consider finite volumes with a single ground state energy, the proof simplifies slightly compared to [118]. To get the constants right, we repeat large parts of the proof but leave out some of the details and the proof of [118, Lemma 3.1]. We first give the proof for (22) and then comment about the small modifications necessary for (23).

First, we observe

$$\begin{aligned}
&\operatorname{Tr}\big(\rho_0 \left[\tau_t^\Lambda(A), B\right]\big) \\
&\quad= \operatorname{Tr}\big(\rho_0\, \tau_t^\Lambda(A)\, (\mathbb{1} - P_0)\, B\big) - \operatorname{Tr}\big(\rho_0\, B\, (\mathbb{1} - P_0)\, \tau_t^\Lambda(A)\big) \\
&\qquad+ \operatorname{Tr}\big(\rho_0\, \tau_t^\Lambda(A)\, P_0\, B\big) - \operatorname{Tr}\big(\rho_0\, B\, P_0\, \tau_t^\Lambda(A)\big) \\
&\quad= \sum_{n>0} \operatorname{Tr}\big(\rho_0\, A\, P_n\, B\big)\, \mathrm{e}^{-\mathrm{i}(E_n - E_0)t} - \sum_{n>0} \operatorname{Tr}\big(\rho_0\, A\, P_n\, B\big)\, \mathrm{e}^{\mathrm{i}(E_n - E_0)t} \\
&\qquad+ \operatorname{Tr}\big(\rho_0\, A\, P_0\, B\big) - \operatorname{Tr}\big(\rho_0\, B\, P_0\, A\big),
\end{aligned} \tag{25}$$

where P_n are the spectral projections for eigenvalues E_n, and we used

$$\mathrm{e}^{\mathrm{i}Ht}\, \rho_0 = \mathrm{e}^{\mathrm{i}E_0 t}\, \rho_0 \qquad \text{and} \qquad \mathrm{e}^{\mathrm{i}Ht}\, P_n = \mathrm{e}^{\mathrm{i}E_n t}\, P_n \quad \text{for all} \quad n.$$

As in [118] we now apply

$$\mathscr{F}(\cdot) := \lim_{T\to\infty} \lim_{\varepsilon\to 0} \frac{\mathrm{i}}{2\pi} \int_{-T}^{T} \frac{(\cdot)\, \mathrm{e}^{-\beta t^2}}{t + \mathrm{i}\varepsilon}\, \mathrm{d}t$$

on both sides of (25). Instead of [118, (3.24)] we use the LRB from (15). Thus, for the left-hand side we bound

$$\left|\frac{\mathrm{i}}{2\pi} \int_{-T}^{T} \frac{\operatorname{Tr}\big(\rho_0 \left[\tau_t^\Lambda(A), B\right]\big)}{t + \mathrm{i}\varepsilon}\, \mathrm{e}^{-\beta t^2}\, \mathrm{d}t\right| \le \frac{C_1}{2\pi}\, F_\alpha(r) \int_{-\infty}^{\infty} \mathrm{e}^{-\beta t^2}\, \mathrm{d}t \le \frac{C_1}{2\sqrt{\pi\beta}}\, F_\alpha(r), \tag{26}$$

with $C_1 = 4\, \|\Psi\|_{F_\alpha}\, \|A\|\, \|B\|\, |X|\, |Y|\, \|\rho_0\|_1$, which also persists the two limits. For the two time-independent terms on the right-hand side, we calculate

$$\lim_{T\to\infty} \lim_{\varepsilon\to 0} \frac{\mathrm{i}}{2\pi} \int_{-T}^{T} \frac{\mathrm{e}^{-\beta t^2}}{t + \mathrm{i}\varepsilon}\, \mathrm{d}t = \frac{1}{2}.$$

And by [118, Lemma 3.1], which – after inspection of the constants – states

$$\left|\mathscr{F}\big(\mathrm{e}^{-\mathrm{i}Et}\big) - 1\right| \le \frac{1}{2}\, \mathrm{e}^{-E^2/(4\beta)} \quad \text{for} \quad E > 0 \tag{27}$$

and

$$\left|\mathcal{F}\left(\mathrm{e}^{-\mathrm{i}Et}\right)\right| \leq \frac{1}{2}\,\mathrm{e}^{-E^2/(4\beta)} \quad \text{for} \quad E < 0, \tag{28}$$

for all $\beta > 0$, the remaining two terms on the right-hand side satisfy

$$\left|\mathcal{F}\Big(\sum_{n>0} \mathrm{Tr}\big(\rho_0\, A\, P_n\, B\big)\, \mathrm{e}^{-\mathrm{i}(E_n-E_0)t}\Big) - \mathrm{Tr}\big(\rho_0\, A\,(\mathbb{1} - P_0)\, B\big)\right| \leq C_2\, \mathrm{e}^{-g^2/(4\beta)}$$

and

$$\left|\mathcal{F}\Big(\sum_{n>0} \mathrm{Tr}\big(\rho_0\, A\, P_n\, B\big)\, \mathrm{e}^{\mathrm{i}(E_n-E_0)t}\Big)\right| \leq C_2\, \mathrm{e}^{-g^2/(4\beta)},$$

with $C_2 = 1/2\,\|A\|\,\|B\|\,\|\rho_0\|_1$. To prove these bounds, we first use triangle inequality and then apply (27) and (28), respectively, together with the upper bound $E_n - E_0 \geq g$. Then it is left to bound

$$\sum_{n>0} \left|\mathrm{Tr}\big(\rho_0\, A\, P_n\, B\big)\right| \leq \sum_m \left|\langle \phi_m, B\,\rho_0\, A\,\phi_m\rangle\right| \leq \|B\,\rho_0\, A\|_1 \leq \|A\|\,\|B\|\,\|\rho_0\|_1,$$

where $\{\phi_m\}$ is any orthonormal energy eigenbasis, i.e. for every n there exists an index set M_n such that $P_n = \sum_{m\in M_n} |\phi_m\rangle\langle\phi_m|$. In the second step we used that for every operator T and every ONB ϕ_m it holds that

$$\sum_m \left|\langle \phi_m, T\,\phi_m\rangle\right| = \sum_m \langle \phi_m, T\,D\,\phi_m\rangle = \mathrm{Tr}(T\,D) \leq \|T\|_1\,\|D\|,$$

where we choose $\theta_m \in \mathbb{C}$ with $|\theta_m| = 1$ such that $\left|\langle\phi_m, T\,\phi_m\rangle\right| = \theta_m\,\langle\phi_m, T\,\phi_m\rangle$ and $D = \sum_m \theta_m\,|\phi_m\rangle\langle\phi_m|$, which satisfies $\|D\| = 1$.

In total, we obtain

$$\begin{aligned}
&\left|\mathrm{Tr}\big(\rho_0\, A\, B\big) - \tfrac{1}{2}\Big(\mathrm{Tr}\big(\rho_0\, A\, P_0\, B\big) + \mathrm{Tr}\big(\rho_0\, B\, P_0\, A\big)\Big)\right| \\
&\quad \leq \frac{C_1}{2\sqrt{\pi\beta}}\, F_\alpha(r) + 2\,C_2\,\mathrm{e}^{-g^2/(4\beta)} \\
&\quad \leq \|A\|\,\|B\|\,\|\rho_0\|_1 \left(\frac{4\,\|\Psi\|_{F_\alpha}\sqrt{\alpha}}{\sqrt{\pi}\,g}\,|X|\,|Y|\,\ln(1+r) + 1\right) F_\alpha(r) \\
&\quad \leq 8\,\|A\|\,\|B\|\,|X|\,|Y|\,\|\rho_0\|_1 \left(\sqrt{\frac{\alpha}{\pi}}\,\frac{\|\Psi\|_{F_\alpha}}{g} + 1\right) \ln(1+r)\, F_\alpha(r),
\end{aligned}$$

where we chose

$$\beta = \frac{g^2}{4\,\alpha\,\ln(1+r)}$$

in the second step. For the last step we use $\ln(1+r)/\ln 2 \geq 1$ to move the logarithm out of the parenthesis, followed by bounding $\ln 2 \leq 1$ and $1/\ln 2 \leq 2$.

To obtain the result for short-range interactions, we only have to modify (26) and the final bound. We choose $b' = (b+\tilde{b})/2 \in (\tilde{b}, b)$ and use the LRB from theorem 4 for $0 < b' < b$. Due to the similar structure of the LRBs, we obtain

$$\left|\mathcal{F}\Big(\mathrm{Tr}\big(\rho_0\,[\tau_t^\Lambda(A), B]\big)\Big)\right| \leq \frac{C_1}{2\sqrt{\pi\beta}}\,F_{b',p}(r)$$

with $C_1 = C_{b,b'}\,\|\Psi\|_{F_{b,p}}\,\|A\|\,\|B\|\,\min\{|X|,|Y|\}\,\|\rho_0\|_1$ and $C_{b,b'}$ the constant from theorem 4. In total, we then obtain

$$\begin{aligned}
&\left|\mathrm{Tr}(\rho_0\,A\,B) - \tfrac{1}{2}\Big(\mathrm{Tr}(\rho_0\,A\,P_0\,B) + \mathrm{Tr}(\rho_0\,B\,P_0\,A)\Big)\right| \\
&\quad\leq \frac{C_1}{2\sqrt{\pi\beta}}\,F_{b',p}(r) + 2\,C_2\,\mathrm{e}^{-g^2/(4\beta)} \\
&\quad\leq \|A\|\,\|B\|\,\min\{|X|,|Y|\}\,\|\rho_0\|_1\left(\sqrt{\frac{\tilde{b}}{\pi}}\,\frac{C_{b,b'}\,\|\Psi\|_{F_{b,p}}}{g}\,r^{p/2}\,F_{b',p}(r) + F_{\tilde{b},p}(r)\right)
\end{aligned}$$

after choosing $\beta = \dfrac{g^2}{4\,\tilde{b}\,r^p}$. With [156, Lemma 7.2.3(b)] we can bound

$$r^{p/2}\,F_{b',p}(r) \leq \big(2\,\mathrm{e}\,(b'-\tilde{b})\big)^{-1/2}\,F_{\tilde{b},p}(r) = \big(\mathrm{e}\,(b-\tilde{b})\big)^{-1/2}\,F_{\tilde{b},p}(r),$$

which leads us to the final bound with

$$C = \left(\frac{\tilde{b}\,C_{b,(b+\tilde{b})/2}}{\pi\,\mathrm{e}\,(b-\tilde{b})}\right)^{1/2} \qquad \text{with } C_{b,(b+\tilde{b})/2} \text{ the constant from theorem 4.} \tag{29}$$

5.2 Proof of Theorem 13

As remarked in the main text, we need a LRB for all Hamiltonians $H + \lambda V$ to apply the results from [214]. But we only need to estimate commutators, where one of the observables is the perturbation itself. We begin with the proof of (24). Without requiring that V and H commute, combining [P4, Lemma 33] and (15), we obtain the bound

$$\left\|\left[\mathrm{e}^{-\mathrm{i}t(H+\lambda V)}\,B\,\mathrm{e}^{\mathrm{i}t(H+\lambda V)}, V\right]\right\| \leq 4\,\|\Psi\|_{F_\alpha}\,\|B\|\,\|V\|\,|X|\,|Y|\,|t|\,\big(1 + \|V\|\,|t|\big)\,F_\alpha\big(d(X,Y)\big). \tag{30}$$

In the language of [214] we have

$$C(r,t) = C\,F_\alpha(r)\,\big(t + t^2\big), \tag{31}$$

where $C = 4\,\|\Psi\|_{F_\alpha}\,\|B\|\,\|V\|\,|X|\,|Y|\,\big(\|V\| + \|V\|^2\big)$. As in [214], we omit the labels λ, since all bounds are uniform in those. Then we calculate, see [214, eq. (11)],

$$\bar{\Omega}(r,y) := \int_0^\infty C(r,t)\,\mathrm{e}^{-yt}\,\mathrm{d}t = C\,F_\alpha(r)\,\frac{y+2}{y^3}.$$

With [214, eq. (13)], we bound

$$\begin{aligned}\ln\left|\Omega_{XY}(0)\right| &\leq \frac{1}{2\pi}\int_0^{2\pi} \ln \bar{\Omega}\big(r, |\rho\sin\theta|\big)\,\mathrm{d}\theta \\ &\leq \ln\big(C\,F_\alpha(r)\big) + \ln\big(\rho+2\big) - \ln\rho^3 - \frac{3}{2\pi}\int_0^{2\pi}\ln|\sin\theta|\,\mathrm{d}\theta \\ &\leq \ln\left(8\,C\,F_\alpha(r)\,\frac{\rho+2}{\rho^3}\right),\end{aligned}$$

for any $\rho \in (0, g)$. In the last step, we used $\int_0^{2\pi}\ln|\sin\theta|\,\mathrm{d}\theta = -\pi\ln 4$. Now, the remaining arguments from [214] and the limit $\rho \to g$ yield the statement.

Since the explicit form of the decay was not used, the proof of (13) is exactly the same, but with

$$C = C_{b,b'}\,\|\Psi\|_{F_{b,b'}}\,\|A\|\,\|B\|\,\min\big\{|X|,|Y|\big\}\left(\|V\| + \|V\|^2\right)$$

and $C(r,t) = C\,F_{b',p}(r)\left(t+t^2\right)$ coming from the LRB given in theorem 4.

5.3 Lieb-Robinson bounds for Hamiltonians with a commuting part

The idea of the proof is to go to the interaction picture where Ψ is the unperturbed part and Φ is the perturbation. The range of the interaction picture interaction Φ^{int} will only be slightly enlarged by the commuting, finite-range interaction Ψ, while the norm of the individual terms are not changed at all by the unitary transformation. This allows to apply known LRBs for non-commuting interactions.

The dynamics for the full Hamiltonian $H(t) := H^{\Psi} + H^{\Phi}(t) := \sum_{Z\subset\Lambda}\Psi(Z) + \sum_{Z\subset\Lambda}\Phi(t,Z)$ is given by the unique solution of

$$-\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\tau_{t,s}(A) = \tau_{t,s}\big(\big[H(t),A\big]\big) \qquad \text{and} \qquad \tau_{s,s} = \mathrm{id} \qquad \text{for all} \qquad s,t\in I. \tag{32}$$

For the interaction picture, we define

$$\Phi^{\mathrm{int}}(t,Z) := \sum_{\substack{X\subset\Lambda:\\ X_R = Z}} \tau_t^{\Psi}\big(\Phi(t,X)\big), \qquad \text{such that} \qquad H^{\mathrm{int}}(t) = \tau_t^{\Psi}\left(H^{\Phi}(t)\right), \tag{33}$$

where $H^{\mathrm{int}}(t) := \sum_{Z\subset\Lambda}\Phi^{\mathrm{int}}(t,Z)$, $H^{\Phi}(s) = \sum_{Z\subset\Lambda}\Phi(t,Z)$, and τ^{Ψ} is the dynamics on Λ generated by Ψ as given in (9). From section 3.1 we know that $\tau_t^{\Psi}\left(\Phi(s,X)\right) \in \mathcal{A}_{X_R}$.

We recall that the dynamics $\tau_{t,s}^{\mathrm{int}}$ generated by Φ^{int} is the solution of the analogue equation to (32) with generator $H^{\mathrm{int}}(t)$. We conclude that $\tau_{t,s} = \tau_{-s}^{\Psi}\circ\tau_{t,s}^{\mathrm{int}}\circ\tau_t^{\Psi}$ by observing that the right-hand side solves (32) since

$$-\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\tau^{\Psi}_{-s}\circ\tau^{\mathrm{int}}_{t,s}\circ\tau^{\Psi}_{t}(A) = \tau^{\Psi}_{-s}\circ\tau^{\mathrm{int}}_{t,s}\Big(\big[H^{\mathrm{int}}(t),\tau^{\Psi}_{t}(A)\big]\Big) + \tau^{\Psi}_{-s}\circ\tau^{\mathrm{int}}_{t,s}\circ\tau^{\Psi}_{t}\big([H^{\Psi},A]\big)$$
$$= \tau^{\Psi}_{-s}\circ\tau^{\mathrm{int}}_{t,s}\circ\tau^{\Psi}_{t}\big([H^{\Psi}+H^{\Phi}(t),A]\big).$$

To obtain a LRB for the full evolution $\tau_{t,s}$ we now look at this decomposition. First, by (33), for $x,y\in\Lambda$, we have

$$\begin{aligned}\sum_{\substack{Z\subset\Lambda:\\ x,y\in Z}}\|\Phi^{\mathrm{int}}(t,Z)\| &\leq \sum_{\substack{Z\subset\Lambda:\\ x,y\in Z}}\ \sum_{\substack{X\subset\Lambda:\\ X_R=Z}}\|\tau^{\Psi}_{t}\big(\Phi(t,X)\big)\| \\ &\leq \sum_{\substack{x'\in B_x(R)\\ y'\in B_y(R)}}\ \sum_{\substack{X\subset\Lambda:\\ x',y'\in Z}}\|\Phi(t,X)\| \\ &\leq \sum_{\substack{x'\in B_x(R)\\ y'\in B_y(R)}}\|\Phi\|_{F(\cdot\,+2R)}\,F\big(d(x',y')+2R\big).\end{aligned}$$

Hence, $\|\Phi^{\mathrm{int}}\|_F \leq (1+\kappa)^2\,R^{2D}\,\|\Phi\|_{F(\cdot\,+2R)}$, since $d(x',y')\geq d(x,y)-2R$. Then, we rewrite

$$\big\|[\tau_{t,s}(A),B]\big\| = \big\|[\tau^{\mathrm{int}}_{t,s}\circ\tau^{\Psi}_{t}(A),\tau^{\Psi}_{s}(B)]\big\| \tag{34}$$

and recall that τ^{Ψ}_{s} does not change the norm, $\|\tau^{\Psi}_{t}(A)\|=\|A\|$ but increases the support, if $A\in\mathscr{A}_X$, then $\tau^{\Psi}_{t}(A)\in\mathscr{A}_{X_R}$. Hence, after applying a LRB for $\tau^{\mathrm{int}}_{t,s}$ we need to subtract $2R$ from the distance.

6 Conclusions

We proved that the quantum many-body dynamics produced by mutually commuting long-range interactions is much more strongly constrained than for general long-range interactions. The difference is fundamental at a physical level, as commutativity directly affects the speed of information propagation and the shape of the light cone. This finding opens up a new structural divide between general long-range interactions and long-range commuting ones.

While commuting interactions are very special, they are of great current interest in physics in the context of quantum error correction [75, 30, 125, 184, 28]. An illustrative example of a long-range toric stabilizer code is given in example 1. The enhanced LRBs place severe limits on the rate of entanglement generation [53] and quantum messaging [85] for these quantum codes.

The results also imply that commuting long-range Hamiltonians with $\alpha>1$ have anomalously slow dynamics and could therefore provide simple test beds for mathematical physics conjectures surrounding many-body localization [1]. For example, they satisfy the main assumption in [208] without disorder-averaging. Finally, it would be interesting to probe the robustness of the methods by assuming only power-law decay of the commutators, in the spirit of [186] and [106], which treated such a question for finite-range and short-range interactions, respectively.

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Author contributions

Marius Lemm: conceptualization (equal); investigation (equal); writing – original draft (equal); writing – review and editing (equal).

Tom Wessel: conceptualization (equal); investigation (equal); writing – original draft (equal); writing – review and editing (equal).

Data availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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Publication P6

Response theory for locally gapped systems

Joscha Henheik [*] Tom Wessel [†]

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Abstract

We introduce a notion of a *local gap* for interacting many-body quantum lattice systems and prove the validity of response theory and Kubo's formula for localized perturbations in such settings. On a high level, our result shows that the usual spectral gap condition, which concerns the system as a whole, is not a necessary condition for understanding local properties of the system.

More precisely, we say that an equilibrium state ρ_0 of a Hamiltonian H_0 is locally gapped in $\Lambda^{\mathrm{gap}} \subset \Lambda$, whenever the Liouvillian $-\mathrm{i}\,[H_0, \cdot\,]$ is almost invertible on local observables supported in Λ^{gap} when tested in ρ_0. To put this into context, we provide other alternative notions of a local gap and discuss their relations.

The validity of response theory is based on the construction of *non-equilibrium almost stationary states* (NEASSs). By controlling locality properties of the NEASS construction, we show that response theory holds to any order, whenever the perturbation ϵV acts in a region which is further than $|\log \epsilon|$ away from the non-gapped region $\Lambda \setminus \Lambda^{\mathrm{gap}}$.

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[*] Institute of Science and Technology Austria (IST Austria),
Am Campus 1, 3400 Klosterneuburg, Austria.
[†] Fachbereich Mathematik, Universität Tübingen,
Auf der Morgenstelle 10, 72076 Tübingen, Germany.

Contents

1 Introduction

Spectral gaps lie at the heart of many areas of physics and mathematics [59, 163, 105, 104, 25]. In particular, questions concerning the spectral gap lie at the center of several of the most challenging problems in physics, e.g. the Berry-Tabor [33] and Bohigas-Giannoni-Schmit [38] conjectures in quantum chaos, the Yang-Mills mass gap problem, or Haldane's conjecture on integer valued antiferromagnetic Heisenberg chains [107, 108].

The great interest in spectral gaps, far beyond these famous conjectures, is rooted in the fact that their existence has tremendous effects on fundamental properties of the system. For example, in quantum many-body physics, the existence of a spectral gap above the ground state eigenvalue has far-reaching consequences for entanglement properties [171, 115] and ground state correlations [118]. On the other hand, the *closing* of a spectral gap is related to the occurrence of a (topological) quantum phase transition [23, 174]. Finally, the assumption of a spectral gap played a crucial role in recent proofs of adiabatic theory and linear response for many-body systems; see Section 1.1 for a detailed discussion.

However, an overall drawback in all of the above examples is that the desired spectral property is of *global* nature, i.e. involving the studied system as a whole, and thus seldomly compatible with a notion of *locality* in an underlying physical space. Therefore, a natural question to ask is:

> How can one express that a system is *locally* gapped, and which consequences that one has for globally gapped systems persist?

In this paper, we study this question in the setting of locally interacting many-body quantum spin lattice systems; see Section 2 for precise definitions. More precisely, in the above spirit of our guiding question, this paper has two main goals:

1. We propose a notion of a *local gap* via a dynamical characterization and exemplarily prove that local perturbations of Hamiltonians with a frustration free product ground state satisfy this condition. Moreover, we study possible alternative notions of local gaps and their relations among each other; see Sections 1.2 and 3.

2. As an application to a physically relevant problem, we show that for Hamiltonians with a local gap, response theory approximately holds to any order and thus justify *Kubo's formula*; see Sections 1.3 and 4.1.

There are only few works in the literature studying many-body quantum systems under a non-standard gap condition, i.e. one differing from the clean separation of eigenvalues: In [69], the authors derive Kubo's formula for two-dimensional disordered systems having only a *mobility gap*. Another recent paper dealing with spectral gaps in presence of disorder is [71]. Together with Teufel, one of us proved [122, Theorem 4.1] that finite systems, whose thermodynamic limit has a usual spectral gap, approximately obey an adiabatic theorem. In another recent work [221], the authors developed a theory of metastable states, characterized by the requirement that local operators raise the energy of such a state by a certain minimal amount. Their condition is similar to an alternative notion of a local gap given in (35) below. Moreover, several months after the first posting of this article the authors of [58] proposed a notion of a local spectral gap in the context of topological materials, modeled by one-body Hamiltonians. Finally, we remark that, in the context of Lie group theory, the notion of a "local gap" has recently been introduced [43] and proved itself to have profound consequences [43, 42].

Next, in Section 1.1, we discuss the problem of justifying linear (and higher order) response theory and Kubo's formula based on adiabatic theory. Afterwards, in Section 1.2, we introduce our *local dynamical gap condition* ($\text{LDG}_{\text{informal}}$). Finally, in Section 1.3, we discuss our main result on response theory.

1.1 Response theory in many-body quantum systems

The purpose of response theory is to express how quantum expectation values change, after a small perturbation is slowly turned on. More precisely, one considers an unperturbed Hamiltonian H_0 with equilibrium state (usually a ground state) ρ_0 and slowly turns on a small additive perturbation εV. Denoting by ρ_ε the state of the system after εV has been turned on, one aims to understand, how the expectation value of an observable B changes, i.e. determine

$$\langle B \rangle_{\rho_\varepsilon} - \langle B \rangle_{\rho_0} = \varepsilon\, \sigma_B + o(\varepsilon) \tag{1}$$

at least to leading order in the strength ε.

A central piece of response theory is *Kubo's formula* [142], which provides a simple expression for the so-called linear response coefficient σ_B in (1). Despite the simplicity and empirical success of Kubo's formula, the problem of justifying it in a very general framework has so far escaped rigorous treatment. The fundamental difficulty lies in the fact that the state ρ_ε is no longer an equilibrium state in general, and therefore determining it is outside the powerful realm of equilibrium statistical mechanics. This problem has prominently been pointed out by Simon [198] in 1984 in his "Fifteen problems in mathematical physics", containing a rigorous justification of Kubo's formula from first principles as problem (4B).

However, in the particular setting of many-body lattice systems with a spectral gap at zero temperature, it has recently been possible [17, 165, 203, 124] to actually prove Kubo's formula and justify the applicability of linear response theory to compute the change in expectation values (1). The more general underlying results establish *generalized super-adiabatic theorems*[1] for short-range interacting Hamiltonians, which can be written as a sum of local terms, and are hence called *SLT operators* [123, 122, P2].

The recent breakthrough, which paved the way for these results, was achieved by Bachmann, De Roeck, and Fraas [17], who proved the first adiabatic theorem for extended (but finite) quantum lattice systems; see [165] for an adaptation to fermionic systems. One key difficulty is that, for macroscopic systems, typical operator norm bounds in adiabatic theory deteriorate due to the *orthogonality catastrophe* and one instead has to formulate the result in a weaker topology by testing against local observables. On a high level, the main ingredient for their proof are the well-known *Lieb-Robinson bounds* (LRBs) [154], which ensure a finite speed of propagation and prevent build-up of long-range entanglement. These LRBs, enabled to prove that the generator of the *spectral flow*, introduced by Hastings and Wen [120], is in fact an SLT operator and hence maintains good locality properties [120, 23] (showing so-called *automorphic equivalence* of ground states).

However, the work [17] had the limitation that the spectral gap of H_0 is assumed to remain open after adding the perturbation εV. To allow εV to close the gap, Teufel [203] combined ideas from space-time adiabatic perturbation theory [182, 180] with locality estimates from [17]. The underlying perturbative scheme is an iterative application of locality preserving Schrieffer-Wolff transformations (a.k.a. Lie-Schwinger block diagonalization [93]), which proved to be a powerful approach in several rather recent works in that direction [17, 203, 123, 122, 220, 71, 221]. In this paper, we carefully exploit locality properties of the operations involved in the perturbative scheme, which allows dealing with *locally* gapped systems; see Section 1.2 below.

The rough physical picture underlying the works [203, 182, 180] is that a gap that is locally intact after adding perturbation should be sufficient for adiabatic theory to be valid (cf. [203, Figure 1]). The results from [17, 165, 203] for large but finite systems were subsequently extended to the thermodynamic limit [123, 122], building on an extension of the spectral flow techniques to infinite systems by Moon and Ogata [167].

[1] This term describes adiabatic theorems for time-dependent Hamiltonians of the form $H_\varepsilon(t) = H_0(t)+\varepsilon V(t)$, where $H_0(t)$ is assumed to have a spectral gap. Now "super-adiabatic" means that for $\varepsilon = 0$, there exists a state $\rho_0^\eta(t)$ close to the instantaneous ground state $\rho_0(t)$ of $H_0(t)$, such that the time-evolution generated by $\eta^{-1}H_0(\,\cdot\,)$ intertwines $\rho_0^\eta(t_0)$ and $\rho_0^\eta(t)$ to any order in η. The term "generalized" means that, even for a gap-closing perturbation V, there exist super-adiabatic non-equilibrium almost-stationary states (NEASSs) $\Pi^{\varepsilon,\eta}(t)$, which are intertwined by the time evolution generated by $\eta^{-1}H_\varepsilon(\,\cdot\,)$ to any order in ε and η. We refer to [165, 203, 123, 122, P2] for details; see also Sections 4.2 and 6.2.

We point out that, contrary to [123], the papers [167, 122] assumed a spectral gap *only* for the GNS Hamiltonian of the infinite system (a *gap in the bulk*). More comprehensive reviews of the developments discussed in this section are given in [124, P2].

In view of the linear response problem and the second of our principal goals, the contribution of this paper is to extend the previous results for uniformly or bulk gapped systems to systems where H_0 is locally gapped. This important extension allows to rigorously treat systems with impurities of gap-closing edge modes; see the discussions in Sections 1.2–1.3 below. Technically, our contribution is to control operations on SLT operators, which are localized on a subregion of the whole lattice; see Section 2.2 and Appendix A.

Lastly, we remark that we only consider finite-dimensional spaces and bounded operators for simplicity of the presentation.

1.2 A local dynamical gap condition

All the previous results on linear response and adiabatic theory mentioned in Section 1.1 above, heavily rely on the range of the initial state $\rho \equiv \rho_0$ being contained in a gapped part of the spectrum of the unperturbed Hamiltonian $H \equiv H_0$.[2] More precisely, these results assume that the spectrum of H can be decomposed as

$$\mathrm{spec}(H) = \sigma_1 \cup \sigma_2 \qquad \text{with} \qquad d(\sigma_1, \sigma_2) \geq g \tag{2}$$

for some gap size $g > 0$ such that $\sigma_1 \subset I \subset \mathbb{R} \setminus \sigma_2$ for some compact interval I. Then, they require $P \rho P = \rho$, where P denotes the spectral projection[3] associated to H onto σ_1.

On a technical level, in all of the works [17, 165, 203, 123, 122], the crucial importance of the gap of H lies in the local invertibility of the Liouvillian. That is, there exists an explicit, locality preserving map $\mathcal{I}_{H,g} = \mathcal{I}_{H,g}[\![\,\cdot\,]\!]$, which is often called *quasi-local inverse of the Liouvillian* on the observable algebra, depending on the SLT Hamiltonian H and the gap size g, which inverts the Liouvillian $\mathcal{L}_H[\![\,\cdot\,]\!] := -\mathrm{i}\,[H, \cdot\,]$ in the projection $\langle\,\cdot\,\rangle_P = \mathrm{tr}\big(P\,\cdot\,\big)$ onto σ_1. More precisely, for all local observables A, B it holds that

$$\big\langle [\mathcal{L}_H \circ \mathcal{I}_{H,g}[\![A]\!] - A, B] \big\rangle_P = 0. \tag{3}$$

Surprisingly, the Hamiltonian H having a spectral gap is actually *equivalent* to the invertibility of the Liouvillian in the above sense, see Proposition 3.1.

The goal of this article is to relax the requirement of a globally spectrally gapped Hamiltonian H and instead work with a so-called *local dynamical gap condition.* This

[2] For ease of notation and since there will be no perturbation εV in the current Section 1.2, we will drop the subscript 0 here.

[3] We will follow the convention that *(orthogonal) projections* will be denoted by P (i.e. satisfying $P^2 = P$ and $P^* = P$), while *states* are denoted by ρ (i.e. satisfying $\rho = \rho^*$ and $0 \leq \rho \leq 1$ with $\mathrm{tr}\,\rho = 1$). Clearly, if P is an orthogonal projection, then $\rho := P/\dim\mathrm{rank}\,P$ is a state.

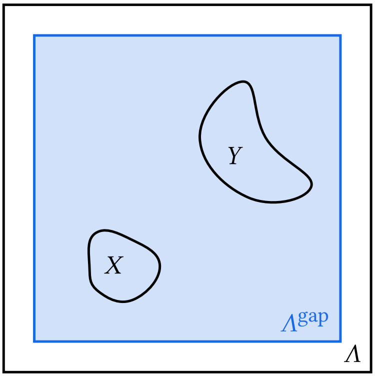

Figure 1. Illustrated is the local dynamical gap condition ($\mathrm{LDG_{informal}}$) in the case where the system is gapped in the bulk of Λ, e.g. due to gap closing edge modes. If the observables $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$ are supported well inside $\Lambda \setminus \Lambda^{\mathrm{gap}}$, the RHS of (4) is small, i.e. the Liouvillian is locally almost invertible.

condition roughly asserts that, the Hamiltonian H behaves as if it had a gap in a spatially localized region, i.e. that the Liouvillian can (almost) be locally inverted in that region. A more formal version of ($\mathrm{LDG_{informal}}$) is formulated in Assumption ($\mathrm{LDG_{main}}$) in Section 3.1.

($\mathbf{LDG_{informal}}$) Local dynamical gap condition (informal version)**.** Let H be an SLT Hamiltonian and ρ an equilibrium state of H, i.e. $[H, \rho] = 0$. We say that ρ is *locally dynamically gapped* of size at least $g > 0$ in a region $\Lambda^{\mathrm{gap}} \subset \Lambda$ if and only if for all observables[4] $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$ localized in $X \subset \Lambda$ and $Y \subset \Lambda$, it holds that

$$\begin{aligned} \left| \big\langle \big[\mathscr{L}_H \circ \mathscr{I}_{H,g}[\![A]\!] - A, B \big] \big\rangle_\rho \right| \le C \, \|A\| \, \|B\| \, \big(\mathrm{diam}(X) + \mathrm{diam}(Y) \big)^{\ell} \\ \times \exp\Big(- \big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}}) + d(Y, \Lambda \setminus \Lambda^{\mathrm{gap}}) \big)^{q} \Big) \end{aligned} \tag{4}$$

for some fixed $\ell \in \mathbb{N}_0$ and constants $C, q > 0$, independent of the sizes $|\Lambda|$ and $|\Lambda^{\mathrm{gap}}|$. $\diamond$

In a nutshell, this means that, within Λ^{gap}, the Hamiltonian H approximately behaves as if it were spectrally gapped – up to an error vanishing (stretched) exponentially fast in the distance to $\Lambda \setminus \Lambda^{\mathrm{gap}}$. On the physical level, one might think of $\Lambda \setminus \Lambda^{\mathrm{gap}}$ as some impurity region causing the global spectral gap to close, or the boundary of Λ and hence allowing for gap-closing edge modes; see Figure 1. Indeed, as we will show in Section 3.5, the local gap condition ($\mathrm{LDG_{informal}}$) is satisfied for ground states of locally in $\Lambda \setminus \Lambda^{\mathrm{gap}}$ (but arbitrarily strongly) perturbed Hamiltonians of certain quantum spin systems, which have a globally gapped ground state.

1.2.1 Verifying the local dynamical gap condition

Despite the supportive examples above, our concrete formulation of a local gap condition ($\mathrm{LDG_{informal}}$) might still seem a bit *ad hoc* at the moment. Therefore, we will

[4] Throughout this paper, $\mathscr{A}_X$ denotes the algebra of observables with support in X.

outline several alternative ways to formulate such a condition and discuss their respective features and relations in Section 3. In particular, in Proposition 3.2 we show the following (the constants q, C, ℓ have the same meaning as in (4) but might take different values):

1. Let the SLT Hamiltonian H with ground state ρ be obtained from a globally gapped SLT Hamiltonian H_* with ground state ρ_* by an SLT perturbation J localized in $\Lambda \setminus \Lambda^{\mathrm{gap}}$, i.e. $H = H_* + J$. If the difference $\rho - \rho_*$ is locally small in the sense that

   $$\|(\rho-\rho_*)\,A\|_{\mathrm{tr}} + \|A\,(\rho-\rho_*)\|_{\mathrm{tr}} \le C\,\|A\|\ \mathrm{diam}(X)^\ell \exp\bigl(-d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})^q\bigr)$$

   for all $X \subset \Lambda$ and $A \in \mathcal{A}_X$, then ρ is locally dynamically gapped.

2. In the same setting as in 1 the following holds: If there exists a norm-preserving automorphism τ on the observable algebra such that $\langle\,\cdot\,\rangle_\rho = \langle \tau[\![\,\cdot\,]\!]\rangle_{\rho_*}$ and

   $$\bigl\|(\tau-\mathbb{1})[\![A]\!]\bigr\| \le C\,\|A\|\ \mathrm{diam}(X)^\ell\ \exp\bigl(-d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})^q\bigr)$$

   for all $X \subset \Lambda$ and $A \in \mathcal{A}_X$, then ρ is locally dynamically gapped.

3. Let H be an SLT Hamiltonian and $\rho = |\psi\rangle\langle\psi|$ its pure *product ground state.* Then, if ρ is effectively gapped well inside Λ^{gap} in the sense that[5]

   $$\mathrm{i}\bigl\langle A^*\mathscr{L}[\![A]\!]\bigr\rangle_\rho \ge g\,\Bigl(1 - C\ \mathrm{diam}(X)^\ell\ \exp\bigl(-d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})^q\bigr)\Bigr)\Bigl(\langle A^*A\rangle_\rho - \bigl|\langle A\rangle_\rho\bigr|^2\Bigr)$$

   for all $X \subset \Lambda$ and $A \in \mathcal{A}_X$, then ρ is locally dynamically gapped.

Items 1 and 2 will be used in Section 3.5.2, to show that ground states of perturbations of gapped frustration free Hamiltonians have a local dynamical gap.

1.3 Discussion of our main result

We can now formulate an informal version of our main result as a showcase application of our local gap condition ($\mathrm{LDG}_{\mathrm{informal}}$) to a physically relevant problem – the validity of response theory. In a nutshell, it says that, even after relaxing the usual condition of a global gap to ($\mathrm{LDG}_{\mathrm{informal}}$), we have *response theory to all orders* for a perturbation localized in Λ^{pert} – provided that $d(\Lambda^{\mathrm{pert}}, \Lambda\setminus\Lambda^{\mathrm{gap}})$ is sufficiently large compared to $|\log(\varepsilon)|^{1/q}$, where $\varepsilon > 0$ is the strength of the perturbation and $q \in (0,1)$ is some small constant; see (10) and Figure 2.

More precisely, let H_0 be an SLT Hamiltonian and ρ_0 an equilibrium state of H_0 that is locally dynamically gapped in Λ^{gap} according to Assumption ($\mathrm{LDG}_{\mathrm{informal}}$). Let V be

[5] For $\Lambda^{\mathrm{gap}} = \Lambda$, this condition for all observables, is actually equivalent to the usual spectral gap condition.

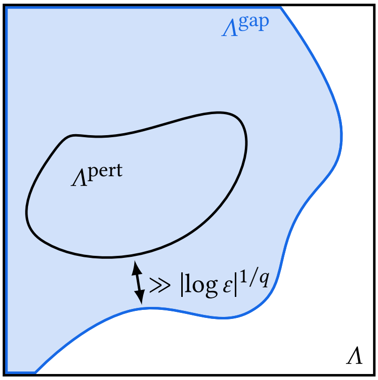

Figure 2. Let H_0 be locally dynamically gapped in $\Lambda \setminus \Lambda^{\text{gap}}$ and consider a small Λ^{pert}-localized perturbation εV, which is adiabatically turned on. Then, if the distance between Λ^{pert} and $\Lambda \setminus \Lambda^{\text{gap}}$ is large compared to $|\log \varepsilon|^{1/q}$, response theory (8) holds to any order.

a Λ^{pert}-localized perturbation by an SLT Hamiltonian as defined in (18), $f\colon \mathbb{R} \to [0,1]$ a smooth switching function with $f(t) = 0$ for $t \le -1$ and $f(t) = 1$ for $t \ge 0$, and define

$$H_\varepsilon(t) := H_0 + \varepsilon f(t) V. \tag{5}$$

Moreover, let $\rho^{\varepsilon,\eta,f}(t)$ be the solution of the time-dependent adiabatic Schrödinger equation

$$\mathrm{i}\,\eta \frac{\mathrm{d}}{\mathrm{d}t} \rho^{\varepsilon,\eta,f}(t) = \left[H_\varepsilon(t), \rho^{\varepsilon,\eta,f}(t)\right] \tag{6}$$

with adiabatic parameter $\eta \in (0,1]$ and initial datum $\rho^{\varepsilon,\eta,f}(t) = \rho_0$ for all $t \le -1$. Finally, for an observable $B \in \mathcal{A}_Y$ define the response to the perturbation as

$$\Sigma_B^{\varepsilon,\eta,f}(t) := \langle B \rangle_{\rho^{\varepsilon,\eta,f}(t)} - \langle B \rangle_{\rho_0}. \tag{7}$$

Main Result (see Theorem 4.1). *For every $j \in \mathbb{N}$ there exists a response coefficient $\sigma_{B,j}$, independent of ε, η and f, such that the following holds: There exists a constant $q \in (0,1)$ and for every $n, m \in \mathbb{N}$ and $T > 0$, there exists a constant $C > 0$, independent of ε, such that for every $t \in [0,T]$ we have that*

$$\sup_{\eta \in [\varepsilon^m, \varepsilon^{\frac{1}{m}}]} \left| \Sigma_B^{\varepsilon,\eta,f}(t) - \sum_{j=1}^{n} \varepsilon^j \sigma_{B,j} \right| \le C\, \|\!|B|\!\|\, \varepsilon^{n+1} \left(1 + \mathrm{e}^{-d(\Lambda^{\text{pert}}, \Lambda \setminus \Lambda^{\text{gap}})^q - C\log(\varepsilon)}\right), \tag{8}$$

where $\|\!|B|\!\|$ measures the norm of B and its support Y.

The first order coefficient is given by Kubo's formula

$$\sigma_{B,1} = -\mathrm{i} \left\langle \left[\mathcal{I}_{H_0,g}[\![V]\!], B\right] \right\rangle_{\rho_0}. \tag{9}$$

When $\Lambda^{\text{gap}} = \Lambda$, the equilibrium state ρ_0 of H_0 is *globally* dynamically gapped and the exponential in (8) is absent since $d(\Lambda^{\text{pert}}, \emptyset) := \infty$. This special case of our result (when $\rho_0 = P_0/\dim\operatorname{rank} P_0$ and P_0 projects onto a gapped spectral patch) has already been proven in [203] with extensions to infinite systems in [123, 122]; see also Remark 4.2 below. Moreover, observe that, whenever

$$d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}}) \gg |\log(\varepsilon)|^{1/q}, \tag{10}$$

with q from our main result, the exponential in (8) is small compared to 1. In particular, this is the case, if $d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}}) \gg \varepsilon^{-\delta}$ for some (arbitrarily small) $\delta > 0$. Hence, our main result gives an effective condition on the distance from the perturbation region, Λ^{pert}, to the non-gapped region, $\Lambda \setminus \Lambda^{\mathrm{gap}}$, in comparison to the perturbation strength ε, which ensures that response theory to all orders is valid; see Figure 2.

It is interesting to compare our main result to the adiabatic theorem [122, Theorem 4.1] for finite systems with a gap in the bulk, which are *locally gapped* in some sense. By an argument as in [203, Proof of Theorem 4.1], [122, Theorem 4.1] yields the analog of (8) again with an *additional error term*: Instead of the exponential in (8), one obtains[6] $\mathcal{O}\big(\big(\varepsilon^C d(Y, \Lambda \setminus \Lambda^{\mathrm{gap}})\big)^{-\infty}\big)$. Therefore, in order to have response theory to all orders in this setting, one needs that

$$d(Y, \Lambda \setminus \Lambda^{\mathrm{gap}}) \gg \varepsilon^{-C}. \tag{11}$$

We point out the following two differences between the conditions in (10) and (11): First, in (10), the relevant distance is between 'where the perturbation V acts', i.e. Λ^{pert}, and 'where we do not have a gap', i.e. $\Lambda \setminus \Lambda^{\mathrm{gap}}$. In contrast to that, the relevant distance in (11) is between 'where the observable B tests', i.e. Y, and 'where we do not have a gap', i.e. $\Lambda \setminus \Lambda^{\mathrm{gap}}$. Second, while in (10) the distance must be much bigger than a power of the *logarithm* of the perturbation strength ε, in (11), the distance must be much bigger than a large *inverse power* of ε.

1.4 Structure of the paper

The rest of this paper is structured as follows. We begin by introducing the mathematical framework, in particular the underlying space and the concept of locality of SLT operators, in Section 2. Afterwards, in Section 3 we discuss the problem of formulating a local gap condition, formulate different variants and explain their connections, and, moreover, show certain exemplary systems to have a local dynamical gap. In Section 4.1, we precisely formulate our main result, Theorem 4.1, and also give its proof based on the NEASS construction. Proofs concerning the formulation of the local gap condition are given in Section 5. Afterwards, in Section 6, we perform the NEASS construction and prove the necessary inputs for Theorem 4.1; several technical lemmata and auxiliary results are deferred to Appendix A.

[6] The bound in [122, eq. (4.1)] essentially means that the unperturbed system on $\Lambda \equiv \Lambda_k$ is gapped within $\Lambda^{\mathrm{gap}} \equiv \Lambda_{\lfloor k(1-o(1))\rfloor}$, where Λ_l denotes the box of side length $2l+1$ in $\mathbb{Z}^d$ centered at zero.

2 Mathematical framework

In this section, we briefly introduce the (standard) mathematical framework used in the formulation of the adiabatic theorems. For similar setups see [203, 123, 122, 17].

2.1 Spatial structure and algebra of observables

We consider a quantum spin system on a finite graph Λ equipped with the graph distance $d(\cdot, \cdot)$. Let $B_r(x) := \{ y \in \Lambda \,|\, d(x,y) \leq r \}$ be the ball of radius r centered at $x \in \Lambda$. The graph is assumed have dimension (at most) $d > 0$, i.e. there exists a constant $C_{\mathrm{vol}} > 0$ such that

$$\sup_{x\in\Lambda} |B_r(x)| \leq 1 + C_{\mathrm{vol}}\, r^d, \tag{12}$$

where $|X|$ denotes the number of sites in $X \subset \Lambda$. The set of all such graphs Λ is denoted by

$$\mathscr{G}(d, C_{\mathrm{vol}}) := \left\{ \Lambda \text{ finite graph} \,\middle|\, \sup_{x\in\Lambda} |B_r(x)| \leq 1 + C_{\mathrm{vol}}\, r^d \text{ for all } r > 0 \right\}. \tag{13}$$

To each vertex $x \in \Lambda$, we associate a single-particle Hilbert space $\mathscr{H}_x$, which we assume to be of finite dimension, $\sup_{x\in\Lambda} \dim \mathscr{H}_x < \infty$. Moreover, for each $X \subset \Lambda$ let $\mathscr{H}_X := \otimes_{x\in X} \mathscr{H}_x$ be the many-particle Hilbert space on X and denote the associated C^*-algebra of observables by $\mathscr{A}_X := \mathscr{L}(\mathscr{H}_X)$. Whenever $X \subset X'$, then $\mathscr{A}_X$ is naturally embedded as a subalgebra of $\mathscr{A}_{X'}$ and we set $\mathscr{A} := \mathscr{A}_\Lambda$.

Since a very similar construction is common for fermionic lattice systems (see e.g. [173, P2]), all the results almost immediately translate to this setting.

2.2 Interactions and SLT operators

An *interaction* is a map

$$\Phi \colon \{ X \subset \Lambda \} \to \mathscr{A},\ X \mapsto \Phi(X) = \Phi(X)^* \in \mathscr{A}_X. \tag{14}$$

With any interaction, one associates a *sum of local terms (SLT) operator* A via

$$A := \sum_{X\subset\Lambda} \Phi(X) \in \mathscr{A}. \tag{15}$$

Note that, while every interaction defines a unique operator, there are multiple interactions realizing the same operator, i.e. the assignment $\Phi \mapsto A(\Phi)$ is not invertible. Note that all interactions and SLT operators are by definition always self-adjoint.

For any $b > 0$ and $p \in (0, 1]$, we consider the stretched exponential function

$$\chi_{b,p} \colon [0, \infty) \to (0, 1]\,,\ x \mapsto \mathrm{e}^{-bx^p} \tag{16}$$

as *localization functions* and define the associated *SLT interaction norm*

$$\|\Phi\|_{b,p} := \sup_{z\in\Lambda} \sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \frac{\|\Phi(Z)\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)} \tag{17}$$

on interactions. The quality of the localization for an interaction will be expressed by the finiteness of a norm $\|\Phi\|_{b,p}$, *independent* of the (size of the) graph Λ. Operators with this property will often be referred to as *(b,p)-localized SLT operators.*

Moreover, in order to further quantify, how well an interaction is localized in a region $\Omega \subset \Lambda$, we introduce the *localized SLT interaction norm*

$$\|\Phi\|_{b,p;\Omega} := \sup_{z\in\Lambda} \sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \frac{\|\Phi(Z)\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)} \tag{18}$$

and refer to operators with the property that $\|\Phi\|_{b,p;\Omega} \le C$ as *(b,p,Ω)-localized SLT operators.* Whenever it is clear from the context, or irrelevant for the discussion, we will often also omit the arguments (b,p), and simply refer to *Ω-localized SLT operators.* Finally, observe that $\|\Phi\|_{b,p;\Lambda} = \|\Phi\|_{b,p}$.

The following simple lemma, whose proof is given in Appendix A.4, relates conceptually easier notions of locality of SLT operators to boundedness of the norm (18).

Lemma 2.1. *Let A be an SLT operator stemming from an interaction Φ, for which we assume that $\|\Phi\|_{b,p} \le C$ for some constant $C > 0$, and let $\Omega \subset \Lambda$.*

(i) *Let A be* strictly Ω-localized, *which means $\Phi(Z) = 0$ whenever $Z \not\subset \Omega$. Then it holds that $\|\Phi\|_{b,p;\Omega} \le C$*

(ii) *Let A be* strongly Ω-localized, *which means $\Phi(Z) = 0$ whenever $Z \cap \Omega = \emptyset$. Then it holds that $\|\Phi\|_{b/2,p;\Omega} \le C$.*

3 How to formulate a local gap condition?

Formulating a (i) physically *and* mathematically transparent, (ii) practically applicable, and (iii) reasonably restrictive condition of a *local gap* for a Hamiltonian H is a non-trivial task. In this section, we discuss several different possible approaches to do so. In principle, there are two main ways: In the *extrinsic formulations* we compare H to a globally gapped reference Hamiltonian H_*, while the *intrinsic formulations* only involve the original Hamiltonian H alone.

In Section 3.1 we introduce the local dynamical gap condition ($\mathrm{LDG}_{\mathrm{main}}$), that we use for the response theory. It is an intrinsic formulation and allows inverting the Liouvillian in a suitable sense. Afterwards, we list different variants of extrinsic formulations in Section 3.2 and other intrinsic formulations in Section 3.3 in view of the above listed three requirements. In Section 3.4 we explain the relations among the formulations. Finally, in Section 3.5 we discuss two exemplary systems, which we prove to have a local dynamical gap in the sense of Assumption ($\mathrm{LDG}_{\mathrm{main}}$).

Throughout the entire section (unless stated differently), we will use C, ℓ, b, p as generic constants satisfying $C > 0$, $\ell \geq 0$, $b > 0$, and $p \in (0,1]$. Their precise values might change from line to line and only depend on the model parameters, i.e. the interaction norms (17)–(18) of the involved SLT Hamiltonians, the lattice parameters in (13), or the gap size $g > 0$ of a reference Hamiltonian.

3.1 Local dynamical gap

The – at least in view of our application – most important property of an SLT Hamiltonian with a gapped part of its spectrum is that its Liouvillian can be *locally* inverted on the off-diagonal part with respect to its gapped part. To explain this in detail, let us first assume that H is globally spectrally gapped, i.e.

$$\mathrm{spec}(H) = \sigma_1 \cup \sigma_2 \qquad \text{with} \qquad d(\sigma_1, \sigma_2) \geq g \tag{19}$$

for some gap size $g > 0$ such that $\sigma_1 \subset I \subset \mathbb{R} \setminus \sigma_2$ for some compact interval I. We denote with P the spectral projection associated to H onto σ_1. Then, there exists an explicit, locality preserving map $\mathscr{I}_{H,g} = \mathscr{I}_{H,g}[\![\cdot]\!] \colon \mathscr{A} \to \mathscr{A}$, which inverts the Liouvillian $\mathscr{L}_H[\![\cdot]\!] := -\mathrm{i}\,[H, \cdot\,]$ in the projection $\langle \cdot \rangle_P = \mathrm{tr}\big(P\,\cdot\big)$ onto σ_1. It is often called *quasi-local inverse of the Liouvillian* and depends on the SLT Hamiltonian H and the gap size g. More precisely, for all $A, B \in \mathscr{A}_\Lambda$ it holds that

$$\big\langle [\mathscr{L}_H \circ \mathscr{I}_{H,g}[\![A]\!] - A, B]\big\rangle_P = 0. \tag{20}$$

Note that such a map cannot be uniquely characterized as a "weak" inverse of $\mathscr{L}_H$, and is thus clearly not unique. However, mostly for concreteness, we will always work with an *explicitly* constructed [120, 23] variant

$$\mathscr{I}_{H,g}[\![A]\!] := \int_{\mathbb{R}} \mathrm{d}t\, w_g(t) \int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{i}Hs}\, A\, \mathrm{e}^{-\mathrm{i}Hs}, \tag{21}$$

and henceforth called the *quasi-local inverse of the Liouvillian.* See Remarks A.10 and A.12 for relaxed, rather abstract conditions on $\mathscr{I}$. The *filter function* $w_g \in L^1(\mathbb{R})$ is positive, normalized to $\int w_g = 1$, and required to have Fourier transform[7] $\widehat{w_g}$ with support

$$\mathrm{supp}(\widehat{w_g}) \subset [-g, g]. \tag{22}$$

Moreover, for the explicitly constructed w_g, we additionally have the bound

$$|w_g(t)| \leq C\, \mathrm{e}^{-|t|^q} \quad \text{for all} \quad q < 1; \tag{23}$$

[7] We use the convention that $\hat{f}(p) := (2\pi)^{-1/2} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}px} f(x)$ for the Fourier transform.

see Lemma A.9 in Appendix A.3. This estimate (23) together with classical Lieb-Robinson bounds [154] for the dynamics generated by H ensure that $\mathscr{I}_{H,g}$ acts as a quasi-local operator. In Appendix A.3 we will briefly recall the construction of $\mathscr{I}_{H,g}$ and report on its properties in more detail.

Interestingly, the spectral gap can be characterized *dynamically* via the quasi-local inverse of the Liouvillian as shown in the next proposition. The proof is given in Section 5.1.

Proposition 3.1 (Dynamical characterization of a spectral gap). *Let H be a self-adjoint operator on a finite-dimensional Hilbert space $\mathscr{H}$. Let $g > 0$, $w_g \in L^1(\mathbb{R})$ be positive, normalized to $\int w_g = 1$ and satisfy* (22) *with $\widehat{w_g}|_{(-g,g)} > 0$. Decompose the spectrum of H as $\mathrm{spec}(H) = \sigma_1 \uplus \sigma_2$ and let P be the spectral projection onto σ_1. Then, we have that*

$$\big\langle [\mathscr{L}_H \circ \mathscr{I}_{H,g}[\![A]\!] - A, B] \big\rangle_P = 0 \quad \forall A, B \in \mathscr{B}(\mathscr{H}) \qquad \Longleftrightarrow \qquad d(\sigma_1, \sigma_2) \geq g. \tag{24}$$

In many applications the local invertibility of the Liouvillian guarantees that the effect of perturbations remains local. More precisely, it allowed to prove automorphic equivalence of gapped ground states [23, 167] and adiabatic theorems in cases where the perturbation is not allowed to close the gap [17, 165] and later also for gap closing perturbations [203, 124, 123, 122].

This motivates the definition of a local gap by weakening (20).

($\mathbf{LDG_{main}}$) Local dynamical gap condition. We say that an equilibrium state ρ of the SLT-operator H, i.e. $[H, \rho] = 0$, is *locally dynamically gapped* of size at least $g > 0$ in a region $\Lambda^{\mathrm{gap}} \subset \Lambda$ with respect to C_{gap}, b, $p > 0$ and $\ell \in \mathbb{N}_0$, if and only if for all $X, Y \subset \Lambda$ and $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$, it holds that

$$\begin{aligned} \Big| \big\langle \big[\mathscr{L}_H \circ \mathscr{I}_{H,g}[\![A]\!] - A, B \big] \big\rangle_\rho \Big| \leq C_{\mathrm{gap}} \, \|A\| \, \|B\| \, \big[\, \mathrm{diam}(X) + \mathrm{diam}(Y) \big]^\ell \\ \times \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}}) + d(Y, \Lambda \setminus \Lambda^{\mathrm{gap}}) \big). \end{aligned} \tag{25}$$

◇

In the special case of $\ell = 0$, by taking a supremum over all observables B with $\|B\| \leq 1$ in (25), we find that our local dynamical gap condition (25) is actually *equivalent* to

$$\big\| \big[\mathscr{L}_H \circ \mathscr{I}_{H,g}[\![A]\!] - A, \rho \big] \big\|_{\mathrm{tr}} \leq C \, \|A\| \, \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}}) \big).$$

Here, we additionally used cyclicity of the trace together with $\sup_{\|D\| \leq 1} |\mathrm{tr}(CD)| = \|C\|_{\mathrm{tr}}$.

Moreover, we point out that, while the RHS of (25) is obviously symmetric in A and B, the LHS is as well. This follows by rewriting

$$\mathscr{L}_H \circ \mathscr{I}_{H,g}[\![A]\!] - A = - \int_{\mathbb{R}} \mathrm{d}t \, w_g(t) \, \mathrm{e}^{\mathrm{i}Ht} \, A \, \mathrm{e}^{-\mathrm{i}Ht} =: \mathscr{J}_{H,g}[\![A]\!] \tag{26}$$

and using $[H, \rho] = 0$ together with the symmetry $w_g(t) = w_g(-t)$.

For the proofs we will use the following slightly weakened, asymmetric version of (25). It can be obtained from ($\mathrm{LDG_{informal}}$) by absorbing $\mathrm{diam}(X)$ with the decay in $d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})$.

(LDG_{weak}) Local dynamical gap condition – weakened version. We say that an equilibrium state ρ of an SLT operator H, i.e. with $[H,\rho] = 0$, is *weakly locally dynamically gapped* of size at least $g > 0$ in a region $\Lambda^{gap} \subset \Lambda$ with respect to C_{gap}, b, p, $\beta > 0$ and $\ell \in \mathbb{N}_0$, if and only if the following holds: For all $X \subset \Lambda$ satisfying $\mathrm{diam}(X) \leq d(X, \Lambda \setminus \Lambda^{gap})^{\beta}$ and observables $A \in \mathcal{A}_X$, and $Y \subset \Lambda$ and observables $B \in \mathcal{A}_Y$, it holds that

$$\left|\left\langle\left[\mathcal{L}_H \circ \mathcal{I}_{H,g}[\![A]\!] - A, B\right]\right\rangle_{\rho}\right| \leq C_{gap} \, \|A\| \, \|B\| \; \mathrm{diam}(Y)^{\ell} \; \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{gap})\big). \tag{27}$$

◇

Finally, we remark that another symmetric (in A and B) bound of the LHS of (25) would be to replace the sum of the distances in the exponent in (25) by $d(X \cup Y, \Lambda \setminus \Lambda^{gap})$ (i.e. take the minimum of the distances to $\Lambda \setminus \Lambda^{gap}$ instead of their maximum). Using quasi-locality estimates for $\mathcal{J}_{H,g}[\![\cdot]\!]$ defined in (26) (see e.g. [174, Lemma 5.1]), this bound can be proven as a consequence of the trace LPPL stated in Section 3.2.1, provided that there is a globally spectrally gapped *parent Hamiltonian* (cf. Section 3.2) H_* for H. However, as explained there, this is not enough to prove our result with the decay in $d(\Lambda^{pert}, \Lambda \setminus \Lambda^{gap})$.

3.2 Extrinsic local gap conditions

In this section, we describe several ways of *extrinsically* expressing that an equilibrium state ρ (e.g. the ground state) of a Hamiltonian H is locally gapped. We call these formulations of a local gap condition *extrinsic*, since the Hamiltonian H and equilibrium state ρ of interest are compared to another reference Hamiltonian H_*, called the *parent Hamiltonian*, with equilibrium state ρ_*, which is globally gapped. Throughout this section, for simplicity of the presentation, we will assume that both ρ and ρ_* are non-degenerate ground states.

The common core of all the different ways to extrinsically formulate a local gap condition, is to assume that H and H_* differ only locally: That is, H_* and H are related as

$$H = H_* + J, \tag{28}$$

where J is some SLT operator, which is localized in $\Lambda \setminus \Lambda^{gap}$ (in one of the senses mentioned in Lemma 2.1). One then assumes that also the equilibrium states ρ and ρ_* only differ locally in a suitable sense. Whenever this is the case, we say that ρ and ρ_* satisfy LPPL. This is a slight abuse of nomenclature, as it is usually a property of the Hamiltonian H_*, which satisfies the *local perturbations perturb locally (LPPL)* principle if the above holds for a broad class of local perturbations J, which has been shown to be the case for *ground states* in several contexts [74, P1, 16]. In Section 3.5 we will also encounter the usual definition, since there we consider H_* and allow the addition of any suitable perturbation J to obtain $H = H_* + J$. But since we only need the relation between two states ρ and ρ_* with Hamiltonians H and H_* for all the following implications, we do not require H_* to satisfy the more general usual LPPL.

3.2.1 Trace LPPL

The simplest way to formulate an *extrinsic* local gap condition for H given by (28) with ground state ρ, is to assume that the parent Hamiltonian H_* is globally gapped and that local expectation values in the ground states ρ and ρ_* are almost the same within the gapped region. More precisely, this means that for all $X \subset \Lambda$ and $A \in \mathcal{A}_X$

$$\big|\langle A\rangle_\rho - \langle A\rangle_{\rho_*}\big| \le C\,\|A\|\;\mathrm{diam}(X)^\ell\;\chi_{b,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big). \tag{29}$$

Because $|\langle A\rangle_\rho - \langle A\rangle_{\rho_*}| = |\mathrm{tr}((\rho-\rho_*)\,A)|$, we call this *trace LPPL*. An estimate of this form has been shown to hold for weakly interacting spin systems in [P1] (based on ideas of [219]) and in [16]. Moreover, in [122, Theorem 4.1], such type of assumption was used to formulate an adiabatic theorem for large but finite systems with a gap in the bulk. In this case (cf. [122, eq. (4.1)]), the additional error term (compared to the case of having a global gap) in the response theory expansion (8) becomes $\mathcal{O}\big(\big(\varepsilon^C\,d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)^{-\infty}\big)$. Moreover, using that H_* and J are SLT operators, (29) implies that a condition similar to ($\mathrm{LDG_{main}}$) holds *but* with the argument of $\chi_{b,p}$ in (25) being the *minimum* of the distances (or the distance of the *union* $X \cup Y$ to $\Lambda\setminus\Lambda^{\mathrm{gap}}$) instead of their sum (i.e. their maximum). Hence, the decay in $d(\Lambda^{\mathrm{pert}},\Lambda\setminus\Lambda^{\mathrm{gap}})$ from our main result, emerging in the course of proving Proposition 4.4, will eventually be lost.

In conclusion, although this way of saying that H has a local gap is quite simple, the resulting error terms are considerably bad and, moreover, it does not quite allow for tracking $d(\Lambda^{\mathrm{pert}},\Lambda\setminus\Lambda^{\mathrm{gap}})$-decays.

3.2.2 Trace norm LPPL and commutator trace norm LPPL

One can strengthen the trace LPPL assumption (29) to a *trace norm LPPL*. This means that ρ and ρ_* are not only close in the weak* sense (29), but instead

$$\|(\rho-\rho_*)\,A\|_{\mathrm{tr}} + \|A\,(\rho-\rho_*)\|_{\mathrm{tr}} \le C\,\|A\|\;\mathrm{diam}(X)^\ell\;\chi_{b,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big) \tag{30}$$

for all $X \subset \Lambda$ and $A \in \mathcal{A}_X$, where $\|\cdot\|_{\mathrm{tr}} := \mathrm{tr}(|\cdot|)$ denotes the trace norm. We remark, that surely (30) implies (29). A somewhat weakened version of (30) is to ask that only the trace norm of the *commutator* $[\rho-\rho_*, A]$ is small

$$\|[\rho-\rho_*, A]\|_{\mathrm{tr}} \le C\,\|A\|\;\mathrm{diam}(X)^\ell\;\chi_{b,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big), \tag{31}$$

As we show in Proposition 3.2 (see also Figure 3), both these ways, (30) and (31), of saying that H has a local gap are sufficient to prove that H has a local gap in the sense of Assumption ($\mathrm{LDG_{main}}$).[8] However, it is a quite restrictive condition that ρ and ρ_* are close in such strong topology as (30) or (31).

[8] This fact is used for proving ($\mathrm{LDG_{main}}$) (which is done in Section 5) for one of the examples discussed in Section 3.5 below.

3.2.3 Intertwining automorphism

Another, way of saying that ρ and ρ_* are close to each other, is to assume that there exists an intertwining norm-preserving $*$-automorphism τ on $\mathscr{A}$, i.e. $\langle \cdot \rangle_\rho = \langle \tau[\![\cdot]\!]\rangle_{\rho_*}$, which satisfies

$$\big\|(\tau - \mathrm{id})[\![A]\!]\big\| \le C\,\|A\|\ \mathrm{diam}(X)^\ell\,\chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big), \tag{32}$$

where id is the identity map. Similarly to (30), (32) implies (29). Moreover, (32) is a sufficient condition for having ($\mathrm{LDG}_{\mathrm{main}}$); see Proposition 3.2. This fact will be used for proving that the examples studied in Section 3.5 satisfy ($\mathrm{LDG}_{\mathrm{main}}$).

3.3 Intrinsic local gap conditions for ground states

In this section, we discuss other ways of *intrinsically* expressing that a ground state ρ of a Hamiltonian H is locally gapped, in contrast to the local dynamical gap conditions, introduced in Section 3.1, they only work for ground states.

In case of ρ being the (unique) *ground state*, a very natural way to connect the notion of locality with spectral analysis is to require that a variational condition characterizing the spectral gap is tested only locally. Generally speaking, assume that ψ_0 is the unique ground state with eigenvalue E_0 of a local Hamiltonian H in some underlying physical space Λ. Then, a spectral gap above E_0 of size (at least) $g > 0$ is characterized by

$$\inf_{\psi \perp \psi_0} \frac{\langle \psi, (H - E_0)\,\psi\rangle}{\langle \psi, \psi\rangle} \ge g. \tag{33}$$

If the minimization in (33) is restricted a smaller set of ψ's, i.e. those which are localized to a region, say, $\Lambda^{\mathrm{gap}} \subset \Lambda$ in some appropriate sense, one could say that H is locally gapped in Λ^{gap}. Alternatively, for every fixed ψ, the gap size g could be assumed to be non-constant but dependent on the distance of the support of ψ to the region Λ^{gap}.

For quantum spin systems on the graph Λ, it can easily be checked that the analog of (33) for an SLT Hamiltonian H with unique ground state ρ is

$$\mathrm{i}\,\langle A^* \mathscr{L}_H[\![A]\!]\rangle_\rho \ge g\left(\langle A^* A\rangle_\rho - \left|\langle A\rangle_\rho\right|^2\right), \tag{34}$$

for all observables $A \in \mathscr{A}$, where $\mathscr{L}_H[\![\cdot]\!] := -\mathrm{i}\,[H, \cdot\,]$ denotes the Liouvillian. We now give two options to turn (34) into a local gap condition.

3.3.1 Gap Decay

As a first option, one could require that the *gap size g decays* as the support X of the observable $A \in \mathscr{A}_X$ approaches the complement of Λ^{gap}, e.g. as

$$\mathrm{i}\,\langle A^* \mathscr{L}_H[\![A]\!]\rangle_\rho \ge g\left(1 - C\ \mathrm{diam}(X)^\ell\,\chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big)\right)\left(\langle A^* A\rangle_\rho - \left|\langle A\rangle_\rho\right|^2\right). \tag{35}$$

In case that $\rho = |\psi\rangle\langle\psi|$ with ψ being a *product state*, variants of Proposition 14 and Lemma 15 in [221] can be used to show the following: The gap decay condition (35) implies the local dynamical gap condition ($\mathrm{LDG}_{\mathrm{weak}}$); see Proposition 3.2(viii).

We further remark that, if one is interested in *taking the thermodynamic limit*, $\Lambda \nearrow \Gamma$ for some infinite graph Γ, and $\Lambda^{\mathrm{gap}} \nearrow \Gamma$ in this limit, e.g. in the scenario of gap-closing edge modes, then (35) yields a *gap in the bulk* of the naturally associated infinite system [P2, Remark 4].[9]

3.3.2 Defective coercivity

Another option is to include a separate *additive* error term on the RHS of (34), e.g. as

$$\mathrm{i}\,\langle A^* \mathscr{L}_H[\![A]\!]\rangle_\rho \geq g\left(\langle A^*A\rangle_\rho - \left|\langle A\rangle_\rho\right|^2\right) - C\,\|A\|^2\,\operatorname{diam}(X)^\ell\,\chi_{b,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big), \quad (36)$$

which we call *defective coercivity* for the following reason.

The global gap characterization can equivalently be rewritten as

$$\mathrm{i}\,\langle \tilde{A}^* \mathscr{L}_H[\![\tilde{A}]\!]\rangle_\rho \geq g\,\langle \tilde{A}^*\tilde{A}\rangle_\rho \quad \text{with} \quad \tilde{A} := A - \langle A\rangle_\rho,$$

which means that on $\mathscr{A}^\perp := \{\rho\}^\perp = \{A - \langle A\rangle_\rho \mid A \in \mathscr{A}\} \subset \mathscr{A}$ the bounded sesquilinear form

$$\mathscr{B}: \mathscr{A}^\perp \times \mathscr{A}^\perp \to \mathbb{C},\ (A,B) \mapsto \mathrm{i}\,\langle A^* \mathscr{L}_H[\![B]\!]\rangle_\rho$$

is *coercive* with respect to the semi-norm $|\!|\!|A|\!|\!| := \sqrt{\langle A^*A\rangle_\rho}$ on $\mathscr{A}^\perp$. Hence, (36) expresses some defect in the original coercivity of (34). We remark that, at least morally, one could use the Lax-Milgram theorem to deduce existence of an inverse of the Liouvillian $\mathscr{L}_H$ given such a coercivity estimate. However, the problem is that the inverse obtained in this way does not necessarily have any nice locality properties, which are crucially used for practical purposes.

Finally, we point out that the defective coercivity (36) for ρ is implied, if there exists a globally gapped parent Hamiltonian H_* with ground state ρ_* that satisfies trace LPPL (29).

3.4 Summary and comparison

In the previous three sections we described several different ways of expressing that a Hamiltonian is locally gapped, distinguishing between *extrinsic* and *intrinsic* formulations.

[9] More precisely, this also requires that an interaction $\Phi = \Phi_H$, associated to the SLT Hamiltonian H, *has a thermodynamic limit* in a suitable sense (see [123, Definition 2.1], [122, Definition 3.1], and [P2, Definitions 2 and 5]). In this case, the linear functional $A \mapsto \operatorname{tr}(PA)$ converges in weak* sense to a *state* on the C^*-algebra of quasi-local observables on Γ. A *gap in the bulk* then means, that the naturally associated GNS Hamiltonian of the infinite system has a spectral gap above zero.

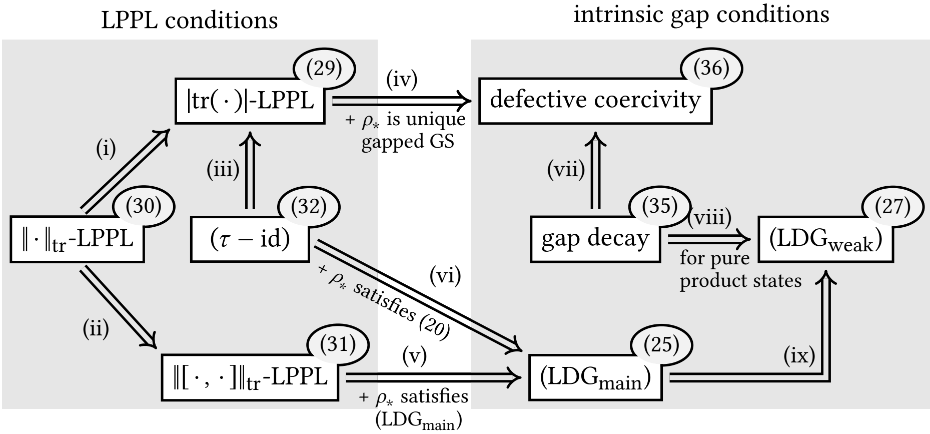

Figure 3. Implications among the various local gap conditions from Sections 3.1–3.3. The numbering refers to the precise statements in Proposition 3.2.

While the extrinsic conditions are easy to formulate, they rely on a reference (parent) Hamiltonian with a global gap satisfying some form of the LPPL principle. Since for a system of interest, it is not guaranteed to have such a well-understood parent Hamiltonian available, it is conceptually more desirable to formulate a local gap condition in an intrinsic way. Or, in other words, saying that a system is (or behaves as if it were) *locally* gapped should not only make sense relative to another *globally* gapped system.

In the intrinsic category, we formulated two local *spectral* gap conditions (35) and (36), which are (i) only meaningful for (non-degenerate) ground states and (ii) although mathematically clean, hardly applicable in physical problems in a direct way (apart from the connection in Proposition 3.2(viii)). These two issues are then resolved by our (intrinsic) local dynamical gap condition (LDG) in (25) and its weakened version in (27).

The following proposition formulates implications between the local gap conditions introduced in the previous sections. The implications are also depicted in Figure 3. The proof is given in Section 5.2.

Proposition 3.2 (Relations among the local gap conditions). *Fix $g > 0$, $d \in \mathbb{N}$, $C_{\mathrm{vol}} > 0$, $\tilde{b} > 0$, $\tilde{p} \in (0,1]$, $C_{\mathrm{int}} > 0$. Let $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$, $\Lambda^{\mathrm{gap}} \subset \Lambda$ and let H_* and J be SLT-operators with corresponding interactions satisfying $\|\Phi_{H_*}\|_{\tilde{b},\tilde{p}} < C_{\mathrm{int}}$ and $\|\Phi_J\|_{\tilde{b},\tilde{p};\Lambda\setminus\Lambda^{\mathrm{gap}}} < C_{\mathrm{int}}$. Let ρ_* be an equilibrium state of H_* and ρ and equilibrium state of $H := H_* + J$.*

Then the following statements hold modulo adjusting the constants C, ℓ, b, and p in a way which only depends on $g, d, C_{\mathrm{vol}}, \tilde{b}, \tilde{p}$, and C_{int}.

Among the LPPL relations, the following implications hold:

(i) *Trace norm LPPL in the sense of* (30) *implies trace LPPL in the sense of* (29).

(ii) *Trace norm LPPL in the sense of* (30) *implies commutator trace norm LPPL in the sense of* (31).

(iii) *If there exists an intertwining automorphism between ρ and ρ_* such that* (32) *holds, then ρ and ρ_* satisfy trace LPPL in the sense of* (29).

If ρ and ρ_* are non-degenerate ground states and ρ_* satisfies an additional gap condition, then the LPPL conditions imply intrinsic gap conditions:

(iv) *If ρ_* is the unique ground states of H_* with a spectral gap of size at least $g > 0$ above its ground state (see* (34)*) and if H_* satisfies the usual LPPL in the sense of* (29)*, then ρ satisfies defective coercivity* (36).

(v) *If ρ_* satisfies* ($\mathrm{LDG_{main}}$) *and if H_* satisfies commutator trace norm LPPL in the sense of* (31)*, then ρ also satisfies* ($\mathrm{LDG_{main}}$)*. The first assumption is satisfied, for example, if ρ_* is a normalized projection onto a gapped spectral patch of H_*; see Proposition 3.1.*

(vi) *If ρ_* has a dynamical gap in the sense that $\langle [\mathscr{L}_{H_*} \circ \mathscr{I}_{H_*,g}[\![A]\!] - A, B]\rangle_{\rho_*} = 0$ for all $A, B \in \mathscr{A}$ and if there exists an intertwining automorphism for ρ_* and ρ which satisfies* (32)*, then ρ satisfies* ($\mathrm{LDG_{main}}$)*. The first assumption is satisfied, for example, if ρ_* is a normalized projection onto a gapped spectral patch of H_*; see Proposition 3.1.*

Moreover, the following relations hold among the intrinsic gap conditions:

(vii) *A decaying gap size in the sense of* (35) *implies defective coercivity in the sense of* (36).

(viii) *If $\rho = |\psi\rangle\langle\psi|$ with ψ a product state and ρ has a decaying gap size in the sense of* (35)*, then ρ satisfies Assumption* ($\mathrm{LDG_{weak}}$).

(ix) *If ρ satisfies Assumption* ($\mathrm{LDG_{main}}$)*, then it also satisfies Assumption* ($\mathrm{LDG_{weak}}$).

3.5 Two exemplary systems with a local dynamical gap

In this section, we discuss two exemplary systems, which we show to satisfy the local dynamical gap ($\mathrm{LDG_{main}}$) by means of Proposition 3.2 (v) and (vi). The first example in Section 3.5.1 is concerned with a local perturbation of a classical Ising model. It is contained in a whole class of examples considered in Section 3.5.2, which are studied based on [16]. We nevertheless discuss it separately, as proving it to satisfy the local dynamical gap condition ($\mathrm{LDG_{main}}$) is elementary, in particular not relying on [16]. The actual proofs for the two examples are given in Section 5.3.

3.5.1 Perturbations of the classical Ising model with weak interaction

As the first example we consider the classical Ising model on $\Lambda \subset \mathbb{Z}^d$

$$H_* = \sum_{x\in\Lambda} \sigma_x^3 + \frac{1}{2} \sum_{x,y\in\Lambda} \lambda(x-y)\, \sigma_x^3 \sigma_y^3, \tag{37}$$

where σ_x^i is the i^{th} Pauli matrix σ^i acting only on the spin on site $x \in \Lambda$. More precisely,

$$\sigma_x^i = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \underbrace{\sigma^i}_{\text{site } x} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \in \mathcal{B}\big(\otimes_{z\in\Lambda}\mathcal{H}_z\big), \tag{38a}$$

where, as usual,

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{38b}$$

We assume that the coupling function $\lambda\colon \mathbb{Z}^d \to \mathbb{R}$ is symmetric, of finite range, i.e. there exists some $R > 0$ such that $\lambda(x) = 0$ for $|x| > R$, and satisfies $\|\lambda\|_1 := \sum_x |\lambda(x)| < 2$. Clearly, for any $p \in (0,1]$ and $b > 0$, the canonical interaction Φ_{H_*} has bounded interaction norm $\|\Phi_{H_*}\|_{b,p} \le C_*$ uniformly in Λ.

For these systems we prove the following in Section 5.3.1: Let J be a strictly $\Lambda \setminus \Lambda^{\text{gap}}$-localized SLT perturbation, i.e. J is given by some interaction Φ_J satisfying $\|\Phi_J\|_{b,p} \le C_J$ for some $p \in (0,1]$, $b, C_J > 0$ and $\Phi_J(Z) = 0$ unless $Z \subset \Lambda \setminus \Lambda^{\text{gap}}$. Then every ground state ρ of $H = H_* + J$ has a local dynamical gap in the sense of Assumption (LDG$_{\text{main}}$).

The example (37) can be generalized to an arbitrary graph $\Lambda \in \mathcal{G}(d, C_{\text{vol}})$. Moreover, our above assertions remain valid for any Hamiltonian with gapped on-site terms and sufficiently weak mutually commuting finite range interactions that can be simultaneously diagonalized with the on-site terms (i.e. for *classical* system).

3.5.2 Perturbations of frustration free product states

As the second basic example, we consider an SLT Hamiltonian H_* of the form

$$H_* = \sum_{Z\subset\Lambda} \Phi(Z), \tag{39}$$

for which there exist $p \in (0,1]$ and $b > 0$ and a constant $C_* > 0$ such that $\|\Phi\|_{b,p} \le C_*$. Moreover, $\Lambda \in \mathcal{G}(d, C_{\text{vol}})$ in (39) is a finite graph as described in Section 2. Apart from the locality of the interaction Φ, we will impose the following further conditions (cf. [16]).

(A1) Frustration-free ground state. All terms in the SLT Hamiltonian (39) are non-negative, i.e. $\Phi(Z) \ge 0$ for all $Z \subset \Lambda$. There exists a unique (up to a phase) normalized vector $\psi_* \in \mathcal{H}$ such that $\psi_* \in \ker \Phi(Z)$ for all $Z \subset \Lambda$. The corresponding ground state (projection) is denoted by $\rho_* = |\psi_*\rangle\langle\psi_*|$. ◇

We note that the frustration free assumption depends on the explicit way (39) the Hamiltonian is written, i.e. on the interaction Φ.

(A2) Product property and regularity. The vector $|\psi_*\rangle \in \mathcal{H}$ factorizes as $|\psi_*\rangle = \otimes_{z\in\Lambda} |\psi_{*,z}\rangle$ and for every $\Omega \subset \Lambda$, the unique ground state vector of

$$H_*\big|_\Omega = \sum_{Z\subset\Omega} \Phi(Z) \tag{40}$$

is given by $|\psi_*|_\Omega\rangle = \otimes_{z\in\Omega}|\psi_{*,z}\rangle$. ◇

The latter condition can be thought of as a strong variant of the common *local topological quantum order (LTQO)*. Moreover, it is possible to relax both, the product property of $|\psi_*\rangle$ and the strong LTQO condition, in the following way: Instead of the product property, we could only assume that $|\psi_*\rangle$ (possibly upon adjoining an auxiliary state after doubling the Hilbert space, see Assumption 4 in [16]) is unitarily conjugate with SLT-generator to a product state. Instead of the strong LTQO property, we could only assume that, upon adjoining a suitable state $|\psi_*|_{\Lambda\setminus\Omega}\rangle \in \otimes_{z\in\Lambda\setminus\Omega}\mathscr{H}_x$, the unique ground state $|\psi_*|_\Omega\rangle$ of (40) is unitarily conjugate to $|\psi_*\rangle$ in the sense that $|\psi_*|_\Omega\rangle \otimes |\psi_*|_{\Lambda\setminus\Omega}\rangle = \mathrm{e}^{\mathrm{i}F}|\psi_*\rangle$, where F is an SLT operator localized at the boundary of Ω. However, we refrain from going into this further generalization of our exemplary system (39) for simplicity.

The final assumption on (39) concerns the spectral gaps

$$\gamma := \inf\big(\mathrm{spec}(H_*) \setminus \{0\}\big) \qquad \text{and} \qquad \gamma(\Omega) := \inf\big(\mathrm{spec}(H_*\big|_\Omega) \setminus \{0\}\big)$$

of H_* and its restrictions $H_*\big|_\Omega$ to some $\Omega \subset \Lambda$, respectively.

(A3) Gap condition. The SLT Hamiltonian H_* from (39) has a spectral gap, i.e. $\gamma > 0$. Moreover, the gap of restrictions (40) of H_* to *balls* $\Omega = B_r(x) = \{\, y \in \Lambda \,|\, d(x,y) \le r \,\}$ in Λ shrink at most polynomially with the radius, i.e. there exist $C_\gamma, d_\gamma > 0$ such that for every $x \in \Lambda$ it holds that

$$\frac{1}{\gamma(B_r(x))} \le C_\gamma\, r^{d_\gamma}. \tag{41}$$

◇

For these systems we prove the following in Section 5.3.2 building on [16]: Let J be a strongly $\Lambda \setminus \Lambda^{\mathrm{gap}}$-localized SLT perturbation, i.e. J is given by some interaction Φ_J satisfying $\|\Phi_J\|_{b,p} \le C_J$ for some $p \in (0,1]$, b, $C_J > 0$ and $\Phi_J(Z) = 0$ whenever $Z \cap \Lambda^{\mathrm{gap}} \neq \emptyset$. Then every ground state ρ of $H = H_* + J$ has a local dynamical gap in the sense of Assumption ($\mathrm{LDG}_{\mathrm{weak}}$).

An exemplary system satisfying all of the above assumptions (up to a constant energy shift) is given by the Heisenberg XXZ model for small enough nearest neighbor interactions $|\lambda_1|$, $|\lambda_3|$ (depending on the dimension d and the constant C_{vol}). The corresponding Hamiltonian is given by

$$H_* = \sum_{x\in\Lambda} \sigma_x^3 + \sum_{(x,y)\in E(\Lambda)} \lambda_1\, \sigma_x^1\sigma_y^1 + \lambda_1\, \sigma_x^2\sigma_y^2 + \lambda_3\, \sigma_x^3\sigma_y^3,$$

where $E(\Lambda)$ denote the edges of Λ and we recall the notations (38).

4 Main result: Response theory for locally gapped systems

In this section, we formulate our main result, the validity of response theory to any order, Theorem 4.1. Its proof, based on the NEASS construction, is given in Section 4.2.

4.1 Response theory

We recall the assumption ($\mathrm{LDG}_{\mathrm{main}}$) from Section 3.3, which we use to formulate our main result. In a nutshell, it says the following: We have validity of *response theory to all orders* under the assumption of ($\mathrm{LDG}_{\mathrm{main}}$) for a perturbation localized in Λ^{pert} – provided that the distance to $\Lambda \setminus \Lambda^{\mathrm{gap}}$ is sufficiently large compared to $|\log(\varepsilon)|^{1/q}$, where $\varepsilon > 0$ is the strength of the perturbation and $q > 0$ is some small constant.

The proof of Theorem 4.1 is given at the end of Section 4.2.

Theorem 4.1 (Response theory to all orders)**.** *Fix $n, m \in \mathbb{N}$ and let $d \in \mathbb{N}$, $C_{\mathrm{vol}} > 0$, $b > 0$, $p \in (0, 1]$, $C_{\mathrm{int}} > 0$ and $g > 0$, $C_{\mathrm{gap}} > 0$, $\ell \in \mathbb{N}_0$, and $C_{\mathrm{switch}} > 0$. Take any $q \in (0, p)$. Then there exist a constant $C_{n,m} > 0$ (in particular depending on n and m) such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$ (recall* (13)*), subsets $\Lambda^{\mathrm{pert}} \subset \Lambda$ and SLT-operators H_0 and V, with corresponding interactions that satisfy $\|\Phi_{H_0}\|_{b,p} < C_{\mathrm{int}}$ and $\|\Phi_V\|_{b,p;\Lambda^{\mathrm{pert}}} < C_{\mathrm{int}}$, respectively, the following holds:*

Assume that the equilibrium state ρ_0 of H_0 is locally dynamically gapped in Λ^{gap} of size at least g and with respect to C_{gap}, b, p and ℓ according to Assumption ($\mathrm{LDG}_{\mathrm{main}}$)*. Let $Y \subset \Lambda$ and $B \in \mathcal{A}_Y$. Then there exist response coefficients $\sigma_{B,j}$ in the following sense: For $\varepsilon > 0$ and smooth switching function $f\colon \mathbb{R} \to [0, 1]$ satisfying $f(t) = 0$ for $t \leq -1$, $f(t) = 1$ for $t \geq 0$, and $\|f\|_{C^{\mathfrak{C}_{n,m}}(\mathbb{R})} < C_{\mathrm{switch}}$ with $\mathfrak{C}_{n,m} := \lceil m(n + 1 + (2d + \ell)/p) \rceil$, consider the time-dependent Hamiltonian*

$$H_\varepsilon(t) := H_0 + \varepsilon\, f(t)\, V. \tag{42}$$

Let $\rho^{\varepsilon,\eta,f}(t)$ be the solution of the time-dependent adiabatic Schrödinger equation

$$\mathrm{i}\,\eta\, \frac{\mathrm{d}}{\mathrm{d}t} \rho^{\varepsilon,\eta,f}(t) = \left[H_\varepsilon(t), \rho^{\varepsilon,\eta,f}(t)\right] \tag{43}$$

with adiabatic parameter $\eta \in (0, 1]$ and initial datum $\rho^{\varepsilon,\eta,f}(t) = \rho_0$ for all $t \leq -1$.

Then, the response to the perturbation, $\Sigma_B^{\varepsilon,\eta,f}(t) := \langle B \rangle_{\rho^{\varepsilon,\eta,f}(t)} - \langle B \rangle_{\rho_0}$, satisfies

$$\begin{aligned}\sup_{\eta \in [\varepsilon^m, \varepsilon^{1/m}]} \left| \Sigma_B^{\varepsilon,\eta,f}(t) - \sum_{j=1}^{n} \varepsilon^j \sigma_{B,j} \right| \leq C_{n,m}\, \|B\|\ \mathrm{diam}(Y)^{(3+n)d+\ell} \left(1 + t\right)^{(2d+\ell)/p+1}\, \varepsilon^{n+1} \\ \times \left(1 + \mathrm{e}^{-d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})^q - (\mathfrak{C}_{n,m}+1)\log(\varepsilon)}\right),\end{aligned} \tag{44}$$

for all $t \geq 0$. The first order coefficient is given by Kubo's formula (9).

Our result can also be extended to infinite systems.

Remark 4.2 (Extension to infinite systems). Following the arguments from [123], it is reasonably straightforward to extend our result to the case of infinite systems. More precisely, in order to do so, we need to

- consider Λ to be part of a sequence of graphs exhausting an infinite graph Γ, e.g. $\Lambda \equiv \Lambda_k := \{-k, \dots, k\}^d \subset \mathbb{Z}^d$ and $\Gamma := \mathbb{Z}^d$;
- assume that the interactions associated to H_0 and V *have a thermodynamic limit* (see [123, Definition 2.1]);
- require that the sequence of equilibrium states $\rho_0 \equiv \rho_0^\Lambda$ satisfies the local dynamical gap condition ($\mathrm{LDG}_{\mathrm{main}}$) with constants independent of Λ and converges (in the weak* sense in the dual to the algebra of quasi-local observables, see [123, Section 2.5]) as $\Lambda \nearrow \Gamma$;
- and suppose that also the perturbation region Λ^{pert} as well as the gapped region Λ^{gap} *converge* (in a suitable sense) to some $\Gamma^{\mathrm{pert}}, \Gamma^{\mathrm{gap}} \subset \Gamma$, respectively, at least ensuring that $d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}}) \to d(\Gamma^{\mathrm{pert}}, \Gamma \setminus \Gamma^{\mathrm{gap}})$ as $\Lambda \nearrow \Gamma$.

Then, (44) also holds for the infinite system but with $d(\Gamma^{\mathrm{pert}}, \Gamma \setminus \Gamma^{\mathrm{gap}})$ in the exponential.

In the special case, where the gapped region exhausts the entire graph, i.e. $\Lambda^{\mathrm{gap}} \nearrow \Gamma$, we find (44) *without* the additional exponential error term. This corresponds to H_0 having a *gap in the bulk* (see [122, P2] and also the discussion around (35) in Section 3), where in the finite systems H_0 could have gap closing edge modes (see Figure 1). ◇

4.2 Non-equilibrium almost stationary states and proof of Theorem 4.1

The main underlying idea of our justification of linear response theory is the construction of so-called *non-equilibrium almost-stationary states (NEASS)* [165, 203, 124, 123, 122] for the dynamics of the perturbed Hamiltonian

$$H_\varepsilon := H_0 + \varepsilon V. \tag{45}$$

The proof of Theorem 4.1 is based on three propositions: Proposition 4.3 provides existence of the NEASS, which is a local dressing of the equilibrium state ρ_0 of H_0, and its almost stationarity under the dynamics generated by (45). Proposition 4.4 shows that adiabatically switching on the Perturbation as in (42) drives the equilibrium state ρ_0 to a state very close to the NEASS. Finally, Proposition 4.5 provides an explicit formula to approximate expectation values of local operators in the NEASS by expectation values of modified operators in the equilibrium state ρ_0.

More precisely, for every $n \in \mathbb{N}$ Proposition 4.3 provides existence a state Π_n^ε, which is obtained from the equilibrium state ρ_0 of H_0 by a unitary transformation with a small SLT generator, in such a way, that it is almost stationary under the dynamics generated by (45) up to order[10] ε^{n+1}. This result is the local gap version of [203, Theorem 3.1] and the proof is given in Section 6.2.

Proposition 4.3 (Non-equilibrium almost-stationary states). *Fix $n \in \mathbb{N}$ and let $d \in \mathbb{N}$, $C_{\mathrm{vol}} > 0$, $b > 0$, $p \in (0,1]$, $C_{\mathrm{int}} > 0$ and $g > 0$, $C_{\mathrm{gap}} > 0$, $\ell \in \mathbb{N}_0$. Take any $q \in (0,p)$. Then there exist a constant $C_n > 0$ (in particular depending on n) such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$ (recall (13)), subsets $\Lambda^{\mathrm{pert}} \subset \Lambda$ and SLT-operators H_0 and V, with corresponding interactions that satisfy $\|\Phi_{H_0}\|_{b,p} < C_{\mathrm{int}}$ and $\|\Phi_V\|_{b,p;\Lambda^{\mathrm{pert}}} < C_{\mathrm{int}}$, respectively, the following holds:*

Assume that the equilibrium state ρ_0 of H_0 is locally dynamically gapped in Λ^{gap} of size at least $g > 0$ and with respect to C_{gap}, b, p and ℓ according to Assumption (LDG$_{\mathrm{main}}$)*. Then, there exists a sequence $(A_\mu)_{\mu\in\mathbb{N}}$ of SLT operators, which are $(1, p', \Lambda^{\mathrm{pert}})$-localized for any $p' < p$, such that the state*

$$\Pi_n^\varepsilon := \mathrm{e}^{\mathrm{i}S_n^\varepsilon}\, \rho_0\, \mathrm{e}^{-\mathrm{i}S_n^\varepsilon} \quad \textit{with} \quad S_n^\varepsilon := \sum_{\mu=1}^{n} \varepsilon^\mu A_\mu \tag{46}$$

is almost-stationary for the dynamics generated by $H_\varepsilon = H_0 + \varepsilon V$ in the following sense: Let $\rho^\varepsilon(t)$ be the solution to the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho^\varepsilon(t) = \left[H_\varepsilon, \rho^\varepsilon(t)\right] \quad \textit{with} \quad \rho^\varepsilon(0) = \Pi_n^\varepsilon. \tag{47}$$

Under these conditions, for all $B \in \mathcal{A}_Y$ with $Y \subset \Lambda$ and $t \geq 0$, it holds that

$$\begin{aligned} \left|\langle B\rangle_{\rho^\varepsilon(t)} - \langle B\rangle_{\Pi_n^\varepsilon}\right| \leq C_{n,m}\, \|B\|\, \mathrm{diam}(Y)^{3d+\ell}\, |t|\left(1 + |t|^{(2d+\ell)/p}\right) \varepsilon^{n+1} \\ \times \left(1 + \mathrm{e}^{-d(\Lambda^{\mathrm{pert}}, \Lambda\setminus\Lambda^{\mathrm{gap}})^q - (n+1)\log(\varepsilon)}\right). \end{aligned} \tag{48}$$

We point out that, similarly to [203, Theorem 3.1], one can improve the bound in (48) by rescaling every t by ε^m for some $m \in \mathbb{N}$, at the cost of increasing the constant in front of $\log(\varepsilon)$.

Next, Proposition 4.4 states that the NEASS constructed in Proposition 4.3 is approached under the dynamics generated by the time-dependent Hamiltonian (42) whose perturbation V is turned on by the switching function f on the adiabatic timescale $1/\eta$ as in (43). It is the local gap version of [203, Proposition 3.2] and the proof is given in Section 6.2.

[10] Following the *resummation procedure* in [123, Appendix E], one could even construct a single (i.e. n independent) state Π^ε, which, for every fixed n, has the same properties as Π_n^ε. We will, however, refrain from doing so for brevity of the presentation.

Proposition 4.4 (Adiabatic switching and the NEASS). *Under the assumptions of Theorem 4.1 (in particular recalling* (43)*) it holds that*

$$\left|\langle B\rangle_{\rho^{\varepsilon,\eta,f}(t)} - \langle B\rangle_{\Pi_n^\varepsilon}\right| \le C\,\|B\|\,\mathrm{diam}(Y)^{3d+\ell}\big(1+t\big)^{(2d+\ell)/p+1}\,\varepsilon\,\frac{\varepsilon^n+\eta^n}{\eta^{(2d+\ell)/p+1}} \times\left(1+\mathrm{e}^{-d(\Lambda^{\mathrm{pert}},\Lambda\setminus\Lambda^{\mathrm{gap}})^q-(n+1)\log(\varepsilon)}\right), \tag{49}$$

for $t \ge 0$, where Π_n^ε is the NEASS (46) *constructed in Proposition 4.3.*

For our application to response theory, it is important to have an explicit expansion of expectation values in the NEASS in powers of ε with coefficients given by expectations in the unperturbed equilibrium state, the linear term constituting the celebrated *Kubo formula.* This is the content of the following proposition, which is the local gap version of [203, Proposition 3.2], and whose proof is given in Section 6.2.

Proposition 4.5 (Asymptotic expansion of the NEASS). *Under the assumptions of Proposition 4.3, there exist linear maps $\mathscr{K}_j\colon \mathscr{A}\to\mathscr{A}$, $j\in\mathbb{N}$, given by nested commutators with the A_μ's in* (46)*, such that for $n \ge m$ it holds that*

$$\left|\langle B\rangle_{\Pi_n^\varepsilon} - \sum_{j=1}^{m}\varepsilon^j\langle\mathscr{K}_j[\![B]\!]\rangle_{\rho_0}\right| \le C_{n,m}\,\varepsilon^{m+1}\,\|B\|\,|Y|^m\,\chi_{b',p}\big(d(\Lambda^{\mathrm{pert}},Y)\big), \tag{50}$$

with b', p the parameters of the localization of the A_μ given in Proposition 4.3. The first two orders of the expansion (50) *are explicitly given by*

$$\langle\mathscr{K}_0[\![B]\!]\rangle_{\rho_0} = \langle B\rangle_{\rho_0} \quad \textit{and} \quad \langle\mathscr{K}_1[\![B]\!]\rangle_{\rho_0} = -\mathrm{i}\Big\langle\big[\mathscr{I}_{H_0,g}[\![V]\!],B\big]\Big\rangle_{\rho_0}, \tag{51}$$

where $\mathscr{I}_{H_0,g}[\![\,\cdot\,]\!]$ is the inverse Liouvillian from (21).

We can finally give the proof of Theorem 4.1.

Proof of Theorem 4.1. Armed with Propositions 4.4–4.5, the proof of Theorem 4.1 follows by a simple application of the triangle inequality when applying Proposition 4.4 with $n\to\tilde n := \lceil m\,(n+(2d+\ell)/p)\rceil$ and Proposition 4.5 for $n\to\tilde n$ and $m\to n$ as in [203, Theorem 4.1]. □

5 Local gap conditions: Proofs for Section 3

This section collects several proofs concerning the local dynamical gap condition ($\mathrm{LDG}_{\mathrm{main}}$) from Section 3.1, which were skipped in earlier sections. More precisely, we will prove the dynamical characterization of a spectral gap in Proposition 3.1 and the relations among the various local gap conditions formulated in Proposition 3.2. Finally, we show that the examples from Section 3.5 satisfy ($\mathrm{LDG}_{\mathrm{main}}$) by means of Proposition 3.2.

5.1 Dynamical characterization of a spectral gap: Proof of Proposition 3.1

Deriving the LHS from the RHS is standard material, see e.g. [174, Lemma 6.8 and Proposition 6.9]. For the other direction, recall (26) and suppose that for all $A, B \in \mathscr{B}(\mathscr{H})$

$$0 = \big\langle \big[\mathscr{L}_H \circ \mathscr{I}_{H,g}[\![A]\!] - A, B\big]\big\rangle_P = \big\langle \big[\mathscr{I}_{H,g}[\![A]\!], B\big]\big\rangle_P = -\operatorname{tr}\big(\big[\mathscr{I}_{H,g}[\![A]\!], P\big]\, B\big).$$

Since B is arbitrary, this means $[\mathscr{I}_{H,g}[\![A]\!], P] = 0$. Moreover, inserting the spectral decomposition $H = \sum_n E_n\, P_n$, this can be written as

$$0 = \big[\mathscr{I}_{H,g}[\![A]\!], P\big] = \sqrt{2\pi}\Bigg(\sum_{\substack{E_n \in \sigma_2 \\ E_m \in \sigma_1}} \widehat{w}(E_m - E_n)\, P_n\, A\, P_m - \sum_{\substack{E_n \in \sigma_1 \\ E_m \in \sigma_2}} \widehat{w}(E_m - E_n)\, P_n\, A\, P_m \Bigg). \tag{52}$$

For *contradiction*, we now assume that $d(\sigma_1, \sigma_2) < g$. Then, since $\widehat{w}|_{(-g,g)} > 0$, one can easily construct an observable A, which violates (52), e.g. $A = P_{n_*} P_{m_*}$ with $E_{n_*} \in \sigma$, $E_{m_*} \in \sigma_2$ satisfying $|E_{n_*} - E_{m_*}| = d(\sigma_1, \sigma_2) < g$.

5.2 Relations among local gap conditions: Proof of Proposition 3.2

We prove the seven implications gathered in Proposition 3.2 one by one. Unless differently stated, we will use the constants C, ℓ, b and p from the formulation of Proposition 3.2 generically, i.e. their precise value might change from line to line. Some technical details are kept brief in this section, more detailed proofs using similar arguments are given in Appendix A.

Proofs of (ii), (i), (iii): All of these are obvious, by application of the estimates

$$\|[\rho - \rho_*, A]\|_{\mathrm{tr}} \le \|(\rho - \rho_*)\, A\|_{\mathrm{tr}} + \|A\, (\rho - \rho_*)\|_{\mathrm{tr}},$$
$$\big|\operatorname{tr}\big((\rho - \rho_*) A\big)\big| \le \|(\rho - \rho_*) A\|_{\mathrm{tr}}\,,$$

and

$$\big|\operatorname{tr}\big(\rho_* (\tau - \mathrm{id})[\![A]\!]\big)\big| \le \big\|(\tau - \mathrm{id})[\![A]\!]\big\|,$$

for (ii), (i) and (iii), respectively.

Proof of (iv): By assumption, it holds that

$$\mathrm{i}\, \langle A^* \mathscr{L}_{H_*}[\![A]\!]\rangle_{\rho_*} \ge g\, \Big[\langle A^* A\rangle_{\rho_*} - \big|\langle A\rangle_{\rho_*}\big|^2 \Big] \tag{53}$$

for all observables $A \in \mathscr{A}$, where $\mathscr{L}_{H_*}[\![\,\cdot\,]\!] := -\mathrm{i}\, [H_*, \,\cdot\,]$ denotes the Liouvillian of H_*. The idea is now to replace $H_* \to H$ and $\rho_* \to \rho$ in (53) and estimate the resulting error in such a way that we arrive at (36).

First, by application of (29), we replace $[\langle A^*A\rangle_{\rho_*} - |\langle A\rangle_{\rho_*}|^2]$ by $[\langle A^*A\rangle_{\rho} - |\langle A\rangle_{\rho}|^2]$ on the RHS of (53) at the cost of an error bounded by $C\,\|A\|^2\,\mathrm{diam}(X)^\ell\,\chi_{b,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)$.

For the LHS of (53), we estimate

$$\begin{aligned}&\Big|\big\langle A^*\mathscr{L}_{H_*}[\![A]\!]\big\rangle_{\rho_*} - \big\langle A^*\mathscr{L}_{H}[\![A]\!]\big\rangle_{\rho}\Big|\\ &\qquad\le \Big|\big\langle A^*\big(\mathscr{L}_{H_*}[\![A]\!] - \mathscr{L}_{H}[\![A]\!]\big)\big\rangle_{\rho_*}\Big| + \big|\mathrm{tr}\big((\rho_*-\rho)A^*\mathscr{L}_{H}[\![A]\!]\big)\big|\end{aligned}\tag{54}$$

by means of the triangle inequality. The first term on the RHS of (54) can now be bounded as (recall that $H = H_* + J$, J is $\Lambda\setminus\Lambda^{\mathrm{gap}}$-localized, and $A\in\mathscr{A}_X$)

$$\Big|\big\langle A^*\big(\mathscr{L}_{H_*}[\![A]\!] - \mathscr{L}_{H}[\![A]\!]\big)\big\rangle_{\rho_*}\Big| \le 2\,\|\Phi_J\|_{b,p;\Lambda\setminus\Lambda^{\mathrm{gap}}}\,\|A\|^2\;\mathrm{diam}(X)^d\,\chi_{b,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)\tag{55}$$

by application of (91) in Lemma A.4.

For the second term on the RHS of (54), we write $\mathscr{L}_H[\![A]\!] = \mathbb{E}_{X_n}\mathscr{L}_H[\![A]\!] + (\mathrm{id}-\mathbb{E}_{X_n})\mathscr{L}_H[\![A]\!]$ for some n to be chosen below, where $X_n := \{x\in\Lambda\,|\,d(x,X)\le n\}$ denotes the n-fattening of the set $X\subset\Lambda$. We now estimate the two terms separately. For the first term, we employ (29) to bound

$$\big|\mathrm{tr}\big((\rho_*-\rho)\,A^*\,\mathbb{E}_{X_n}\mathscr{L}_H[\![A]\!]\big)\big| \le C\,\|A\|^2\big(\mathrm{diam}(X)+n\big)^\ell\,\chi_{b,p}\big(d(X_n,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)\tag{56}$$

where we used that $\mathbb{E}_{X_n}\mathscr{L}_H[\![A]\!]\in\mathscr{A}_{X_n}$ (by definition) and $\|\mathbb{E}_{X_n}\mathscr{L}_H[\![A]\!]\|\le C\,|X|\,\|A\|$ (by (91) from Lemma A.4 and Lemma A.6 (c)). For the second term, we use (92) from Lemma A.4 and Lemma A.6 (e) for estimating the difference $\|(\mathrm{id}-\mathbb{E}_{X_n})\mathscr{L}_H[\![A]\!]\|$ to get

$$\big|\mathrm{tr}\big((\rho_*-\rho)\,A^*\,(\mathrm{id}-\mathbb{E}_{X_n})\,\mathscr{L}_H[\![A]\!]\big)\big| \le C\,\|A\|^2\;\mathrm{diam}(X)^\ell\,\chi_{b,p}(n)\tag{57}$$

Using $d(X_n,\Lambda\setminus\Lambda^{\mathrm{gap}}) \ge d(X,\Lambda\setminus\Lambda^{\mathrm{gap}}) - n$ for (56), we can pick $n = d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})/2$, say, to estimate

$$(56)+(57)\le C\,\|A\|^2\;\mathrm{diam}(X)^\ell\,\chi_{b/2,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big).$$

Combining this with (55), we estimate (54) by $C\,\|A\|^2\,\mathrm{diam}(X)^\ell\,\chi_{b,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)$ and we thus arrive at (36).

Proof of (vii): This is obvious, because $\big|\langle A^*A\rangle_\rho - \big|\langle A\rangle_\rho\big|^2\big|\le 2\|A\|^2$.

Proof of (v): By assumption (25), it holds that, for all observables $A\in\mathscr{A}_X$ and $B\in\mathscr{A}_Y$ localized in $X\subset\Lambda$ and $Y\subset\Lambda$, it holds that (recall (26) for the definition of $\mathscr{J}_{H_*,g}$)

$$\begin{aligned}\Big|\big\langle\big[\mathscr{J}_{H_*,g}[\![A]\!],B\big]\big\rangle_{\rho_*}\Big| &\le C\,\|A\|\,\|B\|\,\big[\,\mathrm{diam}(X)+\mathrm{diam}(Y)\big]^\ell\\ &\qquad\times\chi_{b,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)+d(Y,\Lambda\setminus\Lambda^{\mathrm{gap}}).\end{aligned}\tag{58}$$

Similarly to the proof of (iv), the idea is now to replace $H_*\to H$ and $\rho_*\to\rho$ on the LHS of (58) at the price of an error that is bounded in terms of the RHS of (58). In order to do so, we will heavily exploit the symmetry of (58) in A and B (recall the discussion around (26)). That is, we will prove the bound first only with $d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})$ in the argument of $\chi_{b,p}$ and later obtain their sum (like on the RHS of (58)) by symmetry (modulo changing $b \to b/2$).

To begin with, by the triangle inequality, we have

$$\begin{aligned} &\Big|\big\langle [\mathscr{I}_{H_*,g}[\![A]\!], B]\big\rangle_{\rho_*} - \big\langle [\mathscr{I}_{H,g}[\![A]\!], B]\big\rangle_{\rho}\Big| \\ &\quad \le \Big|\big\langle [(\mathscr{I}_{H_*,g} - \mathscr{I}_{H,g})[\![A]\!], B]\big\rangle_{\rho}\Big| + \big|\mathrm{tr}\big((\rho-\rho_*)[B, \mathscr{I}_{H_*,g}[\![A]\!]]\big)\big|. \end{aligned} \tag{59}$$

For the first term on the RHS of (59), we estimate

$$\|(\mathscr{I}_{H_*,g} - \mathscr{I}_{H,g})[\![A]\!]\| \le \int_{\mathbb{R}} \mathrm{d}t\, w_g(t)\, \|\mathrm{e}^{\mathrm{i}tH_*} A \mathrm{e}^{-\mathrm{i}tH_*} - \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}\|. \tag{60}$$

Recalling $H = H_* + J$, the difference between the two time evolutions can be written as

$$\mathrm{e}^{\mathrm{i}tH_*}\, A\, \mathrm{e}^{-\mathrm{i}tH_*} - \mathrm{e}^{\mathrm{i}tH}\, A\, \mathrm{e}^{-\mathrm{i}tH} = -\mathrm{i}\int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{i}tH}\, \big[J, \mathrm{e}^{\mathrm{i}sH_*}\, A\, \mathrm{e}^{-\mathrm{i}sH_*}\big]\, \mathrm{e}^{-\mathrm{i}tH}.$$

We thus find that

$$\begin{aligned} (60) &\le \int_{\mathbb{R}} \mathrm{d}t\, w_g(t)\, |t| \sup_{s\in[0,t]} \big\|\big[J, \mathrm{e}^{\mathrm{i}sH_*}\, A\, \mathrm{e}^{-\mathrm{i}sH_*}\big]\big\| \\ &\le C\, \|A\|\, \mathrm{diam}(X)^{2d} \Big(\chi_{b,p}\big(d(X, \Lambda\setminus\Lambda^{\mathrm{gap}})\big) \int_I \mathrm{d}t\, w_g(t)\, |t|\, \big(1+|t|\big)^{d/p} \\ &\qquad\qquad + \int_{\mathbb{R}\setminus I} \mathrm{d}t\, w_g(t)\, |t|\, \big(1+|t|\big)^{d/p} \Big) \\ &\le C\, \|A\|\, \mathrm{diam}(X)^{2d}\, \chi_{b,p}\big(d(X, \Lambda\setminus\Lambda^{\mathrm{gap}})\big), \end{aligned} \tag{61}$$

where we denoted $I := \big\{ t \in \mathbb{R} \,\big|\, |t| \le \big(d(X, \Lambda\setminus\Lambda^{\mathrm{gap}})/(2v)\big)^p/2 \big\}$. Here, v is the Lieb-Robinson velocity from Lemma A.7, which we employed in the second step. In the final step, we used the stretched exponential decay of w_g from (23) and possibly adjusted the constants C, b, and p.

The second term on the RHS of (59) can be estimated by means of (31), since

$$\big|\mathrm{tr}\big((\rho-\rho_*)\, \big[B, \mathscr{I}_{H_*,g}[\![A]\!]\big]\big)\big| \le \|B\|\, \big\|\big[\rho-\rho_*, \mathscr{I}_{H_*,g}[\![A]\!]\big]\big\|_{\mathrm{tr}}.$$

For $\mathscr{I}_{H_*,g}[\![A]\!]$ we now apply the local decomposition technique, analogously to the arguments around (56)–(57). More precisely, taking $A \in \mathscr{A}_X$, we now write $\mathscr{I}_{H_*,g}[\![A]\!] = \mathbb{E}_{X_n}\mathscr{I}_{H_*,g}[\![A]\!] + (\mathrm{id} - \mathbb{E}_{X_n})\mathscr{I}_{H_*,g}[\![A]\!]$ for some n to be chosen below, where $X_n := \{ x \in \Lambda \mid d(x, X) \le n \}$ denotes the n-fattening of the set $X \subset \Lambda$. We now estimate the two terms separately. For the first term, we employ (31) to bound

$$\big\|\big[\rho-\rho_*, \mathbb{E}_{X_n}\mathscr{I}_{H_*,g}[\![A]\!]\big]\big\|_{\mathrm{tr}} \le C\, \|A\|\, \big(\mathrm{diam}(X)+n\big)^{\ell}\, \chi_{b,p}\big(d(X_n, \Lambda\setminus\Lambda^{\mathrm{gap}})\big) \tag{62}$$

where we used that $\mathbb{E}_{X_n}\mathcal{I}_{H_*,g}[\![A]\!] \in \mathcal{A}_{X_n}$ (by definition) and $\|\mathbb{E}_{X_n}\mathcal{I}_{H_*,g}[\![A]\!]\| \leq \|A\|$. For the second term, we simply use Lemma A.6 together with (100) applied with $\mathcal{I} \to \mathcal{J}$ for estimating the difference $\|(\mathrm{id} - \mathbb{E}_{X_n})\mathcal{I}_{H_*,g}[\![A]\!]\|$ to get

$$\begin{aligned}\big\|\big[\rho - \rho_*, (\mathrm{id} - \mathbb{E}_{X_n})\mathcal{I}_{H_*,g}[\![A]\!]\big]\big\|_{\mathrm{tr}} \\ \leq 2\|\rho - \rho_*\|_{\mathrm{tr}}\, \|(\mathrm{id} - \mathbb{E}_{X_n})\mathcal{J}_{H_*,g}[\![A]\!]\| \leq C\,\|A\|\, \mathrm{diam}(X)^\ell\, \chi_{b,p}(n)\,.\end{aligned} \tag{63}$$

Using $d(X_n, \Lambda \setminus \Lambda^{\mathrm{gap}}) \geq d(X, \Lambda \setminus \Lambda^{\mathrm{gap}}) - n$ for (62), we can pick $n = d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})/2$, say, to estimate

$$(62) + (63) \leq C\,\|A\|\, \mathrm{diam}(X)^\ell\, \chi_{b/2,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big).$$

Finally, as mentioned above, interchanging the roles of A and B (by symmetry of the original expression; recall the discussion around (26)), we thus arrive at a bound on $\big|\big\langle\big[\mathcal{I}_{H,g}[\![A]\!], B\big]\big\rangle_\rho\big|$ of the form given by the RHS of (58) (modulo changing $b \to b/2$). Therefore, combining (58) with (59), and (60)–(61) as well as (62)–(63), we conclude the desired.

Proof of (vi): Instead of (58), we start with (by assumption)

$$\big\langle\big[\mathcal{I}_{H_*,g}[\![A]\!], B\big]\big\rangle_{\rho_*} = 0 \qquad \text{for all} \qquad A, B \in \mathcal{A}. \tag{64}$$

Apart from this, the idea is identical to (iv) and (v). Hence, by means of the triangle inequality, we obtain the same two terms from (59). The first term can be estimated in exactly the same way as in (60)–(61). The second term in (59) has to be treated a bit differently as in (62)–(63), since we now assumed (32) instead of (31).

In fact, for this term, using $\langle\,\cdot\,\rangle_\rho = \langle\tau[\![\,\cdot\,]\!]\rangle_{\rho_*}$, that τ is a $*$-automorphism, and (64), we get

$$\big|\mathrm{tr}\big((\rho - \rho_*)\big[B, \mathcal{I}_{H_*,g}[\![A]\!]\big]\big)\big| = \big|\mathrm{tr}\big(\rho_*\big[\tau[\![B]\!], (\tau - \mathrm{id}) \circ \mathcal{I}_{H_*,g}[\![A]\!]\big]\big)\big| \leq \|(\tau - \mathrm{id}) \circ \mathcal{I}_{H_*,g}[\![A]\!]\|.$$

This can now be treated exactly as done in the argument around (62)–(63) (i.e. taking $A \in \mathcal{A}_X$, writing $\mathcal{I}_{H_*,g}[\![A]\!] = \mathbb{E}_{X_n}\mathcal{I}_{H_*,g}[\![A]\!] + (\mathrm{id} - \mathbb{E}_{X_n})\mathcal{I}_{H_*,g}[\![A]\!]$, and estimating the two terms separately while optimizing in n).

Proof of (viii): Without loss of generality, we may assume that $X \subset \Lambda$ in (35) is such that $C\, \mathrm{diam}(X)^\ell \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) < 1/2$, say (otherwise there is nothing to prove), and $\mathrm{diam}(X) \leq d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})^\beta$ for some $\beta < 1$. By assumption, for such $X \subset \Lambda$ and $A \in \mathcal{A}_X$, we have

$$\mathrm{i}\langle\psi, A^*\mathcal{L}_H[\![A]\!]\psi\rangle \geq \frac{g}{2}\left(\langle\psi, A^*A\psi\rangle - \big|\langle\psi, A\psi\rangle\big|^2\right) \quad \text{with} \quad |\psi\rangle = \otimes_{x\in\Lambda}|\psi_x\rangle,\ \|\psi_x\| = 1. \tag{65}$$

Our claim will be a consequence of the following lemma.

Lemma 5.1 (cf. Proposition 14 in [221])**.** *Take a Hermitian $A \in \mathcal{A}_X$ as above and assume* (65)*. Denote the n-fattening of X by $X_n := \{z \in \Lambda \,|\, d(z, X) \le n\}$. Then, there exists a Hermitian operator $\widetilde{A} \in \mathcal{A}_{X_n}$ with $n := \lfloor d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})/2 \rfloor$ such that ψ is an eigenvector of $\widetilde{A}$, i.e. $\widetilde{A}\psi = \widetilde{E}\psi$ for some $\widetilde{E} \in \mathbb{R}$, and we have the bound*

$$\left\| \mathbb{E}_{X_n} \mathcal{J}_{H,g}[\![A]\!] - \widetilde{A} \right\| \le C \, \|A\| \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) \tag{66}$$

with $\mathcal{J}_{H,g}$ being defined in (26)*.*

Armed with Lemma 5.1, we now turn to estimating the LHS of (27), which is given by $\langle [\mathcal{J}_{H,g}[\![A]\!], B] \rangle_\rho = \langle \psi, [\mathcal{J}_{H,g}[\![A]\!], B]\psi \rangle$. By the triangle inequality, we have

$$\begin{aligned} &\big|\big\langle \psi, [\mathcal{J}_{H,g}[\![A]\!], B]\psi \big\rangle\big| \\ &\quad \le \big|\big\langle \psi, [(\mathrm{id} - \mathbb{E}_{X_n})\mathcal{J}_{H,g}[\![A]\!], B]\psi \big\rangle\big| + \big|\big\langle \psi, [\mathbb{E}_{X_n}\mathcal{J}_{H,g}[\![A]\!] - \widetilde{A}, B]\psi \big\rangle\big| + \big|\big\langle \psi, [\widetilde{A}, B]\psi \big\rangle\big| \end{aligned} \tag{67}$$

and estimate the three terms separately. The first term in (67) can be treated as in (63), yielding the bound

$$C \, \|A\| \, \|B\| \, \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big), \tag{68}$$

where we additionally used that $\mathrm{diam}(X) \le d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})^\beta$. For the second term, we employ (66), yielding the same bound as for the first term. The third term in (67) vanishes since $\widetilde{A}\psi = \widetilde{E}\psi$. Therefore, (67) is bounded by (68) and we have proven Proposition 3.2(viii).

It thus remains to give the proof of Lemma 5.1.

Proof of Lemma 5.1. The principal idea is similar to [221, Proposition 14]. To start with, we assume w.l.o.g. that $H\psi = 0$, i.e. ψ is an eigenvector to the eigenvalue zero. Then, we note that, since $\mathrm{diam}(X) \le d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})^\beta$, the bound in (63) implies that

$$\left\| (\mathrm{id} - \mathbb{E}_{X_n})\mathcal{J}_{H,g}[\![A]\!] \right\| \le C \, \|A\| \, \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big). \tag{69}$$

We continue by decomposing

$$\big(\mathcal{J}_{H,g}[\![A]\!] - \langle \psi, \mathcal{J}_{H,g}[\![A]\!]\, \psi \rangle\big)|\psi\rangle = |\phi_{\le n}\rangle + |\phi_{>n}\rangle \tag{70}$$

where we defined

$$|\phi_{\le n}\rangle = |\phi_{\le n}\rangle_{X_n} \otimes |\psi\rangle_{X_n^c} := \big(\mathbb{E}_{X_n}\mathcal{J}_{H,g}[\![A]\!] - \langle \psi, \mathbb{E}_{X_n}\mathcal{J}_{H,g}[\![A]\!]\, \psi \rangle\big)|\psi\rangle$$

and $|\phi_{>n}\rangle$, which is defined such that (70) holds, satisfies the bound

$$\begin{aligned} \| |\phi_{>n}\rangle \| &\le \left\| \big((\mathrm{id} - \mathbb{E}_{X_n})\mathcal{J}_{H,g}[\![A]\!] - \langle \psi, (\mathrm{id} - \mathbb{E}_{X_n})\mathcal{J}_{H,g}[\![A]\!]\, \psi \rangle\big)|\psi\rangle \right\| \\ &\le \left\| (\mathrm{id} - \mathbb{E}_{X_n})\mathcal{J}_{H,g}[\![A]\!] \right\| \le C \, \|A\| \, \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big), \end{aligned}$$

where in the last step we employed (69). Then, similarly to [221, Lemma 15, eq. (B49)], one can compute $\langle \phi_{\leq n}, H^2 \phi_{\leq n} \rangle$ and use the Payley-Zygmund inequality to show that the norm of $|\phi_{\leq n}\rangle$ is essentially bounded by the norm of $|\phi_{>n}\rangle$. That is, in our case, we find

$$\| |\phi_{\leq n}\rangle \| \leq C \, \|A\| \, \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big), \tag{71}$$

as always modulo adjusting the constants C, b and p. Hence, defining the Hermitian operator

$$\widetilde{A} := \mathbb{E}_{X_n} \mathscr{I}_{H,g} [\![A]\!] - \big(|\phi_{\leq n}\rangle_{X_n} \langle \psi| + \text{h.c.} \big),$$

supported in X_n we easily see that $\widetilde{A}|\psi\rangle = \langle \psi, \mathbb{E}_{X_n} \mathscr{I}_{H,g} [\![A]\!] \, \psi \rangle |\psi\rangle =: \widetilde{E} |\psi\rangle$ and the bound (66) follows from (71). □

Proof of (ix): This is obvious from the definitions (25) and (27).

This concludes the proof Proposition 3.2. □

5.3 Local dynamical gap for the examples in Section 3.5

In this section, we prove the systems considered in Section 3.5 to have a local dynamical gap.

5.3.1 Perturbations of the classical Ising model with weak interactions

In this section, we prove the claim of a local dynamical gap from Section 3.5.1, where we considered perturbations of the classical Ising model with weak interactions. First, the (unique) ground state vector of (37) and the associated ground state energy is easily found as

$$\psi_* = \otimes_{x \in \Lambda} |\downarrow\rangle \quad \text{satisfying} \quad H_* \, \psi_* = \Big(-|\Lambda| + \tfrac{1}{2} \sum_{x,y} \lambda(x-y) \Big) \psi_*,$$

and the associated spectral projection (ground state) is simply given by $\rho_* = P_* = |\psi_*\rangle\langle\psi_*|$. We note that this is a globally gapped eigenstate of H_*, since the ground state energy corresponding to ψ_* is separated by a spectral gap $g \geq 2 - \|\lambda\|_1 > 0$ from the first excited state.

For the following argument, it is important to observe that, for any given $\Lambda^{\mathrm{gap}} \subset \Lambda$, the ground state projection factorizes, i.e.

$$\rho_* = \Big(\bigotimes_{x \in \Lambda^{\mathrm{gap}}} |\downarrow\rangle\langle\downarrow| \Big) \otimes \Big(\bigotimes_{x \in \Lambda \setminus \Lambda^{\mathrm{gap}}} |\downarrow\rangle\langle\downarrow| \Big) =: \rho_*^{\Lambda^{\mathrm{gap}}} \otimes \rho_*^{\Lambda \setminus \Lambda^{\mathrm{gap}}}. \tag{72}$$

Indeed, since $\|\lambda\|_1 < 2$, every ground state ρ of $H = H_* + J$, where J is a *strictly* $\Lambda \setminus \Lambda^{\mathrm{gap}}$-localized SLT Hamiltonian, as described in Section 3.5.1, also factorizes as

$$\rho = \rho_*^{\Lambda^{\mathrm{gap}}} \otimes \rho^{\Lambda \setminus \Lambda^{\mathrm{gap}}}. \tag{73}$$

In order to see this, first note that there exists an eigenbasis[11] of H for which every eigenvector ψ of H can be written as a linear combination $\sum_j c_j\, e_j \otimes \varphi_j$, where $e_j \in \mathscr{H}_{\Lambda^{\mathrm{gap}}}$ are eigenvectors of

$$H\big|_{\Lambda^{\mathrm{gap}}} := \sum_{x\in\Lambda^{\mathrm{gap}}} \sigma_x^3 + \frac{1}{2}\sum_{x,y\in\Lambda^{\mathrm{gap}}} \lambda(x-y)\,\sigma_x^3\,\sigma_y^3$$

to a *common* eigenvalue and $\varphi_j \in \mathscr{H}_{\Lambda\setminus\Lambda^{\mathrm{gap}}}$. Then, to see (73), it suffices to realize that, for every $x \in \Lambda^{\mathrm{gap}}$, starting from the *unique* ground state vector $\psi_*|_{\Lambda^{\mathrm{gap}}} = \otimes_{x\in\Lambda^{\mathrm{gap}}}|\downarrow\rangle$ of $H\big|_{\Lambda^{\mathrm{gap}}}$, the energy cost for flipping the spin $|\downarrow\rangle$ to $|\uparrow\rangle$ in the first term of (37) is two, whereas the potential gain stemming from the second summand in (37) is bounded by $\|\lambda\|_1 < 2$, yielding (73). In particular, any *overall* pure ground state of H can be obtained by tensorizing the unique separate ground state vector of $H|_{\Lambda^{\mathrm{gap}}}$, i.e. $\psi|_{\Lambda^{\mathrm{gap}}}$, with an appropriate (not necessarily unique) minimizer $\varphi^{\Lambda\setminus\Lambda^{\mathrm{gap}}}$ of

$$\min_{\substack{\varphi\in\mathscr{H}_{\Lambda\setminus\Lambda^{\mathrm{gap}}}\\ \|\varphi\|=1}} \Big\langle \psi_*|_{\Lambda^{\mathrm{gap}}} \otimes \varphi, (H - H|_{\Lambda^{\mathrm{gap}}})\,\psi_*|_{\Lambda^{\mathrm{gap}}} \otimes \varphi \Big\rangle,$$

i.e. by conditioning on the first factor $\psi_*|_{\Lambda^{\mathrm{gap}}}$. The pure ground state is then obtained as $|\psi_*|_{\Lambda^{\mathrm{gap}}}\rangle\langle\psi_*|_{\Lambda^{\mathrm{gap}}}| \otimes |\varphi^{\Lambda\setminus\Lambda^{\mathrm{gap}}}\rangle\langle\varphi^{\Lambda\setminus\Lambda^{\mathrm{gap}}}|$.

Therefore, combining (72) and (73), we have the following: For ρ being a *pure state*, i.e. $\rho^{\Lambda\setminus\Lambda^{\mathrm{gap}}}$ from (73) can be written as $\rho^{\Lambda\setminus\Lambda^{\mathrm{gap}}} = |\varphi^{\Lambda\setminus\Lambda^{\mathrm{gap}}}\rangle\langle\varphi^{\Lambda\setminus\Lambda^{\mathrm{gap}}}|$, there exists a unitary $U \equiv U^{\Lambda\setminus\Lambda^{\mathrm{gap}}} \in \mathscr{A}_{\Lambda\setminus\Lambda^{\mathrm{gap}}}$ such that $|\varphi^{\Lambda\setminus\Lambda^{\mathrm{gap}}}\rangle = U \otimes_{x\in\Lambda\setminus\Lambda^{\mathrm{gap}}} |\downarrow\rangle$. In particular, $\rho = U\rho_* U^*$ and hence we have a norm-preserving $\star$-automorphism $\tau[\![A]\!] := U^* A U$ on $A \in \mathscr{A}$, which intertwines the ground states, i.e. $\langle\,\cdot\,\rangle_\rho = \langle\tau[\![\,\cdot\,]\!]\rangle_{\rho_*}$, and satisfies (32). By means of Proposition 3.2(vi) (note that, since ρ_* is spectrally gapped, it fulfills the additional assumption of Proposition 3.2(vi) by means of Proposition 3.1), we thus find that ρ satisfies Assumption ($\mathrm{LDG}_{\mathrm{main}}$). Finally, for a general (*mixed*) state ρ, we conclude the desired after noticing that Assumption ($\mathrm{LDG}_{\mathrm{main}}$) is invariant under taking convex combinations.

5.3.2 Perturbations of gapped frustration free product states

In this section, we prove the claim of a local dynamical gap from Section 3.5.2, where we considered perturbations of gapped frustration free Hamiltonians with a product ground state.

Similarly to Section 5.3.1, one can easily verify that H_* from (39) is globally gapped with its ground state vector being given by $\otimes_{x\in\Lambda}|\psi_{*,x}\rangle$. The same is true for all restrictions $H_*|_{\Lambda'}$.

Moreover, for $\Lambda^{\mathrm{gap}} \subset \Lambda$ and a fixed exponent $\beta > 0$, consider $X \subset \Lambda$ satisfying $\mathrm{diam}(X) \le d(X, \Lambda\setminus\Lambda^{\mathrm{gap}})^\beta$. Under Assumptions (A1), (A2), and (A3), the authors

[11] This is simply a common eigenbasis of H and $H\big|_{\Lambda^{\mathrm{gap}}} \otimes \mathbb{1}_{\mathscr{A}_{\Lambda\setminus\Lambda^{\mathrm{gap}}}}$, which commute.

of [16] have proven the following: Let $|\psi\rangle$ be a ground state vector of $H = H_* + J$, where J is a *strongly* $\Lambda \setminus \Lambda^{\mathrm{gap}}$-localized SLT Hamiltonian, as described in Section 3.5.2, and $P_{*,X}$ denote the projection onto the ground state vector $|\psi_*|_X\rangle$ of $H_*|_X$. Then it holds that

$$\|\rho - \tilde{\rho}\|_{\mathrm{tr}} \leq C \exp\big(-\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big)^q\big) \tag{74}$$

for some $C, q > 0$ and $\|\cdot\|_{\mathrm{tr}}$ being the trace norm. Here, $\rho \equiv P$ and $\tilde{\rho} \equiv \tilde{P}$ denote the orthogonal projections on $|\psi\rangle$ and $P_{*,X}|\psi\rangle$, respectively, i.e. they are *pure states.*

Due to the product structure of the ground state vector of H_* and its restrictions, we easily see that $\tilde{\rho}$ can be written as

$$\tilde{\rho} = |\psi_*|_X\rangle\langle\psi_*|_X| \otimes \tilde{\rho}^{\Lambda\setminus X}$$

for some state $\tilde{\rho}^{\Lambda\setminus X}$ on $\Lambda \setminus X$. This means that, analogously to Section 5.3.1, $\langle\,\cdot\,\rangle_{\rho_*}$ and $\langle\,\cdot\,\rangle_{\tilde{\rho}}$ can be intertwined by a norm preserving $*$-automorphism τ satisfying (32). In particular, by means of Proposition 3.2(vi) (note that, since ρ_* is spectrally gapped, it fulfills the additional assumption of Proposition 3.2(vi) by means of Proposition 3.1), we thus find that $\tilde{\rho}$ satisfies Assumption (LDG$_{\mathrm{main}}$) – but only for observables supported in $X \subset \Lambda$ with $\mathrm{diam}(X) \leq d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})^\beta$ and without $d(Y, \Lambda \setminus \Lambda^{\mathrm{gap}})$ in the argument of $\chi_{b,p}$; that is, Assumption (LDG$_{\mathrm{weak}}$). This implies, by means of Proposition 3.2 (ii) and (v) (modified to the setting of (LDG$_{\mathrm{weak}}$)) and (74), that ρ satisfies Assumption (LDG$_{\mathrm{weak}}$). Finally, for a general (*mixed*) state ρ, we conclude the desired by taking convex combinations (as at the end of the argument in Section 5.3.1).

6 Construction of the NEASS: Proofs for Section 4

The fundamental conceptual idea behind the proof of Proposition 4.4 is a perturbative scheme, which was called *space-time adiabatic perturbation theory* in [182, 180]. Before going into this expansion in Section 6.2, we show that the weakened local dynamical gap condition (LDG$_{\mathrm{weak}}$) from Section 3.1 carries over to SLT operators. The main technical input for carrying out the space-time adiabatic perturbation scheme, is to show that all the operations involved in the expansion preserve localization of SLT operators as required for Lemma 6.1 to apply. This is the content of several auxiliary technical results in Appendix A.

6.1 Local dynamical gap condition for SLT operators

Throughout the proof, we will work with the weakened version of the local dynamical gap condition (LDG$_{\mathrm{weak}}$) from Section 3.1. As mentioned above, we start with the following basic lemma which will heavily be used in our proof. It says that the local dynamical gap condition naturally carries over to SLT operators.

Lemma 6.1. *Let $b > 0$, $p \in (0,1)$ and H_0 be a $\chi_{b,p}$-SLT operator. Assume that the equilibrium state ρ_0 of H_0 satisfies the weakened local gap condition, Assumption* (LDG$_{\text{weak}}$) *above, with gap size at least $g > 0$ and with respect to C_{gap}, b, p, β and ℓ.*

Then, there exists a constant C, such that for any (b,p,Ω)-localized SLT operator A and observable $B \in \mathscr{A}_Y$, we have that

$$\left|\left\langle\left[\mathscr{L}_{H_0} \circ \mathscr{I}_{H_0,g}[\![A]\!] - A, B\right]\right\rangle_{\rho_0}\right| \leq C \ \mathrm{diam}(Y)^{\ell+d} \, \|B\| \, \|\Phi_A\|_{b,p;\Omega} \, \chi_{b/2,p\min\{\beta,1\}}\big(d(\Omega, \Lambda \setminus \Lambda^{\text{gap}})\big). \tag{75}$$

The proof of Lemma 6.1 is presented in Appendix A.5. The principal idea is to write $A = \sum_{Z \subset \Lambda} \Phi_A(Z)$ and then estimate only the contribution of 'small' Z, i.e. those with $\mathrm{diam}(Z) \leq d(Z, \Lambda \setminus \Lambda^{\text{gap}})^\beta$, by (27). Large Z's, i.e. those with $\mathrm{diam}(Z) > d(Z, \Lambda \setminus \Lambda^{\text{gap}})^\beta$, are treated using Lieb-Robinson bounds and the smallness of $\|\Phi_A(Z)\|$ (by definition (18)).

6.2 The adiabatic perturbation scheme

For the proof of Proposition 4.4 we use the same strategy as in Teufel [203]. However, since we only have a local gap, the lower order terms do not vanish exactly, but can be bounded using Assumption (LDG$_{\text{weak}}$).

The statements in Propositions 4.4, will be deduced from a time-dependent NEASS, which is part of the next Theorem. In contrast to the previous works, it will not include a time-dependent unperturbed Hamiltonian $H_0(t)$, because there is no spectral flow available. That means, we construct a time-dependent NEASS $\Pi_n^\varepsilon(t)$ specifically for the switching Hamiltonian given in (42). For times $t \geq 0$ it will turn out to be time-independent.

In order to formulate the result, we introduce time-dependent interactions

$$\Phi \colon I \times \{X \subset \Lambda\} \to \mathscr{A}, \quad (t, X) \mapsto \Phi(t,X) = \Phi(t,X)^* \in \mathscr{A}_X$$

for $I \subset \mathbb{R}$. We will assume that $t \mapsto \Phi(t,X)$ is smooth for every $X \subset \Lambda$, and we denote the term-wise time derivatives by $\Phi^{(k)}$, i.e. $\Phi^{(k)}(t,X) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\Phi(t,X)$ for every $X \subset \Lambda$. Moreover, we identify $\Phi(t,X) = \big(\Phi(t)\big)(X)$, such that for every fixed $t \in I$, $\Phi(t)$ can be viewed as a time-independent interaction. The notion of SLT operators naturally translates to the time-dependent setting.

Theorem 6.2 (Time-dependent NEASS). *Fix $n \in \mathbb{N}$ and let $d \in \mathbb{N}$, $C_{\text{vol}} > 0$, $b > 0$, $p \in (0,1]$, $C_{\text{int}} > 0$ and $g > 0$, $C_{\text{gap}} > 0$, $\beta > 0$, $\ell \in \mathbb{N}_0$. Take any $q \in (0, p\min\{1,\beta\})$. Then there exist a constant $C_n > 0$ (in particular depending on n) such that for all lattices $\Lambda \in \mathscr{G}(d, C_{\text{vol}})$ (recall* (13)*), subsets $\Lambda^{\text{pert}} \subset \Lambda$, intervals $I \subset \mathbb{R}$ and SLT-operators H_0 and $V(t)$, with corresponding interactions that satisfy $\|\Phi_{H_0}\|_{b,p} < C_{\text{int}}$ and $\sup_{t\in I}\|\Phi_V^{(k)}(t)\|_{b,p;\Lambda^{\text{pert}}} < C_{\text{int}}$ for all $k \leq n$, respectively, the following holds:*

Assume that the equilibrium state ρ_0 of H_0 is locally dynamically gapped in Λ^{gap} of size at least $\mathrm{g} > 0$ and with respect to C_{gap}, b, p and ℓ according to Assumption (LDG$_{\mathrm{weak}}$). *And let*

$$H_\varepsilon(t) := H_0 + \varepsilon V(t) \tag{76}$$

be the perturbed Hamiltonian.

Then, there exists a sequence $(A_\mu^\delta)_{\mu\in\mathbb{N}}$ of polynomials in $\delta \in \mathbb{R}$ with maximal degree $\mu - 1$ and time-dependent SLT operators as coefficients, which are $(1, p', \Lambda^{\mathrm{pert}})$-localized for any $p' < p$. The A_μ^δ are such that the state

$$\Pi_n^{\varepsilon,\eta}(t) := \mathrm{e}^{\mathrm{i}S_n^{\varepsilon,\eta}(t)}\, \rho_0\, \mathrm{e}^{-\mathrm{i}S_n^{\varepsilon,\eta}(t)} \quad \text{with} \quad S_n^{\varepsilon,\eta}(t) := \sum_{\mu=1}^{n} \varepsilon^\mu A_\mu^{\eta/\varepsilon}(t), \tag{77}$$

is almost-stationary for the dynamics generated by $H_\varepsilon(t)$ in the following sense: Let $t_0 \in \mathbb{R}$ and let $\rho^{\varepsilon,\eta,f}(t)$ be the solution of the time-dependent adiabatic Schrödinger equation

$$\mathrm{i}\eta \frac{\mathrm{d}}{\mathrm{d}t}\rho^{\varepsilon,\eta,f}(t) = \left[H_\varepsilon(t), \rho^{\varepsilon,\eta,f}(t)\right] \quad \text{with} \quad \rho^{\varepsilon,\eta,f}(t_0) = \Pi_n^{\varepsilon,\eta}(t_0) \tag{78}$$

with adiabatic parameter $\eta \in (0, 1]$.

Under these conditions, for all $B \in \mathcal{A}_Y$ with $Y \subset \Lambda$ and $t \in \mathbb{R}$, it holds that

$$\begin{aligned}\left|\langle B\rangle_{\rho_{t_0}^{\varepsilon,\eta}(t)} - \langle B\rangle_{\Pi_n^{\varepsilon,\eta}(t)}\right| \leq C_n \,\|B\|\, \mathrm{diam}(Y)^{3d+\ell} \left(1 + \frac{\eta^n}{\varepsilon^n}\right) \varepsilon^{n+1} \\ \times \frac{|t-t_0|}{\eta}\left(1 + \frac{v\,|t-t_0|}{\eta}\right)^{(\ell+2d)/p} \\ \times \left(1 + \mathrm{e}^{-d(\Lambda^{\mathrm{pert}},\Lambda\setminus\Lambda^{\mathrm{gap}})^q - (n+1)\log(\varepsilon)}\right).\end{aligned} \tag{79}$$

Moreover, the operators $A_\mu(t)$ at time t depend only on V and its first μ derivatives at time t.

Before we prove Theorem 6.2, let us deduce the results from Section 4, which will follow by taking $\beta = 1$ in Theorem 6.2.

Proof of Proposition 4.3. We choose $V(t) = V$, which implies that also all $A_\mu^{\eta/\varepsilon}(t) = A_\mu^{\eta/\varepsilon}$ are time-independent (i.e. the time-dependent interaction is constant). Moreover, since there is no adiabatic timescale in (47), we choose $\eta = 1$. To obtain the correct scaling, we inspect the proof of Theorem 6.2: In equation (82), we expand in powers of ε and $R_j^{\varepsilon,\eta}$ are polynomials in η/ε. Here, since $\dot{S}_n^{\varepsilon,\eta} = 0$, these polynomials are just constants, $A_\mu^{\eta/\varepsilon} = A_\mu$, and there is no η in (82) or any of the later expressions. Hence, also the norm estimates $\|\Phi_{\tilde{R}_j^{\varepsilon,\eta}}(s)\| < C(1 + \eta^{j-1}/\varepsilon^{j-1})$ used in the end of the proof simplify to $\|\Phi_{\tilde{R}_j^\varepsilon}\| < C$ uniformly in ε. Hence, $(1 + \eta^n/\varepsilon^n)$ is replaced by 1 in (79). All other η in (79) come from the adiabatic timescale and are thus replaced by 1. □

Proof of Proposition 4.4. We choose $V(t) = f(t)\,V$ and since $V(t)$ constant for $t \le 0$, we obtain $\Pi_n^{\varepsilon,\eta}(t) = \Pi_n^\varepsilon$ for all $t \ge 0$. To compare with the solution of (43), we choose $t_0 = -1$ such that $\Pi_n^{\varepsilon,\eta}(t_0) = \rho_0$. Then $|t - t_0| = 1 + t$ and (79) gives the statement. □

Proof of Proposition 4.5. To prove the asymptotic expansion, we first expand (46) and obtain

$$\operatorname{tr}\big(\Pi_n^\varepsilon\, B\big) = \operatorname{tr}\big(\rho_0\, \mathrm{e}^{\mathscr{L}_{S_n^\varepsilon}} [\![B]\!]\big) = \sum_{k=0}^{m} \frac{1}{k!} \big\langle \mathscr{L}^k_{S_n^\varepsilon} [\![B]\!] \big\rangle_{\rho_0} + \frac{1}{(m+1)!} \operatorname{tr}\big(\rho_0\, \mathrm{e}^{\mathscr{L}_{S_n^{\tilde\varepsilon}}} \circ \mathscr{L}^{m+1}_{S_n^{\tilde\varepsilon}} [\![B]\!]\big)$$

for some $\tilde\varepsilon \in [0, \varepsilon]$. Since the A_μ and thus also the S_n^ε are Λ^{pert}-localized, we use Lemma A.4 to bound the remainder by

$$\frac{1}{(m+1)!} \big\|\mathscr{L}^{m+1}_{S_n^{\tilde\varepsilon}} [\![B]\!]\big\| \le C\, \|B\|\, |Y|^{m+1}\, \chi_{b',p}\big(d(Y, \Lambda^{\mathrm{pert}})\big)\, \|S_n^\varepsilon\|^{m+1}_{b',p,\Lambda^{\mathrm{pert}}},$$

where $\|S_n^\varepsilon\|_{b',p;\Lambda^{\mathrm{pert}}} \le \varepsilon\, \tilde C$. It is clear from the proof of Theorem 6.2, that $\tilde C$ depends only on $n, d, C_{\mathrm{vol}}, b, p, C_{\mathrm{int}}$, and g. We now expand S_n^ε in the first term and group the terms according to the powers in ε. The zero order term clearly is $\langle B\rangle_{\rho_0}$. In first order, we obtain $\varepsilon\,\langle \mathscr{L}_{A_1}\rangle_{\rho_0} = -\varepsilon\, \mathrm{i}\big\langle \big[\mathscr{I}_{H_0,g}[\![V]\!], B\big]\big\rangle_{\rho_0}$ as can be read of from (85). All $\mathscr{K}_j$ for $j \le m$ are constructed in this way. In the end, some higher order terms are left. They all come with multi-commutators of Λ^{pert}-localized A_μ with B and can be bounded as the remainder above using Lemma A.4. □

We now prove the time-dependent NEASS from Theorem 6.2.

Proof of Theorem 6.2. For the proof we first assume the more general form $S_n^{\varepsilon,\eta}(t) := \sum_{\mu=1}^n \varepsilon^\mu A_\mu^{\varepsilon,\eta}(t)$ of (77) (it will become clear from the proof, that $A_\mu^{\varepsilon,\eta}(t)$ is actually a polynomial in η/ε) and then iteratively choose $A_\mu^{\varepsilon,\eta}(t)$ such that the statement holds.

Therefore, let $U_{t,t_0}^{\varepsilon,\eta}$ be the solution of

$$\mathrm{i}\,\eta\, \frac{\mathrm{d}}{\mathrm{d}t} U_{t,t_0}^{\varepsilon,\eta} = H_\varepsilon(t)\, U_{t,t_0}^{\varepsilon,\eta} \qquad \text{with} \qquad U_{t_0,t_0}^{\varepsilon,\eta} = \mathbb{1} \quad \text{for all} \quad t, t_0 \in I, \tag{80}$$

with H_ε given in (42). Then, $\rho_{t_0}^{\varepsilon,\eta}(t) := U_{t,t_0}^{\varepsilon,\eta}\, \Pi_n^{\varepsilon,\eta}(t_0)\, U_{t_0,t}^{\varepsilon,\eta}$ is the solution of (78). To obtain (79) we use the fundamental theorem of calculus, and get

$$\langle B\rangle_{\rho_{t_0}^{\varepsilon,\eta}(t)} - \langle B\rangle_{\Pi_n^{\varepsilon,\eta}(t)} = -\int_{t_0}^t \mathrm{d}s\, \frac{\mathrm{d}}{\mathrm{d}s} \Big\langle \mathrm{e}^{-\mathrm{i}S_n^{\varepsilon,\eta}(s)}\, U_{s,t}^{\varepsilon,\eta}\, B\, U_{t,s}^{\varepsilon,\eta}\, \mathrm{e}^{\mathrm{i}S_n^{\varepsilon,\eta}(s)} \Big\rangle_{\rho_0}. \tag{81}$$

By product rule and Duhamel's formula, the derivative evaluates as

$$\frac{\mathrm{d}}{\mathrm{d}s}\, \mathrm{e}^{-\mathrm{i}S_n^{\varepsilon,\eta}(s)}\, U_{s,t}^{\varepsilon,\eta}\, B\, U_{t,s}^{\varepsilon,\eta}\, \mathrm{e}^{\mathrm{i}S_n^{\varepsilon,\eta}(s)} = -\frac{\mathrm{i}}{\eta} \Big[Q_n^{\varepsilon,\eta}(s), \mathrm{e}^{-\mathrm{i}S_n^{\varepsilon,\eta}(s)}\, U_{s,t}^{\varepsilon,\eta}\, B\, U_{t,s}^{\varepsilon,\eta}\, \mathrm{e}^{\mathrm{i}S_n^{\varepsilon,\eta}(s)}\Big],$$

where

$$
\begin{aligned}
Q_n^{\varepsilon,\eta}(s) &= \eta \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{-\mathrm{i}\lambda S_n^{\varepsilon,\eta}(s)}\, \dot{S}_n^{\varepsilon,\eta}(s)\, \mathrm{e}^{\mathrm{i}\lambda S_n^{\varepsilon,\eta}(s)} + \mathrm{e}^{-\mathrm{i}S_n^{\varepsilon,\eta}(s)} \big(H_0 + \varepsilon\, V(s)\big)\, \mathrm{e}^{\mathrm{i}S_n^{\varepsilon,\eta}(s)} \\
&= \eta \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{\lambda \mathscr{L}_{S_n^{\varepsilon,\eta}(s)}} [\![\dot{S}_n^{\varepsilon,\eta}(s)]\!] + \mathrm{e}^{\mathscr{L}_{S_n^{\varepsilon,\eta}(s)}} [\![H_0 + \varepsilon\, V(s)]\!] \\
&= H_0 + \sum_{j=1}^{n} \varepsilon^j\, R_j^{\varepsilon,\eta}(s) + \varepsilon^{n+1}\, R_{n+1}^{\varepsilon,\eta}(s).
\end{aligned}
\tag{82}
$$

In the last line we expanded in powers of ε and η such that $R_j^{\varepsilon,\eta}(s)$ are polynomials in η/ε of degree $j-1$ with ε- and η-independent SLT operators as coefficients. In this way, the joint power of ε and η in front of the SLT operators collected in $R_j^{\varepsilon,\eta}$ is j and there is at least one ε. By Taylor formula with mean-value form of the remainder, there exist $\theta \in [0,1]$ such that

$$
\begin{aligned}
\mathrm{e}^{\mathscr{L}_{S_n^{\varepsilon,\eta}(s)}} [\![H_0 + \varepsilon\, V(s)]\!] = &\sum_{k=0}^{n} \frac{1}{k!}\, \mathscr{L}^k_{S_n^{\varepsilon,\eta}(s)} [\![H_0 + \varepsilon\, V(s)]\!] \\
&+ \frac{1}{(n+1)!}\, \mathrm{e}^{\theta \mathscr{L}_{S_n^{\varepsilon,\eta}(s)}} \circ \mathscr{L}^{n+1}_{S_n^{\varepsilon,\eta}(s)} [\![H_0 + \varepsilon\, V(s)]\!].
\end{aligned}
\tag{83}
$$

Similarly, for the first term in (82) we expand the integrand using the integral form of the remainder and obtain

$$
\begin{aligned}
&\eta \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{\lambda \mathscr{L}_{S_n^{\varepsilon,\eta}(s)}} [\![\dot{S}_n^{\varepsilon,\eta}(s)]\!] \\
&\quad = \eta \sum_{k=0}^{n-2} \frac{\int_0^1 \mathrm{d}\lambda\, \lambda^k}{k!}\, \mathscr{L}^k_{S_n^{\varepsilon,\eta}(s)} [\![\dot{S}_n^{\varepsilon,\eta}(s)]\!] \\
&\qquad + \eta \int_0^1 \mathrm{d}\lambda \int_0^\lambda \mathrm{d}\mu\, \frac{(\lambda-\mu)^{n-2}}{(n-2)!}\, \mathrm{e}^{\mu \mathscr{L}_{S_n^{\varepsilon,\eta}(s)}} \circ \mathscr{L}^{n-1}_{S_n^{\varepsilon,\eta}(s)} [\![\dot{S}_n^{\varepsilon,\eta}(s)]\!], \\
&\quad = \eta \sum_{k=0}^{n-2} \frac{1}{(k+1)!}\, \mathscr{L}^k_{S_n^{\varepsilon,\eta}(s)} [\![\dot{S}_n^{\varepsilon,\eta}(s)]\!] \\
&\qquad + \eta \int_0^1 \mathrm{d}\mu\, \frac{(1-\mu)^{n-1}}{(n-1)!}\, \mathrm{e}^{\mu \mathscr{L}_{S_n^{\varepsilon,\eta}(s)}} \circ \mathscr{L}^{n-1}_{S_n^{\varepsilon,\eta}(s)} [\![\dot{S}_n^{\varepsilon,\eta}(s)]\!],
\end{aligned}
\tag{84}
$$

From this expansion and (77), we can read of

$$
R_1^{\varepsilon,\eta}(s) = -\mathscr{L}_{H_0} [\![A_1^{\varepsilon,\eta}(s)]\!] + V(s), \tag{85}
$$

$$
R_2^{\varepsilon,\eta}(s) = -\mathscr{L}_{H_0} [\![A_2^{\varepsilon,\eta}(s)]\!] + \tfrac{1}{2}\, \mathscr{L}^2_{A_1^{\varepsilon,\eta}(s)} [\![H_0]\!] + \mathscr{L}_{A_1^{\varepsilon,\eta}(s)} [\![V(s)]\!] + \tfrac{\eta}{\varepsilon}\, \dot{A}_1^{\varepsilon,\eta}(s), \tag{86}
$$

and more generally

$$R_j^{\varepsilon,\eta}(s) = -\mathscr{L}_{H_0}[\![A_j^{\varepsilon,\eta}(s)]\!] + \tilde{R}_j^{\varepsilon,\eta}(s), \tag{87}$$

where the $\tilde{R}_j^{\varepsilon,\eta}(s)$ are sums of iterated commutators of the operators $A_i^{\varepsilon,\eta}(s)$ and $\dot{A}_i^{\varepsilon,\eta}(s)$ for $i < j \leq n$ and $V(s)$. We can now iteratively choose

$$A_j^{\varepsilon,\eta}(s) = \mathscr{I}_{H_0,g}\big[\!\big[\tilde{R}_j^{\varepsilon,\eta}(s)\big]\!\big].$$

Clearly, for all $p' < p$, it holds that $\tilde{R}_1^{\varepsilon,\eta}(s) = V(s)$ is a $(b, p', \Lambda^{\mathrm{pert}})$-localized SLT operator. Hence, by Lemma A.11, $A_1^{\varepsilon,\eta}(s)$ is $(b', p', \Lambda^{\mathrm{pert}})$-localized for any $b' < b$. This step only works for $p' < p$, because Lemma A.11 requires a slightly better localization of the Hamiltonian H_0 compared to the argument of the inverse Liouvillian. Finally, by Lemma A.5 also $\mathscr{L}_{H_0}[\![A_1^{\varepsilon,\eta}(s)]\!]$ and thus $R_1^{\varepsilon,\eta}(s)$ are $(b', p', \Lambda^{\mathrm{pert}})$-localized SLT operator for any slightly smaller b'. The same arguments hold for the higher orders $R_j^{\varepsilon,\eta}(s)$ with $j \leq n$ as well, at each step lowering b'. We point out that clearly the smallest b' can be chosen independently of n by choosing all intermediate b' in an n-dependent equidistant way.

With this expansion it is also immediate, that the $A_\mu^{\varepsilon,\eta}(s)$ only depend on V and its derivatives at time s. Moreover, they are polynomials in η/ε of degree $j-1$ with ε- and η-independent, time-dependent $(b', p', \Lambda^{\mathrm{pert}})$-localized SLT operators as coefficients. Thus, we denote them $A_\mu^{\eta/\varepsilon}(s) = A_\mu^{\varepsilon,\eta}(s)$ in the statement.

Putting everything back together and denoting $\tau_{t,s}[\![B]\!] := U_{s,t}^{\varepsilon,\eta}\, B\, U_{t,s}^{\varepsilon,\eta}$, we find

$$\begin{aligned} &\left|\langle B\rangle_{\rho_{t_0}^{\varepsilon,\eta}(t)} - \langle B\rangle_{\Pi_n^{\varepsilon,\eta}(t)}\right| \\ &\leq \frac{|t-t_0|}{\eta} \sup_{s\in[t_0,t]} \left(\begin{aligned} &\sum_{j=1}^n \varepsilon^j \left|\left\langle\left[\mathscr{L}_{H_0}\circ \mathscr{I}_{H_0,g}\big[\!\big[\tilde{R}_j^{\varepsilon,\eta}(s)\big]\!\big] - \tilde{R}_j^{\varepsilon,\eta}(s), \mathrm{e}^{\mathscr{L}_{S_n^{\varepsilon,\eta}(s)}}\circ\tau_{t,s}[\![B]\!]\right]\right\rangle_{\rho_0}\right| \\ &+\varepsilon^{n+1}\left|\left\langle\left[R_{n+1}^{\varepsilon,\eta}(s), \mathrm{e}^{\mathscr{L}_{S_n^{\varepsilon,\eta}(s)}}\circ\tau_{t,s}[\![B]\!]\right]\right\rangle_{\rho_0}\right|. \end{aligned}\right) \end{aligned} \tag{88}$$

The remainder $R_{n+1}^{\varepsilon,\eta}(s)$ collects all the remaining terms which are (a) the higher order terms from the first lines and (b) the remainder terms from the second lines of (83) and (84). The former are local by the previous arguments. The latter additionally include an evolution by the local operator $S_n^{\varepsilon,\eta}(s)$ and are local by Lemma A.8.

To apply the local gap assumption in the form given in Lemma 6.1 to the lower order terms, we need to decompose the second entry of the commutator into strictly local operators. Therefore, we use the same decomposition as in the proof of Lemma A.7

$$\tau_{t,s}[\![B]\!] = \sum_{k=0}^{\infty} \Delta_k,$$

where

$$\begin{aligned} & && \Delta_0 \in \mathcal{A}_{Y_\delta} \subset \mathcal{A}_{Y_{\delta^{1/p}}}, && \|\Delta_0\| \le \|B\|, \\ & \text{and for } k \in \mathbb{N} && \Delta_k \in \mathcal{A}_{Y_{(\delta+k)^{1/p}}}, && \|\Delta_k\| \le C_{\mathrm{LR}}\, \mathrm{e}\, \|B\|\, |Y|\, \mathrm{e}^{-bk}. \end{aligned}$$

For better readability we abbreviate $\delta = 1 + v\,|t-s|/\eta$ here. The extra $1/\eta$ is due to the scaling in (80). For the outer automorphism, we can use the decomposition from the proof of Lemma A.8, just replace $O = \Delta_k$ there, to obtain, for each k, the decomposition

$$\mathrm{e}^{\mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} [\![\Delta_k]\!] = \sum_{l=0}^{\infty} \Delta_{k,l}$$

with

$$\begin{aligned} &\Delta_{k,0} \in \mathcal{A}_{Y_{(\delta+k)^{1/p}}}, && \|\Delta_{k,0}\| \le \|\Delta_k\|, \quad \text{and} \\ &\Delta_{k,l} \in \mathcal{A}_{Y_{l+(\delta+k)^{1/p}}}, && \|\Delta_{k,l}\| \le C_{\mathrm{LR}}\, \|\Delta_k\|\, \left|Y_{(\delta+k)^{1/p}}\right| \left(\mathrm{e}^{c\,\|\Phi_S\|_{d,q}} - 1\right) \chi_{d',j}(l) \end{aligned}$$

for $l \in \mathbb{N}$. Thus, in total we have

$$\mathrm{e}^{\mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} \circ \tau_{t,s}[\![B]\!] = \sum_{k,l=0}^{\infty} \Delta_{k,l},$$

where the sum is actually finite on finite lattices as discussed in Lemmata A.7 and A.8. Then we use the triangle inequality and Lemma 6.1 to apply (75) and bound

$$\begin{aligned} &\left|\left\langle\left[\mathcal{L}_{H_0} \circ \mathcal{I}_{H_0,g}[\![\tilde{R}_j^{\varepsilon,\eta}(s)]\!] - \tilde{R}_j^{\varepsilon,\eta}(s), \mathrm{e}^{\mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} \circ \tau_{t,s}[\![B]\!]\right]\right\rangle_{\rho_0}\right| \\ &\quad \le C\, \|\Phi_{\tilde{R}_j^{\varepsilon,\eta}}(s)\|_{b',p';\Lambda^{\mathrm{pert}}} \sum_{k,l=0}^{\infty} \mathrm{diam}(Y_{l+(\delta+k)^{1/p}})^{\ell+d}\, \|\Delta_{k,l}\|\, \chi_{b'/2,q'}\big(d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) \\ &\quad \le C\, \|\Phi_{\tilde{R}_j^{\varepsilon,\eta}}(s)\|_{b',p';\Lambda^{\mathrm{pert}}} \sum_{k=0}^{\infty} \mathrm{diam}(Y_{(\delta+k)^{1/p}})^{\ell+2d}\, \|\Delta_k\| \\ &\qquad \times \left(1 + \left(\mathrm{e}^{c\,\|\Phi_S\|_{d,q}} - 1\right) \sum_{l=1}^{\infty} l^{\ell+d}\, \chi_{d',j}(l)\right) \chi_{b'/2,q'}\big(d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) \\ &\quad \le C\, \|\Phi_{\tilde{R}_j^{\varepsilon,\eta}}(s)\|_{b',p';\Lambda^{\mathrm{pert}}}\, \mathrm{e}^{c\,\|\Phi_S\|_{d,q}}\, \mathrm{diam}(Y)^{\ell+3d}\, \|B\|\, \delta^{(\ell+2d)/p} \\ &\qquad \times \sum_{k=0}^{\infty} k^{(\ell+2d)/p}\, \mathrm{e}^{-bk}\, \chi_{b'/2,q'}\big(d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) \\ &\quad \le C\, \|\Phi_{\tilde{R}_j^{\varepsilon,\eta}}(s)\|_{b',p';\Lambda^{\mathrm{pert}}}\, \mathrm{e}^{c\,\|\Phi_S\|_{d,q}}\, \mathrm{diam}(Y)^{\ell+3d}\, \|B\|\, \delta^{(\ell+2d)/p}\, \chi_{b'/2,q'}\big(d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})\big), \end{aligned}$$

where we abbreviated $q' := p' \min\{\beta, 1\}$ (according to Lemma 6.1) and b' is the smallest of the b' such that all $\Phi_{\tilde{R}_j^{\varepsilon,\eta}}(s)$ are $(b', p', \Lambda^{\mathrm{pert}})$-localized. Finally, for any $q < q'$ we bound

$$\chi_{b'/2,q'}\big(d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) \le C\, \chi_{1,q}\big(d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) \tag{89}$$

for some constant $C > 0$ depending only on b', q', and q. Since $p' < p$ was arbitrary, this q can be chosen arbitrarily in the interval $(0, p \min\{\beta, 1\})$.

With the same local decomposition argument as above and using Lemma A.4 we can also bound the remainder

$$\begin{aligned} &\left| \left\langle \left[R_{n+1}^{\varepsilon,\eta}(s), \mathrm{e}^{\mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} \circ \tau_{t,s}[\![B]\!] \right] \right\rangle_{\rho_0} \right| \\ &\qquad \le C\, \|\Phi_{\tilde{R}_{n+1}^{\varepsilon,\eta}}(s)\|_{b',p';\Lambda^{\mathrm{pert}}}\, \mathrm{e}^{c\, \|\Phi_S\|_{d,q}}\, \operatorname{diam}(Y)^{3d}\, \|B\|\, \delta^{2d/p}. \end{aligned}$$

We recall, that like the $R_j^{\varepsilon,\eta}(s)$, also the $\tilde{R}_j^{\varepsilon,\eta}(s)$ are polynomials of degree $j-1$ in η/ε with Λ^{pert}-localized SLT operators as coefficients. Hence, the interaction norms in the above equations can be bounded by a constant depending on the interaction norms of H and $V(t)$ times $1 + (\eta/\varepsilon)^{j-1}$. With this observation, we can insert these bounds into (88) and conclude (79).

To reduce the number of constants in the formulation of the statement, we actually do the proof for $(p+p')/2$ instead of p'. In the end we then estimate $\chi_{b',(p+p')/2} \le C\, \chi_{1,p'}$ for the locality of the operators $A_\mu^{\varepsilon,\eta}(t)$, similarly to (89). □

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A Technical lemmata

In this section, we prove the technical lemmata required for the construction of the NEASS. We begin with some general properties of the functions $\chi_{b,p}$ and Lieb-Robinson bounds for (b, p)-localized SLT-operators in Appendix A.1. In Appendix A.2, we prove that the various operations used in the construction of the NEASS preserve locality. In Appendix A.3 we recall the construction of the quasi-local inverse of the Liouvillian and prove that it also preserves locality of SLT-operators. Finally, Appendices A.4 and A.5 are concerned with the proofs of Lemmata 2.1 and 6.1, respectively.

In all proofs, $C > 0$ is a generic constant that might change within the computations. It can in particular depend on all the parameters chosen in the statements, but it is uniform in the chosen lattice and the operators appearing.

A.1 Properties of the decay function $\chi_{b,p}$

Let us first collect some properties of the decay function $\chi_{b,p}$ we use in the definition of the interaction norm. From [156, Lemma 7.2.3] we have the following Lemma, where we simplified the statements.

Lemma A.1. *For any $b \geq 0$ and $s \in (0, 1]$, the function $\chi_{b,p}$ satisfies the following properties:*

(a) *$\chi_{b,p}$ is logarithmically superadditive, i.e. $\chi_{b,p}(x + y) \geq \chi_{b,p}(x)\,\chi_{b,p}(y)$ for all x, $y \geq 0$.*

(b) *For every $b > 0$ and $k \geq 0$ there exists a constant $C > 0$ such that*

$$\sup_{x\geq 0} x^k\,\chi_{b,p}(x) = C.$$

As a direct consequence of Lemma A.1 (b), we get the following Lemma, which we write out to fix the constant and recall it in later proofs.

Lemma A.2. *Let $d \in \mathbb{N}$, $C_{\mathrm{vol}} > 0$, $b > 0$, $p \in (0, 1]$ and $k \in \mathbb{N}$. Then there exists a constant $\mathscr{C}_{\mathrm{vol},b,p,k} > 0$ such that for all lattices $\Lambda \in \mathscr{G}(d, C_{\mathrm{vol}})$ and sets $Z \subset \Lambda$*

$$|Z|^k\,\chi_{b,p}\big(\mathrm{diam}(Z)\big) \leq \mathscr{C}_{\mathrm{vol},b,p,k}.$$

Before we state the Lieb-Robinson bound, which is a crucial ingredient in the proof, let us briefly recall the time-dependent Heisenberg evolution. For a time-dependent interaction defined on an interval $I \subset \mathbb{R}$ with corresponding SLT operator $H(t)$, let $\tau_{t,s}$ be the unique solution of

$$-\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\tau_{t,s}(A) = \tau_{t,s}\big([H(t), A]\big) \qquad \text{and} \qquad \tau_{s,s} = \mathrm{Id} \qquad \text{for all } s, t \in I.$$

This was already used with a different time scaling for the Hamiltonian $H_\varepsilon(t)$ in the proof of Theorem 6.2. Under locality assumptions on the Hamiltonian, one finds the following Lieb-Robinson bound.

Lemma A.3 (Lieb-Robinson bound [156, Theorem 7.3.3]). *Let $d \in \mathbb{N}$, $C_{\mathrm{vol}} > 0$, $b' > b > 0$, $p \in (0,1]$ and $k \in \mathbb{N}$. There exists constants C and $c > 0$ such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$, intervals $I \subset \mathbb{R}$, time-dependent interactions Φ, disjoint subsets $X, Y \subset \Lambda$, observables $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, and $s, t \in I$ it holds that*

$$\big\|\big[\tau_{t,s}(A), B\big]\big\| \le C\,\|A\|\,\|B\|\left(\mathrm{e}^{b\,v\,|t-s|} - 1\right) D(X,Y), \tag{90}$$

where $v = c\,\|\Phi\|_{b',p}/b$ is the Lieb-Robinson velocity and

$$\begin{aligned} D(X,Y) :=& \min\Big\{\sum_{x\in X} \chi_{b,p}\big(d(x,Y)\big), \sum_{y\in Y} \chi_{b,p}\big(d(y,X)\big)\Big\} \\ \le& \min\big\{|X|, |Y|\big\}\, \chi_{b,p}\big(d(X,Y)\big). \end{aligned}$$

The Lieb-Robinson velocity is defined including the $1/b$ because (90) can be bounded by

$$C\,\|A\|\,\|B\|\,\min\big\{|X|,|Y|\big\}\,\mathrm{e}^{b\left(v\,|t-s| - d(X,Y)^p\right)}.$$

A.2 Commutators and dynamics of localized SLT-operators

Lemma A.4 (Commutator with local observable). *Let $d \in \mathbb{N}$, $C_{\mathrm{vol}} > 0$, $b > 0$, $p \in (0,1]$ and $k \in \mathbb{N}$. There exists constants C and $C_k > 0$ such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$, subsets Ω, $X \subset \Lambda$, SLT operators A_1 and observables $O \in \mathcal{A}_X$ it holds that*

$$\big\|[A_1, O]\big\| \le 2\,\|O\|\,|X|\,\chi_{b,p}\big(d(X,\Omega)\big)\,\|\Phi_{A_1}\|_{b,p;\Omega}. \tag{91}$$

For a second observable $\tilde{O} \in \mathcal{A}_Y$, it holds that

$$\big\|[[A_1, O], \tilde{O}]\big\| \le 4\,\|O\|\,\|\tilde{O}\|\,|X|\,\chi_{b,p}\big(d(X,Y)\big)\,\chi_{b,p}\big(d(X,\Omega)\big)\,\|\Phi_{A_1}\|_{b,p;\Omega}. \tag{92}$$

Finally, if additionally also $A_2, \dots, A_k$ are SLT operators, then

$$\big\|\mathrm{ad}_{A_k} \cdots \mathrm{ad}_{A_1}(O)\big\| \le C_k\,\|O\|\,|X|^k\,\chi_{b,p}\big(d(X,\Omega)\big)\,\|\Phi_{A_1}\|_{b,p;\Omega} \prod_{j=2}^{k} \|\Phi_{A_j}\|_{b,p}. \tag{93}$$

All three bounds, in particular, also hold for $\Omega = \Lambda$, where $\|\cdot\|_{b,p;\Omega} = \|\cdot\|_{b,p}$ and $d(X,\Omega) = 0$.

Proof. We begin with the first statement and write $A = A_1$. Since $[\Phi_A(Z), O]$ vanishes whenever $Z \cap X = \emptyset$, we find

$$\begin{aligned}\big\|[A,O]\big\| &\leq \sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\emptyset}} 2\,\big\|\Phi_A(Z)\big\|\,\|O\| \\ &\leq 2\,\|O\| \sum_{z\in X}\sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \frac{\big\|\Phi_A(Z)\big\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)}\,\chi_{b,p}\big(d(z,\Omega)\big) \\ &\leq 2\,\|O\| \sum_{z\in X} \chi_{b,p}\big(d(z,\Omega)\big)\,\|\Phi_A\|_{b,p;\Omega} \\ &\leq 2\,\|O\|\,|X|\,\chi_{b,p}\big(d(X,\Omega)\big)\,\|\Phi_A\|_{b,p;\Omega},\end{aligned}$$

where we just overcount in the second inequality. Clearly, the same statement also holds with $\Omega = \Lambda$. The proof of (92) is analogous to the proof of (91) and so omitted.

We conclude by proving (93) using induction. Note that the outer operators are all SLT-operators on Λ. The $k = 1$ case is given in (91). We now assume (93) for some fixed k and with $\Omega = \Lambda$. Then, we add a further commutator with A_0 to conclude

$$\begin{aligned}&\big\|\mathrm{ad}_{A_k}\cdots\mathrm{ad}_{A_1}\,\mathrm{ad}_{A_0}(O)\big\| \\ &\quad\leq \sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\emptyset}} \big\|\mathrm{ad}_{A_k}\cdots\mathrm{ad}_{A_1}[\Phi_{A_0}(Z),O]\big\| \\ &\quad\leq \sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\emptyset}} C_k\,\big\|[\Phi_{A_0}(Z),O]\big\|\,|X\cup Z|^k \prod_{j=1}^{k}\|\Phi_{A_j}\|_{b,p} \\ &\quad\leq C_k\,2^k\,\|O\|\,|X|^k \sum_{z\in X}\chi_{b,p}\big(d(z,\Omega)\big)\prod_{j=1}^{k}\|\Phi_{A_j}\|_{b,p} \\ &\qquad\times\sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \frac{\big\|\Phi_{A_0}(Z)\big\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)}\,\chi_{b,p}\big(\mathrm{diam}(Z)\big)\,|Z|^k \\ &\quad\leq \mathscr{C}_{\mathrm{vol},b,p,k}\,C_k\,2^k\,\|O\|\,|X|^{k+1}\,\chi_{b,p}\big(d(X,\Omega)\big)\,\|\Phi_{A_0}\|_{b,p;\Omega}\prod_{j=1}^{k}\|\Phi_{A_j}\|_{b,p},\end{aligned}$$

where we used Lemma A.2 in the last step. This finishes the induction. □

Lemma A.5 (Multi-commutators)**.** *Let $d \in \mathbb{N}$, $C_{\mathrm{vol}} > 0$, $b > 0$, $p \in (0,1]$, $\varepsilon > 0$ and $k \in \mathbb{N}$. There exists a constant $C > 0$ such that for all lattices $\Lambda \in \mathscr{G}(d, C_{\mathrm{vol}})$, subsets*

$\Omega \subset \Lambda$ and SLT operators $A_0, \dots, A_k$ it holds that

$$\left\|\Phi_{\mathrm{ad}_{A_k} \cdots \mathrm{ad}_{A_1}(A_0)}\right\|_{b,p;\Omega} \le C \left\|\Phi_{A_0}\right\|_{b+\varepsilon,p;\Omega} \prod_{j=1}^{k} \left\|\Phi_{A_j}\right\|_{2b+\varepsilon,p}. \tag{94}$$

Proof. For the proof we first need to construct an interaction for the commutator of two SLT operators A and B. It turns out that it can be given as

$$\Phi_{[A,B]}(Z) = \sum_{\substack{X,Y \subset \Lambda: \\ X \cup Y = Z \\ X \cap Y \neq \emptyset}} \left[\Phi_A(X), \Phi_B(Y)\right].$$

Then,

$$\begin{aligned}
&\sum_{\substack{Z \subset \Lambda: \\ z \in Z}} \frac{\left\|\Phi_{[A,B]}(Z)\right\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)\, \chi_{b,p}\big(d(z,\Omega)\big)} \\
&\qquad \le \sum_{\substack{Z \subset \Lambda: \\ z \in Z}} \sum_{\substack{X,Y \subset \Lambda: \\ X \cup Y = Z \\ X \cap Y \neq \emptyset}} \frac{2 \left\|\Phi_A(X)\right\| \left\|\Phi_B(Y)\right\|}{\chi_{b,p}\big(\mathrm{diam}(X)\big)\, \chi_{b,p}\big(\mathrm{diam}(Y)\big)\, \chi_{b,p}\big(d(z,\Omega)\big)},
\end{aligned}$$

where we used $\mathrm{diam}(Z) \le \mathrm{diam}(X) + \mathrm{diam}(Y)$ and the properties of $\chi_{b,p}$. The above sum can be bounded by the sum of the terms where $z \in X$ or $z \in Y$. The latter can be upper bounded by

$$\begin{aligned}
&2 \sum_{\substack{Y \subset \Lambda: \\ z \in Y}} \frac{\left\|\Phi_B(Y)\right\|}{\chi_{b,p}\big(\mathrm{diam}(Y)\big)\, \chi_{b,p}\big(d(z,\Omega)\big)} \sum_{x \in Y} \sum_{\substack{X \subset \Lambda: \\ x \in X}} \frac{\left\|\Phi_A(X)\right\|}{\chi_{b,p}\big(\mathrm{diam}(X)\big)} \\
&\quad \le 2 \sum_{\substack{Y \subset \Lambda: \\ z \in Y}} \frac{\left\|\Phi_B(Y)\right\|}{\chi_{b+\varepsilon,p}\big(\mathrm{diam}(Y)\big)\, \chi_{b+\varepsilon,p}\big(d(z,\Omega)\big)} \chi_{\varepsilon,p}\big(\mathrm{diam}(Y)\big)\, |Y| \left\|\Phi_A\right\|_{b,p} \\
&\quad \le C\, \mathscr{C}_{\mathrm{vol},\varepsilon,p,1} \sum_{\substack{Y \subset \Lambda: \\ z \in Y}} \frac{\left\|\Phi_B(Y)\right\|}{\chi_{b+\varepsilon,p}\big(\mathrm{diam}(Y)\big)\, \chi_{b+\varepsilon,p}\big(d(z,\Omega)\big)} \left\|\Phi_A\right\|_{b,p} \\
&\quad \le C \left\|\Phi_B\right\|_{b+\varepsilon,p;\Omega} \left\|\Phi_A\right\|_{b,p},
\end{aligned}$$

where we used Lemma A.2 in the third step.

Using $\chi_{b,p}\big(d(z,\Omega)\big) \ge \chi_{b,p}\big(\mathrm{diam}(X)\big)\, \chi_{b,p}\big(d(x,\Omega)\big)$ for all $z, x \in X$, the part of the sum where $z \in X$ can be bounded by

$$
\begin{aligned}
&2 \sum_{\substack{X\subset\Lambda:\\ z\in X}} \frac{\left\|\Phi_A(X)\right\|}{\chi_{2b,p}\big(\operatorname{diam}(X)\big)} \sum_{y\in X} \sum_{\substack{Y\subset\Lambda:\\ y\in Y}} \frac{1}{\chi_{b,p}\big(d(y,\Omega)\big)} \frac{\left\|\Phi_B(Y)\right\|}{\chi_{b,p}\big(\operatorname{diam}(Y)\big)} \\
&\quad\le 2 \sum_{\substack{X\subset\Lambda:\\ z\in X}} \frac{\left\|\Phi_A(X)\right\|}{\chi_{2b+\varepsilon,p}\big(\operatorname{diam}(X)\big)} \chi_{\varepsilon,p}\big(\operatorname{diam}(X)\big)\, |X| \left\|\Phi_B\right\|_{b,p;\Omega} \\
&\quad\le C \left\|\Phi_A\right\|_{2b+\varepsilon,p} \left\|\Phi_B\right\|_{b,p;\Omega}.
\end{aligned}
$$

Both bounds together prove the claim for $k = 1$. To proceed by induction we assume that the statement holds for some fixed k. Then, applying first the statement for $k = 1$ and then $k = k$ both with $\varepsilon/2$ we obtain

$$
\begin{aligned}
\left\|\Phi_{\operatorname{ad}_{A_{k+1}}\cdots\operatorname{ad}_{A_1}(A_0)}\right\|_{b,p;\Omega} &\le C \left\|\Phi_{\operatorname{ad}_{A_k}\cdots\operatorname{ad}_{A_1}(A_0)}\right\|_{b+\varepsilon/2,p;\Omega} \left\|\Phi_{A_{k+1}}\right\|_{2b+\varepsilon/2,p} \\
&\le C \left\|\Phi_{A_0}\right\|_{b+\varepsilon,p;\Omega} \prod_{j=1}^{k+1} \left\|\Phi_{A_j}\right\|_{2b+\varepsilon,p}.
\end{aligned}
$$
□

For the following statements we need to approximate the time evolution of local operators, which in principle live on the whole lattice. This can be done by a so-called conditional expectation, which is just the partial trace in our case of finite spin systems. We collect its properties in the following lemma.

Lemma A.6 ([174, Lemma 4.1]). *Let Λ be a lattice and $X \subset \Lambda$. Then there exists a unit-preserving, completely positive linear map $\mathbb{E}_X : \mathscr{A}_\Lambda \to \mathscr{A}_\Lambda$ satisfying*

(a) $\mathbb{E}_X(\mathscr{A}_\Lambda) \subset \mathscr{A}_X$;

(b) $\mathbb{E}_X(ABC) = A\,\mathbb{E}_X(B)\,C$ *for all* $B \in \mathscr{A}_\Lambda$ *and* $A, C \in \mathscr{A}_X$; *This in particular implies* $\mathbb{E}_X(A) = A$ *for all* $A \in \mathscr{A}_X$;

(c) $\|\mathbb{E}_X\| = 1$;

(d) $\mathbb{E}_X \circ \mathbb{E}_Y = \mathbb{E}_{X\cap Y}$, *for* $X, Y \subset \Lambda$;

(e) *If* $A \in \mathscr{A}_\Lambda$ *satisfies*

$$
\big\|[A,B]\big\| \le \eta\,\|A\|\,\|B\| \quad \text{for all} \quad B \in \mathscr{A}_{\Lambda\setminus X}, \tag{95}
$$

for some $\eta > 0$, *then*

$$
\|A - \mathbb{E}_X(A)\| \le \eta\,\|A\|. \tag{96}
$$

Together with the Lieb-Robinson bound we can now obtain the following.

Lemma A.7 (Dynamics). *Let $d \in \mathbb{N}$, $C_{\mathrm{vol}} > 0$, $b, b' > 0$, and $p, p' \in (0,1]$ satisfying $p' < 1$ or $b' < b$. There exists constants C and $c > 0$ such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$, the following holds: Let $I \subset \mathbb{R}$ an interval, the interaction Φ generate the dynamics $\tau_{t,s}$ with Lieb-Robinson velocity $v = c\,\|\Phi\|_{b,p}/b$. For every $\chi_{b',p'}$-SLT operator A, subsets $\Omega, X \subset \Lambda$, observables $O \in \mathcal{A}_X$, and $t, s \in I$ it holds that*

$$\left\|\left[A, \tau_{t,s}(O)\right]\right\| \leq C\,\|O\|\,|X|\,|X_{(2v\,|t-s|)^{1/p}}|\,\|\Phi_A\|_{b',p';\Omega}\,\chi_{b',p'}\big(d(X_{(2v\,|t-s|)^{1/p}}, \Omega)\big)$$

Proof. We use the local decomposition technique similar to [174, Section 5]. Therefore, let

$$\Delta_0 := \mathbb{E}_{X_{v\,|t-s|}}\big(\tau_{t,s}(O)\big)$$

and

$$\Delta_k := \mathbb{E}_{X_{(v\,|t-s|+k)^{1/p}}}\big(\tau_{t,s}(O)\big) - \mathbb{E}_{X_{(v\,|t-s|+k-1)^{1/p}}}\big(\tau_{t,s}(O)\big),$$

so that $\tau_{t,s}(O) = \sum_{k=0}^{\infty} \Delta_k$, where the sum is finite since eventually $X_{(v\,|t-s|+k)^{1/p}} = \Lambda$. By the properties of the conditional expectation

$$\|\Delta_0\| \leq \|\tau_{t,s}(O)\| = \|O\|$$

and for $k \geq 1$ it holds that

$$\Delta_k = \mathbb{E}_{X_{(v\,|t-s|+k)^{1/p}}}\Big(\big(1 - \mathbb{E}_{X_{(v\,|t-s|+k-1)^{1/p}}}\big)\,\tau_{t,s}(O)\Big)$$

and thus

$$\|\Delta_k\| \leq \Big\|\big(1 - \mathbb{E}_{X_{(v\,|t-s|+k-1)^{1/p}}}\big)\,\tau_{t,s}(O)\Big\|.$$

Furthermore, for all $B \in \mathcal{A}_{\Lambda \setminus X_{(v\,|t-s|+k-1)^{1/p}}}$ by the Lieb-Robinson bound (Lemma A.3)

$$\left\|[\tau_{t,s}(O), B]\right\| \leq C_{\mathrm{LR}}\,\|O\|\,\|B\|\,|X|\,\mathrm{e}^{b\big(v\,|t-s| - (v\,|t-s|+k-1)\big)} = C_{\mathrm{LR}}\,\mathrm{e}\,\|O\|\,\|B\|\,|X|\,\mathrm{e}^{-bk}$$

because $d(X, \Lambda \setminus X_{v\,|t-s|+k-1}) \geq v\,|t-s| + k - 1$, and thus by Lemma A.6

$$\|\Delta_k\| \leq C_{\mathrm{LR}}\,\mathrm{e}\,\|O\|\,|X|\,\mathrm{e}^{-bk}.$$

Now we can apply Lemma A.4 to each of the summands in the decomposition

$$
\begin{aligned}
\|[A,\tau_{t,s}(O)]\| &\leq \sum_{k=0}^{\infty} \|[A,\Delta_k]\| \\
&\leq \sum_{k=0}^{\infty} 2\,\|\Delta_k\|\,\big|X_{(v\,|t-s|+k)^{1/p}}\big|\,\chi_{b',p'}\big(d(X_{(v\,|t-s|+k)^{1/p}},\Omega)\big)\,\|\Phi_A\|_{b',p';\Omega} \\
&\leq \tilde{C}\,\|O\|\,|X|\,\|\Phi_A\|_{b',p';\Omega} \sum_{k=0}^{\infty} \big|X_{(v\,|t-s|+k)^{1/p}}\big|\,\chi_{b',p'}\big(d(X_{(v\,|t-s|+k)^{1/p}},\Omega)\big)\,\mathrm{e}^{-bk}. \\
&\leq \tilde{C}\,\|O\|\,|X|\,|X_{(2v\,|t-s|)^{1/p}}|\,\|\Phi_A\|_{b',p';\Omega}\,\chi_{b',p'}\big(d(X_{(2v\,|t-s|)^{1/p}},\Omega)\big) \\
&\qquad \times \sum_{k=0}^{\infty} (1+C_{\mathrm{vol}}\,(2k)^{d/p})\,\chi_{b',p'}^{-1}(k)\,\mathrm{e}^{-bk},
\end{aligned}
$$

where we abbreviated $\tilde{C} = 2\max\{1, C_{\mathrm{LR}}\,\mathrm{e}\}$ and used $(v\,|t-s|+k)^{1/p} \leq (2v\,|t-s|)^{1/p} + (2k)^{1/p}$. To conclude the result, we observe, that the series is bounded for $p' < 1$ or $b' < b$ if $p' = 1$. □

Lemma A.8 (Conjugation with unitaries). *Let $d \in \mathbb{N}$, $C_{\mathrm{vol}} > 0$, $\varepsilon > 0$, $a, b > 0$, and $p, q \in (0,1]$ satisfying $p < q$ or $p = q$ and $a > (2^p + 1)\,b$. There exists constants C and $c > 0$ such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$, the following holds: For SLT operators D and S it holds that $A := \mathrm{e}^{\mathrm{i}\,S}\,D\,\mathrm{e}^{-\mathrm{i}\,S}$ is an SLT operator as well. More precisely, there exists an interaction Φ_A such that*

$$
\|\Phi_A\|_{b,p;\Omega} \leq C\,\mathrm{e}^{c\,\|\Phi_S\|_{a,q}}\,\|\Phi_D\|_{b+\varepsilon,p;\Omega}
$$

Proof. The proof uses the same technique as the proof of Lemma A.7.

First fix $X \subset \Lambda$ and $O \in \mathcal{A}_X$ and denote $\tau(O) = \mathrm{e}^{\mathrm{i}S}\,O\,\mathrm{e}^{-\mathrm{i}S}$. Then define

$$
\Delta_0(O) := \mathbb{E}_X\big(\tau(O)\big)
$$

and

$$
\Delta_k(O) := \mathbb{E}_{X_k}\big(\tau(O)\big) - \mathbb{E}_{X_{k-1}}\big(\tau(O)\big) = \mathbb{E}_{X_k}\Big(\big(\mathrm{id} - \mathbb{E}_{X_{k-1}}\big)\,\tau(O)\Big).
$$

By properties of the conditional expectation, Lemma A.6, and the Lieb-Robinson bound, Lemma A.3, one can bound

$$
\|\Delta_0(O)\| \leq \|O\|
$$

and

$$
\|\Delta_k(O)\| \leq C_{\mathrm{LR}}\,\|O\|\,|X|\,\big(\mathrm{e}^{c\,\|\Phi_S\|_{a,q}} - 1\big)\,\chi_{a',q}(k) \tag{97}
$$

for $k \geq 1$ because $d(X, \Lambda \setminus X_{k-1}) = k$ in our geometry, where we chose $a' < a$.

We now construct an interaction for A. First note, that

$$A = \tau(A) = \sum_{Z\subset\Lambda} \tau\big(\Phi_D(Z)\big) = \sum_{Z\subset\Lambda}\sum_{k=0}^{\infty} \Delta_k\big(\Phi_D(Z)\big)$$

where $\Delta_k\big(\Phi_A(Z)\big) \in \mathscr{A}_{Z_k}$ and the sum is actually finite. For any function $f\colon \{\Omega \subset \Lambda\} \to \mathscr{A}_\Lambda$ and $k \geq 0$ it holds that

$$\sum_{Y\subset\Lambda} f(Y) = \sum_{Y\subset\Lambda}\Big(\sum_{Z\subset\Lambda} \mathbb{1}_{Z=Y_k}\Big) f(Y) = \sum_{Z\subset\Lambda}\sum_{\substack{Y\subset\Lambda:\\ Y_k=Z}} f(Y).$$

Applying this with $f\colon Z \mapsto \Delta_k(\Phi_D(Z))$ for each k we find

$$A = \sum_{k=0}^{\infty}\sum_{Z\subset\Lambda}\sum_{\substack{Y\subset\Lambda:\\ Y_k=Z}} \Delta_k\big(\Phi_D(Y)\big) = \sum_{Z\subset\Lambda}\Phi_A(Z) \quad \text{with} \quad \Phi_A(Z) := \sum_{k=0}^{\infty}\sum_{\substack{Y\subset\Lambda:\\ Y_k=Z}} \Delta_k\big(\Phi_D(Y)\big).$$

With this interaction for A and any $z \in \Lambda$ we bound

$$\begin{aligned}\sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \frac{\|\Phi_A(Z)\|}{\chi_{b,p}\big(\operatorname{diam}(Z)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)} &\leq \sum_{\substack{Z\subset\Lambda:\\ z\in Z}}\sum_{k=0}^{\infty}\sum_{\substack{Y\subset\Lambda:\\ Y_k=Z}} \frac{\big\|\Delta_k\big(\Phi_D(Y)\big)\big\|}{\chi_{b,p}\big(\operatorname{diam}(Z)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)}\\ &= \sum_{k=0}^{\infty}\sum_{Y\subset\Lambda} \mathbb{1}_{z\in Y_k} \frac{\big\|\Delta_k\big(\Phi_D(Y)\big)\big\|}{\chi_{b,p}\big(\operatorname{diam}(Y_k)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)}.\end{aligned}$$

The $k = 0$ term is bounded by $\|\Phi_D\|_{b,p;\Omega}$. For $k \geq 0$ we use (97). Moreover, $\operatorname{diam}(Y_k) \leq \operatorname{diam}(Y) + 2k$ and, since $z \in Y_k$, there exists $y \in B_z(k) \cap Y$, such that $d(z,\Omega) \leq k + d(y,\Omega)$. Hence, the remaining sum is bounded by

$$\begin{aligned}&C_{\mathrm{LR}}\left(\mathrm{e}^{c\,\|\Phi_S\|_{a,q}} - 1\right)\sum_{k=1}^{\infty}\sum_{y\in B_z(k)}\sum_{\substack{Y\subset\Lambda:\\ y\in Y}} \frac{|Y|\,\|\Phi_D(Y)\|}{\chi_{b,p}\big(\operatorname{diam}(Y)\big)\,\chi_{b,p}\big(d(y,\Omega)\big)}\,\frac{\chi_{a',q}(k)}{\chi_{b,p}(2k)\,\chi_{b,p}(k)}\\ &\quad= C_{\mathrm{LR}}\left(\mathrm{e}^{c\,\|\Phi_S\|_{a,q}} - 1\right) C\,\|\Phi_D\|_{b+\varepsilon,p;\Omega}\sum_{k=1}^{\infty}\big(1 + C_{\mathrm{vol}}\,k^d\big)\,\frac{\chi_{a',q}(k)}{\chi_{(2^p+1)b,p}(k)},\end{aligned}$$

for some $C > 0$. The remaining sum is bounded if $q > p$, or $q = p$ and $a' > (2^p + 1)\,b$. The last condition is equivalent to $a > (2^p + 1)\,b$, by our choice of a'. To conclude the statement, we choose the total constant larger than 1 and add the $k = 0$ term $\|\Phi_D\|_{b,p;\Omega} \leq \|\Phi_D\|_{b+\varepsilon,p;\Omega}$. □

A.3 Quasi-local inverse of the Liouvillian

In this section, we briefly recall the construction of the quasi-local inverse of the Liouvillian $\mathscr{I}$ and the related operator $\mathscr{J}$ introduced in (21) and (26). Both of them use certain properties of a *filter function* w_g, which one can construct explicitly.

Lemma A.9 (Explicit filter function, cf. Lemma 2.3 from [23]). *Let $g > 0$ and consider the sequence $(a_n)_{n\geq 1}$ of positive numbers, defined as $a_n = a_1(n(\log n)^2)^{-1}$ for $n \geq 2$ and a_1 chosen such that $\sum_{n\geq 1} a_n = \gamma/2$. Then, the positive function $w_g \in L^1(\mathbb{R})$ defined via the infinite product*

$$w_g(t) := c_g \prod_{n=1}^{\infty} \left(\frac{\sin(a_n t)}{a_n t} \right)^2 \tag{98}$$

and $c_g > 0$ chosen such that $\int_{\mathbb{R}} \mathrm{d}t\, w_g(t) = 1$, has Fourier transform $\widehat{w_g}$ with compact support $\operatorname{supp}(\widehat{w_g}) \subset [-g, g]$ *(cf.* (22)*) and satisfies the bound $|w_g(t)| \leq C\,\mathrm{e}^{-|t|^q}$ for every $q < 1$ (cf.* (23)*).*

Given the explicit filter function (98), the quasi-local inverse of the Liouvillian $\mathscr{I}_{H,g}[\![\,\cdot\,]\!] : \mathscr{A} \to \mathscr{A}$ of the Hamiltonian H with gap parameter $g > 0$, acting on $A \in \mathscr{A}$, is then defined as

$$\mathscr{I}_{H,g}[\![A]\!] := \int_{\mathbb{R}} \mathrm{d}t\, w_g(t) \int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{i}Hs} A\, \mathrm{e}^{-\mathrm{i}Hs}. \tag{99}$$

Remark A.10 (On the filter function). We point out, that, in principle and unless additional conditions are given, any map $\mathscr{I}_{H,g}$ with the properties (21)–(23) would work for all of our proofs in this paper, in particular including the statements from Section 3. ◇

Together with the map $\mathscr{J}_{H,g} : \mathscr{A} \to \mathscr{A}$, again depending on the Hamiltonian H and gap parameter $g > 0$, with action on $A \in \mathscr{A}$ defined as

$$\mathscr{J}_{H,g}[\![A]\!] := -\int_{\mathbb{R}} \mathrm{d}t\, w_g(t)\, \mathrm{e}^{\mathrm{i}Ht} A\, \mathrm{e}^{-\mathrm{i}Ht},$$

one then has (recalling the Liouvillian $\mathscr{L}_H[\![\,\cdot\,]\!] = -\mathrm{i}\,[H, \cdot\,]$) the identity $\mathscr{L}_H \circ \mathscr{I}_{H,g}[\![A]\!] - A = \mathscr{J}_{H,g}[\![A]\!]$ for all $A \in \mathscr{A}$; see (26).

The inverse Liouvillian $\mathscr{I}_{H,g}$ is called *quasi-local*, since, if H satisfies the Lieb-Robinson bound from Lemma A.3, then it holds that, for $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$ (see e.g. [174, Example 5.7])

$$\big\| \big[\mathscr{I}_{H,g}[\![A]\!], B\big] \big\| \leq C\, \|A\|\, \|B\|\, \min\{|X|, |Y|\} \chi_{\tilde{b},\tilde{p}}\big(d(X, Y)\big) \tag{100}$$

for some $\tilde{b} > 0$ (depending on the Lieb-Robinson velocity v from Lemma A.3) and $\tilde{p} \in (0, 1)$, which can be chosen as p from (90) if $p < 1$. The estimate (100) holds verbatim with $\mathscr{I}$ replaced by $\mathscr{J}$.

Beside the classical quasi-locality estimate (100), the inverse Liouvillian even preserves locality of SLT operators. This is the content of the following lemma, the special case for $p = 1$ without localization (i.e. for Λ-localized SLT operators) already appeared in [156, Theorem 7.5.6] and is based on [23, Theorem 4.8].

Lemma A.11 (Quasi-local inverse of the Liouvillian on SLT operators)**.** *Let $d \in \mathbb{N}$, $C_{\mathrm{vol}} > 0$, $\varepsilon > 0$, $a, b > 0$, $p, q \in (0,1]$ satisfying $q < p$, and $C_{\mathrm{int}} > 0$. There exist a constant $C > 0$ such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$, the following holds: For SLT operators H and D it holds that $\mathcal{I}_{H,g}(D)$ is an SLT operator as well. More precisely if $\|\Phi_H\|_{b,p} \leq C_{\mathrm{int}}$, then there exists an interaction $\Phi_{\mathcal{I}_{H,g}(D)}$ such that*

$$\|\Phi_{\mathcal{I}_{H,g}(D)}\|_{a,q;\Omega} \leq C \left\|\Phi_D\right\|_{a+\varepsilon,q;\Omega}.$$

The statement holds verbatim when replacing $\mathcal{I}$ by $\mathcal{J}$ from (26).

In the proof of the Lemma we use the equality

$$\mathcal{I}_{H,g}[\![A]\!] = \int_{\mathbb{R}} \mathrm{d}t\, \mathscr{W}_g(t)\, \mathrm{e}^{\mathrm{i}Ht} A \mathrm{e}^{-\mathrm{i}Ht},$$

where the function $\mathscr{W}_g$ is given by $\mathscr{W}_g(t) = -\int_{-\infty}^{t} \mathrm{d}s\, w_g(s) + \mathbb{1}_{[0,\infty)}(t)$ with $\mathbb{1}_{[0,\infty)}$ being the characteristic function of $[0,\infty)$. In many works, e.g. [23, 17, 165, 203, 174], this is used as a definition for the inverse Liouvillian. It can easily be checked that, by Lemma A.9, also $\mathscr{W}_g$ satisfies $|\mathscr{W}_g(t)| \leq C\, \mathrm{e}^{-|t|^q}$.

Proof. The proof uses the same technique as the proofs of Lemma A.7 and A.8.

First fix $X \subset \Lambda$ and $O \in \mathcal{A}_X$ and denote $\tau(O) = \mathrm{e}^{\mathrm{i}H}\, O\, \mathrm{e}^{-\mathrm{i}H}$. Then, define

$$\Delta_0(O) = \mathrm{i} \int_{\mathbb{R}} \mathrm{d}s\, \mathscr{W}_g(s)\, \mathbb{E}_X\big(\tau_s(O)\big)$$

and

$$\begin{aligned}\Delta_k(O) &= \mathrm{i} \int_{\mathbb{R}} \mathrm{d}s\, \mathscr{W}_g(s) \Big(\mathbb{E}_{X_k}\big(\tau_s(O)\big) - \mathbb{E}_{X_{k-1}}\big(\tau_s(O)\big)\Big) \\ &= \mathrm{i} \int_{\mathbb{R}} \mathrm{d}s\, \mathscr{W}_g(s)\, \mathbb{E}_{X_k}\Big(\big(\mathrm{id} - \mathbb{E}_{X_{k-1}}\big)\big(\tau_s(O)\big)\Big)\end{aligned}$$

for $k \geq 1$. Then $\mathcal{I}_{H,g}(O) = \sum_{k \in \mathbb{N}} \Delta_k(O)$ where the sum is eventually finite.

For $k = 0$ we have

$$\|\Delta_0(O)\| \leq \|\mathscr{W}_g\|_{L^1}\, \|O\|.$$

For $b' \in (0,b)$, $k \geq 1$ and some $T > 0$ to be chosen, Lemma A.3 and the properties of the conditional expectation, yield

$$\begin{aligned}
&\left\| \mathrm{i} \int_{-T}^{T} \mathrm{d}s\, \mathscr{W}_g(s)\, \mathbb{E}_{X_k}\Big(\big(\mathrm{id} - \mathbb{E}_{X_{k-1}}\big)\big(\tau_s(O)\big)\Big) \right\| \\
&\quad \le \|\mathscr{W}_g\|_{L^\infty} \int_{-T}^{T} \mathrm{d}s\, \big\| \big(\mathrm{id} - \mathbb{E}_{X_{k-1}}\big)\big(\tau_s(O)\big)\big\| \\
&\quad \le C_{\mathrm{LR}}\, \|O\|\, |X|\, \|\mathscr{W}_g\|_{L^\infty} \int_{-T}^{T} \mathrm{d}s\, (\mathrm{e}^{b'\, v\,|s|} - 1)\, \chi_{b',p}(k) \\
&\quad = 2\, C_{\mathrm{LR}}\, \|O\|\, |X|\, \|\mathscr{W}_g\|_{L^\infty} \frac{\mathrm{e}^{b' vT} - b' vT - 1}{b'\, v}\, \chi_{b',p}(k) \\
&\quad \le 2\, C_{\mathrm{LR}}\, \|O\|\, |X|\, \|\mathscr{W}_g\|_{L^\infty}\, \chi_{b'/2,p}(k)/(b'\, v)
\end{aligned}$$

where we chose $T = k^p/(2v)$ for the last step. Furthermore, by Lemma A.9 and after integrating twice, for any $0 < \tilde{p} < 1$ there exists C and $\tilde{b} > 0$ such that

$$\begin{aligned}
&\left\| \mathrm{i} \int_{|s|\ge T} \mathrm{d}s\, \mathscr{W}_g(s)\, \mathbb{E}_{X_k}\Big(\big(\mathrm{id} - \mathbb{E}_{X_{k-1}}\big)\big(\tau_s(O)\big)\Big) \right\| \\
&\quad \le 2\, \|O\| \int_{|s|\ge T} \mathrm{d}s\, |\mathscr{W}_g(s)| \\
&\quad \le C\, \chi_{\tilde{b},\tilde{p}}(T) \\
&\quad \le C\, \chi_{\tilde{b}',\tilde{p}p}(k),
\end{aligned}$$

where $\tilde{b}' = \big(1/(2v)\big)^{\tilde{p}}\, \tilde{b}$. Then, for all $p' \in (0, p)$ we can choose $\tilde{p} = p'/p < 1$ and combine the two bounds. Then, there exists C and $\eta > 0$ such that

$$\|\varDelta_k(O)\| \le C\, |X|\, \|O\|\, \chi_{\eta,p'}(k) \qquad \text{for all} \qquad k \ge 0. \tag{101}$$

An interaction for $A := \mathscr{I}_{H,g}(D)$ is given by

$$\varPhi_A(Z) = \sum_{k=0}^{\infty} \sum_{\substack{Y\subset\varLambda:\\ Y_k = Z}} \varDelta_k(\varPhi_D(Y)).$$

It follows that

$$\begin{aligned}
\sum_{\substack{Z\subset\varLambda:\\ z\in Z}} \frac{\|\varPhi_A(Z)\|}{\chi_{a,q}\big(\mathrm{diam}(Z)\big)\, \chi_{a,q}\big(d(z,\varOmega)\big)} &\le \sum_{\substack{Z\subset\varLambda:\\ z\in Z}} \sum_{k=0}^{\infty} \sum_{\substack{Y\subset\varLambda:\\ Y_k = Z}} \frac{\big\|\varDelta_k\big(\varPhi_D(Y)\big)\big\|}{\chi_{a,q}\big(\mathrm{diam}(Z)\big)\, \chi_{a,q}\big(d(z,\varOmega)\big)} \\
&= \sum_{k=0}^{\infty} \sum_{Y\subset\varLambda} \mathbb{1}_{z\in Y_k} \frac{\big\|\varDelta_k\big(\varPhi_D(Y)\big)\big\|}{\chi_{a,q}\big(\mathrm{diam}(Y_k)\big)\, \chi_{a,q}\big(d(z,\varOmega)\big)}.
\end{aligned}$$

The $k = 0$ term is bounded by $\|\mathscr{W}_g\|_{L^1}\|\Phi_D\|_{a,q;\Omega}$. For $k \geq 1$ and $z \in Y_k$ there exists $y \in B_z^\Lambda(k) \cap Y$ such that $d(z,\Omega) \leq k + d(y,\Omega)$. Furthermore, $\operatorname{diam}(Y_k) \leq 2k + \operatorname{diam}(Y)$. Hence, using (101) the rest of the sum is bounded by

$$
\begin{aligned}
& C \sum_{k=1}^{\infty} \chi_{\eta,p'}(k) \sum_{y \in B_z^\Lambda(k)} \sum_{\substack{Y \subset \Lambda: \\ y \in Y}} \frac{\|\Phi_D(Y)\|\,|Y|}{\chi_{a,q}\big(\operatorname{diam}(Y_k)\big)\,\chi_{a,q}\big(d(z,\Omega)\big)} \\
& \quad \leq C \sum_{k=1}^{\infty} \frac{\chi_{\eta,p'}(k)}{\chi_{a,q}(2k)\,\chi_{a,q}(k)} \sum_{y \in B_z^\Lambda(k)} \mathscr{C}_{\mathrm{vol},\varepsilon,q,1} \sum_{\substack{Y \subset \Lambda: \\ y \in Y}} \frac{\|\Phi_D(Y)\|}{\chi_{a+\varepsilon,q}\big(\operatorname{diam}(Y)\big)\,\chi_{a,q}\big(d(y,\Omega)\big)} \\
& \quad \leq C\,\mathscr{C}_{\mathrm{vol},\varepsilon,q,1}\,\|\Phi_D\|_{a+\varepsilon,q;\Omega} \sum_{k=1}^{\infty} \frac{\chi_{\eta,p'}(k)}{\chi_{(2^q+1)a,q}(k)}\,\big(1 + C_{\mathrm{vol}}\,k^d\big).
\end{aligned}
$$

The remaining sum is bounded if $p' > q$, which we can ensure if $p > q$ by choosing $p' \in (q,p)$. Thus, there exists C such that

$$
\sum_{\substack{Z \subset \Lambda: \\ z \in Z}} \frac{\|\Phi_A(Z)\|}{\chi_{a,q}\big(\operatorname{diam}(Z)\big)\,\chi_{a,q}\big(d(z,\Omega)\big)} \leq C\,\|\Phi_D\|_{a+\varepsilon,q;\Omega}
$$

for some constant C depending on $\|\Phi_H\|_{p,b}$, p, b, q, a, ε and d which finishes the proof. □

Remark A.12 (Abstract properties of $\mathscr{I}$ needed in the proof of our main result). For the purpose of proving our main result, Theorem 4.1, it is not necessary to work with the explicit $\mathscr{I}_{H,g}$ from (99). In fact, by inspecting the proof of Proposition 4.4 in Section 6, which is the key input for our main result, we realize the following: For Theorem 4.1 being valid (up to minor adjustments of constants), one only needs that there exists some operator $\widetilde{\mathscr{I}} : \mathscr{A} \to \mathscr{A}$ for which Assumption ($\mathrm{LDG}_{\mathrm{informal}}$) is satisfied and such that

1. for $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$ it holds that (cf. (104))

$$
\Big|\Big\langle\big[\mathscr{L}_H \circ \widetilde{\mathscr{I}}[\![A]\!] - A, B\big]\Big\rangle_{\rho_0}\Big| \leq C\,\|A\|\,\|B\|\,\operatorname{diam}(X)^\ell \exp(-d(X,Y)^q)
$$

   for some positive constants $C, q, \ell > 0$, i.e. the composition $\mathscr{L}_H \circ \widetilde{\mathscr{I}}$ behaves as a quasi-local operator if tested in the above way;

2. Lemma A.11 holds, i.e. $\widetilde{\mathscr{I}}$ maps localized SLT operators to localized SLT operators.

These relaxed abstract conditions are, however, not sufficient for showing the relations among the various gap conditions outlined in Section 3. ◇

A.4 Localized SLT operators: Proof of Lemma 2.1

For (i), since Φ is strictly Ω-localized, $\Phi(Z) \neq 0$ only if $Z \subset \Omega$ and then $\chi_{b,p}\big(d(z,\Omega)\big) = 1$ for all $z \in Z$. Hence,

$$\begin{aligned}\|\Phi\|_{b,p;\Omega} &= \sup_{z\in\Lambda} \sum_{\substack{Z\subset\Omega:\\ z\in Z}} \frac{\|\Phi(Z)\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)} \\ &\le \sup_{z\in\Lambda} \sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \frac{\|\Phi(Z)\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)} \\ &= \|\Phi\|_{b,p} \le C.\end{aligned}$$

Next, for (ii) and strongly Ω-localized Φ, we have that

$$\begin{aligned}\|\Phi\|_{b/2,p;\Omega} &= \sup_{z\in\Lambda} \sum_{\substack{Z\subset\Lambda:\\ Z\cap\Omega\neq\emptyset,\, z\in Z}} \frac{\|\Phi(Z)\|}{\chi_{b/2,p}\big(\mathrm{diam}(Z)\big)\,\chi_{b/2,p}\big(d(z,\Omega)\big)} \\ &\le \sup_{z\in\Lambda} \sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \frac{\|\Phi(Z)\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)} = \|\Phi\|_{b,p} \le C\end{aligned}$$

since $d(z,\Omega) \le \mathrm{diam}(Z)$ for $z \in Z$ and $Z \cap \Omega \neq \emptyset$, together with monotonicity of $\chi_{b/2,p}$ and using $(\chi_{b/2,p})^2 = \chi_{b,p}$. This concludes the proof of Lemma 2.1. □

A.5 Assumption (LDG_{weak}) for SLT operators: Proof of Lemma 6.1

We write $A = \sum_{Z\subset\Lambda} \Phi_A(Z)$ (cf. (15)) and estimate

$$\begin{aligned}&\Big|\Big\langle\big[\mathscr{L}_{H_0} \circ \mathscr{I}_{H_0,g}[\![A]\!] - A, B\big]\Big\rangle_{\rho_0}\Big| \\ &\qquad \le \sum_{Z\subset\Lambda}\Big|\Big\langle\big[\mathscr{L}_{H_0} \circ \mathscr{I}_{H_0,g}[\![\Phi_A(Z)]\!] - \Phi_A(Z), B\big]\Big\rangle_{\rho_0}\Big|.\end{aligned} \tag{102}$$

For the 'small' $Z \subset \Lambda$ satisfying $\mathrm{diam}(Z) \le d(Z, \Lambda\setminus\Lambda^{\mathrm{gap}})^\beta$, we bound

$$\begin{aligned}&\Big|\Big\langle\big[\mathscr{L}_{H_0} \circ \mathscr{I}_{H_0,g}[\![\Phi_A(Z)]\!] - \Phi_A(Z), B\big]\Big\rangle_{\rho_0}\Big| \\ &\qquad \le C_{\mathrm{gap}}\,\|\Phi_A(Z)\|\,\|B\|\,\mathrm{diam}(Y)^\ell\,\chi_{b,p}\big(d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)\end{aligned} \tag{103}$$

by means of Assumption (LDG_{weak}). Additionally, we need the following alternative estimate on (103) (recall (26)):

$$\begin{aligned}
&\Big|\Big\langle\big[\mathscr{L}_{H_0}\circ\mathscr{I}_{H_0,g}[\![\Phi_A(Z)]\!]-\Phi_A(Z),B\big]\Big\rangle_{\rho_0}\Big|\\
&\qquad\le\int_{\mathbb{R}}\mathrm{d}t\,w_g(t)\,\big\|\big[\mathrm{e}^{\mathrm{i}tH_0}\,\Phi_A(Z)\,\mathrm{e}^{-\mathrm{i}tH_0},B\big]\big\|\\
&\qquad\le C\,\|\Phi_A(Z)\|\,\|B\|\ \mathrm{diam}(Z)^d\left(\chi_{b/2,p}\big(d(Z,Y)\big)\int_I\mathrm{d}t\,w_g(t)+\int_{\mathbb{R}\setminus I}\mathrm{d}t\,w_g(t)\right)\\
&\qquad\le C\,\|\Phi_A(Z)\|\,\|B\|\ \mathrm{diam}(Z)^d\ \chi_{b/2,p}\big(d(Z,Y)\big)
\end{aligned}\tag{104}$$

where we denoted $I:=\big\{t\in\mathbb{R}\,\big|\,v|t|\le d(Z,Y)^p/3\big\}$. Here, v is the Lieb-Robinson velocity from Lemma A.3, which we employed in the second step with $b\to 3b/4$. In the final step, we used the stretched exponential decay of w_g from Lemma A.9. Note that (103) and (104) track two different relevant distances, namely those of Z to $\Lambda\setminus\Lambda^{\mathrm{gap}}$ and Y, respectively.

In fact, a weighted geometric mean of (103) and (104), that combines these two effects, can now be summed up as (neglecting the factor $C\,\|B\|\ \mathrm{diam}(Y)^\ell$, which will be put back in (106))

$$\begin{aligned}
&\sum_{\substack{Z\subset\Lambda:\\ \mathrm{diam}(Z)\le d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})^\beta}}\mathrm{diam}(Z)^d\ \chi_{b-\varepsilon,p}\big(d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)\ \chi_{\varepsilon/2,p}\big(d(Z,Y)\big)\,\|\Phi_A(Z)\|\\
&\quad\le\sum_{z\in\Lambda}\sum_{\substack{Z\subset\Lambda:\\ z\in Z}}\|\Phi_A(Z)\|\frac{\chi_{b-\varepsilon,p}\big(\mathrm{diam}(Z)+d(z,\Omega)+d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)}\ \chi_{\varepsilon/2,p}\big(d(z,Y)\big)\\
&\quad\le\|\Phi_A\|_{b,p;\Omega}\,\chi_{b-\varepsilon,p}\big(d(\Omega,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)\sum_{z\in\Lambda}\chi_{\varepsilon/2,p}\big(d(z,Y)\big)\\
&\quad\le C\ \mathrm{diam}(Y)^d\,\|\Phi_A\|_{b,p;\Omega}\,\chi_{b-\varepsilon,p}\big(d(\Omega,\Lambda\setminus\Lambda^{\mathrm{gap}})\big).
\end{aligned}\tag{105}$$

For the first bound, we used logarithmic superadditivity of $\chi_{b,p}$ together with elementary monotonicity properties from Lemma A.1 (a) and estimated $\mathrm{diam}(Z)^d$ by $1/\chi_{\varepsilon,p}\big(\mathrm{diam}(Z)\big)$. For the second bound, we used the definition of $\|\Phi_A\|_{b,p;\Omega}$ from (18) and the fact that, for $z\in Z$, we have $\mathrm{diam}(Z)+d(z,\Omega)+d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})\ge d(\Omega,\Lambda\setminus\Lambda^{\mathrm{gap}})$. In the final step, we employed summability of $\chi_{\varepsilon/2,p}\big(d(z,Y)\big)$.

Therefore, combining (102) with (105), the contribution of those $Z\subset\Lambda$, for which $\mathrm{diam}(Z)\le d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})^\beta$, to (102) is bounded by

$$C\ \mathrm{diam}(Y)^{d+\ell}\,\|B\|\,\|\Phi_A\|_{b,p;\Omega}\,\chi_{b-\varepsilon,p}\big(d(\Omega,\Lambda\setminus\Lambda^{\mathrm{gap}})\big).\tag{106}$$

For the 'large' $Z\subset\Lambda$ that satisfy $\mathrm{diam}(Z)>d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})^\beta$, we simply use the estimate from (104), which we can sum up as

$$\begin{aligned}
&\sum_{\substack{Z\subset\Lambda:\\ \operatorname{diam}(Z)>d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})^\beta}} \operatorname{diam}(Z)^d\, \chi_{b/2,p}\big(d(Z,Y)\big)\,\|\Phi_A(Z)\| \\
&\quad\le \sum_{z\in\Lambda}\sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \|\Phi_A(Z)\|\, \frac{\chi_{b/2,p\beta}\big(\operatorname{diam}(Z)+d(z,\Omega)+d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)}{\chi_{b,p}\big(\operatorname{diam}(Z)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)}\,\chi_{b/2,p}\big(d(z,Y)\big) \\
&\quad\le C\, \operatorname{diam}(Y)^d\, \|\Phi_A\|_{b,p;\Omega}\, \chi_{b/2,p\beta}\big(d(\Omega,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)
\end{aligned} \tag{107}$$

analogously to (105). In the second step we used that $\operatorname{diam}(Z) > d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})^\beta$ and elementary monotonicity properties of $\chi_{b,p}$ in b, p.

Therefore, by means of (107), also the large Z's contribute only in a way that is controlled in terms of (106) (but with worse constants $b/2$ and $p\min\{\beta,1\}$). This concludes the proof of Lemma 6.1. □

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Lieb-Robinson bounds, automorphic equivalence and LPPL for long-range interacting fermions

Stefan Teufel [*] Tom Wessel [*]

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Abstract

We prove a Lieb-Robinson bound for lattice fermion models with polynomially decaying interactions, which can be used to show the locality of the quasi-local inverse Liouvillian. This allows us to prove automorphic equivalence and the local perturbations perturb locally (LPPL) principle for these systems. The proof of the Lieb-Robinson bound is based on the work of Else et al. [84], and our results also apply to spin systems. We explain why some newer Lieb-Robinson bounds for long-range spin systems cannot be used to prove the locality of the quasi-local inverse Liouvillian, and in some cases may not even hold for fermionic systems.

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[*] Fachbereich Mathematik, Universität Tübingen,
Auf der Morgenstelle 10, 72076 Tübingen, Germany

Contents

1 Introduction

Locality is an important feature of many-body quantum systems. And a key tool to characterize locality in quantum lattice systems are Lieb-Robinson bounds [154], which underlie the existence of the thermodynamic limit of the Heisenberg dynamics [154, 170], decay of correlations for gapped ground states [118, 171, 214], the characterization of topological phases [120, 23], the adiabatic theorem [17, 165] and generalized response theory [203, 122].

Lieb-Robinson bounds measure locality in lattice systems, where the Hamiltonian is given as a sum of local terms

$$H = \sum_{Z \subset \Lambda} \Phi(Z),$$

where each $\Phi(Z)$ is only acting on the sites in Z. Here, Λ is a finite lattice and the Hilbert space could either describe a spin system, which means at each site there are some spin degrees of freedom, or a fermionic lattice system, describing fermions moving on the lattice. Denoting with τ_t the time evolution generated by H, a Lieb-Robinson bound is an upper bound for

$$\|[\tau_t(A), B]\|,$$

where the observables A and B are supported on disjoint sets X and $Y \subset \Lambda$, respectively, and, for fermionic systems, at least one of them is even. As the commutator would exactly vanish if $\tau_t(A)$ was supported in $\Lambda \setminus Y$, Lieb-Robinson bounds provide a measure of locality of $\tau_t(A)$.

For *short-range* interactions, i.e. if $\|\Phi(Z)\|$ decays exponentially in $\mathrm{diam}(Z)$, the Lieb-Robinson bounds have the form [118, 171, 172, 174]

$$\|[\tau_t(A), B]\| \leq C \, \min\{|X|, |Y|\} \, \|A\| \, \|B\| \, \mathrm{e}^{b(vt-r)}. \tag{1}$$

This implies, in particular, that the time-evolved operator is supported inside a ball of radius $r \leq vt$, up to exponentially decaying terms outside. Accordingly, this region is often called the *light cone*, and more specifically the Lieb-Robinson bound for short-range interactions (1) exhibits a *linear light cone*, as the bound is small up to times $vt \sim r$.

The main purpose of this paper is to prove locality for the quasi-local inverse Liouvillian and automorphic equivalence of gapped ground states in the same phase, which we explain later, for *long-range* interactions, where $\|\Phi(Z)\|$ decays polynomially in $\mathrm{diam}(Z)$, in fermionic lattice systems. The standard proofs of Lieb-Robinson bounds also give bounds for long-range interactions [118], but they only yield a logarithmic light cone and are not strong enough to prove automorphic equivalence [174]. Recently, there was a focus on proving Lieb-Robinson bounds with linear light cones for very slow polynomial decay [89, 149, 209, 210]. However, these works typically focus on large times and two-body interactions, making them not suitable for our application. We need to apply the Lieb-Robinson bound to the generator of the spectral flow, which contains general many-body interactions, and we need the bounds also for small times. Moreover, as we explain in section 6, some of the proofs [209, 210] only work for spin systems but not for fermions.

Hence, we focus on two results [160, 84] for long-range interactions, with comparably short proofs, which can easily be generalized to fermionic systems. These results have a root-like light cone, but that is no problem for our application. For our long-term goal of proving the adiabatic theorem and generalized response theory, these bounds have a problematic scaling in the size of $|X|$, which unlike the other cases appears with a possibly high power. To mitigate this, we first improve these results to the common linear prefactor $|X|$, see theorem 6.

Based on the improved Lieb-Robinson bounds, we prove locality of the quasi-local inverse Liouvillian and automorphic equivalence in these systems, following the corresponding proof strategies for short-range interactions [120, 23, 173, 17]. For simplicity, we restrict to two-body interactions on the square lattice to explain some of these results here, while the main text covers arbitrary, finite, surface-regular graphs and many-body interactions. For the moment, let $\Lambda \subset \mathbb{Z}^D$ be a finite lattice and $s \mapsto H(s) = \sum_{x,y\in\Lambda} \Phi(\{x,y\},s)$ be a smooth family of gapped Hamiltonians comprised of two-body interactions with decay $\|\Phi(\{x,y\},s)\| \leq C\,(1+d(x,y))^{\alpha_{\mathrm{tb}}}$ for some[1] $\alpha_{\mathrm{tb}} > 3D+1$, and similar decay for $\dot{\Phi}$. Then, an automorphism that connects the instantaneous ground states exists. It is generated by an interaction with a sufficiently high-power polynomial decay and still satisfies Lieb-Robinson bounds. Our result immediately implies automorphic equivalence for gapped ground states of interactions with super-polynomial decay, with the automorphisms being generated by interactions with super-polynomial decay as well. These Lieb-Robinson bounds are also used to prove automorphic equivalence in [31] and an adiabatic theorem in [32] for infinite volume systems with super-polynomially decaying interactions and a gap in the bulk. Moreover, they might be useful for proving stability of the gap for super-polynomially decaying interactions, as implicitly suggested by the comments in [174, Appendix E].

Automorphic equivalence can also be shown using the known bounds from [84], albeit with slightly different assumptions. But repeated applications of the inverse Liouvillian, as needed in the proof of super-adiabatic theorems [17, 165, 203], yield a much worse decay with the Lieb-Robinson bound of [84]. See the discussion after theorem 9. Unfortunately, even with the current Lieb-Robinson bounds and their interplay with various other steps in the proofs, one still requires very high power-law decay to obtain super-adiabatic theorems. Hence, we leave this discussion until further improvements of the Lieb-Robinson bounds[2] and the involved steps are developed.

Using the same ideas underlying automorphic equivalence, we also prove the local perturbations perturb locally principle for these systems, see theorem 10. Again as a simplified special case, consider $H(s) = \sum_{x,y\in\Lambda} \Phi(\{x,y\}) + s\,W$ for some two-body interaction Φ with decay exponent $\alpha_{\mathrm{tb}} > 2D$ and W supported on $X \subset \Lambda$. Moreover, assume that the ground state of the Hamiltonian $H(s)$ is gapped, with the gap bounded below uniformly in s and denote with $P(s)$ the ground state projection. Then, for every $\beta < \alpha_{\mathrm{tb}} - D$ and observable A supported on $Y \subset \Lambda$,

$$\left|\mathrm{tr}\big(P(s)\,A\big) - \mathrm{tr}\big(P(0)\,A\big)\right| \lesssim \|A\|\,|Y|\,\|\dot{H}(t)\|\,\big(d(X,Y)+1\big)^{-\beta}.$$

While our result applies also to general many-body interactions, we note that a different strategy was used in [214] to obtain a similar LPPL statement with better decay, albeit only for two-body interactions. See the discussion after theorem 10 for details.

[1] As we restrict to two-body interactions here, we follow the convention that α_{tb} characterises the decay of each two-body term. The power α used in the main text, which appears in the definition of the interaction norm (5), has a different meaning. See the discussion after theorem 10.

[2] We later argue that an improved light cone with same spatial decay as achieved in [210] does not improve the results on automorphic equivalence in theorem 9. However, it can still improve other necessary steps in the proofs of generalized super-adiabatic theorems.

The main text is organized as follows. In section 2 we discuss the lattices, operator algebras and interaction norms we use. Afterwards we state the Lieb-Robinson bounds in section 3 including the proof of our main improvement. For completeness and convenience of the reader, all intermediate steps which are similar to previous results are proven in the appendix. Using the Lieb-Robinson bounds, we prove automorphic equivalence in section 4 and discuss LPPL in section 5. Eventually, we discuss why a trick commonly used for spin systems, which would simplify our proof and underlies the sharpest Lieb-Robinson bounds [209, 210], seems to not work for fermionic systems.

2 Mathematical framework

This section introduces the mathematical framework that we use to describe long-range interacting fermions on a lattice.

2.1 Lattice and operator algebras

As lattices, we consider finite graphs (Λ, E) with graph distance $d(\cdot,\cdot)$ such that there exist constants $D \in \mathbb{N}_+$ and $\mathscr{C}_\mathrm{A} \geq 1$ such that

$$\left|S_y(R)\right| := \left|\{ x \in \Lambda \,|\, d(y,x) = R \}\right| \leq \mathscr{C}_\mathrm{A}\, R^{D-1} \quad \text{for all } y \in \Lambda \text{ and } R \geq 1. \tag{2}$$

These graphs are called *surface-regular*, and the set of all D-dimensional surface-regular graphs with growth constant $\mathscr{C}_\mathrm{A}$ is denoted $\mathscr{G}(D, \mathscr{C}_\mathrm{A})$. As the graphs are assumed to be finite, the existence of these constants is trivial. The crucial point is that all bounds we derive will be uniform for all graphs in the class $\mathscr{G}(D, \mathscr{C}_\mathrm{A})$, i.e. they only depend on the dimension D and the growth constant $\mathscr{C}_\mathrm{A}$, but not the specific Λ. In particular, they are independent of $|\Lambda|$. Hence, one can always take an increasing sequence $\Lambda_k \subset \Lambda_{k+1}$ and obtain bounds independent of k whenever $\mathscr{C}_\mathrm{A}$ and D are uniformly bounded in k.

All the following definitions implicitly depend on the specific lattice $\Lambda \in \mathscr{G}(D, \mathscr{C}_\mathrm{A})$ which will be clear from the context later on. If Λ is not explicitly specified, the statements and definitions hold for all $D \in \mathbb{N}_+$, $\mathscr{C}_\mathrm{A} > 0$ and $\Lambda \in \mathscr{G}(D, \mathscr{C}_\mathrm{A})$.

A simple integration shows that every surface-regular graph is also a regular graph in the sense that

$$\left|B_y(R)\right| := \left|\{ x \in \Lambda \,|\, d(y,x) \leq R \}\right| \leq \mathscr{C}_\mathrm{V}\, (R+1)^D \quad \text{for all } y \in \Lambda \text{ and } R \geq 0, \tag{3}$$

with $\mathscr{C}_\mathrm{V} \leq \max\{1, \mathscr{C}_\mathrm{A}/D\}$. One could use the more general class of regular graphs, but all the examples we have in mind are surface-regular and one obtains stronger results under this restriction, compare [160].

The above conditions in particular include the square lattices

$$\Lambda := \{-k, \dots, k\}^D \subset \mathbb{Z}^D \quad \text{for some } k \in \mathbb{N}$$

with ℓ^1-metric and additionally allow for torus or cylinder geometries. In these geometries, points on opposite ends of the lattice Λ, would have distance 1.

We use $d(\cdot,\cdot)$ also to denote the distance $d(x, Y) := \inf_{y\in Y} d(x, y)$ between a point $x \in \Lambda$ and a set $Y \subset \Lambda$ and $d(X, Y) := \inf_{x\in X, y\in Y} d(x, y)$ between two sets $X, Y \subset \Lambda$. We denote with

$$\operatorname{diam}(X) := \sup\{ d(x, y) \,|\, x, y \in X \}$$

the diameter and with $|X|$ the cardinality of any set $X \subset \Lambda$. Furthermore, for $X \subset \Lambda$ and $m \geq 0$, denote by X_m the *fattening*

$$X_m := \{ y \in \Lambda \,|\, d(y, X) \leq m \}. \tag{4}$$

2.2 Operator Algebra for fermions

For a fermion with spin $\mathfrak{s} \in \mathbb{N}$ on the lattice Λ, the one particle Hilbert space is $\mathscr{H} := \ell^2(\Lambda, \mathbb{C}^{\mathfrak{s}})$. For $N \in \mathbb{N}_+$, the N-particle Hilbert space is the antisymmetric tensor product $\mathscr{H}_N := \bigwedge_{k=1}^N \mathscr{H}$, and the fermionic Fock space is $\mathscr{F} := \bigoplus_{N=0}^{\mathfrak{s}|\Lambda|} \mathscr{H}_N$, where $\mathscr{H}_0 := \mathbb{C}$. These Hilbert spaces are all finite-dimensional and hence all linear operators on them are bounded. The C*-algebra $\mathscr{A}_\Lambda = \mathscr{B}(\mathscr{F})$ of bounded operators on $\mathscr{F}$ is generated by the fermionic annihilation and creation operators $a_{z,i}$ and $a^*_{z,i}$. They satisfy the canonical anti-commutation relations (CAR)

$$\{a_{x,i}, a_{y,j}\} = 0 = \{a^*_{x,i}, a^*_{y,j}\} \quad \text{and} \quad \{a_{x,i}, a^*_{y,j}\} = \delta_{i,j}\,\delta_{x,y}\,\mathbb{1}$$

for all $i, j \in \{1, \dots, \mathfrak{s}\}$ and $x, y \in \Lambda$, where $\{A, B\} := AB + BA$ denotes the anti-commutator. For each subset $Z \subset \Lambda$ one defines the subalgebra $\mathscr{A}_Z$ as the one generated by annihilation and creation operators supported in Z. This yields a natural inclusion $\mathscr{A}_{Z'} \subset \mathscr{A}_Z$ for $Z' \subset Z \subset \Lambda$.

By $\mathscr{A}_Z^+ \subset \mathscr{A}_Z$ we denote the subalgebra generated by products of an even number of creation and annihilation operators located in Z. For disjoint subsets $X, Y \subset \Lambda$ and operators $A \in \mathscr{A}_X^+$, $B \in \mathscr{A}_Y$, it then holds that $[A, B] = 0$ by the CAR. Moreover, if $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$ and $[A, B] = 0$, then $A \in \mathscr{A}_X^+$ or $B \in \mathscr{A}_Y^+$ must hold true [173, Proposition 2.1].

The restriction to even operators is natural for most applications, because in standard physical models no single fermionic particles are created or annihilated. For our purpose, we are actually often interested in operators conserving the particle number. And indeed, the subset $\mathscr{A}_Z^{\mathscr{N}} \subset \mathscr{A}_Z$ of elements commuting with the number operator

$$\mathscr{N}_Z = \sum_{z\in Z} \sum_{i=1}^{\mathfrak{s}} a^*_{z,i}\, a_{z,i}$$

is a subalgebra $\mathscr{A}_Z^{\mathscr{N}} \subset \mathscr{A}_Z^+$ of the even algebra.

2.3 Interactions and operator-families

An *interaction* is a function

$$\Phi\colon \{Z \subset \Lambda\} \to \mathcal{A}_\Lambda^+, \quad Z \mapsto \Phi(Z) \in \mathcal{A}_Z^+ \quad \text{where} \quad \Phi(Z) = \Phi(Z)^* \text{ for all } Z \subset \Lambda.$$

We denote the space of interactions defined on Λ as $\mathcal{S}_\Lambda$. Associated to each interaction is a *sum-of-local-terms* (SLT) operator

$$S := \sum_{Z \subset \Lambda} \Phi(Z).$$

Note that each interaction term is assumed to be even and self-adjoint. Hence, also the corresponding SLT operator is even and self-adjoint.

To control the locality of an interaction, we use the *interaction norm*

$$\|\Phi\|_{\alpha,n} := \sup_{z\in\Lambda} \sum_{\substack{Z\subset\Lambda:\\ z\in Z}} |Z|^n \frac{\|\Phi(Z)\|}{F_\alpha\big(\operatorname{diam}(Z)\big)}, \tag{5}$$

where $n \in \mathbb{N}$, $\alpha \geq 0$ and $F_\alpha(r) := (r+1)^{-\alpha}$, and we write $\|\cdot\|_\alpha := \|\cdot\|_{\alpha,0}$ for short.

For an interval $I \subset \mathbb{R}$, *time-dependent interactions* are functions $\Phi(\cdot,\cdot)$ where $\Phi(\cdot,t)$ is an interaction for each $t \in I$ and $\Phi(Z,\cdot)$ is norm continuous for all $Z \subset \Lambda$. We denote the space of time-dependent interactions defined on Λ for an interval I as $\mathcal{S}_\Lambda(I)$. For $\Phi \in \mathcal{S}_\Lambda(I)$ we write $\|\Phi(t)\|_{\alpha,n} := \|\Phi(\cdot,t)\|_{\alpha,n}$ and define

$$\|\Phi\|_{\alpha,n} := \sup_{t\in I} \|\Phi(t)\|_{\alpha,n}. \tag{6}$$

Furthermore, for any function $F\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$, denote

$$\|F\|_\Lambda := \sup_{x\in\Lambda} \sum_{z\in\Lambda} F\big(d(x,z)\big). \tag{7}$$

Then, note that

$$\|F_\alpha\|_\Lambda = \sup_{x\in\Lambda} \sum_{r=0}^{\infty} \sum_{\substack{z\in\Lambda:\\ r=d(x,z)}} F_\alpha\big(d(x,z)\big) \leq \mathscr{C}_\mathrm{A} \sum_{r=0}^{\infty} (r+1)^{D-1-\alpha} < \infty \tag{8}$$

for all $\Lambda \in \mathcal{G}(D, \mathscr{C}_\mathrm{A})$ and $\alpha > D$ uniformly in $|\Lambda|$.

It is important to note that all the constants will only depend on D and $\mathscr{C}_\mathrm{A}$ and the interaction norm, even though our setting is only dealing with finite lattices $\Lambda \in \mathcal{G}(D, \mathscr{C}_\mathrm{A})$. In particular, given an interaction Φ on an infinite surface regular lattice Γ, e.g. $\Gamma = \mathbb{Z}^D$, and defining an interaction norm on Γ as in (5) with the sum over finite $Z \Subset \Gamma$, the restrictions $\Phi|_\Lambda$ onto finite subsets $\Lambda \subset \Gamma$ satisfy our statements uniformly in Λ, because $\|\Phi|_\Lambda\|_{\alpha,n} \leq \|\Phi\|_{\alpha,n}$.

3 Improved Lieb-Robinson bounds for time-dependent long-range interactions

In this section we prove Lieb-Robinson bounds for long-range interacting fermions, improving and extending the previous results from [84, 160]. We begin by setting up some more notation, which will be used throughout the rest of the paper.

We say that $H_0 : I \to \mathscr{A}_\Lambda^+$ is an on-site Hamiltonian if $H_0(t) = \sum_{z\in\Lambda} h_z(t)$ where each $h_z : I \to \mathscr{A}_{\{z\}}^+$ is norm continuous and pointwise self-adjoint and even.

In this section, whenever there is the time-dependent interaction $\Phi \in \mathcal{S}_\Lambda(I)$ and an on-site Hamiltonian H_0, we associate with them the operators

$$H(t) = \sum_{Z\subset\Lambda} \Phi(Z,t) + H_0(t) \quad \text{and} \quad H_{<R}(t) = \sum_{\substack{Z\subset\Lambda:\\ \operatorname{diam}(Z)<R}} \Phi(Z,t) + H_0(t). \tag{9}$$

From continuity, it follows that there exists a unique solution $U(t,s)$ of

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} U(t,s) = H(t)U(t,s) \quad \text{with} \quad U(s,s) = \mathbb{1} \quad \text{for all} \quad t,s \in I,$$

which is a two-parameter evolution family of unitary operators, i.e.

$$U(t,r)U(r,s) = U(t,s) \quad \text{and} \quad U(t,s)^{-1} = U(t,s)^* = U(s,t)$$

for all $t,r,s \in I$. We call $\tau_{t,s} : \mathscr{A}_\Lambda \to \mathscr{A}_\Lambda$ defined by

$$\tau_{t,s}(A) = U(t,s)^* \, A \, U(t,s) \quad \text{for all } A \in \mathscr{A}_\Lambda$$

the *dynamics* of H. Similarly, $\tau_{t,s,<R}$ denotes the dynamics of $H_{<R}$.

Extending the finite-range Lieb-Robinson bound from [160] to time-dependent Hamiltonians, we obtain the following proposition. Within the statement, we make explicit the trivial bound $\left\|[\tau_{t,s,<R}(A), B]\right\| \leq 2\,\|A\|\,\|B\|$ which holds independently of the supports of A and B. For the convenience of the reader, the proof, a slightly stronger bound and some comments are given in appendix A.

Proposition 1. *Let $D \in \mathbb{N}_+$, $\mathscr{C}_\mathrm{A} > 0$, $\alpha > D$ and $\Lambda \in \mathscr{G}(D, \mathscr{C}_\mathrm{A})$. Moreover, let $I \subset \mathbb{R}$ be an interval, $\Phi \in \mathcal{S}_\Lambda(I)$ be a time-dependent interaction and H_0 be an on-site Hamiltonian. For all $X, Y \subset \Lambda$, $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$, with A or B even, and $R \geq 1$ it holds that*

$$\left\|[\tau_{t,s,<R}(A), B]\right\| \leq \|A\|\,\|B\|\,\Delta\Big(2\min\{|X|,|Y|\}\,\mathrm{e}^{v|t-s|-d(X,Y)/R}\Big),$$

where

$$\Delta(u) = \begin{cases} 2 & u > 2 \\ u & \text{otherwise} \end{cases}$$

and $v = 2\mathrm{e}\,\|F_\alpha\|_\Lambda\,\|\Phi\|_\alpha$.

It is important to note that proposition 1 is a statement about truncated long-range interactions. This allows to have a Lieb-Robinson velocity v uniform in R. For finite range interactions with uniformly bounded interaction terms, $\sup_{Z\subset\Lambda}\|\Phi(Z)\| < J$, the velocity would increase with R. And for exponentially decaying interactions, a similar bound without dependence on R holds [174]. However, as can be seen by choosing large R, the bound is not optimal, and we will prove better bounds for truncated long-range interactions using lemma 3.

The key ingredient to obtain the long-range Lieb-Robinson bounds based on proposition 1 is [160, Lemma 3.1], whose extension to time-dependent interactions reads as follows. The extension to time-dependent interactions works as usual, and we only give the proof in appendix B for the convenience of the reader and because it is very short with the tools presented in appendix A.

Lemma 2. *Let $D \in \mathbb{N}_+$, $\mathscr{C}_\mathrm{A} > 0$ and $\Lambda \in \mathscr{G}(D, \mathscr{C}_\mathrm{A})$. Moreover, let $I \subset \mathbb{R}$ be an interval, $\Phi \in \mathcal{S}_\Lambda(I)$ be a time-dependent interaction and H_0 be an on-site Hamiltonian. Furthermore, let $A, B \in \mathscr{A}_\Lambda$, and $1 < R' < R$. Then,*

$$\big\|\big[\tau_{t,s,<R}(A), B\big]\big\| \le \big\|\big[\tau_{t,s,<R'}(A), B\big]\big\| + 2\,\|B\| \sum_{\substack{Z\subset\Lambda:\\ R'\le \mathrm{diam}(Z)<R}} \int_{\min\{s,t\}}^{\max\{s,t\}} \big\|\big[\tau_{t,\theta,<R'}(A), \Phi(Z,\theta)\big]\big\|\,\mathrm{d}\theta. \quad (10)$$

To obtain the bound for the complete dynamics $\tau_{t,s}$ one can take R large enough.

The rough idea is to use the interaction picture and consider the long-range part as a perturbation of the short-range Hamiltonian. We obtain contributions of A evolved by the short-range Hamiltonian and additional corrections due to the long-range part. The same idea was used before in [89, 160, 84], it is depicted in figure 1.

Lemma 2 directly allows to prove a long-range Lieb-Robinson bound as in theorem 4, by choosing R large and bounding the individual summands using the finite range bound from proposition 1. Instead, we first formulate the following lemma, which allows to improve the finite-range bound and can thus be applied iteratively. After the first application, one arrives at theorem 4, and the iterations lead to theorem 6.

Lemma 3. *Let $D \in \mathbb{N}_+$, $\mathscr{C}_\mathrm{A} > 0$, $\alpha > D$ and $\Lambda \in \mathscr{G}(D, \mathscr{C}_\mathrm{A})$. Moreover, let $I \subset \mathbb{R}$ be an interval, $\Phi \in \mathcal{S}_\Lambda(I)$ be a time-dependent interaction and H_0 be an on-site Hamiltonian. Fix s, $t \in I$ and assume that we have a Lieb-Robinson bound*

$$\big\|\big[\tau_{t,\theta,<R}(A), B\big]\big\| \le \lambda_R\big(d(X,Y)\big)\min\{|X|,|Y|\}\,\|A\|\,\|B\| \quad (11)$$

for all $X, Y \subset \Lambda$, $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$ with A or B even, and $\theta \in [\min\{s,t\}, \max\{s,t\}]$. Then, for any $R' > 1$, (11) *also holds with λ_R replaced by*

$$\tilde{\lambda}_R(r) = \Delta\Big(2\,\mathrm{e}^{v|t-s|-r/R'} + 2\,\mathbb{1}_{R>R'}\,|t-s|\,\|\Phi\|_{\alpha,1}\,F_\alpha(R')\,\|\lambda_{R'}\|_\Lambda\Big), \quad (12)$$

where Δ is as in proposition 1 and $\|\cdot\|_\Lambda$ is defined in (7).

A similar result was proven in [160, Theorem 2.1 and Lemma B.1] and [84, Lemma 1]. Our main improvement is to use the $\|\cdot\|_{\alpha,1}$-norm instead of the $\|\cdot\|_{\alpha}$-norm. This requires a slightly stronger assumption on Φ, but it allows getting rid of a factor $|X|$ in the second summand, which was present in the previous works (where λ_R also had to depend on $|X|$). In the proof below, the difference is to use the Lieb-Robinson bound with $|Z|$ instead of $|X|$ to arrive at (13). The improvement in particular gives a much better result in the iteratively improved Lieb-Robinson bound in theorem 6, where one applies lemma 3 multiple times. Additionally, we allow for time-dependent interactions and additional on-site Hamiltonians.

Proof of lemma 3. The surrounding Δ is obtained from the trivial estimate for the commutator $\|[\tau_{t,\theta,<R}(A), B]\| \leq 2\,\|A\|\,\|B\|$ whenever it is better or $X \cap Y \neq \emptyset$ and noticing that the first term in the argument of Δ is larger than 2 for $r = 0$. It remains to concentrate on the argument of Δ.

For $R' \geq R$ we simply apply the Lieb-Robinson bound from proposition 1. For $R' < R$ we apply lemma 2 and then bound the individual summands from (10). The first summand is bounded using proposition 1 by

$$\left\|\left[\tau_{t,s,<R'}(A), B\right]\right\| \leq 2\,\mathrm{e}^{v\,|t-s|-d(X,Y)/R'}\,|X|\,\|A\|\,\|B\|.$$

The second summand is bounded using the Lieb-Robinson bound from (11) as follows. Note that Φ is even, and we can thus apply the Lieb-Robinson bound (11) for any $A \in \mathscr{A}_X$. To simplify notation we assume $s < t$ and otherwise exchange their symbols.

$$\begin{aligned}
2\,\|B\| \sum_{\substack{Z\subset\Lambda:\\ R'\leq \mathrm{diam}(Z)<R}} \int_s^t \left\|\left[\tau_{t,\theta,<R'}(A), \Phi(Z,\theta)\right]\right\| \mathrm{d}\theta \qquad\qquad& \\
\leq 2\,\|B\| \sum_{\substack{Z\subset\Lambda:\\ R'\leq \mathrm{diam}(Z)<R}} \lambda_{R'}\big(d(X,Z)\big)\,|Z|\,\|A\| \int_s^t \|\Phi(Z,\theta)\|\,\mathrm{d}\theta& \\
\leq 2\,|t-s|\,\|A\|\,\|B\| \sum_{z\in\Lambda} \lambda_{R'}\big(d(X,z)\big) \sup_{\theta\in[s,t]} \sum_{\substack{Z\subset\Lambda:\\ z\in Z\\ R'\leq \mathrm{diam}(Z)<R}} |Z|\,\|\Phi(Z,\theta)\|& \qquad (13)\\
\leq 2\,|t-s|\,\|A\|\,\|B\| \sum_{z\in\Lambda} \lambda_{R'}\big(d(X,z)\big) \|\Phi\|_{\alpha,1} F_\alpha(R')& \\
\leq 2\,|t-s|\,\|A\|\,\|B\|\,\|\Phi\|_{\alpha,1}\, F_\alpha(R') \sum_{x\in X}\sum_{z\in\Lambda} \lambda_{R'}\big(d(x,z)\big)& \\
\leq 2\,|t-s|\,|X|\,\|A\|\,\|B\|\,\|\Phi\|_{\alpha,1}\, F_\alpha(R')\,\|\lambda_{R'}\|_\Lambda.&
\end{aligned}$$

This proves the claim with $|X|$ instead of the minimum in (11). Applying it to the RHS of

$$\left\|\left[\tau_{t,s,<R}(A), B\right]\right\| = \left\|\left[\tau_{s,t,<R}(B), A\right]\right\|,$$

yields the same bound with $|Y|$, which concludes the proof. □

Before discussing the iteration, let us derive the bound from [160] incorporating our improvements. To do so, we need to bound $\|\lambda_{R'}\|_\Lambda$ for the finite range Lieb-Robinson bound $\lambda_{R'}(r) = \Delta\big(2\exp(v|t-s| - r/R')\big)$ from proposition 1.

Following the strategy from [160], for any $\rho \geq 1$, we find

$$\begin{aligned}\sum_{z\in\Lambda} \lambda_{R'}\big(d(x,z)\big) &\leq \sum_{\substack{z\in\Lambda:\\ d(x,z)\leq\rho}} 2 + \sum_{k=1}^{\infty} \sum_{\substack{z\in\Lambda:\\ d(x,z)=\rho+k}} 2\mathrm{e}^{v|t-s|-d(x,z)/R'} \\ &\leq 2\,\mathscr{C}_{\mathrm{V}}\,(\rho+1)^D + 2\,\mathrm{e}^{v|t-s|} \sum_{k=1}^{\infty} \mathscr{C}_{\mathrm{A}}(\rho+k)^{D-1}\mathrm{e}^{-(\rho+k)/R'} \\ &\leq 2\,\mathscr{C}_{\mathrm{V}}\,(\rho+1)^D + 2\,\mathrm{e}^{v|t-s|}\,\mathscr{C}_{\mathrm{A}} \int_{\rho}^{\infty} (q+1)^{D-1}\mathrm{e}^{-q/R'}\,\mathrm{d}q \\ &\leq 2\,\mathscr{C}_{\mathrm{V}}\,(\rho+1)^D + \mathscr{C}_{\mathrm{LR}}\,(\rho+R')^{D-1}\,R'\,\mathrm{e}^{v|t-s|-\rho/R'}.\end{aligned}$$

Here $\mathscr{C}_{\mathrm{LR}}$ is a constant depending only on D and $\mathscr{C}_{\mathrm{A}}$. The bound of the integral is elementary and alternatively one can use lemma 17. Now, if we choose R large enough and set $\rho = d(X,Y)$, we obtain the first long-range Lieb-Robinson bound.[3]

Theorem 4. *Let $D \in \mathbb{N}_+$, $\mathscr{C}_{\mathrm{A}} > 0$, $\alpha > D$ and $\Lambda \in \mathscr{G}(D, \mathscr{C}_{\mathrm{A}})$. Moreover, let $I \subset \mathbb{R}$ be an interval, $\Phi \in \mathscr{S}_\Lambda(I)$ be a time-dependent interaction and H_0 be an on-site Hamiltonian. Furthermore, let $X, Y \subset \Lambda$, $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$, with A or B even, and $R' \geq 1$. Then, for any $s, t \in I$ it holds that*

$$\begin{aligned}\big\|\big[\tau_{t,s}(A), B\big]\big\| \leq{}& 2\,\|A\|\,\|B\| \min\{|X|, |Y|\} \\ &\times \Big(\mathrm{e}^{v|t-s|-r/R'} + 2\,\mathscr{C}_{\mathrm{V}}\,\|\Phi\|_{\alpha,1}\,|t-s|\,(R'+1)^{-\alpha}\,(r+1)^D \\ &\quad + \mathscr{C}_{\mathrm{LR}}\,\|\Phi\|_{\alpha,1}\,|t-s|\,(R'+1)^{-\alpha}\,(r+R')^{D-1}\,R'\,\mathrm{e}^{v|t-s|-r/R'}\Big),\end{aligned}$$

where $r := d(X,Y)$, $v = 2\mathrm{e}\,\|F_\alpha\|_\Lambda\,\|\Phi\|_\alpha$ and $\mathscr{C}_{\mathrm{LR}}$ is defined above.

Furthermore, choosing $R' = r^\sigma$ with $0 < \sigma < 1$ we obtain the following Lieb-Robinson bound.

Corollary 5. *Let $D \in \mathbb{N}_+$, $\mathscr{C}_{\mathrm{A}} > 0$, $\alpha > D$ and $\Lambda \in \mathscr{G}(D, \mathscr{C}_{\mathrm{A}})$. Moreover, let $I \subset \mathbb{R}$ be an interval, $\Phi \in \mathscr{S}_\Lambda(I)$ be a time-dependent interaction, H_0 be an on-site Hamiltonian,*

[3] Here, and in the following the results for $X \cap Y \neq \emptyset$, i.e. $r = 0$, follow directly from the trivial bound. And we have $d(X,Y) \geq 1$ otherwise, since we use a graph distance.

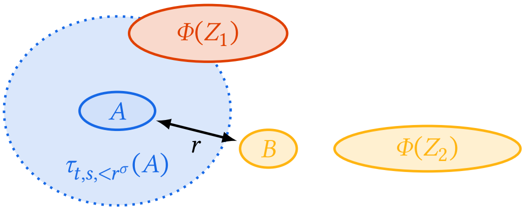


Figure 1. Situation after applying lemma 2 once, as in corollary 5 or after applying lemma 3 once. The Hamiltonian is split into a short- and long-range part. Terms are considered short-range, if their diameter is smaller than $R' = r^\sigma$, where $r = d(A,B)$. Depicted are some terms appearing in (10): $\tau_{t,s,<r^\sigma}(A)$ is the Heisenberg time evolution of A according to the Hamiltonian $H_{<R'}$. $\Phi(Z_1)$ and $\Phi(Z_2)$ are examples of different long-range contributions, i.e. $\mathrm{diam}(Z_1), \mathrm{diam}(Z_2) \geq R' = r^\sigma$, appearing in (10).

and $\sigma \in (0,1)$. *Furthermore, let* $X, Y \subset \Lambda$, $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$ *with* A *or* B *even. Then, for any* $s,t \in I$ *it holds that*

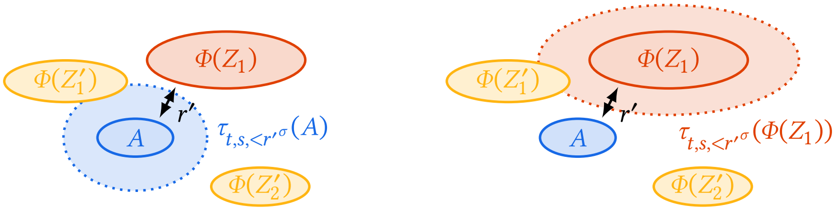
$$\big\|\big[\tau_{t,s}(A), B\big]\big\| \leq 2\,\|A\|\,\|B\| \min\{|X|,|Y|\} \times \left(\mathrm{e}^{v|t-s|-r^{1-\sigma}} + C\,\|\Phi\|_{\alpha,1}\,|t-s|\,(r+1)^{D-\alpha\sigma}\left(1+\mathrm{e}^{v|t-s|-r^{1-\sigma}}\right)\right), \tag{14}$$

where v *and* r *are the same as in theorem 4 and* $C = \max\{2\,\mathscr{C}_\mathrm{V}, 2^{D(1-\sigma)}\,\mathscr{C}_\mathrm{LR}\}$.

The rough idea for the bound can be seen from figure 1. For the explanation it is easier to use lemma 2 directly, instead of lemma 3. Hence, we apply lemma 2 to bound the commutator $\|[\tau_{t,s}(A), B]\|$. The first term in (14) stems from the commutator $[\tau_{t,s,<R'}(A), B]$ in (10) after setting $R' = r^\sigma$. The second term stemming from the sum in (10) has two contributions: Those terms where Z intersects the light cone of $\tau_{t,s,<r^\sigma}(A)$, see $\Phi(Z_1)$ in figure 1, are bounded using the trivial estimate. The others, see $\Phi(Z_2)$ in figure 1, are bounded using the finite-range Lieb-Robinson bound again and lead to the contribution containing the exponential.

To further improve the dependence on r, one can either apply lemma 2 multiple times to the second term in (10), or apply lemma 3 iteratively. Both paths result in the same bound. For the formal proof it is simpler to follow the iteration of lemma 3. But here we explain the intuition using lemma 2 multiple times.

As before, we split the norm using lemma 2 and use the finite range Lieb-Robinson bound for the first term in (10). The sum in the second term contains two types of terms. The first type intersects the light cone of $\tau_{t,s,<r^\sigma}(A)$ with $d(X,Z_1) \leq d(X,Y)$, the others lie outside $d(X,Z_2) < d(X,Y)$, see figure 1. For the terms $\Phi(Z)$ very close to X we use the trivial bound and the number of these terms can be bounded by $C\,|X|\,(1 + v\,|t-s|^{D/(1-\sigma)})$. But there are more terms, where applying lemma 2 again with $R' = d(X,Z)^\sigma$ could improve the bound, see $\Phi(Z_1)$ in figure 2. Indeed, [84] proceed by



(a) Second step in the iteration from [84].

(b) Second step in our iteration after applying lemma 2 with a twist.

Figure 2. Depicted is the situation after the second iteration step in the proof of theorem 6. The goal is to reduce contribution of the terms like $\Phi(Z_1)$ in figure 1 which intersect the light cone of $\tau_{t,s,<r^\sigma}(A)$, by applying (10) again. The situation Else et al. [84] obtain is pictured in (a). By introducing a new sum, which scales like $|X|$, in each step, they have the bad decay in $|X|$. We apply (10) with a twist to obtain the situation shown in (b). The additional sums then scale with $|Z_1|$ which can be absorbed in the interaction norm. See the main text for more details.

directly applying lemma 2, the resulting contributions are depicted in figure 2(a). The problem is that this introduces a double sum, each contributing a factor $|X|$. Hence, our improvement is to apply lemma 2 with a twist, namely to

$$\left\|\left[\tau_{s,t,<R'}(\Phi(Z,\theta)), A\right]\right\| = \left\|\left[\tau_{t,s,<R'}(A), \Phi(Z,\theta)\right]\right\|.$$

See the corresponding sets in figure 2(b). This sum introduces a factor $|Z_1|$ which can be absorbed in the interaction norm. We can repeat this iteratively, e.g. to obtain better bounds for the contribution stemming from $\Phi(Z_1')$ in figure 2.

Theorem 6. *Let $D \in \mathbb{N}_+$, $\mathscr{C}_\mathrm{A} > 0$ and $\alpha > D$. Then there exists a constant $C > 0$ such that for all lattices $\Lambda \in \mathscr{G}(D, \mathscr{C}_\mathrm{A})$, all $\sigma \in \big((D+1)/(\alpha+1), 1\big)$, all intervals $I \subset \mathbb{R}$, interactions $\Phi \in \mathcal{S}_\Lambda(I)$, on-site Hamiltonians H_0, $X, Y \subset \Lambda$, $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$ with A or B even, and $s, t \in I$ the following holds*

$$\begin{aligned}\left\|\left[\tau_{t,s}(A), B\right]\right\| &\leq 2\,\|A\|\,\|B\| \min\{|X|, |Y|\} \\ &\quad \times \left(\mathrm{e}^{v|t-s|-r^{1-\sigma}} + C_\sigma\,(r+1)^{-\sigma\alpha}\, v|t-s| \left(1 + (v|t-s|)^{D/(1-\sigma)}\right)\right).\end{aligned}$$

Here $v = \max\{2\mathrm{e}\,\|F_\alpha\|_\Lambda\,\|\Phi\|_\alpha, \|\Phi\|_{\alpha,1}\}$, $r = d(X,Y)$ and $C_\sigma = C\left(\sigma - \frac{D+1}{\alpha+1}\right)^{-2} \frac{1}{1-\sigma}\,\Gamma\!\left(\frac{D}{1-\sigma}\right)$ with Γ the gamma function.

Our main improvement here compared to [84] is to obtain $\min\{|X|, |Y|\}$ instead of a factor $|X|^n$ with $n = \lceil \sigma D/(\sigma\alpha - D)\rceil + 2 \to \infty$ for $\alpha \to D$ in front of the second summand. While this does not improve the light cone shape, it is advantageous for some applications using Lieb-Robinson bounds, e.g. for repetitive applications of the quasi-local inverse of the Liouvillian as it is used in the generalized adiabatic theorem, see section 4.

Some readers might wonder whether the same result could be concluded using [209, Lemma 4], which basically allows replacing $|X|^m$ with $|X|^1$ in any existing Lieb-Robinson bound. We explain their trick in our language using the conditional expectation in section 6 and point out why it does not work for fermions.

4 Automorphic equivalence

In this section we use the Lieb-Robinson bound from theorem 6 to prove locality of the quasi-local inverse Liouvillian and thus locality of the spectral flow [120, 23, 173]. To do so, we consider a smooth family of polynomially localized Hamiltonians $H(s)$ with a uniformly gapped part of the spectrum and show that the corresponding spectral projections $P(s)$ are connected by a polynomially localized unitary $P(s) = U(s)\,P(0)\,U(s)^*$.

More precisely, we require the Hamiltonians to satisfy the following assumptions additionally to certain decay properties which we impose later.

Assumption 7. For a differentiable family of Hamiltonians $s \mapsto H(s)$ with $s \in I$ and I a closed interval, we assume that for all $s \in I$ the spectrum $\sigma(H(s))$ has a gapped part $\sigma_*(s) \subset \sigma(H(s))$ satisfying the following: There exist $g > 0$ and continuous functions $f_\pm : I \to \mathbb{R}$ such that

(i) $f_\pm(s) \in \mathbb{R} \setminus \sigma(H(s))$,

(ii) $\big[f_-(s), f_+(s)\big] \cap \sigma(H(s)) = \sigma_*(s)$ and

(iii) $\operatorname{dist}\big(\sigma_*(s), \sigma(H(s)) \setminus \sigma_*(s)\big) \geq g$

for all $s \in I$. ◇

Note that, using the standard argument involving the Cauchy formula for the resolvent, assumption 7 implies also differentiability of $s \mapsto P(s)$. Taking a derivative of $P(s)^2 = P(s)$ one finds

$$\mathrm{i}\,\dot{P}(s) = \big[\mathrm{i}[\dot{P}(s), P(s)], P(s)\big] = [G_{\mathrm{Kato}}(s), P(s)],$$

with the self-adjoint Kato-generator $G_{\mathrm{Kato}}(s) := \mathrm{i}\,[\dot{P}(s), P(s)]$. Hence, the solution of

$$\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}s}\,U_{\mathrm{Kato}}(s) = G_{\mathrm{Kato}}(s)\,U_{\mathrm{Kato}}(s) \quad \text{with} \quad U_{\mathrm{Kato}}(0) = \mathbb{1}$$

is unitary and satisfies

$$P(s) = U_{\mathrm{Kato}}(s)\,P(0)\,U_{\mathrm{Kato}}(s)^*. \tag{15}$$

However, in general the Kato-generator $G_{\mathrm{Kato}}(s)$ does not come from a local interaction. But G is not uniquely determined by the desired property (15) and one can use the Hastings generator $G(s) = -\mathcal{I}_{H,g,0}(\dot{H}(s))$ instead, where $\mathcal{I}_{H,g,\delta}$ is the *quasi-local* *inverse of the Liouvillian* which we discuss in appendix D. In contrast to the Kato-generator, it can be given by an interaction with bounded interaction norm. For exponentially decaying interactions this was known before [17]. And the following proposition, which we prove in appendix D, gives locality for polynomially decaying interactions.

Proposition 8. *Let $D \in \mathbb{N}_+$, $\mathscr{C}_\mathrm{A} > 0$, $n \in \mathbb{N}$, $\beta > 0$, $\alpha > (n+1)D + 1 + \beta$, $g > \delta \geq 0$, and $v_* \geq 1$. Then there exists a constant $C > 0$ such that for all lattices $\Lambda \in \mathscr{G}(D, \mathscr{C}_\mathrm{A})$ the following holds:*

Let $\Phi \in \mathcal{S}_\Lambda$ be an interaction such that $v = \max\{2\mathrm{e}\,\|F_\alpha\|_\Lambda\,\|\Phi\|_\alpha, \|\Phi\|_{\alpha,1}\} \leq v_*$, and let H_0 be an on-site Hamiltonian. Then the quasi-local inverse of the Liouvillian preserves locality in the following way: Let K be given by an interaction $\Phi_K \in \mathcal{S}_\Lambda$. Then,

$$\|\Phi_{\mathscr{I}_{H,g,\delta}(K)}\|_{\beta,n} \leq C\,\|\Phi_K\|_{\beta,n+1}.$$

In contrast to the previous results, where all constants were independent of the interactions, C includes a non-linear dependence on the norm of Φ through v_* and also depends on the spectral properties of the Hamiltonian H through g and δ. Only the linear dependence in the SLT operator K is specified explicitly.

Note, moreover, that there is no time-dependence in the statement. For time-dependent Hamiltonians H and operator families K, we can define a time-dependent version

$$\big(\mathscr{I}_{H,g,\delta}(K)\big)(t) := \mathscr{I}_{H(t),g,\delta}\big(K(t)\big),$$

which does not depend on H or K at other times.

Automorphic equivalence with an automorphism satisfying Lieb-Robinson bounds then directly follows.

Theorem 9. *Let $D \in \mathbb{N}_+$, $\mathscr{C}_\mathrm{A} > 0$, $I = [0,1]$, $n \in \mathbb{N}$, $\beta > 0$, $\alpha > (n+1)D + 1 + \beta$, $g > \delta \geq 0$, and $v_* \geq 1$. Then there exists a constant $C > 0$ such that for all lattices $\Lambda \in \mathscr{G}(D, \mathscr{C}_\mathrm{A})$ the following holds:*

Let $\Phi \in \mathcal{S}_\Lambda(I)$ be an interaction such that $v = \max\{2\mathrm{e}\,\|F_\alpha\|_\Lambda\,\|\Phi\|_\alpha, \|\Phi\|_{\alpha,1}\} \leq v_*$, and let H_0 be a time-independent on-site Hamiltonian. Furthermore, assume that the corresponding operator family H satisfies assumption 7. Then the projections $P(s)$ associated to $\sigma_*(s)$ (see assumption 7) are unitarily equivalent

$$P(s) = U(s)\,P(0)\,U(s)^*, \tag{16}$$

where the unitaries have a local generator $G = -\mathscr{I}_{H,g,0}(\dot{H})$, which is given by an interaction Ψ, satisfying $\|\Psi(t)\|_{\beta,n} \leq C\,\|\dot{\Phi}(t)\|_{\beta,n+1}$.

As it was known before that the ground states could be connected by a unitary evolution, the importance of this statement lies in the fact that the generator can be given by a long-range interaction. As such, one has locality in terms of Lieb-Robinson bounds for the unitary mapping $A \mapsto U(s)^* A U(s)$. To use the Lieb-Robinson bounds from section 3 for this purpose, one needs $n = 1$ and $\beta > D$. To obtain the smallest α, for which one has locality of the unitary mapping, one can fall back to other Lieb-Robinson. This is reasonable here, as we are only interested in "times" up to $s = 1$ and hence do not need good light cones. One can use [118], which only requires $\|\Psi\|_{\beta,1} < \infty$ for some $\beta > 0$ and then provides $\|[U(s)^* A U(s), B]\| \lesssim \|A\| \|B\| |X| |Y| F_\beta\big(d(X,Y)\big)$. Alternatively, one can use the bounds from [84, 160], which provide some slower polynomial decay in $d(X,Y)$ without the prefactor $|Y|$ given $\|\Psi\|_{\beta,0} < \infty$ for some $\beta > D$.

Similar results hold, if one starts with the previously known Lieb-Robinson bounds from [160] or [84]. In both cases, the weaker assumption $v = 2\mathrm{e}\, \|F_\alpha\|_\Lambda \, \|\Phi\|_\alpha \le v_*$ is enough[4]. However, using the bound from [84] we would only obtain a bound $\|\Psi(t)\|_{\beta,n} \le C \, \|\dot{\Phi}(t)\|_{\beta,n+m}$, where m can be quite large depending on the parameters α and β, see the discussion after theorem 6. And with the bound from [160], one would only obtain $\|\Psi(t)\|_{\beta,n} \le C \, \|\dot{\Phi}(t)\|_{\beta,n+2}$, but additionally need to assume that $\alpha > (n+2)\, D + 1 + \beta$ which in turn is a stronger assumption on Φ.

The result obtained by using the Lieb-Robinson bound from [84] might not be a problem when dealing with k-body interactions for finite k when considering explicit Hamiltonians. However, we want to use the results to prove generalized super-adiabatic theorems, where one iteratively applies proposition 8, i.e. we want to apply proposition 8 to SLT operators K which by construction are given by many-body interactions. There the following problem arises: For simplicity, let us consider $A = \mathcal{I}_{H,g,\delta}\big(\mathcal{I}_{H,g,\delta}(K)\big)$, and try to bound $\|\Phi_A\|_{\beta,n}$ of the corresponding interaction Φ_A. With our bounds, we obtain

$$\|\Phi_A\|_{\beta,n} \le C \, \|\Phi_{\mathcal{I}_{H,g,\delta}(K)}\|_{\beta,n+1} \le C \, \|\Phi_K\|_{\beta,n+2},$$

where, due to the second bound, C depends on $\|\Phi\|_{\alpha,1}$ with $\alpha > (n+2)\, D + 1 + \beta$. In comparison, with the Lieb-Robinson bounds from [84], one would only obtain

$$\|\Phi_A\|_{\beta,n} \le C \, \|\Phi_{\mathcal{I}_{H,g,\delta}(K)}\|_{\beta,n+m} \le C \, \|\Phi_K\|_{\beta,n+2m},$$

where C depends on $\|\Phi\|_{\alpha,0}$ with $\alpha > (n+m+1)\, D + 1 + \beta$, where m can be quite large. Hence, iteratively applied, the new bound is also significantly better concerning the scaling of the Hamiltonian.

5 Local perturbations perturb locally

With the results from the previous section we can also prove a *local perturbation perturb locally* (LPPL) principle as introduced by Bachmann et al. [23]. Given a time-independent interaction $\Phi \in \mathcal{S}_\Lambda([0,1])$, an on-site Hamiltonian H_0 and a perturbation

[4] While [160, 84] do not explicitly provide the Lieb-Robinson velocities, following the argument of lemma 2, this is the Lieb-Robinson velocity and condition in our setting.

$W \in \mathcal{A}_X^+$ with $X \subset \Lambda$ one considers the Hamiltonian

$$H(s) = H_0 + \sum_{Z \subset \Lambda} \Phi(Z) + s\,W. \tag{17}$$

Supposing that H satisfies the conditions from theorem 9, we know that the projections $P(1)$ and $P(0)$ are unitarily equivalent. Moreover, one can show that the generator of this unitary can be well approximated by a strictly local operator in $\mathcal{A}_{X_R}^+$, such that $P(0)$ and $P(1)$ (almost) agree far away from X. More precisely, we find the following theorem where we allow more general perturbations by including them in the interaction Φ.

Theorem 10. *Let $D \in \mathbb{N}_+$, $\mathscr{C}_\mathrm{A} > 0$, $n \in \mathbb{N}$, $\alpha > D$, $g > \delta \geq 0$, $v_* \geq 1$ and $\varepsilon \in (0, \alpha - D)$. Then there exists a constant $C > 0$ such that for all lattices $\Lambda \in \mathcal{G}(D, \mathscr{C}_\mathrm{A})$ the following holds:*

Let $\Phi \in \mathcal{S}_\Lambda(I)$ be an interaction such that $v = \max\{2\mathrm{e}\,\|F_\alpha\|_\Lambda\,\|\Phi\|_\alpha, \|\Phi\|_{\alpha,1}\} \leq v_*$, $\dot{\Phi}(s) \in \mathcal{A}_X^+$ for some $X \subset \Lambda$ and, and let H_0 be a time-independent on-site Hamiltonian. Furthermore, assume that the corresponding operator family H satisfies assumption 7. Then, for all $A \in \mathcal{A}_Y$ with $Y \subset \Lambda$ and $Y \cap X = \emptyset$ it holds that

$$\left|\mathrm{tr}\big(P(s)\,A\big) - \mathrm{tr}\big(P(0)\,A\big)\right| \leq C\,s\,\mathrm{rank}\big(P(0)\big)\,|Y|\,\|A\|\,\sup_{t\in I}\|\dot{H}(t)\|\,\big(d(X,Y)+1\big)^{-\alpha+\varepsilon} \tag{18}$$

and

$$\left|\mathrm{tr}\big(P(s)\,A\big) - \mathrm{tr}\big(P(0)\,A\big)\right| \leq C\,s\,\mathrm{rank}\big(P(0)\big)\,|Y|\,\|A\|\,\|\dot{\Phi}\|_{0,1}\,\big(d(X,Y)+1\big)^{-\alpha+D+\varepsilon}. \tag{19}$$

We recall that with the prime example from (17), $\sup_{t\in I}\|\dot{H}(t)\| = \|W\|$ and, if $W = \sum_{Z\subset X}\Phi_W(Z)$ is given by an interaction, $\|\dot{\Phi}\|_{0,1} = \|\Phi_W\|_{0,1}$. While the underlying idea is similar to theorem 9, the assumptions are relaxed, because we never need to bound an interaction norm of the generator G of U. More precisely, for (18) we consider the whole perturbation $\dot{H}(t) = \sum_{Z\subset X}\dot{\Phi}(t)$ at once. Hence, its operator norm is present in the bound, which is fine, if X is a fixed region. For extensive perturbations, the bound (19) is better, as it only includes an interaction norm of $\dot{\Phi}$, but with a slower decay. The proof is given in appendix D.

The result is similar to the one recently obtained by Wang and Hazzard [214], who prove an LPPL principle using complex analysis and avoiding the Hastings generator. They deal with two-body interactions only, denote with α_tb the power of the decay of each individual term, and require $\alpha_\mathrm{tb} > 2D$. Such interactions are included in our results for $\alpha = \alpha_\mathrm{tb} - D > D$, which is exactly, what we require. However, they achieve a scaling with exponent α_tb (minus logarithmic corrections) in (18) instead of our $\alpha - \varepsilon = \alpha_\mathrm{tb} - D - \varepsilon$. But due to the specific long-range Lieb-Robinson bound they use, they limit themselves to two-body interactions.

6 Comments on spin systems and the conditional expectation

In this section, we want to briefly outline, how slightly better bounds can be obtained directly from the previous bounds [160, 84] for spin systems. To do so, one uses a simple trick, which is relatively common in the quantum information community. We formulate it here using the conditional expectation and argue, why we believe that the same argument does not work for lattice fermions. This also poses the question, how linear light cones as in [209, 210] could be obtained for lattice fermions, as the proofs rely on the same trick.

6.1 Mathematical framework

In this section we consider the same lattice geometries as in the rest of the work. But instead of lattice fermions, we consider lattice spins, which requires us to change the involved Hilbert spaces and operator algebras. On each lattice site $z \in \Lambda$, we attach $\mathfrak{s} \in \mathbb{N}$ spin degrees of freedom such that the local Hilbert space is $\mathcal{H}_z^{\mathrm{spin}} = \mathbb{C}^{\mathfrak{s}}$. For any $Z \subset \Lambda$ we then define the Hilbert space and algebra of bounded linear operators

$$\mathcal{H}_Z^{\mathrm{spin}} = \bigotimes_{x\in Z} \mathcal{H}_x^{\mathrm{spin}} \qquad \text{and} \qquad \mathcal{A}_Z^{\mathrm{spin}} = \mathcal{B}(\mathcal{H}_Z^{\mathrm{spin}}).$$

For any $Z' \subset Z \subset \Lambda$, we have $\mathcal{A}_{Z'}^{\mathrm{spin}} \subset \mathcal{A}_Z^{\mathrm{spin}}$ due to the trivial identification of $A \in \mathcal{A}_{Z'}^{\mathrm{spin}}$ with $A \otimes \mathbb{1}_{Z\setminus Z'} \in \mathcal{A}_Z^{\mathrm{spin}}$. In comparison to the fermionic systems, for X, $Y \subset \Lambda$ disjoint, $[A, B] = 0$ for all $A \in \mathcal{A}_X^{\mathrm{spin}}$ and $B \in \mathcal{A}_Y^{\mathrm{spin}}$. Hence, for spin systems there is no need to define an algebra of even operators and all the statements of the rest of the work hold after replacing $\mathcal{A} \to \mathcal{A}^{\mathrm{spin}}$ and $\mathcal{A}^+ \to \mathcal{A}^{\mathrm{spin}}$.

6.2 Conditional expectation to improve Lieb-Robinson bounds in spin systems

In most cases, there is no difference between the analysis of lattice spin systems or fermions on a lattice, as long as one is concerned with even operators in the latter. In our analysis of the lattice fermions, we do restrict to even interactions and even observables to make the analysis possible and this is justified by physical reasons.

We now explain a useful trick using the conditional expectation to improve Lieb-Robinson bounds for spin systems. In section 6.3 we then explain, why this does not work well enough for fermions, due to the distinction between even and odd observables. For spin systems, the conditional expectation is given by the partial trace, and can be generalized to infinite systems [174]. It satisfies the following lemma, which is exactly what we also have for fermionic systems, but without the restriction to even operators in some of the statements, see lemma 19.

Lemma 11 ([174, Lemma 4.1]). *Let $X \subset \Lambda$. Then there exists a unit-preserving, completely positive linear map $\mathbb{E}_X\colon \mathscr{A}_\Lambda^{\mathrm{spin}} \to \mathscr{A}_\Lambda^{\mathrm{spin}}$ satisfying*

(i) $\mathbb{E}_X(\mathscr{A}_\Lambda^{\mathrm{spin}}) \subset \mathscr{A}_X^{\mathrm{spin}}$;

(ii) $\mathbb{E}_X(ABC) = A\,\mathbb{E}_X(B)C$ *for all* $B \in \mathscr{A}_\Lambda^{\mathrm{spin}}$ *and* $A, C \in \mathscr{A}_X^{\mathrm{spin}}$; *This in particular implies* $\mathbb{E}_X(A) = A$ *for all* $A \in \mathscr{A}_X^{\mathrm{spin}}$;

(iii) $\|\mathbb{E}_X\| = 1$;

(iv) $\mathbb{E}_X \circ \mathbb{E}_Y = \mathbb{E}_{X\cap Y}$, *for* $X, Y \subset \Lambda$;

(v) *If* $A \in \mathscr{A}_\Lambda^{\mathrm{spin}}$ *satisfies*

$$\|[A,B]\| \le \eta\,\|A\|\,\|B\| \quad \text{for all} \quad B \in \mathscr{A}_{\Lambda\setminus X}^{\mathrm{spin}}, \tag{20}$$

for some $\eta > 0$, *then*

$$\|A - \mathbb{E}_X(A)\| \le \eta\,\|A\|. \tag{21}$$

The conditional expectation is especially useful to approximate the support of a time evolved observable, therefore let $A \in \mathscr{A}_X^{\mathrm{spin}}$. Then, it is standard to approximate $\tau_{t,s}(A)$ with the strictly local $\mathbb{E}_{X_r}\big(\tau_{t,s}(A)\big) \in \mathscr{A}_{X_r}^{\mathrm{spin}}$ using lemma 11(v) and Lieb-Robinson bound to obtain

$$\big\|\tau_{t,s}(A) - \mathbb{E}_{X_r}\big(\tau_{t,s}(A)\big)\big\| \le \sup_{B\in\mathscr{A}_{\Lambda\setminus X_r}:\|B\|=1} \big\|\big[\tau_{t,s}(A), B\big]\big\|.$$

This indeed also works for lattice fermions and is used in the proofs.

Moreover, the conditional expectation allows for a trick which is often used for spin systems [149, 209, 147, 210]. We present it here using the conditional expectation, which we have not seen before. Usually, the calculation is performed in an explicit realization of a partial trace, which obscures the key idea.

Let $Y \subset \Lambda$ and choose an enumeration $Y = \{y_1, \dots, y_n\}$. Then, by lemma 11(iv),

$$\mathrm{id} - \mathbb{E}_{\Lambda\setminus Y} = \sum_{j=1}^{n} \mathbb{E}_{\Lambda\setminus\{y_1,\dots,y_{j-1}\}} - \mathbb{E}_{\Lambda\setminus\{y_1,\dots,y_j\}} = \sum_{j=1}^{n} \mathbb{E}_{\Lambda\setminus\{y_1,\dots,y_{j-1}\}} \circ (\mathrm{id} - \mathbb{E}_{\Lambda\setminus\{y_j\}}),$$

where we understand $\{y_1, \dots, y_0\} = \emptyset$. And hence,

$$\big\|(\mathrm{id} - \mathbb{E}_{\Lambda\setminus Y})(A)\big\| \le \sum_{y\in Y} \big\|(\mathrm{id} - \mathbb{E}_{\Lambda\setminus\{y\}})(A)\big\| \tag{22}$$

by lemma 11(iii). To bound the right-hand side, we can again use lemma 11(v), where we only need to consider operators $B \in \mathscr{A}_{\{y\}}^{\mathrm{spin}}$ for each summand.

Therefore, let f be an arbitrary function and assume the following Lieb-Robinson bound

$$\big\|\big[\tau_{t,s}(A), B\big]\big\| \le \|A\|\,\|B\|\, f\big(|t-s|, d(X,Y), \min\{|X|, |Y|\}\big) \tag{23}$$

for all $X, Y \subset \Lambda$, $A \in \mathscr{A}_X^{\text{spin}}$ and $B \in \mathscr{A}_Y^{\text{spin}}$. For $X, Y \subset \Lambda$, $A \in \mathscr{A}_X^{\text{spin}}$ and $B \in \mathscr{A}_Y^{\text{spin}}$ one can then bound

$$\begin{aligned} \big\|\big[\tau_{t,s}(A), B\big]\big\| &\le 2\,\|B\|\,\big\|(\mathrm{id} - \mathbb{E}_{\Lambda\setminus Y})\big(\tau_{t,s}(A)\big)\big\| \\ &\le 2\,\|B\| \sum_{y\in Y} \sup_{O_y \in \mathscr{A}_{\{y\}}:\|O_y\|=1} \big\|\big[\tau_{t,s}(A), O_y\big]\big\| \\ &\le 2\,\|A\|\,\|B\| \sum_{y\in Y} f\big(|t-s|, d(X,y), 1\big) \\ &\le 2\,\|A\|\,\|B\|\,|Y|\, f\big(|t-s|, d(X,Y), 1\big), \end{aligned} \tag{24}$$

where we used $[\mathbb{E}_{\Lambda\setminus Y}(\tau_{t,s}(A)), B] = 0$. Applying the same argument to $\|[\tau_{t,s}(A), B]\| = \|[A, \tau_{s,t}(B)]\|$, one obtains the same result with $|Y|$ replaced by $|X|$. Hence, under assumption (23), we are able to prove the Lieb-Robinson bound

$$\big\|\big[\tau_{t,s}(A), B\big]\big\| \le 2\,\|A\|\,\|B\| \min\{|X|, |Y|\}\, f\big(|t-s|, d(X,Y), 1\big) \tag{25}$$

for all $X, Y \subset \Lambda$, $A \in \mathscr{A}_X^{\text{spin}}$ and $B \in \mathscr{A}_Y^{\text{spin}}$. For spin systems, this exactly allows changing the factor $|X|^n$ to $2\,|X|$ in the Lieb-Robinson bounds from [84], i.e. for spin systems one can obtain theorems 4 and 6 and corollary 5 with $\|\Phi\|_{\alpha,1}$ replaced by $\|\Phi\|_\alpha$ in the assumptions, after multiplying the bound with 2.

One can even apply the same principle to (24) again. First, use $\|[\tau_{t,s}(A), O_{\{y\}}]\| = \|[A, \tau_{s,t}(O_{\{y\}})]\|$ there, and then apply the bound (24) to this commutator to obtain

$$\begin{aligned} \big\|\big[\tau_{t,s}(A), B\big]\big\| &\le 4\,\|A\|\,\|B\| \sum_{y\in Y} \sup_{O_y \in \mathscr{A}_{\{y\}}:\|O_y\|=1} \sum_{x\in X} \sup_{O_x \in \mathscr{A}_{\{x\}}:\|O_x\|=1} \big\|\big[\tau_{t,s}(O_x), O_y\big]\big\| \\ &\le 4\,\|A\|\,\|B\| \sum_{x\in X}\sum_{y\in Y} f\big(|t-s|, d(x,y), 1\big). \end{aligned} \tag{26}$$

This for example allows deducing a Lieb-Robinson bound

$$\big\|\big[\tau_{t,s}(A), B\big]\big\| \le C\,\|A\|\,\|B\|\,|X|\,\mathrm{e}^{v|t-s|}\, F_{\alpha-D}\big(d(X,Y)\big)$$

for all $X, Y \subset \Lambda$, $A \in \mathscr{A}_X^{\text{spin}}$ and $B \in \mathscr{A}_Y^{\text{spin}}$ from the Lieb-Robinson bound from [118], which scales with $|X|\,|Y|$ and thus seems useless for bounding $\|(\mathrm{id} - \mathbb{E}_{X_r})\big(\tau_{t,s}(A)\big)\|$ using the conditional expectation at first glance.

6.3 Difference in lattice fermions

We now explain, why the above argument does not apply to fermionic systems. Closely comparing the statements of lemmata 11 and 19, the major complication for fermions lies in the fact that the bound in lemma 19(v) only holds for *even* A but requires a bound on $[A, B]$ for *all* $B \in \mathscr{A}_{\Lambda\setminus X}$. Moreover, the commutator Lieb-Robinson bounds can only give a bound if at least one of the observables is even, otherwise one needs to consider anti-commutators, see [173]. Hence, following the steps described in section 6.2, from a Lieb-Robinson bound as in (23) with the restriction that A or B are even, one obtains the Lieb-Robinson bound (25) for A *and* B even. Due to the latter constraint, such a bound is not enough to characterize the localization of $\tau_{t,s}(A)$ via the conditional expectation. Moreover, we see no chance to prove a bound as in (26) for fermions, because the argument requires a Lieb-Robinson bound for arbitrary $O_{\{x\}}$ and $O_{\{y\}}$, which does not hold for fermions.

If the goal is to obtain a good bound on the local approximation of $\tau_{t,s}(A)$, one might try to use (22) directly and aim for a bound which scales with $|X|$. Therefore, let us again assume a Lieb-Robinson bound as in (23) with the restriction that A or B must be even. Then, one can bound

$$\left\|\left(\mathrm{id}-\mathbb{E}_{X_r}\right)\left(\tau_{t,s}(A)\right)\right\| \leq \sum_{y\in\Lambda\setminus X_r}\left\|\left(\mathrm{id}-\mathbb{E}_{\Lambda\setminus\{y\}}\right)\left(\tau_{t,s}(A)\right)\right\| \leq \|A\| \sum_{y\in\Lambda\setminus X_r} f\left(|t-s|, d(X,y), 1\right).$$

This is in contrast to the approach for spin systems, where one first improves the Lieb-Robinson bound using the conditional expectation trick, and then directly bounds

$$\left\|\left(\mathrm{id}-\mathbb{E}_{X_r}\right)\left(\tau_{t,s}(A)\right)\right\| \leq 2\,\|A\|\,|X|\, f\left(|t-s|, d(X,Y), 1\right).$$

Remark 12. Instead of localizing $\tau_{t,s}(A)$ for $A \in \mathscr{A}_X$ with the conditional expectation, one can also use the truncated evolution $\tau_{t,s}^{X_r}(A) \in \mathscr{A}_{X_r}$, generated by the interaction $\Phi^{X_r}(Z) = \Phi(Z)$ if $Z \subset X_r$ and $\Phi^{X_r}(Z) = 0$ otherwise. Similarly to the proof of lemma 2, one is then left to bound

$$\left\|\tau_{t,s}(A) - \tau_{t,s}^{X_r}(A)\right\| \leq \sum_{\substack{Z\subset\Lambda:\\ Z\cap X_r\neq\emptyset\\ Z\cap\Lambda\setminus X_r\neq\emptyset}} \int_{\min\{s,t\}}^{\max\{s,t\}} \left\|\left[\Phi(Z,\theta), \tau_{\theta,s}^{X_r}(A)\right]\right\| \mathrm{d}\theta.$$

Here, one can absorb the factor $|Z|^n$ (instead of $|X|^n$) into the interaction norm $\|\Phi\|_{\alpha,n}$, similarly to the proof of lemma 3. However, while the Lieb-Robinson bound one obtains this way only scales linearly in $|X|$, the assumptions on Φ are much stronger than the ones we obtain in section 3. ◇

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A Proof of the finite-range Lieb-Robinson bound

Instead of the statement given in the main text, we will prove the following slightly stronger bound. It is stronger for small $|t-s| \ll 1/v$ and varying $\|\Phi\|_\alpha(r)$, which we cannot assume in the main text.

Proposition 13. *Let $D \in \mathbb{N}_+$, $\mathscr{C}_\mathrm{A} > 0$, $\alpha > D$ and $\Lambda \in \mathscr{G}(D, \mathscr{C}_\mathrm{A})$. Moreover, let $I \subset \mathbb{R}$ be an interval, $\Phi \in \mathscr{S}_\Lambda(I)$ be a time-dependent interaction and H_0 be an on-site Hamiltonian. Furthermore, let $X, Y \subset \Lambda$ be disjoint, $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$, with A or B even, and $R \geq 1$. Then, for any $s, t \in I$*

$$\left\|\left[\tau_{t,s,<R}(A), B\right]\right\| \leq 2\,\|A\|\,\|B\| \min\{|X|,|Y|\} \left(\mathrm{e}^{I_{t,s}(\Phi)} - 1\right) \mathrm{e}^{-d(X,Y)/R},$$

where

$$I_{t,s}(\Phi) = 2\mathrm{e}\,\|F_\alpha\|_\Lambda \int_{\min\{t,s\}}^{\max\{t,s\}} \|\Phi\|_\alpha(\theta)\,\mathrm{d}\theta \leq v|t-s|$$

and $v = 2\mathrm{e}\,\|F_\alpha\|_\Lambda\,\|\Phi\|_\alpha$.

Combining this with the trivial bound $\|[\tau_{t,s,<R}(A), B]\| \leq 2\,\|A\|\,\|B\|$ yields proposition 1. In particular, for $d(X,Y) = 0$, the argument of Δ in proposition 1 is larger than 2 and we use the trivial bound.

The proof of the Lieb-Robinson bound in proposition 13 uses the same strategy as the proof in [174] adjusted to our interaction norms. Additionally, we allow time-dependent on-site contributions which do not influence the bound. In [174] only time-independent but possibly unbounded on-site contributions were allowed by utilizing the interaction picture. To incorporate the range R in the bound, we use the trick from [160] to see that (28) vanishes for $m\,R \leq d(X,Y)$.

One might wonder why the bound does not seem to improve for larger α. Indeed, it is possible to obtain an extra factor $F_{\alpha-D}\big(d(X,Y)\big)$ after changing the constants. Therefore, one adjusts the bound in (31) and uses the convolution property of F_α, see [174]. However, since this does not improve the iteration and complicates the constants, we use the version given in proposition 13.

Before we prove proposition 13, let us give a lemma from [173]. It is proven using, variation of parameters and the fact that the dynamics of a self-adjoint operator are norm preserving.

Lemma 14 ([173, Lemma 3.2]). *Let $\mathscr{H}$ be a complex Hilbert space, $I \subset \mathbb{R}$ an interval and $A, B\colon I \to \mathscr{B}(\mathscr{H})$ be norm-continuous with A pointwise self-adjoint. Then, for any $s \in I$, the solution of*

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \mathrm{i}\left[A(t), f(t)\right] + B(t) \quad \text{with} \quad f(s) = f_0 \in \mathscr{B}(\mathscr{H})$$

satisfies

$$\|f(t)\| \leq \|f(s)\| + \int_{\min\{s,t\}}^{\max\{s,t\}} \|B(\theta)\| \,\mathrm{d}\theta.$$

Based on this, we find the following extension of [173, Lemma 3.3]. To state the lemma we need two more definitions. For $k \in \mathbb{N}$ and $X \subset \Lambda$ let

$$S(X) := \left\{ Z \subset \Lambda \,\middle|\, Z \cap X \neq \emptyset \text{ and } Z \cap (\Lambda \setminus X) \neq \emptyset \right\} \tag{27}$$

be the set of *boundary sets* of X. And for two sets $X, Y \subset \Lambda$, let $\delta_{X,Y} = 0$ if $X \cap Y = \emptyset$, and $\delta_{X,Y} = 1$ otherwise.

Lemma 15 (extension of [173, Lemma 3.3]). *Let $D \in \mathbb{N}_+$, $\mathscr{C}_\mathrm{A} > 0$ and $\Lambda \in \mathscr{G}(D, \mathscr{C}_\mathrm{A})$. Moreover, let $I \subset \mathbb{R}$ be an interval, $\Phi \in \mathcal{S}_\Lambda(I)$ be a time-dependent interaction and H_0 be an on-site Hamiltonian. Then, for all $X, Y \subset \Lambda$, $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$ with A or B even, it holds that*

$$\left\|\left[\tau_{t,s,<R}(A), B\right]\right\| \leq 2\,\|A\|\,\|B\|\,\delta_{X,Y} + 2\,\|A\| \sum_{\substack{Z \subset S(X):\\ \mathrm{diam}(Z)<R}} \int_{\min\{t,s\}}^{\max\{t,s\}} \left\|\left[\tau_{\theta,s,<R}\big(\Phi(Z,\theta)\big), B\right]\right\| \mathrm{d}\theta.$$

Proof. For better readability, we fix $R \in \mathbb{N}$ throughout the proof and drop the subscript '$_{<R}$'.

As before, let $H(t) = \sum_{Z \subset \Lambda : \mathrm{diam}(Z)<R} \Phi(Z,t) + H_0(t)$, $U(\cdot,\cdot)$ be the solution of the corresponding differential equation, and $\tau_{t,s}$ be the dynamics. Furthermore, let $H_X(t) = \sum_{Z \subset X : \mathrm{diam}(Z)<R} \Phi(Z,t) + \sum_{z \in X} h_z(t)$, $U_X(\cdot,\cdot)$ be the solution of the corresponding differential equation, and $\tau_{t,s,X}$ its dynamics. Fix $s, t \in I$ and define

$$f(\theta) = \left[\tau_{\theta,s} \circ \tau_{s,\theta,X} \circ \tau_{t,s,X}(A), B\right]$$

such that $f(t) = [\tau_{t,s}(A), B]$. It follows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta} f(\theta) &= \mathrm{i}\left[\tau_{\theta,s}\Big(\big[H(\theta) - H_X(\theta), \tau_{s,\theta,X} \circ \tau_{t,s,X}(A)\big]\Big), B\right] \\
&= \mathrm{i} \sum_{\substack{Z \subset S(X):\\ \mathrm{diam}(Z)<R}} \left[\left[\tau_{\theta,s}\big(\Phi(Z,\theta)\big), \tau_{\theta,s} \circ \tau_{s,\theta,X} \circ \tau_{t,s,X}(A)\right], B\right] \\
&= \mathrm{i} \sum_{\substack{Z \subset S(X):\\ \mathrm{diam}(Z)<R}} \left(\left[\tau_{\theta,s}\big(\Phi(Z,\theta)\big), f(\theta)\right] + \left[\tau_{\theta,s} \circ \tau_{s,\theta,X} \circ \tau_{t,s,X}(A), \left[B, \tau_{\theta,s}\big(\Phi(Z,\theta)\big)\right]\right]\right).
\end{aligned}$$

For the second equality, we use that the commutator of all terms of $H(\theta) - H_X(\theta)$ supported in $\Lambda \setminus X$ commute with $\tau_{s,\theta,X} \circ \tau_{t,s,X}(A) \subset \mathscr{A}_X$ since they are even. This especially includes all remaining on-site terms $h_z(\theta)$. For the last equality we used the Jacobi identity and the definition of $f(\theta)$. Applying lemma 14 to this equation and using the trivial bound for the outer commutator of the inhomogeneous term we find

$$\|f(t)\| \leq \left\|\left[\tau_{t,s,X}(A), B\right]\right\| + 2\,\|A\| \sum_{\substack{Z \subset S(X): \\ \operatorname{diam}(Z) < R}} \int_{\min\{s,t\}}^{\max\{s,t\}} \left\|\left[\tau_{\theta,s}(\Phi(Z,\theta)), B\right]\right\| \mathrm{d}\theta.$$

The lemma follows using the trivial bound for the first summand. □

We can now give the proof of the finite-range Lieb-Robinson bound.

Proof of proposition 13. For better readability, we fix $R \in \mathbb{N}$ throughout the proof and drop the subscript "$_{<R}$". Additionally, assume that $s < t$ and otherwise flip the integration boundaries in the proof.

Applying lemma 15 iteratively $N \in \mathbb{N}$ times and using the trivial bound for the integrand in the last step and $\delta_{X,Y} = 0$, we find

$$\left\|\left[\tau_{t,s}(A), B\right]\right\| \leq 2\,\|A\|\,\|B\| \left(\sum_{m=1}^{N} a_m(t) + R_{N+1}(t)\right)$$

where

$$a_m(t) = 2^m \sum_{\substack{Z_1 \subset S(X): \\ \operatorname{diam}(Z_1) < R}} \sum_{\substack{Z_2 \subset S(Z_1): \\ \operatorname{diam}(Z_2) < R}} \cdots \sum_{\substack{Z_m \subset S(Z_{m-1}): \\ \operatorname{diam}(Z_m) < R}} \delta_{Y,Z_m} \int_s^t \int_s^{\theta_1} \cdots \int_s^{\theta_{m-1}} \left(\prod_{j=1}^{m} \|\Phi(Z_j,\theta_j)\|\right) \mathrm{d}\theta_m \cdots \mathrm{d}\theta_1 \tag{28}$$

and

$$R_{N+1}(t) = 2^{N+1} \sum_{\substack{Z_1 \subset S(X): \\ \operatorname{diam}(Z_1) < R}} \sum_{\substack{Z_2 \subset S(Z_1): \\ \operatorname{diam}(Z_2) < R}} \cdots \sum_{\substack{Z_{N+1} \subset S(Z_N): \\ \operatorname{diam}(Z_{N+1}) < R}} \times \int_s^t \int_s^{\theta_1} \cdots \int_s^{\theta_N} \left(\prod_{j=1}^{N+1} \|\Phi(Z_j,\theta_j)\|\right) \mathrm{d}\theta_{N+1} \cdots \mathrm{d}\theta_1. \tag{29}$$

We proceed with bounds for $a_m(t)$. In the same way, one shows that $R_{N+1}(t) \to 0$ as $N \to \infty$. Noting that the sets $X, Z_1, \ldots, Z_m, Y$ form a chain of pairwise intersecting sets, one can estimate

$$\sum_{\substack{Z_1\subset S(X):\\ \mathrm{diam}(Z_1)<R}} \sum_{\substack{Z_2\subset S(Z_1):\\ \mathrm{diam}(Z_2)<R}} \cdots \sum_{\substack{Z_m\subset S(Z_{m-1}):\\ \mathrm{diam}(Z_m)<R}} \delta_{Y,Z_m} *$$
$$\leq \sum_{w_1\in X} \sum_{w_2,\dots,w_m\in\Lambda} \sum_{w_{m+1}\in Y} \sum_{\substack{Z_1\subset\Lambda:\\ w_1,w_2\in Z_1,\\ \mathrm{diam}(Z_1)<R}} \sum_{\substack{Z_2\subset\Lambda:\\ w_2,w_3\in Z_2,\\ \mathrm{diam}(Z_2)<R}} \cdots \sum_{\substack{Z_m\subset\Lambda:\\ w_m,w_{m+1}\in Z_m,\\ \mathrm{diam}(Z_m)<R}} * \tag{30}$$

where the $*$ denotes arbitrary positive terms depending on all Z_j. With that and bounding sums of products with products of sums one arrives at terms

$$\sum_{\substack{Z_j\subset\Lambda:\\ w_j,w_{j+1}\in Z_j,\\ \mathrm{diam}(Z_j)<R}} \|\Phi(Z_j,\theta_j)\| \leq \|\Phi(\theta_j)\|_\alpha\, F_\alpha\big(d(w_j,w_{j+1})\big).$$

Putting everything together, we obtain

$$\begin{aligned} a_m(t) &\leq 2^m \sum_{w_1\in X} \sum_{w_2,\dots,w_m\in\Lambda} \sum_{w_{m+1}\in Y} \int_s^t \int_s^{\theta_1} \cdots \int_s^{\theta_{m-1}} \left(\prod_{j=1}^m \|\Phi(\theta_j)\|_\alpha F_\alpha\big(d(w_j,w_{j+1})\big) \right) \mathrm{d}\theta_m \cdots \mathrm{d}\theta_1 \\ &\leq \min\{|X|,|Y|\}\, \frac{1}{m!} \left(2\,\|F_\alpha\|_\Lambda \int_s^t \|\Phi(\theta)\|_\alpha\, \mathrm{d}\theta \right)^m . \end{aligned} \tag{31}$$

But from (28) it is also apparent that $a_m(t) = 0$ for $m\,R \leq d(X,Y)$. Hence, taking the limit $N\to\infty$ we have

$$\begin{aligned} \big\|\big[\tau_{t,s}(A),B\big]\big\| &\leq 2\,\|A\|\,\|B\| \min\{|X|,|Y|\} \sum_{m>d(X,Y)/R} \frac{1}{m!} \left(2\,\|F_\alpha\|_\Lambda \int_s^t \|\Phi(\theta)\|_\alpha\, \mathrm{d}\theta \right)^m \mathrm{e}^{m-d(X,Y)/R} \\ &\leq 2\,\|A\|\,\|B\| \min\{|X|,|Y|\} \sum_{m\geq 1} \frac{1}{m!} \left(2\mathrm{e}\,\|F_\alpha\|_\Lambda \int_s^t \|\Phi(\theta)\|_\alpha\, \mathrm{d}\theta \right)^m \mathrm{e}^{-d(X,Y)/R}, \end{aligned}$$

where $\mathrm{e}^{m-d(X,Y)/R} \geq 1$ was added to each summand in the first step. □

B Proof of the range-splitting lemma

The proof of lemma 2 uses the techniques from appendix A.

Proof of lemma 2. We again fix $R\in\mathbb{N}$ throughout the proof and drop the subscript "$_{<R}$". Let

$$H_{<R'}(t) = \sum_{\substack{Z\subset\Lambda:\\ \mathrm{diam}(Z)<R'}} \Phi(Z,t) + H_0(t) \quad\text{and}\quad H_{\geq R'}(t) = \sum_{\substack{Z\subset\Lambda:\\ R'\leq\mathrm{diam}(Z)<R}} \Phi(Z,t).$$

Furthermore, let $U(\cdot,\cdot)$ and $U_{<R'}(\cdot,\cdot)$ be the solution of the differential equations corresponding to $H(t) = H_{<R'}(t) + H_{\geq R'}(t)$ and $H_{<R'}$, respectively, and $\tau_{t,s}$ and $\tau_{t,s,<R'}$ their dynamics. Let $\mathscr{U}(t,s) = U_{<R'}(t,s)^* \, U(t,s)$. Fix $s,t \in I$ and define

$$f(\theta) = \left[\tau_{t,s,<R'}(A), \mathscr{U}(\theta,s)\, B\, \mathscr{U}(\theta,s)^*\right].$$

Then $\|f(t)\| = \|[\tau_{t,s}(A), B]\|$. Furthermore, from

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta} f(\theta) = {}& \left[-\mathrm{i}\, \tau_{\theta,s}\big(H_{\geq R'}(\theta)\big), f(\theta)\right] \\ & + \mathrm{i}\Big[\mathscr{U}(\theta,s)\, B\, \mathscr{U}(\theta,s)^*, \left[\tau_{t,s,<R'}(A), \tau_{\theta,s,<R'}\big(H_{\geq R'}(\theta)\big)\right]\Big]\end{aligned}$$

it follows with lemma 14 that

$$\begin{aligned}\big\|\left[\tau_{t,s}(A), B\right]\big\| &\leq \big\|\left[\tau_{t,s,<R'}(A), B\right]\big\| + 2\,\|B\| \int_{\min\{s,t\}}^{\max\{s,t\}} \big\|\left[\tau_{t,s,<R'}(A), \tau_{\theta,s,<R'}(H_{\geq R'}(\theta))\right]\big\| \,\mathrm{d}\theta \\ &\leq \big\|\left[\tau_{t,s,<R'}(A), B\right]\big\| + 2\,\|B\| \sum_{\substack{Z\subset\Lambda:\\ R'\leq \mathrm{diam}(Z)<R}} \int_{\min\{s,t\}}^{\max\{s,t\}} \big\|\left[\tau_{t,\theta,<R'}(A), \Phi(Z,\theta)\right]\big\| \,\mathrm{d}\theta.\end{aligned}$$

□

C Proof of the iteratively improved Lieb-Robinson bound

For the proof of theorem 6, we follow the strategy from [84] and iteratively apply lemma 3 to obtain an improved Lieb-Robinson bound. We repeat the proof here to achieve the better dependence on $|X|$, adjust the proof to our definition of the graphs and metrics and find the correct scaling of the constant depending on σ.

The lattice parameters $\mathscr{C}_{\mathrm{A}}$ and D as well as the decay exponent α are fixed in the beginning, and all the constants might depend on them. Importantly, the constants do not depend on the specific lattice $\Lambda \in \mathscr{G}(D, \mathscr{C}_{\mathrm{A}})$, the interaction or the parameter σ.

Proof of theorem 6. First notice that the theorem is trivially satisfied for $r = 0$ by using the trivial bound. Hence, we assume $r \geq 1$ in the following.

The following lemma is similar to [84, Lemma 2] adopted to our geometries. It allows bounding $\|\lambda\|_{\Lambda}$ by an integral as

$$\|\lambda\|_{\Lambda} \leq \lambda(0) + \mathscr{C}^{\mathrm{Lem.\,16}} \int_{1/2}^{\infty} \lambda(\rho)\, \rho^{D-1}\, \mathrm{d}\rho. \tag{32}$$

Lemma 16. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a monotonically decreasing function, $\mathscr{C}_{\mathrm{A}} > 0$, $D \in \mathbb{N}$, $\Lambda \in \mathcal{G}(D, \mathscr{C}_{\mathrm{A}})$, $x \in \Lambda$ and $R \geq 1$. Then, it holds that*

$$\sum_{\substack{z \in \Lambda: \\ 1 \leq d(x,z) \leq R}} f\big(d(x,z)\big) \leq \mathscr{C}^{\mathrm{Lem.\,16}} \int_{1/2}^{R} f(r)\, r^{D-1}\, \mathrm{d}r,$$

with $\mathscr{C}^{\mathrm{Lem.\,16}} = 2^D \mathscr{C}_{\mathrm{A}} \geq 1$.

Proof. Since $d(\cdot,\cdot)$ is integer valued and f is monotonically decreasing we find

$$\begin{aligned}
\sum_{\substack{z \in \Lambda: \\ 1 \leq d(x,z) \leq R}} f\big(d(x,z)\big) &= \sum_{j=1}^{R} \sum_{\substack{z \in \Lambda: \\ d(x,z) = j}} f\big(d(x,z)\big) \\
&\leq \mathscr{C}_{\mathrm{A}} \sum_{j=1}^{R} j^{D-1} f(j) \\
&\leq 2\, \mathscr{C}_{\mathrm{A}} \int_{1/2}^{R} f(r)\, (r + 1/2)^{D-1}\, \mathrm{d}r \\
&\leq 2^D \mathscr{C}_{\mathrm{A}} \int_{1/2}^{R} f(r)\, r^{D-1}\, \mathrm{d}r.
\end{aligned}$$ □

To bound the integral in (32), we also need the following lemma from [84]. Since we want to provide explicit scaling, we state the result with an explicit constant that can easily be deduced from the original proof in [84].

Lemma 17 ([84, Lemma 3])**.** *For $\mu \in \mathbb{R}$ and $\nu > 0$ and*

$$\mathscr{C}^{\mathrm{Lem.\,17}}_{\mu,\nu} = \tfrac{1}{\nu} \max\Big\{1, \mathrm{e}\, \Gamma\Big(\tfrac{\mu+1}{\nu}\Big)\Big\},$$

where Γ is the gamma function, it holds that

$$\int_{\rho}^{\infty} \mathrm{e}^{-x^{\nu}} x^{\mu}\, \mathrm{d}x \leq \mathscr{C}^{\mathrm{Lem.\,17}}_{\mu,\nu}\, \mathrm{e}^{-\rho^{\nu}} \big(1 + \rho^{\mu-\nu+1}\big) \quad \textit{for all } \rho > 0.$$

Now fix t and s, and write $\delta = |t - s|$. We begin with the finite range Lieb-Robinson bound from proposition 1, $\lambda^{(0)}_{R'}(r) = \Delta(2\, \mathrm{e}^{v\delta - r/R'})$. The trivial bound is in particular better for $r \leq R' v\delta$. Hence, for $R' v\delta > 1/2$, we find

$$\begin{aligned}
\|\lambda^{(0)}_{R'}\|_{\Lambda} &\leq 2 + 2\, \mathscr{C}^{\mathrm{Lem.\,16}} \left(\int_{1/2}^{R' v\delta} \rho^{D-1}\, \mathrm{d}\rho + \int_{R' v\delta}^{\infty} \mathrm{e}^{v\delta - \rho/R'}\, \rho^{D-1}\, \mathrm{d}\rho \right) \\
&\leq 2\, \mathscr{C}^{\mathrm{Lem.\,16}} \Big(1 + (R' v\delta)^D + \mathscr{C}^{\mathrm{Lem.\,17}}_{D-1,1}\, \mathrm{e}^{v\delta} R'^D\, \mathrm{e}^{-v\delta} \big(1 + (v\delta)^{D-1}\big)\Big) \\
&\leq 6\, \mathscr{C}^{\mathrm{Lem.\,16}} \mathscr{C}^{\mathrm{Lem.\,17}}_{D-1,1}\, R'^D \big(1 + (v\delta)^D\big)
\end{aligned}$$

where we substituted $\rho/R' \to \rho$ and used lemma 17 in the second step. For the last step, we restricted to $R' \geq 1$. For $R' v\delta \leq 1/2$, we do not split the integral and instead obtain

$$\begin{aligned}\|\lambda_{R'}^{(0)}\|_\Lambda &\leq 2 + 2\,\mathscr{C}^{\text{Lem. 16}} \int_{1/2}^{\infty} \mathrm{e}^{v\delta-\rho/R'}\,\rho^{D-1}\,\mathrm{d}\rho \\ &\leq 2\,\mathscr{C}^{\text{Lem. 16}}\Big(1 + \mathscr{C}_{D-1,1}^{\text{Lem. 17}}\,\mathrm{e}^{v\delta} R'^D\,\mathrm{e}^{-\frac{1}{2R'}}\Big(1+\big(\tfrac{1}{2R'}\big)^{D-1}\Big)\Big) \\ &\leq 6\,\mathscr{C}^{\text{Lem. 16}}\,\mathscr{C}_{D-1,1}^{\text{Lem. 17}}\,R'^D\,\big(1+(v\delta)^D\big).\end{aligned}$$

In the last step, we added the term $(v\delta)^D$ to obtain the same bound as for $R' v\delta > 1/2$.

Setting $R' = r^\sigma$, lemma 3 yields the improved Lieb-Robinson bound

$$\lambda_R^{(1)}(r) \leq \Delta\Big(2\,\mathrm{e}^{v\delta-r^{1-\sigma}} + 2\,C_1\,\mathbb{1}_{R>r^\sigma}\,\delta\,\|\Phi\|_{\alpha,1}\,(r+1)^{-\sigma(\alpha-D)}\,\big(1+(v\delta)^D\big)\Big),$$

with $C_1 = 2^{\alpha(1-\sigma)+1}\,3\,\mathscr{C}^{\text{Lem. 16}}\,\mathscr{C}_{D-1,1}^{\text{Lem. 17}}$. The additional prefactor comes from bounding $F_\alpha(r^\sigma) \leq 2^{\alpha(1-\sigma)}\,(1+r)^{\sigma\alpha}$.

We now proceed by induction. Assume that we have a Lieb-Robinson bound with

$$\lambda_R^{(n)}(r) \leq \Delta\Big(2\,\mathrm{e}^{v\delta-r^{1-\sigma}} + 2\,\mathbb{1}_{R>r^\sigma} \sum_{i=1,2} f_i^{(n)}(v\delta)\,(r+1)^{\mu_i^{(n)}}\Big). \tag{33}$$

For $n = 1$ this is satisfied for

$$\begin{aligned} f_1^{(1)}(\tau) &= C_1\,(\tau + \tau^{D+1}), & \mu_1^{(1)} &= \sigma\,(-\alpha + D), \\ f_2^{(1)}(\tau) &= 0 \quad \text{and} & \mu_2^{(1)} &= -\sigma\alpha, \end{aligned}$$

when we redefine $v = \max\{2\mathrm{e}\,\|F_\alpha\|_\Lambda\,\|\Phi\|_\alpha, \|\Phi\|_{\alpha,1}\}$ to keep the constant independent of Φ.

We now calculate $\|\lambda_{R'}^{(n)}\|_\Lambda$. Only looking at the exponential term in (33), we notice that the trivial bound is better at least for $r < (v\delta)^{1/(1-\sigma)}$. We first consider the case $v\delta \geq 1$. Additionally, we assume $D + \mu_i^{(n)} \neq 0$ which we ensure later by changing the iteration at the right step. Then

$$\|\lambda_{R'}^{(n)}\|_\Lambda \leq 2 + 2\,\mathscr{C}^{\text{Lem. 16}} \int_{1/2}^{(v\delta)^{1/(1-\sigma)}} \rho^{D-1}\,\mathrm{d}\rho + \mathscr{C}^{\text{Lem. 16}} \int_{(v\delta)^{1/(1-\sigma)}}^{\infty} \rho^{D-1}\,\lambda_{R'}^{(n)}(\rho)\,\mathrm{d}\rho$$

(34)

$$\begin{aligned}
&\leq 2\,\mathscr{C}^{\text{Lem. 16}}\Bigg(1+(v\delta)^{\frac{D}{1-\sigma}}+\int_{(v\delta)^{1/(1-\sigma)}}^{\infty}\rho^{D-1}\,\mathrm{e}^{v\delta-\rho^{1-\sigma}}\,\mathrm{d}\rho \\
&\qquad\qquad+\sum_{i=1,2}\int_{(v\delta)^{1/(1-\sigma)}}^{R'^{1/\sigma}} f_i^{(n)}(v\delta)\,(\rho+1)^{D-1+\mu_i^{(n)}}\,\mathrm{d}\rho\Bigg) \qquad (35)\\
&\leq 2\,\mathscr{C}^{\text{Lem. 16}}\Bigg(1+(v\delta)^{\frac{D}{1-\sigma}}+\mathscr{C}^{\text{Lem. 17}}_{D-1,1-\sigma}\left(1+(v\delta)^{\frac{D}{1-\sigma}-1}\right)\\
&\qquad\qquad+\sum_{i=1,2}\frac{f_i^{(n)}(v\delta)}{D+\mu_i^{(n)}}\,(\rho+1)^{D+\mu_i^{(n)}}\Big|_{(v\delta)^{1/(1-\sigma)}}^{R'^{1/\sigma}}\Bigg)\\
&\leq 4\,\mathscr{C}^{\text{Lem. 16}}\,\mathscr{C}^{\text{Lem. 17}}_{D-1,1-\sigma}\left(1+(v\delta)^{\frac{D}{1-\sigma}}\right)\\
&\qquad+2\,\mathscr{C}^{\text{Lem. 16}}\sum_{i=1,2}\frac{f_i^{(n)}(v\delta)}{\left|D+\mu_i^{(n)}\right|}\begin{cases}\left(R'^{1/\sigma}+1\right)^{D+\mu_i^{(n)}} & D+\mu_i^{(n)}>0\\ \left(v\delta\right)^{(D+\mu_i^{(n)})/(1-\sigma)} & D+\mu_i^{(n)}<0.\end{cases}
\end{aligned}$$

In the last step, we bounded the last summand by the upper limit of the integration if $D+\mu_i^{(n)}>0$ and by the lower if $D+\mu_i^{(n)}<0$ because the other limit gives a negative contribution in each case. In the latter case we moreover removed the " $+\,1$" in the parenthesis.

Using lemma 3 and setting $R'=r^{\sigma}$, we obtain $\lambda_R^{(n+1)}$ as in (33) with

$$\begin{aligned}
f_1^{(n+1)}(\tau)&=\tfrac{C_0}{|D+\mu_1^{(n)}|}\,\tau\,f_1^{(n)}(\tau), & \mu_1^{(n+1)}&=-\sigma\,\alpha+D+\mu_1^{(n)} && \text{if } D+\mu_1^{(n)}>0,\\
f_1^{(n+1)}(\tau)&=\tfrac{C_0}{|D+\mu_1^{(n)}|}\,\tau^{1+(D+\mu_1^{(n)})/(1-\sigma)}f_1^{(n)}(\tau), & \mu_1^{(n+1)}&=-\sigma\alpha && \text{if } D+\mu_1^{(n)}<0,
\end{aligned}$$

and

$$f_2^{(n+1)}(\tau)=C_2\left(\tau+\tau^{1+D/(1-\sigma)}\right)+\tfrac{C_0}{|D-\sigma\alpha|}\,\tau^{1+(D-\sigma\alpha)/(1-\sigma)}f_2^{(n)}(\tau),\qquad \mu_2^{(n+1)}=-\sigma\alpha.$$

The constants are given by $C_0=2^{\alpha(1-\sigma)+1}\,\mathscr{C}^{\text{Lem. 16}}$ and $C_2=2^{\alpha(1-\sigma)+2}\,\mathscr{C}^{\text{Lem. 16}}\,\mathscr{C}^{\text{Lem. 17}}_{D-1,1-\sigma}$ and for $f_2^{(n)}$ we already used that $\mu_2^{(n)}=-\sigma\alpha$ is constant.

Let $\varepsilon=\sigma\alpha-D>0$, then the iteration directly yields

$$f_2^{(n)}(\tau)=C_2\,(\tau+\tau^{1+D/(1-\sigma)})\sum_{j=0}^{n-2}\left(\frac{C_0}{\varepsilon}\right)^{j}\tau^{-j\frac{\sigma(\alpha+1)-(D+1)}{1-\sigma}},$$

so that we can expect a polynomial scaling in τ with exponent at least $1+D/(1-\sigma)$.

Now assume that $D+\mu_1^{(1)}>0$, then

$$\mu_1^{(n)}=-n\sigma\alpha+(n-1+\sigma)\,D$$

decreases by ε in each iteration step until $\mu_1^{(n)} - D \leq 0$. To avoid small $\left|D + \mu_1^{(n)}\right|$, which would give large multiplicative constants, we tweak the iteration as follows: First choose $\eta \in (0, 1/2)$ and

$$n_* = \left\lceil \frac{\sigma D}{\sigma\alpha - D} - (1-\eta) \right\rceil < \frac{\sigma D}{\sigma\alpha - D} + \eta$$

so that the basic iteration from above gives $D + \mu_1^{(n_*)} \in \left(-\varepsilon\,\eta, \varepsilon\,(1-\eta)\right]$.

If $D + \mu_1^{(n_*)} \in \left(\varepsilon\,\eta, \varepsilon\,(1-\eta)\right]$, we just continue the iteration and obtain $D + \mu_1^{(n_*+1)} = -\varepsilon\,\eta < 0$ and

$$f_1^{(n_*+2)}(\tau) \leq \left(\tfrac{C_0}{\varepsilon}\right)^{n_*+1} \tfrac{1}{\eta^2}\, \tau^{1+\frac{D}{1-\sigma}}\, C_1 \left(\tau^{-D} + 1\right),$$

because

$$1 + \frac{D + \mu_1^{(n_*+1)}}{1-\sigma} + (n_* + 1) + D = 1 + \frac{D}{1-\sigma} - \frac{n_* + 1}{\sigma - 1}\big(\sigma\,(\alpha+1) - (D+1)\big) \leq 1 + \frac{D}{1-\sigma}.$$

Thus, the scaling in τ is already better than that of $f_2^{(n)}$ and also the spatial decay does not improve in further steps.

Otherwise, if the basic iteration yields $D + \mu_1^{(n_*)} \in \left(-\varepsilon\,\eta, \varepsilon\,\eta\right]$, we instead choose $\mu_1^{(n_*)} = -D + \varepsilon\,\eta$ by bounding the term in (33). Then, $D + \mu_1^{(n_*+1)} = -\varepsilon\,(1-\eta) < 0$. And the iteration results in

$$f_1^{(n_*+2)}(\tau) \leq \left(\tfrac{C_0}{\varepsilon}\right)^{n_*+1} \tfrac{4}{\eta}\, \tau^{1+\frac{D}{1-\sigma}+\eta\left(1+\frac{\sigma\,\alpha - D}{1-\sigma}\right)}\, C_1 \left(\tau^{-D} + 1\right).$$

After a further iteration step and choosing $\eta = \sigma - \frac{D+1}{\alpha+1}$, we obtain

$$f_1^{(n_*+3)}(\tau) \leq \left(\tfrac{C_0}{\varepsilon}\right)^{n_*+2} \tfrac{4}{\eta}\, \tau^{1+\frac{D}{1-\sigma}}\, C_1 \left(\tau^{-D} + 1\right),$$

which again has a better scaling in τ than $f_2^{(n)}$.

Since we can bound

$$\varepsilon > \tfrac{\alpha - D}{\alpha+1} > 0 \qquad \text{and} \qquad n_* < \tfrac{2\,\alpha\,D}{\alpha - D} + \tfrac{3}{2},$$

the only σ-dependence of the constants left is in

$$\mathscr{C}^{\text{Lem. 17}}_{D-1,1-\sigma} < \tfrac{1}{1-\sigma}\,\Gamma\!\left(\tfrac{D}{1-\sigma}\right) \qquad \text{and} \qquad \eta = \sigma - \tfrac{D+1}{\alpha+1}$$

and leads to divergences at the boundaries of the interval of allowed σ. This completes the first part of the proof.

We are left with the case $v\delta < 1$, where we do not split the integral in (34) and instead obtain

$$\mathcal{I}[\lambda_{R'}^{(n)}] \leq 16\, \mathscr{C}^{\text{Lem. 16}}\, \mathscr{C}^{\text{Lem. 17}}_{D-1,1-\sigma} + 2\, \mathscr{C}^{\text{Lem. 16}} \sum_{i=1,2} \frac{f_i^{(n)}(v\delta)}{\left|D + \mu_i^{(n)}\right|} \begin{cases} \left(R'^{1/\sigma} + 1\right)^{D+\mu_i^{(n)}} & D + \mu_i^{(n)} > 0 \\ \left(\frac{3}{2}\right)^{D+\mu_i^{(n)}} & D + \mu_i^{(n)} < 0. \end{cases}$$

This results in

$$f_1^{(n+1)}(\tau) = \frac{C_0}{|D+\mu_1^{(n)}|}\, \tau\, f_1^{(n)}(\tau), \qquad \begin{aligned} \mu_1^{(n+1)} &= -\sigma\,\alpha + D + \mu_1^{(n)} && \text{if } D + \mu_1^{(n)} > 0, \\ \mu_1^{(n+1)} &= -\sigma\,\alpha && \text{if } D + \mu_1^{(n)} < 0, \end{aligned}$$

and

$$f_2^{(n+1)}(\tau) = 4\, C_2\, \tau + \frac{C_0}{|D-\sigma\alpha|}\, \tau\, f_2^{(n)}(\tau), \quad \mu_2^{(n+1)} = -\sigma\alpha$$

with the same adjustment of the n_* step as before, and yields

$$f_1^{(n)}(\tau) = \left(\frac{C_0}{\varepsilon}\right)^{n-1} \frac{2}{\eta^2}\, C_1\, \tau^n \left(1 + \tau^D\right), \qquad f_2^{(n)}(\tau) = 4\, C_2 \sum_{j=1}^{n-1} \left(\frac{C_0}{\varepsilon}\right)^j \tau^j$$

and $\mu_1^{(n)} = \mu_2^{(n)} = -\sigma\alpha$ for $n \geq n_* + 2$. We again stop at $n = n_* + 2$ or $n = n_* + 3$ and the lowest power in τ (recall that $\tau < 1$ here) is 1. Putting everything together proves the theorem. □

D The spectral flow and its decay properties

In this section we define the spectral flow $\mathcal{I}_{H,g,\delta}$ and prove its decay properties as stated in proposition 8. Therefore, let $g > \delta \geq 0$ and $\mathscr{W}_{g,\delta} \in L^1(\mathbb{R})$ be a function satisfying

$$\sup_{|s|>1} |s|^n\, |\mathscr{W}_{g,\delta}(s)| < \infty \quad \text{for all } n \in \mathbb{N}$$

with Fourier transform $\hat{\mathscr{W}}_{g,\delta} \in C^\infty(\mathbb{R})$ satisfying

$$\hat{\mathscr{W}}_{g,\delta}(\omega) = \frac{-\mathrm{i}}{\sqrt{2\pi}\,\omega} \quad \text{for all } |\omega| \geq g \qquad \text{and} \qquad \hat{\mathscr{W}}_{g,\delta}(\omega) = 0 \quad \text{for all } |\omega| \leq \delta.$$

For $\delta = 0$ such a function was constructed in [23], it additionally satisfies $\|\mathscr{W}_{g,0}\|_{L^1} = 1$ and $\|\mathscr{W}_{g,0}\|_\infty = 1/2$, and is enough to prove automorphic equivalence. This function was then modified in [165, 203] for $\delta > 0$, because the spectral flow has additional properties for larger δ, see proposition 18.

On a general Hilbert space we now find the following proposition. The proof and more details are given in [165, 203]. The general idea goes back to the construction of the spectral flow in [113, 120], see also [23, 17].

For any self-adjoint Hamiltonian $H \in \mathcal{A}_\Lambda^+$, we can define

$$\mathcal{I}_{H,g,\delta} \colon \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda, \quad \mathcal{I}_{H,g,\delta}(A) = \int_{\mathbb{R}} \mathcal{W}_{g,\delta}(s)\, \mathrm{e}^{\mathrm{i}Hs}\, A\, \mathrm{e}^{-\mathrm{i}Hs}\, \mathrm{d}s \tag{36}$$

which by the properties of $\mathcal{W}$ satisfies the following.

Proposition 18. *Let $H \in \mathcal{A}_\Lambda^+$ be self-adjoint and assume that the spectrum $\sigma(H)$ has a gapped part $\sigma_* \subset \sigma(H)$ such that $\sigma_* \subset I$, $\sigma(H) \setminus \sigma_* \subset \mathbb{R} \setminus I$ and $\operatorname{dist}(\sigma_*, \sigma(H) \setminus \sigma_*) \geq g$ for some interval $I \subset \mathbb{R}$ and $g > 0$. Let P be the spectral projection corresponding to σ_*. Then*

$$A = -\mathrm{i}\left[H, \mathcal{I}_{H,g,\delta}(A)\right]$$

for all $A \in \mathcal{A}_\Lambda$ satisfying $A = PAP^\perp + P^\perp AP$, where $P^\perp = \mathbb{1} - P$. I.e. $\mathcal{I}_{H,g,\delta}$ is the inverse of the Liouvillian $A \mapsto -\mathrm{i}\left[H, A\right]$ for off-diagonal (w.r.t. P) A. Moreover, if $\operatorname{diam}(\sigma_*) < \delta$ then

$$P\, \mathcal{I}_{H,g,\delta}(A)\, P = 0 \quad \textit{for all} \quad A \in \mathcal{A}_\Lambda.$$

As a simple consequence (see [17, Corollary 4.2]) $G(s) = \mathcal{I}_{H(s),g,\delta}(\dot{H}(s))$ indeed generates the spectral flow as a simple calculation shows

$$-\mathrm{i}\,\dot{P} = -\left[H, \mathcal{I}_{H,g,\delta}(\dot{P})\right] = -\mathcal{I}_{H,g,\delta}\left(\left[H, \dot{P}\right]\right) = \mathcal{I}_{H,g,\delta}\left(\left[\dot{H}, P\right]\right) = \left[\mathcal{I}_{H,g,\delta}(\dot{H}), P\right],$$

where we only used proposition 18 and off-diagonality of $\dot{P}$ in the first step and the integral form of $\mathcal{I}_{H,g,\delta}$ and $[H, P] = 0$ in all the other steps.

It is left to show that $\mathcal{I}_{H(t),g,\delta}(\dot{H}(t))$ can be given by a polynomially decaying interaction. Therefore, we approximate the individual terms using a conditional expectation. For fermionic lattice systems, such a conditional expectation was constructed in [173]. We collect its properties in the following lemma.

Lemma 19 ([173, Lemma 4.1, 4.2 and 4.3])**.** *Let $X \subset \Lambda$. Then there exists a unit-preserving, completely positive linear map $\mathbb{E}_X \colon \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ satisfying*

(i) $\mathbb{E}_X(\mathcal{A}_\Lambda^+) \subset \mathcal{A}_X^+$;

(ii) $\mathbb{E}_X(ABC) = A\, \mathbb{E}_X(B)\, C$ *for all* $B \in \mathcal{A}_\Lambda^+$ *and* $A, C \in \mathcal{A}_X^+$; *This in particular implies* $\mathbb{E}_X(A) = A$ *for all* $A \in \mathcal{A}_X^+$;

(iii) $\|\mathbb{E}_X\| = 1$;

(iv) $\mathbb{E}_X \circ \mathbb{E}_Y = \mathbb{E}_{X \cap Y}$, *for* $X, Y \subset \Lambda$;

(v) *If* $A \in \mathcal{A}_\Lambda^+$ *satisfies*

$$\left\|[A, B]\right\| \leq \eta\, \|A\|\, \|B\| \quad \textit{for all} \quad B \in \mathcal{A}_{\Lambda \setminus X}, \tag{37}$$

for some $\eta > 0$, *then*

$$\|A - \mathbb{E}_X(A)\| \leq \eta\, \|A\|. \tag{38}$$

This allows to construct a decaying interaction representing $\mathscr{I}_{H,g,\delta}(K)$.

Proof of proposition 8. For the proof we first fix D, $\mathscr{C}_{\mathrm{A}}$, n, β, g, δ, v_*, and choose α large enough (the possible values will be clear later on). In the following we allow the constant C to change in each step depending on all above-mentioned constants. Importantly, they do not depend on Λ nor Φ directly.

To prove the claim, we use the local decomposition technique: Each term $\tau_s(\Phi_K(Z))$ is split into a sum of terms in $\mathscr{A}^+_{Z_j}$ with norm decreasing rapidly in j, where Z_j denotes the fattening defined in (4).

For any $O \in \mathscr{A}_\Omega$ define

$$\Delta_0(O) = \mathrm{i} \int_{\mathbb{R}} \mathscr{W}_{g,\delta}(s)\, \mathbb{E}_\Omega\big(\tau_s(O)\big)\, \mathrm{d}s$$

and, for $j \geq 1$

$$\begin{aligned}\Delta_j(O) &= \mathrm{i} \int_{\mathbb{R}} \mathscr{W}_{g,\delta}(s)\, \mathbb{E}_{\Omega_j}\big(\tau_s(O)\big) - \mathbb{E}_{\Omega_{j-1}}\big(\tau_s(O)\big)\, \mathrm{d}s \\ &= \mathrm{i} \int_{\mathbb{R}} \mathscr{W}_{g,\delta}(s)\, \mathbb{E}_{\Omega_j} \circ \big(\mathbb{1} - \mathbb{E}_{\Omega_{j-1}}\big)\big(\tau_s(O)\big)\, \mathrm{d}s.\end{aligned}$$

Then $\mathscr{I}_{H,g,\delta}(O) = \sum_{j=0}^\infty \Delta_j(O)$ where the sum is finite because Λ is finite.

For $j = 0$ we have the trivial bound

$$\|\Delta_0(O)\| \leq \|O\|\, \|\mathscr{W}_{g,\delta}\|_{L^1}.$$

For $j \geq 1$ and some $(D+1)/(\alpha+1) < \sigma < 1$ to be chosen, theorem 6 and the properties of the conditional expectation, yield

$$\begin{aligned}\Big\|\mathbb{E}_{\Omega_j} \circ \big(\mathbb{1} - \mathbb{E}_{\Omega_{j-1}}\big)\big(\tau_s(O)\big)\Big\| &\leq \Big\|\big(\mathbb{1} - \mathbb{E}_{\Omega_{j-1}}\big)\big(\tau_s(O)\big)\Big\| \\ &\leq 2\, \|O\|\, |\Omega| \Big(\mathrm{e}^{v|s| - j^{1-\sigma}} + C_\sigma\, (j+1)^{-\sigma\alpha}\, v|s|\, \big(1 + (v|s|)^{D/(1-\sigma)}\big)\Big)\end{aligned}$$

because $d(\Omega, \Lambda \setminus \Omega_{j-1}) = j$.

Thus, for $T = j^{p(1-\sigma)}/v_*$ with $p \in (0,1)$ we can bound

$$\begin{aligned}&\bigg\|\mathrm{i} \int_{-T}^{T} \mathscr{W}_{g,\delta}(s)\, \mathbb{E}_{\Omega_j} \circ \big(\mathbb{1} - \mathbb{E}_{\Omega_{j-1}}\big)\big(\tau_s(O)\big)\, \mathrm{d}s\bigg\| \\ &\qquad\leq 4\, \|O\|\, |\Omega|\, \|\mathscr{W}_{g,\delta}\|_\infty \int_0^T \mathrm{e}^{vs - j^{1-\sigma}} + C_\sigma\, (j+1)^{-\sigma\alpha}\, vs\, \big(1 + (vs)^{D/(1-\sigma)}\big)\, \mathrm{d}s \\ &\qquad\leq 4\, \|O\|\, |\Omega|\, \|\mathscr{W}_{g,\delta}\|_\infty \Big(\mathrm{e}^{j^{p(1-\sigma)} - j^{1-\sigma}} + C_\sigma\, (j+1)^{-\sigma\alpha}\, j^{2p(1-\sigma)}\, \big(1 + j^{pD}\big)\Big) \\ &\qquad\leq 8\, \|O\|\, |\Omega|\, \|\mathscr{W}_{g,\delta}\|_\infty \Big(\mathrm{e}^{j^{p(1-\sigma)} - j^{1-\sigma}} + C_\sigma\, (j+1)^{-\sigma\alpha + 2p(1-\sigma) + pD}\Big).\end{aligned}$$

Furthermore, for any $m > 0$ there exists C_m such that

$$\begin{aligned}\left\| \mathrm{i} \int_{|s|>T} \mathscr{W}_{g,\delta}(s)\, \mathbb{E}_{\Omega_j} \circ \big(\mathbb{1} - \mathbb{E}_{\Omega_{j-1}}\big)\big(\tau_s(O)\big)\, \mathrm{d}s \right\| &\le 2\,\|O\| \int_{|s|\ge T} \mathscr{W}_{g,\delta}(s)\, \mathrm{d}s \\ &\le 8\,\|\mathscr{W}_{g,\delta}\|_\infty\, C_m\, \|O\|\, v_*^m\, j^{-mp(1-\sigma)}\end{aligned}$$

by the properties of $\mathscr{W}_{g,\delta}$. Hence, altogether we find

$$\|\Delta_j(O)\| \le 8\,\|\mathscr{W}_{g,\delta}\|_\infty\, \|O\|\, |\Omega| \left(\mathrm{e}^{j^{p(1-\sigma)} - j^{1-\sigma}} + C_\sigma\, (j+1)^{-\sigma\alpha + 2p(1-\sigma) + pD} + C_m\, v_*^m\, j^{-mp(1-\sigma)} \right).$$

An interaction for $A := \mathscr{I}_{H,g,\delta}(K)$ is given by

$$\Phi_A(Z) = \sum_{j=0}^{\infty} \sum_{\substack{Y \subset \Lambda: \\ Y_j = Z}} \Delta_j(\Phi_K(Y)).$$

It follows that

$$\begin{aligned}\sum_{\substack{Z \subset \Lambda: \\ z \in Z}} \frac{|Z|^n\, \|\Phi_A(Z)\|}{F_\beta\big(\mathrm{diam}(Z)\big)} &\le \sum_{\substack{Z \subset \Lambda: \\ z \in Z}} \sum_{j=0}^{\infty} \sum_{\substack{Y \subset \Lambda: \\ Y_j = Z}} \frac{|Z|^n\, \|\Delta_j(\Phi_K(Y))\|}{F_\beta\big(\mathrm{diam}(Z)\big)} \\ &\le \sum_{j=0}^{\infty} \sum_{Y \subset \Lambda} \mathbb{1}_{z \in Y_j} \frac{|Y_j|^n\, \|\Delta_j(\Phi_K(Y))\|}{F_\beta\big(\mathrm{diam}(Y_j)\big)}.\end{aligned} \tag{39}$$

The $j = 0$ term is bounded by $\|\mathscr{W}_{g,\delta}\|_{L^1}\, \|\Phi_K\|_{\beta,n}$. For $j \ge 1$ and $z \in Y_j$, there exists $y \in B_z(j) \cap Y$. Moreover, $|Y_j| \le |Y|\, \mathscr{C}_{\mathrm{V}}\, (j+1)^D$ and $\mathrm{diam}(Y_j) \le 2j + \mathrm{diam}(Y)$. Hence, the rest of the sum is bounded by

$$\begin{aligned}&\sum_{j=1}^{\infty} \sum_{y \in B_z(j)} \sum_{\substack{Y \subset \Lambda: \\ y \in Y}} \frac{|Y|^n\, \mathscr{C}_{\mathrm{V}}^n\, (j+1)^{nD}\, \|\Phi_K(Y)\|\, |Y|}{F_\beta\big(\mathrm{diam}(Y)\big)\, F_\beta(2j)} \\ &\qquad \times 8\,\|\mathscr{W}_{g,\delta}\|_\infty \Big(\mathrm{e}^{j^{p(1-\sigma)} - j^{1-\sigma}} + C_\sigma\, (j+1)^{-\sigma\alpha + 2p(1-\sigma) + pD} \\ &\qquad\qquad + C_m\, v_*^m\, j^{-mp(1-\sigma)} \Big) \\ &\le 2^{3+\beta}\, \|\mathscr{W}_{g,\delta}\|_\infty\, \mathscr{C}_{\mathrm{V}}^n\, \|\Phi_K\|_{\beta,n+1} \\ &\qquad \times \sum_{j=1}^{\infty} \sum_{y \in B_z(j)} (j+1)^{\beta + nD} \Big(\mathrm{e}^{j^{p(1-\sigma)} - j^{1-\sigma}} + C_\sigma\, (j+1)^{-\sigma\alpha + 2p(1-\sigma) + pD} \\ &\qquad\qquad + C_m\, v_*^m\, j^{-mp(1-\sigma)} \Big).\end{aligned}$$

The remaining sum over y can be bounded by $|B_z(j)| \leq \mathscr{C}_{\mathrm{V}}\,(j+1)^D$. The infinite sum over the first term in the parenthesis, is then bounded for $p, \sigma \in (0,1)$. The same holds for the last term by choosing m appropriately. The sum over the second term is bounded whenever $\sigma\,\alpha > 1 + (n+1)\,D + \beta + 2p\,(1-\sigma) + pD$, i.e. as long as $\alpha > (n+1)\,D + 1 + \beta$ by choosing σ appropriately. The final constant thus depends on D, α, β, n (after choosing σ and p optimally), but also on the v_* and g and δ and thus directly on the norm of the interaction Φ and its spectral properties. □

Proving theorem 10 uses very similar ideas, but is simpler because we only consider strictly localized $\dot{H}(s) \in \mathscr{A}_X$ and far apart observables.

Proof of theorem 10. Let $W(s) = \sum_{Z \subset X} \dot{\Phi}(Z,s)$, then the generator of U from theorem 9 is $G = \mathscr{I}_{H,g,\delta}(W)$. For the proof, we approximate $U(s)$ by a unitary $V_Y(s)$, which acts like the identity on $\mathscr{A}_Y$, where B is supported. Therefore, let $V_Y(s)$ be defined by its generator

$$G_Y(t) := \mathrm{i} \int_{\mathbb{R}} \mathscr{W}_{g,\delta}(s)\, \mathbb{E}_{\Lambda\setminus Y}\Big(\tau_s\big(W(t)\big)\Big)\,\mathrm{d}s. \tag{40}$$

As in the proof of proposition 8, and abbreviating $r = d(X,Y)$, we obtain

$$\begin{aligned} &\big\|G_Y(t) - G(t)\big\| \\ &\quad \leq C\,\|W(t)\|\,|Y|\,\Big(\mathrm{e}^{(r+1)^{p(1-\sigma)} - (r+1)^{1-\sigma}} + (r+2)^{-\sigma\alpha + 2p(1-\sigma) + pD} + (r+1)^{-mp(1-\sigma)}\Big), \end{aligned}$$

using the Lieb-Robinson bound from theorem 6, which requires a uniform bound on $\|\Phi\|_{\alpha,1} < \infty$ for some $\alpha > D$. We can further bound this by $C\,\|W(t)\|\,|Y|\,F_{\alpha-\varepsilon}(r)$ for all $\varepsilon > 0$ by choosing $p, \sigma \in (0,1)$ and m appropriately. By the fundamental theorem of calculus,

$$U(s) - V_Y(s) = -U(s) \int_0^s \frac{\mathrm{d}}{\mathrm{d}t} U(t)^*\, V_Y(t)\,\mathrm{d}t = \mathrm{i}\, U(s) \int_0^s U(t)^*\,\big(G(t) - G_Y(t)\big)\, V_Y(t)\,\mathrm{d}t.$$

And hence $\big\|U(s) - V_Y(s)\big\| \leq s\, \sup_{t\in I} \big\|G(t) - G_Y(t)\big\|$ for all $s \in I$. Since $V_Y(s) \in \mathscr{A}^+_{\Lambda\setminus Y}$, it satisfies $[V_Y(s), A] = 0$. Hence, after using cyclicity of the trace, we have

$$\begin{aligned} \big|\mathrm{tr}\big(P(s)\,A\big) - \mathrm{tr}\big(P(0)\,A\big)\big| &= \big|\mathrm{tr}\big(P(0)\,U(s)^*\,\big[A, U(s) - V_Y(s)\big]\big)\big| \\ &\leq 2\ \mathrm{tr}\big(P(0)\big)\,\|A\|\,s\,C\,\sup_{t\in I}\|W(t)\|\,|Y|\,F_{\alpha-\varepsilon}\big(d(X,Y)\big), \end{aligned}$$

and conclude (18).

To obtain (19), we split $W(t) = \sum_{Z \subset X} \dot{\Phi}(Z,t)$ in (40) and apply the localization to each summand. Following the remaining arguments, we then obtain

$$
\begin{aligned}
&\left|\operatorname{tr}\big(P(s)\,A\big) - \operatorname{tr}\big(P(0)\,A\big)\right| \\
&\qquad \leq 2\ \operatorname{tr}(P(0))\,\|A\|\,s\,C\ \sup_{t\in I}\ \sum_{Z\subset X} \|\dot{\Phi}(Z,t)\|\,|Z|\,F_{\alpha-\varepsilon}\big(d(Z,Y)\big) \\
&\qquad \leq 2\ \operatorname{tr}(P(0))\,\|A\|\,s\,C\ \sum_{z\in X}\sum_{y\in Y}\ \sup_{t\in I}\ \sum_{\substack{Z\subset X:\\ z\in Z}} \|\dot{\Phi}(Z,t)\|\,|Z|\,F_{\alpha-\varepsilon}\big(d(z,y)\big) \\
&\qquad \leq 2\ \operatorname{tr}(P(0))\,\|A\|\,s\,C\,\mathscr{C}_{\mathrm{A}}\ \tfrac{\alpha-\varepsilon-D+1}{\alpha-\varepsilon-D}\ |Y|\,\|\dot{\Phi}\|_{0,1}\,F_{\alpha-D-\varepsilon}\big(d(X,Y)\big),
\end{aligned}
$$

if $\alpha - \varepsilon > D$, where similarly to (32) we used that for $\beta > D$ and $R \geq 1$

$$
\sum_{\substack{z\in\Lambda:\\ d(z,y)\geq R}} F_\beta\big(d(z,y)\big) \leq \mathscr{C}_{\mathrm{A}}\left(F_\beta(R) + \int_R^\infty F_{\beta-D+1}(r)\,\mathrm{d}r\right) \leq \mathscr{C}_{\mathrm{A}}\ \tfrac{\beta-D+1}{\beta-D}\ F_{\beta-D}(R). \qquad \square
$$

Publication P8

Uniform-in-temperature locality estimates for weakly interacting quantum systems

Arka Adhikari [*] Joscha Henheik [†] Marius Lemm [‡]

Tom Wessel [‡]

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Abstract

The locality of thermal quantum states has emerged as a key input for applications to thermalization, response theory, and efficient simulability. Locality is either captured by the decay of correlations or by local indistinguishability, which allows to approximate local expectation values by those of local thermal states. Most techniques for deriving locality bounds deteriorate at small temperature, a physically highly relevant regime and so it is of interest to identify conditions for uniform-in-temperature bounds. Here we prove that a class of weakly interacting quantum Hamiltonians satisfies exponential decay of correlations and local indistinguishability *uniformly in the temperature*. The proof uses a low-temperature cluster expansion and a quantum version of a probabilistic swapping trick developed by the first author and Cao [3] in the context of lattice gauge theories.

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[*] Department of Mathematics, University of Maryland-College Park, College Park, MD, USA.

[†] University of Geneva, Rue du Conseil Général 7-9, 1205 Geneva, Switzerland.

[‡] Department of Mathematics, University of Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany.

Contents

1 Introduction

A central problem in the study of equilibrium quantum many-body physics is to determine the locality properties of thermal (Gibbs) states

$$\rho^\beta = \frac{\mathrm{e}^{-\beta H}}{\mathrm{tr}(\mathrm{e}^{-\beta H})}$$

where H is the Hamiltonian operator, assumed to be a sum of local interaction terms, and $\beta = 1/T$ denotes the inverse temperature. The traditional way to express locality is through exponential decay of correlations (DoC), also known as "clustering" of correlations. DoC is the statement that two bounded local observables A and B supported on distinct spatial regions X and Y, respectively, satisfy

$$\left|\mathrm{tr}(A\,B\,\rho^\beta) - \mathrm{tr}(A\,\rho^\beta)\ \mathrm{tr}(B\,\rho^\beta)\right| \lesssim \|A\|\,\|B\|\,\mathrm{e}^{-d(X,Y)/\xi(\beta)} \tag{1}$$

for a suitable temperature-dependent decay rate $\xi(\beta) > 0$, whose sharp value is called the inverse correlation length. As in classical statistical mechanics, clustering of correlations is intimately connected to the absence of phase transitions, a longstanding and notoriously difficult topic in mathematical physics (see e.g. [99, 189, 77]).

A new perspective has emerged in the past 15 years originating from quantum information theory. From this new perspective, the locality of thermal states is essential to break up a quantum many-body system into local pieces, which can be separately prepared, simulated, manipulated, or measured – depending on the precise practical task at hand. Thus, locality provides a powerful tool to address the fundamental curse of dimensionality that arises in any quantum many-body problem. Celebrated examples of such uses of locality include the efficient classical simulability of thermal states [164, 109, 6, 64, 50, 88], their efficient tomography [194] and sampling [201, 76], as well as the rapid equilibration of open quantum systems [148, 27, 26, 141], which is relevant for exploring possible quantum memories and for the efficient preparation of thermal states on quantum devices [45, 131, 60, 193]. A strong form of clustering was shown to imply the area law for the entanglement entropy [44]. Related locality bounds have also been used in condensed matter physics to prove quantization of the Hall conductance [119, 100, 12, 129, 216].

It turns out that for many of these applications one does not directly require decay of correlations, but instead a different locality property of the thermal state which is called *local indistinguishability* (LI) [139, 45, 50, 36, P4], which will play a prominent role in this paper. LI says that when calculating the expectation value of a local observable, one can replace the global thermal state by a local thermal state. More precisely, given nested spatial regions $Y \subset \Lambda' \subset \Lambda$ and a bounded local observable B supported on region Y, LI is a bound of the difference between expectation values in the thermal states ρ_Λ^β and $\rho_{\Lambda'}^\beta$ on the different regions, i.e.

$$\left|\mathrm{tr}(B\,\rho_\Lambda^\beta) - \mathrm{tr}(B\,\rho_{\Lambda'}^\beta)\right| \lesssim \|B\|\,\mathrm{e}^{-d(Y,\Lambda'\setminus\Lambda)/\xi_{\mathrm{LI}}(\beta)}. \tag{2}$$

Here, $\xi_{\mathrm{LI}}(\beta)$ is another suitable temperature-dependent decay rate, which may or may not be the same as $\xi(\beta)$ in (1). LI was originally introduced by Kliesch et al. as establishing the *locality of temperature* [139], because it indeed allows measuring the system's global temperature based on local information only. LI also has many applications to address the curse of dimensionality, especially for simulating thermal states, either on a classical or on a quantum device [45] and for efficiently preparing thermal states [62, Section III.B] as we discuss further in section 1.3. For further background on LI and its uses, we refer to the reviews [140, 5].

Since the locality properties of thermal states are of such fundamental importance for many applications, their mathematical derivation has developed into a large subfield in its own right. Since locality is intimately connected to the absence of phase transitions, most existing results treat the case of zero temperature for gapped ground states [118, 171, 73, 214], 1D systems [9, 36, 185, 144], or the high temperature regime by cluster expansions [183, 139, 98, 140, 146, 132].

The low-temperature regime is comparatively less studied, even though it is highly relevant for condensed matter physics and modern quantum simulation platforms in ultracold gases [34]. Of course, locality at low temperatures in dimension greater than one is a more subtle topic because phase transitions can occur. Proving DoC at low temperature for suitable systems is a longstanding topic in mathematical physics with many significant contributions, both for quantum spin systems [205, 161, 40, 66, 67, 94] and fermions [112, 118, 73, 94, 102]. However, proving local indistinguishability (LI) uniformly at low temperature does not seem to have been considered so far either in the mathematical physics or quantum information theory literature.

This raises the following problem.

Problem. *Identify conditions under which DoC and LI hold uniformly in temperature for quantum systems in any spatial dimension.*

In particular, the challenge is to obtain bounds on the *correlation lengths* ξ in (1) and ξ_{LI} in (2) that are uniform in β and thus bounded in the low-temperature $\beta \to \infty$ regime. By analogy with classical statistical mechanics, it is clear that the relevant condition should exclude phase transitions at low temperature and we discuss this point in detail in section 1.2.

Our goal in this paper is to tackle this problem head-on by presenting conceptually simple and robust proofs of both DoC and LI with a decay rate that is independent of β and so, in particular, uniform as $\beta \to \infty$. To achieve this, we develop a new analytical argument that combines a low-temperature cluster expansion with a quantum version of a swapping trick originating in probability theory [3]; see section 3.1 for a brief sketch of the main idea.

Remark 1.1 (Relation between DoC and LI). Let us comment on the relation between DoC and LI and why one may consider LI as the stronger property. First, a general form of LI implies DoC with the same constants because the local truncation of the thermal states decorrelates the observables; see remark 2.4 for more details. Conversely, it has been shown [45, P4] that at every *fixed* temperature T DoC uniformly[1] in Λ' implies LI. Unfortunately, by this route, the constants in the resulting LI statement depend adversely on T. More precisely, uniform-in-temperature DoC implies LI with a prefactor and correlation length that scale polynomially in β as $\beta \to \infty$, which poses a problem for the applications we described before. The underlying reason for this effect is that the main tool of [45, P4] is quantum belief propagation [115], a differential equation to describe deformed Gibbs states, and this is well-known to produce constants that diverge as $T \to 0$. To summarize, uniform-in-temperature LI implies uniform-in-temperature DoC, while, conversely, uniform-in-temperature DoC implies LI with constants that diverge as $T \to 0$. ◇

[1] This kind of "uniform" DoC is standard and sometimes referred to as "uniform clustering", where uniformity refers to the choice of $\Lambda' \subset \Lambda$. For the systems we consider, we indeed prove results uniform in Λ.

1.1 Summary of main results

In this paper, we prove exponential DoC and LI at all inverse temperatures $\beta \in [0, \infty)$ for lattice systems on $\Lambda \Subset \mathbb{Z}^D$ with finite-dimensional on-site Hilbert spaces. The considered Hamiltonians are of the form

$$H = H^0 + V = \sum_{x \in \Lambda} h_x + \sum_{\substack{x \in \Lambda : \\ B_R(x) \subset \Lambda}} v_x.$$

Here, every $h_x \geq 0$ acts only on the site $x \in \Lambda$ and has a unique ground state separated from the rest of the spectrum by a gap of size at least one, and every v_x is a small finite-range interaction supported on $B_R(x)$, a ball of radius R around site x. In particular, we do not assume any sort of periodicity or translation invariance.

Additionally, we assume that every v_x is relatively form bounded with respect to the sum of all h_x in $B_R(x)$ in the following sense: There exists a small $a \in (0, 1)$ such that

$$|\langle \psi, v_x \psi \rangle| \leq a \left\langle \psi, \sum_{z \in B_R(x)} h_z \psi \right\rangle \qquad \text{for all states } \psi. \tag{3}$$

We explain the motivation for studying these Hamiltonians in section 1.2.

Roughly speaking, the form bound (3) ensures that the spectrum of H is contained in $[0, \infty)$ (as is the case for H^0), the ground state of H remains a gapped product state, and all the (non-negative) eigenvalues $\{E_j^0\}$ of H^0 remain "of the same order" when perturbed by V. In fact, a simple application of the min-max-principle (or alternatively of [133, Theorem 3.6 in Chapter VII]) shows that the j^{th} eigenvalue of H lies in the interval $[(1-a)\, E_j^0, (1+a)\, E_j^0]$. This ensures, in particular, that the many-body density of states (mbDoS) of H behaves similarly to the mbDoS of the non-interacting Hamiltonian H^0: The low- and high-energy states have a small mbDoS, while for intermediate energies the mbDoS is large.

As our main results, we obtain the following bounds (see theorems 2.2 and 2.3).

Main Result (informal). *For small enough $a > 0$ as in* (3)*, there exist constants $C_1, C_2\ \xi, \xi_{\mathrm{LI}} > 0$, such that the system satisfies decay of correlations (DoC)*

$$|\mathrm{tr}(A\, B\, \rho_\Lambda^\beta) - \mathrm{tr}(A\, \rho_\Lambda^\beta)\ \mathrm{tr}(B\, \rho_\Lambda^\beta)| \leq C_1\, \mathrm{e}^{C_2\, (|X|+|Y|)}\, \|A\|\, \|B\|\, \mathrm{e}^{-d(X,Y)/\xi}$$

and local indistinguishability (LI)

$$|\mathrm{tr}(B\, \rho_\Lambda^\beta) - \mathrm{tr}(B\, \rho_{\Lambda'}^\beta)| \leq C_1\, \mathrm{e}^{C_2\, |Y|}\, \|B\|\, \mathrm{e}^{-d(Y, \Lambda \setminus \Lambda')/\xi_{\mathrm{LI}}}$$

for all $\Lambda' \subset \Lambda \Subset \mathbb{Z}^D$, $\beta > 0$ and observables A and B supported on X and Y, respectively.

Notice that ξ and ξ_{LI} are independent of β. We remark that there is a third notion of locality of Gibbs states known as the *local perturbations perturb locally* (LPPL) principle, which asserts that local perturbations of the Hamiltonian affect the associated thermal state only locally [23, 74, P1, 16, P4]. Our method adapts to LPPL and we discuss this further in section 4.

1.2 Discussion

As in classical statistical mechanics, if exponential DoC for ρ^β breaks down at a certain critical β_c, then this indicates a phase transition occurring at this inverse temperature. This explains why proving DoC and LI for Gibbs states is necessarily a temperature-dependent and thus somewhat delicate task – especially in higher dimensions. Therefore, to prove DoC and LI, it is necessary to identify a condition which excludes a phase transition in the temperature regime under consideration. There are different approaches to excluding phase transitions.

- In arbitrary dimensions, results are usually about sufficiently *high temperature*, i.e. for β below a certain (universal) critical inverse temperature β_*, no phase transitions are possible and DoC can be derived by cluster expansion techniques [183, 139, 98, 140].
- A special case arises in one spatial dimension, where it is folklore wisdom that phase transitions can only occur at zero temperature. Indeed, Araki [9] showed in pioneering work in 1969 that exponential DoC holds at any positive temperature in one-dimensional, finite-range, translation-invariant systems; see [36, 185, 136] for recent extensions. While the one-dimensional case covers any positive temperature, the possibility of a phase transitions at zero temperature affects the $T \to 0$ (or $\beta \to \infty$) behaviour of the bound. This is expressed through the correlation length $\xi(\beta)$ which features as the inverse decay rate in (1). The best known upper bound $\xi(\beta) \lesssim \exp(c\,\beta)$ was proved recently in [136] and divergence as $\beta \to \infty$ can indeed occur. For example, in the classical one-dimensional Ising model with coupling strength $J > 0$, the correlation length of the two-point spin-spin covariance can explicitly be computed to be given by $\xi(\beta) = -1/\log(\tanh(\beta J)) \sim \frac{1}{2}\,\mathrm{e}^{2J\beta}$ as $\beta \to \infty$. This shows that deriving a uniform-in-temperature bound on the correlation length is subtle even in one dimension.

We aim for a uniform bound on the correlation length in any dimension and so these two standard approaches to exclude phase transitions are *unavailable to us*.

One thus needs a different idea to exclude phase transitions and this is to suitably "perturb" a stable classical phase with a non-commuting perturbation. A stable classical phase can be implemented for example by breaking a symmetry. Indeed, taking another hint from the classical one-dimensional Ising model, notice that its behaviour drastically changes when breaking the symmetry through a constant external field of strength $h > 0$. This breaks the two-fold ground state degeneracy present for $h = 0$ and leads to a correlation length that is uniformly bounded in the low temperature regime. The idea of adding quantum perturbations to classical systems and deriving DoC was first implemented for special models [205, 161]. A general approach was then developed around the same time by Borgs, Kotecký and Ueltschi [40] and Datta, Fröhlich and Fernandez [66, 67] for translation-invariant systems through quantum versions of the machinery of Pirogov-Sinai theory. These works obtain DoC in dimension at least two for translation-invariant quantum perturbations of classical Hamiltonians with finitely many periodic gapped ground states. Intermediate temperature ranges have been considered as well [94], also by a modified Pirogov-Sinai theory.

Our model is also a quantum perturbation of a classical Hamiltonian as in [40, 66, 67]. However, there are a few important differences: We consider the case that the classical Hamiltonian has a unique ground state, while these works consider classical Hamiltonian with multiple phases which are then distinguished by suitable boundary conditions. Importantly, our method is completely different because it tackles DoC and LI directly through a swapping trick (see section 3.1), while these works develop an involved quantum version of Pirogov-Sinai theory, whose convergence then implies DoC in their setting. Our new approach via the swapping trick leads to a fully self-contained and, we believe, conceptually simple proof of DoC and the first proof of LI uniformly in temperature.

Another advantage of the fact that our new method is very direct is that the proof is robust in new ways. First, our proof also works in *one spatial dimension.* This means our result also improves the correlation length in recent DoC results in the low-temperature regime [36, 185, 144] for our class of weakly interacting quantum Hamiltonians, and we obtain the first uniform bound on the correlation length $\xi(\beta)$. Second, we do not assume that the classical Hamiltonian and the quantum perturbation are *translation-invariant*, which was needed for the Pirogov-Sinai theory [40, 66]. This robustness arises because the swapping trick exploits local cancellations that are adapted to the local structure of the system. In particular, we can treat for the first time *disordered* Hamiltonians, like the disordered XXZ chain that we describe now.

Example 1.2. A paradigmatic example Hamiltonian satisfying our conditions is the XXZ model with (random) external field on some $\Lambda \Subset \mathbb{Z}^D$. More precisely, let σ_x^i be the i-th Pauli matrix acting only on site $x \in \Lambda$, and define the *ladder operators* and the *number operator* acting on $x \in \Lambda$ as $\sigma_x^\pm = \frac{1}{2}(\sigma_x^1 \pm \mathrm{i}\sigma_x^2)$ and $\mathcal{N}_x = \frac{1}{2}(\mathbb{1}_x - \sigma_x^3)$, respectively. Then the XXZ Hamiltonian with (random) external field acting on the Hilbert space $\bigotimes_{x\in\Lambda} \mathbb{C}^2$ is given by

$$H_{\mathrm{XXZ}} = H^0 + V$$

with

$$\begin{aligned} H^0 &:= \sum_{x\in\Lambda} (1 + \lambda\,\omega_x)\,\mathcal{N}_x \qquad \text{and} \\ V &:= \sum_{x,y\in\Lambda} J_{12}(x,y)\,(\sigma_x^+\sigma_y^- + \sigma_x^-\sigma_y^+) + \sum_{x,y\in\Lambda} J_3(x,y)\,\mathcal{N}_x\,\mathcal{N}_y, \end{aligned} \tag{4}$$

where $\{\omega_x\}_{x\in\Lambda}$ is a collection of random variables supported on $[0,1]$ and the parameter $\lambda \geq 0$ modulates the strength of the randomness. The self-adjoint coupling matrices J_{12} and J_3, i.e. satisfying $J_{12}(y,x) = \overline{J_{12}(x,y)}$ and $J_3(y,x) = \overline{J_3(x,y)}$, are assumed to be of *finite range*. That is, there exists $R > 0$ such that $J_{12}(x,y) = J_3(x,y) = 0$ whenever $d(x,y) \geq R$.

By construction, every summand of H^0 in (4) satisfies $(1+\lambda\,\omega_x)\,\mathcal{N}_x \geq 0$ with ground state eigenvalue 0 and excited eigenvalue $1+\lambda\,\omega_x \geq 1$. Moreover, if $\sup_{x,y\in\Lambda}|J_{12}(x,y)|$ and $\sup_{x,y\in\Lambda}|J_3(x,y)|$ are small enough, we find a relative form bound as in (3) for the interaction V in (4) in terms of H^0. Hence, the XXZ Hamiltonian H_{XXZ} as in (4) satisfies all the assumptions in our main result, and hence satisfies DoC and LI. ◇

Remark 1.3. At zero temperature, the existence of a spectral gap above the ground state implies DoC [118, 171, 73, 214]. Naively, one may therefore think that a suitable condition to tackle the problem at low temperature is that there exists a unique gapped ground state. However, there are classical examples (an Ising chain perturbed by a single on-site field) which show that a spectral gap is insufficient to even tackle small positive temperature. ◇

1.3 Applications and outlook

Our result on uniform LI, theorem 2.3, implies that the "locality of temperature" found in [139] (i.e., the fact that the temperature of the global thermal state equals the temperature of its local approximation) is in fact a uniform property that does not deteriorate at small temperature. Second, our uniform LI impacts the efficient preparation of thermal states, as can be seen by replacing the use of Theorem III.2 with our theorem 2.3 in the proof of Corollary III.3 of [62]. Third, LI also improves classical simulability results for thermal states. More precisely, let us focus on the problem of efficiently simulating local observables (e.g., Problem 3 in [50]), which also allows for efficient simulation of free energies (see Problem 1 and Lemma 12 in [50]). Here, efficient simulation refers to algorithms that produce errors that are polynomially small in the number of sites $|\Lambda|$ in runtimes that are also (only) polynomially small in $|\Lambda|$. One can use LI to obtain such an efficient algorithm for expectation values of local observables by proceeding as follows: One first applies LI with the distance parameter $d(Y, \Lambda' \setminus \Lambda)$ logarithmic in the total system size $|\Lambda|$, which yields a polynomial error in $|\Lambda|$. This step can now be performed uniformly in temperature for our system. Afterwards, one runs a classical algorithm on the subsystem whose runtime versus error scales with the subsystem dimension which (on account of the logarithm) will be polynomial in $|\Lambda|$. It is an interesting question in this context if the second, classical step can also be performed uniformly in temperature by further exploiting the cluster expansion method we devise here, but we leave its investigation to future work.

Apart from the various applications of LI to quantum information science partly discussed above, our original motivation to derive LI uniformly at low temperature was that we have identified it as a key property for developing a robust response theory for low-temperature interacting quantum systems, a topic whose mathematical investigation has only begun recently [102, 127]. To keep this paper focused on the general properties of DoC and LI, we will present these consequences of our theorem 2.3 for response theory elsewhere [121].

2 Setup and main results

After introducing the necessary mathematical framework in section 2.1, we present our main results on decay of correlations (theorem 2.2) and local indistinguishability (theorem 2.3) in section 2.2.

2.1 Mathematical setup

Consider the regular lattice $\mathbb{Z}^D$, for fixed $D \in \mathbb{N}$, equipped with the ℓ^1-metric $d\colon \mathbb{Z}^D \times \mathbb{Z}^D \to \mathbb{N}_0$. We denote arbitrary subsets as $\Lambda \subset \mathbb{Z}^D$ (including equality) and finite subsets by $\Lambda' \Subset \Lambda$ (again including equality if Λ is finite). The set of all finite subsets is denoted $\mathscr{P}_0(\Lambda)$. The cardinality of a set $\Lambda \Subset \mathbb{Z}^D$ is denoted by $|\Lambda|$. Given any two subsets $X, Y \subset \mathbb{Z}^D$ we denote by $d(X,Y)$ their distance with respect to the metric d. Likewise, we denote by $\operatorname{diam}(X) := \sup_{x,y\in X} d(x,y)$ the diameter of X.

With every site $x \in \mathbb{Z}^D$ we associate a finite-dimensional local Hilbert space $\mathscr{H}_x := \mathbb{C}^q$, with $q \geq 2$, and the corresponding space of linear operators $\mathscr{A}_x := \mathscr{B}(\mathscr{H}_x)$. For each $\Lambda \Subset \mathbb{Z}^D$ we define the Hilbert space $\mathscr{H}_\Lambda := \bigotimes_{x\in\Lambda} \mathscr{H}_x$, and denote the algebra of bounded linear operators on $\mathscr{H}_\Lambda$ by $\mathscr{A}_\Lambda := \mathscr{B}(\mathscr{H}_\Lambda)$. Due to the tensor product structure, we have $\mathscr{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathscr{B}(\mathscr{H}_x)$. Hence, for $X \subset \Lambda \Subset \mathbb{Z}^D$, any $A \in \mathscr{A}_X$ can be viewed as an element of $\mathscr{A}_\Lambda$ by identifying A with $A \otimes \mathbb{1}_{\Lambda\setminus X} \in \mathscr{A}_\Lambda$, where $\mathbb{1}_{\Lambda\setminus X}$ denotes the identity in $\mathscr{A}_{\Lambda\setminus X}$. Using this identification, we define the algebra of local operators

$$\mathscr{A}^{\mathrm{loc}} := \bigcup_{\Lambda\Subset\mathbb{Z}^D} \mathscr{A}_\Lambda \qquad \text{and its completion} \qquad \mathscr{A} := \overline{\mathscr{A}^{\mathrm{loc}}}^{\|\cdot\|}.$$

We consider Hamiltonians composed of two parts $H_\Lambda := H_\Lambda^0 + V_\Lambda$. The first is a sum of on-site terms $h_x \in \mathscr{A}_{\{x\}}$, each with unique gapped ground state $\Omega_x \in \mathscr{H}_x$ with gap at least 1, i.e. we have

$$\langle \Omega_x, h_x\, \Omega_x \rangle = 0 \qquad \text{and} \qquad \inf_{\substack{\psi\in\mathscr{H}_x \\ \langle\psi,\Omega_x\rangle=0}} \frac{\langle \psi, h_x\, \psi\rangle}{\langle \psi, \psi\rangle} \geq 1.$$

In particular,

$$H_\Lambda^0 := \sum_{x\in\Lambda} h_x$$

has ground state $\Omega_\Lambda = \bigotimes_{x\in\Lambda} \Omega_x$ with gap 1.

For the second part of the Hamiltonian, let

$$B_r(x) := \{ y \in \mathbb{Z}^D \mid d(x,y) \le r \}$$

denote the ball in $\mathbb{Z}^D$. We also abbreviate $B_r = B_r(0)$ which will mainly be used for the size $|B_r| = |B_r(x)| \le (2r+1)^D$. Then, fix a range $R \in \mathbb{N}$ and define

$$V_\Lambda := \sum_{\substack{x\in\Lambda:\\ B_R(x)\subset\Lambda}} v_x$$

as a sum of local terms $v_x \in \mathscr{A}_{B_R(x)}$. We denote $\|v\|_\infty = \sup_{x\in\mathbb{Z}^D}\|v_x\|$. Moreover, we assume that there exists $a < 1$ such that each v_x is relatively form bounded w.r.t. $H^0_{B_R(x)}$ in the sense that

$$|\langle\psi, v_x\,\psi\rangle| \le \frac{a}{|B_R|}\,\langle\psi, H^0_{B_R(x)}\,\psi\rangle \qquad \text{for all } \psi \in \mathscr{H}_{B_R(x)}. \tag{5}$$

A class of examples of Hamiltonians satisfying our assumption are Heisenberg XXZ Hamiltonians in a sufficiently strong external field in any dimension (see example 1.2 where the disordered case was presented). Then, it immediately follows that V_Λ is relatively form-bounded w.r.t. H^0_Λ, namely

$$|\langle\psi, V_\Lambda\,\psi\rangle| \le \frac{a}{|B_R|}\sum_{\substack{x\in\Lambda:\\ B_R(x)\subset\Lambda}}\langle\psi, H^0_{B_R(x)}\,\psi\rangle \le a\,\langle\psi, H^0_\Lambda\,\psi\rangle \qquad \text{for all } \psi \in \mathscr{H}_\Lambda.$$

We point out that, at the cost of adjusting the constant a in the form bound, we can assume w.l.o.g. that $v_x \le 0$ for all $x \in \Lambda$. In fact, denoting $\tilde a := a/|B_R|$, we can write

$$H^0_{B_R(x)} + v_x = (1+\tilde a)\,H^0_{B_R(x)} + (-\tilde a\,H^0_{B_R(x)} + v_x),$$

where now $(-\tilde a\,H^0_{B_R(x)} + v_x) \le 0$. Moreover, $(1+\tilde a)\,h_x$ satisfies the same conditions as h_x and $(-a\,H^0_{B_R(x)} + v_x)$ satisfies the same conditions as v_x, although the form bound is now satisfied with $2\,\tilde a/(1+\tilde a)$ instead of a. We will henceforth assume that $v_x \le 0$ without further mentioning this.

Finally, we denote the partition functions

$$Z^0_\Lambda = \operatorname{tr}\big(e^{-\beta H^0_\Lambda}\big) \qquad \text{and} \qquad Z_\Lambda = \operatorname{tr}\big(e^{-\beta H_\Lambda}\big) \tag{6}$$

and the Gibbs states

$$\rho^0_\Lambda = \frac{e^{-\beta H^0_\Lambda}}{Z^0_\Lambda} \qquad \text{and} \qquad \rho_\Lambda = \frac{e^{-\beta H_\Lambda}}{Z_\Lambda}, \tag{7}$$

of the unperturbed and the full system, respectively. Here and in the following, unless specified differently, every trace (e.g., the ones used in the definition of the partition functions in (6)) is understood to be taken over $\mathscr{A}_\Lambda$.

To state the results, we also need to introduce the notion of R-connected sets. They naturally appear later as part of the cluster decomposition, see definitions 3.5 and 3.6.

Definition 2.1 (R-connected sets)**.** A subset $X \subset \Lambda$ is an *R-connected set*, if for every two points $x, y \in X$ there exists a sequence of points $z_1 = x, z_2, \dots, z_m \in X, z_{m+1} = y$ such that $d(z_i, z_{i+1}) \leq 2R$ for all $i \in \{1, \dots, m\}$. ◇

2.2 Main results

We are now ready to formally state our main results, and begin with decay of correlations (DoC) uniformly in the temperature. Recall that D denotes the spatial dimension and q denotes the dimension of the local Hilbert space.

Theorem 2.2 (Decay of correlations)**.** *Let $D, q, R \in \mathbb{N}$ and $C_{\mathrm{int}} > 0$. Then there exist $a \in (0,1)$ and $C_1, C_2, \xi > 0$ such that the following holds. Consider the lattice $\Lambda \Subset \mathbb{Z}^D$ and a Hamiltonian $H_\Lambda^0 + V_\Lambda$ as defined in section 2.1 with $\|h\|_\infty, \|v\|_\infty \leq C_{\mathrm{int}}$, v_x of range $R \in \mathbb{N}$, and v_x relatively a-bounded w.r.t. H_Λ^0 in the sense* (5)*. Then the Gibbs state ρ_Λ at any inverse temperature $\beta \in (0, \infty)$ satisfies*

$$\begin{aligned} &|\mathrm{tr}(A\,B\,\rho_\Lambda) - \mathrm{tr}(A\,\rho_\Lambda)\,\mathrm{tr}(B\,\rho_\Lambda)| \\ &\qquad \leq C_1\,\|A\|\,\|B\|\,\exp\bigl(C_2\,(|X| + |Y|)\bigr)\,\exp\bigl(-d(X,Y)/\xi\bigr) \end{aligned} \tag{8}$$

for all R-connected sets $X, Y \subset \Lambda$ and observables $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$.

We stress that all the constants a, C_1, C_2 and ξ in the above theorem are *independent* of the lattice Λ and in particular its size. Moreover, they are also independent of the inverse temperature β and hence the correlation length ξ is uniformly bounded away from zero and infinity, indicating the absence of any kind of phase transition. In particular, we have decay of correlations, uniformly for all temperatures.

We now state our second main result, local indistinguishability (LI) uniformly in temperature.

Theorem 2.3 (Local indistinguishability)**.** *Let $D, q, R \in \mathbb{N}$ and $C_{\mathrm{int}} > 0$. Then there exist $a \in (0,1)$ and $C_1, C_2, \xi_{\mathrm{LI}} > 0$ such that the following holds. Consider the lattice $\Lambda \Subset \mathbb{Z}^D$ and a Hamiltonian $H_\Lambda^0 + V_\Lambda$ as defined in section 2.1 with $\|h\|_\infty, \|v\|_\infty \leq C_{\mathrm{int}}$, v_x of range $R \in \mathbb{N}$, and v_x relatively a-bounded w.r.t. H_Λ^0 in the sense* (5)*. Moreover, let $\Lambda' \subset \Lambda$ and denote with $H_{\Lambda'}$ the Hamiltonian restricted to Λ'. Then the Gibbs states ρ_Λ and $\rho_{\Lambda'}$ at any inverse temperature $\beta \in (0, \infty)$ satisfy*

$$|\mathrm{tr}(B\,\rho_\Lambda) - \mathrm{tr}(B\,\rho_{\Lambda'})| \leq C_1\,\|B\|\,\exp\bigl(C_2\,|Y|\bigr)\,\exp\bigl(-d(Y, \Lambda \setminus \Lambda')/\xi_{\mathrm{LI}}\bigr) \tag{9}$$

for all R-connected sets $Y \subset \Lambda$ and observables $B \in \mathscr{A}_Y$.

As mentioned in the introduction, while one could obtain LI from DoC using quantum belief propagation [P4], the resulting bound would scale exponentially in β. Conversely, one can easily recover DoC from LI, i.e. LI is a stronger property than DoC.

Remark 2.4 (LI implies DoC). Theorem 2.3 also holds for unions $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are R-connected sets but Y is not. In particular for the case that $B = B_1 B_2$ with $B_i \in \mathscr{A}_{Y_i}$ is a product observable, the proof is straightforwardly adjusted[2]. Then, one can easily recover DoC from LI as follows. Given R-connected sets $X, Y \subset \Lambda$ and observables $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$, we let $\ell = \lfloor d(X,Y)/2 \rfloor - 1$ and define the ℓ-fattening $X_\ell = \{ z \in \Lambda \,|\, d(z,X) \leq \ell \}$ and analogously Y_ℓ.[3] Then we apply LI with $\Lambda' = X_\ell \cup Y_\ell$ to all expectation values in (8). Notice that $H_{\Lambda'} = H_{X_\ell} + H_{Y_\ell}$ by the choice of ℓ and hence the Gibbs state $\rho^\beta_{\Lambda'} = \rho^\beta_{X_\ell} \otimes \rho^\beta_{Y_\ell}$ factors and the correlations $\mathrm{Cov}_{\rho^\beta_{\Lambda'}}(A,B) = 0$ vanish. The error terms introduced by LI sum up to a bound as in (8) with different constants. ◇

A consequence of LI is that the finite volume states converge in the thermodynamic limit, and the resulting limit is a β-KMS state that satisfies DoC and LI.

Corollary 2.5 (DoC and LI in the thermodynamic limit)**.** *Using the conditions and notations from theorems 2.2 and 2.3, the following holds in the thermodynamic limit $\Lambda \nearrow \mathbb{Z}^D$.*

- *(i) For all $A \in \mathscr{A}^{\mathrm{loc}}$ the limit $\tau_t^{\mathbb{Z}^D}(A) := \lim_{\Lambda \nearrow \mathbb{Z}^D} \tau_t^\Lambda(A)$ exists and defines a strongly continuous one-parameter family of automorphisms on $\mathscr{A}$.*
- *(ii) For all $A \in \mathscr{A}^{\mathrm{loc}}$ the limit $\omega^\beta_{\mathbb{Z}^D}(A) := \lim_{\Lambda \nearrow \mathbb{Z}^D} \mathrm{tr}(A\,\rho^\beta_\Lambda)$ exists and defines a state, that is a normalized positive linear functional, on $\mathscr{A}$. Moreover, the state $\omega^\beta_{\mathbb{Z}^D}$ satisfies LI as in theorem 2.3 with Λ replaced by $\mathbb{Z}^D$ and for all finite $\Lambda' \Subset \mathbb{Z}^D$.*
- *(iii) The limit state $\omega^\beta_{\mathbb{Z}^D}$ satisfies DoC as in theorem 2.2 with the same constants.*
- *(iv) The state $\omega^\beta_{\mathbb{Z}^D}$ is a $(\tau^{\mathbb{Z}^D}, \beta)$-KMS state.*

We conjecture that the state $\omega^\beta_{\mathbb{Z}^D}$ obtained in this way is the only thermodynamic limit β-KMS state.

Proof. First, note that all limits can be understood along any increasing exhausting sequence $(\Lambda_n)_n$, i.e. $\Lambda_n \subset \Lambda_{n+1}$ and for every $X \subset \mathbb{Z}^D$ there exist $n \in \mathbb{N}$ such that $X \subset \Lambda_n$, and the limits do not depend on the sequence.

[2] Theorems 2.2 and 2.3 can be generalized to X and Y having a finite number of connected components. However, we expect the constant C_1 to grow faster than exponential in the number of connected components of X and Y. We present the full proof only for connected observables, because that is the physically relevant setting and it reduces the technicalities.

[3] Note that here, Y_ℓ is the ℓ-fattened version of the R-connected set Y, not Y_1 or Y_2 from before.

The convergence statement (i) is a standard result from the literature involving Lieb-Robinson bounds, see e.g. [174, Theorem 3.5].

For (ii), note that $(\mathrm{tr}(A\,\rho^{\beta}_{\Lambda_n}))$ is a Cauchy sequence by theorem 2.3, which has a unique limit. In this way, one obtains a state on $\mathscr{A}^{\mathrm{loc}}$ that can uniquely be extended to $\mathscr{A}$ by the Hahn-Banach theorem. To obtain LI for $\omega^{\beta}_{\mathbb{Z}^D}$, we choose a sequence $\Lambda_n \nearrow \mathbb{Z}^D$ with $\Lambda' \subset \Lambda_1$. Then

$$\begin{aligned}\left|\omega^{\beta}_{\mathbb{Z}^D}(B) - \mathrm{tr}(B\,\rho^{\beta}_{\Lambda'})\right| &\leq \limsup_{n\to\infty}\Big(\left|\omega^{\beta}_{\mathbb{Z}^D}(B) - \mathrm{tr}(B\,\rho^{\beta}_{\Lambda_n})\right| + \left|\mathrm{tr}(B\,\rho^{\beta}_{\Lambda_n}) - \mathrm{tr}(B\,\rho^{\beta}_{\Lambda'})\right|\Big)\\ &\leq C_1\,\|B\|\,\exp\big(C_2\,|Y|\big)\,\exp\big(-d(Y,\mathbb{Z}^D\setminus\Lambda')/\xi_{\mathrm{LI}}\big)\end{aligned}$$

because the first term vanishes in the limit $\Lambda_n \nearrow \mathbb{Z}^D$ and the second is uniformly bounded.

Since the constants in theorem 2.2 are independent of Λ, (iii) follows immediately from the convergence of the states.

Finally, (iv) follows from the convergence of the dynamics and states by standard results [49, Proposition 5.3.23]. □

3 Proofs

Both results, theorems 2.2 and 2.3, are proven using the same overall strategy. We first prove theorem 2.2 in three main steps. First, in section 3.2, we rewrite the exponential of the Hamiltonian via an inclusion-exclusion argument and estimate the terms in the decomposition exploiting analyticity of $\mathbb{C} \ni z \mapsto z\,v_x$ for every $x \in \Lambda$, as done in [217]. Next, in section 3.3, we adapt the probabilistic cluster expansion technique from [3] (see also [103, 4]) to the quantum setting in order to (algebraically) unravel cancellations in the covariance. The idea behind the underlying *swapping trick* is explained in section 3.1. The remaining terms then naturally carry ratios of partition functions of H_S and H^0_S for some $S \subset \Lambda$, which are then estimated in section 3.4. Finally, the arguments provided in sections 3.2 to 3.4 are combined in section 3.5, where we complete the proof of theorem 2.2, additionally using some combinatorial percolation estimate. We then explain the necessary modifications to prove theorem 2.3 in section 3.6.

3.1 Proof strategy and the swapping trick

We briefly describe at a high level the idea of the crucial swapping trick in the case of DoC. We start by rewriting the truncated correlation function as follows

$$\mathrm{tr}(A\,B\,\rho^{\beta}) - \mathrm{tr}(A\,\rho^{\beta})\,\mathrm{tr}(B\,\rho^{\beta}) = \frac{\mathrm{tr}(A\,B\,\mathrm{e}^{-\beta H})\,\mathrm{tr}(\mathrm{e}^{-\beta H}) - \mathrm{tr}(A\,\mathrm{e}^{-\beta H})\,\mathrm{tr}(B\,\mathrm{e}^{-\beta H})}{\mathrm{tr}(\mathrm{e}^{-\beta H})^2}.$$

We cluster expand each of the four terms in the numerator; call them (AB), (1), (A), (B), respectively. The expansion expresses the term (AB) as a sum over connected clusters, with those distinguished that intersect the support of A, the support of B, or both. We similarly distinguish special clusters in the expansions of the terms (A) and (B). Next, we write out the product $(AB) \times (1)$ as a sum over the connected components, which we call *superclusters*. Superclusters are obtained by collating two overlapping clusters in the two expansions that make up $(AB) \times (1)$; see definition 3.8 for their formal definition. We then derive an analogous supercluster representation for the product $(A) \times (B)$.

Now we can perform the swapping trick. Supercluster configurations contributing to $(AB) \times (1)$ *in which A and B lie in different superclusters* can be relabelled to match exactly all the terms in the supercluster expansion of $(A)\times(B)$. Therefore, the truncated correlation function is *exactly equal* to the sum over supercluster configurations in which A and B lie in the same supercluster. This is formalized in theorem 3.9. A similar swapping trick can be performed in the LI case; see theorem 3.13.

We see that the swapping trick is an exact algebraic equality that takes care of the truncation part of the truncated correlation function.

3.2 Analyticity bound

Throughout the proof, we consider both the inverse temperature β and the underlying lattice $\Lambda \Subset \mathbb{Z}^D$ to be fixed. In particular, every trace is understood to be taken over $\mathscr{A}_\Lambda$. To start, we introduce the following notations: For any $M \subset \Lambda$ we define their *interior* and *closure* (relative to Λ) as

$$M^\circ := \left\{ x \in M \,\middle|\, B_R(x) \subset \Lambda \right\} \qquad \text{and} \qquad \overline{M} := \Lambda \cap \bigcup_{x \in M} B_R(x),$$

such that, in particular,

$$V_\Lambda = \sum_{x \in \Lambda^\circ} v_x \qquad \text{and} \qquad \sum_{x \in M} v_x \in \mathscr{A}_{\overline{M}} \quad \text{for all } M \subset \Lambda^\circ.$$

We now follow the expansion from [217]. First, note that for any function f defined on $\mathscr{P}_0(\mathbb{Z}^D)$ and every $\Gamma \Subset \mathbb{Z}^D$ it holds that

$$f(\Gamma) = \sum_{I \subset \Gamma} \sum_{M \subset I} (-1)^{|I|-|M|} f(M),$$

because on the right hand side all terms except $f(\Gamma)$ cancel exactly. We now apply this to

$$f(M) = \exp\Big(-\beta \left(H_\Lambda^0 + \sum_{x \in M} v_x \right) \Big)$$

to obtain

$$e^{-\beta H_\Lambda} = f(\Lambda^\circ) = \sum_{I\subset\Lambda^\circ} T_I \qquad \text{with} \qquad T_I = \sum_{M\subset I}(-1)^{|I|-|M|} f(M).$$

In T_I, the second term in the exponential is supported on $\overline{M} \subset \overline{I}$ and thus commutes with $H^0_{\Lambda\setminus\overline{I}} = H^0_\Lambda - H^0_{\overline{I}}$, which can thus be factored out of the exponential. Hence,

$$T_I = e^{-\beta H^0_{\Lambda\setminus\overline{I}}}\, T_I^{\overline{I}} \qquad \text{with} \qquad T_{I_1}^{I_2} = \sum_{M\subset I_1}(-1)^{|I_1|-|M|}\, e^{-\beta\left(H^0_{I_2}+\sum_{x\in M} v_x\right)} \in \mathscr{A}_{I_2\cup\overline{I_1}}.$$

With this notation, $T_I = T_I^\Lambda$.

Lemma 3.1. *Using the above notations, it holds that*

$$\|T_I^{\overline{I}}\| \leq (2a)^{|I|}.$$

Proof. For $M \subset I \subset \Lambda^\circ$ consider the function

$$\mathbb{C}^{|M|} \to \mathscr{A}_{\overline{I}}, \quad z = (z_x)_{x\in M} \mapsto g_M(z) = H^0_{\overline{I}} + \sum_{x\in M} z_x\, v_x,$$

which agrees with $H^0_{\overline{I}} + \sum_{x\in M} v_x$ for $z = (1,\dots,1)$ and is analytic. Moreover, by (5)

$$\left|\left\langle\psi, \sum_{x\in M} z_x\, v_x\, \psi\right\rangle\right| \leq \sum_{x\in M} |z_x|\, \frac{a}{|B_R|}\, \left\langle\psi, H^0_{B_R(x)}\,\psi\right\rangle \leq \max_{x\in M} |z_x|\, a\left\langle\psi, H^0_{\overline{I}}\,\psi\right\rangle$$

for all $\psi \in \mathscr{H}_{\overline{I}}$. Hence, for all z such that $\max_x |z_x| < 1/a$ it holds that

$$\operatorname{Re}\langle\psi, g_M(z)\,\psi\rangle \geq 0,$$

which by the Hille-Yosida theorem implies that

$$\|e^{-\beta g_M(z)}\| \leq 1.$$

Now, for $I \subset \Lambda^\circ$ consider the function

$$\mathbb{C}^{|I|} \to \mathscr{A}_{\overline{I}}, \quad z = (z_x)_{x\in I} \mapsto T_I^{\overline{I}}(z) = \sum_{M\subset I}(-1)^{|I|-|M|}\, e^{-\beta\, g_M\left((z_x)_{x\in M}\right)}.$$

This function is also analytic, satisfies $\|T_I^{\overline{I}}(z)\| \leq 2^{|I|}$ if $\max_{x\in I}|z_x| < 1/a$ by the above bounds, and $T_I^{\overline{I}}\left((1)_{x\in I}\right) = T_I^{\overline{I}}$. Moreover, if $z_y = 0$ for some $y \in I$, then $T_I^{\overline{I}}(z) = 0$. To understand this, note that for any $M \subset I\setminus\{y\}$

$$g_{M\cup\{y\}}\left((z_x)_{x\in M\cup\{y\}}\right) = g_M\left((z_x)_{x\in M}\right)$$

and due to the different signs in $T_I^{\overline{I}}(z)$ all terms cancel. The result follows by applying [217, Lemma 2]. □

3.3 Cluster expansion

Using Yarotsky's decomposition from last section, for any $\Omega \subset \Lambda$ and $O \in \mathscr{A}_\Omega$ we can decompose

$$\begin{aligned}\frac{\operatorname{tr}\big(O\, \mathrm{e}^{-\beta H_\Lambda}\big)}{Z_\Lambda^0} &= \sum_{I \subset \Lambda^\circ} \frac{\operatorname{tr}\big(O\, T_I\big)}{Z_\Lambda^0} \\ &= \sum_{I \subset \Lambda^\circ} \frac{\operatorname{tr}\big(O\, T_I^{\bar{I}}\, \mathrm{e}^{-\beta H^0_{\Omega \setminus \bar{I}}}\, \mathrm{e}^{-\beta H^0_{\Lambda \setminus (\bar{I} \cup \Omega)}}\big)}{\operatorname{tr}\big(\mathrm{e}^{-\beta H^0_{\bar{I}}}\, \mathrm{e}^{-\beta H^0_{\Omega \setminus \bar{I}}}\, \mathrm{e}^{-\beta H^0_{\Lambda \setminus (\bar{I} \cup \Omega)}}\big)} \\ &= \sum_{I \subset \Lambda^\circ} \frac{\operatorname{tr}\big(O\, T_I^{\bar{I} \cup \Omega}\big)}{Z^0_{\bar{I} \cup \Omega}},\end{aligned}$$

where due to O, the trace only factorizes between $\mathscr{A}_{\bar{I} \cup \Omega}$ and $\mathscr{A}_{\Lambda \setminus (\bar{I} \cup \Omega)}$. For $O = \mathbb{1}$, one can choose $\Omega = \emptyset$ and obtain

$$\frac{\operatorname{tr}\big(\mathrm{e}^{-\beta H_\Lambda}\big)}{Z_\Lambda^0} = \sum_{I \in \Lambda^\circ} \frac{\operatorname{tr}\big(T_I^{\bar{I}}\big)}{Z^0_{\bar{I}}}.$$

At this point, we would like to interpret the subset I that is being summed over on the right-hand side of the equations above as a "configuration" that is produced by the partition function. This interpretation allows us to phrase the methods we will later use in this paper in the language of statistical physics. Accordingly, we will also give the following definition to link our quantities to concepts from statistical physics.

Definition 3.2 (Configuration, Support, and Weight)**.** We defined a configuration I to be any subset of Λ°. The support of the configuration is equivalent to the subset itself. The weight of the configuration is $\operatorname{tr}(T_I^{\bar{I}})/Z^0_{\bar{I}}$. ◇

Remark 3.3 (On the h_x's being on-site operators). In implementing the decomposition above and using the concepts introduced in definition 3.2, we crucially used that the operators h_x act only on individual sites. This is necessary to introduce the notion of "weights" and use that the weight of union $I \cup J$ factorizes, if I and J are not *R-connected* to each other (see definition 3.5 later). ◇

Remark 3.4 (Comparison to the probabilistic setting). The only notion presented above that would distinguish our present setup from standard statistical physics is the fact that the weight associated to each configuration is not necessarily positive. Moreover, we remark that the notion of "support" is only mentioned to complete the analogy, but not used in the argument. ◇

A critical idea in the paper [217] and statistical physics in general is the idea of cluster and the compatibility between clusters. Indeed, one can very strongly characterize properties of the Gibbs measures in statistical physics if one knows that configurations can be broken apart into distinct clusters and the weight of a configuration is a simple product of the weight of each individual cluster.

Definition 3.5 (R-connectedness)**.** Let x and y be two points of Λ. We say that x and y are *R-connected* if $B_R(x) \cap B_R(y) \neq \emptyset$. Similarly, we say that two sets I_1, $I_2 \subset \Lambda$ are *R-connected to each other*, if there exist R-connected points $x_1 \in I_1$ and $x_2 \in I_2$. ◇

Given this R-connectedness, we define clusters as R-connected sets as in definition 2.1.

Definition 3.6 (Cluster)**.** A subset $I \subset \Lambda$ is a *cluster* or equivalently an *R-connected set*, if for every two points $x, y \in I$ there exists a sequence of points $z_1, \dots, z_m \in I$ such that z_i is R-connected to z_{i+1} for $i \in \{1, \dots, m-1\}$ and x is R-connected to z_1 and y is R-connected to z_m. ◇

The following Lemma shows that our configurations and weights satisfy the fundamental properties of a cluster expansion.

Lemma 3.7. *Let $I_1, I_2 \subset \Lambda^\circ$ and $\Omega_1, \Omega_2 \subset \Lambda$. Assume that $I_1 \cup \Omega_1$ and $I_2 \cup \Omega_2$ are not R-connected to each other. Then, for every $O_1 \in \mathscr{A}_{\Omega_1}$ and $O_2 \in \mathscr{A}_{\Omega_2}$, we have that*

$$\frac{\operatorname{tr}(O_1\, O_2\, T_{I_1 \cup I_2}^{\overline{I_1 \cup I_2} \cup \Omega_1 \cup \Omega_2})}{Z^0_{\overline{I_1 \cup I_2} \cup \Omega_1 \cup \Omega_2}} = \frac{\operatorname{tr}(O_1\, T_{I_1}^{\overline{I_1} \cup \Omega_1})}{Z^0_{\overline{I_1} \cup \Omega_1}} \, \frac{\operatorname{tr}(O_2\, T_{I_2}^{\overline{I_2} \cup \Omega_2})}{Z^0_{\overline{I_2} \cup \Omega_2}}. \tag{10}$$

Proof. We first write

$$T_{I_1 \cup I_2}^{\overline{I_1 \cup I_2} \cup \Omega_1 \cup \Omega_2} = T_{I_1 \cup I_2}^{\overline{I_1 \cup I_2}}\, \mathrm{e}^{-\beta H^0_{(\Omega_1 \cup \Omega_2) \setminus \overline{I_1 \cup I_2}}}.$$

Since I_1 and I_2 are not R-connected to each other, we must have that $\overline{I_1} \cap \overline{I_2} = \emptyset$. Thus, $\overline{I_1 \cup I_2} = \overline{I_1} \cup \overline{I_2}$ is a disjoint union. Furthermore, I_1 and I_2 are disjoint and any subset $M \subset I_1 \cup I_2$ can be uniquely decomposed as $M = M_1 \cup M_2$, where $M_1 \subset I_1$ and $M_2 \cup I_2$. As a consequence, we see that

$$\begin{aligned} T_{I_1 \cup I_2}^{\overline{I_1 \cup I_2}} &= \sum_{M \subset I_1 \cup I_2} (-1)^{|I_1 \cup I_2| - |M|} \exp\Bigl(-\beta \Bigl(H^0_{\overline{I_1 \cup I_2}} + \sum_{x \in M} v_x\Bigr)\Bigr) \\ &= \sum_{M \subset I_1 \cup I_2} (-1)^{|I_1| - |M_1|} (-1)^{|I_2| - |M_2|} \exp\Bigl(-\beta \Bigl(H^0_{\overline{I_1}} + \sum_{x \in M_1} v_x\Bigr)\Bigr) \\ &\qquad \times \exp\Bigl(-\beta \Bigl(H^0_{\overline{I_2}} + \sum_{x \in M_2} v_x\Bigr)\Bigr) \\ &= T_{I_1}^{\overline{I_1}}\, T_{I_2}^{\overline{I_2}}. \end{aligned}$$

In the second line, we first decomposed the subset $M = M_1 \cup M_2$ as above. The operators $\sum_{x\in M_1} v_x$ and $H^0_{\overline{I_1}}$ are in $\mathscr{A}_{\overline{I_1}}$ while $\sum_{x\in M_2} v_x$ and $H^0_{\overline{I_2}}$ are in $\mathscr{A}_{\overline{I_2}}$. Thus, the operators $H^0_{\overline{I_1}}+\sum_{x\in M_1} v_x$ and $H^0_{\overline{I_2}}+\sum_{x\in M_2} v_x$ commute with each other. Since H^0 has only on-site terms, we also have

$$\mathrm{e}^{-\beta H^0_{(\Omega_1\cup\Omega_2)\setminus\overline{I_1\cup I_2}}} = \mathrm{e}^{-\beta H^0_{\Omega_1\setminus\overline{I_1}}}\,\mathrm{e}^{-\beta H^0_{\Omega_2\setminus\overline{I_2}}}$$

and

$$\mathrm{e}^{-\beta H^0_{\overline{I_1\cup I_2}\cup\Omega_1\cup\Omega_2}} = \mathrm{e}^{-\beta H^0_{\overline{I_1}\cup\Omega_1}}\,\mathrm{e}^{-\beta H^0_{\overline{I_2}\cup\Omega_2}}.$$

Using factorization of the trace between $\overline{I_1}\cup\Omega_1$ and $\overline{I_2}\cup\Omega_2$ gives (10). □

With these consequences in hand, we can now get to the crux of the matter.

Definition 3.8 (Supercluster)**.** Let I and J be two configurations. A supercluster decomposition of $I\cup J\cup X\cup Y$ is a decomposition

$$I\cup J\cup X\cup Y = S_1\cup S_2\cup\cdots\cup S_n,$$

where the sets S_i are all maximally R-connected subsets, i.e. the S_i are R-connected sets that are not R-connected to each other. ◇

We can provide configuration expansions for $\mathrm{tr}(\mathrm{e}^{-\beta H_\Lambda})/Z^0_\Lambda$, $\mathrm{tr}(A\,B\,\mathrm{e}^{-\beta H_\Lambda})/Z^0_\Lambda$ as well as $\mathrm{tr}(A\,\mathrm{e}^{-\beta H_\Lambda})/Z^0_\Lambda$ and $\mathrm{tr}(B\,\mathrm{e}^{-\beta H_\Lambda})/Z^0_\Lambda$. The product of the configuration decompositions of $\mathrm{tr}(\mathrm{e}^{-\beta H_\Lambda})/Z^0_\Lambda$ and $\mathrm{tr}(A\,B\,\mathrm{e}^{-\beta H_\Lambda})/Z^0_\Lambda$ will lead to a supercluster decomposition which will, in some instances, cancel with appropriate terms that appear in a supercluster decomposition involving $\mathrm{tr}(A\,\mathrm{e}^{-\beta H_\Lambda})/Z^0_\Lambda$ and $\mathrm{tr}(B\,\mathrm{e}^{-\beta H_\Lambda})/Z^0_\Lambda$. This is the content of the following theorem.

Theorem 3.9. *Let X and $Y\subset\Lambda$ each be an R-connected set and $A\in\mathscr{A}_X$ and $B\in\mathscr{A}_Y$. Let I be a configuration produced in the decomposition of $\mathrm{tr}(\mathrm{e}^{-\beta H_\Lambda})/Z^0_\Lambda$ and J be a configuration produced in the decomposition of $\mathrm{tr}(A\,B\,\mathrm{e}^{-\beta H_\Lambda})/Z^0_\Lambda$. We consider the supercluster decomposition of $I\cup J\cup X\cup Y$ and let $E(X,Y)$ be the event that the associated supercluster decomposition does not contain X and Y in the same supercluster. As a shorthand, we write such terms in the product as*

$$\left.\frac{\mathrm{tr}(\mathrm{e}^{-\beta H_\Lambda})}{Z^0_\Lambda}\,\frac{\mathrm{tr}(A\,B\,\mathrm{e}^{-\beta H_\Lambda})}{Z^0_\Lambda}\right|_{E(X,Y)}.$$

We consider a similar supercluster decomposition for

$$\left.\frac{\mathrm{tr}(A\,\mathrm{e}^{-\beta H_\Lambda})}{Z^0_\Lambda}\,\frac{\mathrm{tr}(B\,\mathrm{e}^{-\beta H_\Lambda})}{Z^0_\Lambda}\right|_{E(X,Y)}.$$

The two quantities above are equal.

Proof. Consider a supercluster expansion for

$$\frac{\operatorname{tr}(\mathrm{e}^{-\beta H_\Lambda})}{Z^0_\Lambda}\,\frac{\operatorname{tr}(A\,B\,\mathrm{e}^{-\beta H_\Lambda})}{Z^0_\Lambda}\bigg|_{E(X,Y)}.$$

This supercluster expansion can be written as $S_1 \cup S_2 \cup \cdots \cup S_m$ and without loss of generality we can assume that $X \subset S_1$ and $Y \subset S_2$. We now define

$$I' = (J \cap S_1) \cup \big(I \cap (S_2 \cup S_3 \cup \cdots \cup S_m)\big)$$

and

$$J' = (I \cap S_1) \cup \big(J \cap (S_2 \cup S_3 \cup \cdots \cup S_m)\big).$$

On an intuitive level, we switch the parts of the configurations I and J that are part of the first supercluster S_1, which contains X.

We remark that the supercluster decomposition of $X \cup Y \cup I' \cup J'$ is still $S_1 \cup S_2 \cup \cdots \cup S_m$. The union of $I' \cup J'$ must be the same as the union of $I \cup J$ by our construction and, thus, we could not have changed the maximally R-connected subsets that appear in the decomposition. Furthermore, Y will still be contained in the supercluster S_2 and X is contained in the supercluster S_1. As a result, we see that this construction is an involution. We only have to check that

$$\frac{\operatorname{tr}(T_I^{\overline{I}})}{Z^0_{\overline{I}}}\,\frac{\operatorname{tr}(A\,B\,T_J^{\overline{J\cup X\cup Y}})}{Z^0_{\overline{J\cup X\cup Y}}} = \frac{\operatorname{tr}(A\,T_{I'}^{\overline{I'\cup X}})}{Z^0_{\overline{I'\cup X}}}\,\frac{\operatorname{tr}(B\,T_{J'}^{\overline{J'\cup Y}})}{Z^0_{\overline{J'\cup Y}}}. \tag{11}$$

Since the superclusters $S_1,\ S_2, \ldots,\ S_m$ are mutually not R-connected to each other and $X \subset S_1$ and $Y \subset S_2$, we can apply lemma 3.7 to each of the fractions, yielding

$$\frac{\operatorname{tr}(T_I^{\overline{I}})}{Z^0_{\overline{I}}} = \prod_{k=1}^{m} \frac{\operatorname{tr}(T_{I\cap S_k}^{\overline{I\cap S_k}})}{Z^0_{\overline{I\cap S_k}}},$$

$$\frac{\operatorname{tr}(A\,B\,T_J^{\overline{J\cup X\cup Y}})}{Z^0_{\overline{J\cup X\cup Y}}} = \frac{\operatorname{tr}(A\,T_{J\cap S_1}^{\overline{J\cap S_1\cup X}})}{Z^0_{\overline{J\cap S_1\cup X}}}\,\frac{\operatorname{tr}(B\,T_{J\cap S_2}^{\overline{J\cap S_2\cup Y}})}{Z^0_{\overline{J\cap S_2\cup Y}}}\prod_{k=3}^{m}\frac{\operatorname{tr}(T_{J\cap S_k}^{\overline{J\cap S_k}})}{Z^0_{\overline{J\cap S_k}}}$$

for the left-hand-side and

$$\begin{aligned}\frac{\operatorname{tr}(A\,T_{I'}^{\overline{I'\cup X}})}{Z^0_{\overline{I'\cup X}}} &= \frac{\operatorname{tr}(A\,T_{I'\cap S_1}^{\overline{I'\cap S_1\cup X}})}{Z^0_{\overline{I'\cap S_1\cup X}}}\,\frac{\operatorname{tr}(T_{I'\cap S_2}^{\overline{I'\cap S_2}})}{Z^0_{\overline{I'\cap S_2}}}\prod_{k=3}^{m}\frac{\operatorname{tr}(T_{I'\cap S_k}^{\overline{I'\cap S_k}})}{Z^0_{\overline{I'\cap S_k}}}\\ &= \frac{\operatorname{tr}(A\,T_{J\cap S_1}^{\overline{J\cap S_1\cup X}})}{Z^0_{\overline{J\cap S_1\cup X}}}\,\frac{\operatorname{tr}(T_{I\cap S_2}^{\overline{I\cap S_2}})}{Z^0_{\overline{I\cap S_2}}}\prod_{k=3}^{m}\frac{\operatorname{tr}(T_{I\cap S_k}^{\overline{I\cap S_k}})}{Z^0_{\overline{I\cap S_k}}},\end{aligned}$$

$$\frac{\operatorname{tr}(B\,T_{J'}^{\overline{J'\cup Y}})}{Z^0_{\overline{J'\cup Y}}} = \frac{\operatorname{tr}(T_{J'\cap S_1}^{\overline{J'\cap S_1}})}{Z^0_{\overline{J'\cap S_1}}}\,\frac{\operatorname{tr}(B\,T_{J'\cap S_2}^{\overline{J'\cap S_2\cup Y}})}{Z^0_{\overline{J'\cap S_2\cup Y}}}\prod_{k=3}^{m}\frac{\operatorname{tr}(T_{J'\cap S_k}^{\overline{J'\cap S_k}})}{Z^0_{\overline{J'\cap S_k}}}$$
$$= \frac{\operatorname{tr}(T_{I\cap S_1}^{\overline{I\cap S_1}})}{Z^0_{\overline{I\cap S_1}}}\,\frac{\operatorname{tr}(B\,T_{J\cap S_2}^{\overline{J\cap S_2\cup Y}})}{Z^0_{\overline{J\cap S_2\cup Y}}}\prod_{k=3}^{m}\frac{\operatorname{tr}(T_{J\cap S_k}^{\overline{J\cap S_k}})}{Z^0_{\overline{J\cap S_k}}}$$

for the right-hand-side. Observing that these terms are the same proves (11) and shows that we have equality of the supercluster expansions on the event $E(X,Y)$. □

At this point, it remains to control the product of the cluster expansions when X and Y are part of the same supercluster. Namely, what we will do is to first fix a given supercluster S that connects X and Y and consider all pairs of configurations I and J such that S is a supercluster of $X\cup Y\cup I\cup J$. We will then sum over all of these pairs of configurations such that I and J are superclusters of this configuration. More formally, what we will actually do is fix the intersection $I\cap S$ and $J\cap S$, and sum over all pairs of clusters with such a fixed value of $I\cap S$ and $J\cap S$. We have the following Lemma.

Lemma 3.10. *Fix subsets I_0, $J_0\subset\Lambda^\circ$ such that $S_0 := I_0\cup J_0\cup X\cup Y$ is an R-connected set. Denote with $\mathscr{C}_{I_0,J_0}$ the set of pairs (I,J), with $I,J\subset\Lambda^\circ$, whose supercluster decomposition $I\cup J\cup X\cup Y = S_0\cup S_1\cup\cdots\cup S_m$ contains S_0 and satisfies $I\cap S_0 = I_0$ and $J\cap S_0 = J_0$. Then, we have that*

$$\sum_{(I,J)\in\mathscr{C}_{I_0,J_0}}\frac{\operatorname{tr}(T_I^{\overline{I}})}{Z^0_{\overline{I}}}\,\frac{\operatorname{tr}(A\,B\,T_J^{\overline{J\cup X\cup Y}})}{Z^0_{\overline{J\cup X\cup Y}}} = \frac{\operatorname{tr}(T_{I_0}^{\overline{I_0}})}{Z^0_{\overline{I_0}}}\,\frac{\operatorname{tr}(A\,B\,T_{J_0}^{\overline{J_0\cup X\cup Y}})}{Z^0_{\overline{J_0\cup X\cup Y}}}\left(\frac{Z_{\Lambda\setminus\overline{S_0}}}{Z^0_{\Lambda\setminus\overline{S_0}}}\right)^2 \tag{12}$$

and

$$\sum_{(I,J)\in\mathscr{C}_{I_0,J_0}}\frac{\operatorname{tr}(A\,T_I^{\overline{I\cup X}})}{Z^0_{\overline{I\cup X}}}\,\frac{\operatorname{tr}(B\,T_J^{\overline{J\cup Y}})}{Z^0_{\overline{J\cup Y}}} = \frac{\operatorname{tr}(A\,T_{I_0}^{\overline{I_0\cup X}})}{Z^0_{\overline{I_0\cup X}}}\,\frac{\operatorname{tr}(B\,T_{J_0}^{\overline{J_0\cup Y}})}{Z^0_{\overline{J_0\cup Y}}}\left(\frac{Z_{\Lambda\setminus\overline{S_0}}}{Z^0_{\Lambda\setminus\overline{S_0}}}\right)^2. \tag{13}$$

Proof. First, note that S_0 and $S_1\cup\cdots\cup S_m$ are not R-connected to each other, and $X,Y\subset S_0$. Hence, we can decompose $\mathscr{C}_{I_0,J_0}$ as

$$\begin{aligned}\mathscr{C}_{I_0,J_0} &= \left\{(I_0\cup I', J_0\cup J')\,\middle|\, I',\ J'\subset\Lambda^\circ \text{ such that } \overline{I'}\cap\overline{S_0}=\emptyset \text{ and } \overline{J'}\cap\overline{S_0}=\emptyset\right\}\\ &= \left\{(I_0\cup I', J_0\cup J')\,\middle|\, I',\ J'\subset\Lambda^\circ\setminus\overline{\overline{S_0}} = \left(\Lambda\setminus\overline{S_0}\right)^\circ\right\}.\end{aligned}$$

Using this and lemma 3.7, we obtain

$$\sum_{(I,J)\in\mathscr{C}_{I_0,J_0}} \frac{\operatorname{tr}(T_I^{\overline{I}})}{Z^0_{\overline{I}}} \frac{\operatorname{tr}(A\,B\,T_J^{\overline{J}\cup X\cup Y})}{Z^0_{\overline{J}\cup X\cup Y}} = \frac{\operatorname{tr}(T_{I_0}^{\overline{I_0}})}{Z^0_{\overline{I_0}}} \frac{\operatorname{tr}(A\,B\,T_{J_0}^{\overline{J_0}\cup X\cup Y})}{Z^0_{\overline{J_0}\cup X\cup Y}} \sum_{I',J'\subset(\Lambda\setminus\overline{S_0})^\circ} \frac{\operatorname{tr}(T_{I'}^{\overline{I'}})}{Z^0_{\overline{I'}}} \frac{\operatorname{tr}(T_{J'}^{\overline{J'}})}{Z^0_{\overline{J'}}}.$$

As in the decomposition of section 3.2, we have

$$\sum_{I'\subset(\Lambda\setminus\overline{S_0})^\circ} \frac{\operatorname{tr}(T_{I'}^{\overline{I'}})}{Z^0_{\overline{I'}}} = \sum_{I'\subset(\Lambda\setminus\overline{S_0})^\circ} \frac{\operatorname{tr}(T_{I'}^{\Lambda\setminus\overline{S_0}})}{Z^0_{\Lambda\setminus\overline{S_0}}} = \frac{\operatorname{tr}(\mathrm{e}^{-\beta H_{\Lambda\setminus\overline{S_0}}})}{Z^0_{\Lambda\setminus\overline{S_0}}} = \frac{Z_{\Lambda\setminus\overline{S_0}}}{Z^0_{\Lambda\setminus\overline{S_0}}}$$

and (12) follows. Equation (13) follows in exactly the same way. □

3.4 Ratios of partition functions

In the previous section, we extracted important cancellations between various terms arising in the cluster expansion (see theorem 3.9). The remaining terms (see lemma 3.10) involve ratios of partition functions associated to the Hamiltonians H_Λ and H^0_Λ, which we control with the aid of the following lemma.

Lemma 3.11. *Let D, q, $R \in \mathbb{N}$ and $a \in (0,1)$. Then there exist $C > 0$ such that the following holds. Consider the lattice $\Lambda \Subset \mathbb{Z}^D$ with local Hilbert space $\mathscr{H}_x \equiv \mathbb{C}^q$, $q \geq 2$, at each $x \in \Lambda$, and a Hamiltonian $H^0_\Lambda + V_\Lambda$ as defined in section 2.1 with v_x relatively a-bounded w.r.t. H^0_Λ in the sense* (5) *and $v_x \leq 0$. Then for all $\beta \in (0,\infty)$ and all R-connected sets $S \subset \Lambda$*

$$\frac{Z^0_\Lambda}{Z_\Lambda}\frac{Z_{\Lambda\setminus\overline{S}}}{Z^0_{\Lambda\setminus\overline{S}}} = \frac{Z_{\Lambda\setminus\overline{S}}\,Z^0_{\overline{S}}}{Z_\Lambda} \leq 1.$$

Proof. The proof relies on the following basic trace inequality. For any two $n \times n$ self-adjoint matrices H and H' satisfying the operator inequality $H \leq H'$, we have the monotonicity

$$\operatorname{tr}\big(\mathrm{e}^{H}\big) \leq \operatorname{tr}\big(\mathrm{e}^{H'}\big). \tag{14}$$

To see that (14) holds, write $\lambda_1 \leq \cdots \leq \lambda_n$ and $\lambda'_1 \leq \cdots \leq \lambda'_n$ for the ordered eigenvalues of H and H', respectively. By the min-max principle and $H \leq H'$, we have $\lambda_i \leq \lambda'_i$ for all i and so the spectral theorem implies (14).

Note, that the equality in the statement just follows by the factorization of Z^0_Λ and $Z^0_{\Lambda\setminus\overline{S}}$. And by monotonicity we find

$$Z_{\Lambda\setminus\overline{S}}\,Z^0_{\overline{S}} = \operatorname{tr}\Big(\mathrm{e}^{-\beta\,(H^0_\Lambda + V_{\Lambda\setminus\overline{S}})}\Big) \leq \operatorname{tr}\Big(\mathrm{e}^{-\beta\,H_\Lambda}\Big) = Z_\Lambda, \tag{15}$$

since

$$H_\Lambda^0 + V_{\Lambda\setminus\overline{S}} \geq H_\Lambda^0 + V_\Lambda = H_\Lambda$$

as we add terms $v_x \leq 0$. And the statement follows. □

3.5 Proof of Theorem 2.2

We can now put everything together and prove theorem 2.2. By the decomposition introduced in section 3.2, we have

$$\begin{aligned}
&|\mathrm{tr}(A\,B\,\rho_\Lambda) - \mathrm{tr}(A\,\rho_\Lambda)\;\mathrm{tr}(B\,\rho_\Lambda)| \\
&\quad = \left(\frac{Z_\Lambda^0}{Z_\Lambda}\right)^2 \left| \frac{\mathrm{tr}(\mathrm{e}^{-\beta H_\Lambda})}{Z_\Lambda^0}\,\frac{\mathrm{tr}(A\,B\,\mathrm{e}^{-\beta H_\Lambda})}{Z_\Lambda^0} - \frac{\mathrm{tr}(A\,\mathrm{e}^{-\beta H_\Lambda})}{Z_\Lambda^0}\,\frac{\mathrm{tr}(B\,\mathrm{e}^{-\beta H_\Lambda})}{Z_\Lambda^0}\right| \\
&\quad = \left(\frac{Z_\Lambda^0}{Z_\Lambda}\right)^2 \left| \sum_{I,J\subset\Lambda^\circ} \frac{\mathrm{tr}(T_I^{\overline{I}})}{Z_{\overline{I}}^0}\,\frac{\mathrm{tr}(A\,B\,T_J^{\overline{J}\cup X\cup Y})}{Z_{\overline{J}\cup X\cup Y}^0} - \frac{\mathrm{tr}(A\,T_I^{\overline{I}\cup X})}{Z_{\overline{I}\cup X}^0}\,\frac{\mathrm{tr}(B\,T_J^{\overline{J}\cup Y})}{Z_{\overline{J}\cup Y}^0}\right|.
\end{aligned}$$

And by theorem 3.9, the part of the sum where the supercluster decomposition of $I \cup J \cup X \cup Y$ has X and Y in two disjoint clusters vanishes. What remains is a sum over $I, J \subset \Lambda^\circ$ such that the supercluster decomposition $I \cup J \cup X \cup Y = S_0 \cup S_1 \cup \cdots \cup S_m$ satisfies $X, Y \subset S_0$. In this cluster S_0, we still have the freedom to choose $I_0 = I \cap S_0$ and $J_0 = J \cap S_0$. Then, using the notation from lemma 3.10, we obtain the upper bound

$$\begin{aligned}
&\left(\frac{Z_\Lambda^0}{Z_\Lambda}\right)^2 \sum_{\substack{S_0\subset\Lambda:\\ S_0\ R\text{-connected}\\ X,Y\subset S_0}} \sum_{\substack{I_0,J_0\subset\Lambda^\circ:\\ I_0\cup J_0\cup X\cup Y=S_0}} \left| \sum_{(I,J)\in\mathscr{C}_{I_0,J_0}} \frac{\mathrm{tr}(T_I^{\overline{I}})}{Z_{\overline{I}}^0}\,\frac{\mathrm{tr}(A\,B\,T_J^{\overline{J}\cup X\cup Y})}{Z_{\overline{J}\cup X\cup Y}^0} - \frac{\mathrm{tr}(A\,T_I^{\overline{I}\cup X})}{Z_{\overline{I}\cup X}^0}\,\frac{\mathrm{tr}(B\,T_J^{\overline{J}\cup Y})}{Z_{\overline{J}\cup Y}^0}\right| \\
&\quad \leq \left(\frac{Z_\Lambda^0}{Z_\Lambda}\right)^2 \sum_{\substack{S_0\subset\Lambda:\\ S_0\ R\text{-connected}\\ X,Y\subset S_0}} \left(\frac{Z_{\Lambda\setminus\overline{S_0}}}{Z_{\Lambda\setminus\overline{S_0}}^0}\right)^2 \sum_{\substack{I_0,J_0\subset\Lambda^\circ:\\ I_0\cup J_0\cup X\cup Y=S_0}} \left|\frac{\mathrm{tr}(T_{I_0}^{\overline{I_0}})}{Z_{\overline{I_0}}^0}\,\frac{\mathrm{tr}(A\,B\,T_{J_0}^{\overline{J_0}\cup X\cup Y})}{Z_{\overline{J_0}\cup X\cup Y}^0}\right| + \left|\frac{\mathrm{tr}(A\,T_{I_0}^{\overline{I_0}\cup X})}{Z_{\overline{I_0}\cup X}^0}\,\frac{\mathrm{tr}(B\,T_{J_0}^{\overline{J_0}\cup Y})}{Z_{\overline{J_0}\cup Y}^0}\right|.
\end{aligned}$$

For each $I, \Omega \subset S_0$ and $O \in \mathscr{A}_\Omega$, we now use lemma 3.1 to control the two summands in the last line as

$$\left|\frac{\mathrm{tr}(O\,T_I^{\overline{I}\cup\Omega})}{Z_{\overline{I}\cup\Omega}^0}\right| = \left|\frac{\mathrm{tr}_{\overline{I}\cup\Omega}(O\,T_I^{\overline{I}\cup\Omega})}{\mathrm{tr}_{\overline{I}}(\mathrm{e}^{-\beta H_{\overline{I}}^0})\;\mathrm{tr}_{\Omega\setminus\overline{I}}(\mathrm{e}^{-\beta H_{\Omega\setminus\overline{I}}^0})}\right| \leq \|O\|\left(2\,a\,q^{(2R+1)^D}\right)^{|I|}.$$

Here, we explicitly took the trace only over $\mathscr{A}_{\overline{I}\cup\Omega}$ and then estimated

$$|\mathrm{tr}_{\overline{I}\cup\Omega}(O\,T_I^{\overline{I}\cup\Omega})| \leq \|O\|\,\|T_I^{\overline{I}}\,\mathrm{e}^{-\beta H_{\Omega\setminus\overline{I}}^0}\|_{\mathrm{Tr}_{\overline{I}\cup\Omega}} \leq \|O\|\,\|T_I^{\overline{I}}\|_{\mathrm{Tr}_{\overline{I}}}\,\|\mathrm{e}^{-\beta H_{\Omega\setminus\overline{I}}^0}\|_{\mathrm{Tr}_{\Omega\setminus\overline{I}}}$$

and

$$\|T_I^{\overline{I}}\|_{\mathrm{Tr}_{\overline{I}}} \leq \|T_I^{\overline{I}}\| \, \|\mathbb{1}_{\overline{I}}\|_{\mathrm{Tr}_{\overline{I}}} \leq (2a)^{|I|} \, q^{|\overline{I}|} \leq (2a)^{|I|} \, q^{(2R+1)^D \, |I|}$$

by lemma 3.1. Putting everything together, we obtain the bound

$$\begin{aligned} &|\mathrm{tr}(A \, B \, \rho_\Lambda) - \mathrm{tr}(A \, \rho_\Lambda) \; \mathrm{tr}(B \, \rho_\Lambda)| \\ &\quad \leq 2 \, \|A\| \, \|B\| \sum_{\substack{S_0 \subset \Lambda: \\ S_0 \ R\text{-connected} \\ X, Y \subset S_0}} \left(\frac{Z_\Lambda^0}{Z_\Lambda} \, \frac{Z_{\Lambda \setminus \overline{S_0}}}{Z^0_{\Lambda \setminus \overline{S_0}}} \right)^2 \sum_{\substack{I_0, J_0 \subset \Lambda^\circ: \\ I_0 \cup J_0 \cup X \cup Y = S_0}} \left(2 \, a \, q^{(2R+1)^D} \right)^{|I_0| + |J_0|}. \end{aligned}$$

The ratio of partition functions in the squared parenthesis is upper bounded by 1 due to lemma 3.11, recalling that $v_x \leq 0$ w.l.o.g., as explained in section 2.1

Next, to control the inner sum over I_0 and J_0, we abbreviate $p := 2 \, a \, q^{(2R+1)^D}$ and $M := S_0 \setminus (X \uplus Y)$. Then, after replacing $X \to X \cap \Lambda^\circ$ and $Y \to Y \cap \Lambda^\circ$, we have

$$\begin{aligned} \sum_{\substack{I_0, J_0 \subset \Lambda^\circ: \\ I_0 \cup J_0 \cup X \cup Y = S_0}} p^{|I_0| + |J_0|} &= \sum_{\substack{I_M, J_M \subset M: \\ I_M \cup J_M = M}} \sum_{\substack{I_X, J_X \subset X \\ I_Y, J_Y \subset Y}} p^{|I_M \uplus I_X \uplus I_Y| + |J_M \uplus J_X \uplus J_Y|} \\ &= \sum_{\substack{I_M, J_M \subset M: \\ I_M \cup J_M = M}} p^{|I_M| + |J_M|} \, (1 + p)^{2|X| + 2|Y|} \\ &= \sum_{I_M \subset M} \sum_{\tilde{J} \subset I_M} p^{|I_M| + |\tilde{J} \uplus M \setminus I_M|} \, (1 + p)^{2|X| + 2|Y|} \\ &= p^{|M|} \sum_{I_M \subset M} \sum_{\tilde{J} \subset I_M} p^{|\tilde{J}|} \, (1 + p)^{2|X| + 2|Y|} \\ &= (2p)^{|M|} \, (1 + p)^{|M|} \, (1 + p)^{2|X| + 2|Y|}, \end{aligned}$$

where we repeatedly used

$$\sum_{E \subset F} p^{|E|} = \sum_{n=0}^{|F|} \binom{|F|}{n} p^n \, 1^{|F| - n} = (1 + p)^{|F|}.$$

In this way, we obtain

$$\begin{aligned}
&|\mathrm{tr}(A\,B\,\rho_\Lambda) - \mathrm{tr}(A\,\rho_\Lambda)\ \mathrm{tr}(B\,\rho_\Lambda)| \\
&\quad \leq 2\,\|A\|\,\|B\|\,(1+p)^{2|X|+2|Y|} \sum_{\substack{S_0 \subset \Lambda: \\ S_0\ R\text{-connected} \\ X,Y\subset S_0}} \big(2p\,(1+p)\big)^{|S_0 \setminus (X \uplus Y)|} \\
&\quad \leq 2\,\|A\|\,\|B\| \left(\frac{1}{2} + \frac{1}{2p}\right)^{|X|+|Y|} \sum_{\substack{S_0 \subset \Lambda: \\ S_0\ R\text{-connected} \\ X,Y\subset S_0}} \big(2p\,(1+p)\big)^{|S_0|}.
\end{aligned} \tag{16}$$

Finally, since S_0 in (16) is an R-connected set and contains X and Y, it must have at least $k = d(X,Y)/(2R)$ sites, since there must be $z_0, z_1, \dots, z_k \in S_0$ with $z_0 \in X$, $z_k \in Y$ and $d(z_i, z_{i+1}) \leq 2R$. Then, by the following lemma, whose proof is given at the end of the section, there are at most C^k such S_0 with $|S_0| = k$.

Lemma 3.12. *Let $D \in \mathbb{N}$ and $R \in \mathbb{N}$. Then there exists a constant $C > 0$ such that for any $v_1 \in \mathbb{Z}^D$ the number R-connected subsets $S \subset \mathbb{Z}^D$ containing $v_1 \in S$ and satisfying $|S| = k \in \mathbb{N}$ is bounded by C^k.*

Therefore, we find that

$$\begin{aligned}
&|\mathrm{tr}(A\,B\,\rho_\Lambda) - \mathrm{tr}(A\,\rho_\Lambda)\ \mathrm{tr}(B\,\rho_\Lambda)| \\
&\quad \leq 2\,\|A\|\,\|B\| \left(\frac{1}{2} + \frac{1}{2p}\right)^{|X|+|Y|} \sum_{k=d(X,Y)/(2R)}^{\infty} \big(2p\,(1+p)\,C\big)^{k} \\
&\quad \leq C\,\|A\|\,\|B\| \left(\frac{1}{2} + \frac{1}{2p}\right)^{|X|+|Y|} \big(2p\,(1+p)\,C\big)^{d(X,Y)/(2R)}.
\end{aligned}$$

Here, in the last step, we chose a such that $2p\,(1+p)\,C < 1$ (recall the shorthand notation $p = 2\,a\,q^{(2R+1)^D}$). This completes the proof of theorem 2.2. □

It remains to give the proof of lemma 3.12.

Proof of lemma 3.12. We will use the following algorithm to count the number of R-connected subsets of $\mathbb{Z}^D$ that contain a specific point v_1. Therefore, fix any well-ordering $v_1 \prec v_2 \prec v_3 \prec \cdots$ on $\mathbb{Z}^D$.

Given any R-connected set $S \subset \mathbb{Z}^D$ of size k that contains v_1, we construct an ordered list $Q = (q_1, q_2, \dots, q_k)$ according to the following algorithm. We begin the algorithm by setting $n = 1$, $i = 1$ and $q_1 = v_1$. In each step of the algorithm we do the following: If the set

$$M := S \cap B_R(q_i) \setminus \{q_1, \dots, q_n\}$$

of vertices, which are in S and R-connected to q_i but not yet in Q, is empty, we increase i by one. Otherwise, we set the next element in Q to the lowest element of M according to the chosen well-ordering $q_{n+1} = \min_{\prec} M$ and then increase n by one. We stop the algorithm after $2k-2$ steps, when $n = i = k$ and $S = \{q_1, \dots, q_k\}$.

Conversely, we can now count the number of R-connected sets $S \subset \mathbb{Z}^D$ containing v_1 and having $|S| = k$ sites by counting the number of ways this algorithm could run. Therefore, we observe that there are less than $\binom{2(k-1)}{k-1}$ possible ways of splitting between the two branches in the algorithm. Moreover, in the second branch, one has less than $|B_R(q_i)| \le (2R+1)^D$ possibilities of choosing q_{n+1}. Hence, in total there can be at most

$$\binom{2\,(k-1)}{k-1}(2R+1)^{D(k-1)} \le \left(2\,\mathrm{e}\,(2R+1)^D\right)^{k-1}$$

R-connected sets of size k that contain v_1. □

3.6 Proof of Theorem 2.3

We now adjust the proof for local indistinguishability. Therefore, fix $\Lambda' \subset \Lambda$. We use the same notation as before for H^0_Λ and H_Λ and all derived quantities. Additionally, we define an interaction with support in Λ', by defining local terms

$$\tilde{v}_x = \begin{cases} v_x & \text{if } B_R(x) \subset \Lambda' \\ 0 & \text{otherwise.} \end{cases}$$

All symbols derived with $\tilde{v}_x$ instead of v_x are also denoted with a tilde. In this way, $\tilde{V}_\Lambda = V_{\Lambda'}$ and $\tilde{H}_\Lambda = H^0_\Lambda + \tilde{V}_\Lambda$. Since the Hamiltonian H^0_Λ only has on-site contributions, the exponential $\mathrm{e}^{-\beta \tilde{H}_\Lambda} = \mathrm{e}^{-\beta H_{\Lambda'}}\,\mathrm{e}^{-\beta H^0_{\Lambda\setminus\Lambda'}}$ factors, and $\mathrm{tr}(A\,\rho_{\Lambda'}) = \mathrm{tr}(A\,\tilde{\rho}_\Lambda)$ for all $A \in \mathscr{A}_{\Lambda'}$. Hence, we only need to compare $\tilde{\rho}_\Lambda$ and ρ_Λ in the following.

Clearly, also the interaction $\tilde{v}_x$ satisfies the assumptions from section 2.1, and thus the derived quantities $\tilde{T}^{I_2}_{I_1}$ satisfy lemmata 3.1, 3.7 and 3.11. We are left to adjust the cluster expansion. Therefore, we write

$$\begin{aligned} &|\mathrm{tr}(B\,\rho_\Lambda) - \mathrm{tr}(B\,\tilde{\rho}_\Lambda)| \\ &\quad= \frac{(Z^0_\Lambda)^2}{Z_\Lambda\,\tilde{Z}_\Lambda}\left|\frac{\mathrm{tr}(B\,\mathrm{e}^{-\beta H_\Lambda})}{Z^0_\Lambda}\,\frac{\mathrm{tr}(\mathrm{e}^{-\beta\tilde{H}_\Lambda})}{Z^0_\Lambda} - \frac{\mathrm{tr}(B\,\mathrm{e}^{-\beta\tilde{H}_\Lambda})}{Z^0_\Lambda}\,\frac{\mathrm{tr}(\mathrm{e}^{-\beta H_\Lambda})}{Z^0_\Lambda}\right| \\ &\quad= \frac{(Z^0_\Lambda)^2}{Z_\Lambda\,\tilde{Z}_\Lambda}\left|\sum_{I,J\subset\Lambda^\circ}\frac{\mathrm{tr}(B\,T^{\overline{I\cup Y}}_I)}{Z^0_{\overline{I\cup Y}}}\,\frac{\mathrm{tr}(\tilde{T}^{\overline{J}}_J)}{Z^0_{\overline{J}}} - \frac{\mathrm{tr}(B\,\tilde{T}^{\overline{I\cup Y}}_I)}{Z^0_{\overline{I\cup Y}}}\,\frac{\mathrm{tr}(T^{\overline{J}}_J)}{Z^0_{\overline{J}}}\right| \end{aligned}$$

for any $B \in \mathscr{A}_Y$ and then observe the equivalent statement to theorem 3.9.

Theorem 3.13. *Let $Y \subset \Lambda'$ be an R-connected set and $B \in \mathscr{A}_Y$. Let I be a configuration produced in the decomposition of $\operatorname{tr}(B\,\mathrm{e}^{-\beta H_\Lambda})/Z_\Lambda^0$ and J be a configuration produced in the decomposition of $\operatorname{tr}(\mathrm{e}^{-\beta \tilde{H}_\Lambda})/Z_\Lambda^0$. We consider the supercluster decomposition of $I \cup J \cup Y$ and let $E(Y, \Lambda'^\circ)$ be the event that the associated supercluster decomposition contains Y in a cluster that itself is contained in Λ'°. As a shorthand, we write such terms in the product as*

$$\frac{\operatorname{tr}(B\,\mathrm{e}^{-\beta H_\Lambda})}{Z_\Lambda^0}\,\frac{\operatorname{tr}(\mathrm{e}^{-\beta \tilde{H}_\Lambda})}{Z_\Lambda^0}\bigg|_{E(Y,\Lambda'^\circ)}.$$

We consider the same supercluster decomposition for

$$\frac{\operatorname{tr}(B\,\mathrm{e}^{-\beta \tilde{H}_\Lambda})}{Z_\Lambda^0}\,\frac{\operatorname{tr}(\mathrm{e}^{-\beta H_\Lambda})}{Z_\Lambda^0}\bigg|_{E(Y,\Lambda'^\circ)}.$$

The two quantities are equal.

Proof. We consider a supercluster expansion $I \cup J \cup Y = S_1 \cup \cdots \cup S_m$. Without loss of generality, we assume $Y \subset S_1$. If $S_1 \subset \Lambda'^\circ$, then $\tilde{T}_{I\cap S_1}^{\overline{I\cap S_1 \cup Y}} = T_{I\cap S_1}^{\overline{I\cap S_1 \cup Y}}$ and $\tilde{T}_{J\cap S_1}^{\overline{J\cap S_1}} = T_{J\cap S_1}^{\overline{J\cap S_1}}$ and there is another configuration

$$\begin{aligned} I' &= (I \cap S_1) \cup \big(J \cap (S_2 \cup S_3 \cup \cdots \cup S_m)\big),\\ J' &= (J \cap S_1) \cup \big(I \cap (S_2 \cup S_3 \cup \cdots \cup S_m)\big) \end{aligned}$$

with the same supercluster decomposition, such that

$$\begin{aligned} &\frac{\operatorname{tr}(B\,T_I^{\overline{I\cup Y}})}{Z^0_{\overline{I\cup Y}}}\,\frac{\operatorname{tr}(\tilde{T}_J^{\overline{J}})}{Z^0_{\overline{J}}}\\ &\quad= \frac{\operatorname{tr}(B\,T_{I\cap S_1}^{\overline{I\cap S_1\cup Y}})}{Z^0_{\overline{I\cap S_1\cup Y}}}\,\frac{\operatorname{tr}(\tilde{T}_{J\cap S_1}^{\overline{J\cap S_1}})}{Z^0_{\overline{J\cap S_1}}}\prod_{k=2}^m \frac{\operatorname{tr}(T_{I\cap S_k}^{\overline{I\cap S_k}})}{Z^0_{\overline{I\cap S_k}}}\,\frac{\operatorname{tr}(\tilde{T}_{J\cap S_k}^{\overline{J\cap S_k}})}{Z^0_{\overline{J\cap S_k}}}\\ &\quad= \frac{\operatorname{tr}(B\,\tilde{T}_{I'\cap S_1}^{\overline{I'\cap S_1\cup Y}})}{Z^0_{\overline{I'\cap S_1\cup Y}}}\,\frac{\operatorname{tr}(T_{J'\cap S_1}^{\overline{J'\cap S_1}})}{Z^0_{\overline{J'\cap S_1}}}\prod_{k=2}^m \frac{\operatorname{tr}(T_{J'\cap S_k}^{\overline{J'\cap S_k}})}{Z^0_{\overline{J'\cap S_k}}}\,\frac{\operatorname{tr}(\tilde{T}_{I'\cap S_k}^{\overline{I'\cap S_k}})}{Z^0_{\overline{I'\cap S_k}}}\\ &\quad= \frac{\operatorname{tr}(B\,\tilde{T}_{I'}^{\overline{I'\cup Y}})}{Z^0_{\overline{I'\cup Y}}}\,\frac{\operatorname{tr}(T_{J'}^{\overline{J'}})}{Z^0_{\overline{J'}}}. \end{aligned}$$

This concludes the proof. □

Hence, as for DoC, what remains is a sum over $I, J \subset \Lambda^\circ$ such that the supercluster decomposition $I \cup J \cup Y = S_0 \cup \cdots \cup S_m$ satisfies $Y \subset S_0$ and $S_0 \cap (\Lambda \setminus \Lambda'^\circ) \neq \emptyset$. We now apply lemma 3.10 with $X = \emptyset$ and $A = \mathbb{1}$, to obtain the upper bound

$$\frac{(Z^0_\Lambda)^2}{Z_\Lambda \tilde{Z}_\Lambda} \sum_{\substack{S_0 \subset \Lambda: \\ S_0 \ R\text{-connected} \\ Y \subset S_0 \\ S_0 \cap (\Lambda \setminus \Lambda'^\circ) \neq \emptyset}} \frac{Z_{\Lambda \setminus \overline{S_0}} \tilde{Z}_{\Lambda \setminus \overline{S_0}}}{(Z^0_{\Lambda \setminus \overline{S_0}})^2} \sum_{\substack{I_0, J_0 \subset \Lambda^\circ: \\ I_0 \cup J_0 \cup Y = S_0}} \left| \frac{\operatorname{tr}(B\, T_{I_0}^{\overline{I_0 \cup Y}})}{Z^0_{\overline{I_0 \cup Y}}} \frac{\operatorname{tr}(\tilde{T}_{J_0}^{\overline{J_0}})}{Z^0_{\overline{J_0}}} \right| + \left| \frac{\operatorname{tr}(B\, \tilde{T}_{I_0}^{\overline{I_0 \cup Y}})}{Z^0_{\overline{I_0 \cup Y}}} \frac{\operatorname{tr}(T_{J_0}^{\overline{J_0}})}{Z^0_{\overline{J_0}}} \right|.$$

Following the arguments in the proof of theorem 2.2, we obtain

$$\begin{aligned}
&|\operatorname{tr}(B\,\rho_\Lambda) - \operatorname{tr}(B\,\tilde{\rho}_\Lambda)| \\
&\quad \leq 2\,\|B\| \sum_{\substack{S_0 \subset \Lambda: \\ S_0 \ R\text{-connected} \\ Y \subset S_0 \\ S_0 \cap (\Lambda \setminus \Lambda'^\circ) \neq \emptyset}} \frac{(Z^0_\Lambda)^2}{Z_\Lambda \tilde{Z}_\Lambda} \frac{Z_{\Lambda \setminus \overline{S_0}} \tilde{Z}_{\Lambda \setminus \overline{S_0}}}{(Z^0_{\Lambda \setminus \overline{S_0}})^2} \sum_{\substack{I_0, J_0 \subset \Lambda^\circ: \\ I_0 \cup J_0 \cup Y = S_0}} p^{|I_0| + |J_0|} \\
&\quad \leq 2\,\|B\|\,(1+p)^{2|Y|} \sum_{\substack{S_0 \subset \Lambda: \\ S_0 \ R\text{-connected} \\ Y \subset S_0 \\ S_0 \cap (\Lambda \setminus \Lambda'^\circ) \neq \emptyset}} \frac{Z^0_\Lambda\, Z_{\Lambda \setminus \overline{S_0}}}{Z_\Lambda\, Z^0_{\Lambda \setminus \overline{S_0}}} \frac{\tilde{Z}^0_\Lambda\, \tilde{Z}_{\Lambda \setminus \overline{S_0}}}{\tilde{Z}_\Lambda\, \tilde{Z}^0_{\Lambda \setminus \overline{S_0}}} \big(2p\,(1+p)\big)^{|S_0 \setminus Y|} \\
&\quad \leq 2\,\|B\| \left(\frac{1}{2} + \frac{1}{2p}\right)^{|Y|} \sum_{\substack{S_0 \subset \Lambda: \\ S_0 \ R\text{-connected} \\ Y \subset S_0 \\ S_0 \cap (\Lambda \setminus \Lambda'^\circ) \neq \emptyset}} \big(2p\,(1+p)\big)^{|S_0|} \\
&\quad \leq 2\,\|B\| \left(\frac{1}{2} + \frac{1}{2p}\right)^{|Y|} \big(2p\,(1+p)\,C\big)^{d(Y, \Lambda \setminus \Lambda'^\circ)/(2R)},
\end{aligned}$$

with $p = 2\,a\,q^{(2R+1)^D}$ and $2p\,(1+p)\,C < 1$ for a small enough.

4 Local perturbations perturb locally

In this section, we provide a result on the local perturbations perturb locally (LPPL) principle. Compared to our main results on the more important notions of DoC and LI, the LPPL bound deteriorates exponentially as $\beta \to \infty$. We conjecture that for the systems we consider, the LPPL bound actually holds uniformly in temperature, but we are unable to prove it with the new method in this paper.

Theorem 4.1 (Local perturbations perturb locally). *Let D, q, $R \in \mathbb{N}$ and $C_{\mathrm{int}} > 0$. Then there exist $a \in (0,1)$ and $C_1, C_2, \xi_{\mathrm{LPPL}} > 0$ such that the following holds. Consider the lattice $\Lambda \Subset \mathbb{Z}^D$ and a Hamiltonian $H^0_\Lambda + V_\Lambda$ as defined in section 2.1 with $\|h\|_\infty$, $\|v\|_\infty \leq C_{\mathrm{int}}$, v_x of range $R \in \mathbb{N}$, and v_x relatively a-bounded w.r.t. H^0_Λ in the sense* (5). *Moreover, let $X \subset \Lambda$ be an R-connected set, $W \subset \mathcal{A}_X$ self-adjoint and $\tilde{H}_\Lambda = H_\Lambda + W$. Then the Gibbs states ρ_Λ and $\tilde{\rho}_\Lambda$ of H_Λ and $\tilde{H}_\Lambda$, respectively, at any inverse temperature $\beta \in (0,\infty)$ satisfy*

$$|\mathrm{tr}(B\,\rho_\Lambda) - \mathrm{tr}(B\,\tilde{\rho}_\Lambda)| \leq C_1\,\|B\|\,\mathrm{e}^{2\beta\|W\|}\,\exp\big(C_2\,(|X|+|Y|)\big)\,\exp\big(-d(X,Y)/\xi_{\mathrm{LPPL}}\big) \tag{17}$$

for all R-connected sets $Y \subset \Lambda$ and observables $B \in \mathcal{A}_Y$.

Notably, this statement also holds, if the perturbed Hamiltonian $H_\Lambda + W$ does not satisfy the conditions from section 2.1. If $H_\Lambda + W$ was of the same type, applying theorem 2.3 for both Hamiltonians together with the triangle inequality gives a uniform-in-temperature bound $2\,C_1\,\|B\|\,\exp\big(C_2\,|Y|\big)\,\exp\big(-d(X,Y)/\xi_{\mathrm{LPPL}}\big)$.

Remark 4.2. The β-dependence is still better than what one would obtain from our theorem 2.2 on DoC with the *circle of equivalences* from [P4]. Similarly as in the case of LI, the problem with the circle of equivalences arises from quantum belief propagation. First, it produces constants that diverge as $\beta \to \infty$. Second, one can only obtain stretched exponential decay because one needs to use DoC for the observable B and an approximation of the quantum belief propagation operator that lives on a suitably enlarged region X_r with r a free parameter that can be optimized. As a consequence, the form of LPPL that one obtains in this way from the good DoC bound theorem 2.2 through the circle of equivalences is suboptimal. Instead, using our cluster expansion approach, we are able to obtain LPPL with exponential decay in theorem 4.1, but the constants diverge as $\beta \to \infty$. ◇

4.1 Proof of Theorem 4.1

We focus on the modifications necessary to prove theorem 4.1. We begin by adjusting the quantities from section 3.2 to the counterparts for $\tilde{H}_\Lambda = H_\Lambda + W$. Again, we denote all modified symbols with an additional tilde. First, let

$$\tilde{f}(M) = \exp\Big(-\beta\,\Big(H^0_\Lambda + W + \sum_{x\in M} v_x\Big)\Big),$$

which clearly satisfies $\mathrm{e}^{-\beta\tilde{H}} = \tilde{f}(\Lambda^\circ)$. Due to the inclusion-exclusion principle, we can decompose it as $\tilde{f}(\Lambda^\circ) = \sum_{I\subset\Lambda^\circ}\tilde{T}_I$ with

$$\tilde{T}_I = \sum_{M\subset I}(-1)^{|I|-|M|}\,\tilde{f}(M)$$

and

$$\tilde{T}_I = \mathrm{e}^{-\beta H^0_{\Lambda\setminus(\bar{I}\cup X)}}\, \tilde{T}_I^{\bar{I}\cup X}, \tag{18}$$

where

$$\tilde{T}_{I_1}^{I_2} = \sum_{M\subset I_1} (-1)^{|I_1|-|M|}\, \mathrm{e}^{-\beta\left(H^0_{I_2}+W+\sum_{x\in M} v_x\right)} \in \mathscr{A}_{I_2\cup\overline{I_1}}.$$

Similarly to lemma 3.1, we find

Lemma 4.3. *Using the above notations, it holds that*

$$\|\tilde{T}_I^{\bar{I}}\| \le (2a)^{|I|}\, \mathrm{e}^{\beta\|W\|}.$$

Proof. The proof follows the one from lemma 3.1, but due to the additional W in $\tilde{g}_M(z)$, we only have

$$\mathrm{Re}\,\langle\psi, \tilde{g}_M(z)\,\psi\rangle \ge \mathrm{Re}\,\langle\psi, W\,\psi\rangle \ge \inf_{\phi}\langle\phi, W\,\phi\rangle.$$

Hence,

$$\|\mathrm{e}^{-\beta\tilde{g}_M(z)}\| \le \mathrm{e}^{-\beta\inf_\phi\langle\phi,W\,\phi\rangle} \le \mathrm{e}^{\beta\|W\|}$$

and the remaining arguments together with [217, Lemma 2] yield the statement. □

When we split the exponential into products of clusters, we need to assign one cluster to have the perturbation W. Hence, we need to modify lemma 3.7 for the perturbed system. Notably, we choose $\tilde{H}^0_\Lambda = H^0_\Lambda + W$, so that $\tilde{Z}^0_\Lambda$ includes the perturbation W.

Lemma 4.4. *Let $I_1, I_2 \subset \Lambda^\circ$ and $\Omega_1, \Omega_2 \subset \Lambda$. Assume that $I_1 \cup \Omega_1 \cup X$ and $I_2 \cup \Omega_2$ are not R-connected to each other. Then, for every $O_1 \in \mathscr{A}_{\Omega_1}$ and $O_2 \in \mathscr{A}_{\Omega_2}$, we have that*

$$\frac{\mathrm{tr}(O_1\, O_2\, \tilde{T}_{I_1\cup I_2}^{\overline{I_1\cup I_2}\cup\Omega_1\cup\Omega_2\cup X})}{\tilde{Z}^0_{\overline{I_1\cup I_2}\cup\Omega_1\cup\Omega_2\cup X}} = \frac{\mathrm{tr}(O_1\, \tilde{T}_{I_1}^{\overline{I_1}\cup\Omega_1\cup X})}{\tilde{Z}^0_{\overline{I_1}\cup\Omega_1\cup X}}\, \frac{\mathrm{tr}(O_2\, T_{I_2}^{\overline{I_2}\cup\Omega_2})}{Z^0_{\overline{I_2}\cup\Omega_2}}. \tag{19}$$

Proof. The proof follows exactly the proof of lemma 3.7, only that W is attributed to one term on each side, and the terms h_x for $x \in X$ cannot be factored out. □

As in the proof of DoC and LI, we obtain the expansion

$$\begin{aligned}&|\mathrm{tr}(B\,\rho_\Lambda) - \mathrm{tr}(B\,\tilde{\rho}_\Lambda)|\\ &\quad= \frac{Z^0_\Lambda\,\tilde{Z}^0_\Lambda}{Z_\Lambda\,\tilde{Z}_\Lambda}\left|\sum_{I,J\subset\Lambda^\circ} \frac{\mathrm{tr}(B\,T_I^{\bar{I}\cup Y})}{Z^0_{\bar{I}\cup Y}}\, \frac{\mathrm{tr}(\tilde{T}_J^{\bar{J}\cup X})}{\tilde{Z}^0_{\bar{J}\cup X}} - \frac{\mathrm{tr}(B\,\tilde{T}_I^{\bar{I}\cup Y\cup X})}{\tilde{Z}^0_{\bar{I}\cup Y\cup X}}\, \frac{\mathrm{tr}(T_J^{\bar{J}})}{Z^0_{\bar{J}}}\right|.\end{aligned}$$

Equivalently to theorem 3.9, we find that only superclusters that contain X and Y in the same cluster contribute.

Theorem 4.5. *Let X and $Y \subset \Lambda$ each be an R-connected set and $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$. Let I be a configuration produced in the decomposition of $\operatorname{tr}(B\,\mathrm{e}^{-\beta H_\Lambda})/Z^0_\Lambda$ and J be a configuration produced in the decomposition of $\operatorname{tr}(\mathrm{e}^{-\beta \tilde{H}_\Lambda})/Z^0_\Lambda$. We consider the supercluster decomposition of $I \cup J \cup X \cup Y$ and let $E(X,Y)$ be the event that the associated supercluster decomposition does not contain X and Y in the same supercluster. As a shorthand, we write such terms in the product as*

$$\frac{\operatorname{tr}(B\,T_I^{\overline{I \cup Y}})}{Z^0_{\overline{I \cup Y}}}\,\frac{\operatorname{tr}(\tilde{T}_J^{\overline{J \cup X}})}{\tilde{Z}^0_{\overline{J \cup X}}}\bigg|_{E(X,Y)}.$$

We consider a similar supercluster decomposition for

$$\frac{\operatorname{tr}(B\,\tilde{T}_I^{\overline{I \cup Y \cup X}})}{\tilde{Z}^0_{\overline{I \cup Y \cup X}}}\,\frac{\operatorname{tr}(T_J^{\overline{J}})}{Z^0_{\overline{J}}}\bigg|_{E(X,Y)}.$$

The two quantities above are equal.

Proof. We consider a supercluster expansion $I \cup J \cup Y \cup X = S_1 \cup S_2 \cup \cdots \cup S_m$ for the event $E(X,Y)$, and without loss of generality, we can assume $X \subset S_1$ and $Y \subset S_2$. We then split each cluster using lemmata 3.7 and 4.4. By denoting $S_\mathrm{c} = S_3 \cup \cdots \cup S_m$ we obtain,

$$\begin{aligned}
&\frac{\operatorname{tr}(B\,T_I^{\overline{I \cup Y}})}{Z^0_{\overline{I \cup Y}}}\,\frac{\operatorname{tr}(\tilde{T}_J^{\overline{J \cup X}})}{\tilde{Z}^0_{\overline{J \cup X}}} \\
&\quad= \frac{\operatorname{tr}(T_{I\cap S_1}^{\overline{I\cap S_1}})}{Z^0_{\overline{I\cap S_1}}}\,\frac{\operatorname{tr}(B\,T_{I\cap S_2}^{\overline{I\cap S_2 \cup Y}})}{Z^0_{\overline{I\cap S_2\cup Y}}}\,\frac{\operatorname{tr}(T_{I\cap S_\mathrm{c}}^{\overline{I\cap S_\mathrm{c}}})}{Z^0_{\overline{I\cap S_\mathrm{c}}}}\,\frac{\operatorname{tr}(\tilde{T}_{J\cap S_1}^{\overline{J\cap S_1 \cup X}})}{\tilde{Z}^0_{\overline{J\cap S_1\cup X}}}\,\frac{\operatorname{tr}(T_{J\cap S_2}^{\overline{J\cap S_2 \cup X}})}{Z^0_{\overline{J\cap S_2\cup X}}}\,\frac{\operatorname{tr}(T_{J\cap S_\mathrm{c}}^{\overline{J\cap S_\mathrm{c} \cup X}})}{Z^0_{\overline{J\cap S_\mathrm{c}\cup X}}} \\
&\quad= \frac{\operatorname{tr}(T_{J'\cap S_1}^{\overline{J'\cap S_1}})}{Z^0_{\overline{J'\cap S_1}}}\,\frac{\operatorname{tr}(B\,T_{I'\cap S_2}^{\overline{I'\cap S_2 \cup Y}})}{Z^0_{\overline{I'\cap S_2\cup Y}}}\,\frac{\operatorname{tr}(T_{I'\cap S_\mathrm{c}}^{\overline{I'\cap S_\mathrm{c}}})}{Z^0_{\overline{I'\cap S_\mathrm{c}}}}\,\frac{\operatorname{tr}(\tilde{T}_{I'\cap S_1}^{\overline{I'\cap S_1 \cup X}})}{\tilde{Z}^0_{\overline{I'\cap S_1\cup X}}}\,\frac{\operatorname{tr}(T_{J'\cap S_2}^{\overline{J'\cap S_2 \cup X}})}{Z^0_{\overline{J'\cap S_2\cup X}}}\,\frac{\operatorname{tr}(T_{J'\cap S_\mathrm{c}}^{\overline{J'\cap S_\mathrm{c} \cup X}})}{Z^0_{\overline{J'\cap S_\mathrm{c}\cup X}}} \\
&\quad= \frac{\operatorname{tr}(B\,\tilde{T}_{I'}^{\overline{I' \cup Y \cup X}})}{\tilde{Z}^0_{\overline{I' \cup Y \cup X}}}\,\frac{\operatorname{tr}(T_{J'}^{\overline{J'}})}{Z^0_{\overline{J'}}},
\end{aligned}$$

where

$$\begin{aligned}
I' &= (J \cap S_1) \cup \big(I \cap (S_2 \cup S_3 \cup \cdots \cup S_m)\big),\\
J' &= (I \cap S_1) \cup \big(J \cap (S_2 \cup S_3 \cup \cdots \cup S_m)\big)
\end{aligned}$$

is another supercluster in $E(X,Y)$. The statement follows. □

The remaining terms have X and Y in the same supercluster and as for DoC we obtain the bound

$$
\begin{aligned}
&|\mathrm{tr}(B\,\rho_\Lambda) - \mathrm{tr}(B\,\tilde\rho_\Lambda)| \\
&\quad\le \frac{Z^0_\Lambda\,\tilde Z^0_\Lambda}{Z_\Lambda\,\tilde Z_\Lambda} \sum_{\substack{S_0\subset\Lambda:\\ S_0\ R\text{-connected}\\ X,Y\subset S_0}} \frac{Z_{\Lambda\setminus\overline{S_0}}\,\tilde Z_{\Lambda\setminus\overline{S_0}}}{Z^0_{\Lambda\setminus\overline{S_0}}\,\tilde Z^0_{\Lambda\setminus\overline{S_0}}} \sum_{\substack{I_0,J_0\subset\Lambda^\circ:\\ I_0\cup J_0\cup X\cup Y=S_0}} \left|\frac{\mathrm{tr}(B\,T_{I_0}^{\overline{I_0}\cup Y})}{Z^0_{\overline{I_0}\cup Y}}\,\frac{\mathrm{tr}(\tilde T_{J_0}^{\overline{J_0}\cup X})}{\tilde Z^0_{\overline{J_0}\cup X}}\right| + \left|\frac{\mathrm{tr}(B\,\tilde T_{I_0}^{\overline{I_0}\cup Y\cup X})}{\tilde Z^0_{\overline{I_0}\cup Y\cup X}}\,\frac{\mathrm{tr}(T_{J_0}^{\overline{J_0}})}{Z^0_{\overline{J_0}}}\right|.
\end{aligned}
$$

We continue the proof analogously to the proof of theorem 2.2, but adjust for the perturbation W. As before, we write for each $I, \Omega \subset S_0$ and $O \in \mathscr{A}_\Omega$,

$$
\left|\frac{\mathrm{tr}(O\,\tilde T_I^{\bar I\cup\Omega\cup X})}{\tilde Z^0_{\bar I\cup\Omega\cup X}}\right| = \left|\frac{\mathrm{tr}_{\bar I\cup\Omega\cup X}(O\,\tilde T_I^{\bar I\cup\Omega\cup X})}{\mathrm{tr}_{\bar I\cup X}(\mathrm{e}^{-\beta\tilde H^0_{\bar I\cup X}})\ \mathrm{tr}_{\Omega\setminus(\bar I\cup X)}(\mathrm{e}^{-\beta H^0_{\Omega\setminus(\bar I\cup X)}})}\right|,
$$

by reducing to a trace over $\mathscr{A}_{\bar I\cup\Omega\cup X}$. The numerator is bounded by

$$
\begin{aligned}
|\mathrm{tr}_{\bar I\cup\Omega\cup X}(O\,\tilde T_I^{\bar I\cup\Omega\cup X})| &\le \|O\|\,\|\tilde T_I^{\bar I\cup X}\,\mathrm{e}^{-\beta H^0_{\Omega\setminus(\bar I\cup X)}}\|_{\mathrm{Tr}_{\bar I\cup\Omega\cup X}} \\
&\le \|O\|\,\|\tilde T_I^{\bar I\cup X}\|\,\|\mathbb{1}_{\bar I\cup X}\|_{\mathrm{Tr}_{\bar I\cup X}}\,\|\mathrm{e}^{-\beta H^0_{\Omega\setminus(\bar I\cup X)}}\|_{\mathrm{Tr}_{\Omega\setminus(\bar I\cup X)}} \\
&\le \|O\|\,(2a)^{|I|}\,\mathrm{e}^{\beta\|W\|}\,q^{|\bar I\cup X|}\ \mathrm{tr}_{\Omega\setminus(\bar I\cup X)}(\mathrm{e}^{-\beta H^0_{\Omega\setminus(\bar I\cup X)}}),
\end{aligned}
$$

using lemma 4.3 in the last step. The second term in the denominator cancels with the last term and for the first term we observe $\tilde H^0_{\bar I\cup X} \le H^0_{\bar I\cup X} + \sup_\psi\langle\psi, W\,\psi\rangle$ and use monotonicity as in the proof of lemma 3.11 to bound

$$
\mathrm{tr}(\mathrm{e}^{-\beta\tilde H^0_{\bar I\cup X}}) \ge \mathrm{tr}(\mathrm{e}^{-\beta H^0_{\bar I\cup X}})\,\mathrm{e}^{-\beta\sup_\psi\langle\psi,W\,\psi\rangle} \ge \mathrm{e}^{-\beta\|W\|}.
$$

Moreover, we observe that lemma 3.11 also holds for the tilde variants as long as $X \subset \overline{S_0}$, as the proof only uses monotonicity to add terms $v_x \ge 0$.

Putting everything together, we obtain the bound

$$
\begin{aligned}
&|\mathrm{tr}(B\,\rho_\Lambda) - \mathrm{tr}(B\,\tilde\rho_\Lambda)| \\
&\quad\le 2\,\|B\|\,\mathrm{e}^{2\beta\|W\|} \sum_{\substack{S_0\subset\Lambda:\\ S_0\ R\text{-connected}\\ X,Y\subset S_0}} \sum_{\substack{I_0,J_0\subset\Lambda^\circ:\\ I_0\cup J_0\cup X\cup Y=S_0}} (p)^{|I_0|+|J_0|}\,q^{|X\setminus I_0|} \\
&\quad\le 2\,\|B\|\,\mathrm{e}^{2\beta\|W\|}\,q^{|X|}\left(\frac{1}{2}+\frac{1}{2p}\right)^{|X|+|Y|} \sum_{\substack{S_0\subset\Lambda:\\ S_0\ R\text{-connected}\\ X,Y\subset S_0}} \big(2p\,(1+p)\big)^{|S_0|}
\end{aligned}
$$

$$\leq 2\,\|B\|\,\mathrm{e}^{2\beta\|W\|}\,q^{|X|}\left(\frac{1}{2}+\frac{1}{2p}\right)^{|X|+|Y|}\big(2p\,(1+p)\,C\big)^{d(X,Y)/(2R)},$$

where, as before, $p = 2\,a\,q^{(2R+1)^D}$ and $2p\,(1+p)\,C < 1$ for a small enough.

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Publication P9

Gaussian filters in quantum lattice systems: Applications to spectral flow, local perturbations, clustering, and the quantum Hall effect

Sven Bachmann [*] Zhiqian (Simon) Du [†] Martin Fraas [†]

Tom Wessel [‡]

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Abstract

We consider the locality and spectral properties of the smearing

$$\tau_f(A) = \int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\, \tau_t(A)$$

when applied to the dynamics τ_t of quantum spin systems. While recent applications of this map have used superpolynomially but not exponentially decaying functions f to ensure exact spectral properties, we use here Gaussian filters. This improves the locality at the expense of errors on the spectral side. We propose a number of concrete applications, from quasi-adiabatic continuation to correlation decay, and exponential stability away from impurities. Finally, we discuss an application to the quantum Hall effect.

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[*] Department of Mathematics, The University of British Columbia
1984 Mathematics Road, Vancouver, BC, Canada V6T 1Z2

[†] Department of Mathematics, University of Califonia, Davis
Davis, 95616, USA

[‡] Fachbereich Mathematik, Universität Tübingen,
Auf der Morgenstelle 10, 72076 Tübingen, Germany.

Contents

1 Introduction

Physical time evolution flows $t \mapsto \tau_t(A) = \mathrm{e}^{\mathrm{i}Ht} A \,\mathrm{e}^{-\mathrm{i}Ht}$ are local, where the precise meaning of locality depends on the particular setup. Propagation is strictly within the light cone when the underlying equation is the wave equation, it is diffusive for the heat equation, and the Lieb-Robinson bound [154] provides an effective ballistic propagation bound for the Schrödinger equation on sufficiently regular lattices. Among others, these results are essential in proving well-posedness for these equations. The locality that originates in the Lieb-Robinson bound has proved crucial to understand both thermal phases and (topologically ordered) ground state phases, see [114] for many examples. In this context, the smearing map[1]

$$\tau_f(A) = \int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\, \tau_t(A) \tag{1}$$

has proved particularly useful, since the decay of f controls the locality of $A \mapsto \tau_f(A)$ while its Fourier transform $\widehat{f}$ controls the spectral properties of the map. In fact, the (Arveson) spectrum of τ is defined as the smallest subset $\sigma \subset \mathbb{R}$ such that τ_f vanishes for all f such that $\operatorname{supp}(\widehat{f}) \cap \sigma = \emptyset$, see [20, Definition A.1].

Although it was not phrased exactly as such in its original formulation [110], smearing of the type (1) is central to quasi-adiabatic continuation, also known as the spectral flow [114, 23], a tool that has become one of the cornerstones of the analysis and classification of topological phases of matter, and of the related questions of adiabaticity and linear response theory [17, 165, 203]. Understanding quasi-particle excitations of topologically ordered states also relies on this technique [20, 12]. In another guise,

[1] When the subscript of τ is a real number, it denotes the time evolution and when the subscript is a function, it denotes this integral.

such smearings are instrumental in proving rapid decay of correlations for gapped quantum lattice systems [118, 171], and to relate this to stability of ground state [16].

In the example of the spectral flow, the Fourier transform $\widehat{f}$ is required to be discontinuous, which implies that f cannot decay exponentially at infinity. While stretched-exponential or even superpolynomial decay is often sufficient and convenient for locality arguments [174], one may wish to use an exponentially decaying function. In fact, the original formulation of quasi-adiabatic continuation used a Gaussian filter. The price to pay is that the spectral properties are not exact anymore. Exponentially decaying filter functions have also proved useful in treating thermal states, see e.g. [115, 80, P4].

In this paper, we consider the use of Gaussian filter functions, providing exact bounds both on the spatial locality and on the spectral errors. In particular, we define a spectral flow which is exponentially local and almost exact. Our Ansatz is similar to the one used in [120]. Here, the smearing is applied term-by-term on an interaction and the width of the Gaussian must be chosen in a spatially inhomogeneous way. As expected, better locality yield worse spectral mapping properties. The same methods allow us to provide an exponential version of the "local perturbations perturb locally" property. What is more, a properly chosen Gaussian filter convolved with a step function yields exponential decay of correlations for gapped spectral patches whose width does not need to vanish in the thermodynamic limit. These results are similar to the those of [214], although the methods are completely different. As a corollary of all the above, we conclude that the Hall conductance is quantized in finite systems up to errors that are exponentially small in the system size.

This paper is organized as follows. After a brief review of the Lieb-Robinson bound, we introduce in Section 3 the almost inverse Liouvillian built using a Gaussian filter. We prove bounds characterizing both its quasi-locality properties and its spectral property. In Section 4, we use it to construct an exponentially local almost spectral flow and show that local perturbations that do not close the gap have exponentially small effect away from the perturbation. Parallel ideas are developed in Section 5 to prove exponential clustering for finite volume spectral patches of finite width. Finally, Section 6 discusses the application of these tools to the quantum Hall effect.

2 Mathematical setup

We consider spin systems on finite D-regular graphs. Therefore, let Λ be a finite set and denote by d the graph distance. Clearly, there exist constants $D \in \mathbb{N}_+$ and $\mathscr{C}_{\mathrm{vol}} > 0$ such that the volume of all balls

$$B_x(r) := \{ z \in \Lambda \,|\, d(z,x) \le r \}$$

scales such that

$$|B_x(r)| \le \mathscr{C}_{\mathrm{vol}} (r+1)^D \qquad \text{for all } x \in \Lambda \text{ and } r \ge 0.$$

The set of graphs with this scaling for fixed D and $\mathscr{C}_{\mathrm{vol}}$ is denoted $\mathscr{G}(D, \mathscr{C}_{\mathrm{vol}})$. With this definition, all finite subsets $\Lambda \subset \mathbb{Z}^D$ are in the same $\mathscr{G}(D, \mathscr{C}_{\mathrm{vol}})$, and they can even have periodic boundary conditions in one or more directions.

For later purposes, we note that there exists a constant $\mathscr{C}_{\mathrm{vol},b,k} \geq 1$ such that for all sets $Z \subset \Lambda$

$$|Z|^k \, \mathrm{e}^{-b \operatorname{diam}(Z)} \leq \mathscr{C}_{\mathrm{vol},b,k}.$$

If $b \leq D$, then

$$\mathscr{C}_{\mathrm{vol},b,k} \leq \mathscr{C}_{\mathrm{vol}}^k \left(\tfrac{kD}{\mathrm{e}}\right)^{kD} b^{-kD} \, \mathrm{e}^b. \tag{2}$$

Indeed, for any $z \in Z$, we have

$$|Z|^k \, \mathrm{e}^{-b \operatorname{diam}(Z)} \leq |B_z(\operatorname{diam}(Z))|^k \, \mathrm{e}^{-b \operatorname{diam}(Z)} \leq \mathscr{C}_{\mathrm{vol}} \sup_{n \geq 0} (n+1)^{kD} \, \mathrm{e}^{-bn},$$

and the bound (2) follows since the supremum is attained at $n = \min\{0, k\,D/b - 1\}$.

With every site $x \in \Lambda$ one associates a finite-dimensional local Hilbert space $\mathscr{H}_x := \mathbb{C}^q$ with the corresponding space of linear operators denoted by $\mathscr{A}_x := \mathscr{B}(\mathbb{C}^q)$. Moreover, we define the Hilbert space $\mathscr{H}_\Lambda := \bigotimes_{x \in \Lambda} \mathscr{H}_x$, and denote the algebra of bounded linear operators on $\mathscr{H}_\Lambda$ by $\mathscr{A}_\Lambda := \mathscr{B}(\mathscr{H}_\Lambda)$. Due to the tensor product structure, we have $\mathscr{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathscr{A}_x$. Hence, for $X \subset \Lambda$, any $A \in \mathscr{A}_X$ can be viewed as an element of $\mathscr{A}_\Lambda$ by identifying A with $A \otimes \mathbb{1}_{\Lambda \setminus X} \in \mathscr{A}_\Lambda$, where $\mathbb{1}_{\Lambda \setminus X}$ denotes the identity in $\mathscr{A}_{\Lambda \setminus X}$. This identification is always understood implicitly and for $B \in \mathscr{A}_\Lambda$ we denote by $\operatorname{supp}(B)$ the smallest $Y \subset \Lambda$ such that $B \in \mathscr{A}_Y$.

Remark 1. While we formulate all our results for spin systems, one can use the same approach for fermionic lattice systems provided the algebra is that of even elements of the CAR, see for example [173]. In particular, theorems 12, 13 and 17 also hold for fermionic lattice systems. ◇

Let $I \subset \mathbb{R}$ be a compact interval. A (time-dependent) interaction on Λ is a function

$$\Phi \colon I \times \{Z \subset \Lambda\} \to \mathscr{A}_\Lambda, \quad (t, Z) \mapsto \Phi(t, Z) \in \mathscr{A}_Z \quad \text{with} \quad \Phi(t, Z) = \Phi(t, Z)^*. \tag{3}$$

For $b \geq 0$, we define interaction norms

$$\|\Phi\|_b := \sup_{t \in I} \sup_{z \in \Lambda} \sum_{\substack{Z \subset \Lambda: \\ z \in Z}} \|\Phi(t, Z)\| \, \mathrm{e}^{b \operatorname{diam}(Z)}. \tag{4}$$

An interaction Φ gives rise to the corresponding operator

$$H(t) := \sum_{Z \subset \Lambda} \Phi(t, Z),$$

which generates the Heisenberg time-evolution $\tau_{s,t}$ defined as the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} \tau_{s,t}(A) = \tau_{s,t}\big(\mathrm{i}\,[H(t), A]\big), \qquad \tau_{s,s}(A) = A, \tag{5}$$

for any $A \in \mathscr{A}_\Lambda$.

An important property of the time-evolution is its locality as captured by Lieb-Robinson bounds, which originated in [154] and were generalized in [171] and many other works. We here state a version for the norm (4), whose time-independent version appeared in [22, Theorem A.1], and the time-dependent one is in [156, Theorem 7.3.3].

Lemma 2 (Lieb-Robinson bound). *Let $D \in \mathbb{N}$, $\mathscr{C}_{\mathrm{vol}} > 0$, $\Lambda \in \mathscr{G}(D, \mathscr{C}_{\mathrm{vol}})$ be finite, and $b' > b > 0$. Then, for all intervals $I \subset \mathbb{R}$, time-dependent interactions Φ such that $\|\Phi\|_{b'} < \infty$, disjoint subsets $X, Y \subset \Lambda$, observables $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$, and $s, t \in I$ it holds that*

$$\left\|\left[\tau_{s,t}(A), B\right]\right\| \leq 2\,\mathscr{C}_{\mathrm{vol},1,b'-b}^{-1}\,\|A\|\,\|B\|\left(\mathrm{e}^{b\,v\,|t-s|} - 1\right) D(X,Y), \tag{6}$$

where $v = 2\,\mathscr{C}_{\mathrm{vol},1,b'-b}\,\|\Phi\|_{b'}/b$ is the Lieb-Robinson velocity and

$$\begin{aligned} D(X,Y) &:= \min\left\{\sum_{x\in X} \mathrm{e}^{-b\,\mathrm{dist}(x,Y)}, \sum_{y\in Y} \mathrm{e}^{-b\,\mathrm{dist}(y,X)}\right\} \\ &\leq \min\left\{|X|, |Y|\right\} \mathrm{e}^{-b\,\mathrm{dist}(X,Y)}. \end{aligned}$$

In this statement and the rest of the paper, the interaction norm and the evolution implicitly depend on Λ and this dependence is understood from the context. Importantly, all the constants will be independent of the specific $\Lambda \in \mathscr{G}(D, \mathscr{C}_{\mathrm{vol}})$ and only depend on the lattice through D and $\mathscr{C}_{\mathrm{vol}}$. In fact, one can define an interaction Φ and the interaction norm $\|\cdot\|_b$ on an infinite lattice $\Gamma = \mathbb{Z}^D$ and the restrictions $\Phi|_\Lambda$ to finite lattices $\Lambda \subset \Gamma$ satisfy $\|\Phi|_\Lambda\|_b \leq \|\Phi\|_b$. Our results are then uniform in $\Lambda \subset \Gamma$.

3 The almost inverse Liouvillian

For $\beta > 0$ let

$$\phi_\beta(t) := \frac{\beta}{\sqrt{\pi}}\,\mathrm{e}^{-\beta^2 t^2}.$$

It is such that $\int \phi_\beta = 1$ and the Fourier transform is

$$\hat{\phi}_\beta(\omega) := \frac{1}{\sqrt{2\pi}} \int \mathrm{d}t\,\phi_\beta(t)\,\mathrm{e}^{-\mathrm{i}t\omega} = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{\omega^2}{4\beta^2}} \tag{7}$$

for which $\int \hat{\phi}_\beta = \beta\sqrt{\frac{2}{\pi}}$. Then, for any Hamiltonian H and observable A we define the *almost inverse Liouvillian*

$$\mathscr{I}_{H,\beta}(A) := \int_{-\infty}^{\infty} \mathrm{d}t\,\phi_\beta(t) \int_0^t \mathrm{d}s\,\tau_s^H(A), \tag{8}$$

where $\tau_s^H(A) = \mathrm{e}^{\mathrm{i}Hs} A \, \mathrm{e}^{-\mathrm{i}Hs}$. To make contact with the previous section, we imagine here a time-dependent interaction and the corresponding Hamiltonian in a finite volume Λ.

To quantify the properties of the almost inverse Liouvillian, it will be helpful to define the exact inverse Liouvillian $\mathscr{I}_H$ along the same lines, see [23]. Instead of ϕ_β, the map $\mathscr{I}_H$ uses a non-negative function w with Fourier transform $\hat{w} \in C_0^1((-\delta, \delta))$ and $\int w = 1$. Note that w can be chosen to decay faster than any inverse power but not exponentially. Whenever $\mathscr{I}_H$ involves a gapped Hamiltonian with gap γ, we implicitly choose $\delta < \gamma$.

Using standard techniques, we see that the almost inverse Liouvillian has good locality properties.

Lemma 3 (Locality of the almost inverse Liouvillian). *Let $D \in \mathbb{N}$, $\mathscr{C}_{\mathrm{vol}} > 0$ and $\Lambda \in \mathscr{G}(D, \mathscr{C}_{\mathrm{vol}})$ be finite. Let $b' > b > 0$, Φ be an interaction such that $\|\Phi\|_{b'} < \infty$ and H the corresponding Hamiltonian. Then, for all disjoint $X, Y \subset \Lambda$, $A \in \mathscr{A}_X$ and $B \in \mathscr{A}_Y$,*

$$\big\|\big[\mathscr{I}_{H,\beta}(A), B\big]\big\| \leq 2 \min\{|X|, |Y|\} \, \|A\| \, \|B\| \inf_{T>0} \left(\frac{2\beta}{\sqrt{\pi}\, b^2 v^2} \, \mathrm{e}^{b(vT - \mathrm{dist}(X,Y))} + \frac{1}{\sqrt{\pi}\,\beta} \, \mathrm{e}^{-\beta^2 T^2} \right), \tag{9}$$

where v is the Lieb-Robinson velocity from lemma 2.

One may choose $T = \frac{d(X,Y)}{2v}$ and then use $\mathrm{dist}(X,Y)^2 \geq \mathrm{dist}(X,Y)$ to get

$$\big\|\big[\mathscr{I}_{H,\beta}(A), B\big]\big\| \leq 2 \min\{|X|, |Y|\} \, \|A\| \, \|B\| \left(\frac{2\beta}{\sqrt{\pi}\, b^2 v^2} + \frac{1}{\sqrt{\pi}\,\beta} \right) \mathrm{e}^{-b(\beta) d(X,Y)}, \tag{10}$$

where $b(\beta) = \min\left\{\frac{b}{2}, \frac{\beta^2}{4v^2}\right\}$.

Remark 4. The last expression (10), in particular $b(\beta)$, emphasizes the double origin of the quasi-locality of the map $\mathscr{I}_{H,\beta}$, namely the locality of the Hamiltonian through the Lieb-Robinson bound yielding the dependence on b and the locality expressed by the Gaussian filter yielding the dependence on β. In particular, the locality cannot be improved further than β of the order of $\sqrt{b}$, and we shall later restrict our attention to this situation, see (16) below. ◇

Proof. We bound

$$\big\|\big[\mathscr{I}_{H,\beta}(A), B\big]\big\| \leq \int_{-\infty}^{\infty} \mathrm{d}t \, \phi_\beta(t) \int_0^t \mathrm{d}s \, \big\|\big[\tau_s^H(A), B\big]\big\| = \inf_{T>0} (I_{<T} + I_{\geq T}),$$

and split the integral into a part $I_{<T}$ where $|t| < T$ and the rest $I_{\geq T}$ where $|t| \geq T$. For the first part we use $\phi_\beta(t) \leq \beta/\sqrt{\pi}$ and the Lieb-Robinson bound for τ^H, lemma 2, where we bound $\mathscr{C}^{-1}_{\mathrm{vol},1,b'-b} \le 1$, to obtain

$$\begin{aligned} I_{<T} &\le \frac{\beta}{\sqrt{\pi}} 4 \min\{|X|,|Y|\} \|A\| \|B\| \mathrm{e}^{-b\,\mathrm{dist}(X,Y)} \int_0^T \mathrm{d}t \int_0^t \mathrm{d}s \left(\mathrm{e}^{bvs} - 1\right) \\ &= \frac{\beta}{\sqrt{\pi}} 4 \min\{|X|,|Y|\} \|A\| \|B\| \frac{1}{(bv)^2}\left(\mathrm{e}^{bvT} - 1 - bvT - \tfrac{1}{2}(bvT)^2\right) \mathrm{e}^{-b\,\mathrm{dist}(X,Y)}. \end{aligned}$$

For the second term we use the decay of ϕ_β and the trivial bound $\left\|\left[\tau_s^H(A), B\right]\right\| \le 2\|A\|\|B\|$ to get

$$I_{\ge T} \le 4\|A\|\|B\| \frac{\beta}{\sqrt{\pi}} \int_T^\infty \mathrm{d}t\, \mathrm{e}^{-\beta^2 t^2}\, t \le 2\|A\|\|B\| \frac{1}{\sqrt{\pi}\,\beta} \left(\mathrm{e}^{-\beta^2 T^2} - 1\right).$$

Combining both bounds concludes the proof. □

Assumption 5 (Gap Assumption)**.** Let Λ be finite. Let Φ be an interaction satisfying $\|\Phi\|_{b'} < \infty$ for some $b' > 0$, and let H be the corresponding Hamiltonian. We assume that the spectrum of H is of the form

$$\sigma(H) = \sigma_0 \cup \sigma_1$$

with $\gamma = \mathrm{dist}(\sigma_1, \sigma_0) > 0$. ◇

From here onwards, we shall drop the subscript Λ for notational clarity. We denote $P = \chi_{\sigma_0}(H)$ the spectral projection of H corresponding to the patch σ_0.

Proposition 6. *Let H satisfy the Gap Assumption and assume that $\sigma_0 = \{E_0\}$ is a single eigenvalue. Then*

$$\mathscr{I}_{H,\beta}(P\,A\,P) = 0,$$

for all $A \in \mathscr{A}_\Lambda$.

Proof. One calculates

$$\mathscr{I}_{H,\beta}(P\,A\,P) = \int_{-\infty}^{\infty} \mathrm{d}t\, \phi_\beta(t) \int_0^t \mathrm{d}s\, \tau_s^H(P\,A\,P) = \int_{-\infty}^{\infty} \mathrm{d}t\, \phi_\beta(t)\, t\, P\,A\,P = 0,$$

because the time-evolution is trivial and ϕ_β is an even function. □

Remark 7. The fact that $\mathscr{I}_{H,\beta}$ vanishes exactly on the range of P depends on the assumption $\sigma_0 = \{E_0\}$. For spectral patches with $\delta = \mathrm{diam}(\sigma_0) > 0$ sufficiently small with respect to γ, a similar result could be obtained, although with an error bound, by replacing $\hat{\phi}_\beta$ with two Gaussians centred at $\pm\frac{\delta+\gamma}{2}$. ◇

Proposition 8. *Let H satisfy the Gap Assumption. Then $\mathscr{I}_{H,\beta}$ is an almost inverse of the Liouvillian $\mathscr{L}_H = -\mathrm{i}\,[H,\cdot]$ on off-diagonal operators. More precisely, for all $A \in \mathscr{A}$ such that $A = P\,A\,P^\perp$ or $A = P^\perp\,A\,P$, and for all $q \in [1,\infty]$,*

$$\left\|\mathscr{I}_{H,\beta} \circ \mathscr{L}_H(A) - A\right\|_q \leq \|P_\mu\|_1\,\|A\|\,\mathrm{e}^{-\frac{\gamma^2}{4\beta^2}}.$$

Moreover, for all $A \in \mathscr{A}$ and $q \in [1,\infty]$, it holds that

$$\left\|\left[\mathscr{I}_{H,\beta} \circ \mathscr{L}_H(A) - A, P\right]\right\|_q \leq 2\,\|P_\mu\|_1\,\|A\|\,\mathrm{e}^{-\frac{\gamma^2}{4\beta^2}}.$$

Proof. For the first statement, since the calculation is exactly analogous, we only consider the case $A = P\,A\,P^\perp$. From the definition of the almost spectral flow, the fact that $\tau_s^H(\mathscr{L}_H(A)) = -\frac{\mathrm{d}}{\mathrm{d}s}\,\tau_s^H(A)$ and the spectral theorem for H, we obtain

$$\begin{aligned}
\left(\mathscr{I}_{H,\beta} - \mathscr{I}_H\right) \circ \mathscr{L}_H(A) &= \int_{-\infty}^{\infty} \mathrm{d}t\,\left(w(t) - \phi_\beta(t)\right)\left(\tau_t^H(A) - A\right) \\
&= \sum_{\mu\in\sigma_0}\sum_{\nu\in\sigma_1}\int_{-\infty}^{\infty} \mathrm{d}t\,\left(w(t) - \phi_\beta(t)\right)\mathrm{e}^{\mathrm{i}t(\mu-\nu)}\,P_\mu\,A\,P_\nu \\
&= \sqrt{2\pi}\sum_{\mu\in\sigma_0} P_\mu\,A\sum_{\nu\in\sigma_1}\hat{\phi}_\beta(\nu-\mu)\,P_\nu,
\end{aligned}$$

where we used $\int w = 1 = \int \phi_\beta$ and $\hat{w}(\nu-\mu) = 0$ because $|\nu-\mu| \geq \gamma$ by the Gap Assumption. By the triangle and Hölder inequalities, in particular using that $\|P_\mu\|_q \leq \|P_\mu\|_1$ for all q and μ and $\sum_{\mu\in\sigma_0}\|P_\mu\|_1 = \|P\|_1$ we obtain,

$$\begin{aligned}
\left\|\mathscr{I}_{H,\beta} \circ \mathscr{L}_H(A) - A\right\|_q &\leq \sqrt{2\pi}\sum_{\mu\in\sigma_0}\|P_\mu\|_q\,\|A\|\left\|\sum_{\nu\in\sigma_1}\hat{\phi}_\beta(\nu-\mu)\,P_\nu\right\| \\
&\leq \sqrt{2\pi}\,\|P\|_q\,\|A\|\,\hat{\phi}_\beta(\gamma).
\end{aligned}$$

With (7), this yields the claim.

For the second statement, note that

$$\left[\mathscr{I}_{H,\beta} \circ \mathscr{L}_H(A) - A, P\right] = \mathscr{I}_{H,\beta} \circ \mathscr{L}_H([A,P]) - [A,P]$$

because H and P commute. The statement follows by applying the first part since $[A,P] = P^\perp\,A\,P - P\,A\,P^\perp$. □

Remark 9. Since we are dealing with a fixed finite volume here, the rank of P is finite. For the applications we have in mind however, it is crucial to have that P_Λ remains uniformly bounded as $\Lambda \to \Gamma$. ◇

It will later also be helpful to compare the almost inverse Liouvillian to the exact inverse Liouvillian directly.

Lemma 10. *Let Λ finite and $H \in \mathscr{A}_\Lambda$ satisfy the Gap Assumption. For all $A \in \mathscr{A}$ such that $A = P\,A\,P^\perp$ or $A = P^\perp\,A\,P$, and for all $q \in [1, \infty]$,*

$$\left\| \mathscr{I}_{H,\beta}(A) - \mathscr{I}_H(A) \right\|_q \leq \|P\|_1 \, \|A\| \, \gamma^{-1} \, \mathrm{e}^{-\frac{\gamma^2}{4\beta^2}}.$$

Moreover, for all $A \in \mathscr{A}$ and $q \in [1, \infty]$,

$$\left\| \left[\mathscr{I}_{H,\beta}(A) - \mathscr{I}_H(A), P \right] \right\|_q \leq 2 \, \|P\|_1 \, \|A\| \, \gamma^{-1} \, \mathrm{e}^{-\frac{\gamma^2}{4\beta^2}}.$$

Proof. As in the proof of proposition 8, we obtain for $A = P\,A\,P^\perp$

$$\begin{aligned} \mathscr{I}_{H,\beta}(A) - \mathscr{I}_H(A) &= \sum_{\mu \in \sigma_0} \sum_{\nu \in \sigma_1} \int_{-\infty}^{\infty} \mathrm{d}t \, \big(\phi_\beta(t) - w(t) \big) \int_0^t \mathrm{d}s \, \mathrm{e}^{\mathrm{i}(\mu - \nu)s} \, P_\mu \, A \, P_\nu \\ &= \sum_{\mu \in \sigma_0} P_\mu \, A \sum_{\nu \in \sigma_1} \frac{\sqrt{2\pi}}{\mathrm{i}(\mu - \nu)} \, \hat{\phi}_\beta(\nu - \mu) \, P_\nu, \end{aligned}$$

by the Gap Assumption. It follows that

$$\left\| \mathscr{I}_{H,\beta}(A) - \mathscr{I}_H(A) \right\|_q \leq \sqrt{2\pi} \sum_{\mu \in \sigma_0} \|P_\mu\|_q \, \|A\| \left\| \sum_{\nu \in \sigma_1} \frac{\hat{\phi}_\beta(\nu - \mu)}{\nu - \mu} \, P_\nu \right\| \leq \sqrt{2\pi} \, \|P_\mu\|_1 \, \|A\| \, \frac{\hat{\phi}_\beta(\gamma)}{\gamma}.$$

The second statement follows as in proposition 8. □

4 The almost spectral flow and LPPL

Under the Uniform Gap assumption below, $\mathscr{I}_{H(t)}\big(\dot{H}(t)\big)$ generates an automorphism $\alpha_{0,s}$ which provides a mapping between the instantaneous ground states of $H(0)$ and $H(s)$, namely

$$\omega_s(A) = \omega_0 \circ \alpha_{0,s}(A) \tag{11}$$

for all $A \in \mathscr{A}_\Lambda$, where $\omega_s(A) = \mathrm{Tr}(P_s)^{-1} \, \mathrm{Tr}(P_s \, A)$, see [23]. However, as explained in the introduction, this automorphism α cannot be exponentially local. Instead, we shall use the almost inverse Liouvillian through $\mathscr{I}_{H(t),\beta}\big(\dot{H}(t)\big)$ to obtain an exponentially local almost spectral flow α^β.

Similarly to the locality of $\mathscr{I}_{H(t),\beta}(A)$ for strictly local operators $A \in \mathscr{A}_X$ discussed in lemma 3, standard arguments show that $\mathscr{I}_{H(t),\beta}\big(\dot{H}(t)\big)$ is given by an exponentially local interaction if H and $\dot{H}$ are, see lemma 15. We first concentrate to the spectral mapping properties of $\alpha^\beta_{0,s}$.

Assumption 11 (Uniform Gap assumption)**.** Let $H(s)$ with $s \in [0,1]$ be a Hamiltonian given by a smooth time-dependent interaction $s \mapsto \Phi(s)$ satisfying $\sup_{s\in[0,1]}\|\Phi(s)\|_{b'} < \infty$ for some $b' > 0$. We assume that $H(s)$ satisfies the Gap Assumption. Moreover, we assume that there exists compact intervals $I(s)$ with endpoints depending smoothly on s such that $\sigma_0(s) \subset I(s) \subset \mathbb{R} \setminus \sigma_1(s)$. ◇

To characterize the almost spectral flow α^β, we wish to compare it to the exact spectral flow α, using the standard identity

$$\alpha^1_{s,t}(A) - \alpha^2_{s,t}(A) = \sum_{Z\subset\Lambda} \int_s^t \mathrm{d}\lambda\, \alpha^1_{s,\lambda}\Big(\Big[\Psi_1(\lambda,Z) - \Psi_2(\lambda,Z), \alpha^2_{\lambda,t}(A)\Big]\Big), \tag{12}$$

for any two automorphisms generated by interactions Ψ_1 and Ψ_2. One then uses the Lieb-Robinson bound to obtain an estimate which grows in $\mathrm{supp}(A)$ but is not extensive in Λ. As outlined above, the locality of the interaction $\mathcal{I}_{H(t)}\big(\dot H(t)\big)$ is insufficient for our purpose as $\mathcal{I}_{H(t)}$ is not exponentially local. An exponential bound can be obtained directly using lemma 10 with A being replaced by the full $\dot H(t)$, but the resulting bound on $\|\alpha_{0,s}(A) - \alpha^\beta_{0,s}(A)\|$ is proportional to $\sup_{s\in[0,1]}\|\dot H(s)\|$, which in general is extensive in Λ.

The solution we propose below is to construct another flow $\alpha^{\beta,X}_{t,s}$ where $X = \mathrm{supp}(A)$, which is adapted to the support of the observable, and use it as an intermediate to compare the exact spectral flow α with its exponentially local but approximate cousin α^β. The generator of $\alpha^{\beta,X}$ is given by

$$\sum_{Z\subset\Lambda} \mathcal{I}_{H(t),\beta_{X,Z}}\big(\dot\Phi(t,Z)\big), \tag{13}$$

where the width of the Gaussian is modulated as follows

$$\frac{1}{\beta^2_{X,Z}} = \frac{1}{\beta^2} + \mathbb{1}_{\mathrm{dist}(X,Z)\geq\ell}\ \mathrm{dist}(X,Z), \tag{14}$$

and the parameter ℓ will be chosen appropriately later. We note that a very similar choice was also proposed in the original [120]. With this, we shall prove the following.

Theorem 12. *Let $D \in \mathbb{N}$, $\mathscr{C}_{\mathrm{vol}} > 0$, $b' > b > 0$, $C^{\mathrm{int}} > 0$, $\gamma > 0$. Then there exist constants C and $c > 0$ such that the following holds. For all $\Lambda \in \mathscr{G}(D, \mathscr{C}_{\mathrm{vol}})$ finite, smooth Hamiltonians H that satisfy the Uniform Gap assumption with gap γ and are given by interactions Φ such that $\|\Phi\|_{b'} < C^{\mathrm{int}}$ and $\|\dot\Phi\|_{b'} < C^{\mathrm{int}}$, the flow $\alpha^\beta_{0,s}$ generated by $\mathcal{I}_{H(s),\beta}\big(\dot H(s)\big)$ is an almost spectral flow in the sense that*

$$\big|\omega_s(A) - \omega_0 \circ \alpha^\beta_{0,s}(A)\big| \leq C\,|X|^2\,\|A\|\,\mathrm{e}^{-c\beta^{-2}}, \tag{15}$$

for all $X \subset \Lambda$ and $A \in \mathscr{A}_X$ and

$$\beta \in \left(0, \min\{1, \sqrt{2b}\, v\}\right), \tag{16}$$

where v is the Lieb-Robinson velocity from lemma 2.

Besides proving automorphic equivalence (11) itself, the spectral flow can also be used to prove the local perturbations perturb locally (LPPL) principle. In this case, $\dot{H}(s) \in \mathscr{A}_{\Lambda^{\mathrm{pert}}}$ is strictly localized in a perturbation region $\Lambda^{\mathrm{pert}} \subset \Lambda$. The strategy sketched above will yield the following result.

Theorem 13. *Let $D \in \mathbb{N}$, $\mathscr{C}_{\mathrm{vol}} > 0$, $b' > 0$, $C^{\mathrm{int}} > 0$, $\gamma > 0$. Then there exist constants C and $c > 0$ such that the following holds. For all $\Lambda^{\mathrm{pert}} \subset \Lambda \in \mathscr{G}(D, \mathscr{C}_{\mathrm{vol}})$ finite, smooth Hamiltonians H that satisfy the Uniform Gap assumption with gap γ and are given by interactions Φ such that $\|\Phi\|_{b'} < C^{\mathrm{int}}$, $\|\dot{\Phi}\|_{b'} < C^{\mathrm{int}}$, and $\dot{\Phi}(Z) = 0$ unless $Z \subset \Lambda^{\mathrm{pert}}$,*

$$\left|\omega_s(A) - \omega_0(A)\right| \leq C\, |X|^2\, \|A\|\, \mathrm{e}^{-c\, \mathrm{dist}(X, \Lambda^{\mathrm{pert}})}, \tag{17}$$

for all $s \in [0, 1]$, $X \subset \Lambda$ and $A \in \mathscr{A}_X$.

Remark 14. A few remarks are in order.

(i) The uniform gap assumption is of course crucial for the result and in general very difficult to verify, except for weak perturbations of frustration-free systems, see [162, 175].

(ii) An LPPL for ground states of gapped quantum spin systems was originally proposed in [23], but the use of the exact spectral flow there meant that the error in (17) could not be proved to be exponential. An exponential decay was obtained in [74] for Hamiltonians that are perturbations of free spins but under no additional gap assumption. We refer to the discussion in [74] for previous perturbative results of similar nature. An LPPL result similar to ours was also obtained in [214] using a completely different method.

(iii) The constants do not depend on Λ^{pert}. In particular, in the scenario of an increasing sequence of volumes $\Lambda \to \Gamma$, the perturbation may be extensive.

◇

Proofs

Within the proofs, we use $C > 0$ and $c > 0$ as generic constants that might change from line to line. They can depend on the same parameters that determine C and c in the statements. Typically, they depend on D, $\mathscr{C}_{\mathrm{vol}}$, b and b' but not on Λ.

As outlined before, one crucial ingredient to the construction of the almost spectral flow is locality of the automorphism α^{β}. For the Lieb-Robinson bound from lemma 2 to be sufficiently sharp, we first provide an exponentially local interaction for the generator $\mathscr{I}_{H(t),\beta}(\dot{H}(t))$. With the locality provided by Lemma 3, the proof is rather standard.

Lemma 15. *Let $D \in \mathbb{N}$, $\mathscr{C}_{\mathrm{vol}} > 0$, $D > b' > b > 0$. Then there exists a constant $C > 0$ such that the following holds. For all finite $\Lambda \in \mathscr{G}(D, \mathscr{C}_{\mathrm{vol}})$, $\beta > 0$, smooth Hamiltonians H given by interactions Φ such that $\|\Phi\|_{b'} < \infty$ and $\|\dot{\Phi}\|_{b'} < \infty$, there exists an interaction Ψ_β such that*

$$\mathscr{I}_{H(t),\beta}\big(\dot{H}(t)\big) = \sum_{Z \subset \Lambda} \Psi_\beta(Z) \qquad \textit{and} \qquad \|\Psi_\beta\|_{b(\beta)/3} < C(\beta)\, \|\dot{\Phi}\|_{b'},$$

where

$$b(\beta) = \min\left\{\frac{b}{2}, \frac{\beta^2}{4v^2}\right\}, \qquad C(\beta) = 1 + C\, b(\beta)^{-(D+1)}(\beta + \beta^{-1}),$$

and v is the Lieb-Robinson velocity of Φ as defined in lemma 2.

Proof. Fix t and let $\Omega \subset \Lambda$ and $O \in \mathscr{A}_\Omega$. Then, denote $\mathscr{I}_\beta(O) := \mathscr{I}_{H(t),\beta}(O)$ and let

$$\Delta_0(O) := \mathbb{E}_\Omega\big(\mathscr{I}_\beta(O)\big) \in \mathscr{A}_\Omega$$

and

$$\Delta_k(O) := \mathbb{E}_{\Omega_k}\big(\mathscr{I}_\beta(O)\big) - \mathbb{E}_{\Omega_{k-1}}\big(\mathscr{I}_\beta(O)\big) \in \mathscr{A}_{\Omega_k} \quad \text{for} \quad k \geq 1.$$

Here, $\mathbb{E}_Z$ denotes the conditional expectation as defined in [174], which is a standard tool to approximate almost local operators by strictly local ones. Then $\mathscr{I}_\beta(O) = \sum_{k=0}^\infty \Delta_k(O)$ and the sum is finite since eventually $\Omega_k = \Lambda$. By the properties of the conditional expectation

$$\|\Delta_0(O)\| \leq \|O\|$$

and together with lemma 3 and (10)

$$\|\Delta_k(O)\| \leq \big\|\big(\mathrm{id} - \mathbb{E}_{\Omega_{k-1}}\big)\, \mathscr{I}_\beta(O)\big\| \leq \tilde{C}(\beta)\, |\Omega|\, \|O\|\, \mathrm{e}^{-b(\beta)\, k},$$

where

$$\tilde{C}(\beta) = 2\left(\frac{2\,\beta}{\sqrt{\pi}\, b^2\, v^2} + \frac{1}{\sqrt{\pi}\, \beta}\right).$$

Then, Ψ_β can be chosen as

$$\Psi_\beta(Z) = \sum_{k=0}^\infty \sum_{\substack{Y \subset \Lambda: \\ Y_k = Z}} \Delta_k\big(\dot{\Phi}(Y)\big).$$

To estimate the interaction norm $\|\Psi_\beta\|$, for any $z \in \Lambda$ we bound

$$\begin{aligned}\sum_{\substack{Z \subset \Lambda: \\ z \in Z}} \|\Psi_\beta(Z)\|\, \mathrm{e}^{p\, \mathrm{diam}(Z)} &\leq \sum_{\substack{Z \subset \Lambda: \\ z \in Z}} \sum_{k=0}^\infty \sum_{\substack{Y \subset \Lambda: \\ Y_k = Z}} \big\|\Delta_k\big(\dot{\Phi}(Y)\big)\big\|\, \mathrm{e}^{p\, \mathrm{diam}(Z)} \\ &= \sum_{k=0}^\infty \sum_{Y \subset \Lambda} \mathbb{1}_{z \in Y_k}\, \big\|\Delta_k\big(\dot{\Phi}(Y)\big)\big\|\, \mathrm{e}^{p\, \mathrm{diam}(Y_k)}.\end{aligned}$$

The $k = 0$ term is bounded by $\|\dot{\Phi}\|_p \le \|\dot{\Phi}\|_{b'}$. For $k \ge 1$ and $z \in Y_k$, there exists $y \in B_z(k) \cap Y$ such that

$$\begin{aligned}\sum_{Y \subset \Lambda} & \mathbb{1}_{z \in Y_k} \left\| \Delta_k\big(\dot{\Phi}(Y)\big) \right\| \mathrm{e}^{p\,\mathrm{diam}(Y_k)} \\ &\le \sum_{y \in B_z(k)} \sum_{\substack{Y \subset \Lambda: \\ y \in Y}} \left\| \Delta_k\big(\dot{\Phi}(Y)\big) \right\| \mathrm{e}^{p\,\mathrm{diam}(Y)}\, \mathrm{e}^{2pk} \\ &\le \tilde{C}(\beta)\, \mathrm{e}^{(2p - b(\beta))\,k} \sum_{y \in B_z(k)} \sum_{\substack{Y \subset \Lambda: \\ y \in Y}} |Y|\, \|\dot{\Phi}(Y)\|\, \mathrm{e}^{p\,\mathrm{diam}(Y)} \\ &\le \tilde{C}(\beta)\, \mathrm{e}^{(2p + 2\varepsilon - b(\beta))\,k}\, \mathscr{C}_{\mathrm{vol},\varepsilon,1}\, \mathscr{C}_{\mathrm{vol},b'-p,1}\, \|\dot{\Phi}\|_{b'},\end{aligned}$$

for any $\varepsilon > 0$ and $p < b'$. The sum over k is then finite for $p < b(\beta)/2$, and we choose $p = b(\beta)/3$ and $\varepsilon = b(\beta)/12$. Hence, the constant $C(\beta)$ from the statement is

$$1 + \sum_{k=1}^{\infty} \tilde{C}(\beta)\, \mathrm{e}^{-b(\beta)\,k/6}\, \mathscr{C}_{\mathrm{vol},b(\beta)/12,1}\, \mathscr{C}_{\mathrm{vol},b'-b(\beta)/3,1}.$$

By definition, $b(\beta) \in (0, b/2)$. Thus, (2) implies that $\mathscr{C}_{\mathrm{vol},b'-b(\beta)/3,1}$ is uniformly bounded in β, while $\mathscr{C}_{\mathrm{vol},b(\beta)/12,1} \le C\, b(\beta)^{-D}$. Moreover, $\tilde{C}(\beta) \le C\,(\beta + \beta^{-1})$, and we conclude with $\sum_{k=1}^{\infty} \mathrm{e}^{-b(\beta)k/6} \le \frac{6}{b(\beta)}$. □

We can now continue with the proof of the almost spectral flow. First of all, we note the following simple bounds: For every $p, c > 0$

$$\sup_{r \ge 0} r^p\, \mathrm{e}^{-cr} \le \left(\frac{p}{\mathrm{e}}\right)^p c^{-p}, \tag{18}$$

and for every $c > 0$ and L

$$\sum_{n \ge L} \mathrm{e}^{-cn} \le \frac{\mathrm{e}^c}{c}\, \mathrm{e}^{-c\lceil L \rceil}. \tag{19}$$

Proof of theorem 12. Let α be the exact spectral flow generated by $\mathscr{I}_{H(t)}\big(\dot{H}(t)\big)$ such that $\omega_s = \omega_0 \circ \alpha_{0,s}$ and α^β be the almost spectral flow. Moreover, fix $X \subset \Lambda$ and let $\alpha^{\beta,X}$ be the automorphism generated by (13) where we choose $\beta_{X,Z}$ as in (14) with ℓ to be chosen later. We then use triangle inequality to bound

$$\left|\omega_s(A) - \omega_0 \circ \alpha_{0,s}^{\beta}(A)\right| \le \left|\omega_0 \circ \big(\alpha_{0,s} - \alpha_{0,s}^{\beta,X}\big)(A)\right| + \left|\omega_0 \circ \big(\alpha_{0,s}^{\beta,X} - \alpha_{0,s}^{\beta}\big)(A)\right|. \tag{20}$$

To bound the first term in (20), we use (12) to obtain

$$\begin{aligned}&\left|\mathrm{tr}\Big(P(0)\big(\alpha_{0,s} - \alpha_{0,s}^{\beta,X}\big)(A)\Big)\right| \\ &\quad \le s \sup_{t \in [0,s]} \sum_{Z \subset \Lambda} \left|\mathrm{tr}\Big(P(t)\left[\mathscr{I}_{H(t)}\big(\dot{\Phi}(Z,t)\big) - \mathscr{I}_{H(t),\beta_{X,Z}}\big(\dot{\Phi}(Z,t)\big), \alpha_{t,s}^{\beta,X}(A)\right]\Big)\right|.\end{aligned} \tag{21}$$

Now, for all projections P and operators V and W

$$\left|\operatorname{tr}\big(P\,[V,W]\big)\right| = \left|\operatorname{tr}\big([P,V]\,W\big)\right| \leq \|[P,V]\|_1\,\|W\|,$$

by cyclicity of the trace and since the Hilbert space is finite-dimensional. Applying this and lemma 10 to each summand in (21) gives

$$\left|\operatorname{tr}\Big(P(0)\big(\alpha_{0,s} - \alpha_{0,s}^{\beta,X}\big)(A)\Big)\right| \leq 2\,s\,\|P\|_1\,\|A\|\,\gamma^{-1} \sup_{t\in[0,s]} \sum_{Z\subset\Lambda} \|\dot{\Phi}(Z,t)\|\,\mathrm{e}^{-\frac{\gamma^2}{4\beta_{X,Z}^2}}.$$

It remains to control the sum. With (14), we can factor out $\mathrm{e}^{-\frac{\gamma^2}{4\beta^2}}$ and organize the sum as

$$\sum_{Z\subset\Lambda} \|\dot{\Phi}(Z,t)\|\,\mathrm{e}^{-\frac{\gamma^2}{4}\mathbb{1}_{\operatorname{dist}(X,Z)\geq\ell}\,\operatorname{dist}(X,Z)} = \sum_{n=0}^{\infty} \sum_{\substack{Z\subset\Lambda:\\ \operatorname{dist}(Z,X)=n}} \|\dot{\Phi}(Z,t)\|\,\mathrm{e}^{-\frac{\gamma^2}{4}\mathbb{1}_{n\geq\ell}n}.$$

Since

$$\sum_{\substack{Z\subset\Lambda:\\ \operatorname{dist}(Z,X)=n}} \|\dot{\Phi}(Z,t)\| \leq \sum_{x\in X} \sum_{\substack{z\in\Lambda:\\ \operatorname{dist}(z,x)=n}} \sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \|\dot{\Phi}(Z,t)\| \leq |X|\,|B_x(n)|\,\|\dot{\Phi}\|_0,$$

we conclude that

$$\begin{aligned}&\sum_{Z\subset\Lambda} \|\dot{\Phi}(Z,t)\|\,\mathrm{e}^{-\frac{\gamma^2}{4}\mathbb{1}_{\operatorname{dist}(X,Z)\geq\ell}\,\operatorname{dist}(X,Z)} \\ &\qquad\leq |X|\,\|\dot{\Phi}\|_0\,\mathscr{C}_{\mathrm{vol}}\Big(\sum_{n=0}^{\lfloor\ell\rfloor}(n+1)^D + \sum_{n=\lfloor\ell\rfloor+1}^{\infty}(n+1)^D\,\mathrm{e}^{-\frac{\gamma^2}{4}n}\Big).\end{aligned}$$

The first sum is bounded by $C\,\ell^{D+1}$. For the second one, we use (18) and (19) to conclude that

$$\sum_{n=\lfloor\ell\rfloor+1}^{\infty}(n+1)^D\mathrm{e}^{-\frac{\gamma^2}{4}n} \leq C\,\frac{\mathrm{e}^{\frac{\gamma^2}{8}}}{\gamma^{2(D+1)}}\,\mathrm{e}^{-\frac{\gamma^2}{8}\ell} \leq C\,\mathrm{e}^{-c\ell}. \tag{22}$$

Altogether,

$$\left|\operatorname{tr}\Big(P(0)\,\big(\alpha_{0,s} - \alpha_{0,s}^{\beta,X}\big)(A)\Big)\right| \leq C\,\|P\|_1\,\|A\|\,|X|\,\|\dot{\Phi}\|_0\,\big(\ell^{D+1} + \mathrm{e}^{-c\ell}\big)\,\mathrm{e}^{-\frac{\gamma^2}{8\beta^2}}. \tag{23}$$

To bound the second term of (20), we use (12) again to obtain

$$\begin{aligned}&\left|\operatorname{tr}\Big(P(0)\,\big(\alpha_{0,s}^{\beta,X} - \alpha_{0,s}^{\beta}\big)(A)\Big)\right| \\ &\quad\leq s \sup_{t\in[0,s]} \sum_{\substack{Z\subset\Lambda:\\ \operatorname{dist}(X,Z)\geq\ell}} \left|\operatorname{tr}\Big(P(t)\Big[\mathscr{I}_{H(t),\beta_{X,Z}}\big(\dot{\Phi}(Z,t)\big) - \mathscr{I}_{H(t),\beta}\big(\dot{\Phi}(Z,t)\big), \alpha_{t,s}^{\beta}(A)\Big]\Big)\right|,\end{aligned}$$

where the restriction is due to the fact that otherwise $\beta_{X,Z} = \beta$ and the terms cancel exactly. To obtain a bound for this term, we only use the locality of both flows. Denoting $r = \operatorname{dist}(X, Z)$, we estimate the commutator as

$$\begin{aligned} &\left\| \left[\mathcal{I}_{H(t),\beta_{X,Z}}\big(\dot{\Phi}(Z,t)\big) - \mathcal{I}_{H(t),\beta}\big(\dot{\Phi}(Z,t)\big), \alpha_{t,s}^{\beta}(A) \right] \right\| \\ &\quad \leq 4 \left\|\dot{\Phi}(Z,t)\right\| \left\|\big(\mathrm{id} - \mathbb{E}_{X_{r/2}}\big)\big(\alpha_{t,s}^{\beta}(A)\big)\right\| && \text{(24a)} \\ &\qquad + \left\|\left[\mathcal{I}_{H(t),\beta}\big(\dot{\Phi}(Z,t)\big), \mathbb{E}_{X_{r/2}}(\alpha_{t,s}^{\beta}(A))\right]\right\| && \text{(24b)} \\ &\qquad + \left\|\left[\mathcal{I}_{H(t),\beta_{X,Z}}\big(\dot{\Phi}(Z,t)\big), \mathbb{E}_{X_{r/2}}(\alpha_{t,s}^{\beta}(A))\right]\right\|, && \text{(24c)} \end{aligned}$$

where $\mathbb{E}_X$ is the conditional expectation as in the proof of lemma 15. We shall use repeatedly below that in applications of lemmata 3 and 15

$$b(\beta) = \frac{\beta^2}{4v^2}$$

and that $b(\beta)$ and β are bounded because of (16).

The first term (24a) is bounded using locality of $\alpha_{t,s}^{\beta}$, lemma 15: The flow α^{β} is generated by an interaction Ψ_β such that $\|\Psi_\beta\|_{b(\beta)/3} \leq \big(1 + C\, b(\beta)^{-(D+1)}\,(\beta + \beta^{-1})\big)\,\|\dot{\Phi}\|_{b'}$. We then use lemma 2 with $b' \to \frac{b(\beta)}{3}$ and $b \to \frac{2b(\beta)}{9}$ to obtain for all $|t - s| \leq 1$

$$4 \left\|\dot{\Phi}(Z,t)\right\| \left\|\big(\mathrm{id} - \mathbb{E}_{X_{r/2}}\big)\, \alpha_{t,s}^{\beta}(A)\right\| \leq C_1(\beta)\, \|\dot{\Phi}(Z,t)\|\, \|A\|\, |X|\, \mathrm{e}^{-\frac{b(\beta)}{9} r}, \tag{25}$$

with

$$C_1(\beta) \leq 8\, \mathrm{e}^{2\mathscr{C}_{\mathrm{vol},1,b(\beta)/9}\, \|\Psi_\beta\|_{b(\beta)/3}} \leq 8\, \mathrm{e}^{c\, b(\beta)^{-D}(1 + b(\beta)^{-(D+1)}(\beta + \beta^{-1}))} \leq C\, \mathrm{e}^{c\, \beta^{-(4D+3)}},$$

for some $C > 0$, where we used the bound $\mathscr{C}_{\mathrm{vol},b(\beta)/9,1} \leq C\, b(\beta)^{-D}$, see (2).

To bound the second term (24b) we use lemma 3. Specifically, (10) yields

$$\left\|\left[\mathcal{I}_{H(t),\beta}\big(\dot{\Phi}(Z,t)\big), \mathbb{E}_{X_{r/2}}(\alpha_{t,s}^{\beta}(A))\right]\right\| \leq C\, \beta^{-1}\, \|\dot{\Phi}(Z,t)\|\, \|A\|\, |X|\, \mathrm{e}^{-\frac{b(\beta)}{2} r}, \tag{26}$$

for some $C > 0$.

To control the last term (24c), we use the same lemma 3, this time in the form (9), because β is replaced by $\beta_{X,Z}$. We shall choose again $T = \frac{r}{2v}$. With $\beta_{X,Z} \leq \beta$, the first term of (9) is bounded by $\beta\, \mathrm{e}^{-\frac{b}{2} r}$, up to the prefactors. For the second term, we use the Gaussian decay. For $r \geq \ell$, we have that

$$\beta_{X,Z}^2\, T^2 = \frac{1}{\beta^{-2} + r}\, \frac{r^2}{(2v)^2} \geq \frac{\beta^2}{1 + r}\, \frac{r^2}{(2v)^2} \geq \frac{b(\beta)\, r}{2}$$

and in turn

$$\frac{1}{\beta_{X,Z}}\,\mathrm{e}^{-\beta_{X,Z}^2 T^2} \leq \frac{1}{\beta}\sqrt{1+r}\,\mathrm{e}^{-\frac{b(\beta)}{2}r},$$

where we used $\beta \leq 1$ and $r \geq 1$ repetitively. We then conclude that

$$\left\|\left[\mathcal{I}_{H(t),\beta_{X,Z}}\big(\dot{\Phi}(Z,t)\big), \mathbb{E}_{X_{r/2}}\big(\alpha_{t,s}^{\beta}(A)\big)\right]\right\| \leq C\,|X|\,\|\dot{\Phi}(Z,t)\|\,\|A\|\left(\mathrm{e}^{-\frac{b}{2}r} + \beta^{-1}\sqrt{r}\,\mathrm{e}^{-\frac{b(\beta)}{2}r}\right).$$

Combining the three terms and handling the sum $Z \subset \Lambda$ with $d(X,Z) \geq \ell$ as in the bound for (21), we obtain

$$\begin{aligned}
&\left|\mathrm{tr}\Big(P(0)\,\big(\alpha_{0,s}^{\beta,X} - \alpha_{0,s}^{\beta}\big)(A)\Big)\right| \\
&\quad \leq C\,\|P\|_1\,\|A\|\,|X|^2\,C^{\mathrm{int}} \sum_{n=\ell}^{\infty}\Big(\mathrm{e}^{c\,\beta^{-(4D+3)}}\,\mathrm{e}^{-\frac{b(\beta)}{9}n} + \beta^{-1}\big(1+\sqrt{n}\big)\,\mathrm{e}^{-\frac{b(\beta)}{2}n} + \mathrm{e}^{-\frac{b}{2}n}\Big) \\
&\quad \leq C\,\|P\|_1\,\|A\|\,|X|^2\left(\mathrm{e}^{c\,\beta^{-(4D+3)}}\,\beta^{-2}\,\mathrm{e}^{-\frac{b(\beta)}{9}\ell} + \beta^{-4}\,\mathrm{e}^{-\frac{b(\beta)}{4}\ell} + \mathrm{e}^{-\frac{b}{2}\ell}\right),
\end{aligned}$$

by using (18), (19) and the properties of $b(\beta)$ and β. We now choose $\ell = \frac{9}{b(\beta)}\,(\beta^{-2} + c\,\beta^{-(4D+3)})$ and absorb the polynomial dependence on β^{-1} in the exponential to obtain the upper bound

$$C\,\|P\|_1\,\|A\|\,|X|^2\,\mathrm{e}^{-c\,\beta^{-2}}.$$

Plugging this choice of ℓ into (23) and similarly absorbing the polynomial dependence, we obtain the same upper bound. □

It remains to prove LPPL.

Proof of theorem 13. We first use triangle inequality

$$\big|\omega_s(A) - \omega_0(A)\big| \leq \big|\omega_s(A) - \omega_0 \circ \alpha_{0,s}^{\beta}(A)\big| + \big|\omega_0 \circ \alpha_{0,s}^{\beta}(A) - \omega_0(A)\big|.$$

To bound the first summand, we again write

$$\begin{aligned}
&\left|\mathrm{tr}\Big(P(0)\,\big(\alpha_{0,s} - \alpha_{0,s}^{\beta}\big)(A)\Big)\right| \\
&\quad \leq s \sup_{t\in[0,s]} \sum_{\substack{Z\subset\Lambda: \\ \mathrm{dist}(X,Z)\geq\mathrm{dist}(X,\Lambda^{\mathrm{pert}})}} \left|\mathrm{tr}\Big(P(t)\left[\mathcal{I}_{H(t)}\big(\dot{\Phi}(Z,t)\big) - \mathcal{I}_{H(t),\beta}\big(\dot{\Phi}(Z,t)\big), \alpha_{t,s}^{\beta}(A)\right]\Big)\right|,
\end{aligned}$$

since $\|\dot{\Phi}(Z,t)\| = 0$ if $\mathrm{dist}(X,Z) < \mathrm{dist}(X,\Lambda^{\mathrm{pert}})$ by assumption. Following the proof of theorem 12, we obtain

$$\big|\omega_s(A) - \omega_0 \circ \alpha_{0,s}^{\beta}(A)\big| \leq C\,|X|^2\,\|A\|\,\mathrm{e}^{-c\,\mathrm{dist}(X,\Lambda^{\mathrm{pert}})}.$$

For the second part, we use that α^β acts almost trivially away from Λ^{pert}. For this, we consider another automorphism $\tilde{\alpha}^\beta$ generated by

$$\tilde{\Psi}_\beta(Z,t) = \begin{cases} \Psi_\beta(Z,t) & \text{if } Z \cap X = \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

which is such that $\tilde{\alpha}^\beta_{t,s}(A) = A$. Moreover, $\Psi_\beta(Z,t)$, $\tilde{\Psi}_\beta(Z,t) = 0$ if $Z \cap \Lambda^{\mathrm{pert}} = \emptyset$ by construction of Ψ_β in the proof of lemma 15 and the assumption that $\dot{\Phi}(Z,t) = 0$ unless, $Z \subset \Lambda^{\mathrm{pert}}$. Then,

$$\begin{aligned}
\left|\omega_0 \circ \alpha^\beta_{0,s}(A) - \omega_0(A)\right| &\leq \|P(0)\|_1 \left\|\alpha^\beta_{0,s}(A) - \tilde{\alpha}^\beta_{0,s}(A)\right\| \\
&\leq s\,\|P(0)\|_1 \sup_t \sum_{Z \subset \Lambda} \left\|\left[\Psi(Z,t) - \tilde{\Psi}(Z,t), \tilde{\alpha}^\beta_{t,s}(A)\right]\right\| \\
&\leq 2\,s\,\|P(0)\|_1\,\|A\| \sup_t \sum_{x \in X} \sum_{z \in \Lambda^{\mathrm{pert}}} \sum_{\substack{Z \subset \Lambda: \\ x,z \in Z}} \|\Psi(Z,t)\| \\
&\leq 2\,s\,\|P(0)\|_1\,\|A\|\,|X|\,\|\Psi\|_{b(\beta)/3} \sup_{x \in \Lambda} \sum_{z \in \Lambda^{\mathrm{pert}}} \mathrm{e}^{-b(\beta)\,\mathrm{dist}(x,z)/3}.
\end{aligned}$$

Bounding this sum and combining both bounds gives the result. □

Note that the choice of Gaussian width β could be optimized to give the sharpest decay rate (following the above, β should be of order $d(X, \Lambda^{\mathrm{pert}})^{-\frac{1}{2}}$).

5 Exponential clustering revisited

Finally, Gaussian filters were already used in previous proofs of the exponential clustering theorem: Ground state correlations decay in the presence of a spectral gap. In the present context of quantum spin systems, exponential decay of correlations were proved in [171] in the infinite-volume limit (the gap refers in this case to the gap of the GNS Hamiltonian), in the limit of finite volumes in [118], where the splitting of the ground state energies in finite volume is assumed to vanish in the limit, while the latter assumption in removed in [14] but the decay is only superpolynomial. In this section, we prove that correlations decay indeed exponentially under the sole assumption of a spectral gap, even if there is eigenvalue splitting in the ground state. A similar result was also obtained in [214] using a complex analysis proof.

Assumption 16. Let Λ be finite. Let Φ be an interaction satisfying $\|\Phi\|_{b'} < \infty$ for some $b' > 0$, and let H be the corresponding Hamiltonian. We assume that the spectrum of H is of the form

$$\sigma(H) = \sigma_0 \cup \sigma_1$$

with $\inf(\sigma_1) - \sup(\sigma_0) \geq \gamma > 0$ and $\mathrm{diam}(\sigma_0) \leq \Delta < \gamma/4$. ◇

As above, we denote by P the spectral projection associated with the spectral patch σ_0. Note that the condition $\Delta < \gamma/4$ is not tight but will simplify the estimates. What is needed in the proof is that $-\Delta + \gamma/2 \geq c > 0$.

Theorem 17. *Let $D \in \mathbb{N}$, $\mathscr{C}_{\text{vol}} > 0$, $b' > 0$, $C^{\text{int}} > 0$, $\gamma > 0$ and $\Delta > 0$. Then there exist constants $C, c > 0$, such that the following holds. For all $\Lambda \in \mathscr{G}(D, \mathscr{C}_{\text{vol}})$ finite and Hamiltonians H that satisfy assumption 16 with gap γ and width Δ and are given by interactions Φ such that $\|\Phi\|_{b'} < C^{\text{int}}$ the following holds:*

For any normalized state $\Omega \in \operatorname{Ran}(P)$,

$$|\langle \Omega, A\, B\, \Omega\rangle - \langle \Omega, A\, P\, B\, \Omega\rangle| \leq C\, \|P\|_1 \|A\|\, \|B\|\, \mathrm{e}^{-cd(X,Y)}$$

for all disjoint $X, Y \subset \Lambda$ and $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$.

Proof. For the proof, we define the filter function

$$g_\beta(t) = \frac{\mathrm{e}^{-\mathrm{i}t\frac{\gamma}{2}}}{\sqrt{2}\beta} \left(\sqrt{\frac{\pi}{2}}\, \delta_0(t) + \frac{\mathrm{i}}{\sqrt{2\pi}}\ \mathrm{p.\,v.}\Big(\frac{1}{t}\Big)\right) \phi_\beta(t),$$

where $\mathrm{p.\,v.}\big(\frac{1}{t}\big)$ denotes the principal value distribution, and let

$$\mathscr{I}_{H,\beta}(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}t\, g_\beta(t)\, \tau_t^H(A).$$

The Fourier transform of $\hat{g}_\beta$ is given by

$$\hat{g}_\beta(\omega) = \frac{1}{\sqrt{2}\beta} (\Theta_{\gamma/2} \star \hat{\phi}_\beta)(\omega) = \frac{1}{2\beta\sqrt{\pi}} \int_{-\infty}^{\omega-\gamma/2} \mathrm{d}\xi\, \mathrm{e}^{-\frac{\xi^2}{4\beta^2}} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{\omega-\gamma/2}{2\beta}} \mathrm{d}x\, \mathrm{e}^{-x^2},$$

where Θ_a is the Heaviside step function with discontinuity at a.

Before we continue the proof, we note some fact about the Gaussian error function. We have

$$1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} \mathrm{d}x\, \mathrm{e}^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{z}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2} \leq \frac{1}{2\sqrt{\pi}\, z} \int_{z}^{\infty} \mathrm{d}x\, 2\, x\, \mathrm{e}^{-x^2} = \frac{1}{2\sqrt{\pi}} \frac{\mathrm{e}^{-z^2}}{z}$$

whenever $z > 0$, and similarly

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} \mathrm{d}x\, \mathrm{e}^{-x^2} \leq \frac{1}{2\sqrt{\pi}} \frac{\mathrm{e}^{-z^2}}{|z|}$$

if $z < 0$. Hence, the Fourier transform of the filter function satisfies

$$\hat{g}_\beta(\omega) \leq \frac{\beta}{\sqrt{\pi}\, |\omega - \gamma/2|}\, \mathrm{e}^{-\frac{(\omega-\gamma/2)^2}{4\beta^2}} \qquad \text{for } \omega < \gamma/2$$

and

$$1 - \hat{g}_\beta(\omega) \le \frac{\beta}{\sqrt{\pi}\,(\omega - \gamma/2)}\, \mathrm{e}^{-\frac{(\omega-\gamma/2)^2}{4\beta^2}} \qquad \text{for } \omega > \gamma/2.$$

We now decompose the correlation into three terms

$$\begin{aligned}
\langle \Omega, A\, B\Omega\rangle - \langle \Omega, A\, P\, B\, \Omega\rangle &= \langle \Omega, A\, P^\perp\, B\, \Omega\rangle \\
&= \big\langle \Omega, [\mathcal{J}_{H,\beta}(A), B]\, \Omega\big\rangle && (27a)\\
&\quad + \big\langle \Omega, B\, \mathcal{J}_{H,\beta}(A)\, \Omega\big\rangle && (27b)\\
&\quad + \big\langle \Omega, \big(P\, A\, P^\perp - P\, \mathcal{J}_{H,\beta}(A)\big)\, B\, P\, \Omega\big\rangle. && (27c)
\end{aligned}$$

In the following we will show that each term decays exponentially in $d(X,Y)$ with the choice $\beta = \frac{\gamma}{2\sqrt{d(X,Y)}}$.

We start by bounding (27b), for which we observe

$$\mathcal{J}_{H,\beta}(A)\, P = \sum_{\mu \in \sigma(H)} \sum_{\nu \in \sigma_0} \hat{g}_\beta(\nu - \mu)\, P_\mu\, A\, P_\nu.$$

Since $\nu - \mu \le \Delta < \gamma/4$, we obtain the bound

$$\|\mathcal{J}_{H,\beta}(A)\, P\| = \|A\| \sum_{\nu \in \sigma_0} \Big\| \sum_{\mu \in \sigma(H)} \hat{g}_\beta(\nu - \mu)\, P_\mu \Big\| \le \frac{4\,|\sigma_0|}{\sqrt{\pi}}\, \frac{\beta}{\gamma}\, \mathrm{e}^{-\frac{1}{64}\left(\frac{\gamma}{\beta}\right)^2}\, \|A\|. \tag{28}$$

For (27c) we bound

$$\|P\, A\, P^\perp - P\, \mathcal{J}_{H,\beta}(A)\, P^\perp\| = \Big\| \sum_{\mu \in \sigma_0} \sum_{\nu \in \sigma_1} \big(1 - \hat{g}_\beta(\nu - \mu)\big)\, P_\mu\, A\, P_\nu \Big\| \le \frac{2\,|\sigma_0|}{\sqrt{\pi}}\, \frac{\beta}{\gamma}\, \mathrm{e}^{-\frac{1}{16}\left(\frac{\gamma}{\beta}\right)^2}\, \|A\|$$

because $\nu - \mu \ge \gamma$. And together with (28), we obtain

$$\|P\, A\, P^\perp - P\, \mathcal{J}_{H,\beta}(A)\| \le \frac{6\,|\sigma_0|}{\sqrt{\pi}}\, \frac{\beta}{\gamma}\, \mathrm{e}^{-\frac{1}{64}\left(\frac{\gamma}{\beta}\right)^2}\, \|A\|. \tag{29}$$

Finally, the commutator $[\mathcal{J}_{H,\beta}(A), B]$ in (27a) is bounded using an argument similar to that of Lemma 3. As we do there, we decompose the integral defining $\mathcal{J}_{H,\beta}(A)$ into $|t| \le T$ and $|t| > T$ and use the Lieb-Robinson bound, Lemma 2, to estimate the short time part while the long time contribution can be bounded by the Gaussian decay. Since $d(X,Y) > 0$, we have that $[A,B] = 0$ and so the δ_0-contribution vanishes. Therefore,

$$\begin{aligned}
&\Big\| \int_{|t|<T} \mathrm{d}t\, g_\beta(t)\, [\tau_t^H(A), B] \Big\| \\
&\quad \le \frac{2\, \mathscr{C}_{\mathrm{vol},1,b'-b}^{-1}}{\beta\sqrt{\pi}}\, \|A\|\, \|B\|\, \min\big\{|X|, |Y|\big\}\, \sup_t |\phi_\beta(t)|\, \mathrm{e}^{-b\, \mathrm{dist}(X,Y)}\, \lim_{\epsilon \to 0} \int_\epsilon^T \mathrm{d}t\, \frac{1}{t}\, \big(\mathrm{e}^{bvt} - 1\big).
\end{aligned}$$

The mean value theorem implies that $t^{-1}\,(\mathrm{e}^{bvt}-1) \le b\,v\,\mathrm{e}^{bvt}$ and so

$$\lim_{\epsilon\to 0}\int_{\epsilon}^{T} \mathrm{d}t\,\frac{1}{t}\,\big(\mathrm{e}^{bvt}-1\big) \le \big(\mathrm{e}^{bvT}-1\big).$$

Since $\sup|\phi_\beta(t)| = \frac{\beta}{\sqrt{\pi}}$, we conclude that

$$\left\|\int_{|t|<T} \mathrm{d}t\, g_\beta(t)\,[\tau_t^H(A),B]\right\| \le \frac{2\,\mathscr{C}_{\mathrm{vol},1,b'-b}^{-1}\,\|A\|\,\|B\|}{\pi}\,\min\big\{|X|,|Y|\big\}\,\mathrm{e}^{-b\,\mathrm{dist}(X,Y)}\,\big(\mathrm{e}^{bvT}-1\big). \tag{30}$$

For $|t| \ge T$, we use the simple norm bound on the commutator and

$$\int_T^\infty \mathrm{d}t\,\frac{\phi_\beta(t)}{t} \le \frac{1}{T}\int_T^\infty \mathrm{d}t\,\phi_\beta(t) \le \frac{1}{2\,T^2\,\sqrt{\pi}}\,\mathrm{e}^{-\beta^2T^2}$$

to conclude that

$$\left\|\int_{|t|>T} \mathrm{d}t\, g_\beta(t)\,[\tau_t^H(A),B]\right\| \le \frac{2\,\|A\|\,\|B\|}{T^2\,\sqrt{\pi}}\,\mathrm{e}^{-\beta^2T^2}. \tag{31}$$

Together, (30), (31) and the choice $T=\frac{d(X,Y)}{2v}$ yield the following bound on (27a):

$$\left\|\int_{-\infty}^{\infty} \mathrm{d}t\, g_\beta(t)\,[\tau_t^H(A),B]\right\| \le \frac{2\,\|A\|\,\|B\|}{\pi}\left(\frac{\min\big\{|X|,|Y|\big\}}{\mathscr{C}_{\mathrm{vol},1,b'-b}}\,\mathrm{e}^{-\frac{bd(X,Y)}{2}} + \frac{4\,v^2\,\sqrt{\pi}}{d(X,Y)^2}\,\mathrm{e}^{-\frac{\beta^2 d(X,Y)^2}{4v^2}}\right). \tag{32}$$

Gathering (32), (28) and (29) to bound (27a), (27b) and(27c), the claim of the theorem follows by bounding $|\sigma_0| \le \|P\|_1$ and setting

$$\beta = \frac{\gamma}{2\sqrt{d(X,Y)}},$$

which makes all exponents proportional to $d(X,Y)$, and using that $d(X,Y)\ge 1$ in the prefactors. □

6 Putting it all together: the quantum Hall effect

We briefly recall the setting of [14] applied to the quantum Hall effect. The lattice is a sequence of discrete tori $\Lambda_L = (\mathbb{Z}/L\mathbb{Z})^2$. The Hamiltonian is given by a finite range interaction Φ that is invariant under a strictly local $U(1)$-action. This means that there is a family $q_x = q_x^* \in \mathscr{A}_x$ with integer spectrum and such that $[Q_{\Lambda_L},\Phi(Z)] = 0$ for all $Z\subset\Lambda_L$. The q_x are the *local charges*. The "ground state space" is the range of a spectral projection P_L of H_{Λ_L} whose dimension is constant and equal to p for all L

large enough. Moreover, H_{Λ_L} is assumed to satisfy the Gap Assumption, uniformly in L for L large enough.

The proof of quantization of the Hall conductance in [14] relies heavily on the inverse Liouvillian on the one hand, and on clustering on the other hand. The inverse Liouvillian is used to construct a unitary U_L describing a magnetic flux threading and its locality allows for the definition of a charge transport operator T_L across a fiducial line of the torus. Replacing the exact inverse Liouvillian by the almost inverse Liouvillian introduced in Section 3 yields a unitary U_L^β and in turn an exponentially localized charge transport operator T_L^β. This improved localization and the exponential clustering of Section 5 yield the following.

Theorem 18. *Let $\beta = L^{-1/2}$. There is an integer $n_L \in \mathbb{Z}$ and constants $C, c > 0$ such that*

$$\left|n_L - \operatorname{tr}\left(P_L T_L^\beta\right)\right| \le C\,\mathrm{e}^{-cL}$$

for all L. If, moreover, the sequence of states $p^{-1}\operatorname{Tr}(P_L(\cdot))$ is convergent, then $n_L = n$ for L large enough and $2\pi\kappa = \frac{n}{p}$, where κ is the Hall conductance.

We now explain the arguments with a focus on the changes from using the almost inverse Liouvillian, referring to [14, Section IV] for more details. The geometric setting is described in [19, Section 2], see in particular Figure 1 therein, where η_- and ν_- correspond to what we will call the lower boundary of the upper half and the left boundary of the right half, respectively, of the torus. We again drop the L dependence to simplify notations.

Similarly to [19, Section 2] we first construct a unitary U^β that models the threading of one unit of flux through the torus. In our case it is exponentially localized near a line along the torus. Therefore, let Q_{U} be the charge on the upper half of the torus and let

$$\overline{Q}_{\mathrm{U}}^\beta = Q_{\mathrm{U}} - \mathscr{I}_{H,\beta} \circ \mathscr{L}_H(Q_{\mathrm{U}}).$$

By the $U(1)$-invariance and the finite range condition of the Hamiltonian, the operator $\mathscr{L}_H(Q_{\mathrm{U}})$ is strictly localized in two strips of finite width around the boundary of the half-torus, which we denote $\mathscr{L}_H(Q_{\mathrm{U}})_{\mathrm{lower}}$ and $\mathscr{L}_H(Q_{\mathrm{U}})_{\mathrm{upper}}$, respectively. The locality result, Lemma 3, implies that $\mathscr{I}_{H,\beta}(\mathscr{L}_H(Q_{\mathrm{U}})_{\mathrm{lower}})$ is exponentially localized around the lower boundary. Specifically, the choice $\beta = L^{-1/2}$ in (10) yields a localization estimate of the form $C\,\mathrm{e}^{-cL}$ for $\operatorname{dist}(X, Y)$ of order L, namely

$$\mathscr{I}_{H,\beta} \circ \left(\mathscr{L}_H(Q_{\mathrm{U}})_{\mathrm{lower}}\right)$$

can be approximated by an operator that is strictly localized in a strip of width $\frac{L}{4}$ at the lower boundary (or any other smaller fraction of L), up to errors that are exponentially small in L. This and the fact that Q_{U} has integer spectrum imply that the unitary $\mathrm{e}^{2\pi \mathrm{i}\overline{Q}_{\mathrm{U}}^\beta}$

factorizes up to exponentially small errors into parts at the lower and upper boundary. We let

$$U^\beta = \big(\mathrm{e}^{2\pi\mathrm{i}\overline{Q}_{\mathrm{U}}^\beta}\big)_{\mathrm{lower}} \tag{33}$$

be the factor localized on the lower boundary of the half-torus.

We briefly pause the argument to compare explicitly with [14]. There, $\overline{Q}_{\mathrm{U}}$ is defined using the exact inverse Liouvillian $\mathcal{I}_H$ rather than $\mathcal{I}_{H,\beta}$. As a result, $[\overline{Q}_{\mathrm{U}}, P] = 0$ while here $\|[\overline{Q}_{\mathrm{U}}^\beta, P]\| \leq C\,L^2\,\mathrm{e}^{-c\beta^{-2}} = C\,\mathrm{e}^{-cL}$, by Proposition 8. The same holds for the exponentials, and by exponential clustering, Theorem 17, for their restrictions to the lower boundary of the half-torus. What is more, Lemma 10 implies that $\overline{Q}_{\mathrm{U}}^\beta$ and $\overline{Q}_{\mathrm{U}}$ are exponentially close and therefore so are U^β and U. As a consequence, U^β almost preserves the ground state space and almost implements 2π-flux threading.

Next, we consider the charge transport along the torus. Therefore, let Q_{R} be the charge on the right half of the torus. By charge conservation and again the locality lemma, the operator $(U^\beta)^*\,Q_{\mathrm{R}}\,U^\beta - Q_{\mathrm{R}}$ decomposes in two contributions at each boundary of the right half-torus. We define the operator of charge transport as the left one

$$T^\beta = \Big((U^\beta)^*\,Q_{\mathrm{R}}\,U^\beta - Q_{\mathrm{R}}\Big)_{\mathrm{left}}. \tag{34}$$

More precisely, $\mathrm{tr}\big(P_L\,T_L^\beta\big)$ measures the amount of charge transported in the ground state across one fiducial line across the torus after threading one unit of flux in the torus – up to exponentially small errors in L, because the splittings we used are only unique up to exponentially small terms.

With these definitions, the proof of quantization of this quantity follows exactly the original argument of [14, Section IV.B]. We only recall the main steps and illustrate where the results of the previous sections yield improved bounds. The unitary

$$Z^\beta(\phi) = (U^\beta)^*\,\mathrm{e}^{\mathrm{i}\phi\overline{Q}_{\mathrm{R}}^\beta}\,U^\beta\,\mathrm{e}^{-\mathrm{i}\phi\overline{Q}_{\mathrm{R}}^\beta}$$

factorizes as $Z^\beta(\phi) = Z^\beta(\phi)_{\mathrm{left}}\,Z^\beta(\phi)_{\mathrm{right}}$ as in the discussion above. Note that the "−" and "+" in [14] are the analogues of "left" and "right" here, respectively. Since $\mathcal{I}_{H,\beta}$ is an almost inverse Liouvillian, Proposition 8, $Z^\beta(\phi)$ commutes with P (in operator norm) up to exponentially small errors in $\beta^{-2} = L$. By exponential clustering, Theorem 17, it follows that $[Z^\beta(\phi)_{\mathrm{left}}, P] \overset{\exp L}{=} 0$, where we use the notation $\overset{\exp L}{=}$ to indicate equality up to exponentially small errors in L. From there on, the argument runs without a change, but the errors are always exponential rather than superpolynomial, yielding that

$$P\,Z^\beta(\phi)_{\mathrm{left}}\,P \overset{\exp L}{=} \mathrm{e}^{\mathrm{i}\phi\left(P\left(T^\beta-((U^\beta)^*K^\beta_{\mathrm{left}}U^\beta-K^\beta_{\mathrm{left}})\right)P+\overline{Q}_{\mathrm{R}}^\beta\right)}\,\mathrm{e}^{\mathrm{i}\phi\overline{Q}_{\mathrm{R}}^\beta}\,P, \tag{35}$$

where $K^{\beta}_{\mathrm{left}} = \mathscr{I}_{H,\beta} \circ \big(\mathscr{L}_H(Q_{\mathrm{R}})_{\mathrm{left}}\big)$. At $\phi = 2\pi$, an independent argument yields that $\det_P Z^{\beta}(2\pi)_{\mathrm{left}} \overset{\exp L}{=} 1$, where the determinant is on the range of P. With this, the claim that $\mathrm{tr}\big(P\,T^{\beta}\big)$ is exponentially close to an integer follows from (35) by computing the trace of the exponent and using the unitary invariance of the trace.

Finally, if the sequence of states is convergent, then $p^{-1}\ \mathrm{tr}(P_L\,T_L)$ is convergent because T_L is a sufficiently local operator, see [12, Corollary 2.3]. Note that T_L is obtained with the exact inverse Liouvillian here. The Laughlin argument then implies that the limit of $p^{-1}\ \mathrm{tr}(P_L\,T_L)$ is equal to $2\,\pi\,\kappa$ where κ is the Hall conductance, see [12, Theorem 3.2]. By Lemma 10, we further have that $\|T_L^{\beta} - T_L\| \to 0$ as $L \to \infty$, again with $\beta^{-2} = L$. We conclude that $p^{-1}\ \mathrm{tr}\big(P_L(T_L^{\beta} - T_L)\big)$ converges to 0, and hence that $p^{-1}\ \mathrm{tr}(P_L\,T_L^{\beta})$ is convergent. In particular the sequence $\big(\frac{n_L}{p}\big)_L$ is eventually constant and equal to $2\pi\kappa$.

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Publication P10

Dynamics generated by spatially growing derivations on quasi-local algebras

Stefan Teufel * Marius Wesle * Tom Wessel *

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Abstract

We prove global existence and uniqueness of dynamics on the quasi-local algebra $\mathscr{A}$ of a quantum lattice system for spatially growing derivations $\mathscr{L}_\Phi = \sum_x [\Phi_x, \cdot]$. Existing results assume that the local terms $\Phi_x \in \mathscr{A}$ of the generator are uniformly bounded in space with respect to appropriate weighted norms $\|\Phi_x\|_{G,x}$. Analogous to the global existence result for first order ODEs, we show that global existence and uniqueness persist if the size of the local terms $\|\Phi_x\|_{G,x}$ grows at most linearly in space. This considerably enlarges the class of derivations known to have well-defined dynamics. Moreover, we obtain Lieb-Robinson bounds with exponential light cones for such dynamics.

For the proof, we assume Lieb-Robinson bounds with linear light cones for dynamics, whose generators have uniformly bounded local terms. Such bounds are known to hold, for example, if the local terms are of finite range or exponentially localized.

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* Fachbereich Mathematik, Universität Tübingen,
Auf der Morgenstelle 10, 72076 Tübingen, Germany

Contents

1 Introduction

In this work we consider interactions $\sum_{x\in\Gamma}\Phi_x$ defined on the CAR algebra $\mathscr{A}$ of lattice fermions on some discrete metric space (Γ, d) with D-dimensional volume growth (think of $\mathbb{Z}^D$ as the standard example), or on the quasi-local algebra of a spin system on Γ, for which the local terms $\Phi_x \in \mathscr{A}$ are not bounded uniformly, but instead satisfy a linear growth bound of the form

$$\|\Phi_x\|_{G,x} \leq C_\Phi \left(1 + d(x, x_0)\right) \quad \text{for all} \quad x \in \Gamma. \tag{1}$$

Here G is a fixed decay function, $\|\cdot\|_{G,x}$ a weighted norm centred at x that quantifies the decay around x, and x_0 a fixed point in Γ. We prove that if G decays fast enough such that interactions $\sum_{x\in\Gamma}\Psi_x$ with uniformly bounded $\|\cdot\|_{G,x}$-norms, i.e. with

$$|||\Psi|||_G := \sup_x \|\Psi_x\|_{G,x} < \infty,$$

satisfy a Lieb-Robinson bound with linear light cone and Lieb-Robinson velocity proportional to $|||\Psi|||_G$, then Φ generates a unique one-parameter group of automorphisms of $\mathscr{A}$ with exponential light cones. Previous results in this direction are known to us only for one-dimensional systems, where the existence of the dynamics for linearly growing generators with exponentially decaying terms can also be concluded from [49, Theorem 6.2.6] via bounds on the surface energy.

Before going into details, let us briefly sketch the underlying heuristic picture. For uniformly bounded interactions with $|||\Psi|||_G < \infty$, Lieb-Robinson bounds control the speed at which the Heisenberg dynamics generated by such an interaction effectively spreads the support of observables uniformly in space. If, for example, the local terms have uniform finite range or decay exponentially, then the support of any local observable can spread at most with speed $v_{\mathrm{LR}} \sim c_{\mathrm{LR}}\,|||\Psi|||_G$, the so called Lieb-Robinson velocity. Such Lieb-Robinson bounds first proved in finite volume can then be used to prove existence of the dynamics in infinite volume, see for example [174] and references therein. The situation is vaguely analogous to global existence of solutions to first order ODEs on $\mathbb{R}^D$. If the velocity field $v\colon \mathbb{R}^D \to \mathbb{R}^D$ is locally Lipschitz continuous and bounded, unique local solutions extend to unique global solutions, as integral curves can only travel finite distances in finite time. However, for first order ODEs the local Lipschitz condition, together with a linear upper bound, is sufficient to guarantee global existence, boundedness of the velocity field v is not needed. Even when the velocity field grows linearly in space, integral curves can not reach infinity in finite time, instead the distance to the starting point can grow at most exponentially in time. Our results establish a similar behaviour for the dynamics generated by interactions that satisfy (1) and may additionally be also time-dependent: We prove global existence and uniqueness of dynamics and exponential light cones for such interactions, see theorem 8.

While we consider our results interesting in their own right because they considerably extend the class of interactions known to generate global dynamics on $\mathscr{A}$, let us briefly mention the application that motivated our study of this question: Consider the Hamiltonian $H^B = \sum_x \Phi_x^B$ of a fermion system subject to a constant magnetic field B. While H^B is typically a bounded interaction, the derivative $\partial_B H^B$ of H^B with respect to B is an interaction with linearly growing local terms. And $\partial_B H^B$, or more precisely its image $\mathscr{I}(\partial_B H^B)$ under the quasi-local inverse $\mathscr{I}$ of the Liouvillian $\mathscr{L}_{H^B}$, is expected to generate the spectral flow for gapped ground states of H^B. So our result is the basis for showing that the spectral flow exists as a cocycle of locally generated automorphisms of $\mathscr{A}$ for gapped phases of matter with varying magnetic fields. We refer to [216] for a short discussion of this problem and to [167, 31] for the spectral flow of gapped ground states in infinite volume.

Finally, our result provides a class of automorphisms that are Fréchet continuous but not of Lieb-Robinson type in the sense of [24]. These automorphisms provide interesting examples for the second part of their Theorem 1.1. However, we provide bounds on the commutator in (5), suggesting that the notion of Lieb-Robinson type in [24] might be too restrictive.

Our paper is organized as follows. Section 2 presents the general setup, and section 3 states the precise assumptions and results. The proofs are given in section 4.

2 Mathematical setup

In the following we will denote by (Γ, d) a countable metric space that is D-regular, i.e. there is a constant C_{vol}, such that for all $x \in \Gamma$ and $r > 0$ we have

$$|B_r(x)| \leq C_{\mathrm{vol}}\,(1+r)^D, \quad \text{where} \quad B_r(x) := \{\, y \in \Gamma \,|\, d(y,x) \leq r \,\}$$

denotes the closed ball of radius r. Standard examples for Γ are $\mathbb{Z}^D$ or any other Delone set in $\mathbb{R}^D$ with the restriction of the Euclidean metric from $\mathbb{R}^D$.

The antisymmetric (or fermionic) Fock space over Γ with local space $\mathbb{C}^n$, $n \in \mathbb{N}$, is

$$\mathscr{F}(\Gamma, \mathbb{C}^n) := \bigoplus_{N=0}^{\infty} \ell^2(\Gamma, \mathbb{C}^n)^{\wedge N}.$$

We use $a^*_{x,i}$ and $a_{x,i}$ for $x \in \Gamma$, $i \in \{1, \dots, n\}$, to denote the fermionic creation and annihilation operators associated to the standard basis of $\ell^2(\Gamma, \mathbb{C}^n)$ and recall that they satisfy the canonical anti-commutation relations (CAR). The number operator at site $x \in \Gamma$ is defined by

$$n_x := \sum_{i=1}^{n} a^*_{x,i}\, a_{x,i}.$$

The algebra of all bounded operators on $\mathscr{F}(\Gamma, \mathbb{C}^n)$ is denoted by $\mathscr{B}(\mathscr{F}(\Gamma, \mathbb{C}^n))$. For each $M \subseteq \Gamma$ let $\mathscr{A}_M$ be the unital C*-subalgebra of $\mathscr{B}(\mathscr{F}(\Gamma, \mathbb{C}^n))$ generated by

$$\left\{ a^*_{x,i} \,\middle|\, x \in M,\ i \in \{1, \dots, n\} \right\}.$$

The C*-algebra $\mathscr{A} := \mathscr{A}_\Gamma$ is the CAR-algebra, which we also call the *quasi-local* algebra. We write $P_0(\Gamma) := \{ M \subseteq \Gamma \,|\, |M| < \infty \}$ and call

$$\mathscr{A}_{\mathrm{loc}} := \bigcup_{M \in P_0(\Gamma)} \mathscr{A}_M \subseteq \mathscr{A}$$

the *local* algebra, which is dense in $\mathscr{A}$. An operator is called quasi-local if it lies in $\mathscr{A}$ and local if it lies in $\mathscr{A}_{\mathrm{loc}}$. There is a unique automorphism[1] Θ of $\mathscr{A}$, such that

$$\Theta(a^*_{x,i}) = -\,a^*_{x,i}, \quad \text{for all} \quad x \in \Gamma \text{ and } i \in \{1, \dots, n\}.$$

One defines the set of even quasi-local operators

$$\mathscr{A}^+ := \{ A \in \mathscr{A} \,|\, \Theta(A) = A \}.$$

It is equal to the norm closure of the set of all linear combinations of products of an even number of annihilation or creation operators. We denote its part in $M \subseteq \Gamma$ by $\mathscr{A}^+_M := \mathscr{A}^+ \cap \mathscr{A}_M$. For disjoint regions $M_1, M_2 \subseteq \Gamma$, all operators $A \in \mathscr{A}^+_{M_1}$ and $B \in \mathscr{A}_{M_2}$ commute, $[A, B] = 0$.

Positive linear functionals of the quasi-local algebra $\omega\colon \mathscr{A} \to \mathbb{C}$ of norm 1 are called states. In order to define quantitative notions of localization for quasi-local operators, one makes use of the fact that one can localize operators to given regions by means of

[1] In the following the term *automorphism* is used in the sense of a $*$-automorphism as defined for example in [48].

the fermionic conditional expectation. To this end first note that $\mathscr{A}$ has a unique state ω^{tr} that satisfies

$$\omega^{\mathrm{tr}}(AB) = \omega^{\mathrm{tr}}(BA)$$

for all $A, B \in \mathscr{A}$, called the tracial state (e.g. [11, Definition 4.1, Remark 2]).

Proposition 1 ([11, Theorem 4.7], [216, Proposition 2.1])**.** *For each $M \subseteq \Gamma$ there exists a unique linear map*

$$\mathbb{E}_M \colon \mathscr{A} \to \mathscr{A}_M,$$

called the conditional expectation *with respect to ω^{tr}, such that*

$$\forall A \in \mathscr{A}\ \forall B \in \mathscr{A}_M : \quad \omega^{\mathrm{tr}}(AB) = \omega^{\mathrm{tr}}(\mathbb{E}_M(A)B). \tag{2}$$

It is unital, positive and has the properties

$$\begin{aligned} \forall M \subseteq \Gamma\ \forall A, C \in \mathscr{A}_M\ \forall B \in \mathscr{A} &: \quad \mathbb{E}_M(A\,B\,C) = A\,\mathbb{E}_M(B)\,C \\ \forall M_1, M_2 \subseteq \Gamma &: \quad \mathbb{E}_{M_1} \circ \mathbb{E}_{M_2} = \mathbb{E}_{M_1 \cap M_2} \\ \forall M \subseteq \Gamma &: \quad \mathbb{E}_M \mathscr{A}^+ \subseteq \mathscr{A}^+ \\ \forall M \subseteq \Gamma\ \forall A \subseteq \mathscr{A} &: \quad \|\mathbb{E}_M(A)\| \leq \|A\|. \end{aligned}$$

Remark 2. Note that [11, Theorem 4.7] discusses only the case of $\Gamma = \mathbb{Z}^D$. The proof however applies in the same way to our setting. ◇

Note that $\|(1 - \mathbb{E}_{B_r(x)})A\| \to 0$ as $r \to \infty$ for all $A \in \mathscr{A}$ by density of $\mathscr{A}^{\mathrm{loc}}$ in $\mathscr{A}$. We now introduce subspaces of $\mathscr{A}$ for which one can explicitly control the rate of convergence in this limit in terms of decay functions.

Definition 3. We call a bounded function $F \colon [0,\infty) \to (0,\infty)$ a *decay function* and define

$$\nu_F := \sup\Big\{ \nu \geq 0 \,\Big|\, \sup_{r \geq 0} F(r)\,(1+r)^{\nu} < \infty \Big\} \in [0,\infty) \cup \{\infty\}.$$

◇

Definition 4. Let F be a decay function. We say an observable $A \in \mathscr{A}$ is *F-localized* if for all $x \in \Gamma$ it holds that

$$\|A\|_{F,x} := \|A\| + \sup_{r \geq 0} \frac{\|(1 - \mathbb{E}_{B_r(x)})A\|}{F(r)} < \infty.$$

We denote the space of all F-localized observables with $\mathscr{A}_F$. For $\nu \geq 0$ and $F(r) := (1+r)^{-\nu}$ we abbreviate $\|\cdot\|_{\nu,x} := \|\cdot\|_{F,x}$ and $\mathscr{A}_\nu := \mathscr{A}_F$. ◇

We included $F \equiv 1$ in the class of decay functions, because then the quasi-local algebra $\mathscr{A}$ itself appears in the scale of spaces $\mathscr{A}_\nu$ at $\nu = 0$. More precisely, we have $\mathscr{A}_0 = \mathscr{A}$ and $\|A\|_{0,x} \leq 3\,\|A\|$. Also note that for decay functions F with exponential or slower decay and all $x_1, x_2 \in \Gamma$, the norms $\|\cdot\|_{F,x_1}$ and $\|\cdot\|_{F,x_2}$ are equivalent. Nevertheless, it is useful to define the family of norms with varying centre in order to express localization of observables more quantitatively [204].

Definition 5. Let $I \subseteq \mathbb{R}$ be an interval. A *time-dependent zero-chain on I* is a map

$$\Phi\colon I \times \Gamma \to \mathscr{A}^+,\ (t,x) \mapsto \Phi_x(t),$$

such that for all $(t,x) \in I \times \Gamma$ the operator $\Phi_x(t)$ is self-adjoint, for each $x \in \Gamma$, the map $I \to \mathscr{A}^+,\ t \mapsto \Phi_x(t)$ is norm-continuous and for each $t \in I$ and $A \in \mathscr{A}_{\mathrm{loc}}$ the sum

$$\mathscr{L}_{\Phi(t)}\, A := \sum_{x \in \Gamma} [\Phi_x(t), A]$$

converges unconditionally.

Let F be a decay function. A time-dependent zero-chain Φ on an interval I is *uniformly F-local* if

$$|\!|\!|\Phi|\!|\!|_F := \sup_{t \in I}\, \sup_{x \in \Gamma} \|\Phi_x(t)\|_{F,x} < \infty.$$

We denote the space of all uniformly F-local time-dependent zero-chains on I with $\mathscr{Z}_{F,I}$. ◇

In the analysis of quantum lattice systems, it is more common to specify the generators by so-called interactions, which associate a strictly local operator to each finite set $M \subseteq \Gamma$. While there is no canonical identification of the set of interactions with the set of zero-chains, there are several natural maps that preserve the associated derivation and decay-properties. For example for each $x \in \Gamma$ one can sum all terms of an interaction that are centred around x in a suitable sense to obtain a zero-chain and one can cut each quasi-local term of a zero-chain in a telescopic fashion via the conditional expectation to obtain an interaction. See, for example, [31] for more details on these procedures and references [129, 130] for the motivation behind the term “zero-chains”. Also note that the sets of derivations on $\mathscr{A}_{\mathrm{loc}}$ obtained from interactions and from zero-chains are exactly the same, namely the antisymmetric $*$-derivations from $\mathscr{A}_{\mathrm{loc}}$ to $\mathscr{A}$ that commute with the parity automorphism g_π. This can be seen as follows: Each such derivation is given by an interaction as is shown in [11]. Each interaction has an associated zero-chain with the same derivation, e.g. [31]. And from the definition above it is easy to see that every interaction coming from a zero-chain again satisfies the properties mentioned above. In this work, we use zero-chains, because they allow for a very clear characterization of linearly growing generators in assumption 1.

Finally, let us define what it means for a zero-chain to generate a cocycle of automorphisms.

Definition 6. Let $I \subseteq \mathbb{R}$ be an interval. A *cocycle of automorphisms on* I is a family $(\alpha_{s,t})_{s,t\in I}$ of automorphisms on $\mathscr{A}$, such that for all $s, t, u \in I$

$$\alpha_{s,t}\, \alpha_{t,u} = \alpha_{s,u}.$$

Let Φ be a time-dependent zero-chain on I. We say the cocycle of automorphisms $(\alpha_{s,t})_{s,t\in I}$ is *generated by* Φ if for all $s, t \in I$ and $A \in \mathscr{A}_{\mathrm{loc}}$ it holds that

$$\partial_t\, \alpha_{s,t}\, A = \alpha_{s,t}\, \mathrm{i}\, \mathscr{L}_{\Phi(t)}\, A. \qquad \diamond$$

3 Results

From now on we fix a time-dependent zero-chain Φ on an interval $I \subseteq \mathbb{R}$ and decay functions F, G with $\nu_F > 2D + 2$ and $\nu_G > D + 2$ (cf. definition 3). We then assume that the terms of Φ grow at most linearly.

Assumption 1. There is an $x_0 \in \Gamma$ and a constant $C_\Phi > 0$, such that

$$\sup_{t\in I} \|\Phi_x(t)\|_{G,x} \le C_\Phi \left(1 + d(x, x_0)\right) \quad \text{for all} \quad x \in \Gamma. \qquad \diamond$$

Moreover, we assume a Lieb-Robinson bound with a linear light cone for all *bounded* zero-chains that have the same decay as Φ.

Assumption 2. There exist constants C_{LR}, $c_{\mathrm{LR}} > 0$, such that for all time-dependent zero-chains $\Psi \in \mathscr{Z}_{G,I}$ with associated cocycle of automorphisms $(\alpha_{s,t})_{s,t\in I}$ and all $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y^+$, and $s, t \in I$ it holds that

$$\left\|\left[\alpha_{s,t}\, A,\, B\right]\right\| \le C_{\mathrm{LR}}\, \|A\|\, \|B\|\, |X|\, F\!\left(\left[d(X,Y) - c_{\mathrm{LR}}\, \|\!|\Psi|\!\|_G\, |t - s|\right]_+\right),$$

where $[x]_+ = x$ if $x \ge 0$ and $[x]_+ = 0$ if $x < 0$. $\diamond$

This assumption is in particular satisfied for exponential localization, with the decay functions $G(r) = \mathrm{e}^{-br}$ and $F(r) = \mathrm{e}^{-b'r}$ for some $b > b' > 0$. To see this, one constructs an associated interaction, which is exponentially decaying in the sense of [173] with the function $r \mapsto F(r)\,(1 + r)^{D+1+\varepsilon}$. The result then follows by [173, Theorem 3.1]. Moreover, we expect it to be satisfied for polynomial localization as well. Indeed, for spin systems with time-independent interactions and polynomial decay, linear light cones for large times have been shown in [149]. For Fermions, only Lieb-Robinson bounds with algebraic light cones, i.e. where the above bound holds with $d(X, Y)$ replaced by $d(X, Y)^\sigma$ for some $\sigma \in (0, 1)$, are known [P7]. In this case, our results hold if one assumes $\sup_{t\in I} \|\Phi_x(t)\|_{G,x} \le C_\Phi \left(1 + d(x, x_0)\right)^\sigma$ instead of assumption 1.

We will apply assumption 2 to approximations of Φ on finite subsets of Γ. For this purpose, for each $k \in [0, \infty)$ we define the time-dependent zero-chain Φ^k by

$$\Phi^k_x(t) := \begin{cases} \mathbb{E}_{B_{k/2}(x)} \Phi_x(t) & \text{for } x \in B_{k/2}(x_0), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

It is defined such that $\sum_{x\in\Gamma} \Phi^k_x$ is strictly localized in $B_k(x_0)$ and $\Phi^k \in \mathcal{Z}_{G,I}$ with $\|\Phi^k\|_G \leq C_\Phi (1 + \frac{k}{2})$. We denote the cocycle generated by Φ^k as $(\alpha^k_{s,t})_{s,t\in I}$.

We then obtain existence and uniqueness of the infinite volume dynamics for short times with an additional explicit Lieb-Robinson type estimate.

Theorem 7. *Let $\tau := 1/(4\, c_{\mathrm{LR}}\, C_\Phi)$. For all $s, t \in I$ with $|t - s| \leq \tau$ and $A \in \mathcal{A}$,*

$$\alpha_{s,t} A := \lim_{k\to\infty} \alpha^k_{s,t} A$$

exists in norm and the convergence is uniform in s and t. Moreover, for subintervals $I' \subset I$ with $|I'| \leq \tau$, $(\alpha_{s,t})_{s,t\in I'}$ is the unique cocycle of automorphisms generated by the restriction $\Phi|_{I'\times\Gamma} : I' \times \Gamma \to \mathcal{A}$, $(t, x) \mapsto \Phi_x(t)$ of the time-dependent zero-chain Φ to I'.

Setting $\mu := \min\big(\nu_F - (2D + 2), \nu_G - (D + 2)\big)$, it holds that for each $\nu \in (0, \mu)$, there exists $\gamma_\nu > 0$, such that for all $s, t \in I$ with $|t - s| \leq \tau$ and $A \in \mathcal{A}_\nu$ we have the bound

$$\|\alpha_{s,t} A\|_{\nu,x_0} \leq \gamma_\nu \|A\|_{\nu,x_0}. \tag{3}$$

In particular, it holds that $\alpha_{s,t} A \in \mathcal{A}_\nu$. The constant γ_ν does not depend on Φ.

While we excluded $\nu = 0$ in the bound (3), from the convergence and properties of the automorphisms $\alpha^k_{s,t}$, one immediately has $\|\alpha_{s,t} A\| \leq \|A\|$ for all $A \in \mathcal{A}$ and $s, t \in I$ with $|t - s| \leq \tau$. Let us also stress that the bound (3) is influenced by C_Φ as it only holds for $|t - s| \leq \tau = 1/(4\, c_{\mathrm{LR}}\, C_\Phi)$, even though γ_ν can be chosen uniformly for all Φ with the specified decay functions.

The idea of the proof is the following. For the part of A that is localized in $B_{k/8}(x_0)$, the restricted evolution $\alpha^k_{s,s+\delta t}$ with Lieb-Robinson velocity $v_{\mathrm{LR}} = c_{\mathrm{LR}}\, C_\Phi (1 + \frac{k}{2})$ is a good approximation of $\alpha_{s,s+\delta t}$ for large k, as long as the enlarged support $B_{k/8+v_{\mathrm{LR}}\,\delta t}(x_0)$ is far from the boundary of $B_{k/2}(x_0)$. And this is the case for $c_{\mathrm{LR}}\, C_\Phi\, \delta t \leq \frac{1}{4}$. The actual proof is technically more difficult, because the Lieb-Robinson velocity only captures the growth of the support of most of the observable and one has to estimate the tails carefully.

The short-time result can then be extended by concatenation to existence and uniqueness for all times and a Lieb-Robinson type estimate with an exponential light cone.

Theorem 8. *The time-dependent zero-chain Φ generates a unique cocycle of automorphisms $(\alpha_{s,t})_{s,t\in I}$. The cocycle $(\alpha^k_{s,t})_{s,t\in I}$ converges strongly to this cocycle as $k \to \infty$, in the sense that for all $A \in \mathcal{A}$ and $s, t \in I$ one has $\alpha^k_{s,t} A \to \alpha_{s,t} A$ as $k \to \infty$.*

Setting $\mu := \min\big(\nu_F - (2D+2), \nu_G - (D+2)\big)$, *it holds that for each* $\nu \in (0,\mu)$, *there are* $C_\nu > 0$, $\gamma_\nu > 0$ *that do not depend on* Φ, *such that for all* $s,t \in I$ *and* $A \in \mathscr{A}_\nu$ *we have the bound*

$$\|\alpha_{s,t}\, A\|_{\nu,x_0} \leq C_\nu\, \mathrm{e}^{\gamma_\nu\, C_\Phi\, |t-s|}\, \|A\|_{\nu,x_0}. \tag{4}$$

In particular, it holds that $\alpha_{s,t}\, A \in \mathscr{A}_\nu$.

As for the short-time result, we immediately obtain $\|\alpha_{s,t}\, A\| \leq \|A\|$ for all $A \in \mathscr{A}$ and $s,t \in I$.

Moreover, the bound (4) implies the usual commutator Lieb-Robinson bound with an exponential light cone: For all $k \geq 0$ and $Y \subset \Gamma$ with $B_k(x_0) \cap Y = \emptyset$ and all $A \in \mathscr{A}_{B_k(x_0)}$, $B \in \mathscr{A}_Y^+$, denoting $r = d\big(B_k(x_0), Y\big)$, one has for all $0 < c < 1$

$$\begin{aligned}
\|[\alpha_{s,t}\, A, B]\| &\leq 2\,\|(1 - \mathbb{E}_{B_{k+cr}(x_0)})\, \alpha_{s,t}\, A\|\, \|B\| + \|[\mathbb{E}_{B_{k+cr}(x_0)}\, \alpha_{s,t}\, A, B]\| \\
&\leq 2\,(1+k+cr)^{-\nu}\, \|\alpha_{s,t}\, A\|_{\nu,x_0}\, \|B\| \\
&\leq 2\,(1+k+cr)^{-\nu}\, (1+k)^\nu\, C_\nu\, \mathrm{e}^{\gamma_\nu\, C_\Phi\, |t-s|}\, \|A\|\, \|B\| \\
&\leq 2\, \|A\|\, \|B\|\, C_\nu\, \mathrm{e}^{\gamma_\nu\, C_\Phi\, |t-s| - \nu \ln(1+cr/(1+k))}
\end{aligned}$$

and therefore

$$\|[\alpha_{s,t}\, A, B]\| \leq 2\, \|A\|\, \|B\|\, C_\nu\, \mathrm{e}^{\gamma_\nu\, C_\Phi\, |t-s| - \nu \ln(1+r/(1+k))}. \tag{5}$$

This bound is referred to as a Lieb-Robinson bound with exponential light cone, since the right-hand side is small whenever

$$r \gg (1+k)\, \mathrm{e}^{C_\Phi\, \gamma_\nu\, \nu^{-1}\, |t-s|}.$$

4 Proofs

We provide the proof of theorems 7 and 8 in the following sections. Some technical lemmas, which are necessary for the proofs, are given in appendix A.

4.1 Existence and uniqueness for short times: Proof of theorem 7

We first show that for every $A \in \mathscr{A} = \mathscr{A}_0$ and all $s,t \in I$ with $|t-s| \leq \tau$ the sequence $(\alpha_{s,t}^k\, A)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{A}$ with respect to the operator norm. By completeness, it has a limit, which we denote $\alpha_{s,t}\, A$. And since this convergence is actually uniform in s and t, we can later conclude that $\alpha_{s,t}$ is the unique cocycle generated by Φ. To prove the estimate (3) for $\nu \in (0,\mu)$, we need a similar estimate for $\|\alpha_{s,t}^l\, A - \alpha_{s,t}^k\, A\|$ with explicit decay of the form $(1+k)^{-\nu}$ for all $l \geq k$ and $A \in \mathscr{A}_\nu$. To not do the same calculation twice, we treat all $\nu \in [0,\mu)$ at once.

Cauchy type estimate

Let $\nu \in [0,\mu)$, $A \in \mathscr{A}_\nu$ and $s, t \in I$ with $|t-s| \le \tau$, where, without loss of generality, we assume $s \le t$. For any $k, l \in [0,\infty)$, with $k \le l$ we find

$$\begin{aligned}
&\|\alpha^l_{s,t}\, A - \alpha^k_{s,t}\, A\| \\
&\quad \le \|(\alpha^l_{s,t} - \alpha^k_{s,t})\,(1-\mathbb{E}_{B_{k/8}(x_0)})\,A\| + \|(\alpha^l_{s,t} - \alpha^k_{s,t})\,\mathbb{E}_{B_{k/8}(x_0)}\,A\| \\
&\quad \le 2\,\|(1-\mathbb{E}_{B_{k/8}(x_0)})\,A\| + \int_s^t \mathrm{d}u\,\|\partial_u\,\alpha^l_{s,u}\,\alpha^k_{u,t}\,\mathbb{E}_{B_{k/8}(x_0)}\,A\| \\
&\quad \le 2\,\|(1-\mathbb{E}_{B_{k/8}(x_0)})\,A\| + \int_s^t \mathrm{d}u \sum_{x\in B_{l/2}(x_0)} \|[\Phi^l_x(u) - \Phi^k_x(u), \alpha^k_{u,t}\,\mathbb{E}_{B_{k/8}(x_0)}\,A]\| \\
&\quad \le 2\,\|(1-\mathbb{E}_{B_{k/8}(x_0)})\,A\| && \text{(6a)}\\
&\qquad + \int_s^t \mathrm{d}u \sum_{x\in B_{k/2}(x_0)} \|[(\mathbb{E}_{B_{l/2}(x)} - \mathbb{E}_{B_{k/2}(x)})\,\Phi_x(u), \alpha^k_{u,t}\,\mathbb{E}_{B_{k/8}(x_0)}\,A]\| && \text{(6b)}\\
&\qquad + \int_s^t \mathrm{d}u \sum_{x\in B_{l/2}(x_0)\setminus B_{k/2}(x_0)} \|[\mathbb{E}_{B_{l/2}(x)}\,(1-\mathbb{E}_{B_{d(x,x_0)/4}(x)})\,\Phi_x(u), \alpha^k_{u,t}\,\mathbb{E}_{B_{k/8}(x_0)}\,A]\| && \text{(6c)}\\
&\qquad + \int_s^t \mathrm{d}u \sum_{x\in B_{l/2}(x_0)\setminus B_{k/2}(x_0)} \|[\mathbb{E}_{B_{l/2}(x)}\,\mathbb{E}_{B_{d(x,x_0)/4}(x)}\,\Phi_x(u), \alpha^k_{u,t}\,\mathbb{E}_{B_{k/8}(x_0)}\,A]\|. && \text{(6d)}
\end{aligned}$$

We bound each of the four terms separately. For $\nu = 0$, the term (6a) converges to 0 because A is quasi-local, as explained before definition 3. For $\nu > 0$, we have $A \in \mathscr{A}_\nu$ and thus (6a) is bounded by

$$\text{(6a)} \le 2\,\frac{1}{(1+\frac{k}{8})^\nu}\,\|A\|_{\nu,x_0} \le 2\,\frac{8^\nu}{(1+k)^\nu}\,\|A\|_{\nu,x_0}.$$

The remaining estimates all work for $\nu \ge 0$. The second and third terms are bounded using only the decay of the quasi-local terms of Φ. In both cases we use the trivial bound for the commutator. The term (6b) is bounded by

$$\begin{aligned}
\text{(6b)} &\le 2\,\tau \sup_{u\in I} \sum_{x\in B_{k/2}(x_0)} \|\mathbb{E}_{B_{l/2}(x)}\,(1-\mathbb{E}_{B_{k/2}(x)})\,\Phi_x(u)\|\,\|A\| \\
&\le 2\,\tau \sup_{u\in I} \sum_{x\in B_{k/2}(x_0)} \|\Phi_x(u)\|_{G,x}\,G(k/2)\,\|A\| \\
&\le 2\,\tau \sum_{x\in B_{k/2}(x_0)} C_\Phi\,\big(1+d(x,x_0)\big)\,G(k/2)\,\|A\| \\
&\le 2\,\tau\,C_{\mathrm{vol}}\,C_\Phi\,\frac{(1+k/2)^{D+1+\nu+\varepsilon}\,G(k/2)}{(1+k/2)^{\nu+\varepsilon}}\,\|A\|
\end{aligned}$$

$$\leq \frac{C_{\mathrm{vol}}}{2\,c_{\mathrm{LR}}}\,\frac{2^{\nu+\varepsilon}\,C}{(1+k)^{\nu+\varepsilon}}\,\|A\|,$$

for some $C > 0$ and an $\varepsilon > 0$, such that $D+1+\nu+\varepsilon < \nu_G$. Here we used that $D+1+\nu+\varepsilon < \nu_G$ and therefore $k \mapsto (1+\frac{k}{2})^{D+1+\nu+\varepsilon}\,G(\frac{k}{2})$ is bounded. For (6c) we apply the same bounds to the commutator and then use the decay of G together with the volume-growth assumption to treat the infinite sum and obtain the upper bound

$$\begin{aligned}
(6c) &\leq 2\,\tau \sup_{u\in I} \sum_{x\in\Gamma\setminus B_{k/2}(x_0)} \|(1-\mathbb{E}_{B_{d(x,x_0)/4}(x)})\,\Phi_x(u)\|\,\|A\| \\
&\leq 2\,\tau \sum_{x\in\Gamma\setminus B_{k/2}(x_0)} C_\Phi\,\big(1+d(x,x_0)\big)\,G\big(d(x,x_0)/4\big)\,\|A\| \\
&\leq 2\,\tau\,C_\Phi \sup_{m\geq k/2}\,(1+m)^{D+2+\varepsilon}\,G(m/4) \sum_{x\in\Gamma} \frac{1}{\big(1+d(x,x_0)\big)^{D+1+\varepsilon}}\,\|A\| \\
&\leq \frac{1}{2\,c_{\mathrm{LR}}}\,\frac{C}{(1+k)^{\nu+\varepsilon}}\,\|A\|,
\end{aligned}$$

for some $C > 0$ and an $\varepsilon > 0$, such that $D+2+\nu+2\varepsilon < \nu_G$. The last sum converges due to lemma 9, and we used that the map $m \mapsto (1+m)^{D+2+\nu+2\varepsilon}\,G(\frac{m}{4})$ is bounded. To bound (6d) we apply the Lieb-Robinson bound from assumption 2 for the cocycle of automorphisms α^k, which is generated by the uniformly G-localized time-dependent zero-chain Φ^k. For this, we first note that

$$\begin{aligned}
d\big(B_{d(x,x_0)/4}(x), B_{k/8}(x_0)\big) &- c_{\mathrm{LR}}\,\|\Phi^k\|_G\,|t-s| \\
&\geq \tfrac{3}{4}\,d(x,x_0) - k/8 - c_{\mathrm{LR}}\,C_\Phi\,(1+k/2)\,\tau \\
&\geq \tfrac{3}{4}\,d(x,x_0) - k/8 - (1+k/2)/4 \\
&\geq \tfrac{3}{4}\,d(x,x_0) - (k+1)/4.
\end{aligned}$$

Then, (6d) is bounded by

$$\begin{aligned}
(6d) &\leq \tau\,C_{\mathrm{LR}}\,\|A\| \sum_{x\in\Gamma\setminus B_{k/2}(x_0)} \sup_{u\in I} \|\Phi_x(u)\|\,|B_{k/8}(x_0)|\,F\big(\big[\tfrac{3}{4}\,d(x,x_0) - \tfrac{k+1}{4}\big]_+\big) \\
&\leq \tau\,C_{\mathrm{LR}}\,\|A\| \sum_{x\in\Gamma\setminus B_{k/2}(x_0)} C_\Phi\,\big(1+d(x,x_0)\big)\,C_{\mathrm{vol}}\,(1+k/8)^D\,F\big(\big[\tfrac{3}{4}\,d(x,x_0) - \tfrac{k+1}{4}\big]_+\big) \\
&\leq \tau\,C_{\mathrm{LR}}\,C_\Phi\,C_{\mathrm{vol}}\,\|A\| \sum_{x\in\Gamma} \big(1+d(x,x_0)\big)^{-(D+1+\varepsilon)} \sup_{m\geq k/2} (1+m)^{2D+2+\varepsilon}\,F\big(\big[\tfrac{3}{4}\,m - \tfrac{k+1}{4}\big]_+\big) \\
&\leq \frac{C_{\mathrm{LR}}\,C_{\mathrm{vol}}}{4\,c_{\mathrm{LR}}}\,\frac{C}{(1+k)^{\nu+\varepsilon}}\,\|A\|,
\end{aligned}$$

for some $C > 0$ and an $\varepsilon > 0$, such that $2D + 2 + \nu + 2\varepsilon < \nu_F$. This time we used that the map $k \mapsto \sup_{m \geq k/2} (1+m)^{2D+2+\nu+2\varepsilon} F\big(\big[\frac{3}{4} m - \frac{k+1}{4}\big]_+\big)$ is bounded, which we show in lemma 10.

Combining the four bounds for $\nu = 0$ we have shown that for all $A \in \mathscr{A}$

$$\|\alpha^l_{s,t} A - \alpha^k_{s,t} A\| \to 0 \quad \text{uniformly for all } s, t \in I \text{ with } |t - s| \leq \tau.$$

And for $\nu \in (0, \mu)$, we have shown that there is a constant $\tilde{\gamma}_\nu > 0$, that does not depend on Φ, such that for all $A \in \mathscr{A}_\nu$ and $s, t \in I$ satisfying $|t - s| \leq \tau = 1/(4\, c_{\mathrm{LR}}\, C_\Phi)$ and all $l \geq k \in [0, \infty)$ it holds that

$$\|\alpha^l_{s,t} A - \alpha^k_{s,t} A\| \leq \frac{\tilde{\gamma}_\nu}{(1+k)^\nu} \|A\|_{\nu,x_0}. \tag{7}$$

Convergence

By the Cauchy estimate for $\nu = 0$, the sequence $(\alpha^k_{s,t} A)_{k\in\mathbb{N}_0}$ converges for all $A \in \mathscr{A}$, and we denote its limit by $\alpha_{s,t} A$. Moreover, this convergence is uniform for all $s, t \in I$ with $|t - s| \leq \tau$.

Cocycle and generator properties

Let $I' \subset I$ be a subinterval with $|I'| \leq \tau$. It is easy to see that $(\alpha_{s,t})_{s,t\in I'}$ is a strongly continuous cocycle of automorphisms on $\mathscr{A}$, since it inherits all the relevant properties from the approximations $(\alpha^k_{s,t})_{s,t\in I'}$.

To show that this cocycle is generated by the time-dependent zero-chain $\Phi|_{I'\times\Gamma}$, let $s, t \in I'$. Then, note that for all $h \in \mathbb{R}$ such that $t + h \in I'$ and all $A \in \mathscr{A}_{\mathrm{loc}}$, due to the strong continuity of $(\alpha^k_{s,t})_{s,t\in I'}$ and continuity of $u \mapsto \mathscr{L}_{\Phi^k(u)} A$, it holds that

$$\alpha^k_{s,t+h} A - \alpha^k_{s,t} A = \int_t^{t+h} \mathrm{d}u\, \alpha^k_{s,u}\, \mathrm{i}\, \mathscr{L}_{\Phi^k(u)} A.$$

Together with the uniform convergence and lemma 11 this gives us in the limit $k \to \infty$ that

$$\alpha_{s,t+h} A - \alpha_{s,t} A = \int_t^{t+h} \mathrm{d}u\, \alpha_{s,u}\, \mathrm{i}\, \mathscr{L}_{\Phi(u)} A.$$

By the strong continuity of $(\alpha_{s,t})_{s,t\in I'}$ and lemma 12, it follows that

$$\partial_t\, \alpha_{s,t} A = \alpha_{s,t}\, \mathrm{i}\, \mathscr{L}_{\Phi(t)} A.$$

Uniqueness

To show uniqueness, let $(\tilde{\alpha}_{s,t})_{s,t\in I'}$ be any cocycle of automorphisms, generated by $\Phi|_{I'\times\Gamma}$. We can show that $(\alpha^k_{s,t})_{s,t\in I'}$ also converges strongly to it, thereby showing that it must be identical to $(\alpha_{s,t})_{s,t\in I'}$. For this, let $k \in [0,\infty)$, $A \in \mathscr{A}$ and bound $\|\tilde{\alpha}_{s,t}\, A - \alpha^k_{s,t}\, A\|$ exactly as we did to arrive at the terms (6a)–(6d). This results in

$$\begin{aligned}
&\|\tilde{\alpha}_{s,t}\, A - \alpha^k_{s,t}\, A\| \\
&\quad\le \|(\tilde{\alpha}_{s,t} - \alpha^k_{s,t})\,(1 - \mathbb{E}_{B_{k/8}(x_0)})\, A\| + \|(\tilde{\alpha}_{s,t} - \alpha^k_{s,t})\,\mathbb{E}_{B_{k/8}(x_0)}\, A\| \\
&\quad\le 2\,\|(1 - \mathbb{E}_{B_{k/8}(x_0)})\, A\| + \int_s^t \mathrm{d}u\, \|\partial_u\, \tilde{\alpha}_{s,u}\, \alpha^k_{u,t}\, \mathbb{E}_{B_{k/8}(x_0)}\, A\| \\
&\quad\le 2\,\|(1 - \mathbb{E}_{B_{k/8}(x_0)})\, A\| + \int_s^t \mathrm{d}u \sum_{x\in\Gamma} \|[\Phi_x(u) - \Phi^k_x(u), \alpha^k_{u,t}\, \mathbb{E}_{B_{k/8}(x_0)}\, A]\| \\
&\quad\le 2\,\|(1 - \mathbb{E}_{B_{k/8}(x_0)})\, A\| \\
&\qquad + \int_s^t \mathrm{d}u \sum_{x\in B_{k/2}(x_0)} \|[(1 - \mathbb{E}_{B_{k/2}(x)})\, \Phi_x(u), \alpha^k_{u,t}\, \mathbb{E}_{B_{k/8}(x_0)}\, A]\| \\
&\qquad + \int_s^t \mathrm{d}u \sum_{x\in\Gamma\setminus B_{k/2}(x_0)} \|[(1 - \mathbb{E}_{B_{d(x,x_0)/4}(x)})\, \Phi_x(u), \alpha^k_{u,t}\, \mathbb{E}_{B_{k/8}(x_0)}\, A]\| \\
&\qquad + \int_s^t \mathrm{d}u \sum_{x\in\Gamma\setminus B_{k/2}(x_0)} \|[\mathbb{E}_{B_{d(x,x_0)/4}(x)}\, \Phi_x(u), \alpha^k_{u,t}\, \mathbb{E}_{B_{k/8}(x_0)}\, A]\|.
\end{aligned}$$

These four terms can be bounded by the exact same steps used to bound the previous four terms, thus showing that $\|\tilde{\alpha}_{s,t}\, A - \alpha^k_{s,t}\, A\| \to 0$ as $k \to \infty$. Hence, $\tilde{\alpha}_{s,t}\, A = \alpha_{s,t}\, A$ for all $s, t \in I'$.

Growth estimate

Clearly for all $A \in \mathscr{A}_\nu$ and $s, t \in I$ with $|t - s| \le \tau$, it holds that $\|\alpha_{s,t}\, A\| \le \|A\|$. And to prove (3) it is left to estimate the locality of $\alpha_{s,t}\, A$. For this, we bound

$$\begin{aligned}
&\|(1 - \mathbb{E}_{B_k(x_0)})\, \alpha_{s,t}\, A\| \\
&\quad= \|(1 - \mathbb{E}_{B_k(x_0)})\,(\alpha_{s,t} - \alpha^k_{s,t}\, \mathbb{E}_{B_k(x_0)})\, A\| \\
&\quad\le \|(1 - \mathbb{E}_{B_k(x_0)})\,(\alpha_{s,t} - \alpha^k_{s,t})\, A\| + \|(1 - \mathbb{E}_{B_k(x_0)})\, \alpha^k_{s,t}\,(1 - \mathbb{E}_{B_k(x_0)})\, A\| \\
&\quad\le 2\,\|(\alpha_{s,t} - \alpha^k_{s,t})\, A\| + \frac{2\,\|A\|_{\nu,x_0}}{(1+k)^\nu} \\
&\quad\le 2\,\frac{\tilde{\gamma}_\nu\,\|A\|_{\nu,x_0}}{(1+k)^\nu} + \frac{2\,\|A\|_{\nu,x_0}}{(1+k)^\nu},
\end{aligned}$$

for all $k \in [0,\infty)$ using (7) and locality of A. This proves that $\alpha_{s,t}\, \mathscr{A}_\nu \subset \mathscr{A}_\nu$ and

$$\|\alpha_{s,t}\, A\|_{\nu,x_0} \le 3\,(\tilde{\gamma}_\nu + 1)\,\|A\|_{\nu,x_0}.$$

4.2 Existence and uniqueness for all times: Proof of theorem 8

Next, we prove theorem 8 by lifting the results from theorem 7 to all times.

Convergence and cocycle and generator properties

Let $\tau = 1/(4\, c_{\mathrm{LR}}\, C_\Phi)$, let $s, t \in I$ and $A \in \mathscr{A}$. Without loss of generality we assume that $s \leq t$. We choose an $N \in \mathbb{N}_0$ and an increasing tuple $(t_i)_{i\in\{0,\dots,N\}}$ of elements of I, such that $t_0 = s$, $t_N = t$, and $t_{i+1} - t_i \leq \tau$. We know by theorem 7 that for all $i \in \{0, \dots, N-1\}$ the restriction $\Phi|_{[t_i,t_{i+1}]\times\Gamma}$ of Φ to $[t_i, t_{i+1}]$ generates a unique cocycle of automorphisms $(\alpha_{s,t})_{s,t\in[t_i,t_{i+1}]}$ that can be approximated in a strong sense by the cocycle $(\alpha^k_{s,t})_{s,t\in[t_i,t_{i+1}]}$. It holds that

$$\begin{aligned}
&\left\| \alpha^k_{s,t}\, A - \left(\prod_{i=0}^{N-1} \alpha_{t_i,t_{i+1}}\right) A \right\| \\
&\quad = \left\| \left(\prod_{i=0}^{N-1} \alpha^k_{t_i,t_{i+1}}\right) A - \left(\prod_{i=0}^{N-1} \alpha_{t_i,t_{i+1}}\right) A \right\| \\
&\quad \leq \sum_{j=0}^{N-1} \left\| \left(\prod_{i=0}^{j-1} \alpha^k_{t_i,t_{i+1}}\right) \left(\alpha^k_{t_j,t_{j+1}} - \alpha_{t_j,t_{j+1}}\right) \left(\prod_{i=j+1}^{N-1} \alpha_{t_i,t_{i+1}}\right) A \right\| \\
&\quad \leq \sum_{j=0}^{N-1} \left\| \left(\alpha^k_{t_j,t_{j+1}} - \alpha_{t_j,t_{j+1}}\right) \left(\prod_{i=j+1}^{N-1} \alpha_{t_i,t_{i+1}}\right) A \right\| \\
&\quad \to 0 \quad \text{as} \quad k \to \infty,
\end{aligned}$$

because $\left(\prod_{i=j+1}^{N-1} \alpha_{t_i,t_{i+1}}\right) A$ is a fixed element of $\mathscr{A}$ and $\alpha^k_{t_j,t_{j+1}} \to \alpha_{t_j,t_{j+1}}$ strongly on $\mathscr{A}$, by theorem 7. Therefore, $\alpha^k_{s,t}\, A$ converges to $\alpha_{s,t}\, A := \left(\prod_{i=0}^{N-1} \alpha_{t_i,t_{i+1}}\right) A$. In particular, $\alpha_{s,t}\, A$ is independent of the choice of intermediate times $(t_i)_{i\in\{0,\dots,N\}}$. It is now easy to see that $(\alpha_{s,t})_{s,t\in I}$ defines a cocycle of automorphisms and is generated by Φ, since it inherits all the relevant properties from the short time cocycles $(\alpha_{s,t})_{s,t\in[t_i,t_{i+1}]}$.

Uniqueness

For the uniqueness, let $(\tilde\alpha_{s,t})_{s,t\in I}$ be any cocycle generated by Φ. We can split it up in the same way as above

$$\tilde\alpha_{s,t}\, A = \left(\prod_{i=0}^{N-1} \tilde\alpha_{t_i,t_{i+1}}\right) A.$$

We observe that for each $i \in \{0, \dots, N-1\}$ the cocycle $(\tilde\alpha_{s,t})_{s,t\in[t_i,t_{i+1}]}$ is generated by $\Phi|_{[t_i,t_{i+1}]\times\Gamma}$. Together with the uniqueness statement of theorem 7, this lets us conclude that $(\tilde\alpha_{s,t})_{s,t\in I} = (\alpha_{s,t})_{s,t\in I}$.

Growth estimate

To obtain the bound, let $A \in \mathscr{A}_\nu$ for some $\nu \in (0,\mu)$ and choose the tuple from above as $t_i = s + i\,\tau$ for $i \in \{0,\dots,\lfloor\frac{t-s}{\tau}\rfloor\}$ and $t_{\lfloor\frac{t-s}{\tau}\rfloor+1} = t$. With the bound of theorem 7 we now find

$$\|\alpha_{s,t}\, A\|_{\nu,x_0} = \left\| \prod_{i=0}^{\lfloor\frac{t-s}{\tau}\rfloor} \alpha_{t_i,t_{i+1}}\, A \right\|_{\nu,x_0} \leq \gamma_\nu^{\lfloor\frac{t-s}{\tau}\rfloor+1} \|A\|_{\nu,x_0} \leq \gamma_\nu\, \mathrm{e}^{\frac{t-s}{\tau}\ln(\gamma_\nu)} \|A\|_{\nu,x_0}.$$

Replacing $1/\tau = 4\,c_{\mathrm{LR}}\,C_\Phi$ and recalling that γ_ν is independent of Φ, we obtain the estimate (4).

A Technical lemmas

In this section we refer to the decay functions F and G and the time-dependent zero-chains Φ and Φ^k as defined in section 3.

Lemma 9. *For all $\varepsilon > 0$ the sum*

$$\sum_{x\in\Gamma} \frac{1}{\big(1 + d(x,x_0)\big)^{D+1+\varepsilon}}$$

converges absolutely.

Proof. For $k \in \mathbb{N}$ we define $S_k := B_k(x_0) \setminus B_{k-1}(x_0)$ and $S_0 = B_0(x_0)$. Due to the volume growth property of (Γ, d) is holds that $|S_k| \leq C_{\mathrm{vol}}\,(1+k)^D$. From this we conclude

$$\begin{aligned}\sum_{x\in\Gamma} \frac{1}{\big(1 + d(x,x_0)\big)^{D+1+\varepsilon}} &\leq \sum_{k=1}^{\infty}\sum_{x\in S_k} \frac{1}{k^{D+1+\varepsilon}} + S_0 \leq \sum_{k=1}^{\infty} \frac{C_{\mathrm{vol}}\,(1+k)^D}{k^{D+1+\varepsilon}} + C_{\mathrm{vol}} \\ &\leq \sum_{k=1}^{\infty} \frac{C_{\mathrm{vol}}\,2^D}{k^{1+\varepsilon}} + C_{\mathrm{vol}} < \infty.\end{aligned}$$

□

Lemma 10. *Let $0 \leq \nu < \nu_F$. It holds that*

$$\sup_{k\geq 0}\ \sup_{m\geq k/2}\ (1+m)^\nu\, F\!\left(\left[\frac{3m}{4} - \frac{k+1}{4}\right]_+\right) < \infty.$$

Proof. Since the expression is bounded away from ∞ it is sufficient to consider the supremum for $k \geq 2$ and bound

$$\begin{aligned}
\sup_{k\geq 2} \sup_{m\geq k/2} &(1+m)^\nu F\Big(\frac{3m}{4} - \frac{k+1}{4}\Big) \\
&= \sup_{k\geq 2} \sup_{m\geq (k/6-1/3)} \Big(1+m+\frac{k+1}{3}\Big)^\nu F\Big(\frac{3m}{4}\Big) \\
&= \sup_{k\geq 0} \sup_{m\geq k/6} \Big(1+m+\frac{k+3}{3}\Big)^\nu F\Big(\frac{3m}{4}\Big) \\
&\leq \sup_{k\geq 0} \sup_{m\geq k/6} (2+3m)^\nu F\Big(\frac{3m}{4}\Big) \\
&\leq 4^\nu \sup_{m\geq 0} \Big(1+\frac{3m}{4}\Big)^\nu F\Big(\frac{3m}{4}\Big) \\
&< \infty,
\end{aligned}$$

where we substituted $m \to m + (k+1)/3$ and $k \to k+2$ in the second and third step, respectively. □

Lemma 11. *For all $A \in \mathscr{A}_{\mathrm{loc}}$ it holds that*

$$\sup_{t\in I} \| \mathscr{L}_{\Phi(t)} A - \mathscr{L}_{\Phi^k(t)} A \| \to 0 \quad as \quad k \to \infty.$$

Proof. Let $A \in \mathscr{A}_{\mathrm{loc}}$ and $k \in \mathbb{N}_0$. We have

$$\begin{aligned}
\sup_{t\in I} &\| \mathscr{L}_{\Phi(t)} A - \mathscr{L}_{\Phi^k(t)} A \| \\
&\leq \sup_{t\in I} \sum_{x\in B_{k/2}(x_0)} \big\| \big[(1-\mathbb{E}_{B_k(x)})\, \Phi_x(t), A\big] \big\| + \sup_{t\in I} \sum_{x\in \Gamma\setminus B_{k/2}(x_0)} \big\| \big[\Phi_x(t), A\big] \big\|.
\end{aligned}$$

The first term is bounded by

$$\sum_{x\in B_{k/2}(x_0)} \sup_{t\in I} 2\, \|\Phi_x(t)\|_{G,x}\, G(k)\, \|A\| \leq 2\, C_{\mathrm{vol}} \left(1+\tfrac{k}{2}\right)^D C_\Phi \left(1+\tfrac{k}{2}\right) G(k)\, \|A\|,$$

and since $D+1 < \nu_G$ this bound vanishes as $k \to \infty$. For the second term we assume that k is large enough so that A is supported in $B_{k/8}(x_0)$. This allows us to insert a conditional expectation and then bound the second term by

$$\begin{aligned}
&\sup_{t\in I} \sum_{x\in\Gamma\setminus B_{k/2}(x_0)} \big\|\big[\big(1-\mathbb{E}_{B_{d(x,x_0)/4}(x)}\big)\,\Phi_x(t), A\big]\big\| \\
&\quad\le \sum_{x\in\Gamma\setminus B_{k/2}(x_0)} \sup_{t\in I}\, 2\,\|\Phi_x(t)\|_{G,x}\, G\big(d(x,x_0)/4\big)\,\|A\| \\
&\quad\le 2 \sum_{x\in\Gamma\setminus B_{k/2}(x_0)} C_\Phi\,\big(1+d(x,x_0)\big)\, G\big(d(x,x_0)/4\big)\,\|A\| \\
&\quad\le \frac{2\,C_\Phi}{(1+k/2)^\varepsilon} \sup_{m\ge k/2}\, (1+m)^{D+2+2\varepsilon}\, G(m/4) \sum_{x\in\Gamma} \frac{1}{\big(1+d(x,x_0)\big)^{D+1+\varepsilon}}\,\|A\|,
\end{aligned}$$

for an $\varepsilon > 0$, such that $D+2+2\varepsilon < \nu_G$. The final sum converges as shown in lemma 9, the supremum is bounded and hence the expression converges to 0 as $k \to \infty$. □

Lemma 12. *For all $A \in \mathscr{A}_{\mathrm{loc}}$ the map $I \to \mathscr{A}$, $t \mapsto \mathscr{L}_{\Phi(t)} A$ is continuous.*

Proof. For all $k \in \mathbb{N}_0$ and all $A \in \mathscr{A}_{\mathrm{loc}}$, we know that the map $I \to \mathscr{A}$, $t \mapsto \mathscr{L}_{\Phi^k(t)} A$ is continuous, since each of the finitely many quasi-local terms is continuous. Together with the uniform convergence of lemma 11, this implies the claim. □

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