Tropical Quiver Theory and Tropical Enumeration on Ruled Surfaces

Dissertation

der Mathematisch-Naturwissenschaftlichen Fakultät

der Eberhard Karls Universität Tübingen

zur Erlangung des Grades eines

Doktors der Naturwissenschaften

(Dr. rer. nat.)

vorgelegt von

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Tübingen

2024

Gedruckt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Eberhard Karls Universität Tübingen.

Tag der mündlichen Qualifikation:	07.11.2024
Dekan:	Prof. Dr. Thilo Stehle
1. Berichterstatter/-in:	Prof. Dr. Hannah Markwig
2. Berichterstatter/-in:	Prof. Dr. Alex Fink

Abstract. In this thesis we investigate problems in linear and enumerative tropical geometry.

In the realm of linear tropical geometry, we introduce two notions of linear maps, and compare them to two preexisting notions of linear maps in valuated matroid theory. We further establish and prove many new properties of and connections between all four notions. We show that the category of valuated matroids with affine morphisms of valuated matroids has similar properties to the category of matroids with strong maps.

We use the uncovered relations to establish a new research program in tropical geometry, *tropical quiver theory*. This research program aims to bridge the gap between representation theory and linear tropical geometry by connecting the two areas of research through the study of matroidal quiver representations and their parameter spaces. We establish two tropical analogues of quiver representations and quiver Grassmannians — *matroidal quiver representations* and *valuated matroidal quiver representations*, as well as *tropicalized quiver Grassmannians* and *quiver Dressians*. We show their correspondence to classical quiver representations and quiver Grassmannians. We then study when the two tropical analogues of quiver Grassmannians coincide. We apply the new theory explicitly to study linear degenerations of tropical flag varieties, which aids in the pursuit of computing tropical flag varieties. Finally, we start on the quest of finding a polyhedral analogue of this theory, and give a conjecture of the structure of polytopes associated to matroidal quiver representations.

In the realm of enumerative tropical geometry, we tackle a curve counting problem on ruled surfaces. We expand the existing theory by developing tools that allow us to count curves in non-orientable surfaces. For two of these surfaces, we establish a tropical curve count and prove its correspondence to the algebraic-geometric Gromov-Witten invariant. Further, we prove regularity results: we show that the functions returning the Gromov-Witten invariants depending on the tangency conditions are quasi-polynomial, and that the generating series in the vertical part of the bidegree is quasi-modular.

Deutsche Zusammenfassung. In dieser Dissertation beschäftigen wir uns mit verschiedenen Problemen in der linearen und enumerativen tropischen Geometrie.

In der linearen tropischen Geometrie betrachten wir zuerst linearen Abbildungen. Wir führen zwei neue Alternativen für lineare Abbildungen ein und vergleichen diese mit zwei bekannten Definitionen. Wir folgern neue Eigenschaften aller vier alternativen Definitionen. Insbesondere zeigen wir, dass die Kategorie der bewerteten Matroide mit affinen Morphismen bewerteter Matroide sehr ähnlich zur Kategorie der Matroide mit starken Abbildungen ist.

Anschließend verwenden wir die neuen Definitionen und Eigenschaften um ein neues Forschungsprogramm aufzubauen, die *tropische Köchertheorie*. Das Ziel dieses Programms ist es, Ergebnisse aus der Darstellungstheorie und der lineare tropischen Geometrie durch die Forschung an tropischen Köcherdarstellungen und tropischen Köchergrassmannschen zu vereinen, um neue Einsichten in beiden Forschungsgebieten zu erhalten. Wir etablieren jeweils zwei analoge tropische Objekte für Köcherdarstellungen und Köchergrassmannsche — *matroidale* und *bewertete matroidale Köcherdarstellungen*, sowie *tropikalisierte Köchergrassmannsche* und *Köcherdresssche*, und untersuchen die Korrespondenz verschiedener Charakterisierungen. Wir vergleichen die Konzepte und bestimmen, wann sich die tropikalisierte Köchergrassmannsche und die Köcherdresssche unterscheiden. Wir verwenden unsere Resultate zur Konstruktion und Analyse von linear degenerierten tropischen Fahnenvarietäten. Weiterhin beginnen wir mit der Einführung einer analogen polyedrischen Theorie.

In der enumerativen tropischen Geometrie untersuchen wir ein Zählproblem von Kurven auf Regelflächen mithilfe von tropischer Geometrie. Wir konstruieren neue kombinatorische Werkzeuge, die es uns erlauben die existierende Theorie zu erweitern um Kurven auf nichtorientierbaren Flächen zu zählen. Wir wenden unsere neuen Methoden auf zwei solcher Flächen an und beweisen einen Korrespondenzsatz, der besagt dass die tropische Zählung mit der algebraisch-geometrischen Zählung übereinstimmt. Weiterhin zeigen wir Regularitätsaussagen: die Funktion, die Tangentenbedingungen ihre korrespondierende Gromov-Witten-Invariante zuweist, ist quasipolynomiell, und die Erzeugendenfunktion im der vertikalen Komponente des Bigrades ist quasimodular.

Contents

1. Introduction

1.1. Tropical geometry.

Tropical geometry is a flavor of combinatorial algebraic geometry. At its core, it is a field that studies objects and properties in algebraic geometry using combinatorics, mainly using methods from polyhedral geometry, matroid theory and graph theory. The translation between an algebraic object and a combinatorial one happens via *tropicalization*. This process assigns to an algebraic variety a polyhedral complex which carries many properties of the variety.

The draw behind the approach is that it is often significantly easier to understand the tropicalization of a variety than it is to understand the algebraic variety itself. Naturally, there is an information loss when passing to the tropicalization of a variety, but often, the tropical data turns out to be enough to solve problems previously unsolvable in algebraic geometry.

There are multiple different approaches to study tropical geometry. Originally, tropicalizations arise when performing logarithmic degeneration on an algebraic variety. Applying the logarithm to the absolute value of points in the variety, we obtain *amoebas* [66, 104, 105], which already have many of the features of tropical varieties, though they are not yet combinatorial. Their limit obtained by varying the base of the logarithm is the tropical variety.

The more common approach to modern tropical geometry is to instead consider it as algebraic geometry over the tropical semifield $\mathbb{T} = \mathbb{R} \cup \{\infty\}$, where we replace addition by taking the minimum and multiplication by standard addition. The operations on this field mimic the behavior of exponents under applying the logarithm and hence get preserved under logarithmic degeneration. This allows for a more axiomatic approach.

Sometimes, for more sophisticated algebraic spaces, tropicalization is not defined. However, it is often possible to instead find a *tropical analogue* which mimics enough properties of a tropicalization so that tropical methods can be adapted to it.

Historically, tropical geometry has been particularly useful in enumerative geometry. Through Mikhalkin's correspondence theorem [106] (and variations of it), many enumerative questions in algebraic geometry can be solved by just counting tropical curves satisfying the same properties with appropriate multiplicities. We elaborate more on this in Section 1.4 and in Part 3.

On the other hand, sometimes tropical geometry allows us to understand combinatorial objects better by understanding the algebraic varieties they model. A particularly interesting example of this is linear tropical geometry. Instead of studying the tropicalization of the Grassmannian, which parametrizes tropicalizations of linear subspaces, we can choose a different tropical analogue. The *Dressian*, arising as the tropical prevariety of the Plücker equations, parametrizes valuated matroids. Further, valuated matroids can be interpreted as polyhedral complexes, called *tropical linear spaces*. These are not necessarily tropicalizations of linear spaces, but carry many similar properties.

In this thesis, we establish results in both of these areas. We introduce different notions of linear maps between tropical linear spaces and study their properties (see Part 1). Afterwards, we introduce tropical prevarieties that serve as parameter spaces for arrangements of tropical linear spaces that can be described by linear maps (see Part 2). Finally, we extend tropical enumerative methods to allow for the counting of curves in non-orientable surfaces (see Part 3).

1.2. **Maps of matroids.** Matroids are foundational objects in combinatorics, generalizing the concept of independence in various different contexts, such as linear algebra, lattice theory and graph theory. Further, matroids have analogues in polyhedral geometry.

These combinatorial objects have a rich history, ranging from Whitney's original definition [140] in the 1930s, via Tutte's work on chromatic polynomials [121] and relations to graph theory [134] in the fifties and sixties to today.

Matroid theory is a vibrant subfield of combinatorics as of today. Recently, it has gained prominence through the proof of Rota's conjecture in matroid theory via algebraic geometry in the work of Adiprasito-Katz-Huh [3, 4], a contribution for which June Huh was awarded his recent Fields medal.

The applications of matroids are numerous and span many different areas of science and mathematics. For instance, matroids can be used in economics to model logistics processes and optimize them through Greedy algorithms, which terminate optimally on matroids. Further, matroids can be used in particle physics. Matroids whose strata in the Grassmannian intersect the totally nonnegative Grassmannian nontrivially, can be used to describe the combinatorics of soliton solutions to special wave equations [93]. Subclasses of matroids and otherwise enriched analogues can be used to model many further processes.

One of these enrichments, which we will focus on in this thesis, is that of a valuation function on matroids. These combinatorial objects coincide with tropical linear spaces in tropical geometry. They are foundational objects which are of intrinsic importance for the area of tropical geometry [131, 132]. Here, they appear as the building blocks for tropical manifolds and tropical ideals [100], and parametrize hyperplane arrangements. Further, they have applications in computational biology as the parameter spaces of phylogenetic trees, [50, 101], and in mathematical physics, describing different combinatorial types of scattering amplitudes [9, 10].

While linear maps between linear spaces are covered in first-year linear algebra classes and are subsequently of great importance to many areas of mathematics, their matroidal counterparts are significantly harder to study. There are different notions of linear maps between matroids, *bimatroids, strong maps, and weak maps*, mainly going back to work of Kung [95, 96, 97] in the seventies and eighties. However, they have not nearly been as well-studied as their linear algebra counterparts. Nevertheless, there exist some works on their structure, especially focused on the structure of strong maps [79]. In particular, all of Kung's constructions have advantages and drawbacks, as none of them capture all the properties linear maps in linear algebra satisfy.

In this thesis, we focus on an extension of Kung's theory into linear tropical geometry. We study four different notions of maps between tropical linear spaces, two of which we also introduce. Further, we study the relations these maps between tropical linear spaces induce on their associated valuated matroids. In particular, we study

- ◦ morphisms of valuated matroids (Section 4.1),
- ◦ affine morphisms of valuated matroids (Section 4.2),
- ◦ naive tropical matrix multiplication (Section 5), and
- ◦ valuated bimatroids (Section 6).

Morphisms of valuated matroids and their affine extensions are the most faithful adaptation of Kung's *strong maps* of matroids [96], whereas valuated bimatroids are a valuated version of Kung's *bimatroids* [95]. Naive tropical matrix multiplication is inspired by the linear algebra interpretation of linear maps as matrices.

This thesis shows major new properties for all discussed concepts, and establishes new connections between them. The particular contributions to each of the different notions are listed below.

(Affine) morphisms of valuated matroids. While Brandt-Eur-Zhang [30] mention the possibility of defining morphisms of valuated matroids in a similar fashion to our definition (see Definition 4.3), we were the first to prove that this construction is well-defined (see Proposition 4.1).

We observe that morphisms of valuated matroids can not describe all possible linear maps, as they cannot describe translations of tropical linear spaces. We enrich the definition to that of affine morphisms of valuated matroids (see Definition 4.7). We prove that this notion is well-defined and that it is compatible with strong maps of the underlying matroids: Any (affine) morphism of valuated matroids induces a strong map of the underlying matroids (see Lemma 4.4 and 4.10).

Further, we show that (affine) morphisms of valuated matroids are "as good as" their unvaluated counterparts — the category of valuated matroids with morphisms given by (affine) morphisms of valuated matroids has similar properties to the category of matroids with morphisms given by strong maps, as studied by Heunen-Patta [79]:

Theorem A. *The category* $\mathbf{VMatr}_a$ *of valuated matroids with affine morphisms of valuated matroids has similar properties to the category* $\mathbf{Matr}_o$ *of matroids with strong maps. These are the following:*
- *The category* $\mathbf{VMatr}_a$ *has coproducts (see Proposition 4.13);*
- *The category* $\mathbf{VMatr}_a$ *does not have products (see Proposition 4.14);*
- *Direct sum and deletion of valuated matroids are functorial operations in* $\mathbf{VMatr}_a$ *(see Proposition 4.17);*
- *Taking the dual of a valuated matroid is not a functorial operation in* $\mathbf{VMatr}_a$ *(see Proposition 4.18); and*
- *The category* $\mathbf{VMatr}_a$ *has a zero object (see Proposition 4.19).*

Naive tropical matrix multiplication is the most encompassing of all the notions of maps between linear spaces. However, the naive product of a tropical linear space with a tropical matrix is not necessarily a tropical linear space anymore. Thus, this operation is not well-defined on the level of valuated matroids. Nevertheless, the resulting sets have some structure which can help in understanding matrix multiplication as a tropical linear map. For instance, we show that they are tropically convex sets (see Proposition 5.3). Attempts to fix the resulting structure have recently been made by Mundinger [109], using work on modifications of tropical linear spaces by Shaw [129]. In our discussion, naive tropical matrix multiplication mainly comes in as a reference point for comparison of other notions of morphisms. In particular, we establish a connection to (affine) maps of valuated matroids, by proving the following theorem:

Theorem B (Proposition 5.18)**.** *Let* μ *be a valuated matroid on* $[m]$ *and* f *a map of sets into* $[m]$*. The induced matroid* $f^{-1}(\mu)$ *of* μ *under* f *can be expressed via tropical matrix multiplication:*

$$\overline{\mathrm{trop}}\big(f^{-1}(\mu)\big) = \mathrm{val}(A_f) \odot \overline{\mathrm{trop}}(\mu).$$

This is an extension of my joint work with Giulia Iezzi [82], where we prove the same result for the case $m = n$, i.e., for square matrices.

Valuated bimatroids. Finally, the idea behind bimatroids [95] is to use the interpretation of linear maps of linear spaces as linear spaces themselves. Bimatroids are matroids on a larger ground set. Ordinary matroids only take into account the independence relations between columns of a matrix. Instead, bimatroids capture the independence relations between both rows and columns after normalization of the matrix. On the tropical side, valuated bimatroids have been introduced by Murota [110] (see Definition 6.2). We construct a different equivalent characterization of valuated bimatroids.

Bimatroids are equivalent to *linking systems* defined by Schrijver [128], which are the more natural interpretation when trying to describe discrete optimization problems. *Valuated linking systems* have been studied by Frenk [60]. Just like bimatroids and linking systems are equivalent, his notion is equivalent to ours.

We express *Stiefel tropical linear spaces* [58] as valuated bimatroids (see Example 6.7). Further, we prove some of their basic properties, in particular, we define products of valuated bimatroids as a tropical analogue of the Cauchy-Binet formula. These products are again valuated bimatroids due to Frenk ([60]). Further, we define a partial ordering on valuated bimatroids of the same size. We then establish some properties of the product and the ordering:

Theorem C.
- *Square valuated bimatroids form a monoid (see Proposition 6.15).*
- *The realizable bimatroid obtained from the matrix product of two matrices is greater than or equal to the bimatroid product of the bimatroids obtained from the matrices (see Proposition 6.17).*
- *The rank of the product* $\mathsf{A} \cdot \mathsf{B}$ *can be expressed in terms of the ranks of* A *and* B *(see Proposition 6.18).*

We conclude by conjecturing the compatibility of valuated bimatroids and morphisms of valuated matroids, and point out a potential avenue towards a proof of that compatibility.

To be able to do computational experiments, we provide implementations for morphisms of matroids and strong maps (see Appendix B). Our code checks whether a given map of sets is a morphism of matroids (or a strong map), and automatically generates all morphisms of matroids between two matroids.

Subsets of this part of the thesis are based on joint work with Alessio Borzì [27], in particular, the definition of morphisms of valuated matroids and their well-definedness. Further, the basic definition of affine morphisms of valuated matroids and their connections to matrix multiplication is contained in joint work with Giulia Iezzi [82]. All work related to valuated bimatroids is part of ongoing work-in-progress with Jeffrey Giansiracusa, Felipe Rincón and Martin Ulirsch. All categorical statements and proofs, as well as the connections to strong maps of the underlying matroids and the computational implementations of these, are my own independent work.

1.3. **Quivers in tropical geometry.** Now that we understand different notions of linear maps between tropical linear spaces, we want to apply this new knowledge to algebraic geometry. It turns out that many interesting spaces, in fact all projective varieties, can be described as the parameter spaces of linear spaces contained inside of each other after application of a linear map [122]. To study these spaces tropically, we now want to study parameter spaces of tropical linear spaces contained inside of each other after applying a tropical linear map in the sense of the previous section. From a more combinatorial point of view, this new theory also allows us to parametrize different interesting arrangements of tropical linear spaces.

The algebraic framework we use to describe such parameter spaces is that of quiver representation theory. A priori, a quiver is just a finite directed graph Q, with vertices V and arrows A. We may endow it with algebraic data by defining a Q-representation R, which is an assignment of vector spaces V_i to the vertices $i \in V$.

Quiver representations have been an object of study in representation theory, as they allow for combinatorial models used for understanding the representation theory of Lie groups. In this thesis, we will focus on the parameter spaces of the arrangements of linear spaces that quiver representations can be used to describe.

Formally, we can describe the arrangements of linear spaces associated to a quiver representation as follows. Each arrow α is assigned a linear map f_α and each vertex i is assigned a vector subspace $U_i \subseteq V_i$ of fixed dimension. Such an assignment of vector spaces is called a *quiver subrepresentation* if $f_\alpha(U_i) \subseteq U_j$ for all α with source U_i and target U_j.

Quiver Grassmannians are projective varieties parametrizing such subrepresentations. They first appeared in [48, 127] and have since been extensively studied. They have been employed in cluster algebra theory [35] as well as for studying linear degenerations of the flag variety [41, 42, 56, 57]. Notably, every projective variety is isomorphic to a quiver Grassmannian [122, 124], and every quiver Grassmannian can be embedded in a product of Grassmannians.

As a set, quiver Grassmannians can be described as the zero set of modified Plücker relations, given in [98]. As is the case for ordinary Grassmannians [132] and flag varieties [30, 76, Theorem A and Theorem 1 respectively], there are two reasonable tropical analogues of quiver Grassmannians. Since they are projective varieties, one can simply tropicalize them. We show that the resulting tropical variety, called the *tropicalized quiver Grassmannian* (see Definition 9.1) parametrizes tropicalizations of quiver subrepresentations, and is compatible with valuated matroidal quiver representations:

Theorem D (Theorem 9.3)**.** *Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)$ be realizable valuated matroids and R be a quiver representation. The following statements are equivalent:*

(a) $\boldsymbol{\mu}$ is a point in the tropicalized quiver Grassmannian of R
(b) $\boldsymbol{\mu}$ is the valuation of a Plücker vector of a quiver subrepresentation.

If further, all maps of the quiver are sufficiently simple, this is equivalent to

(c) $\boldsymbol{\mu}$ is a valuated matroidal quiver subrepresentation where all morphisms are simultaneously realizable.

On the other hand, we can simply tropicalize the Plücker relations generating the variety and take the intersections of their hypersurfaces. This way we obtain a tropical prevariety called the *quiver Dressian* (see Definition 9.5). A priori, it is unclear that this construction yields a reasonable result. However, we show that this polyhedral complex parametrizes tropical linear spaces contained inside of each other after naive tropical matrix multiplication. Combined with the results from the previous section, this allows us to characterize some quiver Dressians in purely matroidal terms, using morphisms of valuated matroids:

Theorem E (Theorem 9.6)**.** *Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ be valuated matroids, Q be a finite quiver, and R be a quiver representation.*

(a) $\boldsymbol{\mu}$ *is a point in the quiver Dressian of R*
(b) $\mathrm{val}(A) \odot \overline{\mathrm{trop}}(\mu_i) \subseteq \overline{\mathrm{trop}}(\mu_j)$ *for each matrix A associated to an arrow between μ_i and μ_j.*

If further, all maps associated to arrows are sufficiently simple, the above are equivalent to

(c) $\boldsymbol{\mu}$ *is a valuated matroidal quiver subrepresentation.*

Since we have two different tropical analogues, we wish to understand when they coincide. While tropical Grassmannians and Dressians coincide up to ambient dimension 7 and tropical flag varieties coincide with flag Dressians up to ambient dimension 5 [30, Theorem 5.2.1], quiver Dressians behave less well. Even in ambient dimension 2 we can find a quiver whose tropicalized quiver Grassmannian and quiver Dressian differ:

Theorem F (Theorem 9.8)**.** *There exists a quiver Q with representations R for every dimension $n \geq 2$ such that the tropicalized quiver Grassmannian and the quiver Dressian differ.*

Aiming to find a more satisfying answer to this question, we give a conjectural relation between the structure of the quiver and the potential for the associated tropical analogues to be equal:

Conjecture G (Conjecture 9.1)**.** *Let Q be a finite quiver and let $n \geq 2$. If the quiver Dressian and the quiver Grassmannian coincide for all Q-representations R, then the underlying quiver Q has a special structure: it has finite isomorphism classes, i.e., each connected component is a Dynkin quiver of type A, D or E (see Example 7.7).*

The flag variety is a particularly important quiver Grassmannian, and has thus been studied extensively using both classical methods [28, 40, 43], and tropical methods [14, 30]. It can be realized as the quiver Grassmannian of an A-type quiver, see below, where all maps on the arrows are chosen to be identity maps.

$$K^n \xrightarrow{id} K^n \xrightarrow{id} \ldots \xrightarrow{id} K^n \xrightarrow{id} K^n$$

On the tropical side, however, tropical flag varieties and flag Dressians get extremely computationally expensive very quickly. Due to the fact that the tropical flag variety and the flag Dressian coincide for complete flags until ambient dimension 5, the case of ambient dimension 6 would be a particularly interesting example to compute. However, most computers are incapable of completing such a task within a reasonable amount of time. Further, while there are some partial computational results for ambient dimension 6 [14], there are no further results for higher ambient dimension (nor are they expected, due to the complexity of the dimension 6 case).

Thus, we try to understand tropical flag varieties and flag Dressians for higher ambient dimension using the quiver structure of the flag variety. Instead of assigning identity maps to the arrows, we assign projection maps. On the algebraic side, this process yields a *linear degeneration* of the flag variety, which has been studied for instance in [28, 40, 41]. On the tropical side, this leads to a less complicated tropical structure that nevertheless captures partial information of the tropical flag variety and flag Dressian (see Section 10). We obtain first results for the structure of the associated quiver Dressians and study the observed connections in more detail for some explicit examples.

Many matroidal concepts have an equivalent characterization in terms of specific polytopes and their properties. For instance, every matroid can be equivalently characterized as a *matroid polytope*, and every valuated matroid can be characterized as a subdivision of the matroid polytope of the underlying matroid. Naturally, we aim to characterize quivers in terms of associated polyhedral constructions. While we are not there yet, we provide a conjectural polyhedral structure:

Conjecture H (Conjecture 11.1)**.** *Let $Q = (V, A, s, t)$ be a finite quiver without directed cycles, and let R be a matroidal Q-representation. A point configuration $\mathscr{P}$ is the quiver point configuration of a matroidal Q-subrepresentation (c.f. Definition 11.5) if and only if the convex hull of $\mathscr{P}$ is a generalized permutohedron and any point $p \in \mathscr{P}$ is in the orbit of integer vectors determined by the structure of the quiver.*

We obtain this conjecture by computationally analyzing multiflag quivers. Further, we analyze the behavior that occurs for quivers containing cycles by studying the loop quiver.

The structural results for linear degenerate flag varieties and quiver Dressians were obtained in joint works with Alessio Borzì [27] and Giulia Iezzi [82] respectively. The work on quiver polytopes is my own, albeit inspired by various conversations with both of my coauthors.

1.4. **Enumerations of curves in ruled surfaces.** Counting objects satisfying different constraints is a classical problem in mathematics. In algebraic geometry, the main object of interest one aims to count is curves in different surfaces. Here, one of the major developments of the last century were discoveries by Witten [141], relating intersection numbers on the moduli space of curves (corresponding to curve counts) to the KdV-equation in mathematical physics. Five years later, Kontsevich proved Witten's conjecture [94], leading to a purely combinatorial count of curves in $\mathbb{P}^2$.

One of the first major successes of tropical geometry was its utility for systematically and combinatorially counting curves in $\mathbb{P}^2$, and to extend approaches by Kontsevich to more complicated surfaces on which one wishes to count curves. The idea here is to count tropical curves, arising as degenerations of algebraic curves, and use the combinatorial curve counting results on the tropical side to infer counts of algebraic curves. The main ingredient for the translation of tropical curve counts to algebraic ones is Mikhalkin's correspondence theorem [106], which shows that counting tropical curves with certain multiplicity is equivalent to counting algebraic curves.

Shortly afterwards, Gathmann-Markwig adapted the lattice path algorithm introduced by Mikhalkin to count tropical curves of arbitrary genus [64] passing through $3d + g - 1$ points in general position and showed a tropical version of Kontsevich's formula for the number of rational curves of degree d passing through n points [65]. Through the translation by Mikhalkin's correspondence theorem, this allowed for a combinatorially driven curve count.

Since then, tropical methods have been employed in enumerative geometry with great success. One reason for that is that tropical geometry is independent of the characteristic of the field K, where the curve one wants to tropicalize is defined over. Thus, counting curves tropically allows giving a combinatorial baseline for a curve count for fields over many different characteristics.

However, it is not as simple as literally translating the curve counts: tropical curves have to be counted with multiplicity, and this multiplicity heavily depends on the field the curve originates from. Nevertheless, tropical geometry has been utilized successfully to compute plenty of curve counts over the real numbers: for instance in [11]. Refined invariants, interpolating between real and complex Gromov-Witten invariants, have been counted tropically since their inception [16, 108]. Recently, a theory of tropical A_1-curve counts has been established, building on the results for complex and real tropical curve counts and extending them beyond those two fields [117, 118, 119].

Apart from the lattice path counts used in early tropical enumerative geometry, different additional combinatorial tools have been introduced to simplify the counting process. Floor diagrams, introduced by Brugallé-Mikhalkin [33], are another combinatorial gadget, modeling tropical curves passing through horizontally stretched point configurations. This gadget is particularly useful when trying to compute refined invariants [16, 17, 130], and when trying to count curves in other types of surfaces [18, 19, 20, 21].

Another way tropical geometry has been used in enumerative geometry is by allowing to change the requirement that curves pass through a fixed number of points to other types of conditions, and providing a combinatorially easy way to deal with the new conditions. For instance, one can replace a point condition by sufficiently many tangency ([64]), line or cross-ratio ([68, 69]) conditions. This means that instead of requiring the tropical curve to pass through $3d + g - 1$ points, one instead requires the curve to pass through fewer points, but instead have tangency at the boundary divisors of a specific order, or require the curve to pass through a line instead, or require four points to have a fixed cross ratio. All these a priori algebraic conditions translate to tropical conditions, hence can be counted using tropical geometry, by developing the right combinatorial tools and proving a variation of Mikhalkin's correspondence theorem for the new problem.

Furthermore, tropical geometry streamlines the counting of curves in different toric surfaces. Tropical curve counts in $\mathbb{P}^2$ correspond to the counting of specific subdivisons of lattice triangles. Other toric surfaces correspond to different lattice polygons. Thus, tropical methods allow for a relatively straightforward generalization — instead of counting subdivisions of lattice triangles, one can count subdivisions of other lattice polygons — rectangles (corresponding to tropical curves in $\mathbb{P}^1 \times \mathbb{P}^1$, see [47]), or subdivisions of specific trapezes (corresponding to tropical curves in the Hirzebruch surfaces $\mathbb{F}_n$ [59]), or arbitrary smooth polytopes (corresponding to smooth projective toric surfaces [107]).

Further, one can extend the curve-counting procedures to ruled surfaces. For instance, one can enumerate curves in a cylinder, i.e., curves inside the projectivization of a line bundle over an elliptic curve [21, 24]. In [21], the author establishes floor diagram counting techniques for these line bundles, whereas in [24], the authors relate the resulting tropical curves to Feynman diagrams.

Until now, results for tropical curve counting have been achieved for orientable surfaces. In this thesis, we extend the theory to two surfaces with non-orientable lattice structures. The two surfaces we study, $\mathbb{C}M_0$ and $\mathbb{C}M_1$ (see Sections 12.5.2 and 12.5.3), are ruled surfaces over an elliptic curve. We call them the *complex Möbius strips* as their tropical analogues $\mathbb{T}M_0$ and $\mathbb{T}M_1$ have the familiar Möbius strip structure — they can be given as a real strip $\mathbb{R} \times [0, l]$ whose two sides $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{l\}$ are glued in reverse orientation (see Proposition 13.1).

In our setting, the surfaces $\mathbb{C}M_0$ and $\mathbb{C}M_1$ are no longer toric. Instead, they are only locally toric — every neighborhood looks like $(\mathbb{C}^*)^2$. This requires us to prove a different version of the correspondence theorem, suitable to our case. To this end, we use that we can construct a logarithmic degeneration of tropical Möbius strips into toric varieties glued along their toric boundaries. This enables us to use the Abramovich-Chen-Gross-Siebert decomposition formula [2], which in turn allows us to compute the tropical invariants on the toric components and translates our result into an algebraic statement.

Theorem I (Theorem 15.2)**.** *The tropical Gromov-Witten invariant and the logarithmic Gromov-Witten invariant of the Möbius strips agree.*

On the tropical side (see Proposition 13.1), this requires an adaptation of the previously discussed methods: We need to track which ends of tropical curves are allowed to be glued to which other ends. This turns out to be a bit more complicated than the cylinder case, as there are two types of cycles appearing in each tropical curve (and indeed each classical curve on the classical Möbius strips we discuss in Section 12.5): there are *orienting* and *disorienting cycles*, since the Möbius strip is a non-orientable surface (for examples, see Figures 20 and 21). Roughly, orienting cycles get treated similarly to their counterparts in the cylinder case, while disorienting cycles require special treatment and carry an additional multiplicity factor of 2.

These two tropical surfaces have a structure not previously considered in tropical geometry — they contain disorienting cycles. We adapt our tool set to deal with this, introducing different types of floors to the classical floor diagrams (see Definition 14.9). These new adjustments require separate treatment when trying to compute multiplicities.

After defining tropical curves in tropical Möbius strips and describing their multiplicities, we proceed with showing that the curve counts are invariant for a compatible choice of tropically general point constraints (see Theorem 13.24). We then introduce an adaptation of floor diagrams (see Definition 14.9), which describe the setting we find ourselves in, and which can handle disorienting cycles. These correspond to a newly introduced floor type called *ground floors*. Further, we introduce *joints*, which correspond to elevators passing through the center (or *soul*) of the Möbius strip. We define multiplicities for the new version of floor diagrams and show that their count coincides with the tropical count (see Proposition 14.24).

We conclude by analyzing the regularity properties of the enumerative invariants, using the combinatorics of floor diagrams to our advantage. In particular, our new methods allow us to analyze *generating series* of Gromov-Witten invariants with respect to different parameters.

The first set of functions we consider is a function in the tangency constraints for each possible horizontal degree b, fixing the vertical degree a. We show the following regularity property:

Theorem J (Theorem 15.5)**.** *The function assigning the corresponding Gromov-Witten invariant to a set of varying tangency conditions is quasi-polynomial.*

If instead, we consider the generating series where we fix the genus, tangency profiles and horizontal degree b, but vary the vertical degree a, we obtain a different kind of regularity:

Theorem K (Theorem 15.12)**.** *The generating series in a are quasi-modular forms for a finite index subgroup of $SL_2(\mathbb{Z})$.*

All work on the tropical enumeration of curves in ruled surfaces is based on joint work with Thomas Blomme [22].

1.5. **Structure.** The thesis begins with a general section reviewing basic concepts in tropical geometry. Afterwards, the thesis is separated into three main parts, covering different aspects of my projects. Each part begins with a more specialized literature review, after which we introduce new theory. In Part 1, we cover maps between tropical linear spaces and valuated matroids. In Part 2, we lay the groundwork for a quiver representation theory in tropical geometry. Finally, in Part 3, we provide (tropical) combinatorial tools to count curves in ruled surfaces. In detail, the sections cover the following:

In **Section 2**, we review notions from tropical geometry. In particular, we introduce the tropicalization of (affine) varieties. We discuss notions of tropical projective spaces and tropicalizations of varieties contained inside of them. Our focus will be on tropicalizations of linear spaces and curves, as these two topics will be relevant for the remaining parts of the thesis.

In **Part 1**, we develop a theory of morphisms for valuated matroids and tropical linear spaces. We contrast different possibilities for morphisms and show some of their properties.

Section 3 starts with a review of basic notions in matroid theory. Further, we recall the existing notions of maps between matroids. We cover matroid quotients, morphisms of matroids and strong maps, and bimatroids and bond matroids. We conclude the preliminaries (for now) by a subsection discussing tropical analogues of matroid theory, starting from valuated matroids and building up to tropical linear spaces and their parametrization spaces, the tropical Grassmannian and the Dressian.

We start our original contributions in **Section 4**. Here, we introduce morphisms of valuated matroids. We continue by enriching them with additional valuative structure, suitable to capture translations of tropical linear spaces. We prove that the category of valuated matroids with affine morphisms of valuated matroids possesses many of the properties the category of matroids with strong maps possesses. In this section, you find Theorem A. The definitions of morphisms and affine morphisms were obtained in joint work with Alessio Borzì [27] and Giulia Iezzi [82] respectively. Most proofs (and a substantial amount of the statements) in this section expand upon the joint works and have not been published elsewhere.

We continue our study of maps between matroids in **Section 5**, where we study interactions of the above introduced definitions with tropical matrix multiplication and prove their compatibility. In particular, we prove Theorem B. This section is again an extension of joint work with Giulia Iezzi [82] and contains the proofs to some core results underpinning tropical quiver theory in Part 2. The main generalization this thesis provides to [82] is the extension of all statements and proofs to non-square matrices.

We wrap up our study of maps between matroids in **Section 6** by investigating valuative structures on bimatroids and show some of their basic properties. We further conjecture relations to the other notions of tropical linear maps given in this thesis. The main results of this section are summarized in Theorem C This section is part of a joint work in progress with Jeffrey Giansiracusa, Felipe Rincón and Martin Ulirsch.

This concludes Part 1. We continue with **Part 2**, in which we establish a new theory of tropical quiver representations, connecting the classical representation theory of quivers with tropical geometry.

In **Section 7**, we review some classical quiver representation theory, covering quiver Grassmannians, some historical facts, and the relation to representation theory of Lie groups.

We resume our original work in **Section 8**. Here, we introduce (valuated) matroidal quiver representations as a matroidal analogue of the classical objects introduced in the previous section. This section is my own, though it is mainly a reformulation of previous work with Giulia Iezzi [82] in more classical terms.

In **Section 9**, we establish the main results of this part of the thesis. We use the classical theory and our notions of morphisms of valuated matroids to introduce two moduli spaces for tropical quiver representations: the *tropicalized quiver Grassmannian* and the *quiver Dressian*. We prove different characterizations of these spaces, summarized in Theorems D and E. Further, we investigate under which conditions the two tropical analogues coincide: we prove Theorem F and discuss Conjecture G. This section is based on joint work with Giulia Iezzi [82].

We apply tropical quiver theory to another tropical concept in **Section 10**. We describe linear degenerate flag varieties and their cover relations (joint with Alessio Borzì [27]).

We conclude the study of tropical quiver representations in **Section 11** by constructing a polyhedral analogue of some particularly nice matroidal quiver representations. In the process, we recap some background on Coxeter matroid polytopes. We summarize our observations in Conjecture H, and identify challenges for establishing a polyhedral theory for quivers containing loops.

In **Part 3** we shift gears to enumerative geometry. We study curve counts on tropical ruled surfaces, focusing on two new ones, the two tropical Möbius strips, which we also define. All results in this part were obtained in joint work with Thomas Blomme [22].

We start this part by reviewing the basic notions of tropical enumerative geometry in **Section 12**, including the basics of tropical curve counting, floor diagrams and classical, logarithmic, refined and tropical Gromov-Witten invariants. Further, we review the classification of ruled surfaces and existing literature on counting tropical curves in them.

We continue in **Section 13** by defining our object of study: the two tropical Möbius strips. We define curves and divisors on them. We study their structure and their relations to the classical ruled surfaces discussed in the previous section, and define relative tropical Gromov-Witten invariants on them.

To compute the number of curves in tropical Möbius strips, we introduce floor diagrams for their curves, and study their properties in **Section 14**. In particular, we prove that the weighted tropical curve count with the multiplicities introduced in the previous section is equal to a weighted count of the associated floor diagrams.

Finally, in **Section 15**, we prove the main theorems of this part: that the tropical curve counts on Möbius strips correspond to the logarithmic Gromov-Witten invariant on the classical counterpart (Theorem I), and that the generating series of the invariants are quasi-polynomial (Theorem J) and quasi-modular (Theorem K).

There are two appendices — **Appendix A** contains a table of all notation used in this thesis for quick reference, and **Appendix B** contains all code for computations carried out in this thesis.

Acknowledgements. First, I'd like to thank my advisor Hannah for (almost) three wonderful years. Most of all, I'm grateful for you giving me the freedom to find my own way to do research, and for teaching me the skills and providing the tools and resources to help me along my path.

I'd like to thank my coauthors, who all taught me something valuable:

... Alessio, who taught me about matroids and how to start a project, and
... Bernd, who taught me how to finish one.
... Clemens, for hours spent reading old German papers and working through Hartshorne exercises together, and for encouraging me to learn about stacks.
... Daniele, for teaching me about elliptic fibrations and divisors.
... Felipe, for his uncanny talent to make insightful comments and to find useful papers.
... Giulia, for teaching me about Schubert varieties and answering all my questions about quiver representations.
... Javier, for discussions about real involutions and del Pezzo surfaces, and for commiseration during the writing period for both our theses.
... Jeff, for teaching me about exterior products and the enthusiasm brought to group discussions.
... Lukas, for explaining parallel computing frameworks to me, and for massive amounts of tea.
... Martin, for the never-ending energy and optimism when our project hit a wall.
... Thomas, for his patience explaining Gromov-Witten invariants and correspondence theorems.

Apart from my coauthors, I profited a lot from being able to learn from many amazing mathematicians who generously shared their knowledge with me. Significant insight was gained after discussions with Lara Bossinger, Chris Eur, Alex Fink, Ghislain Fourier, Manoel Jarra, Tong Jin, Michael Joswig, Arne Kuhrs, Kevin Kühn, Oliver Lorscheid, Matilde Manzaroli, Yue Ren, Felix Röhrle, Rob Silversmith, Ben Smith, Eduardo Vital, and many more.

I'm grateful for my family and friends, for their support and patience when I was traveling and unreachable, for providing me a bed to sleep in when there was a conference close-by, for being there to listen when I needed to complain and for providing distraction when necessary.

Finally, I want to thank Diane, for introducing me to tropical geometry, and for all the support and advice along the way, especially during challenging times.

2. Preliminaries: Tropical geometry

In this section, we review the basic notions of tropical geometry. We start out by fixing some standard notation. For a full table of notation, we refer to Appendix A.

Notation. Throughout the thesis, we write $[n]$ for the set $\{1, \dots n\}$. In general, we write tuples or vectors in bold font: $\mathbf{x} = (x_1, \dots, x_n)$. For a set $S \subseteq [n]$, we define the *indicator vector* e_S by

$$(e_S)_i = \begin{cases} 1 & i \in S \\ 0 & \text{otherwise.} \end{cases}$$

Further, to keep consistent with the standard notation in the literature, we write $\mathbf{1}$ for the vector $e_{[n]}$ of appropriate length.

The set $\binom{[n]}{k}$ denotes all subsets of $[n]$ of size k. For readability purposes, for $S = \{s_1, s_2, \dots, s_k\} \in \binom{[n]}{k}$, we sometimes write $S = s_1 s_2 \dots s_k$, omitting the commas and the set brackets. For the complement of a subset $I \subseteq [n]$, we write $I^c = [n] \setminus I$ if the set $[n]$ is clear from context.

Throughout, K is a non-Archimedean field with valuation.

Structure. In **Section 2.1**, we review the notions of tropicalization in very affine varieties, i.e., subvarieties of $(K^*)^n$. We focus on the examples of tropicalizations of curves, as these will be the main objects of study in Part 3.

For Parts 1 and 2, the tropicalization process needs to be extended to subvarieties of multiprojective space. In **Section 2.2**, we cover this extension of the classical tropical theory. Here, our main examples are tropicalizations of linear spaces, which will be prime examples of the objects studied in Parts 1 and 2.

2.1. **Tropicalization.**

We refer to [101] for more details and proofs of the claims made, in particular to Chapter 3. Throughout the thesis, we use the min-convention for all tropical, matroidal and polyhedral operations.

2.1.1. *Tropical varieties.*

Tropical geometry studies combinatorial objects that arise from degenerations of algebraic varieties. This degeneration process is called *tropicalization* and was historically defined as taking the logarithmic limit of an algebraic variety, [34, 104, 105].

Systematically, the behavior of coefficients under logarithmic degeneration can be described by applying a non-Archimedean valuation. This way, we can consider tropical geometry as algebraic geometry over the tropical semifield.

Definition 2.1. Set $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ and define $a \oplus b = \min\{a, b\}$ and $a \odot b = a + b$ for every $a, b \in \mathbb{T}$. Then $(\mathbb{T}, \oplus, \odot)$ is a semifield, called the *tropical semifield.* To translate algebraic geometry concepts to tropical geometry, we need to be able to consider polynomials. A *tropical polynomial* is an element of the semiring $\mathbb{T}[x_1, \dots, x_n]$ in the variables $x_1, \dots, x_n$ with coefficients in $\mathbb{T}$.

We can pass from an element of the field K to an element of the tropical semiring by applying a non-Archimedean valuation map.

Definition 2.2. A (non-Archimedean) *valuation* is a map $\mathrm{val} : K \to \mathbb{T}$ satisfying

(1) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$ for all $a, b \in K$;

(2) $\mathrm{val}(a+b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$ for all $a, b \in K$; and
(3) $\mathrm{val}(a) = \infty$ if and only if $a = 0$.

The following are the two main examples of valuations we will use throughout the thesis.

Example 2.3. For any field K we can define a valuation $\mathrm{val} : K \to \mathbb{T}$ by setting $\mathrm{val}(a) = 0$ if $a \neq 0$ and $\mathrm{val}(0) = \infty$. This valuation is called the *trivial valuation* on K.

Example 2.4. A more sophisticated example is the t*-adic* valuation on the *field of Puiseux series* $\mathbb{C}\{\{t\}\}$. The elements of $\mathbb{C}\{\{t\}\}$ are formal power series $f = \sum_{\alpha \in \mathbb{Q}} c_\alpha t^\alpha$ satisfying that

- there exists a lowest exponent α_f in f, and
- all α have a common denominator.

The t-adic valuation on $\mathbb{C}\{\{t\}\}$ is defined as $\mathrm{val} : \mathbb{C}\{\{t\}\} \to \mathbb{T}$ with $\mathrm{val}(f) = \alpha_f$. That is, every f is mapped to its lowest exponent. The field of Puiseux series is algebraically closed (see [101, Theorem 2.1.5]) and of characteristic 0. The following are some examples of Puiseux series and their valuations:

f	$3t^2$	7	$1+t^2+t^5$	0
$\mathrm{val}(f)$	2	0	0	∞

The t-adic valuation on the Puiseux series can be considered as modeling the behavior of an element $f \in \mathbb{C}\{\{t\}\}$ under taking its logarithmic limit. This ties back to the original construction of tropical curves using amoebas.

We can pass from the algebraic world into the tropical world by applying a process called *tropicalization*. Studying tropicalizations of polynomials over the field of Puiseux series allows us to describe the logarithmic limit set of the family of curves defined by such a polynomial.

Definition 2.5. Let K be a field with valuation $\mathrm{val} : K \to \mathbb{T}$. The *tropicalization* of a polynomial $f = \sum_{u \in \mathbb{N}^n} a_u x^u \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is the tropical polynomial

$$\mathrm{trop}(f) \;=\; \bigoplus_{u \in \mathbb{N}^n} \mathrm{val}(a_u) \odot x^u \;\in\; \mathbb{T}[x_1, \ldots, x_n].$$

Further, the *tropicalization* $\mathrm{trop}(I)$ of an ideal $I \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is the set of tropical polynomials generated by the tropicalizations of all polynomials in I:

$$\mathrm{trop}(I) \;=\; \left\{ \mathrm{trop}(f) : f \in I \right\} \;\subseteq\; \mathbb{T}[x_1, \ldots, x_n].$$

Example 2.6. First, let us consider two polynomials in $\mathbb{C}[x, y, z]$, using the trivial valuation.

f	$x+y+z$	$x^2+xy+xz+yz+y^2$
$\mathrm{trop}(f)$	$x \oplus y \oplus z$	$x^2 \oplus xy \oplus xz \oplus yz \oplus y^2$
	$\min(x, y, z)$	$\min(2x, x+y, x+z, y+z, 2y)$

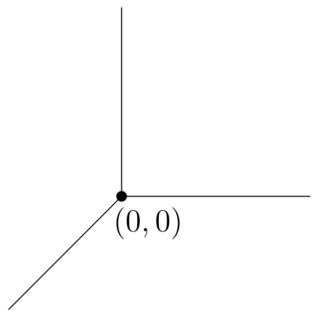
Since the trivial valuation sends every element of $\mathbb{C}^*$ to 0, to obtain nontrivial coefficients in the tropicalization of the polynomial, we have to work over a different field. Over $\mathbb{C}\{\{t\}\}[x, y, z]$ with the Puiseux valuation, we obtain more complicated coefficients:

f	$t^2x+ty+z$	$3tx^2+7xy+(1+t^2+t^5)xz+(7+t^4)yz+3t^2y^2$
$\mathrm{trop}(f)$	$(2 \odot x) \oplus (1 \odot y) \oplus z$	$(1 \odot x^2) \oplus x \odot y \oplus x \odot z \oplus y \odot z \oplus (2 \odot y^2)$
	$\min(x+2, y+1, z)$	$\min(2x+1, x+y, x+z, y+z, 2y+2)$

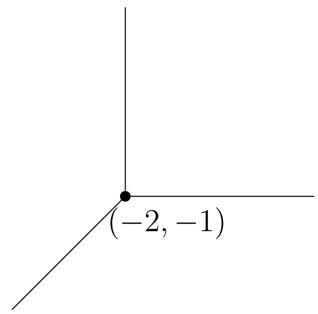
Definition 2.7. The *tropical hypersurface* of a tropical polynomial $F = \bigoplus_{u \in \mathbb{N}^n} c_u \odot x^u \in \mathbb{T}[x_1, \dots, x_n]$ is

$$V(F) \;=\; \left\{ x \in \mathbb{R}^n/\mathbb{R}\mathbf{1} : \min_{u \in \mathbb{N}^n} \left\{ c_u + \sum_{i=1}^{n} u_i \cdot x_i \right\} \text{is achieved at least twice} \right\}.$$

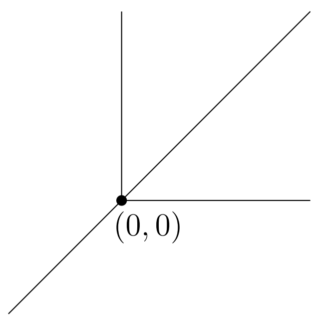
Example 2.8. Let us continue Example 2.6 and draw the tropical hypersurfaces of the tropical polynomials given there:



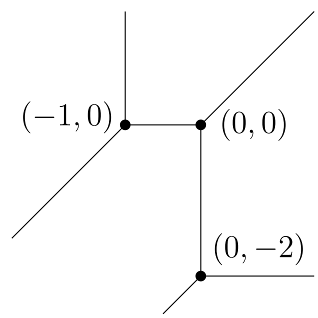
$V(x \oplus y \oplus z)$



$V\big((2 \odot x) \oplus (1 \odot y) \oplus z\big)$



$V(x^2 \oplus xy \oplus xz \oplus yz \oplus y^2)$



$V\big((1 \odot x^2) \oplus x \odot y \oplus x \odot z \oplus y \odot z \oplus (2 \odot y^2)\big)$

All tropical hypersurfaces above are examples of *tropical curves*. We will study these in more detail in Section 12.2. Further, they will be the main object of study in Part 3. In addition, the curves in the top row are examples of *tropicalized linear spaces*, which are prime examples of the objects we study in Parts 1 and 2.

Note that the tropical hypersurfaces obtained by using the trivial valuation on the left above are polyhedral fans. This is not a coincidence — tropical hypersurfaces (and, in fact, tropical varieties) obtained via tropicalization over a field with trivial valuation are always polyhedral fans.

Definition 2.9. Let $S \subseteq \mathbb{T}[x_1, \dots, x_n]$ be a set of tropical polynomials. The *tropical prevariety* $V(S)$ is defined by

$$V(S) \;=\; \bigcap_{F \in S} V(F) \subseteq \mathbb{R}^n/\mathbb{R}\mathbf{1}.$$

If S is an ideal, $V(S)$ has additional polyhedral structure and is called a *tropical variety.*

Over algebraically closed fields with nontrivial valuation, there are alternative characterizations of tropical varieties, which are shown to be equivalent in the *fundamental theorem of tropical geometry.*

Theorem 2.10 ([101, Theorem 3.2.3])**.** *Let K be an algebraically closed field with nontrivial valuation, let I be an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and let $X = V(I)$ be its algebraic variety in $(K^*)^n$. Then,*

$$V(\mathrm{trop}(I)) \;=\; \bigcap_{F \in I} V(F) \;=\; \overline{\big\{(\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)) \mid \mathbf{x} \in X\big\}} =: \mathrm{val}(X) \;\subseteq\; \mathbb{R}^n/\mathbb{R}\mathbf{1},$$

where the closure is with respect to the Euclidean topology induced on $\mathbb{R}^n/\mathbb{R}\mathbf{1}$.

The above theorem implies that tropical varieties can be obtained in two equivalent ways: we can first tropicalize the ideal, and then take its tropical variety $V(\mathrm{trop}(I))$, or we can consider the variety $V(I)$ and take the valuation of its points to obtain $\mathrm{trop}(V(I))$.

2.1.2. *Polyhedral structure of tropical varieties.* We begin by introducing the basic polyhedral notions we need to describe tropical (pre)varieties as polyhedral objects. For a thorough introduction to polyhedral theory, we refer to [142].

Definition 2.11. A *polyhedron* P is the intersection of finitely many closed half-spaces in an ambient space $\mathbb{R}^n$. *Polytopes* are bounded polyhedra and can be alternatively given as the convex hull of points in $\mathbb{R}^n$. The *face* of P in direction $\mathbf{w} \in \mathbb{R}^n$ is the set

$$\mathrm{face}_{\mathbf{w}}(P) \;=\; \{\mathbf{x} \in P \mid \mathbf{w} \cdot \mathbf{x} \leq \mathbf{w} \cdot \mathbf{y} \text{ for all } \mathbf{y} \in P\}.$$

Intuitively, $\mathrm{face}_{\mathbf{w}}(P)$ is the closest side of P observed when standing outside the polytope in direction $-\mathbf{w}$ and looking back at the polytope in direction $\mathbf{w}$. *Facets* are faces maximal with respect to containment.

Definition 2.12. Let $P, Q \subset \mathbb{R}^n$ be two polyhedra. Their *Minkowski sum* is the polyhedron

$$P + Q \;=\; \{\mathbf{p} + \mathbf{q} \mid \mathbf{p} \in P \text{ and } \mathbf{q} \in Q\} \;\subseteq\; \mathbb{R}^n.$$

Tropical (pre)varieties are particularly nice gluings of multiple polyhedra.

Definition 2.13. A *polyhedral complex* Σ is a collection of polyhedra satisfying that

(a) if $P \in \Sigma$, then any face of P is in Σ, and
(b) if both P and Q are in Σ, then their intersection $P \cap Q$ is either empty or a face of both P and Q.

A *fan* is a polyhedral complex whose only finite polyhedron is a vertex at the origin.

The *dimension* of a polyhedral complex is the maximal dimension of a polytope contained in it. We say that Σ is *pure* if all maximal polytopes in Σ are of the same dimension. The *support* of a polyhedral complex Σ is its set of points

$$\mathrm{supp}(\Sigma) \;=\; \{x \in \mathbb{R}^n \mid x \in P \text{ for some } P \in \Sigma\}.$$

The *star* of a cell σ in Σ is the fan $\mathrm{star}_\Sigma(\sigma)$ with cones

$$\overline{\tau} \;=\; \{\lambda(\mathbf{x} - \mathbf{y}) \mid \lambda > 0, \mathbf{x} \in \sigma, \mathbf{y} \in \tau\}$$

for all τ in Σ that contain σ as a face.

While tropical prevarieties are polyhedral complexes, tropical varieties have additional properties. We can assign them integer weights in a *balanced* way.

Definition 2.14 ([101, Definition 3.3.1])**.** Let $\Sigma \subseteq \mathbb{R}^n$ be a one-dimensional rational fan. To each ray $\sigma \in \Sigma$ we assign a positive integer weight $m(\sigma)$ and denote the first lattice point on the ray σ by v_σ. We say that Σ is *balanced* if

$$\sum_{\sigma \in \Sigma} m(\sigma) v_\sigma \;=\; 0.$$

For a (pure) d-dimensional rational fan Σ, we extend this by quotienting out by $(d-1)$-dimensional subspaces, as follows. Let τ be a $(d-1)$-dimensional cone and let L be the linear space parallel to τ. We define $N(\tau) = \mathbb{Z}^n/(L \cap \mathbb{Z}^n)$. Then, for each $\sigma \in \Sigma$ with $\tau \subsetneq \sigma$, the quotient $(\sigma + L)/L$ is a one-dimensional cone in $N(\tau) \otimes_{\mathbb{Z}} \mathbb{R}$. We write v_σ for the first lattice point on this ray. Now, we say that the fan Σ is balanced at τ if this one-dimensional fan is balanced, i.e., if

$$\sum_{\tau \subsetneq \sigma} m(\sigma) v_\sigma \;=\; 0.$$

Finally, Σ is balanced if it is balanced at all $(d-1)$-dimensional rays.

Now, we can use the definition of the star of cells (c.f. Definition 2.13) to extend the balancing of fans to a balancing of polyhedral complexes. We say that a polyhedral complex Σ is balanced if the fan $\text{star}_{\Sigma(\tau)}$ is balanced for all $\tau \in \Sigma$ of dimension $d-1$.

Theorem 2.15. [101, Theorem 3.3.5] *Let $X = V(I)$ be an irreducible subvariety of $(K^*)^n$ of dimension d. Then, the tropical variety* $\text{trop}(V(I))$ *is the support of a pure d-dimensional, rational, weighted balanced polyhedral complex that is connected through codimension one.*

2.1.3. *Newton subdivisions and tropical hypersurfaces.* Determining the weights that make the tropicalization of an algebraic variety balanced can be a challenging endeavor. For hypersurfaces, however, the problem can be translated into polyhedral geometry and solved there. Hypersurfaces are of particular interest to us, as all curves in two-dimensional spaces are hypersurfaces. Thus, understanding hypersurfaces in two-dimensional spaces will be important for our study of enumerative problems in surfaces.

Definition 2.16. Let $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial. The *Newton polytope* of f is the convex hull of the exponent vectors of the monomials of f, i.e.,

$$\mathcal{N}_f \;=\; \text{conv}\{\mathbf{e} \in \mathbb{Z}^n \mid x^{\mathbf{e}} \text{ is a monomial of } f\}.$$

The valuation of the coefficients of the monomials now induce a subdivision on $\mathcal{N}_f$ as follows. Consider the polytope

$$\mathcal{P}_f \;=\; \text{conv}\{(\mathbf{e}, \text{val}(c_{\mathbf{e}})) \in \mathbb{R}^{n+1} \mid x^{\mathbf{e}} \text{ is a monomial of } f\},$$

where $c_{\mathbf{e}}$ is the coefficient of the monomial $x^{\mathbf{e}}$ in f. Now, the subdivision Δ_f of $\mathcal{N}_f$ induced by the valuation of coefficients, called *Newton subdivision,* is the collection of the projections of lower faces of $\mathcal{P}_f$ onto the first n coordinates. Here, *lower faces* are faces whose inner normal vector has positive last coordinate.

An example of a Newton subdivision and its dual tropical curve can be seen in Figure 1.

Proposition 2.17 ([101, Proposition 3.1.6 and Proposition 3.3.2])**.** *Let $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial. The tropical hypersurface* $V(\text{trop}(f))$ *is the support of a polyhedral complex Σ_f which is dual to the Newton subdivision. In this thesis, as a convention, we*

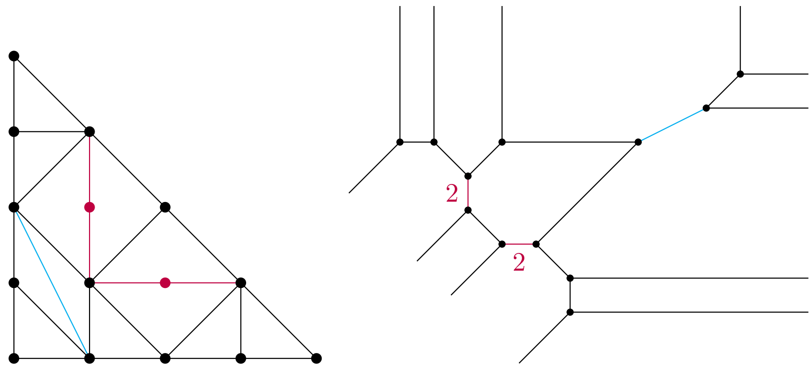


FIGURE 1. A tropical curve and its dual Newton subdivision. The tropical curve is balanced with respect to the weights assigned to the edges, where we omit all weights that are equal to 1. The weighting corresponds to the lattice length of edges in the subdivision, as discussed in Proposition 2.17. Note that the cyan edge has lattice length 1.

use the intersection of the normal fan of the extended Newton polytope $\mathcal{P}_f$ with the plane $x_{n+1} = 1$.

Let F be a facet of $V(\mathrm{trop}(f))$. Then, there exists an edge e_F in the Newton subdivision dual to F. Now, we assign to each facet F the edge-length of its dual edge in the subdivision, $\mathrm{length}(e_F)$. *The polyhedral complex Σ_f is balanced with respect to these weights.*

2.2. **Tropicalization in multiprojective space.** A thorough discussion of tropical projective spaces can be found in [129]. Since then, much of the theory of tropicalization has been extended to subvarieties of multiprojective space. For the general theory, we refer to [101, Chapter 6] and [89, 116]. For the case of linear subspaces, we also refer to [30, 129].

2.2.1. *(Multi-)projective tropical varieties and compactification.*

Definition 2.18. The *tropical projective space* is $\mathbb{P}(\mathbb{T}^n) = \big(\mathbb{T}^n \setminus \{(\infty,\ldots,\infty)\}\big)/\mathbb{R}\mathbf{1} = (\mathbb{T}^n \setminus \{(\infty,\ldots,\infty)\})/\sim$. Here, $\sim$ is the equivalence relation $\mathbf{u} \sim \mathbf{v}$ if $\mathbf{u} = \mathbf{v} + c\mathbf{1}$ for some $c \in \mathbb{R}$. The *tropical multiprojective space* $\mathbb{P}(\mathbb{T}^{n_1}) \times \cdots \times \mathbb{P}(\mathbb{T}^{n_k})$ is a product of tropical projective spaces. That is,

$$\mathbb{P}(\mathbb{T}^{n_1}) \times \cdots \times \mathbb{P}(\mathbb{T}^{n_k}) \;=\; (\mathbb{T}^{n_1} \setminus \{(\infty,\ldots,\infty)\} \times \cdots \times \mathbb{T}^{n_k} \setminus \{(\infty,\ldots,\infty)\})/\sim,$$

where $\sim$ is the equivalence relation $\mathbf{u} \sim \mathbf{v}$ if $\mathbf{u} = \mathbf{v} + (c_1 \cdot e_{[n_1]} + \cdots + c_k \cdot e_{[\sum_{i=1}^{k} n_i]\setminus[\sum_{i=1}^{k-1} n_i]})$ for $c_1, \ldots, c_k \in \mathbb{R}$.

Considering varieties in tropical (multi)-projective space necessitates the treatment of coordinate entries that are infinite. We thus extend the notion of tropical varieties by allowing for infinite entries as follows:

Definition 2.19. Let $n = n_1 + \cdots + n_k$ be a sum of integers. The *(multiprojective) tropical hypersurface* of a tropical polynomial $F = \bigoplus_{u\in\mathbb{N}^n} c_u \odot x^u \in \mathbb{T}[x_1,\ldots,x_n]$ is

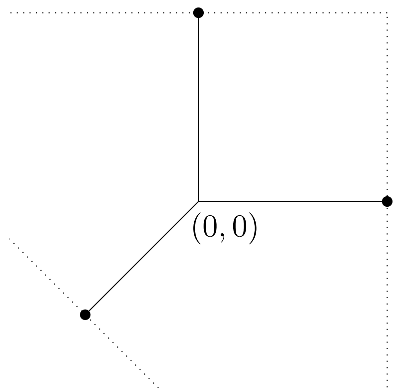
$$V(F) \;=\; \left\{ x \in \mathbb{P}(\mathbb{T}^{n_1}) \times \cdots \times \mathbb{P}(\mathbb{T}^{n_k}) : \min_{u\in\mathbb{N}^n} \left\{ c_u + \sum_{i=1}^{n} u_i \cdot x_i \right\} \text{is achieved at least twice} \right\},$$

where whenever $\min_{u\in\mathbb{N}^n}\{c_u + \sum_{i=1}^n u_i \cdot x_i\} = \infty$, by convention, the minimum is achieved at least twice, even if the expression is a tropical monomial. The *(multiprojective) tropical prevariety* of a set of multihomogeneous tropical polynomials $J \subseteq \mathbb{T}[x_1, \dots, x_n]$ is defined by

$$V(J) \;=\; \bigcap_{F\in J} V(F) \subseteq \mathbb{P}(\mathbb{T}^{n_1}) \times \cdots \times \mathbb{P}(\mathbb{T}^{n_k}).$$

Again, if J is the tropicalization of an ideal, this polyhedral complex has a special structure we say that $V(J)$ is a *(multiprojective) tropical variety.*

Example 2.20. In Example 2.8, we had seen the tropicalization of the tropical line $x + y + z$ inside $\mathbb{R}^3/\mathbb{R}\mathbf{1}$. Now, we will compute the tropicalization of the tropical line in the tropical projective space $\mathbb{P}(\mathbb{T}^3)$.



In the interior of $\mathbb{P}(\mathbb{T}^3)$, the tropical line does not change under this change of ambient space. Instead, we are introducing three boundary points. In particular, the tropical line contained inside tropical projective space is compact.

We can again give a tropicalization in terms of the valuations of points of the variety.

Definition 2.21. Over an algebraically closed base field K with a non-trivial valuation, the *tropicalization* $\overline{\text{trop}}(X)$ of a subvariety $X \subseteq \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_k-1}$ is defined by

$$\overline{\text{trop}}(X) \;=\; \overline{\left\{\big((\text{val}(x_0^1), \dots, \text{val}(x_{n_k}^k)\big) \in \mathbb{P}(\mathbb{T}^{n_1}) \times \cdots \times \mathbb{P}(\mathbb{T}^{n_k}) \right.} \\ \overline{\left.\text{where } \big((x_0^1 : \cdots : x_{n_1}^1), \dots, (x_0^k : \cdots : x_{n_k}^k)\big) \in X\right\}}$$

where the closure is with respect to the Euclidean topology induced on $\mathbb{P}(\mathbb{T}^{n_1}) \times \cdots \times \mathbb{P}(\mathbb{T}^{n_k})$ from the Euclidean topology on $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$.

Tropical varieties in tropical multiprojective space have many of the properties of their very affine counterparts. In particular, the Fundamental Theorem extends to this setting (and the more general setting of smooth toric varieties)!

Theorem 2.22 ([101, Corollary 6.2.16])**.** *Let Y be a subvariety of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and let I be its (multihomogeneous) ideal in $K[x_1, \dots, x_{n_1+\cdots+n_k}]$. Then, the following subsets of $\mathbb{P}(\mathbb{T}^{n_1}) \times \cdots \times \mathbb{P}(\mathbb{T}^{n_k})$ coincide:*

$$\bigcap_{f\in I} \overline{\text{trop}}(V(f)) \;=\; \overline{\text{trop}}(Y) \;=\; \bigcup_{\sigma\in\Sigma} \overline{\text{trop}}(Y \cap \mathcal{O}_\sigma)$$

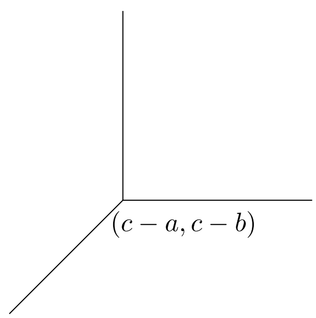
where Σ *is the fan of the toric variety* $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$*, and* $\mathcal{O}_\sigma$ *is the orbit corresponding to the cone* $\sigma \in \Sigma$*.*

The second description in the theorem allows us to do for arbitrary varieties exactly what we had seen in Example 2.20: We can first compute the classical tropicalization, corresponding to the origin in the fan of multiprojective space, and then add in the components of the tropical variety corresponding to the intersections with the boundary components, which correspond to the orbits of the cones in Σ that are not the origin.

Tropical varieties of this form have been studied in the literature, as they are a common way of *compactifying* a given (affine) tropical variety.

2.2.2. *Tropicalized linear spaces.* Many examples we will see in Part 1 of the thesis are tropicalizations of linear spaces (or are arrangements of tropicalizations of linear spaces). We give a few examples in this section, demonstrating the techniques we discussed above.

Example 2.23. In $\mathbb{P}(\mathbb{T}^3)$, up to translation of the center vertex, there is only one tropicalization of a generic linear space, corresponding to $V(a \odot x \oplus b \odot y \oplus c \odot z)$. It consists of three rays in the three coordinate directions and one vertex at $(c-a, c-b)$. Under trivial valuation, there exists only one tropical line — the one with vertex $(0,0)$.



Example 2.24. Now, let us consider tropicalizations of lines (i.e., 1-dimensional tropicalized linear spaces or tropical lines) in $\mathbb{P}(\mathbb{T}^4)$. A generic tropical line in $\mathbb{P}(\mathbb{T}^4)$ is a one dimensional, balanced polyhedral complex with one bounded edge and four unbounded edges, two adjacent to each vertex of the bounded edge. The four distinct directions of the unbounded edges of L_p are given by the images of e_i in $\mathbb{P}(\mathbb{T}^4)$ for $i \in [4]$. Since L_p is balanced, the directions of the unbounded edges are completely determined by the direction of the bounded edge, which is of the form $e_i + e_j$ for $i, j \in [4]$ distinct. As $e_a + e_b = -(e_c + e_d)$ in $\mathbb{P}(\mathbb{T}^4)$ for all $a, b, c, d \in [4]$ distinct, there are three such choices. We depict them all in Figure 2. There also exists a degenerate tropical line in $\mathbb{P}(\mathbb{T}^4)$: The tropicalization of $x_1 + x_2 + x_3 + x_4$ consists only of four rays in all four coordinate directions. It does not have a bounded edge. It corresponds to the dotted lines in Figure 2.

Example 2.25. Finally, let us consider planes (i.e., 2-dimensional projective linear spaces) in $\mathbb{P}(\mathbb{T}^4)$. Up to change of the center point, all planes have the same combinatorial type. They are pure two-dimensional fans whose top-dimensional cones are spanned by pairs of vectors in coordinate direction. We depict an example of a tropical plane in $\mathbb{P}(\mathbb{T}^4)$ in Figure 3.

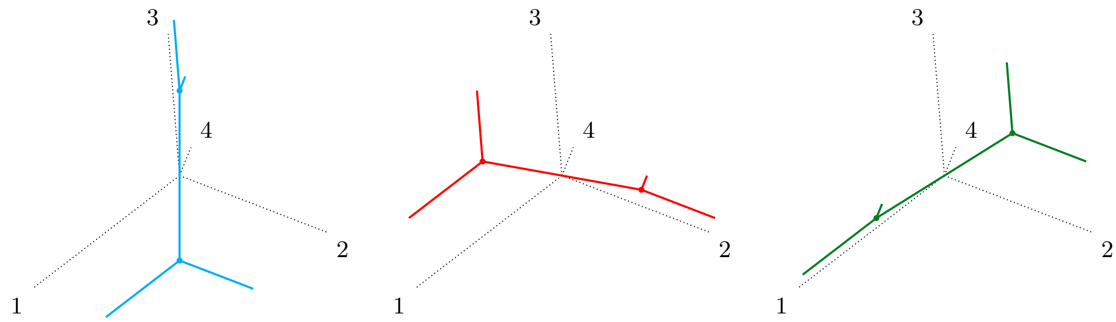


FIGURE 2. The three different combinatorial types of general lines in $\mathbb{P}(\mathbb{T}^4)$.

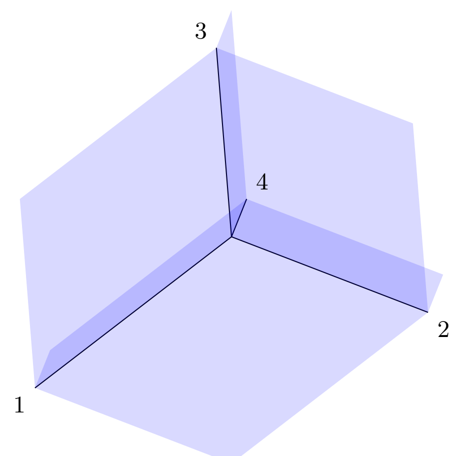


FIGURE 3. A tropical plane in $\mathbb{P}(\mathbb{T}^4)$.

Part 1. Morphisms of valuated matroids

In linear algebra, linear maps between linear spaces take a central role. In this part, we introduce different notions of analogues of linear maps for valuated matroids. We compare and contrast them, discuss their properties, advantages, and disadvantages.

We begin by discussing the necessary background in **Section 3**. We cover basics of matroid theory, including their different cryptomorphic definitions via bases, independent sets, rank functions, circuits, cocircuits and lattices of flats. Further, we discuss matroid representability. Then, we cover morphisms of matroids to lay the foundation for the valuative analogues we aim to construct later on. Next, we discuss valuated matroids and tropical linear spaces. We review their characterizations via basis valuation functions, independent set axioms, and circuit, cocircuit, and vector elimination axioms. We conclude this section by discussing parameter spaces for valuated matroids and valuated flag matroids, namely tropical Grassmannians, Dressians, tropicalized flag varieties and flag Dressians.

In **Section 4**, we then introduce our first notion of linear maps for valuated matroids — morphisms of valuated matroids and their affine extensions, and establish some of their properties. In particular, we show that they are compatible with strong maps of the underlying matroids. Further, we prove that the category of valuated matroids where affine morphisms of valuated matroids shares many of the properties which are known for the category of matroids with strong maps as morphisms. Some of this section is based on joint work with Alessio Borzì [27] and with Giulia Iezzi [82], and some of it is new work done by myself.

We continue by contrasting (affine) morphisms of valuated matroids with naive pointwise tropical matrix multiplication of the corresponding tropical linear space in **Section 5**. For nice enough matrices, we show that the two notions are equivalent. All results in this section are extensions of joint work with Alessio Borzì [27] and with Giulia Iezzi [82].

Finally, in **Section 6**, we study a different notion — valuated bimatroids. These provide an alternative approach to generalize matrices in matroid theory and linear tropical geometry. We prove a tropical Cauchy-Binet formula and conjecture a relation to affine morphisms of valuated matroids and naive tropical matrix multiplication. This section is based on current work-in-progress, joint with Jeffrey Giansiracusa, Felipe Rincón, and Martin Ulirsch.

3. Preliminaries: Linear tropical geometry

In this section, we cover all preliminaries necessary to study linear tropical geometry in the next two parts. We begin in **Section 3.1** by giving an overview over the many different cryptomorphic axiom systems for matroids we will encounter in this thesis: bases, independent sets, rank functions, circuits, cocircuits and lattices of flats. We discuss how the different axiom systems are related, and observe a first connection to tropical geometry via Bergman fans.

Next, in **Section 3.2**, we study maps between matroids: we cover matroid quotients, morphisms and strong maps of matroids. Further, we discuss some known category-theoretical properties for the category of matroids with strong maps, and discuss some properties specific to projection maps.

In **Section 3.3** we then proceed by studying valuated matroids. Again, we cover different characterizations — we discuss valuation maps defined on bases and their extensions to independent sets, and discuss valuated circuits, cocircuits and vectors. Then, we use the latter characterizations to construct equivalent tropical linear spaces.

We finish the introduction to linear tropical geometry in **Section 3.4** by discussing different tropical analogues for the Grassmannian, the tropicalized Grassmannian and the Dressian, which provide parameter spaces for tropicalizations of linear spaces and tropical linear spaces respectively. We conclude by providing the same analysis for the tropical analogues of the flag variety, the tropicalized flag variety and the flag Dressian.

3.1. Matroids.

Matroids are foundational objects in combinatorics. They were introduced by Whitney in 1935 [140] to unify notions of independence in linear algebra and graph theory, and have applications all over mathematics, for instance in algebraic geometry and optimization. All topics discussed in this section are standard definitions and results in matroid theory, covered for instance in [115, 138, 139].

As matroids generalize many notions, there are multiple different *cryptomorphic* axiom systems defining them. We will now give an overview over some of these systems, which are foundational components of the results we will introduce in Parts 1 and 2 of the thesis.

We start in Section 3.1.1 with the notions generalizing concepts in linear algebra: bases, independent sets and rank functions. Next, in Section 3.1.2, we cover the notions generalizing independence in graph theory: circuits and cocircuits. Matroids can be constructed from other matroids. In Section 3.1.3, we cover two such constructions — deletion and contraction.

Then, we discuss lattices of flats and their associated Bergman fans as a first appearance of a tropical object in matroid theory in Section 3.1.4. We finish our coverage of matroids in Section 3.1.5 by discussing basic notions of representability, i.e., when matroids correspond to actual linear spaces, and when they merely are combinatorial objects.

3.1.1. *Bases, independent sets and rank functions.* One of the original motivations for defining matroids was to provide an abstract framework which generalizes linear independence in linear algebra.

Definition 3.1. A *matroid* M is a set $[n]$ together with a set of *bases* $\mathcal{B}(M) \subseteq 2^{[n]}$ satisfying

- (B1) $\mathcal{B}(M) \neq \emptyset$, and
- (B2) for all $A, B \in \mathcal{B}(M)$ and $a \in A \setminus B$ there exists $b \in B \setminus A$ such that both $(A \setminus a) \cup b$ and $(B \setminus b) \cup a$ are bases, i.e., are contained in $\mathcal{B}(M)$. In the future, in cases where the set A is clear, we will omit brackets and write $A \setminus a \cup b$ for $(A \setminus a) \cup b$.

The latter axiom is sometimes called the *symmetric exchange axiom.*

Remark 3.2. A set of bases of a matroid can be obtained from a set of vectors generating a linear space. Let $v_1, \dots, v_n \in K^m$ be vectors, and let $L = \mathrm{span}(v_1, \dots, v_n)$ be a linear subspace of K^m. Then, the set of maximal independent sets of vectors

$$\mathcal{B} \; = \; \{I \mid \mathbf{v}_I \text{ is a basis of } L\}$$

forms the set of bases of a matroid over $[n]$, which we denote by $M(L)$ (or $M(\mathbf{v})$).

In the following example, we carry out the above procedure for a concrete linear space.

Example 3.3. Consider the set of vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \tag{1}$$

Each pair of vectors is linearly independent, but $v_1 + v_2 - v_3 = 0$, so the triple of vectors is not. Thus, $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are maximal sets of independent vectors. Over the ground set $[3]$, the sets made up of indices of the vectors above form a matroid: $12, 13$ and 23 are the bases of the matroid $U_{2,3}$.

Example 3.4. The matroid $U_{2,3}$ discussed in the previous example is part of a special family of matroids. It is a *uniform matroid.* In general, the uniform matroid $U_{r,n}$ is the matroid where the bases are all subsets of size r of the ground set $[n]$, i.e., $\mathcal{B}(M) = \binom{[n]}{r}$.

Definition 3.5. Just like bases of linear spaces, all bases of matroids have the same size. This size is called the *rank* of the matroid and denoted by $\mathrm{rk}(M)$.

Instead of describing matroids in terms of bases, we can describe them in terms of independent sets, again modeled after the notion from linear algebra.

Definition 3.6. A matroid M is a ground set $[n]$ together with a collection of *independent sets* $\mathrm{Ind}(M) \subseteq 2^{[n]}$, satisfying

- (I1) $\emptyset \in \mathrm{Ind}(M)$,
- (I2) if $I \in \mathrm{Ind}(M)$ and $J \subseteq I$, then $J \in \mathrm{Ind}(M)$, and
- (I3) if $I, J \in \mathrm{Ind}(M)$, and $|I| = |J| + 1$, there exists $i \in I \setminus J$ such that $J \cup i \in \mathrm{Ind}(M)$.

Remark 3.7. Bases and independent sets can be stated in terms of each other. First, assume M is defined by a set of bases $\mathcal{B}(M)$ as in Definition 3.1. Then, independent sets are all subsets of bases, and we have

$$\operatorname{Ind}(M) \;=\; \{I \subseteq B \mid B \in \mathcal{B}(M)\}.$$

Analogously, if M is defined by its set of independent sets $\operatorname{Ind}(M)$, bases are exactly the inclusion-maximal sets in $\operatorname{Ind}(M)$.

Example 3.8. We continue Example 3.3. Writing down all independent sets of vectors in Equation (1), we obtain

$$\operatorname{Ind}(U_{2,3}) \;=\; \{\emptyset, 1, 2, 3, 12, 13, 23\}.$$

Note that as claimed in Remark 3.7, these are exactly all subsets of bases in $\mathcal{B}(M)$. For a general uniform matroid $U_{r,n}$, the independent sets are

$$\operatorname{Ind}(U_{r,n}) \;=\; \{I \subseteq [n] \mid |I| \leq r\}.$$

The final concept of this section (and actually Whitney's original notion of a matroid in [140]) is that of rank functions. Rank functions model dimension properties of subspaces spanned by vectors.

Definition 3.9. A matroid M is a set $[n]$ together with a *rank function* $\operatorname{rk} : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ satisfying for any set $S, T \subseteq [n]$

(R1) $\operatorname{rk}_M(S) = 0$ if $S = \emptyset$;
(R2) $\operatorname{rk}_M(S \cup i) = \operatorname{rk}_M(S) + k$ for any $i \notin S$ and $k \in \{0, 1\}$; and
(R3) $\operatorname{rk}_M(S \cup T) + \operatorname{rk}_M(S \cap T) \leq \operatorname{rk}_M(S) + \operatorname{rk}_M(T)$.

The axiom (R2) implies that the rank function is *monotonic*, and (R3) is saying that it is *submodular*.

We can state some additional properties for rank functions of matroids:

Lemma 3.10. *Let M be a matroid on $[n]$. Then, the rank function satisfies*

(1) if $A \subseteq B \subseteq [n]$, then $\operatorname{rk}_M(A) \leq \operatorname{rk}_M(B)$*; and*
(2) for all $A \subseteq [n]$, $0 \leq \operatorname{rk}_M(A) \leq |A|$.
(3) the rank of the matroid as described in Definition 3.5 can be described as $\operatorname{rk}(M) = \max_{S \subseteq [n]} \operatorname{rk}_M(S)$.

Remark 3.11. Again, we can characterize independent sets of a matroid M on $[n]$ in terms of its rank function. Independent sets are precisely the sets where the second inequality in Lemma 3.10(2) is an equality, i.e., $A \subseteq [n]$ is an independent set of M if and only if $\operatorname{rk}_M(A) = |A|$.

3.1.2. *Circuits and Cocircuits.* In the last section, we discussed how matroids generalize notions in linear algebra. In this section, we approach the topic with the goal of generalizing graph theory. Again, the notions in this section are due to Whitney's original paper [140], and have since been used extensively in matroid theory. For instance, they are the main point of view Tutte [121, 134] uses in his works, and will be our main tool to define tropical linear spaces in Section 3.3.2.

Definition 3.12. A matroid M is a set $[n]$ together with a collection $\mathcal{C}(M)$ of *circuits*, satisfying that

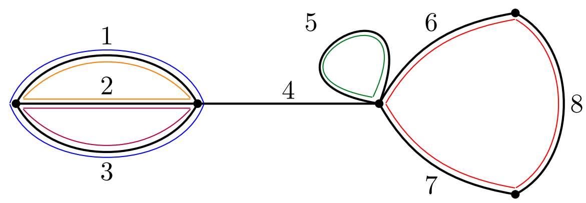
(C1) if $C, D \in \mathcal{C}(M)$ and $C \subseteq D$, then $C = D$; and
(C2) for $C, D \in \mathcal{C}(M)$ and $c \in C \cap D$, then there exists $E \in \mathcal{C}(M)$ such that $E \subseteq (C \cup D) \setminus c$.

The axiom (C2) is sometimes also referred to as the *circuit elimination axiom*.

Remark 3.13. Let $G = (V, E)$ be an undirected graph with sets of vertices V and edges E. We can construct a matroid $M(G)$ on $[|E|]$ as follows. We enumerate all edges. Then, the circuits of $M(G)$ are the (graph-theoretic) simple cycles of G.

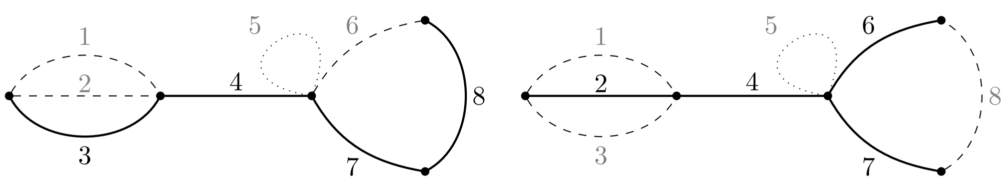
In this setting, a dependent set is a set containing a cycle, whereas an independent set is a forest. The bases of $M(G)$ are the spanning forests of G.

Example 3.14. Let us explicitly construct a graphic matroid. We consider the graph below, and draw in the circuits in colors.



We can verify the circuit elimination axiom: Consider the two circuits $C = 12$ and $D = 13$. Then $C \cap D = 1$ and $(C \cup D) \setminus 1 = 23$, which is also a circuit. The analogue holds true for any choice of C and D in $\{12, 13, 23\}$. Since the other two circuits 5 and 678 have no nonempty intersections with each other or the other circuits, this concludes the verification.

From the graph above, it is clear that 4 needs to be contained in every basis, and that 5 can never be contained in a basis. The bases are the sets that contain one of $1, 2$ and 3; 4; and two of $6, 7$ and 8. We list them all in Example 3.21. Below, we draw two examples of bases, which correspond to the solid-color edges in the graphs.



In the above example, we have seen that some matroids contain elements that are contained in every basis, and elements that are contained in none.

Definition 3.15. Let M be a matroid on $[n]$ and $i \in [n]$. If i is not contained in any basis, i is called a *loop*. This corresponds to the graphical picture: If M is a matroid arising from a graph, any loop of the matroid is a loop of the graph and vice versa. If i is contained in every basis, i is called a *coloop*. Further, two non-loops $i, j \in [n]$ are called *parallel* if $\{i, j\}$ is dependent in M. For matroids arising from a graph, this corresponds to the edges i and j being parallel.

Example 3.16. The matroid $M(G)$ we considered in Example 3.14, has one loop, the element 5 corresponding to the loop edge in the graph G, and one coloop, 4, corresponding to the bridge edge in G. Furthermore, three elements are (pairwise) parallel, the elements $1, 2$ and 3, which correspond to the respective parallel edges of G.

Remark 3.17. Circuits and independent sets can be stated in terms of each other. Circuits are *minimal dependent sets*, i.e., the set of circuits of a matroid M is the set

$$\mathcal{C}(M) \;=\; \{S \subseteq [n] \mid (S \setminus s) \in \mathrm{Ind}(M) \text{ for all } s \in S\}.$$

Definition 3.18. Let M be a matroid on the ground set $[n]$ with bases $\mathcal{B}(M)$. Its *dual matroid* M^* is a matroid on the same ground set $[n]$. It is defined by its set of bases

$$\mathcal{B}(M^*) \;=\; \{B^c \mid B \in \mathcal{B}(M)\},$$

i.e., the bases of $M*$ are the complements of the bases of M. We call the set $\mathcal{B}(M^*)$ the *cobases* of M. Here, B^c denotes the complement of the set B in $[n]$.

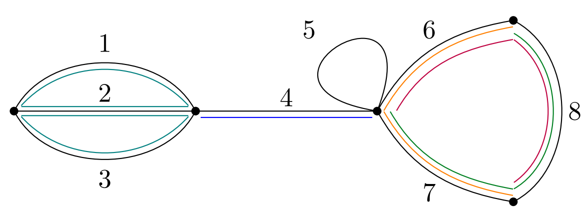
Example 3.19. The dual of the uniform matroid $U_{2,3}$ considered in Example 3.3 is the uniform matroid $U_{1,3}$. For a general uniform matroid $U_{r,n}$, its dual matroid is $U_{n-r,n}$. We construct the dual of the matroid in Example 3.14 below in Example 3.21.

Definition 3.20. Let M be a matroid. The set of *cocircuits* $\mathcal{C}^*(M)$ is the set of circuits of the dual matroid M^*.

Example 3.21. We again consider the matroid $M(G)$ arising from the graph in Example 3.14. We give its bases and cobases below.

$\mathcal{B}(M)$	1467	1468	1478	2467	2468	2478	3467	3468	3478
$\mathcal{B}(M^*)$	2358	2357	2356	1358	1357	1356	1258	1257	1256

Using Remark 3.17, we can determine its cocircuits to be 123, 4, 67, 68 and 78, colored in on the graph below. Observe that cocircuits are in general *not* the complements of circuits!



3.1.3. *Constructions on matroids.* There are many different ways in which one can construct one matroid from another. One such construction is that of the dual matroid we considered in Definition 3.18. We continue with two different ways to remove an element of the base set.

Definition 3.22. Let M be a matroid on $[n]$ and let $S \subseteq [n]$. The set

$$\mathcal{I} \;=\; \{I \subseteq S \mid I \in \mathrm{Ind}(M)\}$$

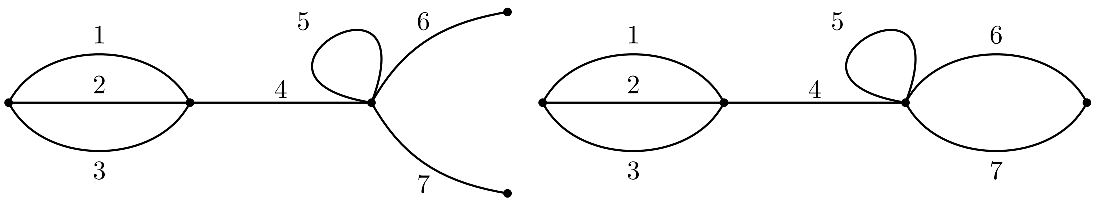
is the set of independent sets of a matroid. We write $M|_S$ (or $M \setminus S^c$) for this matroid and call it the *restriction* of M to S (or, alternatively, the *deletion* of S^c from M).

If M is a matroid arising from a graph, deletion on the level of matroids corresponds to the deletion of edges of a graph, and then taking the matroid of the resulting graph. Another way of getting rid of an edge in graph theory is by contracting it. This has an interpretation in matroid theory, too:

Definition 3.23. Let M be a matroid on $[n]$ and $T \subseteq [n]$. The *contraction* of T in M is the matroid

$$M/T \;=\; (M^* \setminus T)^*.$$

Example 3.24. We continue with the matroid from Example 3.14. On the left, we depict the graph obtained from deleting 8 and on the right, we depict the graph obtained from contracting 8.



We can compute the bases of the two matroids:

$\mathcal{B}(M \setminus 8)$	1467	2467	3467			
$\mathcal{B}(M/8)$	146	147	246	247	346	347

Note that the two matroids are very different: $M \setminus 8$ has rank 4 whereas $M/8$ has rank 3 — any basis of $M \setminus 8$ contains 67, whereas 67 is a dependent set in $M/8$.

We conclude by defining direct sums of matroids, which allow us to construct a matroid on a larger ground set.

Definition 3.25. Let M and N be matroids over the ground sets $[m]$ and $[n]$ respectively. Their *direct sum* is the matroid $M \oplus N$ on the disjoint union of the ground sets $[m] \sqcup [n]$, given by its bases

$$\mathcal{B}(M \oplus N) \;=\; \big\{B_1 \sqcup B_2 \mid B_1 \in \mathcal{B}(M) \text{ and } B_2 \in \mathcal{B}(N).\big\}$$

3.1.4. *Flats and Bergman fans.* Another concept we can use in linear algebra to characterize independence is that of closed subspaces. For matroids, *flats* model this behavior.

Definition 3.26. Let M be a matroid over $[n]$ and let rk_M be its rank function. Then, the set of flats $\mathcal{L}_M$ consists of all sets that are closed under the rank function, i.e., $F \subseteq [n]$ is a flat if and only if $\mathrm{rk}_M(F) < \mathrm{rk}_M(F \cup i)$ for any $i \in F^c$.

The flats of a matroid have additional combinatorial structure. They form a *geometric lattice*.

Definition 3.27. A *lattice* is a partially ordered set $\mathcal{L} = (\mathcal{L}, \leq)$ such that for any two *points* x and y in $\mathcal{L}$

- (L1) the set $\{z \in \mathcal{L} \mid z \leq x \text{ and } z \leq y\}$ has a unique maximal element, called the *meet* of x and y, and
- (L2) the set $\{z \in \mathcal{L} \mid z \geq x \text{ and } z \geq y\}$ has a unique minimal element, called the *join* of x and y.

We say that x *covers* y if $x \neq y$, $x \geq y$ and if $x \geq z \geq y$, then $z = x$ or $z = y$. If $\mathcal{L}$ has a unique minimal element o, we call x an *atom* of $\mathcal{L}$ if x covers o. Intuitively, atoms are the smallest nontrivial elements of the lattice.

A lattice $\mathcal{L}$ is *geometric* if it additionally satisfies the following properties:

(L3) $\mathcal{L}$ is *finite*, i.e., $|L| < \infty$;
(L4) $\mathcal{L}$ is *semimodular*, i.e., if x and y cover their meet, then their join covers both x and y; and
(L5) every point of $\mathcal{L}$ that is not o is the join of atoms.

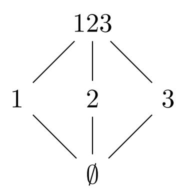
Geometric lattices give us yet another characterization of matroids:

Theorem 3.28 ([138, Theorem 3.1]). *A lattice $\mathcal{L}$ is isomorphic to a lattice of flats of a matroid if and only if it is geometric.*

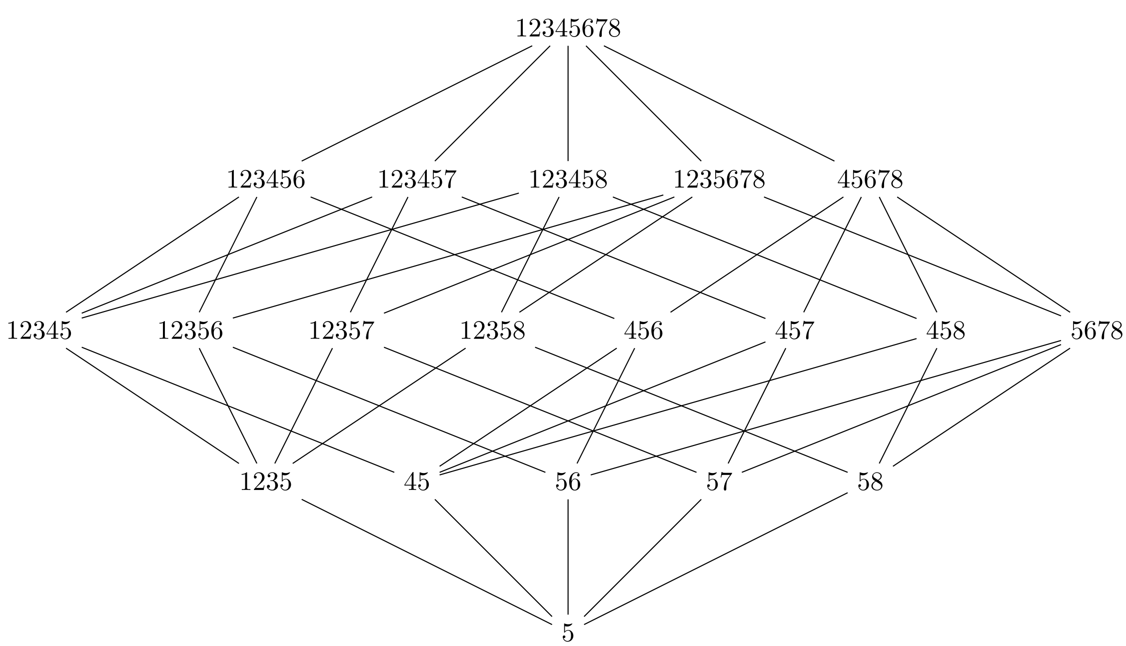
Remark 3.29. Let M be a matroid. We can make a few direct observations about its lattice of flats $\mathcal{L}_M$:

(a) Maximal flags of flats (i.e., maximal chains of flats contained inside of each other) have length $\mathrm{rk}(M) + 1$.
(b) The lowest element o is the union of all loops of M.
(c) If A and B are parallel elements in M, then any flat F containing A also contains B.

Example 3.30. We construct two examples. We begin by reconsidering the matroid $U_{2,3}$ from Example 3.3. Its lattice of flats is the lattice below.



In the above lattice, the atoms are 1, 2 and 3. Each atom can be written as the join of the atom with itself (i.e., 1 is the join of 1 and 1), and 123 is the join of 1, 2 and 3. Hence, ($L5$) is satisfied. For a more complicated example, we compute the lattice of flats of the matroid $M(G)$ from Example 3.14. We can either check them manually, or use the `Oscar` [114] command `flats(M)` to obtain them computationally. Using either method, we obtain the lattice below:



In this lattice, let us consider the points 1235 and 458. Their join is the flat 123458 and their meet is the smallest flat, which is 5. Their meet 5 is covered by 1235, but not by 458, as there exist the two intermediate flats 45 and 58. Analogously, their join 123458 is a cover of 458 but not a cover of 1235, as there exists the intermediate flats 12345 and 12358 in the lattice. While 1235 is an atom itself, the flat 458 can be written as the join of the two atoms 45 and 58.

We can observe the properties discussed in Remark 3.29:

(a) In Example 3.14 we observed that the matroid $M(G)$ has rank 4, and all maximal flags of flats, depicted by paths from the minimal element of the lattice to the maximal element of the lattice are of length 5.
(b) In Example 3.16 we computed that $M(G)$ has one loop, the element 5. This corresponds to the minimal element o of $\mathcal{L}_{M(G)}$.
(c) Again, in Example 3.16, we remarked that $M(G)$ has three parallel elements, the elements $1, 2$ and 3. In the above lattice, we can observe that each flat either contains 123 or does not contain any of these elements.

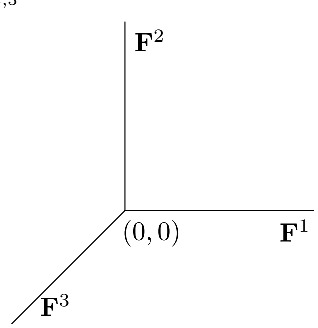
Bergman fans. Now, we give a construction that connects matroids to linear tropical geometry. The *Bergman fan* is a polyhedral fan that can be constructed from the lattice of flats of a matroid and its support coincides with the tropicalization of any realization of the matroid as a linear subspace over a field with trivial valuation. It was originally introduced by Bergman [15] and brought to the tropical world by Ardila-Klivans [8] and Feichtner-Sturmfels [55].

Definition 3.31 (For the formulation used here, see [80])**.** Let M be a matroid with lattice of flats $\mathcal{L}_M$. Its *Bergman fan* $\Sigma_M \subseteq \mathbb{R}^n/\mathbb{R}\mathbf{1}$ is the polyhedral fan spanned by flags of nonempty flats. That is, for each flag of flats $\mathbf{F} = F_1 \subseteq \ldots, F_k$ in $\mathcal{L}_M$, the fan Σ_M has a cone

$$u_{\mathbf{F}} \;=\; \mathrm{Cone}(e_{F_1}, \ldots, e_{F_k}),$$

where $\mathrm{Cone}(e_{F_1}, \ldots, e_{F_k})$ denotes the polyhedral cone spanned by the vectors $e_{F_1}, \ldots, e_{F_k}$.

Example 3.32. We return to $U_{2,3}$. There are three maximal chains in the lattice of flats we computed in Example 3.30, $\mathbf{F}^1 = \{\emptyset \subseteq 1 \subseteq 123\}$, $\mathbf{F}^2 = \{\emptyset \subseteq 2 \subseteq 123\}$ and $\mathbf{F}^3 = \{\emptyset \subseteq 3 \subseteq 123\}$. Thus, the Bergman fan $\Sigma_{U_{2,3}}$ has the three cones depicted below.



3.1.5. *Representability.* In the previous sections, we have seen multiple different definitions of matroids arising as generalizations of concepts in linear algebra and graph theory. We will conclude our review of matroid theory by discussing the concept of *representability.* Intuitively, we want to know whether a matroid can arise as the matroid of a linear space or of a graph, or whether it is simply a purely combinatorial object. Later, we will study similar phenomena in tropical geometry: we will investigate polyhedral complexes and wish to identify whether they arise as the tropicalization of an algebraic variety or not.

Definition 3.33. A matroid M is called *representable* (in characteristic k) if there exists a linear space L (of characteristic k) such that $M = M(L)$ (as constructed in Remark 3.2). We call L a *representation* of M. Otherwise, we call M *nonrepresentable.*

If M is representable in any characteristic, we say that M is *regular.* Further, M is called *graphic* if there exists a graph G such that $M = M(G)$.

Lemma 3.34 ([120] and [138, Theorem 9.5.1])**.** *Every graphic matroid is regular, i.e., representable over any characteristic.*

Now, we give some notable examples of nonrepresentable matroids.

Example 3.35. We begin by constructing the *Fano matroid* F_7. It is a matroid used for describing the incidence geometry of lines and points in a projective plane given in Figure 4(a). Here, each vertex describes a point in some projective space, and each line describes the collinearity of points: Points that lie on a line are collinear. For instance, the points $1, 4$ and 7 are collinear, whereas the points $1, 3$ and 4 do not lie on the same line. Sometimes, a matroid arising in this way is called a *geometric matroid.*

We can read off a matroid from the geometric representation in Figure 4(a) as follows. Collinear points are linearly dependent, so independent sets are sets of vertices such that there is no line between them, and bases are the maximal such sets. Then, the bases are all subsets of size 3 except the sets corresponding to vertices lying on one of the six lines or the circle, i.e.,

$$\mathcal{B}(F_7) \;=\; \binom{[n]}{3} \setminus \{123, 147, 156, 246, 257, 345, 367\}.$$

This geometric arrangement is representable over $\mathbb{F}_2$ using all possible vectors in $\mathbb{F}_2^3$ assigned to the vertices of the diagram above as follows:

$$v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, v_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, v_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}, v_4 = \begin{bmatrix}0\\1\\1\end{bmatrix}, v_5 = \begin{bmatrix}0\\0\\1\end{bmatrix}, v_6 = \begin{bmatrix}1\\0\\1\end{bmatrix} \text{ and } v_7 = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

The arrangement is not representable in any other characteristic.

Now, let's consider the *non-Fano* matroid $\overline{F}_7$. The only difference to the Fano matroid is that now, $2, 4$ and 6 are not collinear. This changes representability drastically: the line arrangement cannot be attained in a projective space of characteristic 2. Since there are only seven distinct vectors in $\mathbb{P}^3_{\mathbb{F}_2}$ and they satisfy the geometric arrangement given by the Fano matroid, the points associated to $2, 4$ and 6 are collinear. Instead, $\overline{F}_7$ is representable in any other characteristic.

Finally, to obtain a matroid that is not representable over any characteristic, we take the direct sum of the two matroids. A smaller (but less intuitive) example of a non-representable matroid is the *Vámos matroid*, introduced by Vámos in [135] and described in [138, Theorem 11.3.7], and the *non-Pappus matroid*, shown to be nonrepresentable in [25, Example 1.1.3]. The non-Pappus matroid NP_9 is the geometric matroid depicted in Figure 5. Intuitively, its obstruction to representability is similar to the one for the Fano matroid: the collinearity

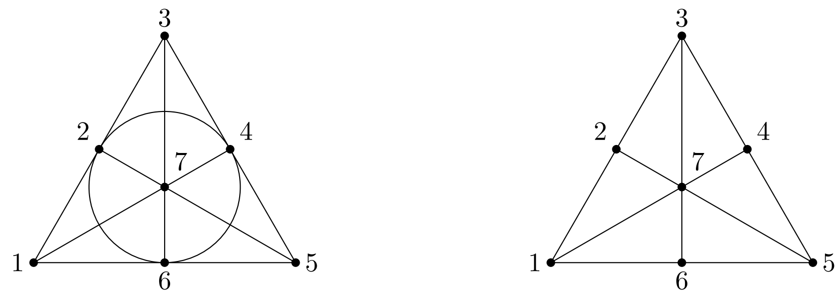


FIGURE 4. The geometric representation of the Fano matroid on the left and the non-Fano matroid on the right.

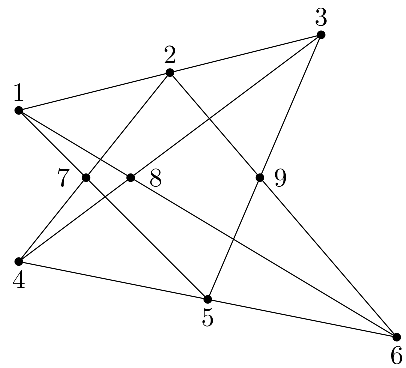


FIGURE 5. The geometric representation of the non-Pappus matroid.

conditions imposed on all points force the points 7, 8 and 9 to be collinear, but the diagram forces them to be independent.

Representability of matroids and tropicalizations of linear spaces have many connections we will see in more detail in Section 3.3. The simplest such connection is the correspondence between Bergman fans of realizable matroids and the tropicalizations of linear spaces using the trivial valuation.

Theorem 3.36 ([55])**.** *Let L be a linear space over a field k and* $\operatorname{trop}(L)$ *its tropicalization (as defined in 2.5) using the trivial valuation. Then, the support of* $\operatorname{trop}(L)$ *equals the support of the Bergman fan $\Sigma_{M(L)}$ of the underlying matroid $M(L)$ of L. Conversely, if the Bergman fan Σ is obtainable as the tropicalization* $\operatorname{trop}(L)$ *of a linear space L using the trivial valuation, then the matroid of Σ is representable.*

We have already observed the behavior stated in the theorem: compare the fans we obtained in Example 2.23 and Example 3.32.

3.2. **Morphisms of matroids.** In this section, we will study maps between matroids and analyze their properties. We will mainly focus on *strong maps*, which we will extend to morphisms of valuated matroids later. While most of the theory introduced in this section is standard, some statements here have no easy direct references in the literature. Wherever this is the case, we supply a proof for completeness' sake.

3.2.1. *Matroid quotients.* To start defining maps of matroids, we first need to define what it means for two matroids to be "contained inside of each other".

Definition 3.37. Let M and N be two matroids over the same ground set $[n]$. We say that N is a *matroid quotient* of M and write $N \twoheadleftarrow M$ if any flat of N is a flat of M.

Lemma 3.38. [139, Proposition 8.1.6] *Let M and N be two matroids on $[n]$. The following are equivalent:*

(a) N is a quotient of M.
(b) Any circuit of M is a union of circuits of N.
(c) M^* is a matroid quotient of N^*.
(d) For any $A \subseteq B \subseteq [n]$, $\mathrm{rk}_N(B) - \mathrm{rk}_N(A) \leq \mathrm{rk}_M(B) - \mathrm{rk}_M(A)$.
(e) There exists a set $[n']$ and a matroid M' on $[n] \cup [n']$ such that $N = M'/[n']$ and $M = M' \setminus [n']$.

Matroid quotients model the linear algebra concept of two vector spaces being contained inside each other. If $L_1 \subseteq L_2 \subseteq K^n$ are two linear spaces, then their matroids form a quotient, $M(L_1) \twoheadleftarrow M(L_2)$. We will see this explicitly in the next example.

Example 3.39. Consider the row spaces L_1 and L_2 of the two matrices

$$M_1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \text{ and } M_2 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}. \tag{2}$$

Then $L_1 \subseteq L_2$. The flats of M_1 are the loop 3 and the full set 123. We have seen in Example 3.30 that the flats of M_2 are $\emptyset, 1, 2, 3$ and 123. Thus, the flats of M_1 are also flats of M_2 and $M_1 \twoheadleftarrow M_2$.

Definition 3.40. Let M_1 and M_2 be two matroids over $[n]$ and let M_1 be a matroid quotient of M_2. We say that the quotient $M_1 \twoheadleftarrow M_2$ is representable if there exist representations L_1 of M_1 and L_2 of M_2 such that $L_1 \subseteq L_2$.

Representability for matroid quotients is more complicated than it is for matroids. We give an example of two matroids that are individually representable and form a quotient, but that are not representable as a quotient.

Example 3.41 (Extended version of [25, Example 1.7.7]). In Example 3.35, we had seen that the non-Pappus matroid NP_9 is not representable over any characteristic. Now, consider the matroid $M_2 = NP_9 \setminus 9$ obtained by deleting 9 from the non-Pappus matroid and the matroid $M_1 = NP_9/9$, obtained by contracting 9 in the non-Pappus matroid. The deletion M_2 is a representable matroid in characteristic ≥ 3, for instance represented as the matroid of the linear space

$$L_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}.$$

Further, the contraction M_1 is representable in characteristic ≥ 5, it can be represented by

$$L_1 = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 & 2 & 1 & 1 & 2 \end{bmatrix}.$$

By Lemma 3.38(e), the two matroids form a quotient $M_1 \twoheadleftarrow M_2$. The matroid M_1 has rank two, and only two subsets of size 2 are linearly dependent: 26 and 35. Assume there is a subspace U_1 of L_2 realizing M_1. We then perform row operations on L_2 until U_1 is the submatrix corresponding to the first two rows of the resulting matrix. We call this resulting matrix A. Since M_1 contains no loops, U_1 contains no zero columns, hence we may assume that e_3 is not a column of A.

We append e_3 as a column to the matrix A and consider the 9×3 matrix $(A|e_3)$. Then, the matroid of $(A|e_3)$ is the non-Pappus matroid NP_9 we had depicted in Figure 5, with e_3 corresponding to the point 9. This can be seen as follows. First, as M_2 is $NP_9 \setminus 9$, all independent sets of M_2 (and hence A) coincide with the independent sets of NP_9 not containing 9. Thus, we only need to confirm that the newly added column e_3 satisfies all the independence properties of the element 9 in NP_9.

By assumption, A has no column equal to e_3. As U_1 realizes M_1, the columns 2 and 6, and 3 and 5 are pairwise dependent in U_1. Then, in $(A|e_3)$, the sets $\{A_2, A_6, e_3\}$ and $\{A_3, A_5, e_3\}$ are linearly dependent as well (here, A_i denotes the i-th column of A). As all other columns in U_1 are pairwise independent, so is $\{A_i, A_j, e_3\}$ for all other pairwise distinct i and j. Thus, e_3 satisfies all properties of the ground set element 9 in NP_9. But then NP_9 is represented by the row space of the matrix $(A|e_3)$, contradicting its nonrepresentability we had discussed in Example 3.35.

3.2.2. *Morphisms and strong maps.* In linear algebra, matrices and linear maps of linear spaces are an important object of study. Generalizing linear maps of linear spaces to matroids is harder, as all generalizations in literature are less well-behaved and only cover aspects of the properties of linear maps in linear algebra. Thus, there are multiple different notions of what a morphism of matroids should be. Among those, the most well-known are *weak* and *strong maps* of matroids [97, 96]. In this thesis, we will focus on strong maps of matroids, which we will generalize to valuated matroids later on. We begin by introducing *morphisms of matroids*, which underlie the theory of strong maps. A morphism of matroids is defined as a map of ground sets that satisfies compatibility conditions with the corresponding matroids.

Definition 3.42. Let M and N be matroids over the ground sets $[m]$ and $[n]$ respectively. A *morphism* $f : M \to N$ is a function of sets from $[m]$ to $[n]$ satisfying

$$\mathrm{rk}_N(f(T_2)) - \mathrm{rk}_N(f(T_1)) \;\leq\; \mathrm{rk}_M(T_2) - \mathrm{rk}_M(T_1) \;\text{ for all }\; T_1 \subseteq T_2 \subseteq [m].$$

Example 3.43. We have secretly already seen our first example of morphisms of matroids: matroid quotients can be naturally interpreted as morphisms of matroids where f is the identity map. This follows directly from Lemma 3.38(d).

We give two explicit examples of morphisms of matroids in Example 3.47. Before we do so, we give different equivalent characterizations of morphisms of matroids, which will allow us to check the morphism properties more easily. To this end, we define the *induced matroid*, which will take the place of a classical preimage. For completeness' sake, we include a proof of the well-known fact that this is a matroid.

Proposition-Definition 3.44. *Let N be a matroid over the ground set $[n]$ and let $m \in \mathbb{N}$. Let $f : [m] \to [n]$ be a function of sets. Then, we can define a matroid $f^{-1}(N)$ on $[m]$, called the* induced matroid*, defined by the rank function*

$$\mathrm{rk}_{f^{-1}(N)}(T) \;=\; \mathrm{rk}_N(f(T)) \textit{ for } T \subseteq [n].$$

Proof. We show that $\mathrm{rk}_{f^{-1}(N)}$ is a rank function. First, we note that since N is a matroid, $\mathrm{rk}_{f^{-1}(N)}(T) = \mathrm{rk}_N(f(T)) \in \mathbb{Z}_{\geq 0}$. The other criteria laid out in Definition 3.9 can be seen as follows.

(R1) We have $\mathrm{rk}_{f^{-1}(N)}(\emptyset) = \mathrm{rk}_N(f(\emptyset)) = \mathrm{rk}_N(\emptyset) = 0$.
(R2) Since f is a function, $f(T) \subseteq f(T \cup x)$ for all $x \in [m]$. Thus, by rk_N being monotonic, $\mathrm{rk}_{f^{-1}(N)}(T) = \mathrm{rk}_N(f(T)) \leq \mathrm{rk}_N(f(T \cup x)) = \mathrm{rk}_{f^{-1}(N)}(T \cup x)$. Finally, since $f(T \cup x) = f(T) \cup f(x)$, $\mathrm{rk}_{f^{-1}(N)}(T \cup x) \leq \mathrm{rk}_{f^{-1}(N)}(T) + 1$.
(R3) Let $T_1, T_2 \subseteq [n]$. We show submodularity as follows.

$$\begin{aligned}
\mathrm{rk}_{f^{-1}(N)}(T_1 \cup T_2) + \mathrm{rk}_{f^{-1}(N)}(T_1 \cap T_2) &= \mathrm{rk}_N(f(T_1 \cup T_2)) + \mathrm{rk}_N(f(T_1 \cap T_2)) \\
&\leq \mathrm{rk}_N(f(T_1) \cup f(T_2)) + \mathrm{rk}_N(f(T_1) \cap f(T_2)) \\
&\leq \mathrm{rk}_N(f(T_1)) + \mathrm{rk}_N(f(T_2)) \\
&= \mathrm{rk}_{f^{-1}(N)}(T_1) + \mathrm{rk}_{f^{-1}(N)}(T_2),
\end{aligned}$$

where the first inequality follows from f being a function of sets and rk_N being monotonic and the second inequality follows from the submodularity of rk_N (see Definition 3.9(R3)). ☐

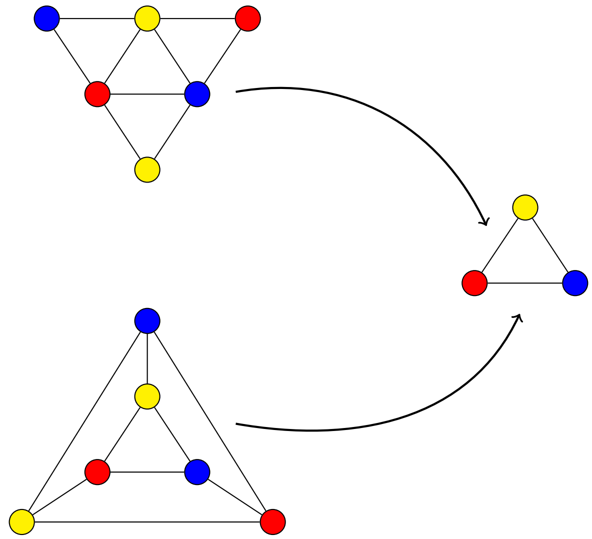
Lemma 3.45 ([53, Definition 2.1, Lemma 2.4])**.** *Let M and N be matroids over the ground sets $[m]$ and $[n]$ respectively, and let $f : [m] \to [n]$ be a function of sets. The following are equivalent:*

(a) f is a morphism of matroids;
(b) The induced matroid $f^{-1}(N)$ is a quotient of M, i.e., $f^{-1}(N) \twoheadleftarrow M$.
(c) If $T \subseteq [n]$ is a cocircuit of N, then $f^{-1}(T)$ is a union of cocircuits of M;
(d) If $T \subseteq [n]$ is a flat of N, then $f^{-1}(T)$ is a flat of M;
(e) If $T \subseteq [n]$ is a circuit of M, then T is a union of circuits of $f^{-1}(N)$, i.e., $f(T)$ is a union of circuits of N.

Proof. The characterizations not explicitly laid out in [53, Lemma 2.4] are a direct consequence of the equivalent characterizations of matroid quotients, see Lemma 3.38. ☐

Remark 3.46. If we are mainly interested in graphical matroids, this *is* the correct generalization of a homomorphism. Any homomorphism of graphs induces a morphism of the underlying matroids (see, for instance, [53, Remark 1.5]). We describe this below in Example 3.47.

Example 3.47. Recall that a homomorphism of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a map of vertices $V_1 \to V_2$ that preserves incidences: Adjacent vertices are mapped to adjacent vertices. The two pictures below can be understood as follows. On the right, we have the graphical representation of $U_{2,3}$ which we introduced in Example 3.3 and studied in various examples in Section 3.1. On the left, we have two graphs representing matroids. The color of a vertex indicates which vertex it gets mapped to. The bottom graph on the left is [53, Example 1.5].



While checking that these maps are morphisms is possible using the first definition we gave, it is much more convenient to use Lemma 3.45: Note that any simple cycle in the graphs on the left has at least one vertex each that is red, yellow, and blue. Thus, any circuit maps onto the sole circuit on the right, and Lemma 3.45(e) now ensures the map is a morphism. We will discuss a more systematic approach to verifying whether a map of ground sets is a morphism of matroids in Example 3.51.

We can verify that $f : M \to N$ is an isomorphism of matroids in the "categorical sense" (i.e., if there exists a morphism $g : M \to N$ such that $f \circ g = \mathrm{id}_N$ and $g \circ f = \mathrm{id}_M$), then the two matroids M and N are isomorphic, i.e., their bases coincide up to consistent relabeling of the ground set.

Lemma 3.48 (Isomorphisms)**.** *Let M and N be matroids over the ground sets $[m]$ and $[n]$ respectively. Let $f : M \to N$ be a morphism of matroids. Then, f satisfies the following properties:*

(a) Let $T \subseteq [m]$. Then $\mathrm{rk}_N(f(T)) \leq \mathrm{rk}_M(T)$.
(b) Assume further that $f^{-1} : N \to M$ is well-defined and a morphism. Then, $M \cong N$, i.e., f is a bijection that preserves independent sets (see [138, Definition 1.2]).

Proof. (a). We have $\emptyset \subseteq T$. Further, $\mathrm{rk}(\emptyset) = 0$ for all matroids. Thus,

$$\mathrm{rk}_N(f(T)) - \mathrm{rk}_N(f(\emptyset)) \leq \mathrm{rk}_M(T) - \mathrm{rk}_M(\emptyset) \quad \Leftrightarrow \quad \mathrm{rk}_N(f(T)) - 0 \leq \mathrm{rk}_M(T) - 0.$$

(b). By f^{-1} being a well-defined map, $f : [m] \to [n]$ is a bijection of sets, and, in particular, $[m] = [n]$. Further, for all $T_1 \subseteq T_2 \subseteq [m]$:

$$\begin{aligned} \mathrm{rk}_N\big(f(T_2)\big) - \mathrm{rk}_N\big(f(T_1)\big) &\leq \mathrm{rk}_M(T_2) - \mathrm{rk}_M(T_1) = \mathrm{rk}_M\big(f^{-1}(f(T_2))\big) - \mathrm{rk}_M\big(f^{-1}(f(T_1))\big) \\ &\leq \mathrm{rk}_N\big(f(T_2)\big) - \mathrm{rk}_N\big(f(T_1)\big), \end{aligned}$$

where the first inequality follows by f being a morphism, the equality by $f^{-1}(f(T)) = T$ and the last inequality is due to f^{-1} being a morphism. Thus, f preserves flats:

Let U be a flat of M. Then, for all $i \in [m] \setminus U$, $1 = \mathrm{rk}_M(U \cup i) - \mathrm{rk}_M(U) = \mathrm{rk}_N\big(f(U \cup i)\big) - \mathrm{rk}_N\big(f(U)\big) = \mathrm{rk}_N\big(f(U) \cup f(i)\big) - \mathrm{rk}_N\big(f(U)\big)$. Now, since f is a bijection of sets, $f([m] \setminus U) = f([m]) \setminus f(U) = [n] \setminus f(U)$, so U is a flat of N. Analogously, any flat of N is a flat of M, so f preserves flats and hence independent sets. □

One of the main obstructions of viewing morphisms of matroids as the appropriate generalization of linear maps of linear spaces is the fact that morphisms of matroids cannot encode coordinate projections: there is no a priori concept of a "zero vector" in matroid theory that vectors could get mapped to. This can be dealt with by adding a loop o to each matroid to encode a zero vector.

Definition 3.49. Let M be a matroid over the ground set $[n]$. Its associated *pointed matroid* M_o is a matroid over the ground set $[n] \cup \{o\}$ with the same bases as M, i.e., $\mathcal{B}(M_o) = \mathcal{B}(M)$. Alternatively, we can describe M_o as $M \oplus U_{0,1}$.

A *strong map* $f : M \to N$ of matroids is a morphism $f : M_o \to N_o$ of matroids (in the sense of Definition 3.42) satisfying that $f(o) = o$.

Remark 3.50. Let M and N be matroids over the ground sets $[m]$ and $[n]$ respectively.

(a) By [79, Lemma 3.2] any morphism of matroids $f : M \to N$ can be extended to a strong map $f_o : M_o \to N_o$ by setting $f_o(i) = f(i)$ for all $i \in [m]$ and $f_o(o) = o$. In Example 3.51 we will see that the converse of this is not true, and that there exist pairs of matroids that have strong maps between them but not morphisms.

(b) Using strong maps instead of morphisms is not a serious issue for the interpretation we gave in Example 3.47 for graphic matroids — if $f : G_1 \to G_2$ is a homomorphism of graphs, then any loop on G_1 gets mapped to a loop on G_2 incident to the correct vertex by the definition of graph homomorphisms. It is just a less natural construction in this setting.

Example 3.51. We give an example of two matroids that have a strong map between them, but no morphisms. We consider the matroid $M = U_{0,1}$ of rank 0 on [1] whose basis is the empty set, and the uniform matroid $N = U_{2,4}$. Then, there exist no morphisms of matroids between M and N, but there exists a strong map — the map sending both loops of M_o to the unique loop o in N_o.

For computations checking whether a given map of sets is a morphism of matroids or a strong map and for generating all morphisms, resp. strong maps between two matroids M and N, we have written `Oscar` [114] code. The code is supplied in Appendix B.1 and can be found in executable form at `https://victoriaschleis.github.io/thesis/`. Below, you can see the computational check that the above map is a morphism of matroids. Computationally, we construct the map as a dictionary, which records the image of elements of the ground set under the map f. For instance, below we consider a map from the ground set [1] to [4] (resp, $[4] \cup \{o\}$, where `5` denotes o). The map sends the sole element `1` to `4`, resp. `5`.

```
julia> M = matroid_from_bases([[]],1);

julia> N = uniform_matroid(2,4);

julia> f = Dict(1=>4);

julia> is_morphism_of_matroids(M, N, f)
false

julia> f2 = Dict(1=>5);
```

```
julia> is_strong_map(M, N, f2)
true
```

Using the function `find_all_morphisms` and `find_all_strong_maps` from Appendix B.1 on the two matroids discussed above yields the assertion made above:

```
julia> find_all_morphisms(M, N)
Dict{Int64, Int64}[]
julia> find_all_strong_maps(M, N)
1-element Vector{Dict{Int64, Int64}}:
Dict(2 => 5, 1 => 5)
```

Just as every linear space has an associated matroid, every linear map of linear spaces has an associated strong map of matroids.

Lemma 3.52. [53, 79] *Let V and W be two linear spaces, and let $f : V \to W$ be a map. Then, f induces a strong map of the underlying matroids, $f_M : M(V) \to M(W)$.*

3.2.3. *Projections.* We now consider an important subclass of examples: projections. These are interesting for many reasons: for instance, as we had seen in Lemma 4.10, all realizable strong maps of matroids are equivalent to projection maps. Further, projection maps will be one of the important special types of maps in the valuated setting in Section 5.1, and their parameter spaces will be our main object of study in Section 10.

Definition 3.53. Let M and N be matroids over a common ground set $[n]$. Let $S \subseteq [n]$. Then, we define the *projection map* $\mathrm{pr}_S : M_o \to N_o$ as the map of sets

$$\mathrm{pr}_S : [n]_o \to [n]_o$$
$$\mathrm{pr}_S(i) = \begin{cases} i & \text{if } i \notin S, \\ o & \text{if } i \in S. \end{cases}$$

We call a projection map that is additionally a strong map a *projection morphism*.

Not all naive projection maps defined in this way are projection morphisms:

Example 3.54. We consider the map $\mathrm{pr}_3 : U_{2,3_o} \to U_{2,3_o}$. Here, pr_3 is not a morphism: we consider the sets $13 \subseteq 123$. Then

$$1 \;=\; \mathrm{rk}(12) - \mathrm{rk}(1) \;=\; \mathrm{rk}(\mathrm{pr}_3(123)) - \mathrm{rk}(\mathrm{pr}_3(13)) \;>\; \mathrm{rk}(123) - \mathrm{rk}(13) \;=\; 0,$$

so pr_3 is not a morphism. This should not surprise us: Consider the representation of $U_{2,3_o}$ given in matrix form on the left side below, where $U_{2,3_o}$ is the row space of the matrix and the columns of the matrix are the elements of the matroid. Then, there exists no linear map representing the naive projection of the third column to zero, i.e., no matrix A satisfying

$$A \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \;=\; \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Further, not all morphisms of matroids can be represented as projection morphisms:

Example 3.55. We consider $f : U_{3,3_o} \to U_{3,3_o}$ given by $f(o) = o, f(1) = 1, f(2) = 2$ and $f(3) = 2$.

First, we note that this is a morphism of matroids since every map of sets $g : [3]_o \to [3]_o$ from $U_{3,3_o}$ to itself that sends o to itself is a morphism of matroids as every flat of $g^{-1}(N)$

contains o by construction and the flats of $U_{3,3_o}$ are all sets that contain o. Now $f^{-1}(U_{3,3})$ is a matroid of rank 2 with only loop o. Assume f is represented as a projection morphism pr_S. Then, $S \neq \emptyset$ as $\mathrm{rk}(f^{-1}(U_{3,3})) = 2$ whereas $\mathrm{rk}(U_{3,3}) = 3$. So, assume $i \in S$. Then i is a loop in $f^{-1}(U_{3,3})$, contradiction.

We can characterize bases of induced matroids under projection maps in the following way.

Lemma 3.56. *Let $\mathrm{pr}_S : M \to N$ be a projection morphism in the sense of Definition 3.53. Let I be a basis of $\mathrm{pr}_S^{-1}(N)$. Then, I satisfies the following properties:*

(a) I is an independent set of both M and N, and $I \cap S = \emptyset$;
(b) Any basis $B \supset I$ of N is of the form $B = I \cup (B \cap S)$.

Proof. (a) Since I is a basis of $\mathrm{pr}_S^{-1}(N)$,

$$|I| \ = \ \mathrm{rk}_{\mathrm{pr}_S^{-1}(N)}(I) \ = \ \mathrm{rk}_N(\mathrm{pr}_S(I)) \ \leq \mathrm{rk}_M(I) \ \leq \ |I|,$$

where the first inequality follows by Lemma 3.48(a). Thus, $|I| = \mathrm{rk}_M(I)$, so I is an independent set of M. Further, since $|I| = \mathrm{rk}_N(\mathrm{pr}_S(I)) \leq |\mathrm{pr}_S(I)|$, $|I| = |\mathrm{pr}_S(I)|$, so $I \cap S = \emptyset$. Now, this implies that $|I| = \mathrm{rk}_N(\mathrm{pr}_S(I)) = \mathrm{rk}_N(I)$, so I is an independent set of N.

(b) Assume not. Let B be a basis of N that contains $j \in B$ where $j \notin I$ and $j \notin S$. We consider $I \cup j$. By construction, $\mathrm{rk}_N(I \cup j) = \mathrm{rk}_N(I) + 1$. Since M and N are matroids over the same ground field and pr_S is the identity outside S, j and hence $I \cup j$ can also be interpreted over M.

First, assume $I \cup j$ is independent over M. Then,

$$|I| + 1 \ = \ |I \cup j| \ = \ \mathrm{rk}_M(I \cup j) \ = \ \mathrm{rk}_N(I \cup j) \ = \ \mathrm{rk}_N\left(\mathrm{pr}_S(I \cup j)\right),$$

as $I \cup j = \mathrm{pr}_S(I \cup j)$. But then I is not a basis of pr_S^{-1}, contradiction.

So, assume $I \cup j$ is dependent over M. Since I is a basis of $\mathrm{pr}_S^{-1}(N)$, thus by (a) an independent set of M, $\mathrm{rk}_M(I \cup j) = \mathrm{rk}_M(I)$. Thus,

$$1 \ = \ \mathrm{rk}_N(I \cup j) - \mathrm{rk}_N(I) \ = \ \mathrm{rk}_N(\mathrm{pr}_S(I \cup j)) - \mathrm{rk}_N(\mathrm{pr}_S(I)) \ > \ \mathrm{rk}_M(I \cup j) - \mathrm{rk}_M(I) \ = \ 0,$$

contradiction to pr_S being a morphism. Thus, $I \cup j$ is dependent over N, hence $B \supset I \cup j$ is not a basis.

□

3.2.4. *Categorical Properties.* We conclude this section by giving a brief review of the categorical properties of the category of matroids with morphisms of matroids **Matr** and the category of matroids with strong maps $\mathbf{Matr}_o$. Most of the properties reviewed here were shown by Heunen-Patta in [79], though some properties go back to Kung [139, Chapter 8]. Many of the categorical properties are just restatements of properties we showed in Section 3.2, expressed in the language of category theory.

Proposition 3.57 ([79, Figure 1.1 and Propositions 3.2-3.7])**.** *The categories* **Matr** *and* $\mathbf{Matr}_o$ *have coproducts which are the direct sums of matroids we defined in Definition 3.25. They do not in general have products, pullbacks, or pushouts. Further, addition and deletion are functors from the categories to themselves, whereas dualization is not.*

The category $\mathbf{Matr}_o$ has slightly more structure than $\mathbf{Matr}$: it has a zero object, the matroid $U_{0,1}$ we discussed in Example 3.51. The category $\mathbf{Matr}$ has no zero element: the matroid $U_{0,1}$ is still a terminal object, but it is not initial. This is exactly the difference we observed in Example 3.51. Thus, the two categories are not equivalent.

Further, the category $\mathbf{Matr}_o$ is a reflexive subcategory of $\mathbf{Matr}$, where the inclusion of $\mathbf{Matr}_o$ into $\mathbf{Matr}$ has a left adjoint $(\cdot)_o$, given by the procedure described in Remark 3.50(a), see [79, Lemma 3.2].

3.3. **Valuated matroids and tropical linear spaces.** In this section, we enrich the structure of matroids with an additional value assigned to each basis. The structure we obtain was first defined by Dress-Wenzel [51]. Since then, many different cryptomorphic sets of axioms have been developed, mainly by Murota [111, 112]. One of the motivating ideas behind matroids is that for linear spaces, matroids encode the nonsingular minors of the matrix defined by their generating sets. For valuated matroids, we retain more structure: we do not just remember which minors are zero or non-zero, instead we remember the valuation of the minor. It turns out that this theory can be translated to tropical geometry: valuated matroids are in one-to-one correspondence with *tropical linear spaces* [132], which are a generalization of the tropicalized linear spaces we have encountered in Section 2.2.2.

3.3.1. *Bases and independent sets.* We begin by defining valuated matroids as a function on bases. This is the original definition, due to Dress-Wenzel [51].

Definition 3.58. A *valuated matroid* of rank r on the ground set $[n]$ is a function $\mu : \binom{[n]}{r} \to \mathbb{T}$ such that $\mu(B) \neq \infty$ for some $B \in \binom{[n]}{r}$ and, for all $I, J \in \binom{[n]}{r}$ and $i \in I \setminus J$, there exists $j \in J \setminus I$ satisfying

$$\mu(I) + \mu(J) \ \geq \ \mu(I \setminus i \cup j) + \mu(J \setminus j \cup i).$$

We say that two valuated matroids μ and ν on a common ground set $[n]$ are *equivalent* if there exists $a \in \mathbb{R}$ such that $\mu(B) = \nu(B) + a$ for every $B \in \binom{[n]}{r}$. In other words, every equivalence class of a valuated matroid $\nu : \binom{[n]}{r} \to \mathbb{T}$ can be seen as a point in $\mathbb{P}\left(\mathbb{T}^{\binom{n}{r}}\right)$. Throughout, we only consider valuated matroids up to equivalence.

As in the case of matroids, every linear space over a field with valuation has an associated matroid. It can be constructed as follows.

Remark 3.59. Consider a field K with valuation $\mathrm{val} : K \to \mathbb{T}$ and an r-dimensional vector subspace L of K^n given as the minimal row span of a matrix A. We denote by $(p_I)_I$ the *Plücker coordinates* of A, where p_I is the minor of A indexed by $I \in \binom{[n]}{r}$. Then, the function $\mu(A) : \binom{[n]}{r} \to \mathbb{T}$ defined by $I \mapsto \mathrm{val}(p_I)$ is a valuated matroid.

Example 3.60. We return to the matroid $U_{2,3}$ we considered extensively in Section 3.1. This time, we choose a different set of vectors to define the matroid — the column vectors of the following matrix defined over the field of Puiseux series $\mathbb{C}\{\{t\}\}$ we discussed in Definition 2.4:

$$L \ = \ \begin{bmatrix} t^b & -t^a & 0 \\ 1 & 0 & -t^a \end{bmatrix}.$$

Computing the minors of the above matrix, we obtain the valuated matroid $\mu(L) : \binom{[3]}{2} \to \mathbb{T}$ with individual values $\mu(12) = \mathrm{val}(t^a) = a$, $\mu(13) = \mathrm{val}(t^{a+b}) = a + b$ and $\mu(23) =$

$\mathrm{val}(t^{2a}) = 2a$. This valuated matroid is equivalent to the matroid $\tilde{\mu}$ obtained by subtracting a from all values of $\mu(L)$. For this matroid, the values are $\tilde{\mu}(12) = 0$, $\tilde{\mu}(13) = b$ and $\tilde{\mu}(23) = a$. In the following, we will identify the matroids $\tilde{\mu}$ and $\mu(L)$.

Definition 3.61. Given a valuated matroid $\nu : \binom{[n]}{r} \to \mathbb{T}$, the set $\{B \in \binom{[n]}{r} : \nu(B) \neq \infty\}$ forms the bases of a matroid N, which we call the *underlying matroid* of ν.

Remark 3.62. Recall from Section 2.1 that every field has a valuation using the trivial valuation. The information of a valuated matroid obtained from a linear space over a field with trivial valuation is precisely that of a matroid — it only recalls whether a set is a basis or is not.

As for matroids, there are again a multitude of different ways to define valuated matroids. One way that will be helpful to study morphisms of valuated matroids is via independent sets. This theory was developed by Murota in [111].

Definition 3.63. A function $\overline{\mu} : 2^{[n]} \to \mathbb{T}$ is called a *valuated matroid* if for all $G, H \subseteq [n]$, the following are satisfied:

(VI1) If $G \subseteq H$, then $\overline{\mu}(G) \leq \overline{\mu}(H)$
(VI2) For $G \in \mathrm{Ind}(M) \setminus \mathcal{B}(M)$, there exists $v \in [n] \setminus G$ such that $\overline{\mu}(G) = \overline{\mu}(G \cup v)$.
(VI3) If $|G| = |H|$, for all $g \in G \setminus H$ there exists $h \in H \setminus G$ such that $\overline{\mu}(G) + \overline{\mu}(H) \geq \overline{\mu}(G \setminus g \cup h) + \overline{\mu}(H \setminus h \cup g)$.
(VI4) If $|G| = |H| - 1$, there exists $h \in H \setminus G$ such that $\overline{\mu}(G) + \overline{\mu}(H) \geq \overline{\mu}(G \cup h) + \overline{\mu}(H \setminus h)$.

Sometimes, we call axiom (VI2) the *extension axiom*, axiom (VI3) the *independent set exchange axiom*, and axiom (VI4) the *augmentation axiom*.

Remark 3.64 ([111, Construction 3.1, Theorem 3.2]). We can connect valuations on independent sets again with valuations on bases. Assume we have a valuated matroid μ defined on bases. Then, we can define a valuation $\overline{\mu}(\cdot)$ on independent sets G by setting

$$\overline{\mu}(G) \;:=\; \min_{\substack{B \in \mathcal{B}(M) \\ G \subseteq B}} \mu(B).$$

On dependent sets, the valuation is formally set to be ∞.

Example 3.65. In Example 3.60 we had seen the valuated matroid $\mu : \binom{[3]}{2} \to \mathbb{T}$ with individual values $\mu(12) = 0$, $\mu(13) = b$ and $\mu(23) = a$. Using our new notion of valuation, we can extend μ to

$$\overline{\mu} \;:\; 2^{[n]} \to \mathbb{T}$$

where the values are given as below:

S	$\emptyset$	1	2	3	12	13	23	123
$\overline{\mu}(S)$	$\min\{0, a, b\}$	$\min\{0, b\}$	$\min\{0, a\}$	$\min\{a, b\}$	0	b	a	∞

3.3.2. *Valuated circuits, cocircuits and tropical linear spaces.* In linear tropical geometry, definitions of valuated matroids via their circuits and cocircuits play an important role, as this is the way one usually constructs tropical linear spaces. As the theory of valuations on independent sets, valuations on circuits and cocircuits were first defined by Murota [112], and have since been studied in various tropical contexts, for instance in [100]. In the projective setting, they were first considered in [129], and have since been studied, for instance, in [30].

Definition 3.66. Let μ be a valuated matroid of rank r on $[n]$. For each $I \in \binom{[n]}{r+1}$ define an element $C_\mu(I) \in \mathbb{T}^n$ by

$$C_\mu(I)_i \;=\; \begin{cases} \mu(I \setminus i) & i \in I, \\ \infty & i \notin I. \end{cases}$$

The set of *valuated circuits* $\mathcal{C}(\mu)$ of μ is defined as the image of

$$\left\{ C_\mu(I) : I \in \binom{[n]}{r+1} \right\} \setminus \{(\infty, \dots, \infty)\}.$$

in $\mathbb{P}(\mathbb{T}^n)$.

Definition 3.67. Let μ be a valuated matroid on $[n]$. The *tropical linear space* of μ is the tropical prevariety

$$\overline{\text{trop}}(\mu) \;=\; \bigcap_{C \in \mathcal{C}(\mu)} V\left(\bigoplus_{i \in [n]} C_i \odot x_i \right) \;\subseteq\; \mathbb{P}(\mathbb{T}^n),$$

which is the intersection of tropical hypersurfaces of the linear forms whose coefficients are the circuit entries.

Remark 3.68. Since we allow for matroids with loops, our matroids will correspond to *projective* tropical linear spaces, i.e., tropical linear spaces are tropical prevarieties living in $\mathbb{P}(\mathbb{T}^n)$. For some tropical linear spaces, these will correspond to tropicalizations of linear spaces in projective space as we had introduced in Section 2.2.

Example 3.69. Consider again the matroid $U_{2,3}$ with the valuation we defined in Example 3.60. We will now compute its tropical linear space. To this end, we compute the valuation of its sole circuit 123. We have

$$C_\mu(123)_1 = \mu(123 \setminus 1) = \mu(23) = a,$$
$$C_\mu(123)_2 = \mu(123 \setminus 2) = \mu(13) = b, \text{ and}$$
$$C_\mu(123)_3 = \mu(123 \setminus 3) = \mu(12) = 0,$$

thus $C_\mu(123) = (a, b, 0)$. Now, its tropical linear space is $V(a \odot x_1 \oplus b \odot x_2 \oplus 0 \odot x_3)$. The associated polyhedral complex is a translation of the classical tropical line, depicted in Figure 6.

Definition 3.70. Let μ be a valuated matroid of rank r over the ground set $[n]$. For each $I \in \binom{[n]}{r-1}$, we define $C^*_\mu(I) \in \mathbb{T}^n$ by

$$C^*_\mu(I)_i \;=\; \begin{cases} \mu(I \cup i) & i \notin I \\ \infty & i \in I \end{cases}$$

The *valuated cocircuits* of μ are defined as the image of

$$\mathcal{C}^*(\mu) \;=\; \left\{ C^*_\mu(I) : I \in \binom{[n]}{r-1} \right\} \setminus \{(\infty, \dots, \infty)\}$$

‘ in $\mathbb{P}(\mathbb{T}^n)$. The support of a cocircuit $C^* \in \mathcal{C}^*(\mu)$ is the set $\text{supp}(C^*) = \{i \in [n] : C^*_i \neq \infty\}$.

Proposition 3.71 (Originally [112, Theorem 3.4], for the formulation here, see [100, Section 2.1])**.** *Let μ be a valuated matroid of rank r on $[n]$. The tropical linear space $\overline{\text{trop}}(\mu)$ can be equivalently defined in terms of the cocircuits of a matroid,*

$$\overline{\text{trop}}(\mu) \;=\; \left\{ \bigoplus_{C^* \in \mathcal{C}^*(\mu)} \lambda_{C^*} \odot C^* : \lambda_{C^*} \in \mathbb{R} \right\}.$$

Example 3.72. We recompute the tropical linear space of $U_{2,3}$ we had computed in Example 3.69, using valuated cocircuits this time. The valuated cocircuits can be computed as follows.

$$\begin{array}{c||c||c} C^*_\mu(1)_1 = \infty & C^*_\mu(2)_1 = \mu(12) = 0 & C^*_\mu(3)_1 = \mu(13) = b \\ C^*_\mu(1)_2 = \mu(12) = 0 & C^*_\mu(2)_2 = \infty & C^*_\mu(3)_2 = \mu(23) = a \\ C^*_\mu(1)_3 = \mu(13) = b & C^*_\mu(2)_3 = \mu(23) = a & C^*_\mu(3)_3 = \infty. \end{array}$$

Thus, the three valuated cocircuits are (∞, b, a), (b, ∞, b) and (a, b, ∞). We draw the resulting tropical linear space in Figure 6.

As instead of considering linear spans for cocircuits, we can also define a valuated matroid by giving a set of vectors corresponding to linear spans of the circuits. These are defined as follows:

Definition 3.73. Let μ be a valuated matroid of rank r over $[n]$. A *vector* (or *valuated cycle*) of μ is any element of $\mathbb{P}(\mathbb{T}^n)$ tropically generated by the valuated circuits. We write the family of vectors as

$$\mathcal{V}(\mu) \;=\; \left\{ \bigoplus_{C \in \mathcal{C}(\mu)} \lambda_C \odot C : \lambda_C \in \mathbb{T}, \lambda_C \neq \infty \right\}.$$

The tropical span of cocircuits of a valuated matroid equivalently defines $\overline{\text{trop}}(\mu)$, see [30, Theorem B].

Vectors satisfy the following axiom system.

Proposition 3.74 (Vector elimination, originally [112], for the formulation here, see [100])**.** *A subset $\mathcal{V} \subseteq \mathbb{T}^n$ is the set of vectors of a valuated matroid if and only if*

(VE1) it is a subsemimodule of $\mathbb{T}^n$, i.e., closed under tropical addition and tropical scalar multiplication; and

(VE2) for any $\mathbf{g}, \mathbf{h} \in \mathcal{V}$ and $i \in [n]$ such that $g_i = h_i \neq \infty$, there exists $\mathbf{f} \in \mathcal{V}$ such that $f_i = \infty$, $f_j \geq g_j \oplus h_j$ for all $j \in ([n] \setminus i)$ and $f_j = g_j \oplus h_j$ for all $j \in ([n] \setminus i)$ where $g_j \neq h_j$.

The notion of representability carries over from classical matroid theory to valuated matroids and tropical linear spaces. Here, it coincides with the realizability of the tropical prevariety as a tropical variety.

Definition 3.75. Let μ be a valuated matroid over $[n]$. We say that μ called *representable* (over K) if there exists a linear space $L \subseteq K^n$ such that $\mu = \mu(L)$ as in Remark 3.59.

A tropical linear space $\overline{\text{trop}}(\mu)$ is called *realizable* (over K) if it is realizable as a tropical prevariety, i.e., if there exists a linear space $L \subseteq K^n$ such that $\overline{\text{trop}}(L) = \overline{\text{trop}}(\mu)$.

Proposition 3.76 ([131])**.** *Let μ be a valuated matroid over $[n]$ and L a linear subspace of K^n. The following are equivalent:*

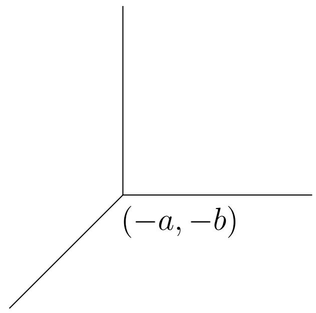


FIGURE 6. The tropical linear space of $U_{2,3}$ with the valuation given in Example 3.60, under the assumption $0 < a, b$.

(a) The valuated matroid μ is representable by L, i.e., $\mu(L) = \mu$.
(b) The tropical and the tropicalized linear space coincide, i.e., $\overline{\text{trop}}(L) = \overline{\text{trop}}(\mu)$.

3.3.3. *Constructions on valuated matroids.* We can extend all the constructions we had considered for matroids to the valuated setting. We start by giving a valuation function for the dual matroid:

Definition 3.77 ([51, Proposition 1.4])**.** Let μ be a valuated matroid of rank r on $[n]$. We define the *dual valuated matroid* μ^* of rank $n-r$ on $[n]$ by setting

$$\mu^* \; : \binom{[n]}{n-r} \to \mathbb{T}$$
$$S \mapsto \mu(S^c).$$

Now, we consider deletions and contractions.

Definition 3.78 ([51, Proposition 1.2])**.** Let μ be a matroid of rank r on $[n]$ with underlying matroid M, and let $S \subseteq [n]$. Let k be the rank of the deletion $M \setminus S$. Choose $I \in \binom{S}{r-k}$ such that $(S^c) \cup I$ has rank r in M. Then, the map $\mu \setminus S : \binom{S^c}{k} \to \mathbb{T}$ defined by

$$(\mu \setminus S)(B) \;=\; \mu(B \cup I)$$

is a valuated matroid, with underlying matroid $M \setminus S$. Further, $\mu \setminus S$ is compatible with equivalence, and different choices of I give rise to equivalent valuated matroids. The valuated matroid $\mu \setminus S$ is called the *deletion* of μ by $S \subseteq [n]$.

This construction can be equivalently stated in terms of the valuated circuits of the matroid:

Definition 3.79. Let μ be a valuated matroid on $[n]$ and $S \subseteq [n]$. The valuated circuits of the deletion $\mu \setminus S$ are

$$\mathcal{C}(\mu \setminus S) = \left\{ C_{|S^c} : C \in \mathcal{C}(\mu), \text{supp}(C) \subseteq S^c \right\}. \tag{3}$$

This is a direct consequence of [112, Theorem 3.1]. For the formulation used here, see [30, Theorem 3.1.6].

Definition 3.80. Let μ be a valuated matroid on $[n]$ and let $S \subseteq [n]$. We define the *contraction* of μ by S as $\mu/S = (\mu^* \setminus S)^*$.

Finally, just as in the case of matroids, we can define the direct sum of two valuated matroids.

Definition 3.81 ([81, Definition 2.6])**.** Let μ and ν be two valuated matroids over the ground sets $[n]$ and $[n']$. Their direct sum can be given as

$$\mu \oplus \nu \;:\; \binom{[n] \sqcup [n']}{\mathrm{rk}(\mu)+\mathrm{rk}(\nu)} \to \mathbb{T}$$
$$\mu \oplus \nu(X) \;=\; \mu(X \cap [n]) + \nu(X \cap [n']).$$

3.4. **Parameter spaces.** In algebraic geometry, one often considers *parameter spaces*, i.e., spaces whose points correspond to all objects satisfying a specific property. In this thesis, parameter spaces will be some of the most important objects of study: Later, we will introduce quiver Dressians as parameter spaces for more complicated arrangements of linear spaces (see Part 2), and in Part 3 we will study different moduli spaces of algebraic, logarithmic and tropical curves. In this section, we will discuss parameter spaces for linear spaces (see Section 3.4.1) and for flags of linear spaces (see Section 3.4.2).

3.4.1. *The tropical Grassmannian and the Dressian.* In algebraic geometry, the parameter space of r-dimensional subspaces of n-space is the *Grassmannian*. It can be characterized as follows.

Definition 3.82. Let $r \leq n$ be a nonnegative integer. The *Grassmann-Plücker relations* are polynomials in the variables $\{p_I : I \in \binom{[n]}{r}\}$ with coefficients in K,

$$\mathscr{P}_{r;n} \;=\; \left\{ \sum_{j \in J\setminus I} \mathrm{sign}(j; I, J) p_{I\cup j} p_{J \setminus j} : I \in \binom{[n]}{r-1}, J \in \binom{[n]}{r+1} \right\},$$

where $\mathrm{sign}(j; I, J) = (-1)^{|\{j' \in J : j < j'\}| + |\{i \in I : i > j\}|}$. These relations define the image of the Grassmannian $\mathrm{Gr}(r; n)$ in the projective space $\mathbb{P}^{\binom{n}{r}-1}$ via the Plücker embedding,

$$\mathrm{Gr}(r; n) \;=\; V\big(\langle \mathscr{P}_{r;n} \rangle\big).$$

Example 3.83. The Grassmannian $\mathrm{Gr}(2; 4)$ is cut out by a single Plücker relation

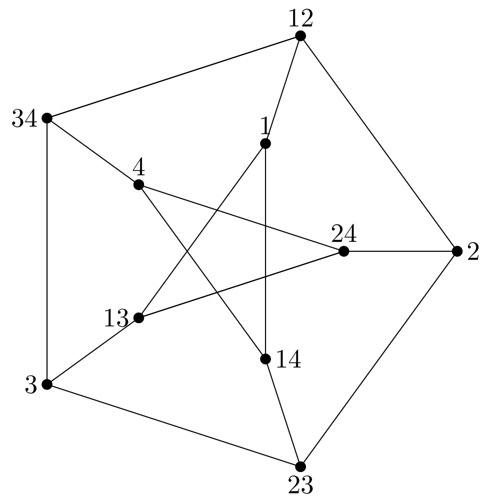
$$f \;=\; p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}.$$

After tropicalizing f as defined in Section 2.1 and quotienting out by the lineality space, the tropical hypersurface $V(\overline{\mathrm{trop}}(f))$ is a tropical line which we have already seen in Example 2.23. Its three rays correspond to the three different combinatorial types of tropical lines we covered in Example 2.24 and depicted in Figure 2.

The Grassmannian $\mathrm{Gr}(2; 5)$ is more complicated, it is cut out by the Plücker relations

$$\begin{array}{lll} p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} & p_{12}p_{35} - p_{13}p_{25} + p_{15}p_{23} & p_{12}p_{45} - p_{14}p_{25} + p_{15}p_{24} \\ p_{13}p_{45} - p_{14}p_{35} + p_{15}p_{34} & p_{23}p_{45} - p_{24}p_{35} + p_{25}p_{34}. & \end{array}$$

Its tropicalization after quotienting by the lineality space and cutting with the sphere with radius 1 is the Petersen graph, depicted below. Just like the tropical line, the Petersen graph is a picture that will reoccur in all parts of this thesis.



In tropical geometry, there exist two tropical analogues of the Grassmannian: the *Tropical Grassmannian* and the *Dressian*.

Definition 3.84. Let $r \leq n$ be a nonnegative integer. The first tropical analogue of the Grassmannian, the *tropical Grassmannian* $\overline{\text{trop}}(\text{Gr}(r;n))$ is the tropicalization of the Grassmannian $\text{Gr}(r;n)$, i.e.,

$$\overline{\text{trop}}(\text{Gr}(r;n)) \quad = \bigcap_{f \in \langle \mathscr{P}_{r;n} \rangle} V(\overline{\text{trop}}(f)) \ \subseteq \ \mathbb{P}(\mathbb{T}^{\binom{n}{r}}).$$

Lemma 3.85 ([131, Theorem 3.8])**.** *Let K be an algebraically closed field with nontrivial valuation and let $0 < r \leq n$ be natural numbers. The tropical Grassmannian* $\text{trop}(\text{Gr}(r;n))$ *is the parameter space of all tropicalizations of rank r linear subspaces of K^n.*

The *Dressian* parametrizes more: its points correspond to tropical linear spaces (i.e., tropical linear spaces).

Definition 3.86. Let $r \leq n$ be a nonnegative integer. The *Dressian* $\text{Dr}(r;n)$ is the tropical prevariety cut out by the Plücker relations. It is

$$\text{Dr}(r;n) \quad = \bigcap_{f \in \mathscr{P}_{r;n}} V(\overline{\text{trop}}(f)) \ \subseteq \ \mathbb{P}(\mathbb{T}^{\binom{n}{r}}).$$

Lemma 3.87. [132] *Let $0 < r \leq n$ be natural numbers. Then, the Dressian* $\text{Dr}(r;n)$ *is the parameter space of all tropical linear spaces of rank r over $[n]$.*

Remark 3.88 ([78, Theorem 3.1.3])*.* Additionally, for a matroid M one can define its Dressian $\text{Dr}(M)$ as the parameter space of all tropical linear spaces with underlying matroid M under the Plücker embedding. With this notation, we have

$$\text{Dr}(r;n) \quad = \bigcup_{\substack{M \text{ is a matroid} \\ \text{of rank } r \text{ on } [n]}} \text{Dr}(M).$$

3.4.2. *Valuated flag matroids and tropical flag varieties.* In this section, we will describe valuated flag matroids and their associated tropicalized flag varieties and flag Dressians. The main references for this section are [30, 76]. Haque originally introduced the concept of the flag Dressian, and Brandt-Eur-Zhang provided realizability statements and polyhedral characterizations. Even though Haque already tried to introduce polyhedral characterizations, his turn out to be incorrect.

Definition 3.89. Let μ and ν be two valuated matroids on same the ground set $[n]$ of rank $r \leq s$ respectively. We say that μ is a *valuated matroid quotient* of ν, denoted $\mu \twoheadleftarrow \nu$, if for every $I \in \binom{[n]}{r}$, $J \in \binom{[n]}{s}$ and $i \in I \setminus J$, there exists $j \in J \setminus I$ such that

$$\mu(I) + \nu(J) \ \geq \ \mu(I \cup j \setminus i) + \nu(J \cup i \setminus j).$$

Valuated matroid quotients induce quotients on the underlying matroids: If $\mu \twoheadleftarrow \nu$ is a valuated matroid quotient, and M and N are the underlying matroids of μ and ν respectively, then $M \twoheadleftarrow N$.

Definition 3.90. A sequence of valuated matroids $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)$ on a common ground set $[n]$, is a *(valuated) flag matroid* if $\mu_i \twoheadleftarrow \mu_j$ for every $1 \leq i \leq j \leq k$. Analogously, a sequence of matroids $\mathbf{M} = (M_1, \dots, M_k)$ on $[n]$ is a *flag matroid* if $M_i \twoheadleftarrow M_j$ for every $1 \leq i \leq j \leq n$.

Remark 3.91. As in the case of valuated matroids and matroid quotients, there is a notion of *realizable flag matroids*. If $L_1 \subseteq L_2$ are two linear subspaces of K^n, then we have $\mu(L_1) \twoheadleftarrow \mu(L_2)$, see [30, Example 4.1.2].

Definition 3.92. Let $r \leq s \leq n$ be nonnegative integers. The *incidence Plücker relations* are the polynomials in the variables $\{p_I : I \in \binom{[n]}{r}\} \cup \{p_J : J \in \binom{[n]}{s}\}$ with coefficients in K:

$$\mathscr{P}_{r,s;n} \ = \ \left\{ \sum_{j \in J \setminus I} \operatorname{sign}(j; I, J) p_{I \cup j} p_{J \setminus j} : I \in \binom{[n]}{r-1}, J \in \binom{[n]}{s+1} \right\},$$

where $\operatorname{sign}(j; I, J) = (-1)^{\#\{j' \in J : j < j'\} + \#\{i \in I : i > j\}}$. The tropicalizations of the incidence Plücker relations are denoted by $\mathscr{P}^{\text{trop}}_{r,s;n}$. If $r = s$, we recover the Grassmann-Plücker relations.

The incidence-Plücker relations, combined with the Grassmann-Plücker relations, are the equations cutting out flag varieties in multiprojective space.

Definition 3.93. Let $r_1 \leq \dots \leq r_k \leq n$ be nonnegative integers. The *flag variety* of rank $r_1, \dots, r_k$ is the multiprojective variety

$$\operatorname{Fl}(r_1, \dots, r_k; n) \ = \ \bigcap_{0 < i \leq j \in [k]} \left(\bigcap_{f \in \langle \mathscr{P}_{r_i, r_j; n} \rangle} V(f) \right) \ \subseteq \ \mathbb{P}^{\binom{[n]}{r_1}} \times \dots \times \mathbb{P}^{\binom{[n]}{r_k}}$$

cut out by the Grassmann-Plücker and the incidence Plücker relations.

Points in the flag variety $\operatorname{Fl}(r_1, \dots, r_k; n)$ parametrize flags of linear spaces $L_1 \subseteq \dots \subseteq L_k$ in K^n where $\dim L_i = r_i$.

As was the case for the Grassmannian, the flag variety has two different tropical analogues.

Definition 3.94. Its tropicalization is the *tropical flag variety*, parametrizing *realizable* flags of tropical linear spaces, that is, sequences $\overline{\operatorname{trop}}(L_1) \subseteq \dots \subseteq \overline{\operatorname{trop}}(L_k)$ where L_i is a linear subspace of K^n satisfying $\dim L_i = r_i$.

If we only consider the tropical prevariety cut out by the Plücker relations instead of tropicalizing the whole flag variety, we obtain the other tropical analogue.

Definition 3.95. The *flag Dressian* of rank $(r_1, \dots, r_k)$ on the ground set $[n]$ is the tropical prevariety of the corresponding incidence-Plücker and Grassmann-Plücker relations,

$$\mathrm{FlDr}(r_1, \dots, r_k; n) \;=\; \bigcap_{0<i\leq j\in[k]} \left(\bigcap_{f\in \mathscr{P}_{r_i,r_j;n}} V(\overline{\mathrm{trop}}(f)) \right) \;\subseteq\; \mathbb{P}\Big(\mathbb{T}^{\binom{[n]}{r_1}}\Big) \times \cdots \times \mathbb{P}\Big(\mathbb{T}^{\binom{[n]}{r_k}}\Big).$$

As in the case of the Dressian, the points in the flag Dressian can be characterized tropically and combinatorially.

Theorem 3.96 ([30, Theorem A], [76, Theorem 1], [109, Lemma 2.6])**.** *Let* $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)$ *be a sequence of valuated matroids on a common ground set* $[n]$ *of ranks* $\boldsymbol{r} = (r_1, \dots, r_k)$ *respectively. The following are equivalent:*

(a) $\boldsymbol{\mu}$ *is a point in* $\mathrm{FlDr}(\boldsymbol{r}; n) = \bigcap_{1\leq i\leq k} V(\mathscr{P}^{\mathrm{trop}}_{r_i;n}) \cap \bigcap_{1\leq i<j\leq k} V(\mathscr{P}^{\mathrm{trop}}_{r_i,r_j;n})$,
(b) $\boldsymbol{\mu}$ *is a valuated flag matroid,*
(c) $\overline{\mathrm{trop}}(\mu_1) \subseteq \cdots \subseteq \overline{\mathrm{trop}}(\mu_k)$.

Example 3.97. In this example, we describe the tropicalization of the flag variety $\mathrm{Fl}(1,2;4)$, parametrizing flags of points in lines in $\mathbb{P}^3$, with respect to the trivial valuation. By definition, $\mathrm{Fl}(1,2;4) = V(\mathscr{P}_{2;4} \cup \mathscr{P}_{1,2;4})$, since $\mathscr{P}_{1;4}$ contains just the zero polynomial. The tropicalizations of the equations defining the ideal, given below, form a tropical basis, thus $\mathrm{FlDr}(1,2;4) = \mathrm{trop}(\mathrm{Fl}(1,2;4))$ (see [30, Theorem 5.2.1]).

$$\mathscr{P}^{\mathrm{trop}}_{2;4} = \{p_{1,4}p_{2,3} \oplus p_{1,3}p_{2,4} \oplus p_{1,2}p_{3,4}\}\,,$$

$$\mathscr{P}^{\mathrm{trop}}_{1,2;4} = \left\{ \begin{array}{l} p_1p_{2,3} \oplus p_2p_{1,3} \oplus p_3p_{1,2}, \\ p_4p_{1,2} \oplus p_2p_{1,4} \oplus p_1p_{2,4}, \\ p_4p_{1,3} \oplus p_1p_{3,4} \oplus p_3p_{1,4}, \\ p_4p_{2,3} \oplus p_2p_{3,4} \oplus p_3p_{2,4}. \end{array} \right\}$$

The tropicalization $\mathrm{trop}\big(\mathrm{Fl}(1,2;4)\big)$ is a 9-dimensional simplicial fan in $\mathbb{P}(\mathbb{T}^4) \times \mathbb{P}(\mathbb{T}^6) \times \mathbb{P}(\mathbb{T}^4)$. It can be computed in Macaulay2 [71] using code obtained in joint work with Alessio Borzì, which can be found in Appendix B.3 and is available online in [26]. We compute the variety and determine some of its properties below using our code:

```
i1 : n = 4;
i2 : G = flatten for i from 1 to n-1 list(sort subsets(n, i));
i3 : R = QQ[ for I in G list p_I ];
i4 : L = {{1,4}, {2,4}, {1,2,4}};
i5 : I = pluckerRelations L;
i6 : Fl4 = tropicalVariety(I);
i7 : ambDim Fl4
o7 = 14
i8 : dim Fl4
o8 = 11
i9 : rank linealitySpace Fl4
o9 = 9
i10 : fVector Fl4
o10 = {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 10, 15}
```

The tropical variety modulo lineality space and after cutting with the sphere with radius 1 is a Petersen graph (see Figure 7). A point p in a top-dimensional cone (corresponding to an edge of the Petersen graph) parametrizes a point v_p contained in a generic tropical line L_p in the three-dimensional tropical projective space $\mathbb{P}(\mathbb{T}^4)$. In Example 2.24 we had seen that there are three combinatorial types of generic tropical lines in $\mathbb{P}(\mathbb{T}^4)$, each having five edges, four of which are unbounded. The position of the point v_p on the line L_p can be freely chosen, and there are 5 choices, one for each edge of L_p. This amounts to $3 \times 5 = 15$ possible choices, corresponding to the maximal cones.

In the Petersen graph found in Figure 7, these cones are indexed as follows. The three (blue) edges connecting the vertices (ab) to (cd), for $a, b, c, d \in [4]$ distinct, correspond to the three cases where the point lies on the bounded edge in direction $e_a + e_b$, for an example see the red flag in Figure 7. The remaining twelve (black) edges, connecting (a) to (ab) for $a, b \in [4]$ distinct, correspond to the cases where the point lies on an unbounded edge in direction e_a, and the bounded edge is in direction $e_a + e_b$. For an example, see the cyan flag in Figure 7.

The length of the bounded edge of L_p and the position of the point v_p in L_p can be freely chosen, amounting to two degrees of freedom. These are the two degrees of freedom remaining after taking the quotient with the lineality space. The choice of the point v_p in $\mathbb{P}(\mathbb{T}^4)$ corresponds to three further degrees of freedom. In total, this generates a cone of (projective) dimension five.

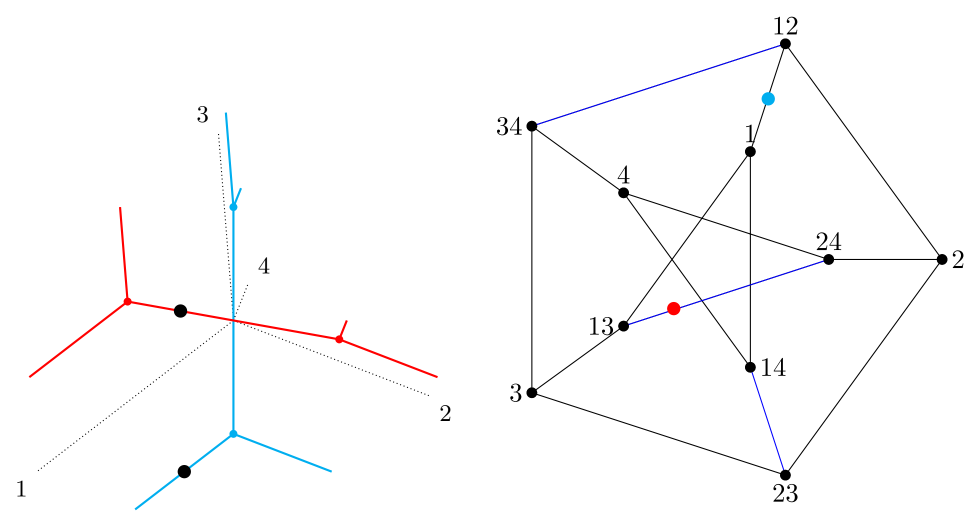


FIGURE 7. On the left, two tropical flags in $\mathrm{trop}(\mathrm{Fl}(1, 2; 4)) \subseteq \mathbb{P}(\mathbb{T}^4)$. On the right, a Petersen graph representing the tropical flag variety $\mathrm{trop}(\mathrm{Fl}(1, 2; 4))$ after quotienting by its lineality space and cutting with the radius 1 sphere. The red point corresponds to the red flag on the left, the cyan point is the cyan flag.

4. Affine morphisms of valuated matroids

In this section, we will introduce morphisms of valuated matroids and their affine extensions. **Section 4.1** is an extended version of joint work with Alessio Borzì [27], introducing morphisms of valuated matroids, focusing on projections as a main application of the theory.

In **Section 4.2**, we extend this definition to generalize affine linear maps. The results of this section are extensions of joint work with Giulia Iezzi [82].

Finally, in **Section 4.3**, we show that the resulting category $\mathbf{VMatr}_a$ of valuated matroids with affine morphisms has many of the same properties as are known for the category $\mathbf{Matr}_o$ we discussed in Section 3.2.4. The results of this section are solely my own.

The main upgrade this thesis provides in comparison with [27, 82] is that we define and prove results for maps of matroids where source and target are over different ground sets. Further, we provide a more explicit proof that the induced valuation is a valuation in the sense of Definition 3.58, and prove that (affine) morphisms of valuated matroids induce strong maps of the underlying matroids. Finally, we prove that every affine linear map of linear spaces induces a realizable morphism of the associated valuated matroids.

4.1. Morphisms of valuated matroids.

In Section 3.2 we have seen that if we have a map $f : [n] \to [m]$ and two matroids N on $[n]$ and M on $[m]$, f is a morphism of matroids if and only if $f^{-1}(M) \twoheadleftarrow N$ is a matroid quotient. As there already exists a notion of a valuated matroid quotient (see Definition 3.89), we use this characterization by constructing an *induced valuation* on the induced matroid $f^{-1}(N)$. The idea to do this was first mentioned in [30, Remark 4.4.3], and a less in-depth proof of the below result was obtained in joint work with Alessio Borzì [27].

Proposition-Definition 4.1. *Let μ be a valuated matroid over $[m]$ and let M be its underlying matroid. Let $f : [n] \to [m]$ be a function of sets. Then, μ induces a valuation $f^{-1}(\mu)$ on the induced matroid $f^{-1}(M)$ (see Definition-Proposition 3.44), given as $f^{-1}(\mu)(B) = \overline{\mu}(f(B))$. Here, $\overline{\mu}$ denotes the valuation on independent sets (see Definition 3.63).*

Proof. By Proposition 3.44, $f^{-1}(M)$ is a matroid. Let B be a basis of $f^{-1}(M)$. While $f(B)$ is not necessarily a basis of M, it is an independent set:

$$B \text{ is a basis of } f^{-1}(M) \implies \mathrm{rk}_{f^{-1}(M)}(B) = |B| \implies \mathrm{rk}_M(f(B)) = |B|.$$

Since M is a matroid, $\mathrm{rk}_M(f(B)) \leq |f(B)|$ and since f is a function, i.e., assigns each element of $[n]$ only one element in $[m]$, we have that $|f(B)| \leq |B|$. Thus, $|B| = \mathrm{rk}_M(f(B)) \leq |f(B)| \leq |B|$, i.e., $f(B)$ is an independent set of M. Further, by $f^{-1}(M)$ being a matroid and equicardinality of bases as remarked in Definition 3.5, we note that all images of bases are independent sets of the same size. Finally, we note that $f|_B$ is a bijection.

This construction defines a valuation on $f^{-1}(M)$: Let A, B be two bases of $f^{-1}(N)$. Then

$$f^{-1}(\mu)(A) + f^{-1}(\mu)(B) \; = \; \overline{\mu}(f(A)) + \overline{\mu}(f(B)).$$

Let $a \in A \setminus B$, and consider $f(a)$. There are two cases.

Case 1: $f(a) \in f(A) \setminus f(B)$. Then, by Definition 3.63(VI3), for all $a' \in f(A) \setminus f(B)$ there exists $b' \in f(B) \setminus f(A)$ such that

$$\overline{\mu}(f(A)) + \overline{\mu}(f(B)) \; \geq \; \overline{\mu}(f(A) \setminus a' \cup b') + \overline{\mu}(f(B) \setminus b' \cup a').$$

Since f is a bijection from B to $f(B)$ by the above analysis, there exists a unique $a \in A$ such that $f(a) = a'$ and a unique $b \in B$ satisfying $f(b) = b'$, so

$$\begin{aligned}\overline{\mu}(f(A) \setminus a' \cup b') + \overline{\mu}(f(B) \setminus b' \cup a') &= \overline{\mu}(f(A) \setminus f(a) \cup f(b)) + \overline{\mu}(f(B) \setminus f(b) \cup f(a)) \\ &= \overline{\mu}(f(A \setminus a \cup b)) + \overline{\mu}(f(B \setminus b \cup a)) \\ &= f^{-1}(\mu)(A \setminus a \cup b) + f^{-1}(\mu)(B \setminus b \cup a).\end{aligned}$$

Case 2: $f(a) \notin f(A) \setminus f(B)$. Then, $f(a) \in f(B)$, i.e., there exists $b \in B$ such that $f(b) = f(a)$. Then, $\overline{\mu}(f(A \setminus a \cup b)) = \overline{\mu}(f(A) \setminus f(a) \cup f(b)) = \overline{\mu}(f(A))$, and an analogous statement holds for B. Thus,

$$f^{-1}(\mu)(A) + f^{-1}(\mu)(B) = f^{-1}(\mu)(A \setminus a \cup b) + f^{-1}(\mu)(B \setminus b \cup a).$$ □

Analogously to the unvaluated case, we again want to model the properties of linear maps of linear spaces. Thus, we aim to construct an analogue of strong maps, which will enable us to study projections. To this end, we extend our definition of pointed matroids to the valuated setting.

Definition 4.2. Let μ be a valuated matroid over $[n]$. The *pointed valuated matroid* μ_o over $[n] \cup \{o\}$ is the valuated matroid $\mu \oplus U_{0,1}$, obtained by adding a loop o to the matroid μ. By a slight abuse of notation, we set $\overline{\text{trop}}(\mu_o) := \overline{\text{trop}}(\mu_o)|_{[n]} = \overline{\text{trop}}(\mu)$, since the tropical linear spaces only differ by removing the ∞-entry in the o-coordinate.

In the above definition, note that $U_{0,1}$ has exactly one valuation up to equivalence — the valuation assigning $\emptyset \mapsto 0$, as all other assignments are equivalent by Definition 3.58. Thus, the above construction is well-defined.

Definition 4.3. Let μ and ν be valuated matroids over the ground sets $[m]$ and $[n]$. Let $f : [n] \cup \{o\} \to [m] \cup \{o\}$ be a map of sets. We say that $f : \nu \to \mu$ is a *morphism of valuated matroids* if $f^{-1}(\mu)_o \twoheadleftarrow \nu_o$ is a quotient of valuated matroids, i.e., for all $I \in \mathcal{B}(f^{-1}(M)_o)$, $J \in \mathcal{B}(N_o)$ and $i \in I \setminus J$, there exists $j \in J \setminus I$ such that

$$f^{-1}(\mu_o)(I) + \nu_o(J) \geq f^{-1}(\mu_o)(I \cup j \setminus i) + \nu_o(J \cup i \setminus j).$$

By definition, the above is equivalent to requiring that for all $I \in \mathcal{B}(f^{-1}(M)_o)$, $J \in \mathcal{B}(N_o)$ and $i \in I \setminus J$, there exists $j \in J \setminus I$ such that

$$\overline{\mu_o}(f(I)) + \nu_o(J) \geq \overline{\mu_o}(f(I \cup j \setminus i)) + \nu_o(J \cup i \setminus j).$$

In the above definition, note that both $f^{-1}(\mu)_o$ and ν are valuated matroids over $[n]$, so the valuated matroid quotient is well-defined.

Lemma 4.4. *Let ν and μ be valuated matroids over $[n]$ and $[m]$ respectively with underlying matroids N and M. Let $f : \nu \to \mu$ be a morphism of valuated matroids. Then, $f : N \to M$ is a strong map of matroids.*

Proof. By Proposition 4.1, $f^{-1}(\mu)$ is a valuated matroid with underlying matroid $f^{-1}(M)$, and we have $f^{-1}(\mu) \twoheadleftarrow \mu$. By [30, Remark 4.2.5], thus $f^{-1}(M) \twoheadleftarrow N$, so by Lemma 3.45, f is a morphism of matroids. □

To study morphisms of valuated matroids, we often consider valuated matroid quotients. Sometimes it can be helpful to have a cryptomorphic definition of valuated matroid quotients on independent sets. The following equivalent characterization is in the spirit of Murota's cryptomorphic description of valuated matroids via independent sets.

Lemma 4.5. *Let μ and ν be valuated matroids over the ground set $[n]$ of rank r and s respectively with $r < s$. Let $S \subseteq [n]$. Then, the following are equivalent:*

(a) For all $I \in \binom{[n]}{r}$, $J \in \binom{[n]}{s}$ and $i \in I \setminus (J \cup S)$ there exists $j \in J \setminus (I \cup S)$ such that

$$\mu(I) + \nu(J) \ \geq \ \mu(I \cup j \setminus i) + \nu(J \cup i \setminus j)$$

(b) For all $G \in \mathrm{Ind}(\mu)$, $J \in \binom{[n]}{s}$ and $g \in G \setminus (J \cup S)$ there exists $h \in J \setminus (G \cup S)$ such that

$$\overline{\mu}(G) + \nu(J) \ \geq \ \overline{\mu}(G \cup j \setminus g) + \nu(J \cup g \setminus j)$$

Proof. $(b) \Rightarrow (a)$, as $\mu(I) \neq \infty$ if and only if I is a basis, all bases are independent sets of equal size and for all bases I, $\overline{\mu}(I) = \mu(I)$.

$(a) \Rightarrow (b)$: If $\overline{\mu}(G) = \infty$, then G is not contained in a basis, and thus not an independent set. Thus, we may assume $\overline{\mu}(G) < \infty$. Now, if $|G| = r$, then G is a basis and satisfies the inequality directly by application of (a). Thus, we can assume $|G| < r$.

By Definition 3.63(VI2), there exists $G' \subseteq [n] \setminus G$, $|G'| = r - |G|$ such that $\overline{\mu}(G \cup G') = \overline{\mu}(G)$ and $G \cup G' \in \binom{[n]}{r}$. Further, $G \cup G'$ is a basis, as $\overline{\mu}(G \cup G') = \overline{\mu}(G) < \infty$.

We now apply the exchange property (a) for all $g \in G \setminus (J \cup S) \subseteq (G \cup G') \setminus (J \cup S)$, i.e., there exists $j \in J \setminus (G \cup G' \cup S) \subseteq J \setminus (G \cup S)$ such that

$$\begin{aligned} \overline{\mu}(G) + \nu(J) &= \overline{\mu}(G \cup G') + \nu(J) \geq \overline{\mu}(G \cup G' \cup j \setminus g) + \nu(J \cup g \setminus j) \\ &\geq \overline{\mu}(G \cup j \setminus g) + \nu(J \cup g \setminus j). \end{aligned}$$

where the second inequality follows from Definition 3.63(VI1). □

Example 4.6. In the following, we give some examples of maps and morphisms of matroids.

Let μ be a valuated matroid over $[n]$. Then, the identity map $\mathrm{id} : \mu \to \mu$ given as the identity map on the ground set is always a morphism of matroids. This is because in this case, the inequalities in Definition 4.3 are precisely the inequalities in the Definition 3.58, describing valuated matroids.

There are various matroidal operations we can describe in terms of maps of matroids.

- *Permutation.* The permutation of elements in the matroid can be described by a map of matroids. An example of such a permutation is the following map:

  $$f : [3] \cup \{o\} \to [3] \cup \{o\}; \quad 1 \mapsto 2, \quad 2 \mapsto 3, \quad 3 \mapsto 1, \quad o \mapsto o$$

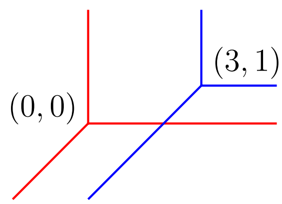
  On the tropical side, this corresponds to a swapping of ray directions. Higher dimensional cones are shifted accordingly — under the permutation above, the cone spanned by $\{e_1, e_2\}$ gets mapped to a cone $\{e_2, e_3\}$.
- *Projection/Deletion.* The projection map pr_S we constructed in Definition 3.53 directly translates to the valuated setting. When defining the projection to a lower cardinality set, this can correspond to matroid deletion, which we considered in Definition 3.79. An example of such a projection is the following map:

  $$f : [3] \cup \{o\} \to [2] \cup \{o\}; \quad 1 \mapsto 1, \quad 2 \mapsto 2, \quad 3 \mapsto o, \quad o \mapsto o$$

- *Contraction.* While the projection map above can be used to describe matroid deletion, we can analogously describe matroid contraction using maps. An example is the following map:

$$f : [3] \cup \{o\} \to [2] \cup \{o\}; \quad 1 \mapsto 1, \quad 2 \mapsto 2, \quad 3 \mapsto 2, \quad o \mapsto o$$

However, some maps we like to consider in linear algebra can not be expressed in terms of maps of matroids. Consider the valuated matroid μ arising from Example 3.8 by substituting $a = 0$ and $b = 0$, and the valuated matroid ν obtained by instead substituting $a = -3$ and $b = -1$. Then, $\mu(12) = \mu(13) = \mu(23) = 0$, and $\nu(12) = 0$, $\nu(13) = -1$ and $\nu(23) = -3$. Plotting their associated tropical linear spaces, we observe that they are just an affine translation of each other.



Nevertheless, there exists no rank-preserving morphism of valuated matroids between the two valuated matroids: The identity map is not a morphism, as

$$-3 \;=\; \mu(12) + \nu(13) \;<\; \mu(13) + \nu(12) \;=\; 0.$$

The analogous contradiction occurs for all bijections.

4.2. **Affine morphisms of valuated matroids.** In this section, we will extend the notion of the previous section, keeping track of additional scaling factors. These are necessary to be able to describe translations of tropical linear spaces, for instance. For an example of this, we refer to Example 5.6.

Definition 4.7. Let μ be a valuated matroid on the ground set $[m]$. We consider a map f defined as

$$(f_1, f_2) : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{T}.$$

We restrict to maps satisfying $f_2(i) = \infty$ if and only if $f_1(i) = o$, and define the *affine induced valuated matroid* as

$$f^{-1}(\mu)(B) \;=\; \mu|_{f_1([n]\cup\{o\})}\big(f_1(B)\big) + \sum_{i\in B} f_2(i),$$

where $\mu|_{f_1([n]\cup\{o\})}$ denotes the restriction of μ_o to the set $f_1([n] \cup \{o\})$. The affine induced valuated matroid is a pointed valuated matroid as in Definition 4.2, hence its tropical linear space is defined as $\overline{\operatorname{trop}}\big(f^{-1}(\mu)\big) := \overline{\operatorname{trop}}\big(f^{-1}(\mu)\big)|_{[n]}$.

Lemma 4.8. *Let μ be a valuated matroid on the ground set $[m]$ and $f : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{T}$. Then, the affine induced valuated matroid is a valuated matroid as defined in Definition 3.58.*

Proof. By Proposition 4.1, the map of sets f_1 induces a valuated matroid $f_1^{-1}(\mu)(B) = \mu|_{f_1([n]\cup\{o\})}\big(f_1(B)\big)$ with underlying matroid $f_1^{-1}(M)$. Since $f_1^{-1}(\mu)$ is a valuated matroid on

$[n]$, for all $I, J \in \binom{[n]}{r}$ and $i \in I \setminus J$ there exists $j \in J \setminus I$ satisfying

$$\begin{aligned} & \mu|_{f_1([n]\cup\{o\})}\big(f_1(I)\big) + \mu|_{f_1([n]\cup\{o\})}\big(f_1(J)\big) \\ \geq\ & \mu|_{f_1([n]\cup\{o\})}\big(f_1(I \setminus i \cup j)\big) + \mu|_{f_1([n]\cup\{o\})}\big(f_1(J \setminus j \cup i)\big). \end{aligned}$$

Now

$$\sum_{i' \in I} f_2(i') + \sum_{j' \in J} f_2(j') \ = \sum_{i' \in ((I\setminus i)\cup j)} f_2(i') + \sum_{j' \in ((J \setminus j)\cup i)} f_2(j'),$$

thus

$$\begin{aligned} & \mu|_{f_1([n]\cup\{o\})}\big(f_1(I)\big) + \sum_{i' \in I} f_2(i') + \mu|_{f_1([n]\cup\{o\})}(f_1(J)) + \sum_{j' \in J} f_2(j') \\ \geq\ & \mu|_{f_1([n]\cup\{o\})}(f_1((I \setminus i) \cup j)) + \sum_{i' \in ((I\setminus i)\cup j)} f_2(i') \\ & + \mu|_{f_1([n]\cup\{o\})}(f_1((J \setminus j) \cup i)) + \sum_{j' \in ((J\setminus j)\cup i)} f_2(j'), \end{aligned}$$

and $f^{-1}(\mu)$ defines a valuation by Definition 3.58. □

Definition 4.9. Let μ and ν be valuated matroids over $[m]$ and $[n]$ respectively, and let

$$f : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{T}$$

be a map. By abuse of notation, we say that $f : \nu \to \mu$ is an *affine morphism of valuated matroids* if $f^{-1}(\mu) \twoheadleftarrow \nu$ as in Definition 4.7 is a quotient of valuated matroids.

If additionally both μ and ν are *realizable*, we say that f is a *realizable* affine morphism of valuated matroids if both μ and the quotient $f^{-1}(\mu) \twoheadleftarrow \nu$ are realizable.

Lemma 4.10. *Let μ and ν be valuated matroids over $[m]$ and $[n]$ respectively, and let $f : \nu \to \mu$ be an affine morphism of valuated matroids. Then, there exists an underlying strong map of underlying matroids $f_1 : N \to M$.*

Proof. Since f is an affine morphism of valuated matroids, by definition $f^{-1}(\mu) \twoheadleftarrow \nu$ is a valuated matroid quotient. Thus, by [30, Remark 4.2.5], the underlying matroids of ν and $f^{-1}(\mu)$ form a matroid quotient. It remains to be shown that the underlying matroid of $f^{-1}(\mu)$ is $f_1^{-1}(M)$.

By definition, if $f_1(B)$ is a basis of the underlying matroid of the (non-affine) induced matroid $f_1^{-1}(\mu)$ if and only if

$$f_1^{-1}(\mu)(B) \ = \ \mu|_{f_1([n]\cup\{o\})}\big(f_1(B)\big) \ \neq \ \infty.$$

Since $f_1^{-1}(\mu)(B) \neq \infty$ implies that $o \notin B$, by definition of the affine map we have

$$\sum_{i \in B} f_2(i) \ \neq \ \infty.$$

Thus, $f_1^{-1}(\mu)(B) \neq \infty$ if and only if $f^{-1}(\mu)(B) \neq \infty$. Hence, both $f_1^{-1}(\mu)$ and $f^{-1}(\mu)$ have the same underlying matroid.

As a result, the underlying matroid of the affine induced valuated matroid is the induced matroid $f_1^{-1}(M)$ as introduced in Definition 4.7. Thus, $f_1^{-1}(M) \twoheadleftarrow N$, which implies that $f_1 : N \to M$ is a strong map of matroids. □

Lemma 4.11. *Let μ and ν be valuated matroids on $[m]$ and $[n]$ respectively. Every morphism of valuated matroids $f : \nu \to \mu$ can be extended to an affine morphism of valuated matroids using the* trivial map

$$\begin{aligned} triv\colon [n] \cup \{o\} &\to \mathbb{T} \\ i &\mapsto \begin{cases} 0 & i \neq o \\ \infty & i = o. \end{cases} \end{aligned}$$

as the second component.

Proof. Let $I \in \mathcal{B}(f^{-1}(M)_o)$, $J \in \mathcal{B}(N_o)$ and $i \in I \setminus J$. As I is a basis of $f^{-1}(M)_o$, $o \notin f_1(I)$, so $o \notin I$. Thus, $\sum_{i' \in I} \operatorname{triv}(i') = 0$. This implies that

$$\mu|_{f([n]\cup\{o\})}\big(f(I)\big) + \sum_{i' \in I} \operatorname{triv}(i') + \nu(J) \;=\; \mu|_{f([n]\cup\{o\})}\big(f(I)\big) + \nu(J).$$

Now, since f is a morphism of valuated matroids, there exists $j \in J \setminus I$ such that

$$\mu|_{f([n]\cup\{o\})}(f(I)) + \nu(J) \;\geq\; \mu(f(I \cup j \setminus i)) + \nu(J \cup i \setminus j).$$

If $I \cup j \setminus i$ is not a basis of $\mathcal{B}(f^{-1}(M)_o)$, we have that $\mu(f(I \cup j \setminus i)) = \infty$, contradicting the inequality. So, $I \cup j \setminus i$ is again a basis of $\mathcal{B}(f^{-1}(M)_o)$, so as above, $o \notin I \cup j \setminus i$. Then, we again have that $\sum_{i' \in I \cup j \setminus i} \operatorname{triv}(i') = 0$, so

$$\mu|_{f([n]\cup\{o\})}\big(f(I)\big) + \nu(J) \;=\; \mu|_{f([n]\cup\{o\})}\big(f(I)\big) + \sum_{i' \in I \cup j \setminus i} \operatorname{triv}(i') + \nu(J).$$

Combining the three equations, we obtain that (f, triv) is an affine morphism of valuated matroids. □

4.3. **Categorical properties.** We conclude this discussion by showing some properties of morphisms of valuated matroids and analyzing the category of valuated matroids with morphisms given by affine morphisms of valuated matroids. This is in the spirit of work by Jarra-Lorscheid-Vital [86], but developed independently. All proofs and statements are my own independent work, except for Proposition 4.15 which is substantially similar to [27, Proposition 3.14] and was developed in joint work with Alessio Borzì.

Definition 4.12. We denote by **VMatr** the category where the objects are valuated matroids and the morphisms are morphisms of valuated matroids as introduced in Definition 4.3. Additionally, we write $\mathbf{VMatr}_a$ for the category whose objects are again valuated matroids, but whose morphisms are affine morphisms of valuated matroids as introduced in Definition 4.7.

In the following, we will show that the category $\mathbf{VMatr}_a$ satisfies the properties of the category $\mathbf{Matr}_o$ we had discussed in Section 3.2.4.

Proposition 4.13. *The category $\mathbf{VMatr}_a$ has a coproduct, given by the direct sum of valuated matroids.*

Proof. Let μ and ν be two valuated matroids over the ground sets $[m]$ and $[n]$ respectively. Then, the direct sum $\mu \oplus \nu$ over $[m] \sqcup [n]$ is the categoretical coproduct, where the maps into the coproduct, i_μ and i_ν are given as follows. For $i_\mu : \mu \hookrightarrow \mu \oplus \nu$, $(i_\mu)_1$ is the identity on $[m]$ and $(i_\mu)_2$ is the trivial map from Lemma 4.11. For $i_\nu : \nu \hookrightarrow \mu \oplus \nu$, $(i_\nu)_1$ is the identity on

$[n]$ and $(i_\nu)_2$ is again the trivial map from Lemma 4.11. By the definition of the direct sum of valuated matroids, Definition 3.81, the two maps are morphisms of valuated matroids.

Now, let ω be another valuated matroid, and let $f : \mu \to \omega$ and $g : \nu \to \omega$ be two affine morphisms of valuated matroids. We then define

$$\begin{aligned} f + g : \mu \oplus \nu &\to \omega \\ S &\mapsto f(S \cap [m]) + g(S \cap [n]). \end{aligned}$$

We now show that the above map is an affine morphism of valuated matroids, i.e., that $(f+g)^{-1}(\omega) \twoheadleftarrow \mu \oplus \nu$. Let $S, S' \in \binom{[m]\sqcup[n]}{\mathrm{rk}(\mu)+\mathrm{rk}(\nu)}$. Then,

$$\begin{aligned} (f+g)^{-1}(\omega)(S) &= \omega|_{(f+g)_1(([m]\sqcup[n])\cup\{o\})}\big((f+g)_1(S)\big) + \sum_{i\in S}(f+g)_2(i) \\ &= \omega|_{(f_1([m]\cup\{o\})\cup g_1([n]\cup\{o\})}\big(f_1(S\cap[m]) + g_1(S\cap[n])\big) \\ &\quad + \sum_{i\in S\cap[m]} f_2(i) + \sum_{i\in S\cap[n]} g_2(i) \\ &= \omega|_{(f_1([m]\cup\{o\})}\big(f_1(S\cap[m])\big) + \sum_{i\in S\cap[m]} f_2(i) \\ &\quad \omega|_{(g_1([n]\cup\{o\})}\big(g_1(S\cap[n])\big) + \sum_{i\in S\cap[n]} g_2(i) \\ &= f^{-1}(\omega)(S\cap[m]) + g^{-1}(\omega)(S\cap[n]). \end{aligned}$$

Now, let $s \in S \setminus S'$. Without limitation of generality, assume $s \in [m]$. As f is an affine morphism of valuated matroids, there exists $s' \in S' \cap [m] \subseteq S'$ such that

$$\begin{aligned} (f+g)^{-1}(\omega)(S) + \mu\oplus\nu(S') &= f^{-1}(\omega)(S\cap[m]) + g^{-1}(\omega)(S\cap[n]) \\ &\quad + \mu(S'\cap[m]) + \nu(S'\cap[n]) \\ &\geq f^{-1}(\omega)((S\setminus s\cup s')\cap[m]) + g^{-1}(\omega)(S\cap[n]) \\ &\quad + \mu((S'\setminus s'\cup s)\cap[m]) + \nu(S\cap[n]) \\ &= f^{-1}(\omega)((S\setminus s\cup s')\cap[m]) + g^{-1}(\omega)((S\setminus s\cup s')\cap[n]) \\ &\quad + \mu((S'\setminus s'\cup s)\cap[m]) + \nu((S'\setminus s'\cup s)\cap[n]). \end{aligned}$$

Thus, $f+g$ is an affine morphism of valuated matroids. □

Proposition 4.14. *The category* $\mathbf{VMatr}_a$ *does not have products.*

Proof. Assume $\mathbf{VMatr}_a$ had products. Let μ and ν be two valuated matroids with underlying matroids M and N. In Lemma 4.10, we had seen that any affine morphism of valuated matroids induces a unique strong map on the underlying matroids. Thus, the underlying matroid $M \times N$ would be a product in the category $\mathbf{Matr}_o$. This contradicts the nonexistence of a product in the category of matroids, [79, Proposition 3.5]. The same argument works for the category $\mathbf{VMatr}$. □

To show functoriality of valuated matroid deletion we had discussed in Definition 3, we first characterize valuated matroid deletion via the associated tropical linear spaces. The following statement and proof are an adaptation of [27, Proposition 3.14].

Proposition 4.15. *Let μ be a valuated matroid on the ground set $[n]$ and let $S \subseteq [n]$. Then*

$$\overline{\text{trop}}\,(\mu \setminus S) \;=\; \overline{\text{trop}}(\mu)|_{\mathbb{P}(\mathbb{T}^{[n]\setminus S})}.$$

Proof. First, we note that we can restrict to the case $S = \{s\}$ and obtain the result for arbitrary S by inductively re-applying the one-element case.

Let $v \in \overline{\text{trop}}(\mu)$. Then the minimum in $\{C_i + v_i\}_{i\in[n]}$ is achieved at least twice for every $C \in \mathcal{C}(\mu)$. In particular, the minimum in $\{C_i + v_i\}_{i\in[n]\setminus s}$ is achieved at least twice for every $C \in \mathcal{C}(\mu)$ where $\text{supp}(C) \subseteq [n]\setminus s$. By Definition 3.79, this implies that $v|_{\mathbb{P}(\mathbb{T}^{[n]\setminus S})} \in \overline{\text{trop}}(\mu\setminus s)$. This proves the first inclusion.

For the reverse inclusion, let $v \in \overline{\text{trop}}(\mu \setminus s) \subseteq \mathbb{P}(\mathbb{T}^{[n]\setminus s})$. Then, the minimum in $\{C_i + v_i\}_{i\in[n]\setminus s}$ is achieved at least twice for every $C \in \mathcal{C}(\mu \setminus s)$. By Definition 3.79, this means it is achieved at least twice for every $C \in \mathcal{C}(\mu)$ with $\text{supp}(C) \subseteq [n] \setminus s$. Now we want to find some $t \in \mathbb{T}$ such that the vector $\tilde{v} = (v_1, \ldots, v_{s-1}, t, v_{s+1}, \ldots, v_n) \in \mathbb{P}(\mathbb{T}^n)$ is in $\overline{\text{trop}}(\mu)$. Then $v = \tilde{v}|_{\mathbb{P}(\mathbb{T}^{[n]\setminus S})} \in \overline{\text{trop}}(\mu)|_{\mathbb{P}(\mathbb{T}^{[n]\setminus S})}$.

If for every $C \in \mathcal{C}(\mu)$ the minimum in $\{C_i + v_i\}_{i\in[n]\setminus s}$ is achieved at least twice, we can set $t = \infty$ and we are done. Therefore, we assume that there exists a circuit $C \in \mathcal{C}(\mu)$ such that the minimum in $\{C_i + v_i\}_{i\in[n]\setminus s}$ is achieved only once. Choose $t \in \mathbb{R}$ such that $t + C_s = \min_{i\in[n]\setminus s}\{C_i + v_i\}$. Then, the minimum in $\{C_i + \tilde{v}_i\}_{i\in[n]}$ is achieved twice. We claim that $\tilde{v} \in \overline{\text{trop}}(\mu)$.

We proceed by contradiction. Let $C' \in \mathcal{C}(\mu)$ and assume that the minimum in $\{C'_i + \tilde{v}_i\}_{i\in[n]}$ is achieved only once at the index $j \in [n]$. Up to tropical scalar multiplication, we can assume that $C'_s = C_s \neq \infty$. Suppose first that $j \neq s$. By construction, $v_i + C_i \geq t + C_s = t + C'_s > v_j + C'_j$ for every $i \in [n]$, in particular $C_j \neq C'_j$. On the other hand, we have $v_i + C'_i > v_j + C'_j$ for every $i \neq j$, therefore $v_i + \min(C_i, C'_i) > v_j + C'_j$ for every $i \neq j$. By the vector elimination axiom, Proposition 3.74, there exists a vector C'' of μ such that $C''_s = \infty$, $C''_i \geq \min\{C_i, C'_i\}$ for all $i \in [n]$ with equality whenever $C_i \neq C'_i$, in particular $C''_j = C'_j$. But now $\text{supp}(C'') \subseteq [n] \setminus s]$, so the minimum in $\{C''_i + v_i\}_{i\in[n]\setminus s}$ has to be achieved at least twice, contradicting $v_i + C''_i \geq v_i + \min(C_i, C'_i) > v_j + C'_j$ for every $i \neq j$.

Now suppose that $j = s$. Let $k \in [n]$ be the index at which the minimum in $\{v_i + C_i\}_{i\in[n]\setminus s}$ is achieved. Then we have $v_k + C_k = t + C_s = t + C'_s < v_i + C'_i$ for every $i \neq s$, in particular $C_k < C'_k$. Now, applying vector elimination (i.e., Proposition 3.74) again, we obtain a vector C'' such that $C''_s = \infty$, $C''_i \geq \min\{C_i, C'_i\}$ for all $i \in [n]$ with equality whenever $C_i \neq C'_i$, in particular $C''_k = C_k$. Then, $v_k + C''_k = v_k + C_k < v_i + C'_i$ and further $v_k + C''_k = v_k + C_k < v_i + C_i$ for every $i \neq s, k$. This contradicts the fact that the minimum in $\{v_i + C''_i\}_{i\in[n]\setminus s]}$ is achieved at least twice. □

Using the correspondence between tropical linear spaces and valuated matroids, this now allows us to prove a purely matroidal statement on quotients of deletions.

Lemma 4.16. *Let μ and ν be valuated matroids over the ground set $[n]$ of rank r and s respectively with $r < s$. Then, the following are equivalent:*

(a) μ is a matroid quotient of ν, and
(b) $\mu \setminus S$ is a matroid quotient of $\nu \setminus S$ for all $S \in 2^{[n]}$.

Proof. $(b) \Rightarrow (a)$ follows by using (b) on the set $S = \emptyset$.

To show $(a) \Rightarrow (b)$, we use the characterization we just proved in Proposition 4.15. By Theorem 3.96, μ is a matroid quotient of ν if and only if $\overline{\text{trop}}(\mu) \subseteq \overline{\text{trop}}(\nu)$.

Let $S \subseteq [n]$. Now, if $\overline{\operatorname{trop}}(\mu) \subseteq \overline{\operatorname{trop}}(\nu)$, then $\overline{\operatorname{trop}}(\mu)|_{\mathbb{P}(\mathbb{T}^{[n]\setminus S})} \subseteq \overline{\operatorname{trop}}(\nu)|_{\mathbb{P}(\mathbb{T}^{[n]\setminus S})}$. By Proposition 4.15, this is equivalent to $\overline{\operatorname{trop}}(\mu \setminus S) \subseteq \overline{\operatorname{trop}}(\nu \setminus S)$. We can again use the equivalent characterization in Theorem 3.96, and obtain that this is equivalent to $\mu \setminus S$ being a quotient of $\nu \setminus S$. □

Proposition 4.17. *Direct sum and deletion are functorial operations in* $\mathbf{VMatr}_a$.

Proof. Let ν, μ, and ω be valuated matroids over the sets $[n_1]$, $[n_2]$, and $[n_3]$ respectively, and let $f : \nu \to \mu$ be an affine morphism of valuated matroids.

Direct sum. We consider the direct sum of both μ and ν with ω respectively. Then, f extends to a map

$$\begin{aligned} f + \mathrm{id} : \nu \oplus \omega &\to \mu \oplus \omega \\ B &\mapsto f(B \cap [n_1]) + \mathrm{id}(B \cap [n_3]), \end{aligned}$$

where the sum is taken component-wise. We observe that $f + \mathrm{id}$ is an affine morphism of valuated matroids analogous to the proof of Proposition 4.13, as f is assumed to be an affine morphism and the identity map from a valuated matroid into itself extended via the trivial map is always an affine morphism of valuated matroids, see Example 4.6.

Deletion. Now, we consider $\mu \setminus S$ and $\nu \setminus S$. Then, f restricts to a map

$$\begin{aligned} f|_{S^c} : \nu \setminus S &\to \mu \setminus S \\ B &\mapsto (f_1(B)|_{S^c}, f_2(i)). \end{aligned}$$

We now show that $f|_{S^c}$ is an affine morphism of valuated matroids. We first show that $f|_{S^c}^{-1}(\mu \setminus S) = f^{-1}(\mu) \setminus S$.

Let M be the underlying matroid of μ, let $T \subseteq S$ be a set such that $\operatorname{rk}_{f^{-1}(M)}([n] \setminus S \cup T) = \operatorname{rk}(f^{-1}(M))$ and let $B \in \mathcal{B}(f^{-1}(M) \setminus S)$. By Definition 3.78,

$$\begin{aligned} (f^{-1}(\mu) \setminus S)(B) &= f^{-1}(\mu)(B \cup T) \\ &= \mu|_{f_1([n]\cup\{o\})}(f_1(B \cup T)) + \sum_{i \in B \cup T} f_2(i) \\ &= \mu|_{f_1([n]\cup\{o\})}(f_1(B) \cup f_1(T)) + \sum_{i \in B} f_2(i) + \sum_{i \in T} f_2(i). \end{aligned}$$

By the above discussion, we know that $\operatorname{rk}_M([m] \setminus S \cup f_1(T)) = \operatorname{rk}(M)$, thus, we can rewrite $f^{-1}|_{S^c}(\mu \setminus S)(B)$ as follows.

$$\begin{aligned} f^{-1}|_{S^c}(\mu \setminus S)(B) &= (\mu \setminus S)|_{f_1|_{S^c}(S^c \cup \{o\})}\big(f_1(B)\big) + \sum_{i \in B} f_2(i) \\ &= \mu|_{f_1([n]\cup\{o\})}\big(f_1(B) \cup f_1(T)\big) + \sum_{i \in B} f_2(i). \end{aligned}$$

Then, the two matroids $f^{-1}(\mu \setminus S)$ and $f^{-1}(\mu) \setminus S$ only differ by the constant $\sum_{i \in T} f_2(i)$ on all bases, thus, the two matroids are equivalent.

Now, we use Lemma 4.16: Since $f : \nu \to \mu$ is an affine morphism of valuated matroids, $f^{-1}(\mu) \twoheadleftarrow \nu$. Thus, by Lemma 4.16 and the above discussion, for any $S \subseteq [m]$, $f|_{S^c}^{-1}(\mu \setminus S) = f^{-1}(\mu) \setminus S \twoheadleftarrow \nu \setminus S$. This is precisely saying that $f|_{S^c}$ satisfies the definition of affine morphisms of valuated matroids. □

Proposition 4.18. *Taking the dual of a matroid is not a functorial operation in* $\mathbf{VMatr}_a$.

Proof. The proof follows as the proof of Proposition 4.14: Affine morphisms of valuated matroids induce strong maps of the underlying matroids, and taking the dual is not functorial in that category, [79, Proposition 3.6]. □

Proposition 4.19. *The category* $\mathbf{VMatr}_a$ *has a zero object, the valuated matroid*

$$\omega : \binom{[1]}{0} \to \mathbb{T}, \emptyset \mapsto 0.$$

Proof. Let μ be a matroid of rank r over a set $[n]$. We first show that ω is initial, i.e., that there exists a unique affine morphism of valuated matroids $f : \omega \to \mu$. The map $f : o \mapsto (o, \infty)$ is trivially an affine morphism of valuated matroids. Let $S \subseteq [1]$. Then, $f^{-1}(\mu) = \mu(f_1(S)) + \infty = \infty$, so the only basis of $f^{-1}(\mu)$ is the empty set. We have to check that $f^{-1}(\mu) \twoheadleftarrow \omega$. For $I = \emptyset$ and $J \in \binom{[1]}{0}$, there exists no $i \in I \setminus J = \emptyset \setminus J$, so $f^{-1}(\mu) \twoheadleftarrow \omega$ is trivially satisfied.

Now we show that ω is terminal, i.e., that there exists a unique affine morphism of valuated matroids $f : \mu \to \omega$. There is only one way to construct such a map: $f : \mu \to \omega, i \mapsto (o, \infty)$. Since every set $S \subseteq [n]$ has valuation $f^{-1}(\omega)(S) = \mu(o) + \infty = \infty$, the matroid $f^{-1}(\omega)$ has rank 0, so again the only basis is $\emptyset$. Thus, there is no $i \in \emptyset \setminus I$ for any set $I \subseteq [n]$, and again there are no exchange conditions to check, so $f^{-1}(\omega) \twoheadleftarrow \mu$ is trivially satisfied.

Thus, ω is both initial and terminal, and hence a zero object. □

5. Morphisms and naive matrix multiplication

In this section, we discuss the relationship of naive tropical matrix multiplication with a tropical linear space with the affine morphisms of valuated matroids we defined and studied in Section 4. Most of this section is a generalization of joint work with Giulia Iezzi [82]. In Section 5.1, we discuss properties of the images of tropical linear spaces under matrix multiplication. In particular, we prove that while these sets are not necessarily tropical linear spaces themselves, they still are tropically convex sets. In Section 5.2, we restrict to tropical matrix multiplication with weakly monomial matrices. In this case, we show that the image under matrix multiplication is again a tropical linear space, and construct an associated affine map of the corresponding valuated matroid.

5.1. Images of linear spaces under matrix multiplication.

In linear algebra, after fixing bases, linear spaces can be represented as matrices, and applying a linear map is equivalent to matrix multiplication with an appropriate matrix. Hence, it is natural to consider a tropical analogue of this in our aim to construct tropical and matroidal analogues for linear maps.

To this end, we study the behavior of tropical linear spaces under matrix multiplication. For a matrix $A \in K^{n\times m}$, we write $\operatorname{val}(A) := \big(\operatorname{val}(A_{ij})\big)_{ij} \in \mathbb{T}^{n\times m}$ for the matrix with valuation applied to all entries. We call $\operatorname{val}(A)$ the *valuation* of A. If μ is a valuated matroid on $[m]$, we write $\operatorname{val}(A) \odot \overline{\operatorname{trop}}(\mu) \subseteq \mathbb{R}^n$ for the pointwise naive tropical matrix multiplication of $\operatorname{val}(A)$ with $\overline{\operatorname{trop}}(\mu)$.

The first property we want to show for the image of a tropical linear space $\operatorname{trop}(\mu)$ under matrix multiplication is *tropical convexity*. This allows us to infer properties of the whole set by just showing them on the images of the valuated cocircuits of $\text{trop}(\mu)$, as they are a spanning set for the image of the tropical linear space. Tropical convexity is a classical concept in tropical geometry and has been used for similar purposes multiple times, for instance in [50].

Definition 5.1 ([50, Introduction])**.** Let $S \subseteq \mathbb{P}(\mathbb{T}^m)$ be a set of vectors. We say that S is *tropically convex* if for all $\tilde{v}, \tilde{w} \in S$ and $\lambda, \rho \in \mathbb{R}$, the linear combination $\lambda \odot \tilde{v} \oplus \rho \odot \tilde{w}$ is also contained in S.

To establish that images of tropical linear spaces are tropically convex, we first need to show that tropical linear spaces themselves are. In [75, Theorem 1.1], it is shown that tropical linear spaces of loopless valuated matroids, considered as subsets of $\mathbb{R}^m/\mathbb{R}\mathbf{1}$ are tropically convex. We, however, consider valuated matroids with loops, and consider them as subsets of $\mathbb{P}(\mathbb{T}^m)$.[1]

Lemma 5.2. *Let μ be a valuated matroid over $[m]$. Then, $\overline{\text{trop}}(\mu) \subseteq \mathbb{P}(\mathbb{T}^m)$ is tropically convex.*

Proof. The proof for the extension of [75, Proposition 2.14], and hence [75, Theorem 1.1] to subsets of $\mathbb{P}(\mathbb{T}^m)$ follows line by line identical to the original proof, observing that Hampe's argument never uses the finiteness of non-minimal coordinate entries of circuits. □

Proposition 5.3. *Let $\overline{\text{trop}}(\mu)$ be a tropical linear space in $\mathbb{P}(\mathbb{T}^m)$ and $A \in K^{n\times m}$. Then $\text{val}(A) \odot \overline{\text{trop}}(\mu)$ is tropically convex.*

Proof. Let $\tilde{v}, \tilde{w} \in \text{val}(A) \odot \overline{\text{trop}}(\mu)$ and $\lambda, \rho \in \mathbb{R}$. We need to show that $\lambda \odot \tilde{v} \oplus \rho \odot \tilde{w} \in \text{val}(A) \odot \overline{\text{trop}}(\mu)$. By definition of $\text{val}(A) \odot \overline{\text{trop}}(\mu)$, there exist v, w such that $\tilde{v} = \text{val}(A) \odot v$ and $\tilde{w} = \text{val}(A) \odot w$. By Lemma 5.2, tropical linear spaces are tropically convex, thus $\lambda \odot v \oplus \rho \odot w \in \overline{\text{trop}}(\mu)$. Since tropical matrix multiplication is distributive and commutes with tropical scalar multiplication,

$$\begin{aligned}(\lambda \odot \tilde{v}) \oplus (\rho \odot \tilde{w}) &= \big(\lambda \odot \big(\text{val}(A) \odot v\big)\big) \oplus \big(\rho \odot \big(\text{val}(A) \odot w\big)\big) \\ &= \text{val}(A) \odot (\lambda \odot v \oplus \rho \odot w) \in \text{val}(A) \odot \overline{\text{trop}}(\mu).\end{aligned}$$ □

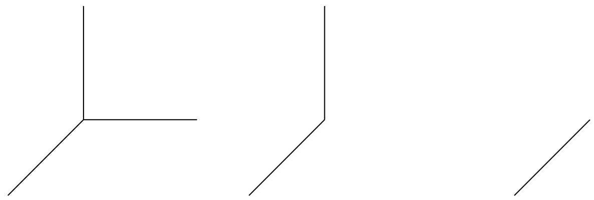
In general, images of tropical linear spaces under pointwise matrix multiplication are not tropical linear spaces, as can be seen in the next example.

Example 5.4. We consider the trivially valued matroid with underlying matroid $U_{2,3}$ introduced in Example 3.3 given by the map $\mu : \binom{[3]}{2} \to \mathbb{R} \cup \{\infty\}, I \mapsto 0$, and the matrices

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The tropical linear space $\overline{\text{trop}}(\mu)$ and the polyhedral complexes $\text{val}(A_1) \odot \overline{\text{trop}}(\mu)$ and $\text{val}(A_2) \odot \overline{\text{trop}}(\mu)$ are depicted below. Both $\text{val}(A_1) \odot \overline{\text{trop}}(\mu)$ and $\text{val}(A_2) \odot \overline{\text{trop}}(\mu)$ are not tropical linear spaces, as these polyhedral complexes cannot be assigned balanced weights.

[1]The author thanks Michael Joswig for pointing her to this gap in the original argument presented in [82].



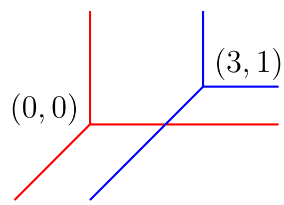
$\overline{\mathrm{trop}}(\mu)$ $\quad$ $\mathrm{val}(A_1) \odot \overline{\mathrm{trop}}(\mu)$ $\quad$ $\mathrm{val}(A_2) \odot \overline{\mathrm{trop}}(\mu)$

5.2. **Weakly monomial matrices and associated affine maps of matroids.** As images of tropical linear spaces under pointwise tropical matrix multiplication are not necessarily tropical linear spaces, tropical matrix multiplication can not be fully compatible with (affine) morphisms of valuated matroids. Intuitively, this boils down to tropicalization commuting with *monomial*, but not linear maps. Nevertheless, we can identify a subclass of matrices which can be expressed in terms of affine morphisms of valuated matroids.

Definition 5.5. Let $A \in K^{n\times m}$. We call A *weakly monomial* if A has at most one non-zero entry in each row.

Example 5.6. We consider the trivially valued matroid $U_{2,3}$ from Example 5.4. Its tropical linear space is the classical tropical line with vertex at $(0,0,0)$ in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ depicted in red below, which we had defined in Section 3.3 and computed explicitly in Examples 2.8. Multiplying by the tropicalization of a weakly monomial matrix $\mathrm{val}(A)$ below yields a shifted tropical linear space to the right, depicted in blue. The permutation further switches the rays in directions y and z.

$$\mathrm{val}(A) = \begin{bmatrix} 3 & \infty & \infty \\ \infty & \infty & 1 \\ \infty & 0 & \infty \end{bmatrix}$$



In the following, we will show several preliminary results with the ultimate goal of proving Proposition 5.18, which asserts that maps $f : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{T}$ correspond to tropical matrix multiplication of a tropical linear space $\overline{\mathrm{trop}}(\mu) \subseteq \mathbb{P}(\mathbb{T}^m)$ with the valuation of a weakly monomial matrix $A \in K^{n\times m}$. To this end, we characterize the tropical linear space of the induced matroid $f^{-1}(\mu)$ (see Definition 4.7) for any matroid μ on $[m]$. Note that if $m \neq n$, the two tropical linear spaces live in different ambient spaces: $\overline{\mathrm{trop}}(\mu)$ is a subset of $\mathbb{P}(\mathbb{T}^m)$, whereas $\overline{\mathrm{trop}}\big(f^{-1}(\mu)\big)$ is a subset of $\mathbb{P}(\mathbb{T}^n)$. We start out by describing the cocircuits of $f^{-1}(\mu)$.

Lemma 5.7. *Let μ be a valuated matroid on $[m]$ and $f : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{T}$. Let $I \in \binom{[n]}{\mathrm{rk}(f^{-1}(\mu))-1}$. Then, the coordinate entries of the valuated cocircuits of $f^{-1}(\mu)$ are given as follows:*

$$C^*_{f^{-1}(\mu)}(I)_i \;=\; \begin{cases} C^*_{\mu|_{f_1([n]\cup\{o\})}}\big(f_1(I)\big)_{f_1(i)} \odot \bigodot_{k\in I\cup i} f_2(k) & o \notin f_1(I \cup i), \\ \infty & o \in f_1(I\cup i). \end{cases}$$

Proof. Since o is a loop, every set containing o has valuation ∞. Now, assume $o \notin f_1(I \cup i)$. If $i \in I$, then $f_1(i) \in f(I)$ and $C^*_{f^{-1}(\mu)}(I)_i = C^*_{\mu|_{f_1([n]\cup\{o\})}}\big(f_1(I)\big)_{f_1(i)} \odot \bigodot_{k\in I\cup i} f_2(k) = \infty$ by Definition 3.71. Further, by Definition 3.71, if $i \notin I$, $C^*_{f^{-1}(\mu)}(I)_i = f^{-1}(\mu)(I \cup i)$. Now, by

Definition 4.7,

$$\begin{aligned} f^{-1}(\mu)(I\cup i) &= \mu|_{f_1([n]\cup\{o\})}\big(f_1(I\cup i)\big) + \sum_{k\in I\cup i} f_2(k) \\ &= \mu|_{f_1([n]\cup\{o\})}\big(f_1(I)\cup f_1(i))\big) + \sum_{k\in I\cup i} f_2(k) \\ &= C^*_{\mu|_{f_1([n]\cup\{o\})}}\big(f_1(I)\big)_{f_1(i)} \odot \bigodot_{k\in I\cup i} f_2(k). \end{aligned}$$ □

Definition 5.8. Let $A_f \in K^{n\times m}$ be a weakly monomial matrix. We define an *associated map* $f : [n]\cup\{o\} \to [m]\cup\{o\}\times\mathbb{T}$ by

$$i \mapsto \begin{cases} (o,\infty) & i = o \text{ or } A_{ij} = 0 \text{ for all columns } j \\ \big(j, \mathrm{val}(A_{ij})\big) & A_{ij} \neq 0. \end{cases}$$

If $f : [n]\cup\{o\} \to [m]\cup\{o\}\times\mathbb{T}$ is a map of sets such that for all $i \in [n]\cup\{o\}$, $f_2(i) = \mathrm{val}(k)$ for some $k \in K$, we construct an *associated matrix* $A_f \in K^{n\times m}$ by setting

$$A_{ij} = \begin{cases} k & \text{if } f_1(i) = j \text{ and } i,j \neq o \\ 0 & \text{otherwise.} \end{cases}$$

Remark 5.9. In the above definition, the associated map of a weakly monomial matrix is unique. The associated matrix $A_f \in K^{n\times m}$ is not unique, but its valuation $\mathrm{val}(A_f)$ is. This is due to the fact that there can be multiple $k \neq k' \in K$ with valuation $\mathrm{val}(k) = \mathrm{val}(k')$.

Remark 5.10. We construct the associated map for some special matrices:

(*a*) If A_f is a *permutation matrix*, then it is quadratic and the associated map f consists of a permutation map f_1 that fixes o, and $f_2(i) = 0$ for $i \in [n]$, and $f_2(o) = \infty$.
(*b*) If A_f is a *projection matrix* of rank $s < n$, then the associated map f is given as $f(i) = (i, 0)$ on $[s]$ and $f(i) = (o,\infty)$ otherwise.
(*c*) If A_f is a *diagonal matrix*, it is again quadratic and the associated map f is $f(i) = (i, \mathrm{val}(A_{ii}))$ for $i \in [n]$ and $f(o) = (o,\infty)$.

For the types of maps associated to matrices given above, we can again construct a (potentially not unique) associated matrix, provided that for every $i \in [n]$, there exists a $k \in K$ such that $\mathrm{val}(k) = f_2(i)$.

(a') If f_1 is a permutation map fixing o and $f_2(i) = 0$ for $i \in [n]$ and $f_2(o) = \infty$, then A_f can be chosen as the (square) permutation matrix associated to the permutation f_1.
(b') If pr_S is a projection map satisfying $f(i) = (i, 0)$ for $i \in S^c$ and $f(i) = (o,\infty)$ for $i \in S\cup\{o\}$, a matrix associated to pr_S is given as the projection matrix $A_{S_{ii}} = 1$ if $i \notin S$, and $A_{ij} = 0$ otherwise.
(c') If f_1 is the identity map, an associated (square) matrix A_f is a diagonal matrix with entries $A_{f,ii} = k_i$ for $k_i \in K$ with $\mathrm{val}(k_i) = f_2(i)$.

Example 5.11. We construct the associated matrices to the affine morphisms we constructed in Example 4.6 below.

Permutation	Projection/Deletion	Contraction
$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

We will now develop the correspondence between affine morphisms of valuated matroids and matrix multiplication with weakly monomial matrices. For the proof, we will decompose each weakly monomial matrix (and analogously, each map) into a product of matrices.

Lemma 5.12. *Let $g : [n_1] \cup \{o\} \to [n_2] \cup \{o\} \times \mathbb{T}$ and $h : [n_2] \cup \{o\} \to [n_3] \cup \{o\} \times \mathbb{T}$ be maps with $g(o) = h(o) = (o, \infty)$, and let $A_g \in K^{n_1 \times n_2}$ and $A_h \in K^{n_2 \times n_3}$ be their associated weakly monomial matrices. Assume that for any tropical linear space $\overline{\text{trop}}(\mu)$, $\text{val}(A_g) \odot \overline{\text{trop}}(\mu) = \overline{\text{trop}}(g^{-1}(\mu))$ and $\text{val}(A_h) \odot \overline{\text{trop}}(\mu) = \overline{\text{trop}}(h^{-1}(\mu))$. Then, for $h \circ g(i) = (h_1(g_1(i)), g_2(i) + h_2(g_1(i)))$, we have $\text{val}(A_g \cdot A_h) \odot \overline{\text{trop}}(\mu) = \overline{\text{trop}}\big((h \circ g)^{-1}(\mu)\big) \subseteq \mathbb{P}(\mathbb{T}^{n_3})$.*

Proof. By assumption,

$$\begin{aligned} \text{val}(A_g \cdot A_h) \odot \overline{\text{trop}}(\mu) &= \text{val}(A_g) \odot \text{val}(A_h) \odot \overline{\text{trop}}(\mu) \\ &= \text{val}(A_g) \odot (\overline{\text{trop}}(h^{-1}(\mu))) = \overline{\text{trop}}(g^{-1}(h^{-1}(\mu))) \end{aligned}$$

Now, we show that $g^{-1}(h^{-1}(\mu)) = (h \circ g)^{-1}(\mu)$. Let $r = \text{rk}(\mu)$, and $I \in \binom{[n_1]}{r}$. Then,

$$\begin{aligned} (h \circ g)^{-1}(\mu)(I) &= \mu|_{(h \circ g)_1([n_1] \cup \{o\})}\big((h \circ g)_1(I)\big) + \sum_{i \in I} (h \circ g)_2(i) \\ &= \mu|_{(h_1(g_1([n_1] \cup \{o\})))}\big(h_1(g_1(I)\big) + \sum_{i \in I} \big(g_2(i) + h_2(g_1(i))\big) \\ &= \mu|_{(h_1(g_1([n_1] \cup \{o\})))}\big(h_1(g_1(I)\big) + \sum_{i \in I} h_2(g_1(i)) + \sum_{i \in I} g_2(i) \\ &= h^{-1}(\mu)\big(\mu|_{g_1([n_1] \cup \{o\})}(g_1(I))\big) + \sum_{i \in I} g_2(i) = g^{-1}\big(h^{-1}(\mu)\big)(I). \end{aligned}$$

Above, the last and the second to last equalities are obtained by explicitly writing out the maps using Definition 4.7. Since the valuated matroids are equal for each basis, the associated tropical linear spaces coincide. We remark that the reversal of order here is compatible with the contravariant properties we describe for affine morphisms of valuated matroids in Section 4. □

To prepare for showing the general case, we consider permutation and projection matrices and maps.

Lemma 5.13. *Let μ be a valuated matroid over $[n]$ and let $f : [n] \cup \{o\} \to [n] \cup \{o\} \times \mathbb{R}$ be a permutation map on the first coordinate and the trivial map (see Lemma 4.11) on the second coordinate, and let A_f be its associated matrix as in Definition 5.8. Alternatively, let A_f be a permutation matrix and f its associated map. Then,*

$$\overline{\text{trop}}(f^{-1}(\mu)) \;=\; \text{val}(A) \odot \overline{\text{trop}}(\mu).$$

Proof. By assumption, we either start with a permutation matrix A and its induced map f_A, or we start with a permutation map f and its associated matrix A_f. Both situations can be handled simultaneously: If we start with a permutation map f, by Remark 5.10(a'), the associated matrix A_f is a permutation matrix. Hence, we may assume that we have a permutation map f and an associated permutation matrix A_f. Conversely, if we start with a permutation matrix A_f, by Remark 5.10(a), f_1 is a permutation, and f_2 is the trivial map (see Lemma 4.11).

Since f_2 is a permutation map and f_2 is trivial, the map permutes entries of vectors of the valuated matroid μ uniformly, without changing any valuation. Similarly, pointwise multiplication by the permutation matrix A_f permutes coordinate entries of points of the associated tropical linear space $\overline{\text{trop}}(\mu)$ in the same fashion. Since vectors of μ correspond to points in $\overline{\text{trop}}(\mu)$, we have $\overline{\text{trop}}(f^{-1}(\mu)) = \text{val}(A) \odot \overline{\text{trop}}(\mu)$. □

Lemma 5.14. *Let μ be a valuated matroid over $[m]$, let $\text{pr}_S : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{R}$ be a projection map, and let A_{pr_S} be its associated matrix as in Definition 5.8. Alternatively, let A_{pr_S} be a projection matrix and pr_S be its associated map. Then,*

$$\overline{\text{trop}}(\text{pr}_S^{-1}(\mu)) \;=\; \text{val}(A_{\text{pr}_S}) \odot \overline{\text{trop}}(\mu).$$

Proof. We can directly see that the naive tropical matrix multiplication of $\overline{\text{trop}}(\mu)$ with $\text{val}(A_{pr_S})$ (see Remark 5.10(b)) results in the set $\overline{\text{trop}}(\mu)|_{\mathbb{P}(\mathbb{T}^{[n]\setminus S})} \times \infty^{\{([m]\setminus[n])\cup S\}}$. Then, by Proposition 4.15,

$$\begin{aligned} \text{val}(A)_{\text{pr}_S} \odot \overline{\text{trop}}(\mu) &= \overline{\text{trop}}(\mu)|_{\mathbb{P}(\mathbb{T}^{[n]\setminus S})} \times \infty^{\{([m]\setminus[n])\cup S\}} \\ &= \overline{\text{trop}}(\mu \setminus S) \times \infty^{\{([m]\setminus[n])\cup S\}} \\ &= \overline{\text{trop}}(\mu \setminus S \oplus U_{0,|S|} \oplus U_{0,[m]\setminus[n]}), \end{aligned}$$

i.e., the valuated matroid arising as the deletion of S from μ, substituting all deleted elements with loops and filling in loops for all elements in $[m]$ that are not in $[n]$. Now, by the construction of pr_S and Definition 4.9, $\overline{\text{trop}}\big(\text{pr}_S^{-1}(\mu)\big) = \overline{\text{trop}}(\mu \setminus S \oplus U_{0,|S|} \oplus U_{0,[m]\setminus[n]})$, thus $A_{\text{pr}_S} \odot \overline{\text{trop}}(\mu) = \overline{\text{trop}}\big(\text{pr}_S^{-1}(\mu)\big)$. □

Remark 5.15 (Normal form for weakly monomial matrices)*.* Let $A \in K^{n\times m}$ be a weakly monomial matrix with entries in $\{0,1\}$. Then, A has n rows and m columns. By definition, A has at most one non-zero entry in each row. Thus, A has at most n non-zero columns, and if $m > n$, after applying permutation maps, we obtain the normal form

$$A = \left[\begin{array}{cccc|c} \mathbf{1}_{k_1} & \mathbf{0}_{k_1} & \cdots & \mathbf{0}_{k_1} & \\ \mathbf{0}_{k_2} & \mathbf{1}_{k_2} & & \mathbf{0}_{k_2} & \\ \dots & & \ddots & \dots & 0 \\ \mathbf{0}_{k_l} & \mathbf{0}_{k_l} & \cdots & \mathbf{1}_{k_l} & \\ \hline & & 0 & & 0 \end{array}\right],$$

which has at least $m - n$ zero columns. Analogously, if $n \le m$, we can transform the matrix into the same normal form. Now, since $\text{rk}(A) \le \min\{m, n\}$, we have at least $n - m$ zero columns.

We apply the results of the previous lemmas to obtain the result for all matrices with entries in $\{0,1\}$. These correspond to the morphisms of valuated matroids we studied in Section 4.1, where we did not consider affine linear maps, and encompass permutations, projections and contractions, as well as compositions of these maps.

Lemma 5.16. *Let μ be a valuated matroid over $[n]$. Let $f : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{T}$ be a map where $f_2(i) = 0$ if $f_1(i) \neq o$. Then, for any associated weakly monomial matrix A_f (as constructed in Definition 5.8),*

$$\overline{\operatorname{trop}}\big(f^{-1}(\mu)\big) \;=\; \operatorname{val}(A_f) \odot \overline{\operatorname{trop}}(\mu). \tag{4}$$

Conversely, if $A_f \in K^{n\times m}$ is a weakly monomial matrix with entries in $\{0,1\}$, the associated map f satisfies $\overline{\operatorname{trop}}\big(f^{-1}(\mu)\big) = \operatorname{val}(A_f) \odot \overline{\operatorname{trop}}(\mu)$.

Proof. Let A_f be an arbitrary weakly monomial matrix with entries in $\{0,1\}$. Then, by Remark 5.15, after multiplication with a permutation matrix, A_f can be assumed to be in normal form

$$A_f = \left[\begin{array}{cccc|c} \mathbf{1}_{k_1} & \mathbf{0}_{k_1} & \cdots & \mathbf{0}_{k_1} & \\ \mathbf{0}_{k_2} & \mathbf{1}_{k_2} & & \mathbf{0}_{k_2} & \\ \dots & & \ddots & \dots & 0 \\ \mathbf{0}_{k_l} & \mathbf{0}_{k_l} & \cdots & \mathbf{1}_{k_l} & \\ \hline & & 0 & & 0 \end{array}\right],$$

where $\mathbf{1}_k$ denotes the vector $(1,\dots,1) \in \mathbb{R}^k$, and by Lemma 5.12, it is sufficient to show it for this form. Let $A_{f'}$ be the matrix obtained from A_f by setting all except the first non-zero entry in each column to 0. Up to permutation, this is a projection matrix, thus, by Lemma 5.14, $\overline{\operatorname{trop}}\big(f'^{-1}(\mu)\big) = \operatorname{val}(A_{f'}) \odot \overline{\operatorname{trop}}(\mu)$.

Now, let $w \in \operatorname{val}(A_f) \odot \overline{\operatorname{trop}}(\mu)$. Then, there exists $v \in \overline{\operatorname{trop}}(\mu)$ such that $w = \operatorname{val}(A_f) \odot v$. Let $w' = \operatorname{val}(A_{f'}) \odot v \in \overline{\operatorname{trop}}\big(f'^{-1}(\mu)\big)$. Now, $w_i = (\operatorname{val}(A_f) \odot v)_i = w'_i$ if $A_{f',i}$ is a non-zero row. Otherwise, $w'_i = \infty$ and

$$w_i \;=\; (\operatorname{val}(A_f) \odot v)_i \;=\; (\operatorname{val}(A_f) \odot v)_j \;=\; (\operatorname{val}(A_{f'}) \odot v)_j \;=\; w'_j$$

for a row j with unique non-zero entry in the same column as the row i.

The map f associated to A_f is defined by $f_1(j) = i$ for each $j \in \{i, \dots, i + k_c\}$ where i is the first non-zero entry of A_f for the column c, and by $f_2(j) = \infty$ for $j \in \{\sum_{p=1}^{l} k_p, \dots, n\}$. Consequently, by Lemma 5.7, each cocircuit $C^* = C^*_{f^{-1}(\mu)}(I) \in \mathcal{C}^*(f_1^{-1}(\mu))$ has coordinates

$$\begin{aligned} C^*_j \;&=\; C^*_{f^{-1}(\mu)}(I)_j \;=\; \mu|_{f_1([n]\cup\{o\})}\big(f_1(I) \cup f_1(j)\big) \;=\; \mu|_{f_1([n]\cup\{o\})}\big(f_1(I) \cup i\big) \\ &=\; \mu|_{f_1([n]\cup\{o\})}\big(f_1(I) \cup f_1(i)\big) \;=\; C^*_i \;=\; C^*_{f'^{-1}(\mu)}(I')_i \end{aligned} \tag{5}$$

for all $j \in \{i, \dots, i + k_c\}$, where I' is the set containing the first non-zero entries of each column that each element in I belongs to. As $\overline{\operatorname{trop}}\big(f^{-1}(\mu)\big)$ and $\overline{\operatorname{trop}}\big(f'^{-1}(\mu)\big)$ are tropically generated by their cocircuits by Proposition-Definition 3.71, we can write w' as the tropical sum of cocircuits, $w'_i = \bigoplus_{C^* \in \mathcal{C}^*(f'^{-1}(\mu))} \lambda_{C^*} \odot C^*_i$, thus

$$w_j \;=\; w'_i \;=\; \bigoplus_{C^* \in \mathcal{C}^*(f'^{-1}(\mu))} \lambda_{C^*} \odot C^*_i \;\overset{(5)}{=}\; \bigoplus_{C^* \in \mathcal{C}^*(f^{-1}(\mu))} \lambda_{C^*} \odot C^*_j,$$

where the last equality follows by using Equation 5, and that every cocircuit of $f'^{-1}(\mu)$ is already a cocircuit of $f^{-1}(\mu)$. Thus, $w \in \overline{\operatorname{trop}}\big(f'^{-1}(\mu)\big)$. The reverse direction follows analogously, thus (4) holds. $\square$

We commence with our last case of matrices in the decomposition: diagonal matrices with entries that are different to 0 and 1, i.e., affine morphisms of valuated matroids with an identity map as the first coordinate map, and a nontrivial second coordinate map. In the realizable case, these are the maps that correspond to translations.

Lemma 5.17. *Let μ be a valuated matroid over $[n]$. Let $f : [n] \cup \{o\} \to [n] \cup \{o\} \times \mathbb{T}$ be a map satisfying $f_1(i) = i$. Further, assume that for all $i \in [n]$ there exists $k \in K$ such that $f_2(i) = \mathrm{val}(k)$. Then, for the associated weakly monomial matrix A_f defined in Definition 5.8, $\overline{\mathrm{trop}}\big(f^{-1}(\mu)\big) = \mathrm{val}(A_f) \odot \overline{\mathrm{trop}}(\mu)$. Conversely, if $A_f \in K^{n\times n}$ is a full rank diagonal matrix, the associated map f satisfies $\overline{\mathrm{trop}}\big(f^{-1}(\mu)\big) = \mathrm{val}(A_f) \odot \overline{\mathrm{trop}}(\mu)$.*

Proof. Assume A_f is a full-rank diagonal matrix, then the map of sets f_1 is the identity. Then, for each valuated cocircuit $C^*_{f^{-1}(\mu)}(J)$ of $f^{-1}(\mu)$, by Lemma 5.7 there exists a valuated cocircuit $C^*_\mu(J)$ with equal support J such that

$$C^*_{f^{-1}(\mu)}(J)_i \;=\; C^*_\mu(J)_i \odot \bigodot_{k\in J\cup i} f_2(k), \tag{6}$$

and vice versa. Let $v \in \overline{\mathrm{trop}}\big(f^{-1}(\mu)\big)$. By Proposition 3.71, the point v can be written as a tropical linear combination of cocircuits of $f^{-1}(\mu)$, thus

$$\begin{aligned} v_i &= \bigoplus_{C^*\in\mathcal{C}^*(f^{-1}(\mu))} \lambda_{C^*} \odot C_i \overset{(6)}{=} \bigoplus_{C^*\in\mathcal{C}^*(\mu)} \Big(\lambda_{C^*} \odot C^*_i \odot \bigodot_{k\in\mathrm{supp}(C^*)\cup i} f_2(k)\Big) \\ &= f_2(i) \odot \Bigg(\bigoplus_{C^*\in\mathcal{C}^*(\mu)} \Big(\lambda_{C^*} \odot C^*_i \odot \bigodot_{k\in\mathrm{supp}(C^*)} f_2(k)\Big)\Bigg) \\ &= \mathrm{val}(A_{f,ii}) \odot \Bigg(\bigoplus_{C^*\in\mathcal{C}^*(\mu)} \Big(\big(\lambda_{C^*} \odot \bigodot_{k\in\mathrm{supp}(C^*)} f_2(k)\big) \odot C^*_i\Big)\Bigg). \end{aligned}$$

We have $\lambda_{C^*} \odot \bigodot_{k\in\mathrm{supp}(C^*)} f_2(k) \in \mathbb{R}$, thus, by Proposition 3.71, the vector v' with entries $v'_i = \big(\lambda_{C^*} \odot \bigodot_{k\in\mathrm{supp}(C^*)} f_2(k)\big) \odot C^*_i$ is in $\overline{\mathrm{trop}}(\mu)$. As $v = \mathrm{val}(A_f) \odot v'$, $v \in \mathrm{val}(A_f) \odot \overline{\mathrm{trop}}(\mu)$.

Now, let $w \in \overline{\mathrm{trop}}(\mu)$. Then $w = \bigoplus_{C^*\in\mathcal{C}^*(\mu)} \tilde{\lambda}_{C^*} \odot C^*$ for some fixed $\tilde{\lambda}_{C^*} \in \mathbb{R}$ for each cocircuit C^*, and we have

$$\begin{aligned} (\mathrm{val}(A_f) \odot w)_i &= (\mathrm{val}(A_f) \odot \bigoplus_{C^*\in\mathcal{C}^*(\mu)} \tilde{\lambda}_{C^*} \odot C^*)_i \overset{(*)}{=} \bigoplus_{C^*\in\mathcal{C}^*(\mu)} \big(\mathrm{val}(A_{f,ii}) \odot \tilde{\lambda}_{C^*} \odot C^*_i\big) \\ &\overset{(**)}{=} \bigoplus_{C^*\in\mathcal{C}^*(\mu)} \big(\tilde{\lambda}_{C^*} \odot f_2(i) \odot C^*_i\big) \\ &= \bigoplus_{C^*\in\mathcal{C}^*(\mu)} \Big(\big(\tilde{\lambda}_{C^*} \oslash \bigodot_{k\in\mathrm{supp}(C^*)} f_2(k)\big) \odot C^*_i \odot f_2(i) \bigodot_{k\in\mathrm{supp}(C^*)} f_2(k)\Big) \\ &\overset{(6)}{=} \bigoplus_{C^*\in\mathcal{C}^*(f^{-1}(\mu))} \Big(\big(\tilde{\lambda}_{C^*} \oslash \bigodot_{k\in\mathrm{supp}(C^*)} f_2(k)\big) \odot C^*_i\Big) \end{aligned}$$

where $(*)$ follows by A_f being diagonal and $(**)$ follows by the construction of f_2. Thus, $\mathrm{val}(A_f) \odot w$ can be written as the tropical sum of scalar multiples of cocircuits of $\overline{\mathrm{trop}}\big(f^{-1}(\mu)\big)$, hence $\mathrm{val}(A_f) \odot w \in \overline{\mathrm{trop}}\big(f^{-1}(\mu)\big)$. $\square$

Proposition 5.18. *Let μ be a valuated matroid over $[m]$. Let $f : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{T}$ be a map such that for all $i \in [n]$ there exists $k \in K$ satisfying $f_2(i) = \mathrm{val}(k)$. Then, for the associated weakly monomial matrix A_f defined in Definition 5.8, $\overline{\mathrm{trop}}\big(f^{-1}(\mu)\big) = \mathrm{val}(A_f) \odot \overline{\mathrm{trop}}(\mu)$. Conversely, if $A_f \in K^{n\times m}$ is a weakly monomial matrix, the associated map f satisfies $\overline{\mathrm{trop}}\big(f^{-1}(\mu)\big) = \mathrm{val}(A_f) \odot \overline{\mathrm{trop}}(\mu)$.*

Proof. By Lemma 5.16, the statement holds for a weakly monomial matrix with entries in $\{0, 1\}$, and by Lemma 5.17 it holds for diagonal matrices of full rank. Since every weakly monomial matrix can be written as the product of these two types, by Lemma 5.12, the claim follows for all matrices. □

Corollary 5.19. *Let μ be a valuated matroid over $[m]$ and let $A \in K^{n\times m}$ be a weakly monomial matrix with no zero rows. Then, $\mathrm{val}(A) \odot \overline{\mathrm{trop}}(\mu) \subseteq \mathbb{P}(\mathbb{T}^n)$ is a tropical linear space.*

Example 5.20. In Example 5.6, we saw that matrix multiplication with a weakly monomial matrix induced a permutation and translation of the tropical linear space. Its matrix can be decomposed into a permutation matrix and a full rank diagonal matrix,

$$\mathrm{val}(A) = \begin{bmatrix} 3 & \infty & \infty \\ \infty & \infty & 1 \\ \infty & 0 & \infty \end{bmatrix} = \begin{bmatrix} 3 & \infty & \infty \\ \infty & 1 & \infty \\ \infty & \infty & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & \infty & \infty \\ \infty & \infty & 0 \\ \infty & 0 & \infty \end{bmatrix}.$$

The associated map of matroids is

$$\begin{aligned} f : [3] \cup \{o\} &\to [3] \cup \{o\} \times \mathbb{T}; \\ 1 &\mapsto (1, 3) \\ 2 &\mapsto (3, 1) \\ 3 &\mapsto (2, 0) \\ o &\mapsto (o, \infty). \end{aligned}$$

The map f is the composition of a permutation map g and a map h composed of an identity map h_1 with non-infinite values in h_2,

$$\begin{aligned} g : [3] \cup \{o\} &\to [3] \cup \{o\} \times \mathbb{T} & h : [3] \cup \{o\} &\to [3] \cup \{o\} \times \mathbb{T} \\ 1 &\mapsto (1, 0) & 1 &\mapsto (1, 3) \\ 2 &\mapsto (3, 0) & 2 &\mapsto (2, 1) \\ 3 &\mapsto (2, 0) & 3 &\mapsto (3, 0) \\ o &\mapsto (o, \infty), & o &\mapsto (o, \infty), \end{aligned}$$

and $f = h \circ g$ as defined above. Hence, the valuations of sets in $f^{-1}(\mu)$ are $f^{-1}(\mu)(12) = 0 + 3 + 1 = 4$, $f^{-1}(\mu)(13) = 0 + 3 + 0 = 3$, $f^{-1}(\mu)(23) = 0 + 0 + 1 = 1$, and $f^{-1}(\mu)(I) = \infty$ if $o \in I$. The following table gives the cocircuits of $f^{-1}(\mu)$ and the vectors $\mathrm{val}(A) \odot C^*_\mu(I)$.

I	$C^*_{f^{-1}(\mu)}(I)$	$\mathrm{val}(A) \odot C^*_\mu(I)$
1	$(\infty, 4, 3)$	$(\infty, 1, 0)$
2	$(4, \infty, 1)$	$(3, 1, \infty)$
3	$(3, 1, \infty)$	$(3, \infty, 0)$

Note that $C^*_{f^{-1}(\mu)}(1) = \mathrm{val}(A) \odot C^*_\mu(1) \odot 3 = \mathrm{val}(A) \odot C^*_\mu(1) \odot f_2(1)$, and that analogously, $C^*_{f^{-1}(\mu)}(2) = \mathrm{val}(A) \odot C^*_\mu(3) \odot 1 = \mathrm{val}(A) \odot C^*_\mu(1) \odot f_2(2)$ and $C^*_{f^{-1}(\mu)}(2) = \mathrm{val}(A) \odot C^*_\mu(2) = \mathrm{val}(A) \odot C^*_\mu(1) \odot f_2(3)$, as in the proof of Lemma 5.17.

6. Valuated bimatroids and tropicalization of matrices

We finish our study of maps between matroids by considering an alternative approach to matrices in matroid theory and tropical geometry. This alternative approach is via the use of valuated bimatroids, which model some properties of matrices more faithfully than taking the valuations of entries does.

In Section 6.1, we discuss Murota's notion of a valuated bimatroid, introduce two alternative characterizations, and discuss some of their properties. In Section 6.2, we are mainly concerned with studying the product of two bimatroids: we define the product as a tropical version of the Cauchy-Binet formula, give bounds on the ranks of the resulting bimatroids, and show that they form a monoid. We conclude this section by providing an outlook on how to connect the approaches in this section with (affine) morphisms of valuated matroids.

This section is part of joint work in progress with Jeffrey Giansiracusa, Felipe Rincón and Martin Ulirsch.

Remark 6.1. There exist different characterizations of bimatroids in the literature. The unvaluated concept goes back to Kung [95], and has an equivalent alternative characterization as *linking systems* due to Schrijver [128].

On the valuated side, Murota introduces valuations on bimatroids in [110]. His notion is equivalent to ours, which we show below. In addition, Frenk introduces an equivalent concept, *valuated linking systems*, in his thesis [60]. In particular, he shows that products of linking systems are again linking systems in [60, Proposition 4.2.21].

6.1. **Valuated bimatroids.**

Definition 6.2. A *valuated bimatroid* A on the rows $[n]$ and columns $[m]$ is a valuated matroid $\widehat{\mu}$ of rank n on the ground set $[n] \sqcup [m]$ such that $\widehat{\mu}([n]) = 0$. We call $\widehat{\mu}$ the *extended valuated matroid* associated to A.

This definition is motivated by the non-valuated definition of bimatroids due to Kung in [139, Chapter 8]. In the following, we will denote by $\binom{[n]}{*} \times \binom{[m]}{*}$ the set of pairs (I, J) of subsets $I \subseteq [n]$ and $J \subseteq [m]$ of arbitrary but equal cardinality.

For a pair $(I, J) \in \binom{[n]}{*} \times \binom{[m]}{*}$ we write

$$\mu_{\mathsf{A}}(I, J) \ := \ \widehat{\mu}\big(([n] \setminus I) \sqcup J\big) \ .$$

We call this function $\mu_{\mathsf{A}} \colon \binom{[n]}{*} \times \binom{[m]}{*} \to \mathbb{T}$ the *minor valuation function* of A. We say that (I, J) is a *regular minor* of A if $\mu_{\mathsf{A}}(I, J) \neq \infty$.

We can characterize a valuated bimatroid in terms of its minor valuation function. This is Murota's original notion of a valuated bimatroid, [110, Section 2]. We show that the two notions are equivalent:

Proposition 6.3. *A function $\mu_{\mathsf{A}} \colon \binom{[n]}{*} \times \binom{[m]}{*} \to \mathbb{T}$ is the minor valuation function of a bimatroid if and only if it satisfies the following axioms:*

(BM0) $\mu_{\mathsf{A}}(\emptyset, \emptyset) = 0$.

(BME) *For all* $(I,J),(I',J') \in \binom{[n]}{*} \times \binom{[m]}{*}$ *we have:*

(i) *For every* $i' \in I' \setminus I$ *at least one of the following two statements holds:*

∘ *there exists* $i \in I \setminus I'$ *such that*

$$\mu_{\mathsf{A}}(I,J) + \mu_{\mathsf{A}}(I',J') \quad \geq \quad \mu_{\mathsf{A}}(I \setminus i \cup i', J) + \mu_{\mathsf{A}}(I' \setminus i' \cup i, J'),$$

∘ *there exists* $j' \in J' \setminus J$ *such that*

$$\mu_{\mathsf{A}}(I,J) + \mu_{\mathsf{A}}(I',J') \quad \geq \quad \mu_{\mathsf{A}}(I \cup i', J \cup j') + \mu_{\mathsf{A}}(I' \setminus i', J' \setminus j').$$

(ii) *For every* $j \in J \setminus J'$ *then at least one of the following two statements holds:*

∘ *there exists* $i \in I \setminus I'$ *such that*

$$\mu_{\mathsf{A}}(I,J) + \mu_{\mathsf{A}}(I',J') \quad \geq \quad \mu_{\mathsf{A}}(I \setminus i, J \setminus j) + \mu_{\mathsf{A}}(I' \cup i, J' \cup j),$$

∘ *there exists* $j' \in J' \setminus J$ *such that*

$$\mu_{\mathsf{A}}(I,J) + \mu_{\mathsf{A}}(I',J') \quad \geq \quad \mu_{\mathsf{A}}(I, J \setminus j \cup j') + \mu_{\mathsf{A}}(I', J' \setminus j' \cup j).$$

Proof. We first show that μ_{A} satisfies (BM0) and (BME) if A is a bimatroid, i.e., if $\widehat{\mu}$ is a valuated matroid of rank n such that $\widehat{\mu}([n]) = 0$.

Property (BM0) is expressing the fact that $\mu_{\mathsf{A}}([n]) = 0$. Property (BME) follows directly from the fact that $\widehat{\mu}$ is a valuated matroid (see Definition 3.58): Let $(I,J),(I',J') \in \binom{[n]}{*} \times \binom{[m]}{*}$. Let $i' \in I' \setminus I$. Then, $i' \in (([n] \setminus I) \sqcup J) \setminus (([n] \setminus I') \sqcup J')$, so by $\widehat{\mu}$ being a valuated matroid, there exists $k \in (([n] \setminus I') \sqcup J') \setminus (([n] \setminus I) \sqcup J)$ such that

$$\begin{aligned}\mu_{\mathsf{A}}(I,J) + \mu_{\mathsf{A}}(I',J') \quad &= \quad \widehat{\mu}\big(([n] \setminus I) \sqcup J\big) + \widehat{\mu}\big(([n] \setminus I') \sqcup J'\big) \\ &\geq \quad \widehat{\mu}\big((([n] \setminus I) \sqcup J) \setminus i' \cup k\big) + \widehat{\mu}\big((([n] \setminus I') \sqcup J') \setminus k \cup i'\big).\end{aligned}$$

There are two cases: Either, $k \in ([n] \setminus I') \setminus ([n] \setminus I) = I \setminus I'$, or $k \in J' \setminus J$.

Case 1: If $k \in ([n] \setminus I') \setminus ([n] \setminus I) = I \setminus I'$,

$$\begin{aligned}&\widehat{\mu}\big((([n] \setminus I) \sqcup J) \setminus i' \cup k\big) + \widehat{\mu}\big((([n] \setminus I') \sqcup J') \setminus k \cup i'\big) \\ = \;& \widehat{\mu}\big(([n] \setminus (I \setminus i' \cup k)) \sqcup J\big) + \widehat{\mu}\big(([n] \setminus (I' \setminus k \cup i')) \sqcup J'\big) \\ = \;& \mu_{\mathsf{A}}(I \setminus k \cup i', J) + \mu_{\mathsf{A}}(I' \setminus i' \cup k, J').\end{aligned}$$

This corresponds to the first option in axiom (BME)(i) above.

Case 2: If $k \in J' \setminus J$, then

$$\begin{aligned}&\widehat{\mu}\big((([n] \setminus I) \sqcup J) \setminus i' \cup k\big) + \widehat{\mu}\big((([n] \setminus I') \sqcup J') \setminus k \cup i'\big) \\ = \;& \widehat{\mu}\big((([n] \setminus I) \setminus i') \sqcup (J \cup k)\big) + \widehat{\mu}\big((([n] \setminus I') \cup i') \sqcup (J' \setminus k)\big) \\ = \;& \mu_{\mathsf{A}}(I \cup i', J \cup k) + \mu_{\mathsf{A}}(I' \setminus i', J' \setminus k).\end{aligned}$$

Property (BME)(ii) follows analogously for $j \in J \setminus J'$.

It remains to be shown that if μ_A satisfies (BM0) and (BME), then $\widehat{\mu}$ is a valuated matroid. Let $B, B' \subseteq [n] \sqcup [m]$ be sets of size n. We can decompose

$$B \;=\; (B \cap [n]) \sqcup (B \cap [m]) \;=\; ([n] \setminus ([n] \setminus (B \cap [n]))) \sqcup (B \cap [m])$$

and denote $I = [n] \setminus (B \cap [n])$ and $J = B \cap [m]$. Analogously, we decompose B' and denote $I' = [n] \setminus (B' \cap [n])$ and $J' = B' \cap [m]$. Then,

$$\widehat{\mu}(B) + \widehat{\mu}(B') \;=\; \widehat{\mu}([n] \setminus I \sqcup J) + \widehat{\mu}([n] \setminus I' \sqcup J') \;=\; \mu_{\mathsf{A}}(I, J) + \mu_{\mathsf{A}}(I', J').$$

Consider $b \in B \setminus B'$. Then, either $b \in [n]$ or $b \in [m]$. We assume $b \in [m]$. Then, we have $b \in J \setminus J'$. By (BME)(ii), there are again two cases. Assume the first option of (BME)(ii) holds. Then, there exists $i \in I \setminus I' \subseteq B' \setminus B$ such that

$$\begin{aligned} \mu_{\mathsf{A}}(I, J) + \mu_{\mathsf{A}}(I', J') \;&\geq\; \mu_{\mathsf{A}}(I \setminus i, J \setminus b) + \mu_{\mathsf{A}}(I' \cup i, J' \cup b) \\ &=\; \widehat{\mu}\big(([n] \setminus (I \setminus i)) \sqcup (J \setminus b)\big) + \widehat{\mu}\big(([n] \setminus (I' \cup i)) \sqcup (J' \cup b)\big) \\ &=\; \widehat{\mu}\big(([n] \setminus ([n] \setminus (B \cap [n]) \setminus i)) \sqcup (B \cap [m] \setminus b)\big) \\ &\quad + \widehat{\mu}\big(([n] \setminus ([n] \setminus (B' \cap [n]) \cup i)) \sqcup ((B' \cap [m]) \cup b)\big) \\ &=\; \widehat{\mu}(B \setminus b \cup i) + \widehat{\mu}(B' \setminus i \cup b). \end{aligned}$$

Now, the other cases follow analogously. □

Definition 6.4. We denote by $\mathrm{BMat}^{n\times m}$ the set of all valuated bimatroids on the ground set $[n] \sqcup [m]$.

Example 6.5 (Realizable bimatroids). Let $A \in K^{n\times m}$ be an $n \times m$ matrix over a valued field K. Then the function

$$\mu_A(I, J) \;=\; \mathrm{val}\big(\det[A]_{I,J}\big)$$

for $I \times J \in \binom{[n]}{*} \times \binom{[m]}{*}$ naturally defines a valuated bimatroid A^{trop} on the ground set $[n] \sqcup [m]$.

In order to see this, we consider the matrix

$$\left[\begin{array}{c|c} I_n & A \end{array}\right] = \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & A \\ & & 1 & \end{array}\right] \in K^{n\times n+m}$$

that is extended by the $n \times n$ identity matrix I_n and note that

$$\det[I_n|A]_{[n],I^c\sqcup J} \;=\; \det[A]_{I,J}\ .$$

The valuations of the maximal minors in $[I_n|A]$ form a valuated matroid such that $\mu(I_n) = 0$. Hence, the valuations of all minors of A form a valuated bimatroid.

We refer to A^{trop} as the *tropicalization* of A and say that a valuated bimatroid arising in this fashion is *realizable* by the matrix A over the field K. Note that the tropicalization A^{trop} is different from the valuation $\mathrm{val}(A)$, which we studied in the previous section.

Definition 6.6. Given a square matrix

$$A = [a_{ij}]_{1\leq i,j\leq n} \;\in\; \mathbb{T}^{n\times n}$$

for $n \geq 0$, the *tropical determinant* is defined by

$$\det A \;=\; \min_{\sigma\in S_n} \big\{a_{1\sigma(1)} + \cdots + a_{n\sigma(n)}\big\}\ .$$

We note that, unlike for the classical determinant of a square matrix over a field K, the tropical determinant is invariant under both row and column exchanges.

Example 6.7 (Valuated bimatroids of Stiefel type). Given a matrix $A \in \mathbb{R}^{n\times m}$, the map $\mu_A\colon \binom{[n]}{*} \times \binom{[m]}{*} \to \mathbb{T}$ given by

$$\mu_A(I,J) \;=\; \det[A]_{I,J}$$

for $(I,J) \in \binom{[n]}{*} \times \binom{[m]}{*}$ defines a valuated bimatroid $\mathsf{St}(A)$ on the ground set $[n] \sqcup [m]$, whose rank is the maximal $1 \le r \le \min\{|[n]|, |[m]|\}$ such that there exists $(I,J) \in \binom{[n]}{r} \times \binom{[m]}{r}$ with $\det[A]_{I,J} \neq \infty$.

The argument that this is a valuated bimatroid follows as in [58]: We choose a sufficiently generic lift of A to a matrix over some non-Archimedean field and apply the reasoning in Example 6.5. Valuated bimatroids of the form $\mathsf{St}(A)$ for a matrix $A \in \mathbb{T}^{n\times m}$ are realizable and are said to be of *Stiefel type*.

Generalizing the corresponding notions for matrices, we can define the rank and the nullity of a valuated bimatroid.

Definition 6.8. We define the *rank* of a bimatroid $\mathsf{A} \in \mathrm{BMat}^{n\times m}$ (written as $\mathrm{rk}(\mathsf{A})$) to be the maximal number r for which there exist $I \subseteq [n]$ and $J \subseteq [m]$ with $|I| = |J| = r$ such that $I \times J$ is a non-singular minor of A. The *nullity* of a bimatroid is defined by $\mathrm{nul}(\mathsf{A}) = n - l$, where l is the minimal number for which none of the $I \times J$ with $|I| = |J| = l$ are non-singular minors.

Both the rank and the nullity are set up such that the *rank-nullity formula*

$$\mathrm{rk}(\mathsf{A}) + \mathrm{nul}(\mathsf{A}) \;=\; n$$

automatically holds.

Example 6.9. There is exactly one bimatroid $\mathsf{0}_{n\times m}$ on a ground set $[n] \sqcup [m]$ of rank $= 0$,

$$\mu_{\mathsf{0}_{n\times m}}(I,J) \;=\; \begin{cases} 0 & \text{if } I = J = \emptyset \\ \infty & \text{else.} \end{cases}$$

The valuated bimatroid $\mathsf{0}_{n\times m}$ is called the *zero bimatroid* and is the analogue of the matroid $U_{0,1}$ discussed in Example 3.51. When $[n] = [m] = \emptyset$, we abbreviate the zero valuated bimatroid $\mathsf{0}_{\emptyset\times\emptyset}$ on the ground set $\emptyset \times \emptyset$ simply by $\mathsf{0}$.

Many properties of matrices directly carry over to valuated bimatroids. For instance, we can define the transpose of a bimatroid as follows.

Definition 6.10. The *transpose* A^T of a bimatroid $\mathsf{A} \in \mathrm{BMat}^{n\times m}(A)$ is defined by the minor valuation function $\mu^T\colon \binom{[m]}{*} \times \binom{[n]}{*} \to \mathbb{T}$ given by

$$\mu_{\mathsf{A}^T}(J,I) \;=\; \mu_{\mathsf{A}}(I,J)$$

for $J \times I \in \binom{[m]}{*} \times \binom{[n]}{*}$.

It is an immediate consequence of this definition that $\mathrm{rk}(\mathsf{A}^T) = \mathrm{rk}(\mathsf{A})$ as well as $(\mathsf{A}^T)^T = \mathsf{A}$ for a valuated bimatroid $\mathsf{A} \in \mathrm{BMat}^{n\times m}$. These facts allow us to deduce the following bimatroidal analogue of the Laplace expansion formula for determinants.

Proposition 6.11. *Let* A *be a valuated bimatroid on the ground set* $[n] \sqcup [m]$ *and let* $I \times J \in \binom{[n]}{*} \times \binom{[m]}{*}$.

(i) *For every* $i \in I$ *we have*

$$\mu(I,J)_{\mathsf{A}} \;\geq\; \min_{j\in J}\Big(\mu_{\mathsf{A}}(i,j)+\mu_{\mathsf{A}}\big(I-i,J-j\big)\Big)$$

and

(ii) *for every* $j \in J$ *we have*

$$\mu(I,J)_{\mathsf{A}} \;\geq\; \min_{i\in I}\Big(\mu_{\mathsf{A}}(i,j)+\mu_{\mathsf{A}}\big(I-i,J-j\big)\Big)\ .$$

Proof. Part (i) is a consequence of Axiom (BME)^T and Part (ii) a consequence of Axiom (BME), applied with $I' \times J' = \emptyset \times \emptyset$. □

The columns and rows of a valuated bimatroid naturally define valuated matroids.

Proposition 6.12. *Let* A *be a non-trivial valuated bimatroid of rank* r *on the ground set* $[n] \sqcup [m]$.

(i) *The map* $\mu_{\mathrm{Row}(\mathsf{A})}\colon \binom{[n]}{r} \to \mathbb{T}$ *given by*

$$\mu_{\mathrm{Row}(\mathsf{A})}(I) \;=\; \min_{J\in\binom{[m]}{r}} \mu_{\mathsf{A}}(I\times J)$$

defines a valuated matroid $\mu_{\mathrm{row}}(\mathsf{A})$ *of rank* r *on the ground set* $[n]$.

(ii) *The map* $\mu_{\mu_{\mathrm{col}}(\mathsf{A})}\colon \binom{[m]}{r} \to \mathbb{T}$ *given by*

$$\mu_{\mathrm{Col}(\mathsf{A})}(J) \;=\; \min_{I\in\binom{[n]}{r}} \mu_{\mathsf{A}}(I\times J)$$

defines a valuated matroid $\mathrm{Col}(\mathsf{A})$ *of rank* r *on the ground set* $[m]$.

The valuated matroids $\mathrm{Col}(\mathsf{A})$ and $\mathrm{Row}(\mathsf{A})$ are called the *column matroid* and the *row matroid* of A, respectively.

Proof of Proposition 6.12. For Part (i) we consider $I, I' \in \binom{[n]}{r}$. There are $J, J' \in \binom{[m]}{r}$ such that

$$\mu_{\mathrm{row}(\mathsf{A})}(I) \;=\; \mu_{\mathsf{A}}(I,J) \qquad \text{and} \qquad \mu_{\mathrm{row}(\mathsf{A})}(I') = \mu_{\mathsf{A}}(I',J')\ .$$

Now let $i \in I \setminus I'$. Apply Axiom (BME)^T (noting that $|I| = \mathrm{rk}(A)$) to find $i' \in I' \setminus I$ such that

$$\mu_{\mathsf{A}}(I,J)+\mu_{\mathsf{A}}(I',J') \;\geq\; \mu_{\mathsf{A}}(I\setminus i\cup i',J)+\mu_{\mathsf{A}}(I'\setminus i'\cup i,J')\ .$$

This immediately implies

$$\begin{aligned}\mu_{\mathrm{row}(\mathsf{A})}(I)+\mu_{\mathrm{row}}((\mathsf{A})(I') \;&=\; \mu_{\mathsf{A}}(I,J)+\mu_{\mathsf{A}}(I',J')\\ &\geq\; \mu_{\mathsf{A}}(I\setminus i\cup i',J)+\mu_{\mathsf{A}}(I'\setminus i'\cup i,J')\\ &\geq\; \mu_{\mathrm{row}(\mathsf{A})}(I\setminus i\cup i')+\mu_{\mathrm{row}(\mathsf{A})}(I'\setminus i'\cup i)\ ,\end{aligned}$$

which means that $\mu_{\mathrm{row}(\mathsf{A})}$ defines a valuated matroid of rank r. Part (ii) follows with an analogous argument, switching rows and columns, and using Axiom (BME) instead of (BME)^T. □

6.2. **The tropical Cauchy-Binet formula.** A central feature of (valuated) bimatroids is that, just like matrices, but unlike (valuated) matroids, they admit a product. In this section, we introduce this product for valuated bimatroids. We refer the reader to [95, Section 6] for the non-valuated case, and to [60] for the equivalent notion of a product of valuated linking systems.

In order to motivate the product, we recall the (generalized) Cauchy-Binet formula from linear algebra. Given two matrices $A \in K^{n_1\times n_2}$ and $B \in K^{n_2\times n_3}$, it tells us that

$$\det\big([AB]_{I,K}\big) \;=\; \sum_{J\in\binom{[n_2]}{d}} \pm\det\big([A]_{I,J}\big)\cdot\det\big([B]_{J,K}\big)$$

for $I\times K \in \binom{[n_1]}{d}\times\binom{[n_3]}{d}$. The product of two valuated bimatroids is defined by the tropical analogue of this formula.

Definition 6.13. Let $\mathsf{A}\in \mathrm{BMat}^{n_1\times n_2}$ and $\mathsf{B}\in\mathrm{BMat}^{n_2\times n_3}$ be two valuated bimatroids with minor valuation functions μ_{A} and μ_{B} respectively. We define the *product* $\mathsf{A}\cdot\mathsf{B}\in\mathrm{BMat}^{n_1\times n_3}$ to be the valuated bimatroid, whose minor valuation function is given by

$$\mu_{\mathsf{A}\cdot\mathsf{B}}(I,K) \;=\; \min_{J\in\binom{[n_2]}{d}}\big\{\mu_{\mathsf{A}}(I,J)+\mu_{\mathsf{B}}(J,K)\big\}$$

for $I\times K\in\binom{[n_1]}{d}\times\binom{[n_3]}{d}$.

For this definition to be reasonable, we need to verify that the product of two valuated bimatroids is again a valuated bimatroid. This result is proven, formulated slightly differently by Frenk in [60]:

Proposition 6.14 ([60, Proposition 4.2.21])**.** *Given two valuated bimatroids* $\mathsf{A}\in\mathrm{BMat}^{n_1\times n_2}$ *and* $\mathsf{B}\in\mathrm{BMat}^{n_2\times n_3}$*, the* product $\mathsf{A}\cdot\mathsf{B}$ *is also a valuated bimatroid, on the ground set* $[n_1]\sqcup[n_3]$*.*

Proposition 6.15. *The set of (square) valuated bimatroids over* $[n]\sqcup[n]$ *with the product outlined in Definition 6.13 forms a monoid. Further, multiplication of compatible bimatroids is associative, and has left and right identity elements. In other words:*

(i) There are unique valuated bimatroids $\mathsf{I}_{[n_1]}\in\mathrm{BMat}^{n_1\times n_1}$ *and* $\mathsf{I}_{[n_2]}\in\mathrm{BMat}^{n_2\times n_2}$ *such that*

$$\mathsf{I}_{[n_1]}\cdot\mathsf{A} \;=\; \mathsf{A} \qquad \textit{and} \qquad \mathsf{A}\cdot\mathsf{I}_{[n_2]} \;=\; \mathsf{A}$$

for all $\mathsf{A}\in\mathrm{BMat}^{n_1\times n_2}$*.*

(ii) The bimatroid product is associative: Given three valuated bimatroids $\mathsf{A}\in\mathrm{BMat}^{n_1\times n_2}$*,* $\mathsf{B}\in\mathrm{BMat}^{n_2\times n_3}$*, and* $\mathsf{C}\in\mathrm{BMat}^{[n_3]\times[n_4]}$*, we have*

$$(\mathsf{A}\cdot\mathsf{B})\cdot\mathsf{C} \;=\; \mathsf{A}\cdot(\mathsf{B}\cdot\mathsf{C})\ .$$

Proof. (i) We define the *identity* $\mathsf{I}_{[n_1]}$ by setting

$$\mu_{\mathsf{I}_{[n_1]}}(I,J) \;=\; \begin{cases}0 & \text{if } I=J\\ \infty & \text{if } I\neq J.\end{cases}$$

for $I\times J\in\binom{[n_1]}{*}\times\binom{[n_1]}{*}$. Then we automatically have $\mathsf{I}_{[n_1]}\cdot\mathsf{A}=\mathsf{A}$ for all $\mathsf{A}\in\mathrm{BMat}^{n_1\times n_2}$. The same construction provides us with $\mathsf{I}_{[n_2]}\in\mathrm{BMat}^{n_2\times n_2}$ such that

$\mathsf{A}\cdot\mathsf{I}_{[n_2]} = \mathsf{A}$ for all $\mathsf{A} \in \mathrm{BMat}^{n_1\times n_2}$. In order to show the uniqueness of $\mathsf{I}_{[n_1]}$ we consider another $\mathsf{I}'_{[n_1]}$ with the same property and note that then $\mathsf{I}'_{[n_1]} = \mathsf{I}'_{[n_1]}\cdot\mathsf{I}_{[n_1]} = \mathsf{I}_{[n_1]}$.

(ii) To show associativity, we observe that for $I \times L \in \binom{[n_1]}{d} \times \binom{[n_4]}{d}$ we have:

$$\begin{aligned}
\mu_{(\mathsf{A}\cdot\mathsf{B})\cdot\mathsf{C}}(I,L) &= \min_{K\in\binom{[n_3]}{d}} \left\{\mu_{\mathsf{A}\cdot\mathsf{B}}(I,K) + \mu_{\mathsf{C}}(K,L)\right\} \\
&= \min_{K\in\binom{[n_3]}{d}} \left\{ \min_{J\in\binom{[n_2]}{d}} \left\{\mu_{\mathsf{A}}(I,J) + \mu_{\mathsf{B}}(J,K)\right\} + \mu_{\mathsf{C}}(K,L)\right\} \\
&= \min_{J\in\binom{[n_2]}{d}} \left\{\mu_{\mathsf{A}}(I,J) + \min_{K\in\binom{[n_3]}{d}} \left\{\mu_{\mathsf{B}}(J,K) + \mu_{\mathsf{C}}(K,L)\right\}\right\} \\
&= \min_{J\in\binom{[n_3]}{d}} \left\{\mu_{\mathsf{A}}(I,J) + \mu_{\mathsf{B}\cdot\mathsf{C}}(J,L)\right\} \\
&= \mu_{\mathsf{A}\cdot(\mathsf{B}\cdot\mathsf{C})}(I,L)\ .
\end{aligned}$$

□

Products of valuated bimatroids are compatible with taking transposes.

Proposition 6.16. *We have*

$$\mathsf{I}_{[n_1]}^T \;=\; \mathsf{I}_{[n_1]} \qquad \textit{as well as} \qquad (\mathsf{A}\cdot\mathsf{B})^T \;=\; \mathsf{B}^T\cdot\mathsf{A}^T$$

for valuated bimatroids $\mathsf{A} \in \mathrm{BMat}^{n_1\times n_2}$ *and* $\mathsf{B} \in \mathrm{BMat}^{n_2\times n_3}$.

Proof. The equality $\mathsf{I}_{[n_1]}^T = \mathsf{I}_{[n_1]}$ is an immediate consequence of

$$\mu_{\mathsf{I}_{[n_1]}^T}(I,I) \;=\; \mu_{\mathsf{I}_{[n_1]}}(I,I) \qquad \text{and} \qquad \mu_{\mathsf{I}_{[n_1]}^T}(I,J) \;=\; \mu_{\mathsf{I}_{[n_1]}}(J,I) = \infty$$

for $I, J \in \binom{[n_1]}{*}$ with $I \neq J$. The second equality follows by observing that for $K \times I \in \binom{[n_3]}{d} \times \binom{[n_1]}{d}$ we have:

$$\begin{aligned}
\mu_{\mathsf{B}^T\cdot\mathsf{A}^T}(K,I) &= \min_{J\in\binom{[n_2]}{d}} \left\{\mu_{\mathsf{B}^T}(K,J) + \mu_{\mathsf{A}^T}(J,I)\right\} \\
&= \min_{J\in\binom{[n_2]}{d}} \left\{\mu_{\mathsf{A}}(I,J) + \mu_{\mathsf{B}}(J,K)\right\} \\
&= \mu_{(A\cdot B)^T}(K,I)\ .
\end{aligned}$$

□

For two valuated bimatroids $\mathsf{A}, \mathsf{B} \in \mathrm{BMat}^{n_1\times n_2}$ we write $\mathsf{A} \leq \mathsf{B}$ if the inequality

$$\mu_{\mathsf{A}}(I,J) \;\leq\; \mu_{\mathsf{B}}(I,J)$$

holds for all $I \times J \in \binom{[n_1]}{*} \times \binom{[n_2]}{*}$. The relation $\leq$ defines a partial order on the set $\mathrm{BMat}^{n_1\times n_2}$, whose unique maximal element is the trivial bimatroid $0_{n_1\times n_2}$. The Stiefel bimatroids discussed in Example 6.7 are minimal with respect to this partial order.

Proposition 6.17. *We have*

$$(I_{[n_1]})^{\mathrm{trop}} \;=\; \mathsf{I}_{[n_1]} \qquad \textit{as well as} \qquad (AB)^{\mathrm{trop}} \;\geq\; A^{\mathrm{trop}}\cdot B^{\mathrm{trop}}$$

for matrices $A \in K^{n_1\times n_2}$ *and* $B \in K^{n_2\times n_3}$.

Proof. The equality $I^{\text{trop}}_{[n_1]} = \mathsf{I}_{[n_1]}$ is trivial, since a minor $[I_{[n_1]}]_{I\times J}$ of the identity matrix has determinant $= 1$ if and only if $I = J$ and $= 0$ otherwise. To show that $(AB)^{\text{trop}} \geq A^{\text{trop}} \cdot B^{\text{trop}}$ for matrices $A \in K^{n_1\times n_2}$ and $B \in K^{n_2\times n_3}$, we use the (generalized) Cauchy-Binet formula, which tells us that

$$\det\big([AB]_{I,K}\big) \;=\; \sum_{J\in\binom{[n_2]}{d}} \det\big([A]_{I,J}\big)\cdot \det\big([B]_{J,K}\big) \tag{7}$$

for $I \times K \in \binom{[n_1]}{d} \times \binom{[n_3]}{d}$. This formula is a consequence of the well-known fact that for linear maps $f\colon V_1 \to V_2$ and $g\colon V_2 \to V_3$ between finite-dimensional vector spaces V_i (for $i = 1, 2, 3$), the induced linear maps on the exterior powers fulfil $\bigwedge(g \circ f) = \bigwedge(g) \circ \bigwedge(f)$.

If we apply the valuation on K to both sides of (7), we find

$$\text{val}\Big(\det\big([AB]_{I,K}\big)\Big) \;\geq\; \min_{J\in\binom{[n_2]}{d}} \Big\{\text{val}\Big(\det\big([A]_{I,J}\big)\Big) + \text{val}\Big(\det\big([B]_{J,K}\big)\Big)\Big\}$$

for all $I \times K \in \binom{[n_1]}{*} \times \binom{[n_3]}{*}$. This is equivalent to $(AB)^{\text{trop}} \geq A^{\text{trop}} \cdot B^{\text{trop}}$. □

The rank of a product of two valuated bimatroids obeys the following rule.

Proposition 6.18. *Let* $\mathsf{A} \in \text{BMat}^{n_1\times n_2}$ *and* $\mathsf{B} \in \text{BMat}^{n_2\times n_3}$ *be two valuated bimatroids. Then we have*

$$\text{rk}(\mathsf{A}\cdot\mathsf{B}) \;=\; \min_{S\subseteq[n_2]} \big\{ \text{rk}_{\mu_{\text{col}}(\mathsf{A})}(S) + \text{rk}_{\mu_{\text{row}}(\mathsf{B})}([n_2]\setminus S)\big\}\ .$$

In particular, setting $S = \emptyset$ and $S = [n_2]$, we always have

$$\text{rk}(\mathsf{A}\cdot\mathsf{B}) \;\leq\; \min\big\{\text{rk}(\mathsf{A}), \text{rk}(\mathsf{B})\big\}\ .$$

Proof of Proposition 6.18. Write $\text{Ind}\big(\mu_{\text{col}}(\mathsf{A})\big)$ and $\text{Ind}\big(\mu_{\text{row}}(\mathsf{B})\big)$ for the set of independent subsets of $\mu_{\text{col}}(\mathsf{A})$ and $\mu_{\text{row}}(\mathsf{B})$ respectively. It is now an immediate consequence of the definitions that the rank of $\mathsf{A}\cdot\mathsf{B}$ is the cardinality of a maximal set in $\text{Ind}\big(\mu_{\text{col}}(\mathsf{A})\big) \cap \text{Ind}\big(\mu_{\text{row}}(\mathsf{B})\big)$. The equality then follows from the matroid intersection theorem. □

We conjecture that it is possible to describe every affine morphism of valuated matroids as a valuated bimatroid in the following way.

Conjecture 6.1 (Correspondence between valuated bimatroids and affine morphisms)**.** *Let* $\mathsf{A} \in \text{BMat}^{n_1\times n_2}$ *be a valuated bimatroid on* $[n_1] \sqcup [n_2]$ *with column matroid* $\mu = \text{col}(\mathsf{A})$*, and let and* $\mathsf{B} \in \text{BMat}^{n_2\times n_3}$ *be another valuated bimatroid. Then, if* $M = \text{col}(\mathsf{A}\cdot\mathsf{B})$ *is the column matroid of the product, we have a natural affine morphism of valuated matroids* $\nu \to \mu$*. Further, every affine morphism of valuated matroids can be represented in this way.*

We justify the conjecture as follows. In [95, Theorem 4], Kung establishes the analogous result for the unvaluated case, leading to our conjecture. On examples, the conjecture works out. Kung's proof of Theorem 4, however, uses the Higgs factorization theorem. This theorem asserts that every matroid quotient can be written as an extension followed by a contraction and is characterization (e) in Lemma 3.38). To our knowledge, at this point in time there exists no valuated analogue of this theorem. Thus, to prove Conjecture 6.1, one would likely have to come up with a valuated analogue of Higgs factorization first.

Part 2. Quivers in tropical geometry

In this part of the thesis, we will use the theory of linear maps in tropical geometry we developed in Part 1 to build a new theory in tropical geometry: that of tropical quiver representations.

Quiver representation theory is a subfield of representation theory that started around 1980. A main appeal is that quiver representations allow for a more combinatorial approach to the study of representations of Lie algebras. In 1992, Schofield introduced quiver Grassmannians [127] in an effort to study general representations of quivers. Since then, quiver Grassmannians have become an interesting object of study in their own right — they are geometrically interesting spaces with rich combinatorial structure and connect to many areas of math, for instance algebraic geometry and the theory of cluster algebra [35].

Every projective variety can be represented as a quiver Grassmannian [122]. When a variety is the quiver Grassmannian of a particularly well-behaved quiver, one can use the quiver representation to control and construct degenerations of the variety. A particularly nice example of this is the flag variety, whose linear degenerations were studied for instance in [40, 41, 54].

The main goal of this part is to lay a foundation for a tropical analogue of quiver representation theory. We have dealt with one of the main obstacles in establishing such a theory already in Part 1, where we introduced, studied and compared different notions of linear maps of tropical linear spaces and their associated valuated matroids.

In this part, we introduce all other notions necessary for such a tropical analogue: we introduce *(valuated) matroidal quiver representations* as an analogue of the classical concept, and construct two different tropical analogues of the quiver Grassmannian: the *tropicalized quiver Grassmannian* and the *quiver Dressian*.

We then prove the main results of this part. We show that quiver Dressians are the parameter spaces of tropical linear spaces contained under naive tropical matrix multiplication, and, requiring that the quiver representations can be described by matrices that are weakly monomial, of valuated matroidal quiver representations, see Theorem E. Analogously, we derive that the tropicalized quiver Grassmannian parametrizes tropicalizations of linear spaces contained under naive tropical matrix multiplication, and, with the same restriction of requiring the representation matrices to be weakly monomial, of realizable valuated matroidal quiver representations, see Theorem D. We compare the two tropical analogues and discuss when they can coincide, see Theorem F and Conjecture G.

Further, we discuss applications of the theory for constructions of linear degenerations of tropical flag varieties, and start with establishing a polyhedral analogue of matroidal quiver representations.

This part is structured as follows. We give an overview of quiver representation theory and all definitions necessary for our purposes in **Section 7**.

We resume our original content in **Section 8**, where we introduce matroidal quiver representations and their valuated analogues. In essence, this section is based on joint work with Giulia Iezzi [82]. However, this section includes new definitions, in an expository effort to tie tropical quiver Grassmannians more closely to the classical theory.

In **Section 9**, we introduce two tropical analogues of quiver Grassmannians, one for realizable quiver subrepresentations, and one for arbitrary tropical quiver subrepresentations.

We show that the analogues are compatible with the notions of morphisms of matroids and maps of tropical linear spaces we introduced in Part 1, and study their relationship with each other. We prove the main theorems of this part of the thesis, Theorems E, D and 9.8. This section is based on joint work with Giulia Iezzi [82], with some supplementary examples. Some examples were computed with the aid of `gfan` [88], `Oscar` [114] and `Macaulay2` [71]. The code used to generate these examples can be found in Appendix B.3 and B.5.

In **Section 10**, we use the main theory developed in the previous section to study a problem in tropical geometry. We investigate the linear degenerations of flag varieties via quivers from a tropical point of view, and establish first results on their cover relations. This will aid in computing flag varieties more efficiently. The section is based on joint work with Alessio Borzì [27].

Finally, in **Section 11**, we begin the study of polyhedral analogues of quiver Dressians. We discuss how morphisms of matroids behave on the associated matroid polytopes, show some subcases and establish a general conjecture, which we verify computationally for examples.

7. Preliminaries: Representation theory of quivers

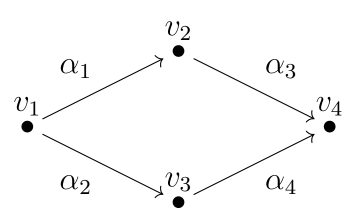
In this section, we review some basics of quiver representation theory. We focus on quiver Grassmannians as the parameter spaces of more complicated arrangements of linear spaces, which we will generalize to tropical geometry later in this part. Further, we give a glimpse into the representation-theoretic background, aimed at providing some insight as to why this area of research is interesting. Standard references for quiver representation theory are [31, 49, 125]. In this section, we mainly follow [31].

Definition 7.1. A finite *quiver* $Q = (V, A, s, t)$ is a directed graph given by a finite set of vertices V, a finite set of arrows A and two maps

$$s, t : A \to V$$

assigning to each arrow its source, resp. target.

Example 7.2. For example, consider the quiver given below.



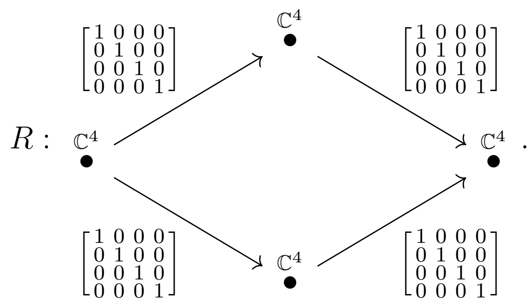
It has four vertices $V = \{v_1, v_2, v_3, v_4\}$ and four arrows $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. The source map assigns $s(\alpha_1) = s(\alpha_2) = v_1$, $s(\alpha_3) = v_2$ and $s(\alpha_4) = v_3$, whereas the target map assigns $t(\alpha_1) = v_2$, $t(\alpha_2) = v_3$ and $t(\alpha_3) = t(\alpha_4) = v_4$.

Now, we assign linear data to the a priori combinatorial objects.

Definition 7.3. Given a quiver Q, we define a finite-dimensional *Q-representation* R over a field K as the ordered pair $\big((R_i)_{i\in V}, (R^\alpha)_{\alpha\in A}\big)$, where R_i is a finite-dimensional K-vector space attached to a vertex $i \in V$ and $R^\alpha : R_{s(\alpha)} \to R_{t(\alpha)}$ is a K-linear map for any $\alpha \in A$.

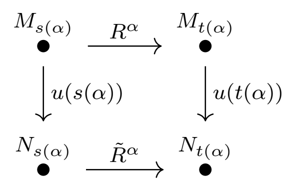
The *dimension vector* of R is $\dim(R) \coloneqq (\dim_K(R_i))_{i\in V} \in \mathbb{Z}^V_{\geq 0}$.

Example 7.4. We continue with the quiver we considered in Example 7.2, and assign a representation. We fix a basis $\mathcal{B} = \{b_1, b_2, b_3, b_4\}$ of $\mathbb{C}^4$ and consider the following representation $R = \big((\mathbb{C}^4)_{i\in[4]}, (\mathrm{id})_{j\in[4]}\big)$:



The linear maps and vertices assigned above now allow us to describe more sophisticated arrangements of linear spaces.

Definition 7.5. Let $Q = (V, A, s, t)$ be a quiver and let $M = \big((M_i)_{i\in V}, (R^\alpha)_{\alpha\in A}\big)$ and $N = \big((N_i)_{i\in V}, (\tilde{R})_{\alpha\in A}\big)$ be two different representations of Q. A *morphism* of representations $u : M \to N$ is a family of linear maps of vector spaces $(u_i : M_i \to N_i)_{i\in V}$ such that the diagram



commutes for all $\alpha \in A$. For any two morphisms, their composition is a morphism. Thus, quiver representations with morphisms of representations form a category. Further, composition is associative and the map $(id_v)_{v\in V}$, which is the identity map on all vertices, is the identity morphism. Thus, the category is abelian.

A common problem in quiver representation theory is the classification of isomorphism classes of (finite-dimensional) quiver representations after fixing the dimensions of the vector spaces assigned to vertices. In other words, we fix a *dimension vector* $\mathbf{d}$ and set $\dim(N_v) = d_v$ for all $v \in V$, and consider all possible maps we can assign to arrows α, determining their isomorphism classes.

Example 7.6. In the following, we will consider three quivers, one with finitely many isomorphism classes of representations, and two with infinitely many such classes.

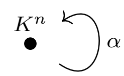
First, we consider the equioriented quiver corresponding to the *Dynkin diagram of type* A_n:

$$\overset{v_1}{\bullet} \xrightarrow{\alpha_1} \overset{v_2}{\bullet} \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} \overset{v_n}{\bullet}.$$

It has finitely many isomorphism classes. After change of bases, every linear map can be represented as a coordinate projection. Successively changing bases allows us to represent each of the arrows above by a projection of the correct rank. We assume that all vertices correspond to a vector space K^m of the same dimension m.

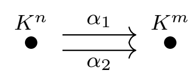
Now, each isomorphism class is an assignment of arrows $(\alpha_1, \ldots, \alpha_{n-1})$ which corresponds to a set of rank vectors $\mathbf{r} = (r_{ij})_{1<j\leq i\leq n-1}$ by setting $\operatorname{rank}(\alpha_{j-1} \circ \cdots \circ \alpha_i) = r_{ij}$. There is only a finite number of rank vectors $\mathbf{r}$, hence there is only a finite number of isomorphism classes for a fixed dimension of the vector spaces assigned to vertices.

We consider the quiver given by a *loop*:



The isomorphism classes of representations are given by $n \times n$ matrices A. Changing bases amounts to conjugation by an invertible matrix $B \in K^{n\times n}$, thus, we have $[B^{-1}AB] \cong [A]$. All conjugacy classes of $n \times n$ matrices can be characterized uniquely by their Jordan normal form. Over an infinite field, there are infinitely many possible Jordan normal forms, hence there are infinitely many isomorphism classes of representations.

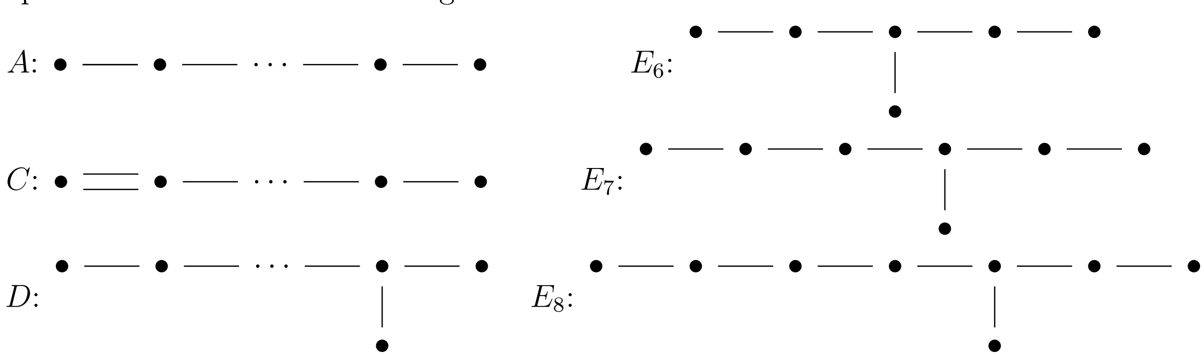
Next, we consider the *Kronecker quiver*, which is a *tame quiver*, but nevertheless a pathological counter-example in quiver representation theory:



The isomorphism classes of representations consist of tuples of $n \times m$ matrices, up to simultaneous multiplication by the same invertible $n \times n$ matrix on the left and $m \times m$ matrix on the right. The classification here is much more involved and due to Kronecker. However, we can see that there are infinitely many isomorphism classes, in the following way:

We assume the dimension of both vector spaces to be equal, i.e., $m = n$, and consider the subset of isomorphism classes of representations where R_{α_1} is invertible. Up to choice of bases, we can then assume that R_{α_1} is the identity matrix. However, this fixes the matrix R_{α_2}. Hence, there are infinitely many isomorphism classes of representations, corresponding to the infinitely many choices of the matrix R_{α_2}.

Example 7.7. In quiver representation theory, *Dynkin quivers* play a special role. These are quivers whose underlying (undirected) graph is a Dynkin diagram. The Dynkin diagrams of importance for this thesis are diagrams listed below:



Here, the number of vertices of a Dynkin diagram of type A is arbitrary, whereas Dynkin diagrams of type C are required to have at least 3 and Dynkin diagrams of type D are required to have at least 4 vertices.

Theorem 7.8 (Gabriel's theorem, [62]). *A quiver has a finite number of isomorphism classes of quiver representations if and only if each connected component of its underlying undirected* *graph is a simply-laced Dynkin diagram, i.e., a quiver associated to a Dynkin diagram of types A, D or E listed in Example 7.7.*

Definition 7.9. A *subrepresentation* of R is a Q-representation $N = \big((N_i)_{i\in V}, (R^\alpha|_{N_{s(\alpha)}})_{\alpha\in A}\big)$ such that $N_i \subseteq R_i$ for all $i \in V$ and $R^\alpha(N_{s(\alpha)}) \subseteq N_{t(\alpha)}$ for all $\alpha \in A$.

Example 7.10. We return to the setting in Example 7.2. Recall that we had fixed a basis $\mathcal{B} = \{b_1, b_2, b_3, b_4\}$ such that we could describe all arrows as the identity matrices. For the quiver representation R, we consider

$$N \;=\; \big((\langle b_1\rangle, \langle b_1, b_2\rangle, \langle b_1, b_4\rangle, \langle b_1, b_2, b_4\rangle), ([\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}1\\0\end{smallmatrix}], \left[\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix}\right])\big).$$

It is an R-subrepresentation with dimension vector $\dim(N) = (1, 2, 2, 3)$. The matrices representing the linear maps appearing in N are the restrictions of the identity maps to the chosen subspaces.

In this way, the quiver we have discussed in the examples in this section describes the arrangement of different linear spaces: It describes the containment of a point inside of two independent lines, which are contained in a common plane.

A sequence of subspaces that cannot belong to any subrepresentation of R is, for instance, $(\langle b_1\rangle, \langle b_1, b_2\rangle, \langle b_1, b_4\rangle, \langle b_1, b_2, b_3\rangle)$, where the b_i are arbitrary choices of bases for $\mathbb{C}^4$, because $\mathrm{id}(\langle b_1, b_4\rangle) \nsubseteq \langle b_1, b_2, b_3\rangle$.

Definition 7.11 (Quiver Grassmannians)**.** Consider a quiver Q, a Q-representation R and a dimension vector $\mathbf{d} \in \mathbb{Z}_{\geq 0}^V$ such that $d_i \leq n_i$ for all $i \in V$. The *quiver Grassmannian* in dimension $\mathbf{n} = (n_i)_{i\in V}$, denoted by $\mathrm{QGr}(R, \mathbf{d}; \mathbf{n})$, is defined as the collection of all subrepresentations N of R with $\dim N_i = d_i$ for all $i \in V$.

Example 7.12 (The flag variety). In [42, Proposition 2.7], the authors realize the (linear degenerate) flag variety as the quiver Grassmannian associated to representations of the equioriented quiver of type A_n. In particular, the complete flag variety can be realized as follows.

Consider the quiver with n vertices, ordered from 1 to n, and $n-1$ arrows of the form $i \to i+1$. We fix the dimension vector $d = (1, 2, \ldots, n)$ and the representation R with $R_i = \mathbb{C}^{n+1}$ for $i = 1, \ldots, n$ and $R^\alpha = \mathrm{id}$ for all $\alpha \in A$:

$$\overset{\mathbb{C}^{n+1}}{\bullet} \xrightarrow{\ \mathrm{id}\ } \overset{\mathbb{C}^{n+1}}{\bullet} \xrightarrow{\ \mathrm{id}\ } \cdots \xrightarrow{\ \mathrm{id}\ } \overset{\mathbb{C}^{n+1}}{\bullet}.$$

The quiver Grassmannian $\mathrm{QGr}(R, \mathbf{d}; (n+1, \ldots, n+1)))$ consists precisely of the subrepresentations N of R with $\dim(N_i) = i$ and $N_i \subseteq N_{i+1}$, i.e., flags of vector subspaces.

Analogously to Grassmannians and flag varieties, quiver Grassmannians can be realized pointwise as subvarieties of products of projective spaces, via the closed embedding

$$\iota \;:\; \mathrm{QGr}(R, \mathbf{d}; \mathbf{n}) \;\to\; \prod_{i\in V} \mathrm{Gr}(d_i, K) \;\subseteq\; \mathbb{P}_K^{\binom{n_1}{d_1}} \times \mathbb{P}_K^{\binom{n_2}{d_2}} \times \cdots \times \mathbb{P}_K^{\binom{n_{|V|}}{d_{|V|}}}$$

which sends a subrepresentation N of R to the collection of d_i-dimensional subspaces N_i of R_i.

In this thesis, we will consider quiver Grassmannians as the zero locus of quiver Plücker relations (see Definition 7.13), since they coincide pointwise and their scheme-theoretic structure as (possibly not reduced) projective varieties is not relevant for our purposes.

Definition 7.13 (Quiver Plücker relations, [98]). Let $Q = (V, A, s, t)$ be a quiver and R a Q-representation. After fixing bases for all vertices, let $M_\alpha \in K^{n\times m}$ be the matrix of the map of $\alpha \in A$. Let $r = \dim(s(\alpha))$ and $s = \dim(t(\alpha))$. For each arrow α, the *quiver Plücker relations* are the polynomials in the variables $\{p_I : I \in \binom{[m]}{r}\} \cup \{p_J : J \in \binom{[n]}{s}\}$ with coefficients in K:

$$\mathscr{P}_{\alpha;(n,m)} = \left\{ \sum_{j\in[n]\setminus I, i\in J} \operatorname{sign}(j; I, J)(M_\alpha)_{i,j} p_{I\cup j} p_{J\setminus i} : I \in \binom{[m]}{r-1}, J \in \binom{[n]}{s+1} \right\}$$

where $\operatorname{sign}(j; I, J) = (-1)^{\#\{j'\in J: j<j'\}+\#\{i\in I: i>j\}}$. Their tropicalization will be denoted by $\mathscr{P}^{\text{trop}}_{\alpha;(n,m))}$. The Plücker relations corresponding to the vertices are the standard Grassmann-Plücker relations of the associated vector spaces. In [98], it is shown that the vanishing set of the quiver Plücker relations coincides with the associated quiver Grassmannian.

Pointwise, the definition of the quiver Grassmannian as the parameter space of quiver subrepresentations coincides with the zero locus of the quiver Plücker relations. However, their scheme-theoretic structure can differ. In particular, the scheme associated to the quiver Plücker relations might not be reduced.

8. Matroidal quiver representations

In this section, we use the previously established theory of affine morphisms of valuated matroids to construct a matroidal analogue of quiver (sub)representations. This section is independent work, but is mainly a re-framing and an expansion of joint work with Giulia Iezzi [82] in the context of quiver representations.

Definition 8.1. Let $Q = (V, A, s, t)$ be a quiver. We define a *matroidal Q-representation R* as the ordered pair $\big(([n_i])_{i\in V}, (f^\alpha)_{\alpha\in A}\big)$, where n_i is a positive integer attached to a vertex $i \in V$ and $f^\alpha : [n_{s(\alpha)}] \cup \{o\} \to [n_{t(\alpha)}] \cup \{o\}$ is a map of sets fixing o for any $\alpha \in A$.

To define the analogue for valuated matroids, in keeping with the definition of affine morphisms of valuated matroids (see Definition 4.9) we attach different maps to the arrows.

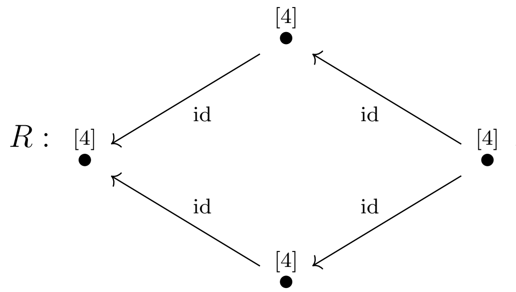
Definition 8.2. Let $Q = (V, A, s, t)$ be a quiver. A *valuated matroidal Q-representation R* is an ordered pair $\big(([n_i])_{i\in V}, (f^\alpha)_{\alpha\in A}\big)$, where n_i is a positive integer attached to a vertex $i \in V$ and $f^\alpha : [n_{s(\alpha)}] \cup \{o\} \to [n_{t(\alpha)}] \cup \{o\} \times \mathbb{T}$ is a map satisfying $f^\alpha(o) = (o, \infty)$ for any $\alpha \in A$.

Remark 8.3. As we had previously noticed when studying (affine) morphisms of valuated matroids, maps of matroids are contravariant to their classical analogues arising via matroid multiplication. Thus, for examples we will usually consider the contravariant quiver, where we reverse all arrows.

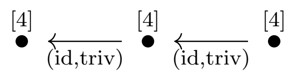
Example 8.4. We construct some corresponding matroidal Q-representations for the quivers discussed in the classical case in Examples 7.2 and 7.6:

(a) Let us first consider the diamond shaped quiver we discussed in Example 7.2. We obtain its matroidal analogue by substituting every vector space $\mathbb{C}^4$ by the set of size $\dim(\mathbb{C}^4) = 4$, and the matrices by their associated affine maps of sets we had constructed in Definition 5.8. This yields the identity map of sets id : $[4] \cup \{o\} \to$

$[4] \cup \{o\}$ for all arrows. Finally, we reverse all arrows. We end up with the following quiver with associated matroidal quiver representation:

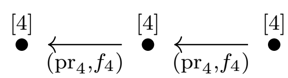


(b) Now, we consider the A_3-Dynkin quiver we had encountered in Example 7.6. We will consider two different valuated matroidal quiver representations. First, we investigate the representation below, where each arrow is assigned the identity map on $[4] \cup \{o\}$ in the first coordinate and the trivial map we discussed in Lemma 4.11 in the second coordinate. We will see later that the parameter space associated to this valuated matroidal quiver representation is the flag Dressian we have already seen in Definition 3.93.

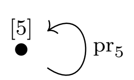


Now, we give a different valuated matroidal quiver representation associated to the same quiver, which assigns projection maps to each arrow. Coordinate-wise, in the first coordinate, the map is given as the projection map we had discussed in Definition 3.53. In the second coordinate, it is the map

$$f_4 : i \mapsto \begin{cases} 0 & i \in \{1,2,3\} \\ \infty & i \in \{4,o\}. \end{cases}$$

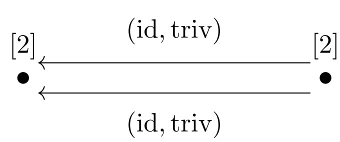


(c) Next, we construct a matroidal quiver representation of the loop quiver,



The map on the arrow is the projection map pr_5 we discussed in Definition 3.53.

(d) Finally, we will consider a valuated matroidal quiver representation of the Kronecker quiver, where the maps are given coordinate-wise as the identity on $[2] \cup \{o\}$ and the trivial map defined in Lemma 4.11.



The choice why we assign twice the same map will remain a mystery for a while. We will see the reason for this construction much later, when we discuss realizability in Example 9.10

Analogously to the classical case, we are mainly interested in (valuated) matroidal quiver subrepresentations and their ambient spaces.

Definition 8.5. Let Q be a quiver and R a matroidal Q-representation. Let $\mathbf{M} = (M_i)_{i \in V}$ be a collection of matroids, such that M_i is a matroid over the ground set $[n_i]$ for each $i \in V$. We say that $\mathbf{M}$ is a *matroidal quiver subrepresentation* if f^α is a strong map of matroids for every $\alpha \in A$.

Analogously, if R is a valuated matroidal Q-representation and $\boldsymbol{\nu} = (\nu_i)_{i \in V}$ a collection of valuated matroids over the ground sets $[n_i]$ for each $i \in V$ respectively, we say that $\boldsymbol{\nu}$ is a *valuated matroidal quiver subrepresentation* if f^α is an affine morphism of valuated matroids for every $\alpha \in A$.

The *rank vector* of a (valuated) matroidal Q-subrepresentation $\mathbf{M}$ (or $\boldsymbol{\nu}$) is $\operatorname{rk}(\mathbf{M}) := (\operatorname{rk}(M_i))_{i \in V}$.

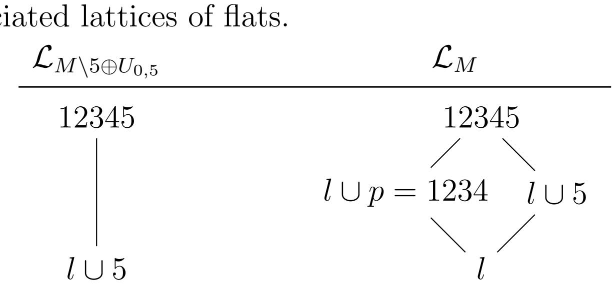
Remark 8.6. By Lemma 4.10, we know that every morphism of valuated matroids induces a strong map of the underlying matroids, which is given by the first coordinate map of the affine map. This means that if we have a valuated matroidal quiver subrepresentation $\boldsymbol{\nu}$, we can obtain an *underlying matroidal quiver subrepresentation* $\mathbf{N}$ by taking the underlying matroids for each valuated matroid, and restricting the map to its first coordinate map.

Example 8.7. In Example 8.4, we had constructed some (valuated) matroidal quiver representations. Now, we will give some examples of the subrepresentations of the (valuated) matroidal quiver representations constructed in Example 8.4(b) and (c).

- We aim to find all matroidal quiver subrepresentations of rank 2 of the matroidal loop quiver representation we constructed in Example 8.4(c). This means that we need to construct all matroids M of rank 2 over [5] for which the projection map pr_5 is a morphism of M into itself. By Definition 3.42 and Proposition 5.14, this means we need to identify all matroids M of rank 2 over 5 such that

$$M \setminus 5 \oplus U_{0,5} \twoheadleftarrow M.$$

If 5 is a loop of M, then $M \setminus 5 \oplus U_{0,5} = M$, so M is a matroidal subrepresentation. Further, all matroids that have 5 as a coloop, and where all other elements of [4] can be decomposed into a set of loops l and a set of parallel elements p. To see that these matroids are matroidal quiver subrepresentations, we use Definition 3.37 and construct the associated lattices of flats.



We directly observe that each flat on the left is also a flat on the right, so by Definition 3.37, $M \setminus 5 \oplus U_{0,5} \twoheadleftarrow M$, thus M is a quiver subrepresentation.

We want to show that no other matroid of rank 2 on [5] is a subrepresentation. We assume that 5 is not a loop. Let l denote the set of loops of M. Then, $l \cup 5$ is the set of loops of $M \setminus 5 \oplus U_{0,\{5\}}$, and hence a flat in $\mathcal{L}_{M\setminus 5 \oplus U_{0,\{5\}}}$. Thus, by M being a matroidal quiver subrepresentation of the loop quiver, $l \cup 5$ is also a flat of M.

Since $\mathrm{rk}(M) = 2$, by Remark 3.29, all maximal flags of flats are of length 3. Now, l, $l \cup 5$ and [5] form a maximal chain of flats, as $5 \notin l$ by assumption. This directly implies that no other flats of M can contain 5: Every flat contains l, and if F is a flat that additionally contains 5, by $l \subsetneq l \cup 5 \subsetneq [5]$ being a maximal chain of flats, F is either $l \cup 5$ or [5].

Observe that

$$\mathrm{rk}_{M\setminus 5 \oplus U_{0,\{5\}}}(S) = \mathrm{rk}_M(S \setminus 5)$$

by definition. Now, if F is a flat of M that does not contain 5, then $F \cup 5$ is a flat of $M \setminus 5 \oplus U_{0,\{5\}}$. By M being a matroidal quiver subrepresentation of the loop quiver, this implies that $F \cup 5$ is also a flat of M, hence $F = l$ or $F = 1234$. Now, this is exactly the description of the lattice of flats we obtained for our examples.

- Now, we construct the valuated matroidal quiver subrepresentation of the matroidal analogue of the quiver representation generating the flag variety, which is the first valuated matroidal quiver representation in Example 8.4(b). Unlike the previous example, we will not construct all subrepresentations, as there are usually infinitely many of them.

  We can directly observe that the row spaces of the matrices below are contained inside of each other.

$$\mathrm{rowsp}\begin{bmatrix} t^b & -t^a & 0 & 1 \end{bmatrix} \subset \mathrm{rowsp}\begin{bmatrix} t^b & -t^a & 0 & 1 \\ 1 & 0 & -t^a & 1 \end{bmatrix} \subset \mathrm{rowsp}\begin{bmatrix} t^b & -t^a & 0 & 1 \\ 1 & 0 & -t^a & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

  Assume that $0 > a > b$. Then, the matroids correspond to the following *Plücker vector*, recording the valuation of bases in lexicographical ordering:

$$\begin{aligned} \mu_1 &= (b, a, \infty, 0) \\ \mu_2 &= (a, a+b, b, 2a, a, a) \\ \mu_3 &= (a+b, b, b, 2a). \end{aligned}$$

  Since their associated linear spaces are contained inside of each other, the three valuated matroids form a flag, which means that they are a valuated matroidal quiver representation. Further, as we observed in Remark 8.6, their underlying matroids $U_{1,3} \oplus U_{0,1}$, $U_{2,4}$ and $U_{3,4}$ are a matroidal quiver representation for the same quiver with the maps restricted to their first coordinate.

Using the connection between tropical matrix multiplication and affine morphisms of valuated matroids, we can establish a direct connection between matroidal quiver representations and quiver Dressians.

Proposition 8.8. *Let $M_f \in K^{n\times m}$ be a weakly monomial matrix, and let μ and ν be matroids of ranks r and s over $[n]$. Then, $\mathrm{val}(M_f) \odot \overline{\mathrm{trop}}(\mu) \subseteq \overline{\mathrm{trop}}(\nu)$ if and only if $f : \nu \to \mu \times \mathbb{R}$ as constructed in Proposition 5.18 is an affine morphism of valuated matroids.*

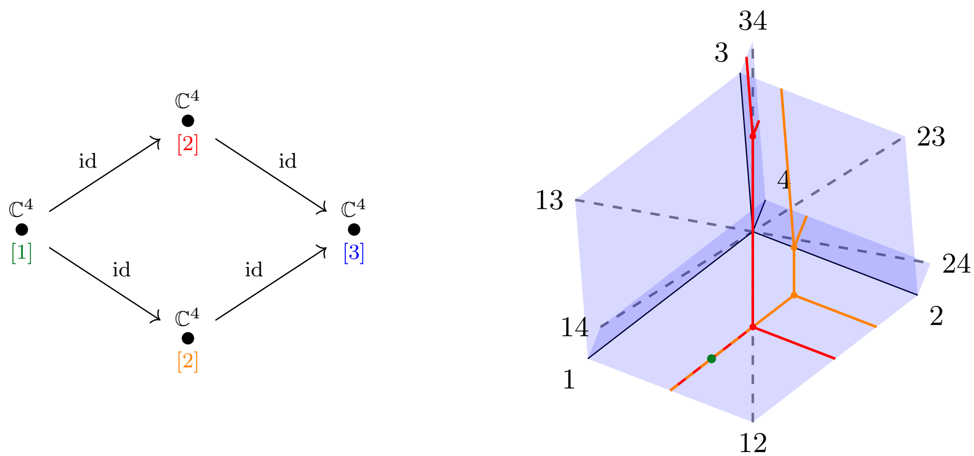


FIGURE 8. The numbers below the vertices of the quiver represent the fixed dimensions of the corresponding subspaces. Any subrepresentation consisting of such subspaces describes the containment of a point in two lines, which are both contained in a common plane. On the right, a collection of tropical linear spaces satisfying these conditions.

Further, $\operatorname{val}(M_f) \odot \overline{\operatorname{trop}}(\mu) \subseteq \overline{\operatorname{trop}}(\nu)$ *is realizable if and only if* $f : \nu \to \mu \times \mathbb{R}$ *is a realizable affine morphism of valuated matroids.*

Proof. By Proposition 5.18, there exists a map $f : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{T}$ such that $\operatorname{val}(M_f) \odot \overline{\operatorname{trop}}(\mu) = \overline{\operatorname{trop}}\big(f^{-1}(\mu)\big)$. By [30, Theorem A],

$$\overline{\operatorname{trop}}\big(f^{-1}(\mu)\big) \;=\; \operatorname{val}(M_f) \odot \overline{\operatorname{trop}}(\mu) \;\subseteq\; \overline{\operatorname{trop}}(\nu)$$

implies that $f^{-1}(\mu)|_{f_1([n])} \twoheadleftarrow \nu$, i.e., by Definition 4.9, that f^{-1} is an affine morphism of valuated matroids. The realizability statement follows from Definition 4.9. ☐

9. Tropical quiver Grassmannians

In this section, we construct the two tropical analogues of quiver Grassmannians. We obtain the *tropicalized quiver Grassmannian* as the tropicalization of the quiver Grassmannian and show that it is the parameter space of tropicalizations of quiver subrepresentations. Further, we construct the *quiver Dressian* as the tropical prevariety of the quiver Plücker relations, and show that it is the parameter space of tropical linear spaces contained inside of each other after naive tropical matrix multiplication. For appropriate quiver representations, we then connect these constructions to the valuated matroidal quiver subrepresentations discussed in the previous section. These correspondence results are two of the three main results of this part of the thesis, Theorem E for the characterization of quiver Dressians, and D for the characterization of tropicalized quiver Grassmannians. We conclude by studying when the two analogues coincide, and prove the third main result of this part of the thesis, Theorem F.

9.1. **Tropicalized quiver Grassmannians.**

Definition 9.1. Let Q be a quiver and let M be a Q-representation with quiver Grassmannian $\mathrm{QGr}(M, \mathbf{d}; \mathbf{n})$. The *tropicalized quiver Grassmannian*

$$\overline{\mathrm{trop}}\big(\mathrm{QGr}(M, \mathbf{d}; \mathbf{n})\big) \subseteq \mathbb{P}\big(\mathbb{T}^{\binom{n_1}{d_1}}\big) \times \cdots \times \mathbb{P}\big(\mathbb{T}^{\binom{n_{|V|}}{d_{|V|}}}\big)$$

is the (multihomogeneous) tropicalization of $\mathrm{QGr}(M, \mathbf{d}; \mathbf{n})$.

Example 9.2. Let us come back to the loop quiver we discussed in Example 8.7. Analogously to the construction in Lemma 4.11, we can attach the trivial map as the second coordinate. This allows us to construct a valuated matroidal quiver representation out of a matroidal one, and to investigate its subrepresentations. We compute the quiver Plücker relations using our code in Appendix B.4.

```
julia> n,m,r,s = (5,5,2,2);

julia> B = [1 0 0 0 0; 0 1 0 0 0; 0 0 1 0 0; 0 0 0 1 0; 0 0 0 0 0];

julia> p1 = quiver_pluecker_relations(r,s,M)
```

The output is a 50 element vector of Plücker relations. Because the loop quiver has the same matroid as the source and the target vertex for its lone arrow, we need to identify the Plücker variables assigned to source and target with each other. We can do this as follows.

```
julia> Iset = subsets(collect(1:m),r-1);

julia> Jset = subsets(collect(1:n),s+1);

julia> m_c_r =subsets(collect(1:m),r);

julia> n_c_s =subsets(collect(1:n),s);

julia> R,x,y = polynomial_ring(QQ, "x"=>m_c_r, "y"=>n_c_s)
(Multivariate polynomial ring in 20 variables over QQ, QQMPolyRingElem[x[[1,
↪  5]], x[[2, 5]], x[[3, 5]], x[[4, 5]], x[[1, 4]], x[[2, 4]], x[[3, 4]],
↪  x[[1, 3]], x[[2, 3]], x[[1, 2]]], QQMPolyRingElem[y[[1, 5]], y[[2, 5]],
↪  y[[3, 5]], y[[4, 5]], y[[1, 4]], y[[2, 4]], y[[3, 4]], y[[1, 3]], y[[2,
↪  3]], y[[1, 2]]])

julia> p1 = [evaluate(i,vcat(x,x)) for i in p1]
```

We note that we can still simplify the relations — there are some relations that got eliminated by the identification, and some relations that are equal up to sign. We can eliminate them as below, and compute the dimension of their ideal and its primary decomposition.

```
julia> p1 = remove_redundant_polys(p1);

julia> I = ideal(p1)

julia> dim(I)
15
```

```
julia> decI = primary_decomposition(I)
(Ideal with 5 generators, Ideal with 5 generators)
 (Ideal (x[[1, 2]], x[[2, 3]], x[[1, 3]], x[[3, 4]], x[[2, 4]], x[[1, 4]]),
↪  Ideal (x[[1, 2]], x[[2, 3]], x[[1, 3]], x[[3, 4]], x[[2, 4]], x[[1,
↪  4]]))
```

The second component is just the irrelevant ideal, which is due to the fact that we are working with homogeneous ideals. The first component is more interesting.

```
julia> decI[1][1]
Ideal generated by
  x[[1, 4]]*x[[2, 3]] - x[[2, 4]]*x[[1, 3]] + x[[3, 4]]*x[[1, 2]]
  x[[4, 5]]
  x[[3, 5]]
  x[[2, 5]]
  x[[1, 5]]
```

First, we know that this set of generators is a tropical basis (i.e., the tropical prevariety cut out by the generators given above is again a tropical variety). The four monomials tell us that the tropical variety lives on the boundary component where

$$p_{15} = p_{25} = p_{35} = p_{45} = \infty.$$

The other relation tells us that the variety is something we have already seen before — the tropicalized Grassmannian $\mathrm{Gr}(2,4)$, which we have seen in Example 3.83.

Now, we investigate what happens if we instead consider a contraction map, given by the matrix

$$A^{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Again, we compute the quiver Plücker relations using our code in Appendix B.4 and simplify as before. After getting rid of relations producing redundant information, the following quiver Plücker relations remain for the sole connected component:

$$\begin{array}{ccc} & p_{14}p_{23} - p_{24}p_{13} + p_{34}p_{12} & p_{45} \\ p_{35} + p_{34} & p_{25} + p_{24} & p_{15} + p_{14} \end{array}$$

This tells us that as before, the tropicalized quiver Grassmannian lives on the boundary, though this time, it only has to satisfy that $p_{45} = \infty$. Further, the binomial relations tell us that for any point p in the tropicalized quiver Grassmannian,

$$\mathrm{val}(p_{34}) = \mathrm{val}(p_{35}), \qquad \mathrm{val}(p_{24}) = \mathrm{val}(p_{25}), \quad \text{and} \quad \mathrm{val}(p_{14}) = \mathrm{val}(p_{15}).$$

We now aim to prove Theorem D, which can be more explicitly stated in the terms below.

Theorem 9.3. *Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ be realizable valuated matroids and Q be a finite quiver. Let each arrow $\alpha \in A_Q$ of Q be represented by a matrix A^α, and let $s(\alpha)$ denote its source and $t(\alpha)$ its target vertex. The following statements are equivalent:*

(a) $\boldsymbol{\mu} \in \overline{\text{trop}}(\text{QGr}(R, \mathbf{d}; \mathbf{n}))$;
(b) $\text{val}(A^\alpha) \odot \overline{\text{trop}}(\mu_{s(\alpha)}) \subseteq \overline{\text{trop}}(\mu_{t(\alpha)})$ *for all* $\alpha \in A_Q$ *and there is a quiver subrepresentation* $(N_i)_{i\in V}$ *of Q such that* $\overline{\text{trop}}(\mu_i) = \overline{\text{trop}}(N_i)$.

If further, all realizations A^α are weakly monomial matrices, the above are equivalent to:

(c) Each arrow $\alpha \in A_Q$ with $s(\alpha) = V_i$ and $t(\alpha) = V_j$ has an associated contravariant realizable affine *morphism of valuated matroids ϕ_α (c.f. Definition 4.9) with $s(\phi_\alpha) = \mu_j$ and $t(\phi_\alpha) = \mu_i$. That is, $\boldsymbol{\mu}$ is a valuated matroidal quiver subrepresentation where all valuated matroids on vertices and all affine morphisms of matroids on arrows are realizable.*

The equivalence $(b) \Leftrightarrow (c)$ was already established in Proposition 8.8. We now show $(a) \Leftrightarrow (b)$ for the above theorem, i.e., that the tropicalized quiver Grassmannian parametrizes containment of tropicalized linear spaces under tropical matrix multiplication.

Proposition 9.4. *Let K be an algebraically closed field with nontrivial valuation, and let M be a quiver representation of a quiver Q with quiver Grassmannian* $\text{QGr}(M, \mathbf{d}; \mathbf{n})$*, for some dimension vector d. Then, $p \in \overline{\text{trop}}\big(\text{QGr}(M, \mathbf{d}; \mathbf{n})\big)$ if and only if there exists a tropical linear space $\overline{\text{trop}}(\mu_i)$ for each vertex $i \in V$ such that $\text{val}(M_f) \odot \overline{\text{trop}}(\mu_{s(f)}) \subseteq \overline{\text{trop}}(\mu_{t(f)})$ for each arrow f, and there exists a quiver subrepresentation $N = \big((N_i)_{i\in V}, (M^\alpha|_{N_{s(\alpha)}})_{\alpha\in A}\big)$ over K such that $\overline{\text{trop}}(\mu_i) = \overline{\text{trop}}(N_i)$ for all $i \in V$.*

Proof. For ease of notation, we restrict to the case where Q is a graph with two vertices and one arrow f, and we write $\text{QGr}(M_f, \mathbf{d}; \mathbf{n})$ for the corresponding quiver Grassmannian. All other cases follow similarly. If $\mu \times \nu \in \overline{\text{trop}}(\text{QGr}(M_f, \mathbf{d}; \mathbf{n}))$, from the Fundamental Theorem of Tropical Geometry [101, Theorem 6.2.15], there exist realizations U of μ and V of ν such that the Plücker coordinates of U and V are a point of $\text{QGr}(M_f, \mathbf{d}; \mathbf{n})$. By the main theorem in [98], points in $\text{QGr}(M_f, \mathbf{d}; \mathbf{n})$ satisfy $M_f \cdot U \subseteq V$, thus $\text{val}(M_f) \odot \overline{\text{trop}}(U) \subseteq \overline{\text{trop}}(V)$.

Now conversely assume that $\text{val}(M_f) \odot \overline{\text{trop}}(U) \subseteq \overline{\text{trop}}(V)$, and that there exist realizations U and V such that $M_f \cdot U \subseteq V$. Then, $U \times V \in \text{QGr}(M_f, \mathbf{d}; \mathbf{n})$, hence $p_{\overline{\text{trop}}(U)} \times p_{\overline{\text{trop}}(V)} \in \overline{\text{trop}}(\text{QGr}(M_f, \mathbf{d}; \mathbf{n})))$ by the Fundamental Theorem 2.22. ☐

9.2. **Quiver Dressians.** Instead of parametrizing tropicalized linear spaces and their containment relations, we now consider parameter spaces of tropical linear spaces contained under naive tropical matrix multiplication.

Definition 9.5. Let $Q = (V, A, s, t)$ be a quiver. The *quiver Dressian*

$$\text{QDr}(R, \mathbf{d}; \mathbf{n}) \ \subseteq \ \mathbb{P}\big(\mathbb{T}^{\binom{n_1}{d_1}}\big) \times \cdots \times \mathbb{P}\big(\mathbb{T}^{\binom{n_{|V|}}{d_{|V|}}}\big)$$

is the tropical prevariety cut out by the tropical Plücker relations and the tropical quiver Plücker relations, $\{\mathscr{P}^{\text{trop}}_{d_i;n}\}_{i\in V} \cup \{\mathscr{P}^{\text{trop}}_{\alpha;\mathbf{n}}\}_{\alpha\in A}$ (see Definition 7.13).

Now, our goal will be to establish the correspondences in Theorem E, which is the analogue of Theorem 9.3/D we just discussed, but for the case of Dressians. In more explicit terms, we aim to prove:

Theorem 9.6. *Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)$ be valuated matroids and Q be a finite quiver. Let each arrow $\alpha \in A_Q$ of Q be represented by a matrix A^α, and let $s(\alpha)$ denote its source and $t(\alpha)$ its target vertex. The following statements are equivalent:*

(a) $\boldsymbol{\mu} \in \mathrm{QDr}(R, \mathbf{d}; \mathbf{n})$*;*
(b) $\mathrm{val}(A^\alpha) \odot \overline{\mathrm{trop}}(\mu_{s(\alpha)}) \subseteq \overline{\mathrm{trop}}(\mu_{t(\alpha)})$ *for all* $\alpha \in A_Q$*.*

If further, all matrices A^α are weakly monomial matrices, the above are equivalent to:

(c) Each arrow $\alpha \in A_Q$ with $s(\alpha) = V_i$ and $t(\alpha) = V_j$ has an associated contravariant affine *morphism of valuated matroids ϕ_α (c.f. Definition 4.9) with $s(\phi_\alpha) = \mu_j$ and $t(\phi_\alpha) = \mu_i$. That is, $\boldsymbol{\mu}$ is a valuated matroidal quiver subrepresentation.*

Again, we have already seen a proof of $(b) \Leftrightarrow (c)$ in Proposition 8.8. Now we show $(a) \Leftrightarrow (b)$, i.e., that the quiver Dressian parametrizes containment of tropical linear spaces under matrix multiplication.

Theorem 9.7. *Let μ and ν be valuated matroids over the ground sets $[m]$ and $[n]$ and of rank r and s respectively, and let Q be a quiver consisting of two vertices connected by one arrow f. Let M denote a Q-representation assigning the matrix $M_f \in K^{n\times m}$ to f. Then,*

$$\mu \times \nu \in \mathrm{QDr}(M, (r,s); (m,n)) \quad \Leftrightarrow \quad \mathrm{val}(M_f) \odot \overline{\mathrm{trop}}(\mu) \subseteq \overline{\mathrm{trop}}(\nu).$$

Proof. The standard Grassmann-Plücker relations associated to the vertices vanish if and only if μ and ν are valuated matroids. Thus, we only focus on the quiver Plücker relations. By definition, $\mu \times \nu \in \mathrm{QDr}(M, (r,s); (m,n))$ if and only if for all $I \in \binom{[m]}{r-1}$ and $J \in \binom{[n]}{s+1}$, the minimum in

$$\bigoplus_{j\in[n]\setminus I, i\in J} \Big(\mathrm{val}\big((M_f)_{i,j}\big) \odot p_{I\cup j} \odot p_{J\setminus i}\Big)$$

is attained at least twice. Equivalently, for all I and J as above, the minimum in

(8)
$$\bigoplus_{\substack{j\in[n]\setminus I,\\ i\in J}} \Big(\mathrm{val}\big((M_f)_{i,j}\big) \odot \mu(I\cup j) \odot \nu(J\setminus i)\Big) = \bigoplus_{\substack{j\in[n]\setminus I,\\ i\in J}} \Big(\mathrm{val}\big((M_f)_{i,j}\big) \odot C^*_\mu(I)_j \odot C_\nu(J)_i\Big)$$

is attained at least twice. We write $\mathrm{val}(M_f) \odot C^*_\mu(I)$ for the vector with coordinate entries

$$\big(\mathrm{val}(M_f) \odot C^*_\mu(I)\big)_j := \big(\mathrm{val}(m_{f,1,j}) \odot C^*_\mu(I)_j\big) \oplus \cdots \oplus \big(\mathrm{val}(m_{f,n,j}) \odot C^*_\mu(I)_j\big).$$

By distribution, the minimum in (8) is attained twice if and only if

$$\mathrm{val}(M_f) \odot C^*_\mu(I) \in V\Big(\bigoplus_{i\in[n]} C_\nu(J)_i \odot x_i\Big) = \overline{\mathrm{trop}}(\nu).$$

Finally, by Proposition 5.3, $\mathrm{val}(M_f) \odot \overline{\mathrm{trop}}(\mu)$ is tropically convex. Using Proposition-Definition 3.71, the above is thus equivalent to

$$\Big\{ \bigoplus_{C^*_\mu(I)\in\mathcal{C}^*(\mu)} \lambda_{C^*_\mu(I)} \odot \mathrm{val}(M_f) \odot C^*_\mu(I) \ : \ \lambda_{C^*_\mu(I)} \in \mathbb{R} \Big\} = \mathrm{val}(M_f) \odot \overline{\mathrm{trop}}(\mu) \subseteq \overline{\mathrm{trop}}(\nu).$$

□

9.3. **Realizability of points in quiver Dressians.** In Section 3.4.1, we remarked the difference between intrinsically tropical and tropicalized objects, and distinguished the Dressian, parametrizing *tropical* linear spaces, from the tropicalized Grassmannians, parametrizing *tropicalized* linear spaces. We observe a similar distinction for quiver Dressians and tropicalized quiver Grassmannians.

The first example of a nonrealizable tropical linear space, i.e., a tropical linear space that is not the tropicalization of any linear space, occurs in ambient dimension 8. The first nonrealizable flag of tropical linear spaces already occurs for ambient dimension 6 (see [30, Example 5.2.4]). For arbitrary quivers, the ambient dimension of the first nonrealizable quiver subrepresentation is even smaller.

9.3.1. *Nonrealizable quiver Dressians over fields with nontrivial valuation.* This subsection will be devoted to the proof of Theorem F, which we give in more details below.

Theorem 9.8. *Let Q be a finite quiver. For $\mathbf{n}$ where $n_i \geq 2$ for all $i \in V$,*

$$\overline{\text{trop}}(\text{QGr}(R, \mathbf{d}; \mathbf{n})) \;\subseteq\; \text{QDr}(R, \mathbf{d}; \mathbf{n})$$

and there exist quiver representations R where the containment is strict. For $\mathbf{n} = \mathbf{1}$,

$$\overline{\text{trop}}(\text{QGr}(R, \mathbf{d}; \mathbf{1})) \;=\; \text{QDr}(R, \mathbf{d}; \mathbf{1})$$

for any quiver Q and any Q-representation R.

Remark 9.9. For ambient dimension 1, there are no classical Plücker relations, and the only quiver Plücker relations are the monomial relations corresponding to the coordinates of the only point in the quiver Grassmannian. On the algebraic side, $\text{QGr}(R, \mathbf{d}, \mathbf{1})$ is a point, hence $\overline{\text{trop}}(\text{QGr}(R, \mathbf{d}, \mathbf{1})) \subseteq \mathbb{P}(\mathbb{T}^1) \times \cdots \times \mathbb{P}(\mathbb{T}^1)$ is a point. Since $\mathbb{P}(\mathbb{T}^1) \times \cdots \times \mathbb{P}(\mathbb{T}^1)$ is also just a point, the containment is an equality, and therefore $\overline{\text{trop}}(\text{QGr}(R, \mathbf{d}, \mathbf{1})) = \text{QDr}(R, \mathbf{d}, \mathbf{1})$.

Example 9.10. We construct an example for $\overline{\text{trop}}(\text{QGr}(R, \mathbf{d}; \mathbf{2})) \neq \text{QDr}(R, \mathbf{d}; \mathbf{2})$. The quiver we consider is known as the Kronecker quiver; we define its representation R as shown in Figure 9, with quiver Grassmannian $\text{QGr}(R, (1, 1); \mathbf{2})$ (see for instance [83, Example 5]; in this case, we replace $\mathbb{C}$ with $\mathbb{C}\{\{t\}\}$, the field of Puiseux series). It is an example of a reduced quiver Grassmannian of dimension 0 with two connected components (the two eigenspaces of the map corresponding to the lower arrow).

Let v_1 and v_2 denote the Plücker variables of the space corresponding to the left vertex, and let w_1 and w_2 denote the Plücker variables of the right vertex. Since $\text{Gr}(1; 2)$ and $\text{Gr}(2; 2)$ have no Grassmann-Plücker relations, the only relations are the quiver Plücker relations (see Definition 7.13), which are $v_1 w_2 + v_2 w_1$ and $v_1 w_2 + (1 + t) v_2 w_1$. We have

$$V(\langle v_1 w_2 + v_2 w_1, v_1 w_2 + (1 + t) v_2 w_1 \rangle) = \{\big((1 : 0), (1 : 0)\big), \big((0 : 1), (0 : 1)\big) \subseteq \mathbb{P}^1 \times \mathbb{P}^1\}, \text{ and}$$

$$\overline{\text{trop}}(\text{QGr}(R, (1, 1), 2)) \;=\; \{\big((0 : \infty), (0 : \infty)\big), \big((\infty : 0), (\infty : 0)\big) \subseteq \mathbb{P}(\mathbb{T}^2) \times \mathbb{P}(\mathbb{T}^2)\}.$$

Tropicalizing the generators, we have that $V(\overline{\text{trop}}(v_1 w_2 + v_2 w_1))$ is the set

$$W \;=\; \{((v_1 : v_2), (w_1 : w_2)) \in \mathbb{P}(\mathbb{T}^2) \times \mathbb{P}(\mathbb{T}^2) \;|\; \min(v_1 + w_2, v_2 + w_1) \text{ is attained at least twice}\}.$$

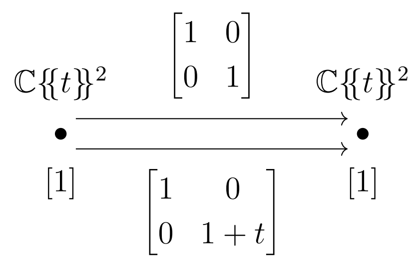


FIGURE 9. A quiver Q with Q-representation R for $\mathbf{n} = (2,2)$ where $\overline{\text{trop}}(\text{QGr}(R,(1,1);(2,2))) \neq \text{QDr}(R,(1,1);(2,2))$.

Since $\text{val}(1+t) = \text{val}(1) = 0$, we further have $W = V(\overline{\text{trop}}(v_1w_2 + (1+t)v_2w_1))$, thus $W = \text{QDr}(R,(1,1),2)$. Now, W can be rewritten as

$$W = \{((v_1 : v_2),(w_1 : w_2)) \in \mathbb{P}(\mathbb{T}^2) \times \mathbb{P}(\mathbb{T}^2) \mid v_1 + w_2 = v_2 + w_1\},$$

which is a connected 1-dimensional space (that contains the two points above), whereas $\overline{\text{trop}}(\text{QGr}(R,(1,1),2))$ is not.

Example 9.10 relies on the nontrivial valuation of the base field. Similar constructions can be given for higher ambient dimension, as described in Example 9.11. However, the examples we construct afterwards for higher equal ambient dimension ($n \geq 4$) already occur for fields with trivial valuation, and for quivers without parallel edges.

Example 9.11. To obtain an analogous example for ambient dimension 3, we can consider the same quiver as in Example 9.10. We assign $\mathbb{C}\{\{t\}\}^3$ to each vertex, the matrix $\left[\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right]$ to the upper arrow and $\left[\begin{smallmatrix}1&0&0\\0&1+t&0\\0&0&1+t^2\end{smallmatrix}\right]$ to the lower arrow. Again, $\text{Gr}(1;3)$ has no Grassmann-Plücker relations, so the only Plücker relations are

$$v_1w_2 - v_2w_1, \quad v_1w_3 + v_3w_1, \quad v_2w_3 - v_3w_2$$

for the upper arrow, and

$$v_1w_2 - (1+t)v_2w_1, \quad v_1w_3 + (1+t^2)v_3w_1, \quad (1+t)v_2w_3 - (1+t^2)v_3w_2$$

for the lower arrow. The zero locus of the six equations is zero-dimensional and consists of three points: $((1:0:0),(1:0:0))$, $((0:1:0),(0:1:0))$ and $((0:0:1),(0:0:1))$, so the tropicalization of the quiver Grassmannian does, too. Again, as the valuations of all non-zero matrix entries is zero, the quiver Dressian is the set

$$\{(\mathbf{v},\mathbf{w}) \in \mathbb{P}(\mathbb{T}^3) \times \mathbb{P}(\mathbb{T}^3) \mid v_1 + w_2 = v_2 + w_1, v_1 + w_3 = v_3 + w_1 \text{ and } v_2 + w_3 = w_2 + v_3\}.$$

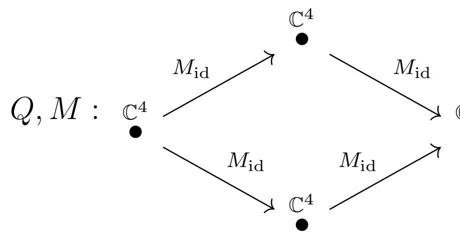
This set is 1-dimensional, thus the tropicalized quiver Grassmannian and the quiver Dressian differ. This example can similarly be extended to an example for higher ambient dimension n. Here, we assign $\mathbb{C}\{\{t\}\}^n$ to both vertices, consider the dimension vector $(1,1)$ and assign the matrices to the two arrows as follows: one arrow is assigned the identity matrix, and the other arrow gets the diagonal matrix with entries $(1, 1+t, 1+t^2, \ldots, 1+t^{n-1})$.

If we wish to assign different ambient dimensions n and m to the two vertices, we can again construct an analogous example by appending zero rows (resp. zero columns) to both matrices.

An example for a quiver representation over a quiver with no parallel edges over a trivially valued field can also be found, though it is significantly more complicated. We give such an example in Section 9.3.2.

Now we give an example of a quiver representation R satisfying $\overline{\text{trop}}(\text{QGr}(R, \mathbf{d}; \mathbf{4})) \neq \text{QDr}(R, \mathbf{d}; \mathbf{4})$ over a field with trivial valuation, using a quiver without parallel edges. Afterwards, we will extend this to a family of such examples for $n > 4$.

Example 9.12. We return to the quiver given in Example 7.4.



$$, \ M_{\text{id}} = \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}, \ d = (1, 2, 2, 3).$$

This quiver Grassmannian parametrizes the arrangement of four tropical objects: two tropical lines that are contained in a common tropical plane, and a common point lying on all of them. In Figure 8, we give an example of such an arrangement.

From Definition 7.13, we obtain the following equations for the quiver Grassmannian $\text{QGr}(M, \mathbf{d}; \mathbf{4})$ inside the product of Grassmannians $\prod \text{Gr}(d_i; \mathbf{4})$:

$$\begin{array}{ccc}
p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} & p'_{12}p'_{34} - p'_{13}p'_{24} + p'_{14}p'_{23} & p_{12}p_{134} + p_{13}p_{124} + p_{14}p_{123} \\
p_{12}p_{234} + p_{23}p_{124} + p_{24}p_{123} & p_{13}p_{234} + p_{23}p_{134} + p_{34}p_{123} & p_{14}p_{234} + p_{24}p_{134} + p_{34}p_{124} \\
p'_{12}p_{134} + p'_{13}p_{124} + p'_{14}p_{123} & p'_{12}p_{234} + p'_{23}p_{124} + p'_{24}p_{123} & p'_{13}p_{234} + p'_{23}p_{134} + p'_{34}p_{123} \\
p'_{14}p_{234} + p'_{24}p_{134} + p'_{34}p_{124} & p_1p_{23} + p_2p_{13} + p_3p_{12} & p_1p_{24} + p_2p_{14} + p_4p_{12} \\
p_1p_{34} + p_3p_{14} + p_4p_{13} & p_2p_{34} + p_3p_{24} + p_4p_{23} & p_1p'_{23} + p_2p'_{13} + p_3p'_{12} \\
p_1p'_{24} + p_2p'_{14} + p_4p'_{12} & p_1p'_{34} + p_3p'_{14} + p_4p'_{13} & p_2p'_{34} + p_3p'_{24} + p_4p'_{23}
\end{array}$$

where we denote by p'_{ij} the Plücker coordinates corresponding to the two-dimensional subspace in the bottom row, denoted by $\text{Gr}(d_3; 4)$.

We use the code provided in the Appendix B to compute the quiver Dressian and the tropicalized quiver Grassmannian in `gfan` [88], and do some auxiliary computations in Oscar [114].

The quiver Dressian has dimension 12 and f-Vector $(1, 58, 466, 1156, 858, 3)$. The tropicalized quiver Grassmannian has dimension 10, as does the ideal generated by the polynomials. Since the dimensions of the tropical (pre-)varieties differ, they cannot be equal, showing the second part of Theorem 9.8 for $n = 4$.

As a polyhedral complex, the tropicalized quiver Grassmannian is the union of the tropicalization of the 46 primary components of the quiver Grassmannian. Of these components, 37 tropicalize to linear components of dimensions 8, 7, 6 and 5 in different coordinate directions. Each of the remaining nine components has, after quotienting out lineality, six rays and ten facets, whose incidences are depicted in the graph in Figure 10.

Corollary 9.13. *For* $\mathbf{n}$ *where* $n_i \geq 4$ *for all* $i \in V$, *there exists a quiver* Q *with a representation* M' *such that* $\overline{\text{trop}}(\text{QGr}(M', \mathbf{d}; \mathbf{n})) \subsetneq \text{QDr}(M', \mathbf{d}; \mathbf{n})$.

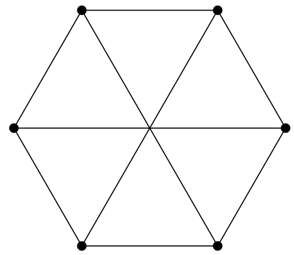


FIGURE 10. The nonlinear irreducible components of Example 9.12 are linear spaces of dimension 10 and 8 over the graph above.

Proof. Let $n \geq 4$. We consider the quiver representation M of Example 9.12, and construct a quiver representation M' by substituting each base set on the vertices by $[n_i]$. For each matrix, we append an appropriate amount of zero rows or columns. This way, $\mathrm{QGr}(M', \mathbf{d}; \mathbf{n})$ has the same Plücker relations as $\mathrm{QGr}(M, \mathbf{d}; \mathbf{4})$. Since

$$\dim(\overline{\mathrm{trop}}(\mathrm{QGr}(M, \mathbf{d}; \mathbf{4})) \ < \ \dim(\mathrm{QDr}(M, \mathbf{d}; \mathbf{4})),$$

we obtain that

$$\dim(\overline{\mathrm{trop}}(\mathrm{QGr}(M', \mathbf{d}; \mathbf{n}))) \ < \ \dim(\mathrm{QDr}(M', \mathbf{d}; \mathbf{n}))$$

Hence,

$$\overline{\mathrm{trop}}(\mathrm{QGr}(M', \mathbf{d}; \mathbf{n})) \ \subsetneq \ \mathrm{QDr}(M', \mathbf{d}; \mathbf{n}).$$

□

This concludes the proof of Theorem 9.8 (and hence the proof of Theorem F).

9.3.2. *An example with trivial valuation.* We construct a quiver and quiver representation such that the quiver Grassmannian is strictly contained in the quiver Dressian for $n = 3$ over a field with trivial valuation.

We obtain a nonrealizable quiver representation using the nonrealizable Q-representation R for $n = 4$, from [82, Example 4.4], as follows. Inspired by the fact that a matroid over [4] can be written as the matroid quotient of two matroids over [3], we replace each vertex V_i of Q with a new quiver, as depicted in Figure 11, obtaining the quiver Q'.

Now, Q' is given as four copies of the quiver (a) in Figure 11, which we call *layers* $L_1, \ldots L_4$. Each vertex in L_i is connected with an arrow to the vertex in the same position in L_{i+1} (modulo 4).

We define the Q'-representation D as follows. We assign to each vertex of Q' the vector space $\mathbb{C}^3$, and to each arrow inside a layer (i.e., all the arrows in Figure 11(a)) the identity map. To each arrow connecting two layers L_i and L_{i+1} (i.e., the vertical arrows in Figure 11(b) and (c)), we assign a rank 2 projection such that the composition with the incoming map connecting L_{i-1} and L_i has rank 1. Denoting by $V_{i,j}$ the red vertices in 11(b) and by $W_{i,j}$ the blue ones, we assign the same linear map to the arrows $\alpha : V_{i,j} \to V_{i,j+1}$ and $\alpha : W_{i,j} \to W_{i,j+1}$ that correspond to moving up layers.

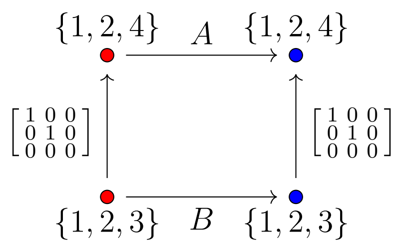
Finally, we consider a dimension vector $\mathbf{d}$ whose entries are in $\{1, 2\}$, corresponding to the vertices depicted in Figure 11(a) — red and black vertices are assigned dimension 1, blue vertices are assigned dimension 2.

We now show that the quiver Grassmannian $\mathrm{QGr}(D, \mathbf{d}; \mathbf{3})$ has the same Plücker relations as the quiver Grassmannian in [82, Example 4.4].

Each arrow in a layer corresponds to a flag of dimension $(1, 2)$ and thus produces exactly one three-term Plücker relation, of the form $p_{v,1}p_{w,23} - p_{v,2}p_{w,13} + p_{v,3}p_{w,12}$. For each red or black vertex (corresponding to a 1-dimensional subspace), the arrow connecting the layers leaves two Plücker variables p_i and p_j unchanged, and replaces the third variable p_k by a new variable p_l. For each blue vertex (corresponding to a 2-dimensional subspace), the arrow fixes the Plücker variable p_{ij} and replaces the variables p_{ik} and p_{jk} with p_{il} and p_{jl} respectively.

Moving up along the four layers, since the layers form a cycle, the only four variables arising from subspaces of dimension 1 are exactly p_1, p_2, p_3 and p_4. For the rightmost vertex in Figure 11 (a), as $\mathrm{Gr}(1, 4) \cong \mathrm{Gr}(3, 4)$, we can rename the four variables $p_{123}, p_{124}, p_{134}$ and p_{234}.

The six variables arising from subspaces of dimension 2 corresponding to the blue vertices are $p_{12}, p_{13}, p_{14}, p_{23}, p_{24}$ and p_{34}. Since the layer maps for the vertices $V_{i,j}$ and $W_{i,j}$ corresponding to the red and blue vertices are equal, we can interpret the coordinates associated to the red vertices exactly as the complementary ones associated to the blue vertices on the same layer. For instance, consider the following subgraph of Figure 11(b), as depicted in (d).



(d)

The identity map A induces the quiver Plücker relation $p_1p_{23} - p_2p_{13} + p_3p_{12}$. Now, we rename the Plücker variables p_i to p_{i4} for all $i \in \{1, 2, 3\}$ and obtain the relation $p_{14}p_{23} - p_{24}p_{13} + p_{34}p_{12}$. Analogously, for the identity map B we have the relation $p_1p_{24} - p_2p_{14} + p_4p_{12}$, and we replace the variables p_i by p_{i3} for all $i \in \{1, 2, 4\}$, yielding again the relation $p_{14}p_{23} - p_{24}p_{13} + p_{34}p_{12}$, i.e., both maps A and B induce the same Plücker relation.

Now, by the construction of Q', we obtain exactly the same Plücker relations as in [82, Example 4.4]. Thus, the two quiver representations with the respective dimension vectors produce the same tropicalized quiver Grassmannian and quiver Dressian, and hence $\overline{\mathrm{trop}}(\mathrm{QGr}(D, \mathbf{d}; \mathbf{3}) \neq \mathrm{QDr}(D, \mathbf{d}; \mathbf{3})$.

9.3.3. *Realizability of valuated matroidal quiver subrepresentations.* In this section, we discuss realizability of valuated matroidal quiver subrepresentations and work towards a more general result on realizability. This is recent work-in-progress.

In Example 9.10, we observed that even for relatively simple quivers we can construct quiver representations such that the tropicalizations of the associated quiver Grassmannians do not coincide with their associated quiver Dressians. In light of this, we conjecture that a guaranteed equality of the two tropical parameter spaces requires an even more specific structure:

Conjecture 9.1. *Let Q be a quiver, and let K be a field with non-trivial valuation. Assume that* $\mathrm{QDr}(R, \mathbf{d}; \mathbf{n}) = \overline{\mathrm{trop}}(\mathrm{QGr}(R, \mathbf{d}; \mathbf{n}))$ *for all Q-representations R over K. Then, either*

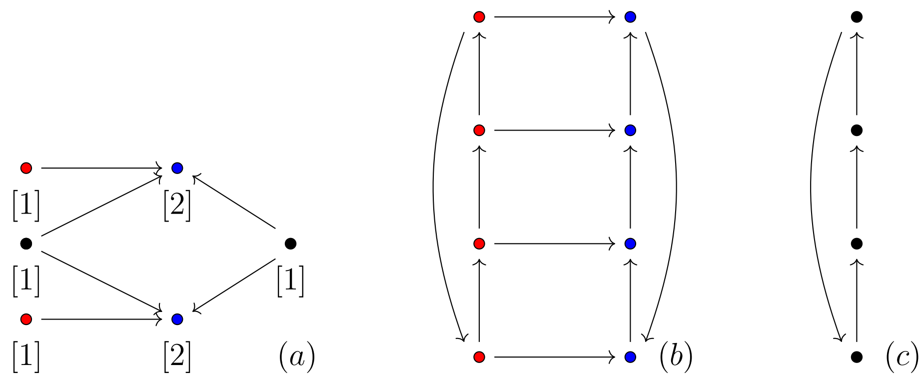


Figure 11. Replacement of vertices. (a) depicts one layer of the resulting quiver Q', (b) depicts the quiver replacing the middle vertices in [82, Example 4.4], i.e., the vertices which were assigned subspaces of dimension 2, and (c) depicts the replacement of the other vertices.

$\mathbf{n}$ *only has entries* $n_i \leq 2$ *or* Q *is of finite orbit type, i.e., an equioriented Dynkin quiver of type* A, D *or* E *we had listed in Example 7.7.*

This can be understood as a tropical version of Gabriel's theorem. We can verify that there are quivers Q satisfying $\mathrm{QDr}(R, \mathbf{d}; \mathbf{n}) = \overline{\mathrm{trop}}(\mathrm{QGr}(R, \mathbf{d}; \mathbf{n}))$ for all Q-representations R over K. For instance, in Section 10 we will study different quiver representations of the A_3-quiver over $\mathbb{C}^4$ and see that the equality holds for all of them.

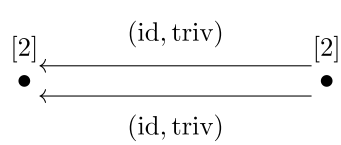
Further, results by Rincón [123, Theorem 4.6] suggest that such an equality could be true for Dynkin quivers of type D, whereas Balla-Olarte [12, Example A] show that for the symplectic case, associated to Dynkin quivers of type C, this is likely impossible. The Dynkin quivers of types C and D are depicted in Example 7.7.

Next, we discuss the topic of realizability with a view towards valuated matroidal quiver subrepresentations, which we introduced in Section 8. For this, we need the following standard concept in tropical geometry.

Definition 9.14. Let $A \in \mathbb{T}^{n \times m}$ be a matrix. Let K be a field with valuation. We say that $B \in K^{n \times m}$ is a *lifting* of A if $\mathrm{val}(B) = A$.

While each matrix over K has a unique valuation, a tropical matrix can potentially have an arbitrary number of liftings over K, depending on the field K and its valuation. We observe the impact of the choice of lifting on whether the tropicalized quiver Grassmannian and the quiver Dressian coincide, as follows.

Remark 9.15. In Example 9.10 we had considered the associated quiver Dressian and an associated tropicalized quiver Grassmannian of the valuated matroidal quiver representation associated to the Kronecker quiver we had discussed in Example 8.4(d), depicted below.



We observe that the quiver Dressian associated to this valuated matroidal quiver representation is unique after fixing the dimension vector and determined by the tropical matrix

$$\operatorname{val}(I_2) \;=\; \begin{bmatrix} 0 & \infty \\ \infty & 0 \end{bmatrix} \in \mathbb{T}^{2\times 2}.$$

However, there are infinitely many different possible liftings of the matrix $\operatorname{val}(I_2)$ to $\mathbb{C}\{\{t\}\}^{2\times 2}$, i.e., there are infinitely many matrices $A \in \mathbb{C}\{\{t\}\}^{2\times 2}$ such that $\operatorname{val}(A) = \operatorname{val}(I_2)$.

In Example 9.10, the liftings $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1+t \end{smallmatrix}\right]$ yielded a tropicalized quiver Grassmannian and a quiver Dressian which did not coincide. If we choose the liftings $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ for both arrows instead, the associated tropicalized quiver Grassmannian and the quiver Dressian coincide.

Observation 9.16. From this example and the definition of the quiver Dressian (Definition 9.5), we observe that the tropicalized quiver Grassmannian is dependent on the specific choice of matrices assigned to the arrows, whereas the quiver Dressian only depends on the valuation of the matrix.

For valuated matroidal quiver representations, this means that there exists a unique associated quiver Dressian, but a priori many different associated tropicalizations of quiver Grassmannians, varying by the chosen lifting of the associated matrix to the field K.

This motivates the following definition.

Definition 9.17. Let Q be a quiver and R a valuated matroidal quiver representation. We say that R is *realizable* over K if there exist liftings $(A_K^\alpha)_{\alpha\in A}$ for all associated matrices to arrows $A^\alpha_{\alpha\in A}$ such that

$$\operatorname{QDr}(((K^{n_i})_{i\in V}, (A_K^\alpha)_{\alpha\in A}), \mathbf{d}; \mathbf{n}) = \overline{\operatorname{trop}}(\operatorname{QGr}(((K^{n_i})_{i\in V}, (A_K^\alpha)_{\alpha\in A}), \mathbf{d}; \mathbf{n})).$$

Further, we say that R is *fully realizable (over K)*, if the equality holds for every lifting.

Example 9.18. We have already seen examples for fully realizable, realizable, and non-realizable valuated matroidal quiver representations:

- The valuated A_3-quiver representation associated to the flag Dressian (Example 8.4(b)) is fully realizable over $\mathbb{C}$, as is the valuated A_3-quiver subrepresentation with the projection map we discuss in the same example.
- The valuated Kronecker quiver representation introduced in Example 8.4 and discussed above is fully realizable over $\mathbb{C}$, but not over $\mathbb{C}\{\{t\}\}$ by Example 9.10. However, it is still realizable over $\mathbb{C}\{\{t\}\}$.
- The valuated matroidal quiver representation associated to the diamond shaped quiver discussed in Example 8.4 is not realizable over $\mathbb{C}$ by Example 9.12.

10. Applications of quiver Dressians

In this section, we will apply the main theory developed in the last sections to a series of specific examples which are of particular importance in tropical geometry. We study linear degenerate flag varieties, their potential for speeding up computations of flag varieties, and their relations to the relative realizability problem.

Quiver Dressians allow us to construct degenerations of tropical varieties in a controlled fashion. If we know the quiver representation of an algebraic variety, we can degenerate the linear maps on the quiver to obtain smaller, more computable degenerations. One particular example of this is flag varieties and their linear degenerations. This subsection is based on joint work with Alessio Borzì, [27].

While we studied quiver representations with arbitrary dimensions of the vector spaces on the vertices, we will now restrict to quiver representations where all vector spaces have the same dimension n.

In Example 7.12, we had seen that the flag variety can be represented as a quiver Grassmannian in a natural way. In this section, we discuss how different quiver representations associated to the same quiver (but with different maps on the arrows) behave. We focus on the quiver Grassmannians we can obtain out of the flag variety. A *linear degenerate flag variety* is a quiver Grassmannian associated to a quiver of type A:

$$K^n \xrightarrow{f_1} K^n \xrightarrow{f_2} \dots \xrightarrow{f_{n-2}} K^n \xrightarrow{f_{n-1}} K^n$$

We will give a proper definition using the appropriate Plücker relations in Definition 10.3

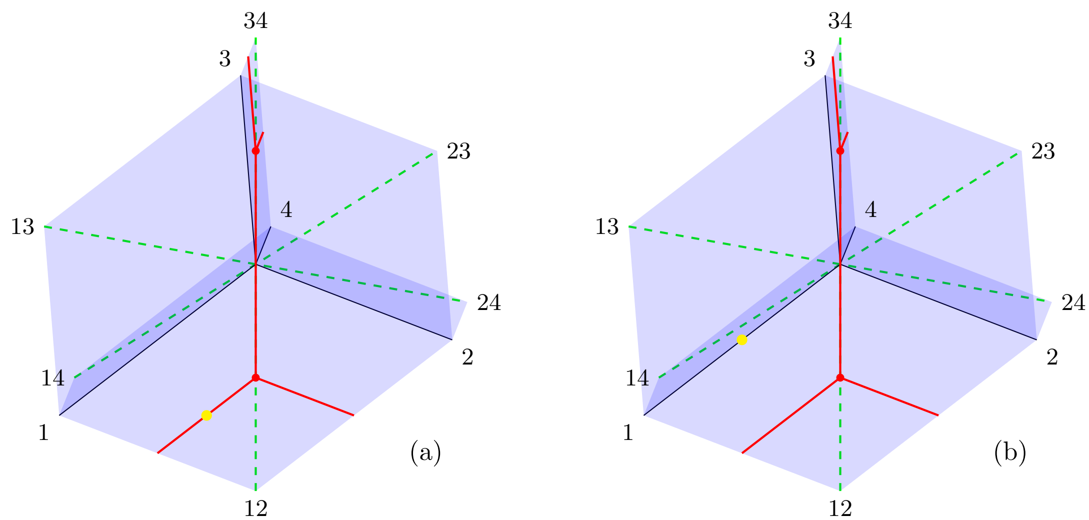


FIGURE 12. (a): A tropical flag in $\operatorname{trop}\big(\operatorname{Fl}((1,2,3);4)\big)$. (b): A tropical linear degenerate flag in $\operatorname{trop}\big(\operatorname{LFl}\big((1,2,3),(\{1\},\emptyset);4\big)\big)$. Both are made of a yellow point, a red tropical line, and a blue tropical plane. The additional subdivision given by the green dashed rays on the tropical plane is useful for describing the (linear degenerate) tropical flag varieties, see Examples 10.8 and 10.9.

It is possible to check that any linear degenerate flag variety $\operatorname{Fl}(\mathbf{r};n)(V)$ can be represented by a fiber of a sequence of projections (see, for instance, [62, Lemma 2.6]). Therefore, in order to study linear degenerate flag varieties, we can always restrict to the case where all linear maps $(f_1,\dots,f_{n-1})$ are projections $\operatorname{pr}_S : V \to V$ for some $S \subseteq [n]$, where pr_S is the linear map that sets the coordinates indexed by $S \subseteq [n]$ to zero. Here, we use the term *linear degenerate flag* to describe a sequence of K-vector spaces $(U_1,\dots,U_n)$ together with a sequence of projections $(\operatorname{pr}_{S_1},\dots,\operatorname{pr}_{S_n-1})$ such that $\operatorname{pr}_{S_i}(U_i) \subseteq U_{i+1}$ for all $1 \le i \le n-1$, that is, we consider A-type quivers decorated with projection maps.

The linear degenerate flag varieties are given by the linear degenerate Plücker relations.

Definition 10.1 (Linear degenerate Plücker relations)**.** Let $r \leq s \leq n$ be nonnegative integers and let $S \subseteq [n]$. The *linear degenerate Plücker relations* are the polynomials in the variables $\{p_I : I \in \binom{[n]}{r}\} \cup \{p_J : J \in \binom{[n]}{s}\}$ with coefficients in K:

$$\mathscr{P}_{r,s;S;n} = \left\{ \sum_{j \in J \setminus (I \cup S)} \operatorname{sign}(j; I, J) p_{I \cup j} p_{J \setminus j} : I \in \binom{[n]}{r-1}, J \in \binom{[n]}{s+1} \right\}$$

where $\operatorname{sign}(j; I, J) = (-1)^{\#\{j' \in J : j < j'\} + \#\{i \in I : i > j\}}$. Their tropicalization will be denoted by $\mathscr{P}^{\text{trop}}_{r,s;S;n}$.

While the linear degenerate Plücker relation can be derived from the quiver Plücker relations we investigated in Definition 7.13, we now give a proof that is based on linear algebra.

Proposition 10.2. *Let U and V be vector subspaces of K^n of dimension $r \leq s$ respectively, and let $S \subseteq [n]$. We have $\operatorname{pr}_S(U) \subseteq V$ if and only if the Plücker coordinates of U and V satisfy the linear degenerate Plücker relations $\mathscr{P}_{r,s;S;n}$.*

Proof. Suppose that $\operatorname{pr}_S(U) \subseteq V$. Let $A \in K^{r,n}$ be a matrix whose rows are a basis of U, and let $A' \in K^{r,n}$ be the matrix obtained from A by substituting the columns indexed by S with columns of zeros. Note that the rows of A' are a set of generators for $\operatorname{pr}_S(U)$. Let $B \in K^{s,n}$ be a matrix whose rows are a basis of V, obtained by extending a basis of $\operatorname{pr}_S(U)$ consisting of rows of A'. Fix $I = \{i_1 < \cdots < i_{r-1}\} \in \binom{[n]}{r-1}$ and $J = \{j_1 < \cdots < j_{s+1}\} \in \binom{[n]}{s+1}$. The column vectors $B_{j_1}, \ldots, B_{j_{s+1}}$ are linearly dependent, satisfying the dependency relation

$$\sum_{k=1}^{s+1} (-1)^k \det(B_{j_1}, \ldots, B_{j_{k-1}}, B_{j_{k+1}}, \ldots, B_{j_{s+1}}) \cdot B_{j_k} = 0.$$

In particular, from the construction of A' and B we also obtain

$$\sum_{k=1}^{s+1} (-1)^k \det(B_{j_1}, \ldots, B_{j_{k-1}}, B_{j_{k+1}}, \ldots, B_{j_{s+1}}) \cdot A'_{j_k} = 0.$$

Substituting the above expression of the (r-dimensional) zero vector into the equation $\det(0, A_{i_1}, \ldots, A_{i_{r-1}}) = 0$ we obtain

$$\begin{aligned} &\det\left(\sum_{k=1}^{s+1} (-1)^k \det(B_{j_1}, \ldots, B_{j_{k-1}}, B_{j_{k+1}}, \ldots, B_{j_{s+1}}) \cdot A'_{j_k}, A_{i_1}, \ldots, A_{i_{r-1}} \right) \\ = &\sum_{k=1}^{s+1} (-1)^k \det(B_{j_1}, \ldots, B_{j_{k-1}}, B_{j_{k+1}}, \ldots, B_{j_{s+1}}) \cdot \det(A'_{j_k}, A_{i_1}, \ldots, A_{i_{r-1}}) = 0. \end{aligned}$$

Now by construction we have

$$A'_{j_k} = \begin{cases} A_{j_k} & \text{if } j_k \notin S, \\ 0 & \text{if } j_k \in S. \end{cases}$$

Thus, by substituting the above in the previously displayed equation, and by reordering the column vectors keeping track of sign changes, the above relations become the desired linear degenerate Plücker relations, up to a possible (global) change of sign that depends on r and s.

Conversely, suppose that the Plücker coordinates of U and V satisfy the linear degenerate Plücker relations. Let A and A' be as above, and let $B \in K^{s,n}$ be a matrix whose rows are a basis of V. We need to show that the rows of A' are spanned by the rows of B. Let $I = \{i_1 < \cdots < i_{r-1}\} \in \binom{[n]}{r-1}$ and $J = \{j_1 < \cdots < j_{s+1}\} \in \binom{[n]}{s+1}$. Proceeding similarly as above, from the incidence Plücker relations we can write

$$\det\left(\sum_{k=1}^{s+1}(-1)^k \det(B_{j_1},\dots,B_{j_{k-1}},B_{j_{k+1}},\dots,B_{j_{s+1}})\cdot A'_{j_k}, A_{i_1},\dots,A_{i_{r-1}}\right) \;=\; 0. \tag{9}$$

Since A has maximal rank, we can choose a subset $I' \in \binom{[n]}{r}$ such that the columns of A indexed by I' form a basis. By choosing all possible cardinality $r-1$ subsets $I \subseteq I'$ in (9), we have that the first vector in the argument of the determinant in (9) is in the span of the spaces generated by the vectors indexed by all such sets I. This is possible only for the zero vector. Therefore, we obtain

$$\sum_{k=1}^{s+1}(-1)^k \det(B_{j_1},\dots,B_{j_{k-1}},B_{j_{k+1}},\dots,B_{j_{s+1}})\cdot A'_{j_k} \;=\; 0.$$

Let C be the matrix consisting of the rows of B plus an additional row of A'. By using the Laplace expansion for computing the determinant of the square submatrix of C with columns indexed by J with respect to the row of A', the above dependencies imply that the rank of C is equal to the rank of B, i.e., that the row of A' in C is a linear combination of the rows of B. □

Definition 10.3 (Linear degenerate flag variety)**.** The *linear degenerate flag variety* of rank $\mathbf{r}$ and degeneration type $\mathbf{S}$ is the subvariety of $\mathbb{P}^{\binom{n}{r_1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{r_k}-1}$ cut out by the linear degenerate Plücker equations,

$$\mathrm{LFl}(\mathbf{r},\mathbf{S};n) \;=\; V\big(\{\mathscr{P}_{r_i;n}\}_{1\le i\le k} \cup \{\mathscr{P}_{r_i,r_j;S_{ij};n}\}_{1\le i<j\le k}\big).$$

We call its tropicalization the *linear degenerate tropical flag variety.*

Example 10.4. In this example, we describe the tropicalization of the linear degenerate flag variety $\mathrm{LFl}((1,2),\{1\};4)$, parametrizing tuples (v,L) of points v and lines L in $\mathbb{P}^3$ such that $\mathrm{pr}_1(v) \subseteq L$. By definition $\mathrm{LFl}((1,2),\{1\};4) = V(\mathscr{P}_{2;4} \cup \mathscr{P}_{1,2;\{1\};4})$, since $\mathscr{P}_{1;4}$ contains just the zero polynomial. We verify computationally that the tropicalizations of the linear degenerate Plücker relations given below form a tropical basis, i.e., generate $\mathrm{trop}(\mathrm{LFl}((1,2),\{1\};4))$.

$$\mathscr{P}^{\mathrm{trop}}_{2;4} = \{p_{1,4}p_{2,3} \oplus p_{1,3}p_{2,4} \oplus p_{1,2}p_{3,4}\},$$

$$\mathscr{P}^{\mathrm{trop}}_{1,2;\{1\};4} = \left\{\begin{array}{l} p_3p_{1,2} \oplus p_2p_{1,3},\\ p_4p_{1,2} \oplus p_2p_{1,4},\\ p_4p_{1,3} \oplus p_3p_{1,4},\\ p_4p_{2,3} \oplus p_3p_{2,4} \oplus p_2p_{3,4}. \end{array}\right\}$$

Note that the polynomials in $\mathscr{P}^{\mathrm{trop}}_{1,2;\{1\};4}$ are obtained from those in $\mathscr{P}^{\mathrm{trop}}_{1,2;4}$ by deleting all monomials containing p_1 (compare with Example 3.97). The tropicalization of the linear degenerate flag variety $\mathrm{trop}\big(\mathrm{LFl}((1,2),\{1\};4)\big)$ can be computed in Macaulay2 [71] and is a 5-dimensional simplicial fan in $\mathbb{P}(\mathbb{T}^4) \times \mathbb{P}(\mathbb{T}^6)$. Its lineality space has dimension 4 and the quotient of the variety by the lineality space has f-vector $(1,3)$. A point p in the tropical linear degenerate flag variety $\operatorname{trop}\big(\mathrm{LFl}((1,2),\{1\};4)\big)$ corresponds to an arrangement of a point $v_p = (v_1 : v_2 : v_3 : v_4)$ and a general tropical line L_p in $\mathbb{P}(\mathbb{T}^4)$ such that $\mathrm{pr}_{e_1}(v_p) \in L_p$. This last condition implies that L_p contains the point $(\infty : v_2 : v_3 : v_4)$. This occurs only if L_p has an unbounded edge l_1 in (projective) coordinate direction e_1 at $(x : v_2 : v_3 : v_4)$ for some $x \in \mathbb{R}$. The direction of the unbounded edge adjacent to l_1 can be chosen to be any of (projective) coordinate directions e_2, e_3 and e_4. The balancing condition fixes the directions of the remaining unbounded edges, as in Example 3.97. The three choices for the direction vector correspond to the three maximal cones.

Since linear degenerate flag varieties are a particular subtype of quiver Grassmannians, we can analogously define the linear degenerate flag Dressian.

Definition 10.5 (Linear degenerate flag Dressian)**.** The *linear degenerate flag Dressian* of rank $\mathbf{r}$ and degeneration type $\mathbf{S}$ is the tropical prevariety in $\mathbb{T}(\mathbb{P}^{\binom{n}{r_1}}) \times \cdots \times \mathbb{T}(\mathbb{P}^{\binom{n}{r_k}})$ cut out by the tropicalizations of the linear degenerate Plücker equations,

$$\mathrm{LFlDr}(\mathbf{r},\mathbf{S};n) \;=\; V\big(\{\overline{\operatorname{trop}}(\mathscr{P}_{r_i;n})\}_{1\le i\le k} \cup \{\overline{\operatorname{trop}}(\mathscr{P}_{r_i,r_j;S_{ij};n})\}_{1\le i<j\le k}\big).$$

Further, the properties of quiver Dressians we had shown in Section 9.2 also hold for linear degenerate flag varieties:

Corollary 10.6 ([27, Theorem A])**.** *Let $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_k)$ be a sequence of valuated matroids. The following statements are equivalent:*

(a) $\boldsymbol{\mu} \in \mathrm{LFlDr}(\mathbf{r},\mathbf{S};n)$;
(b) $\mathrm{pr}^{\mathrm{trop}}_{S_i}\big(\overline{\operatorname{trop}}(\mu_i)\big) \subseteq \overline{\operatorname{trop}}(\mu_{i+1})$ for all $i \in \{1,\ldots,k-1\}$; and
(c) every projection $\mathrm{pr}_{S_i} : \mu_{i+1} \to \mu_i$ is a morphism of valuated matroids (Definition 4.3).

Proof. The equivalence $(b) \Leftrightarrow (c)$ is a direct consequence of Lemma 5.14 and Proposition 5.18. The equivalence $(a) \Leftrightarrow (c)$ then follows from Theorem 9.7. ☐

Linear degenerate flag varieties can be arranged in a poset in a natural way as follows. Fix $k, n \in \mathbb{N}$ and a sequence $\mathbf{r} = (r_1,\ldots,r_k)$ of nonnegative integers such that $r_1 \le \cdots \le r_k \le n$. Now, we consider the following set of linear degenerate flag varieties:

$$\mathcal{L} \;=\; \Big\{\mathrm{LFl}(\mathbf{r},\mathbf{S};n) : \mathbf{S} = (S_1,\ldots,S_{k-1}) \text{ with } S_i \subseteq [n]\Big\}.$$

We can define an order relation $\preceq$ on $\mathcal{L}$ given by $\mathrm{LFl}(\mathbf{r},\mathbf{S};n) \preceq \mathrm{LFl}(\mathbf{r},\mathbf{S}';n)$ if and only if $S_i \subseteq S'_i$ for every $i \in \{1,\ldots,k\}$, where $\mathbf{S}' = (S'_1,\ldots,S'_{k-1})$. Note that $(\mathcal{L},\preceq)$ is a finite lattice isomorphic to the product of lattices $\prod_{i=1}^k 2^{[n]}$, where $2^{[n]}$ is the power set of $[n]$ ordered by set inclusion.

The maximum of $\mathcal{L}$ is the linear degenerate flag variety with $S_i = [n]$ for every i, in other words at each step of the flag we are projecting all the coordinates, so the linear spaces of the flag do not have any relation to each other, and the variety we obtain is just a product of Grassmannians:

$$\mathrm{LFl}(\mathbf{r},([n],\ldots,[n]);n) \;=\; G(r_1;n) \times \cdots \times G(r_k;n).$$

On the other hand, the minimum of $\mathcal{L}$ is the linear degenerate flag variety with $S_i = \emptyset$ for every i. Here, we are not degenerating the flag variety as at each step the projection is an identity map, thus all linear degenerate flags are flags:

$$\mathrm{LFl}(\mathbf{r}, (\emptyset, \ldots, \emptyset); n) \;=\; \mathrm{Fl}(\mathbf{r}; n).$$

Analogously, we can arrange linear degenerate tropical flag varieties, and linear degenerate flag Dressians in lattices isomorphic to $\prod_{i=1}^{k} 2^{[n]}$.

Example 10.7. We give the partial lattice of Plücker relations for $n = [4], r = 2$ and $s = 3$, degenerating only with respect to 0, 1 and 2.

$$\begin{gathered} P_{0,3}P_{1,2} \oplus P_{0,2}P_{1,3} \oplus P_{0,1}P_{2,3} \\ P_{0,3}P_{0,1,2} \oplus P_{0,2}P_{0,1,3} \oplus P_{0,1}P_{0,2,3} \\ P_{1,3}P_{0,1,2} \oplus P_{1,2}P_{0,1,3} \oplus P_{0,1}P_{1,2,3} \\ P_{2,3}P_{0,1,2} \oplus P_{1,2}P_{0,2,3} \oplus P_{0,2}P_{1,2,3} \\ P_{2,3}P_{0,1,3} \oplus P_{1,3}P_{0,2,3} \oplus P_{0,3}P_{1,2,3} \end{gathered}$$

$$\begin{gathered} P_{0,3}P_{1,2} \oplus P_{0,2}P_{1,3} \oplus P_{0,1}P_{2,3} \\ P_{0,3}P_{0,1,2} \oplus P_{0,2}P_{0,1,3} \oplus P_{0,1}P_{0,2,3} \\ P_{1,3}P_{0,1,2} \oplus P_{1,2}P_{0,1,3} \\ P_{2,3}P_{0,1,2} \oplus P_{1,2}P_{0,2,3} \\ P_{2,3}P_{0,1,3} \oplus P_{1,3}P_{0,2,3} \end{gathered}
\qquad
\begin{gathered} P_{0,3}P_{1,2} \oplus P_{0,2}P_{1,3} \oplus P_{0,1}P_{2,3} \\ P_{0,3}P_{0,1,2} \oplus P_{0,2}P_{0,1,3} \\ P_{1,3}P_{0,1,2} \oplus P_{1,2}P_{0,1,3} \oplus P_{0,1}P_{1,2,3} \\ P_{2,3}P_{0,1,2} \oplus P_{0,2}P_{1,2,3} \\ P_{2,3}P_{0,1,3} \oplus P_{0,3}P_{1,2,3} \end{gathered}
\qquad
\begin{gathered} P_{0,3}P_{1,2} \oplus P_{0,2}P_{1,3} \oplus P_{0,1}P_{2,3} \\ P_{0,3}P_{0,1,2} \oplus P_{0,1}P_{0,2,3} \\ P_{1,3}P_{0,1,2} \oplus P_{0,1}P_{1,2,3} \\ P_{2,3}P_{0,1,2} \oplus P_{1,2}P_{0,2,3} \oplus P_{0,2}P_{1,2,3} \\ P_{1,3}P_{0,2,3} \oplus P_{0,3}P_{1,2,3} \end{gathered}$$

$$\begin{gathered} P_{0,3}P_{1,2} \oplus P_{0,2}P_{1,3} \oplus P_{0,1}P_{2,3} \\ P_{0,3}P_{0,1,2} \oplus P_{0,2}P_{0,1,3} \\ P_{1,3}P_{0,1,2} \oplus P_{1,2}P_{0,1,3} \\ P_{2,3}P_{0,1,2} \\ P_{2,3}P_{0,1,3} \end{gathered}
\qquad
\begin{gathered} P_{0,3}P_{1,2} \oplus P_{0,2}P_{1,3} \oplus P_{0,1}P_{2,3} \\ P_{0,3}P_{0,1,2} \oplus P_{0,1}P_{0,2,3} \\ P_{1,3}P_{0,1,2} \\ P_{2,3}P_{0,1,2} \oplus P_{1,2}P_{0,2,3} \\ P_{1,3}P_{0,2,3} \end{gathered}
\qquad
\begin{gathered} P_{0,3}P_{1,2} \oplus P_{0,2}P_{1,3} \oplus P_{0,1}P_{2,3} \\ P_{0,3}P_{0,1,2} \\ P_{1,3}P_{0,1,2} \oplus P_{0,1}P_{1,2,3} \\ P_{2,3}P_{0,1,2} \oplus P_{0,2}P_{1,2,3} \\ P_{0,3}P_{1,2,3} \end{gathered}$$

$$\begin{gathered} P_{0,3}P_{1,2} \oplus P_{0,2}P_{1,3} \oplus P_{0,1}P_{2,3} \\ P_{0,3}P_{0,1,2} \\ P_{1,3}P_{0,1,2} \\ P_{2,3}P_{0,1,2} \end{gathered}$$

Linear degenerate tropical flag varieties with $n = 4$. Now, we want to take a closer look at the lattice of linear degenerate tropical flag varieties for the case $n = 4$.

We used Macaulay2 [71] to compute the linear degenerate Plücker relations, using code available in the GitHub repository [26]; and used the package `Tropical.m2` [6] to compute the respective linear degenerate tropical flag varieties. We did some additional computations in `gfan` [88] and Oscar [114].

For the rest of this section, we will consider varieties of *complete flags* in $\mathbb{C}^4$. More precisely, we fix $\mathbf{r} = (1, 2, 3)$, and omit $\mathbf{r}$ in our notation. For instance, we denote $\mathrm{Fl}(4) := \mathrm{Fl}(\mathbf{r}; 4)$ and $\mathrm{LFl}\big((\{1\}, \emptyset); 4\big) := \mathrm{LFl}\big(\mathbf{r}, (\{1\}, \emptyset); 4\big)$. To simplify the notation, we will use $\mathrm{LFl}(S_1, S_2; 4)$ in place of $\mathrm{LFl}((S_1, S_2); 4)$.

A point p in the tropicalization of (linear degenerate) flag varieties corresponds to an arrangement of a point v_p, a general tropical line L_p and a general tropical plane P_p in $\mathbb{P}(\mathbb{T}^4)$. As in previous examples, L_p has two vertices connected by a bounded edge, and four unbounded edges in the coordinate directions, two adjacent to each vertex. A general tropical plane in $\mathbb{P}(\mathbb{T}^4)$ consists of a vertex, four adjacent rays in all four coordinate directions, and all two-dimensional cones spanned by pairs of these rays (see Figure 12).

Example 10.8. We begin by analyzing the tropicalization $\mathrm{trop}(\mathrm{Fl}(4))$ of the complete flag variety $\mathrm{Fl}(4)$. By [30, Theorem 5.2.1], $\mathrm{trop}(\mathrm{Fl}(4)) = \mathrm{FlDr}(4)$, i.e., the Plücker relations form a tropical basis and all tropical flags of length 4 are realizable.

The tropical variety trop(Fl(4)) is a six-dimensional simplicial fan with lineality dimension three in $\mathbb{P}(\mathbb{T}^4)\times\mathbb{P}(\mathbb{T}^6)\times\mathbb{P}(\mathbb{T}^4)$. It has f-vector $(1, 20, 79, 78)$ after quotienting by the lineality space. This variety was computed in [28, Theorem 4] and a sketch of it was given in [87, Figure 9], which we report here in Figure 13.

A point p in trop(Fl(4)) corresponds to a complete tropical flag, i.e., $v_p \subseteq L_p \subseteq P_p \subseteq \mathbb{P}(\mathbb{T}^4)$. An example of a complete flag was given in Figure 12(a).

After quotienting by the lineality space, trop(Fl(4)) can be seen as a "tropical line bundle" over the Petersen graph (as explained in [87, Paragraph 3.3.3]). There are 15 combinatorially distinct ways to arrange a generic tropical line inside a fixed generic tropical plane in $\mathbb{P}(\mathbb{T}^4)$, corresponding to the edges of the Petersen graph in Figure 13. In fact, this is dual to a fixed point inside a generic tropical line in $\mathbb{P}(\mathbb{T}^4)$, see Example 3.97.

The three blue edges in Figure 13(a), connecting (ab) to (cd) for distinct $a, b, c, d \in [4]$ correspond to the case when one vertex of the tropical line L_p lies on the ray of the tropical plane P_p in direction $e_a + e_b$, and the other vertex of L_p lies on the ray of P_p in direction $e_c + e_d$. This situation is depicted in Figure 12(a), where the dashed green rays are those in direction $e_a + e_b$. Further, the twelve black edges of the Petersen graph in Figure 13(a) connecting (a) to (ab) for distinct $a, b \in [4]$, correspond to p where one vertex of L_p lies on the ray of P_p in direction e_a and the other vertex of L_p lies in the cone of P_p spanned by e_a and e_b. Finally, the remaining degrees of freedom for the position of the point p in the line bundle determine the position of the point v_p on L_p.

Note that both of the cases discussed above correspond to maximal cells: in the first case, both vertices of L_p vary in the one-dimensional space in direction $e_a + e_b = -(e_c + e_d)$, whereas in the second case, the vertex of L_p inside the span of e_a and e_b can be freely chosen, but then fixes the choice of the other vertex on e_a by the balancing condition. The polyhedral structure of the tropical line bundle differs accordingly, see Figure 13(b) for blue edges and Figure 13(b') for black edges respectively.

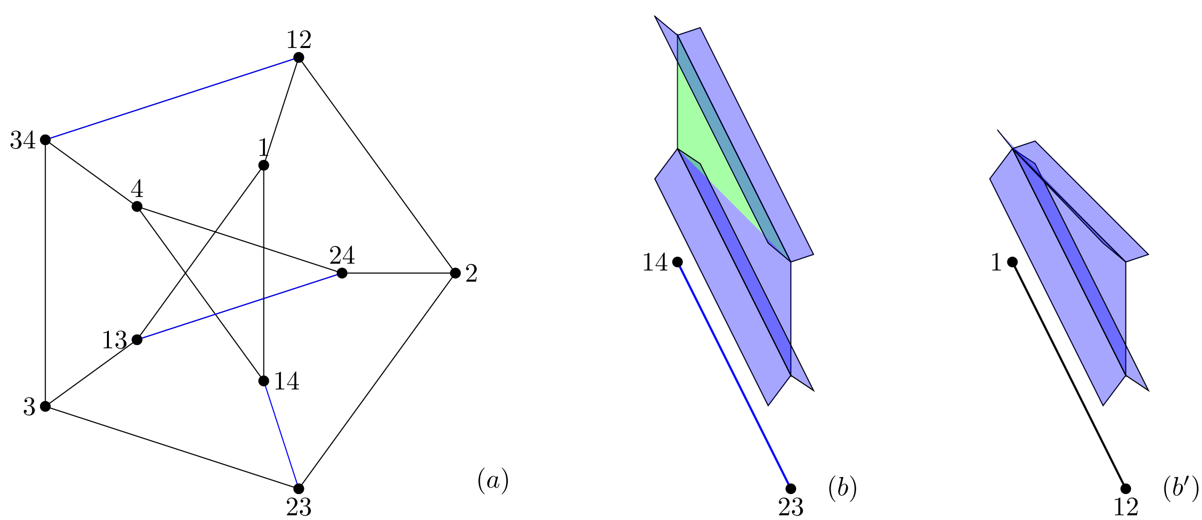


FIGURE 13. The tropical flag variety $\text{trop}(\text{Fl}_4)$ after quotienting by its lineality space, interpreted as a "tropical line bundle" over the Petersen graph, see also [87, Paragraph 3.3.3 and Figure 9].

Example 10.9. We continue by considering the two linear degenerate tropical flag varieties trop(LFl($\{1\}, \emptyset; 4$)) and trop(LFl($\emptyset, \{1\}; 4$)). The tropical variety trop(LFl($\{1\}, \emptyset; 4$)) parametrizes tropical flags such that $\mathrm{pr}_1(v_p) \subseteq L_p \subseteq P_p$. An example of a point in trop(LFl($\{1\}, \emptyset; 4$)) is depicted in Figure 12 (b).

The linear degenerate tropical flag variety trop(LFl($\{1\}, \emptyset; 4$)) is a six-dimensional simplicial fan in $\mathbb{P}(\mathbb{T}^4)\times\mathbb{P}(\mathbb{T}^6)\times\mathbb{P}(\mathbb{T}^4)$ with lineality dimension four. It has a familiar combinatorial structure: after quotienting by the lineality space, we obtain a fan over the Petersen graph. Again, the fifteen cones over the Petersen graph correspond to the fifteen combinatorially distinct ways of arranging a general line in a general plane (see Example 10.8). This time, the position of the point v_p imposes no additional combinatorial constraints, as by $\mathrm{pr}_1(v_p) \subseteq L_p$, v_p is contained in the span of the e_1-ray of L_p.

Comparing to Example 10.8, the situation is much easier: the fan is a "usual line bundle" (as opposed to a "tropical line bundle" in Example 10.8) over the Petersen graph. In Figure 14, we depict this degeneration.

We can understand trop(LFl($\emptyset, \{1\}; 4$)) similarly. The two different types of linear degenerate flags described above are dual to each other, in fact LFl($\{1\}, \emptyset; 4$) $\simeq$ LFl($\emptyset, \{1\}; 4$), and trop(LFl($\{1\}, \emptyset; 4$)) and trop(LFl($\emptyset, \{1\}; 4$)) have the same polyhedral structure with different lineality spaces.

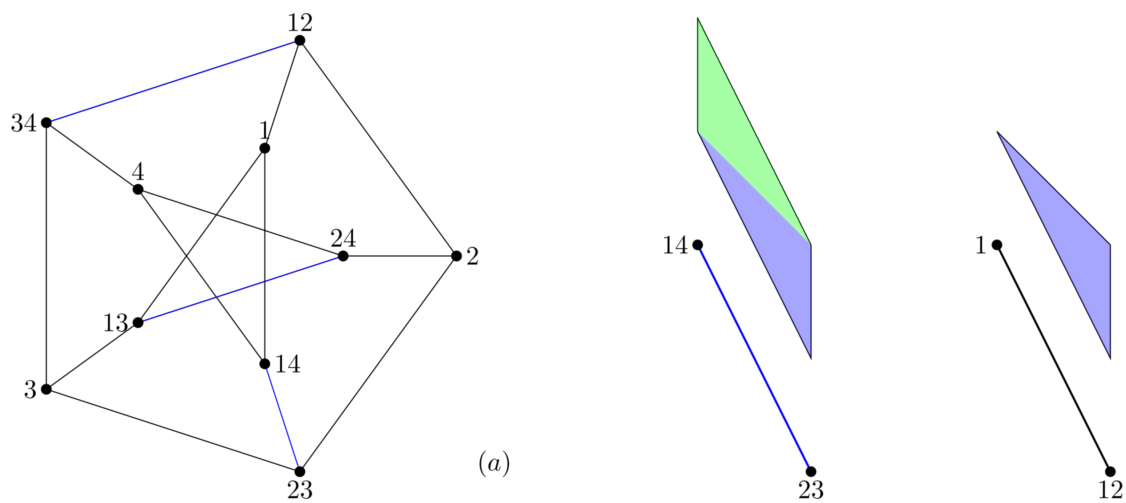


FIGURE 14. The linear degenerate tropical flag variety trop(LFl($\emptyset, \{1\}$); 4) can be interpreted as a "line bundle" over the Petersen graph.

Example 10.10. To finish, we include an additional degeneration step and study the linear degenerate tropical flag variety trop(LFl($\{1\}, \{1\}; 4$)). It is a six-dimensional simplicial fan with lineality dimension five in $\mathbb{P}(\mathbb{T}^4) \times \mathbb{P}(\mathbb{T}^6) \times \mathbb{P}(\mathbb{T}^4)$ and f-vector $(1, 3)$ after quotienting out the lineality space, i.e., a six-dimensional "line bundle" over a tropical line. We depict it in Figure 15. Here, maximal cells can be interpreted as follows. As before, the first linear degeneration condition implies that v_p lies on the span of the ray of L_p in direction e_1. The second degeneration implies that one vertex of the tropical line lies on the linear span of e_1 (and thus that the other vertex lies on one of the 3 coordinate half-planes spanned by e_1).

Note that this does not imply that L_p is contained in P_p. Thus, there are three maximal cells.

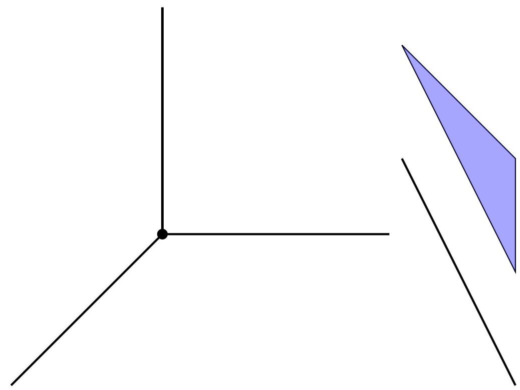


FIGURE 15. The linear degenerate tropical flag variety $\mathrm{trop}(\mathrm{LFl}(\{1\},\{1\});4)$ can be interpreted as a "line bundle" over the tropical line.

One possible application of the poset of linear degenerate tropical flag varieties would be to reduce the problem of computing a tropical flag variety to the problem of computing (a product of) tropical Grassmannians. Recall that the tropical flag variety is the minimum of the poset $\mathcal{L}$ of linear degenerate tropical flag varieties, while its maximum is a product of tropical Grassmannians. Therefore, one could try to start from the top of $\mathcal{L}$, and, by descending the poset $\mathcal{L}$ step by step, reconstruct the structure of the tropical flag variety. In order to do that, it would be enough to understand what happens at the *covers* of the poset $\mathcal{L}$, that is, to (fully or partially) reconstruct the structure of $\mathrm{trop}(\mathrm{LFl}(\mathbf{r},\mathbf{S};n))$ from another linear degenerate tropical flag variety $\mathrm{trop}(\mathrm{LFl}(\mathbf{r},\mathbf{S}';n))$ that covers it, i.e., $\mathbf{S}$ is obtained from $\mathbf{S}'$ by adding one element in one of the sets S_i.

Question 10.11. *Can we reconstruct the structure of* $\mathrm{trop}(\mathrm{LFl}(\mathbf{r},\mathbf{S};n))$ *from a cover?*

The examples we have seen above already provide some insight into what the answer is for complete flags with $n=4$. A common behavior that we observe is that the lineality space increases in dimension after each linear degeneration. In fact, this can be shown in more generality. The next result shows that for an ideal $I \subseteq k[x_0,\dots,x_n]$, the lineality space of a tropical variety $\mathrm{trop}(V(I))$ contains the homogeneity space of I, which is the linear subspace of vectors $v \in \mathbb{R}^{n+1}$ such that I is homogeneous with respect to the grading $\deg(x_i)=v_i$.

Lemma 10.12. *Let $I \subseteq k[x_0,\dots,x_n]$ be an ideal, where k is a field with the trivial valuation. Let $v=(v_0,\dots,v_n)\in\mathbb{R}^{n+1}$. If I is homogeneous with respect to the grading* $\deg(x_i)=v_i$ *then v is in the lineality space of* $\mathrm{trop}(V(I))$.

Proof. If I is homogeneous with respect to the grading $\deg(x_i)=v_i$, then $\mathrm{in}_v(f)=f$ for every $f\in I$. This implies that $\mathrm{in}_{v+w}(f)=\mathrm{in}_w(f)$, as for every monomial m of f, we are adding the same weight to the scalar product of the exponent vector of m and w. In particular, $\mathrm{in}_{w+v}(I)=\mathrm{in}_w(I)$ for every $w\in\mathbb{R}^{n+1}$. Hence, $w\in\mathrm{trop}(V(I))$ if and only if $w+v\in\mathrm{trop}(V(I))$, that is, v is in the lineality space of $\mathrm{trop}(V(I))$. □

Corollary 10.13. *The lineality space of a linear degenerate tropical flag variety is contained in the lineality space of every linear degenerate tropical flag variety in $\mathcal{L}$ that covers it.*

Proof. The claim follows from the structure of the linear degenerate Plücker relations. In fact, for a fixed grading of the Plücker variables, if the polynomials in $\mathscr{P}_{r,s;S;n}$ are homogeneous with respect to this grading, then polynomials in $\mathscr{P}_{r,s;S';n}$ are also homogeneous with respect to this grading for every $S' \supseteq S$, thus, by Lemma 10.12, the lineality space is contained. □

By looking at the previous examples, one might be tempted to conjecture that a cover relation on the poset implies set inclusion on the tropical varieties. In general, this is false, as the following example shows.

Example 10.14. In this example, we are going to show that

$$\operatorname{trop}\big(\operatorname{LFl}((1,2),\emptyset;4)\big) \nsubseteq \operatorname{trop}\big(\operatorname{LFl}((1,2),\{1\};4)\big).$$

We already described the above tropical varieties in Example 3.97 and Example 10.4. Now, assume that our base field K is the *field of Laurent series* $\mathbb{K}((t))$, that is the quotient field of the DVR $\mathbb{K}[[t]]$ of formal power series with coefficients in a field $\mathbb{K}$ in the variable t. Then, K has valuation $v : K \to \mathbb{T}$ where $v(f(t))$ is the minimum of the exponents appearing in f.

Now let $a, b \in \mathbb{Q}$ with $b > a > 0$, and consider the two matrices

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ t^a & 0 & t^b & 1 \end{pmatrix}.$$

Let $L_1, L_2 \subseteq \mathbb{K}^4$ be the two linear spaces generated by the rows of the matrices A_1 and A_2 respectively. By construction, $L_1 \subseteq L_2$, but $\operatorname{pr}_{\{1\}}(L_1) \nsubseteq L_2$. We can see this through the Plücker equations computed in Example 10.4 as follows. The valuations of the Plücker coordinates of L_1 and L_2 are

$$(p_1, p_2, p_3, p_4) = (0, 0, 0, 0),$$
$$(p_{1,2}, p_{1,3}, p_{1,4}, p_{2,3}, p_{2,4}, p_{3,4}) = (a, a, 0, b, 0, 0).$$

In particular, the minima in all the tropical polynomials in $\mathscr{P}^{\text{trop}}_{2;4}$ and $\mathscr{P}^{\text{trop}}_{1,2;4}$ are achieved at least twice, while the minimum in, for instance, the second tropical polynomial of $\mathscr{P}^{\text{trop}}_{1,2;\{1\};4}$ listed in Example 10.4, $p_4 p_{1,2} \oplus p_2 p_{1,4}$ is not achieved twice:

$$p_4 p_{1,2} = 0 \odot a = a > 0 = 0 \odot 0 = p_2 p_{1,4}.$$

While we do not obtain containment on tropical flag varieties or Dressians in the poset of linear degenerations, from the definition of the linear degenerate Plücker relations, we obtain the following containment on some boundary components.

Corollary 10.15. *Let* $\operatorname{LFlDr}(r, r', S \cup \{s\}, n) \prec \operatorname{LFlDr}(r, r', S, n)$ *be a cover in the poset of linear degenerate flag Dressians. Set*

$$\mathscr{B} = \left\{ (p_I) \in \mathbb{T}^{\binom{n}{r}} \times \mathbb{T}^{\binom{n}{r'}} : p_I = \infty \text{ for every } I \in \binom{[n]}{r} \text{ such that } s \in I \right\}.$$

Then we have

$$\operatorname{LFlDr}(r, r', S, n) \cap \mathscr{B} \subseteq \operatorname{LFlDr}(r, r', S \cup \{s\}, n).$$

Another interesting application of the poset of linear degenerate flag varieties concerns *relative realizability*. We say that two realizable tropical linear spaces $T_1 \subseteq T_2$ are *relatively realizable* if there exist realizations L_1 of T_1 and L_2 of T_2 such that $L_1 \subseteq L_2$. Let $\mathcal{L}$ be the poset of linear degenerate tropical flag varieties with flags of length 2 in $\mathbb{P}(\mathbb{T}^n)$ and rank vector

(r, s). Then, accurately describing the cover relations of $\mathcal{L}$ provides us a way to solve the relative realizability problem: the maximal element of $\mathcal{L}$ is $\operatorname{trop}(\operatorname{Gr}(r; n)) \times \operatorname{trop}(\operatorname{Gr}(s; n))$ in which we impose no conditions on either containment or relative realizability, whereas the minimal element $\operatorname{trop}(\operatorname{Fl}(r, s; n))$ of $\mathcal{L}$ does. Thus, if we could explicitly reconstruct $\operatorname{trop}(\operatorname{Fl}(r, s; n))$ from $\operatorname{trop}(\operatorname{Gr}(r; n)) \times \operatorname{trop}(\operatorname{Gr}(s; n))$, we would have an explicit solution to the relative realizability problem by tracking elements in the cover relations.

11. Quiver polytopes

We conclude this part of the thesis by providing an outlook for future research. Matroids and their valuated analogues can be described in terms of polyhedral geometry, as matroid polytopes and their subdivisions. In this section, we lay the foundation for an extension of this theory to the much more general setting of quiver polytopes and point configurations. This section is based on current (sole author) work-in-progress. Much of this section is based on computational experiments, using the code provided in Appendix B.1 and B.2 on suitable examples.

11.1. **Preliminaries: Polytopes in matroid theory and tropical geometry.** In Section 3.1, we have seen many different characterizations of (valuated) matroids, in terms of bases, independent sets, circuits, and more. In this section, we will add another characterization to this list — matroid polytopes.

Definition 11.1. Let M be a matroid over the ground set $[n]$ with a set of bases $\mathcal{B}(M)$. We define the *matroid polytope* $\mathscr{P}(M) = \operatorname{conv}(e_B \mid B \in \mathcal{B}(M))$.

We draw the base polytopes of $U_{1,3}$ and $U_{2,3}$ in Figure 16.

Matroid polytopes are *Coxeter polytopes* of type A, i.e., their edges are parallel to $e_i - e_j$ for $i, j \in [n]$. Further, by definition, their vertices only have entries in $\{0, 1\}$. In fact, every polytope satisfying these properties is a matroid polytope:

Proposition 11.2. *A polytope $P \subseteq \mathbb{R}^n$ is a matroid polytope if and only if*

(P1) all vertices of P have entries in $\{0, 1\}$ and
(P2) all edges of P are copies of $e_i - e_j$ for distinct i and j in $[n]$.

An analogous polyhedral characterization exist for flag matroids, see, for instance, [25].

Proposition-Definition 11.3. *A sequence of matroids $\mathbf{M} = M_1, \dots, M_k$ over $[n]$ of ranks $\mathbf{r} = r_1 < \cdots < r_k$ is a* flag matroid $M_1 \twoheadleftarrow M_2 \twoheadleftarrow \cdots \twoheadleftarrow M_k$ *if and only if*

$$\mathscr{P}_{\mathbf{M}} = \mathscr{P}_{M_1} + \cdots + \mathscr{P}_{M_k}$$

is a

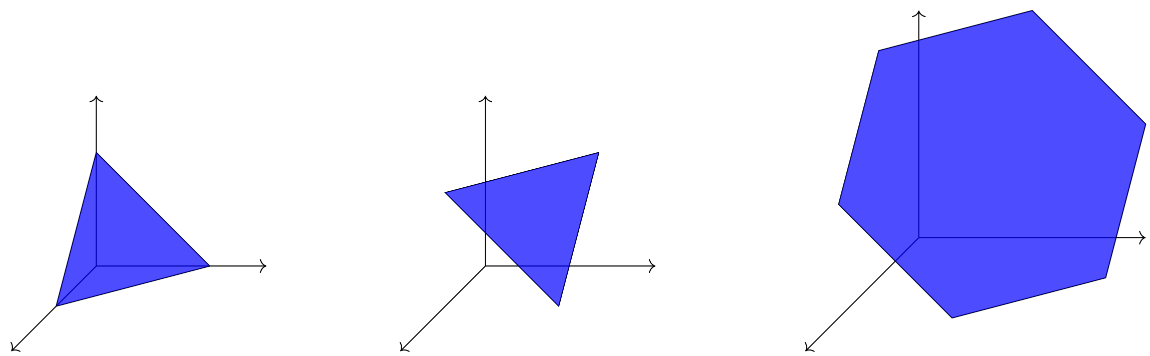
- *generalized permutahedron, such that*
- *all vertices of $\mathscr{P}_{\mathbf{M}}$ are in the S_n-orbit of*

$$e_{\mathbf{r}} = e_{[r_1]} + e_{[r_2]} + \cdots + e_{[r_k]}.$$

Example 11.4. Let us consider the uniform matroids $U_{1,3}$ and $U_{2,3}$. We depict their base polytopes in Figure 16 and construct their Minkowski sum, which is depicted in the same figure and is the polytope with vertices

$$(2, 1, 0),\ (2, 0, 1),\ (1, 2, 0),\ (0, 2, 1),\ (1, 0, 2) \text{ and } (0, 1, 2).$$

Using Proposition 11.3 we can now easily observe that the two matroids form a quotient.

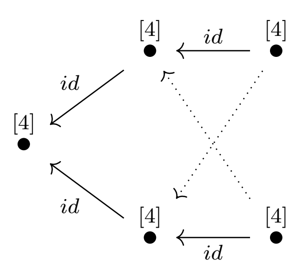


$\mathscr{P}_{U_{1,3}}$ $\mathscr{P}_{U_{2,3}}$ $\mathscr{P}_{(U_{1,3},U_{2,3})} = \mathscr{P}_{U_{1,3}} + \mathscr{P}_{U_{2,3}}$

FIGURE 16. The matroid polytopes of $U_{1,3}$ and $U_{2,3}$, and their Minkowski sum, which is the flag matroid polytope of the flag matroid $U_{1,3} \twoheadleftarrow U_{2,3}$.

There exists a characterization of valuated matroids via matroid polytopes, which is used in linear tropical geometry, see, for instance, [30]. Valuated matroids correspond to *matroid subdivisions* of matroid polytopes, and valuated flag matroids correspond to *flag matroid subdivisions*. However, in this section we will restrict ourselves to the non-valuated case.

11.2. **Case study: the multiflag quiver.** To start our study of quiver polytopes, we extend flags in the following way. We consider a "multiflag" quiver below, where all arrows are still identity maps, but we have a more complicated underlying undirected graph, which may contain undirected (but not directed) cycles. Here, the dotted arrows are arrows which might or might not be required to be identity maps in the quiver. Our idea is that we wish to distinguish the polyhedral structures associated to matroidal quiver subrepresentations which satisfy additional morphism constraints from those who do not, and record our findings below.



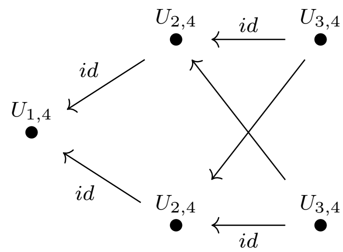
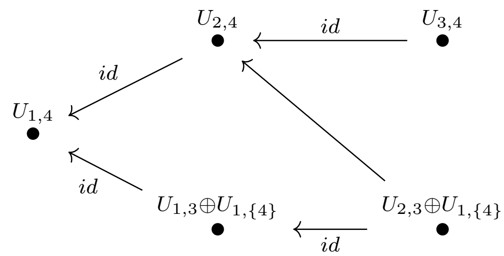
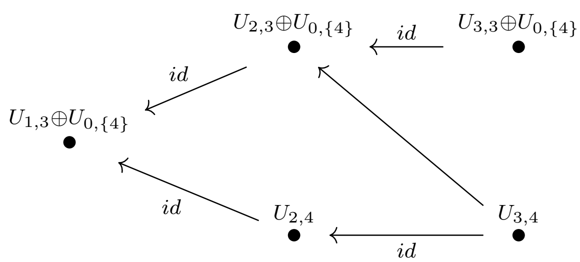
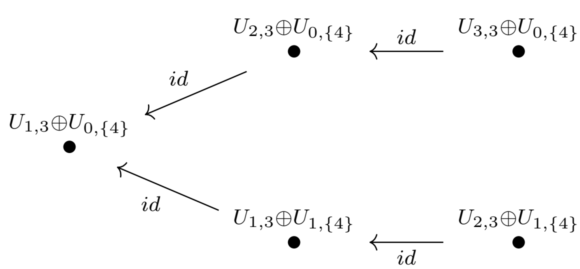
Analogously to the flag matroid case, we define the associated quiver polytope as the sum of the base polytopes of the matroids assigned to the vertices of the quiver.

Definition 11.5. Let Q be a quiver, R a Q-representation and $(M_i)_{i\in V}$ a matroidal quiver subrepresentation of R. Its *quiver point configuration* is the point configuration given as

$$\mathscr{P}_{\mathbf{M}} = \Big\{ \sum_{i\in V} w_i \mid w_i \text{ is a vertex of } \mathscr{P}_{M_i} \Big\}.$$

The convex hull of $\mathscr{P}_{\mathbf{M}}$ is the *quiver polytope*, and can be characterized as

$$\operatorname{conv}\big(\mathscr{P}_{\mathbf{M}}\big) = \sum_{i\in V} \mathscr{P}_{M_i}.$$

Matroidal quiver subrepresentation	**Lattice boundary points of quiver polytope**
	**Vertices:** S_4 orbit of $(5,4,2,0)$ **Other lattice points:** S_4 orbit of $(5,4,1,1)$ S_4 orbit of $(5,3,3,0)$ S_4 orbit of $(5,3,2,1)$ S_4 orbit of $(5,2,2,2)$ S_4 orbit of $(4,4,3,0)$
	**Vertices:** S_4 orbit of $(5,4,2,0)$ $S_{\{1,2,3\}}$ orbit of $(5,3,0,3)$ $S_{\{1,2,3\}}$ orbit of $(5,3,1,2)$ **Other lattice points:** $S_{\{1,2,3\}}$ orbit of $(5,1,1,4)$ $S_{\{1,2,3\}}$ orbit of $(4,1,1,5)$ remaining S_4 orbit of $(5,3,3,0)$ remaining S_4 orbit of $(5,3,2,1)$ S_4 orbit of $(5,2,2,2)$ $S_{\{1,2,3\}}$ orbit of $(4,4,0,3)$ $S_{\{1,2,3\}}$ orbit of $(4,4,0,3)$ $S_{\{1,2,3\}}$ orbit of $(4,3,2,2)$ $S_{\{1,2,3\}}$ orbit of $(4,4,1,2)$
	**Vertices:** $S_{\{1,2,3\}}$ orbit of $(5,4,2,0)$ $S_{\{1,2,3\}}$ orbit of $(5,4,1,1)$ $S_{\{1,2,3\}}$ orbit of $(5,3,1,2)$ **Other lattice points:** $S_{\{1,2,3\}}$ orbit of $(5,3,3,0)$ $S_{\{1,2,3\}}$ orbit of $(5,2,2,2)$ $S_{\{1,2,3\}}$ orbit of $(5,3,2,1)$ $S_{\{1,2,3\}}$ orbit of $(4,4,3,0)$ $S_{\{1,2,3\}}$ orbit of $(4,4,1,2)$ $S_{\{1,2,3\}}$ orbit of $(4,3,2,2)$
	**Vertices:** $S_{\{1,2,3\}}$ orbit of $(5,3,1,2)$ **Other lattice points:** $S_{\{1,2,3\}}$ orbit of $(5,2,2,2)$ $S_{\{1,2,3\}}$ orbit of $(4,4,1,2)$

The code used for generating this table is omitted for brevity as it is rather repetitive and straightforward, but can be found at `https://victoriaschleis.github.io/thesis.html`.

Observation 11.6. There are a few observations we can make on these examples.

(i) Every vertex has a unique coordinate entry 5, the orbit of which is determined by the symmetry group of the leftmost matroid.
(ii) The sum of the other vertex coordinates is always 6, and any 3-partition of 6 occurs, except $(2,2,2)$. However, all quiver polytopes above have a boundary point that is a permutation of $(5,2,2,2)$.
(iii) There are boundary lattice points in which 5 is not a coordinate entry. These can likely be disregarded for a polyhedral characterization of matroidal quiver representations as they do not correspond to a multiflag of bases of the assigned matroids, i.e., a collection $(B_i)_{i\in V}$ of bases $B_i \in \mathcal{B}(M_i)$ such that $B_{t(\alpha)} \subseteq B_{s(\alpha)}$.
(iv) The amount of different vertex orbits is bounded from above by the number of maximal paths in the quiver.

We summarize the observations of the examples considered above in the following characterization.

Conjecture 11.1. *Let $Q = (V, A, s, t)$ be a finite quiver without directed cycles, let R be a matroidal Q-representation, i.e., a Q-representation where all maps are given by weakly monomial matrices, and $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{|V|}$ be a dimension vector.*

A point configuration $\mathscr{P}$ is the quiver point configuration of a matroidal quiver subrepresentation of Q (c.f. Definition 11.5) if and only if the convex hull of $\mathscr{P}$ is a generalized permutohedron and any point $p \in \mathscr{P}$ is in the orbit of an element of the set

$$\{e_\alpha = \sum_{\alpha_i \in \alpha} f_{\alpha,1}(\sigma(e_{[n_{s(\alpha)}]})) \mid \alpha \text{ is a maximal path in } Q\}$$

under the action of a group $G_Q \leq S_{\max_{i\in V}([n_i])}$, and $\sigma_{s(\alpha)} \in S_{n_{s(\alpha)}}$. Here, f_α again denotes the map of valuated matroids induced by the matrix A^α.

11.3. **Case study: the loop quiver.** Now, we will consider what happens if a quiver contains loops. To this end, we study the loop quiver, which is the easiest such quiver, and allow for arbitrary maps. We observe that the situation is far more complicated than in the multiflag case.

We consider the matroidal quiver representation we had already seen in 8.4(c),

$$Q = \overset{[5]}{\bullet} \circlearrowleft \alpha \, ,$$

and will consider different maps α throughout this section.

First, let us choose α to be a permutation map $f : M \to M$. We observe that f is a morphism of matroids if and only if f maps bases to bases. Since f is a bijection, this means that $\mathscr{P}_{f^{-1}(M)} = \mathscr{P}_M$. In polyhedral terms, $\mathscr{P}_M$ is a matroid polytope associated to a matroidal quiver subrepresentation if and only if the coordinate permutation f corresponds to an element of the symmetry group of the matroid polytope.

Now, let us instead consider as α the projection map $f : \mathrm{pr}_5 : M \to M$. In Example 8.7 we have already computed all associated matroidal quiver subrepresentations, which were

(a) matroids in which 5 is a loop, and
(b) matroids in which 5 is a coloop and all other non-loop elements are parallel.

On the level of matroid polytopes, the corresponding matroids are the following

(a) all matroid polytopes in $\mathbb{R}^5$ that are contained in $\mathbb{R}^5|_{x_5=0}$; and
(b) all matroid polytopes in $\mathbb{R}^5$ that are contained in $\mathbb{R}^5|_{x_5=1}$.

Observation 11.7. Observe that for both types of matroid polytopes, the combinatorial structure is preserved under the coordinate projection pr_5 on $\mathbb{R}^5$. From these two examples, we might expect this to always be the case and conjecture that a map $f : M \to M$ is a strong map if and only if the corresponding map on $\mathbb{R}^n$ preserves the combinatorial structure of the matroid polytope, i.e., if the matroid polytope of the induced matroid $f^{-1}(M)$ is equal to the matroid polytope of M. We will now see that this is *not* the case.

To this end, we wish to study arbitrary maps on the loop quiver computationally. For computational expediency, we change matroidal quiver representations and consider the analogous representation, where we assign [4] to the vertex instead of [5].

Example 11.8. We consider the matroid $M = U_{2,3} \oplus U_{1,1}$, which has bases 14, 24 and 34. Its associated matroid polytope is the polytope

$$P_M = \operatorname{conv}\big((1,0,0,1),(0,1,0,1),(0,0,1,1)\big),$$

which is combinatorially equivalent to the triangle that is the base polytope of $U_{1,3}$ which we had seen on the left in Figure 16. We can construct it computationally, using standard `Oscar` methods and the additional code we implemented in Appendix B.1, and determine all strong maps of the matroid into itself.

```
julia> M = normalize_groundset(direct_sum(uniform_matroid(1,3),
↪  uniform_matroid(1,1)))
Matroid of rank 2 on 4 elements
julia> l = find_all_strong_maps(M, M)
julia> length(l)
145
```

Doing this, we observe that there are 145 distinct strong maps of matroids from M into itself. The full list can easily be generated, using the code above.

We now compute the matroid polytopes of all 145 induced matroids $f^{-1}(M)$. While there are 145 different strong maps, their induced matroids, and hence the associated matroid polytopes may coincide. In fact, there are only 5 different distinct associated matroid polytopes. We give them by their vertices below. We then compute how often they occur.

```
julia> p = [matroid_polytope(induced_matroid(add_loop(M),i)) for i in l];
julia> polytope_list = [(sum(vertices(i)[1]),vertices(i)) for i in p];
julia> u = unique(polytope_list)
5-element Vector{Tuple{QQFieldElem,
↪  SubObjectIterator{PointVector{QQFieldElem}}}}:
 (1, [[0, 0, 0, 0, 1], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 1, 0, 0, 0]])
 (2, [[0, 1, 0, 0, 1], [0, 1, 1, 0, 0], [0, 1, 0, 1, 0]])
 (1, [[0, 1, 0, 0, 0]])
 (1, [[0, 0, 0, 0, 1], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0]])
 (0, [[0, 0, 0, 0, 0]])
```

```
julia> occurrences = [length(findall(i->i==j,polytope_list)) for j in u]
5-element Vector{Int64}:
 82
 30
  4
 28
  1
```

Next, we compute an example morphism with respect to each possible occurring induced matroid polytope, where 5 denotes the loop o.

```
julia> ex_mo = [l[findfirst(i->i==j,polytope_list)] for j in u]
5-element Vector{Dict{Int64, Int64}}:
 Dict(5 => 5, 4 => 1, 2 => 1, 3 => 1, 1 => 1)
 Dict(5 => 5, 4 => 1, 2 => 4, 3 => 4, 1 => 4)
 Dict(5 => 5, 4 => 1, 2 => 5, 3 => 5, 1 => 5)
 Dict(5 => 5, 4 => 5, 2 => 1, 3 => 1, 1 => 1)
 Dict(5 => 5, 4 => 5, 2 => 5, 3 => 5, 1 => 5)
```

We stick with the loop quiver and perform the same computations for the matroids $U_{2,4}$ and $U_{3,4}$, and obtain the two tables recording the same information we just computed for the matroid $U_{1,3} \oplus U_{1,\{4\}}$. For the matroid $U_{2,4}$, we obtain the following maps:

Induced matroid	**Induced matroid polytope**	**# of morphisms**	**Morphisms**
$U_{2,4}$	$(0,1,0,1),(0,0,1,1)$ $(1,0,0,1),(0,1,1,0)$ $(1,1,0,0),(1,0,1,0)$	24	All permutations in S_4
$U_{1,4}$	$(1,0,0,0),(0,1,0,0),$ $(0,0,1,0),(0,0,0,1)$	4	all elements mapped to one element
$U_{1,\{1,2,3\}} \oplus U_{0,\{4\}}$	$(1,0,0,0),(0,1,0,0),$ $(0,0,1,0)$	4	pr_4, all other elements mapped to one element
$U_{1,\{1,2,4\}} \oplus U_{0,\{3\}}$	$(1,0,0,0),(0,1,0,0),$ $(0,0,0,1)$	4	pr_3, all other elements mapped to one element
$U_{1,\{1,3,4\}} \oplus U_{0,\{2\}}$	$(1,0,0,0),(0,0,1,0),$ $(0,0,0,1)$	4	pr_2, all other elements mapped to one element
$U_{1,\{2,3,4\}} \oplus U_{0,\{1\}}$	$(0,1,0,0),(0,0,1,0),$ $(0,0,0,1)$	4	pr_1, all other elements mapped to one element
$U_{0,4}$	$(0,0,0,0)$	1	$\mathrm{pr}_{[4]}$

Analogous to the case of $U_{2,4}$ where we had different symmetrical polytopes associated to the matroid $U_{1,\{a,b,c\}} \oplus U_{0,\{d\}}$ for $\{a, b, c, d\} = [4]$, we observe symmetrical polytopes for $U_{3,4}$. We record them up to symmetry due to the amount of cases. We obtain the following table:

Induced matroid	**Induced matroid polytope**	**# of morphisms**	**Morphisms**
$U_{3,4}$	$(1,1,1,0),(1,1,0,1)$ $(1,0,1,1),(0,1,1,1)$	24	All permutations in S_4
$U_{1,\{a,b\}} \oplus U_{1,\{c,d\}}$	$e_a + e_c, e_b + e_c,$ $e_a + e_d, e_b + e_d$	$4 \cdot 12$	a, c get mapped to the same element i b, d get mapped to the same element j $i \neq j$
$U_{1,4}$	$(1,0,0,0),(0,1,0,0),$ $(0,0,1,0),(0,0,0,1)$	4	all elements mapped to one element
$U_{1,\{a,b,c\}} \oplus U_{0,\{d\}}$	e_a, e_b, e_c	$4 \cdot 4$	pr_d, a, b, c get mapped to the same element
$U_{1,\{a,b\}} \oplus U_{0,\{c,d\}}$	e_a, e_b	$6 \cdot 4$	pr_{cd}, a, b get mapped to the same element
$U_{0,4}$	$(0,0,0,0)$	1	$\mathrm{pr}_{[4]}$

Observation 11.9 (label = ∘)**.** In summary, we observe that the polytopes associated to the induced matroids $f^{-1}(M)$ are not very well behaved. Nevertheless, there are a few observations we can still make:

- Permutations preserve the matroid polytope if and only if they are morphisms.
- The matroid polytope of the induced matroid has less than or equal the amount of vertices as the original matroid polytope.

Part 3. Enumeration of curves in ruled surfaces

In the last part of this thesis, we concern ourselves with a classical problem in algebraic geometry. Counting how many algebraic curves of fixed degree and genus pass through a fixed number of general points in some variety is a complicated problem in algebraic geometry. The most classical case is the count of rational curves passing through $3d-1$ points in the complex projective plane $\mathbb{P}^2$. The solution to this problem was conjectured by Witten [141] in 1991 to have deep connections to the KdV-equation in particle physics. In 1995, it was proven by Kontsevich [94], including a recursive combinatorial formula to determine these numbers. Since then, various generalizations of this problem have been achieved. Notably, the case of counting plane curves with higher fixed genus g passing through $3d+g-1$ points was later studied by Caporaso-Harris, and they established another recursive combinatorial determining these curve counts, see [36]. Generally, the count of curves of fixed degree d and genus g on an algebraic surface S passing through a fixed number of points in general position is called a *Gromov-Witten invariant.*

About ten years after the proofs by Kontsevich and Caporaso-Harris, Mikhalkin showed that the algebraic counts correspond to analogous counts in tropical geometry [106, Theorem 1.1]. Further, Gathmann-Markwig proved combinatorial formulas for the Gromov-Witten invariants using tropical geometry [64, 65].

Since then, these results have been substantially generalized. Through tropical curve counting, many results in enumerative geometry can be extended to fields of different characteristic or even the real numbers. Here, tropical geometry can be used to obtain recursive combinatorial formulas for the real analogue of Gromov-Witten invariants, the so-called *Welschinger invariants* [11, 136, 137].

Further, tropical curve counting can be successfully implemented and shown to be equal to algebraic Gromov-Witten invariants over other algebraic surfaces. For instance, in the past tropical geometry has been successfully used to count curves in Hirzebruch surfaces [59], cylinders and particular ruled surfaces [18]; and tori [19]. In this part, we extend the latter study to fill in all gaps in the tropical enumeration of ruled surfaces.

The structure is as follows. In **Section 12** we recall the basics of algebraic, logarithmic, and tropical enumerative geometry. In the process, we discuss the Abramovich-Chen-Gross-Siebert decomposition formula as the main translation tool of logarithmic Gromov-Witten invariants on more complicated surfaces to tropical Gromov-Witten invariants. Further, we discuss refined Gromov-Witten invariants as an extension of complex counts to the real numbers. We conclude by studying ruled surfaces: We recall the classification of (complex) ruled surfaces, give and first results on tropical enumerative geometry on tropicalizations of these surfaces.

We continue in **Section 13** by defining our main objects of study, tropical Möbius strips. We determine some of their algebraic features and describe embeddings of abstract tropical curves into the strips. We then proceed by defining multiplicities of tropical curves embedded into Möbius strips. We conclude by proving that the tropical Gromov-Witten invariant is invariant for tropically general point configurations.

In **Section 14** discuss floor diagrams in the plane and then extend them to work in our setting. We define their multiplicities and show that counting floor diagrams with multiplicity corresponds to counting curves with multiplicity.

Finally, in **Section 15** we prove the main theorems of this part: We show the correspondence between the tropical and the logarithmic Gromov-Witten invariant, and we determine the regularity properties of functions in the invariants and their generating series: we show that the functions returning the Gromov-Witten invariants depending on the tangency conditions are quasi-polynomial, and that the generating series in the vertical part of the bidegree is quasi-modular.

12. Preliminaries: Complex and tropical enumerative geometry

In this section, we give an overview of algebraic, logarithmic, and tropical enumerative geometry. Since modern developments in enumerative geometry have been significantly intertwined with the development of tropical methods, we go back and forth between the different points of view.

We begin by defining Gromov-Witten invariants for counts of curves in $\mathbb{P}^2$ in Section 12.1. We then introduce our first bit of tropical enumerative geometry in Section 12.2 — abstract and parametrized tropical curves and their moduli spaces. The first two sections lay the groundwork for the modern approach of logarithmic Gromov-Witten theory, which we introduce in Section 12.3. In particular, in Section 12.3.4 we cover the Abramovich-Chen-Gross-Siebert decomposition formula, which is the main ingredient for correspondence theorems of curve counts on non-toric surfaces. Following that, in Section 12.4, we consider refined Gromov-Witten invariants. These are Laurent polynomials that interpolate between the real and the complex count of curves in $\mathbb{P}^2$.

Next, we lay the foundations of our extension of the theory to ruled surfaces in the later sections of this part. We discuss a classification theorem for complex ruled surfaces in Section 12.5, and introduce the two main examples of ruled surfaces we will study in this thesis, $\mathbb{C}M_0$ and $\mathbb{C}M_1$. We finish in Section 12.6 by discussing first approaches of tropical analogues of ruled surfaces: tropical cylinders and curve counts on them.

12.1. Gromov-Witten invariants: Algebraic.

In this section, we will recall the general setup of enumerative problems. We start by defining the objects we wish to count - curves of genus g with n marked points. In the process, we define the moduli spaces $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$. Next, we discuss parametrized curves and their parameter spaces $\mathcal{M}_{g,n}(\mathbb{P}^2, d)$ and introduce Gromov-Witten invariants as a property of an evaluation map of the space. As a concrete example, we consider the count of rational parametrized curves.

12.1.1. *Curves and their moduli spaces.* We begin our journey by introducing curves and their moduli spaces. There are many good introductory texts to this topic. We base our exposition mainly on [126, Section 3] and refer to [5] for a more in-depth discussion.

Definition 12.1. A *curve* C over $\mathbb{C}$ is a reduced separated scheme of finite type over $\mathrm{Spec}(\mathbb{C})$, such that all irreducible components have dimension 1.

We will soon see that the full parameter space of curves is very complicated. To make our lives easier, we restrict ourselves to counting curves with mild singularities and automorphism groups.

Definition 12.2. We say that a curve C is *smooth* if every closed point $p \in C$ is smooth. A closed point $p \in C$ is a *node* if there exists a neighborhood of p in C which is analytically isomorphic to a neighborhood of the origin of the locus $\{(x, y) \mid x \cdot y = 0\} \subseteq \mathbb{C}^2$. We say that C is *nodal* if every closed point $p \in C$ is either smooth or a node. A connected, nodal curve C is called *stable* if its group of automorphisms is finite.

In general, there are infinitely many curves of a given degree and genus. To obtain a well-posed counting problem, we thus need to impose conditions on which curves we want to count. The easiest way to do that is to *mark* some special points on the curve, whose positions we will later investigate further and impose restrictions on.

Definition 12.3. We say that $(C, p_1, \ldots, p_n)$ is an *n-marked curve* if all p_i are distinct, smooth points on C. The properties of curves discussed in the previous definition can be directly adapted to n-marked curves: We say that an n-marked curve is nodal if its underlying curve C is. We say that it is stable if the group of automorphisms of C fixing all p_i is finite.

Finally, this allows us to "define" a moduli space of curves:

Definition 12.4.
- The parameter space of all *smooth* n-marked curves of genus g is the (non-compact) moduli space $\mathcal{M}_{g,n}$.
- The parameter space of all *stable* n-marked curves of genus g is the (compact) moduli space $\overline{\mathcal{M}}_{g,n}$.

We will later see some intuition for the fact that $\overline{\mathcal{M}}_{g,n}$ it is the compactification of the moduli space $\mathcal{M}_{g,n}$.

Remark 12.5. To be precise, we would have to define the above moduli spaces as moduli functors, ([126, Definition 3.17]). For a thorough theory, we also need to consider the spaces not as varieties or even schemes, but as Deligne-Mumford stacks. In the interest of time, nerves and readability, we omit this theory and point to Alper [5] and Schmitt [126].

Remark 12.6. Why are these the correct curves to parametrize in our moduli space, especially when we want to tackle enumerative problems? To summarize the intuition:

Nodes. The interior of the moduli space is going to consist of the smooth points. For counting purposes, we will need a compact space. In the compactification process, we will need to allow singularities for our curves. Nodes are just particularly mild singularities, and they interact well with tropical geometry, too!

Stability. Automorphisms cause trouble in parameter spaces, because we want to consider two automorphic curves with different input data as the same point. We will later see this explicitly in the construction of the moduli space of the tropical analogue, the moduli space of abstract tropical curves, where automorphisms change the shape of the cones in the polyhedral complex. As with nodes, once we start considering curves with genus > 0, we cannot escape automorphisms completely. But finite automorphism groups are easier to deal with than infinite ones.

Remark 12.7. What if we don't have stability? Infinite automorphism groups bring trouble. But what if we still want to study parameter spaces of these curves? It turns out that the parameter space $\mathfrak{M}_{g,n}$ of nodal curves of genus g with n marked points is a far more general stack — it is an *Artin stack*, and it is not separated. We will catch up with this train of thought at the end of Section 12.3, once we have defined logarithmic structures.

12.1.2. *Counting curves in the projective plane.* In the previous section we discussed an abstract notion of curves. Now, our goal will be to count curves in $\mathbb{P}^2$. Our exposition here is standard, following the exposition in [39, Chapters 1 and 4].

We can parametrize curves in the projective plane as the image of the abstract curves defined in the previous section.

Definition 12.8. A *parametrized curve of degree* d (to $\mathbb{P}^2$) is a morphism

$$f : C \to \mathbb{P}^2$$

of degree d where C is a smooth curve. Two parametrized curves are isomorphic if they can be stated in terms of each other by a reparametrization of the source curve.

The parameter space of all isomorphism classes of parametrized curve of degree d, the *moduli space of parametrized curves* $\mathcal{M}_g(\mathbb{P}^2, d)$, has expected dimension $3d + g - 1$, see [13, 92, Section 8.1.1 and Computation 2.3.5 respectively].

Heuristically, introducing a *point condition* (i.e., requiring a curve to pass through a fixed point p) on the target of a parametrized curve f reduces the dimension of the parameter space by 1. Since we are interested in a finite count of curves, we want the parameter space of all parametrized curve with n fixed points to be zero-dimensional. This implies that we should impose $n = 3d+g-1$ point conditions to get a generically finite, but non-zero number of curves satisfying the conditions. We will continue by studying parametrized curves with $3d + g - 1$ general fixed points, which arise as the parametrization of smooth curves with $3d + g - 1$ marked points.

Definition 12.9. The *moduli space of parametrized curves of degree* d *with* n *marked points* $\mathcal{M}_{g,n}(\mathbb{P}^2, d)$ is the parameter space of all isomorphism classes of parametrized curves with n marked points

$$f : (C, q_1, \dots, q_n) \to \mathbb{P}^2,$$

of degree d. Two parametrized curves with n marked points are isomorphic if they can be stated in terms of each other by a reparametrization of the source curve, such that the marked points get reparametrized accordingly as well.

The number of rational curves in $\mathbb{P}^2$ passing through $3d + g - 1$ points can be restated as a property of the moduli space $\mathcal{M}_{g,3d+g-1}(\mathbb{P}^2, d)$. It is the degree of the evaluation map, evaluating marked points of the curve at their image under the parametrization map.

The degree of this map is only well-defined after passing to a compactification of the space (see [61]), and obtain a compact moduli space of rational parametrized curves, $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^2, d)$. The compactification of the moduli space of rational parametrized curves now contains additional types of curves - they are not necessarily smooth, but still stable and nodal, just as for the moduli spaces $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$. Explicitly, we can now define the the evaluation map as follows.

Definition 12.10. We define the *evaluation map*

$$\mathrm{ev} : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^2, d) \to (\mathbb{P}^2)^n,$$

which evaluates an element f at the n special points: $\mathrm{ev}\big([f]\big) = \big(f(q_1), \dots, f(q_n)\big)$.

After showing that the evaluation map for $n = 3d + g - 1$ has generically finitely many preimages, [92, Lemma 4.1.3], we can rephrase our original problem as follows:

Theorem 12.11 ([61, Theorem 2 and Section 1.3])**.** *Let* $n = 3d + g - 1$. *On the compact moduli space* $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^2, d)$ *of parametrized curves of degree* d *with* n *marked points, the degree* *of the evaluation map* deg(ev) *is well-defined. Consequently, the number of curves genus g and degree d passing through n points coincides with the degree of the evaluation map:*

$$N_{d,g} \;=\; \deg(\mathrm{ev}).$$

We can determine the degree of the evaluation map by computing intersection numbers of the pullbacks of general points of the evaluation map.

Remark 12.12 (Virtual fundamental classes). Most (algebraic and later logarithmic) moduli spaces we consider in this thesis are not varieties: they are proper Deligne-Mumford stacks, which can be singular and consist of multiple irreducible components of potentially differing dimension. This complicates the intersection theory involved in defining Gromov-Witten invariants (see for instance Theorem 12.13).

In particular, pullbacks of points under the evaluation map are potentially not of the expected dimension, thus do not form fundamental cycles where we would have a well-defined intersection theory. To tackle this issue, one can introduce *virtual fundamental classes*, whose intersection theory mimics that of fundamental cycles. For a cycle α in any of the moduli spaces we discuss, we denote by $[\alpha]^{\mathrm{vir}}$ its virtual fundamental class. For details on this construction, we refer to [13, Sections 7 and 8].

We can instead re-formulate the Gromov-Witten invariant as an intersection number of virtual fundamental classes on a moduli space, as follows.

Theorem 12.13 ([92, Proposition 4.1.5])**.** *The classical Gromov-Witten invariant N_d can be expressed as intersection numbers of pullbacks of points under the evaluation map:*

$$N_{d,g} \;=\; \int_{[\overline{\mathcal{M}}_n(\mathbb{P}^2,d)]^{vir}} \prod_{1}^{n} \mathrm{ev}_i^*(\mathrm{pt}),$$

where pt *is the cohomology class Poincaré dual to a point.*

Here, ev_i *denotes the i-th coordinate entry of the evaluation map, and the integral symbol denotes the count of the points in the (0-dimensional) intersection in the virtual fundamental class, while the product denotes the intersection product of cohomology classes.*

12.2. **Abstract tropical curves and their moduli spaces.** In this section, we will focus on tropicalizing the theory discussed in the previous section. We have already seen first examples of tropical curves in Example 2.8 arising as the tropicalization of algebraic curves. Now, we will additionally review a more abstract point of view on tropical curves in Section 12.2.1. Analogous to the algebraic case, we will first discuss abstract curves and their moduli spaces. In Section 12.2.2, we will discuss parametrized tropical curves on $\mathbb{R}^2$ and their moduli spaces as the tropical analogue to the parametrized curves discussed in the previous section. This will allow us to set up tropical Gromov-Witten invariants. We conclude with Section 12.2.3, where we introduce tangency conditions for tropical curves and tropical relative Gromov-Witten invariants, which will serve as the backdrop for the discussion of logarithmic Gromov-Witten invariants afterwards.

12.2.1. *Abstract tropical curves.* In this section, we discuss abstract tropical curves and their moduli spaces. A main reference for this section is [45], with some additional motivation and application as presented in [103].

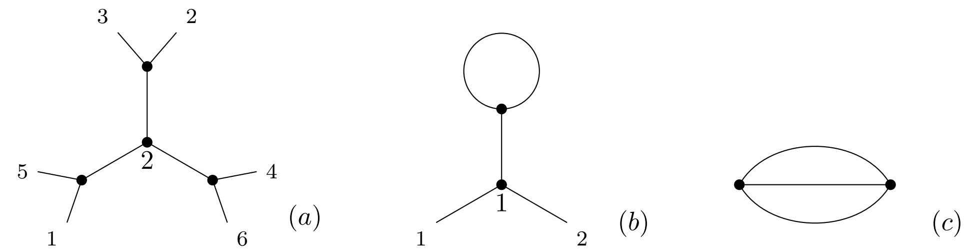


Figure 17. Some abstract tropical curves, omitting the length of edges. Vertices without labels are by convention of genus 0.

Definition 12.14. An *abstract tropical curve* is a metric graph $\Gamma = (V, E, \ell, \gamma)$ consisting of

- a collection of vertices V,
- a set of edges E, which contains both finite edges and half-rays we call *ends* (or legs),
- a length function $\ell : E \to \mathbb{R}_{>0} \cup \{\infty\}$ on edges, where the length of ends is infinite, and the only edges of infinite length are ends, and
- a genus function $\gamma : V \to \mathbb{Z}_{\geq 0}$ on the vertices.

When specifying the set of vertices or edges of an abstract or later parametrized tropical curve Γ, we write $V(\Gamma)$ for the set of vertices and $E(\Gamma)$ for the set of edges. Further, we write $E_b(\Gamma)$ for the set of bounded and $E_{ub}(\Gamma)$ for the set of unbounded edges of Γ.

In Figure 17, we give a few examples of abstract tropical curves.

Remark 12.15. Each tropical curve arising as the tropicalization of a tropical curve as discussed in Section 2.1 has an underlying abstract tropical curve, obtained by "forgetting the embedding", i.e., taking the underlying metric graph of a tropicalization. Here, the ends correspond precisely to the rays going off to infinity. Note, however, that this underlying tropical curve is not necessarily unique — we can insert as many vertices into edges as we like. We will soon see that uniqueness can be attained by requiring a tropical version of stability.

Remark 12.16. We say that two edges of Γ are *parallel* if they are parallel in the graph theoretic sense, i.e., if they have the same adjacent vertices.

Definition 12.17. If Γ is connected, we say that the tropical curve is *irreducible*. Let Γ be an irreducible tropical curve. The *genus* $g(\Gamma)$ of an abstract tropical curve is its genus as a graph together with the sum of the vertex genera, i.e.,

$$g(\Gamma) \ = \ |E| - |V| + 1 + \sum_{v \in V} \gamma(V).$$

If Γ is not irreducible but consists of connected components $\Gamma_1, \ldots, \Gamma_k$, the genus is

$$g(\Gamma) \ = \ \sum_{i=1}^{k} g(\Gamma_i) - k + 1.$$

If all the vertex genera are zero, the genus of the curve can equivalently be expressed in terms of the Euler characteristic $\chi(\Gamma)$ of the underlying graph. We have $g = 1 - \chi(\Gamma)$. We say that Γ *has n marked points* if Γ has n distinct, labeled ends.

Example 12.18. All curves in Figure 17 are of genus 2 — curve (a) through its inner vertex with $\gamma(v) = 2$, curve (b) because it contains one vertex of genus 1 and one cycle as a graph, and curve (c), because it has three edges and two vertices, i.e., $g(\Gamma_{(c)}) = 3 - 2 + 1 = 2$. The curve (a) has six marked points, the curve (b) has two and the curve (c) has zero.

Definition 12.19. The *combinatorial type* C_Γ of an abstract tropical curve Γ is its underlying graph (V, E), forgetting all metric information, but remembering the markings on the ends and the genus on vertices. We say that a tropical curve Γ is *stable* if for every vertex $v \in V$

$$\text{valence}(v) \;=\; \begin{cases} \geq 3 & \text{if } \gamma(v) = 0 \\ \geq 1 & \text{if } \gamma(v) = 1 \\ \geq 0 & \text{otherwise.} \end{cases}$$

A loop at a vertex contributes two to the valence of the vertex.

Remark 12.20. We can stabilize many abstract tropical curves by removing superfluous vertices. If a vertex v has valence 2, the two edges adjacent to v are distinct, and at least one has finite length, we delete the vertex and join the two adjacent edges e_1 and e_2 into a joint edge of length $\ell(e_1) + \ell(e_2)$.

Definition 12.21. The *tropical moduli space* $\mathcal{M}_{g,n}^{\text{trop}}$ is the parameter space of stable tropical curves of genus g with n marked points. Each combinatorial type C_Γ occurring for tropical curves of genus g with n marked points has an associated cone

$$\mathcal{C}_\Gamma \;=\; \mathbb{R}^k_{>0}/\sim$$

where k is the number of edges of finite length in the combinatorial type, and the equivalence is given by metric graph automorphisms of the representatives. For instance, if an abstract tropical curve contains two parallel finite edges e_1 of length l_1 and e_2 of length l_2 (see Remark 12.16), this curve is equivalent to the curve with the same combinatorial type and the same edge-lengths, except $\ell(e_1) = l_2$ and $\ell(e_2) = l_1$. Now, the tropical moduli space $\mathcal{M}_{g,n}^{\text{trop}}$ is given as

$$\mathcal{M}_{g,n}^{\text{trop}} \;=\; \coprod_{\substack{\text{combinatorial types} \\ C_\Gamma}} \overline{\mathcal{C}_\Gamma}/\sim,$$

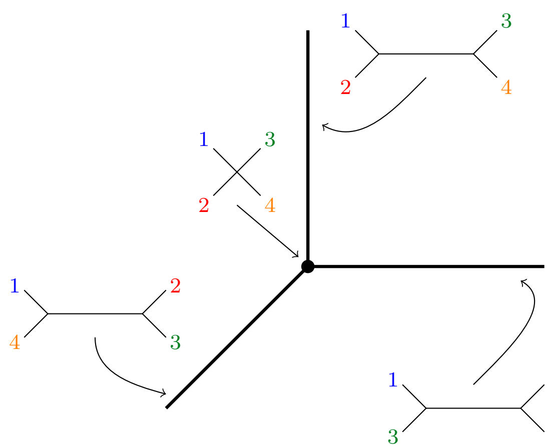
where two points $[\Gamma] \in \mathcal{C}_\Gamma$ and $[\Gamma'] \in \mathcal{C}_\Gamma$ are identified if they are equal after contracting all edges of length zero, and $\overline{\mathcal{C}_\Gamma}$ denotes the closure with respect to the Euclidean topology. It is a cone complex whose cones correspond to the different possible combinatorial types of tropical curves with the prescribed genus and number of marked points.

Remark 12.22. In the previous section we had defined stable algebraic curves as algebraic curves with finite automorphism group. Here, we can see how this becomes relevant in the construction of the tropical moduli space.

In general, taking equivalence classes with respect to a finite number of automorphisms is a polyhedral operation, roughly corresponding to folding the cone onto itself. If we do this an infinite number of times, the resulting equivalence class might not be a polyhedral cone anymore, making its structure harder to analyze. We can see such an infinite process resulting in a more complicated space in the case of the moduli space of principally polarized abelian varieties, [44, Section 4].

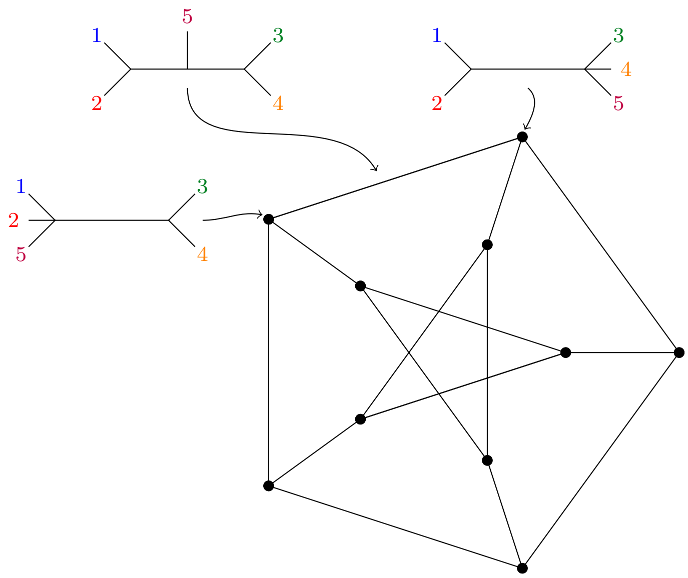
Example 12.23. In this example, we will construct two moduli spaces of rational tropical curves explicitly. By definition, points in these moduli spaces correspond to stable metric trees with n marked points.

We begin with the moduli space $\mathcal{M}_{0,4}^{\text{trop}}$ of stable, rational tropical curves. There are four combinatorial types of stable tropical curves of genus 0 with four marked points, three of which are maximal. The moduli space $\mathcal{M}_{0,4}^{\text{trop}}$ is a generic tropical line, depicted below, indicating the combinatorial type associated to each cone.



We observe that abstract tropical curves with four ends are, up to a shift, tropical linear spaces in $\mathbb{P}(\mathbb{T}^4)$, which we studied in Example 2.24. This means that, up to lineality, the tropical moduli space $\mathcal{M}_{0,4}^{\text{trop}}$ coincides with the Dressian $\text{Dr}(2,4)$ we computed in Example 3.83.

Next, we will construct the tropical moduli space $\mathcal{M}_{0,5}^{\text{trop}}$ of stable tropical curves. It is the cone over the Petersen graph. In the below picture, we annotate one of the maximal cells and two of the codimension-one cells. Note that each of the $10 = \binom{5}{2} = \binom{5}{3}$ codimension-one cells corresponds to a graph where one vertex v_1 has three adjacent ends and one vertex v_2 has two adjacent ends, and that each of the $15 = \binom{5}{2} + \binom{5}{1}$ maximal cells corresponds to a graph with three vertices, where the two exterior vertices each have two adjacent ends and the interior vertex has one. Each codimension-one cell is adjacent to the three maximal cells, corresponding to fixing the vertex v_2 and choosing one of the three ends of v_1 to be the end adjacent to the interior vertex. You can see one such concrete example in the below picture.



Similar to the observation we just made for abstract tropical curves with four ends, we observe that abstract tropical curves with five ends are, up to a shift, tropical linear spaces in $\mathbb{P}(\mathbb{T}^5)$, and that, up to lineality, the tropical moduli space $\mathcal{M}_{0,5}^{\text{trop}}$ coincides with the Dressian $\text{Dr}(2,5)$ we computed in Example 3.83.

Remark 12.24. When studying the two moduli spaces $\mathcal{M}_{0,4}^{\text{trop}}$ and $\mathcal{M}_{0,5}^{\text{trop}}$ in Example 12.23, we observed that the moduli spaces have a very familiar structure: They look just like the Dressians $\text{Dr}(2;4)$ and $\text{Dr}(2;5)$ we had computed in Example 3.83, after quotienting by lineality! This is not a coincidence, but one instance of a much richer connection between line Grassmannians $\text{Gr}(2,n)$ and algebraic moduli spaces of stable n-pointed rational curves $\overline{\mathcal{M}}_{0,n}$ — one can give equations for the moduli space $\overline{\mathcal{M}}_{0,n}$ using the Plücker embedding, and $\overline{\mathcal{M}}_{0,n} = \text{Gr}(2;n)/\!\!/_{Ch}\, T^{n-1}$ is the Chow quotient of the Grassmannian by a rescaling action of an $(n-1)$-dimensional torus (see Kapranov [90, Theorem 4.1.8]). On the open part of the Grassmannian, Gibney-Maclagan show that this is a GIT-quotient, and that all GIT-quotients of $\text{Gr}(2;n)$ can be derived from $\overline{\mathcal{M}}_{0,n}$ (see [67, Theorem 6.3 and Theorem 7.1] respectively).

12.2.2. *Basics of tropical curve-counting.* Now, let us set up the problem of counting curves passing through $3d+g-1$ points in tropical geometry. Explicitly, this is done similar to the algebraic case by considering parametrized (tropical) curves, defining evaluation maps and computing their degrees at general points. Most of this section is based on [39, Chapters 9, 10 and 11], with some additional input from [106] and [107].

We now construct parametrizations for abstract tropical curves. These correspond to the parametrized curves we have already seen in the algebraic context in the previous section.

Definition 12.25. An *n-marked parametrized tropical curve* (also known as a *stable tropical map*) $h : \Gamma \to \mathbb{R}^2$ is a tuple (Γ, h), where

- $\Gamma \in \mathcal{M}_{g,n+m}^{\text{trop}}$ is a stable abstract tropical curve with $n+m$ ends such that $n \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{\geq 2}$, and
- $h : \Gamma \to \mathbb{R}^2$ is a continuous, piecewise integral affine function which restricts to an integral affine function on each edge of Γ. We then have $h|_e = \mathbf{m}_e t + \mathbf{a}$ for each edge $e \in E(\Gamma)$, where $t \in [0, \ell(e)]$ and $\mathbf{m}_e \in \mathbb{Z}^2$.

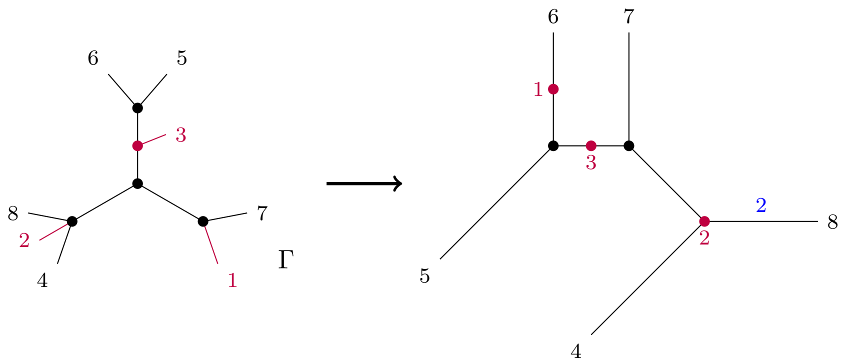


Figure 18. A parametrized tropical curve from the abstract tropical curve Γ to $\mathbb{R}^2$. The ends 4 and 5 get mapped in direction $-(e_1+e_2)$, the ends 6 and 7 get mapped in direction e_2 and the end 8 gets mapped in direction $2e_1$. This corresponds to the edge having weight 2 in Proposition 2.17.

- every vertex v satisfies the *balancing condition*, i.e.,

$$\sum_{e \ni v} \mathbf{m}_e \;=\; 0,$$

  where $\mathbf{m}_e \in \mathbb{Z}^2$ denotes the slope of an edge e under h as defined above.

Further, we require that the map h is constant on the first n ends (soon to be *marked points*), and that it is not constant on the remaining m *ends*.

Remark 12.26. The image $h(\Gamma) \subseteq \mathbb{R}^2$ of a parametrized tropical curve $h : \Gamma \to \mathbb{R}^2$ is dual to a regular Newton subdivision in the sense of Definition 2.16.

The image is a balanced polyhedral complex in the sense of Definition 2.14, where the weight of an edge e is determined by its slope: $w_e \cdot \mathbf{p}_e = \mathbf{m}_e$, i.e., w_e is the scalar when expressing $\mathbf{m}_e$ as the multiple of a primitive vector.

Since the image is dual to a regular Newton subdivision, there exists a Nonarchimedean field with valuation such that $h(\Gamma)$ is a tropical hypersurface in the sense of Definition 2.7. However, this *lifting* is not necessarily unique.

Note that we do not require the images of vertices of Γ to be vertices in the polyhedral complex $h(\Gamma)$ — they may lie on top of edges or coincide with other vertices.

A main advantage of this new point of view is that it will allow us to construct tropical analogues of curves in other varieties, or using other lattice structures, more easily. The drawback is that such a definition is only well-studied for curves, and not for arbitrary varieties, contrary to Definition 2.7.

In Figure 18, you can see an example of such a parametrized tropical curve.

We now consider parametrizations of abstract tropical curves into $\mathbb{P}^2$ that correspond to tropicalizations of curves. For such an algebraic counterpart to exist, there must be an appropriate amount of ends in the correct directions.

Definition 12.27. We write $\mathcal{M}_{g,n}^{\mathrm{trop}}(\mathbb{R}^2, \Delta)$ for the *moduli space of n-marked parametrized tropical curves of degree* Δ. This space parametrizes n-marked parametrized tropical curves with d ends of the image curve of weight 1, which are pointing in the directions e_1, e_2 and

$-(e_1+e_2)$ respectively. We say that the parametrized tropical curves in this moduli space are of *degree*

$$\Delta \;=\; \big\{d\cdot e_1, d\cdot e_2, d\cdot -(e_1+e_2)\big\}.$$

We say that the rational parametrized tropical curve $h:\Gamma\to\mathbb{R}^2$ is *simple* if

- ∘ Γ is 3-valent at all vertices,
- ∘ h is an immersion,
- ∘ for any $y\in\mathbb{R}^2$, the inverse image $h^{-1}(y)$ consists of at most two points, and
- ∘ if $a,b\in\Gamma$ such that $h(a)=h(b)$ but $a\neq b$, then neither a nor b are vertices of Γ.

Proposition 12.28 ([39, Proposition 4.4.2])**.** *There exists a* forgetful map

$$\begin{aligned} ft:\mathcal{M}^{\mathrm{trop}}_{g,n}(\mathbb{R}^2,\Delta) &\to \mathcal{M}_{g,n+3d+g}\\ (h:\Gamma\to\mathbb{R}^2) &\mapsto \Gamma, \end{aligned}$$

forgetting the embedding h but remembering the underlying stable rational abstract tropical curve Γ.

The positions of the n ends that get contracted to points under the map $h:\Gamma\to\mathbb{R}^2$ play the same role as the n marked points in the algebraic setting (see Definition 12.9): We use them to define evaluation maps.

Definition 12.29. We define the evaluation map

$$\mathrm{ev}_{\mathrm{trop}}:\mathcal{M}^{\mathrm{trop}}_{g,n}(\mathbb{R}^2,\Delta)\to(\mathbb{R}^2)^n;\quad (h:\Gamma\to\mathbb{R}^2)\mapsto\big(h(1),\dots,h(n)\big)$$

where $h(i)$ denotes the image of the ith contracted end.

Together, the evaluation map and the forgetful map allow us to see $\mathcal{M}^{\mathrm{trop}}_{g,n}(\mathbb{R}^2,\Delta)$ as a cone complex — the product

$$\mathrm{ft}\times\mathrm{ev}_1:\mathcal{M}^{\mathrm{trop}}_{g,n}(\mathbb{R}^2,\Delta)\to\mathcal{M}_{g,n+3d+g}\times\mathbb{R}^2$$

is a bijection, [37, Theorem 5.1], and $\mathcal{M}_{g,n+3d+g}\times\mathbb{R}^2$ is a cone complex by the definition of $\mathcal{M}_{g,n+3d+g}$.

Again, to obtain a finite but non-zero count we want to count parametrized tropical curves whose $3d-1$ points in general position.

Definition 12.30. A point configuration $\{p_1,\dots,p_{3d+g-1}\}:=\mathscr{P}\subseteq\mathbb{R}^2$ consisting of $3d+g-1$ points is called *tropically general* if the preimage $\mathrm{ev}^{-1}_{\mathrm{trop}}(p_1\times\cdots\times p_{3d+g-1})$ only consists of simple tropical curves.

One can show that the set of tropically general point configurations is dense in $(\mathbb{R}^2)^{3d+g-1}$. For *rational* parametrized tropical curves and their moduli spaces, the evaluation maps above are maps of fans embedded in a vector space, so one can compute their degree and show that it is constant for a tropically general point configuration, see [63]. This allows us to define a tropical analogue of the Gromov-Witten invariant:

Definition 12.31. Let $\mathscr{P}$ be a tropically general point configuration consisting of $3d-1$ points in $\mathbb{R}^2$, and let $\Delta=\big\{d\cdot e_1,d\cdot e_2,d\cdot-(e_1+e_2)\big\}$. We define the tropical Gromov-Witten invariants as the degree of the evaluation map (see [63]),

$$N^{\mathrm{trop}}_{0,\Delta} \;:=\; \deg(\mathrm{ev}_{\mathrm{trop}}).$$

For higher genus, the degree computation is possible, but more complicated. However, computing the degree of rational tropical curves, we observe that the degree of the tropical evaluation map can be computed as a count of parametrized tropical curves whose marked ends evaluate to a tropically generic point configuration, when counted with appropriate multiplicity. This definition now extends to curves of higher genus:

Definition 12.32. Let $h : \Gamma \to \mathbb{R}^2$ be a simple parametrized tropical curve. Its *multiplicity* is given by the product over the (normalized) area of triangles in the Newton subdivision dual to the image $h(\Gamma)$ (see Definition 2.16 for a definition of the Newton subdivision):

$$m(\Gamma, h) \; = \prod_{\substack{v \in V(\Gamma) \\ \text{trivalent}}} \operatorname{Area}(\Delta_v),$$

where Δ_v denotes the cell in the Newton subdivision associated to the vertex v of the tropical hypersurface $h(\Gamma)$. Since we only sum over the trivalent vertices, the associated cells are triangles.

This allows us to define tropical Gromov-Witten invariants for counting problems in higher genus:

Definition 12.33. Let $\mathscr{P}$ be a tropically general point configuration consisting of $3d+g-1$ points in $\mathbb{R}^2$, and let $\Delta = \{d \cdot e_1, d \cdot e_2, d \cdot -(e_1+e_2)\}$. We define the tropical Gromov-Witten invariants as sum of multiplicities of parametrized tropical curves $h : \Gamma \to \mathbb{R}^2$ of degree Δ whose marked points evaluate to $\mathscr{P}$,

$$N^{\text{trop}}_{g,\Delta} \; := \sum_{h:\Gamma \to \mathbb{R}^2} m(\Gamma, h).$$

One of the foundational results of early tropical geometry is that the tropical Gromov-Witten invariants can be used in (algebraic) enumerative geometry: *Mikhalkin's correspondence theorem*, shows that the tropical Gromov-Witten invariant defined in this way is equal to the algebraic one.

Theorem 12.34 ([106, Theorem 1])**.** *Let* $\Delta = \{d \cdot e_1, d \cdot e_2, d \cdot -(e_1 + e_2)\}$. *Then,*

$$N^{\text{trop}}_{g,\Delta} \; = \; N_{g,d},$$

where $N_{g,d}$ *is the algebraic Gromov-Witten invariant we had discussed in Theorem 12.11.*

12.2.3. *Imposing tangency conditions.* Instead of counting parametrized tropical curves with images that have $3d$ outgoing ends of weight 1, we now wish to count tropical curves where the ends are allowed to have arbitrary fixed weights. On the algebraic side, this will correspond to *imposing tangency conditions with the toric boundary of* $\mathbb{P}^2$, which we will see in more detail in the next section, and give rises to new counts, called *relative Gromov-Witten invariants.*

To keep track of the tangency conditions on the tropical side, we introduce the following notation:

Notation 12.35. Let α be a sequence of non-negative integers. We define $|\alpha| = \sum_{i=0}^{\infty} \alpha_i$.

Further, we write I for the sequence $(1, 2, 3, \dots)$, and write

$$I\alpha \; = \; \sum_{i=1}^{\infty} (\alpha_i \cdot i) \text{ and } I^{\alpha} \; = \; \prod_{i=1}^{\infty} i^{\alpha_i}.$$

Since we will later only be required to consider tangency conditions in one direction, we will restrict ourselves to this setting for notational ease. Tangency conditions in multiple different directions can be considered analogously.

Definition 12.36. Let d and $g \in \mathbb{Z}_{\geq 0}$ and let α and β be sequences such that $I\alpha + I\beta = d$, where I denotes the sequence $(1, 2, \dots)$. Let $h : \Gamma \to \mathbb{R}^2$ be a simple parametrized tropical curve of genus g and degree

$$\big\{(\alpha_i + \beta_i) \cdot (i \cdot e_1), d \cdot e_2, d \cdot -(e_1 + e_2)\big\}.$$

That is, $h(\Gamma)$ has $\alpha_i + \beta_i$ ends of weight i to the right.

We write $\mathcal{M}^{\text{trop}}_{g,n}(\mathbb{R}^2, d, \alpha, \beta)$ for the moduli space of parametrized tropical curves of degree $\big\{(\alpha_i + \beta_i) \cdot (i \cdot e_1), d \cdot e_2, d \cdot -(e_1 + e_2)\big\}$ of genus g with n marked points.

In the above definition, we split the tangency conditions into two sequences α and β. We will use α to denote *fixed* tangency conditions, where we not only fix the multiplicity of the end of the tropical curve, but also its position. We will use β to denote *free* tangency conditions, where we only fix the multiplicity of the end but not its position.

As before, to set up a tropical counting problem for arbitrary genus, we need to adapt the multiplicity of parametrized tropical curves to this new setting.

Definition 12.37. Let $h : \Gamma \to \mathbb{R}^2$ be a simple parametrized tropical curve with tangency conditions (α, β). Its multiplicity is given by the product over the (normalized) area of triangles in the Newton subdivision dual to the image $h(\Gamma)$ (see Definition 2.16 for a definition of the Newton subdivision):

$$m(\Gamma, h) \;=\; \frac{1}{I^\alpha} \prod_{\substack{v \in V(\Gamma) \\ \text{trivalent}}} \text{Area}(\Delta_v),$$

where Δ_v denotes the cell in the Newton subdivision associated to the trivalent vertex v of the tropical hypersurface $h(\Gamma)$.

Now, we need to determine the dimension of the moduli space of parametrized tropical curves with tangency conditions to determine how many marked points we need to fix to obtain a generically finite count.

Remark 12.38. Let d and $g \in \mathbb{Z}_{\geq 0}$ and let α and β be sequences such that $I\alpha + I\beta = d$. The dimension of $\mathcal{M}^{\text{trop}}_{g,n}(\mathbb{R}^2, d, \alpha, \beta)$ can be computed to be $2d + g + |\beta| - 1$.

Definition 12.39. The *refined tropical Gromov-Witten invariant* can be defined as the sum of multiplicities of parametrized tropical curves whose whose $2d + g + |\beta| - 1$ marked ends get contracted to a tropically general point configuration $\mathscr{P}$,

$$N^{\text{trop}}_{g,\Delta} \;=\; \sum_{h : \Gamma \to \mathbb{R}^2} m(\Gamma, h).$$

12.3. **Gromov-Witten invariants: Logarithmic.** In the previous sections we have studied curves in algebraic and tropical geometry. Now, we discuss a field that can be seen as an enrichment of algebraic geometry by tropical enumerative information — *logarithmic enumerative geometry.* This will allow us to discuss the algebraic analogue of tropical relative Gromov-Witten invariants which we studied in the last section.

To make things easier for us, we start by rewriting the case of counting curves in the projective plane in our new language, and introduce tangency conditions in the algebraic context (see Section 12.3.1). Then, we set up the same theory for arbitrary logarithmic schemes, focusing on the case of toric varieties together with their toric boundaries (see Section 12.3.2). We continue with the setup in the setting for toric varieties (see Section 12.3.3), and conclude with an important theorem which will later allow us to show the correspondence theorem on ruled surfaces: the decomposition formula (see Section 12.3.4).

12.3.1. *Rational logarithmic parametrized curves in the plane.* Parametrized logarithmic curves and logarithmic Gromov-Witten invariants were introduced in [1, 46, 73]. We will later phrase our own enumerative problem in this language. Our exposition here is mostly based on [39, Chapter 4 and 5].

Definition 12.40. We fix coordinates on $\mathbb{P}^2$ by fixing three hyperplanes H_0, H_1 and H_2. A *non-degenerate rational parametrized logarithmic curves* of degree d (with n marked points $s_1, \ldots, s_n$) is a map

$$f : (\mathbb{P}^1, p_1, \ldots, p_d, q_1, \ldots, q_d, r_1, \ldots, r_d, s_1, \ldots, s_n) \to (\mathbb{P}^2, H_0, H_1, H_2)$$

such that all tangency with the three hyperplanes H_0, H_1 and H_2 is transverse and

$$f^{-1}(H_0) = \sum_{i=1}^{d} p_i, \quad f^{-1}(H_1) = \sum_{i=1}^{d} q_i, \quad \text{and} \quad f^{-1}(H_2) = \sum_{i=1}^{d} r_i.$$

Again, we say that two parametrized logarithmic curves are equivalent if they can be stated in terms of each other by reparametrization of the source curve. The parameter space of equivalence classes of rational parametrized logarithmic curves is the moduli space $\mathcal{M}_{0,n}^{\log}(\mathbb{P}^2, d)$.

The moduli space of rational parametrized logarithmic curves has a forgetful map ft to the moduli space of rational parametrized curves, forgetting the extra information, i.e., the position of the $3d$ transverse boundary tangency points, but remembering the map and the n marked points. This map has degree $(d!)^3$ (see [39, Lemma 4.1.1]). It further allows us to define an evaluation map on the moduli space of rational parametrized logarithmic curves:

$$\mathrm{ev} : \mathcal{M}_{0,n}^{\log}(\mathbb{P}^2, d) \to (\mathbb{C}^*)^2 \ := \ \mathrm{ev}_{\mathcal{M}_{0,n}(\mathbb{P}^2,d)} \circ \mathrm{ft}.$$

Its image is contained in $(\mathbb{C}^*)^{2n}$ as all boundary points of the curve are marked and get forgotten after applying the forgetful map.

12.3.2. *Logarithmic schemes.* The construction defined above is the first example of the extension we are aiming at. We now introduce the most general case we will need — *logarithmic schemes* — before coming back to the more manageable curve counts of genus g curves in $\mathbb{P}^2$ with some fixed tangency conditions, whose tropical analogue we have already seen in Section 12.2.3.

Definition 12.41. Let Y be a scheme. A *logarithmic structure* on Y is a sheaf of monoids M_Y (corresponding to the *monomial functions* on Y) together with a map of monoids to the sheaf of regular functions $\mathcal{O}_Y$,

$$\alpha : M_Y \to \mathcal{O}_Y,$$

such that $\alpha^{-1}(\mathcal{O}_Y^*)$ is isomorphic to $\mathcal{O}_Y^*$ using α. A *logarithmic scheme* is a scheme with a logarithmic structure, we write (Y, M_Y).

Example 12.42. Consider a smooth projective toric variety X and its boundary divisor D, with irreducible components $D_1, \ldots, D_r$. Then, on an open set U, the sheaf M_X is the subsheaf of $\mathcal{O}_X(U \setminus D)$ consisting of invertible functions. That is, functions that are locally expressed as monomials in the defining equations of the boundary.

More generally, the analogue is true for any smooth normal crossings pair (Y, E): the sheaf M_Y consists of functions that are locally monomial in the defining equations of the component of E.

Example 12.43. A more explicit example of this is the compactification of the moduli space of stable genus g curves with n marked points, $\overline{\mathcal{M}}_{g,n}$. Here, the logarithmic structure is defined as the divisors of singular curves. (One can see that this is a smooth normal crossings divisor in the étale topology.)

Consider a small open neighborhood around a point $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$. Recall that any curve in $\overline{\mathcal{M}}_{g,n}$ has at worst nodal singularities. Passing to an étale cover U, we can then label the nodes of C as $q_1, \ldots, q_r$. The curve C will be smooth at a generic point in U. But, there are divisors D_i in U where q_i is still a node. The functions cutting out the D_i are the monomial functions in U and thus extend to the locally monomial functions on $\overline{\mathcal{M}}_{g,n}$. Hence, this sheaf is a sheaf of monomial functions on $\overline{\mathcal{M}}_{g,n}$ and endows it with a logarithmic structure.

Definition 12.44. A *morphism* of logarithmic schemes $f : (Y, M_Y) \to (Z, M_Z)$ (or *logarithmic morphism*) is a morphism of schemes $f : Y \to Z$ together with a compatible morphism of sheaves of monoids $f^\flat : f^*(M_Z) \to M_Y$.

Example 12.45. We had already mentioned $\mathfrak{M}_{g,n}$ as the Artin stack of not necessarily stable curves of genus g with n marked points in Remark 12.7. It has a logarithmic structure similar to the logarithmic structure we just discussed for $\overline{\mathcal{M}}_{g,n}$ — the sheaf of monoids is defined by the divisors of singular curves just as above, using the topology defined by smooth maps instead of étale maps. Further, the universal curve $\mathfrak{C} \to \mathfrak{M}_{g,n}$ also has logarithmic structure, and the morphism between the two is logarithmic.

Definition 12.46. Let (Y, M_Y) be a logarithmic scheme. A logarithmic curve is a logarithmic morphism $Y \to \mathfrak{M}_{g,n}$. Pulling back the universal curve $\mathfrak{C}$ yields a family of curves over Y which have logarithmic structures. We denote this family by Y/S.

We can now use this to define a parametrized logarithmic curves:

Definition 12.47 ([39, Definition 5.3.2])**.** Let (Y, M_Y) be a logarithmic scheme. A *pre-parametrized logarithmic curve* to Y over another logarithmic scheme S is a logarithmic curve C/S together with a logarithmic map from the total space of curves to (Y, M_Y). A pre-parametrized logarithmic curve is a logarithmic curve if the underlying scheme theoretic map $C \to Y$ is a parametrized curve in the sense of Section 12.1.

In most of our discussion later, we will choose S to be the logarithmic general point.

We will now restrict ourselves again to the case of curves in smooth projective varieties, as this allows us to give a more descriptive definition in line with our previous approaches.

Definition 12.48. Let X be a smooth projective variety and D a simple normal crossings (SNC) divisor on X, with irreducible components $D_1, \ldots, D_r$. A *non-degenerate logarithmic* *map* is a smooth curve with $n+m$ marked points $(C, p_1, \dots, p_n, q_1, \dots, q_m)$ together with a map

$$F : (C, p_1, \dots, p_n, q_1, \dots, q_m) \to (X, D),$$

where the preimage of D is a union of points in $p_1, \dots, p_n$. Further, each boundary component D_j has a fixed *tangency order* t_{ij} at each point p_i, obtained by pulling back the defining equation of D_j to p_i and computing the order of vanishing.

We fix all discrete data given here, i.e., the genus g of the curve C, the number of tangency points n and the number of marked points m, the homology class of the pushforward of the curve $F_*([C]) \in H_2(X, \mathbb{Z})$ and all tangency profiles at all points and divisors and denote it shorthand by $\beta = (g, n, F_*([C]), T = (t_{ij})_{ij})$. This allows us to define a moduli space $\mathcal{M}_\beta^{\log}(X \mid D)$.

The boundary divisor of any smooth projective toric variety X is SNC. Further, let us get more specific and re-write the original rational plane case in Definition 12.40 in terms of the newly defined notation.

Example 12.49. We consider the smooth projective toric variety $\mathbb{P}^2$ with its three standard boundary divisors $H), H_1$ and H_2. We set $g = 0$ and $n = 3d$. Further, we set the matrix T to be the $3 \times 3d$ matrix

$$\begin{bmatrix} 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 1 \end{bmatrix}.$$

12.3.3. *Logarithmic Gromov-Witten invariants.* Now that we have defined parametrized logarithmic curves, we can set up a counting problem of parametrized curves whose m marked points get mapped to a fixed tropically generic point configuration under the evaluation map, which we extend to account both for points on the variety X and for the positions of points on the divisor D.

Definition 12.50. Let X be a smooth projective variety and D a simple normal crossings divisor. Further, let β denote suitable discrete data as in Definition 12.48 and let $\mathcal{M}_\beta^{\log}(X \mid D)$ denote the moduli space of parametrized logarithmic curves.

For each of the m marked points, the moduli space is equipped with an *evaluation map*

$$\mathrm{ev}_i : \mathcal{M}_\beta^{\log}(X \mid D) \longrightarrow X \setminus D$$

It maps a parametrized logarithmic curve to the image of the i-th marked point. Moreover, for each of the n marked points mapped to the boundary divisor, we have an additional evaluation map with values in the corresponding divisor:

$$\widehat{\mathrm{ev}_i} : \mathcal{M}_\beta^{\log}(X \mid D) \longrightarrow D.$$

Proposition-Definition 12.51. *For a suitable set of discrete data β as in Definition 12.48, we can define the* logarithmic Gromov-Witten invariant $\mathcal{N}_\beta$ *as the number of points in the (0-dimensional) intersection of pullbacks of points in the virtual fundamental class,*

$$\mathcal{N}_\beta \;=\; \int_{[\mathcal{M}_\beta]^{vir}} \prod_1^n \mathrm{ev}_i^*(\mathrm{pt}) \prod_1^{|\mu|} \widehat{\mathrm{ev}_i}^*(\mathrm{pt}),$$

where pt *is the cohomology class Poincaré dual to a point.*

Construction 12.52. Every logarithmic scheme Y we consider (i.e., logarithmic schemes that are fine, saturated, and of finite type) has a *tropicalization* defined by taking the dual cone over the monoid stalk at a logarithmically generic point η, $M_{Y,\eta}$, due to [73, Appendix B] and [2, Section 2.5]. This notion of tropicalization is more general than ours, relating logarithmic schemes to colimits of diagrams in the category of cones with face maps as morphisms. As in our case, this theory allows us to consider embeddings of abstract tropical curves into these complexes, and allows us to define and characterize parametrized logarithmic curves via this discrete data.

In many cases, we can extend Mikhalkin's correspondence theorem from classical Gromov-Witten invariants to logarithmic Gromov-Witten invariants. We will see one approach to proving such correspondence theorems in the next section, and will do so later ourselves in Theorem 15.2.

12.3.4. *The decomposition formula.* In this part of the thesis, our goal is to compute logarithmic Gromov-Witten invariants of ruled surfaces. We will see later that we can decompose the (non-toric) complex ruled surfaces associated to tropical Möbius strips into a collection of toric surfaces, glued along their toric boundary divisors. We can use this information to aid us.

Remark 12.53 (Computing logarithmic Gromov-Witten invariants by degeneration). As we have discussed in the previous sections, counting tropical curves in toric surfaces and keeping track of their boundary behaviors is doable. So, how do we approach surfaces that are not toric? In this thesis, we will focus on two ruled surfaces which can be deformed into a union of toric surfaces, glued along their toric boundaries. In this case, we can use the results of the previous section, laid out as follows.

We construct a logarithmically smooth family of surfaces with a central fiber that is just the union of the toric surfaces. Now, the main ingredient of our correspondence theorem is the fact that logarithmic Gromov-Witten invariants are constant in logarithmically smooth families, and the relation of the logarithmic Gromov-Witten invariants to tropical enumerative invariants, which we will recall in this section. These results were proven by Abramovich-Chen-Gross-Siebert [2].

Notation 12.54. As before, throughout this section we assume all logarithmic schemes to be fine and saturated. Further, throughout this section we will denote the discrete data of a logarithmic Gromov-Witten invariant as in Definition 12.48 by β.

We consider a logarithmically smooth and projective morphism $X \to B$, where B is a logarithmically smooth curve with a single closed point $b_0 \in B$ where the logarithmic structure is nontrivial. We denote by $\mathcal{M}^{\log}_{\beta}(X \,|\, B)$ the moduli space of parametrized logarithmic curves defined in Definition 12.48

For any point b, we define $j_b : \{b\} \hookrightarrow B$. Now, we can consider a smooth family of logarithmic stacks obtained by taking fibers over the points b. We write X_b for a fiber of the deformation corresponding to b, and denote by $X_0 = X_{b_0}$ the central fiber.

Proposition 12.55 ([2, Theorem 1.1])**.** *Logarithmic Gromov-Witten invariants are deformation invariant. In particular, the logarithmic Gromov-Witten invariants of fibers of the deformation agree, that is,* $\mathcal{N}_{\beta}(X_b) = \mathcal{N}_{\beta}(X_0)$.

We can now use this information and combine it with a correspondence theorem to obtain an expression of the logarithmic Gromov-Witten invariant as a sum of logarithmic invariants with discrete data determined by parametrized tropical curves.

Theorem 12.56 ([2, Theorem 5.4])**.** *Assume Y is a smooth logarithmic stack. Let $X_0 \to b_0$ be a logarithmically smooth morphism with a family X_0/b_0 of logarithmic curves discussed in Definition 12.46. Then,*

$$[\mathcal{M}^{\log}_{\beta}(X_0 \,|\, b_0)]^{vir} \;=\; \sum_{h:\Gamma\to\mathrm{trop}(X)} [\mathcal{M}^{\log}_{\beta,h}(X_0)]^{vir},$$

where $[\mathcal{M}^{\log}_{\beta,h}(X_0)]^{vir}$ is the virtual fundamental class corresponding to the parametrized logarithmic curves to the central fiber X_0 whose combinatorial type is encoded by a parametrized tropical curve $h : \Gamma \to \mathrm{trop}(X)$.

In other words, the dual graph of the source curve is the abstract tropical curve Γ, and the component corresponding to a vertex v is mapped to the irreducible component of X_0 corresponding to $h(v)$.

Corollary 12.57. *Let $X_0 \to b_0$ be a full degeneration into a family of toric surfaces glued along their toric boundaries such that the associated tropical curves are all 3-valent without vertex genus. The logarithmic Gromov-Witten invariant $\mathcal{N}_\beta(X_0 \,|\, b_0)$ splits into a sum*

$$\mathcal{N}_\beta(X_0 \,|\, b_0) \;=\; \sum_{h:\Gamma\to\mathrm{trop}(X)} \int_{[\mathcal{M}^{\log}(X_\Gamma,h)]^{vir}} 1.$$

12.4. **Gromov-Witten invariants: Refined.** When trying to compute curve counts for curves in surfaces over different fields, our methods have to be adapted. Refined invariants are a variation of Gromov-Witten invariants, related to string theory. They were introduced in the algebraic context by Göttsche-Shende in [70] and soon after computed tropically by Block-Göttsche in [16], and Mikhalkin-Itenberg [85]. Instead of numbers, they are Laurent polynomials in one variable q, and can be used to interpolate between Gromov-Witten invariants, counting complex curves (substituting 1 for q) and Welschinger invariants (see [136]), counting real curves (substituting -1 for q), see [16, Theorem 3.10 and 3.11], when considering the same general enumerative problem. In this section, we mainly follow the exposition of the original three papers.

12.4.1. *Gromov-Witten invariants with fixed numbers of nodes and refined invariants.* In the following, we consider curves with singularities. We have already restricted ourselves to nodal curves, so the finer restriction here will be in the form of fixing the number of nodes.

Definition 12.58. We write N^δ_d for the number of δ-nodal curves passing through $\frac{(d+3)d}{2} - \delta$ general points in $\mathbb{P}^2$. It is a Gromov-Witten invariant, i.e., independent of the specific choice of general points.

Remark 12.59. The Gromov-Witten invariant N^δ_d can also be expressed as the degree of the Severi variety, which parametrizes plane curves with δ simple nodes. In fact, using this definition, the invariance of the point condition can be observed.

We can naturally extend the nodal counts N^δ_d to other projective algebraic surfaces:

Definition 12.60. We write $N^{\delta}_{(S,\mathscr{L})}$ for the number of δ-nodal reduced curves in a toric projective algebraic surface S and a line bundle $\mathscr{L}$ on S is which are in the complete linear system $|\mathscr{L}| = \mathbb{P}\big(H^0(S,\mathscr{L})\big)$ and are passing through $\dim(\mathscr{L}) - \delta$ general points on S not containing a toric boundary divisor as a component. In other words, the number $N^{\delta}_{(S,\mathscr{L})}$ is a logarithmic Gromov-Witten invariant (see Definition 12.51).

Dropping the restriction to toric surfaces (and hence, additionally, the restriction to curves not containing a toric boundary component), we may again view this number as the degree of the associated Severi variety.

The numbers $N^{\delta}_{(S,\mathscr{L})}$ can be put in relation to the Euler characteristic of the Severi variety as follows. Let C be a smooth projective curve of genus g, and $C^{[n]}$ its relative Hilbert scheme of n points. Then, from Macdonald [99] and the generating series for the Euler characteristic, we obtain

$$\sum_{i=0}^{\infty} t^{i-g+1}\chi\big(C^{[i]}\big) \;=\; \left(\frac{t}{(1-t)^2}\right)^{1-g}.$$

This sequence can be analogously formulated in the surface setting, i.e., with a projective algebraic surface S and a line bundle $\mathscr{L}$ on S. Let $u_{\mathcal{C}} : \mathcal{C} \to \mathbb{P}^{\delta}$ be the universal curve on a general δ-dimensional subspace of the complete linear system $|\mathscr{L}|$, i.e., the subscheme

$$\mathcal{C} \;=\; \big\{(p,C) \mid p \in [C]\big\} \;\subseteq\; S \times \mathbb{P}^{\delta},$$

where $[C]$ denotes the point corresponding to C in $\mathbb{P}^{\delta}$. That is, the fiber of $u_{\mathcal{C}}$ over $[C] \in \mathbb{P}^{\delta}$ is the curve C. Let $(\mathcal{C}/\mathbb{P}^{\delta})^{[n]}$ denote the relative Hilbert scheme of n points. Then, we can rewrite the generating series involving the Euler characteristic in terms of enumerative invariants:

$$\sum_{i=0}^{\infty} t^{i-g+1}\chi\big((\mathcal{C}/\mathbb{P}^{\delta})^{[n]}\big) \;=\; \sum_{j=0}^{\infty} N^{j}_{(S,\mathscr{L})} \cdot \left(\frac{t}{(t-1)^2}\right)^{j+1-g} \tag{10}$$

Now, we *refine* the invariant by exchanging the Euler characteristic with a more sophisticated "genus count" — the *normalized χ_{-q}-genus*. It is defined similarly to the Euler characteristic of a variety (interpreted as a scheme), replacing the usual dimension of the i-th $\mathbb{Z}$-cohomology group with Hodge numbers, and introducing a variable q.

Definition 12.61. The *normalized χ_{-q}-genus* of a variety X is defined as

$$\chi_{-q}(X) \;=\; \sum_{a,b,\geq 0} (-1)^{a+b} q^b h^{a,b}(X).$$

In the cases discussed above, the χ_{-q}-genus is a polynomial in q, giving us a similar formula as Equation (10):

Proposition 12.62 ([16, Proposition 2.2])**.** *Assume $(\mathcal{C}/\mathbb{P}^{\delta})^{[n]}$ is nonsingular for all n. Then, there exist polynomials $n_0(q), \dots, n_{g(\mathscr{L})}(q)$ such that*

$$\sum_{i=0}^{\infty} \chi_{-q}\big((\mathcal{C}/\mathbb{P}^{\delta})^{[n]}\big)t^n \;=\; \sum_{r=0}^{g(\mathscr{L})} n_r(q) t^r \big((1-t)(1-tq)\big)^{g(\mathscr{L})-r-1}.$$

Here, $g(\mathscr{L}) = \frac{\mathscr{L}(\mathscr{L}+K_S)}{2}$ denotes the genus of a curve in $|\mathscr{L}|$.

Now, this allows us to define refined invariants.

Definition 12.63. The *refined invariants* $BG^{\delta}_{(S,\mathscr{L})}$ are the normalizations of the polynomials n_δ in Proposition 12.62. That is,

$$BG^{\delta}_{(S,\mathscr{L})} \;=\; \frac{n_\delta(q)}{q^\delta}.$$

There exist versions of refined invariants where we drop the assumption of the nonsingularity of the relative Hilbert schemes, and where we can even define the refined invariant for $\delta > g(L)$ respectively, see [70, Proposition 47].

12.4.2. *Tropical refined invariants.* From their inception, refined invariants have been computed using tropical enumerative geometry. To establish a similar tropical theory, we first need to define what it means for a tropical curve to be nodal.

Definition 12.64. Let $h : \Gamma \to \mathbb{R}^2$ be a simple parametrized tropical curve. We say that (Γ, h) is *nodal* if its Newton subdivision only consists of triangles and parallelograms.

Counting refined invariants now boils down to a count of nodal tropical curves with different *refined* multiplicities — we simply replace the standard multiplicity discussed in Definition 12.32 with its *quantum number*.

Definition 12.65. Let $n \in \mathbb{Z}$. The *quantum number* $[n]_q$ is the Laurent polynomial

$$[n]_q \;=\; \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}.$$

Let $h : \Gamma \to \mathbb{R}^2$ be a parametrized tropical curve. The *refined multiplicity* of (Γ, h) is

$$m^q(\Gamma, h) \;=\; \prod_{\substack{v \in V(\Gamma)\\ \text{trivalent}}} [\mathrm{Area}(\Delta_v)]_q,$$

where Δ_v is the triangle in the Newton subdivision dual to $h(\Gamma)$ associated to the vertex v, and $[\mathrm{Area}(\Delta_v)]_q$ is the quantum number of $\mathrm{Area}(\Delta_v)$.

Counting tropical curves with δ nodes corresponds to a tropical enumerative invariant.

Definition 12.66 ([16, Definition 3.7])**.** Let Δ be a lattice polygon, and $\delta \in \mathbb{Z}_{\geq 0}$. The *tropical refined Gromov-Witten invariant* $BG^{\delta,\mathrm{trop}}_{\Delta}$ is defined as

$$BG^{\delta,\mathrm{trop}}_{\Delta} \;:=\; \sum_{h:\Gamma\to\mathbb{R}^2} m^q(\Gamma, h),$$

where we sum over all simple parametrized tropical curves $h : \Gamma \to \mathbb{R}^2$ with an appropriate amount of marked points mapped to a tropically general point configuration in $\mathbb{R}^2$. For the case of $S = \mathbb{P}^2$, this will be $\frac{(d+3)d}{2} - \delta$ points. The tropical refined Gromov-Witten invariant is an invariant and a Laurent polynomial in q.

There exists a correspondence between the refined and the tropical refined invariant.

Theorem 12.67 ([16, Theorem 1.1])**.** *Let $S = \mathbb{P}^2$ and Δ be its corresponding lattice triangle of length d. Then,*

$$BG^{\delta,\mathrm{trop}}_{\Delta} \;=\; BG^{\delta}_{(\mathbb{P}^2,\mathcal{O}(d))}.$$

Again, there exist analogues of this theorem for counting curves in more complicated spaces.

Now we have introduced all types of Gromov-Witten invariants we will consider in the remainder of Part 3. We now study the complex versions of the ruled surfaces on which we want to count, and introduce tropical cylinders as tropical analogues for some ruled surfaces.

12.5. **Complex ruled surfaces.** In the following, we give some background on complex ruled surfaces. These are going to be the classical counterpart of the theory we develop in later chapters. We start by giving an overview of the theory of ruled surfaces. This part of the preliminaries is based on [77, Chapter V.2], with some exposition of intersection theory inspired by [52]. The main examples arising will be the complex counterparts to the tropical theory we will develop in later sections, and were obtained in joint work with Thomas Blomme [22].

12.5.1. *A classification of ruled surfaces.* Ruled surfaces are an important object of study in algebraic geometry. While algebraic geometry on surfaces is hard in general, through their construction, ruled surfaces inherit a lot of properties from the algebraic curves underlying them. This allows us to study both them and the curves lying on them more easily. The discussion here elaborates on the discussion in [77, Chapter V.2].

Definition 12.68. A *ruled surface* is a (nonsingular, projective) surface X together with a surjective morphism $\pi : X \to C$ to a nonsingular curve C, such that each fiber X_y is isomorphic to $\mathbb{P}^1$ for every point $y \in C$, and such that π admits a section.

Equivalently, a ruled surface X can be described as the projectivization of a locally free sheaf $\mathscr{E}$ of rank 2 on C. We write $X = \mathbb{P}(\mathscr{E})$ for this description.

Example 12.69. We give two examples of families of ruled surfaces:

- An easy (but important) example of ruled surfaces are the *Hirzebruch surfaces* $\mathbb{F}_d$. These are ruled surfaces over $\mathbb{P}^1$, arising as the projectivization of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}(-d)$ for some $d \in \mathbb{Z}_{\geq 0}$.
- For every nonsingular projective curve, we can construct a ruled surface by just taking $\mathbb{P}^1 \times C$ and defining π to be the projection onto the second coordinate. There are two examples of this construction that are particularly approachable:
  - The multiprojective space $\mathbb{P}^1 \times \mathbb{P}^1$ (which is also the zeroth Hirzebruch surface), and
  - the cylinder $\mathbb{P}^1 \times E$, where E is a nonsingular elliptic curve.

The surfaces we will be interested in later are ruled surfaces, too. We will analyze them in detail after covering the general theory, see Sections 12.5.2 and 12.5.3.

Lemma 12.70 ([77, Proposition V.2.8])**.** *If X is a ruled surface over C, it is possible to write X as the projectivization of a locally free sheaf $\mathscr{E}$ on C such that $H^0(\mathscr{E}) \neq 0$ and $H^0(\mathscr{E} \otimes \mathscr{L}) = 0$ for all invertible sheaves $\mathscr{L}$ (i.e., line bundles) on C with* $\deg(\mathscr{L}) < 0$. *Its negative degree $e = -\deg(\mathscr{E})$ is an invariant of X.*

Remark 12.71. If we write X as the projectivization of a bundle $\mathscr{E}$ as above, we say that $\mathscr{E}$ is *normalized.* Note that the description of X as the projectivization of a normalized bundle is not necessarily unique — only the degree of the bundle is.

We can characterize all ruled surfaces in terms of the description as the projectivization of the bundle over C.

Theorem 12.72 (Characterization of ruled surfaces, [77, Theorem 2.12])**.** *Let X be a ruled surface over a curve C of genus g, and let X be given as the projectivization of a normalized locally free sheaf $\mathscr{E}$.*

(a) If $\mathscr{E}$ is decomposable into a direct sum of two invertible sheaves, then $\mathscr{E} \cong \mathcal{O}_C \oplus \mathscr{L}$, where $\mathscr{L}$ is invertible and $\deg(\mathscr{L}) \leq 0$*. Further, $e \geq 0$, for e as defined in Lemma 12.70.*

(b) If $\mathscr{E}$ is indecomposable, we have $-2g \leq e \leq 2g-2$.

Example 12.73. The characterization allows us to give a full classification of ruled surfaces over $\mathbb{P}^1$ and over elliptic curves.

- Since every locally free sheaf $\mathscr{E}$ of rank 2 over $\mathbb{P}^1$ is decomposable by Birkhoff-Grothendieck [74, Theorem 2.1], every ruled surface over $\mathbb{P}^1$ is (isomorphic to) a Hirzebruch surface.
- This is not true for ruled surfaces over nonsingular elliptic curves E. Here, both cases exist:
  - Cylinders and twisted cylinders, corresponding to decomposable plane bundles,
  - Two additional ruled surfaces arising as the projectivization of an indecomposable plane bundle $\mathscr{E}$,
    - One for $e = 0$, satisfying the non-split short exact sequence
    $$0 \to \mathcal{O}_E \to \mathscr{E} \to \mathcal{O}_E \to 0, \text{ and}$$
    - another for $e = -1$, satisfying the non-split short exact sequence
    $$0 \to \mathcal{O}_E \to \mathscr{E} \to \mathcal{O}(p) \to 0,$$
    for some point $p \in C$.

In the following two sections, we will study two examples of ruled surfaces in detail. The surface $\mathbb{C}M_0$ is the projectivization of a decomposable plane bundle. The second surface $\mathbb{C}M_1$ is the surface corresponding to the indecomposable plane bundle characterized by $e = -1$.

We call these two ruled surfaces the classical *Möbius strips*, due to their constructions as quotients of the trivial plane bundle under swapping endomorphisms. These quotients correspond to the gluing along the vertical boundary with reverse orientation. We denote them with $\mathbb{C}$ in the front, in order to notationally distinguish them from their tropical analogues, $\mathbb{T}M_0$ and $\mathbb{T}M_1$, which will be the main focus of study in the sequel. If we want to refer to both classical Möbius strips, we write $\mathbb{C}M_\delta$.

12.5.2. *The surface* $\mathbb{C}M_0$. Let $\lambda \in \mathbb{C}^*$. Consider the trivial plane bundle τ over $\mathbb{C}^*$ and the swapping endomorphism

$$\Phi_0 : (z, u_0, u_1) \longmapsto (\lambda z, u_1, u_0) \in \mathbb{C}^* \times \mathbb{C}^2.$$

The quotient of τ by Φ_0 descends to a plane bundle $\mathscr{E}_0$ over the elliptic curve E. As a manifold, E can be expressed as a quotient $E = \mathbb{C}^*/\langle\lambda\rangle$, and we call λ its *length*. Its projectivization $\mathbb{P}(\mathscr{E}_0)$ is the ruled surface $\mathbb{C}M_0$. It can be equivalently constructed as the quotient of $\mathbb{C}^* \times \mathbb{P}^1_{\mathbb{C}}$ by the action

$$\varphi_0\big(z, [u_0 : u_1]\big) \;=\; \big(\lambda z, [u_1 : u_0]\big).$$

We set $w = u_1/u_0$. The dense torus $(\mathbb{C}^*)^2 \subset \mathbb{C}^* \times \mathbb{P}^1_{\mathbb{C}}$ is stable under the action which restricts to

$$\phi_0 : (z, w) \mapsto \left(\lambda z, \frac{1}{w}\right).$$

The ruled surface $\mathbb{C}M_0$ is thus a compactification of the quotient of $(\mathbb{C}^*)^2$ by the action of ϕ_0. We will later in Definition 13.1 see the tropical analogue of this construction when we define the tropical Möbius strip $\mathbb{T}M_0$.

Taking the quotient by Φ_0^2 instead (resp. φ_0^2, ϕ_0^2), we obtain a plane (resp. $\mathbb{P}^1_{\mathbb{C}}$, $\mathbb{C}^*$) bundle over the elliptic curve $\widehat{E} = \mathbb{C}^*/\langle\lambda^2\rangle$. Projectivization yields a ruled surface $\mathbb{C}C_0$ which is a two-to-one cover of $\mathbb{C}M_0$. The surface $\mathbb{C}C_0$ is the total space of the trivial $\mathbb{P}^1_{\mathbb{C}}$-bundle, and the complex analogue of the tropical cylinder $\mathbb{T}C_0$ we will see later in Definition 12.79.

To finish, we need to identify $\mathbb{C}M_0$ in the classification of ruled surfaces over an elliptic curve. The plane bundle $\mathscr{E}_0$ splits into a sum of two line bundles. Indeed, we can consider the sections $s_\pm : z \in \mathbb{C}^* \mapsto (1, \pm 1) \in \mathbb{C}^2$ of the trivial bundle. Both are non-vanishing. Hence, each defines a line bundle inside the trivial plane bundle, and both of the intersections are stable under the action of Φ_0. Thus, both sections induce line bundles over E, yielding two non-intersecting sections of $\mathbb{C}M_0$. Thus, the plane bundle $\mathscr{E}_0$ is split, and $\mathbb{C}M_0$ is the projective completion of a line bundle over E.

By a change of coordinates $w' = \frac{w-1}{w+1}$, $\mathbb{C}M_0$ can also be written as the quotient of $\mathbb{C}^* \times \mathbb{P}^1_{\mathbb{C}}$ by the action of

$$\varphi_0' : (z, w') \longmapsto (\lambda z, -w'), \tag{11}$$

which is the projective completion of a 2-torsion line bundle over $\mathbb{C}E$.

12.5.3. *The surface* $\mathbb{C}M_1$. We now construct the surface $\mathbb{C}M_1$ analogously, considering the trivial plane bundle over $\mathbb{C}^*$ with the twisted swapping action

$$\Phi_1 : (z, u_0, u_1) \longmapsto (\lambda z, z u_1, u_0) \in \mathbb{C}^* \times \mathbb{C}^2,$$

where $\lambda \in \mathbb{C}^*$. The quotient by the action yields a plane bundle $\mathscr{E}_1$ over E, where $E = \mathbb{C}^*/\langle\lambda\rangle$ is the elliptic curve of length λ. Its projectivization $\mathbb{P}(\mathscr{E}_1)$ is the ruled surface $\mathbb{C}M_1$. Projectivizing first, we also obtain $\mathbb{C}M_1$ as the quotient of $\mathbb{C}^* \times \mathbb{P}^1_{\mathbb{C}}$ by the action

$$\varphi_1\big(z, [u_0 : u_1]\big) \;=\; \big(\lambda z, [z u_1 : u_0]\big).$$

The action on the dense torus is given by

$$\phi_1 : (z, w) \mapsto \left(\lambda z, \frac{z}{w}\right). \tag{12}$$

We have $\phi_1^2(z, w) = (\lambda^2 z, \lambda w)$, and the quotient $\mathbb{C}C_1 = \mathbb{C}^* \times \mathbb{P}^1_{\mathbb{C}}/\langle\phi_1^2\rangle$ is the complex counterpart to the tropical cylinder $\mathbb{T}C_1$ we will see later in Definition 12.79. It is the total space of a 2-torsion line bundle over $\widehat{E} = \mathbb{C}^*/\langle\lambda^2\rangle$.

The surface $\mathbb{C}M_1$ is a ruled surface if it has a section. Sections of $\mathbb{C}M_1 \to E$ are sections of $\mathbb{C}C_1 \to \widehat{E}$ invariant under the action induced by φ_1. There are many sections of $\mathbb{C}C_1 \to \widehat{E}$, but two among them are special: As $\mathbb{C}C_1$ is the projective completion of a 2-torsion line bundle, we consider the 0 and ∞ section. Unfortunately, neither of them is invariant under the action induced by φ_1: they are exchanged by the action and thus form a multisection of $\mathbb{C}M_1 \to \mathbb{C}E$. To find sections invariant under the φ_1-action, we have to investigate further.

Take the meromorphic function $\theta : \mathbb{C}^* \to \mathbb{C}$ given by $\theta(z) = \sum_{-\infty}^{\infty} \lambda^{n^2} z^n$. It is the θ-function (see [72]) on $\widehat{E} = \mathbb{C}^*/\langle \lambda^2 \rangle$, and it satisfies

$$\theta(\lambda^2 z) \;=\; \frac{1}{\lambda z}\theta(z) \text{ and } \theta\left(\frac{1}{z}\right) \;=\; \theta(z).$$

It is the only meromorphic function satisfying the first equation up to multiplication by a scalar. Using both equations, we can see that $\theta(-\lambda) = -\theta(-\lambda)$, thus $\theta(-\lambda) = 0$. Moreover, $-\lambda$ is the only zero of θ modulo multiplication by λ^2. Any quotient $f(z) = \frac{\theta(\alpha z)}{\theta(\alpha\lambda z)}$ gives a section of $\mathbb{C}C_1$ since it satisfies

$$f(\lambda^2 z) \;=\; \frac{\lambda \cdot \lambda\alpha z}{\lambda \cdot \alpha z}\frac{\theta(\alpha z)}{\theta(\lambda\alpha z)} \;=\; \lambda f(z).$$

Moreover,

$$\phi_1\left(z, \mu\frac{\theta(\alpha z)}{\theta(\lambda\alpha z)}\right) \;=\; \left(\lambda z, \frac{z}{\mu}\frac{\theta(\alpha\lambda z)}{\theta(\alpha z)}\right) \;=\; \left(\lambda z, \frac{z}{\mu\lambda\alpha z}\frac{\theta(\alpha \cdot \lambda z)}{\theta(\lambda\alpha \cdot \lambda z)}\right).$$

Thus, if $\mu^2\lambda\alpha = 1$, it gives a section of $\mathbb{C}M_1 \to E$ and $\mathbb{C}M_1$ is a ruled surface over E.

Sections of $\mathbb{C}M_1$ pull back to sections of $\mathbb{C}C_1$. These sections always intersect apart from the 0 and ∞-sections of $\mathbb{C}C_1$, which form a multisection by the previous analysis. Thus, we cannot find disjoint sections of $\mathbb{C}M_1$, which prevents $\mathscr{E}_1$ from being split. As the line bundle $\Lambda^2\mathscr{E}_1$ on E is of odd degree, the classification given by [77] ensures that $\mathbb{C}M_1$ is the unique non-split ruled surface of degree 1, given as $\mathbb{P}(\mathscr{E})$ where $\mathscr{E}$ is a non-split vector bundle of rank 2 over an elliptic curve fitting in a short exact sequence

$$0 \to \mathcal{O} \to \mathscr{E} \to \mathcal{O}(p) \to 0.$$

12.5.4. *Curves and divisors.* We conclude by investigating the divisors of ruled surfaces, and curves in them.

The divisor groups of ruled surfaces are particularly simple: they just depend on the surjection onto the underlying curve C.

Lemma 12.74 ([77, Proposition V.2.2])**.** *Let X be a ruled surface with the surjective morphism $\pi : X \to C$, and let $C_0 \subseteq X$ be a section. Then,*

$$\operatorname{Pic}(X) \;\cong\; \mathbb{Z} \oplus \pi^*\big(\operatorname{Pic}(C)\big),$$

where $\mathbb{Z}$ is generated by C_0.

Example 12.75. The Picard group of the Hirzebruch surface $\mathbb{F}_d$ is $\operatorname{Pic}(\mathbb{F}_d) = \mathbb{Z}a \oplus \mathbb{Z}b$, where a and b are two divisors with $a^2 = 0, a \cdot b = 1$ and $b^2 = d$.

Example 12.76 (curves in $\mathbb{C}M_0$ and $\mathbb{C}M_1$). A basis of the second homology group is provided by the class of a section and the class of a fiber F. We prefer to replace the class of a section by the class of the boundary divisor E, which is a 2-section. As $E \cdot F = 2$, they do not form a basis of the $H_2(\mathbb{C}M_\delta, \mathbb{Z})$. Computing the intersection number of a section with E and F, we see that $H_2(\mathbb{C}M_\delta, \mathbb{Z})$ is given as follows.

$$H_2(\mathbb{C}M_\delta, \mathbb{Z}) \;=\; \left\{aE + bF \mid a, b \in \frac{1}{2}\mathbb{Z}, 2b \equiv 2a\delta \bmod 2\right\}.$$

Definition 12.77. Let S_g be a genus g Riemann surface. We say that a curve $\varphi : S_g \to \mathbb{C}M_\delta$ is of bidegree (a, b) on a Möbius strip $\mathbb{C}M_\delta$ if $\varphi_*\big([S_g]\big) = aE + bF \in H_2(\mathbb{C}M_\delta, \mathbb{Z})$.

Note that the boundary divisor is not anticanonical: due to the non-orientability of the Möbius strip, the meromorphic 2-form $\frac{\mathrm{d}z}{z} \wedge \frac{\mathrm{d}w}{w}$ on $(\mathbb{C}^*)^2$ does not induce a 2-form on the quotient $\mathbb{C}M_\delta$. However, it is possible to construct such a 2-form using θ-functions, proving the boundary divisor is numerically equivalent to the canonical class.

In the case of $\mathbb{C}M_0$ ($\mathbb{C}M_1$ is treated similarly), we have the meromorphic 2-form $\Omega_{\mathbb{C}C_0} = \frac{\mathrm{d}z}{z} \wedge \frac{\mathrm{d}w}{w}$ on $\mathbb{C}^*/\langle\lambda^2\rangle \times \mathbb{P}^1_{\mathbb{C}}$, which satisfies $\varphi_0^*\Omega_{\mathbb{C}C_0} = -\Omega_{\mathbb{C}C_0}$. Now, let $\lambda^{1/2}$ be a square root of λ and $\theta(z) = \sum(\lambda^{1/2})^{n^2} z^n$ be the associated θ-function on E, which satisfies $\theta(\lambda z) = \frac{1}{\lambda^{1/2} z}\theta(z)$. Thus, the function $f(z) = \frac{\theta(z)}{\theta(-z)}$ satisfies $f(\lambda z) = -f(z)$, and $f(\lambda^2 z) = f(z)$ and descends to a meromorphic function on $\widehat{E}$. Thus, the 2-form $f(z)\Omega_{\mathbb{C}C_0}$ is φ_0-invariant and descends to a meromorphic 2-form on $\mathbb{C}M_0$. It has poles along the boundary divisor and along a fiber, and zeros along one of the fibers. The poles and zeros along the fibers correspond to poles and zeros of f.

12.6. **Tropical cylinders.** In this section, we will start on pursuing tropical counts on tropical ruled surfaces. We introduce first tropical analogues of ruled surfaces, *tropical cylinders*. Next, we discuss how one can construct tropical curves in the cylinder. Further, we count them (and their algebro-geometric analogue) by adapting the evaluation map to this new setting, and computing its degree. We will mainly focus on their structural properties, as these are needed to study Möbius strips later. This is due to the fact that tropical Möbius strips have a 2-to-1 cover by tropical cylinders. This section is mainly based on [21, Section 3 and 5].

Tropical cylinders correspond to line bundles over tropical elliptic curves .

Definition 12.78. A *tropical elliptic curve* $\mathbb{T}E$ is the quotient of $\mathbb{R}$ by a positive real multiple of $\mathbb{Z}$, i.e., $\mathbb{T}E = \mathbb{R}/l\mathbb{Z}$, where $l \in \mathbb{R}_{>0}$ is called the *length* of $\mathbb{T}E$.

Now, the tropical line bundle over $\mathbb{T}E$ corresponding to a tropical cylinder $\mathbb{T}C$ can be constructed as a quotient of $\mathbb{R}^2$ by a diffeomorphism.

Definition 12.79. Let $\mathbb{T}E$ be a tropical elliptic curve of length l. We construct a tropical cylinder over $\mathbb{T}E$ by considering $\mathbb{Z}$-action on $\mathbb{R}^2$ induced by the diffeomorphism

$$\varphi : (x, y) \mapsto (x + l, y + \delta x - \alpha),$$

where $\delta \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$.

Now, we say that the space $\mathbb{R}^2/\langle\varphi\rangle$ is a *tropical cylinder*.

Example 12.80. No matter which α and δ we choose, the space $\mathbb{R}^2/\langle\varphi\rangle$ is a topological cylinder of infinite height. Further, the space $\mathbb{R}^2/\langle\varphi\rangle$ has a lattice structure, induced by the standard lattice structure on $\mathbb{R}^2$ and determined by its behavior under the map φ. Two of these cases are of particular importance to us:

- The tropical cylinder $\mathbb{T}C_0$, generated as the quotient $\mathbb{R}^2/\langle\varphi\rangle$ where $\alpha = \delta = 0$, i.e., where $(0, y) \sim (l, y)$, and a line crossing the boundary on the left enters the tropical cylinder with the same slope on the right, and
- the tropical cylinder $\mathbb{T}C_1$, where $\alpha = 0$ and $\delta = 1$. Again, we have an identification along the boundary, $(0, y) \sim (l, y)$, but this time the lattice structure changes. It has

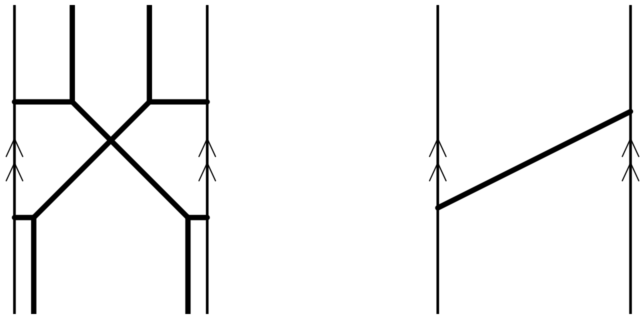


FIGURE 19. On the left, a curve on the cylinder $\mathbb{T}C_0$. On the right, a superabundant loop with slope $\ell/2$ on a cylinder $\mathbb{T}C_{0,\ell/2}$.

monodromy $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, that is a line with slope (a, b) passing through the boundary on the right comes back from the left with slope $(a, b-1)$.

Tropical cylinders correspond to the tropicalization of cylinders in algebraic geometry, which are the ruled surfaces corresponding to decomposable plane bundles, see Example 12.73.

We can define embeddings of abstract tropical curves in tropical cylinders.

Definition 12.81. Let $\mathbb{T}C$ be a tropical cylinder and let Γ be an abstract tropical curve. Then, $h : \Gamma \to \mathbb{T}C$ is a *parametrized tropical curve* on $\mathbb{T}C$ if it satisfies the same requirements as Definition 12.25. Note that the tropical cylinder $\mathbb{T}C$ has a lattice structure different to $\mathbb{R}^2$, which will make a difference in determining the slopes of edges!

We say that a parametrized tropical curve $h : \Gamma \to \mathbb{R}^2$ is of *bidegree* (d_1, d_2) if it has d_2 unbounded horizontal edges in direction $(0, 1)$ (i.e., to the top of the cylinder) and d_1 intersection points (counted with intersection multiplicity) with a vertical line. We encode the tangency profile in direction $(0, 1)$ (i.e., in top horizontal direction) with $|\mu_\infty|$ and the tangency profile in direction $(0, -1)$ (i.e., in bottom horizontal direction) with $|\mu_0|$.

In Figure 19, we see an example of a tropical curve embedded in a tropical cylinder.

Example 12.82. One peculiarity we encounter when studying curves in ruled surfaces is the existence of embedded, balanced curves without ends. In Figure 19, this is the curve on the right. These curves are special cases when trying to count as they vary in a space with strictly larger dimension than we would expect them to. We call them *superabundant loops*.

Remark 12.83. We can construct a parametrized tropical curve in the projective plane, $\widehat{h} : \widehat{\Gamma} \to \mathbb{R}^2$ on $\mathbb{R}^2$ from a parametrized tropical curve on a tropical cylinder, $h : \Gamma \to \mathbb{T}C$. To do so, we remove a finite set of points $\mathscr{Q}$ from Γ such that h restricted to the complement of $\mathscr{Q}$ can be lifted to $\mathbb{R}^2$, the universal cover of any tropical cylinder. Such a set is called *admissible*. We then extend edges cut in the process to get ends. We refer to the corresponding sections of [21] and [18] for more details.

For an example of such an admissible set, consider the points of the left curve in Figure 19 that are intersection points with the left and right side where the two sides are glued together.

This cutting procedure provides some additional structure on the tropical curves. Since the resulting curves are tropical curves in $\mathbb{R}^2$, they fulfill the *Menelaus relation* [108, Proposition 39]. More precisely, let $h(\Gamma)$ be the image of a parametrized tropical curve in $\mathbb{R}^2$. Then, each end e has an associated *moment*, defined as the determinant of the outgoing slope of e and any point $p \in e$. The Menelaus theorem asserts that the sum of moments is 0. This is a consequence of the balancing condition on embedded tropical curves.

Definition 12.84. We write $\mathcal{M}^{\text{trop}}_{g,n}\big(\mathbb{T}C_\delta, (d_1, d_2), \mu_0, \mu_\infty, \nu_0, \nu_\infty\big)$ for the moduli space of parametrized tropical curves in the tropical cylinder $\mathbb{T}C_\delta$ of bidegree (d_1, d_2) and genus g, which have free tangency ν_0 and ν_∞ at the two boundaries E_0 and E_∞, and fixed tangency μ_0 and μ_∞ at the two boundaries E_0 and E_∞ respectively.

To set up a curve counting problem on tropical cylinders, we need to adapt the multiplicity of a parametrized tropical curve to our new setting. Since the image of the parametrized tropical curve wraps around the cylinder, it do not have a dual Newton subdivision anymore. However, we can mimic the area computation of triangles in the Newton subdivision by defining the multiplicity of a vertex as follows:

Definition 12.85. Let $h : \Gamma \to \mathbb{T}C$ be a simple parametrized tropical curve and v a vertex of $h(\Gamma)$. Then, the *multiplicity* $m_v = |\det(a_v, b_v)|$, where a_v and b_v are the slopes of two out of the three different edges adjacent to v. In the planar case, this coincides with the multiplicity we had discussed in Definition 12.32. Now, as before, the multiplicity of the tropical curve is the product over the multiplicities of its vertices, $m(\Gamma, h) = \prod_{v\in V} m_V$.

The moduli space $\mathcal{M}^{\text{trop}}_{g,n}\big(\mathbb{T}C_\delta, (d_1, d_2), \mu_0, \mu_\infty, \nu_0, \nu_\infty\big)$ of parametrized tropical curves on $\mathbb{T}C_\delta$, has dimension $|\mu_0| + |\mu_\infty| + g - 1$ (see [21]). Thus, we can set up the tropical Gromov-Witten invariant in a familiar way:

Definition 12.86. Let d_1 and d_2 be two positive integers, and let μ_0, ν_0, μ_∞ and ν_∞ be four sets such that $\mu_0 + \nu_0$ and $\mu_\infty + \nu_\infty$ are two partitions of d_2. We obtain the *tropical Gromov-Witten invariant on* $\mathbb{T}C$ as

$$N^{\mathbb{T}C}_{g,(d_1,d_2)}(\mu_0, \mu_\infty, \nu_0, \nu_\infty) \;=\; \frac{1}{I^{\mu_0} I^{\mu_\infty}} \sum_{h:\Gamma\to\mathbb{T}C} m(\Gamma, h),$$

where we sum over all parametrized tropical curves whose ends map to a tropically generic point configuration $\mathscr{P} \subseteq \mathbb{T}C$, consisting of $|\nu_0| + |\nu_\infty| + g - 1$ points in the interior, and $|(\mu_0)_i|$ points on E_0 and $|(\mu_\infty)_i|$ points on E_∞ for each i (see [21]).

As in the enumerative problems considered previously, the tropical count corresponds to a classical count of curves in the corresponding ruled surfaces via a correspondence theorem [21, Theorem 4.13].

13. Tropical Möbius strips

We begin by defining our main object of study in this part: tropical Möbius strips, which are the tropical analogues of the two ruled surfaces $\mathbb{C}M_0$ and $\mathbb{C}M_1$ we studied in Sections 12.5.2 and 12.5.3. We continue by defining parametrized tropical curves in them and analyze their multiplicities. Then, we compute the dimension of their moduli space and discuss the positions of points on the curves. All results in this section were obtained in joint work with Thomas Blomme [22].

13.1. **Tropical structures on Möbius strips.** Similar to the treatment of tropical cylinders in Section 12.6, tropical Möbius strips are obtained from a strip $[0;l] \times \mathbb{R}$ by gluing the two boundary components. However, this time we are reversing the orientation. The quotient obtained in this way is a non-orientable surface. To endow it with a lattice structure, we regard it as a quotient of $\mathbb{R}^2$ by the $\mathbb{Z}$-action generated by a fixed-point free orientation-reversing diffeomorphism. These correspond to the diffeomorphisms ϕ_0 (11) and ϕ_1 (12) generating the surfaces $\mathbb{C}M_0$ and $\mathbb{C}M_1$ discussed in Sections 12.5.2 and 12.5.3.

Proposition 13.1. *Let $l \in \mathbb{R}_{>0}$, and let $\mathbb{T}E = \mathbb{R}/l\mathbb{Z}$ be a tropical elliptic curve as in Definition 12.78. There are two Möbius strips obtained as the quotient of $\mathbb{R}^2$ by a $\mathbb{Z}$-action:*

- *the Möbius strip $\mathbb{T}M_0$, obtained as the quotient of $\mathbb{R}^2$ by the action of*
$$\varphi_0 : (x, y) \longmapsto (x + l, -y), \text{ and}$$
- *the Möbius strip $\mathbb{T}M_1$, obtained as the quotient of $\mathbb{R}^2$ by the action of*
$$\varphi_1 : (x, y) \longmapsto (x + l, -y + x).$$

Proof. To induce a lattice structure on the quotient using the natural lattice structure of $\mathbb{R}^2$, we consider lattice preserving diffeomorphisms $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}^2$, i.e., whose derivative lies in $GL_2(\mathbb{Z})$. Up to a change of coordinates, this forces φ to be affine. As we additionally require it to be fixed point free, it is of the form

$$\varphi_{l,\pm,\delta,\alpha} : (x, y) \longmapsto (x + l, \pm y + \delta x + \alpha)$$

for some $l > 0$, $\delta \in \mathbb{Z}$, choice of sign and $\alpha \in \mathbb{R}$. The choice of $+y$ induces orientation preserving diffeomorphisms and corresponds to the tropical cylinders discussed in Section 12.6. We are left with the choice of $-y$, giving the possible lattice structures on Möbius strips.

The conjugation by an invertible affine map preserving the vertical direction gives

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \delta + 2a & -1 \end{pmatrix}.$$

Thus, for non-orientable diffeomorphisms, only the value of δ mod 2 matters. Up to a change of coordinates, given by the translation $y = \tilde{y} + \alpha/2$, α can then be assumed to be 0. Thus, we have two families of tropical Möbius strips that only differ by a scaling factor inside each family. □

Notation 13.2. In the following, we will use $\mathbb{T}M_\delta$ to mean one of the two Möbius strips $\mathbb{T}M_0$ and $\mathbb{T}M_1$.

Concretely, both Möbius strips are obtained by gluing the two boundary components of $[0;l] \times \mathbb{R}$ via $(0, y) \sim (l, -y)$, i.e., orientation-reversing. The strip $[0;l] \times \mathbb{R}$ is a fundamental domain for the action of φ_δ on $\mathbb{R}^2$. However, the monodromy of the lattice structure under the quotient differs: it is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ for $\mathbb{T}M_0$ and $\left(\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right)$ for $\mathbb{T}M_1$. For instance, a curve with horizontal slope crossing the right boundary comes back from the left boundary with slope 0 for $\mathbb{T}M_0$ but with slope 1 for $\mathbb{T}M_1$, compare Figures 20 and 21.

The projection onto the first coordinate induces a fibration of the Möbius strips over the tropical elliptic curve $\mathbb{T}E = \mathbb{R}/l\mathbb{Z}$. Adding the points at top and bottom infinity, i.e., considering the action extended to $\mathbb{R} \times [-\infty; +\infty]$, they can thus be seen as $\mathbb{TP}^1$-bundles over a tropical elliptic curve, just as the cylinders in Definition 12.79. It is natural to expect that they arise as the tropicalization of $\mathbb{P}^1_{\mathbb{C}}$-bundles over an elliptic curve. This is true — we can see that the action φ_0 is the tropical analogue of the action ϕ_0 in (11), and that the action φ_1 is similarly the tropical analogue of the action ϕ_1 in (12). Similarly, the space $\mathbb{C}^2$ tropicalizes to the space $\mathbb{R}^2$. Thus, we can see $\mathbb{T}M_0$ as the tropicalization of $\mathbb{C}M_0$ and $\mathbb{T}M_1$ as the tropicalization of $\mathbb{C}M_1$.

Both strips have two to one covers by tropical cylinders: $\mathbb{T}M_0$ is covered by $\mathbb{T}C_0 = \mathbb{R}^2/\langle\varphi_0^2\rangle$, which is the total space of the trivial line bundle over $\mathbb{T}\widehat{E} = \mathbb{R}/2l\mathbb{Z}$ (see Section 12.5.2); and $\mathbb{T}M_1$ is covered by $\mathbb{T}C_1$, the total space of the unique 2-torsion tropical line bundle over $\mathbb{T}\widehat{E}$ (see Section 12.5.3).

13.2. **Tropical curves in Möbius strips.** In all that follows, we will only consider abstract tropical curves without vertex genus. We adapt the definition of parametrized tropical curves (see Definition 12.25) to our setting.

Definition 13.3. A *parametrized tropical curve* $h : \Gamma \to \mathbb{T}M_\delta$ is a tuple (Γ, h) of an abstract tropical curve Γ and a piecewise affine map $h : \Gamma \to \mathbb{T}M_\delta$ satisfying the balancing condition (see Definition 12.25) at each vertex.

The lattice structure on the tropical Möbius strip defined above induces a monodromy. This induced symmetry preserves the direction of some vectors in $\mathbb{R}^2$, determined by the eigenspaces of the matrices determining the diffeomorphism in Proposition 13.1. For both $\mathbb{T}M_\delta$, the vertical direction $\binom{0}{1}$ is preserved. For $\mathbb{T}M_0$, the other eigenspace is the span of $\binom{1}{0}$, and for $\mathbb{T}M_1$ it is the span of $\binom{2}{1}$. Thus, considering the *vertical direction* for tropical curves is well-defined, and by the *horizontal direction*, we mean the other eigenspace. Hence, the horizontal direction for $\mathbb{T}M_0$ is $\binom{1}{0}$, and for $\mathbb{T}M_1$ is $\binom{2}{1}$.

Example 13.4. In Figure 20, we sketch six tropical curves in the Möbius strip $\mathbb{T}M_0$. We have two natural choices of fundamental domain for $\mathbb{T}M_0$. First, we can choose the fundamental domain to be $[0; l] \times \mathbb{R}$, depicted in (a), (b) and (c): the two sides of the strip are identified by a reflection along the horizontal axis and a translation, and slopes change accordingly.

Alternatively, we can choose $[0; 2l] \times \mathbb{R}_{\geq 0}$ as a fundamental domain. It is obtained from the first fundamental domain by cutting along the dots in the first row of Figure 20, and gluing it back along the right vertical side. We obtain a domain resembling that of the cylinders in Section 12.6, but infinite only in one direction. This is depicted in (a'), (b') and (c'). We call the bottom finite side of the Möbius strip its *soul*. This fundamental domain comes from the intuition that $\mathbb{T}M_0$ has a 2-to-1 cover by the tropical cylinder $\mathbb{T}C_0$ defined in Example 12.80.

Example 13.5. We proceed similarly for $\mathbb{T}M_1$. Taking $[0; l] \times \mathbb{R}$ as a fundamental domain, the gluing is the same as for $\mathbb{T}M_0$, but the monodromy of the lattice structure is different. We give examples in the first row of Figure 21. Choosing the fundamental domain corresponding to the cover by the cylinder, or, alternatively, the cut-up of the tropical Möbius strip along its soul, we obtain the sets $\{0 \leq x \leq 2l, 2y \geq x\}$, or $\{0 \leq x \leq 2l, y \geq \max(0, x - l)\}$ as fundamental domain. For this, we give examples in the second row of Figure 21.

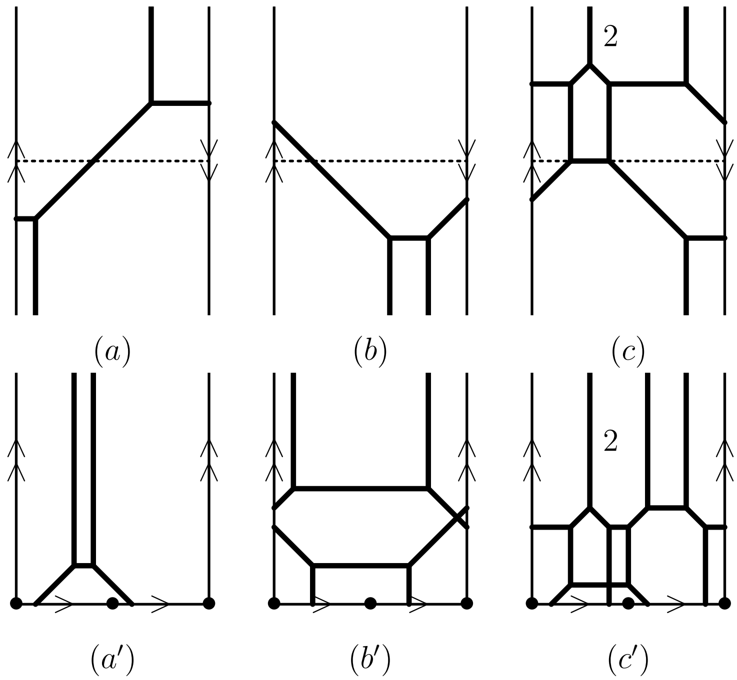


FIGURE 20. Examples of tropical curves inside the Möbius strip $\mathbb{T}M_0$. In the first row we take the fundamental domain to be $[0; l] \times \mathbb{R}$, and in the second row $[0; 2l] \times \mathbb{R}_{\geq 0}$.

13.2.1. *Degree of a tropical curve.* The degree of a parametrized tropical curve is defined as the class inside the tropical homology group $H_{1,1}(\mathbb{T}M_\delta, \mathbb{Z})$ realized by the image of the parametrized tropical curve. See [84] for a definition of the tropical homology groups. In our case, this homology group is isomorphic to $\mathbb{Z}^2$. Moreover, this group has a well-defined intersection form which is unimodular by tropical Poincaré duality. It is thus possible to recover the degree of a tropical curve by intersecting it with cycles that form a basis of $H_{1,1}(\mathbb{T}M_\delta, \mathbb{Z})$.

Both homology groups contain the class F of a fiber (red in Figure 22), and the class E (blue in Figure 22) of the boundary of the Möbius strip. These two classes satisfy $E \cdot F = 2$ since a fiber intersects the boundary of the Möbius strip twice. Thus, E and F span an index 2 sublattice of $H_{1,1}(\mathbb{T}M_\delta, \mathbb{Z})$. They also form a basis of the group if we allow the coefficients to be integer multiples of $\frac{1}{2}$. To describe $H_{1,1}(\mathbb{T}M_\delta, \mathbb{Z})$ fully, we need to consider the cases of $\mathbb{T}M_0$ and $\mathbb{T}M_1$ separately.

- For $\mathbb{T}M_0$, the Möbius strip contains the tropical elliptic curve that goes around the strip once (in horizontal direction), contained in the soul of the Möbius strip. As this curve intersects the fiber F only once and does not intersect the boundary, its class can be expressed as $\frac{1}{2}E$. Thus, elements of $H_{1,1}(\mathbb{T}M_0, \mathbb{Z})$ can be written as

$$aE + bF,\ a \in \frac{1}{2}\mathbb{Z}, b \in \mathbb{Z}.$$

- For $\mathbb{T}M_1$, there exists a curve in the class C_0 which has intersection 1 with both E and F. Thus, $C_0 = \frac{E+F}{2}$. We depict a representative in Figure 21 (a). Hence,

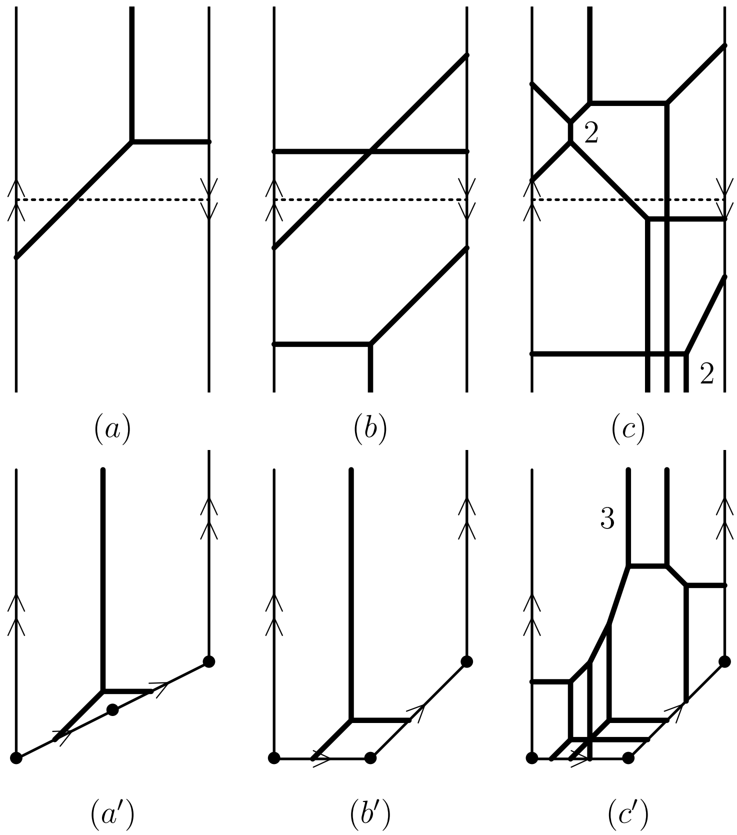


FIGURE 21. Examples of tropical curves inside the Möbius strip $\mathbb{T}M_1$.

elements in $H_{1,1}(\mathbb{T}M_1, \mathbb{Z})$ can be written as

$$aE + bF,\ \ a, b \in \frac{1}{2}\mathbb{Z}, a + b \in \mathbb{Z}.$$

The description can be unified in the following way: the lattice is the set of $aE + bF$ with $a, b \in \frac{1}{2}\mathbb{Z}$ such that

$$2b \equiv 2\delta a \mod 2.$$

In both cases, a curve in the class $aE + bF$ has $2b$ weighted ends since its intersection with E is equal to $2b$, and its intersection with F is equal to $2a$. Here, both intersection numbers are taken with multiplicity.

Example 13.6. We consider the curves in Figure 20 on $\mathbb{T}M_0$. We compute their degree by intersecting with E and F, obtaining respectively

$$\begin{array}{llll} (a) & \frac{1}{2}E + F & (a') & \frac{1}{2}E + F \\ (b) & \frac{1}{2}E + F & (b') & 2E + F \\ (c) & E + 2F & (c') & \frac{3}{2}E + 2F. \end{array}$$

Notice that when considering the second fundamental domain, fibers have two ends going to top infinity, both meeting the top cycle, as depicted in Figure 22.

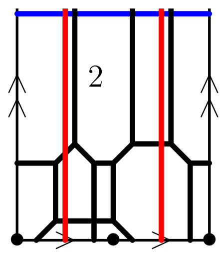


FIGURE 22. A curve in the class $\frac{3}{2}E + 2F$ in $\mathbb{T}M_0$, along with a (red) fiber of the projection to $E = \mathbb{R}/l\mathbb{Z}$ and the boundary E_∞ marked in blue.

Example 13.7. The classes of the curves in $\mathbb{T}M_1$ depicted in Figure 21 are as follows:

(a)	$C_0 = \frac{1}{2}E + \frac{1}{2}F$	(a')	C_0
(b)	$E + C_0 = \frac{3}{2}E + \frac{1}{2}F$	(b')	C_0
(c)	$E + 2F + C_0 = \frac{3}{2}E + \frac{5}{2}F$	(c')	$2E + 2F.$

Definition 13.8. A parametrized tropical curve in the class $aE+bF$ is said to have tangency profile $\mu \vdash 2b$ (i.e., μ is a partition of $2b$) if it has μ_i ends of weight i for each i. This is the generalization of Definition 12.36 to our setting.

Example 13.9. The curves in Figure 20 (c) and (c') have tangency profile $\mu = 2^1 1^2$, and the curve in Figure 21 (c') has tangency profile $\mu = 3^1 1^1$.

Remark 13.10. As in case of cylinders (see Remark 12.83) and abelian surfaces [18], we can construct a parametrized tropical curve $\widehat{h} : \widehat{\Gamma} \to \mathbb{R}^2$ on $\mathbb{R}^2$ from a parametrized tropical curve $h : \Gamma \to \mathbb{T}M_\delta$. To do so, by removing a finite set of points $\mathscr{Q}$ from Γ such that h restricted to the complement of $\mathscr{P}$ can be lifted to $\mathbb{R}^2$, just as we did in Remark 12.83.

In the toric setting as well as in the cylinder case, we have a Menelaus relation between the positions of the ends of a tropical curve, see Remark 12.83. Since these relations come from the existence of an area form, there is a priori no Menelaus relation in the Möbius strip case.

Using the cutting procedure outlined as above and as discussed for cylinders in Remark 12.83, we can still use the information of the Menelaus relation. We use that the positions of ends in vertical direction in $\mathbb{R}^2$ have Menelaus relations. This induces relations on the ends, which we call the *induced Menelaus relations* on the Möbius strip.

13.3. **Dimension of the moduli space of curves.** We compute the dimension of the moduli space of parametrized tropical curves in a Möbius strip $\mathbb{T}M_\delta$. As before, we will only consider simple tropical curves. The definition carries over from the case of tropical curves over $\mathbb{R}^2$, see Definition 12.27:

Definition 13.11. A parametrized tropical curve $h : \Gamma \to \mathbb{T}M_\delta$ is *simple* if it satisfies the conditions of Definition 12.27 for parametrized tropical curves on $\mathbb{R}^2$.

In particular, a simple tropical curve has no contracted edges nor flat vertices, i.e., a vertex where adjacent edges have the same slope.

As previously done for cylinders in Example 12.82, we can characterize curves without ends.

Proposition 13.12. *Let $h : \Gamma \to \mathbb{T}M_\delta$ be a simple tropical curve without ends. Then $h(\Gamma)$ has genus 1, a unique (bounded) edge of slope $\binom{w}{0}$ if on $\mathbb{T}M_0$, or slope $\binom{2w}{w}$ if on $\mathbb{T}M_1$.*

Proof. Let $\mathbb{T}C_\delta \to \mathbb{T}M_\delta$ be the two-to-one cover of the Möbius strip by a tropical cylinder $\mathbb{T}C_\delta$. Up to taking a two-to-one cover of Γ, we can lift the tropical curve $h : \Gamma \to \mathbb{T}M_\delta$ to a tropical curve $\widehat{h} : \widehat{\Gamma} \to \mathbb{T}C_\delta$. The lift $\widehat{h}$ is an immersion and $\widehat{\Gamma}$ still has no end. We have already characterized them as superabundant loops in Example 12.82. Concretely, for the two cylinders $\mathbb{T}C_0$ and $\mathbb{T}C_1$ we consider, they are the following:

- For $\mathbb{T}C_0$, they are of the form $t \mapsto (w \cdot t, c) \in \mathbb{T}C_0$, where $c \in \mathbb{R}$ and $n \in \mathbb{N}$. It is a curve that goes around the cylinder direction k times with slope $\binom{w}{0}$.
- For $\mathbb{T}C_1$, they are of the form $t \mapsto (2w \cdot t, w \cdot t)$.

Both maps are periodic, and can be made compact by quotienting by a sublattice of the periods. □

In particular, we get two types of genus 1 curves without ends: fixed curves meeting the soul of the Möbius strip an odd number of times, and curves that meet the soul of the Möbius strip an even number of times. The latter are obtained as the images of curves without ends in the cover of $\mathbb{T}M_\delta$ by $\mathbb{T}C_\delta$.

For what follows, let $\mathcal{M}^{\text{trop}}_{g,n}(\mathbb{T}M_\delta, (a,b), \mu)$ be the moduli space of genus g parametrized tropical curves in the class $aE+bF$ with tangency profile $\mu \vdash 2b$ and n marked points in $\mathbb{T}M_\delta$. It is a tropical moduli space of parametrized curves, analogous to the one we considered in Definition 12.25.

We will compute the dimension of $\mathcal{M}^{\text{trop}}_{g,n}(\mathbb{T}M_\delta, (a,b), \mu)$ by considering *deformations* of the source abstract tropical curve, and investigate how this deformation influences the image curve $h(\Gamma)$. In particular, the dimension in which we can deform while staying simple coincides with the dimension of that component of the tropical moduli space.

Proposition 13.13. *The dimension of the subspace of $\mathcal{M}^{\text{trop}}_{g,n}(\mathbb{T}M_\delta, (a,b), \mu)$ parametrizing simple tropical curves is $|\mu| + g - 1 + n$.*

Proof. The proof is similar to the computation of the dimension in [21]. Let $h : \Gamma \to \mathbb{T}M_\delta$ be a simple tropical curve. Using the cutting procedure with an admissible set $\mathscr{Q}$, we obtain a parametrized tropical curve $\widehat{h} : \widehat{\Gamma} \to \mathbb{R}^2$. It has new marked points, coming from the points $\mathscr{Q}$ on the non-vertical ends that get identified through the quotient map. A small deformation ofΓ is equivalent to a small deformation of $\widehat{\Gamma}$ where marked points keep their identification under the quotient.

Let e and e' be two ends identified by the quotient map, and let μ_e and $\mu_{e'}$ be their moments. Let $\lambda_e \in \mathbb{Z}$ be the class in $\pi_1(\mathbb{T}M_\delta)$ realized by the loop obtained as the push-down of a path between e and e' in $\widehat{\Gamma}$ to $\mathbb{T}M_\delta$. The condition for ends to keep being identified throughout the deformation of $\widehat{\Gamma}$ can be written as $\mu_e + (-1)^{\lambda_e}\mu_{e'} = 0$.

We have an analogous constraint for each pair of ends. As there is at least one vertical end, the above impose $|\mathscr{Q}|$ linearly independent relations. By the computation of the dimension in the planar case (see [106]), $(\widehat{\Gamma}, h)$ varies in a space of dimension $(|\mu|+2|\mathscr{Q}|)+(g-|\mathscr{Q}|)-1+n$. Subtracting $|\mathscr{Q}|$ yields the expected dimension. □

13.3.1. *Enumerative problems.* To set up the enumerative problems we want to consider, we first give an adaptation of the tropical evaluation maps to our current setting. This will allow us to verify that we only need to consider simple parametrized tropical curves. The moduli space $M_{g,n}^{\text{trop}}\big(\mathbb{T}M_\delta,(a,b)\big)$ of parametrized tropical curves in $\mathbb{T}M_\delta$ has the evaluation map

$$\text{ev}_{\text{trop}}:\mathcal{M}_{g,n}^{\text{trop}}\big(\mathbb{T}M_\delta,(a,b)\big)\longrightarrow \mathbb{T}M_\delta^n,$$

evaluating the position of images of marked points.

If we additionally take into account tangency conditions imposed by two partitions μ and ν, we need to account for marked points indexed by μ: We obtain an exteded evaluation map

$$\widehat{\text{ev}}_{\text{trop}}:\mathcal{M}_{g,n}^{\text{trop}}\big(\mathbb{T}M_\delta,(a,b),\mu\big)\longrightarrow \mathbb{T}M_\delta^n\times\mathbb{T}E^{|\mu|},$$

evaluating the position of images of marked points and ends.

In each case, solving the problem will amount to finding the preimages of a general point in the codomain of the evaluation map. We can now show that this number is finite:

Proposition 13.14. *For a tropically general choice of constraints satisfying the conditions above, there is a finite number of preimages of* $\text{ev}_{\text{trop}}\times\widehat{\text{ev}}_{\text{trop}}$ *in* $\mathcal{M}_{g,n}^{\text{trop}}\big(\mathbb{T}M_\delta,(a,b),\mu\big)$*, all of which are simple.*

Proof. Let $h:\Gamma\to\mathbb{T}M_\delta$ be a tropical curve. Up to a reparametrization by merging parallel edges (in the sense of Remark 12.16), we can assume that h is an immersion. Merging parallel edges of Γ replaces Γ by an abstract tropical curve that is either of smaller genus, has fewer ends, or at least one non-trivalent vertex, but does not change the dimension of the image under the evaluation map. Thus, the evaluation map is not surjective for these combinatorial types and does not map the marked points to a tropically general point configuration. Hence, only simple combinatorial types provide solutions.

For simple combinatorial types, as the codomain has the same dimension, the set of preimages is discrete. Indeed, if the derivative of ev is not injective, it is not surjective either, and a general choice of constraints would not be in the image.

The other combinatorial types where h is an immersion vary in a space of strictly smaller dimension, thus cannot contribute any solution, hence the result. □

13.3.2. *Position of the images of marked points on the solutions.* Analogous to previous enumerative problems, we want to characterize the position of images of marked points on the image of a parametrized tropical curve. For the classical problem in $\mathbb{R}^2$ and the case of cylinders, given a parametrized curve meeting the constraints above, the complement of images of marked points on it has no cycle and each connected component contains a unique end. This is due to the fact that both cycles and paths relating two distinct ends can deform in a dimension one space, which is prohibited by finiteness of the number of solutions. Thus, the complement of images of marked points is a forest of trees, each rooted at an end, whose leaves are images of marked points.

To obtain a similar characterization for Möbius strips, we have to distinguish between two types of cycles, *orienting* cycles and *disorienting* cycles.

Definition 13.15. A *disorienting* tropical cycle on a tropical Möbius strip $\mathbb{T}M_\delta$ is a tropical cycle passing through the soul of $\mathbb{T}M_\delta$. Conversely, an *orienting* cycle is a tropical cycle on $\mathbb{T}M_\delta$ that does not meet the soul.

We will see now that this influences the structure of the cells in the moduli space, as disorienting cycles cannot be deformed without moving at least one of the adjacent edges.

Proposition 13.16. *Let $h : \Gamma \to \mathbb{T}M_\delta$ be a simple parametrized tropical curve of genus g in the class $aE + bF$ which maps marked points to a general configuration of $2b + g - 1$ points $\mathscr{P}$. Then, each component of the complement of images of marked points $h(\Gamma) \setminus \mathscr{P}$ satisfies either of the following conditions:*

(i) It contains a unique end and no cycle, or
(ii) It contains a unique disorienting cycle and no end.

Proof. We obtain a space $\Gamma_{\mathscr{P}}$ by disconnecting the curve at each image of a marked point p, replacing the edge e it lies on by two closed half-edges e_p and e'_p. The connected components of $\Gamma_{\mathscr{P}}$ correspond exactly to the connected components of $\Gamma \setminus h^{-1}(\mathscr{P})$. As there are $2b+g-1$ points in $\mathscr{P}$, and the Euler characteristic of Γ is $1 - g - 2b$, we have

$$\chi(\Gamma_{\mathscr{P}}) \;=\; \chi(\Gamma) + 2b + g - 1 = 1 - g - 2b + 2b + g - 1 = 0.$$

Let Γ_i be a connected component of $\Gamma_{\mathscr{P}}$. The Euler characteristic of Γ_i is $1 - g_i - x_i$, where g_i is the first Betti number of the component, and x_i the number of ends that it contains.

If we have $g_i = x_i = 0$, the component is a tree with a point of $\mathscr{P}$ at each of its outgoing edges since it contains no end. Here, two leaves e_p and e'_p incident to the image of a single marked point p can belong to the same component. As the genus is 0, we can lift the component to $\mathbb{R}^2$. There, we have the tropical Menelaus condition between the position of ends, yielding a relation between the position of the images of marked points, see Remark 12.83 for details. Notice that if e_p and e'_p are leafs of the same component, their contributions to the relation cancel. First, assume the relation is nonempty. Then, this corresponds to a non-general choice of the constraints, contradicting genericity.

Now, assume the relation is empty. Then, if a leaf e_p is in the component, the leaf e'_p has to be as well. Therefore, the component corresponds to a curve without ends. It is thus a genus 1 curve that varies in a 1-dimensional space containing a unique point condition. This concludes the proof if $\Gamma_{\mathscr{P}}$ is connected.

Thus, we can assume that we always have $1 - g_i - x_i \leq 0$. As the sum is 0, each summand is 0 and we have two possibilities: either $g_i = 0$ and $x_i = 1$; or $g_i = 1$ and $x_i = 0$.

The first case corresponds to components without cycles and with a unique end. For the second case, as it is always possible to deform an orienting cycle, the unique cycle has to be disorienting. This concludes the proof. □

Remark 13.17. Proposition 13.16 essentially states that disorienting cycles behave like ends, as they can be deformed to comply with the deformation of an adjacent edge.

Example 13.18. In Figure 23, we can see two tropical curves with a unique disorienting cycle. Contrarily to orienting cycles, we see that it is not possible to deform the cycle while fixing the adjacent edges. Instead, it is possible to deform the cycle while varying only one of the adjacent edges: the right vertical edge for (a) (over $\mathbb{T}M_0$) and the only vertical edge for (b) (over $\mathbb{T}M_1$).

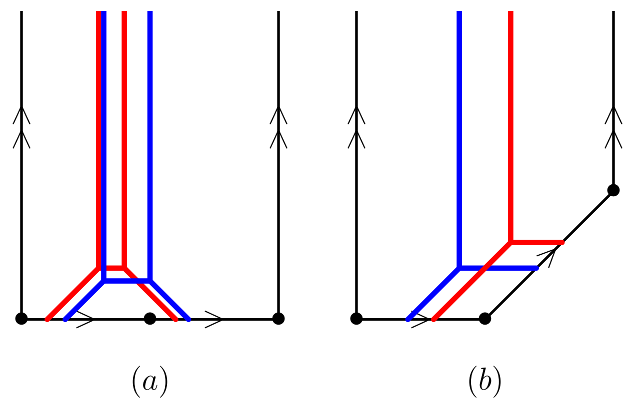


FIGURE 23. A disorienting cycle with a deformation moving only one adjacent edge. (a) is on $\mathbb{T}M_0$ and (b) is on $\mathbb{T}M_1$.

13.4. **Multiplicities.** In the preliminaries for this part, Section 12, we had seen that tropical curve counting usually involves defining correct multiplicities for the curves we wish to count. The case of curves in tropical Möbius strips is no different. Let $h : \Gamma \to \mathbb{T}M_\delta$ be a parametrized tropical curve of genus g, tangency profile $\mu + \nu$ (where μ and ν are partitions) with $n = |\nu| + g - 1$ general fixed images of marked points $\mathscr{P}$, where the ends indexed by μ are fixed. By genericity, h is a simple tropical curve. For an edge e, let u_e be its slope, and $u_e = w(e)\widehat{u_e}$, where $\widehat{u_e}$ is the primitive lattice vector in direction u_e. Using Proposition 13.16, we choose an orientation of the edges of Γ such the edges inside a component of $h(\Gamma) \setminus \mathscr{P}$ point toward the disorienting cycle or the free end in the component. For the edges of disorienting cycles, we choose an arbitrary orientation of the cycle.

Each Möbius strip is endowed with a lattice structure as given in Proposition 13.1. Thus, given a vertex v of Γ, we obtain a rank 2 lattice N_v at the point $h(v) \in \mathbb{T}M$. Moreover, given a bounded edge e in Γ, the lattice structure can be trivialized on the interior of the edge, so that we can define a rank 2 lattice N_e containing the slope u_e. Moreover, for each flag $v \in e$ of Γ, we have a map $N_v \to N_e$ identifying both lattices. Thus, we have the following map of lattices,

$$\begin{array}{rccc} \Theta : & \bigoplus_{v \in V(\Gamma)} N_v & \longrightarrow & \bigoplus_{e \in E_b(\Gamma)} N_e/\langle \widehat{u_e} \rangle \oplus \bigoplus_1^n N_{v_i} \oplus \bigoplus_\mu N_e/\langle \widehat{u_e} \rangle \\ & (\phi_v) & \longmapsto & \big((\phi_{\mathfrak{s}(e)} - \phi_{\mathfrak{t}(e)}), (\phi_{v_i}), (\phi_{\mathfrak{s}(e)}) \big) \end{array},$$

where $\mathfrak{s}(e)$ (resp. $\mathfrak{t}(e)$ if e is bounded) is the source (resp. target) of an edge e, and v_i is the vertex corresponding to the i-th marked point. The domain is indexed by the vertices of the tropical curve (including the images of marked points), and the codomain is indexed by the bounded edges of the curve along with the images of marked points and ends whose position is evaluated. The curve is simple, hence trivalent. Counting the number of pairs $(v \in e)$ in two ways and computing the Euler characteristic, we have two equations:

$$3|V(\Gamma)| \ = \ 2|E_b(\Gamma)| + n + |\mu| + |\nu| \text{ and } |V(\Gamma)| - |E_b(\Gamma)| \ = \ 1 - g.$$

Thus, Γ has $|V(\Gamma)| = |\mu| + |\nu| + 2g - 2 + n$ vertices, and $|E_b(\Gamma)| = 3g - 3 + |\mu| + |\nu| + n$ bounded edges. As $n = |\nu| + g - 1$, both ranks are equal to $2|\mu| + 4|\nu| + 6g - 6$. Thus, we can compute the lattice index of the image of Θ. Taking bases of the lattices, we can see Θ as an integer matrix whose lattice index is equal to $|\det\Theta|$.

Definition 13.19. We define the multiplicity of a parametrized tropical curve to be

$$m(\Gamma, h) \;=\; |\det\Theta| \prod_{e\in E_b(\Gamma)} w(e).$$

The following proposition gives a concrete expression for the multiplicity.

Proposition 13.20. *Let $h:\Gamma\to\mathbb{T}M_\delta$ be a simple parametrized tropical curve of genus g in the class $aE+bF$ of tangency profile $\mu+\nu\vdash 2b$ whose marked ends map to a general configuration of $|\nu|+g-1$ points $\mathscr{P}$ under the evaluation map, and which has the position of the ends indexed by μ fixed. Let k be the number of disorienting cycles in the complement of the images of marked points. Then, the multiplicity of $h:\Gamma\to\mathbb{T}M_\delta$ is given by:*

$$m(\Gamma, h) \;=\; \frac{2^k}{I^\mu} \prod_{\substack{v\in V(\Gamma)\\ \text{trivalent}}} m_v,$$

where m_v is the usual vertex multiplicity, defined as $m_v = |\det(a_v, b_v)|$ if a_v and b_v are two out of the three outgoing slopes (see Definition 12.32), and $I^\mu=\prod_i i^{\mu_i}$ as discussed in Notation 12.35.

Proof. The statement follows by computing the determinant of the lattice map Θ. To write the matrix of Θ, we choose a basis for each N_v. For each e, choosing an identification with $\mathbb{Z}^2$, the linear form $\det(\widehat{u_e}, -)$ provides a coordinate (i.e., a bijection to $\mathbb{Z}$) of the quotient lattice $N_e/\langle\widehat{u_e}\rangle$. The matrix of Θ is constructed as follows.

- For fixed ends, we have a 1×2 block comprised of $\det(\widehat{u_e}, -)$ evaluating the position of the unique adjacent vertex in the quotient.
- For bounded edges, we have two of these blocks, one for each adjacent vertex. Substituting the primitive slope $\widehat{u_e}$ by the actual slope $u_e = w(e)\widehat{u_e}$ divides the determinant by $w(e)$. This cancels with the $w(e)$ in the definition of $m(\Gamma,h)$. Thus, we can assume that the slopes are the true slopes u_e. Hence, we obtain a map Θ' with $m(\Gamma,h)=\frac{1}{I^\mu}|\det\Theta'|$ where we divide by the product of weights of marked points since they do not appear in the original product.
- Evaluating at each of the $|\nu|+g-1$ marked points, contributes $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Applying Laplace expansion for the determinant with respect to these deletes one of the $\det(u_e,-)$ in each row corresponding to an adjacent bounded edge. We are left with the matrix of the lattice map Θ for the tropical curve where each point has been removed, and the bounded edge containing it is replaced by a pair of ends whose positions we evaluate.

Thus, we may assume that there are no images of marked points. The rest of the computation is done recursively in two steps.

The first step is needed to prove that the multiplicity in the sense of Nishinou-Siebert as in [113] coincides with the one defined by Mikhalkin in [106]. We briefly recall it for the sake of completeness. The second step is specific to our case, dealing with disorienting cycles. Notice that the matrix of Θ splits into blocks for the different connected components of Γ, of which there might be several after the last step above.

Step 1. We prune the branches of the complement of the images of marked points. If v is a vertex adjacent to two fixed ends with slopes u_1 and u_2, the column of v only has the following blocks:

$$\begin{array}{|c|}\hline \det(u_1,-) \\ \hline \det(u_2,-) \\ \hline \det(u_e,-) \\ \hline \end{array}.$$

The first two rows correspond to the evaluation of the position of the adjacent ends, and the last row to the position of the remaining adjacent edge e. This row does not appear if e is an end. As the first two rows are the only non-zero elements, we expand $m_v = \det(u_1,u_2)\cdot|\det\Theta_{\Gamma\setminus v}|$ where $\Theta_{\Gamma\setminus v}$ is the lattice of the image of $h|_{\Gamma\setminus v}$, where e disappears if it was unbounded or becomes an end whose position is evaluated.

If there are no cycles in the complement of the images of the marked points, this suffices to compute the determinant.

Step 2. Else, by Proposition 13.16 Γ has a unique disorienting cycle, all of whose adjacent edges are ends whose position is evaluated. Let $v_1,\dots,v_p$ be the vertices on the cycle, $u_1,\dots,u_p$ be the slopes of the edges of the cycle, and $r_1,\dots,r_p$ the slope of the adjacent ends, such that r_i is adjacent to the edges of slopes u_i, r_i and u_{i+1}, indices taken modulo p. We have $u_i + r_i = u_{i+1}$. The matrix has the following form:

$$\left(\begin{array}{ccccc}
\det(u_1,-) & & & & \det(u_1,-) \\
-\det(u_2,-) & \det(u_2,-) & & & \\
 & -\det(u_3,-) & \ddots & & \\
 & & \ddots & \det(u_{p-1},-) & \\
 & & & -\det(u_p,-) & \det(u_p,-) \\
\hline
\det(r_1,-) & & & & \\
 & \det(r_2,-) & \ddots & & \\
 & & \ddots & \det(r_{p-1},-) & \\
 & & & & \det(r_p,-)
\end{array}\right)$$

The columns correspond to the vertices $v_1,\dots,v_p$, the top rows to the edges of the disorienting cycle, and the bottom rows to the evaluation of the fixed ends. As the cycle is disorienting, it is not possible to trivialize the lattice along the cycle. Then, both entries in the last column in the upper block are positive, and we have a non-zero determinant.

For the copy of $\mathbb{Z}^2$ corresponding to v_i, we can take a basis of the form (r_i, r_i') such that the block $\det(r_i,-)$ becomes $(0\ 1)$. We then expand with respect to the bottom rows:

$$\left(\begin{array}{ccccc}
\det(u_1,r_1) & & & & \det(u_1,r_p) \\
-\det(u_2,r_1) & \det(u_2,r_2) & & & \\
 & -\det(u_3,r_2) & \ddots & & \\
 & & \ddots & \det(u_{p-1},r_{p-1}) & \\
 & & & -\det(u_p,r_{p-1}) & \det(u_p,r_p)
\end{array}\right)$$

For each column, as $m_{v_i} = \det(u_i, r_i) = \det(u_{i+1}, r_i)$, we can factor the vertex multiplicity m_{v_i}. Finally, we are left with computing the following determinant:

$$\det \begin{vmatrix} 1 & & & & 1 \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & \ddots & 1 & \\ & & & -1 & 1 \end{vmatrix} = 2.$$

□

Remark 13.21. Note that the multiplicity of the curve depends on the position of the images of the marked points and not just on its combinatorial type. This is similar to the multiplicity of a tropical curve in a linear system on an abelian surface introduced in [19].

Using the above results, we can now finally define *tropical Gromov-Witten invariants on* $\mathbb{T}M_\delta$:

Definition 13.22. We define the *tropical Gromov-Witten invariants on* $\mathbb{T}M_\delta$ as

$$N^\delta_{g,aE+bF}(\mathscr{P}) \;=\; \sum_{h:\Gamma\to\mathbb{T}M_\delta} m(\Gamma,h),$$

where we sum over parametrized tropical curves of genus g in the class $aE + bF$ in the Möbius strip $\mathbb{T}M_\delta$ whose marked points evaluate to a tropically generic point configuration $\mathscr{P}$. Further, we can define *tropical Gromov-Witten invariants on* $\mathbb{T}M_\delta$

$$N^\delta_{g,aE+bF}(\mu,\nu) = \sum_{h:\Gamma\to\mathbb{T}M_\delta} m(\Gamma,h),$$

where curves additionally have the tangency behavior prescribed by μ and ν. Note that we already included the division by the multiplicity of fixed ends in the definition of $m(\Gamma, h)$, so we do not have to divide again here.

Now, we wish to compute refined invariants. Analogous to the planar case we had discussed in Section 12.4, we can define refined multiplicities for curves in $\mathbb{T}M_\delta$ by substituting multiplicities by the Laurent polynomial that is their quantum number.

Definition 13.23. Let $h : \Gamma \to \mathbb{T}M_\delta$ be a parametrized tropical curve of genus g in the class $aE + bF$, tangency profile $\mu + \nu \vdash 2b$ meeting the constraints. We set

$$m(\Gamma,h)^q \;=\; \frac{1}{I^\mu_q} 2^k \prod_{\substack{v\in V(\Gamma)\\ \text{trivalent}}} [m_v]_q,$$

where k is the number of disorienting cycles in the complement of the images of the marked points, $[a]_q = \frac{q^{a/2}-q^{-a/2}}{q^{1/2}-q^{-1/2}}$ denotes the q-analog defined in Definition 12.65, and we set $I^\mu_q = \prod [i]^{\mu_i}_q$.

We then consider the refined counts

$$BG^\delta_{g,aE+bF}(\mathscr{P}) \;=\; \sum_{(\Gamma,h)} m(\Gamma,h)^q,$$

,

where we sum over all parametrized tropical curves whose marked points get mapped to the tropically generic point configuration $\mathscr{P}$. Further, we consider the relative refined count $BG^{\delta}_{g,aE+bF}(\mu,\nu)(\mathscr{P})$.

13.5. **Invariance.** To show that the numbers we defined in the previous section are reasonable objects to study, we need to show that they are invariants. As refined multiplicities specialize on classical multiplicities when q goes to 1, we can show both invariance statements at the same time.

Theorem 13.24. *The counts $BG^{\delta}_{g,aE+bF}(\mathscr{P})$ and $BG^{\delta}_{g,aE+bF}(\mu,\nu)(\mathscr{P})$ do not depend on the choice of the constraints as long as they are general.*

Proof. We reduce to the planar case [85, Theorem 1], and proceed similarly to the proof of analogous statements in [18, 19, 21]. Let $(\mathscr{P}_t)_{t\in[0;1]}$ be a generic path between two general choices of constraints $\mathscr{P}_0$ and $\mathscr{P}_1$ in $\mathbb{T}M_\delta$. We assume that along the path $(\mathscr{P}_t)_{t\in[0;1]}$, the constraints move one at a time: either a unique point or the position of a fixed end moves. Let $h:\Gamma\to\mathbb{T}M_\delta$ be a simple solution. By Proposition 13.16, when removing all marked points and fixed ends from $h(\Gamma)$ except the moving constraint, we are in one of the following situations:

- we have a unique orienting cycle,
- we have a path linking two non-fixed ends,
- we have a component with two disorienting cycles,
- we have a component with a disorienting cycle and an non-fixed end.

When slightly moving the constraint, the solutions slightly deform accordingly. As long as no edge length goes to 0, both combinatorial type and multiplicity of the curves remain the same, hence we have local invariance. It remains to check what happens when an edge length goes to 0, i.e., $\mathscr{P}_t$ becomes non-general for some value of t. We call this *crossing a wall*.

Assume $\mathscr{P}_{t_*}$ is non-general. Hence, there is a solution with at least a quadrivalent vertex, obtained by deformation of a simple solution. Moreover, for every $t\in]t_*-\varepsilon;t_*+\varepsilon[$ with $t\neq t_*$, $\mathscr{P}_t$ is general again and the solutions are simple. Let $h:\Gamma\to\mathbb{T}M_\delta$ be a simple solution near the wall. To reduce to the planar setting, we use the cutting procedure from Remark 13.10. As there is a minimal length for non-contractible cycles given by the length l of the underlying tropical elliptic curve $\mathbb{T}E=\mathbb{R}/l\mathbb{Z}$, we can choose an admissible set $\mathscr{Q}\subset h(\Gamma)$ such that none of the cut edges has a length that vanishes under deformation. Thus, we are left with deformation of tropical curves inside $\mathbb{R}^2$. The choice of $\mathscr{Q}$ ensures that for $t\in]t_*-\varepsilon;t_*+\varepsilon[$, $\mathscr{Q}$ deforms and keeps being admissible through the deformation. According to [85, Lemmas 3.1 and 3.3], the following can occur during the deformation:

- a rectangular shaped cycle containing the image of a marked point gets contracted to a pair of quadrivalent vertices linked by a pair of parallel edges,
- a quadrivalent vertex appears,
- the image of a marked point merges with a vertex.

The first two cases are treated as in [85, Lemmas 3.1 and 3.3]. In the first case, there is one curve on each side of the wall with the same multiplicity, as the marked point just jumps from one side of the cycle to the other (see [85, Lemma 3.1]). This is depicted in Figure 24(a). For quadrivalent vertices, there are three adjacent combinatorial types, and the invariance is already proved in [21, 85], and depicted in Figure 24(b) and (b').

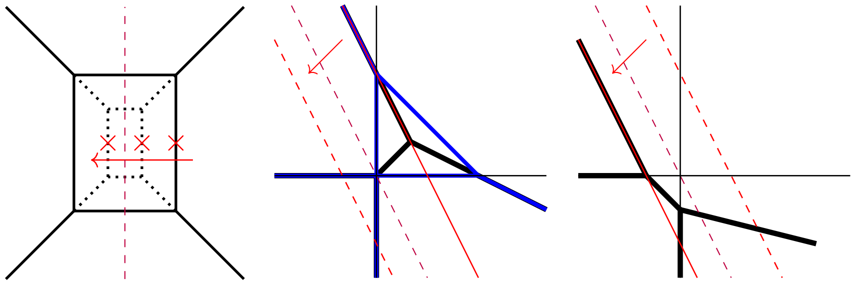


FIGURE 24. In (a), a cycle gets contracted by moving the marked point. Deforming further, the cycle opens again and the marked point has changed sides. In (b) and (b'), the movement of the red line forces the solution to pass through a quadrivalent vertex. If the red line passes into the upper right quadrant, we have two tropical solutions depicted in (b) in black and blue. If not, we have one tropical solution depicted in (b').

We are left with the case where a marked point meets a vertex during the deformation. In the classical case, there are only two adjacent combinatorial types that provide solutions since the complement of marked points is a forest (a set of rooted trees). Here, due to the different description of solutions given in Proposition 13.16, we have more cases, just as in the case of curves in abelian surfaces, see [19].

Let v be the vertex and p the marked point that meet through the deformation. Let a, b and c be the three edges adjacent to v and $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be the connected components of $\Gamma - \big(h^{-1}(\mathscr{P}\backslash\{p\}) \cup \{v\}\big)$. Some of these components may coincide if v lies on a cycle.

By Proposition 13.16, $\mathscr{A} \cup \mathscr{B} \cup \mathscr{C} \cup \{v\}$ contains exactly two free ends, two disorienting cycles or one of each, and the marked point p separates the two (or lies on the orienting cycle in case the two disorienting cycles have non-disjoint support).

(i) Assume that all three components are distinct. By the discussion above, exactly one component is bounded, say $\mathscr{A}$, while the other components contain either a free end or a disorienting cycle. Then p must lie on either b or c. Thus, degeneration leads to two different combinatorial types with the same multiplicity, depending on which side of the line spanned by a the point p lies. This corresponds to Figure 25(b).

(ii) Now, assume that $\mathscr{B} = \mathscr{C}$. Then, $\mathscr{B}$ has at least one cycle γ passing through b and c. If γ is orienting, p needs to lie on b or c and we conclude as in (i). If γ is disorienting, there are three combinatorial types. The first corresponds to $p \in a$, and the other to $p \in b$ or c. If p lies on either b or c, the disorienting cycle γ is now cut by the image of a marked point. Since $\mathscr{B} = \mathscr{C}$, both combinatorial types this contributes lie on the other side of the wall, and the invariance of multiplicities follows by

$$2^{k-1}\prod m_v + 2^{k-1}\prod m_v \;=\; 2^k \prod m_v.$$

This case is depicted on Figure 25(c).

(iii) Finally, assume $\mathscr{A} = \mathscr{B} = \mathscr{C}$. For each pair of edges in a, b, c there exists a cycle γ_{ij} passing through both. We have $\gamma_{ab} = \gamma_{ac} + \gamma_{bc}$ in $H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$. Taking the image in $H_1(\mathbb{T}M_\delta, \mathbb{Z}/2\mathbb{Z})$, we see that at least one cycle is orienting. By Proposition 13.16, at

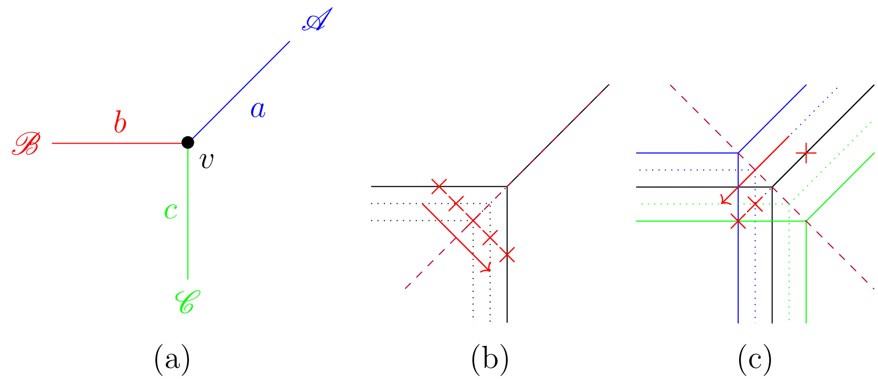


Figure 25. In (a) the three components cut by the removal of vertex v. In (b), the wall where only two out of the three adjacent combinatorial types are allowed. The marked point moves from one edge to the other, deforming the curve. In (c), the wall where all three adjacent combinatorial types are allowed. When the marked point crosses the wall, the black tropical curve becomes the pair of curves given in blue and green. The black curve has an additional disorienting cycle which gets cut on the other side of the wall.

most one cycle is orienting, thus exactly one, say γ_{bc} is orienting. Hence, p needs to lie on γ_{bc}, i.e., in b or c, leading to two adjacent combinatorial types, and we conclude as in (i).

□

Remark 13.25. It would be interesting to see if the factor 2^k appearing in the complex multiplicity can also be refined.

14. Floor diagrams on tropical Möbius strips

In this section we discuss floor diagrams as a counting tool for tropical curves. We begin with a review of the literature in Section 14.1, and continue afterwards by adapting the existing tool to work on tropical Möbius strips. We prove the analogue of Theorem 14.8, that floor diagram counts with the suitable multiplicities coincide with the tropical curve counts discussed in the previous section.

14.1. **Preliminaries: Floor diagrams in the plane.**

Now that we have determined different objects we wish to count, we introduce techniques on *how* to count. We construct and study *floor diagrams*, combinatorial tools used to count tropical, and, using Mikhalkin's correspondence theorem (Theorem 12.34), algebraic curves.

Floor diagrams are a useful combinatorial tool for counting tropical curves. They were introduced by Brugallé-Mikhalkin in [32, 33] and have since been used countless times in tropical enumerative geometry. Floor diagrams are combinatorially easier than tropical curves, but can similarly be counted with multiplicity to recover the tropical (and hence the algebraic) curve count. We mainly follow the exposition in [39, Chapter 11] and [37, Day 5].

Definition 14.1. We say that a point configuration $\mathscr{P}$ is in *horizontally stretched* position if it is contained in a small horizontal strip and all points are very far apart.

A parametrized tropical curve $h : \Gamma \to \mathbb{R}^2$ where the image of the (in our current setting $2d + g - 1 + |\beta|$) contracted ends under the evaluation map is in horizontal position is called *floor decomposed*. That is, the Newton subdivision associated to Γ contains all possible vertical edges.

We give an example of a floor decomposed curve in Figure 26. We have already seen multiple examples of non-floor decomposed curves, for instance in Figure 1.

Definition 14.2. Let $h : \Gamma \to \mathbb{R}^2$ be a floor-decomposed parametrized tropical curve. We call the horizontal edges of $h(\Gamma)$ the *elevators* of $h(\Gamma)$. They correspond to the vertical line segments in the Newton subdivision dual to $h(\Gamma)$.

A connected component of $h(\Gamma)$ without its elevators is called a *floor*. In the image $h(\Gamma)$ of the left parametrized tropical curve in Figure 26, floors are the black connected components.

Lemma 14.3 ([39, Lemma 11.2.2])**.** *Let $h : \Gamma \to \mathbb{R}^2$ be a floor decomposed parametrized tropical curve whose images of contracted ends (in our current setting, $2d+g-1+|\beta|$) form a horizontally stretched point configuration $\mathscr{P}$. Then, each floor and each elevator of $h(\Gamma)$ contains precisely one point in $\mathscr{P}$.*

This means that for the solution of our tropical counting problem it is only relevant which elevators and floors the parametrization of a tropical curve possesses, their multiplicities, and which point (i.e., contracted end) lies on which elevator. All of this information gets encoded in a new combinatorial gadget: *floor diagrams*.

Definition 14.4. A *floor diagram* $\mathfrak{D}$ of degree d, genus g and tangency profile (α, β) (where $I\alpha + I\beta = d$) is a bipartite weighted directed graph of genus g on an ordered vertex set such that all vertices are colored either black (corresponding to elevators) or white (corresponding to floors) and that

- there are d white and $d + g + |\beta| - 1$ black vertices,
- there are $\alpha_i + \beta_i$ unbounded ends of weight i to the right,
- each white vertex is of divergence 1 (i.e., has one fewer (weighted) incoming edge from the left than outgoing edges on the right), and
- each black vertex is of valence 2 and divergence 0 (i.e., has the same weighted number of incoming and outgoing edges).

Construction 14.5. We can construct a floor diagram $\mathfrak{D}_{\Gamma,h}$ from the image of a parametrized tropical curve $h : \Gamma \to \mathbb{R}^2$ of degree d and genus g as follows.

(1) Turn each image of a marked point of Γ on an elevator of $h(\Gamma)$ into a black vertex.
(2) Contract each floor of $h(\Gamma)$ into a white vertex.
(3) Weight edges in the floor diagram with the multiplicity of the corresponding elevator in $h(\Gamma)$.

The result is, in fact, a floor diagram of degree d and genus g: Since the image of every floor-decomposed parametrized tropical curve has exactly d floors, each with one contracted end on them, the diagram $\mathfrak{D}_{\Gamma,h}$ has d white vertices. The other $2d+g-1+|\beta|-d = d+g-1+|\beta|$ images of marked points have to lie on the elevators, resulting in $d+g-1+|\beta|$ black vertices. Since each image of a marked point on an elevator of the tropical curve has two adjacent edges of the same weight, the divergence condition for black vertices is satisfied. The divergence for white vertices follows by the fact that in the contraction of the floor, the only two ends we contract are in direction $(0, 1)$ and $(-1, -1)$. Thus, by balancing, each floor needs an

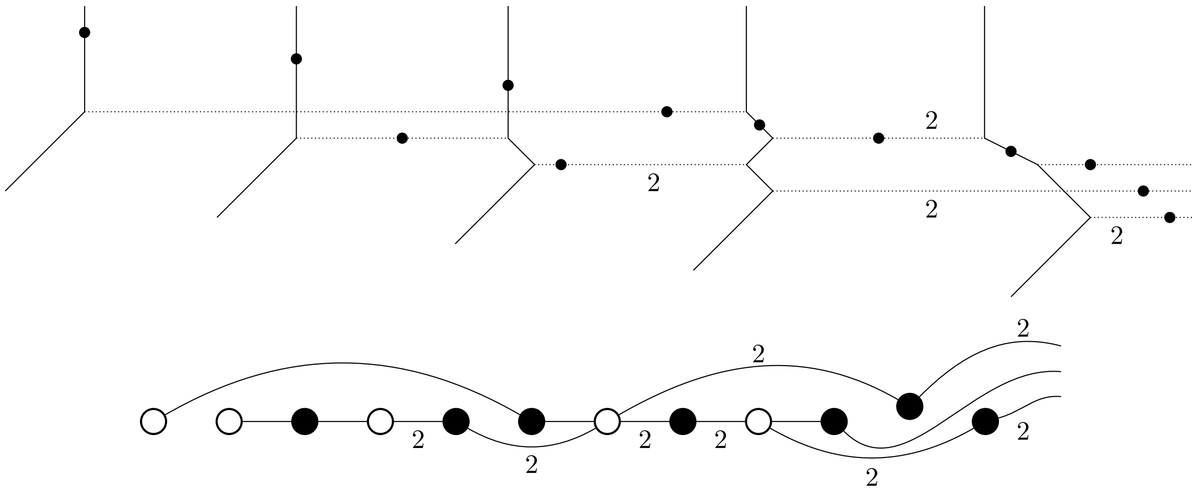


FIGURE 26. A floor decomposed curve of degree 5 and genus 0 with tangency profile $((0,0,\dots),(1,1,2,0,\dots)$ passing through a horizontally stretched point configuration. Elevators are drawn dotted, the floors are the remaining solid connected components. Below, the associated floor diagram.

additional outgoing contribution in direction $(1,0)$, either contributing weight to an existing elevator, or by creating a new one. Hence, the divergence assumption follows.

Conversely, it is possible to reconstruct a tropical curve from a floor diagram when given a horizontally stretched point configuration though we omit the slightly more complicated process here.

We give an example of a floor diagram associated to a tropical curve in Figure 26. As we had done for parametrized tropical curves, we can define a multiplicity for floor diagrams.

Definition 14.6. We define $N^{\text{floor}}\big(d,g,(\alpha,\beta)\big)$ as the number of floor diagrams of degree d, genus g and tangency profile (α,β), counted with multiplicity

$$m(\mathfrak{D}_{\Gamma,h}) \;=\; \frac{1}{I^\alpha}\prod_{e\in E_b(\mathfrak{D}_{\Gamma,h})} w(e),$$

for each floor diagram $\mathfrak{D}_{\Gamma,h}$, where $E_b(\mathfrak{D}_{\Gamma,h})$ denotes the bounded edges of the diagram and $w(e)$ the weight of the edge e.

Example 14.7. In Figure 27, the upper floor diagram has a multiplicity of 64, contributing to the count $N^{\text{floor}}\big(5,0,\big((0,0,\dots),(1,2,0,\dots)\big)\big)$.

The tropical curve count and the count of floor diagrams with multiplicity coincides:

Theorem 14.8. *Let d and g be integers. Then,*

$$N^{\text{floor}}\big(d,g,(\alpha,\beta)\big) \;=\; N^{\text{trop}}\big(d,g,(\alpha,\beta)\big).$$

14.2. **Floor diagrams on Möbius strips.** We start with the extension of abstract *floor diagrams* associated to curves in a Möbius strip $\mathbb{T}M_\delta$. In the next subsection, we describe their relationship to *floor-decomposed* tropical curves.

Definition 14.9. A *floor diagram* in $\mathbb{T}M_\delta$ is an oriented graph with infinite outgoing edges to the right and the following additional data:

(*i*) Vertices are separated into three disjoint sets: *ground floors*, *étages*[2] and *joints*. *Floors* encompass ground floors and étages.
(*ii*) An edge e, also called *elevator*, has an integer weight $w(e) \in \mathbb{N}$.
(*iii*) An étage $\mathcal{F}$ has a degree $a_\mathcal{F} \in \mathbb{N}$.
(*iv*) A ground floor $\mathcal{G}$ has a degree $a_\mathcal{G} \in \frac{1}{2}\mathbb{N}$.

The additional data has to satisfy the following conditions:

(A) Ground floors have no incoming edges. Moreover, if $\mathcal{G}$ is a ground floor, we have the balancing condition:

$$\sum_{e \ni \mathcal{G}} w(e) \equiv \delta 2a_\mathcal{G} \bmod 2 \;=\; \begin{cases} 0 & \text{for } \mathbb{T}M_0 \\ 2a_\mathcal{G} & \text{for } \mathbb{T}M_1. \end{cases}$$

(B) Étages have zero divergence: the sum of weights of incoming edges is equal to the sum of weights of outgoing edges.
(C) Joints have exactly two outgoing edges of the same weight and no incoming edge.

Remark 14.10. The degrees of the ground floors are allowed to be half-integers analogous to the coefficients of $aE + bF$ in $H_{1,1}(\mathbb{T}M_\delta, \mathbb{Z})$.

Note that unlike classical floor diagrams, vertices are monotonously linearly ordered but all ground floors and joints have the same height. Hence, there is not a strict ordering on vertices, only on étages.

Definition 14.11. Given a floor diagram $\mathfrak{D}$, we define the following:

- The *genus* of $\mathfrak{D}$ is its genus as a graph plus its number of floors.
- The diagram $\mathfrak{D}$ is said to be in the class $aE + bF$ if the sum of the weights of infinite outgoing edges is $2b$, and the sum of degrees of the floors (both ground floors and étages) is equal to a.
- A diagram in the class $aE+bF$ is of tangency profile $\mu \vdash 2b$ if it has μ_i ends of weight i for each i.

Remark 14.12. The newly defined floor diagrams differ from the ones we considered previously on the plane (in Section 14.1) in the following way:

- The vertices of the floor diagrams on Möbius strips only correspond to the white vertices of planar floor diagrams. We will see the analogue of black vertices in this setting in Definition 14.14.
- Analogous to floor diagrams on the cylinder (see [21]), and different to the planar case, floors on the Möbius strip contribute to the genus of the floor diagram.
- Unique to the Möbius strip case is the separation of floors into ground floors and étages. All floors in floor diagrams of curves in the tropical cylinder correspond to étages in our new definition. The novelty is the occurrence of ground floors. These correspond to the *disorienting* cycles on the Möbius strip, which cannot occur on orienting surfaces.

[2]The English language apparently lacks a word for floors which are not ground floors.

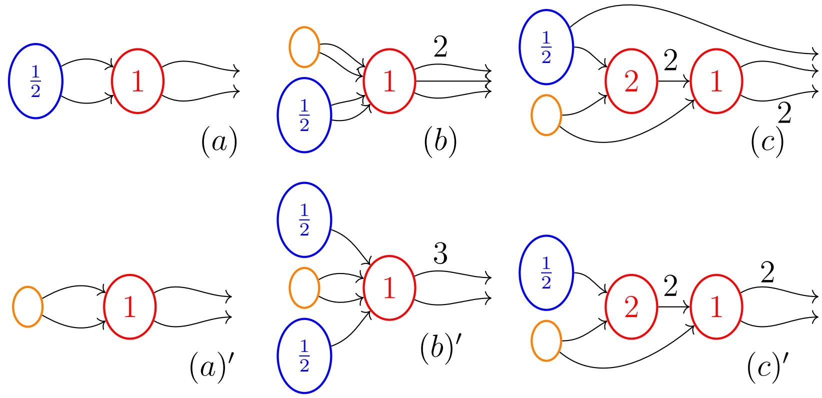


FIGURE 27. Some examples of floor diagrams in $\mathbb{T}M_0$ and $\mathbb{T}M_1$. In the top row, we give examples in $\mathbb{T}M_0$ and in the bottom row, we give some in $\mathbb{T}M_1$.

- Further, unique to the Möbius strip floor diagrams is the existence of joints. We will soon see that these correspond to elevators passing through the soul of the Möbius strip.

Example 14.13. In Figure 27, we give several examples of floor diagrams in $\mathbb{T}M_0$ and $\mathbb{T}M_1$. In all pictures in this section, we draw elevators in black, ground floors in blue, étages in red and joints in orange. Unlabeled edges have weight one. Note that the main difference for floor diagrams for $\mathbb{T}M_0$ and $\mathbb{T}M_1$ is the treatment of ground floors. The floor diagrams in Figure 27 are of the following genus, class, and tangency profiles:

	Genus	Class	Tangency		Genus	Class	Tangency
(a)	3	$\frac{3}{2}E+F$	1^2	$(a)'$	2	$E+F$	1^2
(b)	4	$\frac{3}{2}E+2F$	$1^2 2^1$	$(b)'$	4	$2E+2F$	$3^1 1^1$
(c)	4	$\frac{7}{2}E+2F$	$1^2 2^1$	$(c)'$	4	$\frac{7}{2}E+\frac{3}{2}F$	$2^1 1^1$

Definition 14.14. Let $\mathfrak{D}$ be a floor diagram of genus g and tangency profile μ. A *marking* $\mathfrak{m}$ of a floor diagram $\mathfrak{D}$ is an increasing map $\mathfrak{m} : [\![1; |\mu|+g-1]\!] \to \mathfrak{D}$ (i.e., $\mathfrak{m}(i) \prec \mathfrak{m}(j) \Rightarrow i < j$, where $\prec$ is the order relation on the cycle-free oriented graph $\mathfrak{D}$), such that:

(a) No marking is mapped to a ground floor or a joint, and a unique marking is mapped to each étage.
(b) At most one marking is mapped to an elevator.
(c) Each component of the complement of markings on elevators is of one of the following types:
  - (i) It contains a unique ground floor and no cycles or free ends.
  - (ii) It contains a unique free end, and no cycles or ground floors.
  - (iii) It has a unique cycle containing an odd number of joints and no free ends or ground floors.

Remark 14.15. Markings on the floor diagram correspond to the black vertices of floor diagrams in the planar case (see Definition 14.4).

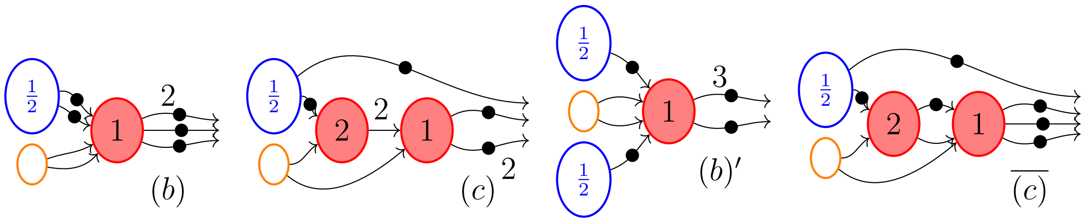


FIGURE 28. Markings on the floor diagrams $(b), (c)$ and $(b)'$ from Figure 27. $(b), (c)$ and $\overline{(c)}$ are floor diagrams on $\mathbb{T}M_0$ and $(b)'$ is on $\mathbb{T}M_1$.

Example 14.16. We give some examples for markings on floor diagrams in Figure 28.

14.3. **Floor diagrams from tropical curves.** We now define a floor diagram for any tropical curve, and show that for a *stretched* choice of constraints, the point conditions induce a marking of the diagram satisfying the condition from Definition 14.14, analogous to the cylinder case, [21]. Let l denote the length of the underlying tropical elliptic curve $\mathbb{T}E$.

Proposition 14.17. *Let $h : \Gamma \to \mathbb{T}M_\delta$ be a parametrized tropical curve in the class $aE+bF$. Then the slope of its edges can only take a finite number of values, and the lengths of the non-vertical edges are bounded by a constant M that only depends on a, b and l.*

Proof. The proof is very similar to the proof of the analogous statements in [21, Lemmas 5.7 and 5.8]. We include it here for completeness' sake anyways.

Let $e \in E_b(\Gamma)$ be a bounded edge of Γ and let $h(e)$ have slope (u, r). Assume $u \neq 0$. The intersection number $|u|$ of u with a fiber F cannot exceed the intersection number of $h(\Gamma)$ and F, i.e., $|u| \leq h(\Gamma) \cdot F = 2a$. Similarly, assuming $r \neq 0$, we can intersect with a section E and obtain $|r| \leq h(\Gamma) \cdot E = 2b$. Since we assumed the slopes to be integers, there is thus only a finite number of possibilities.

Now, assume $\ell(e) > l$. Then, the edge goes around the Möbius strip multiple times. Thus, it intersects the fiber in at least $\lfloor \frac{\ell(e)}{l} \rfloor$ points, each contributing $|u|$ to the intersection number $h(\Gamma) \cdot F$. Thus, $\frac{\ell(e)}{l} \cdot |u| \leq 2a$. Since $|u| \geq 1$ by assumption, we can thus set $M = l(2a+1)$

This shows that there is a uniform bound M on the length of non-vertical edges for curves of a fixed class in $\mathbb{T}M_\delta$. □

We now construct a floor diagram from a tropical curve.

Construction 14.18. Let $h : \Gamma \to \mathbb{T}M_\delta$ be a tropical curve in the class $aE + bF$ with tangency profile μ and genus g. We match the definitions of floor diagrams by calling an edge with *vertical* slope an *elevator*, and a connected component of the complement of elevators a *floor*. Floors containing a disorienting cycle correspond to *ground floors*, while all other floors are *étages*. This differs from the planar case, where the elevators correspond to the *horizontal* edges.

We can now form a floor diagram where

- vertices are the floors of the curve,
- two vertices are linked by an edge if both floors are linked by an elevator,
- joints are inserted in the middle of elevators that meet the soul of the Möbius strip,

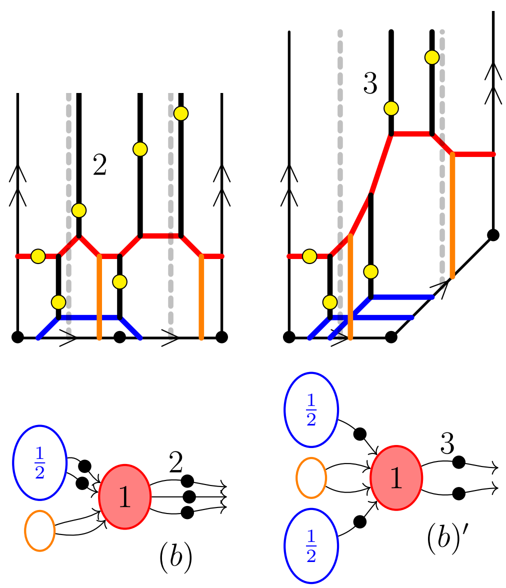


Figure 29. The transformation of (marked) tropical curves into (marked) floor diagrams. In blue, we mark disorienting cycles, which correspond to ground floors, in red we mark étages. In black, we mark elevators and in orange we mark joints. On the left, we give an example for $\mathbb{T}M_0$: curve (c') in Figure 20, whereas on the right, we show it on $\mathbb{T}M_1$, for curve (c') in Figure 21.

- the weight of an elevator is its weight as an edge of $h(\Gamma)$, and
- the degree of a floor is equal to half its intersection number with a fiber.

Moreover, assuming the curve passes through a collection of points $\mathscr{P}$, indexed by $[\![1; |\mu|+g-1]\!]$, we have a map $[\![1; |\mu|+g-1]\!] \to \mathfrak{D}$ that maps the image of a marked point to the part of $\mathfrak{D}$ that contains it.

Example 14.19. In Figure 29 we show how to construct floor diagrams from tropical curves in both Möbius strips for the tropical curves $(c)'$ from Figure 20 and Figure 21 respectively. We color the edges corresponding to étages, ground floors and joints in their corresponding colors. The resulting floor diagrams are the diagrams (b) and $(b)'$ in Figure 27. We note that the class, genus, and tangency profile of the tropical curves computed in Examples 13.6 and 13.7 coincide with the ones of the corresponding floor diagrams computed in Example 14.13.

Definition 14.20. A point configuration $\mathscr{P}$ is *vertically stretched* if the difference between the y-coordinates of the points is far bigger than the length of the Möbius strip l, and they are far away from the soul of the Möbius strip.

When the point configuration $\mathscr{P}$ is stretched, the marked floor diagrams corresponding to curves whose marked point evaluate to $\mathscr{P}$ satisfy the conditions in Definitions 14.9 and 14.14.

Proposition 14.21. *Let $\mathscr{P}$ be a stretched point configuration and $h : \Gamma \to \mathbb{T}M_\delta$ a tropical curve whose marked points evaluate to $\mathscr{P}$. Then, the induced marked diagram $(\mathfrak{D}, \mathfrak{m})$ satisfies the following:*

(*i*) *the degrees of floors satisfy* $\sum_{\mathcal{F}} a_{\mathcal{F}} = a$*, and the sum of weights of the ends is equal to* $2b$*,*
(*ii*) *each floor satisfies the divergence assumptions in Definition 14.9A and B,*
(*iii*) *each étage contains exactly one image of a marked point, and no ground floor contains such an image of a marked point,*
(*iv*) *each elevator contains at most one image of a marked point,*
(*v*) *an étage consists of a unique cycle with adjacent elevators, and a ground floor consists of a unique disorienting cycle with adjacent elevators, and*
(*vi*) *each connected component of the complement of marked elevators is of one of the following types:*
- *It contains a unique ground floor and no cycles or free ends;*
- *It contains a unique free end and no cycles or ground floors;*
- *It has a unique cycle containing an odd number of joints and no free ends or ground floors.*

In particular, $\mathfrak{D}$ *is a floor diagram and* $\mathscr{P}$ *induces a marking* $\mathfrak{m}$ *of* $\mathfrak{D}$*.*

Proof. (*i*) This follows from the definition of the degree of the floors and class of the curve.
(*ii*) The statement for étages follows using the balancing condition in Definition 14.9B. For ground floors, we pass to the two-to-one cover of the Möbius strip, which is $\mathbb{T}E\times\mathbb{R}$ for $\mathbb{T}M_0$ and the total space of the 2-torsion line bundle on $\mathbb{T}\widehat{E} = \mathbb{R}/2l\mathbb{Z}$ for $\mathbb{T}M_1$. As the ground floor contains a disorienting cycle, the preimage of a neighborhood of the ground floor by the two-to-one cover is connected. Each elevator adjacent to the ground floor yields a pair of elevators in the cover, whose coordinates on $\mathbb{T}\widehat{E}$ differ by l and which are in opposite direction. Let $x_e \in \mathbb{R}/2l\mathbb{Z}$ denote the coordinate of an adjacent elevator e. Using the tropical Menelaus relation for the cylinder $\mathbb{T}\widehat{E}\times\mathbb{R}$, see Remark 12.83, we obtain

$$\sum_e w(e)\cdot x_e - w(e)\cdot(x_e + l) \equiv l\cdot\delta\cdot 2a_{\mathcal{G}} \bmod 2l.$$

This yields the desired relation for the divergence of grounds floors.
(*iii*) An étage has to contain at least one image of a marked point, otherwise it is possible to translate it vertically, resulting in a 1-parameter family of solutions. It cannot contain more than one point, as the point conditions are stretched whereas slope and length of non-vertical edges are bounded. Further, as a disorienting cycle intersects the soul of the Möbius strip and the points are chosen very far from it, a ground floor cannot contain an image of a marked point.
(*iv*) As the point configuration is general, no points have the same projection onto E, and an elevator cannot contain more than one image of a marked point.
(*v*) Any cycle contained in an étage is orienting. If an étage contained two cycles, one of them would be without the image of any marked point, contradicting that $\Gamma\backslash h^{-1}(\mathscr{P})$ contains no orienting cycle (see Proposition 13.16). If a ground floor contained two cycles, it would again contain an orienting cycle. By the balancing condition, a disconnecting edge in the étage would have vertical slope, hence there are no disconnecting edges, leading to the statement.

(*vi*) The last statement is the direct translation of Proposition 13.16, following from the observation that disorienting cycles lie either in a ground floor, or can use elevators, as long as they intersect the soul of the Möbius strip an odd number of times in total. □

14.4. **Floor diagram multiplicities.** Now, we recover the marked tropical curves that are solutions of the enumerative problem from the floor diagrams. Definition 14.22 below gives the (refined) multiplicity of a floor diagram so that it matches the sum of (refined) multiplicities of the tropical curves it encodes (see Proposition 14.24). We set

$$\widehat{\sigma_1}(a) \;=\; \sum_{\substack{k|a\\ k \text{ odd}}} \frac{a}{k}. \tag{13}$$

Definition 14.22. Let $(\mathfrak{D}, \mathfrak{m})$ be a marked floor diagram. If $\mathcal{F}$ is an étage and $\mathcal{G}$ is a ground floor, we set

$$\begin{aligned} m(\mathcal{F}) &= a_{\mathcal{F}}^{\mathrm{val}\mathcal{F}-1}\sigma_1(a_{\mathcal{F}})\prod_{e\ni\mathcal{F}} w(e), & m^q(\mathcal{F}) &= \sum_{k|a_f} k^{\mathrm{val}\mathcal{F}-1}\prod_{e\ni\mathcal{F}}\left[\frac{w(e)a_{\mathcal{F}}}{k}\right]_q \\ m(\mathcal{G}) &= 2\cdot(2a_{\mathcal{G}})^{\mathrm{val}\mathcal{G}-1}\widehat{\sigma_1}(2a_{\mathcal{G}})\prod_{e\ni\mathcal{G}} w(e) & m^q(\mathcal{G}) &= 2\sum_{\substack{k|2a_{\mathcal{G}}\\ k \text{ odd}}} k^{\mathrm{val}\mathcal{G}-1}\prod_{e\ni\mathcal{G}}\left[\frac{w(e)\cdot 2a_{\mathcal{G}}}{k}\right]_q \end{aligned} \tag{14}$$

Further, we write $N(\mathfrak{D})$ for the number of cycles in the complement of marked elevators, and $\mathrm{Aut}\mathfrak{D}$ for the group of automorphisms of the diagram. We define the (refined) multiplicity of a marked floor diagram to be

$$m(\mathfrak{D}, \mathfrak{m}) = \frac{2^{N(\mathfrak{D})}}{|\mathrm{Aut}\mathfrak{D}|}\prod_{\mathcal{F}} m(\mathcal{F})\prod_{\mathcal{G}} m(\mathcal{G})\prod_{E_{\mathrm{um}}} w(e),$$
$$m^q(\mathfrak{D}, \mathfrak{m}) = \frac{2^{N(\mathfrak{D})}}{|\mathrm{Aut}\mathfrak{D}|}\prod_{\mathcal{F}} m^q(\mathcal{F})\prod_{\mathcal{G}} m^q(\mathcal{G})\prod_{E_{\mathrm{um}}} w(e).$$

Remark 14.23. The multiplicity of a tropical curve is a product over its vertices. In the classical floor diagrams we discussed in Definition 14.6 as well as for refined floor diagrams, introduced in [16] and mentioned in Section 12.4, the weight of the edges is enough to determine the multiplicity, which is a product over the edges of the diagram. Here, as in the cylinder case discussed in Section 12.6, we have to further account for the floors. In the classical case, it is possible to factor out the weight of the edges from the floors, but not in the refined case.

Proposition 14.24. *The (refined) multiplicity of a marked floor diagram corresponds to the (refined) count of tropical curves that it encodes, counted with (refined) multiplicity.*

Proof. We recover tropical curves from floor diagrams, proceeding as follows. First, we determine the shape of the floors.

- Given an étage $\mathcal{F}$ of degree $a_{\mathcal{F}}$, the situation is handled as for floors on cylinders [21]. An étage consists of a unique cycle realizing an even homology class $2k_{\mathcal{F}} \in H_1(\mathbb{T}M_\delta) \simeq \mathbb{Z}$ in the Möbius strip. The horizontal coordinate $v_{\mathcal{F}}$ of the slope of the edges in the cycle is well-defined. In particular, an elevator of weight $w(e)$ that meets the cycle does so at a vertex with multiplicity $w(e)v_{\mathcal{F}}$. Moreover, intersecting with a fiber yields $2k_{\mathcal{F}}$ intersection points each of multiplicity $v_{\mathcal{F}}$. As the intersection index is by definition $2a_{\mathcal{F}}$, we have $a_{\mathcal{F}} = k_{\mathcal{F}}v_{\mathcal{F}}$, hence $k_{\mathcal{F}}|a_{\mathcal{F}}$.

Given $k_{\mathcal{F}}|a_{\mathcal{F}}$, we can unfold the étage so that the cycles goes around the Möbius strip only twice. This induces a floor if and only if the position of the adjacent elevators satisfy the *Menelaus relation* [21] in $\mathbb{R}/2lk_{\mathcal{F}}\mathbb{Z}$: if x_e is the position of the elevator e, we require

$$\sum_{e\ni\mathcal{F}} \pm w(e)x_e \equiv \delta v_{\mathcal{F}} l \ \in \ \mathbb{R}/2lk_{\mathcal{F}}\mathbb{Z}.$$

- Given a ground floor $\mathcal{G}$ of degree $a_{\mathcal{G}}$, there also is a unique cycle that needs to be disorienting, i.e., it realizes an odd homology class in the Möbius strip. Let $k_{\mathcal{G}} \in H_1(\mathbb{T}M_\delta) \simeq \mathbb{Z}$ be this odd homology class. The horizontal coordinate $v_{\mathcal{G}}$ of the slope of the edges in the cycle is also well-defined. Intersecting with a fiber yields $k_{\mathcal{G}}$ intersection points each of multiplicity $v_{\mathcal{G}}$. By the definition of the degree, we obtain $k_{\mathcal{G}}v_{\mathcal{G}} = 2a_{\mathcal{G}}$. Therefore, $k_{\mathcal{G}}$ is an odd divisor of $2a_{\mathcal{G}}$.

  Thus, we can unfold the ground floor by the cover of degree $k_{\mathcal{G}}$, so that it goes around the Möbius strip only once. Hence, we can assume that $k_{\mathcal{G}} = 1$ up to choosing a lift of the elevators by this cover. Then, we can take the preimage by the two-to-one cover of the Möbius strip by the cylinder. Each adjacent elevator gets two lifts with opposite directions whose horizontal position differs by l. The Menelaus condition on the cylinder is

$$\sum_{e\ni\mathcal{G}} w(e)(x_e - (x_e - l)) \ = \ \delta l \in \mathbb{R}/2l\mathbb{Z}.$$

  This relation is automatically satisfied by the balancing condition, so that we have a unique curve up to translation. If require the curve to be invariant by the deck transformation of the cover, it is fully unique. In the end, we can draw a unique curve for each choice of lift of the elevators.

Now, we recover the positions of the elevators for each floor and fixed $k_{\mathcal{F}}$ as the lattice index of the following map. For each elevator e, let e_+ and e_- be its extremities, which are floors, joints, or points on the boundary of the strip. If e_- is a joint, we set $k_{e_-} = 1$ and if e_+ is a boundary point, we set $k_{e_+} = 1$. We consider the space of positions of the elevators in the unfolded version

$$\prod_e \mathbb{R}/2lk_{e_+}\mathbb{Z} \times \mathbb{R}/2lk_{e_-}\mathbb{Z}.$$

An element $(x_{e_+}, x_{e_-})_e$ can only correspond to a tropical curve if the following are satisfied:

- For each edge e, $x_{e_+} \equiv x_{e_-}$ in $\mathbb{R}/2l\mathbb{Z}$, so both extremities can linked by an elevator.
- For each joint, x_{e_+} and x_{e_-} differ by $l \in \mathbb{R}/2l\mathbb{Z}$.
- For each étage and two adjacent elevators e and e', $(x_{e_+}, x_{e_-})_e$ and $(x_{e'_+}, x_{e'_-})_{e'}$ satisfy the unfolded Menelaus relation in $\mathbb{R}/2lk_{\mathcal{F}}\mathbb{Z}$.
- Images of marked points and fixed ends fix the position of elevators.

Finally, we have the map of real tori

$$\begin{array}{rccc} \Phi : & \prod_e \mathbb{R}/2lk_{e_+}\mathbb{Z} \times \mathbb{R}/2lk_{e_-}\mathbb{Z} & \longrightarrow & \prod_e \mathbb{R}/2l\mathbb{Z} \times \prod_{\mathcal{F}} \mathbb{R}/2lk_{\mathcal{F}}\mathbb{Z} \times \prod_{\mathcal{J}} \mathbb{R}/2l\mathbb{Z} \times \prod_{e \text{ marked}} \mathbb{R}/2l\mathbb{Z}. \\ & (x_{e_+}, x_{e_-}) & \longmapsto & \Big((x_{e_+} - x_{e_-}), (\sum \pm w(e)x_{e_\pm}), (x_{e(\mathcal{J})_-} - x_{e'(\mathcal{J})_-}), (x_{e(i)_+})\Big) \end{array}$$

This is a group homomorphism of real tori of the same dimension. We now determine the number of preimages of an element $((0), (\delta v_{\mathcal{F}} l), (l), (x_i))$ by computing the lattice index of the map between the first homology groups of the tori. The lattice index of Φ_* can be computed similarly as the computation from the multiplicity in Proposition 13.20. We prune the diagram using the Laplace expansion formula for determinants. In the end, we obtain

$$2^{N(\mathfrak{D})} \prod_{\mathcal{F}} k_{\mathcal{F}}^{\mathrm{val}\mathcal{F}-1} \prod_{e \text{ unmarked}} w(e),$$

where $N(\mathfrak{D})$ denotes the number of cycles in the complement of the image of marked points in $\mathfrak{D}$. To conclude, we multiply the lattice index of Φ_* by the multiplicity of a curve encoded by the diagram, and make the sum over the possible divisors $k_{\mathcal{F}}|a_{\mathcal{F}}$ and $k_{\mathcal{G}}|2a_{\mathcal{G}}$, yielding the result. □

Example 14.25. We conclude by applying the floor diagram algorithm in the genus one case. In genus one, every diagram has a unique floor, either a ground floor or an étage.

In the case of a unique ground floor, every elevator is adjacent to it. The contribution is $(2a)^{2b}\widehat{\sigma}_1(2a)$ respectively.

In the case of a unique étage, every elevator is still adjacent to the étage, but might be passing through a joint. By balancing, there are as many elevators directly adjacent to the étage as elevators passing through a joint before going to the étage. Thus, b needs to be even. We count the number of markings by considering the lift under the 2-to-1 cover of the half-line to get a floor diagram in a cylinder, also with a unique floor. The (even) number of ends $2b$ coincides with the number of marked points. There are 2^{2b-1} lifts of the points in the cylinder up to the deck transformation. In the lift there are precisely 2 markings, depending on where the free end lies, as given in [21]. The multiplicity is thus $2^{2b-1} \cdot 2 \cdot a^{2b}\sigma_1(a)$.

Summing over all contributions, we get

$$N^{\delta}_{1,aE+bF} \;=\; (2a)^{2b} \left(\widehat{\sigma}_1(2a) + [a, b \in \mathbb{Z}]\sigma_1(a)\right),$$

where $[a, b \in \mathbb{Z}]$ is 1 if both a and b are integers, and 0 else.

15. Correspondence and Regularity

In this section, we prove the three main results of this part of the thesis. We show that the tropical curve count (and hence the floor diagram count) coincide with the logarithmic Gromov-Witten invariant of curves in the complex ruled surfaces $\mathbb{C}M_0$ and $\mathbb{C}M_1$ we discussed in Sections 12.5.2 and 12.5.3 respectively. We then proceed by proving quasi-polynomiality and quasi-modularity of the generating series of the enumerative invariants.

15.1. **Correspondence Theorem.** We first show the correspondence theorem. To this end, we briefly discuss the logarithmic Gromov-Witten invariants of the surfaces $\mathbb{C}M_\delta$ as the complex counterpart to the tropical problem, similar to the way we proceeded in Section 12.3. As before, we fix the surfaces to be the surfaces $\mathbb{C}M_\delta$ defined in Sections 12.5.2 and 12.5.3, and discuss the discrete data necessary to have a meaningful invariant. Then, we proceed by showing that we can apply the Abramovich-Chen-Gross-Siebert decomposition formula, Theorem 12.56, to obtain a correspondence between the tropical and the logarithmic count.

We consider one of the surfaces $\mathbb{C}M_\delta$ and fix the following discrete data $\beta_{a,\mu,\nu}$:

- positive integers g, n and a half-integer a,
- two partitions μ and ν. We set $2b = \|\mu\| + \|\nu\|$ and assume that $2b \equiv 2a\delta \bmod 2$.

As discussed in Section 12.3, there is a *moduli stack of parametrized logarithmic curves* with logarithmic structure

$$\mathcal{M}^{\log}_{\beta_{a,\mu,\nu}}(\mathbb{C}M_\delta \,|\, \mathbb{C}\widehat{E}),$$

parametrizing parametrized logarithmic curves from a source of genus g with $n+|\mu|+|\nu|$ marked points to $\mathbb{C}M_\delta$. The partitions μ and ν impose the intersection profile with the boundary divisor of $\mathbb{C}M_\delta$, similar to the tangency matrix discussed in Definition 12.40.

As previously discussed for the toric case, for each of the n marked points, the moduli stack is equipped with an *evaluation map*

$$\mathrm{ev}_i : \mathcal{M}^{\log}_{\beta_{a,\mu,\nu}}(\mathbb{C}M_\delta \,|\, \mathbb{C}\widehat{E}) \longrightarrow \mathbb{C}M_\delta.$$

It maps a parametrized logarithmic curve to the image of the i-th marked point. Moreover, for each of the marked points mapped to the boundary divisor, we have an additional evaluation map with values in the corresponding divisor:

$$\widehat{\mathrm{ev}_i} : \mathcal{M}^{\log}_{\beta_{a,\mu,\nu}}(\mathbb{C}M_\delta \,|\, \mathbb{C}\widehat{E}) \longrightarrow \mathbb{C}\widehat{E}.$$

The moduli stack $\mathcal{M}^{\log}_{\beta_{a,\mu,\nu}}(\mathbb{C}M_\delta \,|\, \mathbb{C}\widehat{E})$ is a proper Deligne-Mumford stack equipped with a *virtual fundamental class* $[\mathcal{M}]^{\mathrm{vir}}$ in degree $g-1+|\mu|+|\nu|+n$. We define the logarithmic Gromov-Witten invariants by intersecting the virtual fundamental class with classes provided by the evaluation morphisms. For the dimension counts to agree, we take $n=|\nu|+g-1$.

Definition 15.1. The logarithmic Gromov-Witten invariant of the Möbius strips is defined by

$$\mathcal{N}^\delta_{g,aE+bF}(\mu,\nu) \;=\; \int_{[\mathcal{M}]^{\mathrm{vir}}} \prod_1^n \mathrm{ev}_i^*(\mathrm{pt}) \prod_1^{|\mu|} \widehat{\mathrm{ev}_i}^*(\mathrm{pt}),$$

where pt is the cohomology class Poincaré dual to a point.

We now relate the logarithmic Gromov-Witten invariants to the tropical counts, using the Abramovich-Chen-Gross-Siebert decomposition formula, Theorem 12.56.

Theorem 15.2. *The tropical invariant and the logarithmic Gromov-Witten invariant agree:*

$$\mathcal{N}^\delta_{g,aE+bF}(\mu,\nu) \;=\; N^\delta_{g,aE+bF}(\mu,\nu).$$

Proof. To apply the Abramovich-Chen-Gross-Siebert decomposition formula (see Theorem 12.56), we need to construct a family of surfaces obtained by a logarithmic smooth degeneration, such that $\mathbb{C}M_\delta$ is a fiber of the degeneration and that the central fiber over the logarithmic general point is a union of toric surfaces, glued along their toric boundary divisors. We construct this family using the tropical curves solving the tropical enumerative problem described in Section 13. Our approach is similar to the one used for the correspondence theorem in [38].

Step 1: Constructing a subdivision. Consider one of the tropical Möbius strips $\mathbb{T}M_\delta$, along with a general configuration $\mathscr{P}$ of $|\nu|+g-1$ points and $|\mu|$ points on the boundary divisor. We can assume that they have rational coordinates. As the configuration is general, there is a finite number of tropical curves in the class $aE+bF$ of tangency profile $\mu+\nu$ and matching the point and tangency constraints. We then consider a polyhedral subdivision Ξ of $\mathbb{T}M_\delta$ such that each tropical curve factors through the 1-skeleton of the subdivision.

We can always take such a subdivision by taking the common refinement of subdivisions corresponding to each tropical curve. Up to scaling, we can assume that the coordinates of the vertices of the subdivision along with the length l of the tropical elliptic curve $\mathbb{T}E$ are integers. Moreover, Ξ projects onto a subdivision Σ of $\mathbb{T}E$.

We unfold the polyhedral subdivision Ξ of $\mathbb{T}M_\delta$ to get a polyhedral subdivision $\widehat{\Xi}$ of its universal cover $\mathbb{R}^2$ by taking the preimage. We do the same for Σ to get a polyhedral subdivision $\widehat{\Sigma}$ of $\mathbb{R}$. By construction, $\widehat{\Sigma}$ is stable by translation by l and $\widehat{\Xi}$ is stable under the action of φ_δ, which lifts the translation by l in $\mathbb{R}$.

Step 2: Constructing the family. We consider the cone over the polyhedral subdivisions $\widehat{\Sigma} \times \{1\} \subset \mathbb{R}^2$ and $\widehat{\Xi} \times \{1\} \subset \mathbb{R}^3$. This yields two (infinite) fans endowed with a map to $\mathbb{R}_{\geq 0}$ provided by the projection on the last coordinate. Using the construction of toric varieties for these (infinite) fans, we get two complex manifolds $\mathbb{C}\widehat{\mathscr{E}}$ and $\mathbb{C}\widehat{\mathscr{M}}$ with a map $\mathbb{C}\widehat{\mathscr{M}} \to \mathbb{C}\widehat{\mathscr{E}}$, and a map to $\mathbb{C}$. The threefold $\mathbb{C}\widehat{\mathscr{M}}$ is a partial compactification of $(\mathbb{C}^*)^3$ while $\mathbb{C}\widehat{\mathscr{E}}$ is a partial compactification of $(\mathbb{C}^*)^2$.

The translation action on $\widehat{\Sigma}$ lifts to the fan by $(x,\tau) \mapsto (x+\tau l, \tau)$. It induces a map on the complex surface $\mathbb{C}\widehat{\mathscr{E}}$ that extends the map of the dense torus

$$(z,t) \longmapsto (\lambda t^l z, t).$$

Similarly, the action of φ_δ lifts to the fan by $(x,y,\tau) \mapsto (x+\tau l, \delta z - w, \tau)$, and there is an extension to the threefold $\mathbb{C}\widehat{\mathscr{M}}$ that extends the map of $(\mathbb{C}^*)^3$ to itself

$$(z,w,t) \longmapsto \left(\lambda t^l z, \frac{z^\delta}{w}, t\right).$$

We can then consider the quotient by the above actions to get manifolds $\mathbb{C}\mathscr{E}$ and $\mathbb{C}\mathscr{M}$, along with a map $\mathbb{C}\mathscr{M} \to \mathbb{C}\mathscr{E}$ and maps to $\mathbb{C}$.

The fiber $\mathbb{C}\mathscr{M}_t$ for $t \neq 0$ is the ruled surface $\mathbb{C}M_\delta$ over the base elliptic curve $\mathbb{C}\mathscr{E}_t = \mathbb{C}^*/\langle \lambda t^l \rangle$. The central fiber of the family of elliptic curves $\mathbb{C}\mathscr{E}_0$ is a chain of copies of $\mathbb{P}^1_{\mathbb{C}}$ meeting along their toric divisors, i.e., their respective 0 and ∞. The central fiber $\mathbb{C}\mathscr{M}_0$ is a union of toric surfaces glued along their toric divisors. This construction is just a periodic version of the construction of a family of toric surfaces as done in [113].

Step 3: Applying the decomposition formula. The tropical points in $\mathscr{P}$ and the points on the boundary divisor give rise to sections of the family. By Proposition 12.55, logarithmic Gromov-Witten invariants are constant in families. Thus, we can compute the logarithmic Gromov-Witten invariant for the central fiber $\mathbb{C}\mathscr{M}_0$. Using the decomposition formula, Theorem 12.56, we can intersect the point constraints with the virtual fundamental class $[\mathcal{M}]^{\mathrm{vir}}$ to get a virtual fundamental class $[\mathcal{M}^{\mathscr{P}}]^{\mathrm{vir}}$ of degree 0, and the logarithmic Gromov-Witten invariant is the degree of this 0-cycle. The virtual fundamental class $[\mathcal{M}^{\mathscr{P}}]^{\log}$ splits as a sum over the tropical curves solving the tropical enumerative problem:

$$[\mathcal{M}^{\mathscr{P}}]^{\mathrm{vir}} = \sum_{h:\Gamma\to\mathbb{T}M_\delta} [\mathcal{M}^{h,\mathscr{P}}]^{\mathrm{vir}},$$

where $[\mathcal{M}^{h,\mathscr{P}}]^{\mathrm{vir}}$ is a virtual fundamental class corresponding to the parametrized logarithmic curves to the central fiber $\mathbb{C}\mathscr{M}_0$ whose combinatorial type is encoded by $h : \Gamma \to \mathbb{T}M_\delta$. In other words, the dual graph to the source curve is Γ, and the component corresponding to a vertex V is mapped to the irreducible component of $\mathbb{C}\mathscr{M}_0$ corresponding to $h(V)$. By Corollary 12.57, the logarithmic Gromov-Witten invariant splits as a sum

$$\mathcal{N}^{\delta}_{g,aE+bF}(\mu,\nu) \ = \sum_{h:\Gamma\to\mathbb{T}M_\delta} \int_{[\mathcal{M}^{h,\mathscr{P}}]^{\mathrm{vir}}} 1.$$

Step 4: Gluing formula. As in [38], we are left with the computation of the summands, i.e., the multiplicity $\int_{[\mathcal{M}^{h,\mathscr{P}}]} 1$ of each tropical curve. To do so, we use the gluing formula from [29, Proposition 13], inspired by the proof of [91, Theorem 1.5,1.6]. We can use the formula since we have a logarithmically smooth family degenerating to a union of toric surfaces meeting along their toric divisors. More precisely, if v is a vertex, we have a corresponding moduli stack $\mathcal{M}_v$ parametrizing the genus 0 curves in the toric surface $\mathbb{C}\mathscr{M}_{h(v)}$ (the irreducible component of the central fiber associated to $h(v)$) that have tangency profile with the toric boundary prescribed by the slopes of h on the edges adjacent to V in Γ. It is endowed with a virtual fundamental class $[\mathcal{M}_v]^{\log}$ and evaluation morphisms. We have a cutting morphism

$$\mathcal{M}^{h,\mathscr{P}} \to \prod_v \mathcal{M}_v,$$

that associates to each parametrized logarithmic curve the restriction to the component associated to the vertex v. Moreover, it is mapped to the diagonal Δ by the evaluation morphism

$$\mathrm{ev} : \prod \mathcal{M}_v \to \prod_e D_e^2,$$

where D_e is the divisor associated to the edge of the subdivision to which the edge e of Γ is mapped. The image is called the set of *pre-logarithmic* curves. According to [29], the covering morphism is a covering map of degree $\prod_e w(e)$ where the product is over the bounded edges of Γ. One thus needs to count the pre-logarithmic curves. Genus 0 curves to a toric surface with three punctures are parametrized by $(\mathbb{C}^*)^2$. We thus have the map

$$\prod_v (\mathbb{C}^*)^2 \longrightarrow (\mathbb{C}^*)^{|E_b|} \times (\mathbb{C}^*)^{2n} \times (\mathbb{C}^*)^{|\mu|},$$

which is exactly $\Theta\otimes\mathbb{C}^*$. As in [102, Proposition 4.10], the set of pre-logarithmic curves matching the point constraints is a $\ker\Theta\otimes\mathbb{C}^*$-torsor. Thus, its size is given by the lattice index $|\det\Theta|$ computed in Section 13.2. In the end, we obtain the expected tropical multiplicity. □

Remark 15.3. The proof follows the same steps as the proofs in [29, 38]. The difference to [38] is that we do not have ψ-constraints, which enables us to compute the lattice index in the end. The difference to [29] is that we do not consider λ-classes insertions. In particular, the above proof is a particular case of the proof of [29], to which we refer.

In [29], the decomposition formula is used to compute the logarithmic Gromov-Witten invariants with the insertion of a λ-class, relating them to refined invariants [16]. This can also be applied in our setting to prove an analogous result: with $n = 2b+g_0-1$, substituting

$q = e^{iu}$, one has

$$\Big((-i)(q^{1/2} - q^{-1/2})\Big)^{2b+2g_0-2} BG^{\delta}_{g_0,aE+bF} = \sum_{g \geq g_0} u^{2g-2+2b} \int_{[\mathcal{M}_{g,n}(\mathbb{C}M_\delta, aE+bF)]^{\mathrm{vir}}} \lambda_{g-g_0} \prod_1^n \mathrm{ev}_i^*(\mathrm{pt}).$$

15.2. **Quasi-polynomiality of relative invariants.** In this section we study the regularity of the relative invariants, fixing the number of intersection points but varying their tangency orders. This was previously done in [7] for the relative invariants of Hirzebruch surfaces and in [21] for the case of line bundles over an elliptic curve. Other results on polynomiality have recently been obtained in [47].

We begin by recalling the standard definition of quasi-polynomiality.

Definition 15.4. An expression $f(x) = \sum_{i=0}^d a_i(x)x^i$ is a *quasi-polynomial in x* if all coefficients $a_i(x)$ are periodic functions with integer periods.

15.2.1. *Setup.* We study the function

$$\Phi^\delta_a : (\mu_1, \dots, \mu_n, \nu_1, \dots, \nu_m) \longmapsto N^\delta_{g,aE+bF}(\mu_1, \dots, \mu_n, \nu_1, \dots, \nu_m),$$

defined for $\delta = 0, 1$, and $a \in \frac{1}{2}\mathbb{N}$. To get a non-zero result, we must have $\sum \mu_j + \sum \nu_j = 2b$. Hence, b is chosen accordingly. In particular,

$$\sum \mu_j + \sum \nu_j \equiv 2\delta a \bmod 2.$$

Therefore, we consider the function Φ^δ_a to be defined only on tuples of integers satisfying the above conditions. The study of regularity relies on the existence of floor diagrams, counting each with a polynomial contribution. Different to [7] and [21], diagrams with ground floors and joints often only have *quasi-polynomial* (but not polynomial) contributions.

Theorem 15.5. *There exist piecewise quasi-polynomial functions P^δ_a in $n+m$ variables such that $\Phi^\delta_a(\mu,\nu) = P^\delta_a(\mu,\nu)$.*

Proof. The proof relies on [133, Theorem 1]. We proceed as in [7]. The relative invariant can be written as a sum over the floor diagrams. As the curves are of fixed genus and have a fixed number of ends, up to the weighting of the elevators, there is a finite number of floor diagrams.

Let $\mathfrak{D}$ be a marked floor diagram. We label the ends of $\mathfrak{D}$ by the coordinates (μ,ν). As in [7], we account for the symmetry in the partition ν by dividing by the order of its automorphism group. Our goal is to find the possible weightings of the internal edges of $\mathfrak{D}$. To incorporate the parity conditions on the edges adjacent to ground floors, we modify $\mathfrak{D}$ to get a graph $G_{\mathfrak{D}}$ by adding a vertex v_e adjacent to each end e and an unbounded edge $e_{\mathcal{G}}$ adjacent to each ground floor $\mathcal{G}$.

Let E be the set of edges of $G_{\mathfrak{D}}$ and V its set of vertices. The graph $G_{\mathfrak{D}}$ inherits an orientation from $\mathfrak{D}$. Now, a *weighting* of $\mathfrak{D}$ is a vector in $\mathbb{N}^E$ satisfying the balancing condition at each floor and the equality of weights of edges adjacent to the same joint. Let $\varepsilon_{\mathcal{G}}$ be $2\delta a_{\mathcal{G}}$ modulo 2. Assume that the balancing condition for ground floors is satisfied on G. We obtain a compatible weight $w_{e_{\mathcal{G}}}$ for the new edge $e_{\mathcal{G}}$ by solving

$$2w_{e_{\mathcal{G}}} + \varepsilon_{\mathcal{G}} \ = \sum_{e \in \mathcal{G} \text{ in } \mathfrak{D}} w(e).$$

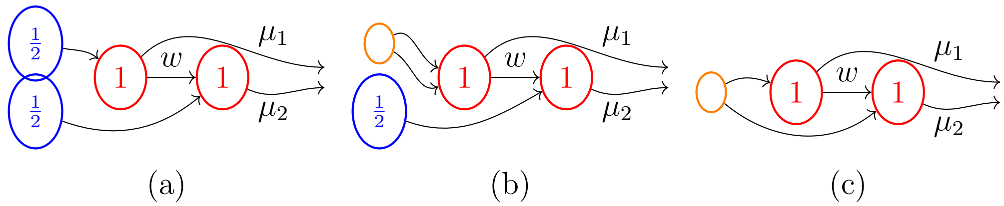


FIGURE 30. Floor diagrams with a non-polynomial contribution.

Let A be the adjacency matrix of the oriented graph $G_{\mathfrak{D}}$. The coefficient of A for $e_{\mathcal{G}} \in \mathcal{G}$ is 2 by balancing. Let $\mathbf{d} \in \mathbb{Z}^V$ be the integer vector whose coordinates are given by

$$d_v = \begin{cases} 0 \text{ if } v \text{ is an étage,} \\ \varepsilon_{\mathcal{G}} \text{ if } v \text{ is a ground floor,} \\ \mu_i \text{ or } \nu_i \text{ if } v = v_e \text{ for some end } e. \end{cases}$$

The multiplicity of a floor diagram is a monomial in the coordinates of the weight vector $\mathbf{w}$. Thus, we count the weight vectors $\mathbf{w} \geq 0$ satisfying $A\mathbf{w} = \mathbf{d}$ with corresponding multiplicity.

If there are no ground floors and every cycle passes through an even number of joints, we can lift the floor diagram to a floor diagram on the cylinder. This corresponds to lifting the tropical curves encoded by the diagram to tropical curves in the two-to-one cylinder cover of the Möbius strip. On the cylinder, the sum of top weights is equal to the sum of bottom weights, thus we can find a weighting only if this balancing condition is satisfied, and polynomiality is due to [21].

If there is at least one ground floor or a cycle passing through an odd number of joints, the map is surjective over $\mathbb{Q}$, i.e., the image of A is of full-dimension.

By [133, Theorem 1], the function is a piecewise quasi-polynomial on the chamber complex of the matrix A. On each chamber, the function is polynomial on vectors with fixed residue modulo all non-zero principal minors of A. Lemma 15.8 gives a more complete description of the minors. □

Remark 15.6. The statement in [133] concerns the count without polynomial multiplicity. The extended statement can be obtained by induction on the degree by considering the graph where some edges have been doubled. For details, see the proof of Theorem 4.2 in [7].

Example 15.7. In the following, we give some examples of floor diagrams with a non-polynomial contribution. Their graphs are depicted in Figure 30.

30(a) Here, for given μ_1 and μ_2, complete the diagram by choosing the weights of the remaining edges. The choice of the interior edge w completely determines the weight of the edges adjacent to the ground floors. However, we have a parity condition on these weights: even (resp. odd) if the diagram encodes curves in $\mathbb{T}M_0$ (resp. $\mathbb{T}M_1$). Thus, w needs to have the same (resp. opposite) parity as μ_1. Recall that the parity of $\mu_1 + \mu_2$ is fixed.

30(b) For the second diagram, we have a similar parity obstruction: since a joint is adjacent to the first étage, we need to have $w \equiv \mu_1 \bmod 2$.

30(c) For the last diagram, we can uniquely solve for the weights of the edges.

15.2.2. *Description of the minors of the adjacency matrix.* Given a marked floor diagram $\mathfrak{D}$ and an adjacency matrix A of $G_{\mathfrak{D}}$, let A_Δ be a principal minor of A corresponding to a subset $\Delta \subseteq E$ of edges and let G_Δ be the subgraph of $G_{\mathfrak{D}}$ containing all the vertices and the edges of Δ. The quotient $\mathbb{Z}^V/\langle A_\Delta \rangle$ is computed by the following lemma.

Lemma 15.8. *In the above notation,*

- (i) *If the determinant of the minor is non-zero, then G_Δ is a disjoint union of connected subgraphs which are either trees, containing a unique $e_{\mathcal{G}}$, or which contain a unique cycle and no edge $e_{\mathcal{G}}$.*
- (ii) *The group $\mathbb{Z}^V/\langle A_\Delta \rangle$ is a sum over the connected components of the subgraph G_Δ, where the group associated to a component is $\mathbb{Z}/2\mathbb{Z}$ if the component is a tree containing some $e_{\mathcal{G}}$, or a cycle passing through an odd number of joints, and* 0 *else (i.e., a cycle passing through an even number of joints).*

Proof. Choosing a principal minor amounts to choosing a subset $\Delta \subseteq E$ of size $|V|$. If the principal minor is non-zero, Δ has a bijection to V that assigns each vertex one to of its adjacent edges in Δ. The expansion of the determinant is now a signed weighted sum over these bijections. As the number of vertices and edges in G_Δ is the same, G_Δ has Euler characteristic 0, and the same holds true for each of its connected components. Thus, for each connected component there are two possibilities: Either it is a tree rooted at some edge corresponding to a ground floor, or it contains a unique cycle.

This proves (i). To show (ii), note that the minor A_Δ splits as a block-diagonal matrix corresponding to its connected components. We claim that the determinant of the block corresponding to a connected component is either 1 or 2, so that the quotient is a sum of copies of $\mathbb{Z}/2\mathbb{Z}$. We can compute each determinant by pruning the branches of the components, which amounts to Laplace expansion with respect to the row corresponding to the vertex we prune. The coefficient of the vertex being ± 1, we end up with one of these two cases:

- For a component without cycle and rooted at an end $e_{\mathcal{G}}$, the determinant is 2 since the coefficient of the latter in A is 2.
- If the component has a unique cycle, pruning the branches, we are left with the cycle. When expanding the determinant, we have exactly two terms according to the choice of vertex-edge assignment. The value of this determinant is 0 if the cycle passes through an even number of joints and 2 if this number is odd.

□

15.2.3. *Piecewise polynomiality in some special circumstances.* We now use this fact to prove polynomiality in several special cases. First, we consider curves that have a unique tangency point with maximal order on the boundary. Afterwards, we show the analogue for curves of small genus.

Corollary 15.9. *The relative invariants having a unique intersection point with the boundary are piecewise polynomial.*

Proof. There is a unique variable $\mu = 2b \in \mathbb{N}$. By Theorem 15.5, for each floor diagram, the contribution is piecewise quasi-polynomial, and the quasi-polynomiality is determined by the residue of the divergence vector $\mathbf{d}$ modulo the principal minors of the adjacency matrix A. To conclude, we only need to show that these residues are constant. Let us consider a principal

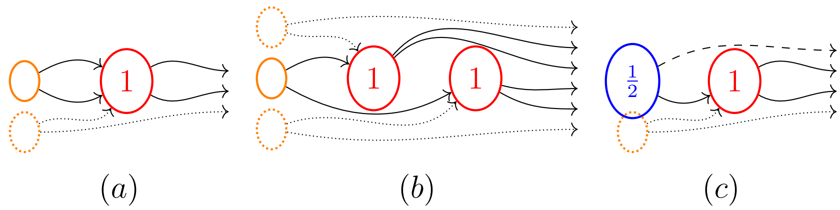


FIGURE 31. All floor diagrams of genus 2 and tangency profile 1^{2a}. An arbitrary degree $2a$ can be reached by attaching ends via joints as indicated by the dotted part of the pictures. Floor diagrams (a) and (b) contribute for both $\mathbb{T}M_0$ and $\mathbb{T}M_1$, the diagram in (c) contributes with the dashed end for $\mathbb{T}M_0$ and without the dashed end for $\mathbb{T}M_1$.

minor A_Δ corresponding to a subgraph G_Δ. According to Lemma 15.8, the cokernel is a sum of copies of $\mathbb{Z}/2\mathbb{Z}$ corresponding to the components with either a cycle with an odd number of joints, or an end $e_{\mathcal{G}}$ at a ground floor. Thus, only the residue mod 2 of the divergence vector $\mathbf{d}$ matter. All of its coordinates are constant except μ, which is either 0 or $\varepsilon_{\mathcal{G}}$. As $\mu = 2b \equiv 2\delta a \bmod 2$, its residue mod 2 is fixed. The function is thus a true polynomial. □

Corollary 15.10. *The relative invariants of genus* 1 *and* 2 *are piecewise polynomials.*

Proof. We proceed similarly to prove that all the residues are fixed. Assume the diagram is of genus 1. It has at most one floor; As it needs to have at least one, the floor is unique.

Assume the unique floor is an étage. Every end is adjacent to it, possibly through some joint. Note that this diagram only contributes if the balancing condition at the étage is satisfied, which imposes a condition on the ends. Then the matrix is unimodular, which, by [7], implies that the contribution is polynomial.

If the unique floor is a ground floor, every end is (directly) adjacent to it. There is a unique minor and the quotient is $\mathbb{Z}/2\mathbb{Z}$. However, as we have $\sum \mu_i = 2b \equiv 2\delta a \bmod 2$, the residue is constant, implying polynomiality.

In case of genus 2, the possibilities for the floor diagrams are given in Figure 31:

31(a) If the diagram has a unique floor, it needs to be an étage with a joint whose both extremities are adjacent to it. Further, every end is adjacent to the étage, after potentially passing through some joint. As there is a cycle passing through an odd number of joints (i.e., 1), the determinant is 2. The residue is given by the sum of the entries, which is fixed by a since $2b \equiv 2\delta a \bmod 2$. Thus, we get polynomiality.

31(b) Assume the diagram has two floors, both of which are étages. The matrix is unimodular, again implying polynomiality.

31(c) Assume the diagram has a ground floor and an étage. The ground floor is linked to the étage by an unique edge. Thus, we have exactly one principal minor with determinant 2, and the residue is given by the sum of the entries. Thus, it is also fixed and we get polynomiality.

□

15.3. **Quasi-modularity.** Quasi-modularity statements for the generating series of invariants of degree 0 line bundle over an elliptic curve have been proven in [21]. Quasi-modularity is a desirable property since it implies a strong control over the coefficients, bounding their growth polynomially. We consider the generating series of $N^{\delta}_{g,aE+bF}(\mu,\nu)$ in a. For $\mathbb{T}M_0$, given b and partitions $\mu+\nu\vdash 2b$, we set

$$F^0_{g,b}(\mu,\nu)(y) \;=\; \sum_{a\in\frac{1}{2}\mathbb{N}} N^0_{g,aE+bF}(\mu,\nu)y^{2a},$$

where $g\geq 1$ and $b\in\mathbb{N}$. We consider exponents $2a$ of the variable y since a is a half-integer. Similarly, for $\mathbb{T}M_1$, as b can be an integer or a half-integer, we have two generating series: we set

$$\begin{aligned} F^1_{g,b}(\mu,\nu)(y) &= \sum_{a\in\mathbb{N}} N^1_{g,aE+bF}(\mu,\nu)y^{2a} \text{ if } b\in\mathbb{N},\\ F^1_{g,b}(\mu,\nu)(y) &= \sum_{a\in\frac{1}{2}+\mathbb{N}}^{\infty} N^1_{g,aE+bF}(\mu,\nu)y^{2a} \text{ if } b\notin\mathbb{N}.\end{aligned}$$

In other words, we consider the generating series of relative invariants, fixing the intersection profile with the boundary and varying the intersection number $(aE+bF)\cdot F=2a$, which is the exponent of the series variable.

Before proving the regularity result on the above generating series, we introduce the following auxiliary functions:

- $G_2(y)=\sum_{n=1}^{\infty}\sigma_1(n)y^n$, the usual Eisenstein series up to an affine transformation,
- $H(y)=\sum_{n=1}^{\infty}\widehat{\sigma_1}(n)y^n$, the generating series of $\widehat{\sigma_1}(n)=\sum_{\substack{k\mid n\\ k \text{ odd}}}\frac{n}{k}$,
- $H_0(y)=\sum_{n=1}^{\infty}\widehat{\sigma_1}(2n)y^{2n},\ H_1(y)=\sum_{n=0}^{\infty}\widehat{\sigma_1}(2n+1)y^{2n+1}$, the odd and even parts of $H(y)$.

Lemma 15.11. *The functions H, H_0 and H_1 are quasi-modular forms for some finite index subgroup of $SL_2(\mathbb{Z})$.*

Proof. We start with the function H. We have:

$$\begin{aligned} H(y) &= \sum_{n=1}^{\infty}\left(\sum_{\substack{k\mid n\\ k\text{ odd}}}\frac{n}{k}\right)y^n \;=\; \sum_{n=1}^{\infty}\left(\sum_{k\mid n}\frac{n}{k}\right)y^n-\sum_{n=1}^{\infty}\left(\sum_{\substack{k\mid n\\ k\text{ even}}}\frac{n}{k}\right)y^n\\ &\overset{(*)}{=} G_2(y)-\sum_{n'=1}^{\infty}\left(\sum_{k'\mid n'}\frac{2n'}{2k'}\right)y^{2n'}=G_2(y)-G_2(y^2).\end{aligned}$$

To see $(*)$, note that for even k and n we can write $k=2k'$, $n=2n'$, and $k|n$ if and only if $k'|n'$. Then, H_0 and H_1 are just the even and odd parts of H:

$$\begin{aligned} H_0(y) &= \tfrac12\big(H(y)+H(-y)\big) & H_1(y) &= \tfrac12\big(H(y)-H(-y)\big)\\ &= \tfrac12\big(G_2(y)+G_2(-y)\big)-G_2(y^2), & &= \tfrac12\big(G_2(y)-G_2(-y)\big).\end{aligned}$$

Using [20, Lemma 2.1], they are quasi-modular forms for finite index subgroups of $SL_2(\mathbb{Z})$. ∎

Theorem 15.12. *The generating series $F^0_{g,b}(\mu,\nu)$ and $F^1_{g,b}(\mu,\nu)$ are quasi-modular forms for some finite index subgroup of $SL_2(\mathbb{Z})$.*

Proof. In the section on polynomiality 15.2, we consider the floor diagrams and forget the elevators weights. Here, we instead forget the degrees of the floors. Given b and partitions μ and ν such that $\mu+\nu\vdash 2b$, there are only a finite number of genus g floor diagrams up to the degree of the floors. Then, for a given $a\in\frac{1}{2}\mathbb{N}$, we find the diagrams contributing to $N^\delta_{g,aE+bF}(\mu,\nu)$ by constructing partitions of a satisfying such that for each étage $\mathcal{F}$, $a_\mathcal{F}\in\mathbb{N}$, for a ground floor $\mathcal{G}$, $a_\mathcal{G}\in\frac{1}{2}\mathbb{N}$ and $\delta\cdot 2a_\mathcal{G}\equiv\sum_{e\ni\mathcal{G}}w(e)$ mod 2, and such that $\sum a_\mathcal{G}+\sum a_\mathcal{F}=a$.

Let $\mathfrak{D}$ be a marked floor diagram without floor degrees, and for a ground floor let $\varepsilon_\mathcal{G}\equiv\sum_{e\ni\mathcal{G}}w(e)$. Given a family $(a_\mathcal{F},a_\mathcal{G})$ of degrees for the floors, let $\mathfrak{D}(a_\mathcal{F},a_\mathcal{G})$ be the corresponding diagram. By Definition 14.22, its multiplicity is

$$m\big(\mathfrak{D}(a_\mathcal{F},a_\mathcal{G})\big) = W\prod_\mathcal{F} a_\mathcal{F}^{\mathrm{val}\mathcal{F}-1}\sigma_1(a_\mathcal{F})\prod_\mathcal{G}(2a_\mathcal{G})^{\mathrm{val}\mathcal{G}-1}\widehat{\sigma_1}(2a_\mathcal{G}),$$

where W accounts for the contribution of the elevators weights and the 2^k term. The sum over all values of a factors as follows:

$$\sum_a\left(\sum_{\Sigma a_\mathcal{F}+\Sigma a_\mathcal{G}=a} m\big(\mathfrak{D}(a_\mathcal{F},a_\mathcal{G})\big)\right)y^{2a}$$
$$= W\prod_\mathcal{F}\left(\sum_{a_\mathcal{F}=1}^{\infty} a_\mathcal{F}^{\mathrm{val}\mathcal{F}-1}\sigma_1(a_\mathcal{F})y^{2a_\mathcal{F}}\right)\prod_\mathcal{G}\left(\sum_{\substack{a_\mathcal{G}\in\frac{1}{2}\mathbb{N}\\ \delta 2a_\mathcal{G}\equiv\varepsilon_\mathcal{G}\bmod 2}}^{\infty}(2a_\mathcal{G})^{\mathrm{val}\mathcal{G}-1}\widehat{\sigma_1}(2a_\mathcal{G})y^{2a_\mathcal{G}}\right).$$

The series in the product over the étages are quasi-modular forms, since they are equal to $(D^kG_2)(y^2)$ for a derivation of order k. The product over the ground floors depends on δ:

In $\mathbb{T}M_0$, there is no parity condition on the sum, thus we recover some derivative of the generating function $H(y)$ and obtain quasi-modularity. In $\mathbb{T}M_1$, depending on the value of $\varepsilon_\mathcal{G}$, we sum over the odd or even values, yielding $H_{\varepsilon_\mathcal{G}}(y)$ in any case. This also results in the quasi-modularity of the series. ☐

Remark 15.13. For $\mathbb{T}M_0$, $\mu=\emptyset$ and $\nu=1^{2b}$, we recover the non-relative invariants of the degree 0 cylinder, for which quasi-modularity has already been proven in [21] and [23].

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Appendix A. Table of Notation

The following is a list of all notation used in this thesis, intended for a quick overview. In the following, K is a valued field, d, g, k and n denote natural numbers, α and β denote sequences of natural numbers, M and N denote matroids, μ and ν denote valuated matroids, Q denotes a quiver and R denotes a Q-representation, and Γ denotes an abstract tropical curve. The table is subdivided roughly by topics.

Symbol	Meaning	Reference
$[n]$	$\{1, \dots, n\}$	
$\mathbf{1}$	$(1, \dots, 1)$	
$\mathbf{v}$	$(v_1, \dots, v_n)$	
e_S	Indicator vector of the set S	
S^c	S^c	
$I \cup j \setminus i$	Shorthand for $(I \cup j) \setminus i$	
$I \setminus i \cup j$	Shorthand for $(I \setminus i) \cup j$	
I	$(1, 2, 3, \dots)$	
$\lvert\alpha\rvert$	$\sum_{i=0}^{\infty} \alpha_i$	
$I\alpha$	$\sum_{i=0}^{\infty} (\alpha_i \cdot i)$	
I^α	$\prod_{i=0}^{\infty} i^{\alpha_i}$	
val	Valuation map on K	2.2
$\mathbb{C}\{\{t\}\}$	Field of Puiseux series	2.4
$\mathbb{T}$	Tropical semifield $\mathbb{R} \cup \{\infty\}$	2.1
$\mathbb{P}(\mathbb{T}^n)$	Tropical projective space	2.18
$\operatorname{trop}(f)$	Tropicalization of the polynomial $f \in K[x_1^{\pm 1}, \dots x_n^{\pm 1}]$	2.5
$\operatorname{trop}(V)$	Tropicalization of the variety V, i.e., $\cap_{f \in V} \operatorname{trop}(f) \subset \mathbb{R}^n / \mathbb{R}\mathbf{1}$	2.5
$\mathcal{N}_f$	Newton polytope of a polynomial $f \in K[x_1^{\pm 1}, \dots x_n^{\pm 1}]$	2.16
Δ_f	Newton subdivision of a polynomial $f \in K[x_1^{\pm 1}, \dots x_n^{\pm 1}]$	2.16
$\overline{\operatorname{trop}}(f)$	Tropicalization of the polynomial $f \in K[x_1, \dots x_n]$	2.19
$\overline{\operatorname{trop}}(V)$	(Multiprojective) tropicalization of the variety V	2.19
$\mathcal{B}(M)$	Bases of M	3.1
$M(L)$	Column matroid of a linear space L	3.2
$\operatorname{Ind}(M)$	Independent sets of M	3.6
rk_M	Rank function on M	3.9
$\mathcal{C}(M)$	Circuits of M	3.12
$M(G)$	Cycle matroid of a graph G	3.13
M^*	Dual matroid of M	3.18
$\mathcal{C}^*(M)$	Cocircuits of M	3.20

$M\|_S$	Restriction of M to a set S	3.22
M/S	Contraction of M by a set S	3.23
$\mathcal{L}_M$	Lattice of flats of M	3.27
Σ_M	Bergman fan of M	3.31
$N \twoheadleftarrow M$	N is a matroid quotient of M	3.38
$f^{-1}(M)$	Induced matroid of M under f	3.44
M_o	Pointed matroid of M	4.2
pr_S	Projection of sets	3.53
$\overline{\mu}$	Extension of μ to independent sets	3.63
$\mathcal{C}(\mu)$	Valuated circuits of μ	3.66
$\mathcal{C}^*(\mu)$	Valuated cocircuits of μ	3.70
$\mathcal{V}(\mu)$	Vectors of μ	3.71
$\mathrm{trop}(\mu)$	Tropical linear space of μ	3.67
$\mu\|_S$	Restriction of μ to a set S	3.79
μ/S	Contraction of μ by a set S	3.80
$\mu \oplus \nu$	Direct sum of μ and ν	3.81
$\nu \twoheadleftarrow \mu$	ν is a valuated matroid quotient of μ	3.89
$\mathrm{Gr}(k;n)$	Grassmannian	3.82
$\mathrm{Dr}(k;n)$	Dressian	3.86
$\mathscr{P}_{r;n}$	Grassmann-Plücker relations	3.82
$\mathscr{P}_{r,s;n}$	Incidence Plücker relations	3.92
$\mathrm{Fl}(\mathbf{d};n)$	Flag variety	3.93
$\mathrm{FlDr}(\mathbf{d};n)$	Flag Dressian	3.93
$f^{-1}(\mu)$	(Affine) induced valuated matroid of μ under f	4.1, 4.7
$\mathrm{QGr}(R,\mathbf{d};n)$	Quiver Grassmannian of R	7.11
$\mathscr{P}_{\alpha;n}$	Quiver Plücker relations	7.13
$\mathrm{QDr}(R,\mathbf{d};n)$	Quiver Dressian of R	9.5
$\mathrm{LFl}(\mathbf{S},\mathbf{d};n)$	Linear degenerate flag variety	10.3
$\mathrm{LFlDr}(\mathbf{S},\mathbf{d};n)$	Linear degenerate flag Dressian	10.5
$\mathscr{P}(M)$	Matroid polytope of M	11.1
$\mathscr{P}_{\mathcal{M}}$	Quiver point configuration of a quiver matroid $\mathcal{M}$	11.5
$\mathcal{M}_{g,n}$	Moduli space of smooth n-marked curves of genus g	12.4
$\mathcal{M}_{g,n}^{\mathrm{trop}}$	Moduli space of stable tropical n-marked curves of genus g	12.23
$\mathfrak{M}_{g,n}$	Moduli space of n-marked curves of genus g	12.7
$\mathcal{M}_n(\mathbb{P}^2,d)$	Moduli space of rational n-marked parametrized curves of degree d	12.27

$\mathcal{M}^{\mathrm{trop}}_{0,n}(\mathbb{R}^2, \Delta)$	Moduli space of tropical rational n-marked parametrized curves of degree d	12.25
$\mathcal{M}^{\log}_{\beta}(X \mid D)$	Moduli space of parametrized logarithmic curves	12.48
$[\mathcal{M}^{\log}_{\beta}(X_0)]^{\log}$	virtual fundamental class of parametrized logarithmic curves to X_0	12.55
ev	Evaluation map	12.10
$\mathrm{ev}_{\mathrm{trop}}$	Tropical evaluation map	12.29
ft	Forgetful map	12.28
N_d	Degree d Gromov-Witten invariant on $\mathbb{P}^2$	12.13
$N^{\mathrm{trop}}_{\Delta}$	Tropical degree d Gromov-Witten invariant on $\mathbb{R}^2$	12.33
$\mathcal{N}_{\beta}$	Logarithmic Gromov-Witten invariant of β	12.51
$N^{\mathrm{trop}}(d, g, (\alpha, \beta))$	Tropical relative degree d, genus g Gromov-Witten invariant on $\mathbb{R}^2$	12.39
$BG^{\delta}_{S,\mathscr{L}}$	Refined δ-Gromov-Witten invariant	12.63
$BG^{\delta,\mathrm{trop}}_{\Delta}$	Refined tropical δ-Gromov-Witten invariant	12.66
$m(\Gamma, h)$	Multiplicity of Γ	12.32
$[n]_q$	Quantum number/q-analog of n	12.65
$m^q(\Gamma, h)$	Refined multiplicity of Γ	12.32
$\mathbb{C}M_{\delta}$	A complex Möbius strip	12.5.2, 12.5.3
$\mathbb{T}E$	A tropical elliptic curve	12.78
$\mathbb{T}C$	A tropical cylinder	12.79
$\mathbb{T}M_{\delta}$	A tropical Möbius strip	13.1

Appendix B. Computations

Here, we give supplementary code from different chapters of the thesis. The source code can be accessed in executable format at `https://victoriaschleis.github.io/thesis.html`. All code in this section is my own, except for some code in Section B.5, which was jointly written with Alessio Borzì and Giulia Iezzi respectively for the articles [27] and [82]. All `Oscar` [114] code was written and tested in `Version 1.0.0`, which was the current version on submission of this thesis.

B.1. **Code for morphisms of matroids.**

In this section, we introduce `Oscar` [114] code, dealing with morphisms of matroids.

We begin by defining a function which checks whether two matroids M and N form a quotient $M \twoheadleftarrow N$, by checking whether the set of flats of M is a subset of the set of flats of N, see Definition 3.37.

```
  function is_flag_matroid(M::Matroid, N::Matroid)
    return issubset(sort.(flats(M)), sort.(flats(N)))
end
```

Our next step in computationally verifying whether a map is a morphism of matroids is to define the induced matroid $f^{-1}(M)$. In `Oscar`, this can be achieved as follows. First, we compute the rank r of the image of the whole set $[n]$ in M. Then, bases of $f^{-1}(M)$ are all subsets of $[n]$ of size r whose image under the map `f` has rank r. The map `f` here is technically implemented as a Dictionary. Then, we create a new matroid consisting of exactly the sets we computed.

```
function induced_matroid(M::Matroid, f::Dict)
    new_groundset = keys(f)
    im_groundset = unique!([get!(f, ele, 0) for ele in new_groundset])
    new_rk = rank(M, im_groundset)
    b_m = bases(uniform_matroid(new_rk, length(new_groundset)))
    new_bases = Vector{Vector{Int64}}([])
    for b in b_m
        im_b = [get!(f, elem, 0) for elem in b]
        if rank(M, im_b) == new_rk
            push!(new_bases, b)
        end
    end
    return matroid_from_bases(new_bases, new_groundset)
end
```

Combining both functions, we can then easily determine whether a given map `f`: $M \to N$ is a morphism of matroids.

```
function is_morphism_of_matroids(M::Matroid, N::Matroid, f::Dict)
    return is_flag_matroid(induced_matroid(N, f), M)
end
```

Alternatively, we might wish to check whether a given map is a strong map of matroids. To this end, we use an auxiliary function that adds a loop to each matroid.

```
function add_loop(M::Matroid)
    return matroid_from_bases(bases(M), M.groundset[length(M.groundset)]+1)
end
```

Now, we use the function adding a loop and then extend the map `f` to account for the new loops added to both matroids.

```
function is_strong_map(M::Matroid, N::Matroid, f::Dict)
    loopmap = Dict((length(M.groundset)+1) => (length(N.groundset)+1))
    pointed_f = merge(f, loopmap)
    return is_flag_matroid(induced_matroid(add_loop(N), pointed_f),
    ↪  add_loop(M))
end
```

We can now use these methods to verify the examples in Section 3.2. Further, for small enough matroids we can compute all possible morphisms between them matroids, and we can determine the endo- and isomorphisms of a matroid into itself, with the latter being possible for much larger ground set sizes. We give the code for these procedures below.

To this end, we need to first be able to generate all possible maps between two ground sets, and all possible maps between two ground sets, mapping the distinguished loop o of M_o to the distinguished loop o of N_o. The following function does both of these things through careful iterator manipulation, enabling the second case when setting `sm = true`.

```
function all_maps_m_to_n(m::Int, n::Int, sm::Bool)
    n+=sm
    k = 1:n
    for i in 2:m
        k = Iterators.product(k, 1:n)
    end
    c = collect(k)
    f = Iterators.flatten(c)
    for i in 2:m
        f = Iterators.flatten(f)
    end
    long_vec = collect(f)
    l = Int(length(long_vec)/m)
    mat_im = reshape(long_vec, m, l)
    if sm
        add_row = reshape(collect(Iterators.flatten([n  for i in 1:l])), 1,
        ↪  l)
        m+=sm
        mat_im = vcat(mat_im, add_row)
    end
    images = cols(mat_im)
    return [Dict(zip(collect(1:m), img)) for img in images]
end
```

Here, the function `cols()` is the following auxiliary function, extracting the columns of a `Matrix` as a `Vector{Vector{Int}}`.

```
function cols(c)
    return collect(c[:, i] for i in 1:size(c)[2])
end
```

Using this function, we can easily compute all morphisms and strong maps `f`: $M \to N$, using the two methods below, provided the ground sets $[m]$ and $[n]$ are small enough to be computationally feasible.

```
  function find_all_morphisms(M::Matroid, N::Matroid)
    all_maps = all_maps_m_to_n(length(M.groundset), length(N.groundset),
    ↪  false)
    return filter((f) -> is_morphism_of_matroids(M, N, f), all_maps)
end

function find_all_strong_maps(M::Matroid, N::Matroid)
    all_maps = all_maps_m_to_n(length(M.groundset), length(N.groundset),
    ↪  true)
    return filter((f) -> is_strong_map(M, N, f), all_maps)
end
```

B.2. **Code for matroid and quiver polytopes.** In the previous section, we had introduced methods for dealing with morphisms of matroids in `Oscar`. In this section, we now concern ourselves with `Oscar` code which helps us analyze matroid polytopes and quiver polytopes, and experimenting with which lead to Conjecture 11.1.

First, we give two basic methods, to convert matroids into matroid polytopes, and vice versa. The first method, `matroid_polytope` allows us to compute a matroid polytope from a matroid:

```
function matroid_polytope(M::Matroid)
    b = bases(M)
    v = Vector{Vector{Int64}}([])
    for i in b
        push!(v, Vector{Int64}(map(j->(j in i), M.groundset)))
    end
    return convex_hull(transpose(reshape(collect(Iterators.flatten(v)),
    ↪  (length(v[1]), length(v)))))
end
```

Now, we give a method allowing us to go in the other direction. It allows us to compute a matroid from a matroid polytope.

```
function matroid_from_polytope(P::Polyhedron)
    verts = Vector{Vector{QQFieldElem}}(collect(vertices(P)))
    bases = Vector{Vector{Int64}}([])
    for v in verts
        v = Int.(v)
        push!(bases, findall(i -> i == 1,v))
    end
    return matroid_from_bases(bases, ambient_dim(P))
end
```

As a slight caution — the function `matroid_polytope(M)` and many of the other functions in this section and the last (in particular, `induced_matroid` and functions using this) only work if the ground set of `M` is an integer range. If it is not, we give the following auxiliary function, which converts the ground set into the necessary integer range.

```
function normalize_groundset(M::Matroid)
    nb = Vector{Vector{Int64}}([])
    for b in bases(M)
        push!(nb, [findfirst(i->i==j, M.groundset) for j in b])
    end
    return matroid_from_bases(nb, length(M.groundset))
end
```

Now that we can easily compute matroid polytopes from matroids, we can take the Minkowski sum of two polytopes with the standard command `+`, enabling us to obtain candidates for quiver polytopes. Now, to test our conjectures, we need to determine different polyhedral properties on the resulting polytopes. For instance, for the polytopes `P` of loop quivers discussed in Section 11.3, we determine the symmetry group of the vertices, using the `Oscar` commands `describe(combinatorial_symmetries(P))`, and the group generated by the linear symmetries of the vertices using `describe(linear_symmetries(P))`. For small enough polytopes, this gives us a complete description of the symmetry group of the vertices.

When working with quivers containing more than one vertex, we take sums of the matroid polytopes associated to each vertex. Per Conjecture 11.1, we now need to not only take into account the combinatorial symmetries of the vertices, but also that of special lattice points on the inside of faces. We can extract these and determine their orbits under the action of the symmetric group S^n from our sum of polytopes as follows:

```
  function boundary_point_orbits(P::Polyhedron)
    return
    ↪  unique!(sort!.(Vector{Vector{Int64}}(boundary_lattice_points(P))))
end
```

When working with projections of polytopes, as we did in Section 11 when discussing the quiver polytopes corresponding to linear degenerate flag varieties, we sometimes want to determine whether a given polytope is a generalized permutohedron. We implement a function that does this, given as follows.

```
  function is_generalized_permutohedron(P::Polyhedron)
    f = collect(faces(P,1))
    res = []
    for i in f
        j = collect(vertices(i))
        dif_vert = j[1] - j[2]
        dif_vert = div.(dif_vert, gcd(dif_vert))
        push!(res, dif_vert)
    end
    return(res)
end
```

B.3. **Code for the generation of linear degenerate Plücker relations.** Below you can find `Macaulay2` [71] code developed jointly with Alessio Borzì for [27]. It is primarily concerned with the generation of (linear degenerate) Plücker incidence relations as discussed in Definitions 3.92 and 10.3 for the classical and the linear degenerate case respectively.

Below is the core of our code, which generates the (linear degenerate) Plücker incidence relations. In this code, the sets `I` and `J` are the sets I and J in the Plücker relations. The set `S` is the projection set necessary to compute linear degenerate Plücker relations.

```
pluckerRelations = method(TypicalValue => Ideal);

pluckerRelations(ZZ, ZZ, List, ZZ) := (r, s, S, n) -> (
  P := for k from 1 to r list(
    for J in subsets(n, s) list(
      for I in subsets(n, r) list(
        M := new List; --list of deg_S(X_Ialpha)
        L := new List; --list of tuples (monomial, power of t)
        M = M | {#((set I) * (set S))};
        L = L | {( - p_I * p_J, #((set I) * (set S)))};
        for alpha in subsets(s, k) do(
          Jalpha := new MutableList from J;
          for i from 0 to #alpha-1 do(Jalpha#(alpha_i) = I_i;);
          Jalpha = toList Jalpha;
          Ialpha := J_alpha | I_(toList(k..r-1));
          condition := (#Ialpha == #(unique Ialpha)) and (#Jalpha ==
↪  #(unique Jalpha));
          if condition then (
            (signPermutations(Ialpha) * p_(sort Ialpha) *
↪  signPermutations(Jalpha) * p_(sort Jalpha));
            L = L | {(signPermutations(Ialpha) * p_(sort Ialpha) *
↪  signPermutations(Jalpha) * p_(sort Jalpha), #((set Ialpha) * (set S)))};
            M = M | {#((set Ialpha) * (set S))};
          );
        );
        m = min M;
        f = 0; --output polynomial
        for l in L do( if (l_1 - m == 0) then(f = f + l_0); );
        f
      )
    )
  );
  return ideal unique flatten flatten P;
);
pluckerRelations(ZZ, ZZ, ZZ) := (r, s, n) ->(return pluckerRelations(r, s,
↪  {}, n););
pluckerRelations(ZZ, ZZ) := (r, n) ->(return pluckerRelations(r, r, n););
pluckerRelations(List) := L -> (return sum for l in L list (
↪  pluckerRelations toSequence l ););
```

We further implemented the below auxiliary method, which computes the sign of terms in a Plücker relation.

```
signPermutations = method(TypicalValue => ZZ);

signPermutations(List) := (l) -> (
  signCount := 0;
    for i from 0 to #l-1 do(
      for j from i to #l-1 do(
        if l_i > l_j then( signCount = signCount +1; )
      );
    );
  return (-1)^(signCount);
);
```

B.4. **Code for the generation of quiver Plücker relations.** For the quiver Plücker relations we discussed in Definition 7.13, we have written the following `Oscar` [114] code. It is a rather direct implementation of the definition. We remark that `r` and `s` denote the dimensions in the dimension vector of the subrepresentation, that `m_c_r` and `n_c_s` denote the sets $\binom{[m]}{r}$ and $\binom{[n]}{s}$ respectively, and that `Icomp` is the complement of `I`. Further, `I_c_j` denotes the set $I \cup j$ and `J_m_i` denotes $J \setminus i$.

```
function quiver_pluecker_relations(r::Int, s::Int, A::Matrix{Int64})
    n,m = size(A)
    Iset = subsets(collect(1:m),r-1)
    Jset = subsets(collect(1:n),s+1)
    m_c_r =subsets(collect(1:m),r)
    n_c_s =subsets(collect(1:n),s)
    R,x,y = polynomial_ring(QQ, "x"=>m_c_r, "y"=>n_c_s)
    pluecker_list = Vector{MPolyRingElem}([])
    for I in Iset
        Icomp = filter!(k->!(in(k,I)), collect(1:m))
        for J in Jset
            f = R(0)
            for j in Icomp
                sj = sign_qpl(j,I,J)
                for i in J
                    I_c_j = sort([I;j]);
                    J_m_i = setdiff(J,[i]);
                    f = f + sj*A[i,j]*x[findfirst(k -> k == I_c_j,
                     ↪  m_c_r)]*y[findfirst(l -> l == J_m_i, n_c_s)]
                end
            end
            push!(pluecker_list,f)
        end
    end
    return unique!(pluecker_list)
end
```

We remark that in the above method, we are giving the implementation for integer matrices. The analogous implementation also works for other fields, where we substitute all occurrences of `QQ` with the respective field. We aim to improve this in an upcoming version.

Again, we compute the sign straightforwardly using the method below:

```
function sign_qpl(j::Int,I::Vector{Int},J::Vector{Int})
    return (-1)^(length(findall(k -> k>j, J))+length(findall(i -> i>j, I)))
end
```

When processing the output of the previous computations, we might want to eliminate polynomials from our list that are just multiples of each other. We can do this using the following two functions, which normalize all polynomials in a list and omit redundant polynomials after normalization.

```
function normalize_polynomial(f::MPolyRingElem)
    coeffs_f = collect(coefficients(f))
    f = f/gcd(coeffs_f)
    if coeffs_f[findfirst(i->i != 0,coeffs_f)]<0
        f = (-1)*f
    end
    return f
end

function remove_redundant_polys(p1::Vector{MPolyRingElem})
    unique!(filter!(i->i != 0, p1))
    return unique!([normalize_polynomial(i) for i in p1])
end
```

B.5. **Code for examples of non-realizable points in quiver Dressians.** For examples corresponding to linear degenerate flag varieties we also refer to the `Macaulay2` [71] code presented in Appendix B.3. Since the computations in this section require heavy Gröbner fan computations, we use `gfan` [88] and `Macaulay2` [71].

Example 9.12. The equations used to generate Example 9.12 were given directly therein and can alternatively be obtained using the code in the previous section. We can compute the associated quiver Dressian using the input `example_n4_nonrealizable` available at `https://victoriaschleis.github.io/thesis/` using the `gfan` [88] command

```
gfan_tropicalintersection <example_n4_nonrealizable >out
```

The pre-packaging from our previous output is necessary due to the formatting requirements of input to `gfan`. The computation yields an output file `out`, which is available at the same website. We record the most important contents of the output here:

```
AMBIENT_DIM
20

DIM
12

F_VECTOR
1 58 466 1156 858 3
```

After inputting the same equations into `Macaulay2` [71], we can compute the dimension of the ideal they generate to be 10.

Using Theorem 2.15, this already implies that the Dressian and the Grassmannian differ. Due to the fact that the quiver Grassmannian has many irreducible components which are partially generated by monomials, associated tropical variety is partially generated by monomials as well, and hence not directly computable using `gfan` or the implementation of `tropicalVariety` in `Macaulay2`, due to the fact that the latter is based on the former.

Hence, to understand the structure, we need to compute a primary decomposition of the quiver Plücker ideal, and then tropicalize the individual components, after removing the monomials in the generating sets. The list of primary components and their tropicalizations can be found at `https://victoriaschleis.github.io/thesis/`, or easily be computed by plugging in the output of the primary decomposition computation into `gfan`, using the following command:

```
gfan_tropicalbruteforce <n4_comp_i >out
```

where `i` is substituted by the number of the component we wish to investigate.