A Realization for Minimal Sullivan Algebras with Quadratic Differential

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Pascal Zeller
aus Reutlingen

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1. Berichterstatter:	Prof. Dr. Frank Loose
2. Berichterstatter:	Prof. Dr. Christoph Bohle

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Die Topologie hat ihren Ursprung im 18. Jahrhundert, als sich Euler mit dem Königsberger Brückenproblem beschäftigte. Die Fragestellung war hierbei, ob es einen Weg gibt, der genau einmal über jede der sieben Brücken führt. Für die Lösung des Problems waren Lage und Länge der Brücken nicht von Bedeutung. Benötigt wurde lediglich die Information, welche Uferabschnitte über welche Brücken miteinander verbunden sind. Euler fasste gemeinsame Uferabschnitte als Knoten auf, die mittels Kanten (die den Brücken entsprechen) miteinander verbunden waren, und konnte so das Problem lösen. Verglichen den klassischen Fragestellungen, die man damals aus der Geometrie kannte, hatte das zu untersuchende Objekt eine deutlich vereinfachte Struktur. Die Tatsache, dass in der Topologie weniger Struktur verlangt wird als beispielsweise in der Geometrie, erschwert jedoch die Untersuchung und Klassifizierung topologischer Räume.
Abhilfe schaffen hier algebraische Methoden, die bei der Untersuchung topologischer Räume Anwendung finden, um Struktur in das Chaos zu bringen. Die Grundidee der algebraischen Topologie ist es, topologischen Räumen in geeigneter Weise algebraische Strukturen, wie beispielsweise Gruppen zuzuordnen. Dabei soll sich möglichst viel Information über den betrachteten Raume auf vereinfachter und zugänglicher Weise in der algebraischen Struktur wiederfinden.
In den meisten Fällen lässt die in der algebraischen Struktur codierte Information eine vollständige Klassifizierung der zugrundeliegenden Räume nicht zu. Allerdings können so oftmals zumindest manche topologische Räume voneinander unterschieden werden. Ein Beispiel hierfür sind die Homotopiegruppen $\pi_k(X)$ eines topologischen Raumes X, $k \in \mathbb{N}_0$. Diese sind invariant unter dem sogenannten Homotopietyp von X, was bedeutet, dass zwei Räume mit unterschiedlichen Homotopiegruppen nicht homotopieäquivalent sein können. Wenngleich die Homotopiegruppen nur in Spezialfällen den Raum X bis auf Homotopieäquivalenz charakterisieren können, ist ihre Berechnung seit Jahrzehnten Bestandteil aktueller mathematischer Forschung. Die Tatsache, dass die Homotopiegruppen $\pi_k(X)$ so schwer zu bestimmen sind, obwohl sie im Allgemeinen nur einen kleinen Teil der Information über den Homotopietyp des zugrundeliegenden Raumes X enthalten, zeigt die Komplexität der Untersuchung topologische Räume auf.
Eine gängige Aufgabenstellung in der algebraischen Topologie ist auch die Rekonstruktion topologischer Räume aus algebraischen Objekten. Bei den hierbei erhaltenen Räumen handelt es sich im Allgemeinen um CW Komplexe, also Räume, die durch das Ankleben von Zellen steigender Dimension konstruiert werden können. Die grundsätzliche Frage ist dann meist, wie aus den algebraische Daten die Information über Zellanzahl und Anklebeabbildungen gewonnen werden kann. Ein prominentes Beispiel ist die Konstruktion sogenannter Eilenberg-MacLane Räume $K(G, n)$. Zu gegebenem $n \geq 1$ und gegebener Gruppe (abelsch, falls $n \geq 2$) lässt sich ein CW Komplex X konstruieren, sodass $\pi_n(X) = G$ gilt und alle anderen Homotopiegruppen verschwinden. Hier ist X dann sogar, bis auf Homotopieäquivalenz, eindeutig. Dies bedeutet, dass jeder andere Raum mit ebendiesen Homotopiegruppen zum konstruierten X homotopieäquivalent ist. Das gilt im Allgemeinen schon dann nicht mehr, wenn zwei oder mehr Homotopiegruppen nicht trivial sind. Auch dann lässt sich ein Raum X konstruieren, der diese realisiert. Jedoch kann es noch viele andere Räume mit passenden Homotopiegruppen geben, sodass man von *einer* anstelle von *der* Realisierung sprechen muss.
Betrachtet man nun Räume, für die $\pi_1(X)$ verschwindet, und wechselt von den Gruppen $\pi_k(X)$ zu den rationalen Vektorräumen $\pi_k(X) \otimes \mathbb{Q}$, verbessert sich die Lage deutlich. Zum einen sind die sogenannten rationalen Homotopiegruppen $\pi_k(X) \otimes \mathbb{Q}$ deutlich leichter zu berechnen, zum anderen gibt es für diesen Vorgang ein topologisches Pendant. Zu gegebenem X gibt es einen Raum $X_\mathbb{Q}$, dessen Homotopiegruppen auf natürliche Weise den Vektorräumen $\pi_k(X) \otimes \mathbb{Q}$ entsprechen. Auf topologischer Seite lässt sich so ein neuer, gröberer Äquivalenzbegriff formulieren, für den die rationalen Homotopiegruppen eine geeignete Invariante sind. Die Untersuchung topologischer Räume bis auf ebendiesen Äquivalenzbegriff, auch rationaler Homotopietyp von X genannt, mittels geeigneter algebraischer Methoden ist Mittelpunkt der rationalen Homotopietheorie.
Was die Klassen von topologischen Räumen betrifft, stellt der Übergang zum rationalen Homoto-

pietyp eine Vereinfachung dar. Dennoch reichen die rationalen Homotopiegruppen für die Klassifizierung topologischer Räume im Sinne der rationalen Homotopietheorie im Allgemeinen nicht aus. Deutlich bessere Chancen hat man bei der Betrachtung von sogenannten minimalen Sullivan Modellen. Darunter versteht man eine bestimmte Klasse freier kommutativ graduierter Differenzialalgebren, sogenannte minimale Sullivan Algebren, die schon auf Kokettenlevel als Modell für die rationale Kohomologie von X fungieren. Wie sich herausstellt, codieren minimale Sullivan Modelle den rationalen Homotopietyp des zugrundeliegenden Raumes unter milden Voraussetzungen komplett. Insbesondere gibt es zu gegebener minimaler Sullivan Algebra eine passende Realisierung, die dann auch bis auf Äquivalenz eindeutig ist.
Vergleicht man nun diese Realisierung mit der oben angesprochenen Variante für Homotopiegruppen und Eilenberg-MacLane Räume, so stellt man fest, dass der genaue Zusammenhang zwischen algebraischen Daten und zellulärer Struktur des konstruierten Raumes unterwegs verloren geht. Die Realisierung ist, wenngleich für die Theorie ausschlaggebend, nicht oder nur schwierig auf Beispiele anwendbar. Dies motiviert die Suche nach einer alternativen Konstruktion, die, verglichen mit der schon bestehenden, eher von geometrischer Natur ist. Im Idealfall soll ein großer Teil der CW Struktur des Raumes direkt aus den Eigenschaften der minimalen Sullivan Algebra herauszulesen sein. Dies ist Ziel der vorliegenden Arbeit.

Im ersten Kapitel werden die algebraischen Grundlagen geschaffen. Die meisten algebraischen Strukturen, die in der algebraischen Topologie auftreten, tragen eine Graduierung. Dies bedeutet an so mancher Stelle kleine Änderungen für die vielleicht schon bekannten Welten der Algebren, Koalgebren und Lie Algebren, weshalb die wichtigsten Grundlagen hier nochmals erklärt werden. Darüber hinaus werden zentrale algebraische Objekte, wie (freie) kommutativ graduierte Differenzialalgebren und (freie) graduierte Lie Algebren eingeführt.
Das zweite Kapitel beschäftigt sich dann mit der topologischen Welt. Der Begriff der Lokalisierung, das topologische Pendant zur Lokalisierung von Gruppen durch Tensorbildung mit Teilringen von $\mathbb{Q}$, wird definiert und die Existenz und Eindeutigkeit bewiesen. Besonderes Augenmerk wird dabei auch auf der zellulären Lokalisierung liegen, da sie ein sehr anschauliches und geometrisches Bild dieses (doch sehr komplexen) Vorgangs liefert. Schließlich wird erklärt, was man unter dem rationalen Homotopietyp eines topologischen Raumes versteht.
Damit wurde die Zielsetzung der rationalen Homotopietheorie, die Untersuchung von Räumen bis auf rationalen Homotopietyp, klargestellt. Unter Verwendung der algebraischen Grundlagen wird es dann im dritten Kapitel darum gehen, eine der wichtigsten algebraischen Hilfsmittel für ebendiese Untersuchung einzuführen. Minimale Sullivan Modelle werden definiert und ihre Existenz und Eindeutigkeit bewiesen. Anhand von Beispielen wird klar werden, weshalb sie so ein hilfreiches Werkzeug in der rationalen Homotopietheorie sind. Darüber hinaus wird die Realisierung von Sullivan Algebren thematisiert und dabei der eins zu eins Bezug zwischen einer bestimmten Klasse von Räumen und minimalen Sullivan Algebren hergestellt.
Das vierte Kapitel beschreibt eine alternative Art von Modellen, die sogenannten freien Lie Modelle. Deren Vorteil ist, dass sie eine deutlich zugänglichere Realisierung zulassen als Sullivan Algebren. Anschließend wird eine wichtige Brücke zwischen der Welt der kommutativ graduierten Differenzialalgebren und graduierten Differenzial Lie Algebren hergestellt.
Das letzte Kapitel beschreibt eine alternative Methodik der Realisierung von Sullivan Algebren mit quadratischem Differenzial. Wir verwenden die Realisierung von freien graduierten Differenzial Lie Algebren und transferieren diese in die Welt der kommutativ graduierten Differenzialalgebren. Dabei wird es möglich sein, bestimmte algebraische Daten einer gegebenen minimalen Sullivan Algebra durch die auftretenden Konstruktionen zu verfolgen, um schließlich Aussagen über Teile der Zellstruktur des realisierenden Raumes machen zu können.

Abstract

In classic homotopy theory, we often study those properties of topological spaces X and continuous maps f between them which are only dependent of the homotopy type of X or the homotopy class of f. The typical approach to this task in algebraic topology is to make use of algebra to bring some structure into the chaos. In practice, this most commonly involves the definition of some algebraic quantity associated with a topological space X or a continuous map f which is invariant in the the homotopy type or homotopy class. While this almost never allows for a full classification up to homotopy type, one at least gets algebraic invariants which allow to tell some spaces and maps apart from each other. Among the most prominent examples are the singular homology and cohomology groups $H_k(X)$, $H^k(X)$ and homotopy groups $\pi_k(X, x_0)$, latter being homotopy classes of maps $(\mathbb{S}^k, s_0) \to (X, x_0)$. While homology groups are always abelian, the homotopy groups are for $k \geq 2$, which makes it possible to rationalize these groups in a purely algebraic sense. This means switching to homology with rational coefficients $H_k(X;\mathbb{Q})$ and the so called rational homotopy groups $\pi_k(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with induced maps $\pi_k(f) \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}}$.
For finitely generated abelian groups, forming the tensor product with $\mathbb{Q}$ makes all torsion parts vanish. Accompanied by this decrease of complexity and information is a notable increase in computability. For example, while the homotopy groups of a sphere are mostly still a mystery, we already know from Serre in 1951, see [23], that for $n \geq 1$

$$\pi_k(\mathbb{S}^{2n+1}) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = 2n+1 \\ 0 & \text{else} \end{cases} \qquad \text{and} \quad \pi_k(\mathbb{S}^{2n}) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = 2n,\ 4n-1, \\ 0 & \text{else.} \end{cases}$$

Quillen introduced the concept of model categories in [19], emphasizing a redefinition of the homotopy category of topological spaces. Instead of the morphisms being homotopy classes of continuous maps, the homotopy category of topological spaces following Quillen is obtained by localizing the category of topological spaces by a class of morphisms, namely the weak homotopy equivalences. The isomorphisms in this category are then exactly the weak homotopy equivalences. Motivated by the simplicity of rational homotopy groups, Quillen localized the category of simply connected spaces at the rational homotopy equivalences. These are morphisms inducing an isomorphism between rational homotopy groups. The result is the rational homotopy category Quillen considered in his article [20] and he could show that it is equivalent to the homotopy category of a certain class of differential graded Lie algebras.
It should be mentioned that in algebraic topology it is very rare to obtain a full classification of topological spaces, even up to some notion of equivalence, through algebraic means. Thus, from the perspective of algebraic topology, this result of Quillen is quite astonishing. However, one might still argue at this point that the definition of the rational homotopy category seems somewhat artificial, as if it was chosen to fit the algebraic models and not the other way around. One does not really get a feeling how this rational homotopy category looks like or what it means for two spaces to be isomorphic in this category. Rational homotopy theory therefore really begins with the discovery of Sullivan that rationalization has an underlying geometric construction. A *rational space* is a topological space whose homotopy groups form a graded rational vector space, implying that its homotopy and rational homotopy groups are naturally isomorphic. In [27], Sullivan shows that any simply connected space X has a rationalization, that is a rational space $X_{\mathbb{Q}}$ together with a continuous map $f\colon X \to X_{\mathbb{Q}}$ inducing an isomorphism

$$\pi_*(f) \otimes \mathrm{id}_{\mathbb{Q}} \colon \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \pi_*(X_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \pi_*(X_{\mathbb{Q}}).$$

Moreover, a continuous map $f\colon X \to Y$ can be rationalized to a map $f_{\mathbb{Q}}\colon X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ which is a weak homotopy equivalence if and only if f induces an isomorphism between rational homotopy groups. Two CW complexes are then rational homotopy equivalent if their rationalizations are homotopy equivalent. Furthermore, the rational homotopy class of a map is the homotopy class of

its rationalization. Rational homotopy theory is then the study of spaces and maps up to rational homotopy type and rational homotopy class.
Regarding the algebraic models, Sullivan used a different approach. By a fundamental result due to Serre and Whitehead, for a continuous map $f\colon X \to Y$ between simply connected spaces it holds

$$\pi_*(f) \otimes_{\mathbb{Z}} \mathbb{Q} \text{ is an isomorphism} \iff H^*(f;\mathbb{Q}) \text{ is an isomorphism.}$$

Moreover, by the theory of Serre classes, the homotopy groups $\pi_*(X)$ of a simply connected space are a graded rational vector space if and only if the singular cohomology $H^*(X, x_0; \mathbb{Z})$ is. This allows to translate the notions of a rational homotopy equivalence and rationalization into the world of commutative graded algebras over $\mathbb{Q}$. Inspired by differential forms on a manifold, Sullivan describes a functor A_{PL} in [26] that assigns a rational commutative differential graded algebra $A_{PL}(X)$ to a given topological space X. It has the crucial property that the cohomology of $A_{PL}(X)$ is naturally isomorphic to $H^*(X;\mathbb{Q})$, the singular cohomology of X with coefficients in $\mathbb{Q}$. The advantage of $A_{PL}(X)$ in contrast to the classical singular cochain algebra $C^*(X;\mathbb{Q})$ lies in the fact that it is commutative, making it possible to define algebraic models of X on cochain level. A *quasi-isomorphism* between commutative differential graded algebras is a morphism which induces an isomorphism between the corresponding homology algebras. One then calls two differential graded algebras weakly equivalent if they are connected by a chain of quasi-isomorphisms. As it turns out, two simply connected spaces with rational homology of finite type are rational homotopy equivalent if and only if $A_{PL}(X)$ and $A_{PL}(Y)$ are weakly equivalent, which outright translates the problem of classifying those spaces up to rational homotopy type into the world of commutative differential graded algebras.
The probably most favorable feature of Sullivan's approach is the fact that the commutative differential graded algebra $A_{PL}(X)$ associated with a simply connected space X allows for a so called *(minimal) Sullivan model*. That is a quasi-isomorphism

$$\varphi\colon (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X),$$

where ΛV is a special free commutative graded algebra. Other than $A_{PL}(X)$, the commutative graded algebra $(\Lambda V, d)$, also referred to as minimal Sullivan algebra, is often computable. In cases where it can even be obtained from the cohomology algebra $H^*(X;\mathbb{Q})$, the space X is called *formal*. Moreover, it holds that the isomorphy type of $(\Lambda V, d)$ is solely determined by the rational homotopy type of X. In more detail, two simply connected spaces X and Y with rational cohomology of finite type have the same rational homotopy type if and only if they have isomorphic minimal Sullivan models. This establishes a one-to-one correspondence between the rational homotopy types of simply connected spaces and isomorphism classes of minimal Sullivan algebras $(\Lambda V, d)$, where on both sides we restrict to rational cohomology of finite type. In view of this correspondence, it is possible to realize minimal Sullivan algebras, meaning we can assign to $(\Lambda V, d)$ a CW complex $|\Lambda V, d|$ such that $(\Lambda V, d)$ is a minimal Sullivan model for $|\Lambda V, d|$. Moreover, any other space X to which $(\Lambda V, d)$ is a model for, has the rational homotopy type of $|\Lambda V, d|$.
On this occasion, it should be pointed out that the simple nature of a minimal Sullivan algebra does not reflect in the CW structure of its realization. While being of central importance to the theory, the realization functor as described by Sullivan and generally used in the literature on rational homotopy theory does not allow an easy description of the obtained CW complex. Our goal in this monograph is therefore to establish a realization for certain minimal Sullivan algebras which has a more geometric flavour. More precisely, given a minimal Sullivan algebra $(\Lambda V, d)$ with $V = V^{\geq 2}$ of finite type and quadratic differential $d = d_1$, we will construct a CW complex X such that $(\Lambda V, d_1)$ is a minimal Sullivan model for X. In doing this, a considerable amount of information on the cell structure of X, such as the number of cells in each dimension, the description of a specific subspace or the classes of certain attaching maps can be directly read off the minimal Sullivan algebra $(\Lambda V, d_1)$. To achieve this, we will make use of so-called *cellular Lie models*. In the world of free differential graded Lie algebras, a more geometrically motivated

realization already exists, and we will use certain functors to translate this realization into the world of Sullivan algebras. To be more precise, associated with any differential graded Lie algebra (L, d) is a differential graded algebra $C^*(L, d)$ which, under mild assumptions, is a Sullivan algebra. Furthermore, given a minimal Sullivan algebra as above, there exists a special *homotopy Lie algebra* L such that $(\Lambda V, d_1) = C^*(L, 0)$. While L itself will most of the time not be free, we can find a free Lie model, that is a morphism $(\mathbb{L}_W, \partial) \xrightarrow{\simeq} (L, 0)$ from a free differential graded Lie algebra which induces an isomorphism on homology. Using functoriality, this is then carried over into a quasi-isomorphism of differential graded algebras, thereby reversing arrows. Finally, for $(\mathbb{L}_W, \partial)$ there then exists an explicit description of a CW complex X such that there exists a quasi-isomorphism $C^*(\mathbb{L}_W, \partial) \xrightarrow{\simeq} A_{PL}(X)$. The composition

$$(\Lambda V, d_1) = C^*(L, 0) \xrightarrow{\simeq} C^*(\mathbb{L}_W, \partial) \xrightarrow{\simeq} A_{PL}(X)$$

then exhibits $(\Lambda V, d_1)$ as minimal Sullivan model of X. A close examination of this whilst tracing algebraic information in $(\Lambda V, d_1)$ through the constructions that are involved allows us to deduce some statements regarding the construction of X.
We can now give a formulation of the main results. Note that the notation will be explained during the course of this monograph. Let $(\Lambda V, d_1)$ be a minimal Sullivan algebra with $V = V^{\geq 2}$ of finite type. Choose a basis $(v_i)_{i \in I}$ of $\ker(d_1)$, then the homology $H^+(\Lambda V, d_1)$ has a basis of the form $\{[v_i], \beta_r\}_{i \in I, r \in R}$.

Theorem. *It holds that $(\Lambda V, d_1)$ is a minimal Sullivan model for a CW complex X for which:*

(i) The number of cells in X^n corresponds to the dimension of $H^n(\Lambda V, d_1)$. More precisely, for each $i \in I$ a cell of dimension $|v_i|$ is attached to the base point, and for each $r \in R$ a cell of dimension $|\beta_r|$ may be attached non-trivially.

(ii) Let $k_1 + 1 = \min\{|\beta_r| : r \in R\}$ and $R_1 \subseteq R$ such that $r \in R_1$ if and only if $|\beta_r| = k_1 + 1$. Then X has a subspace

$$Y := \vee_{i \in I} \mathbb{S}^{|v_i|} \cup_f \dot{\bigcup}_{r \in R_1} \mathbb{B}^{k_1+1} \subseteq X,$$

where f is a family of Whitehead products of inclusions $\iota_k \colon \mathbb{S}^{|v_k|} \hookrightarrow \vee_{i \in I} \mathbb{S}^{|v_i|}$.

In particular, if $R_1 = R$, no further cells are attached and we get $Y = X$. In this case, $(\Lambda V, d_1)$ is the minimal Sullivan model of a wedge of spheres, with cells of dimension $k_1 + 1$ attached by Whitehead products in $\pi_{k_1}(\vee_{i \in I} \mathbb{S}^{|v_i|})$.

In addition to that, in some cases, the products $v_k \wedge v_l$ of two elements in $(v_i)_{i \in I}$ contribute to cells in X that are attached via Whitehead products of bracket length two. To begin with, complement $(v_i)_{i \in I}$ to a basis of V, denoting the additional basis elements by $(w_j)_{j \in J}$. Note that $d_1(w_j) \in \Lambda^2 V$ is generated by products of length two and that the products $v_k \wedge v_l$ may belong to a suitable basis of $\Lambda^2 V$. Thus $d_1(w_j)$ has coordinates $\lambda^j_{k,l}$ with respect to $v_k \wedge v_l$.

Theorem. *Denote by $Y^n \subseteq X^n$ the bouquet of spheres up to dimension n, one for each v_i of degree n or lower. Write $j \colon Y^n \hookrightarrow X^n$ for the inclusion and α_i for the classes represented by the inclusions $\iota_i \colon \mathbb{S}^{|v_i|} \hookrightarrow Y^n$. It holds:*

(i) A basis element $v_k \wedge v_l \in (\Lambda^2 V)^{n+1}$ for which the coefficient $\lambda^j_{k,l}$ in $d_1(w_j)$ is zero for all $j \in J$ implies that a cell $\mathbb{B}^{n+1}$ is attached to X^n by the class

$$-(-1)^{|v_k|} \pi_*(j)([\alpha_l, \alpha_k]_W) \in \pi_n(X^n).$$

Then $[v_k][v_l]$ is an element in a basis of $H^+(\Lambda V, d_1)$.

(ii) Assume that, for $m \geq 2$ distinct basis elements $v_k \wedge v_l \in (\Lambda^2 V)^{n+1}$, there exists exactly one $j \in J$ such that the coefficients $\lambda^j_{k,l}$ in $d_1(w_j)$ are non-zero, $k, l \in I$. Fix one element $v_p \wedge v_q$, then for each of the remaining $m-1$ elements $v_k \wedge v_l$ a cell $\mathbb{B}^{n+1}$ is attached to X^n by the class

$$(-1)^{|v_p|}\mu^j_{p,q}\pi_*(j)([\alpha_q, \alpha_p]_W) - (-1)^{|v_k|}\mu^j_{k,l}\pi_*(j)([\alpha_l, \alpha_k]_W) \in \pi_n(X^n).$$

Then the $m-1$ products $[v_k][v_l]$ are elements in a basis of $H^+(\Lambda V, d_1)$.

Moreover, if the products $[v_k][v_l]$ above in combination with the elements $([v_i])_{i \in I}$ provide a full basis of $H^+(\Lambda V, d_1)$, this concludes the description of X.

This monograph is organized as follows. The first chapter gives an introduction to the world of (differential) graded algebras, (differential) graded coalgebras and (differential) graded Lie algebras, setting the foundation for the algebraic models that will appear in rational homotopy theory. In particular, two very essential constructions will be introduced, namely that of a free commutative graded algebra and that of a free graded Lie algebra. In the second chapter, we will define the idea of localization that is due to Sullivan. This is where the topological side of rational homotopy theory will be discussed, such that we know what world our algebraic models will be applied to. We will introduce the geometric interpretation of localization (rationalization) and define the rational homotopy type of a space. Moreover, we will discuss how given spaces can be localized cell by cell, deepening the understanding of what localization means from a topological viewpoint. The third chapter then gives an overview of the models used by Sullivan. We will describe what a (minimal) Sullivan model is, how it can be obtained, and why they are such a powerful computational tool. Moreover, we will briefly discuss the more general concept of relative Sullivan algebras, which are especially useful in modelling fibrations. In the end of chapter three, we present the realization of a Sullivan algebra as due to Sullivan, implying that under mild assumptions, the full rational homotopy type of a space is encoded in its minimal Sullivan model. Along the way, we will specify how certain information, such as the rational homotopy groups of the space and the Whitehead product, can be obtained from the model. The fourth chapter then introduces (free) Lie models of spaces. While it was Quillen who showed that differential graded Lie algebras provide good models for the rational homotopy type, we will use a functor to translate them into the world of commutative graded algebras such that we can make use of Sullivan's A_{PL} construction. After a description of the functors involved, we can address the realization of certain free differential graded Lie algebras that we will make use of. With that, we are equipped to show the two main theorems as stated above in chapter five before providing some examples, as well as a small outlook on what might further be possible.

Contents

1 Graded Algebra

If we take a look at the invariants that are mainly used in homotopy theory, an immediate observation is that they most often entail a grading. In addition to that, we will see that there commonly exists an operation respecting the grading. So for example, the cohomology groups have the cup product which stems from a comultiplication that can be defined on the homology groups, and on the rational homotopy groups of a space the Whitehead product can be used to define a Lie bracket. We will therefore begin by introducing notions of graded (co-)algebras and graded Lie algebras. Moreover, since the algebraic models we will use later on will not be applied on (co-)homology level but one step prior, we frequently work with objects that have a differential. Therefore, let us start with the introduction of complexes.

1.1 Complexes

Throughout this section, let $\mathbb{K}$ be any commutative unitary ring. When we speak about modules or linearity, we mean $\mathbb{K}$-modules and $\mathbb{K}$-linear maps. Tensor products are to be understood as tensor products over $\mathbb{K}$.

Definition 1.1.1. (Graded modules)
A *graded module* $(V,(V_i)_{i\in\mathbb{Z}})$ is a family (V_i) of modules, such that $V = \oplus_i V_i$. An element $v \in V_i$ is called a *homogeneous element of degree* i, and we write $|v| = i$. If $V_i = 0$ for all $i \notin I$ for some $I \subseteq \mathbb{Z}$, we say V is *concentrated in degrees* $i \in I$ and identify V with $\oplus_{i\in I} V_i$.

We call a graded module V *free* if V_i is a free module for each $i \in \mathbb{Z}$. In this case, we call the disjoint union of bases of the modules V_i a *homogeneous basis* for V. Whenever $\mathbb{K}$ is a *field*, we speak about a *graded vector space*. These are always free, given the axiom of choice.
We say a graded vector space has *finite type* if V_i is of finite dimension for all $i \in \mathbb{Z}$. A vector space V that is concentrated in finitely many degrees is finite dimensional if and only if it has finite type. The dimension of V is the sum of the dimensions of the subspaces V_i.
We chose the denotation V to insinuate that in many cases of interest, we will in fact work with graded vector spaces, or free modules at least.

Example 1.1.2.

(a) We can view $\mathbb{K}$ as a graded module concentrated in degree zero.

(b) For $i \in \mathbb{Z}$ let S_i be any set. We can form the free $\mathbb{K}$-module with basis S_i that is defined as

$$V_i := \bigoplus_{s\in S_i} \mathbb{K}.$$

By identifying each $s \in S_i$ with the element whose s-th entry is the unity and all others being zero, we can think of elements in V_i as being formal linear combinations $\sum_{s\in S_i} k_s s$, where only finitely many $k_s \in \mathbb{K}$ are nonzero. Then $V := \oplus_i V_i$ is a free graded module with basis $S := \cup_i S_i$.

Remark 1.1.3.

(a) We say $V' \subseteq V$ is a (graded) submodule if there exist submodules $V'_i \subseteq V_i$ such that $V' = \oplus_i V'_i$. For example, each module V_i can be regarded as a submodule of V associated with the family (V_j) where $V_j = 0$ for $j \neq i$.

(b) The graded quotient module V/V' associated with a submodule V' is the module that is given by the family (V_i/V'_i), so $V/V' = \oplus_i(V_i/V'_i)$.

(c) We denote by $V_{>k}$ the graded submodule associated with the family $(V_i)_{i>k}$ and define $V_{\geq k}$, $V_{<k}$, $V_{\leq k}$ analogously. In particular, $V_+ := V_{>0}$.

Let $j \in \mathbb{Z}$. A linear map $f\colon V \to W$ between graded modules such that $f(V_i) \subseteq W_{i+j}$ for all $i \in \mathbb{Z}$ is called a *linear map of degree* j. By a morphism between graded modules we mean a linear map of degree zero. The restriction $f|_{V_i}$ will be denoted by f_i, but we may occasionally drop the indices when they are obvious from the context. This way, any linear map f on V defines a family of linear maps (f_i). Further, by the universal property of the direct sum it follows that any such family defines a linear map on V. Therefore, it suffices to specify linear maps on homogeneous elements of V. To emphasize the graded setting when needed, we sometimes use the notation V_* for the direct sum and f_* for linear maps on it.

Example 1.1.4. Associated with a pointed topological space (X, x_0) are its homotopy groups $\pi_k(X, x_0)$, that are homotopy classes of maps $f\colon (\mathbb{S}^k, s_0) \to (X, x_0)$. While $\pi_0(X, x_0)$ is simply the set of path-components of X, for $k \geq 1$ the map $\mathbb{S}^k \to \mathbb{S}^k \vee \mathbb{S}^k$ collapsing the equator $\mathbb{S}^{k-1} \subseteq \mathbb{S}^k$ to a point may be used to define a group operation on $\pi_k(X, x_0)$. For $k \geq 2$ this has the property that it is commutative. Moreover, any continuous map $g\colon (X, x_0) \to (Y, y_0)$ induces a morphism $\pi_k(g)\colon \pi_k(X, x_0) \to \pi_k(Y, y_0)$ via $\pi_k(g)([f]) = [g \circ f]$.
If X is path-connected, the groups $\pi_k(X, x_0)$ are isomorphic for all choices of $x_0 \in X$. In this case one just writes $\pi_k(X)$ for the isomorphy class. Assuming X is actually a simply connected space, the groups $\pi_k(X)$ are all abelian, as $\pi_0(X) = \pi_1(X) = 0$. Thus in this case, $\pi_*(X)$ is a graded $\mathbb{Z}$-module and any map $g\colon X \to Y$ between simply connected spaces induces a morphism $\pi_*(g)\colon \pi_*(X) \to \pi_*(Y)$.

Definition 1.1.5. A *differential* in a graded module is a linear map $d\colon V \to V$ of degree -1, with $d^2 = 0$. That is, d consists of a family of linear maps $(d_i\colon V_i \to V_{i-1})_{i\in\mathbb{Z}}$ such that $d_i \circ d_{i+1} = 0$. The pair (V, d) is then called a *complex*.

By a morphism between two complexes, we mean a linear map $\varphi\colon (V, d_V) \to (W, d_W)$ of degree zero commuting with the differentials, i.e. $\varphi \circ d_V = d_W \circ \varphi$. Any two such morphisms φ, ψ are called *homotopic*, if there exists a linear map $h\colon (V, d_V) \to (W, d_W)$ of degree 1 such that $\varphi - \psi = h \circ d_V + d_W \circ h$. This is then denoted by $\varphi \sim \psi$ and h is called a *chain homotopy*. Moreover, if $\varphi\colon (V, d_V) \to (W, d_W)$ is a morphism such that there exists a second morphism $\psi\colon (W, d_W) \to (V, d_V)$ with $\psi \circ \varphi \sim \mathrm{id}_V$ and $\varphi \circ \psi \sim \mathrm{id}_W$, then φ is called a *chain equivalence*.

Remark 1.1.6. A submodule $V' \subseteq V$ of a complex (V, d) is called a *subcomplex* if the restriction $d|_{V'}$ is a differential in V', meaning $d_i|_{V'_i}$ is a map $V'_i \to V'_{i-1}$ with $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. A subcomplex allows the definition of a differential on the quotient module V/V' that is induced by the differential in V, so $d([v]) := [d(v)]$, where $[v]$ denotes the equivalence class of a homogeneous element $v \in V$. This is well-defined as $d(V') \subseteq V'$ and a differential since $d\colon V \to V$ is one.

Example 1.1.7. Let V, W be graded modules. The direct sum and tensor product of V and W, when given the gradings

$$V \oplus W = \bigoplus_{i\in\mathbb{Z}} (V_i \oplus W_i)$$
$$V \otimes W = \bigoplus_{k\in\mathbb{Z}} \Big(\bigoplus_{i+j=k} (V_i \otimes W_j) \Big),$$

are again graded modules.
If $f\colon V_1 \to V_2$, $g\colon W_1 \to W_2$ are linear maps of degree j, k, we can define a linear map $f \otimes g\colon V_1 \otimes W_1 \to V_2 \otimes W_2$ of degree $j + k$ by

$$(f \otimes g)(v \otimes w) := (-1)^{k|v|} f(v) \otimes g(w).$$

Suppose we are given differentials d_V and d_W respectively in V and W, then there exists a differential d in $V \otimes W$, namely

$$d(v \otimes w) := d_V(v) \otimes w + (-1)^{|v|} v \otimes d_W(w)$$

where $v \in V$, $w \in W$ are homogeneous elements. This means whenever (V, d_V), (W, d_W) are complexes, so is $(V \otimes W, d)$.

Example 1.1.8. Let $\operatorname{Hom}(V, W)$ be the set of linear maps from V to W. There is a natural grading on $\operatorname{Hom}(V, W)$ that is given by the sets $(\operatorname{Hom}(V, W))_i$ consisting of all linear maps of degree i. Together with the pointwise addition and scalar multiplication, each $(\operatorname{Hom}(V, W))_i$ itself is a module, turning $\operatorname{Hom}(V, W)$ into a graded module.
If $f\colon V_1 \to V_2$, $g\colon W_1 \to W_2$ are linear maps of degree j, k, we can define a linear map of degree $j + k$ by

$$\operatorname{Hom}(f, g)\colon \begin{cases} \operatorname{Hom}(V_2, W_1) & \to \operatorname{Hom}(V_1, W_2) \\ \varphi & \mapsto (-1)^{j(k+|\varphi|)} g \circ \varphi \circ f. \end{cases}$$

If (V, d_V), (W, d_W) are complexes, the linear map $d\colon \operatorname{Hom}(V, W) \to \operatorname{Hom}(V, W)$ defined by

$$d(\varphi) := d_W \circ \varphi - (-1)^{|\varphi|} \varphi \circ d_V$$

for a homogeneous element $\varphi \in \operatorname{Hom}(V, W)$ is a differential. This way, $(\operatorname{Hom}(V, W), d)$ is again a complex.

Note that the kernel and the image of linear maps again determine graded submodules. If $f\colon V \to W$ is a linear map of degree j, then $\ker(f) \subseteq V$ and $\operatorname{im}(f) \subseteq W$ consist of the families $(\ker f)_i := \ker f_i \subseteq V_i$ and $(\operatorname{im} f)_i := \operatorname{im} f_{i-j} \subseteq W_i$.
Especially for a complex (V, d) this means $\operatorname{im} d_{i+1} \subseteq \ker d_i \subseteq V_i$, as $d^2 = 0$.

Definition 1.1.9. Let (V, d) be a complex. The graded quotient module

$$H(V, d) := \ker d / \operatorname{im} d$$

is called the *homology* of V, or (V, d) to be more precise. An element in $\ker d$ is called a *cycle*, whereas elements of $\operatorname{im} d$ are referred to as *boundaries*.
The corresponding family is given by the modules $(H_i(V, d))_i$, where $H_i(V, d) = \ker d_i / \operatorname{im} d_{i+1}$ is called the *i-th homology* of V. In the following, we will frequently abbreviate the notation and just write $H_i(V)$, dropping the differential.

Any morphism $\varphi\colon (V, d_V) \to (W, d_W)$ between complexes induces a linear map $H(\varphi)\colon H(V) \to H(W)$ by setting $H(\varphi)([v]) := [\varphi(v)]$, where $v \in V$ is a cycle and the brackets denote the equivalence class. If φ and ψ are homotopic, then they induce the same map $H(\varphi) = H(\psi)$. We call φ a *quasi-isomorphism* if the induced map $H(\varphi)$ is an isomorphism. This will be denoted by $\varphi\colon (V, d_V) \xrightarrow{\simeq} (W, d_W)$.

Remark 1.1.10.

(a) A complex (V, d) where V is concentrated in degrees $i \in \mathbb{N}_0$ is called a *chain complex*. The corresponding homology may sometimes be denoted by $H_*(V)$ to stress the lower grading.

(b) A complex (V, d) where V is concentrated in degrees $i \in \mathbb{Z} \setminus \mathbb{N}$ is called a *cochain complex*. In this case, we use the notation

$$V^i := V_{-i}, \quad d^i := d_{-i}$$

to avoid negative degrees. This way, d is considered a linear map of degree 1 with respect to upper degrees, $d^i\colon V^i \to V^{i+1}$. The corresponding homology may sometimes be denoted by $H^*(V)$ to stress the upper grading. Moreover, it usually will be referred to as *cohomology*. Analogously, the elements of $(\ker d)$ are called *cocycles* and the elements of $(\operatorname{im} d)$ are the *coboundaries*.

Proposition 1.1.11. *Given two complexes* (V, d_V) *and* (W, d_W)*, consider the complexes* $(V \otimes W, d)$ *and* $(\mathrm{Hom}(V, W), d)$ *defined in examples 1.1.7 and 1.1.8. When* $\mathbb{K}$ *is a field, it holds that the natural linear maps*

$$H(V, d_V) \otimes H(W, d_W) \to H(V \otimes W, d), \quad [v] \otimes [w] \mapsto [v \otimes w],$$
$$H(\mathrm{Hom}(V, W), d) \to \mathrm{Hom}(H(V, d_V), H(W, d_W)), \quad [\varphi] \mapsto H(\varphi)$$

are isomorphisms. We say that homology commutes with $\mathrm{Hom}(-,-)$ *and* $- \otimes -$.

Proof. This follows from a more general result in [3] and the fact that $\mathrm{Hom}(-,-)$ is an exact functor when $\mathbb{K}$ is a field. □

Example 1.1.12. Let us consider a special case of example 1.1.7 where we assume (V, d) is a chain complex and $(W, d) = (\mathbb{K}, 0)$ concentrated in degree zero. Then $V \otimes \mathbb{K} = \oplus_{i \geq 0} V_i \otimes \mathbb{K}$ is again a chain complex with differential $d(v \otimes \lambda) = d(v) \otimes \lambda$. A linear map $f\colon V_1 \to V_2$ between chain complexes and $g := \mathrm{id}_{\mathbb{K}}$ induce the map

$$f \otimes \mathrm{id}_{\mathbb{K}}\colon \begin{cases} V_1 \otimes \mathbb{K} & \to V_2 \otimes \mathbb{K} \\ v \otimes \lambda & \mapsto f(v) \otimes \lambda \end{cases}$$

which is a morphism of chain complexes that we will usually denote with $f \otimes \mathbb{K}$.

Example 1.1.13. Let us consider a special case of example 1.1.8 where we assume (V, d) is a chain complex and $(W, d) = (\mathbb{K}, 0)$. Each element in $(\mathrm{Hom}(V, \mathbb{K}))_j$ is a linear map $\varphi\colon V \to \mathbb{K}$ of degree $j \in \mathbb{Z}$, so a family $(\varphi_i\colon V_i \to \mathbb{K}_{i+j})_i$. These maps are trivial for $i \leq 0$, and $i \neq -j$ since V is a chain complex and $\mathbb{K}_{i+j} = 0$ for $i + j \neq 0$. Thus φ is either trivial or can be identified with the only nontrivial member of the family, φ_{-j}, which is an element of degree $j \geq 0$ with respect to upper degrees. It follows that the homogeneous elements of degree j of the graded module $\mathrm{Hom}(V, \mathbb{K})$ are exactly the linear maps $V_j \to \mathbb{K}$, so

$$(\mathrm{Hom}(V, \mathbb{K}))^j = \mathrm{Hom}(V_j, \mathbb{K}).$$

Now for a linear map $f\colon V_1 \to V_2$ between chain complexes and $g := \mathrm{id}_{\mathbb{K}}$ consider the induced linear map $\mathrm{Hom}(f, \mathrm{id}_{\mathbb{K}})\colon \mathrm{Hom}(V_2, \mathbb{K}) \to \mathrm{Hom}(V_1, \mathbb{K})$ defined in example 1.1.8. By definition, it holds

$$\mathrm{Hom}(f, \mathrm{id}_{\mathbb{K}})(\varphi) = (-1)^{|f||\varphi|} \varphi \circ f$$

for any homogeneous element $\varphi \in \mathrm{Hom}(V_2, \mathbb{K})$. Since this map only depends on f, we will denote it by $\mathrm{Hom}(f, \mathbb{K})$. For the special case that $f = d$ is the differential in V, we get $\mathrm{Hom}(d, \mathbb{K})(\varphi) = -(-1)^{|\varphi|} \varphi \circ d$ which defines a differential in $\mathrm{Hom}(V, \mathbb{K})$, making it a cochain complex.
Note that this defines a contravariant functor between chain and cochain complexes. For a morphism $f\colon V_1 \to V_2$ between chain complexes the induced map simplifies to

$$\mathrm{Hom}(f, \mathbb{K})\colon \begin{cases} \mathrm{Hom}(V_2, \mathbb{K}) & \to \mathrm{Hom}(V_1, \mathbb{K}) \\ \varphi & \mapsto \varphi \circ f. \end{cases}$$

Hence $\mathrm{Hom}(f, \mathbb{K})$ is a map of degree zero and defines a morphism of cochain complexes.

Remark 1.1.14. In the case that $\mathbb{K}$ is a field, we will frequently use the shorter notation

$$V^* := \mathrm{Hom}(V, \mathbb{K}) \quad \text{and} \quad f^* := \mathrm{Hom}(f, \mathbb{K})$$

and refer to these objects as *dual space* or respectively *dual map*. In this case, it also follows that $H(V^*, d^*) \cong H(V, d)^*$ via the mapping $[\varphi] \mapsto H(\varphi)$ due to proposition 1.1.11. Note that in this special case $H(\varphi)([v]) = \varphi(v)$, but we will usually express this using a different notation. More precisely, there is a natural pairing between V and its dual V^* given by

$$V^* \times V \to \mathbb{K}, \quad (\varphi, v) \mapsto \langle \varphi; v\rangle = \varphi(v).$$

This pairing is compatible with the corresponding differentials and can be handed down to give a pairing between the cohomology of V^* and the homology of V via $\langle [\varphi], [v]\rangle := \varphi(v)$. Obviously, this is bilinear and we get a map

$$H(V^*, d^*) \to H(V, d)^*, \quad [\varphi] \mapsto \langle [\varphi]; -\rangle,$$

which is exactly the isomorphism described in proposition 1.1.11.

Note that if V is a graded vector space of finite type, then we have $V_i \cong (V_i)^*$ for all i and therefore $V \cong V^*$. A homogeneous basis of V then defines a homogeneous basis in V^* by the dual pairing above, which we will call dual basis.

Preparation 1.1.15. Let X be a topological space and $n, m \in \mathbb{N}$. Denote by e_i, $0 \leq i \leq n, m-1$ the standard basis vectors in $\mathbb{R}^{n+1}$ or $\mathbb{R}^m$ respectively. The standard n-simplex is defined as the subset $\Delta^n := \{\sum_{i=0}^n \lambda_i e_i \mid 0 \leq \lambda_i \leq 1, \sum_{i=0}^n \lambda_i = 1\} \subseteq \mathbb{R}^{n+1}$. For any sequence of elements $x_0, \ldots, x_m \in \mathbb{R}^{n+1}$ there exists an associated linear map

$$\langle x_0, \ldots, x_m\rangle \colon \begin{cases} \Delta^m & \to \mathbb{R}^{n+1} \\ \sum_{i=0}^m \lambda_i e_i & \mapsto \sum_{i=0}^m \lambda_i x_i. \end{cases}$$

In particular, when each x_i is an element of the standard basis of $\mathbb{R}^{n+1}$, we get a map $\Delta^m \to \Delta^n$. Now, for each $0 \leq i, j \leq n+1$ there exist inclusions

$$\delta_i := \langle e_0, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\rangle \colon \Delta^n \hookrightarrow \Delta^{n+1}$$

as well as projections

$$\rho_j := \langle e_0, \ldots, e_j, e_j, \ldots, e_n\rangle \colon \Delta^{n+1} \to \Delta^n.$$

Denote by

$$S_n(X) := \{\sigma \colon \Delta^n \to X \mid \sigma \text{ is continuous}\}$$

the set of singular n-simplices. Then there exist maps $\partial_i \colon S_{n+1}(X) \to S_n(X)$, $\partial_i(\sigma) := \sigma \circ \delta_i$ for all $0 \leq i \leq n+1$ and $s_j \colon S_n \to S_{n+1}(X)$, $s_j(\sigma) := \sigma \circ \rho_j$ for all $0 \leq j \leq n+1$. We set $S(X) := \{S_n(X)\}_{n\geq 0}$ and note that any continuous map $f \colon X \to Y$ between topological spaces induces a family $S(f) := (S_n(f))_{n\geq 0}$, where

$$S_n(f) \colon S_n(X) \to S_n(Y), \quad \sigma \mapsto f \circ \sigma.$$

Remark 1.1.16.

(a) Denote by $C_n(X; \mathbb{K})$ the free $\mathbb{K}$-module with basis $S_n(X)$ as introduced in example 1.1.2. The elements of $C_n(X; \mathbb{K})$ can be written as formal linear combinations $\sum_{\sigma \in S_n(X)} k_\sigma \sigma$ and any map defined on the basis $S_n(X)$ extends uniquely to a linear map on $C_n(X; \mathbb{K})$. Therefore, the maps that are given by $d_n(\sigma) := \sum_{i=0}^n (-1)^i \partial_i(\sigma)$ for $\sigma \in S_n(X)$ define a differential d on

$$C_*(X; \mathbb{K}) := \oplus_{n\geq 0} C_n(X; \mathbb{K}).$$

The resulting chain complex $(C_*(X; \mathbb{K}), d)$ is called the *singular chain complex* of X with coefficients in $\mathbb{K}$. Its homology $H(C_*(X; \mathbb{K})) =: H_*(X; \mathbb{K})$ is the *singular homology* of X with

coefficients in $\mathbb{K}$.
Note that it would be an equivalent construction to consider the free $\mathbb{Z}$-module $C_n(X;\mathbb{Z})$ with basis $S_n(X)$ and then proceed by tensoring with $\mathbb{K}$, so $C_n(X;\mathbb{K}) = C_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{K}$. This works more generally, when $\mathbb{K}$ is not necessarily a ring but an abelian group. The *universal coefficient theorem* then states the existence of a short exact sequence

$$0 \longrightarrow H_k(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{K} \longrightarrow H_k(X;\mathbb{K}) \longrightarrow \operatorname{Tor}(H_{k-1}(X,\mathbb{Z}),\mathbb{K}) \longrightarrow 0.$$

In particular, if $\mathbb{K}$ is a field, the term $\operatorname{Tor}(H_{k-1}(X,\mathbb{Z}),\mathbb{K})$ vanishes for all $k \geq 1$, yielding an isomorphism

$$H_*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{K} \xrightarrow{\cong} H_*(X;\mathbb{K}), \quad [z] \otimes \lambda \mapsto [z \otimes \lambda],$$

Thus, homology with field coefficients is the tensor product of homology with coefficients in $\mathbb{Z}$ with said field and we may identify $H_*(f;\mathbb{K}) = H_*(f;\mathbb{Z}) \otimes \mathbb{K}$.

(b) To the singular chain complex $C_*(X;\mathbb{K})$ we may apply the contravariant functor $\operatorname{Hom}_{\mathbb{K}}(-,M)$, where M is any $\mathbb{K}$-module. This results in a cochain complex

$$C^*(X;M) := \operatorname{Hom}_{\mathbb{K}}(C_*(X;\mathbb{K}),M),$$

with grading $C^n(X;M) = \operatorname{Hom}_{\mathbb{K}}(C_n(X;\mathbb{K}),M)$ that is called the *singular cochain complex* of X with coefficients in M. The differential is the signed dual of the differential in $C_*(X;\mathbb{K})$ and is defined by $d(\varphi) = (-1)^{|\varphi|+1}\varphi \circ d$. The corresponding cohomology is called the *singular cohomology* of X with coefficients in M and denoted by $H^*(X;M)$. Now if $\mathbb{K}$ is a principal ideal domain, the *universal coefficient theorem*, this time for cohomology, states that there exists a short exact sequence

$$0 \longrightarrow \operatorname{Ext}_{\mathbb{K}}(H_{k-1}(X;\mathbb{K}),M) \longrightarrow H^k(X;M) \longrightarrow \operatorname{Hom}_{\mathbb{K}}(H_k(X;\mathbb{K}),M) \longrightarrow 0.$$

In particular, if $\mathbb{K}$ is a field, the term $\operatorname{Ext}_{\mathbb{K}}(H_{k-1}(X;\mathbb{K}),M)$ vanishes for all $k \geq 1$, yielding an isomorphism

$$\mathcal{H}\colon H^*(X;M) \xrightarrow{\cong} \operatorname{Hom}_{\mathbb{K}}(H_*(X;\mathbb{K}),M), \quad \mathcal{H}([\varphi])([z]) := \varphi(z).$$

In particular, for $M = \mathbb{K}$ we get $H^*(X;\mathbb{K}) \cong (H_*(X;\mathbb{K}))^*$. This implies that cohomology with field coefficients is the dual of homology and we may identify $H^*(f;\mathbb{K}) = (H_*(f;\mathbb{K}))^*$.

More details on this can be found in most literature on algebraic topology, for example in [13]. Note also that these isomorphisms are a special case of proposition 1.1.11.

1.2 Graded Algebras

Throughout this section, let $\mathbb{K}$ be any commutative unitary ring. When we speak about modules or linearity, we mean $\mathbb{K}$-modules and $\mathbb{K}$-linear maps. Tensor products are to be understood as tensor products over $\mathbb{K}$.

Definition 1.2.1. (Graded algebra)
A *graded algebra* is a graded module A together with a linear map of degree zero, $A \otimes A \to A$, $a \otimes b \mapsto ab$ that is associative and has an identity element $1_A \in A_0$. As the notation insinuates, we will often refer to this map as a multiplication.
A morphism $\varphi\colon A \to B$ between graded algebras is a morphism between graded modules that is an algebra homomorphism, i.e. $\varphi(ab) = \varphi(a)\varphi(b)$ for $a,b \in A$ and $\varphi(1_A) = 1_B$.

Example 1.2.2. We can view $\mathbb{K}$ as a graded algebra concentrated in degree zero. Any morphism $\varphi\colon A \to \mathbb{K}$ is called an *augmentation*.

Remark 1.2.3.

(a) We call $A' \subseteq A$ a *graded subalgebra* if it is a graded submodule containing the identity element, such that $ab \in A'$ for any $a, b \in A'$.

(b) A *left ideal* in a graded algebra A is a graded submodule I such that for all $a \in A$ and $b \in I$ it holds $ab \in I$. Analogously, *Right ideals* are defined by requiring $ba \in I$. When we talk about *ideals*, we mean a graded submodule that is both right and left ideal.

(c) For an ideal $I \subseteq A$ we can equip the quotient A/I with the grading (A_i/I_i), turning it into a graded algebra.

(d) If A is a graded algebra, then we call a graded module V a *left A-module* if there exists a linear map $A \otimes V \to V$, $a \otimes v \mapsto av$ which is associative and suffices $1v = v$ for each $v \in V$. A right module is defined in the same way, and we say V is an *A-Module* if it is both right and left module.
A map $f\colon V \to W$ between (left) A-modules is called a linear map of degree k, if $|f| = k$ and $f(av) = (-1)^{|f||a|} af(v)$ holds for all $a \in A$, $v \in V$. The space of A-linear maps is denoted by $\mathrm{Hom}_A(V, W)$ and is a graded submodule of $\mathrm{Hom}(V, W)$.

(e) Note that any morphism $\varphi\colon A \to B$ between graded algebras makes B into an A-Module by using the multiplication in B and setting $ab := \varphi(a)b$ and $ba := b\varphi(a)$ for $a \in A$, $b \in B$.

(f) Given a right A-module V and left A-module W of a graded algebra A, we can define the *tensor product* to be the quotient space $V \otimes_A W := (V \otimes W)/L$, where L is the submodule generated by the set

$$\{va \otimes w - v \otimes aw \mid v \in V, w \in W, a \in A\}.$$

The equivalence class of an element $v \otimes w$ is denoted by $v \otimes_A w$. Obviously this is again a graded module, but in general it is not an A-module in any natural way.

Note that for a graded algebra $A = \oplus_i A_i$, each A_i is a submodule of A but not necessarily a subalgebra. In fact, for homogeneous elements $a \in A_i$, $b \in A_j$ it holds $ab \in A_{i+j}$, as $|a \otimes b| = i + j$. This means we can think of graded algebras as associative algebras with an unity element, allowing for a grading which is respected by its multiplication.

Example 1.2.4. Let A, B be graded algebras. We can define a multiplication on the graded module $A \otimes B$ making it a graded algebra, namely

$$(a_1 \otimes b_1)(a_2 \otimes b_2) := (-1)^{|b_1||a_2|}(a_1a_2 \otimes b_1b_2)$$

for homogeneous elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

Example 1.2.5. Assume $\varphi\colon A \to B$ to be a morphism of graded algebras and W to be a free A-module, that is $W \cong A \otimes V$ where V is a free graded module. As an A-module, A is just generated by the identity, so a basis of V naturally provides a basis for W. It then holds

$$B \otimes_A W = B \otimes_A (A \otimes V) = B \otimes V,$$

as $B \otimes_A A \cong B$ via the map $b \otimes_A a \mapsto b$, which has an inverse $b \mapsto b \otimes_A 1$. Thus, $B \otimes_A W$ is a free B-module with the same basis as V.

Example 1.2.6. Let V be a graded module, then $\mathrm{Hom}(V, V)$ is a graded algebra, with multiplication given by the composition of maps. Similarly, if V is an A-module for a graded algebra A, then $\mathrm{Hom}_A(V, V)$ is a graded algebra.

Example 1.2.7. Assume we are given linear maps $f_1, f_2 \in \mathrm{Hom}(V,V)$ and $g_1, g_2 \in \mathrm{Hom}(W,W)$, then a simple calculation using the formula of example 1.1.7 shows

$$(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (-1)^{|g_1||f_2|}(f_1 \circ f_2) \otimes (g_1 \circ g_2).$$

In other words, the induced graded algebra structure on the tensor product $\mathrm{Hom}(V,V) \otimes \mathrm{Hom}(W,W)$ in the sense of example 1.2.4 coincides with the multiplication on $\mathrm{Hom}(V \otimes W, V \otimes W)$ that is the composition of maps.

Definition 1.2.8. A linear map $d\colon A \to A$ of degree k on a graded algebra A is called a *derivation of degree k* if it holds

$$d(ab) = d(a)b + (-1)^{k|a|}ad(b)$$

for any two homogeneous elements $a, b \in A$. This is sometimes referred to as *Leibniz rule*. When we speak of a *derivation*, we mean a derivation of degree -1 or 1, depending on whether lower or upper degrees are used in the context.

Definition 1.2.9. We call a graded algebra A *commutative* in the graded sense if for any two homogeneous elements $a, b \in A$ it holds

$$ab = (-1)^{|a||b|}ba.$$

The algebra A is then called *commutative graded algebra* or CGA for short.

Remark 1.2.10. When we assume $\mathbb{K}$ to be a field of characteristic zero, it follows that for a CGA A it holds $a^2 = 0$ whenever $a \in A$ is a homogeneous element of odd degree. This is immediate from the identity $a^2 = -a^2$ implied by the commutativity.

Example 1.2.11. The product structure defined in remark 1.2.4 makes $A \otimes B$ into a CGA if both A and B are commutative graded algebras.
If A is graded commutative, then any left A-module V is automatically a right module and vice versa. For example, a left-multiplication defines a right-multiplication through $va = (-1)^{|v||a|}av$. Suppose W is a second A-module. It follows then that in case of the underlying algebra being commutative, the modules $V \otimes_A W$ and $\mathrm{Hom}_A(V,W)$ are A-modules again, as

$$(af)(v) := af(v) = (-1)^{|a||f|}f(av)$$
$$a(v \otimes_A w) := av \otimes_A w = (-1)^{|a||v|}v \otimes_A aw.$$

In particular, given morphisms $A \to B$ and $A \to C$ of commutative graded algebras, then the product structure on $B \otimes C$ induces a product structure on $B \otimes_A C$ making it into a commutative graded algebra that is an A-module.

Definition 1.2.12. (Commutative differential graded algebra)
A *differential graded algebra*, or DGA for short, is a complex (A,d) where A is a graded algebra and d is a differential in A that is also a derivation.
If the algebra A is also commutative in the graded sense, we call the pair (A,d) a *commutative differential graded algebra*, or CDGA.

Note that the homology $H(A,d)$ of a (C)DGA is itself a (commutative) graded algebra, with the induced representative wise multiplication. We may thus speak of a *homology algebra*. A morphism $\varphi\colon (A,d_A) \to (B,d_B)$ of DGAs is a morphism of graded algebras that is also a morphism of complexes. As before, we will call any such morphism a quasi-isomorphism if the induced map between the homologies is an isomorphism, that is, an isomorphism between graded algebras.
Accordingly, the terms *(commutative) chain algebra* and *(commutative) cochain algebra* refer to a (C)DGA with families $(A_i)_{i\geq 0}$ and $(A^i)_{i\geq 0}$ respectively. When we want to stress lower or upper grading, we may occasionally write $H_*(A,d)$ and $H^*(A,d)$ for the homology.

Example 1.2.13. There exists a product structure on the singular cochain complex $C^*(X, \mathbb{K})$ that is called the *cup product*. It is not commutative and makes $C^*(X, \mathbb{K})$ a differential graded algebra, or simply cochain algebra. As it turns out, the induced multiplication on the singular cohomology algebra $H^*(X; \mathbb{K})$ is commutative, making $H^*(X; \mathbb{K})$ into a commutative graded algebra. For a detailed description of the cup product on cochain level, see [15] and [6].

Example 1.2.14. When (A, d) is a DGA, we speak of a *left* (A, d)*-module* (V, d) when the corresponding linear map $A \otimes V \to V$ respects the differential, meaning

$$d(av) = d(a)v + (-1)^{|a|}ad(v).$$

A right (A, d)-module (W, d) is defined analogously. Now, there exists a differential in the corresponding module $V \otimes_A W$ which is induced by the differential in $V \otimes W$ (see example 1.1.7). This naturally makes $V \otimes_A W$ a complex, which we simply denote by $(V \otimes_A W, d)$ or by $(V, d) \otimes_{(A,d)} (W, d)$ if we want to emphasize that we have a tensor product of (A, d)-modules.
For example, any morphism $\varphi\colon (A, d) \to (B, d)$ of DGAs makes (B, d) into an (A, d)-module in the sense that we have already defined for graded algebras in remark 1.2.3, as φ commutes with the differential. Assume now $(A, d_A) \to (B, d_B)$ and $(A, d_A) \to (C, d_C)$ are morphisms of CDGAs. We know that in this case, $B \otimes_A C$ is a commutative graded algebra, and by the above it also has a differential d. One can now check that this is also a derivation, meaning $(B \otimes_A C, d)$ is also a CDGA.

Example 1.2.15. Let M be a smooth manifold and denote by $A^p_{DR}(M)$ the set of p-forms on M. Equipped with the exterior product, the space $A_{DR}(M) := \bigoplus_{p \geq 0} A^p_{DR}(M)$ is a CGA. Thus, the *de Rham complex* $(A_{DR}(M), d)$, where d is the exterior derivative, is an example of a CDGA - or more precisely a commutative cochain algebra.

Definition 1.2.16. We call two commutative cochain algebras (A, d_A), (B, d_B) *weakly equivalent* if there exists $k \geq 0$ and a sequence of commutative cochain algebras $(C_i, d_i)_{0 \leq i \leq k}$ together with morphisms

$$(A, d_A) \xrightarrow{\simeq} (C_0, d_0) \xleftarrow{\simeq} \dots \xrightarrow{\simeq} (C_k, d_k) \xleftarrow{\simeq} (B, d_B)$$

that are quasi-isomorphisms. This sequence, together with the morphisms, is then called a *weak equivalence* and we may simply write $(A, d_A) \simeq (B, d_B)$.

Being weakly equivalent defines an equivalence relation on the set of commutative cochain algebras.

1.3 Graded Coalgebras

Throughout this section, let $\mathbb{K}$ be any commutative unitary ring. When we speak about modules or linearity, we mean $\mathbb{K}$-modules and $\mathbb{K}$-linear maps. Tensor products are to be understood as tensor products over $\mathbb{K}$.

Definition 1.3.1. For a graded module V we define its *suspension* sV to be the graded module given by $(sV)_i = V_{i-1}$. For $v \in V_{i-1}$ we denote the respective element of degree i by $sv \in (sV)_i$. Vice versa, we can define $s^{-1}V$ to be the graded module $(s^{-1}V)_i = V_{i+1}$, with $s^{-1}v \in (s^{-1}V)_i$ for any $v \in V_{i+1}$. Spoken differently, sV is a copy of V with an upshift of grades, while $s^{-1}V$ is a copy of V with a downshift of grades.

By the *suspension of a complex* (V, d), we mean the complex $s(V, d) := (sV, \overline{d})$, where the differential is defined as $\overline{d}(sv) := -sd(v)$. It is clear that $H(sV, \overline{d}) = sH(V, d)$.

Definition 1.3.2. A graded module C is called a *graded coalgebra* if there exist linear maps of degree zero,

$$\Delta\colon C \to C \otimes C, \qquad \varepsilon\colon C \to \mathbb{K}$$

such that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and $(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$. We then refer to Δ as a *comultiplication* and ε is an *augmentation.*
Further, we say C is *co-commutative* if the comultiplication Δ is invariant under the involution $\tau\colon C \otimes C \to C \otimes C$, $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$, that is $\tau \circ \Delta = \Delta$.

Remark 1.3.3.

(a) By a morphism of graded coalgebras we mean a linear map of degree zero $\varphi\colon C \to D$ such that $(\varphi \otimes \varphi) \circ \Delta_C = \Delta_D \circ \varphi$ and $\varepsilon_C = \varepsilon_D \circ \varphi$.

(b) When we can choose an element $1_C \in C_0$ such that $\varepsilon(1_C) = 1$ and $\Delta(1_C) = 1_C \otimes 1_C$, we call C a *co-augmented* graded coalgebra.

(c) In a co-augmented graded coalgebra, it holds for any element $a \in \ker(\varepsilon)$ that

$$\Delta a - (a \otimes 1 + 1 \otimes a) \in \ker(\varepsilon) \otimes \ker(\varepsilon).$$

(d) A *coderivation of degree k* in a graded coalgebra C is a linear map $f\colon C \to C$ of degree k such that $\Delta \circ f = (f \otimes \mathrm{id} + \mathrm{id} \otimes f) \circ \Delta$ and $\varepsilon \circ f = 0$. When we simply speak of a coderivation, we mean a coderivation of degree -1.

Example 1.3.4. Let $C = C_0$ be the free module with basis S concentrated in degree zero. Define $\Delta(s) := s \otimes s$ and $\varepsilon(s) := 1$ for all $s \in S$. The unique extensions of Δ and ε on $C \otimes C$ and C make it a graded coalgebra that is co-commutative.

Definition 1.3.5. A graded coalgebra C together with a differential d that is also a coderivation is called a *differential graded coalgebra* (C, d), or DGC for short.

Example 1.3.6. Given a graded coalgebra C we can always define a multiplication on the graded module $\mathrm{Hom}(C, \mathbb{K})$ that turns it into a graded algebra. That is, for $f, g \in \mathrm{Hom}(C, \mathbb{K})$ we define $fg\colon C \to \mathbb{K}$ to be the map

$$(fg)(c) := (f \otimes g)(\Delta(c)),$$

where as usual, we understand $f \otimes g$ as a map $C \otimes C \to \mathbb{K}$ making use of the multiplication in $\mathbb{K}$. It follows that the identity element is given by the map $\varepsilon \in \mathrm{Hom}(C, \mathbb{K})$.
Now assume (C, d) is in fact a differential graded coalgebra. Since d is a differential, we get an induced differential $\mathrm{Hom}(d, \mathbb{K})$ on $\mathrm{Hom}(C, \mathbb{K})$ that we, abusing notation, again denote by d. Then the fact that d is also a coderivation implies that this induced differential is a derivation.

Proof. By example 1.2.7 and the definition of the induced differential (see 1.1.13) we see that

$$\begin{aligned} d(fg)(c) &= -(-1)^{|f|+|g|}(fg) \circ d(c) = -(-1)^{|f|+|g|}(f \otimes g) \circ \Delta \circ d(c) \\ &= -(-1)^{|f|+|g|}(f \otimes g) \circ (d \otimes \mathrm{id} + \mathrm{id} \otimes d) \circ \Delta(c) \\ &= -(-1)^{|f|}((f \circ d) \otimes g) \circ \Delta(c) - (-1)^{|f|+|g|}(f \otimes (g \circ d)) \circ \Delta(c) \\ &= -(-1)^{|f|}(f \circ d)g(c) - (-1)^{|f|+|g|} f(g \circ d)(c) \\ &= (d(f)g + (-1)^{|f|} f d(g))(c) \end{aligned}$$

for all $c \in C$ and all $f, g \in \mathrm{Hom}(C, \mathbb{K})$. $\square$

In other words, a differential graded coalgebra (C, d) induces a differential graded algebra

$$(\mathrm{Hom}(C, \mathbb{K}), \mathrm{Hom}(d, \mathbb{K}))$$

that we will simply denote by (C^*, d^*) when $\mathbb{K}$ is a field. Note that for the involution τ and $f, g \in \mathrm{Hom}(C, \mathbb{K})$ it holds $(f \otimes g) \circ \tau = (-1)^{|f||g|} g \otimes f$, since for $a \otimes b \in C \otimes C$ we get

$$(f \otimes g) \circ \tau(a \otimes b) = (f \otimes g)((-1)^{|a||b|} b \otimes a) = (-1)^{|f||g|} f(b)g(a) = (-1)^{|f||g|}(g \otimes f)(a \otimes b),$$

where we used that we may assume $|f| = |b|$ and $|g| = |a|$. It follows that $(\mathrm{Hom}(C, \mathbb{K}), \mathrm{Hom}(d, \mathbb{K}))$ is commutative if C is co-commutative.

Example 1.3.7. There exists a comultiplication on the free graded module $C_*(X; \mathbb{K})$ with basis $S(X)$. For this, consider the Alexander-Whitney map $Q\colon C_*(X \times Y; \mathbb{K}) \to C_*(X; \mathbb{K}) \otimes C_*(Y; \mathbb{K})$ induced by defining

$$Q(\sigma, \tau) := \sum_{k=0}^{n} \sigma \circ \langle e_0, \dots, e_k \rangle \otimes \tau \circ \langle e_k, \dots, e_n \rangle$$

on the singular n-simplices $(\sigma, \tau)\colon \Delta^n \to X \times Y$. See preparation 1.1.15 for the notation. Next, consider the diagonal map $\delta\colon X \to X \times X$, $x \mapsto (x, x)$. As any continuous map, it induces a linear map $C_*(X; \mathbb{K}) \to C_*(X \times X; \mathbb{K})$ via $C_*(\delta, \mathbb{K})(\sigma) = \delta \circ \sigma$, so we can define a map

$$\Delta := Q \circ C_*(\delta)\colon C_*(X; \mathbb{K}) \to C_*(X; \mathbb{K}) \otimes C_*(X; \mathbb{K})$$

that is a comultiplication. A choice of a constant map $c_{x_0}\colon X \to \{x_0\} \subseteq X$ provides a suitable augmentation $\varepsilon := C_*(c_{x_0}; \mathbb{K})\colon C_*(X; \mathbb{K}) \to C_*(\{x_0\}; \mathbb{K}) = \mathbb{K}$. Together with the differential d on $C_*(X; \mathbb{K})$ which is obtained from the maps ∂_i on the simplicial set $S(X)$ we get that the singular chain complex $(C_*(X; \mathbb{K}), d)$ is in fact a differential graded coalgebra.
By applying the contravariant functor $\mathrm{Hom}(-, \mathbb{K})$ to $(C_*(X; \mathbb{K}), d)$ we obtain the singular cochain complex $(C^*(X; \mathbb{K}), d)$ of X. Now, the multiplication induced on $C^*(X; \mathbb{K})$ with regard to example 1.3.6 actually corresponds to the cup product that we mentioned earlier. Hence, we regain the same algebra, that is the singular cochain algebra $(C^*(X; \mathbb{K}), d)$ of X. Again, for more details on this one may check [15].

Example 1.3.8. There is a natural coalgebra structure on the free commutative graded algebra ΛV associated with a free graded module V. The comultiplication is obtained by defining $\Delta(v) := v \otimes 1 + 1 \otimes v$ for all $v \in V$ and extending this to a map $\Delta\colon \Lambda V \to \Lambda V \otimes \Lambda V$. An augmentation is given by $\varepsilon(1) = 1$ and $\varepsilon|_{\Lambda^+ V} = 0$, since $\Lambda^0 V = \mathbb{K}$ it is co-augmented by choosing the unity $1 \in \mathbb{K}$. One easily checks that $\tau \circ \Delta(v) = \Delta(v)$ for all $v \in V$, thus $\tau \circ \Delta = \Delta$ on ΛV, which means it is a co-commutative graded coalgebra.

Construction 1.3.9. Let (C, d) a differential graded coalgebra which is co-augmented, and set $\overline{C} := \ker(\varepsilon)$ such that $C = \mathbb{K} \oplus \overline{C}$. Then by remark 1.3.3, it holds $\Delta c - (c \otimes 1 + 1 \otimes c) \in \overline{C} \otimes \overline{C}$ for any $c \in \overline{C}$. Thus, there exists a so called *reduced comultiplication* in $\overline{C}$ which is defined as

$$\overline{\Delta}\colon \overline{C} \to \overline{C} \otimes \overline{C}, \quad \overline{\Delta}(c) := \Delta c - (c \otimes 1 + 1 \otimes c).$$

Note that $(\overline{\Delta} \otimes \mathrm{id}) \circ \overline{\Delta} = (\mathrm{id} \otimes \overline{\Delta}) \circ \overline{\Delta}$, so this is in fact a comultiplication in $\overline{C}$.
Consider the tensor algebra $T(s^{-1}\overline{C}) =: \Omega C$ on the graded module $s^{-1}\overline{C}$. Obviously, this is a graded algebra, with multiplication given by the tensor product. Let $c \in \overline{C}$, then $\overline{\Delta}(c) \in \overline{C} \otimes \overline{C}$ is a linear combination $\sum_i a_i \otimes b_i$ of elements with $a_i, b_i \in \overline{C}$. Proceed to define linear maps

$$d_0(s^{-1}c) := -s^{-1}d(c),$$
$$d_1(s^{-1}c) := \sum_i (-1)^{|a_i|} s^{-1}a_i \otimes s^{-1}b_i.$$

Then $d := d_0 + d_1$ defines a differential in ΩC, thus making $(\Omega C, d)$ an augmented DGA. In the following, we will refer to this construction as *cobar construction*. For further information, we refer to the work of Adams [1] where the construction was originally introduced.

1.4 Graded Lie Algebras

Throughout this section, let $\mathbb{K}$ be a field of characteristic zero. When we speak about vector spaces or linearity, we mean $\mathbb{K}$-vector spaces and $\mathbb{K}$-linear maps. Tensor products are to be understood as tensor products over $\mathbb{K}$.

Definition 1.4.1. Let $(L, (L_i)_{i\in\mathbb{Z}})$ be a graded vector space over $\mathbb{K}$ and let $L\otimes L \to L,\ x\otimes y \mapsto [x, y]$ be a linear map of degree zero, such that

(a) $[x, y] = -(-1)^{|x||y|}[y, x]$ (anti-symmetry)

(b) $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$ (Jacobi identity)

for all homogeneous elements $x, y, z \in L$. We call the pair $(L, [\ ,\ ])$ a *graded Lie algebra* and the product $[\ ,\ ]$ a *Lie bracket* on L.

A *morphism of graded Lie algebras* is a linear map $\varphi\colon L \to E$ of degree zero between two graded Lie algebras L, E that is Lie bracket preserving, i.e. $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in L$. Note that for any graded Lie algebra L, the anti-symmetry condition immediately implies $[x, x] = -[x, x]$ and hence $[x, x] = 0$ for an element x of even degree. Note further that a graded Lie algebra in general is not a graded algebra, since the Lie bracket will in general not be associative.

Remark 1.4.2.

(a) For two graded subspaces E, F of a graded Lie algebra L, we set

$$[E, F] := \Big\{ \sum_{i=1}^{k} \lambda_i [x_i, y_i] \mid k \in \mathbb{N}, \lambda_i \in \mathbb{K}, x_i \in E, y_i \in F \Big\}$$

for the linear span of the products from elements of E and F. This is again a graded subspace of L.

(b) A *sub Lie algebra* $E \subseteq L$ is a graded subspace that is closed under the Lie bracket, that is $[E, E] \subseteq E$. In similar fashion, a graded subspace $I \subseteq L$ is called an *ideal* if $[L, I] \subseteq I$. In both cases, the restriction of $[\ ,\ ]$ to E, I yields a graded Lie algebra structure on E, I.

(c) Suppose L is a graded Lie algebra and $I \subseteq L$ is an ideal. Then there exists a unique graded Lie algebra structure on the quotient L/I such that the canonical projection $L \to L/I$ is a morphism of graded Lie algebras.

(d) The subspace $[L, L] \subseteq L$ is always an ideal and is called the *derived sub Lie algebra*. We say L is *abelian* if $[L, L] = 0$, meaning the Lie bracket on L is trivial.

(e) The intersection $E \cap F$ of two sub Lie algebras $E, F \subseteq L$ is again a sub Lie algebra. Similarly, the intersection of two ideals is again an ideal in L.

(f) Let $S \subseteq L$ be any subset. The sub Lie algebra *generated by* S is the intersection of all sub Lie algebras of L containing S. It is by construction the smallest sub Lie algebra of L that contains S. Similarly, we may define the sub ideal generated by S.

The so-called *commutator bracket* can make any given graded algebra A (which are associative by our definition) into a graded Lie algebra. It is defined by

$$[x, y] := xy - (-1)^{|x||y|}yx,$$

which immediately implies anti-symmetry. The Jacobi identity then follows by the associativity of A, and the resulting Lie algebra $(A, [\ ,\ ])$ is abelian if and only if A is commutative.

Example 1.4.3. Let A be a graded algebra and consider the graded space $\mathrm{Hom}(A, A)$. This is a graded algebra with multiplication given by the composition of maps, so we can view it as a graded Lie algebra using the commutator bracket. Now, the graded space $\mathrm{Der}(A)$ of derivations of A is a subspace. A simple calculation shows that for any two derivations $\alpha, \beta \in \mathrm{Der}(A)$, the element $[\alpha, \beta] = \alpha \circ \beta - (-1)^{|\alpha||\beta|} \beta \circ \alpha$ is again a derivation. It follows that $\mathrm{Der}(A)$ is in fact a sub Lie algebra of $\mathrm{Hom}(A, A)$.

Remark 1.4.4.

(a) By a *derivation of degree* k in a Lie algebra L we mean a linear map $f\colon L \to L$ of degree k such that $f([x, y]) = [f(x), y] - (-1)^{k|x|}[x, f(y)]$. As above, we get a graded sub Lie algebra $\mathrm{Der}(L) \subseteq \mathrm{Hom}(L, L)$ with the commutator bracket. When only speaking of a derivation, we mean a derivation of degree -1.

(b) The product of two graded Lie algebras E and L is the direct sum $E \oplus L$. The Lie bracket is defined as

$$[(x_1, y_1), (x_2, y_2)] := ([x_1, x_2], [y_1, y_2]),$$

for $x_1, x_2 \in E$ and $y_1, y_2 \in L$. Note that this immediately implies $[(x, 0), (0, y)] = 0$ for all $x \in E$, $y \in L$.

(c) Assume we are given a commutative graded algebra A and a Lie algebra L. We can define a Lie bracket on $A \otimes L$ by setting $[a \otimes x, b \otimes y] := (-1)^{|b||x|} ab \otimes [x, y]$ for homogeneous elements $a, b \in A$ and $x, y \in L$. However, there seems to be no natural, or at least no obvious way to define a Lie bracket on the tensor product of two Lie algebras $L \otimes E$.

Example 1.4.5. Consider the tensor algebra TV of a graded vector space V. This is a graded algebra, so we can regard it as a graded Lie algebra by using the commutator bracket. Then the space $V \subseteq TV$ generates a sub Lie algebra, which we denote by $\mathbb{L}_V$. It is called the *free graded Lie algebra on* V, due to reasons that will become clear soon.

Example 1.4.6. Let us investigate $\mathbb{L}_V$ for the simplest case, that is V being a vector space with a single generator v, $V = \mathbb{K}v$. If the degree of that generator is even, then $[v, v]$ evaluates to zero in TV and thus V, together with the trivial Lie bracket, is a sub Lie algebra of TV. Therefore, we have $\mathbb{L}_V = V$. In case the degree of v is odd, we obtain the element $[v, v] \in TV$. Using the Jacobi identity, the elements $[v, [v, v]]$ and $[[v, v], v]$ are trivial. Further, by anti-symmetry $[[v, v], [v, v]] = 0$ since $[v, v]$ has even degree. It follows $V \oplus \mathbb{K}[v, v]$ is a graded subspace of TV that is closed under the Lie bracket and hence a sub Lie algebra. It is easily seen that it is the smallest sub Lie algebra containing V and therefore

$$\mathbb{L}_V = \begin{cases} \mathbb{K}v & \text{if } |v| = 2n, \\ \mathbb{K}v \oplus \mathbb{K}[v, v] & \text{if } |v| = 2n + 1. \end{cases}$$

Since $\mathbb{L}_V$ only depends on a basis for V we may write $\mathbb{L}_{(v_i)}$ for $\mathbb{L}_V$ when (v_i) is a basis for V.

Now, it follows that as graded spaces $\mathbb{L}_V = \oplus_{k \geq 1}(\mathbb{L}_V \cap T^k V)$, so it is sensible to describe the subspaces $\mathbb{L}_V \cap T^k V$ further. We say an element in $\mathbb{L}_V$ is of *bracket length* k if it is a linear combination of elements of the form $[v_1, \ldots, [v_{k-1}, v_k] \ldots]$ with $v_i \in V$. By the definition of the commutator bracket, any such element is contained in $T^k V$. The other way round, any element in $T^k V$ that can be expressed in terms of the commutator bracket would necessarily have bracket length k, and if it can not be written as a bracket at all it is not contained in $\mathbb{L}_V$ by construction. Therefore, the subspaces $\mathbb{L}_V \cap T^k V =: \mathbb{L}_V^{(k)}$ contain exactly the elements of bracket length k, so $\mathbb{L}_V$ is the sum of its subspaces of bracket length k.

Any graded algebra can be turned into a Lie algebra by equipping it with the commutator bracket. The other way around however, a graded Lie algebra is in general not a graded algebra in our terms, as the bracket is generally not associative. However, one can ask if there exists an algebra structure on L such that the induced commutator bracket is exactly the Lie bracket $[\ ,\ ]$ on L. As it turns out, this is not possible in the general case. Nevertheless, we can always find a graded algebra with said property that contains L as sub Lie algebra.

Construction 1.4.7. Let L be a graded Lie algebra. The tensor algebra TL is a graded algebra, and we consider the ideal $I \subseteq TL$ generated by the set

$$\{x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y] \mid x, y \in L\}.$$

The quotient $UL := TL/I$ is a graded algebra which is called the *universal enveloping algebra of* L. It has the following properties:

(a) The linear map $q_L \colon L \to UL$ obtained as the composition of the inclusion $L \hookrightarrow TL$ with the projection $TL \to UL$ is a morphism of graded Lie algebras when UL is equipped with the commutator bracket, as

$$q_L([x, y]) = [x, y] + I = x \otimes y - (-1)^{|x||y|} y \otimes x + I = [x + I, y + I].$$

(b) The universal enveloping algebra, together with $q_L \colon L \to UL$, hast the following universal property. Suppose A is a graded algebra and $f \colon L \to A$ is a linear map of degree zero that is a Lie algebra morphism when A is regarded as Lie algebra with the commutator bracket. Then there exists a unique morphism $F \colon UL \to A$ of graded algebras, such that $F \circ q_L = f$. Indeed, since f induces a unique morphism $T_f \colon TL \to A$ of graded algebras, we get a commutative diagram

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "f"'] & TL \arrow[r] \arrow[dl, dashed, "T_f"'] & UL \arrow[dll, dashed, "F"] \\
A & &
\end{tikzcd}$$

where the map F on the quotient is uniquely induced by T_f, since

$$\begin{aligned}
&f([x, y]) = [f(x), f(y)] = f(x)f(y) - (-1)^{|x||y|} f(y)f(x), \\
&T_f(x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]) = f(x)f(y) - (-1)^{|x||y|} f(y)f(x) - f([x, y]) = 0.
\end{aligned}$$

(c) Finally, we can combine these observations to obtain the following. Let $\varphi \colon L \to E$ be a Lie algebra morphism. Then the composition $q_E \circ \varphi \colon L \to UE$ is also a morphism of graded Lie algebras, where again UE is endowed with the commutator bracket. It follows with $f := q_E \circ \varphi$ and $A := UE$ that there exists a unique morphism of graded algebras $\Phi \colon UL \to UE$ such that $\Phi \circ q_L = q_E \circ \varphi$, i.e. the diagram

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "\varphi"'] & TL \arrow[r] & UL \arrow[d, dashed, "\Phi"] \\
E \arrow[r] & TE \arrow[r] & UE
\end{tikzcd}$$

commutes.

Remark 1.4.8. Assuming L is abelian, the ideal I is generated by elements $x \otimes y - (-1)^{|x||y|} y \otimes x$, $x, y \in L$ as the Lie bracket vanishes. Hence, we obtain the same ideal as in the definition of the free commutative graded algebra associated with a graded vector space. In other words, if L is abelian, then the universal enveloping algebra is just the free commutative graded algebra on L, $UL = \Lambda L$.

This is not true in general, as the structure on UL becomes more complex when non-trivial Lie brackets appear. However, when regarded simply as graded vector spaces, they coincide by the following.

Theorem 1.4.9. *Let L be any graded Lie algebra, then the linear map $q_L\colon L \to UL$ extends to an isomorphism of graded vector spaces $Q_L\colon \Lambda L \xrightarrow{\cong} UL$.*

Proof. This theorem is essentially due to Poincaré, Birkhoff and Witt. A detailed proof by Serre can be found in [24]. □

We understand Q_L as an extension of q_L in the usual sense that $Q_L \circ j = q_L$, where $j\colon L \to \Lambda L$ is the canonical injection. In particular, q_L is an injective Lie algebra morphism. Therefore, by identifying L with its image $q_L(L) \subseteq UL$ and equipping the latter with the commutator bracket, we may henceforth assume L to be a graded sub Lie algebra of UL.

Example 1.4.10. Remember the free graded Lie algebra $\mathbb{L}_V$ on a graded vector space V is a sub Lie algebra of TV. Using construction 1.4.7, we can extend the inclusion $\mathbb{L}_V \hookrightarrow TV$ to a morphism of graded algebras $U\mathbb{L}_V \to TV$. Vice versa, the composition of inclusions $V \hookrightarrow \mathbb{L}_V \hookrightarrow U\mathbb{L}_V$ as graded subspaces extends to a graded algebra morphism $TV \to U\mathbb{L}_V$ by the universal property of the tensor algebra. One can show that these are in fact inverse isomorphisms, so $TV \cong U\mathbb{L}_V$ as graded algebras and therefore also $(TV, [\ ,\ ]) \cong (U\mathbb{L}_V, [\ ,\ ])$ as graded Lie algebras with commutator brackets.

Remark 1.4.11. We can now proceed to describe the *universal property* of a free graded algebra $\mathbb{L}_V$ on a graded vector space V, which is the reason for its naming. Suppose we are given a second graded Lie algebra L and a linear map of degree zero $\psi\colon V \to L$. Using the universal property of TV, the composition $q_L \circ \psi\colon V \to L \to UL$ extends to a unique algebra morphism $\Psi\colon TV \to UL$. Regarding both algebras as graded Lie algebras with commutator bracket, we get that Ψ is a Lie algebra morphism. Since $L \subseteq UL$ is a sub Lie algebra, the graded subspace $\Psi^{-1}(L) \subseteq TV$ is a graded sub Lie algebra, too. As Ψ is an extension, we have $V \subseteq \Psi^{-1}(L)$ and hence $\mathbb{L}_V \subseteq \Psi^{-1}(L)$ by definition. Thus, the restriction $\Psi|_{\mathbb{L}_V}\colon \mathbb{L}_V \to L$ yields a morphism of graded Lie algebras that is unique since Ψ is.

$$\begin{tikzcd}
V \arrow[r, "\psi"] \arrow[d] & L \arrow[r, "q_L"] & UL \\
\mathbb{L}_V \arrow[ur, dashed, "\Psi|_{\mathbb{L}_V}"] \arrow[d] & & \\
TV \arrow[uurr, "\Psi"'] & &
\end{tikzcd}$$

To summarize, any linear map of degree zero from a graded vector space V to a graded Lie algebra L extends uniquely to a graded Lie algebra morphism $\mathbb{L}_V \to L$.
Note that in a more category theoretical approach, one would use this universal property for the definition of a free graded Lie algebra, then show its uniqueness up to isomorphy. What we have done here then shows that the specific graded Lie algebra $\mathbb{L}_V$ that we defined as sub Lie algebra of TV, together with the inclusion $V \to \mathbb{L}_V$, satisfies this universal property.

Having introduced the universal property of free graded Lie algebras, one might ask how we can determine whether a given graded Lie algebra L is free or not. For this reason, we will now introduce a criterion which, under mild assumptions, allows us to express L as a quotient of a free graded Lie algebra. Moreover, given L is free, it allows for the description of a graded vector space V such that $L \cong \mathbb{L}_V$. For this, assume L is any graded Lie algebra. Let $V \subseteq L$ be a graded subspace that is complementary to the derived sub Lie algebra $[L, L]$, meaning that $L = V \oplus [L, L]$ as subspaces of a graded vector space. We may use this newly established universal property of the free Lie algebra to extend the inclusion $V \hookrightarrow L$ to a graded Lie algebra morphism

$$\sigma\colon \mathbb{L}_V \to L.$$

The other way round, suppose we are provided with a free Lie algebra $\mathbb{L}_V$. Then $[\mathbb{L}_V, \mathbb{L}_V]$ is the graded subspace of elements with bracket length at least two, so we may choose V as complementary subspace. We thus obtain $\mathbb{L}_V = V \oplus [\mathbb{L}_V, \mathbb{L}_V]$.

Proposition 1.4.12. *If L is concentrated in positive degrees, it holds that the morphism σ is surjective. Moreover, L is free if and only if σ is an isomorphism.*

Proof. It holds $\sigma(V) = V$ in L and thus the image $E := \sigma(\mathbb{L}_V)$ is a sub Lie algebra of L that contains V. Since σ is the extension of the inclusion $V \hookrightarrow L$ via the universal property, E is the smallest such sub Lie algebra, i.e. the sub Lie algebra of L that is generated by V. We assumed $L = V \oplus [L, L]$, so since $V \subseteq E$ it holds $L = E + [L, L]$. Substituting L by this expression and using the bilinearity of the bracket and the fact that $[E, E] \subseteq E$ yields $E + [L, [L, L]] = L$. Through iterating this process k times, we end up with

$$L = E + [L, [L, \dots [L, L] \dots]].$$

Now remember that we assumed $L = L_{\geq 1}$, which means each element in $[L, \dots [L, L] \dots]$ has a degree of at least $k + 1$. Letting k grow shows that each element eventually lies in E, in other words $E = L$. $\square$

Remark 1.4.13. In a sense, proposition 1.4.12 above tells us how close L is to being free. For $L = L_{\geq 1}$ the surjection σ induces an isomorphism $L \cong \mathbb{L}_V / \ker(\sigma)$ of graded Lie algebras. We call the elements in $\ker(\sigma)$ the *relations* of L. Since $\mathbb{L}_V = V \oplus [\mathbb{L}_V, \mathbb{L}_V]$, we can deduce from $\sigma|_V = \mathrm{id}_V$ and $\sigma([\mathbb{L}_V, \mathbb{L}_V]) \subseteq [L, L]$ that $\ker(\sigma) \subseteq [\mathbb{L}_V, \mathbb{L}_V]$. Thus, we can think of relations as bracket expressions that, with the Lie bracket in $\mathbb{L}_V$, are non-trivial, but vanish when written with the Lie bracket of L. Note however that in addition to these, brackets in L may evaluate to zero due to the usual relations that any Lie algebra has, generated by anti-symmetry and the Jacobi identity.

Example 1.4.14. As an easy example, consider the graded vector space $L := \mathbb{K}v$ with a single generator v of odd degree ≥ 1. Equipped with the trivial bracket, this is a graded Lie algebra. It is however not free, since $\mathbb{L}_{(v)} = \mathbb{K}v \oplus \mathbb{K}[v, v]$ by example 1.4.6. The element $[v, v]$ is a relation of L.

In analogy to the world of graded algebras, the terminology of a derivation in graded Lie algebras as defined in remark 1.4.4 suggests the following.

Definition 1.4.15. A *differential graded Lie algebra* (L, d), for short DGL, is a graded Lie algebra L equipped with a differential d that is a derivation. When $L = L_{\geq 0}$ we call (L, d) a *chain Lie algebra.* If in addition $L_0 = 0$, we say the chain Lie algebra (L, d) is *connected.*

Remark 1.4.16.

(a) A *morphism of* DGLs is a morphism of graded Lie algebras that commutes with the differential.

(b) Let (L, d) be a DGL. We can extend the differential uniquely to a derivation d in the tensor algebra TL by simply taking the Leibniz rule as definition on the tensor product, as described in example 1.1.7. Then again, $d^2 = 0$ and $d(I) \subseteq I$, where I is the ideal generated by elements $x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]$. Thus, we get an induced differential on the quotient algebra $UL = TL/I$, making it a DGA (UL, d). Note that the linear map q_L becomes a DGL morphism $(L, d) \to (UL, d)$, and (UL, d) together with q_L has the obvious universal property. We call (UL, d) the universal enveloping algebra of (L, d) and denoted it by $U(L, d)$.

(c) The homology $H(L)$ of a DGL (L, d) naturally inherits the structure of a graded Lie algebra. For if $z, w \in L$ are cycles, then by the derivation property we get $d([z, w]) = 0$, so $[z, w]$ is a cycle. We then define $[[z], [w]]$ to be the class of $[z, w]$. If φ is a morphism of differential graded Lie algebras, then the induced map $H(\varphi)$ is a morphism of graded Lie algebras. If $H(\varphi)$ is an isomorphism, we call φ a *quasi-isomorphism*.

2 Rational Homotopy Type

After this introduction to the world of graded algebras, we will move on to address the process of *rationalization*, or more generally, *localization*. For abelian groups G, this means switching to the $\mathbb{K}_P$-module $G \otimes_{\mathbb{Z}} \mathbb{K}_P$, where $\mathbb{K}_P$ is a subring of the rationals. This can also be applied to group homomorphisms, where we just take the identity on the second factor. As we shall see, this forces some, or in the case $\mathbb{K}_P = \mathbb{Q}$ even all torsion terms of G to vanish. To readily apply this to the homotopy groups of a topological space X, we require $\pi_1(X)$ to be abelian. In fact, a non-trivial fundamental group generally tends to make things more complicated in rational homotopy theory. By restricting ourselves to simply connected spaces we circumvent this obstacle, but note that many of the results presented here hold under more general assumptions. A detailed explanation of the theory for the non-simply connected case can be found in [8]. A brief but nevertheless complete overview is presented in [10].

The main focus of this chapter will be the description of an underlying geometric interpretation of the process of localizing the homotopy groups, leading to the term of *localization* of a space which is first introduced by Sullivan in [27]. At this, we first consider simply connected spaces whose homotopy groups already are $\mathbb{K}_P$-modules. Given such space Y, there exists a natural isomorphism $\pi_*(Y) \cong \pi_*(Y) \otimes_{\mathbb{Z}} \mathbb{K}_P$, meaning no information is lost when switching to the tensor product. These spaces can generally be quite weird and complicated, but we will see that any space X can be associated with such a space Y such that $\pi_*(X) \otimes_{\mathbb{Z}} \mathbb{K}_P \cong \pi_*(Y)$. The space Y will usually be denoted by X_P, or in the case that $\mathbb{K}_P = \mathbb{Q}$ by $X_{\mathbb{Q}}$, and is used to define the *rational homotopy type* of X. Moreover, from cellular localization we will see that we can even build X_P from X through attaching cells, which gives the originally algebraic motivated definition of localization a genuine topological meaning. Thus topologically, rationalization involves the modification of a space by cell attachments such that the resulting space realizes the rational homotopy groups via its homotopy groups. Furthermore, a given continuous map $f\colon X \to Y$ can be extended to a continuous map $f_P\colon X_P \to Y_P$ between the localizations.

As working in the context of localization instead of rationalization does not necessarily make things more complicated, we will proceed by introducing some of Sullivan's ideas in their more general context. However, note that for the most part of this monograph, we will most often stick to the rational case. At the end of the section, some detail on the approach due to Quillen will also be provided, which involves the definition of rational homotopy type over a chain of so-called rational homotopy equivalences rather than using rationalizations. Along the way, we will give a very brief summary of the category theoretical context to the subject.

2.1 P-Local Spaces

To specify the above, assume P to be a set of prime numbers in $\mathbb{N}$. Further, consider the set

$$\mathcal{R}(P) := \{n \in \mathbb{Z} \mid \gcd(n, p) = 1 \text{ for all } p \in P\}$$

of integers that are relatively prime to the elements of P. In this section, the ground ring will be $\mathbb{K}_P := \{\frac{m}{n} \mid m \in \mathbb{Z},\ n \in \mathcal{R}(P)\} \subseteq \mathbb{Q}$, associated with a set of primes P. For example, if $P = \varnothing$, then $\mathcal{R}(P) = \mathbb{Z}$ and thus $\mathbb{K}_P = \mathbb{Q}$.

Definition 2.1.1. Let $(G, +)$ be any abelian group and $\mathcal{R}(P)$ as above. If the map

$$\varphi_n\colon G \to G, \quad g \mapsto ng := g + \cdots + g$$

is an isomorphism for all $n \in \mathcal{R}(P)$, then G is called *P-local*.

Note that if G contains a torsion element with respect to some $n \in \mathcal{R}(P)$, it can not be P-local. Vice versa, given G is not P-local we find some $0 \neq g \in G$ and $n \in \mathcal{R}(P)$ such that $ng = 0$. Before we continue, there are some important properties of P-local groups that we will use frequently in the following.

Remark 2.1.2.

(a) It is not difficult to see that G is P-local if and only if it is a $\mathbb{K}_P$-module, or in case of $P = \varnothing$, a rational vector space. For if G is a $\mathbb{K}_P$-module, then in particular $\frac{1}{n}g$ is in G for each $g \in G$ and $n \in \mathcal{R}(P)$, so each φ_n has an inverse φ_n^{-1}. Vice versa, assuming that G is P-local, one can check that $kg := m\varphi_n^{-1}(g)$, where $k = \frac{m}{n} \in \mathbb{K}_P$, defines a suitable scalar multiplication.

(b) Note that whereas G itself might not be P-local, the group $G \otimes_{\mathbb{Z}} \mathbb{K}_P$ always is. Each element in $G \otimes_{\mathbb{Z}} \mathbb{K}_P$ is a sum of terms of the form $g \otimes k$, where $k = \frac{m}{n} \in \mathbb{K}_P$. By finding a common denominator for the fractions, we can always reduce this representation to one with a single term. One can then see that multiplication with $\frac{1}{n}$ is, like in the previous discussion, a suitable inverse of φ_n, as $\mathcal{R}(P)$ is closed under multiplication. Further note that any element $g \otimes k \in G \otimes_{\mathbb{Z}} \mathbb{K}_P$ is equal to $mg \otimes \frac{1}{n}$ for suitable $m \in \mathbb{Z}$, $n \in \mathcal{R}(P)$. Thus, it suffices to consider elements of the form $g \otimes \frac{1}{n}$. That being said, there is a natural homomorphism

$$G \to G \otimes_{\mathbb{Z}} \mathbb{K}_P, \quad g \mapsto g \otimes 1$$

that is called *P-localization*. Whenever G itself happens to be P-local, we see that this homomorphism has an inverse given by $g \otimes \frac{1}{n} \mapsto \varphi_n^{-1}(g)$. It follows that G is P-local if and only if the P-localization is an isomorphism.

(c) A morphism $f\colon A \to B$ between abelian groups induces a homomorphism $f \otimes_{\mathbb{Z}} \mathbb{K}_P \colon A \otimes_{\mathbb{Z}} \mathbb{K}_P \to B \otimes_{\mathbb{Z}} \mathbb{K}_P$ that is supposed to be the identity in the second factor. This makes localization a functor, and as such it is *exact*, i.e. it has the property of preserving exact sequences. Since this is a property that only depends on the underlying module, one often refers to this by saying the module is *flat*. Assume we are given a short exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of abelian groups. Consider the tensor product with $\mathbb{K}_P$ and let $b \otimes \frac{1}{n} \in \ker(g \otimes_{\mathbb{Z}} \mathbb{K}_P)$, so $g(b) \otimes \frac{1}{n}$ is trivial in $C \otimes_{\mathbb{Z}} \mathbb{K}_P$. This can only be true if $g(b)$ has some finite order $m \in \mathcal{R}(P)$, as only the left factor can vanish and a fraction in $\mathbb{K}_P$ can only be expanded by such m. We deduce that $mg(b) = g(mb)$ is trivial and thus $mb \in \ker(g) = \operatorname{im}(f)$. Hence, there exists $a \in A$ with $f(a) = mb$, meaning $(f \otimes_{\mathbb{Z}} \mathbb{K}_P)(a \otimes \frac{1}{mn}) = mb \otimes \frac{1}{mn} = b \otimes \frac{1}{n}$. On the other hand, $g \otimes_{\mathbb{Z}} \mathbb{K}_P$ is trivial on any element of the form $f(a) \otimes \frac{1}{n}$, since $g \circ f = 0$. This means

$$A \otimes_{\mathbb{Z}} \mathbb{K}_P \xrightarrow{f \otimes_{\mathbb{Z}} \mathbb{K}_P} B \otimes_{\mathbb{Z}} \mathbb{K}_P \xrightarrow{g \otimes_{\mathbb{Z}} \mathbb{K}_P} C \otimes_{\mathbb{Z}} \mathbb{K}_P$$

is a short exact sequence, so $\mathbb{K}_P$ is flat.

(d) Given an exact sequence

$$A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E$$

of abelian groups, where A, B, D and E are P-local, the same is true for C. This follows if we map this sequence to itself using the map φ_n above for any $n \in \mathcal{R}(P)$, then applying the five lemma to the resulting commutative diagram.

Now that the terms of P-locality and P-localization have been elaborated for abelian groups, we may use the homotopy group functor $\pi_k(-)$ to carry these notions over to the world of simply connected topological spaces.

Definition 2.1.3. Let X be a simply connected topological space. It is called a *P-local space*, whenever its homotopy groups $\pi_k(X)$ are P-local for all $k \geq 2$. If $\mathbb{K}_P = \mathbb{Q}$, then X is called a *rational space*.

We may rephrase this by saying that a simply connected space X is P-local if and only if its homotopy groups are $\mathbb{K}_P$ modules. Therefore, a rational space is a simply connected space whose homotopy groups are rational vector spaces. The existence of P-local spaces is easily verified, as any Eilenberg-MacLane space $K(G,n)$ with G a $\mathbb{K}_P$-module is P-local. However, these spaces in general have a rather complicated geometry and do not appear among the classical examples for topological spaces. Actually, most mathematicians have never come across a rational space in their entire lives, and we will present a first example shortly.
At this point, the restriction to simple connected spaces seems to be fairly excessive. While we could extend definition 2.1.3 to spaces with an abelian fundamental group without second thought, it will become clearer later on why it is beneficial to stick to the simply connected case. With that being said, as suggested earlier, most of the definitions and results presented here can, with some additional effort, be generalized to the broader class of nilpotent spaces. In fact, the following example of a rational space is not simply connected, so it is not even a rational space by our definition. Nevertheless, as its construction is one of the most vivid among rational spaces, we want to use it as our first example.

Example 2.1.4. Let $k \geq 1$. We begin by considering the maps

$$f_k\colon \begin{cases} \mathbb{S}^1 & \to \mathbb{S}^1 \\ e^{2\pi i t} & \mapsto e^{2\pi i k t} \end{cases}$$

that can be visualized by wrapping the sphere $\mathbb{S}^1$ around itself k times. We start with $X_1 := \mathbb{S}^1 \times I$. Denote by $i, j\colon \mathbb{S}^1 \to \mathbb{S}^1 \times I$ the inclusions of the bottom and top sphere, so $i(x) := (x,0)$ and $j(x) := (x,1)$ for all $x \in \mathbb{S}^1$. Set $j_1 = j$ and inductively let $X_k := (\mathbb{S}^1 \times I) \cup_{j_{k-1}\circ f_k} X_{k-1}$, where the inclusion j_k is given by j followed by the canonical projection $\mathbb{S}^1 \times I \to X_k$. The corresponding pushout diagram is

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r, "j_{k-1}\circ f_k"] \arrow[d, "i"] & X_{k-1} \arrow[d] \\
\mathbb{S}^1 \times I \arrow[r] & X_k,
\end{tikzcd}$$

and comes with an inclusion $i_k\colon X_{k-1} \hookrightarrow X_k$ that is just the canonical projection. Lastly, we finish the construction by setting $X := \cup_{k\geq 1} X_k$.
Our claim is now that X is a rational space. For this, first note that by construction, any space X_k contains $\mathbb{S}^1$ as a strong deformation retraction. We can, for instance, collapse X_k to the top sphere $\mathbb{S}^1 \times \{1\}$ that comes with the lastly attached $\mathbb{S}^1 \times I$. Under this retraction, the inclusion j_k becomes the identity, so we can take $[j_k]$ as generator for $\pi_1(X_k) = \pi_1(\mathbb{S}^1) = \mathbb{Z}$.
Now, the image of any continuous map $\mathbb{S}^n \to X$ hast to lie in some X_k since $\mathbb{S}^n$ is compact. As $\pi_n(X_k) = \pi_n(S^1)$ is trivial for $n \geq 2$, we conclude that the higher homotopy groups of X vanish. For the fundamental group, note that via inclusion $i_k\colon X_{k-1} \hookrightarrow X_k$, the path j_{k-1} corresponds to a k-fold circulating loop in X_k, so i_k induces multiplication by k on the fundamental groups, $\pi_1(i_k)([j_{k-1}]) = k[j_k]$. We can thus define

$$\Phi\colon \pi_1(X) \to \mathbb{Q}, \quad \Phi([j_k]) := \frac{1}{k!}.$$

Again, the image of any continuous map $\mathbb{S}^1 \to X$ has to lie in some X_k, so it suffices to specify Φ on the generators of each $\pi_1(X_k)$. This naturally extends to a group homomorphism: for $l \leq k$ the class $[j_l]$ in $\pi_1(X_l)$ gets mapped to $\frac{k!}{l!}[j_k]$ in $\pi_1(X_k)$ under the inclusion $X_l \hookrightarrow X_k$, which clarifies the well-definedness and the preservation of the group structure, as

$$\Phi([j_l] + [j_k]) = \Phi(\frac{k!}{l!}[j_k] + [j_k]) = (\frac{k!}{l!} + 1)\Phi([j_k]) = \frac{1}{l!} + \frac{1}{k!}.$$

Finally, an inverse $\Psi\colon \mathbb{Q} \to \pi_1(X)$ is given by $\frac{m}{n} \mapsto (n-1)!m[j_n]$.

Remark 2.1.5.

(a) The space X constructed above is not new to us, or more precisely its homotopy type is not, as it is simply an Eilenberg-MacLane space $K(\mathbb{Q}, 1)$. However, the general construction of Eilenberg-MacLane spaces involves the attachment of cells of increasing dimension to make sure higher homotopy groups vanish, so it differs from the method used above.

(b) The sphere $\mathbb{S}^1$ is a subspace of X via the composition of inclusions $i\colon \mathbb{S}^1 \hookrightarrow X_1 \subseteq X$. It is therefore worth to note that we constructed the CW complex X from $\mathbb{S}^1$ by attaching cells such that $\pi_1(X) = \mathbb{Q}$, without changing the higher homotopy groups. This idea of rationalizing a given space will be developed further shortly.

(c) Note that instead of attaching $\mathbb{S}^1 \times I$ via f_k for any positive number k, we can restrict the construction to elements of $\mathcal{R}(P)$ for some set of prime numbers P. The resulting space will be P-local, in fact it will be an Eilenberg-MacLane space $K(\mathbb{K}_P, 1)$. We denote it by $\mathbb{S}^1_P$ and call it the *P-local 1-sphere*. The space X is denoted by $\mathbb{S}^1_{\mathbb{Q}}$ and called the *rational 1-sphere*.

The following construction is the higher dimensional equivalent of example 2.1.4, stated in a more general context so that the resulting space is P-local and not necessarily rational. However, as we shall see most of what follows works analogously, but this time the considered space will be simply connected and thus actually fit into our definition of a P-local space.

Construction 2.1.6. Let P be a set of prime numbers and $(k_j)_{j\in\mathbb{N}}$ be an enumeration of the elements of $\mathcal{R}(P)$. For any $k \geq 0$, take a n-sphere $\mathbb{S}^n_k$ and consider the bouquet $\bigvee_k \mathbb{S}^n_k$. As usual, $\pi_n(\mathbb{S}^n_k) = \mathbb{Z}$, with the identity class $[\mathrm{id}_{\mathbb{S}^n_k}]$ as a generator. Remember that

$$\pi_n\Big(\bigvee_k \mathbb{S}^n_k\Big) \cong \oplus_k \mathbb{Z}\alpha_k,$$

where the isomorphism is induced by the inclusions $i_l\colon \mathbb{S}^n_l \hookrightarrow \bigvee_k \mathbb{S}^n_k$ and $\alpha_l := [i_l] \in \pi_n(\bigvee_k \mathbb{S}^n_k)$. We proceed by gluing a $(n+1)$-disc $\mathbb{B}^{n+1}_j$ to the bouquet for any $j \geq 1$, where the attaching map $f_j\colon \partial\mathbb{B}^{n+1}_j = \mathbb{S}^n \to \mathbb{S}^n_{j-1} \vee \mathbb{S}^n_j$ is any representative of the class $\alpha_{j-1} - k_j\alpha_j \in \pi_n(\mathbb{S}^n_{j-1} \vee \mathbb{S}^n_j) = \mathbb{Z} \oplus \mathbb{Z}$. Finally, this defines a map $f\colon \dot{\bigcup}_j \mathbb{S}^n_j \to \bigvee_k \mathbb{S}^n_k$ and we set

$$\mathbb{S}^n_P := \Big(\bigvee_{k=0}^{\infty} \mathbb{S}^n_k\Big) \cup_f \Big(\dot{\bigcup}_{j=1}^{\infty} \mathbb{B}^{n+1}_j\Big).$$

For each $r \geq 1$, denote by f_r the restriction of f to $\dot{\bigcup}_{j=1}^{r} \mathbb{S}^n_j$ and consider the subcomplex

$$X_r := \Big(\bigvee_{k=0}^{r} \mathbb{S}^n_k\Big) \cup_{f_r} \Big(\dot{\bigcup}_{j=1}^{r} \mathbb{B}^{n+1}_j\Big),$$

which deformation retracts to the sphere $\mathbb{S}^n_r$, as any two spheres $\mathbb{S}^n_{j-1}$ and $\mathbb{S}^n_j$ are now joint by a $(n+1)$-cell $\mathbb{B}^{n+1}_j$. We deduce that $\pi_i(X_r) = 0$ for $i \neq n$ with generator $[\mathrm{id}_{\mathbb{S}^n_r}]$ in degree n, and thus the same is true for $\pi_i(\mathbb{S}^n_P)$ by the same compactness argument used in 2.1.4, as $\mathbb{S}^n_P$ is exactly the union of the subcomplexes X_r. Note that the class of $[f_r]$, as with any attaching map, vanishes in $\pi_n(X_r)$ and thus $\alpha_{r-1} - k_r\alpha_r = 0$. It follows that the induced map of the inclusion $X_{r-1} \hookrightarrow X_r$ maps the class α_{r-1} to $k_r\alpha_r$, thus $\pi_n(\mathbb{S}^n_P) \cong \mathbb{K}_P$ by the same isomorphism used in example 2.1.4. Note that this isomorphism distinguishes the class of the inclusion of the initial sphere $\mathbb{S}^n_0$ as a generator. Finally, for $n \geq 2$ it follows that $\mathbb{S}^n_P$ is simply connected, and thus a P-local space.

There are further reasons for restricting to the simple connected case, one such is given by the following result, which is due to Serre.

Theorem 2.1.7. *Let X be a simply connected topological space, $x_0 \in X$ and P any set of prime numbers. Then it holds that $\pi_*(X)$ is P-local if and only if $H_*(X, x_0; \mathbb{Z})$ is P-local.*

Proof. From remark 2.1.2 it follows P-local groups define a Serre class. The claim is then originally due to [22], but newer variants can be found in [4] and [7]. In the latter, the equivalence is extended to include $H_*(\Omega X, x_0; \mathbb{Z})$, where ΩX is the loop space of X. □

Note that here, both $\pi_*(X)$ and $H_*(X, x_0; \mathbb{Z})$ as direct sums of abelian groups are themselves abelian groups, so definition 2.1.1 applies.

Remark 2.1.8.

(a) Obviously, by theorem 2.1.7 any Moore space $M(\mathbb{K}_P, n)$ with $n \geq 2$ would be an example for a P-local space. Further, the construction of $\mathbb{S}^n_P$ as shown above can be stated in terms of homology, making use of the fundamental classes $[\mathbb{S}^n_k]$ instead of the classes $\alpha_k = [i_k]$ of the inclusions. Using the Hurewicz isomorphism, we can easily switch between homotopy and homology. Translated into the world of homology groups, this construction is sometimes called *infinite telescope* or *mapping telescope* and is actually a common method to construct Moore spaces.

(b) By theorem 2.1.7, in the definition of a rational, or more precisely P-local space, we can exchange homotopy groups for homology groups. Moreover, since $\mathbb{K}_P$ is a flat module, it is an immediate consequence that $\mathrm{Tor}(G; \mathbb{K}_P)$ vanishes for any abelian group G. Thus, when considering the short exact sequence derived from the universal coefficient theorem

$$0 \longrightarrow H_k(X) \otimes_{\mathbb{Z}} \mathbb{K}_P \longrightarrow H_k(X; \mathbb{K}_P) \longrightarrow \mathrm{Tor}(H_k(X); \mathbb{K}_P) \longrightarrow 0$$

for a topological space X, we see that $H_k(X) \otimes_{\mathbb{Z}} \mathbb{K}_P \cong H_k(X; \mathbb{K}_P)$ holds for all $k \geq 1$. Hence, we arrive back at a familiar phrasing we already had for homotopy groups, namely a simple connected space is P-local if and only if its homology groups in positive degree are $\mathbb{K}_P$-modules.

2.2 P-Localization

We have yet to explain how the notion of P-localization is transferred into the topological world. For this, consider a continuous map $f\colon X \to Y$ between simply connected spaces. It induces a morphism

$$\pi_k(f) \otimes_{\mathbb{Z}} \mathbb{K}_P \colon \pi_k(X) \otimes_{\mathbb{Z}} \mathbb{K}_P \to \pi_k(Y) \otimes_{\mathbb{Z}} \mathbb{K}_P, \quad [\varphi] \otimes \lambda \mapsto [f \circ \varphi] \otimes \lambda.$$

Now, provided that Y is P-local, we have $\pi_k(Y) \cong \pi_k(Y) \otimes_{\mathbb{Z}} \mathbb{K}_P$ via $[\varphi] \mapsto [\varphi] \otimes 1$, whose inverse is given by multiplication $[\varphi] \otimes \lambda \mapsto \lambda[\varphi]$. Therefore in this case, $\pi_k(f) \otimes_{\mathbb{Z}} \mathbb{K}_P$ extends uniquely to a map $\pi_k(X) \otimes_{\mathbb{Z}} \mathbb{K}_P \to \pi_k(Y)$ which we again denote with $\pi_k(f) \otimes_{\mathbb{Z}} \mathbb{K}_P$. As usual, we write $\pi_*(f) \otimes_{\mathbb{Z}} \mathbb{K}_P$ for the induced map on the direct sum of the homotopy groups, so all in all it holds

$$\pi_*(f) \otimes_{\mathbb{Z}} \mathbb{K}_P(\alpha \otimes \lambda) = \lambda \pi_*(f)(\alpha) \in \pi_*(Y)$$

for homogeneous $\alpha \otimes \lambda \in \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{K}_P$.

Definition 2.2.1. Let $f\colon X \to X_P$ be a continuous map between topological spaces and assume X_P is a P-local space. If f induces an isomorphism

$$\pi_*(f) \otimes_{\mathbb{Z}} \mathbb{K}_P \colon \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{K}_P \xrightarrow{\cong} \pi_*(X_P),$$

we call it a *P-localization*. When $\mathbb{K}_P = \mathbb{Q}$, then f is called a *rationalization* and we write $f\colon X \to X_{\mathbb{Q}}$ instead.

Remark 2.2.2.

(a) To put the definition above into the context of the P-localization of an abelian group, a P-localization $f\colon X \to X_P$ is a topological map realizing $\pi_*(X) \to \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{K}_P$, $\alpha \mapsto \alpha \otimes 1$ in the sense that the diagram

$$\begin{tikzcd} \pi_*(X) \arrow[r, "\pi_*(f)"] \arrow[d, "\alpha\mapsto\alpha\otimes 1"'] & \pi_*(X_P) \\ \pi_*(X)\otimes_{\mathbb{Z}}\mathbb{K}_P \arrow[r] \arrow[ur, dashed, "\pi_*(f)\otimes_{\mathbb{Z}}\mathbb{K}_P", "\cong"'] & \pi_*(X_P)\otimes_{\mathbb{Z}}\mathbb{K}_P \arrow[u, "\alpha\otimes\lambda\mapsto\lambda\alpha"'] \end{tikzcd}$$

commutes.

(b) Note that, unlike in the classical case, the topological meaning of an element in $\pi_*(X) \otimes_{\mathbb{Z}} \mathbb{K}_P$ is not directly clear. We may, and frequently will use P-localization

$$\pi_*(X) \to \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{K}_P, \quad \alpha \mapsto \alpha \otimes 1$$

to regard α as an element in $\pi_*(X) \otimes_{\mathbb{Z}} \mathbb{K}_P$. That is, by abuse of notation, when writing $\alpha \in \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{K}_P$ we actually refer to the element $\alpha \otimes 1$. This way, a representative $f\colon \mathbb{S}^n \to X$ of α in the usual way can be viewed as a representative of $[f] = \alpha \in \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{K}_P$. Note however, that if $g\colon \mathbb{S}^n \to X$ is continuous such that $[g] =: \beta \in \pi_n(X)$ is torsion with respect to some $k \in \mathcal{R}(P)$, we get $(\alpha + \beta) \otimes 1 = \alpha \otimes 1$, and thus $f + g$ is a suitable representative of $\alpha \in \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{K}_P$, although in general $\alpha \neq \alpha + \beta = [f + g] \in \pi_n(X)$.
In case that X is P-local, we can represent all elements in $\pi_*(X)\otimes_{\mathbb{Z}}\mathbb{K}_P$ this way. For the general case, when given a P-localization $\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{K}_P \cong \pi_n(X_P)$, one can pick a representative $\mathbb{S}^n \to X_P$ provided by this isomorphism to give an arbitrary element in $\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{K}_P$ a topological interpretation.

(c) For a simply connected space X, the Hurewicz homomorphism $h\colon \pi_k(X) \to H_k(X)$ induces a P-local variant

$$h \otimes \mathbb{K}_P \colon \pi_k(X) \otimes_{\mathbb{Z}} \mathbb{K}_P \to H_k(X) \otimes_{\mathbb{Z}} \mathbb{K}_P \cong H_k(X; \mathbb{K}_P)$$

that as usual is the identity in the second factor.

We have already seen a first example of a rationalization.

Example 2.2.3. The inclusion $i\colon \mathbb{S}^1 \hookrightarrow X_1 \subseteq \mathbb{S}^1_{\mathbb{Q}}$ from example 2.1.4 is a rationalization. We have already established $[i] \in \pi_1(\mathbb{S}^1_{\mathbb{Q}})$ as a generator. Now $\pi_1(\mathbb{S}^1) \otimes_{\mathbb{Z}} \mathbb{K}_P$ is generated by $[\mathrm{id}] \otimes 1$ and clearly, $\pi_1(i) \otimes_{\mathbb{Z}} \mathbb{K}_P([\mathrm{id}] \otimes 1) = [i]$.

The following result due to Whitehead and Serre once again illustrates the convenience of restricting to the simple connected case. It lets us switch into the world of homology. Unlike homotopy, where arguments tend to be abstract from time to time, homology frequently provides a more tangible approach to things.

Theorem 2.2.4. *Let $f\colon X \to Y$ be a map between simply connected spaces and P any set of prime numbers. Then it holds that $\pi_*(f) \otimes_{\mathbb{Z}} \mathbb{K}_P \colon \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{K}_P \to \pi_k(Y) \otimes_{\mathbb{Z}} \mathbb{K}_P$ is an isomorphism if and only if $H_*(f; \mathbb{K}_P)\colon H_*(X; \mathbb{K}_P) \to H_*(Y; \mathbb{K}_P)$ is an isomorphism.*

Proof. This result is a consequence of the Whitehead theorem modulo a Serre class. Once again, a proof can be found in [4] or in [7], where in the latter the equivalence is extended to include $H_*(\Omega f; \mathbb{K}_P)$, with Ωf being the induced map between the loop spaces of X and Y. □

Corollary 2.2.5. *The inclusion $i\colon \mathbb{S}^n_0 \hookrightarrow \mathbb{S}^n_P$, $n \geq 1$, of the sphere into the infinite telescope is a P-localization.*

Proof. By the construction 2.1.6 of the P-local sphere $[i]$ is a generator of $\pi_n(\mathbb{S}^n_P) = \mathbb{K}_P$. The Hurewicz map thus provides an isomorphism

$$\pi_n(\mathbb{S}^n_P) \cong H_n(\mathbb{S}^n_P;\mathbb{Z}), \quad [i] \mapsto H_n(i)([\mathbb{S}^n_0]),$$

which implies that $H_n(i)$ maps a generator $[\mathbb{S}^n_0] \in H_n(\mathbb{S}^n_0) = \mathbb{Z}$ to a generator (though not as a $\mathbb{Z}$-module) of $H_n(\mathbb{S}^n_P;\mathbb{Z}) = \mathbb{K}_P$. It follows that i induces an isomorphism $H_n(i)\otimes_\mathbb{Z}\mathbb{K}_P \colon H_n(\mathbb{S}^n_0)\otimes_\mathbb{Z}\mathbb{K}_P \to H_n(\mathbb{S}^n_P)\otimes_\mathbb{Z}\mathbb{K}_P$. Due to the universal coefficient theorem and remark 2.1.2 we get the following diagram

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{S}^n_0)\otimes_\mathbb{Z}\mathbb{K}_P \arrow[r, "\cong"] \arrow[d, "H_n(i)\otimes_\mathbb{Z}\mathbb{K}_P"] & H_n(\mathbb{S}^n_0;\mathbb{K}_P) \arrow[r] \arrow[d, "H_n(i;\mathbb{K}_P)"] & 0 \\
0 \arrow[r] & H_n(\mathbb{S}^n_P)\otimes_\mathbb{Z}\mathbb{K}_P \arrow[r, "\cong"] & H_n(\mathbb{S}^n_P;\mathbb{K}_P) \arrow[r] & 0,
\end{tikzcd}$$

where the vertical homomorphisms are induced by the inclusion. The diagram commutes, as the horizontal isomorphisms from the universal coefficient theorem map $[z]\otimes\lambda$ to $[z\otimes\lambda]$ and for the induced map $H_n(i;\mathbb{K}_P)\colon H_n(\mathbb{S}^n;\mathbb{K}_P) \to H_n(\mathbb{S}^n_P;\mathbb{K}_P)$ it holds by definition $H_n(i;\mathbb{K}_P)([z\otimes\lambda]) = [i(z)\otimes\lambda]$. Now, since $H_n(i)\otimes_\mathbb{Z}\mathbb{K}_P$ is an isomorphism, the same must be true for $H_n(i;\mathbb{K}_P)$. Lastly, from the knowledge of the groups $H_k(\mathbb{S}^n_0)$ and $H_k(\mathbb{S}^n_P)$ we get

$$H_k(\mathbb{S}^n_0;\mathbb{K}_P) = H_k(\mathbb{S}^n_P;\mathbb{K}_P) = \begin{cases} \mathbb{K}_P, & k=0,\ k=n \\ 0, & \text{else,} \end{cases}$$

which implies that $H_*(i;\mathbb{K}_P)$ is an isomorphism. □

Remark 2.2.6. By theorem 2.2.4, a continuous map $f\colon X \to X_P$ between simply connected spaces, where X_P is P-local, is a P-localization if and only if $H_*(f;\mathbb{K}_P)$ is an isomorphism. This rephrases the definition of P-localization in terms of homology.

The idea of topological localization through assigning P-localizations $X \to X_P$ to a space X is attributed to Sullivan. While the examples above show that P-local spaces are geometrically more complicated, from the viewpoint of algebraic topology, it will provide a significant simplification, as we shall see. First, we have to show the existence of P-localizations.

Theorem 2.2.7. *Let X be a simply connected space and let P be any set of prime numbers. Then there exists a P-localization $X \to X_P$.*

There is an explicit construction of X_P from X through attaching cells, such that the inclusion $X \to X_P$ is a P-localization, which is essentially a generalization of the construction of $\mathbb{S}^n_\mathbb{Q}$. We will come back to this later when we introduce cellular localization. As the author has spent some time with the study of Postnikov towers and obstruction theory, we can not resist providing a slightly more elegant way to think of localizations of spaces. For this, remember that any connected CW complex for which the action of $\pi_1(X)$ on $\pi_n(X)$ is trivial for all $n \geq 2$, admits a Postnikov tower of principal fibrations. This means each space X_n in the Postnikov tower is weakly homotopy equivalent to the homotopy fibre of a map

$$k_{n-1}\colon X_{n-1} \to K(\pi_n(X), n+1)$$

which only depends on the homotopy class $[k_{n-1}] \in [X_{n-1}, K(\pi_n(X), n+1)]$. Due to Hurewicz, $[X_{n-1}, K(\pi_n(X), n+1)] \cong H^{n+1}(X_{n-1}, \pi_n(X))$ and the corresponding cohomology classes are called *k-invariants* of X. Furthermore, remember that there exists a weak homotopy equivalence between X and the inverse limit given by the Postnikov tower. The idea is now to localize the stages X_n in a Postnikov tower of X and extend this weak homotopy equivalence to the inverse limit of the localized stages. In the proof, we will make use of the following lemma, which will later be refined for the case that $\mathbb{K}_P = \mathbb{Q}$.

Lemma 2.2.8. *Let $f\colon X \to Y$ be a map between simply connected spaces and P any set of prime numbers. Then for all $k \geq 0$ it holds if $H_k(f;\mathbb{K}_P)\colon H_k(X;\mathbb{K}_P) \to H_k(Y;\mathbb{K}_P)$ is an isomorphism, then $H^k(f;A)\colon H^k(Y;A) \to H^k(X;A)$ is an isomorphism for all $\mathbb{K}_P$-modules A.*

Proof. The universal coefficient theorem for cohomology as stated in remark 1.1.16 provides an isomorphism

$$\mathcal{H}\colon H^k(Y;A) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{K}_P}(H_k(Y;\mathbb{K}_P),A).$$

A cocycle $[\varphi] \in H^k(Y;A)$ is represented by a homomorphism $\varphi \in \mathrm{Hom}_{\mathbb{K}_P}(C_k(Y;\mathbb{K}_P),A)$, and $\mathcal{H}([\varphi])$ is defined via $\mathcal{H}([\varphi])([z]) := \varphi(z)$ for any cocycle $[z] \in H_k(Y;\mathbb{K}_P)$. Obviously, the same holds for X, so this leaves us with a diagram

$$\begin{tikzcd}
H^k(Y;A) \arrow[rr, "\mathcal{H}", "\cong"'] \arrow[d, "H^k(f;A)"] & & \mathrm{Hom}_{\mathbb{K}_P}(H_k(Y;\mathbb{K}_P),A) \arrow[d, "\mathrm{Hom}_{\mathbb{K}_P}(H_k(f;\mathbb{K}_P),A)"] \\
H^k(X;A) \arrow[rr, "\mathcal{H}", "\cong"'] & & \mathrm{Hom}_{\mathbb{K}_P}(H_k(X;\mathbb{K}_P),A).
\end{tikzcd}$$

Suppose $[\varphi] \in H^k(Y;A)$ and let $[z] \in H_k(X;\mathbb{K}_P)$ be arbitrary, then

$$\mathrm{Hom}_{\mathbb{K}_P}(H_k(f;\mathbb{K}_P),A) \circ \mathcal{H}([\varphi])([z]) = \mathcal{H}([\varphi]) \circ H_k(f;\mathbb{K}_P)([z]) = \mathcal{H}([\varphi])([C_k(f)(z)]) = \varphi(C_k(f)(z))$$
$$\mathcal{H} \circ H^k(f;A)([\varphi])([z]) = \mathcal{H}([\varphi \circ c_k(f)])([z]) = \varphi \circ C_k(f)(z) = \varphi(C_k(f)(z)),$$

where by $C_k(f)$ we mean the induced morphism between chain complexes $C_k(X;\mathbb{K}_P) \to C_k(Y;\mathbb{K}_P)$. In other words, the diagram commutes. Assuming now that $H_k(f;\mathbb{K}_P)$ is an isomorphism, by the properties of $\mathrm{Hom}_{\mathbb{K}_P}(-,A)$ as a contravariant functor it follows that $\mathrm{Hom}_{\mathbb{K}_P}(H_k(f;\mathbb{K}_P),A)$ is an isomorphism for any $\mathbb{K}_P$ module A, implying $H^k(f;A)$ has to be an isomorphism as well. ☐

Proof of theorem 2.2.7. By assumption, X omits a Postnikov tower of principal fibrations. We use the abbreviation $\pi_k := \pi_k(X)$ for the homotopy groups of X. Let X_n be the stages in a Postnikov tower of X, so X_2 is a $K(\pi_2,2)$ and from there on X_n is inductively determined by X_{n-1} and the k-invariants. We will construct X_P through sequentially localizing the stages X_n in the Postnikov tower, providing P-localizations $X_n \to (X_n)_P$.
For $k \geq 2$, there exist natural maps $\pi_k \to \pi_k \otimes \mathbb{K}_P$ as seen earlier, with $\pi_k \otimes \mathbb{K}_P$ always being P-local. It is a well-known result used for example in the proof of the uniqueness of Eilenberg-MacLane spaces that, given an abelian group A, $n \geq 2$ and a homomorphism $\Phi\colon A \to A \otimes \mathbb{K}_P$, there exists a continuous map $f\colon K(A,n) \to K(A \otimes \mathbb{K}_P, n)$ realizing Φ, so $\pi_n(f) = \Phi$. For more details on this, see for example [28]. Now obviously any Eilenberg-MacLane space of the form $K(\pi_k \otimes \mathbb{K}_P, n)$ is P-local, thus there exist P-localizations $f_2\colon K(\pi_2,2) \to K(\pi_2 \otimes \mathbb{K}_P, 2)$ and $g_2\colon K(\pi_3,4) \to K(\pi_3 \otimes \mathbb{K}_P,4)$. We can start by setting $(X_2)_P := K(\pi_2 \otimes \mathbb{K}_P, 2)$ and get a diagram

$$\begin{tikzcd}
X_3 \arrow[r, dashed] \arrow[d, dashed, "f_3"] & X_2 \arrow[r, "k_2"] \arrow[d, "f_2"] & K(\pi_3,4) \arrow[d, "g_2"] \\
(X_3)_P \arrow[r, dashed] & (X_2)_P \arrow[r, dashed, "(k_2)_P"] & K(\pi_3 \otimes \mathbb{K}_P, 4)
\end{tikzcd}$$

where the missing dashed parts will be constructed in the following.
By replacing $(X_2)_P$ with the mapping cylinder, we can assume $((X_2)_P, X_2)$ to be a CW pair and f_2 to be the inclusion. Now 2.2.4 implies the induced maps $H_k(f_2;\mathbb{K}_P)\colon H_k(X_2;\mathbb{K}_P) \to H_k((X_2)_P;\mathbb{K}_P)$ are isomorphisms, so by lemma 2.2.8 if follows that $H^k(f_2;A)\colon H^k((X_2)_P;A) \to H^k(X_2;A)$ is an isomorphism for any $\mathbb{K}_P$-module A. From the long exact cohomology sequence of $((X_2)_P, X_2)$ it follows that the relative groups $H^k((X_2)_P, X_2; A) \cong H^k((X_2)_P \cup CX_2; A)$ are zero, meaning all

obstruction classes vanish. For $A := \pi_3 \otimes \mathbb{K}_P$ when $k = 5$, obstruction theory as for instance presented in [28] then implies that the composition $g_2 \circ k_2 \colon X_2 \to K(\pi_3 \otimes \mathbb{K}_P, 4)$ can be extended to a map $(k_2)_P \colon (X_2)_P \to K(\pi_3 \otimes \mathbb{K}_P, 4)$.
In order to proceed inductively, the diagram above should extend to the left as indicated, which we will do using the next stage of the Postnikov tower. By turning k_2 and $(k_2)_P$ into fibrations, we know that X_3 is the homotopy fibre of k_2 up to homotopy equivalence, and we define $(X_3)_P$ to be the homotopy fibre of $(k_2)_P$. From the long exact sequence associated with the fibration $(k_2)_P$ it follows that $(X_3)_P$ is P-local. To see this, let $n \in \mathcal{R}(P)$ and denote by φ_n the homomorphism obtained by multiplication with n, as well as $K := K(\pi_3 \otimes \mathbb{K}_P, 4)$. We get a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & \pi_{k+1}((X_2)_P) \arrow[r] \arrow[d, "\varphi_n"] & \pi_{k+1}(K) \arrow[r] \arrow[d, "\varphi_n"] & \pi_k((X_3)_P) \arrow[r] \arrow[d, "\varphi_n"] & \pi_k((X_2)_P) \arrow[r] \arrow[d, "\varphi_n"] & \pi_k(K) \arrow[r] \arrow[d, "\varphi_n"] & \dots \\
\dots \arrow[r] & \pi_{k+1}((X_2)_P) \arrow[r] & \pi_{k+1}(K) \arrow[r] & \pi_k((X_3)_P) \arrow[r] & \pi_k((X_2)_P) \arrow[r] & \pi_k(K) \arrow[r] & \dots
\end{tikzcd}$$

in which the first two and the last two vertical maps are isomorphisms, since the corresponding groups are P-local. By the five-lemma, φ_n at $\pi_k((X_3)_P)$ has to be an isomorphism as well.
What is left is to show that the natural map $f_3 \colon X_3 \to (X_3)_P$ induced by f_2 through restricting to the fibre, is a P-localization. This is again an application of the long exact sequence, this time for both fibrations k_2 and $(k_2)_P$, where for the long exact sequence of k_2 we use the fact that $- \otimes \mathbb{K}_P$ is an exact functor. Setting $K := K(\pi_3, 4)$ and $K_P := K(\pi_3 \otimes \mathbb{K}_P, 4)$ we get the commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & \pi_{k+1}(K) \otimes \mathbb{K}_P \arrow[r] \arrow[d, "\pi_{k+1}(g_2) \otimes \mathbb{K}_P"] & \pi_k(X_3) \otimes \mathbb{K}_P \arrow[r] \arrow[d, "\pi_k(f_3) \otimes \mathbb{K}_P"] & \pi_k(X_2) \otimes \mathbb{K}_P \arrow[r] \arrow[d, "\pi_k(f_2) \otimes \mathbb{K}_P"] & \dots \\
\dots \arrow[r] & \pi_{k+1}(K_P) \arrow[r] & \pi_k((X_3)_P) \arrow[r] & \pi_k((X_2)_P) \arrow[r] & \dots
\end{tikzcd}$$

Now f_2 and g_2 are assumed to be P-localizations, so using the five-lemma again we get that $\pi_k(f_3) \otimes \mathbb{K}_P$ has to be an isomorphism.
Repeating this argumentation inductively produces a sequence of fibrations $(X_n)_P \to (X_{n-1})_P$, and each stage comes with a P-localization $X_n \to (X_n)_P$. We see that $\pi_k(\varprojlim_n (X_n)_P) \cong \varprojlim_n \pi_k((X_n)_P)$, which can again be found in [28], so by using the fact that being P-local is preserved by the inverse limit we see that $X_P := \varprojlim_n (X_n)_P$ is a P-local space.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] \\
 & X_3 \arrow[r] \arrow[d] & (X_3)_P \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] & X_2 \arrow[r] & (X_2)_P
\end{tikzcd}$$

Now there exists a unique morphism $\Phi \colon X \to X_P$ such that for each n the map $X \to X_n \to (X_n)_P$ is equal to the composition $q_n \circ \Phi$, where the maps $q_n \colon \varprojlim_n (X_n)_P \to (X_n)_P$ denote the natural projections associated with the limit. But $X \to X_n$ induces an isomorphism on homotopy groups up to level n, so we get induced isomorphisms $\pi_k(X) \otimes \mathbb{K}_P \to \pi_k(X_n) \otimes \mathbb{K}_P \to \pi_k((X_n)_P)$ for $k \leq n$. However, this means $\pi_k(q_n \circ \Phi) \otimes \mathbb{K}_P = \pi_k(q_n) \otimes \mathbb{K}_P \circ \pi_k(\Phi) \otimes \mathbb{K}_P$ has to be an isomorphism for all $k \leq n$, so this is in particular true for $\pi_k(\Phi) \otimes \mathbb{K}_P \colon \pi_k(X) \otimes \mathbb{K}_P \to \pi_k(X_P) \otimes \mathbb{K}_P \cong \pi_k(X_P)$. Letting n be arbitrary large, we see that Φ has to be a P-localization. $\square$

Note that by CW approximation, we can always assume X_P to be a CW complex.
Having shown the existence of P-localizations $X \to X_P$ for simply connected spaces, the question arises whether X_P is unique. The following shows that P-localization is a functor and provides an affirmative answer to this question.

Proposition 2.2.9. *Assume $f\colon X \to Y$ is a continuous map between simply connected spaces and $X \to X_P$, $Y \to Y_P$ are P-localizations, then there exists a map $f_P\colon X_P \to Y_P$ such that*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X_P \arrow[d, "f_P"] \\ Y \arrow[r] & Y_P \end{tikzcd}$$

commutes. Moreover, whenever $f \simeq g$, then $f_P \simeq g_P$.

Proof. This will be an immediate consequence of theorem 2.3.9 that we will come to in a bit. □

Remark 2.2.10. In particular, by choosing $Y = X$ and $f = \mathrm{id}_X$, for a given P-localization $X \to X_P$ the identity id_{X_P} makes the diagram commute, so $(\mathrm{id}_X)_P = \mathrm{id}_{X_P}$. Further, assume $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps between simply connected spaces and fix P-localizations for X, Y and Z respectively. When f_P and g_P commute with f respectively g and the localizations, then $f_P \circ g_P$ commutes with $f \circ g$ and the localizations of X and Z, so $(f \circ g)_P = f_P \circ g_P$.
From these functorial properties, it follows if $f\colon X \to Y$ is a homotopy equivalence with homotopy inverse g, then $f_P\colon X_P \to Y_P$ is a homotopy equivalence with homotopy inverse g_P. Hence, the homotopy type of X_P is uniquely determined by the homotopy type of X.

2.3 Cellular Localization

We have already seen some first examples of P-local spaces and P-localizations and have proved that the latter always exist for simply connected spaces. However, to fully grasp the geometric structure of a P-localization, especially when looking at CW complexes, we have to localize cell by cell. Just as CW complexes are formed by successively gluing disks of increasing dimension to an existing skeleton, we can define a P-local analogue using P-local disks.

Definition 2.3.1. The space $\mathbb{S}^n_P$ is called the *P-local n-sphere* and the space $\mathbb{B}^{n+1}_P := \mathbb{S}^n_P \times I / \mathbb{S}^n_P \times \{0\}$ is called the *P-local $(n+1)$-disk*.
If $P = \varnothing$ and thus $\mathbb{K}_P = \mathbb{Q}$, we call $\mathbb{S}^n_{\mathbb{Q}}$ the *rational n-sphere* and accordingly $\mathbb{B}^{n+1}_{\mathbb{Q}}$ the *rational $(n+1)$-disk*.

By construction, $\mathbb{S}^n_P$ is a CW complex consisting of cells of the dimensions n and $n+1$. As the cone of such, the P-local $(n+1)$-disk $\mathbb{B}^{n+1}_P$ is a CW complex with cells of dimension n to $n+2$, containing $\mathbb{S}^n_P$ as subcomplex. Moreover, due to the universal property of the quotient space the inclusion $\mathbb{S}^n_0 \to \mathbb{S}^n_P$ from construction 2.1.6 extends to $\mathbb{B}^{n+1}$, providing an inclusion of CW pairs $(\mathbb{B}^{n+1}, \mathbb{S}^n) \to (\mathbb{B}^{n+1}_P, \mathbb{S}^n_P)$.

Preparation 2.3.2. Assume A is a simply connected topological space. For every $n \geq 1$, let I_n be an index set and consider a family $(\mathbb{B}^{n+1}_{P,i}, \mathbb{S}^n_{P,i})_{i \in I_n}$ of P-local disks together with their corresponding spheres. Given cellular maps $f_{n,i}\colon \mathbb{S}^n_{P,i} \to X^{(n)}$, we can inductively define

$$X^{(n+1)} := X^{(n)} \cup_{f_n} \left(\dot{\bigcup}_{i \in I_n} \mathbb{B}^{n+1}_{P,i} \right),$$

where we start with $X^{(1)} := A$ and $f_n\colon \dot{\bigcup}_{i \in I_n} \mathbb{S}^n_{P,i} \to X^{(n)}$ is induced by the universal property of the disjoint union, so $f_n(x,i) = f_{n,i}(x)$ for $i \in I_n$ and $x \in \mathbb{S}^n_{P,i}$.

Definition 2.3.3. A *relative CW_P complex* is a pair (X, A) of topological spaces where A is simply connected and X is a union of closed subspaces $(X^{(n)})_{n \geq 1}$, where each $X^{(n)}$ is constructed from A as described above. When $A = \{\mathrm{pt}\}$, we call X a *CW_P complex*. Whenever $P = \varnothing$, we write $CW_{\mathbb{Q}}$ instead of CW_P.

Analogue to regular CW complexes, we call the space $X^{(n)}$ the *n-skeleton of the CW_P-structure* on (X, A). It contains all the P-local disks $\mathbb{B}_P^k$ with $k \leq n$ that are attached to X.

Remark 2.3.4. Remember that if $f\colon Y \to Z$ is a cellular map between CW complexes, the pushout $Y \cup_f Z$ carries a cell structure as well. In our case, for each $n \geq 1$, the map f_n is cellular by assumption and $X^{(n+1)}$ is the pushout of $X^{(n)}$ and a CW complex. It follows by induction that the spaces $X^{(n)}$ carry an induced cell structure with respect to $X^{(1)} = A$. Hence (X, A) is a relative CW complex, with a regular skeleton denoted by X^n. Each $X^{(n)}$ is a subcomplex of X and the cells it contains that are not in A have dimension $n+1$ or lower, which can be seen inductively, as $\mathbb{B}_P^n \setminus \mathbb{S}_P^{n-1}$ contains only cells of dimension n and $n+1$. This also shows that $X^{(n)}$ contains all cells of dimension n and lower, since afterwards only cells of dimension $n+1$ and higher are added. We obtain

$$X^n \subseteq X^{(n)} \subseteq X^{n+1} \quad \text{for } n \geq 1,$$

relating the CW and CW_P structure on (X, A).

Proposition 2.3.5. *Assume (X, A) is a relative CW_P complex, where A is P-local. Then X is a P-local space. In particular, any CW_P complex X is P-local.*

Proof. As $X^1 \subseteq X^{(1)} = A$, we see that any cell in $X \setminus A$ is of dimension greater than 1. We conclude from cellular approximation that (X, A) is 1-connected. Thus X is simply connected, since we assumed the same to be true for A. Hence, we may apply theorem 2.1.7 and use tools from homology theory.
As $\mathbb{B}_P^{n+1}$ deformation retracts to a point, its homology vanishes on positive level. We deduce from the long exact sequence of relative homology groups and the calculation of $H_k(\mathbb{S}_P^n; \mathbb{Z})$ that the singular homology of $(\mathbb{B}_P^{n+1}, \mathbb{S}_P^n)$ with coefficients in $\mathbb{Z}$ equals $\mathbb{K}_P$ concentrated in degree $n+1$. Now, as mentioned above, $X^{(n+1)}$ is a (relative) CW complex with subcomplex $X^{(n)}$. In other words, $(X^{(n+1)}, X^{(n)})$ is a CW pair, so in particular the quotient map $(X^{(n+1)}, X^{(n)}) \to (X^{(n+1)}/X^{(n)}, X^{(n)}/X^{(n)})$ induces an isomorphism between homology groups due to the excision theorem. Finally, $X^{(n+1)}$ is obtained by gluing the cones $\mathbb{B}_{P,i}^{n+1}$ to $X^{(n)}$ along the bottom $\mathbb{S}_{P,i}^n$. Thus, by collapsing $X^{(n)}$ to a point we are left with a wedge of suspensions $\vee_{i \in I_n} S(\mathbb{S}_{P,i}^n)$ and we get

$$\begin{aligned} H_k(X^{(n+1)}, X^{(n)}; \mathbb{Z}) &= H_k(X^{(n+1)}/X^{(n)}, X^{(n)}/X^{(n)}; \mathbb{Z}) \\ &= H_k(\vee_{i \in I_n} S(\mathbb{S}_{P,i}^n); \mathbb{Z}) = \oplus_{i \in I_n} H_{k-1}(\mathbb{S}_{P,i}^n; \mathbb{Z}) \end{aligned}$$

for all $k \geq 1$, i.e. $H_*(X^{(n+1)}, X^{(n)}; \mathbb{Z})$ is P-local. Now remember that we assumed $A = X^{(1)}$ to be P-local, so suppose by induction that $X^{(n)}$ is P-local. Consider the long exact homology sequence for the pair $(X^{(n+1)}, X^{(n)})$ at

$$H_{k+1}(X^{(n+1)}, X^{(n)}) \to H_k(X^{(n)}) \to H_k(X^{(n+1)}) \to H_k(X^{(n+1)}, X^{(n)}) \to H_{k-1}(X^{(n)}).$$

As follows from the computation of $H_*(X^{(n+1)}, X^{(n)}; \mathbb{Z})$, we have an isomorphism $H_k(X^{(n)}) \cong H_k(X^{(n+1)})$ for $k \leq n-1$ and $k \geq n+2$. For $k = n$ and $k = n+1$ we may use the fact that, aside from $H_k(X^{(n+1)})$, we know that all homology groups in the exact sequence above are P-local. By remark 2.1.2, it follows that $H_k(X^{(n+1)})$ is P-local, which closes the induction step. Finally, for any CW complex the inclusion $X^n \hookrightarrow X$ induces an isomorphism between homology groups on levels $k \leq n-1$, so the same is true for $X^{(n)} \hookrightarrow X$ since $X^n \subseteq X^{(n)}$. All in all, this shows that $H_k(X; \mathbb{Z})$ is P-local for $k \geq 1$. Since $H_0(X, x_0; \mathbb{Z})$ is trivial for any $x_0 \in X$, we conclude that $H_*(X, x_0; \mathbb{Z})$ and therefore X is P-local. $\square$

We have now seen how P-local CW complexes are built. Next, we will discuss how a given CW complex can be localized cell by cell.

Construction 2.3.6. We begin by considering a relative CW complex (X, A) where $X^0 = X^1 = A$ is assumed to be P-local and simply connected. Again, since X is obtained from A by attaching cells of dimension greater than one, it follows that X is simply connected. However, it need not be P-local. Our goal is now to construct from (X, A) a relative CW$_P$ complex (X_P, A) together with an inclusion $\varphi\colon (X, A) \hookrightarrow (X_P, A)$ that induces an isomorphism between homology groups with coefficients in $\mathbb{K}_P$, therefore being a P-localization. We will achieve this by induction over n, successively replacing n-cells of (X, A) with P-local n-cells such that φ restricts to a map $\varphi_n\colon X^n \to (X_P)^{(n)}$.
Starting with $n = 1$, we have $X^1 = A$ and $\varphi_1\colon A \to A$ the identity. Next, suppose inductively we have $(X_P)^{(n)}$ and φ_n already constructed. Let $\dot{\bigcup}_i \mathbb{B}_i^{n+1}$ be a collection of all $(n+1)$-cells in the relative CW complex (X, A) and let g_n be an attaching map for this collection. Then

$$X^{n+1} = X^n \cup_{g_n} \Big(\dot{\bigcup}_i \mathbb{B}_i^{n+1}\Big).$$

Denote by $g_{i,n}\colon \mathbb{S}_i^n \to X^n$ the attaching map for a single cell $\mathbb{B}_i^{n+1}$. We will now extend the composition $\varphi_n \circ g_{i,n}\colon \mathbb{S}_i^n \to (X_P)^{(n)}$ to a cellular map $h_{i,n}\colon \mathbb{S}_{P,i}^n \to (X_P)^{(n)}$. For this, let i be fixed and remember that

$$\mathbb{S}_P^n := \Big(\bigvee_{k=0}^{\infty} \mathbb{S}_k^n\Big) \cup_f \Big(\dot{\bigcup}_{j=1}^{\infty} \mathbb{B}_j^{n+1}\Big),$$

with attaching maps $f_{j,n}$ of $\mathbb{B}_j^{n+1}$ being represented by $\alpha_{j-1} - k_j\alpha_j$. Here, α_j is the class of the inclusion of the j-th sphere in $\pi_n(\bigvee_{k=0}^{\infty} \mathbb{S}_k^n)$ and $k_j \in \mathcal{R}(P)$. Now, since $\pi_n((X_P)^{(n)})$ is a $\mathbb{K}_P$-module, we find a representative $g_k\colon \mathbb{S}_k^n \to (X_P)^{(n)}$ such that

$$[g_k] = \frac{1}{\prod_{j=1}^{k} k_j}[\varphi_n \circ g_{i,n}]$$

for each $k \geq 1$. For $k = 0$, we set $g_0 := \varphi_n \circ g_{i,n}$ and thereby identify $\mathbb{S}_i^n$ with the initial sphere $\mathbb{S}_0^n$. This defines

$$h'_{i,n} := \bigvee_{k=0}^{\infty} g_k \colon \bigvee_{k=0}^{\infty} \mathbb{S}_k^n \to (X_P)^{(n)}$$

extending $\varphi_n \circ g_{i,n}$ to the bouquet of spheres. For some $r \geq 1$ it now follows that the composition $h'_{i,n} \circ f_{r,n}$ is nullhomotopic, as

$$\begin{aligned}
[h'_{i,n} \circ f_{r,n}] &= \pi_*(h'_{i,n})([f_{r,n}]) = \pi_*(h'_{i,n})(\alpha_{r-1}) - k_r\pi_*(h'_{i,n})(\alpha_r) = [h'_{i,n} \circ i_{r-1}] - k_r[h'_{i,n} \circ i_r] \\
&= \frac{1}{\prod_{j=1}^{r-1} k_j}[\varphi_n \circ g_{i,n}] - \frac{k_r}{\prod_{j=1}^{r} k_j}[\varphi_n \circ g_{i,n}] = 0.
\end{aligned}$$

Therefore, $h'_{i,n}$ extends over the cell $\mathbb{B}_r^{n+1}$ that is attached via $f_{r,n}$. Since this is true for all $r \geq 1$, we may extend it to all of $\mathbb{S}_{P,i}^n$ and denote the resulting map by $h_{n,i}$. The corresponding commutative diagram reads

$$\begin{tikzcd}
\mathbb{S}_i^n \arrow[r] \arrow[d, "g_{n,i}"'] & \bigvee_{k=0}^{\infty} \mathbb{S}_k^n \arrow[r] \arrow[dr, "h'_{i,n}"'] & \mathbb{S}_{P,i}^n \arrow[d, "h_{i,n}"] \\
X^n \arrow[rr, "\varphi_n"'] & & (X_P)^{(n)}.
\end{tikzcd}$$

We do this for all n-cells of (X, A) and then attach cells $\mathbb{B}_{P,i}^{n+1}$ to $(X_P)^{(n)}$ using the maps $h_{i,n}$ to obtain $(X_P)^{(n+1)}$. By denoting the induced map on the disjoint union $\dot{\bigcup}_i \mathbb{S}_{P,i}^n$ of P-local spheres by h_n this reads

$$(X_P)^{(n+1)} := (X_P)^{(n)} \cup_{h_n} \Big(\dot{\bigcup}_i \mathbb{B}_{P,i}^{n+1}\Big).$$

Finally, to define $\varphi_{n+1}\colon X^{n+1} \to (X_P)^{(n+1)}$ extending φ_n, consider the inclusions $\mathbb{B}_i^{n+1} \hookrightarrow \mathbb{B}_{P,i}^{n+1}$. These define a map $\psi_n\colon \dot{\bigcup}_i \mathbb{B}_i^{n+1} \to \dot{\bigcup}_i \mathbb{B}_{P,i}^{n+1}$ and we set

$$\varphi_{n+1} := \varphi_n \cup \psi_n \colon X^n \cup_{g_n} \left(\dot{\bigcup}_i \mathbb{B}_i^{n+1}\right) \to (X_P)^{(n)} \cup_{h_n} \left(\dot{\bigcup}_i \mathbb{B}_{P,i}^{n+1}\right),$$

completing the induction step and therefore the construction of $\varphi\colon (X, A) \to (X_P, A)$.

Lemma 2.3.7. *The map $\varphi\colon (X, A) \to (X_P, A)$ from construction 2.3.6 is a P-localization.*

Proof. It follows at once from proposition 2.3.5 that X_P is P-local. Therefore, to argue that φ is a P-localization, by theorem 2.2.4 it suffices to show that $H_*(\varphi; \mathbb{K}_P)\colon H_*(X; \mathbb{K}_P) \to H_*(X_P; \mathbb{K}_P)$ is an isomorphism. Assume by induction that φ_n induces an isomorphism between homology groups with coefficients in $\mathbb{K}_P$. We may use the same arguments as in proposition 2.3.5 to show that

$$H_*(X^{n+1}, X^n; \mathbb{K}_P) \cong H_*(\vee_i S\mathbb{S}_i^n; \mathbb{K}_P), \quad H_*((X_P)^{(n+1)}, (X_P)^{(n)}; \mathbb{K}_P) \cong H_*(\vee_i S\mathbb{S}_{P,i}^n; \mathbb{K}_P),$$

where the isomorphisms are induced by the quotient maps. Plugging this into the long exact homology sequences of the CW pairs (X^{n+1}, X^n) and $((X_P)^{(n+1)}, (X_P)^{(n)})$ this reads

$$\begin{tikzcd}
\dots \arrow[r] & H_*(X^n; \mathbb{K}_P) \arrow[r] \arrow[d, "H_*(\varphi_n;\mathbb{K}_P)"', "\cong"] & H_*(X^{n+1}; \mathbb{K}_P) \arrow[r] \arrow[d, "H_*(\varphi_{n+1};\mathbb{K}_P)"'] & H_*(\vee_i S\mathbb{S}_i^n; \mathbb{K}_P) \arrow[r] \arrow[d, "\cong"] & \dots \\
\dots \arrow[r] & H_*((X_P)^{(n)}; \mathbb{K}_P) \arrow[r] & H_*((X_P)^{(n+1)}; \mathbb{K}_P) \arrow[r] & H_*(\vee_i S\mathbb{S}_{P,i}^n; \mathbb{K}_P) \arrow[r] & \dots,
\end{tikzcd}$$

where the right-hand isomorphism is induced by the inclusions $(\mathbb{B}_i^{n+1}, \mathbb{S}_i^n) \to (\mathbb{B}_{P,i}^{n+1}, \mathbb{S}_{P,i}^n)$. Note that this diagram commutes by definition of φ_{n+1}, which restricts to φ_n on X^n and is extended over the $(n+1)$-cells using the natural inclusion written above. Applying once more the five lemma, we see that $H_*(\varphi_{n+1}; \mathbb{K}_P)$ is an isomorphism. □

Remark 2.3.8. This concludes the P-localization of CW complexes. The benefits of this approach, when compared to localization via the Postnikov tower, is that the obtained P-localization $X \hookrightarrow X_P$ is an inclusion of a subcomplex. Moreover, the cell structure of X directly determines the cell structure of X_P in the sense that any attaching map $\mathbb{S}^n \to X^n$ gets replaced by a P-local version $\mathbb{S}_P^n \to (X_P)^{(n)}$. Thus, the construction above does a better job in comparing the geometry of X and X_P. That being said, in most cases an explicit description of X_P will not be necessary, since we will soon begin to employ algebraic tools. As we shall see then, the complicated nature of P-localization should not worry us, for it allows the use of rather convenient algebraic models.

Next, we will extend the results of construction 2.3.6 to the case of arbitrary simply connected spaces through the use of CW approximation. As the P-localizations that we have constructed are inclusions, the question of the functoriality of P-localization as stated in proposition 2.2.9 results in finding an extension to X_P for any continuous map on X.

Theorem 2.3.9. *Each simply connected space X can be localized through the attachment of P-local cells. That is:*

(i) There exists a relative CW complex (X_P, X) such that the inclusion $X \hookrightarrow X_P$ is a P-localization.

(ii) If $f\colon X \to Y$ is a continuous map between simply connected spaces and Y is P-local, then there exists $F\colon X_P \to Y$ such that $F|_X = f$.

(iii) Any homotopy $f \simeq g$ extends to a homotopy $F \simeq G$. Hence (X_P, X) is unique up to homotopy equivalence relative X.

Note that this in particular implies the statements from theorem 2.2.7 and proposition 2.2.9. For the latter, assuming Y is not P-local and we are given $f\colon X \to Y$, we may just apply theorem 2.3.9 to the composition $X \to Y \to Y_P$ of f with any P-localization of Y. Finally, the asserted uniqueness of (X_P, X) follows at once from the theorem if we consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & X_P \arrow[d, bend left, "F'"] \\ X \arrow[r, "f'"'] & X'_P. \arrow[u, bend left, "F"] \end{tikzcd}$$

Here, both $f\colon X \hookrightarrow X_P$ and $f'\colon X \hookrightarrow X'_P$ are assumed to be P-localizations of X, and F extends f with respect to X'_P, while F' extends f' with respect to X_P. However, a suitable extension for f with respect to X_P is id_{X_P}, while for f' with respect to X'_P it is $\mathrm{id}_{X'_P}$. It follows that $F' \circ F \simeq \mathrm{id}_{X_P}$ and $F \circ F' \simeq \mathrm{id}_{X'_P}$.

Proof of theorem 2.3.9. (*i*) In order to apply construction 2.3.6, we begin by making use of CW approximation. Let $\psi\colon Z \to X$ be a weak homotopy equivalence, where Z is a CW complex. Since X is simply connected, we may assume that $Z^0 = Z^1$ consists of a single 0-cell. Then the non-relative version of construction 2.3.6 yields an inclusion $\varphi\colon Z \hookrightarrow Z_P$ that is a P-localization. Now consider $Z \times I$ and glue $Z \times \{0\}$ to X using ψ, as well as $Z \times \{1\}$ to Z_P using φ. We obtain

$$X_P := X \cup_\psi (Z \times I) \cup_\varphi Z_P,$$

where we may assume ψ and φ to be cellular maps. It follows that X_P is simply connected since X is and that (X_P, X) is a relative CW complex with cells of dimension two and higher. Now, from the excision theorem, it follows that

$$H_*(X_P, Z_P; \mathbb{Z}) \cong H_*(X \cup_\psi (Z \times I), Z \times \{1\}; \mathbb{Z}).$$

The space $X \cup_\psi (Z \times I)$ that remains is exactly the mapping cylinder M_ψ of ψ. This deformation retracts to X via some retraction $r\colon M_\psi \to X$ and contains Z as subspace with inclusion $i\colon Z \times \{1\} \hookrightarrow M_\psi$. As such, it holds $\psi = r \circ i$, and since $H_*(\psi; \mathbb{Z})$ is an isomorphism the same holds for $H_*(i; \mathbb{Z})$. Consequentially, the long exact homology sequence implies that $H_*(M_\psi, Z \times \{1\}; \mathbb{Z}) = 0$ and thus $H_*(X_P, Z_P; \mathbb{Z}) = 0$. Then again, the long exact homology sequence of (X_P, Z_P) implies $H_*(X_P, x_0; \mathbb{Z}) \cong H_*(Z_P, z_0; \mathbb{Z})$, so X_P is P-local since Z_P is.
Finally, it once more follows from excision that

$$H_*(X_P, X; \mathbb{K}_P) \cong H_*(Z_P, Z; \mathbb{K}_P) = 0,$$

where the second equality is due to the long exact homology sequence and fact that the inclusion $\varphi\colon Z \hookrightarrow Z_P$ is a P-localization, therefore inducing an isomorphism on homology with coefficients in $\mathbb{K}_P$. The long exact homology sequence for (X_P, X) with coefficients in $\mathbb{K}_P$ then shows that the inclusion $X \hookrightarrow X_P$ is a P-localization.
(*ii*) Glue X_P to Y along X using f to obtain $Y \cup_f X_P$. This contains Y as subspace, so $(Y \cup_f X_P, Y)$ is a relative CW complex, and Y is P-local. We may thus apply construction 2.3.6 to obtain a P-localization $(Y \cup_f X_P, Y) \to ((Y \cup_f X_P)_P, Y)$. Now, similar to the above, by using excision we obtain

$$H(Y \cup_f X_P, Y; \mathbb{K}_P) \cong H_*(X_P, X; \mathbb{K}_P) = 0,$$

where again the latter equality holds due to $X \hookrightarrow X_P$ being a P-localization. As follows from the long exact homology sequence, the inclusion $Y \hookrightarrow Y \cup_f X_P$ induces an isomorphism on homology groups with coefficients in $\mathbb{K}_P$. Therefore, the composition

$$\tau\colon Y \hookrightarrow Y \cup_f X_P \hookrightarrow (Y \cup_f X_P)_P$$

of this inclusion with the P-localization mentioned above also induces an isomorphism on homology groups. By theorem 2.2.4, $\pi_*(\tau) \otimes \mathbb{K}_P$ is an isomorphism, and thus even $\pi_*(\tau)$ is an isomorphism since both Y and $(Y \cup_f X_P)_P$ are P-local. Hence τ and in particular the inclusion $Y \hookrightarrow Y \cup_f X_P$ is a weak homotopy equivalence. This implies the existence of a retraction $r\colon Y \cup_f X_P \to Y$ which, together with the canonical projection $X_P \to Y \cup_f X_P$, provides F as claimed.
(*iii*) The last claim follows at once from the fact that the inclusion

$$(X_P \times \{0\}) \cup (X \times I) \cup (X_P \times \{1\}) \hookrightarrow X_P \times I$$

is a P-localization. Therefore, the map composed of F on $X_P \times \{0\}$, the homotopy $f \simeq g$ on $X \times I$ and G on $X_P \times \{1\}$ extends to a homotopy $F \simeq G$ on $X_P \times I$. □

2.4 Rational Homotopy Type

The preceding chapters show that the more integers $\mathcal{R}(P) \subseteq \mathbb{Z}$ contains, the more complicated the construction of a P-localization $X \to X_P$ gets. As for the groups $\pi_*(X_P) = \pi_*(X) \otimes \mathbb{K}_P$, things tend to get easier, since the closer $\mathbb{K}_P$ is to being $\mathbb{Q}$, the more torsion is ignored when compared to $\pi_*(X)$. This trend, where increased chaos on the topological side leads to a simplification on the algebraic side (and vice versa), can be observed on multiple occasions throughout homotopy theory. While rational homotopy theory makes heavy use of this circumstance, the complicated geometry of P-local spaces should not be something to worry about. According to the usual practice in algebraic topology, rather than a space itself one is oftentimes more interested in a class of spaces, which is what we will come to now. When given such a class, one can then still hope to find a representative with relatively easy geometry.
Henceforth, we will focus solely on the special case $P = \varnothing$ and $\mathbb{K}_P = \mathbb{Q}$. This immediately provides another benefit, since with field coefficients cohomology is just the dual of homology by the universal coefficient theorem. Let again $f\colon X \to Y$ be a map between simply connected spaces. We can extend the Whitehead-Serre theorem 2.2.4, which now states that the following conditions

1. $\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism
2. $H_*(f; \mathbb{Q})$ is an isomorphism
3. $H^*(f; \mathbb{Q})$ is an isomorphism

are equivalent.

Definition 2.4.1. A continuous map $f\colon X \to Y$ between simply connected spaces is called a *rational homotopy equivalence* if one and hence all of the above conditions are satisfied.

Remark 2.4.2. Obviously, any rationalization $X \to X_{\mathbb{Q}}$ is a rational homotopy equivalence, and a rational homotopy equivalence $X \to Y$ is a rationalization if and only if Y is a rational space. Furthermore, a weak homotopy equivalence $X \to Y$ is always a rational homotopy equivalence, and the other implication only holds if both X and Y are rational spaces. This is evident from the commutative diagram

$$\begin{tikzcd}
\pi_*(X) \arrow[r, "\pi_*(f)"] \arrow[d, "\cong", "\alpha\mapsto\alpha\otimes 1"'] & \pi_*(Y) \\
\pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r] \arrow[ur, dashed, "\pi_*(f)\otimes_{\mathbb{Z}}\mathbb{Q}"] & \pi_*(Y) \otimes_{\mathbb{Z}} \mathbb{Q}. \arrow[u, "\cong"', "\alpha\otimes\lambda\mapsto\lambda\alpha"]
\end{tikzcd}$$

in which the vertical maps are isomorphisms if and only if X and Y are rational spaces.

Looking back at remark 2.2.10, we have seen that a homotopy equivalence $f\colon X \to Y$ induces a homotopy equivalence $f_{\mathbb{Q}}\colon X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ between localizations. Now as follows from remark 2.4.2 above, with the notably weaker assumption of $f\colon X \to Y$ being a rational homotopy equivalence, by applying the functor $\pi_*(-) \otimes \mathbb{Q}$ to the commutative diagram from proposition 2.2.9 we see that the induced map $f_{\mathbb{Q}}\colon X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is a weak homotopy equivalence. Using CW approximation, we may even suppose $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ to be CW complexes, which means they are homotopy equivalent by Whitehead's theorem. This shows that from the viewpoint of classical homotopy theory, two distinct spaces X and Y might share the same rationalization. In rational homotopy theory, however, two such spaces will share a common class which we shall define now.

Definition 2.4.3. We say two simply connected spaces X, Y have the same *rational homotopy type* if there exists $n \in \mathbb{N}_0$ and simply connected spaces $Z(k)$, $0 \leq k \leq n$ together with a chain

$$X \longleftarrow Z(0) \longrightarrow \cdots \longleftarrow Z(n) \longrightarrow Y$$

of rational homotopy equivalences. We indicate this by writing $X \simeq_{\mathbb{Q}} Y$.

Remark 2.4.4. This defines an equivalence relation on the simply connected spaces, so by the rational homotopy type of a simply connected space X we mean the class of all spaces Y such that $X \simeq_{\mathbb{Q}} Y$. Roughly speaking, two spaces of the same rational homotopy type resemble each other in the sense that they share those very properties which are conveyed under rational homotopy equivalence. Therefore, rational homotopy theory consists in the study of the characteristics of spaces and maps that are invariant under rational homotopy equivalence, meaning they only depend on the rational homotopy type.

Remember that in the weak homotopy category, or more commonly called the standard homotopy category, two spaces are isomorphic if they are connected by a chain $X \leftarrow \cdots \to Y$ of weak homotopy equivalences. Denote the existence of such chain by writing $X \simeq_w Y$. For example, any CW approximation is a chain of weak homotopy equivalences, so CW approximation assigns to a space a CW complex of the same weak homotopy type.
Switching back to rational homotopy theory, a rationalization $X \to X_{\mathbb{Q}}$ of a simply connected space X is obviously a chain of rational homotopy equivalences, so $X \simeq_{\mathbb{Q}} X_{\mathbb{Q}}$. In other words, a rationalization assigns to a space a rational space with the same rational homotopy type. Let Y be a second simply connected space with rationalization $Y \to Y_{\mathbb{Q}}$ and assume that $X_{\mathbb{Q}} \simeq_w Y_{\mathbb{Q}}$. As any weak homotopy equivalence is a rational homotopy equivalence, a chain $X_{\mathbb{Q}} \leftarrow \cdots \to Y_{\mathbb{Q}}$ of weak homotopy equivalences is automatically a chain of rational homotopy equivalences. Put together with the rationalizations, we obtain a chain $X \to X_{\mathbb{Q}} \leftarrow \cdots \to Y_{\mathbb{Q}} \leftarrow Y$ of rational homotopy equivalences, so $X \simeq_{\mathbb{Q}} Y$.
The other way around, assume X and Y are connected by a chain of rational homotopy equivalences described in definition 2.4.3. From the functoriality of rationalization as stated in proposition 2.2.9 it follows that the rationalizations of the appearing spaces in the chain have to be weakly homotopy equivalent. More precisely, we get a commutative diagram

$$\begin{tikzcd}
X \arrow[d] & Z(0) \arrow[l] \arrow[r] \arrow[d] & \ldots & Z(k) \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d] \\
X_{\mathbb{Q}} & Z(0)_{\mathbb{Q}} \arrow[l] \arrow[r] & \ldots & Z(k)_{\mathbb{Q}} \arrow[l] \arrow[r] & Y_{\mathbb{Q}}
\end{tikzcd}$$

in which each map is a rational homotopy equivalence. Applying remark 2.4.2 to the bottom row then yields $X_{\mathbb{Q}} \simeq_w Y_{\mathbb{Q}}$. All in all, this shows the following.

Proposition 2.4.5. *The rational homotopy type of a simply connected space X corresponds to the weak homotopy type of $X_{\mathbb{Q}}$.*

So far, we have ignored most of the category theoretical aspects of rational homotopy theory. While it provides helpful and important tools, for our purpose a more concrete approach to the subject is suitable, as we will generally be interested in the detailed structure of objects and not only their universal properties. In doing so we sacrifice some generality, but are able to work with explicit constructions, which will make future examples and assertions more accessible. Therefore, we will only occasionally use category theory where it contributes insight to the big picture, without claiming to be completely accurate while doing so. The following is one of these cases.

Remark 2.4.6.

(a) Following Quillen, the localization of a category at a collection of morphisms involves the process of uniformly making these morphisms into isomorphisms. So, for example, the standard homotopy category as mentioned above is obtained through localising the category of topological spaces at the continuous maps that are weak homotopy equivalences. To put it simply, the existence of a weak homotopy equivalence $f\colon X \to Y$ is not a symmetric relation, but we can say X and Y have the same weak homotopy type if they are equivalent under the equivalence relation generated by weak homotopy equivalence. In other words, X and Y are weakly homotopy equivalent if they are connected by a chain of weak homotopy equivalences. In definition 2.4.3 we imitate this procedure, this time localizing at the broader class of morphisms that induce an isomorphism between rational homotopy groups.

(b) With the help of CW approximation and Whitehead's theorem one can show that two spaces are weakly equivalent if and only if their respective CW approximations are homotopy equivalent. This means that the standard homotopy category is equivalent to the subcategory of the classical homotopy category that is provided by CW complexes. By incorporating proposition 2.4.5 into this argumentation, we see that the rational homotopy category can be interpreted as the subcategory composed of those CW complexes that are rational spaces.

3 Models via Sullivan

So far, we have established what we wish to study in rational homotopy theory, which is the rational homotopy type of spaces and the rational homotopy class of maps. Moving on, in this chapter we will introduce powerful algebraic models that will help us to do so. Essentially, we will be able to fully classify the rational homotopy type and rational homotopy class of a broad class of spaces and maps through purely algebraic means. This is achieved by so-called minimal Sullivan models, which were originally introduced by Sullivan in [26]. We will show the existence and uniqueness of these models for simply connected spaces, and briefly introduce the more general concept of relative Sullivan algebras. These are used in models of continuous maps and play a central role in the models of fibrations. Moreover, we will learn how information on a simply connected space can be read from its minimal Sullivan model, such as the rational homotopy groups and the Whitehead product on them. Moreover, in the last section the realization functor will be presented, essentially establishing a bijection between isomorphy classes of minimal rational Sullivan algebras of finite type and simply connected spaces with rational cohomology of finite type. Along the way, it will become clear from the definition that the realization functor, while undoubtedly delivering powerful results, provides a quite complicated CW complex in the rational homotopy class that the minimal Sullivan model belongs to. This motivates the hunt for an alternative, more geometric realization, which is the motivation for this monograph.
This section is inspired by the brilliant explanation of the subject provided by Félix, Halperin and Thomas in [7]. Many of the results that we present here can be found in this book, and we will refer to it in some occasions when the proof is too tedious and not really needed for our purpose. We recommend the newer book of Félix, Halperin and Thomas, [8] to everyone who is bothered by the simply connectedness assumption or wants to learn about the theory in a more general setting. Standard literature on the subject is also [9], in which a large variety of examples and applications can be found, as well as [11].

3.1 Free Commutative Cochain Algebra

In general, the cochain algebras that appear naturally in algebraic topology are quite complex. To study them further, we are often forced to replace them by simpler cochain algebras. For this, the most promising candidates would be free algebras, allowing for a facilitating universal property. While there exists a free graded algebra associated with a graded vector space V, the tensor algebra TV, it is in general not commutative. However, we can make it commutative, which will lead us to the free commutative graded algebra denoted by ΛV which is fundamental to the algebraic models used by Sullivan. In the following, $\mathbb{K}$ is a commutative unitary ring, and tensor products as well as linearity are to be understood over $\mathbb{K}$ unless stated otherwise.

Example 3.1.1. Let V be a free graded module. We define its *tensor algebra* to be the graded module

$$TV = \bigoplus_{k\geq 0} T^k V = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

Naturally, the tensor algebra is always a graded module, even if the underlying module is not. However, we will consider TV with the grading that is induced by the grading of V: if $v_1, \dots, v_k$ are homogeneous elements of V, then the element $v_1 \otimes \dots \otimes v_k \in T^k V$ is an element of degree $\sum_{i=1}^k |v_i|$ and *word length* k. This grading makes the tensor product of two elements $v, w \in TV$ a multiplication, turning TV into a graded algebra with $vw := v \otimes w$.

Now obviously, $V = T^1 V$ is a graded submodule of TV. Let $i\colon V \hookrightarrow TV$ be the canonical inclusion. Given a morphism $f\colon V \to A$ of graded modules, where A is a graded algebra, there exists a unique extension $F\colon TV \to A$ that is a morphism of graded algebras. It is given by

$F(v_1 \otimes \cdots \otimes v_k) = f(v_1) \dots f(v_k)$ and makes the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "i"] & A \\ TV \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

commute. This is the universal property of TV and i. Furthermore, given any linear map $d\colon V \to TV$ of degree k, we can extend this to a unique derivation $d\colon TV \to TV$ of degree k. This is by induction on the word length, thereby defining d in accord with the Leibniz rule.

Example 3.1.2. For a free graded module V consider the ideal I on the tensor algebra TV that is generated by the set

$$\{v \otimes w - (-1)^{|v||w|} w \otimes v \mid v, w \in V \text{ are homogeneous}\}.$$

We then call the quotient $\Lambda V := TV/I$ the *free commutative graded algebra* on V.

Remark 3.1.3.

(a) For $v = v_1 \otimes \cdots \otimes v_k$ and $w = w_1 \otimes \cdots \otimes w_n \in TV$ we write $[v] = v_1 \wedge \cdots \wedge v_k$ and $[w] = w_1 \wedge \cdots \wedge w_n$ for the corresponding equivalence classes in ΛV. Multiplication is defined representative wise, so

$$\begin{aligned}(v_1 \wedge \cdots \wedge v_k)(w_1 \wedge \cdots \wedge w_n) = [v][w] := [vw] &= [v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_n] \\ &= v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_n.\end{aligned}$$

This makes ΛV a commutative graded algebra. Roughly speaking, we made the tensor algebra commutative by killing the relations that posed an obstruction to commutativity.

(b) We denote by $\Lambda^k V$ the subspace that is generated by elements of the form $v_1 \wedge \cdots \wedge v_k$, where $v_i \in V$ is homogeneous for all $1 \leq i \leq k$. It follows that $\Lambda V = \bigoplus_{k \geq 0} \Lambda^k V$, which provides a grading on ΛV that is independent of the grading on V. However, as with TV, we usually consider ΛV with the grading that is induced by the grading of V. That is, elements $v_1 \wedge \cdots \wedge v_k \in \Lambda^k V$ have degree $\sum_{i=1}^k |v_i|$ and word length k.

(c) Assume V is a free module and $(v_i)_{i \in I}$, where I is some index set, is a homogeneous basis of V. Then the set

$$\{v_{i_1} \otimes \cdots \otimes v_{i_n} \mid n \in \mathbb{N}, i_k \in I\}$$

provides a homogeneous basis for TV. The corresponding equivalence classes in turn define a set of homogeneous generators of ΛV. In these cases we can thus write $T(v_i)$ and $\Lambda(v_i)$ for TV and ΛV respectively.
For example, if V is generated by a single element v and $\mathbb{K}$ is a field of characteristic zero, a basis of ΛV is given by $\{1, v\}$ if $|v|$ is odd and $\{1, v, v^2, v^3, \dots\}$ if $|v|$ is even.

Any morphism $f\colon V \to A$ of graded modules, where A is a commutative graded algebra, extends to a unique morphism of graded algebras $F\colon \Lambda V \to A$. This follows from the universal properties of TV and the quotient. If $F'\colon TV \to A$ extends f on TV, we get

$$F'(v \otimes w - (-1)^{|v||w|} w \otimes v) = f(v)f(w) - (-1)^{|v||w|} f(w)f(v) = 0$$

and thus $F'(I) = 0$. So F' in turn induces $F\colon \Lambda V \to A$ on the quotient. Finally, let $j\colon V \to \Lambda V$ denote the canonical injection that is the composition of the inclusion $i\colon V \hookrightarrow TV$ with the canonical projection $\pi\colon TV \to \Lambda V$. Thus, we get a commutative diagram

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "i"] \arrow[dd, bend right, "j"'] & A \\
TV \arrow[ur, dashed, "F'"] \arrow[d, "\pi"] & \\
\Lambda V. \arrow[uur, dashed, "F"'] &
\end{tikzcd}$$

The uniqueness of F is an immediate consequence of the universal property of the tensor algebra, since F' is unique and π is surjective. We call this the *universal property of* $(\Lambda V, j)$.

Remark 3.1.4. Note that from there on, a standard argument shows that this universal property determines $(\Lambda V, j)$ up to isomorphy. Namely, if there exists a second commutative graded algebra A' and a morphism $j'\colon V \to A'$ satisfying the universal property, we get a commutative diagram

$$\begin{tikzcd}
V \arrow[r, "j'"] \arrow[d, "j"] & A' \arrow[dl, dashed, bend left, "J"] \\
\Lambda V, \arrow[ur, dashed, "J'"] &
\end{tikzcd}$$

where J extends j and J' extends j'. It follows $j = J \circ J' \circ j$ and $j' = J' \circ J \circ j'$, meaning $J \circ J'$ extends j over itself, while $J' \circ J$ extends j' over itself. However, so do the corresponding identities, and since the induced morphisms in the universal property are unique, we get $J \circ J' = \mathrm{id}_{\Lambda V}$ and $J' \circ J = \mathrm{id}_{A'}$.

Proposition 3.1.5. *Suppose V, W are free graded modules. Then there exists a unique isomorphism $\Lambda(V \oplus W) \cong \Lambda V \otimes \Lambda W$ that is an isomorphism between CGAs. Moreover, if precomposed with the canonical injections $V \to \Lambda(V \oplus W)$, $W \to \Lambda(V \oplus W)$ this yields the corresponding identity on V or W.*

Proof. Besides the inclusion $j\colon V \oplus W \hookrightarrow \Lambda(V \oplus W)$ we may consider the morphism

$$j'\colon V \oplus W \to \Lambda V \otimes \Lambda W, \quad (v, w) \mapsto v \otimes 1 + 1 \otimes w.$$

If we show that $(\Lambda V \otimes \Lambda W, j')$ too suffices the universal property of $(\Lambda(V \oplus W), j)$, remark 3.1.4 above yields the asserted isomorphism J'. Thus, suppose $f\colon V \oplus W \to A$ is a morphism of graded modules, where A is a commutative graded algebra. The restrictions $f|_V\colon V \to A$ and $f|_W\colon W \to A$ given by $f|_V(v) := f(v, 0)$ and $f|_W(w) := f(0, w)$ induce unique morphisms $F_V\colon \Lambda V \to A$ and $F_W\colon \Lambda W \to A$ by the universal properties of ΛV and ΛW. Through these, we may define

$$F := F_V \otimes F_W\colon \Lambda V \otimes \Lambda W \to A, \quad F(a \otimes b) = F_V(a)F_W(b).$$

It follows that $F(v \otimes 1 + 1 \otimes w) = f|_V(v) + f|_W(w) = f(v, w)$ and thus $F \circ j' = f$. For the uniqueness of F, assume $G\colon \Lambda V \otimes \Lambda W \to A$ is a morphism such that $f = G \circ j'$. If we define morphisms $G_V\colon \Lambda V \to A$, $G_V(a) := G(a \otimes 1)$ and $G_W\colon \Lambda W \to A$, $G_W(b) := G(1 \otimes b)$ it follows that

$$G_V \otimes G_W(a \otimes b) = G(a \otimes 1)G(1 \otimes b) = G((a \otimes 1)(1 \otimes b)) = G(a \otimes b).$$

Finally, it holds

$$\begin{aligned} G_V(v) &= G(v \otimes 1) = G \circ j'(v, 0) = f(v, 0) = f|_V(v) \\ G_W(w) &= G(1 \otimes w) = G \circ j'(0, w) = f(0, w) = f|_W(w), \end{aligned}$$

so G_V and G_W extend $f|_V$ and respectively $f|_W$ over the natural inclusions $V \hookrightarrow \Lambda V$, $W \hookrightarrow \Lambda W$. It follows that $G_V = F_V$ and $G_W = F_W$ by the uniqueness in the universal properties of ΛV and ΛW. $\square$

The free commutative graded algebra ΛV associated with a graded module V is relevant in the world of CDGAs, too. More specifically, it is fairly easy to construct a differential d on ΛV, making it a CDGA of the form $(\Lambda V, d)$. These play a major role in Sullivan's approach to rational homotopy theory and will occur frequently in the following.

Remark 3.1.6.

(a) A linear map $d\colon V \to \Lambda V$ of degree k extends to a unique derivation $d\colon \Lambda V \to \Lambda V$ of degree k. As in the case of TV, this is done by inductively extending d on $\Lambda^k V$, making it apply to the Leibniz rule. Thus, any linear map $V \to \Lambda V$ of degree 1 defines a unique derivation, where it suffices to specify the map on a homogeneous basis of V.

(b) Let $f\colon \Lambda V \to A$ be a morphism of commutative graded algebras and consider derivations d_V, d_A in ΛV and A respectively. If $f \circ d_V = d_A \circ f$ holds for elements in V, then by the universal property the maps commute on all of ΛV. Similarly, if $d\colon \Lambda V \to \Lambda V$ is a derivation of degree k with $d^2|_V = 0$, then $d^2 = 0$ on all of ΛV.

It follows that in the case of a free commutative graded algebra ΛV, it suffices to specify a linear map d of degree 1 on a basis of V so that $d^2 = 0$. Any such map then extends to a derivation $d\colon \Lambda V \to \Lambda V$ with $d^2 = 0$ on all of ΛV, resulting in a CDGA $(\Lambda V, d)$.

Remark 3.1.7. In particular, a linear map $d\colon V \oplus W \to \Lambda(V \oplus W)$ of degree 1 with $d^2 = 0$ extends over $j\colon V \oplus W \hookrightarrow \Lambda(V \oplus W)$ and induces a differential d on $\Lambda(V \oplus W)$. Moreover, the inclusions

$$V \hookrightarrow V \oplus W \hookrightarrow \Lambda(V \oplus W) \quad \text{and} \quad W \hookrightarrow V \oplus W \hookrightarrow \Lambda(V \oplus W)$$

extend to inclusions $\Lambda V \hookrightarrow \Lambda(V \oplus W)$ and $\Lambda W \hookrightarrow \Lambda(V \oplus W)$. Assuming d restricts to maps $d|_V\colon V \to \Lambda V$ and $d|_W\colon W \to \Lambda W$, these define differentials d_V and d_W in ΛV and ΛW respectively. It follows that the isomorphism J' induced by j' in proposition 3.1.5 is an isomorphism of commutative cochain algebras $(\Lambda(V \oplus W), d) \cong (\Lambda V \otimes \Lambda W, d_V \otimes d_W)$, since

$$d_V \otimes d_W(j'(v,w)) = d_V(v) \otimes 1 + 1 \otimes d_W(w) = J'(d(v,0) + d(0,w)) = J'(d(v,w)).$$

Example 3.1.8. Consider for any $n \in \mathbb{N}_0$ the free commutative graded algebra

$$(A_P)_n := \Lambda(t_0, \ldots, t_n, dt_0, \ldots, dt_n)$$

with basis elements t_i of degree zero and dt_i of degree one. With regard to proposition 3.1.5, we get $(A_P)_n = \Lambda(V_0 \oplus V_1) = \Lambda V_0 \otimes \Lambda V_1$, where V_0 and V_1 indicate the free modules generated by the elements t_i and dt_i respectively. Note that ΛV_0 is the polynomial ring over $\mathbb{K}$ in $n+1$ variables, so we get $(A_P)_n = \mathbb{K}[t_0, \ldots, t_n] \otimes \Lambda V_1$. In particular, for $n = 0$ we get

$$(A_P)_0 = \Lambda(t, dt) = \left\{ \sum_{i=0}^{k} \alpha_i t^i + \sum_{j=0}^{l} \beta_j t^j dt \mid k, l \in \mathbb{N},\ \alpha_i, \beta_j \in \mathbb{K} \right\}.$$

We can define a linear map $d\colon (A_P)_n \to (A_P)_n$ of degree 1 by setting $d(t_i) := dt_i$ and $d(dt_i) := 0$ for all $0 \leq i \leq n$, which extends to a unique derivation on $(A_P)_n$, making $((A_P)_n, d)$ a commutative cochain algebra.

Remark 3.1.9. This free commutative graded algebra will play a rather central role in the upcoming constructions. Therefore, just to get a better feeling for the object, let us do a quick calculation of its cohomology in the case that $\mathbb{K}$ is a field of characteristic zero. Start with $\Lambda(t, dt)$ and note that for any $k \geq 1$, we see by induction and the Leibniz rule that $d(t^k) = kt^{k-1}dt$. Hence, for the differential of an arbitrary element of degree zero it holds

$$d\left(\sum_{k=0}^{n} \alpha_k t^k\right) = \sum_{k=1}^{n} k\alpha_k t^{k-1} dt.$$

As $\mathbb{K}$ has characteristic zero, this equals zero if and only if $\alpha_k = 0$ for $k \geq 1$, so $\alpha_0 \in \mathbb{K}$ may be arbitrary. Hence, we obtain $H^0(\Lambda(t, dt)) = \mathbb{K}$ as the kernel of the differential.
On elements of degree one the differential always vanishes, as $dt^2 = 0$. However, it holds

$$d\bigg(\sum_{k=0}^{n} \frac{\beta_k}{k+1} t^{k+1}\bigg) = \sum_{k=0}^{n} \beta_k t^k dt,$$

thus any element of degree one is the image of an element of degree zero under the differential and hence a coboundary. It follows $H^1(\Lambda(t, dt)) = 0$. In degrees higher than one the cohomology vanishes trivially, since there are no elements of degree greater than one. This shows $H^*(\Lambda(t, dt)) = \mathbb{K}$, where we consider $\mathbb{K}$ as differential graded algebra concentrated in degree zero, with trivial differential.
Lastly, we extend this calculation to general n, using the isomorphism $(A_P)_n \cong \otimes\Lambda_{i=0}^{n}\Lambda(t_i, dt_i)$ from proposition 3.1.5. Since homology respects the tensor product, see proposition 1.1.11, this immediately yields $H^*((A_P)_n) = \mathbb{K}$.

3.2 Polynomial Differential Forms

On the singular cochain complex $CS^*(X; \mathbb{K})$ of a topological space X there exists a product structure which, on the level of the singular cohomology of X, is called the *cup product*. While the cup product makes $H^*(X; \mathbb{K})$ a commutative graded algebra, the initial product structure on cochain level is generally not commutative. This means that we are likely to run into trouble when attempting to replace $CS^*(X; \mathbb{K})$ by a simpler cochain complex whose homology coincides with the singular cohomology of X. For such model would not be commutative, whereas we require its homology to be a commutative graded algebra. As it turns out, if we assume $\mathbb{K}$ to be a field of characteristic zero, there is a commutative alternative to $CS^*(X; \mathbb{K})$. Inspired by the de Rham complex $A_{DR}(M)$ of a smooth manifold, which happens to be commutative and for which $H(A_{DR}(M)) \cong H^*(X; \mathbb{R})$ holds, in [26] Sullivan introduced a functor which, given some space X, returns a commutative cochain algebra $A_{PL}(X; \mathbb{K})$ for which a natural isomorphism

$$H(A_{PL}(X; \mathbb{K})) \cong H^*(X; \mathbb{K})$$

exists. While $A_{PL}(X; \mathbb{K})$ itself still is quite complicated, we will see soon that it allows for well-behaved algebraic models.
This section begins with a short introduction of simplicial objects and a quick reminder of the singular (co)chain complex and will then proceed with the construction of $A_{PL}(X; \mathbb{K})$.

Definition 3.2.1. (Simplicial objects)
A *simplicial object* K is a sequence of objects $(K_n)_{n\geq 0}$ in a specific category, together with morphisms

$$\begin{aligned} \partial_i &\colon K_{n+1} \to K_n, \quad && 0 \leq i \leq n+1, \\ s_j &\colon K_n \to K_{n+1}, \quad && 0 \leq j \leq n \end{aligned}$$

in the corresponding category such that it holds:

$$\begin{aligned} \partial_i \circ \partial_j &= \partial_{j-1} \circ \partial_i, \quad && \text{for } i < j, \\ s_j \circ s_j &= s_{j+1} \circ s_i, \quad && \text{for } i \leq j, \\ \partial_i \circ s_j &= \begin{cases} s_{j-1} \circ \partial_i, & \text{for } i < j, \\ \mathrm{id}, & \text{for } i = j, \text{ or } i = j+1, \\ s_j \circ \partial_{i-1}, & \text{for } i > j+1. \end{cases} \end{aligned}$$

We will sometimes refer to these as *face and degeneracy maps.*
A *simplicial morphism* $\varphi\colon K \to L$ between two simplicial objects (of the same category) is a

sequence of morphisms $\varphi_n\colon K_n \to L_n$ commuting with the face and degeneracy maps. We write $\mathrm{Hom}_{\mathrm{SIM}}(K,L)$ for the space consisting of all simplicial morphisms between K and L.

$$\begin{tikzcd}
\dots \arrow[r, shift left] & K_n \arrow[l, shift left] \arrow[r, shift left, "s_j^K"] \arrow[d, "\varphi_n"] & K_{n+1} \arrow[l, shift left, "\partial_i^K"] \arrow[r, shift left] \arrow[d, "\varphi_{n+1}"] & \dots \arrow[l, shift left] \\
\dots \arrow[r, shift left] & L_n \arrow[l, shift left] \arrow[r, shift left, "s_j^L"] & L_{n+1} \arrow[l, shift left, "\partial_i^L"] \arrow[r, shift left] & \dots \arrow[l, shift left]
\end{tikzcd}$$

In the following, we will call a simplicial object in the category of sets a *simplicial set*. Similarly, a *simplicial cochain complex* is a simplicial object in the category of cochain complexes. Therefore, a simplicial cochain complex is a family $(V_n, d_n)_{n\geq 0}$ of cochain complexes, so each $V_n = (V_n^k)_{k\geq 0}$ is a graded module with differential $d_n = (d_n^k)_{k\geq 0}$, $d_n^k\colon V_n^k \to V_n^{k+1}$. As usual, the grading on each module V_n will be indicated by uppercase letters. The corresponding face and degeneracy maps $\partial_i\colon V_{n+1} \to V_n$, $s_j\colon V_n \to V_{n+1}$ each consist of families of linear maps $(\partial_i^k)_{k\geq 0}$, $\partial_i^k\colon V_{n+1}^k \to V_n^k$ and $(s_j^k)_{k\geq 0}$, $s_j^k\colon V_n^k \to V_{n+1}^k$ commuting with the differentials.
A family of cochain algebras $(A_n, d_n)_{n\geq 0}$ together with the face and degeneracy maps that are algebra homomorphisms is called a *simplicial cochain algebra*. Hence, when talking about a simplicial cochain complex, or a simplicial cochain algebra $(A_n)_{n\geq 0}$ one should have in mind the following commutative diagram

$$\begin{tikzcd}
A_0^0 \arrow[r, shift left] \arrow[d, "d_0^0"] & A_1^0 \arrow[l, shift left] \arrow[r, shift left, "s_j^0"] \arrow[d, "d_1^0"] & A_2^0 \arrow[l, shift left, "\partial_i^0"] \arrow[r, shift left] \arrow[d, "d_2^0"] & \dots \arrow[l, shift left] \\
A_0^1 \arrow[r, shift left] \arrow[d, "d_0^1"] & A_1^1 \arrow[l, shift left] \arrow[r, shift left, "s_j^1"] \arrow[d, "d_1^1"] & A_2^1 \arrow[l, shift left, "\partial_i^1"] \arrow[r, shift left] \arrow[d, "d_2^1"] & \dots \arrow[l, shift left] \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

A simplicial morphism between two simplicial cochain complexes can then be visualized as a map between two such grids, such that in any square the occurring maps commute.

Remark 3.2.2. Note that each line in the diagram above itself defines a simplicial object: for a simplicial cochain algebra A and fixed $k \geq 0$, the family $A^k := (A_n^k)_{n\geq 0}$ with the corresponding face and degeneracy maps is again a simplicial object in the category of modules.

Example 3.2.3. Remember that for $n \geq 0$ the set of singular n-simplices as introduced in preparation 1.1.15. One can show that the face and degeneracy maps $\partial_i\colon S_{n+1}(X) \to S_n(X)$, $\partial_i(\sigma) := \sigma \circ \delta_i$ and $s_j\colon S_n \to S_{n+1}(X)$, $s_j(\sigma) := \sigma \circ \rho_j$ suffice the identities in the definition of a simplicial object, making $S(X) := \{S_n(X)\}_{n\geq 0}$ a simplicial set. Moreover, a continuous map $f\colon X \to Y$ induces $S(f)\colon S(X) \to S(Y)$ such that S is a functor from the category of topological spaces to the category of simplicial sets.

Henceforth, we assume $\mathbb{K}$ to be a field of characteristic zero.

Example 3.2.4. Let $n \in \mathbb{N}_0$ and consider the commutative cochain algebra $(A_P, d)_n$ from example 3.1.8. Let I_n be the ideal that is generated by the two elements $\sum_{i=0}^n t_i - 1$ and $\sum_{i=0}^n dt_i$. As $d(I_n) \subseteq I_n$ we get an induced derivation on the quotient space

$$(A_{PL})_n := \Lambda(t_0, \dots, t_n, dt_0, \dots, dt_n)/I_n$$

which we will simply denote by d and for which $d(t_i) = dt_i$ and $d(dt_i) = 0$ holds, as before. This makes $(A_{PL})_n$ into a CDGA for each $n \geq 0$. We then define for $0 \leq i \leq n+1$ and $0 \leq j \leq n$ the maps $\partial_i\colon (A_{PL})_{n+1} \to (A_{PL})_n$, $s_j\colon (A_{PL})_n \to (A_{PL})_{n+1}$ as morphisms between DGAs given by

$$\partial_i(t_k) := \begin{cases} t_k & \text{for } k < i, \\ 0 & \text{for } k = i, \\ t_{k-1} & \text{for } k > i, \end{cases} \qquad s_j(t_k) := \begin{cases} t_k & \text{for } k < j, \\ t_k + t_{k+1} & \text{for } j = k, \\ t_{k+1} & \text{for } k > j. \end{cases}$$

Note that it suffices to define the maps on the basis elements t_k, as any morphism of DGAs commutes with the differential by definition, and they are compatible with the ideal.
This turns $A_{PL} := \{(A_{PL})_n\}_{n\geq 0}$ into a simplicial cochain algebra.

Remark 3.2.5. $(A_{PL})_n$ is free commutative, which can be seen by the fact that the inclusions of the basis elements t_i and dt_i for $i \geq 1$ extend to an isomorphism of commutative cochain algebras

$$(A_P)_{n-1} = (\Lambda(t_1, \dots, t_n, dt_1, \dots, dt_n), d) \xrightarrow{\cong} (A_{PL})_n.$$

The inverse is obtained by the morphism that maps $t_0 \mapsto -(\sum_{i=1}^n t_i - 1)$ and $dt_0 \mapsto -\sum_{i=1}^n dt_i$ as well as $t_i \mapsto t_i$ and $dt_i \mapsto dt_i$ for $i \geq 1$, then passing on to the quotient.

If we assume $\mathbb{K} \subseteq \mathbb{R}$, with reference to example 3.1.8 we obtain

$$(A_{PL})_n^0 = \mathbb{K}[t_0, \dots, t_n]/(\sum t_i - 1).$$

This means the elements of $(A_{PL})_n$ of degree zero can be viewed as the subalgebra of the smooth functions $C^\infty(\Delta^n)$ generated by the restrictions t_i of the coordinate functions of $\mathbb{R}^{n+1}$. This allows to identify $(A_{PL})_n$ as a sub cochain algebra of the de Rham algebra $(A_{DR})(\Delta^n)$. Hence, we call elements of $(A_{PL})_n^k$ the *polynomial degree k differential forms* on Δ^n with coefficients in $\mathbb{K}$.

Remark 3.2.6. To investigate A_{PL} a little bit, note that it holds $(A_{PL})_0 = \mathbb{K}$ and $H((A_{PL})_n) = \mathbb{K}$ for each $n \geq 0$, concentrated in degree zero. While the first claim is immediate from the definition, the second follows from remark 3.1.9 and the isomorphism above.

In order to use A_{PL} to define a functor from the category of topological spaces to the category of cochain algebras, we will now introduce a more general construction that provides us with a cochain algebra whenever we specify a simplicial set and a simplicial cochain algebra.

Construction 3.2.7. Suppose we are given a simplicial set $K := (K_n)_{n\geq 0}$ and a simplicial cochain algebra $A := (A_n)_{n\geq 0}$. Then the family

$$A(K) := \{A^k(K)\}_{k\geq 0}$$

consisting of the spaces $A^k(K) := \mathrm{Hom}_{\mathrm{SIM}}(K, A^k)$ is a cochain algebra. A homogeneous element $\varphi \in A^k(K)$ is a family of maps $\varphi_n\colon K_n \to A_n^k$ commuting with the face and degeneracy maps, i.e.

$$\varphi_{n-1} \circ \partial_i^K(\sigma) = \partial_i^A \circ \varphi_n(\sigma) \quad \text{and} \quad \varphi_{n+1} \circ s_j^K(\sigma) = s_j^A \circ \varphi_n(\sigma)$$

for each $\sigma \in K_n$. In the following, the subscription on φ_n will be dropped.
The vector space structure is defined as follows:

1. For $\varphi, \psi \in A^k(K)$ and $\lambda \in \mathbb{K}$, we set $(\varphi + \psi)(\sigma) := \varphi(\sigma) + \psi(\sigma)$ and $(\lambda\varphi)(\sigma) := \lambda\varphi(\sigma)$. This way, the addition and scalar multiplication in A induce an graded vector space structure on $A(K)$.

2. Similarly, for two elements $\varphi \in A^k(K)$ and $\psi \in A^p(K)$, a multiplication is defined by $(\varphi\psi)(\sigma) := \varphi(\sigma)\psi(\sigma)$.

3. A differential in $A(K)$ is given by the differential in A. Let $\varphi \in A^k(K)$ and

$$d^k := (d_n^k)_{n\geq 0}\colon A^k \to A^{k+1}$$

the family of maps that is defined by the differentials $d_n^k\colon A_n^k \to A_n^{k+1}$. Then $d(\varphi) := d^k \circ \varphi$ defines a differential in $A(K)$.

4. Moreover, whenever $f\colon A \to B$ is a morphism of simplicial cochain algebras, then there is an induced morphism $f(K)\colon A(K) \to B(K)$ of DGAs that is $f(K)(\varphi) := f \circ \varphi$.

5. Further, a map $g\colon K \to L$ that is a morphism of simplicial sets induces a morphism

$$A(g)\colon \begin{cases} A(L) & \to A(K) \\ \varphi & \mapsto \varphi \circ g \end{cases}$$

of cochain algebras.

This makes $A(K)$ into a cochain algebra. Note that by definition it holds that $A(K)$ is a commutative cochain algebra whenever the multiplication that is given on each A_n is graded commutative. When we want to stress the underlying coefficient field $\mathbb{K}$, we will write $A(K;\mathbb{K})$ instead of $A(K)$, and usually drop the differential in the notation.
Observe that this construction also works for simplicial cochain complexes A and will in this case provide a cochain complex $A(K)$.

Remark 3.2.8.

(a) The construction of $A(K)$ provides both a covariant functor in A and a contravariant functor in K. Given a topological space X, we write $A(X)$ as abbreviation for $A(S(X))$ and analogously $A(f)\colon A(Y) \to A(X)$ instead of $A(S(f))$ when $f\colon X \to Y$ is a continuous map between topological spaces. Thus, for a fixed simplicial cochain algebra A, we have now established a functor from the category of topological spaces to the category of (C)DGAs that is just the composition of A and S (and, by abuse of language, is again denoted by A).

(b) As an important special case of this construction, consider the simplicial cochain algebra A_{PL} and any topological space X. The obtained commutative cochain algebra $A_{PL}(X)$ is called the algebra of *polynomial differential forms on X with coefficients in* $\mathbb{K}$. A homogeneous element $\varphi \in A_{PL}^k(X)$ is a simplicial map assigning each singular n-simplex of X a polynomial k-form on Δ^n, where $n \geq 0$.

(c) It can be shown that an inclusion $X \hookrightarrow Y$ induces a surjection $A_{PL}(Y) \to A_{PL}(X)$. Thus, for any pair (Y, X) there exists a short exact sequence

$$0 \longrightarrow A_{PL}(Y,X) \longrightarrow A_{PL}(Y) \longrightarrow A_{PL}(X) \longrightarrow 0.$$

Example 3.2.9. For $X = \{\mathrm{pt}\}$ it follows that $A_{PL}(X) = (A_{PL})_0 = \mathbb{K}$.

The commutative cochain algebra $A_{PL}(X;\mathbb{K})$ plays a major role in rational homotopy theory. Its importance originates from the following statement.

Theorem 3.2.10. *Let X be a topological space and $C^*(X;\mathbb{K})$ the associated singular cochain algebra with coefficients in $\mathbb{K}$. There exists a natural chain of quasi-isomorphisms of cochain algebras*

$$C^*(X;\mathbb{K}) \xrightarrow{\simeq} D(X) \xleftarrow{\simeq} A_{PL}(X;\mathbb{K}),$$

where $D(X)$ is a third natural cochain algebra. This induces a natural isomorphism $H^*(X;\mathbb{K}) \cong H(A_{PL}(X;\mathbb{K}))$ and we may identify $H^*(f;\mathbb{K}) = H(A_{PL}(f;\mathbb{K}))$ for any continuous map f.

Proof. A detailed proof of this statement can be found in chapter 10 of [7]. $\square$

This allows us to replace $C^*(X)$ by the cochain algebra $A_{PL}(X)$, whose advantage is that it is commutative. That being said, $A_{PL}(X)$ is in general quite complicated, so we will frequently make use of simpler commutative cochain algebras that work as models. To do so, remember that two simply connected spaces X and Y share the same rational homotopy type if they are connected by a chain

$$X \leftarrow Z(0) \rightarrow \cdots \leftarrow Z(n) \rightarrow Y$$

of rational homotopy equivalences, that are continuous maps inducing an isomorphism on cohomology with coefficients in $\mathbb{Q}$. Now, when we apply the functor $A_{PL}(-)$, we get a chain

$$A_{PL}(X) \xrightarrow{\simeq} A_{PL}(Z(0)) \xleftarrow{\simeq} \dots \xrightarrow{\simeq} A_{PL}(Z(n)) \xleftarrow{\simeq} A_{PL}(Y)$$

of commutative cochain algebras, with the morphisms being quasi-isomorphisms due to theorem 3.2.10. This implies that $A_{PL}(X)$ and $A_{PL}(Y)$ are weakly equivalent, as defined in 1.2.16. In other words, $X \simeq_{\mathbb{Q}} Y$ implies $A_{PL}(X;\mathbb{Q}) \simeq A_{PL}(Y;\mathbb{Q})$, so the weak equivalence class of $A_{PL}(X;\mathbb{Q})$ is invariant under the rational homotopy type of X. This motivates the approach of replacing $A_{PL}(X)$ by a simpler, weakly equivalent commutative cochain algebra.

Definition 3.2.11. A *commutative model* for a topological space X is a commutative cochain algebra (A,d) together with a weak equivalence

$$(A,d) \xrightarrow{\simeq} \dots \xleftarrow{\simeq} A_{PL}(X;\mathbb{K}).$$

Example 3.2.12. For a smooth manifold M, consider the associated commutative cochain algebra $A_{DR}(M)$ that is the de Rham complex. Then $A_{PL}(M;\mathbb{R})$ and $A_{DR}(M)$ are weakly equivalent. Thus, $A_{DR}(M)$ is a commutative model for M and we obtain natural isomorphisms

$$H^*(M;\mathbb{R}) \cong H(A_{PL}(M;\mathbb{R})) \cong H(A_{DR}(M)).$$

3.3 Sullivan Models

As we have seen at the end of the preceding chapter, the weak equivalence class of $A_{PL}(X;\mathbb{Q})$ is an invariant for the rational homotopy type of the simply connected space X. That is, if $X \simeq_{\mathbb{Q}} Y$, then $A_{PL}(X;\mathbb{Q}) \simeq A_{PL}(Y;\mathbb{Q})$. Thus, it would suffice to find commutative models for $A_{PL}(X;\mathbb{Q})$ and $A_{PL}(Y;\mathbb{Q})$ which are not weakly equivalent in order to distinguish the rational homotopy types of X and Y. Motivated by this, we will now study commutative cochain algebras themselves in order to find a model as simple as possible, which leads us to the introduction of Sullivan models. Through the application of $A_{PL}(-)$, we can then use these as models for simply connected spaces. What makes Sullivan models so powerful is that they contain most, in many cases even all the information on the rational homotopy type of X, while still being very computable.

Definition 3.3.1. A commutative cochain algebra (A,d) is called a *Sullivan algebra* if it is a free commutative graded algebra, meaning $(A,d) = (\Lambda V, d)$ for some vector space V, such that

(a) $V = V^+$,

(b) $V = \bigcup_{k=0}^{\infty} V(k)$ for an increasing sequence of graded subspaces $V(0) \subseteq V(1) \subseteq \dots$,

(c) $d|_{V(0)} = 0$ and $d(V(k)) \subseteq \Lambda V(k-1)$ for all $k \geq 1$.

Definition 3.3.2. Let X be a path connected topological space and (A, d_A) a commutative cochain algebra. A *Sullivan model for* (A, d_A) is a Sullivan algebra $(\Lambda V, d_V)$ that comes with a quasi-isomorphism

$$m\colon (\Lambda V, d_V) \xrightarrow{\simeq} (A, d_A).$$

By a *Sullivan model for* X, we mean a Sullivan model for $A_{PL}(X; \mathbb{K})$. A Sullivan algebra or Sullivan model is called *minimal* if

$$\operatorname{im}(d) \subseteq \Lambda^{\geq 2} V$$

holds, where $\Lambda^{\geq 2} V := \oplus_{k \geq 2} \Lambda^k V$.

Note that we may frequently drop the subscript on differentials, simply denoting them by d. It will then be clear from the context which cochain algebra they belong to.

Example 3.3.3.

(a) The cochain algebra $(\Lambda(v_1, v_2, v_3), d)$, where $|v_i| = 1$ and $d(v_1) = v_2 \wedge v_3$, $d(v_2) = v_1 \wedge v_3$ and $d(v_3) = v_1 \wedge v_2$ is not a Sullivan algebra, as there does not exist a sequence of subspaces $V(k)$ satisfying property (c). If there was, we would necessarily have a cocycle of degree 1.

(b) In contrast, any cochain algebra $(\Lambda V, d)$ with $V = V^{\geq 2}$ and $\operatorname{im}(d) \subseteq \Lambda^{\geq 2} V$ is automatically a minimal Sullivan algebra. An increasing sequence is defined by $V(k) := V^{\leq k}$. For a homogeneous element $v \in V^{\leq k}$, the element $d(v)$ is a linear combination of homogeneous basis elements of the form $v_1 \wedge \cdots \wedge v_l$ with $\sum_j |v_j| = |v| + 1 \leq k + 1$. The minimality condition implies $l \geq 2$, so in fact $|v_j| \leq k - 1$ for all $1 \leq j \leq l$. This means $d(v) \in \Lambda V^{\leq k-1}$.

(c) The other way around, for a minimal Sullivan algebra $(\Lambda V, d)$ with $H^1(\Lambda V, d) = 0$ it always holds that $V = V^{\geq 2}$. Let $V = \bigcup_{k \geq 0} V(k)$ and note that $V^1(0) = 0$ since otherwise $d|_{V(0)} = 0$ would imply that $H^1(\Lambda V, d)$ does not vanish. Inductively assume $V^1(k-1) = 0$, then the condition $d(V(k)) \subseteq \Lambda^{\geq 2} V(k-1)$ implies $d|_{V^1(k)} = 0$. However, since the Sullivan algebra is minimal, there are no coboundaries of word length one, so in particular no element in $V^1(k)$ is a coboundary. Once again, $H^1(\Lambda V, d) = 0$ implies that $V^1(k) = 0$, thus by induction $V^1 = 0$.

Remark 3.3.4.

(a) Remember that by the universal property of the free commutative graded algebra, it suffices to define the differential $d\colon \Lambda V \to \Lambda V$ and morphisms $\varphi\colon \Lambda V \to A$ on the vector space V. The image of an element $v_1 \wedge \cdots \wedge v_k \in \Lambda^k V$ of word length k is then determined by the product of the elements $\varphi(v_i)$ or, in case of the differential, by the Leibniz rule of d. Since $\Lambda V = \oplus_{k \geq 0} \Lambda^k V$, we then know φ or respectively d on all of ΛV.

(b) Let $v \in V$. Then $d(v) \in \Lambda V$ is a linear combination of elements $v_1 \wedge \cdots \wedge v_k$, where the word length k may vary. We can decompose $d|_V$ into a sum of maps $d_i\colon V \to \Lambda^{i+1} V$, $i \geq 0$, where each $d_i(v)$ only contains the elements of the linear combination of word length $i+1$. Then $d|_V = d_0 + d_1 + d_2 + \dots$ and d_i raises word length by i. Using the universal property, each d_i extends to a derivation on ΛV that we again denote by d_i and the sum $d = d_0 + d_1 + d_2 + \dots$ decomposes the differential on all of ΛV. The derivation d_0 is then called the *linear part* and d_1 is called the *quadratic part* of the differential d. In this context, the minimality condition of a Sullivan algebra exactly means that $d_0 = 0$.

(c) Suppose $(\Lambda V, d)$ is a minimal Sullivan algebra, so $d = d_1 + d_2 + \dots$. Then d_1^2 raises word length by exactly 2 and $d^2 - d_1^2$ raises word length by at least 3. However, $d^2 = 0$, so we get a contradiction unless $d_1^2 = 0$. This implies d_1 is a differential, and thus $(\Lambda V, d_1)$ is again a minimal Sullivan algebra.

Proposition 3.3.5. *Let (A, d_A) be a commutative cochain algebra. If $H^0(A) = \mathbb{K}$, then there exists a Sullivan model*

$$m\colon (\Lambda V, d_V) \xrightarrow{\simeq} (A, d_A).$$

Proof. We will inductively construct graded vector spaces V_k, $k \geq 0$ such that $V := \oplus_k V_k$ has the desired properties. By the universal property, it suffices to define d and m on the graded vector space V. We will start by setting $d = 0$ on V_0 and then extend d inductively by defining it on each V_k. In doing so, we will see that it actually holds

$$d(V_k) \subseteq \Lambda(\oplus_{i=0}^{k-1} V_i).$$

To begin with, note that $H^+(A)$ is itself a graded vector space concentrated in positive degrees. Hence, we can just define V_0 to be a copy of $H^+(A)$, which we will do in a way that $V_0 \cong H^+(A)$ is induced by a map $m_0\colon V_0 \to A$. More precisely, for a homogeneous basis $(\alpha_i)_{i\in I}$ of $H^+(A)$ let $(v_i)_{i\in I}$ be a replication of this basis, so we want to give each v_i the same degree α_i has. Then V_0 is set to be the graded vector space that is generated by this basis $(v_i)_{i\in I}$. Now for each $\alpha_i \in H^+(A)$ choose a representative, meaning a cocycle $a_i \in A^+$ such that $[a_i] = \alpha_i$. To define the map m_0, let $m_0(v_i) := a_i$ on the basis and then take the unique linear extension on V_0, which is then a linear map of degree zero between graded vector spaces. Finally, we can consider m_0 as morphism between complexes, $m_0\colon (V_0, 0) \to (A^+, d)$ and by construction it holds that $H(m_0)\colon H(V_0) = V_0 \to H^+(A)$ is an isomorphism.
Now, whereas $V_0 = V_0^{\geq 1}$, meaning in particular $V_0^0 = 0$, we have $(\Lambda V_0)^0 = \mathbb{K}$. This is due to the fact that for any element $v \in \Lambda^k V_0$ with $k \geq 1$ it holds $|v| > 0$. Using the unique property of free commutative graded algebras, we can again extend m_0 to get a morphism between commutative cochain algebras,

$$m_0\colon (\Lambda V_0, 0) \to (A, d).$$

The induced map $H(m_0)$ is both injective and surjective in degree 0, as $m_0(1_{\mathbb{K}}) = 1_A \in A^0$ and $H^0(A) = \mathbb{K}$ by assumption. Surely it is also surjective in positive degrees, as the restriction to V_0 is even an isomorphism $V_0 \cong H^+(A)$. Hence, we get a surjection

$$H(m_0)\colon H(\Lambda V_0) = \Lambda V_0 \to H(A).$$

In fact, this is the only point in the proof where the assumption $H^0(A) = \mathbb{K}$ is needed.
For the inductive step, assume m_0 is already extended to a morphism $m_k\colon (\Lambda V(k), d) \to (A, d)$ for some $k \geq 0$, where we set

$$V(k) := \bigoplus_{i=0}^{k} V_i.$$

Then $\ker(H(m_k)) \subseteq H(\Lambda V(k))$ is a graded subspace with some basis $(\beta_j)_{j\in J}$. Choose cocycles $z_j \in \Lambda V(k)$ such that $[z_j] = \beta_j$ for each $j \in J$. By construction $H(m_k)$ is injective in degree 0, so $|z_j| \geq 1$. As before, let $(v_j)_{j\in J}$ be a copy of this basis, where this time we set $|v_j| := |z_j| - 1$ for the degree of each v_j. We can then consider the graded vector space V_{k+1} that is generated by this copy.
In order to extend d and m_k on $\Lambda V(k+1)$, by the universal property, it suffices to define both maps on $V(k+1)$. Since they are already defined on $V(k)$, we only need to construct them on V_{k+1}, wherefore it is enough to specify both of them on the basis elements v_j.
We start with d and set $d(v_j) := z_j$. This extends to a linear map of degree 1 on V_{k+1} with $d(V_{k+1}) \subseteq \Lambda V(k)$. It holds $d^2(v_j) = d(z_j) = 0$ since z_j is a cocycle. Hence $d^2 = 0$ on V_{k+1}, and with that on $V(k+1)$ by the induction hypothesis. This extends to a unique differential on $\Lambda V(k+1)$ that is also a derivation by the properties states in remark 3.1.6.

Now for m_k, we want to obtain an extension m_{k+1} on $V(k+1)$. By choice of the z_j, we get $0 = H(m_k)(\beta_j) = [m_k(z_j)]$, so $m_k(z_j)$ is a coboundary. Hence, for each $j \in J$ there exists $a_j \in A$ such that $m_k(z_j) = d(a_j)$. By setting $m_{k+1}(v_j) := a_j$, it follows that

$$m_{k+1}(d(v_j)) = m_{k+1}(z_j) = d(a_j) = d(m_{k+1}(v_j)).$$

Thus d and m_{k+1} commute on V_{k+1} and they do so on $V(k)$ by induction hypothesis, hence we get $m_{k+1} \circ d = d \circ m_{k+1}$ on all of $V(k+1) = V(k) \oplus V_{k+1}$. Using the universal property as seen in remark 3.1.6, we get a morphism between commutative cochain algebras,

$$m_{k+1} \colon (\Lambda V(k+1), d) \to (A, d).$$

With this we have inductively constructed $m\colon (\Lambda V, d) \to (A, d)$, where $V = \oplus_{k \geq 0} V_k$. Here, m is the extension of the map on V that is given by m_k on V_k for each $k \geq 0$. Since the induced map of the restriction $m|_{\Lambda V_0} = m_0$ is surjective, $H(m)$ is surjective as well. For injectivity, assume $H(m)([z]) = 0$ for some $z \in \Lambda V$. There exists $k \geq 0$ such that $z \in \Lambda V(k)$, so $m(z) = m_k(z)$ and hence $H(m_k)([z]) = 0$. By construction, we find $v \in V_{k+1}$ such that $d(v) = z$, meaning z is a coboundary in $\Lambda V(k+1)$. It follows that $[z] = 0$ in $H(\Lambda V(k+1))$ which implies the same is true for $H(\Lambda V)$. All in all, $H(m)$ is an isomorphism, and m is a quasi-isomorphism as claimed.
It remains to show that $(\Lambda V, d)$ is a Sullivan algebra. Note that the properties (b) and (c) of definition 3.3.1 hold by construction and the definition of $V(k)$. It remains to show that V is concentrated in positive degrees, which follows if the same holds for all V_k, $k \geq 0$. We have already seen that this is true for V_0, so inductively assume that V_i vanishes in degree zero for all $0 \leq i \leq k$. Then for a homogeneous element $v \in \Lambda V(k)$ with $|v| = 1$ it holds that v has word length 1, since there are no elements of degree zero. This means $v \in V(k)^1 = \oplus_{i \leq k} V_i^1$, which is the subspace of $\Lambda V(k)$ containing elements of word length 1. We may thus write v as sum

$$v = v_0 + \cdots + v_k, \quad v_i \in V_i^1.$$

Assume now v is a cocycle, so $[v] \in H^1(\Lambda V(k))$ and $d(v) = 0$. It follows that $d(v_k) = d(e)$ for an element $e \in V(k-1)$, so in particular $[d(v_k)] = 0$ in $H(\Lambda V(k-1))$. This contradicts the construction of V_k, which is generated by elements v_j such that $[d(v_j)] = [z_j]$ is a basis of $\ker(H(m_k)) \subseteq H(\Lambda V(k-1))$. We arrive at $v_k = 0$. Repeating this argument, we obtain $v_i = 0$ for $1 \leq i \leq k$, so we find $v = v_0$. It holds

$$H(m_k)([v]) = H(m_k)([v_0]) = H(m_0)([v_0]) = [m_0(v_0)]$$

which is zero if and only if $v = v_0 = 0$ in V_0, as m_0 defines an isomorphism $V_0 \cong H^+(A)$. This means $\ker(H(m_k))$ does not contain non-trivial elements of degree 1 and is thus concentrated in degrees ≥ 2. From the construction we can then deduce that V_{k+1} is concentrated in degrees ≥ 1, concluding the induction. $\square$

Remark 3.3.6. In particular, each path connected space X has a Sullivan model, since

$$H^0(A_{PL}(X; \mathbb{K})) = H^0(X; \mathbb{K}) = \mathbb{K}$$

holds.

Example 3.3.7. Let $k \geq 1$ and consider the sphere $\mathbb{S}^k$. Recall that a choice of a generator $[\mathbb{S}^k] \in H_k(\mathbb{S}^k; \mathbb{Z}) = \mathbb{Z}$ is called a *fundamental class* of $\mathbb{S}^k$. A unique element

$$\omega \in H^k(A_{PL}(\mathbb{S}^k); \mathbb{K}) \cong H^k(\mathbb{S}^k; \mathbb{K}) \cong \mathrm{Hom}(H_k(\mathbb{S}^k; \mathbb{Z}), \mathbb{K})$$

is defined by setting $\omega([\mathbb{S}^k]) := 1$, where the second isomorphism is due to the universal coefficient theorem and the fact that $\mathbb{K}$ is a field. As $H_n(\mathbb{S}^k; \mathbb{Z})$ is trivial for $n \geq 1$, $n \neq k$, it follows that $\{1, \omega\}$

defines a basis of $H(A_{PL}(\mathbb{S}^k))$. We choose a cocycle $\varphi \in A_{PL}^k(\mathbb{S}^k)$ representing ω and distinguish between two cases.
Assume first k is odd and let V be the graded vector space generated by a single element v of degree k. Then $\{1, v\}$ is a basis of $\Lambda V = \Lambda(v)$. A linear map $m\colon V \to A_{PL}(\mathbb{S}^k)$ of degree zero is defined by setting $m(v) := \varphi$ and extends uniquely to a morphism

$$m\colon (\Lambda(v), 0) \xrightarrow{\simeq} A_{PL}(\mathbb{S}^k)$$

of commutative cochain algebras. This is a quasi-isomorphism, as the induced map is mapping the basis $\{1, v\}$ of $H(\Lambda(v), 0) = (\Lambda(v), 0)$ onto the basis $\{1, \omega\}$ of $H(A_{PL}(\mathbb{S}^k))$.
Suppose otherwise k is even. In this case, the map above does not induce an isomorphism, as a basis of $\Lambda(v)$ is given by $\{1, v, v^2, \dots\}$. However, in this case we can conclude that φ^2 has to be a coboundary, as it represents an element in $H^{2k}(A_{PL}(\mathbb{S}^k)) = 0$. Thus, we find $\psi \in A_{PL}(\mathbb{S}^k)$ such that $d(\psi) = \varphi^2$. Now take a second element w of degree $2k-1$ and define a differential on $\Lambda(v, w)$ by $d(v) := 0$ and $d(w) := v^2$. Finally, set $m(w) := \psi$. Since $m(d(v)) = 0 = d(\varphi)$ and $m(d(w)) = d(\psi) = d(m(w))$, we may again extend m to a unique morphism

$$m\colon (\Lambda(v, w), d) \xrightarrow{\simeq} A_{PL}(\mathbb{S}^k)$$

of commutative cochain algebras. We will now check that this is a quasi-isomorphism. To show that a basis of $H(\Lambda(v, w), d)$ is again given by $\{1, [v]\}$, observe first that there are no powers of w in $\Lambda(v, w)$, since its degree is odd. In fact, the only non-trivial elements arise in degree nk through v^n and in degree $(n+1)k - 1$ through $w \wedge v^{n-1}$, where $n \geq 1$. Since the differential at $(\Lambda(v, w))^k$ is trivial, we get $H^k(\Lambda(v, w)) = (\Lambda(v, w))^k$, which is generated by the class of v. Now, inductively we see that $d(v^{n-1}) = 0$ for $n \geq 1$, which implies $d(w \wedge v^{n-1}) = v^{n+1}$. We conclude that

$$d\colon \begin{cases} (\Lambda(v, w))^{(n+1)k-1} & \to (\Lambda(v, w))^{(n+1)k} \\ w \wedge v^{n-1} & \mapsto v^{n+1} \end{cases}$$

is an isomorphism, meaning the groups $H^{(n+1)k-1}(\Lambda(v, w))$ and $H^{(n+1)k}(\Lambda(v, w))$ are trivial for every $n \geq 1$. This shows m is a quasi-isomorphism.

Remark 3.3.8. Note that both commutative cochain algebras from example 3.3.7 are indeed Sullivan algebras. In the first case this is trivial and in the second case it follows easily by denoting the vector space $V(0)$ to be the subspace generated by v and setting $V(1) = V$. Moreover, in both cases the stated Sullivan algebra is minimal, so the construction provides a Sullivan minimal model for the sphere $\mathbb{S}^k$.

The Sullivan model constructed in the proof of proposition 3.3.5 need not be minimal. The advantage of minimal Sullivan models is that they satisfy a certain uniqueness property, as we shall soon see. While one can show the existence of minimal Sullivan models for commutative cochain algebras (A, d_A) with $H^0(A) = \mathbb{K}$, the general proof requires some knowledge of relative Sullivan algebras. However, under the additional assumption that $H^1(A)$ vanishes, there exists a rather nice construction of a minimal Sullivan model, which we shall present now. Besides, this additional assumption does not pose a restriction in our case, since we will be interested in simply connected spaces.

Construction 3.3.9. Suppose (A, d) is a commutative cochain algebra with $H^0(A) = \mathbb{K}$ and $H^1(A) = 0$.

- For every generator of $H^2(A)$, choose a representing cocycle in A^2. Let V^2 be the vector space generated by basis elements of degree 2, one for each such cocycle in A^2. Further, denote by m_2 the map sending the basis elements of V^2 to the corresponding cocycles and extend it to a morphism $m_2\colon (\Lambda V^2, 0) \to (A, d)$ of cochain algebras. Note that the induced map

$$H^2(m_2)\colon H^2(\Lambda V^2) = V^2 \xrightarrow{\cong} H^2(A)$$

is an isomorphism. By default, the map $H^1(m_2)$ is an isomorphism as well, since $H^1(\Lambda V^2) = 0$ by construction and $H^1(A) = 0$ by assumption. Also, $H^3(m_2)$ is trivially injective, since ΛV^2 does not contain any elements of degree 3 (or any odd degree, in fact).

- By induction, we may assume $m_k\colon (\Lambda V^{\leq k}, d) \to (A, d)$ to already be constructed, such that $H^n(m_k)$ is an isomorphism for $n \leq k$ and injective for $n = k+1$.
Supposing $H^{k+1}(m_k)$ is not surjective, for each generator in $\operatorname{coker}(H^{k+1}(m_k)) \subseteq H^{k+1}(A)$, choose a representing cocycle $a_i \in A^{k+1}$, where i is in some index set I. Moreover, choose cocycles $z_j \in (\Lambda V^{\leq k})^{k+2}$ such that the corresponding cohomology classes provide a generator for $\ker(H^{k+2}(m_k)) \subseteq H^{k+2}(\Lambda V^{\leq k})$ and denote the corresponding index set by J. It holds
$$H^{k+1}(A) = \operatorname{im}(H^{k+1}(m_k)) \oplus \bigoplus_{i\in I} \mathbb{K}\cdot[a_i], \qquad \ker(H^{k+2}(m_k)) = \bigoplus_{j\in J} \mathbb{K}\cdot[z_j].$$
Since $[m_k(z_j)]$ is trivial, for each $j \in J$ there exist $b_j \in A^{k+1}$ such that $m_k(z_j) = d(b_j)$. Let v_i, w_j be basis elements with $|v_i| = |w_j| = k+1$, and denote by V^{k+1} the vector space with basis $\{v_i, w_j \mid i \in I, j \in J\}$.
Note that in order to extend d and m_k from $\Lambda V^{\leq k}$ to $\Lambda V^{\leq k+1}$, it suffices to specify both maps on V^{k+1}, so $d\colon V^{k+1} \to (\Lambda V^{\leq k}) \subseteq (\Lambda V^{\leq k+1})$ and $m_{k+1}\colon V^{k+1} \to A$. As usual, since both maps are by induction already defined on $V^{\leq k}$, we obtain d and m_{k+1} on $V^{\leq k+1}$. This induces a derivation on $\Lambda V^{\leq k+1}$ and a corresponding morphism $\Lambda V^{\leq k+1} \to A$, that (by abuse of notation) will again be named d and m_{k+1} respectively. With that said, we set
$$d(v_i) := 0, \quad d(w_j) := z_j \quad \text{and} \quad m_{k+1}(v_i) = a_i, \quad m_{k+1}(w_j) = b_j.$$
By construction, $d^2 = 0$ on V^{k+1}, as $d(w_j) = z_j$ is a cocycle, and on $\Lambda V^{\leq k}$ this holds by induction hypothesis. Thus $d^2 = 0$ on $\Lambda V^{\leq k+1}$. To see that m_{k+1} commutes with the differential, observe it holds $m_{k+1} \circ d(v_i) = 0 = d(a_i) = d \circ m_{k+1}(v_i)$, as a_i is a cocycle in A. Also, $m_{k+1} \circ d(w_j) = m_{k+1}(z_j) = d(b_j) = d \circ m_{k+1}(w_j)$, meaning the following diagram commutes:
$$\begin{tikzcd} V^{k+1} \arrow[r, "d"] \arrow[d, "m_{k+1}"] & (\Lambda V^{\leq k})^{k+2} \arrow[r, "d"] \arrow[d, "m_{k+1}=m_k"] & (\Lambda V^{\leq k})^{k+3} \\ A^{k+1} \arrow[r, "d"] & A^{k+2}. & \end{tikzcd}$$
Thus $m_{k+1} \circ d = d \circ m_{k+1}$ on V^{k+1}, so the same holds by construction on $V^{\leq k+1}$. Employing once again the universal property, we see that $m_{k+1} \circ d = d \circ m_{k+1}$ on all of $\Lambda V^{\leq k+1}$.

Proposition 3.3.10. *Let (A, d) be a commutative cochain algebra such that $H^0(A) = \mathbb{K}$ and $H^1(A) = 0$ and assume $m\colon (\Lambda V, d) \to (A, d)$ to be the morphism constructed above. It holds*

(i) $m\colon (\Lambda V, d) \xrightarrow{\simeq} (A, d)$ is a Sullivan minimal model.

(ii) For $r := \min\{r \geq 1 \mid H^r(A) \neq 0\}$ it holds $V^i = 0$ for all $1 \leq i < r$ and thus $H^r(\Lambda V) = V^r$. Moreover, the induced map $H^r(m)$ is an isomorphism $V^r \cong H^r(A)$.

(iii) Supposing $H^*(A)$ has finite type, it follows V has finite type, too.

Proof. To verify (i), it remains to show that $(\Lambda V, d)$ is a minimal Sullivan algebra and m is a quasi-isomorphism.
For the first fact set $V(k) := V^{\leq k}$ and observe that $d(V^{k+1}) \subseteq \Lambda V^{\leq k}$, so d is a map $V(\leq k+1) \to \Lambda V(k)$. Notice it holds $V = V^{\geq 2}$. Now, an element $v \in (\Lambda V^{\leq k})^{k+2}$ can be written as sum of elements $v_1 \wedge \cdots \wedge v_l$, where $\sum |v_i| = k+2$ and $2 \leq |v_i| \leq k$. Hence for degree reasons, any of the summands has word length $l \geq 2$, so $v \in \Lambda^+ V^{\leq k} \cdot \Lambda^+ V^{\leq k}$. By construction, the differential d

is a linear map of degree 1b and thus maps to $(\Lambda V^{\leq k})^{k+2}$ if restricted to V^{k+1}. We conclude that $(\Lambda V, d)$ is in fact a Sullivan minimal model.
To prove the second fact, we will show by induction that

$$H^i(m_k) \text{ is } \begin{cases} \text{an isomorphism for } i \leq k \text{ and} \\ \text{injective for } i = k+1, \end{cases}$$

whenever $k \geq 2$, which implies m induces an isomorphism on every level. The assertion is true for $k = 2$, so suppose it to be true for some k. By induction, $H^i(m_k)$ is surjective for $i \leq k$, hence the same holds for the map induced by the extension, $H^i(m_{k+1})$. Further, by construction, $H^{k+1}(m_{k+1})$ is surjective.
It remains so show $H^i(m_{k+1})$ is injective for all $i \leq k+2$. Let $[z] \in \ker(H^i(m_{k+1}))$, so z is a cocycle in $(\Lambda V^{\leq k+1})^i$. Now, for $i \leq k$ this immediately implies $z \in \Lambda V^{\leq k}$ for degree reasons. In case $i = k+2$ the same is true, with similar argumentation. Consider $z_1 \wedge \cdots \wedge z_l$ for some $l \geq 1$ and homogeneous elements $z_j \in V^{\leq k+1}$. Then $2 \leq |z_j| \leq k+1$, and $\sum_{j=1}^{l} |z_j| = k+2$. This implies $l \geq 2$ and thus $|z_j| \leq k$ for all $1 \leq j \leq l$. Since $m_{k+1}|_{\Lambda V^{\leq k}} = m_k$, we deduce that $[z] \in \ker(H^i(m_k))$ in both cases. Using the induction hypothesis, we get $[z] = 0$ for $i \leq k$. If $i = k+2$, the construction implies that z is a coboundary, hence $[z] = 0$ as well.
The case $i = k+1$ needs a little extra argument. In the presentation of $z \in \Lambda V^{\leq k+1}$ we may isolate elements of word length 1 so that $z = u + u'$ with $u \in \Lambda V^{\leq k}$ and $u' \in V^{k+1}$. Writing u' in turn as linear combination of the basis elements as noted in construction 3.3.9, we obtain

$$z = u + \sum_{i \in I} \alpha_i v_i + \sum_{j \in J} \beta_j w_j, \quad \alpha_i, \beta_j \in \mathbb{K},\ u \in \Lambda V^{\leq k}.$$

As $d(z) = 0$ and $d(v_i) = 0$, this leads to the equation $\sum_j \beta_j z_j = -d(u)$, where $d(w_j) = z_j \in (\Lambda V^{\leq k})^{k+2}$ as in construction 3.3.9. This means the linear combination is a coboundary, so $\sum_j \beta_j [z_j] = 0$. Since these elements are linear independent in $H^{k+2}(\Lambda V^{\leq k})$ by construction, we conclude $\beta_j = 0$ for all j. This in turn implies $d(u) = 0$, so u is a cocycle and defines a class $[u] \in H^{k+1}(\Lambda V^{\leq k})$. We observe

$$0 = H^{k+1}(m_{k+1})([z]) = [m_{k+1}(z)] = [m_k(u)] + \sum_{i \in I} \alpha_i [m_{k+1}(v_i)] = H^{k+1}(m_k)([u]) + \sum_{i \in I} \alpha_i [a_i],$$

which by the definition of the a_i can only be true if both terms of the sum are zero. We deduce that $\alpha_i = 0$ for each i and $[u] \in \ker(H^{k+1}(m_k))$. The first observation implies $u = z$, so with the second one it follows $[z] \in \ker(H^{k+1}(m_k))$. By induction, this means $[z] = 0$.
The verification of (ii) is obvious from the construction. For (iii), note that $V^2 = H^2(A)$ is of finite dimension. By induction, we may assume V^i to have finite dimension for all $i \leq k$. Then $\Lambda V^{\leq k}$ has finite type. This means $\ker(H^{k+2}(m_k)) \subseteq H^{k+2}(\Lambda V^{\leq k})$ is finite dimensional. Hence, the construction implies that the basis of V^{k+1} is finite since $H^{k+1}(A)$ has finite dimension. □

Corollary 3.3.11. *Assume X is a simply connected topological space such that $H^k(X; \mathbb{Q})$ has finite dimension for all $k \geq 1$. Then X has a Sullivan minimal model*

$$m \colon (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$$

such that $V = V^{\geq 2}$ and $\dim(V^k) < \infty$ for all k.

Remark 3.3.12. With few extra arguments, one can show that the statement above is also true when the assumption is formulated in terms of homology instead of cohomology. Hence, in particular, any simply connected finite CW complex has a minimal model such that $V^0 = V^1 = 0$ and V^k is finite dimensional for all $k \geq 2$.

Example 3.3.13. Let us describe a minimal Sullivan model for $X := \mathbb{S}^2 \vee \mathbb{S}^2$. Remember that

$$H^*(\mathbb{S}^2 \vee \mathbb{S}^2; \mathbb{K}) = \mathbb{K}e_1 \oplus \mathbb{K}e_2,$$

with e_1, e_2 being dual to the generators of $H_2(\mathbb{S}^2; \mathbb{K})$. This follows from the fact that the homology of a wedge sum is the direct sum of homologies and with coefficients in a field, cohomology is the dual of homology. As in example 3.3.7, we choose cocycles $\varphi_1, \varphi_2 \in A^2_{PL}(\mathbb{S}^2 \vee \mathbb{S}^2)$ representing e_1 and e_2 and start the construction of our model with two elements v_1, v_2 of degree 2, defining $m(v_i) := \varphi_i$. In $(\Lambda(v_1, v_2), 0)$, we get non-trivial products of degree 4 which we have to kill by introducing generators w, w_1, w_2 of degree 3 for which we set

$$d(w) := e_1 \wedge e_2, \quad d(w_1) := e_1^2, \quad d(w_2) := e_2^2.$$

However, this leads to unwanted cocycles $e_1 \wedge w - w_1 \wedge e_2$ and $e_2 \wedge w - w_2 \wedge e_1$ of degree 5 that in turn must be killed by the introduction of generators of degree 4. Continuing in this fashion, we obtain a minimal Sullivan model of the form

$$m\colon (\Lambda(v_1, v_2, w, w_1, w_2, \dots), d) \xrightarrow{\simeq} A_{PL}(\mathbb{S}^2 \vee \mathbb{S}^2),$$

with $m(v_i) := \varphi_i$ and zero on all other generators. In particular, the minimal Sullivan model of $\mathbb{S}^2 \vee \mathbb{S}^2$ has generators in each degree $n \geq 2$.

The example above can be generalized to wedges of spheres of arbitrary dimension, where one has to make the adjustment that in the case of a sphere of odd dimension, the self-products e_i^2 vanish. As one might have recognized, in the examples 3.3.7 and 3.3.13 we could calculate the minimal Sullivan model solely from the cohomology algebra of the spaces. Caution is advised, however, since it is not always this easy to obtain minimal models for a given space.

Definition 3.3.14. A commutative cochain algebra (A, d) for which $H^0(A) = \mathbb{K}$ holds is called *formal* if it is weakly equivalent to the commutative cochain algebra $(H(A), 0)$. We call a path connected space X *formal* if $A_{PL}(X; \mathbb{Q})$ is formal as cochain algebra.

As we shall soon see, weakly equivalent commutative cochain algebras share a common minimal Sullivan model. Since the assumption $H^0(A) = \mathbb{K}$ ensures the existence of a minimal Sullivan model, it then follows that (A, d) or respectively X are formal if and only if their minimal Sullivan models are a direct consequence of cohomology.

Example 3.3.15. Examples 3.3.7 and 3.3.13 exhibit $\mathbb{S}^n$ and $\mathbb{S}^2 \vee \mathbb{S}^2$ as formal spaces. More generally, given X and Y are simply connected formal spaces, then $X \times Y$ is formal. Furthermore, under the additional assumption that X and Y have rational homology of finite type, the wedge product $X \vee Y$ is formal. The first claim follows from example 3.5.1 in a later chapter, but is also proven in [9]. For the second, the reader may take a look at [18].

To be a little more precise, we have to distinguish between intrinsic formality and formality per se. A commutative graded algebra H is intrinsic formal if any commutative cochain algebra (A, d) with $H(A) \cong H$ is formal. We may then call X intrinsic formal if its cohomology algebra is. So for example, the spheres are intrinsic formal. This also becomes evident by the fact that we did not need the knowledge on the formality of $\mathbb{S}^n$ to calculate its minimal Sullivan model, all that was needed was the cohomology algebra. For more detail on formality, article [25] provides a comprehensive survey on the matter. Moreover, in [12], Halperin and Stasheff establish an obstruction theory for formality. Another prominent result is the formality of Kähler manifolds, which can be found in [5].

3.4 Homotopy on Sullivan Algebras

So far, given a commutative cochain algebra (A,d) with $H^0(A) = \mathbb{K}$ and $H^1(A) = 0$, we have established the existence of a minimal Sullivan model $(\Lambda V, d) \xrightarrow{\simeq} (A,d)$. Remember, the incentive in looking at these models is that we wish to consider the much simpler free commutative cochain algebra $(\Lambda V, d)$ instead of the generally quite complicated (A,d). In order for this to be feasible, the model should be unique in some sense. We shall see in this chapter that in fact a minimal model determines the weak equivalence class of (A,d) and therefore the rational homotopy class of a space X. In order to achieve this, we will discuss how morphisms between commutative cochain algebras are modelled. To begin with, we will introduce a notion homotopy for morphisms on Sullivan algebras that is carried over from the topological world.

Preparation 3.4.1. Consider the free commutative graded algebra $\Lambda(t,dt)$ introduced in example 3.1.8, with $|t| = 0$, $|dt| = 1$ and $d(t) = dt$. For $i \in \{0,1\}$ define morphisms

$$\varepsilon_i \colon \Lambda(t,dt) \to \mathbb{K}, \quad \varepsilon_i(t) := i$$

of commutative cochain algebras, where $\mathbb{K}$ is concentrated in degree zero and has trivial differential. Note that this automatically implies $\varepsilon_i(dt) = 0$, since we require the morphisms to commute with the differentials. Given a commutative cochain algebra (A, d_A), we may then define maps $(\mathrm{id} \cdot \varepsilon_i) \colon (A, d_A) \otimes \Lambda(t,dt) \to (A, d_A)$ for which holds

$$(\mathrm{id} \cdot \varepsilon_0)(a \otimes t^k) = \begin{cases} a & \text{if k} = 0, \\ 0 & \text{else} \end{cases} \qquad \text{and} \quad (\mathrm{id} \cdot \varepsilon_1)(a \otimes t^k) = a,$$

while $(\mathrm{id} \cdot \varepsilon_i)(a \otimes dt) = 0$ for both $i = 0$ and $i = 1$.

Definition 3.4.2. We say two morphisms $\varphi, \psi \colon (\Lambda V, d_V) \to (A, d_A)$ between a Sullivan algebra $(\Lambda V, d_V)$ and a commutative cochain algebra (A, d_A) are *homotopic* if there exists a morphism of commutative cochain algebras

$$H \colon (\Lambda V, d_V) \to (A, d_A) \otimes \Lambda(t,dt)$$

such that $(\mathrm{id} \cdot \varepsilon_0) \circ H = \varphi$ and $(\mathrm{id} \cdot \varepsilon_1) \circ H = \psi$. The map H is then called a *homotopy* from φ to ψ and we write $\varphi \simeq \psi$.

It is true, although not trivial that being homotopic defines an equivalence relation on the set of morphisms of the form $\varphi \colon (\Lambda V, d) \to (A, d_A)$ from a Sullivan algebra to an arbitrary commutative cochain algebra. One usually denotes the set of equivalence classes of such morphisms by $[(\Lambda V, d), (A, d_A)]$ and write $[\varphi]$ for the homotopy class of φ.

Example 3.4.3. Consider the special case that $(\Lambda V, d)$ is minimal, $d_A = 0$ and the so-called *constant* morphism $\varphi \colon (\Lambda V, d) \to (A, 0)$ that is given by $\varphi|_V = 0$. Then for every morphism $\varphi' \colon (\Lambda V, d) \to (A, 0)$ with $\varphi' \simeq \varphi$, it actually holds $\varphi' = \varphi$. In other words, any nullhomotopic map to $(A, 0)$ is constant.

We begin by noting that homotopic maps induce the same map on cohomology.

Lemma 3.4.4. *Let (A, d_A) be any commutative cochain algebra and $(\Lambda V, d_V)$ be a Sullivan algebra. If $\varphi_0 \simeq \varphi_1 \colon (\Lambda V, d_V) \to (A, d_A)$, then $H(\varphi_0) = H(\varphi_1)$.*

Proof. Assume $H \colon (\Lambda V, d_V) \to (A, d_A) \otimes \Lambda(t,dt)$ is a homotopy from φ_0 to φ_1. Define a morphism $h \colon (A, d_A) \otimes \Lambda(t,dt) \to (A, d_A)$ of degree -1 by

$$h(a \otimes t^k) := 0, \quad h(a \otimes t^k dt) := \frac{(-1)^{|a|}}{k+1} a.$$

Let $z \in \Lambda V$ and assume $H(z) = a \otimes b$ for some $a \in A$, $b \in \Lambda(t, dt)$. We may write b as linear combination $\sum_{k\geq 0} \alpha_k t^k + \beta_k t^k dt$ with coefficients $\alpha_k, \beta_k \in \mathbb{K}$. Its differential is given by $\sum_{k\geq 1} k\alpha_k t^{k-1} dt$ and thus a quick calculation shows that

$$\begin{aligned} d \circ H(z) &= d_A(a) \otimes (\sum_{k\geq 0} \alpha_k t^k + \beta_k t^k dt) + (-1)^{|a|} a \otimes \sum_{k\geq 1} k\alpha_k t^{k-1} dt, \\ h \circ d \circ H(z) &= \sum_{k\geq 0} (-1)^{|a|+1} \frac{\beta_k}{k+1} d_A(a) + \sum_{k\geq 1} (-1)^{2|a|} \alpha_k a, \end{aligned}$$

where d is the differential in $(A, d_A) \otimes \Lambda(t, dt)$. On the other hand, from the definition of h we get $h \circ H(z) = \sum_{k\geq 0} (-1)^{|a|} \frac{\beta_k}{k+1} a$. It follows that

$$\begin{aligned} (d_A \circ h \circ H + h \circ H \circ d_V)(z) &= d_A \circ h \circ H(z) + h \circ d \circ H(z) \\ &= \sum_{k\geq 0} (-1)^{|a|} \frac{\beta_k}{k+1} d_A(a) + \sum_{k\geq 0} (-1)^{|a|+1} \frac{\beta_k}{k+1} d_A(a) + \sum_{k\geq 1} (-1)^{2|a|} \alpha_k a \\ &= a(\sum_{k\geq 0} \alpha_k - \alpha_0) = (\mathrm{id}_A \cdot \varepsilon_1)(H(z)) - (\mathrm{id}_A \cdot \varepsilon_0)(H(z)) \\ &= (\varphi_1 - \varphi_0)(z). \end{aligned}$$

Thus, $h \circ H$ is a chain homotopy between φ_0 and φ_1, meaning $H(\varphi_0) = H(\varphi_1)$ as claimed. □

Of major importance throughout this section is the following result, which allows us to lift over quasi-isomorphisms up to homotopy.

Proposition 3.4.5. *Assume (A, d_A), (B, d_B) are commutative cochain algebras and $\psi\colon (A, d_A) \to (B, d_B)$ is a quasi-isomorphism. Further, suppose that $(\Lambda V, d)$ is a Sullivan algebra and $\varphi\colon (\Lambda V, d) \to (B, d_B)$ is a morphism between commutative cochain algebras. Then there exists a morphism $\tilde{\varphi}\colon (\Lambda V, d) \to (A, d_A)$ which is unique up to homotopy, such that $\psi \circ \tilde{\varphi} \simeq \varphi$. This means the diagram*

$$\begin{tikzcd} & (A, d_A) \arrow[d, "\simeq"', "\psi"] \\ (\Lambda V, d) \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"] & (B, d_B) \end{tikzcd}$$

is homotopy commutative.

Remark 3.4.6. Just as in the case of topological maps, homotopy of maps $\varphi_0, \varphi_1\colon (\Lambda V, d_V) \to (A, d_A)$ is compatible with the composition of maps. Supposing Φ is a homotopy realizing $\varphi_0 \simeq \varphi_1$ and $\phi\colon (\Lambda W, d_W) \to (\Lambda V, d_V)$ is a morphism between Sullivan algebras, it holds $\varphi_0 \circ \phi \simeq \varphi_1 \circ \phi$ via $\Phi \circ \phi$. This allows to define the map

$$\phi^*\colon [(\Lambda V, d_V), (A, d_A)] \to [(\Lambda W, d_W), (A, d_A)], \quad \phi^*([\varphi]) := [\varphi \circ \phi].$$

Moreover, suppose $\psi\colon (A, d_A) \to (B, d_B)$ is a morphism between commutative cochain algebras. Then $(\psi \otimes \mathrm{id}_{\Lambda(t,dt)}) \circ \Phi$ is a homotopy between $\psi \circ \varphi_0$ and $\psi \circ \varphi_1$. This gives raise to a well-defined map

$$\psi_*\colon [(\Lambda V, d_V), (A, d_A)] \to [(\Lambda V, d_V), (B, d_B)], \quad \psi_*([\varphi]) := [\psi \circ \varphi].$$

In regard to this, proposition 3.4.5 states that this map is actually an isomorphism if ψ is a quasi-isomorphism.

To show the existence of such lift $\tilde{\varphi}$, we will begin with a straightforward proof under the additional assumption that the quasi-isomorphism is surjective. From then on, we will approach the general case using a simple construction which allows us to factor arbitrary morphisms between commutative cochain algebras as a quasi-isomorphism followed by a surjection.

Lemma 3.4.7. *In the situation of proposition 3.4.5, assume the quasi-isomorphism ψ is surjective. Then there exists a morphism $\tilde{\varphi}\colon (\Lambda V, d) \to (A, d_A)$, such that $\psi \circ \tilde{\varphi} = \varphi$. Thus, the above diagram is even commutative and we say $\tilde{\varphi}$ is a lift of φ through ψ.*

Proof. Let $\{V(k)\}_{k\geq 0}$ be a increasing sequence of subspaces in V. For each basis element v_i in $V(0)$ the set $\psi^{-1}(\varphi(v_i))$ is non-empty, so we can define the image of $\tilde{\varphi}(v_i)$ to be any point in this set. Since $d = 0$ in $V(0)$, this automatically extends to the desired morphism. We may thus inductively assume that $\tilde{\varphi}$ is already defined in $V(k)$. Let V_{k+1} be a subspace such that $V(k+1) = V(k) \oplus V_{k+1}$ and denote by (w_j) a basis of V_{k+1}. The induction step is completed when we extend $\tilde{\varphi}$ on the elements w_j in such a way that it commutes with the differentials and $\psi \circ \tilde{\varphi}(w_j) = \varphi(w_j)$ holds. Since $d(w_j) \in \Lambda V(k)$, the map $\tilde{\varphi} \circ d$ is already defined on V_{k+1} and it holds

$$d_A(\tilde{\varphi}(d(w_j))) = \tilde{\varphi}(d^2(w_j)) = 0,$$
$$(\psi \circ \tilde{\varphi})(d(w_j)) = \varphi(d(w_j)) = d_B(\varphi(w_j)).$$

The first equation shows $\tilde{\varphi}(d(w_j))$ is a cocycle in A, therefore defining a class $[\tilde{\varphi}(d(w_j))]$ in $H(A)$. the second equation shows $H(\psi)([\tilde{\varphi}(d(w_j))]) = [d_B(\varphi(w_j))] = 0$, which implies $[\tilde{\varphi}(d(w_j))] = 0$ since ψ is a quasi-isomorphism. It follows that $\tilde{\varphi}(d(w_j))$ is a coboundary, meaning there exists $a_j \in A$ such that $d_A(a_j) = \tilde{\varphi}(d(w_j))$. Then the element $\varphi(w_j) - \psi(a_j)$ is a cocycle. As we assumed ψ to be surjective, thus we find $z_j \in A$ such that $\psi(z_j) = \varphi(w_j) - \psi(a_j)$. Let us for now assume that z_j is a cocycle. Then

$$\psi(z_j + a_j) = \varphi(w_j) \quad \text{and} \quad d_A(z_j + a_j) = d_A(a_j) = \tilde{\varphi}(d(w_j)).$$

We may thus conclude the induction by setting $\tilde{\varphi}(w_j) := z_j + a_j$.
Now, in case that z_j is not a cocycle, we find $z'_j \in A$ with $d_A(z'_j) = 0$ and $[\psi(z'_j)] = H(\psi)([z'_j]) = [\varphi(w_j) - \psi(a_j)]$, since $H(\psi)$ is an isomorphism. It follows that $[\psi(z_j)] = [\psi(z'_j)]$, which implies $[\psi(z_j) - \psi(z'_j)] = 0$. We thus find $b \in B$ such that $d_B(b) = \psi(z_j) - \psi(z'_j)$, and once again for b we find $a \in A$ for which $\psi(a) = b$. This implies $\psi(d_A(a)) = d_B(\psi(a)) = d_B(b) = \psi(z_j) - \psi(z'_j)$, so that $\psi(z_j) = \psi(z'_j + d_A(a))$. Therefore, we may replace z_j by the cocycle $z'_j + d_A(a)$. □

In order to extend this result to arbitrary quasi-isomorphisms, we can employ a little trick. For this, let $W = (W^i)_{i\geq 0}$ be a graded vector space and sW its suspension, that is a copy of W with degree increased by one. This defines a linear map $W \to sW$ of degree 1 which we may extend to a map d on $W \oplus sW$ where we set $d|_{sW} := 0$, implying that $d^2 = 0$. Thus d extends to a differential

$$d\colon \Lambda(W \oplus sW) \to \Lambda(W \oplus sW),$$

yielding a commutative cochain algebra that we will denote by $E(W)$. This is automatically a Sullivan algebra whenever $W^0 = 0$.
Note that this construction contains the commutative cochain algebra $(A_P)_n$ as a special case. Therefore, similar to remark 3.1.9 one shows that $H(E(W)) = \mathbb{K}$ using the fact that $\mathbb{K}$ is a field of characteristic zero. Due to this, commutative cochain algebras of the form $E(W)$ are sometimes called *contractible*.

Preparation 3.4.8. Let $(A, d), (B, d)$ be commutative cochain algebras and consider the inclusion

$$\iota\colon (A, d) \to (A, d) \otimes E(B), \quad a \mapsto a \otimes 1.$$

From the natural isomorphisms $H((A, d) \otimes E(B)) \cong H(A, d) \otimes H(E(B)) \cong H(A, d)$ given by the map $[a \otimes 1] \mapsto [a]$, it follows immediately that ι is a quasi-isomorphism. Moreover, there exists a unique surjective morphism $\sigma\colon E(B) \to (B,d)$, extending the identity on B. For elements $b \in B$, we obviously set $\sigma(b) = b$. Now by definition, for any $sb \in sB$ it holds $d(b) = sb$, thus we define $\sigma(sb) := d(\sigma(b)) = d(b)$ in order for σ to commute with the differentials. As usual, this extends to a morphism $\sigma\colon E(B) = (\Lambda(B \oplus sB), d) \to (B,d)$ with $\sigma|_B = \mathrm{id}_B$, as claimed. Therefore, any given morphism $\varphi\colon (A,d) \to (B,d)$ of commutative cochain algebras factors through $(A,d) \otimes E(B)$ via

$$(A,d) \xrightarrow[\simeq]{\iota} (A,d) \otimes E(B) \xrightarrow{\varphi \cdot \sigma} (B,d).$$

Since σ is surjective and $\varphi(1_A) = 1_B$, the map $\varphi \cdot \sigma$ is surjective, too.

Proof of proposition 3.4.5. By the above, the quasi-isomorphism $\psi\colon (A, d_A) \to (B, d_B)$ factors over the inclusion $\iota\colon (A,d) \to (A,d) \otimes E(B)$ and the surjective quasi-isomorphism $\psi \cdot \sigma\colon (A,d) \otimes E(B) \to (B,d)$. Remember that $E(B) = \Lambda(B \oplus sB)$ and continue by defining a morphism $\varepsilon\colon E(B) \to \mathbb{K}$ through setting $\varepsilon|_B := 0$, so it vanishes anywhere except for $\mathbb{K}$. From the definition of $E(B)$ it immediately follows that ε is a quasi-isomorphism. Now, the morphism $\mathrm{id}_A \cdot \varepsilon\colon (A,d) \otimes E(B) \to (A,d)$ is surjective and a left inverse of ι. We claim now that for any morphism $\phi\colon (\Lambda V, d) \to (A,d) \otimes E(B)$ it holds $\iota \circ (\mathrm{id}_A \cdot \varepsilon) \circ \phi \simeq \mathrm{id}_{(A,d) \otimes E(B)} \circ \phi$. Indeed, $\iota \circ (\mathrm{id}_A \cdot \varepsilon)$ vanishes on elements in $(A,d) \otimes \Lambda^+(B \oplus sB)$ by the definition of ε and it holds $\iota \circ (\mathrm{id}_A \cdot \varepsilon)(a \otimes 1) = a \otimes 1$. Consider the morphism $\eta\colon E(B) \to E(B) \otimes \Lambda(t, dt)$ given by $\eta(b) := b \otimes t$ for all $b \in B$ and note that $\eta(1) = 1 \otimes 1$. Then it holds

$$\begin{aligned}
&(\mathrm{id}_{(A,d)\otimes E(B)} \cdot \varepsilon_0) \circ (\mathrm{id}_{(A,d)} \otimes \eta)(a \otimes 1) = a \otimes 1, \quad &&(\mathrm{id}_{(A,d)\otimes E(B)} \cdot \varepsilon_0) \circ (\mathrm{id}_{(A,d)} \otimes \eta)(a \otimes b) = 0,\\
&(\mathrm{id}_{(A,d)\otimes E(B)} \cdot \varepsilon_1) \circ (\mathrm{id}_{(A,d)} \otimes \eta)(a \otimes 1) = a \otimes 1, \quad &&(\mathrm{id}_{(A,d)\otimes E(B)} \cdot \varepsilon_1) \circ (\mathrm{id}_{(A,d)} \otimes \eta)(a \otimes b) = a \otimes b.
\end{aligned}$$

This shows that $H := (\mathrm{id}_{(A,d)} \otimes \eta) \circ \phi$ is the desired homotopy. Using lemma 3.4.7 to lift φ over the surjective quasi-isomorphism $\psi \cdot \sigma$, we obtain the commutative diagram

$$\begin{tikzcd}
(A,d) \arrow[r, bend left, "\iota"] & (A,d) \otimes E(B) \arrow[l, bend left, "\mathrm{id} \cdot \varepsilon"] \arrow[d, "\psi \cdot \sigma", "\simeq"'] \\
(\Lambda V, d) \arrow[u, dashed, "\tilde{\varphi}"] \arrow[ur, dashed, "\varphi'"'] \arrow[r, "\varphi"'] & (C,d).
\end{tikzcd}$$

Here, $\tilde{\varphi} := \mathrm{id} \cdot \varepsilon \circ \varphi'$ and thus $\iota \circ \tilde{\varphi} \simeq \varphi'$. It now holds

$$\varphi = \psi \cdot \sigma \circ \varphi' \simeq \psi \cdot \sigma \circ \iota \circ \tilde{\varphi} = \psi \circ \tilde{\varphi}.$$

For a proof of the uniqueness, we refer to [7] chapter 12 . ☐

Definition 3.4.9. Consider a morphism $\varphi\colon (\Lambda V, d_V) \to (\Lambda W, d_W)$ between Sullivan algebras. For every $v \in V$, the element $\varphi(v) \in \Lambda W$ is a linear combination of elements $w_1 \wedge \cdots \wedge w_k \in \Lambda^k W$ with variable word length $k \geq 1$. Denote by $Q(\varphi)(v)$ the elements with word length 1, implying that $\varphi(v) - Q(\varphi)(v) \in \Lambda^{\geq 2} W$. This defines a map

$$Q(\varphi)\colon V \to W$$

that we will call the *linear part* of the morphism φ.

If $\psi\colon (\Lambda W, d_W) \to (\Lambda Z, d_Z)$ is a second morphism of Sullivan algebras, then $Q(\psi \circ \varphi) = Q(\psi) \circ Q(\varphi)$ as $\psi(\Lambda^{\geq 2} W) \subseteq \Lambda^{\geq 2} Z$. It follows that $Q(\varphi)$ commutes with the linear part $d_0 = Q(d)$ of the differential, i.e. $Q(\varphi) \circ (d_V)_0 = (d_W)_0 \circ Q(\varphi)$.

Remark 3.4.10. If $d = d_0 + d_1 + \ldots$ is the decomposition of a differential in a Sullivan algebra, it holds that d_0^2 is the only component in d^2 that does not raise the word length. Thus $d^2 = 0$

implies $d_0^2 = 0$. Therefore, if $\varphi\colon (\Lambda V, d) \to (\Lambda W, d)$ is a morphism between Sullivan algebras, then the linear part is a morphism

$$Q(\varphi)\colon (V, d_0) \to (W, d_0)$$

of complexes. As such, it induces $H(Q(\varphi))\colon H(V, d_0) \to H(W, d_0)$, a linear map between graded vector spaces. It is a non-trivial observation that φ is a quasi-isomorphism if and only if $H(Q(\varphi))$ is an isomorphism. We will prove a similar statement in the near future, but will have to refer to [7] for a full proof of this equivalence. However, it is worth to mention that this in particular implies that a quasi-isomorphism $(\Lambda V, d) \xrightarrow{\simeq} (\Lambda W, d)$ between minimal Sullivan algebra yields an isomorphism $V \cong W$, since $d_0 = 0$ in this case.

Proposition 3.4.11. *Let $(\Lambda V, d), (\Lambda W, d)$ be minimal Sullivan algebras.*

(i) If $H^1(\Lambda V, d) = 0$, then $\varphi_0 \simeq \varphi_1\colon (\Lambda V, d) \to (\Lambda W, d)$ implies $Q(\varphi_0) = Q(\varphi_1)$.

(ii) If $H^1(\Lambda V, d) = 0 = H^1(\Lambda W, d)$, then any quasi isomorphism $\varphi\colon (\Lambda V, d) \xrightarrow{\simeq} (\Lambda W, d)$ is in fact an isomorphism $(\Lambda V, d) \cong (\Lambda W, d)$.

Proof. (i) Let $H\colon (\Lambda V, d) \to (\Lambda W, d) \otimes \Lambda(t, dt)$ be a homotopy from φ_0 to φ_1. We showed in example 3.3.3 that for minimal Sullivan algebras, $H^1(\Lambda V) = 0$ implies V^1 to vanish, thus $V = V^{\geq 2}$. Since $\Lambda(t, dt)$ does not contain elements of degree ≥ 2, this implies that $H(V) \subseteq \Lambda^+ W \otimes \Lambda(t, dt)$. As H is an algebra morphism it follows $H(\Lambda^{\geq 2} V) \subseteq \Lambda^{\geq 2} W \otimes \Lambda(t, dt)$, thus we get an induced map of cochain complexes

$$\overline{H}\colon \Lambda^+ V / \Lambda^{\geq 2} V \longrightarrow (\Lambda^+ W / \Lambda^{\geq 2} W) \otimes \Lambda(t, dt).$$

Now, the minimality condition implies that the induced differentials are trivial, such that we may identify $\Lambda^+ V / \Lambda^{\geq 2} V = (V, 0)$ and $\Lambda^+ W / \Lambda^{\geq 2} W = (W, 0)$. It follows that $\overline{H}\colon (V, 0) \to (W, 0) \otimes \Lambda(t, dt)$ is a homotopy from $Q(\varphi_0) = (\mathrm{id} \cdot \varepsilon_0) \circ \overline{H}$ to $Q(\varphi_1) = (\mathrm{id} \cdot \varepsilon_1) \circ \overline{H}$. Then lemma 3.4.4 implies that $Q(\varphi_0) = Q(\varphi_1)$ as claimed.
(ii) Apply proposition 3.4.5 to lift the identity on $(\Lambda W, d)$ over φ to obtain $\tilde{\varphi}\colon (\Lambda W, d) \to (\Lambda V, d)$ such that $\varphi \circ \tilde{\varphi} \simeq \mathrm{id}_{\Lambda W}$. Part (i) now yields $Q(\varphi) \circ Q(\tilde{\varphi}) = \mathrm{id}_W$, so $Q(\varphi)$ is surjective. Hence, for any $w \in W^k$ there exists $v \in V^k$ such that $Q(\varphi)(v) = w$. This implies $\varphi(v) = w + w'$, where $w' \in \Lambda^{\geq 2} W$. We have $|w'| = k$ and $W^0 = 0$, so due to degree reasons $w' \in \Lambda W^{\leq k-1}$. It follows $W^k \subseteq \mathrm{im}(\varphi) + \Lambda W^{\leq k-1}$ for any $k \geq 1$. In fact, even $W^1 = 0$ since $H^1(\Lambda W, d) = 0$, thus for $k = 1$ this yields $W^{\leq 2} = W^2 \subseteq \mathrm{im}(\varphi)$. Assume by induction that $W^{\leq k-1} \subseteq \mathrm{im}(\varphi)$. Then $\Lambda W^{\leq k-1} \subseteq \mathrm{im}(\varphi)$, since φ respects the product structure. We have established that $W^k \subseteq \mathrm{im}(\varphi) + \Lambda W^{\leq k-1}$, which now implies that W^k and thus $W^{\leq k}$ is contained in the image of φ, concluding the induction. Finally, $W^{\leq k} \subseteq \mathrm{im}(\varphi)$ in turn implies $\Lambda W^{\leq k}$, where k may be arbitrary large. This shows that φ is surjective, meaning we may apply lemma 3.4.7 to even chose $\tilde{\varphi}$ such that $\varphi \circ \tilde{\varphi} = \mathrm{id}_{\Lambda W}$. In particular, this means that $\tilde{\varphi}$ is injective. It follows $\varphi \circ \tilde{\varphi} \circ \varphi = \varphi$, so $\tilde{\varphi} \circ \varphi$ lifts φ over itself. But so does the identity on ΛV, thus by the uniqueness of the lift in proposition 3.4.5 we conclude $\tilde{\varphi} \circ \varphi \simeq \mathrm{id}_{\Lambda V}$. From there on, we reuse the arguments from above, starting with $Q(\tilde{\varphi}) \circ Q(\varphi) = \mathrm{id}_V$ due to part (i), which means that $Q(\tilde{\varphi})$ and therefore $\tilde{\varphi}$ is surjective. It follows $\tilde{\varphi}$ is an isomorphism, and because of $\varphi \circ \tilde{\varphi} = \mathrm{id}_{\Lambda W}$ its inverse is exactly φ, which in conclusion is also an isomorphism. $\square$

With all this work put in, we can finally show the uniqueness of minimal Sullivan models for commutative cochain algebras. As mentioned before, the assumption $H^1(A) = 0$ can actually be dropped, but it does not restrict us too much.

Theorem 3.4.12. *Any commutative cochain algebra (A, d) with $H^0(A) = \mathbb{K}$ and $H^1(A) = 0$ has a minimal Sullivan model $m\colon (\Lambda V, d) \xrightarrow{\simeq} (A, d)$ which is unique up to isomorphy. That is, if*

$m' \colon (\Lambda W, d) \xrightarrow{\simeq} (A, d)$ is a quasi-isomorphism and $(\Lambda W, d)$ a minimal Sullivan algebra, then there exists an isomorphism $\varphi \colon (\Lambda V, d) \to (\Lambda W, d)$ such that $m \simeq m' \circ \varphi$.

$$\begin{tikzcd}
(\Lambda W, d) \arrow[dr, "m'", "\simeq"'] & \\
(\Lambda V, d) \arrow[u, "\varphi"', "\cong"] \arrow[r, "m", "\simeq"'] & (A, d)
\end{tikzcd}$$

Proof. Let $m \colon (\Lambda V, d) \xrightarrow{\simeq} (A, d)$ and $m' \colon (\Lambda W, d) \xrightarrow{\simeq} (A, d)$ be minimal Sullivan models. Using proposition 3.4.5 we obtain a lift $\varphi \colon (\Lambda V, d) \to (\Lambda W, d)$ of m over m', i.e. $m \simeq m' \circ \varphi$. By lemma 3.4.4 this implies $H^*(m) = H^*(m') \circ H^*(\varphi)$, so φ is a quasi-isomorphism. Now note that due to the assumption $H^1(A, d) = 0$ we get $H^1(\Lambda V) = 0 = H^1(\Lambda W)$, thus φ is an isomorphism by part (ii) of proposition 3.4.11. $\square$

Previously for commutative cochain algebras we defined the notion of weak equivalence $(A, d_A) \simeq (B, d_B)$, which is the case whenever both algebras are connected by a chain of quasi-isomorphisms of commutative cochain algebras. Let us now put Sullivan models into this context.

Remark 3.4.13. Assuming (A, d_A), (B, d_B) have a common Sullivan model, the chain

$$(A, d_A) \xleftarrow{\simeq} (\Lambda V, d) \xrightarrow{\simeq} (B, d)$$

automatically defines a weak equivalence. The other way around, consider a chain of commutative cochain algebras (C_i, d_i) realizing the weak equivalence and assume $m_A \colon (\Lambda V, d_V) \to (A, d)$ is a Sullivan model for A. We get a diagram

$$\begin{tikzcd}
(\Lambda V, d_V) \arrow[d, "m_A"] \arrow[dr, dashed, "m_0"] \arrow[drrr, dashed, "m_k", "\simeq"'] \arrow[drrrr, dashed, "m_B", "\simeq"'] & & & & \\
(A, d_A) \arrow[r, "\simeq"] & (C_0, d_0) & \ldots \arrow[l, "\simeq"'] \arrow[r] & (C_k, d_k) & (B, d_B), \arrow[l, "\simeq"']
\end{tikzcd}$$

where each m_i is either a direct composition of quasi-isomorphisms, or exists as a lift over a quasi-isomorphism due to proposition 3.4.5. In the latter case, lemma 3.4.4 implies that it is a quasi-isomorphism. We summarize that (A, d_A) and (B, d_B) are weakly equivalent if and only if they share a common Sullivan model.

Note that there may be multiple Sullivan models for a commutative cochain algebra. The remark above states that in case we find a Sullivan algebra that works as a model for (A, d) and (B, d), they are weakly equivalent and vice versa. Here, the uniqueness of minimal Sullivan models is a key advantage. Namely, remark 3.4.13 says that $(A, d) \simeq (B, d)$ if and only if the minimal Sullivan models of (A, d) and (B, d) are isomorphic. Thus, the question whether or not two commutative cochain algebras are weakly equivalent reduces to the calculation of their respective minimal Sullivan models.
Spoken differently, there is a one-to-one correspondence between isomorphy classes of minimal Sullivan algebras on the one side, and weak equivalence classes of commutative cochain algebras on the other. On both sides, we restrict to the case $H^0(-) = \mathbb{K}$ and $H^1(-) = 0$. Let us now transfer these results into the world of topology.

Remark 3.4.14. Assume X, Y are simply connected topological spaces with the same rational homotopy type. Applying A_{PL} to a chain of rational homotopy equivalences yields that $A_{PL}(X; \mathbb{Q})$, $A_{PL}(Y; \mathbb{Q})$ are weakly equivalent. This in turn implies the minimal models of X and Y are isomorphic. In other words, the isomorphy class of the minimal Sullivan model of a simply connected space is invariant under its rational homotopy type.

So far, we have only talked about models for commutative cochain algebras or topological spaces. Proposition 3.4.5 now allows us to assign to morphisms between these commutative cochain algebras a representative between their respective models. As usual, this then extends to continuous maps using $A_{PL}(-)$. Let (A, d), (B, d) be commutative cochain algebras with $H^0(A) = H^0(B) = \mathbb{K}$ and choose Sullivan models

$$m_A\colon (\Lambda V, d_V) \xrightarrow{\simeq} (A, d_A), \quad m_B\colon (\Lambda W, d_W) \xrightarrow{\simeq} (B, d_B).$$

For any morphism $\alpha\colon (A, d_A) \to (B, d_B)$ we may now apply proposition 3.4.5 for the cases $\psi := m_B$ and $\varphi := \alpha \circ m_A$ to obtain a unique homotopy class in $[(\Lambda V, d_V), (\Lambda W, d_W)]$ represented by a map $\tilde{\varphi}$, such that $m_B \circ \tilde{\varphi} \simeq \alpha \circ m_A$.

Definition 3.4.15. Given any morphism $\alpha\colon (A, d_A) \to (B, d_B)$ of commutative cochain algebras and Sullivan models $m_A\colon (\Lambda V, d_V) \xrightarrow{\simeq} (A, d_A)$, $m_B\colon (\Lambda W, d_W) \xrightarrow{\simeq} (B, d_B)$, a morphism $\varphi\colon (\Lambda V, d_V) \to (\Lambda W, d_W)$ allowing for a homotopy $m_B \circ \varphi \simeq \alpha \circ m_A$ is called a *Sullivan representative* for α. A *Sullivan representative for a continuous map* $f\colon X \to Y$ is a Sullivan representative for $A_{PL}(f)$.

$$\begin{tikzcd}
(A, d_A) \arrow[r, "\alpha"] & (B, d_B) \\
(\Lambda V, d_V) \arrow[u, "\simeq"'] \arrow[r, dashed, "\varphi"] & (\Lambda W, d_W) \arrow[u, "\simeq"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A_{PL}(X;\mathbb{K}) & A_{PL}(Y;\mathbb{K}) \arrow[l, "A_{PL}(f)"'] \\
(\Lambda V, d_V) \arrow[u, "\simeq"'] & (\Lambda W, d_W) \arrow[u, "\simeq"'] \arrow[l, dashed, "\varphi_f"']
\end{tikzcd}$$

If $\alpha \simeq \beta\colon (\Lambda V, d_V) \to (B, d_B)$ are homotopic, it immediately follows that any Sullivan representative of α is a Sullivan representative of β and vice versa, hence they have to be homotopic by uniqueness of the lift in proposition 3.4.5. However, a similar result for continuous maps $f, g\colon X \to Y$ is not as trivial, since it is not directly clear how $A_{PL}(f)$ and $A_{PL}(g)$ relate whenever $f \simeq g$. Fortunately, the functor $A_{PL}(-)$ does a good job in translating between the two homotopy concepts as the following proposition states. That being said, note that in reality the situation is actually reversed. Homotopy on Sullivan algebras in fact stems from topological homotopy, which is carried over to the world of commutative cochain algebras under $A_{PL}(-)$.

Proposition 3.4.16. *Let X, Y be topological spaces and $f, g\colon X \to Y$ be continuous maps. Further, suppose $\psi\colon (\Lambda V, d) \to A_{PL}(Y)$ is a morphism of commutative cochain algebras, where $(\Lambda V, d)$ is a Sullivan algebra. Then $f \simeq g$ implies $A_{PL}(f) \circ \psi \simeq A_{PL}(g) \circ \psi$.*

Proof. Remember an element in $A^0_{PL}(I)$ is a map $S_*(I) \to A^0_{PL}$ compatible with the face and degeneracy maps. Let $\varphi \in A^0_{PL}(I)$ such that $\varphi(c^0_n) = 0$ and $\varphi(c^1_n) = 1$ for all $n \geq 0$, where $c^i_n\colon \Delta^n \to \{i\} \subseteq I$ for $i \in \{0, 1\}$. Under the morphism $\Lambda(t, dt) \to A_{PL}(I)$ of commutative cochain algebras that is given by $t \mapsto \varphi$, we may view $\Lambda(t, dt)$ as sub cochain algebra of $A_{PL}(I)$. Next, consider the inclusions

$$j_0\colon X \to X \times I,\ j_0(x) := (x, 0) \quad \text{and} \quad j_1\colon X \to X \times I,\ j_1(x) := (x, 1),$$

as well as the projections $p_I\colon X \times I \to I$ and $p_X\colon X \times I \to X$. Then $p_I \circ j_i$ is the constant map $X \to \{i\} \subseteq I$ and $p_X \circ j_i = \mathrm{id}_X$. Consider now the morphisms

$$\begin{aligned}
A_{PL}(p_X) \cdot A_{PL}(p_I)&\colon A_{PL}(X) \otimes \Lambda(t, dt) \longrightarrow A_{PL}(X \times I) \\
(A_{PL}(j_0), A_{PL}(j_1))&\colon A_{PL}(X \times I) \longrightarrow A_{PL}(X) \times A_{PL}(X),
\end{aligned}$$

where we restrict $A_{PL}(p_I)\colon A_{PL}(I) \to A_{PL}(X \times I)$ to the sub cochain algebra $\Lambda(t, dt)$ and identify φ with t. We claim that

$$(A_{PL}(j_0), A_{PL}(j_1)) \circ (A_{PL}(p_X) \cdot A_{PL}(p_I)) = (\mathrm{id}_{A_{PL}(X)} \cdot \varepsilon_0, \mathrm{id}_{A_{PL}(X)} \cdot \varepsilon_1).$$

A quick calculation of $A_{PL}(p_I \circ j_i)\colon \Lambda(t, dt) \to A_{PL}(X)$ yields

$$A_{PL}(p_I \circ j_i)(t)(\sigma) = \varphi(p_I \circ j_i \circ \sigma) = \varphi(c_n^i) = i$$

for any singular n-simplex $\sigma \in S_n(X)$, and thus $A_{PL}(p_I \circ j_i)(t) = i$. We conclude that $A_{PL}(p_I \circ j_i) = \varepsilon_i$ on $\Lambda(t, dt)$. Moreover, $A_{PL}(p_X \circ j_i) = A_{PL}(\mathrm{id}_X) = \mathrm{id}_{A_{PL}(X)}$. Put together, this shows

$$\begin{aligned}
&(A_{PL}(j_0), A_{PL}(j_1)) \circ (A_{PL}(p_X) \cdot A_{PL}(p_I)) \\
&= (A_{PL}(p_X \circ j_0) \cdot A_{PL}(p_I \circ j_0), A_{PL}(p_X \circ j_1) \cdot A_{PL}(p_I \circ j_1)) \\
&= (\mathrm{id}_{A_{PL}(X)} \cdot \varepsilon_0, \mathrm{id}_{A_{PL}(X)} \cdot \varepsilon_1)
\end{aligned}$$

as claimed. Finally, let f, g, ψ be as in the proposition and assume $H\colon X \times I \to Y$ is a homotopy from f to g. The morphism $A_{PL}(p_X) \cdot A_{PL}(p_I)$ is a quasi-isomorphism, as $H^*(A_{PL}(p_X)) = H^*(p_X; \mathbb{K})$ is an isomorphism and the commutative cochain algebra $\Lambda(t, dt)$ is contractible. We may thus lift $A_{PL}(H) \circ \psi\colon (\Lambda V, d) \to A_{PL}(X \times I)$ over $A_{PL}(p_X) \cdot A_{PL}(p_I)$ to obtain $\tilde{\varphi}\colon (\Lambda V, d) \to A_{PL}(X) \otimes \Lambda(t, dt)$. This means that in the diagram

$$\begin{tikzcd}
 & & A_{PL}(X) \otimes \Lambda(t, dt) \arrow[dd, "A_{PL}(p_X) \cdot A_{PL}(p_I)"'] \arrow[dr, "(\mathrm{id} \cdot \varepsilon_0, \mathrm{id} \cdot \varepsilon_1)"] & \\
 & & & A_{PL}(X) \times A_{PL}(X) \\
(\Lambda V, d) \arrow[r, "\psi"'] \arrow[uurr, dashed, "\tilde{\varphi}"] & A_{PL}(Y) \arrow[r, "A_{PL}(H)"'] & A_{PL}(X \times I) \arrow[ur, "(A_{PL}(j_0), A_{PL}(j_1))"'] &
\end{tikzcd}$$

the right-hand triangle commutes, whereas the left-hand triangle is homotopy commutative. This yields

$$(\mathrm{id} \cdot \varepsilon_0, \mathrm{id} \cdot \varepsilon_1) \circ \tilde{\varphi} \simeq (A_{PL}(j_0), A_{PL}(j_1)) \circ A_{PL}(H) \circ \psi = (A_{PL}(f) \circ \psi, A_{PL}(g) \circ \psi),$$

as $H \circ j_0 = f$ and $H \circ j_1 = g$. Since $(\mathrm{id} \cdot \varepsilon_0) \circ \tilde{\varphi} \simeq (\mathrm{id} \cdot \varepsilon_1) \circ \tilde{\varphi}$ via $\tilde{\varphi}$ itself, the assertion follows. □

Assume now we are given continuous maps $f, g\colon X \to Y$ with $f \simeq g$. Let $m_X\colon (\Lambda V, d) \to A_{PL}(X)$ and $m_Y\colon (\Lambda W, d) \to A_{PL}(Y)$ be Sullivan models. We find Sullivan representatives

$$\varphi_f, \varphi_g\colon (\Lambda W, d) \to (\Lambda V, d),$$

where φ_f lifts $A_{PL}(f) \circ m_Y$ over m_X and φ_g lifts $A_{PL}(g) \circ m_Y$ over m_X. Proposition 3.4.16 now implies $A_{PL}(f) \circ m_Y \simeq A_{PL}(g) \circ m_Y$, thus $\varphi_f \simeq \varphi_g$ by the uniqueness of the lift. It follows that Sullivan representatives of homotopic maps are homotopic morphisms, or in other words, the homotopy class of f determines a unique homotopy class in $[(\Lambda V, d), (\Lambda W, d)]$ of suitable Sullivan representatives. Therefore, when provided an element $\alpha \in [X, Y]$, we may write φ_α for any representative of this class.

3.5 Sullivan Models of Pushouts and Fibrations

In this section, we want to examine in which sense geometric constructions can be algebraically modelled, such that it is possible to directly compute models without requiring additional information. The pushout $X \cup_f Y$ is obtained by attaching X to the space Y along $A \subseteq X$ via $f\colon A \to Y$. One may then ask whether it is possible to compute a commutative model for the pushout from the algebraic data associated with $i\colon A \hookrightarrow X$ and f, which are the corresponding Sullivan representatives. Similarly, given a fibration $p\colon E \to B$, the weak homotopy type of the homotopy fibre F is determined by p. Therefore, it seems natural to ask when and how a Sullivan model of F can be obtained from $A_{PL}(p)$.

Rather than working out every detail, in the following we intend to provide a compact and brief overview over the useful and important methods and constructions. Our goal is then to use these in a later chapter to show a fundamental result in rational homotopy theory, that is the existence of an isomorphism

$$V \cong \mathrm{Hom}_{\mathbb{Z}}(\pi_*(X), \mathbb{K}),$$

where $(\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$ is a minimal Sullivan model. As usual, we work over a field $\mathbb{K}$ of characteristic zero.

Example 3.5.1. As an introductory example, consider Sullivan models $m_X \colon (\Lambda V_X, d) \to A_{PL}(X)$ and $m_Y \colon (\Lambda V_Y, d) \to A_{PL}(Y)$ of two simply connected spaces, where we assume X or Y to have rational homology of finite type. Denote by $p_X \colon X \times Y \to X$ and $p_Y \colon X \times Y \to Y$ the projections, then the morphism

$$A_{PL}(p_X) \cdot A_{PL}(p_y) \colon A_{PL}(X) \otimes A_{PL}(Y) \to A_{PL}(X \times Y)$$

induces the isomorphism $H^*(p_X; \mathbb{K}) \cdot H^*(p_Y; \mathbb{K}) \colon H^*(X; \mathbb{K}) \otimes H^*(Y; \mathbb{K}) \to H^*(X \otimes Y; \mathbb{K})$ on cohomology and is thus a quasi-isomorphism. We may therefore define a quasi-isomorphism

$$(A_{PL}(p_X) \cdot A_{PL}(p_y)) \circ (m_X \otimes m_Y) \colon (\Lambda V_X, d) \otimes (\Lambda V_Y, d) \xrightarrow{\simeq} A_{PL}(X \times Y),$$

which, together with the fact that the tensor product of Sullivan algebras is again a Sullivan algebra, exhibits $(\Lambda V_X, d) \otimes (\Lambda V_Y, d)$ as Sullivan model for $X \times Y$.

Preparation 3.5.2. Suppose (X, A) is a pair of topological spaces and $f \colon A \to Y$ is a continuous map. Then the pushout $X \cup_f Y$ fits into the commutative diagram at the left, while by applying the functor $A_{PL}(-)$ we get a diagram as shown on the right.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "\pi|_Y"] \\ X \arrow[r, "\pi|_X"'] & X \cup_f Y \end{tikzcd} \quad \overset{A_{PL}}{\curvearrowright} \quad \begin{tikzcd} A_{PL}(A) & A_{PL}(Y) \arrow[l, "A_{PL}(f)"'] \\ A_{PL}(X) \arrow[u, "A_{PL}(i)"] & A_{PL}(X \cup_f Y) \arrow[l, "A_{PL}(\pi|_X)"] \arrow[u, "A_{PL}(\pi|_Y)"'] \end{tikzcd}$$

Here $\pi \colon X + Y \to X \cup_f Y$ denotes the canonical projection. By the unique property of the pullback $A_{PL}(X) \times_{A_{PL}(A)} A_{PL}(Y)$ of the morphisms $A_{PL}(f)$ and $A_{PL}(i)$, there must be an induced morphism

$$\Phi \colon A_{PL}(X \cup_f Y) \to A_{PL}(X) \times_{A_{PL}(A)} A_{PL}(Y),$$

depending solely on $A_{PL}(\pi|_X)$ and $A_{PL}(\pi|_Y)$.
Now suppose that $H_*(X, A; \mathbb{K}) \cong H_*(X \cup_f Y, Y; \mathbb{K})$ induced by $\pi|_X$. Since $\mathbb{K}$ is a field of characteristic zero, the cohomology with coefficients in $\mathbb{K}$ is the dual of homology. Thus, the map $\pi|_X \colon (X, A) \to (X \cup_f Y, Y)$ induces an isomorphism between cohomology groups, which can be identified with the induced map of $A_{PL}(\pi|_X) \colon A_{PL}(X \cup_f Y, Y) \to A_{PL}(X, A)$ which hence is a quasi-isomorphism. It follows from the diagram

$$\begin{tikzcd} 0 \arrow[r] & A_{PL}(X \cup_f Y, Y) \arrow[r] \arrow[d, "\simeq"] & A_{PL}(X \cup_f Y) \arrow[r] \arrow[d] & A_{PL}(Y) \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & A_{PL}(X, A) \arrow[r] & A_{PL}(X) \times_{A_{PL}(A)} A_{PL}(Y) \arrow[r] & A_{PL}(Y) \arrow[r] & 0 \end{tikzcd}$$

that the map in the middle hast to be a quasi-isomorphism as well. This means the pullback $A_{PL}(X) \times_{A_{PL}(A)} A_{PL}(Y)$ is a commutative model for the pushout $X \cup_f Y$.

We will now investigate how this relationship translates when we pass on to Sullivan models. For this, assume X, A and Y are path-connected and assume $m_X\colon (\Lambda V, d_V) \to A_{PL}(X)$, $m_A\colon (\Lambda U, d_U) \to A_{PL}(A)$ and $m_Y\colon (\Lambda W, d_W) \to A_{PL}(Y)$ are Sullivan models. Further, suppose $\varphi\colon (\Lambda V, d_V) \to (\Lambda U, d_U)$ and $\psi\colon (\Lambda W, d_W) \to (\Lambda U, d_U)$ are Sullivan representatives for i and f. The situation can be summarized by the following homotopy commutative diagram:

$$\begin{tikzcd}
(\Lambda V, d_V) \arrow[r, "\varphi"] \arrow[d, "m_X"] & (\Lambda U, d_U) \arrow[d, "m_A"] & (\Lambda W, d_W) \arrow[l, "\psi"'] \arrow[d, "m_Y"] \\
A_{PL}(X) \arrow[r, "A_{PL}(i)"] & A_{PL}(A) & A_{PL}(Y) \arrow[l, "A_{PL}(f)"']
\end{tikzcd}$$

Denote by $(\Lambda V \times_{\Lambda U} \Lambda W, d)$ and $A_{PL}(X) \times_{A_{PL}(A)} A_{PL}(Y)$ the corresponding pullbacks. The vertical maps are quasi-isomorphisms and the injectivity of i implies that $A_{PL}(i)$ is surjective. By a more general result on commutative cochain algebras it follows that if one of the maps φ, ψ is surjective, the pullbacks above are weakly equivalent.

Proposition 3.5.3. *Assume $H_*(\pi|_X;\mathbb{K})\colon H_*(X, A;\mathbb{K}) \to H_*(X \cup_f Y, Y;\mathbb{K})$ is an isomorphism (which for example holds when (X, A) is a CW pair), and that one of the Sullivan representatives φ, ψ is surjective. Then the pullback $(\Lambda V \times_{\Lambda U} \Lambda W, d)$ is a commutative model for $X \cup_f Y$.*

Proof. By the above, the pullbacks $(\Lambda V \times_{\Lambda U} \Lambda W, d)$ and $A_{PL}(X) \times_{A_{PL}(A)} A_{PL}(Y)$ are connected by a chain of quasi-isomorphisms. If in addition $\pi|_X$ induces an isomorphism, we can extend the chain by a link and obtain

$$(\Lambda V \times_{\Lambda U} \Lambda W, d) \xrightarrow{\simeq} \dots \xleftarrow{\simeq} A_{PL}(X) \times_{A_{PL}(A)} A_{PL}(Y) \xleftarrow{\simeq} A_{PL}(X \cup_f Y)$$

as claimed. □

We will return to this later in proposition 3.6.13, which provides a simpler commutative model when the pushout $X \cup_f Y$ is the attachment of a cell. Still, even in this special case, the question regarding the calculation of a Sullivan model for $X \cup_f Y$ remains unanswered.
Until now, we used Sullivan algebras $(\Lambda V, d)$ as models for commutative cochain algebra (A, d) up to weak equivalence. In this section, we want to generalize this idea by providing models for morphisms $(B, d) \to (A, d)$ of commutative cochain algebras. This will be achieved by using commutative cochain algebras of the form $(B \otimes \Lambda V, d)$ that are called relative Sullivan algebras. The importance of these algebras lies in the fact that they provide good models for fibrations.

Definition 3.5.4. A *relative Sullivan algebra* is a commutative cochain algebra $(B \otimes \Lambda V, d)$, where V is a vector space and (B, d) is a commutative cochain algebra such that

(a) $V = V^+$,

(b) $V = \cup_{k=0}^{\infty} V(k)$ for an increasing sequence of graded subspaces $V(0) \subseteq V(1) \subseteq \dots$,

(c) $d(V(0)) \subseteq B$ and $d(V(k)) \subseteq B \otimes \Lambda V(k-1)$ for all $k \geq 1$,

(d) (B, d) is a sub cochain algebra of $(B \otimes \Lambda V, d)$ via $B = B \otimes 1$, $d = d \otimes \mathrm{id}$, and it holds $H^0(B) = \mathbb{K}$.

The algebra (B, d) is referred to as *base algebra* of $(B \otimes \Lambda V)$.

We will frequently use the identities $B = B \otimes 1$ and $\Lambda V = 1 \otimes \Lambda V$ to regard these graded algebras as subalgebras of $(B \otimes \Lambda V)$. However, whereas (B, d) is a sub cochain algebra, the same is in general not true for $(\Lambda V, d)$. In other words, the differential of a relative Sullivan algebra does generally not preserve (ΛV).
Note that for $(B, d) = (\mathbb{K}, 0)$, the corresponding relative Sullivan algebra is simply a Sullivan algebra. In this sense, relative Sullivan algebras are a generalization of Sullivan algebras.

Definition 3.5.5. Let $f\colon X \to Y$ be a continuous map, and $\varphi\colon (B,d) \to (A,d)$ a morphism of commutative cochain algebras. A *Sullivan model for* φ is a relative Sullivan algebra $(B \otimes \Lambda V, d)$ that comes with a quasi-isomorphism

$$m\colon (B \otimes \Lambda V, d) \xrightarrow{\simeq} (A,d)$$

such that $m|_B = \varphi$. By a *Sullivan model for* f we mean a Sullivan model for $A_{PL}(f)$. A Sullivan model is called *minimal* if the associated relative Sullivan algebra $(B \otimes \Lambda V, d)$ is minimal, that is, if

$$\mathrm{im}(d) \subseteq B^+ \otimes \Lambda V + B \otimes \Lambda^{\geq 2} V$$

holds, where $\Lambda^{\geq 2} V = \oplus_{k \geq 2} \Lambda^k V$.

Note that a Sullivan (minimal) model of the morphism $\mathbb{K} \to (A,d)$, $1 \mapsto 1$, provides a Sullivan (minimal) model for (A,d). In this sense, the models for morphisms generalize the concept of models for commutative cochain algebras respectively spaces.

Remark 3.5.6.

(a) Regard $\mathbb{K}$ as (B,d)-module via a morphism $\varepsilon\colon (B,d) \to \mathbb{K}$. Any relative Sullivan algebra $(B \otimes \Lambda V, d)$ is by definition a free (B,d)-module. If we consider the associated tensor product, as in example 1.2.5 we see that

$$\mathbb{K} \otimes_{(B,d)} (B \otimes \Lambda V, d) = (\Lambda V, \overline{d})$$

is actually a quotient Sullivan algebra that depends on ε. Here, we renamed the differential to indicate that whereas the resulting free commutative algebra is the same, the differential changed to a differential in ΛV (and is not the initial differential restricted to ΛV, which in general is not a complex in the first place).

(b) To generalize the above, assume $\psi\colon (B,d) \to (B',d')$ is any morphism of commutative cochain algebras with $H^0(B') = 0$. The cochain algebra

$$(B',d') \otimes_{(B,d)} (B \otimes \Lambda V, d) = (B' \otimes \Lambda V, d)$$

is a relative Sullivan algebra with base algebra (B',d') and is called the *pushout of* $(B \otimes \Lambda V, d)$ *along* ψ. The associated commutative diagram is

$$\begin{tikzcd} (B,d) \arrow[r] \arrow[d, "\psi"] & (B \otimes \Lambda V, d) \arrow[d, "\psi\otimes\mathrm{id}"] \\ (B',d') \arrow[r] & (B' \otimes \Lambda V, d), \end{tikzcd}$$

where the horizontal maps are the natural injections of sub cochain algebras. It follows that $\psi \otimes \mathrm{id}$ commutes with the differentials and thus is a morphism. One can then show that if ψ is a quasi-isomorphism, the same is true for $\psi \otimes \mathrm{id}$.

The notion of homotopy between maps from a Sullivan algebra analogously generalizes to the relative case.

Definition 3.5.7. Suppose $\varphi, \psi\colon (B \otimes \Lambda V, d) \to (A,d)$ are morphisms of commutative cochain algebras, where $(B \otimes \Lambda V, d)$ is a relative Sullivan algebra. Further, suppose $\varphi|_B = \psi|_B\colon (B,d) \to (A,d)$. We say φ and ψ are *homotopic relative* B if there exists a morphism of commutative cochain algebras

$$\Psi\colon (B \otimes \Lambda V, d) \to (A,d) \otimes \Lambda(t,dt)$$

such that $(\mathrm{id} \cdot \varepsilon_0) \circ \Psi = \varphi$, $(\mathrm{id} \cdot \varepsilon_1) \circ \Psi = \psi$ and $\Psi(b) = \varphi(b) \otimes 1 = \psi(b) \otimes 1$ for all $b \in B$. The map Ψ is then called a *relative homotopy* from φ to ψ, and we denote this by writing $\varphi \simeq \psi$ rel B.

Remark 3.5.8. As before, homotopic relative B is an equivalence relation in the set of possible morphisms. Further, if $\varphi \simeq \psi$ rel B, then the induced maps on homology level coincide, $H(\varphi) = H(\psi)$.

The question of existence and uniqueness of minimal Sullivan models for morphisms is summarized in the following theorem. Note that the conditions are now weaker as they were when we considered Sullivan models of commutative cochain algebras, or respectively spaces. As minimal Sullivan models for morphisms are the more general concept, this theorem is stronger than many of the results we introduced before.

Theorem 3.5.9. *Let $\varphi\colon (B,d) \to (A,d)$ be a morphism of commutative cochain algebras. Suppose $H^0(B) = \mathbb{K} = H^0(A)$ and $H^1(\varphi)$ is injective. Then the morphism φ has a minimal Sullivan model*

$$m\colon (B \otimes \Lambda V, d) \xrightarrow{\simeq} (A,d).$$

Assuming $m'\colon (B \otimes \Lambda V', d) \xrightarrow{\simeq} (A,d)$ is another minimal Sullivan model for φ, then there exists an isomorphism

$$\alpha\colon (B \otimes \Lambda V, d) \xrightarrow{\simeq} (B \otimes \Lambda V', d)$$

such that $\alpha|_B = \mathrm{id}_B$ and $m' \circ \alpha \simeq m$ rel B.

Proof. This is proven in chapter 14 of [7]. □

Corollary 3.5.10. *Any commutative cochain algebra (A,d) with $H^0(A) = \mathbb{K}$ has a unique minimal Sullivan model. In particular, any path connected topological space X has a unique minimal Sullivan model.*

Remember that for a continuous map $f\colon X \to Y$, the homotopy type of the corresponding homotopy fibre F is determined by the homotopy class of f. We will now see that, in a similar fashion, a Sullivan model for F can be computed from the morphism $A_{PL}(f)\colon A_{PL}(Y) \to A_{PL}(X)$ under the assumption that Y is a simply connected space with rational homology of finite type.

Preparation 3.5.11. Consider a Serre fibration $p\colon X \to Y$ of path connected spaces, with path connected fibres. Let $y_0 \in Y$ and denote by $j\colon F \hookrightarrow X$ the inclusion of the fibre $F := p^{-1}(y_0)$ and by $\pi\colon F \to \{y_0\}$ the projection. Then the functor $A_{PL}(-)$ converts the associated commutative diagram on the left to the one on the right,

$$\begin{array}{ccc} F & \xrightarrow{j} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ \{y_0\} & \xrightarrow[i]{} & Y \end{array} \quad \overset{A_{PL}}{\curvearrowright} \quad \begin{array}{ccc} A_{PL}(F) & \xleftarrow{A_{PL}(j)} & A_{PL}(X) \\ \uparrow{\scriptstyle A_{PL}(\pi)} & & \uparrow{\scriptstyle A_{PL}(p)} \\ \mathbb{K} & \xleftarrow[\varepsilon]{} & A_{PL}(Y) \end{array}$$

where the augmentation $A_{PL}(i)$ is denoted by ε.
We assumed F to be path connected, so from the long exact sequence of the fibration it follows that $\pi_1(p)\colon \pi_1(X) \to \pi_1(Y)$ is surjective. From the Hurewicz theorem it follows that $H_1(p;\mathbb{Z})$ is surjective as well, as the diagram

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{h_X} & H_1(X;\mathbb{Z}) \\ \downarrow{\scriptstyle \pi_1(p)} & & \downarrow{\scriptstyle H_1(p;\mathbb{Z})} \\ \pi_1(Y) & \xrightarrow[h_Y]{} & H_1(Y;\mathbb{Z}) \end{array}$$

commutes and the Hurewicz homomorphisms h_X and h_Y are surjective since X and Y are path connected. Since we are dealing with field coefficients, it follows that $H_1(p;\mathbb{K})$ too is surjective, and thus its dual map $H^1(p;\mathbb{K}) = H^1(A_{PL}(f;\mathbb{K}))$ is injective. In particular, the assumptions in theorem 3.5.9 are satisfied, meaning there exists a (minimal) Sullivan model for p,

$$m\colon (A_{PL}(Y)\otimes \Lambda V, d) \xrightarrow{\simeq} A_{PL}(X).$$

Going by remark 3.5.6, the augmentation ε defines a Sullivan algebra $\mathbb{K}\otimes_{A_{PL}(Y)} (A_{PL}(Y)\otimes \Lambda V, d) = (\Lambda V, \overline{d})$ that is the pushout of $(A_{PL}(Y)\otimes \Lambda V, d)$ under ε. This means we can extend the commutative diagram to

$$\begin{tikzcd}
 & A_{PL}(F) & A_{PL}(X) \arrow[l, "A_{PL}(j)"'] & \\
 & \mathbb{K} \arrow[u, "A_{PL}(\pi)"] \arrow[dl] & A_{PL}(Y) \arrow[u, "A_{PL}(p)"'] \arrow[l, "\varepsilon"] \arrow[dr] & \\
(\Lambda V, \overline{d}) \arrow[uur, dashed, bend left, "\overline{m}"] & & & (A_{PL}(Y)\otimes \Lambda V, d). \arrow[lll, "\varepsilon\cdot\mathrm{id}"] \arrow[uul, bend right, "m"']
\end{tikzcd}$$

The bottom square is just the commutative diagram as seen in remark 3.5.6 associated with the pushout. Note that the map $\varepsilon\otimes\mathrm{id}$ changes to multiplication since $B' = \mathbb{K}$ and we identify $\mathbb{K}\otimes\Lambda V = \Lambda V$.

Next, note that $(\Lambda V, \overline{d})$ is actually obtained as a quotient of $A_{PL}(Y)\otimes \Lambda V$,

$$\mathbb{K}\otimes_{A_{PL}(Y)} (A_{PL}(Y)\otimes\Lambda V) = (\mathbb{K}\otimes(A_{PL}(Y)\otimes\Lambda V))/\sim = (A_{PL}(Y)\otimes\Lambda V)/\sim.$$

Further, $\varepsilon\cdot\mathrm{id}$ is surjective since ε is an algebra morphism and $A_{PL}(j)\circ m(\ker(\varepsilon)\otimes\Lambda V) = 0$. To verify the second claim, let $a\in A_{PL}(Y)$ such that $\varepsilon(a) = 0$ in $\mathbb{K}$. As the inner square and right-hand triangle of the diagram commutes, we get

$$0 = A_{PL}(\pi)\circ\varepsilon(a) = A_{PL}(j)\circ A_{PL}(p)(a) = A_{PL}(j)\circ m(a\otimes 1).$$

Hence $A_{PL}(j)\circ m$ factors over $\varepsilon\cdot\mathrm{id}$. It follows that there exists a unique morphism $\overline{m}\colon (\Lambda V,\overline{d}) \to A_{PL}(F)$, making the outer diagram commute.

Theorem 3.5.12. *Let $p\colon X\to Y$ be a Serre fibration as above. Assume Y is simply connected and one of the graded vector spaces $H_*(Y;\mathbb{K})$, $H_*(F;\mathbb{K})$ has finite type. Then*

$$\overline{m}\colon (\Lambda V,\overline{d}) \to A_{PL}(F)$$

is a quasi-isomorphism.

Proof. This is proven in chapter 15 of [7]. □

Note that, as $(\Lambda V,\overline{d})$ is a Sullivan algebra, this provides a Sullivan model for the fibre F. Remember that any continuous map $f\colon X\to Y$ can be turned into a fibration. Assume X is path connected and Y is simply connected. Then $H^1(Y;\mathbb{K}) = 0$, so $H^1(A_{PL}(f))$ is injective and f has a Sullivan model as above. Supposing now that at least one of the graded spaces $H_*(X;\mathbb{K})$, $H_*(Y;\mathbb{K})$ has finite type, we conclude from 3.5.12 that $(\Lambda V,\overline{d})$ is a Sullivan model for the homotopy fibre of f.

Remark 3.5.13. In the considerations above, it might be useful to replace $A_{PL}(Y)$ by a Sullivan algebra, using a model $m_Y\colon (\Lambda V_Y, d) \xrightarrow{\simeq} A_{PL}(Y)$. Suppose the conditions of theorem 3.5.12 are satisfied. As Y is simply connected, $V_Y^0 = 0$, so $(\Lambda V_Y)^0 = \mathbb{K}$, which means there is a unique augmentation ε that is the identity in degree zero. Consider the following commutative diagram

$$\begin{tikzcd}
A_{PL}(Y) \arrow[r, "A_{PL}(p)"] & A_{PL}(X) \arrow[r, "A_{PL}(j)"] & A_{PL}(F) \\
(\Lambda V_Y, d) \arrow[u, "m_Y", "\simeq"'] \arrow[r, "i"] & (\Lambda V_Y \otimes \Lambda V, d) \arrow[u, "m"] \arrow[r, "\varepsilon \cdot \mathrm{id}"] & (\Lambda V, \overline{d}), \arrow[u, "\overline{m}"]
\end{tikzcd}$$

in which we assume $(\Lambda V_Y \otimes \Lambda V, d)$ to be a relative Sullivan algebra with base algebra $(\Lambda V_Y, d)$ and i to be the inclusion of the base. Assume further that $m\colon (\Lambda V_Y \otimes \Lambda V, d) \to A_{PL}(X)$ is a morphism which restricts to $A_{PL}(p) \circ m_Y$ in $(\Lambda V_Y, d)$. Then, as before, $A_{PL}(j) \circ m$ factors over $\varepsilon \cdot \mathrm{id}$ to yield the morphism $\overline{m}$.

1. Suppose first $m\colon (\Lambda V_Y \otimes \Lambda V, d) \to A_{PL}(X)$ is a quasi-isomorphism, and hence a relative Sullivan model for $A_{PL}(p) \circ m_Y$. In this case, we can deduce two things: $(\Lambda V_Y \otimes \Lambda V, d)$ is a Sullivan algebra and $\overline{m}$ is a quasi-isomorphism. It follows that both m and $\overline{m}$ provide Sullivan models for X and F respectively.
   Note that we may choose $(\Lambda V_Y \otimes \Lambda V, d)$ to be minimal as a relative Sullivan algebra. Then $(\Lambda V, \overline{d})$ will be a minimal Sullivan algebra and thus $\overline{m}$ is a minimal Sullivan model of F. However, $(\Lambda V_Y \otimes \Lambda V, d)$ will in general not be minimal as a Sullivan algebra, so m might not be a minimal Sullivan model for X.

2. Suppose otherwise that $\overline{m}$ is a quasi-isomorphism, and hence a Sullivan model for F. Then it follows that m is a quasi-isomorphism, and thus with the above a Sullivan model for X.

Example 3.5.14. To give an example how powerful of a tool this is, let us calculate the model of an Eilenberg-MacLane space X of type (G, n), $n \geq 1$ where we assume G to be abelian if $n = 1$. By the Hurewicz theorem, $H_k(X; \mathbb{Z}) = 0$ for $0 \leq k \leq n-1$ and there exists an isomorphism $G = \pi_n(X) \xrightarrow{\cong} H_n(X; \mathbb{Z})$. Applying the functor $- \otimes_{\mathbb{Z}} \mathbb{Q}$ turns this into an isomorphism $G \otimes \mathbb{Q} \xrightarrow{\cong} H_n(X; \mathbb{Q})$. Thus, the dual map

$$H^n(X; \mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(H_n(X; \mathbb{Q}), \mathbb{Q}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Q}}(G \otimes \mathbb{Q}, \mathbb{Q}) = \mathrm{Hom}(G, \mathbb{Q})$$

is an isomorphism, too. Let $V^n := \mathrm{Hom}(G, \mathbb{Q})$ and assume now that $G \otimes \mathbb{Q}$ is finite dimensional as a $\mathbb{Q}$-vector space. We will see by induction that a minimal Sullivan model of X is given by

$$m\colon (\Lambda V^n, 0) \xrightarrow{\simeq} A_{PL}(X),$$

where m is a morphism realizing the isomorphism $V^n \cong H^n(X; \mathbb{Q})$ above.
To begin the induction, let $n = 1$ and choose $g_1, \dots, g_r$ representing a basis of $G \otimes \mathbb{Q}$, by which we mean the elements $g_i \otimes 1$ are a basis for $G \otimes \mathbb{Q}$. This defines a homomorphism $\alpha\colon \mathbb{Z}^r \to G$ via $\alpha(z_1, \dots, z_r) = z_1 g_1 + \dots + z_r g_r$. It is a classical result that this can be realized, meaning there exist Eilenberg-MacLane spaces $K(\mathbb{Z}^r, 2)$ and $K(G, 2)$ as well as a map $f\colon K(\mathbb{Z}^r, 2) \to K(G, 2)$ inducing α, that is, $\pi_2(f) = \alpha$. Now $\alpha \otimes \mathbb{Q}$ maps the canonical basis of $\mathbb{Z}^r \otimes \mathbb{Q}$ to a basis of $G \otimes \mathbb{Q}$. Hence $\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism. From the Whitehead-Serre theorem it follows that $H_*(\Omega f; \mathbb{Q})$ and thus $H^*(\Omega f; \mathbb{Q})$ is an isomorphism, where Ωf is the induced map between the loop spaces.
We know that in general $\Omega K(G, n) = K(G, n-1)$, so in particular $\Omega f\colon K(\mathbb{Z}^r, 1) \to K(G, 1)$. Now X is a $(G, 1)$-space and thus has the weak homotopy type of $K(G, 1) = \Omega K(G, 2)$. Further, $\Pi_{k \leq r} \mathbb{S}^1$ is a $K(\mathbb{Z}^r, 1)$. As weak homotopy equivalences induce isomorphisms between cohomology groups, we deduce that

$$H^*(\Pi_{k \leq r} \mathbb{S}^1; \mathbb{Q}) \cong H^*(K(\mathbb{Z}^r, 1); \mathbb{Q}) \cong H^*(K(G, 1); \mathbb{Q}) \cong H^*(X; \mathbb{Q}),$$

induced by a chain of rational homotopy equivalences. In particular, X and $\Pi_{k \leq r} \mathbb{S}^1$ have isomorphic minimal Sullivan models.

By example 3.5.1 minimal models respect products, so we see that a minimal Sullivan model of X is given by $(\Lambda(e_1,\dots,e_r),0)$ with $|e_i|=1$. Clearly, $\Lambda(e_1,\dots,e_r)=\Lambda V^1$ if we set $V^1 := \mathrm{Hom}(G,\mathbb{Q}) = (G\otimes\mathbb{Q})^*$, as the latter is a $\mathbb{Q}$-vector space of dimension r.
Let $n\geq 2$ and assume the claim is true for $n-1$. if X is an Eilenberg-MacLane space of type (G,n), then ΩX has type $(G,n-1)$. We may use the induction hypothesis and assume ΩX has a model of the form $(\Lambda U^{n-1},0)$ with $U^{n-1}\cong \mathrm{Hom}(G,\mathbb{Q})$ which is finite dimensional by assumption. It follows that $(H_*(\Omega X;\mathbb{Q}))^* = H^*(\Omega X;\mathbb{Q})\cong \Lambda U^{n-1}$ has finite type, so $H_*(\Omega X;\mathbb{Q})$ has finite type. Thus, when looking at the path-loop fibration

$$\Omega X \to PX \to X$$

of X, the conditions of theorem 3.5.12 apply. Choose a minimal Sullivan model $(\Lambda V_X,d)$ for X. Note that $H^k(X;\mathbb{Q})=0$ for $1\leq k\leq n-1$ again due to Hurewicz and the duality between homology and cohomology, so $V_X^k=0$ for $1\leq k\leq n-1$. Since $(\Lambda V_X,d)$ is minimal and $n\geq 2$, we can conclude that $d=0$ in V_X^n. We now have the situation as described in the second part of remark 3.5.13, namely

$$\begin{tikzcd}
A_{PL}(X) \arrow[r] & A_{PL}(PX) \arrow[r] & A_{PL}(\Omega X) \\
(\Lambda V_X,d) \arrow[r] \arrow[u, "\simeq", "m_X"'] & (\Lambda V_X\otimes\Lambda U^{n-1},d) \arrow[r] \arrow[u, "m"'] & (\Lambda U^{n-1},0). \arrow[u, "\simeq", "m_{\Omega X}"']
\end{tikzcd}$$

It follows that m is a quasi-isomorphism, which in turn shows that $H^*(\Lambda V_X\otimes\Lambda U^{n-1},d)=\mathbb{Q}$. This distinguishes $(\Lambda V_X\otimes\Lambda U^{n-1},d)$ as contractible Sullivan algebra, so $d\colon U^{n-1}\to V_X$ is an isomorphism, and V_X is concentrated in degree n.

3.6 Whitehead Product and Duality

We have spent quite some effort on the computation of Sullivan models, but have yet to make specific why they are of central importance to rational homotopy theory. So far, we know that they are invariant under the rational homotopy type. In this section, we will learn that they contain another important invariant in rational homotopy theory, the rational homotopy groups. For simply connected spaces X with rational homotopy of finite type, the vector space V in a minimal Sullivan model is the dual of the rational vector space $\pi_*(X)\otimes\mathbb{Q}$. Since we assumed finite type, this means we obtain the rational homotopy groups directly from V. Moreover, minimal Sullivan models are also interesting from the viewpoint of classic homotopy theory. While all information on the torsion part of the groups is lost when we switch from $\pi_*(X)$ to $\pi_*(X)\otimes\mathbb{Q}$, the rational homotopy groups contain information on the non-finite part of the homotopy groups. We can thus detect non-trivial elements in the homotopy groups of a space, directly from the minimal Sullivan model. Given homotopy groups are in general hard to compute, this makes Sullivan models a strong tool throughout algebraic topology. Furthermore, as we shall see, the differential in a minimal Sullivan model contains information regarding a Lie algebra structure on the homotopy groups. The so-called Whitehead product on rational homotopy groups can be obtained from the quadratic part of the differential via the duality of V and $\pi_*(X)\otimes\mathbb{Q}$.

Preparation 3.6.1. We will begin by defining said *Whitehead product*, which is a map

$$[\ ,\ ]_W\colon \pi_k(X)\times\pi_n(X)\to\pi_{k+n-1}(X)$$

on the homotopy groups of a path connected space X. First, for each $k\geq 1$ consider the continuous map $a_k\colon (I^k,\partial I^k)\to(\mathbb{S}^k,s_0^k)$ which collapses the boundary of the cube I^k to a point, yielding the sphere $\mathbb{S}^k$. It defines a continuous map

$$a_k\times a_n\colon (I^{k+n},\partial I^{k+n},y_1)\to(\mathbb{S}^k\times\mathbb{S}^n,\mathbb{S}^k\vee\mathbb{S}^n,s_0)$$

on the product spaces, where we choose $y_1 := (1, 1, \dots, 1)$ and $s_0 = (s_0^k, s_0^n)$. The restriction to $\partial I^{k+n} \cong \mathbb{S}^{k+n-1}$ then defines a continuous map $a_{k,n} \colon (\mathbb{S}^{k+n-1}, y_0) \to (\mathbb{S}^k \vee \mathbb{S}^n, s_0)$, where $y_0 = \frac{1}{\sqrt{k+n}} y_1$. The corresponding homotopy class $[a_{k,n}] \in \pi_{k+n-1}(\mathbb{S}^k \vee \mathbb{S}^n, s_0)$ is called *universal* (k,n)-*Whitehead product.*

Definition 3.6.2. Let $\alpha \in \pi_k(X, x_0)$ and $\beta \in \pi_n(X, x_0)$ be represented by maps $a \colon (\mathbb{S}^k, s_0^k) \to (X, x_0)$ and $b \colon (\mathbb{S}^n, s_0^n) \to (X, x_0)$. Then the composition

$$(a \vee b) \circ a_{k,n} =: [a, b]_W \colon (\mathbb{S}^{k+n-1}, y_0) \to (\mathbb{S}^k \vee \mathbb{S}^n, s_0) \to (X, x_0)$$

represents an element $[\alpha, \beta]_W := [[a, b]_W] \in \pi_{k+n-1}(X, x_0)$ that is called the *Whitehead product of* α *and* β.

The Whitehead product extends naturally to a bilinear map

$$(\pi_k(X) \otimes \mathbb{K}) \times (\pi_n(X) \otimes \mathbb{K}) \to (\pi_{k+n-1}(X) \otimes \mathbb{K}),$$

where $[\alpha \otimes \lambda, \beta \otimes \mu]_W = [\alpha, \beta]_W \otimes \lambda\mu$. Remember that for a simply connected space X the tensor product $\pi_*(X) \otimes \mathbb{K}$ is always a $\mathbb{K}$-vector space. We may thus write $\alpha \otimes \lambda = \lambda(\alpha \otimes 1)$, where an element of the form $\alpha \otimes 1$ was simply denoted by $\alpha \in \pi_*(X) \otimes \mathbb{K}$. For such elements $\alpha, \beta \in \pi_*(X) \otimes \mathbb{K}$ we can choose representatives a, b. The image under the Whitehead product is then again an element $[\alpha, \beta]_W \in \pi_*(X) \otimes \mathbb{K}$, and it is represented by $[a, b]_W$.

Preparation 3.6.3. Let $f \colon (Y, y_0) \to (X, x_0)$ be a continuous map between simply connected spaces and choose minimal Sullivan models

$$m_Y \colon (\Lambda W, d_W) \to A_{PL}(Y), \qquad m_X \colon (\Lambda V, d_V) \to A_{PL}(X).$$

For f there exists a corresponding Sullivan representative $\varphi_f \colon (\Lambda V, d_V) \to (\Lambda W, d_W)$ whose homotopy class only depends on the choice of the minimal Sullivan models and the homotopy class $[f]$ of f. By proposition 3.4.11, the linear part of all Sullivan representatives of f coincides, so $Q(\varphi_f)$ does not depend on the choice of the representative φ_f. This justifies to define the *linear part* of f by

$$Q(f) := Q(\varphi_f) \colon V \to W.$$

In particular, it holds $Q(f_0) = Q(f_1)$ whenever $f_0 \simeq f_1$, since then $\varphi_{f_0} \simeq \varphi_{f_1}$. Further, if $g \colon Z \to Y$ is continuous, then $Q(f \circ g) = Q(g) \circ Q(f)$ as the composition $\varphi_g \circ \varphi_f$ of Sullivan representatives is a Sullivan representative for the composition $f \circ g$. In the following, this will be used to define a natural pairing between V and $\pi_*(X)$.

Construction 3.6.4. Recall from example 3.3.7 that minimal models for the sphere $\mathbb{S}^k$ are given by $m_k \colon (\Lambda(e), 0) \to A_{PL}(\mathbb{S}^k)$, $|e| = k$ if k is odd and $m_k \colon (\Lambda(e, e'), d) \to A_{PL}(\mathbb{S}^k)$, $|e| = k$, $|e'| = 2k - 1$ with $d(e') := e^2$ if k is even. In both cases, m_k was defined by setting $m_k(e)$ to be a cocycle in $A_{PL}^k(\mathbb{S}^k)$ which represents the unique element in $H^k(A_{PL}(\mathbb{S}^k); \mathbb{K}) \cong \operatorname{Hom}(H_k(\mathbb{S}^k; \mathbb{Z}), \mathbb{K})$ that maps the fundamental class to 1. In other words, $H^k(m_k)[e]$ is defined to be the unique basis element dual to $[\mathbb{S}^k]$ via the pairing described in remark 1.1.14.
Let $\alpha \in \pi_k(X)$ and choose a representative $a \colon (\mathbb{S}^k, s_0) \to (X, x_0)$ and suppose $m_X \colon (\Lambda V, d_V) \to A_{PL}(X)$ is a minimal Sullivan model. Then $Q(a)$ is only non-trivial on level k and, if k is even, on $2k - 1$. We get a map

$$Q(a)|_{V^k} \colon V^k \to \mathbb{K} \cdot e$$

that depends only on α and the choice of m_X. Now for $v \in V^k$ we define $\langle v; \alpha \rangle$ to be the unique element $\lambda \in \mathbb{K}$ such that $\lambda \cdot e = Q(a)(v)$. This can be naturally extended for $v \in V$ and $\alpha \in \pi_*(X)$ by setting $\langle v; \alpha \rangle = 0$ if $|v| \neq |\alpha|$ to define a pairing

$$\langle -; - \rangle \colon V \times \pi_*(X) \to \mathbb{K}.$$

Now, if $f\colon (Y,y_0)\to(X,x_0)$ is a continuous map and $\beta\in\pi_k(Y)$ is represented by $b\colon(\mathbb{S}^k,s_0)\to(Y,y_0)$, it holds $Q(f\circ b)=Q(b)\circ Q(f)$. Thus $Q(b)(Q(f)(v))$ and $Q(f\circ b)(v)$ share the same prefactor $\lambda\in\mathbb{K}$. As $\pi_k(f)(\beta)$ is represented by $f\circ b$, it follows that for each $v\in V$ and $\beta\in\pi_*(Y)$ we get the identity

$$\langle v;\pi_*(f)(\beta)\rangle=\langle Q(f)(v);\beta\rangle.$$

Lemma 3.6.5. *The pairing $\langle -;-\rangle\colon V\times\pi_*(X)\to\mathbb{K}$ is bilinear, that is $\mathbb{K}$-linear in V and $\mathbb{Z}$-linear in $\pi_*(X)$.*

Proof. Linearity in V is immediate, as the map $Q(a)$ is $\mathbb{K}$-linear for any map a representing an element $\alpha\in\pi_*(X)$. Let $k\in\mathbb{N}$ and denote by $i_0,i_1\colon\mathbb{S}^k\hookrightarrow\mathbb{S}^k\vee\mathbb{S}^k$ the inclusion of the top and bottom sphere. Then $[i_0]+[i_1]$ is represented by the composition $i_0\vee i_1\circ c$, where $c\colon\mathbb{S}^k\to\mathbb{S}^k\vee\mathbb{S}^k$ collapses the sphere at the equator, and $i_0\vee i_1$ is the induced map on the wedge $\mathbb{S}^k\vee\mathbb{S}^k$ (which, in this case, is just the identity).
As cohomology is the dual of homology with field coefficients, $H^k(\mathbb{S}^k\vee\mathbb{S}^k;\mathbb{K})\cong(H_k(\mathbb{S}^k\vee\mathbb{S}^k;\mathbb{K}))^*$. Remember that the images of the fundamental class under the induced maps of the inclusions, $H_k(i_0)([\mathbb{S}^k])$ and $H_k(i_1)([\mathbb{S}^k])$, form a basis for $H_k(\mathbb{S}^k\vee\mathbb{S}^k;\mathbb{K})=\mathbb{K}\oplus\mathbb{K}$. Denote by $\omega_0,\omega_1\in H^k(\mathbb{S}^k\vee\mathbb{S}^k;\mathbb{K})$ the corresponding dual basis. Then similar to the minimal model of $\mathbb{S}^k$, we need generators e_0,e_1 of degree k and define a morphism m by mapping these to a representing cocycle of ω_0,ω_1 such that $H^k(m)([e_0])=\omega_0$, $H^k(m)([e_1])=\omega_1$. The corresponding minimal model for $\mathbb{S}^k\vee\mathbb{S}^k$ is the of the form

$$m\colon(\Lambda(e_0,e_1,\dots),d)\xrightarrow{\simeq}A_{PL}(\mathbb{S}^k\vee\mathbb{S}^k),$$

where we will not need to worry about the other generators, as these occur in degrees higher than k. This has also been done in example 3.3.13 for the case $k=2$. Next, suppose $\varphi_0\colon(\Lambda(e_0,e_1,\dots),d)\to(\Lambda(e,\dots),d)$ is a Sullivan representative of i_0, where e is the generator of degree k in the minimal model m_k for $\mathbb{S}^k$ as described in construction 3.6.4 above. We arrive at a homotopy commutative diagram

$$\begin{tikzcd}
A_{PL}(\mathbb{S}^k\vee\mathbb{S}^k) \arrow[r, "A_{PL}(i_0)"] & A_{PL}(\mathbb{S}^k) \\
(\Lambda(e_0,e_1,\dots),d) \arrow[u, "m"] \arrow[r, "\varphi_0"] & (\Lambda(e,\dots),d) \arrow[u, "m_k"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H^*(\mathbb{S}^k\vee\mathbb{S}^k;\mathbb{K}) \arrow[r, "H^*(i_0)"] & H^*(\mathbb{S}^k;\mathbb{K}) \\
H(\Lambda(e_0,e_1,\dots)) \arrow[u, "H(m)", "\cong"'] \arrow[r, "H(\varphi_0)"] & H(\Lambda(e,\dots)), \arrow[u, "\cong", "H(m_k)"']
\end{tikzcd}$$

which translates to the commutative diagram on the left, which allows us to identify $H(\varphi_0)$ with $H^*(i_0)$. By the definition of the dual basis ω_0,ω_1 it holds

$$H^k(i_0)(\omega_0)([\mathbb{S}^k])=\omega_0\circ H_k(i_0)([\mathbb{S}^k])=1$$
$$H^k(i_0)(\omega_1)([\mathbb{S}^k])=\omega_1\circ H_k(i_0)([\mathbb{S}^k])=0,$$

so $H^k(i_0)(\omega_0)$ provides a basis $\omega\in H^k(\mathbb{S}^k;\mathbb{K})$ that is dual to the basis $[\mathbb{S}^k]$ of $H_k(\mathbb{S}^k;\mathbb{K})$, and $H^k(i_0)(\omega_1)$ vanishes. As $H^k(m_k)([e])=\omega$ and $H^k(m)([e_i])=\omega_i$ by definition, we deduce from this that $H^k(\varphi_0)[e_0]=[e]$ and $H^k(\varphi_0)[e_1]=0$.
As there are no elements of degree $k-1$, there are no coboundaries on level k, so we conclude that $\varphi_0(e_0)=e$ and $\varphi_0(e_1)=0$. These images already have word length 1, so the corresponding linear part is the same and we get $Q(i_0)(e_0)=e$ and $Q(i_0)(e_1)=0$. In the same way, we see that $Q(i_1)(e_0)=0$ and $Q(i_1)(e_1)=e$, as well as $Q(c)(e_i)=e$ for $i\in\{0,1\}$.
Assume now $\alpha_0,\alpha_1\in\pi_k(X)$ are represented by maps a_0,a_1. Then $\alpha_0+\alpha_1$ is represented by $a_0+a_1=a_0\vee a_1\circ c$. Let V be the vector space associated with a minimal Sullivan model for X and consider $Q(a_0\vee a_1)\colon V^k\to\mathbb{K}\cdot e_0\oplus\mathbb{K}\cdot e_1$. For $v\in V^k$, write $Q(a_0\vee a_1)(v)=\lambda_0e_0+\lambda_1e_1$. Then

$$Q(a_0\vee a_1\circ i_0)(v)=Q(i_0)\circ Q(a_0\vee a_1)(v)=\lambda_0 e,$$
$$Q(a_0\vee a_1\circ i_1)(v)=Q(i_1)\circ Q(a_0\vee a_1)(v)=\lambda_1 e,$$
$$Q(a_0\vee a_1\circ c)(v)=Q(c)\circ Q(a_0\vee a_1)(v)=(\lambda_0+\lambda_1)e.$$

Now, as $a_0 \vee a_1 \circ i_0 = a_0$ and $a_0 \vee a_1 \circ i_1 = a_1$ and $a_0 \vee a_1 \circ c = a_0 + a_1$ it follows $\langle v; \alpha_0 + \alpha_1 \rangle = \lambda_0 + \lambda_1 = \langle v; \alpha_0 \rangle + \langle v; \alpha_1 \rangle$ as claimed. $\square$

Hence, there is an induced natural linear map

$$\nu_X \colon V \to \mathrm{Hom}_{\mathbb{Z}}(\pi_*(X), \mathbb{K}), \quad v \mapsto \langle v; - \rangle.$$

By the identity from construction 3.6.4 it then holds $\nu_X \circ Q(f) = \mathrm{Hom}_{\mathbb{Z}}(\pi_*(f), \mathbb{K}) \circ \nu_Y$, so this is in fact a natural transformation.

Theorem 3.6.6. *If X is simply connected and $H_*(X; \mathbb{K})$ has finite type, then*

$$\nu_X \colon V_X \to \mathrm{Hom}_{\mathbb{Z}}(\pi_*(X), \mathbb{K})$$

is an isomorphism.

Before we turn our attention towards the proof, let us point out what can be derived from this. A canonical isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\pi_*(X), \mathbb{K}) \cong \mathrm{Hom}_{\mathbb{K}}(\pi_*(X) \otimes \mathbb{K}, \mathbb{K}) = (\pi_*(X) \otimes \mathbb{K})^*$ is given by $\varphi \mapsto \varphi \otimes \mathbb{K}$. Thus, an application of theorem 3.6.6 for $\mathbb{K} = \mathbb{Q}$ yields that the dual space of the rational homotopy groups can be read off the minimal Sullivan model. We assumed $H_*(X; \mathbb{Q})$ to be of finite type, so by corollary 3.3.11, V_X and thus $(\pi_*(X) \otimes \mathbb{Q})^*$ has finite type. As any finite dimensional vector space is naturally isomorphic to its dual space, we finally obtain the isomorphy class of rational homotopy groups from the minimal Sullivan model. Even more, the isomorphism of theorem 3.6.6 allows us to identify some topological maps with algebraic equivalents.

Remark 3.6.7. Consider two simply connected spaces X and Y with rational homology of finite type. Choose minimal Sullivan models

$$m_X \colon (\Lambda V_X, d_X) \xrightarrow{\simeq} A_{PL}(X), \qquad m_Y \colon (\Lambda V_Y, d_Y) \xrightarrow{\simeq} A_{PL}(Y),$$

for X and Y and remember that for a continuous map $f \colon X \to Y$ there exists a Sullivan representative $\varphi_f \colon (\Lambda V_Y, d_Y) \to (\Lambda V_X, d_X)$ such that $A_{PL}(f) \circ m_Y \simeq m_X \circ \varphi_f$. As usual, denote by $Q(f) \colon V_Y \to V_X$ the induced linear part. The linear maps

$$\nu_X \colon V_X \to \mathrm{Hom}_{\mathbb{Z}}(\pi_*(X), \mathbb{K}), \quad \nu_Y \colon V_Y \to \mathrm{Hom}_{\mathbb{Z}}(\pi_*(Y), \mathbb{K})$$

are isomorphisms by theorem 3.6.6. In the construction 3.6.4 we argued that ν_X and ν_Y are natural transformations. This implies that the diagram

$$\begin{tikzcd}
V_X \arrow[r, "\nu_X", "\cong"'] & \mathrm{Hom}_{\mathbb{Z}}(\pi_*(X), \mathbb{K}) \\
V_Y \arrow[u, "Q(j)"] \arrow[r, "\nu_Y", "\cong"'] & \mathrm{Hom}_{\mathbb{Z}}(\pi_*(Y), \mathbb{K}). \arrow[u, "\mathrm{Hom}_{\mathbb{Z}}(\pi_*(f),\mathbb{K})"']
\end{tikzcd}$$

commutes. Therefore, using the canonical identification $\mathrm{Hom}_{\mathbb{Z}}(\pi_*(f), \mathbb{K}) = (\pi_*(f) \otimes \mathbb{K})^*$, we may identify $Q(f)$ with the dual of $\pi_*(f) \otimes \mathbb{K}$.

Let $m \colon (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$ be a minimal Sullivan model for a simply connected space X. Then by minimality, we have $\mathrm{im}(d) \subseteq \Lambda^{\geq 2} V$, which means that the projection $\Lambda V \to V \oplus \mathbb{K} = \Lambda V / \Lambda^{\geq 2} V$ is a morphism of complexes $(\Lambda V, d) \to (V \oplus \mathbb{K}, 0)$. As such, it induces a linear map $\zeta \colon H^+(\Lambda V, d) \to V$.

Remark 3.6.8. Note that the Hurewicz homomorphism $h_{\mathbb{K}} \colon \pi_*(X) \otimes \mathbb{K} \to H(X; \mathbb{K}) = H(X; \mathbb{Z}) \otimes \mathbb{K}$ dualizes to a map $h^*_{\mathbb{K}} \colon H^*(X; \mathbb{K}) \to \mathrm{Hom}(\pi_*(X), \mathbb{K})$. For a minimal Sullivan model m and ζ as above, consider the isomorphism $\nu\colon V \to \mathrm{Hom}(\pi_*(X), \mathbb{K})$. Let $n \geq 1$ and fix $\alpha \in \pi_n(X)$ represented by some continuous map a. The minimal Sullivan models for X and $\mathbb{S}^n$ yield a commutative diagram

$$\begin{tikzcd} H^n(\Lambda V, d) \arrow[r, "H^*(a;\mathbb{K})\circ H(m)"] \arrow[d, "\mathrm{id}"'] & H^n(\mathbb{S}^n; \mathbb{K}) \arrow[d, "\cong"] \\ H^n(\Lambda V, d) \arrow[r, "Q(a)\circ\zeta"] & \mathbb{K}e. \end{tikzcd}$$

For a fundamental class $[\mathbb{S}^n]$ we get $H^n(\mathbb{S}^n; \mathbb{K}) = \mathbb{K}[\mathbb{S}^n]$, and the isomorphism on the right reads $[\mathbb{S}^n] \mapsto e$. Thus, for any $[z] \in H^n(\Lambda V, d)$, if $H^*(a; \mathbb{K}) \circ H(m)([z]) = \lambda[\mathbb{S}^n]$, then $Q(a) \circ \zeta([z]) = \lambda e = \langle \zeta([z]); \alpha\rangle e$. By identifying $H^*(a; \mathbb{K}) = (H_*(a; \mathbb{K}))^*$, we get $H^*(a; \mathbb{K}) \circ H(m)([z]) = H(m)([z]) \circ H_*(a; \mathbb{K})\colon H_n(\mathbb{S}^n; \mathbb{K}) \to \mathbb{K}$ and thus

$$\begin{aligned} \nu(\zeta([z]))(\alpha) &= \langle \zeta([z]); \alpha\rangle = H(m)([z]) \circ H_*(a; \mathbb{K})([\mathbb{S}^n]) \\ &= H(m)([z]) \circ h_{\mathbb{K}}([\alpha]) = h^*_{\mathbb{K}}(H(m)([z]))(\alpha). \end{aligned}$$

In other words, the diagram

$$\begin{tikzcd} H^+(\Lambda V, d) \arrow[r, "H(m)", "\cong"'] \arrow[d, "\zeta"] & H^*(X; \mathbb{K}) \arrow[d, "h^*_{\mathbb{K}}"] \\ V \arrow[r, "\nu", "\cong"'] & \mathrm{Hom}(\pi_*(X), \mathbb{K}) \end{tikzcd}$$

commutes, which exhibits ζ as the dual of $h_{\mathbb{K}}$.

Proof of 3.6.6. Let $n := \min\{k \geq 0 \mid \pi_k(X) \neq 0\}$ and note that by assumption $n \geq 2$. Then $\pi_n(X) \cong H_n(X; \mathbb{Z})$ and $H_k(X; \mathbb{Z}) = 0$ for $1 \leq k < n$ by the Hurewicz theorem. Since $\mathbb{K}$ is a field, as usual, we have an isomorphism $H_*(X; \mathbb{K}) \cong H_*(X; \mathbb{Z}) \otimes \mathbb{K}$ and $H^*(X; \mathbb{K}) \cong (H_*(X; \mathbb{K}), \mathbb{K})^*$ mentioned in remark 1.1.14. This gives $H^k(X; \mathbb{K}) = 0$, which by proposition 3.3.10 implies $V_X^k = 0$ for $k < n$. Moreover, since $(\Lambda V_X, d)$ is minimal and $n \geq 2$, it follows that $d = 0$ in V_X^n, so $H^n(\Lambda V_X) = (\Lambda V_X)^n = V_X^n$ and we get an isomorphism $H(m_X)\colon V_X^n \xrightarrow{\cong} H^n(X; \mathbb{K})$. Finally, there exists an isomorphism

$$h^*_{\mathbb{K}}\colon H^n(X; \mathbb{K}) = \mathrm{Hom}_{\mathbb{K}}(H_n(X; \mathbb{K}), \mathbb{K}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{K}}(\pi_n(X) \otimes \mathbb{K}, \mathbb{K}) = \mathrm{Hom}(\pi_n(X), \mathbb{K})$$

that we already used in example 3.5.14. It is essentially the dual of the Hurewicz homomorphism, which is an isomorphism on this level. In particular, the composition $h^*_{\mathbb{K}} \circ H(m_X)\colon V_X^n \to \mathrm{Hom}(\pi_n(X), \mathbb{K})$ is an isomorphism, and by a more general calculation in remark 3.6.8 this isomorphism exactly is ν_X.

This takes care of levels $k \leq n$. For the remaining cases, assume $f\colon X \to Y$ is a weak homotopy equivalence. Then $H^*(f; \mathbb{K})$ is an isomorphism, and hence any Sullivan representative φ_f is a quasi-isomorphism. As such, remark 3.4.10 tells us that $H(Q(f)) = Q(f)\colon V_Y \to V_X$ is an isomorphism. Further, by the Whitehead-Serre theorem, $\pi_*(f) \otimes \mathbb{K}$ and therefore its dual $\mathrm{Hom}(\pi_*(f), \mathbb{K})$ is an isomorphism. By naturality, the diagram in remark 3.6.7 commutes, implying that if ν_Y is an isomorphism then so is ν_X. This means that by CW approximation, we may assume X is a CW complex.

Let $g\colon X \to K(\pi_n(X), n) := K$ be a continuous map such that $\pi_n(g)$ is the identity. We may turn g into a fibration, that is replacing g by a fibration $p_g\colon X_g \to K$ with a space X_g containing X such that the inclusion $X \hookrightarrow X_g$ is a homotopy equivalence, see for instance [28] or [13]. Hence, there exists a fibration $p\colon X \to K$ with fibre F such that $\pi_n(p)$ is an isomorphism. By example 3.5.14, the minimal model of K is of the form

$$m_K\colon (\Lambda V^n, 0) \xrightarrow{\simeq} A_{PL}(K) \quad \text{with} \quad V^n \cong \mathrm{Hom}(\pi_n(X), \mathbb{K}).$$

In particular, $H^*(K;\mathbb{K})$ has finite type, so theorem 3.5.12 implies a minimal Sullivan model for p determines a model of the form $(\Lambda V_F,\overline{d}) \xrightarrow{\simeq} A_{PL}(F)$ for F. The corresponding commutative diagram as in remark 3.5.13 reads

$$\begin{tikzcd}
A_{PL}(K) \arrow[r, "A_{PL}(p)"] & A_{PL}(X) \arrow[r, "A_{PL}(j)"] & A_{PL}(F) \\
(\Lambda V^n, 0) \arrow[u, "\simeq"', "m_K"] \arrow[r, "i"] & (\Lambda V^n \otimes \Lambda V_F, d) \arrow[u, "\simeq"', "m"] \arrow[r, "\varepsilon\cdot \mathrm{id}"] & (\Lambda V_F, \overline{d}). \arrow[u, "\simeq"', "m_F"]
\end{tikzcd}$$

where $j\colon F \hookrightarrow X$ is the inclusion and m is a Sullivan model for $A_{PL}(p)\circ m_K$. Now, by the long exact sequence of the fibration p and the fact that $\pi_n(p)$ is an isomorphism, we get $\pi_k(F)=0$ for all $1\le k\le n$. Thus, applying once again proposition 3.3.10, for the minimal model of F it holds $V_F^k=0$ for all $1\le k\le n$. It follows that the Sullivan algebra $(\Lambda V^n\otimes \Lambda V_F, d)$ is minimal.
Next, note that we can lift $m_X\colon (\Lambda V_X, d)\xrightarrow{\simeq} A_{PL}(X)$ along m to a quasi-isomorphism

$$\varphi\colon (\Lambda V_X, d)\to (\Lambda V^n\otimes \Lambda V_F, d) \quad \text{such that} \quad m\circ\varphi\simeq m_X.$$

Then proposition 3.4.11 distinguishes φ as an isomorphism. Now, by proposition 3.3.10, ΛV_X has finite type since $H^*(X;\mathbb{K})$ does. The isomorphism φ then ensures that ΛV_F has finite type, and thus so does $H^*(F;\mathbb{K})$.
As $m_F\circ(\varepsilon\cdot\mathrm{id})\circ\varphi = A_{PL}(j)\circ m\circ\varphi\simeq A_{PL}(j)\circ m_X$, it follows that $(\varepsilon\cdot\mathrm{id})\circ\varphi$ is a Sullivan representative for $A_{PL}(j)$, i.e. for j. Thus, $Q(j)=Q((\varepsilon\cdot\mathrm{id})\circ\varphi)=Q(\varepsilon\cdot\mathrm{id})\circ Q(\varphi)$. Especially, in degree $n+1$ this means $Q^{n+1}(j)$ is an isomorphism, but the long exact sequence of the fibration ensures that $\pi_{n+1}(j)$ too is an isomorphism. By naturality it holds

$$\nu_X^{n+1}\circ Q^{n+1}(j)=\mathrm{Hom}(\pi_{n+1}(j),\mathbb{K})\circ\nu_F^{n+1}.$$

However, note that $\min\{k\ge 0 \mid \pi_k(F)\neq 0\}=n+1$ and therefore ν_F^{n+1} is an isomorphism by the arguments in the beginning of the proof. It follows that ν_X^{n+1} is an isomorphism. Inductively repeating this process, starting again with a fibration g, we eventually see that ν_X is an isomorphism as well. $\square$

Example 3.6.9. Remember the minimal model for $\mathbb{S}^n$ is given by $(\Lambda(e),0)$ if n is odd and $(\Lambda(e,e'), d(e')=e^2)$ if n is even, $|e|=n$. By theorem 3.6.6 we obtain

$$\pi_n(\mathbb{S}^{2k+1})\otimes\mathbb{Q}=\begin{cases}\mathbb{Q}, & n=2k+1\\ 0, & \text{else}\end{cases} \qquad \pi_n(\mathbb{S}^{2k})\otimes\mathbb{Q}=\begin{cases}\mathbb{Q}, & n=2k,\ 4k-1\\ 0, & \text{else.}\end{cases}$$

It follows immediately from the homotopy commutative diagram associated with the Sullivan representative for $\mathrm{id}\colon\mathbb{S}^n\to\mathbb{S}^n$ that $Q(\mathrm{id})(e)=e$ in both cases, n even and odd. Thus $\langle e;[\mathrm{id}]\rangle=1$ for the class $[\mathrm{id}]\in\pi_n(\mathbb{S}^n)$. By theorem 3.6.6 it follows that $\langle e;-\rangle$ generates $(\pi_n(\mathbb{S}^n)\otimes\mathbb{Q})^*$ for $n=2k+1$ or $n=2k$ respectively. Thus, in both cases $0\neq[\mathrm{id}]$ in $\pi_n(\mathbb{S}^n)\otimes\mathbb{Q}$.

To specify a non-trivial element in $\pi_{4k-1}(\mathbb{S}^{2k})\otimes\mathbb{Q}$, some extra work is needed. Fortunately, the minimal Sullivan model of X does not only provide us with the rational homotopy groups, but also gives us information on some structures of the groups. More precisely, we obtain the Whitehead product from the quadratic part of the differential of the minimal Sullivan algebra.
To see this, assume $m_X\colon(\Lambda V,d)\to A_{PL}(X)$ to be a Sullivan minimal model of a simply connected space X. We can define a trilinear map

$$\begin{aligned}
&\langle\ ;\ ,\ \rangle\colon \Lambda^2V\times\pi_*(X)\times\pi_*(X)\to\mathbb{K},\\
&\langle v\wedge w;\alpha,\beta\rangle := \langle v;\beta\rangle\langle w;\alpha\rangle + (-1)^{|w||\alpha|}\langle w;\beta\rangle\langle v;\alpha\rangle
\end{aligned}$$

using the dual pairing of V and $\pi_*(X)$, where $v,w\in V$ and $\alpha,\beta\in\pi_*(X)$ are homogeneous.

Proposition 3.6.10. *Let $v \in V$, $\alpha \in \pi_k(X)$ and $\beta \in \pi_n(X)$ and denote by d_1 the quadratic part of the differential of $(\Lambda V, d)$. Then it holds*

$$\langle d_1(v); \alpha, \beta \rangle = (-1)^{k+n-1} \langle v; [\alpha, \beta]_W \rangle.$$

In other words, the Whitehead product is dual to the quadratic part of the differential.

Proof. For a proof of this statement, we refer to chapter 13 in [7]. □

Example 3.6.11. As usual, let $(\Lambda(e, e'), d(e') = e^2)$ be the minimal Sullivan model of $\mathbb{S}^{2n}$. We have already seen that for $\alpha := [\mathrm{id}] \in \pi_{2k}(\mathbb{S}^{2k})$ it holds $\langle e; \alpha \rangle = 1$. We may thus deduce by the formula of proposition 3.6.10 that

$$\langle e'; [\alpha, \alpha]_W \rangle = -\langle e^2, \alpha, \alpha \rangle = -2.$$

Since $\langle e'; - \rangle$ generates $(\pi_{4k-1}(\mathbb{S}^{2k}) \otimes \mathbb{Q})^*$, we get that $0 \neq [\alpha, \alpha]_W$ in $\pi_{4k-1}(\mathbb{S}^{2k}) \otimes \mathbb{Q}$. In particular, $[\alpha, \alpha]_W \in \pi_{4k-1}(\mathbb{S}^{2k})$ is not a torsion element.

Example 3.6.12. From the computation of a minimal Sullivan model of $\mathbb{S}^2 \vee \mathbb{S}^2$ in example 3.3.13, it follows using theorem 3.6.6 that $\pi_n(\mathbb{S}^2 \vee \mathbb{S}^2) \otimes \mathbb{Q}$ is non-trivial for all $n \geq 2$. Therefore, $\pi_n(\mathbb{S}^2 \vee \mathbb{S}^2)$ contains elements of infinite order for each $n \geq 2$. In particular,

$$\pi_n(\mathbb{S}^2 \vee \mathbb{S}^2) \neq 0 \quad \text{for all } n \geq 2.$$

This means that for all $n \geq 2$, there are homotopical non-trivial maps $f\colon \mathbb{S}^n \to \mathbb{S}^2 \vee \mathbb{S}^2$.

The dual pairing can also be used to improve the statement in proposition 3.5.3, concerning commutative models for the pushout $X \cup_f Y$. Assume X is simply connected and

$$m_X\colon (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$$

is a minimal Sullivan model for X. Given a continuous map $f\colon \mathbb{S}^n \to X$, we can attach a $(n+1)$-cell to X to obtain the pushout $X \cup_f \mathbb{B}^{n+1}$. It is well known that the homotopy type of this pushout only depends on the class $[f] =: \alpha \in \pi_n(X)$, so one usually writes $X \cup_\alpha \mathbb{B}^{n+1}$.

Proposition 3.6.13. *Consider the cochain algebra $(\Lambda V \oplus \mathbb{K}u_\alpha, d_\alpha)$ given by*

(i) $|u_\alpha| = n + 1$

(ii) $d_\alpha(u_\alpha) = 0$ *and* $d_\alpha(v) = d(v) + \langle v; \alpha \rangle u_\alpha$ *for all* $v \in V$

(iii) the relations $u_\alpha \cdot v = 0 = u_\alpha^2$ *for all* $v \in \Lambda^+ V$,

where ΛV is assumed to be a subalgebra. It holds that $(\Lambda V \oplus \mathbb{K}u_\alpha, d_\alpha)$ is a commutative model for $X \cup_\alpha \mathbb{B}^{n+1}$.

Proof. Just like before, a proof of this statement can be found in chapter 13 of [7]. □

Remark 3.6.14.

(a) Note that the cochain algebra $(\Lambda V \oplus \mathbb{K}u_\alpha, d_\alpha)$ in general is not a minimal model for $X \cup_\alpha \mathbb{B}^{n+1}$. In fact, $\Lambda V \oplus \mathbb{K}u_\alpha$ is only a free commutative graded algebra in the case that $V = 0$ and n is odd.

(b) By the definition of the pairing, we get $d_\alpha(v) = d(v)$ whenever $|v| \neq n$. This also implies that $d_\alpha(z) = d(z)$ for all $z \in \Lambda^{\geq 2} V$ by the derivation property of d_α.

While proposition 3.6.13 is quite useful in numerous occasions, it does not solve the problem of calculating a minimal model for the pushout. Moreover, there does not seem to be a general way how Sullivan models for $X \cup_\alpha \mathbb{B}^{n+1}$ and X can be related. This makes a geometric interpretation of these models rather difficult.

3.7 Spatial Realization

The functor $A_{PL}(-)$ provides a link between topological spaces and commutative cochain algebras that we have used many times. Now, we want to do the reverse and introduce a functor $|-|$ assigning each commutative cochain algebra a topological space, or to be more precise a CW complex. We will also see that when $\mathbb{K} = \mathbb{Q}$, this functor serves as a cohomology-inverse to $A_{PL}(-)$ in the sense that if $(\Lambda V, d)$ is a simply connected rational Sullivan algebra of finite type, then there exists a quasi-isomorphism $(\Lambda V, d) \xrightarrow{\simeq} A_{PL}(|\Lambda V, d|)$. The goal of this section is to provide a description of this realization, alongside with an overview of the results that are achieved with it. We abstain from trying to prove all of these, and advise the wary reader to check chapter 17 of [7].
The construction of $|-|$ is via the composition of two functors, namely

1. Sullivan's simplicial realization functor $\langle - \rangle$: commutative cochain algebras $\to$ simplicial sets
2. Milnor's realization functor $|-|$: simplicial sets $\to$ CW complexes.

We begin by introducing the latter.

Construction 3.7.1. Let K be a simplicial set, with face and degeneracy maps ∂_i and s_j. Recall the standard n-simplex Δ^n as well as the linear maps $\delta_i \colon \Delta^n \to \Delta^{n+1}$, $\rho_j \colon \Delta^{n+1} \to \Delta^n$ defined in preparation 1.1.15. Equip each K_n with the discrete topology. We define the *Milnor realization* of K to be the quotient space

$$|K| := \left(\dot{\bigcup}_n K_n \times \Delta^n \right) / \sim .$$

The equivalence relation $\sim$ is thereby generated by

$$\begin{aligned} (\partial_i(\sigma), x) &\sim (\sigma, \delta_i(x)), \quad \sigma \in K_{n+1},\ x \in \Delta^n \\ (s_j(\sigma), x) &\sim (\sigma, \rho_j(x)), \quad \sigma \in K_n,\ x \in \Delta^{n+1}. \end{aligned}$$

Remark 3.7.2. This construction is functorial in the following sense: given a morphism $f \colon K \to L$ between simplicial sets, $f_n \colon K_n \to L_n$, then we can consider the maps $f_n \times \mathrm{id} \colon K_n \times \Delta^n \to L_n \times \Delta^n$. Since each K_n and L_n has the discrete topology, these are continuous. Denote the equivalence class of $(\sigma, x) \in K_n \times \Delta^n$ by $[\sigma, x]$. Then there exists a continuous map $|f| \colon |K| \to |L|$ defined by $|f|([\sigma, x]) = [(f_n \times \mathrm{id})(\sigma, x)] = [f_n(\sigma), x]$.

Recall that a simplex $\sigma \in K_n$ is called *degenerate* if $\sigma = s_j(\tau)$ for some j and some $\tau \in K_{n-1}$. If σ is not degenerate, we call it *non-degenerate* and denote the subset of non-degenerate n-simplices by NK_n. Further, we say an element $x \in \Delta^n$ is in the *interior* if it is not in the image of the maps δ_i. Thus, if we let $\partial \Delta^n := \cup_i \delta_i(\Delta^{n-1})$ then the interior is given by $\Delta^n \setminus \partial \Delta^n =: \mathring{\Delta}^n$. We want to show that the topological space $|K|$ in fact admits a cell structure. For this, let $n \geq 0$ and consider the subsimplicial set $K(n)$ given by

$$K(n)_k := \begin{cases} K_k, & \text{if } k \leq n \\ \{ s_j(\tau) \mid 0 \leq j \leq n-1, \tau \in K(n)_{k-1} \}, & \text{if } k > n. \end{cases}$$

This is called the *n-skeleton* of K. Then $N(K(n)_k) = NK_k$ when $k \leq n$ and $N(K(n)_k) = \varnothing$ if $k > n$. Further, denote by $q_K \colon \dot{\bigcup}_n K_n \times \Delta^n \to |K|$ the canonical projection.

Proposition 3.7.3. *For each simplicial set K it holds that $|K|$ is a CW complex with n-skeleton $|K(n)|$. The n-cells are given by the non-degenerate n-simplices $\sigma \in NK_n$, with attaching maps and characteristic maps the respective restrictions $q_K|_{\{\sigma\} \times \partial \Delta^n}$ and $q_K|_{\{\sigma\} \times \Delta^n}$. Here, we identify $\partial \Delta^n = \mathbb{S}^{n-1}$ and $\Delta^n = \mathbb{B}^n$.*

The proof is due to the following central lemma.

Lemma 3.7.4. *The restriction* $\tilde{q}_K\colon \dot{\bigcup}_n NK_n \times \mathring{\Delta}^n \to |K|$ *of the quotient map is a bijection.*

Proof. A detailed proof of this statement can be found in chapter 3 of [16]. □

Note that this way, for any subsimplicial set $L \subseteq K$ it follows that $|L| \subseteq |K|$. To quickly verify this, let $\sigma \in L_n$ be degenerate as element in K, so we find $\tau \in K_{n-1}$ such that $\sigma = s_j(\tau)$. But then $\tau = \partial_j \circ s_j(\tau) = \partial_j(\sigma) \in L_{n-1}$, since $\partial_j \circ s_j = \mathrm{id}$ and ∂_j restricts to L. It follows that σ is degenerate as element in L, thus $NL_n \subseteq NK_n$ for each n. By lemma 3.7.4, this implies $|L| \subseteq |K|$. In particular, we get $|K(n)| \subseteq |K|$ for the n skeleton $K(n)$.
Note that $\tilde{q}_K$ restricts to bijections $\tilde{q}_{K(n)}\colon \dot{\bigcup}_{k=0}^n NK_k \times \mathring{\Delta}^k \to |K(n)|$. Hence, we are given an injection $j_{n-1}\colon |K(n-1)| \hookrightarrow |K(n)|$ by the composition

$$|K(n-1)| \xrightarrow{(\tilde{q}_{K(n-1)})^{-1}} \dot{\bigcup}_{k=0}^{n-1} NK_k \times \mathring{\Delta}^k \hookrightarrow \dot{\bigcup}_{k=0}^{n-1} NK_k \times \mathring{\Delta}^k + NK_n \times \mathring{\Delta}^n \xrightarrow{\tilde{q}_{K(n)}} |K(n)|.$$

In particular, we get a continuous map $j_{n-1} + \tilde{q}_{K(n)}\colon |K(n-1)| + NK_n \times \Delta^n \to |K(n)|$.

Proof of proposition 3.7.3. Suppose by induction that $|K(n-1)|$ is a CW complex of dimension $n-1$. Fix $\sigma \in K_n$ and let $y \in \partial\Delta^n$, then $y = \delta_i(x)$ for some $x \in \Delta^{n-1}$. By the equivalence relation, $[\partial_i(\sigma), x] = [\sigma, \delta_i(x)]$ and $\partial_i(\sigma) \in K_{n-1} = K(n-1)_{n-1}$, so $q_K(\sigma, y) = q_{K(n-1)}(\partial_i(\sigma), x) \in |K(n-1)|$. It follows that we can restrict q_K to a continuous map $q_\sigma\colon \{\sigma\} \times \partial\Delta^n \to |K(n-1)|$. Treating NK_n as an index set (equipped with the discrete topology) we obtain a continuous map

$$(q_\sigma)\colon NK_n \times \partial\Delta^n \to |K(n-1)|$$

that we can use to attach n-cells $\mathring{\Delta}^n$ to $|K(n-1)|$ for each $\sigma \in NK_n$. This produces a n-dimensional CW complex $|K(n-1)| \cup_{(q_\sigma)} (NK_n \times \Delta^n)$ together with a natural projection $p_{n-1}\colon |K(n-1)| + (NK_n \times \Delta^n) \to |K(n-1)| \cup_{(q_\sigma)} (NK_n \times \Delta^n)$. Let $x \in |K(n-1)|$ and $(\sigma, y) \in NK_n \times \Delta^n$ and suppose $x \sim (\sigma, y)$, then $x = q_\sigma(\sigma, y)$ and thus

$$j_{n-1}(x) = \tilde{q}_{K(n)} \circ (\tilde{q}_{K(n-1)})^{-1}(q_\sigma(\sigma, y)) = \tilde{q}_{K(n)}(\sigma, y)$$

by the definition of j_{n-1}. It follows that $j_{n-1} + \tilde{q}_{K(n)}$ factors over p_{n-1} to yield a continuous map $q(n)\colon |K(n-1)| \cup_{(q_\sigma)} (NK_n \times \Delta^n) \to |K(n)|$ fitting into a commutative diagram

$$\begin{tikzcd}
|K(n-1)| + (NK_n \times \Delta^n) \arrow[r, "p_{n-1}"] \arrow[dr, "j_{n-1}+\tilde{q}_{K(n)}"'] & |K(n-1)| \cup_{(q_\sigma)} (NK_n \times \Delta^n) \arrow[d, "q(n)"] \\
 & |K(n)|.
\end{tikzcd}$$

Vice versa, $j_{n-1} + \tilde{q}_{K(n)}$ is surely surjective and it is injective on elements (σ, x) with $x \in \mathring{\Delta}^n$. Thus, let $y, y' \in \partial\Delta^n$ and suppose $\tilde{q}_{K(n)}(\sigma, y) = \tilde{q}_{K(n)}(\sigma, y')$. Then obviously $q_\sigma(\sigma, y) = q_\sigma(\sigma, y') = x$ for some $x \in |K(n-1)|$ and thus $p(\sigma, y) = p(\sigma, y')$. From this it follows that $q(n)$ is injective and thus a bijection.
Suppose now $B \subseteq |K(n)|$ is compact, then $(j_{n-1} + \tilde{q}_{K(n)})^{-1}(B)$ is compact. It follows that $q(n)^{-1}(B)$ is compact, and thus that $q(n)$ is a homeomorphism. Hence, $|K(n)|$ is a CW complex of dimension n. Finally, using the maps $q(n)$, we can define a homeomorphism from a CW complex with the desired properties to $|K|$. □

Remark 3.7.5. In the past, we have established a somewhat converse functor $S(-)$, assigning a simplicial set to each topological space. Starting with a simplicial set K, then $S(|K|)$ is again a simplicial set containing K as a subsimplicial set through the inclusion

$$\xi_K\colon K \to S(|K|), \quad \sigma \mapsto q_\sigma,$$

where q_σ is the restriction of q_K to $\{\sigma\} \times \Delta^n$. This induces a DGC morphism $C_*(\xi_K)\colon C_*(K) \to C_*(|K|)$. Moving on to the homologies of $C_*(K)$ and $C_*(|K|)$ one can show that the map ξ_K induces an isomorphism $H(\xi_K)\colon H(C_*(K);\mathbb{K}) \xrightarrow{\cong} H_*(|K|;\mathbb{K})$.
On the other hand, starting with a topological space X, the realization $|S(X)|$ of $S(X)$ is always a CW approximation to X. This is due to the continuous evaluation map

$$ev\colon \dot{\bigcup}_n S_n(X) \times \Delta^n \to X, \quad (\sigma, t) \mapsto \sigma(t)$$

which factors over the projection $q_{S(X)}\colon \dot{\bigcup}_n S_n(X) \times \Delta^n \to |S(X)|$ to give a continuous map

$$s_X\colon |S(X)| \to X$$

that is a weak homotopy equivalence. From the definition, one gets $s_{|K|} \circ |\xi_K| = \mathrm{id}_{|K|}$.

Proposition 3.7.6. *The map induced map $s_X\colon |S(X)| \to X$ is a weak homotopy equivalence. In other words, it is a CW approximation to X.*
Further, if K is a simplicial set such that $|K|$ is simply connected, then $|\xi_K|\colon |K| \to |S(|K|)|$ is a homotopy equivalence.

Proof. The first claim is shown in chapter 3 of [16]. For the second one, we refer to chapter 17 of [7]. □

Remark 3.7.7. Moreover, one can show that the realization respects products, i.e. there exists a natural homeomorphism $|K \times L| \xrightarrow{\cong} |K| \times |L|$.

Construction 3.7.8. Suppose (A, d) is a commutative cochain algebra. Recall that for two simplicial sets K, L the space of simplicial morphisms $K \to L$ was denoted by $\mathrm{Hom}_{SIM}(K, L)$. Similarly, given DGAs $(A, d), (B, d)$, denote the space of DGA-morphisms by $\mathrm{Hom}_{DGA}(A, B)$. Let $\varphi \in \mathrm{Hom}_{DGA}(A, B)$ and define a simplicial set $\langle A, d\rangle$ and a simplicial morphism $\langle\varphi\rangle\colon \langle B, d\rangle \to \langle A, d\rangle$ by

(a) $\langle A, d\rangle_n := \mathrm{Hom}_{DGA}(A, (A_{PL})_n)$,

(b) for $\sigma \in \langle A, d\rangle_n$, let $\partial_i(\sigma) := \partial_i \circ \sigma$ and $s_j(\sigma) := s_j \circ \sigma$, where ∂_i and s_j are the face and degeneracy maps of the simplicial cochain algebra A_{PL},

(c) for $\sigma \in \langle B, d\rangle$, set $\langle\varphi\rangle(\sigma) := \sigma \circ \varphi$.

This makes $\langle -\rangle$ a contravariant functor from the category of commutative cochain algebras to the category of simplicial sets.

Remark 3.7.9. Consider a simplicial set K and any commutative cochain algebra (A, d). Remember that $A_{PL}(K) = \mathrm{Hom}_{SIM}(K, A_{PL})$. Thus, there is a natural bijection

$$\mathrm{Hom}_{DGA}(A, A_{PL}(K)) \xrightarrow{\cong} \mathrm{Hom}_{SIM}(K, \langle A, d\rangle), \quad \varphi \to f,$$

where $f\colon K \to \langle A, d\rangle$ is defined by $f(\sigma)(a) := \varphi(a)(\sigma)$, $a \in A$, $\sigma \in K_n$. In this regard, $A_{PL}(-)$ and $\langle -\rangle$ are adjoint functors. In particular, if we specify $K = \langle A, d\rangle$ and let f be the identity on $\langle A, d\rangle$ we get a canonical morphism of DGAs,

$$\eta_A\colon (A, d) \to A_{PL}(\langle A, d\rangle), \quad \eta_A(a)(\sigma) = \sigma(a),$$

$a \in A$, $\sigma \in \langle A, d\rangle_n$.

Definition 3.7.10. The *spatial realization* of a commutative cochain algebra (A, d) is the CW complex $|A, d| := |\langle A, d\rangle|$. In the same fashion, the spatial realization of a morphism $\varphi\colon (A, d) \to (B, d)$ is the continuous map $|\varphi| := |\langle\varphi\rangle|$.

Remark 3.7.11.

(a) The Sullivan realization functor has the property that $\langle (A,d)\otimes(B,d)\rangle \cong \langle A,d\rangle \times \langle B,d\rangle$. It follows for the spatial realization functor that

$$|(A,d)\otimes(B,d)| \cong |A,d| \times |B,d|.$$

(b) For a Sullivan algebra $(\Lambda V,d)$ it holds $V = V^+$, so there exists a unique augmentation $\varepsilon\colon (\Lambda V,d) \to \mathbb{K}$, that is the identity in $\mathbb{K}$ and trivial on positive degrees. Thus $\langle \Lambda V,d\rangle$ has a unique 0-simplex which in turn gives a unique 0-cell in $|\Lambda V,d|$. We will use this as base point and denote it by ε.

(c) Suppose now $\varphi \simeq \psi\colon (\Lambda V,d) \to (\Lambda W,d)$ as morphisms between Sullivan algebras by some homotopy $H\colon (\Lambda V,d) \to (\Lambda W,d)\otimes\Lambda(t,dt)$. Then the realization is a map $|H|\colon |\Lambda W,d| \times |\Lambda(t,dt)| \to |\Lambda V,d|$. The augmentations ε_i define 0-cells in $|\Lambda(t,dt)|$, while the identity defines a 1-cell joining these. It follows that $|H|$ induces a homotopy such that

$$|\varphi| \simeq |\psi|\colon |\Lambda W,d| \to |\Lambda V,d|$$

as continuous maps between CW complexes.

We now turn our attention towards the realization of a Sullivan algebra $(\Lambda V,d)$. As described in remark 3.7.11, there exists a unique augmentation $\varepsilon\colon (\Lambda V,d) \to \mathbb{K}$ which defines a 0-cell in $|\Lambda V,d|$, so $(|\Lambda V,d|,\varepsilon)$ is a pointed space. The map $\xi_{\langle\Lambda V,d\rangle}\colon \langle\Lambda V,d\rangle \to S(|\Lambda V,d|)$ is a quasi-isomorphism and injective. Hence in particular, $A_{PL}(\xi_{\langle\Lambda V,d\rangle})$ is a quasi-isomorphism and surjective. We may use this and lemma 3.4.7 to lift $\eta_{(\Lambda V,d)}\colon (\Lambda V,d) \to A_{PL}(\langle\Lambda V,d\rangle)$ over $A_{PL}(\xi_{\langle\Lambda V,d\rangle})$ to obtain a morphism $m_{(\Lambda V,d)}\colon (\Lambda V,d) \to A_{PL}(|\Lambda V,d|)$ such that

$$\begin{tikzcd}
 & A_{PL}(|\Lambda V,d|) \arrow[d, "\simeq"', "A_{PL}(\xi_{\langle\Lambda V,d\rangle})"] \\
(\Lambda V,d) \arrow[ur, dashed, "m_{(\Lambda V,d)}"] \arrow[r, "\eta_{(\Lambda V,d)}"'] & A_{PL}(\langle\Lambda V,d\rangle)
\end{tikzcd}$$

commutes. Now let $\alpha \in \pi_n(|\Lambda V,d|)$ be represented by a map $a\colon (\mathbb{S}^n,s_0) \to (|\Lambda V,d|,\varepsilon)$. We may use the bilinear map from construction 3.6.4, $\langle -;-\rangle\colon V\times\pi_*(|\Lambda V,d|) \to \mathbb{K}$ and obtain an induced homomorphism

$$\mu_n\colon \pi_n(|\Lambda V,d|) \to \mathrm{Hom}_{\mathbb{K}}(V^n,\mathbb{K}), \quad \mu_n(\alpha)(v) := (-1)^n\langle v;\alpha\rangle.$$

Theorem 3.7.12. *Let $(\Lambda V,d)$ be a Sullivan algebra such that $H^1(\Lambda V,d)=0$. If $H(\Lambda V,d)$ has finite type, it holds*

(i) $|\Lambda V,d|$ is simply connected and $\mu_n\colon \pi_n(|\Lambda V,d|) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{K}}(V^n,\mathbb{K})$ is an isomorphism, $n\geq 2$,

(ii) when $\mathbb{K}=\mathbb{Q}$ the map $m_{(\Lambda V,d)}\colon (\Lambda V,d) \xrightarrow{\simeq} A_{PL}(|\Lambda V,d|)$ is a quasi-morphism, hence a Sullivan model for $|\Lambda V,d|$.

Proof. See chapter 17 of [7]. □

The first step in proving this statement is to show that the realization $|\Lambda(v),0|$ with $|v| =: n\geq 2$ is an Eilenberg-MacLane space $K(\mathbb{K},n)$, which is done through the calculation of the homotopy groups rather than an explicit construction. In particular, assuming $\mathbb{K}=\mathbb{Q}$ and n is odd, the realization $|\Lambda(v),0|$ does not yield the sphere $\mathbb{S}^n$ but a rational representative with has the same rational homotopy type. This means $|\Lambda(v),0|$ is weakly homotopy equivalent to $\mathbb{S}^n_{\mathbb{Q}}$, and thus even homotopy equivalent since both are CW complexes. More generally, the first part of theorem

3.7.12 states that $|\Lambda V, d|$ is always a rational space, which suggests a rather complicated geometry. Therefore, while the spatial realization undoubtedly is of high significance in rational homotopy theory, it is difficult to obtain information on the explicit CW structure of $|\Lambda V, d|$, limiting its use for computations.

Remark 3.7.13.

(a) In the situation of theorem 3.7.12, the Sullivan model $m_{(\Lambda V,d)} \colon (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(|\Lambda V, d|)$ is called a *canonical Sullivan model* for $|\Lambda V, d|$.

(b) Now suppose X is a simply connected CW complex with rational homology of finite type and assume $m_X \colon (\Lambda W, d) \xrightarrow{\simeq} A_{PL}(X)$ is a minimal Sullivan model. The adjoint formula from remark 3.7.9 for $K = S(X)$ and $(A, d) = (\Lambda W, d)$ gives a natural simplicial map

$$\gamma_X \colon S(X) \to \langle \Lambda W, d \rangle$$

that is adjoint to m_X.
Moreover, the map $s_X : |S(X)| \to X$ is a homotopy equivalence and thus has a homotopy inverse which is unique up to homotopy. Denote the homotopy inverse of s_X by $t_X \colon X \to |S(X)|$ and let

$$h_X := |\gamma_X| \circ t_X \colon X \to |\Lambda W, d|.$$

It then holds: if $\mathbb{K} = \mathbb{Q}$, this map is a rational homotopy equivalence, and hence a rationalization of X since $|\Lambda W, d|$ is a rational space by theorem 3.7.12.

It is a direct consequence of theorem 3.7.12 that any simply connected Sullivan algebra with rational coefficients and finite type is a Sullivan model of a CW complex. In a similar manner, any Sullivan representative $\varphi_f \colon (\Lambda V, d) \to (\Lambda W, d)$ of a continuous map $f \colon Y \to X$ can again be realized as a map $|\varphi_f| \colon |\Lambda V, d| \to |\Lambda W, d|$ between CW complexes. Its relation to the original map f is explained in the following statement.

Theorem 3.7.14. *Let $f \colon Y \to X$ be a continuous map and assume $(\Lambda V, d) \to A_{PL}(X)$, $(\Lambda W, d) \to A_{PL}(Y)$ are minimal Sullivan models. Further, let $\varphi_f \colon (\Lambda V, d) \to (\Lambda W, d)$ be a Sullivan representative of f. Then*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "h_Y"] & X \arrow[d, "h_X"] \\ |\Lambda W, d| \arrow[r, "|\varphi_f|"'] & |\Lambda V, d| \end{tikzcd}$$

is homotopy commutative.

Proof. See chapter 17 of [7]. □

Now, $f \simeq g$ implies that $\varphi_f \simeq \varphi_g$, since Sullivan representatives of homotopic maps are homotopic morphisms. By remark 3.7.11 this implies $|\varphi_f| \simeq |\varphi_g|$ as continuous maps. Similarly, when $\psi_1 \simeq \psi_2$, then $\varphi_{|\psi_1|} \simeq \varphi_{|\psi_2|}$. As promised, if we restrict to rational homology of finite type, we arrive at the one-to-one correspondences:

- Rational homotopy types of simply connected spaces and isomorphy classes of minimal Sullivan algebras over $\mathbb{Q}$ with $V = V^{\geq 2}$. The bijections are provided by $X \mapsto (\Lambda V_X, d)$ and $(\Lambda V, d) \mapsto |\Lambda V, d|$ on level of the representatives.

- Homotopy classes of continuous maps between rational spaces and homotopy classes of morphisms of Sullivan algebras over $\mathbb{Q}$. The bijections are provided by $f \mapsto \varphi_f$ and $\psi \mapsto |\psi|$ on level of the representatives.

4 Models via Quillen

Prior to the one-to-one correspondence between isomorphy classes of minimal Sullivan algebras and simply connected spaces that we established in the previous chapter, Quillen showed in [20] that the rational homotopy category of simply connected spaces is equivalent to the homotopy category of a class of differential graded Lie algebras. Considering this, it should be possible to find a transgression between these two algebraic models, that is a functor from the world of differential graded Lie algebras to the world of differential graded algebras. We will achieve this by first describing a functor from differential graded Lie algebras to differential graded coalgebras and then using the fact that duality transfers this into a differential graded algebra. The description of this functor that we denote by $C(-)$ will play a large role in the following section, and makes it possible to define Lie models for spaces using Sullivan's functor $A_{PL}(-)$. Moreover, we will introduce free Lie models of spaces that will play a major role in the future. That is, for free chain Lie algebras $(\mathbb{L}_V, d)$, connected and of finite type, there exists an explicit description of a CW complex X for which $(\mathbb{L}_V, d)$ is a Lie model. For example, the number of cells in X^n is determined by the dimension of V_n. One then calls $(\mathbb{L}_V, d)$ a cellular Lie model. Moreover, we will see that given a differential graded algebra (L, d), we can construct a free Lie model. That is a free differential graded algebra, together with a quasi-isomorphism of differential graded algebras, similar to Sullivan models being a model for arbitrary commutative cochain algebras.
We begin this section by introducing a certain type of graded Lie algebras, the so called homotopy Lie algebras. We will then move on to the construction of $C^*(-)$ and free Lie models. This will then be applied to provide models for spaces. Lastly, the realization of free graded Lie algebras will be discussed. As in the previous section, many of the elaborations presented here can be found in [7], sometimes in greater and sometimes in lesser detail.

4.1 Homotopy Lie Algebras and Duality

Suppose X is a simply connected space. Remember that there exists an isomorphism $\partial\colon \pi_*(X) \xrightarrow{\cong} \pi_{*-1}(\Omega X)$, which is the connecting homomorphism of the long exact sequence of *path space fibration* $PX \to X$ of X. Using ∂ and the Whitehead product $[\ ,\ ]_W\colon \pi_k(X) \times \pi_n(X) \to \pi_{k+n-1}(X)$, we can define

$$[\alpha, \beta] := -(-1)^{|\alpha|}\partial([\partial^{-1}(\alpha), \partial^{-1}(\beta)]_W)$$

on elements $\alpha \in \pi_{k-1}(\Omega X)$, $\beta \in \pi_{n-1}(\Omega X)$ which then defines a bracket on the graded vector space $L_X := \pi_*(\Omega X) \otimes \mathbb{K}$. It is a well known result due to Milnor and Moore in [17] that this is in fact a Lie bracket, making L_X into a graded Lie algebra concentrated in degrees ≥ 0.
Assume further that $f\colon X \to Y$ is a continuous map between simply connected spaces. Then the induced map $\pi_*(\Omega f) \otimes \mathbb{K}$ preserves the Lie brackets, i.e. is a morphism of graded Lie algebras, so this construction is in fact functorial.

Remark 4.1.1. Note that ∂ allows us to identify the graded vector space $\pi_*(X)\otimes\mathbb{K}$ with $\pi_*(\Omega X)\otimes\mathbb{K}$ by a downshift of grades. More precisely, we may identify $\pi_*(X) \otimes \mathbb{K}$ with the suspension sL_X of $L_X = \pi_*(\Omega X) \otimes \mathbb{K}$ through setting $s\alpha := -(-1)^{|\alpha|}\partial^{-1}(\alpha)$ for each homogeneous element $\alpha \in L_X$.

Definition 4.1.2. The graded Lie algebra $L_X := (\pi_*(\Omega X) \otimes \mathbb{K}, [\ ,\ ])$ is called the *homotopy Lie algebra of* X with coefficients in $\mathbb{K}$. If $\mathbb{K} = \mathbb{Q}$ we call it the *rational homotopy Lie algebra of* X.

So far, we have defined the free graded Lie algebra $\mathbb{L}_W$ associated with a graded vector space W and the homotopy Lie algebra L_X of a simple connected space X. There is also a natural way to obtain a Lie algebra when provided a minimal Sullivan algebra $(\Lambda V, d)$, which will be introduced in the following.

Remark 4.1.3. Let L be a graded Lie algebra and remember that $\Lambda L = \oplus_{k\geq 0}\Lambda^k L$, where $\Lambda^k L$ is generated by elements of the form $x_1 \wedge \cdots \wedge x_k$ with $x_i \in L$ homogeneous. Now ΛL is a commutative graded algebra, thus for any $\sigma \in S_k$ it holds $x_1 \wedge \cdots \wedge x_k = \pm x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k)}$, where the sign depends on σ and the grades $|x_i|$, $1 \leq i \leq k$. In the following, denote by ε_σ the sign corresponding to σ such that the above equation holds, where we suppress the dependency on the grades of the elements x_i in the notation. In other words, for fix $x_1 \wedge \cdots \wedge x_k \in \Lambda L$, given any $\sigma \in S_k$ there exists $\varepsilon_\sigma \in \{\pm 1\}$ such that

$$x_1 \wedge \cdots \wedge x_k = \varepsilon_\sigma x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k)}.$$

As one can check easily, this implies that $\varepsilon_{\tau\circ\sigma} = \varepsilon_\tau \varepsilon_\sigma$ for $\tau, \sigma \in S_k$ and fix $x_1 \wedge \cdots \wedge x_k \in \Lambda L$.

Preparation 4.1.4. Assume V is a graded vector space. Denote its desuspended dual space by

$$L := s^{-1}\operatorname{Hom}(V, \mathbb{K})$$

such that sL is just the dual space of V. Remember, homogeneous elements of degree k in sL are linear maps $V^k \to \mathbb{K}$. Define a pairing by setting

$$\langle\ ;\ \rangle \colon V \times sL \to \mathbb{K}, \quad \langle v; sx\rangle := (-1)^{|v|} sx(v),$$

on the homogeneous elements $v \in V$, extended trivially whenever v and sx have different degree. Using the notation of remark 4.1.3 we can define a $(k+1)-$linear map $\Lambda^k V \times sL \times \cdots \times sL \to \mathbb{K}$ by setting

$$\langle v_1 \wedge \cdots \wedge v_k; sx_k, \ldots, sx_1\rangle := \sum_{\sigma \in S_k} \varepsilon_\sigma \langle v_{\sigma(1)}; sx_1\rangle \ldots \langle v_{\sigma(k)}; sx_k\rangle.$$

Indeed, the linearity is not the problem here. In the first argument, linearity holds by definition, as we have only defined the map on homogeneous basis elements and proceed to the linear extension. In the remaining arguments, the linearity is immediate due to the pairing. A little more subtle is the well-definedness of the map in the first argument, which is the reason we need the factors ε_σ for. Fix any $\tau \in S_k$, then it holds

$$\begin{aligned}\langle v_1 \wedge \cdots \wedge v_k; sx_k, \ldots, sx_1\rangle &= \sum_{\sigma \in S_k} \varepsilon_\sigma \langle v_{\sigma(1)}; sx_1\rangle \ldots \langle v_{\sigma(k)}; sx_k\rangle \\ &= \sum_{\sigma \in S_k} \varepsilon_{\sigma\circ\tau} \langle v_{\sigma\circ\tau(1)}; sx_1\rangle \ldots \langle v_{\sigma\circ\tau(k)}; sx_k\rangle \\ &= \varepsilon_\tau \langle v_{\tau(1)} \wedge \cdots \wedge v_{\tau(k)}; sx_k, \ldots, sx_1\rangle\end{aligned}$$

where the second step is just a rearrangement of the sum.
Now, assuming finite type and making use of the duality of V and sL, we get a dual basis on sL with respect to the pairing $\langle\ ;\ \rangle$. That is, assuming $(v_i)_{i\in I}$ is a basis for V, a unique basis $(sx_j)_{j\in I}$ of sL is defined by setting $\langle v_i; sx_j\rangle = \delta_{ij}$ on the basis of V. By desuspending, we get the corresponding elements x_j in L. As the elements $(sx_j)_{j\in I}$ form a basis of sL, the set $(x_j)_{j\in I}$ is a basis for L. In the future, we will refer to these bases of V and L as *dual bases*.

While this duality can be defined for any graded vector space, we will now see that in the case V is the graded vector space of a minimal Sullivan algebra, we can define a Lie bracket on L that is induced by the quadratic part of the differential.

Construction 4.1.5. Let $(\Lambda V, d)$ be a minimal Sullivan algebra. Remember we can write the differential as sum $d = d_0 + d_1 + d_2 + \ldots$ where d_i increases word length by i. By minimality, $d_0 = 0$. Further, remember we called d_1 the quadratic part of the differential. As seen in 3.3.4, $(\Lambda V, d_1)$ is itself a minimal Sullivan algebra, that is, $d_1^2 = 0$. As in preparation 4.1.4, define sL to

be the dual space of V with pairing $\langle\ ;\ \rangle$.
Now, for $x, y \in L$ we may define $s[x,y] \in sL$ to be the linear map

$$(-1)^{|v|} s[x,y](v) = \langle v; s[x,y]\rangle := (-1)^{|y|+1}\langle d_1(v); sx, sy\rangle$$

for arbitrary $v \in V$. This yields a map $[\ ,\ ]\colon L \times L \to L$ that we will now see is a Lie bracket.

Proposition 4.1.6. *The map defined in construction 4.1.5 is a Lie bracket on L, so $(L, [\ ,\ ])$ is a graded Lie algebra.*

Proof. Indeed, the bracket is bilinear, which is a direct consequence of the trilinearity of the map $\langle\ ;\ ,\ \rangle\colon \Lambda^2 \times sL \times sL \to \mathbb{K}$. Hence, we may identify it with the induced map on the tensor product $L \otimes L$. Now, by definition, the element $\langle v; s[x,y]\rangle$ can only be non-zero if $|v| = |sx| + |sy| - 1$. Thus $s[x,y]$ is a linear map of degree $|sx| + |sy| - 1$, and by downshifting we get $|[x,y]| = |x| + |y|$. For the anti-symmetry condition, $d_1(v)$ is a linear combination of elements $v_1 \wedge v_2$ of word length two. Since $v_1 \wedge v_2 = \varepsilon v_2 \wedge v_1$ with $\varepsilon = (-1)^{|v_1||v_2|}$, we obtain

$$\begin{aligned}\langle v_1 \wedge v_2; sx, sy\rangle &= \langle v_1; sy\rangle\langle v_2; sx\rangle + \varepsilon\langle v_2; sy\rangle\langle v_1; sx\rangle = \varepsilon^2\langle v_2; sx\rangle\langle v_1; sy\rangle + \varepsilon\langle v_1; sx\rangle\langle v_2; sy\rangle \\ &= \varepsilon(\langle v_1; sx\rangle\langle v_2; sy\rangle + \varepsilon\langle v_2; sx\rangle\langle v_1; sy\rangle) = \varepsilon\langle v_1 \wedge v_2; sy, sx\rangle.\end{aligned}$$

Note that we may assume $|v_1| = |sy|$, $|v_2| = |sx|$ or $|v_1| = |sx|$, $|v_2| = |sy|$ since the pairing vanishes otherwise. It follows that $|v_1||v_2| = |sx||sy|$ and thus $\varepsilon = (-1)^{(|x|+1)(|y|+1)}$. This means that $\langle d_1(v); sx, sy\rangle = (-1)^{(|x|+1)(|y|+1)}\langle d_1(v); sy, sx\rangle$ for all $v \in V$, $x, y \in L$ and thus

$$\begin{aligned}\langle v; s[x,y]\rangle &= (-1)^{|y|+1}\langle d_1(v); sx, sy\rangle = (-1)^{|y|+1}(-1)^{(|x|+1)(|y|+1)}\langle d_1(v); sy, sx\rangle \\ &= (-1)^{|y|+1}(-1)^{(|x|+1)|y|}(-1)^{|x|+1}\langle d_1(v); sy, sx\rangle = -(-1)^{|x||y|}\langle v; s[y,x]\rangle\end{aligned}$$

This immediately implies $[x,y] = -(-1)^{|x||y|}[y,x]$ as claimed.
A similar, yet more tedious computation shows

$$\langle d_1^2(v); sx, sy, sz\rangle = (-1)^{|y|}\langle v; s[x,[y,z]] - s[[x,y],z] - (-1)^{|x||y|}s[y,[x,z]]\rangle.$$

Since $d_1^2 = 0$, we see the Jacoby identity holds on the suspended elements and hence on L. $\square$

Definition 4.1.7. The graded Lie algebra $(L, [\ ,\ ])$ is called the *homotopy Lie algebra of* $(\Lambda V, d)$.

Remark 4.1.8. Not only can we assign a graded Lie algebra to each minimal Sullivan algebra, it also holds that for a morphism $\varphi\colon (\Lambda V, d) \to (\Lambda W, d)$ between minimal Sullivan algebras there exists a corresponding Lie algebra morphism $\psi\colon E \to L$, where E and L are the homotopy Lie algebras of ΛW respectively ΛV. This makes the construction of the homotopy Lie algebra a contravariant functor.

For more details on this, remember that $Q(\varphi)\colon V \to W$ is the linear part of the morphism φ. Since $W \subseteq \Lambda W$, we can view $Q(\varphi)$ as linear map to ΛW. By the universal property of the free commutative graded algebra, this again induces a morphism $\Lambda Q(\varphi)\colon \Lambda V \to \Lambda W$ of graded algebras (which in general will be different from φ). Note that in particular, $\Lambda Q(\varphi)$ does not increase the word length of elements. Now, let $v \in V$, so $\Lambda Q(\varphi)(v) = Q(\varphi)(v)$ has word length one. Thus, $\Lambda Q(\varphi) \circ d_1(v)$ and $d_1 \circ \Lambda Q(\varphi)(v)$ have word length two. By the definition of $Q(\varphi)$, it holds $\varphi(v) - Q(\varphi)(v) \in \Lambda^{\geq 2}W$, so we get

$$d_1 \circ \varphi(v) - d_1 \circ \Lambda Q(\varphi)(v) \in \Lambda^{\geq 3}W, \quad \varphi \circ d_1(v) - \Lambda Q(\varphi) \circ d_1(v) \in \Lambda^{\geq 3}W.$$

Hence the sum $(d_1 \circ \varphi - \varphi \circ d_1)(v) + (\Lambda Q(\varphi) \circ d_1 - d_1 \circ \Lambda Q(\varphi))(v)$ too has a word length of at least three, where the first part is the trivial map since φ commutes with d, so the same is true for d_1. However, we have already noted that the latter part of the sum has word length two, so we conclude

$(\Lambda Q(\varphi) \circ d_1 - d_1 \circ \Lambda Q(\varphi))(v) = 0$ for all $v \in V$. In other words, $\Lambda Q(\varphi)\colon (\Lambda V, d_1) \to (\Lambda W, d_1)$ is a DGA morphism between minimal Sullivan algebras.
Using this we can define

$$\psi\colon E \to L,\ x \mapsto \psi(x),$$

dual to $Q(\varphi)$, that is if $x \in E$ then $\psi(x)$ is the linear map on V that is given by

$$\langle v; s\psi(x)\rangle := \langle Q(\varphi)(v); sx\rangle.$$

Proposition 4.1.9. *The map $\psi\colon E \to L$ is a morphism of graded Lie algebras.*

Proof. Observe first that on elements $v_1 \wedge v_2 \in \Lambda^2 V$ it holds

$$\begin{aligned}\langle \Lambda Q(\varphi)(v_1 \wedge v_2); sx, sy\rangle &= \langle Q(\varphi)(v_1); sy\rangle\langle Q(\varphi)(v_2); sx\rangle + (-1)^{|v_1||v_2|}\langle Q(\varphi)(v_2); sy\rangle\langle Q(\varphi)(v_1); sx\rangle \\ &= \langle v_1; s\psi(y)\rangle\langle v_2; s\psi(x)\rangle + (-1)^{|v_1||v_2|}\langle v_2; s\psi(y)\rangle\langle v_1; s\psi(x)\rangle \\ &= \langle v_1 \wedge v_2; s\psi(x), s\psi(y)\rangle\end{aligned}$$

for any $x, y \in L$.
Further, ψ is a linear map of degree zero, since $Q(\varphi)$ is. Let $v \in V$. Piecing everything together we have

$$\begin{aligned}\langle v; s\psi([x,y])\rangle &= \langle Q(\varphi)(v); s[x,y]\rangle = (-1)^{|y|+1}\langle d_1 \circ Q(\varphi)(v); sx, sy\rangle \\ &= (-1)^{|y|+1}\langle \Lambda Q(\varphi) \circ d_1(v); sx, sy\rangle = (-1)^{|y|+1}\langle d_1(v); s\psi(x), s\psi(y)\rangle \\ &= \langle v; s[\psi(x), \psi(y)]\rangle\end{aligned}$$

and thus $\psi([x,y]) = [\psi(x), \psi(y)]$ as claimed. ☐

It is only natural to wonder if and how the homotopy Lie algebras L of $(\Lambda V, d)$ and L_X of X are related in the case that $(\Lambda V, d)$ is the minimal Sullivan algebra of X,

$$m\colon (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X).$$

An affirmative answer to this question is already suggested when we take a look at the appearing brackets: The Lie bracket in L is derived from d_1, whereas the Lie bracket in L_X is being defined using the Whitehead product. Now remember that by proposition 3.6.10 if $(\Lambda V, d)$ is the minimal Sullivan algebra of X then the Whitehead product in $\pi_*(X)$ is dual to d_1 in the sense that

$$\langle d_1(v); \alpha, \beta\rangle = -(-1)^{k+n}\langle v; [\alpha, \beta]_W\rangle,$$

for $v \in V$, $\alpha \in \pi_k(X)$ and $\beta \in \pi_n(X)$, where we used the pairing between V and $\pi_*(X)$ from construction 3.6.4. Remember also that due to this pairing, there is a map $\mu\colon \pi_*(X) \otimes \mathbb{K} \to \mathrm{Hom}(V, \mathbb{K}) = sL$ that is given by $\mu(\alpha)(v) = (-1)^{|\alpha|}\langle v; \alpha\rangle$. Finally, recall that $sL_X = \pi_*(X) \otimes \mathbb{K}$ with $s\alpha = -(-1)^{|\alpha|}\partial^{-1}(\alpha)$. Hence,

$$\mu\colon sL_X \to sL$$

may be used to define a map $L_X \to L$ by desuspending.

Theorem 4.1.10. *Let $\sigma\colon L_X \to L$ be the linear map that is given by $s\sigma(\alpha) = \mu(s\alpha)$ for $\alpha \in L_X$. Then σ is an isomorphism of graded Lie algebras.*

Proof. We have already seen that the pairing $\langle\ ;\ \rangle$ between V and $\pi_*(X)$ is non-degenerate, so μ and as such σ are isomorphisms of graded vector spaces. What remains to show is that σ is Lie bracket preserving. Let $\alpha, \beta \in L_X$ and observe that by the definition of the suspension sL_X and the bracket in L_X we get

$$\begin{aligned} s[\alpha,\beta] &= -(-1)^{|\alpha|+|\beta|}\partial^{-1}([\alpha,\beta]) = -(-1)^{|\alpha|+|\beta|}\partial^{-1}(-(-1)^{|\alpha|}\partial([\partial^{-1}(\alpha),\partial^{-1}(\beta)]_W)) \\ &= (-1)^{|\beta|}[\partial^{-1}(\alpha),\partial^{-1}(\beta)]_W = (-1)^{|\beta|}[-(-1)^{|\alpha|}s\alpha, -(-1)^{|\beta|}s\beta]_W = (-1)^{|\alpha|}[s\alpha, s\beta]_W. \end{aligned}$$

Due to the duality of d_1 and $[\ ,\ ]_W$ we get

$$\langle v; [s\alpha, s\beta]_W\rangle = (-1)^{|\alpha|+|\beta|+1}\langle d_1(v); s\alpha, s\beta\rangle$$

for any $v \in V$. This implies $\langle d_1(v); s\alpha, s\beta\rangle = (-1)^{|\beta|+1}\langle v; s[\alpha,\beta]\rangle$.
By the definition of σ, the equality $\sigma([\alpha,\beta]) = [\sigma(\alpha),\sigma(\beta)]$ that we have to show reads

$$\mu(s[\alpha,\beta])(v) = s[\sigma(\alpha),\sigma(\beta)](v)$$

for any $v \in V$. This in turn reformulates to

$$(-1)^{|s[\alpha,\beta]|}\langle v; s[\alpha,\beta]\rangle = (-1)^{|v|}\langle v; s[\sigma(\alpha),\sigma(\beta)]\rangle$$

by the definition of μ and the dual pairing of V and sL. We may assume that $|v| = |s[\alpha,\beta]| = |\alpha| + |\beta| + 1$, as otherwise the equality is trivial. Hence, we can drop the prefactors on both sides. Now, for the left-hand side

$$\langle v; s[\alpha,\beta]\rangle = (-1)^{|\beta|+1}\langle d_1(v); s\alpha, s\beta\rangle$$

as we have seen already. On the right-hand side, we obtain

$$\langle v; s[\sigma(\alpha),\sigma(\beta)]\rangle = (-1)^{|\beta|+1}\langle d_1(v); s\sigma(\alpha), s\sigma(\beta)\rangle = (-1)^{|\beta|+1}\langle d_1(v); \mu(s\alpha), \mu(s\beta)\rangle$$

by the definition of the Lie bracket in L and σ. Note that in the upper equality we have a map $\Lambda^2 V \times \pi_*(X) \times \pi_*(X)$ based on the pairing $V \times \pi_*(X)$ as defined in 3.6.4, while in the lower equality the map is $\Lambda^2 \times sL \times sL$ induced from the pairing $V \times sL$ as seen in preparation 4.1.4. However, by the nature of μ, this pairing is again based on the pairing $V \times \pi_*(X)$, so all that is left to do is to go into the definitions and see that both sides are in fact equal. As always, $d_1(v)$ is a linear combination of elements $v_1 \wedge v_2 \in \Lambda^2 V$ with $|v_1| + |v_2| = |v| + 1$, so it suffices to verify equality on those. Note that we can further assume that $|v_1| = |\beta| + 1$, $|v_2| = |\alpha| + 1$ or $|v_1| = |\alpha| + 1$, $|v_2| = |\beta| + 1$ holds, otherwise the appearing terms will vanish. It follows

$$\begin{aligned} \langle v_1 \wedge v_2; s\alpha, s\beta\rangle &= \langle v_1; s\beta\rangle\langle v_2; s\alpha\rangle + (-1)^{|v_2|(|\alpha|+1)}\langle v_2; s\beta\rangle\langle v_1; s\alpha\rangle \\ &= \langle v_1; \mu(s\beta)\rangle\langle v_2; \mu(s\alpha)\rangle + (-1)^{|v_1||v_2|}\langle v_2; \mu(s\beta)\rangle\langle v_1; \mu(s\alpha)\rangle \\ &= \langle v_1 \wedge v_2; \mu(s\alpha), \mu(s\beta)\rangle \end{aligned}$$

where we used the fact that $\langle v; \mu(s\gamma)\rangle = (-1)^{|v|+|\gamma|+1}\langle v; s\gamma\rangle = \langle v; s\gamma\rangle$ if $|v| = |s\gamma| = |\gamma| + 1$ or zero otherwise. □

4.2 The Quillen and Cartan-Eilenberg-Chevalley Construction

In the following, two important functorial constructions are introduced that allow to switch between the worlds of differential graded Lie algebras and differential graded coalgebras. Only the ideas behind these constructions and how they relate will be provided, as the primary focus will be on the applications and examples they make possible.

Construction 4.2.1. Quillen described a functorial way to construct a free DGL when provided a co-augmented DGC (C, d) which is also co-commutative. Remember that through the cobar construction 1.3.9 we obtain a DGA $\Omega C = T(s^{-1}\overline{C})$ with differential $d = d_0 + d_1$, which were maps $d_0 \colon s^{-1}\overline{C} \to s^{-1}\overline{C}$ and $d_1 \colon s^{-1}\overline{C} \to s^{-1}\overline{C} \otimes s^{-1}\overline{C}$. Now, since C is assumed to be co-commutative, we can express d_1 in terms of the commutator bracket on ΩC, namely $\overline{\Delta}(c) = \sum_i a_i \otimes b_i = \sum_i (-1)^{|a_i||b_i|} b_i \otimes a_i$ for $c \in \overline{C}$. Thus d_1 becomes

$$\begin{aligned} d_1(s^{-1}c) &= \frac{1}{2} \sum_i (-1)^{|a_i|} (s^{-1}a_i \otimes s^{-1}b_i - (-1)^{(|a_i|-1)(|b_i|-1)} s^{-1}b_i \otimes s^{-1}a_i) \\ &= \frac{1}{2} \sum_i (-1)^{|a_i|} [s^{-1}a_i, s^{-1}b_i]. \end{aligned}$$

This shows $d_1(s^{-1}\overline{C}) \subseteq \mathbb{L}_{s^{-1}\overline{C}}$ and thus the same is true for d. Furthermore, d_1 is a derivation on ΩC as graded Lie algebra with commutator bracket, and so is d. We conclude that d is a differential in the free graded Lie algebra $\mathbb{L}_{s^{-1}\overline{C}}$, i.e. $(\mathbb{L}_{s^{-1}\overline{C}}, d)$ is a DGL. Note that by example 1.4.10 the universal enveloping algebra is given by the tensor algebra on $s^{-1}\overline{C}$, that is, $U\mathbb{L}_{s^{-1}\overline{C}} = \Omega C$. The functoriality of the construction follows by the universal property of the free graded Lie algebra, for if $\varphi \colon (C, d_C) \to (D, d_D)$ is a DGC morphism, then $\varepsilon_c = \varepsilon_D \circ \varphi$ and thus $\varphi(\overline{C}) \subseteq \overline{D}$. Denoting the induced map $s^{-1}C \to s^{-1}D$ by φ as well, we see $\varphi(s^{-1}\overline{C}) \subseteq s^{-1}\overline{D} \subseteq \mathbb{L}_{s^{-1}\overline{D}}$. Now use the universal property stated in remark 1.4.11 to extend this to a morphism $\mathbb{L}_{s^{-1}\overline{C}} \to \mathbb{L}_{s^{-1}\overline{D}}$ of graded Lie algebras. Finally, observe that this morphism commutes with d_0 since φ commutes with the differentials d_C, d_D and it commutes with d_1 since it preserves the Lie bracket. Hence, we get a DGL morphism.

Definition 4.2.2. The DGL $(\mathbb{L}_{s^{-1}\overline{C}}, d)$ is called the *Quillen construction* on the co-commutative, co-augmented DGC (C, d). To emphasize the functoriality, the construction will be denoted by $\mathcal{L}(C, d)$.

Construction 4.2.3. The other way around, there exists a functor which assigns a DGC $C_*(L, d_L)$ to a provided DGL (L, d_L). We begin with the free commutative graded algebra ΛsL, which is a co-commutative graded coalgebra by example 1.3.8. For an element $sx_1 \wedge \cdots \wedge sx_k \in \Lambda^k sL$ and $1 \le i, j \le k$ let $n_i := \sum_{l<i} |sx_l|$. Given $1 \le i < j \le k$, let n_{ij} be a natural number such that $(-1)^{n_{ij}} = \varepsilon_\sigma$,

$$sx_1 \wedge \cdots \wedge sx_k = \varepsilon_\sigma sx_{\sigma(1)} \wedge sx_{\sigma(2)} \wedge \cdots \wedge sx_{\sigma(k)},$$

where we fix $\sigma \in S_k$ to be the permutation for which holds

$$\sigma(1) = i, \quad \sigma(2) = j, \quad \text{and} \quad \sigma(l) = \begin{cases} l+2, & l < i \\ l+1, & i < l < j \\ l, & l > j. \end{cases}$$

We define the following linear maps

$$\begin{aligned} d_0(sx_1 \wedge \cdots \wedge sx_k) &:= \sum_{i=0}^{k} (-1)^{n_i} sx_1 \wedge \cdots \wedge sd_L(x_i) \wedge \cdots \wedge sx_k, \\ d_1(sx_1 \wedge \cdots \wedge sx_k) &:= \sum_{1 \le i < j \le k} (-1)^{|x_i|+1} (-1)^{n_{ij}} s[x_{\sigma(1)}, x_{\sigma(2)}] \wedge sx_{\sigma(3)} \wedge \cdots \wedge sx_{\sigma(k)} \end{aligned}$$

and observe they extend to coderivations in ΛsL. Hence, $d := d_0 + d_1$ is a coderivation in ΛsL and one can show that it satisfies $d^2 = 0$, implying $(\Lambda sL, d)$ is a DGC. The functoriality of this construction is due to the fact that a DGL morphism $\psi \colon (E, d_E) \to (L, d_L)$ induces a DGC morphism $\Lambda\overline{\psi} \colon \Lambda sE \to \Lambda sL$ by setting $\overline{\psi}(sx) := s\psi(x)$.

Definition 4.2.4. The DGC $(\Lambda sL, d)$ is called the *Cartan-Eilenberg-Chevalley construction* on the differential graded Lie algebra (L, d_L). To emphasize the functoriality, the construction will be denoted by $C_*(L, d_L)$.

Remark 4.2.5. It was Quillen who altered the original construction of Cartan, Eilenberg and Chevalley to fit the case of differential graded Lie algebras in [20]. Moreover, he showed that this construction preserves quasi-isomorphisms. That is, assuming $\psi\colon (E, d) \to (L, d_L)$ is a quasi-isomorphism of differential graded Lie algebras, then

$$C_*(\psi)\colon C_*(E, d_E) \to C_*(L, d_L)$$

is a quasi-isomorphism of differential graded coalgebras. If in fact E and L are concentrated in positive degrees, one can show the converse implication holds as well.

Theorem 4.2.6. *Suppose (L, d_L) is a connected chain Lie algebra, i.e. $L = (L_i)_{i\geq 1}$, and (C, d_C) is a co-commutative differential graded coalgebra with $C = \mathbb{K} \oplus_{i\geq 2} C_i$. Then there exist natural quasi-isomorphisms*

$$\varphi\colon (C, d_C) \xrightarrow{\simeq} C_*(\mathcal{L}(C, d_C)) \quad \textit{and} \quad \psi\colon \mathcal{L}(C_*(L, d_L)) \xrightarrow{\simeq} (L, d_L)$$

of differential graded coalgebras or, respectively, differential graded Lie algebras.

Proof. We do not rely on this result, but wanted to include it since it provides fruitful insight to the theory. For proof, we advise the reader to check appendix B of [20] or chapter 22 of [7]. □

As usual, constructions and definitions on chain complexes can be transferred to the world of cochain complexes by applying the $\mathrm{Hom}(-, \mathbb{K})$ functor. We can hence make use of the Quillen and Cartan-Eilenberg-Chevalley constructions to build a bridge between DGLs and CDGAs, or more specific connected chain Lie algebras and commutative cochain algebras.

Remark 4.2.7. Consider a DGL (L, d_L) and with it the associated DGC $C_*(L, d_L)$. By example 1.3.6,

$$C^*(L, d_L) := \mathrm{Hom}(C_*(L, d_L), \mathbb{K})$$

is a differential graded algebra. Now, since $C_*(L, d_L)$ is co-commutative, the DGA $C^*(L, d_L)$ is commutative. Furthermore, in case that (L, d_L) is a connected chain Lie algebra, it follows by construction that $C^*(L, d_L)$ is a cochain algebra. Thus, we find ourselves back in a well-known world in which strong machinery like that of Sullivan models already has been established.

These constructions, while powerful, involve a multitude of different steps and sub constructions, making them quite complicated and limiting their practical use. However, in case that (L, d_L) is a connected chain Lie algebra with L of finite type, we will now provide a simpler description of $C^*(L, d_L)$ that exhibits it as a Sullivan algebra.

Preparation 4.2.8. We begin by noting that $C^*(L) = (\Lambda sL)^*$ by definition. Now there is a natural surjection $\varphi\colon \Lambda sL \to sL$ which is the identity on elements of word length one and otherwise zero. It dualizes to an injection $\varphi^*\colon (sL)^* \hookrightarrow C^*(L)$ which extends to a morphism of graded algebras $\Lambda\varphi^*\colon \Lambda(sL)^* \to C^*(L)$ that, as we will see now, is an isomorphism. Set $V := (sL)^*$ with dual pairing $\langle\ ;\ \rangle\colon V \times sL \to \mathbb{K}$, $\langle v; sx\rangle := v(sx)$. Then, for any $v \in V$ it immediately holds $\varphi^*(v)(sx) = \langle v; sx\rangle$ for all $sx \in sL$, whereas $\varphi^*(v)$ is trivial on $\Lambda^{\geq 2} sL$ by definition. We may thus abuse language and identify v with its image $\varphi^*(v)$ in $C^*(L) = (\Lambda sL)^*$.
Our goal is to identify elements $v_1 \wedge \cdots \wedge v_k \in \Lambda^k V$ with their image in $C^*(L)$ under $\Lambda\varphi^*$. Let $v, w \in V$ and consider the element $\Lambda\varphi^*(v \wedge w) = \varphi^*(v)\varphi^*(w) \in C^*(L)$. The comultiplication Δ in $C_*(L)$ is the unique algebra morphism $\Lambda sL \to \Lambda sL \otimes \Lambda sL$ obtained by $\Delta(sx) = sx \otimes 1 + 1 \otimes sx$,

where the multiplication on $\Lambda sL \otimes \Lambda sL$ is the usual multiplication on the tensor product of two graded algebras. Thus,

$$\begin{aligned}\Delta(sx \wedge sy) &= (sx \otimes 1 + 1 \otimes sx)(sy \otimes 1 + 1 \otimes sy) \\ &= sx \wedge sy \otimes 1 + sx \otimes sy + (-1)^{|sx||sy|} sy \otimes sx + 1 \otimes sx \wedge sy\end{aligned}$$

for $sx, sy \in sL$. In combination with the definition of the multiplication in $C^*(L)$ this yields

$$\begin{aligned}\Lambda\varphi^*(v \wedge w)(sx \wedge sy) &= (\varphi^*(v) \otimes \varphi^*(w))(\Delta(sx \wedge sy)) \\ &= \langle v; sx\rangle\langle w; sy\rangle + (-1)^{|sx||sy|}\langle v; sy\rangle\langle w; sx\rangle.\end{aligned}$$

for any $sx \wedge sy \in \Lambda^2 sL$, which we define to be $\langle v \wedge w; sx \wedge sy\rangle$. Then $\langle v \wedge w; -\rangle := \Lambda\varphi^*(v \wedge w) \in C^*(L)$ vanishes on the subspace $\Lambda^{\geq 3}$ by definition of φ^*. Further, we also see that $\langle v \wedge w; -\rangle|_{sL} = 0$, since L is connected and hence V^0 vanishes. Hence we may naturally identify $\langle v \wedge w; -\rangle \in (\Lambda^2 sL)^* \subseteq C^*(L)$, and the same argumentation shows that $v_1 \wedge \cdots \wedge v_k \in \Lambda^k V$ defines an element in $(\Lambda^k sL)^*$ for $k \geq 1$ and $v_i \in V$. We now claim that for $v_i \in V$, $sx_i \in sL$ and any $k \in \mathbb{N}$

$$\langle v_1 \wedge \cdots \wedge v_k; sx_1 \wedge \cdots \wedge sx_k\rangle = \sum_{\sigma \in S_k} \varepsilon_\sigma \langle v_1; sx_{\sigma(1)}\rangle \ldots \langle v_k; sx_{\sigma(k)}\rangle,$$

where we proceed to write $\langle v_1 \wedge \cdots \wedge v_k; -\rangle := \Lambda\varphi^*(v_1 \wedge \cdots \wedge v_k)$. Here, ε_σ is defined as usual, depending on σ and the degrees of the elements $sx_i \in sL$. Indeed, assume the claim to be true for k and observe it holds

$$\Delta(sx_1 \wedge \cdots \wedge sx_m) = \sum_{n=0}^{m} \frac{1}{n!}\frac{1}{(m-n)!} \sum_{\sigma \in S_m} \varepsilon_\sigma sx_{\sigma(1)} \wedge \cdots \wedge sx_{\sigma(n)} \otimes sx_{\sigma(n+1)} \wedge \cdots \wedge sx_{\sigma(m)}$$

for any $m \geq 1$. Then by this formula and the induction hypothesis we get for $v := v_1 \wedge \cdots \wedge v_{k+1}$ and $sx := sx_1 \wedge \cdots \wedge sx_{k+1}$

$$\begin{aligned}\Lambda\varphi^*(v)(sx) &= (\varphi^*(v_1) \otimes \Lambda\varphi^*(v_2 \wedge \cdots \wedge v_{k+1}))(\Delta(sx_1 \wedge \cdots \wedge sx_{k+1})) \\ &= \frac{1}{k!}\sum_{\sigma \in S_{k+1}} \varepsilon_\sigma \langle v_1; sx_{\sigma(1)}\rangle\langle v_2 \wedge \cdots \wedge v_{k+1}; sx_{\sigma(2)} \wedge \cdots \wedge sx_{\sigma(k+1)}\rangle \\ &= \frac{1}{k!}\sum_{\sigma \in S_{k+1}} \varepsilon_\sigma \langle v_1; sx_{\sigma(1)}\rangle \sum_{\tau \in S_k} \varepsilon_\tau \langle v_2; sx_{\tau\circ\sigma(2)}\rangle \ldots \langle v_{k+1}; sx_{\tau\circ\sigma(k+1)}\rangle \\ &= \frac{1}{k!}\sum_{\tau \in S_k}\sum_{\sigma \in S_{k+1}} \varepsilon_\tau\varepsilon_\sigma \langle v_1; sx_{\sigma(1)}\rangle\langle v_2; sx_{\tau\circ\sigma(2)}\rangle \ldots \langle v_{k+1}; sx_{\tau\circ\sigma(k+1)}\rangle \\ &= \sum_{\sigma \in S_{k+1}} \varepsilon_\sigma \langle v_1; sx_{\sigma(1)}\rangle \ldots \langle v_{k+1}; sx_{\sigma(k+1)}\rangle.\end{aligned}$$

Assume now $(sx_i)_{i \in \mathbb{N}}$ is a Basis for sL and let $(v_i)_{i \in \mathbb{N}}$ be the dual basis, that is $\langle v_j; sx_i\rangle = \delta_{ij}$. By the evaluation formula above, we immediately see that $\langle v_i^{n_i}; sx_i^{n_i}\rangle = n_i!$ where $n_i \geq 2$ only if $|v_i|$ is even. It follows

$$\langle v_1^{n_1} \wedge \cdots \wedge v_k^{n_k}; sx_1^{n_1} \wedge \cdots \wedge sx_k^{n_k}\rangle = n_1! \ldots n_k!$$

whereas any other element in ΛsL that is not a scalar multiple evaluates to zero. Remember that we assumed L to have finite type, so ΛsL is finite dimensional in each degree. Since elements of the form $sx_1^{n_1} \wedge \cdots \wedge sx_k^{n_k}$ provide a basis of ΛsL, this shows $\langle v_1^{n_1} \wedge \cdots \wedge v_k^{n_k}; -\rangle = \Lambda\varphi^*(v_1^{n_1} \wedge \cdots \wedge v_k^{n_k})$ is a basis for $(\Lambda sL)^* = C^*(L)$. In other words, $\Lambda\varphi^*$ is an isomorphism.

This identifies $C^*(L, d_L)$ as cochain algebra of the form $(\Lambda V, d)$, where the differential d remains unchanged. Observe $V = V^{\geq 2}$, since $L = L_{\geq 1}$ and $V = (sL)^*$. This automatically exhibits $(\Lambda V, d)$ as Sullivan algebra with $V(0) := V \cap \ker(d)$ and $V(k) := V \cap d^{-1}(\Lambda V(k-1))$ for $k \geq 1$.

Proposition 4.2.9.

(i) Suppose (L, d_L) is a connected chain Lie algebra of finite type. Then $C^*(L, d_L) \cong (\Lambda V, d)$ is a Sullivan algebra, where $V = (sL)^*$ and $d = d_0 + d_1$. Here, d_0 is defined by d_L and d_1 is defined by the Lie bracket in L via the dual pairing form preparation 4.2.8,

$$\langle d_0(v); sx\rangle = (-1)^{|v|}\langle v; sd_L(x)\rangle, \quad \langle d_1(v); sx \wedge sy\rangle = (-1)^{|y|+1}\langle v; s[x, y]\rangle.$$

(ii) Suppose $(\Lambda V, d)$ is a Sullivan algebra with $d = d_0 + d_1$ and $V = V^{\geq 2}$ has finite type. Then $(\Lambda V, d) \cong C^*(L, d_L)$ for a unique connected chain Lie algebra (L, d_L) with $sL = V^*$. Here, d_L is defined by d_0 and the Lie bracket is defined by d_1 via the dual pairing from preparation 4.1.4,

$$\langle v; sd_L(x)\rangle = (-1)^{|v|}\langle d_0(v); sx\rangle, \quad \langle v; s[x, y]\rangle = (-1)^{|y|+1}\langle d_1(v); sx, sy\rangle.$$

Proof. The first claim is immediate from preparation 4.2.8, since the differential d in ΛV is by definition the dual of the differential $d' = d_0' + d_1'$ in $C_*(L, d_L)$. So by the definition of the dual differential $d = \mathrm{Hom}(d', \mathbb{K})$ and the definition of the differential d' in $C_*(L, d_L)$ we obtain

$$\begin{aligned}\langle d_0(v); sx\rangle = d_0(v)(sx) = -(-1)^{|v|}v(d_0'(sx)) = (-1)^{|v|}v(sd_L(x)) = (-1)^{|v|}\langle v; sd_L(x)\rangle,\\ \langle d_1(v); sx \wedge sy\rangle = -(-1)^{|v|}v(d_1'(sx \wedge sy)) = (-1)^{|v|}v((-1)^{|x|}s[x, y]) = (-1)^{|y|+1}\langle v; s[x, y]\rangle,\end{aligned}$$

where in the last step we assumed $|v| = |sx| + |sy| - 1$, as otherwise the pairing vanishes.
For the second claim, associated with $(\Lambda V, d_1)$ we obtain the homotopy Lie algebra $(L, [\ ,\ ])$ where L was the desuspension of V^* and whose Lie bracket was exactly defined by the above formula, using the fact that $d_1^2 = 0$. For the differential d_L, observe that $d^2 = 0$ implies that $d_0 \circ d_1 + d_1 \circ d_0 = 0$ and $d_0^2 = 0$, since they rise word length by one respectively zero. From the first identity it follows that d_L is a Lie derivation, as

$$\begin{aligned}\langle v; sd_L([x, y])\rangle &= (-1)^{|v|}\langle d_0(v); s[x, y]\rangle = (-1)^{|v|}(-1)^{|y|+1}\langle d_1 \circ d_0(v); sx, sy\rangle\\ &= (-1)^{|v|+1}(-1)^{|y|+1}\langle d_0 \circ d_1(v); sx, sy\rangle\\ &= (-1)^{|y|+1}\langle d_1(v); sd_L(x), sy\rangle + (-1)^{|v|+1}\langle d_1(v); sx, sd_L(y)\rangle\\ &= \langle v; s[d_L(x), y]\rangle + (-1)^{|x|}\langle v; s[x, d_L(y)]\rangle,\end{aligned}$$

where we used the fact that $|v| = |x| + |y|$ in the last step. The second to last step is due to the fact that $\langle d_0(w); sx, sy\rangle = \langle w; sx, sd_L(y)\rangle + (-1)^{|w|}\langle w; sd_L(x), sy\rangle$ for any $w \in \Lambda^2 V$, which follows by the definition of the pairing and the derivation property of d_0. Finally, we can directly deduce that d_L is a differential from the fact that $d_0^2 = 0$, so (L, d_L) is a connected chain Lie algebra. $\square$

Remark 4.2.10. Given any minimal Sullivan algebra $(\Lambda V, d)$ where $V = V^{\geq 2}$ has finite type, remember $(\Lambda V, d_1)$ is itself a minimal Sullivan algebra. The connected chain Lie algebra L that we obtain from $(\Lambda V, d_1)$ through the identity

$$C^*(L, 0) = (\Lambda V, d_1)$$

as described in proposition 4.2.9 is exactly the homotopy Lie algebra of $(\Lambda V, d)$. Vice versa, if L is the homotopy Lie algebra of some Sullivan algebra $(\Lambda V, d)$ and we view it as a DGL with zero differential, then the corresponding cochain algebra $C^*(L, 0)$ is exactly the minimal Sullivan algebra $(\Lambda V, d_1)$. This is due to the fact that V has finite type, so each V^i is a reflexive vector space. We thus get a natural isomorphism $j\colon V \to (V^*)^* = (sL)^*$, which is given by $j(v)(sx) = \langle v; sx\rangle$ with pairing defined as in preparation 4.2.8. Hence, we may again identify v with $\langle v; -\rangle$ and, this way, identify $C^*(L, 0)$ with $(\Lambda V, d_1)$.

Remark 4.2.11. Note that the expressions in proposition 4.2.9 exactly mean that d_0 is the suspended dual of d_L and, the other way around, d_L is the desuspended dual of d_0. That is, given (L, d_L) and $V := (sL)^*$, the suspended dual is the differential $\overline{d}_L^*\colon V \to V$ given by

$$\langle \overline{d}_L^*(v); sx\rangle = (-1)^{|v|}\langle v; -\overline{d}_L(sx)\rangle = (-1)^{|v|}\langle v; sd_L(x)\rangle = \langle d_0(v); sx\rangle,$$

which shows $d_0 = \overline{d}_L^*$. Vice versa, given $(\Lambda V, d)$ and $sL := V^*$, we see that for the dual $d_0^*\colon sL \to sL$ it holds

$$\langle v; d_0^*(sx)\rangle = (-1)^{|sx|}\langle d_0(v); sx\rangle = \langle v; -sd_L(x)\rangle = \langle v; \overline{d}_L(sx)\rangle,$$

and therefore $\overline{d}_L = d_0^*$.

Example 4.2.12. We have seen in 1.4.6 that the free Lie algebra $\mathbb{L}_{(v)}$ on a single generator is given by

$$\mathbb{L}_{(v)} = \begin{cases} \mathbb{K}v & \text{if } |v| = 2n, \\ \mathbb{K}v \oplus \mathbb{K}[v, v] & \text{if } |v| = 2n+1, \end{cases}$$

for some $n \geq 1$. By proposition 4.2.9, in the first case $C^*(\mathbb{L}_{(v)}, 0)$ is just the Sullivan algebra in one generator e dual to v, meaning e is the unique linear map on $\mathbb{K}v$ that is given by $\langle v; e\rangle = 1$. The differential in $C^*(L)$ is trivial, as the bracket and differential in $\mathbb{L}_{(v)}$ vanish. Since the vector space generating $C^*(L)$ is the suspended dual of L, we arrive at $|e| = 2n+1$. In the second case, the bracket is not trivial and we get an element $[v, v]$ of degree $4n+2$. We again denote the element dual to v by e, this time $|e| = 2n+2$, and get a second generator e' which is dual to $[v, v]$, so $|e'| = 4n+3$. Then $d_1(e')$ is the element defined by $\langle d_1(e'); sv\wedge sv\rangle = \langle e'; s[v, v]\rangle = 1$, as $sv\wedge sv$ is the only element in $\Lambda s\mathbb{L}_{(v)}$ of degree $4n+4$. However, it holds $\langle e^2; sv\wedge sv\rangle = \langle e; sv\rangle^2 + (-1)^{|sv||sv|}\langle e; sv\rangle^2 = 2$ and thus $2d_1(e') = e^2$. We may then switch to $2e'$ as generator, naming it e' again, to arrive at $d_1(e') = e^2$. As before, $d_0 = 0$ since there is no differential in $\mathbb{L}_{(v)}$, and so $d = d_1$. To summarize, we get

$$C^*(\mathbb{L}_{(v)}, 0) = \begin{cases} (\Lambda(e), 0) & \text{with } |e| = 2n+1, \\ (\Lambda(e, e'), d(e') = e^2) & \text{with } |e| = 2n+2. \end{cases}$$

Suppose now (E, d_E) is an arbitrary, connected chain Lie algebra of finite type and let $(\Lambda V, d_V)$ be the minimal Sullivan algebra from the minimal Sullivan model

$$m\colon (\Lambda V, d_V) \xrightarrow{\simeq} C^*(E, d_E)$$

of the Sullivan algebra $C^*(E, d_E) = (\Lambda W, d_0 + d_1)$ with $W = (sE)^*$ by proposition 4.2.9. As before, let L be the homotopy Lie algebra of $(\Lambda V, d_V)$. The linear part of m is the map

$$Q(m)\colon V \to W,$$

which commutes with the linear part of the differentials. Since $(\Lambda V, d_V)$ is minimal, the linear part of d_V vanishes and we are left with $d_0 \circ Q(m) = 0$. In other words, $Q(m)$ as a morphism of complexes $(V, 0) \to (W, d_0)$ and hence induces $H(Q(m))\colon V \to H(W, d_0)$. Finally, it holds $H(W, d_0) = H((sE)^*, \overline{d}_E^*) = sH(E, d_E)^*$ and $V^* = sL$, which means $H(Q(m))$ dualizes to a map

$$H(Q(m))^*\colon sH(E, d_E) \to sL.$$

Proposition 4.2.13. *The map $Q(m)$ induces an isomorphism $H(Q(m))\colon V \xrightarrow{\cong} H(W, d_0)$ of graded vector spaces, which desuspends to a dual isomorphism*

$$s^{-1}H(Q(m))^*\colon H(E, d_E) \xrightarrow{\cong} L$$

of graded Lie algebras.

Proof. We begin by noting that as mentioned in remark 3.4.10, it holds that $H(Q(m))$ is an isomorphism, thus $s^{-1}H(Q(m))^*$ surely is an isomorphism of graded vector spaces. Now, by comparison of word length, the fact that $(d_0 + d_1)^2 = 0$ implies $d_0 \circ d_1 = -d_1 \circ d_0$. The sign can be corrected by setting $d_1'(w) := (-1)^{|w|} d_1(w)$ for $w \in W$ and denoting the resulting derivative on ΛW again by d_1. Note that this does not change the isomorphy type of $(\Lambda W, d_0 + d_1)$ and we end up with $d_0 \circ d_1 = d_1 \circ d_0$. In other words, $d_1 \colon (\Lambda W, d_0) \to (\Lambda W, d_0)$ induces

$$H(d_1)\colon H(\Lambda W, d_0) \to H(\Lambda W, d_0).$$

Consider $C^*(H(E), 0) = (\Lambda s H(E)^*, d_1')$, where the differential d_1' is defined by the Lie bracket in $H(E)$. The natural isomorphism $H(W) = H(sE^*) \cong sH(E)^*$ given by $[w] \mapsto H(w)$ identifies $H(d_1)$ with d_1'. Thus $C^*(H(E), 0) = (\Lambda H(W, d_0), H(d_1))$, which exhibits $H(E)$ as homotopy Lie algebra of $(\Lambda H(W, d_0), H(d_1))$.
To show that homotopy Lie algebras are a functorial construction, in proposition 4.1.9 we argued that a morphism $\Lambda Q(\varphi)\colon (\Lambda V, d_1) \to (\Lambda W, d_1)$ induces a morphism of graded Lie algebras $\psi = s^{-1}Q(\varphi)^*\colon E \to L$ between the homotopy Lie algebras. The same then applies to

$$\Lambda H(Q(m))\colon (\Lambda V, (d_V)_1) \to (\Lambda H(W, d_0), H(d_1))$$

and shows that $s^{-1}H(Q(m))^*\colon H(E) \to L$ is a morphism of graded Lie algebras. $\square$

Remark 4.2.14. Proposition 4.2.13 states that, for a connected chain Lie algebra of finite type, the minimal Sullivan model via $C^*(-)$ contains the homology Lie algebra in the form of the homotopy Lie algebra of the model. Moreover, it shows that if we replace $C^*(E, d_E) = (\Lambda W, d_0 + d_1)$ by a minimal Sullivan algebra $(\Lambda V, d_V)$, we get rid of d_0 and obtain $(d_V)_1 = H(d_1)$. However, the differential d_V may now have arbitrary many terms $(d_V)_i$ of higher order, raising word length by $i \geq 1$.

We have covered how to translate from connected chain Lie algebras to commutative cochain algebras. For the sake of completeness, let us briefly discuss the other way round. By utilizing $\mathcal{L}$ on the dual of a commutative cochain algebra, we obtain a transition from the commutative cochain algebras to connected chain Lie algebras.

Remark 4.2.15. Suppose we have a commutative cochain algebra (A, d) of finite type with $A^0 = \mathbb{K}$ and $A^1 = 0$. Set $(C, d_C) := \mathrm{Hom}(A, \mathbb{K})$ as defined in general for complexes, i.e. $d_C(\varphi) = -(-1)^{|\varphi|}\varphi \circ d$ and (C, d_C) is a chain complex. Note that the dual of the multiplication is given by the map $\Delta\colon C \to C \otimes C$, $\Delta(\varphi)(a \otimes b) := ab$, $\varphi \in C$, $a, b \in A$. This is in fact a comultiplication, making (C, d_C) a co-commutative differential graded coalgebra. Hence, we may apply the Quillen construction and denote the obtained free DGL by

$$\mathcal{L}_{(A,d)} := \mathcal{L}(C, d_C) = \mathcal{L}(\mathrm{Hom}(A, \mathbb{K})).$$

We observe $\mathcal{L}_{(A,d)}$ is a free connected chain Lie algebra of finite type by the assumptions made for (A, d). Finally, applying the $\mathrm{Hom}(-, \mathbb{K})$ functor to the quasi-isomorphism from theorem 4.2.6, we obtain a quasi-isomorphism of commutative cochain algebras $C^*(\mathcal{L}_{(A,d)}) \xrightarrow{\simeq} (A, d)$ which is in fact a Sullivan model for (A, d) by proposition 4.2.9.

Example 4.2.16. Assume $(\Lambda V, d)$ is a minimal Sullivan algebra, where the underlying graded vector space is of finite type and $V = V^{\geq 2}$. Then by remark 4.2.15 $\mathcal{L}_{(\Lambda V, d)}$ is a free connected chain Lie algebra of finite type, and $C^*(\mathcal{L}_{(\Lambda V, d)})$ is a Sullivan algebra with differential $d_0 + d_1$ by proposition 4.2.9. Further, we get a quasi-isomorphism $\eta\colon C^*(\mathcal{L}_{(\Lambda V, d)}) \xrightarrow{\simeq} (\Lambda V, d)$, so if $(\Lambda V, d)$ is a Sullivan model of some space then so is $C^*(\mathcal{L}_{(\Lambda V, d)})$. Note, however, that in general $C^*(\mathcal{L}_{(\Lambda V, d)})$ will not be minimal, as d_0 is defined by the dual of the differential d in ΛV due to the cobar construction 1.3.9.
Since $(\Lambda V, d)$ is a Sullivan algebra, we find a lift $\varphi\colon (\Lambda V, d) \to C^*(\mathcal{L}_{(\Lambda V, d)})$ of the identity on ΛV over η by proposition 3.4.5. Since $\eta \circ \varphi \simeq \mathrm{id}$ as morphisms on Sullivan algebras, we have $H(\eta) \circ H(\varphi) = \mathrm{id}$, exhibiting φ as quasi-isomorphism. Thus, $(\Lambda V, d)$ is a minimal Sullivan model for $C^*(\mathcal{L}_{(\Lambda V, d)})$.

4.3 Free Lie Models

As with Sullivan algebras and connected cochain algebras, a major advantage of free graded Lie algebras is that they provide models for arbitrary connected chain Lie algebras. Note here that $\mathbb{L}_V$ is a connected chain Lie algebra if and only if $V = V_+$.

Definition 4.3.1. Let (L, d_L) be a connected chain Lie algebra. A *free model of* (L, d_L) is a quasi-isomorphism

$$m\colon (\mathbb{L}_V, d) \xrightarrow{\simeq} (L, d),$$

where $V = V_+$ is a graded vector space.

The existence of free models for connected chain Lie algebras is an immediate consequence of theorem 4.2.6, since $\mathcal{L}(C_*(L, d_L))$ is a free connected chain Lie algebra and hence

$$\psi\colon \mathcal{L}(C_*(L, d_L)) \xrightarrow{\simeq} (L, d_L)$$

is a natural free model of (L, d_L). For our purpose, however, it will be essential to have a more detailed description of free Lie models.

Construction 4.3.2. A more constructive approach to obtain a free model involves the extension of a given morphism $\psi\colon (\mathbb{L}_W, d) \to (L, d_L)$ to a free model of the form $m\colon (\mathbb{L}_{W\oplus V}, d) \xrightarrow{\simeq} (L, d_L)$. For this, assume m is already constructed on $\mathbb{L}_{W\oplus V_{\leq k}}$ for $k \geq 0$ such that $H_r(m)$ is an isomorphism for $r < k$ and is surjective for $r = k$.

- Suppose $H_{k+1}(m)$ is not surjective and let $(\beta_j)_{j\in J}$ be a basis of $\operatorname{coker}(H_{k+1}(m))$, where J is some index set. For each β_j, choose a representing cycle $z_j \in L$. Let V^1_{k+1} be the vector space generated by elements $(w_j)_{j\in J}$ with $|w_j| = k+1$ and define $d(w_j) := 0$, $m(w_j) := z_j$.

- Suppose further $H_k(m)$ is not injective and let $(\alpha_i)_{i\in I}$ be a basis of $\ker(H_k(m))$, where I is some index set. For each α_i, choose a representing cycle $x_i \in \mathbb{L}_{W\oplus V_{\leq k}}$. Since $[m(x_i)] = 0$, we find $y_i \in L$ of degree $k+1$ such that $d_L(y_i) = m(x_i)$. Let V^2_{k+1} be the vector space generated by elements $(v_i)_{i\in I}$ with $|v_i| = k+1$ and define $d(v_i) := x_i$, $m(v_i) := y_i$.

Note that this also contains the first step of the construction when we specify $m = \psi$ and $k = 0$. Finally, set $V_{k+1} := V^1_{k+1} \oplus V^2_{k+1}$. We can use this construction to inductively extend m to a map on $\mathbb{L}_{V\oplus W}$. The arguments for the induction beginning and the induction step are the same, so assume m has been extended to a map $m\colon \mathbb{L}_{W\oplus V_{\leq k}} \to L$. The construction above defines m on V_{k+1}, thus we get $m\colon W \oplus V_{\leq k+1} \to L$. We may then use the universal property of the free graded algebra to extend m to a map on $\mathbb{L}_{W\oplus V_{\leq k+1}}$. The corresponding commutative diagram is

$$\begin{tikzcd}
W \oplus V_{\leq k} \arrow[r] \arrow[d] & W \oplus V_{\leq k+1} \arrow[r, "m"] \arrow[d] & L \\
\mathbb{L}_{W\oplus V_{\leq k}} \arrow[r] & \mathbb{L}_{W\oplus V_{\leq k+1}} \arrow[ur, dashed] &
\end{tikzcd}$$

where the maps in the left square are the inclusions. This shows the newly defined map on $\mathbb{L}_{W\oplus V_{\leq k+1}}$ is indeed an extension on the previous map on $\mathbb{L}_{W\oplus V_{\leq k}}$. In a similar manner, extend d to a differential in $\mathbb{L}_{W\oplus V_{\leq k+1}}$ and observe it commutes with m by construction. Iterating this procedure and defining $V := \oplus V_k$ yields a quasi-isomorphism $m\colon (\mathbb{L}_{W\oplus V}, d) \to (L, d_L)$ together with a commutative diagram

$$\begin{tikzcd}
(\mathbb{L}_{W\oplus V}, d) \arrow[rr, "m", "\simeq"'] & & (L, d_L) \\
& (\mathbb{L}_W, d) \arrow[ul] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

where again the unlabeled arrow is the inclusion.

Remark 4.3.3.

(a) Just as with Sullivan algebras, the differential d in a free connected chain Lie algebra $(\mathbb{L}_V, d)$ decomposes to a sum $d_i\colon V \to \mathbb{L}_V^{(i+1)}$, where $\mathbb{L}_V^{(i+1)}$ is the space containing the elements of bracket length $i+1$. The map $d_0\colon V \to V$ is then defined by the condition $d(v) - d_0(v) \in \mathbb{L}_V^{(\geq 2)}$ and is a differential in V, as $d_0^2 = 0$. It is called the *linear part of the differential.*

(b) Analogously, for a morphism $\varphi\colon (\mathbb{L}_V, d) \to (\mathbb{L}_W, d)$ of free connected chain Lie algebras, define the *linear part of* φ to be the chain complex morphism

$$\varphi_0\colon (V, d_0) \to (W, d_0)$$

for which $\varphi(v) - \varphi_0(v) \in \mathbb{L}_W^{(\geq 2)}$ for all $v \in V$.

(c) Let $(\mathbb{L}_V, d)$ be a free differential graded Lie algebra. Remember that $\mathbb{L}_V = V \oplus [\mathbb{L}_V, \mathbb{L}_V]$, where $[\mathbb{L}_V, \mathbb{L}_V] = \mathbb{L}_V^{(\geq 2)}$ is the subspace of elements with bracket length at least two. Clearly $[\mathbb{L}_V, \mathbb{L}_V]$ is an ideal preserved by the differential, and the linear part d_0 of the differential is obtained by factoring out the parts of d that lie in $[\mathbb{L}_V, \mathbb{L}_V]$. Hence, dividing by $[\mathbb{L}_V, \mathbb{L}_V]$ yields a surjective chain complex morphism

$$\eta\colon (\mathbb{L}_V, d) \to (V, d_0).$$

Similarly to the case of Sullivan algebras, for the uniqueness of the free Lie model, additional assumptions regarding the differential have to be made. Remember that a Sullivan model is called minimal if the linear part of the differential vanishes. Analogously, we make the following definition.

Definition 4.3.4. A free connected chain Lie algebra $(\mathbb{L}_V, d)$ with $d_0 = 0$ is called *minimal.* A quasi-isomorphism

$$m\colon (\mathbb{L}_V, d) \xrightarrow{\simeq} (L, d_L)$$

of connected chain Lie algebras, where $(\mathbb{L}_V, d)$ is minimal, is called a *minimal free model of* (L, d_L).

Theorem 4.3.5. *Every connected chain Lie algebra (L, d_L) has a minimal free Lie model*

$$m\colon (\mathbb{L}_V, d) \xrightarrow{\simeq} (L, d_L)$$

that is unique up to isomorphism.

Proof. The existence of a free Lie model $(\mathbb{L}_W, d) \xrightarrow{\simeq} (L, d_L)$ is proven in construction 4.3.2 above. We will outline the main ideas on how this can be improved to a minimal free Lie model. First, decompose W as direct sum of subspaces $V \oplus U \oplus d_0(U)$ such that $d_0|_V = 0$. Denote by $I \subseteq \mathbb{L}_W$ the ideal that is generated by U and $d_0(U)$, then I is preserved by d and we obtain the quotient connected chain Lie algebra $(\mathbb{L}_W/I, \overline{d})$. One can check that the quotient map on the one hand defines an isomorphism $\mathbb{L}_W/I \cong \mathbb{L}_V$ and on the other may be used to lift the identity on $\mathbb{L}_V$ to a quasi-isomorphism $(\mathbb{L}_V, \overline{d}) \xrightarrow{\simeq} (\mathbb{L}_W, d)$. The composition with $(\mathbb{L}_W, d) \xrightarrow{\simeq} (L, d_L)$ then exhibits $(\mathbb{L}_V, \overline{d})$ as minimal free Lie model of X. For a more detailed explanation and for the uniqueness part, the reader may check chapter 22 of [7]. □

Example 4.3.6. Based on the situation of example 4.2.16, given a minimal Sullivan model $(\Lambda V, d)$, assume $m\colon (\mathbb{L}_W, \partial) \xrightarrow{\simeq} \mathcal{L}_{(\Lambda V, d)}$ is a minimal free Lie model. As mentioned in remark 4.2.5 that means $C_*(m)$ is a quasi-isomorphism, and hence the same is true for its dual $C^*(m)\colon C^*(\mathcal{L}_{(\Lambda V,d)}) \to C^*(\mathbb{L}_W, \partial)$. Precomposing this with the quasi-isomorphism φ from example 4.2.16 we get a quasi-isomorphism

$$\psi\colon (\Lambda V, d) \xrightarrow{\simeq} C^*(\mathbb{L}_W, \partial).$$

In other words, if $(\mathbb{L}_W, \partial)$ is a minimal free Lie model of $\mathcal{L}_{(\Lambda V, d)}$, then $(\Lambda V, d)$ is a minimal Sullivan model of $C^*(\mathbb{L}_W, \partial)$.

Now, given a free differential graded Lie algebra $(\mathbb{L}_V, d)$, one can suspend the linear part d_0 to yield a differential $\overline{d}$ in sV by defining $\overline{d}(sv) := -sd_0(v)$.

Proposition 4.3.7. *There exists a natural quasi-isomorphism of complexes*

$$\theta\colon C_*(\mathbb{L}_V, d) \xrightarrow{\simeq} (sV \oplus \mathbb{K}, \overline{d})$$

that is the composition $\Lambda s\mathbb{L}_V \to s\mathbb{L}_V \oplus \mathbb{K} \to sV \oplus \mathbb{K}$ of the division by $\Lambda^{\geq 2}s\mathbb{L}_V$ followed by the division by $s[\mathbb{L}_V, \mathbb{L}_V]$.

Proof. This is proven in chapter 22 of [7]. □

Remark 4.3.8. In particular, this yields an isomorphism

$$H(\theta)\colon H(C_*(\mathbb{L}_V, d)) \xrightarrow{\cong} H(sV \oplus \mathbb{K}, \overline{d}) = sH(V, d_0) \oplus \mathbb{K}$$

which dualizes to an isomorphism

$$H(\theta)^*\colon sH(V, d_0)^* \oplus \mathbb{K} \xrightarrow{\cong} H(C^*(\mathbb{L}_V, d))$$

of graded vector spaces. Moreover, assuming that $\varphi\colon (\Lambda W, d) \xrightarrow{\simeq} C^*(\mathbb{L}_V, d)$ is a minimal Sullivan model, the composition of $H(\varphi)$ with the inverse of $H(\theta)^*$ yields an isomorphism

$$\Phi := (H(\theta)^*)^{-1} \circ H(\varphi)\colon H^+(\Lambda W, d) \xrightarrow{\cong} sH(V, d_0)^*$$

of graded vector spaces. Now, precomposing θ with the inclusion $s\mathbb{L}_V \hookrightarrow \Lambda s\mathbb{L}_V$ exactly yields the surjection $s\eta\colon s\mathbb{L}_V \to sV$. Moreover, we have the canonical projections $\Lambda W \to \Lambda W/\Lambda^{\geq 2}W = W \oplus \mathbb{K}$ and $C^*(\mathbb{L}_V) = \Lambda s\mathbb{L}_V^* \to \Lambda s\mathbb{L}_V^*/\Lambda^{\geq 2}s\mathbb{L}_V^*$, where the second projection can be identified as the dual of the inclusion $s\mathbb{L}_V \hookrightarrow \Lambda s\mathbb{L}_V$. It follows that the induced linear maps ζ and ζ' on homology fit into the commutative diagram

$$\begin{tikzcd}
H^+(\Lambda W, d) \arrow[r, "H(\varphi)", "\cong"'] \arrow[d, "\zeta"'] \arrow[rr, bend left, "\Phi", "\cong"'] & H^+(C^*(\mathbb{L}_V, d)) \arrow[d, "\zeta'"] & sH(V, d_0)^* \arrow[l, "H(\theta)^*"', "\cong"] \arrow[dl, "sH(\eta)^*"] \\
W \arrow[r, "H(Q(\varphi))", "\cong"'] & sH(\mathbb{L}_V, d)^* &
\end{tikzcd}$$

where $H(Q(\varphi))$ is an isomorphism as mentioned in remark 3.4.10.

In the special case that $(\mathbb{L}_V, d)$ is minimal free connected chain Lie algebra, we have that $(\mathbb{L}_V)_0 = 0$ and $d_0 = 0$. This means the induced isomorphisms $H(\theta)$ and $H(\theta)^*$ read

$$H(C_*(\mathbb{L}_V, d)) \cong sV \oplus \mathbb{K} \quad \text{and} \quad sV^* \oplus \mathbb{K} \cong H(C^*(\mathbb{L}_V, d)).$$

and Φ becomes an isomorphism $H^+(\Lambda W, d) \xrightarrow{\cong} sV^*$.

4.4 Lie Models of Spaces

One might already guess how this theory can be applied to obtain models for topological spaces, since the functors $A_{PL}(-)$ and $C^*(-)$ connect the world of topological spaces and, respectively, connected chain Lie algebras with the world of commutative cochain algebras. So as before, this category will be the playground for Lie models, and many strategies and statements from the theory of Sullivan models can be used. However, while we do switch to the category of DGAs for the definition of a Lie model, the underlying Lie algebra as such does contain itself useful and accessible information on the space it is modelling. For example, we will see that the homology Lie algebra of the DGL that provides a model for some space X is naturally isomorphic to the homotopy Lie algebra of X, and free chain Lie algebras allow for a direct description of a fitting CW complex they provide a model for.

Definition 4.4.1.

(a) A *Lie model* for a simply connected space X with rational homology of finite type is a connected chain Lie algebra (L, d_L) together with a DGA quasi-isomorphism

$$m\colon C^*(L, d_L) \xrightarrow{\simeq} A_{PL}(X).$$

In case $L = \mathbb{L}_V$ is a free graded Lie algebra, we speak of a *free model* for X. If (L, d_L) is minimal as a DGL, we say the model is *minimal*.

(b) Assume $f\colon X \to Y$ is a continuous map between simply connected spaces with rational homology of finite type, and $m\colon C^*(L, d_L) \xrightarrow{\simeq} A_{PL}(X)$, $n\colon C^*(E, d_E) \xrightarrow{\simeq} A_{PL}(Y)$ are Lie models. Then a *Lie representative* of f is a DGL morphism

$$\varphi\colon (L, d_L) \to (E, d_E)$$

such that $m \circ C^*(\varphi) \simeq A_{PL}(f) \circ n$.

Example 4.4.2. For the free Lie algebra in one generator $\mathbb{L}_{(v)}$, we have seen that $C^*(\mathbb{L}_{(v)}) = (\Lambda(e), 0)$ if $|v| = 2n$ and $C^*(\mathbb{L}_{(v)}) = (\Lambda(e, e'), d(e') = e^2)$ if $|v| = 2n+1$, where $|e| = |v| + 1$. Thus, in both cases we get the minimal Sullivan algebras from the model of the sphere $\mathbb{S}^{|v|+1}$, so we may use the usual quasi-isomorphism and obtain a minimal free Lie model

$$C^*(\mathbb{L}_{(v)}) \xrightarrow{\simeq} A_{PL}(\mathbb{S}^{k+1}), \quad \text{with } k = |v|.$$

Remark 4.4.3. As the preceding example shows, Lie models can be obtained by finding a connected chain Lie algebra (L, d_L) such that $C^*(L, d_L)$ is the Sullivan algebra in a Sullivan model of X. This is due to proposition 4.2.9 which exhibits $C^*(L, d_L)$ as Sullivan algebra. However, the corresponding candidates must be Sullivan models of the form $(\Lambda V, d = d_0 + d_1)$. In the special case that X has a minimal model of the form $(\Lambda V, d_1)$, remark 4.2.10 exhibits the homotopy Lie algebra of the minimal model and hence the homotopy Lie algebra of X as Lie model for X.
Note that the other way round, a Lie model for X automatically provides a Sullivan model of the form $(\Lambda V, d = d_0 + d_1)$ for X, as $C^*(L, d_L)$ is of this form by proposition 4.2.9. However, as discussed in remark 4.2.14, the differential of the minimal Sullivan model of X can have arbitrary many higher terms.
Furthermore, from the definition one sees that a morphism φ is a Lie representative of f if and only if $C^*(\varphi)$ is a Sullivan representative.

Remark 4.4.4. Suppose X has a commutative model (A, d) of finite type with $A^0 = \mathbb{K}$ and $A^1 = 0$. The Quillen construction yields a free connected chain Lie algebra $\mathcal{L}_{(A,d)}$ of finite type. As mentioned in remark 4.2.15 there exists a natural quasi-isomorphism $C^*(\mathcal{L}_{(A,d)}) \xrightarrow{\simeq} (A, d)$ that provides a Sullivan model for (A, d). Now, since (A, d) and $A_{PL}(X)$ are weakly equivalent, this means $C^*(\mathcal{L}_{(A,d)})$ is also a Sullivan model for X. Remember, a corresponding quasi-isomorphism $C^*(\mathcal{L}_{(A,d)}) \xrightarrow{\simeq} A_{PL}(X)$ was obtained by lifting through the chain of quasi-isomorphisms connecting (A, d) and $A_{PL}(X)$.
Moreover, assume the connected chain Lie algebra (L, d_L) in a Lie model for X is linked to a second connected chain Lie algebra (E, d_E) by a chain of DGL quasi-isomorphisms. Then $C^*(-)$ preserves quasi-isomorphisms, so the Sullivan algebras $C^*(L, d_L)$ and $C^*(E, d_E)$ are weakly equivalent. Lifting through the chain of DGA quasi-isomorphisms and composition with the quasi-isomorphism from the Lie model (L, d_L) yields a quasi-isomorphism $C^*(E, d_E) \xrightarrow{\simeq} A_{PL}(X)$. Therefore, (E, d_E) is a Lie model for X.

Proposition 4.4.5. *Assume X, Y are simply connected topological spaces with rational homology of finite type. Further, assume (L, d_L) is a connected chain Lie algebra of finite type. It holds:*

(i) X has a minimal free Lie model $(\mathbb{L}_V, d)$ that is unique up to isomorphism. Every continuous map $f\colon X \to Y$ has a Lie representative.

(ii) If $\mathbb{K} = \mathbb{Q}$, then (L, d_L) is the Lie model of a simply connected CW complex that is unique up to rational homotopy equivalence.

Proof. We will only show the existence part of the first assertion, to provide a feeling for how these statements can be derived from the theory of Sullivan models. For a complete proof of the statement, see chapter 24 in [7]. Let $m\colon (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$ be a minimal Sullivan model for X. By the assumptions made for X we deduce that $V = V^{\geq 2}$ of finite type. Then by remark 4.2.15 we obtain a natural quasi-isomorphism $\eta\colon C^*(\mathcal{L}_{(\Lambda V,d)}) \xrightarrow{\simeq} (\Lambda V, d)$ of Sullivan algebras. Hence, the composition $m \circ \eta$ is a quasi-isomorphism and exhibits $\mathcal{L}_{(\Lambda V,d)}$ as free Lie model of X. While this need not be minimal, by the same arguments used in the proof of theorem 4.3.5 there exists a free minimal chain Lie algebra $(\mathbb{L}_V, d)$ that is connected and of finite type, together with a quasi-isomorphism $\mathcal{L}_{(\Lambda V,d)} \xrightarrow{\simeq} (\mathbb{L}_V, d)$ of DGLs. Applying the functor $C^*(-)$ then shows that $(\mathbb{L}_V, d)$ is in fact a minimal free Lie model for X. $\square$

Let X be a simply connected space with rational homology of finite type, and let (E, d_E) be the DGL from a Lie model for X. Then a minimal Sullivan model $(\Lambda V, d) \xrightarrow{\simeq} C^*(E, d_E)$ for $C^*(E, d_E)$ automatically provides a minimal Sullivan model for X, since the composition

$$(\Lambda V, d) \xrightarrow{\simeq} C^*(E, d_E) \xrightarrow{\simeq} A_{PL}(X)$$

is a quasi-isomorphism.
The other way around, assume $(\Lambda V_X, d)$ to be the minimal Sullivan algebra from the Sullivan model of X and let (E, d_E) as before. Then, by lifting the minimal Sullivan model over the quasi-isomorphism from the Lie model, we obtain a quasi-isomorphism $(\Lambda V_X, d) \xrightarrow{\simeq} C^*(E, d_E)$ and thus a minimal Sullivan model for $C^*(E, d_E)$. The homotopy commutative diagram is

$$\begin{tikzcd} & C^*(E, d_E) \arrow[d, "\simeq"] \\ (\Lambda V_X, d) \arrow[ur, dashed, "\simeq"] \arrow[r, "\simeq"'] & A_{PL}(X). \end{tikzcd}$$

In this sense, the minimal Sullivan models for X and for the Lie model of X are the same, and we may write $V = V_X$ in the following.

Remark 4.4.6. Associated with X and respectively $(\Lambda V_X, d)$, we obtain the homotopy Lie algebras L_X and L, where $L_X = \pi_*(\Omega X) \otimes \mathbb{K}$ with a bracket defined by the Whitehead product and $sL = V_X^*$ with a bracket defined by d_1. In theorem 4.1.10 we showed that these are actually isomorphic as Lie algebras. We may now apply proposition 4.2.13 to observe that $H(E, d_E) \cong L$, thus the composition yields an isomorphism

$$\sigma_E\colon H(E, d_E) \cong L \cong L_X = \pi_*(\Omega X) \otimes \mathbb{K}$$

as graded Lie algebras. That is, if $[\ ,\ ]$ is the inherited Lie bracket on $H(E, d_E)$, it holds $\sigma_E([z_1, z_2]) = (-1)^{|z_1|+1}\partial([\partial^{-1}(\sigma_E(z_1)), \partial^{-1}(\sigma_E(z_2))]_W)$ for all $z_1, z_2 \in H(E)$ due to the definition of the bracket in L_X. Finally, remember that we identified $sL_X = \pi_*(X) \otimes \mathbb{K}$ through the suspension $s\alpha = -(-1)^{|\alpha|}\partial^{-1}(\alpha)$, with ∂ being the connecting homomorphism from the long exact sequence of the path space fibration of X. Thus, by suspending the isomorphism above, we obtain an isomorphism

$$\tau_E\colon sH(E, d_E) \xrightarrow{\cong} \pi_*(X) \otimes \mathbb{K}.$$

Now, since σ_E is Lie bracket preserving, this means for τ_E on the suspended bracket elements that

$$\begin{aligned}
\tau_E(s[z_1, z_2]) = s\sigma_E([z_1, z_2]) = s[\sigma_E(z_1), \sigma_E(z_2)] &= -(-1)^{|z_1|+|z_2|}\partial^{-1}([\sigma_E(z_1), \sigma_E(z_2)]) \\
&= -(-1)^{|z_1|+|z_2|}\partial^{-1}((-1)^{|z_1|+1}\partial([\partial^{-1}(\sigma_E(z_1)), \partial^{-1}(\sigma_E(z_2))]_W) \\
&= (-1)^{|z_2|}[-(-1)^{|z_1|}s\sigma_E(z_1), -(-1)^{|z_2|}s\sigma_E(z_2)]_W \\
&= (-1)^{|z_1|}[\tau_E(sz_1), \tau_E(sz_2)]_W,
\end{aligned}$$

where we made use of the definition of the suspension in sL_X and the bracket in L_X. We see that a Lie model for X does not only carry the information on rational homotopy groups, but also, like Sullivan algebras, allows a description of the Whitehead product.

While the preceding discussion is true for arbitrary Lie models (E, d_E), free Lie models can yield some advantages. Remember that a minimal Sullivan model contains the information on homotopy groups in its underlying vector space. As we shall see now, with any minimal free Lie model $(\mathbb{L}_V, d)$ of X, we may read off the homology of X directly from the vector space V. Moreover, dual to the case of Sullivan models, a free Lie model contains the Hurewicz homomorphism.

Preparation 4.4.7. Suppose therefore $m\colon C^*(\mathbb{L}_V, d) \xrightarrow{\simeq} A_{PL}(X)$ is a free Lie model. Then, since $\mathrm{Hom}(-, \mathbb{K})$ commutes with homology, the isomorphism $H(m)\colon H(C^*(\mathbb{L}_V, d)) \xrightarrow{\cong} H^*(X; \mathbb{K})$ dualizes to an isomorphism

$$(H(m))^*\colon H_*(X; \mathbb{K}) \xrightarrow{\cong} H(C_*(\mathbb{L}_V, d)).$$

By proposition 4.3.7 there exists a quasi-isomorphism $\theta\colon C_*(\mathbb{L}_V, d) \xrightarrow{\simeq} (sV \oplus \mathbb{K}, \overline{d})$. Here $\overline{d}$ was defined by $\overline{d}(sv) = -sd_0(v)$, with d_0 being the linear part of the differential d. Hence $H(sV \oplus \mathbb{K}, \overline{d}) = sH(V, d_0) \oplus \mathbb{K}$. Altogether this yields an isomorphism $H(\theta) \circ (H(m))^*\colon H_*(X; \mathbb{K}) \xrightarrow{\cong} sH(V, d_0) \oplus \mathbb{K}$ whose inverse we will denote by

$$\Psi\colon sH(V, d_0) \oplus \mathbb{K} \xrightarrow{\cong} H_*(X; \mathbb{K}).$$

It simplifies to $H_*(X; \mathbb{K}) \cong sV \oplus \mathbb{K}$ when $(\mathbb{L}_V, d)$ is a minimal free differential graded Lie algebra and as such $d_0 = 0$. Moreover, remember there exists a linear map $\eta\colon (\mathbb{L}_V, d) \to (V, d_0)$ which is obtained through division by $[\mathbb{L}_V, \mathbb{L}_V]$.

Proposition 4.4.8. *Assume $(\mathbb{L}_V, d)$ is a free model for X and let $\tau\colon sH(\mathbb{L}_V, d) \xrightarrow{\cong} \pi_*(X) \otimes \mathbb{K}$ be the isomorphism from remark 4.4.6. Further, let η and Ψ be the linear maps described in preparation 4.4.7 and let $h_{\mathbb{K}}\colon \pi_*(X) \otimes \mathbb{K} \to H_+(X; \mathbb{K})$ be the Hurewicz homomorphism. Then the diagram*

$$\begin{tikzcd}
sH(\mathbb{L}_V, d) \arrow[r, "\tau", "\cong"'] \arrow[d, "sH(\eta)"'] & \pi_*(X) \otimes \mathbb{K} \arrow[d, "h_{\mathbb{K}}"] \\
sH(V, d_0) \arrow[r, "\Psi", "\cong"'] & H_+(X; \mathbb{K})
\end{tikzcd}$$

commutes, allowing to identify $h_{\mathbb{K}}$ with $sH(\eta)$.

Proof. We use the same argument as in remark 4.3.8. The composition of the quasi-isomorphism

$$\theta\colon C_*(\mathbb{L}_V, d) \xrightarrow{\simeq} (sV \oplus \mathbb{K}, \overline{d})$$

with the inclusion $j\colon s\mathbb{L}_V \hookrightarrow C_*(\mathbb{L}_V)$ precisely gives $s\eta$. The diagram thus reads

$$\begin{tikzcd}
sH(\mathbb{L}_V, d) \arrow[r, "\tau", "\cong"'] \arrow[d, "H(j)"] \arrow[dd, bend right=60, "sH(\eta)"'] & \pi_*(X) \otimes \mathbb{K} \arrow[d, "h^{\mathbb{K}}"] \\
H(C_*(\mathbb{L}_V, d)) \arrow[r, "(H(m)^*)^{-1}", "\cong"'] \arrow[d, "H(\theta)"] & H_+(X; \mathbb{K}) \\
sH(V, d_0) \arrow[ur, "\cong", "\Psi"'] &
\end{tikzcd}$$

where the lower triangle commutes by definition. Thus, it remains to show that the square commutes, but it is the dual to the commutative diagram in remark 3.6.8. $\square$

In the situation of preparation 4.4.7, a minimal Sullivan model $\varphi\colon (\Lambda W, d) \xrightarrow{\simeq} C^*(\mathbb{L}_V, d)$ automatically provides a minimal Sullivan model for X. We may thus combine the statements of remarks 3.6.8 and 4.3.8 to obtain a commutative diagram

$$\begin{tikzcd}
H^*(X, \mathbb{K}) \arrow[d, "h^*_{\mathbb{K}}"] & H^+(\Lambda W, d) \arrow[l, "H(m\circ\varphi)"', "\cong"] \arrow[r, "\Phi", "\cong"'] \arrow[d, "\zeta"] & sH(V, d_0)^* \arrow[d, "sH(\eta)^*"] \\
\mathrm{Hom}(\pi_*(X), \mathbb{K}) & W \arrow[l, "\nu", "\cong"'] \arrow[r, "H(Q(\varphi))", "\cong"'] & sH(\mathbb{L}_V, d)^*
\end{tikzcd}$$

which is the exact dual of the commutative diagram in the proposition above.

4.5 Cellular Lie Models

Assume X is a simply connected topological space with rational homology of finite type. Let I be an index set and suppose $f_i\colon (\mathbb{S}^{n_i}_i, s_0) \to (X, x_0)$ is a continuous map for each $i \in I$, where $n_i \geq 1$. We set $f := (f_i)_{i\in I}$ and attach cells $(\mathbb{B}^{n_i+1}_i)_{i\in I}$ to X via f to form the adjunction space

$$Y := X \cup_f \left(\dot{\bigcup}_{i\in I} \mathbb{B}^{n_i+1}_i \right).$$

We will now see that a free Lie model for X can be extended to a free Lie model for Y using the classes of the attaching maps. It is worth to mention that a comparable strategy does not exist for Sullivan models, since the best we can achieve so far is the construction of a commutative model for Y as introduced in proposition 3.6.13. This will unlock a way of constructing a CW complex X for any given free rational and connected chain Lie algebra $(\mathbb{L}_V, d)$ of finite type, such that $(\mathbb{L}_V, d)$ provides a free Lie model for X. Thereby, it will be clear for each cell what algebraic data relating $(\mathbb{L}_V, d)$ is responsible for its attachment, giving this construction a much more geometric flavour than the spatial realization of Sullivan algebras.

Preparation 4.5.1. Let

$$m_X\colon C^*(\mathbb{L}_V, d) \xrightarrow{\simeq} A_{PL}(X)$$

be a free Lie model for X. Associated with it is the isomorphism

$$\tau\colon sH(\mathbb{L}_V, d) \xrightarrow{\cong} \pi_*(X) \otimes \mathbb{K}.$$

Now for every class $\alpha_i := [f_i] \in \pi_{n_i}(X)$ there exists the class $s\beta_i \in sH(\mathbb{L}_V, d)$ such that $\tau(s\beta_i) = \alpha_i$. Note that this is an abuse of notation, since the expression $\alpha_i \in \pi_{n_i}(X) \otimes \mathbb{Q}$ actually refers to the canonical image $\alpha_i \otimes 1$ of α_i in $\pi_{n_i}(X) \otimes \mathbb{Q}$. Especially, if α_i is torsion, then $0 = \alpha_i \in \pi_{n_i}(X) \otimes \mathbb{Q}$. For every such class β_i, choose a representing cocycle $z_i \in \mathbb{L}_V$ such that $[z_i] = \beta_i$. Note that any z_i has degree $n_i - 1$. For every $i \in I$, introduce a generator w_i with $|w_i| := n_i$ and denote by W the graded vector space with basis $(w_i)_{i\in I}$. Extend the differential on $\mathbb{L}_V$ to $\mathbb{L}_{V\oplus W}$ by setting $d(w_i) := z_i$.

Theorem 4.5.2. *There exists a quasi-isomorphism $m_Y\colon C^*(\mathbb{L}_{V\oplus W}, d) \xrightarrow{\simeq} A_{PL}(Y)$ and thus the chain Lie algebra $(\mathbb{L}_{V\oplus W}, d)$ is a Lie model for Y. Furthermore, there exists a homotopy commutative diagram*

$$\begin{tikzcd}
C^*(\mathbb{L}_V) \arrow[d, "m_X"'] & C^*(\mathbb{L}_{V\oplus W}) \arrow[l, "C^*(\lambda)"'] \arrow[d, "m_Y"] \\
A_{PL}(X) & A_{PL}(Y) \arrow[l, "A_{PL}(j)"]
\end{tikzcd}$$

where $j\colon X \hookrightarrow Y$ and $\lambda\colon \mathbb{L}_V \hookrightarrow \mathbb{L}_{V\oplus W}$ denote the corresponding inclusions.

Proof. Only a sketch of the proof will be provided, roughly summarizing the main ideas. Let $(\Lambda V_X, d) \xrightarrow{\simeq} A_{PL}(X)$ be a minimal Sullivan model and remember that the minimal Sullivan models of $A_{PL}(X)$ and $C^*(\mathbb{L}_V)$ coincide, so we get a quasi-isomorphism $(\Lambda V_X, d) \xrightarrow{\simeq} C^*(\mathbb{L}_V, d)$. Denote by U the graded vector space with basis $(u_i)_{i\in I}$, where $|u_i| := n_i + 1$. Then, due to proposition 3.6.13 the cochain algebra $A := (\Lambda V_X \oplus U, d_A)$ defined by

$$d_A(v) = d(v) + \sum_{i\in I} \langle v; \alpha_i \rangle, \quad d_A(u_i) = 0$$

and $u_i a := 0$ for all $i \in I$, $a \in A^+$ with ΛV_X being a subalgebra, is a commutative model for Y. The argumentation in remark 4.4.4 yields a quasi-isomorphism $C^*(\mathcal{L}_A) \xrightarrow{\simeq} A_{PL}(Y)$, identifying $C^*(\mathcal{L}_A)$ as Sullivan model for Y and thus $(\mathcal{L}_A, d)$ as Lie model for Y. From there on, one can construct a DGL quasi-isomorphism $(\mathcal{L}_A, d) \xrightarrow{\simeq} (\mathbb{L}_{V\oplus W}, d)$, see chapter 24 of [7]. The composition

$$C^*(\mathbb{L}_{V\oplus W}, d) \xrightarrow{\simeq} C^*(\mathcal{L}_A, d) \xrightarrow{\simeq} A_{PL}(Y)$$

is then a quasi-isomorphism of cochain algebras and exhibits $(\mathbb{L}_{V\oplus W}, d)$ as Lie model for Y. □

Construction 4.5.3. Assume now $\mathbb{K} = \mathbb{Q}$ and suppose we are given:

- A connected CW complex X which has only finitely many cells in each dimension and no cells of dimension one. This means X can be obtained from a point by attaching cells in dimensions ≥ 2, whereby only finitely many cells are attached for each dimension. Thus, theorem 4.5.2 can be applied to obtain a free Lie model for X.
  To do so, we will proceed inductively over the dimension of the cells. A 0-cell $\{x_0\}$ has a trivial free Lie model $L = 0$ since $C^*(0) = \mathbb{Q}$ and $A_{PL}(\{x_0\}) = \mathbb{Q}$. Nevertheless, we will carry out the argumentation for the attachment of 2-cells as well before proceeding with the induction step. By assumption, $X^1 = X^0 = \{x_0\}$, so let $f_i \colon \mathbb{S}^1_i \to \{x_0\}$ be the attaching maps of the 2-cells of X, where one may replace 2 by the minimal $n \geq 2$ such that $X^n \neq X^0$ in case X has no cells of dimension 2. By assumption, $i \in I_1$ for some finite index set I_1 counting the 2-cells of X. Proceed by introducing v_i with $|v_i| := 1$ for each attaching map f_i and denote by V_1 to be the graded vector space with basis $(v_i)_{i\in I_1}$, concentrated in degree 1. Since $\{x_0\}$ has a trivial model, the corresponding homology vanishes and we set $d(v_i) = 0$. Then the free connected chain Lie algebra $(\mathbb{L}_{V_1}, 0)$ is a free model for X^2 by theorem 4.5.2.
  Assume now by induction that X^n has a free Lie model of the form $(\mathbb{L}_{V_{<n}}, d)$ and that we are provided with a family of attaching maps $(f_i \colon \mathbb{S}^n_i \to X^n)_{i\in I_n}$ for the $(n+1)$-cells of X. As before, I_n is some finite index set, and we are provided with the isomorphism $\tau \colon sH(\mathbb{L}_{V_{<n}}, d) \xrightarrow{\cong} \pi_*(X^n) \otimes \mathbb{Q}$. Choose $z_i \in \mathbb{L}_{V_{<n}}$ such that $s[z_i] = \tau^{-1}([f_i])$ and denote by V_n the graded vector space concentrated in degree n with basis $(v_i)_{i\in I_n}$, $|v_i| := n$. Then $V_{<n+1} := V_{<n} \oplus V_n$ and we can extend d to a differential on $\mathbb{L}_{V_{<n+1}}$ through $d(v_i) := z_i$. By theorem 4.5.2, $(\mathbb{L}_{V_{<n+1}}, d)$ is a free Lie model for X^{n+1}.
  All in all, we obtain a free Lie model $(\mathbb{L}_V, d)$ of X such that each $(\mathbb{L}_{V_{<n}}, d)$ provides a model for X^n. It is called the *cellular Lie model* for X. Note that for the basis elements v_i of V^n it holds $s[d(v_i)] = \tau^{-1}([f_i])$, where $n \geq 1$ and $i \in I_n$.

- A free connected chain Lie algebra $(\mathbb{L}_V, d)$ of finite type. Using once again theorem 4.5.2 we are able to construct a CW complex X such that $(\mathbb{L}_V, d)$ provides a cellular Lie model for X. Since $\mathbb{L}_V$ is connected, we see that $V = V_+$, so $\mathbb{L}_{V_{<1}}$ is trivial and thus provides a Lie model for $\{x_0\} = X^0 = X^1$. Further, V is of finite type, since $\mathbb{L}_V$ is. Thus, we may choose a basis $(v_i)_{i\in I_1}$ of V_1, where I_1 is a finite index set. Now for each $i \in I_1$ let $f_i \colon \mathbb{S}^1_i \to X^1 = \{x_0\}$ be the constant map and attach a 2-cell $\mathbb{B}^2_i$ to X^1 using f_i to obtain

$$X^2 := X^1 \cup_{(f_i)} \left(\dot{\bigcup}_{i\in I_1} \mathbb{B}^2_i \right) = \vee_{i\in I_1} \mathbb{S}^2_i.$$

Now by degree reasons, $d(v_i) = 0$ for all $i \in I_1$ and $\pi_1(X^1) = 0$. Thus by theorem 4.5.2, $(\mathbb{L}_{V_1} = \mathbb{L}_{V_{<2}}, 0)$ is a Lie model for X^2.
Suppose by induction that X^n is already constructed such that $(\mathbb{L}_{V_{<n}}, d)$ is a cellular Lie model for X^n. Again, this means there exists an isomorphism $\tau\colon sH(\mathbb{L}_{V_{<n}}, d) \xrightarrow{\cong} \pi_*(X^n) \otimes \mathbb{Q}$. A choice for a basis of V_n is a finite family $(v_i)_{i\in I_n}$. Now the elements $d(v_i)$ are cycles of degree $n-1$, and thus we get elements $s[d(v_i)] \in sH(\mathbb{L}_{V_{<n}}, d)$ of degree n. Then $\tau(s[d(v_i)]) \in \pi_n(X^n)\otimes\mathbb{Q}$, so by linearity, we may choose v_i such that $\tau(s[d(v_i)]) = \alpha_i \otimes 1$ for some $\alpha_i \in \pi_n(X^n)$. Thus, there exist continuous maps $f_i\colon \mathbb{S}^n_i \to X^n$ such that $[f_i] = \tau(s[d(v_{n,i})])$. Finally, for each $i \in I_n$ attach a $(n+1)$-cell $\mathbb{B}^{n+1}_i$ to X^n using f_i to obtain

$$X^{n+1} := X^n \cup_{(f_i)} \left(\dot{\bigcup}_{i\in I_n} \mathbb{B}^{n+1}_i\right).$$

By construction and using again theorem 4.5.2 it follows that $(\mathbb{L}_{V_{<n+1}}, d)$ is a cellular Lie model for X^{n+1}. Proceeding inductively we arrive at $X = \cup_{k\geq 0} X^k$. Further, there exists the homotopy commutative diagram

$$\begin{tikzcd}
 & C^*(\mathbb{L}_V, d) \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\cdots & C^*(\mathbb{L}_{V_{<n}}, d) \arrow[l] \arrow[d, "\simeq"'] & C^*(\mathbb{L}_{V_{<n+1}}, d) \arrow[l] \arrow[d, "\simeq"'] & \cdots \arrow[l] \\
\cdots & A_{PL}(X^n) \arrow[l] & A_{PL}(X^{n+1}) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

in which the horizontal maps are induced by the corresponding inclusions and the vertical arrows denote the quasi-isomorphisms from the cellular Lie models. The maps $C^*(\mathbb{L}_V, d) \to C^*(\mathbb{L}_{V_{<k}}, d)$ are also induced by the inclusions and are isomorphisms in degrees lower than $k+1$. This induces a quasi-isomorphism $C^*(\mathbb{L}_V, d) \xrightarrow{\simeq} A_{PL}(X)$ which exhibits $(\mathbb{L}_V, d)$ as cellular Lie model for X.

The description of a cellular Lie model becomes considerably more difficult when we allow X to contain 1-cells. Nevertheless, some work has been done in trying to extend the above to the case of arbitrary CW complexes. In [14], the authors describe a free Lie model for the interval, together with a geometric interpretation of the construction. From there on, identifying the endpoints and describing a corresponding implementation in the algebraic model, a Lie model for the circle is provided.

5 Realization of Minimal Sullivan Algebras with Quadratic Differential

Suppose $(\Lambda V, d = d_1)$ is a minimal Sullivan algebra of finite type with $V = V^{\geq 2}$. Then, as mentioned in remark 4.2.10, it holds $C^*(L,0) = (\Lambda V, d_1)$, where L is the homotopy Lie algebra of $(\Lambda V, d_1)$. Since $V = V^{\geq 2}$ of finite type, it holds L is a connected chain Lie algebra of finite type. Consequently, there exists a free Lie model $m\colon (\mathbb{L}_W, \partial) \xrightarrow{\simeq} (L, 0)$ for $(L,0)$. As follows from the construction of free Lie models, the free connected chain Lie algebra $\mathbb{L}_W$ is of finite type, since L is. This means there exists a CW complex X such that $(\mathbb{L}_W, \partial)$ is a cellular Lie model for X and we get a quasi-isomorphism $\varphi\colon C^*(\mathbb{L}_W, \partial) \xrightarrow{\simeq} A_{PL}(X)$. Therefore, the composition

$$\varphi \circ C^*(m)\colon (\Lambda V, d_1) = C^*(L,0) \xrightarrow{\simeq} C^*(\mathbb{L}_W, \partial) \xrightarrow{\simeq} A_{PL}(X)$$

of DGA quasi-isomorphisms implies that $(\Lambda V, d_1)$ is a minimal Sullivan model for X.
In the following, we will use three constructions:

- construction 4.1.5, which describes how L is obtained from $(\Lambda V, d_1)$,
- the constructive method on how to obtain $(\mathbb{L}_W, \partial)$ from $(L, 0)$ as carried out in 4.3.2,
- the construction of a space X for which $(\mathbb{L}_W, \partial)$ is a cellular Lie model, as seen in 4.5.3.

All in all, this will provide us with a constructive description of a space X for which $(\Lambda V, d_1)$ is a minimal Sullivan model. The goal is then to investigate this construction, trying to form a direct connection between algebraic information in $(\Lambda V, d_1)$ and geometric structures of X. In the best case one could hope to formulate some sort of dictionary, translating between algebra and geometry, which allows us to directly read off the cell structure of the realization X after having identified a sufficient amount of algebraic information in $(\Lambda V, d_1)$. In general, this seems to be too ambitious. However, we can describe a part of this dictionary, which will allow us to make statements regarding the amount of cells in each dimension or the description of the initial steps in the construction of X. Finally, both will be a direct consequence of accessible information in $(\Lambda V, d_1)$.
In the following, we will recall the above-mentioned constructions before applying them to some examples. However, instead of presenting the topics in a general context again, the formulation of some statements might be adjusted to the cases we need. The proofs of our main theorems will be conducted during the course of this chapter. On the way, it will become clear how the information translates from algebra to topology, and we will be able to observe where the difficulties arise when it comes to computing things. To bypass these and to make things easier, we will make some assumptions, but we will also try to provide an insight into what might still be possible in a more general context.

5.1 Translating from a Sullivan to a Lie Algebra

We start with a minimal Sullivan algebra $(\Lambda V, d_1)$ of finite type with $V = V^{\geq 2}$. Throughout this section, this will be the general condition we impose on all Sullivan algebras that we want to realize. The homotopy Lie algebra of such a minimal Sullivan algebra is the Lie algebra L which satisfies the equality

$$C^*(L,0) \cong (\Lambda V, d_1).$$

More precisely, L is given by $sL = V^*$ and the bracket $[\ ,\ ]$ is defined by d_1 through the identity

$$\langle v; s[x,y]\rangle = (-1)^{|y|+1}\langle d_1(v); sx, sy\rangle$$

for all $v \in V$.

Preparation 5.1.1. Let $(v_i)_{i\in I}$ be a basis of $\ker(d_1) \subseteq V$ and choose a subspace $W \subseteq V$ such that $V = \ker(d_1) \oplus W$. Then any basis $(w_j)_{j\in J}$ of W complements the family $(v_i)_{i\in I}$ to a basis of V and it holds $d_1(v_i) = 0$ and $d_1(w_j) \neq 0$ for all $i \in I$, $j \in J$. We may assume the elements v_i, w_j to be homogeneous and consider the corresponding dual basis in sL. That is, there exist elements $(sx_i)_{i\in I}$ and $(sy_j)_{j\in J}$ in sL such that $\langle v_i; sx_i\rangle = 1 = \langle w_j; sy_j\rangle$, but they vanish on all other elements of the basis of V. As before, we denote by $A \subseteq L$ a subspace complementary to $[L, L]$, such that $L = A \oplus [L, L]$ and hence $sL = sA \oplus s[L, L]$. However, we will see that the choice of a basis for V already specifies a suitable space A.
We will frequently write u_p for basis elements of V when we do not want to distinguish whether the elements are part of $(v_i)_{i\in I}$ or $(w_j)_{j\in J}$. The corresponding dual basis element in sL will be denoted by sz_p.

We may assume that the chosen basis of V is well-ordered, so in particular the index sets I and J are well-ordered. In the following, we will take the ordinal sum $I + J$ as order for the basis on V, that is the topological sum $I + J$ with the convention that $i \leq j$ if $i \in I$ and $j \in J$. Orders within each index set I and J respectively stay unchanged. A basis of $\Lambda^2 V$ is then given by the set

$$\{v_{i_1} \wedge v_{i_2}, v_i \wedge w_j, w_{j_1} \wedge w_{j_2} \mid i_1, i_2, i \in I,\ j_1, j_2, j \in J,\ i_1 \leq i_2,\ j_1 \leq j_2\},$$

where $i_1 = i_2$ or $j_1 = j_2$ is only possible if $|v_{i_1}|$ respectively $|w_{j_1}|$ is even. In particular, for an element $u \in V$ we can write

$$d_1(u) = \sum_{i_1,i_2} \lambda_{i_1,i_2} v_{i_1} \wedge v_{i_2} + \sum_{i,j} \kappa_{i,j} v_i \wedge w_j + \sum_{j_1,j_2} \mu_{j_1,j_2} w_{j_1} \wedge w_{j_2},$$

where the restrictions on the pairs i_1, i_2 and j_1, j_2 will not be mentioned explicitly for the sake of neatness. When the distinction of basis elements is to be ignored, this simplifies to

$$d_1(u) = \sum_{p,q} \lambda_{p,q} u_p \wedge u_q$$

with $p, q \in I + J$, $p \leq q$ and $p = q$ only if $|u_p|$ is even.
Naturally, we define the order of the dual basis in L to be given by the ordinal sum $I + J$, too. Furthermore, for a bracket $[z_l, z_k]$ of basis elements, we agree to consider the case that $k \leq l$. By anti-symmetry, the corresponding statements for $k > l$ follow immediately, possibly after a change of sign. Using the Jacobi identity, this can be extended to brackets of higher length, where the innermost elements are defined to have the smallest order.

Remark 5.1.2. We can now express the differential d_1 in terms of linear combinations of a designated basis of $\Lambda^2 V$. This representation of d_1 will allow us to examine the brackets in L. Therefore, we first need to evaluate the basis elements $u_p \wedge u_q$ using the pairing $\Lambda^2 V \times sL \times sL$ from preparation 4.1.4. It immediately follows from the definition of the pairing that

$$\langle u_p \wedge u_q; sz_l, sz_k\rangle = \begin{cases} 1, & \text{if } k = p < q = l \\ 2, & \text{if } k = p = q = l, \\ 0 & \text{else.} \end{cases}$$

Lemma 5.1.3. *For any $j \in J$, use the basis $(u_p \wedge u_q)_{p,q}$ of $\Lambda^2 V$ to write*

$$d_1(w_j) = \sum_{p,q} \lambda^j_{p,q} u_p \wedge u_q$$

with unique coefficients $\lambda^j_{p,q} \in \mathbb{Q}$. Then for any two indexes $k, l \in I + J$, $k \leq l$ we can express the bracket of the two corresponding basis elements $z_l, z_k \in L$ through the coefficients $\lambda^j_{k,l}$ that appear in $d_1(w_j)$ in front of the product $u_k \wedge u_l$ for each $j \in J$, namely

$$[z_l, z_k] = \begin{cases} (-1)^{|u_k|} \sum_j \lambda^j_{k,l} y_j, & \text{if } k \neq l \\ 2 \sum_j \lambda^j_{k,k} y_j & \text{if } k = l. \end{cases}$$

In particular, $[z_l, z_k] = 0$ *if and only if* $\lambda^j_{k,l} = 0$ *for all* $j \in J$.

Proof. Let $x, y \in L$ and write $s[x,y] = \sum_i \lambda_i sx_i + \sum_j \mu_j sy_j$. Then $d_1(v_i) = 0$ by definition implies $\lambda_i = \langle v_i; s[x,y]\rangle = 0$ for all $i \in I$. It follows that any suspended bracket $s[x,y]$ is generated by the elements $(sy_j)_{j\in J}$ and vanishes on cocycles. Therefore, write $s[z_l, z_k] = \sum_j \mu_j sy_j$ and consider any basis element w_j. Express the element $0 \neq d_1(w_j)$ in terms of the basis of $\Lambda^2 V$ that is provided by the basis $(u_p)_{p\in I+J}$ of V as described above. Then use the identity in remark 5.1.2 to verify

$$\mu_j = \langle w_j; s[z_l, z_k]\rangle = (-1)^{|z_k|+1}\langle d_1(w_j); sz_l, sz_k\rangle = \begin{cases} (-1)^{|u_k|}\lambda^j_{k,l} & \text{if } k \neq l, \\ 2\lambda^j_{k,k} & \text{if } k = l. \end{cases}$$

□

Let us now observe how the basis elements x_i and y_j correlate to a fitting subspace $A \subseteq L$ such that $L = A \oplus [L, L]$.

Remark 5.1.4. The space $s[L, L]$ is generated by elements $s[x, y]$ with $x, y \in L$. By lemma 5.1.3, any such element is in turn a linear combination of $(sy_j)_{j\in J}$, which implies that $[L, L]$ lies in the subspace of L generated by $(y_j)_{j\in J}$. On the other hand, for w_j there exists some coefficient $\lambda^j_{k,l} \neq 0$ such that

$$\mu s[z_l, z_k] = \lambda^j_{k,l} sy_j + \sum_{r\neq j} \lambda^r_{k,l} sy_r, \quad \text{and thus} \quad y_j = \frac{\mu}{\lambda^j_{k,l}}[z_l, z_k] - \sum_{r\neq j} \frac{\lambda^r_{k,l}}{\lambda^j_{k,l}} y_r,$$

with μ being $(-1)^{|u_k|}$ or $\frac{1}{2}$. Thus, for any choice of A such that $L = A \oplus [L, L]$ it follows that $y_j \notin A$ For all $j \in J$, which by linear independence essentially implies that $y_j \in [L, L]$. In other words, $[L, L]$ is the subspace generated by $(y_j)_{j\in J}$ and we may assume A to be the complementary subspace generated by $(x_i)_{i\in I}$.

We obtain a simpler expression of the brackets $[z_l, z_k]$ when we put the following restriction on d_1. Suppose that for a fixed basis element $u_k \wedge u_l$, the corresponding coefficients $\lambda^j_{k,l}$ in $d_1(w_j)$ are only non-zero for exactly one index $j \in J$. Then lemma 5.1.3 states that $[z_l, z_k] = (-1)^{|u_k|}\lambda^j_{k,l} y_j$ or respectively $[z_k, z_k] = 2\lambda^j_{k,k} y_j$. In particular, if this is true for every basis element of $\Lambda^2 V$, then each $y_j \in [L, L]$ equals, up to a scalar to multiple, to just one bracket of basis elements $(z_p)_{p\in I+J}$. A minimal Sullivan algebra $(\Lambda V, d_1)$ whose differential satisfies this additional property will be called *separated*, as this exactly means that the representations of the elements $d_1(w_j)$ all use separate basis elements of $\Lambda^2 V$. The following statement summarizes what we have done so far.

Proposition 5.1.5. *Let* $(\Lambda V, d_1)$ *be a minimal Sullivan algebra of finite type with* $V = V^{\geq 2}$ *and basis as described. Let* $L = A \oplus [L, L]$ *be the homotopy Lie algebra of* $(\Lambda V, d_1)$ *with* A *as before. It holds:*

(i) A basis element v_i *corresponds to a basis element* x_i *in* $A \subseteq L$.

(ii) A basis element $u_k \wedge u_l \in \Lambda^2 V$ *for which the coefficient* $\lambda^j_{k,l}$ *in* $d_1(w_j)$ *is zero for all* $j \in J$ *implies that* $[z_l, z_k] = 0$.

(iii) A basis element $u_k \wedge u_l \in \Lambda^2 V$ *for which the coefficient* $\lambda^j_{k,l}$ *in* $d_1(w_j)$ *is non-zero for exactly one* $j \in J$ *implies that*

$$[z_l, z_k] = \begin{cases} (-1)^{|u_k|}\lambda^j_{k,l} y_j, & \text{if } k \neq l, \\ 2\lambda^j_{k,k} y_j, & \text{if } k{=}l. \end{cases}$$

Proof. The first claim follows from the proof of lemma 5.1.3. The remaining statements are an immediate consequence of the preceding discussion and the statements of lemma 5.1.3. □

Remark 5.1.6. Instead of just expressing a bracket $[z_l, z_k]$ in terms of the basis $(y_j)_{j\in J}$, we will more frequently express the basis elements y_j in terms of brackets, especially when we can do so with a single bracket. Therefore, assume we are given a separated minimal Sullivan algebra. To abbreviate notation, we combine the appearing factors and write $y_j = \mu^j_{k,l}[z_l, z_k]$. Clearly, we have

$$\mu^j_{k,l} = \frac{(-1)^{|u_k|}}{\lambda^j_{k,l}} \quad \text{for } k \neq l \text{ and } \quad \mu^j_{k,k} = \frac{1}{2\lambda^j_{k,k}},$$

but we will henceforth not distinguish between the cases $k = l$ and $k \neq l$. Note however that normally, j will be a fix index and $k, l \in I + J$ may be variable, but the restrictions from the indexing of the basis of $\Lambda^2 V$, such as $k \leq l$, apply. Note further that this allows us to express y_j as a bracket in L, but there will be multiple such bracket expressions in case that various basis elements of $\Lambda^2 V$ share the same index $j \in J$. More precisely, whenever we find $j \in J$ such that for $u_k \wedge u_l$ and $u_p \wedge u_q$ it holds $\lambda^j_{k,l} \neq 0$ and $\lambda^j_{p,q} \neq 0$ for a separated minimal Sullivan algebra, we have that $\mu^j_{k,l}[z_l, z_k] = y_j = \mu^j_{p,q}[z_q, z_p]$, relating these brackets.

The differential d_1 defines the Lie bracket in L. Thus, in some sense, a more complicated d_1 provides more brackets in L, making it closer to being free. Of special interest to us are brackets in the elements $x_i \in A$ that evaluate to zero in L, although they would not vanish if L was free. They will lead to the introduction of generators in a free Lie model of L, which we will come to in a bit. For now, remember that such brackets were called relations of L and lie in the kernel of a natural surjection $\mathbb{L}_A \to L$, see remark 1.4.13. The second and third statement of proposition 5.1.5 provide criteria for what behaviour of d_1 may lead to the absence of bracket expressions. To clarify what we mean, let us now discuss some examples.

Example 5.1.7. We start with the easiest minimal Sullivan algebra, $(\Lambda(v), 0)$ with $|v|$ odd. The corresponding homotopy Lie algebra is generated by a single element x that has even degree, and $[x, x] = 0$ in L. This is also true in the free graded Lie algebra $\mathbb{L}_{(x)}$ generated by x, due to degree reasons. Hence, in this case, $L = \mathbb{L}_{(x)}$ is a free graded Lie algebra.
The case that $|v|$ is even is at the first glance similar. Again, L is generated by an element x, and the Lie bracket is trivial since the differential in $(\Lambda(v), 0)$ is. This time however, $|x|$ is odd, meaning the bracket $[x, x]$ is a relation and L is not free.

Example 5.1.8. Consider the minimal Sullivan algebra $(\Lambda(v, w), d_1(w) = v^2)$, where $|v| \geq 2$ is even. Then L has the basis $\{x, y\}$, where $|x|$ is odd. Using proposition 5.1.5 for the products $v \wedge w$ and v^2 we find that $[y, x] = 0$ and $[x, x] = 2y$ in L. In particular, the bracket $[x, x]$ is now non-trivial and thus not a relation in L. Note further that while one might suspect $[y, x]$ to be a relation in L, in reality it holds $[y, x] = \frac{1}{2}[[x, x], x] = 0$, which already vanishes due to the Jacobi identity. Therefore, we get $L = \mathbb{L}_{(x)} = \mathbb{Q}x \oplus \mathbb{Q}[x, x]$, which is free as a graded Lie algebra.

Example 5.1.9. For a minimal Sullivan algebra $(\Lambda(v_1, v_2), 0)$ with $|v_1|, |v_2| \geq 2$ it immediately follows that $[x_2, x_1] = 0$, since the differential is trivial. Further, also $[x_1, x_1] = [x_2, x_2] = 0$, but we have to distinguish whether this is due to degree reasons (which is the case if $|v_1|$ or $|v_2|$ are odd, so the products v_1^2 or v_2^2 vanish too) or not. In the latter case, the brackets provide relations in L.

Example 5.1.10. Assume we are given a vector space $V = V^{\geq 2}$ of finite type with basis $(v_i)_{i\in I}$. Write $k_i := |v_i|$ for $i \in I$. Then the corresponding homotopy Lie algebra is the graded vector space L generated by elements x_i, $|x_i| = k_i - 1$, with all brackets being trivial. As a graded Lie algebra, excluding the case that V has only one generator of odd degree, L is far from being free. However, if we consecutively add generators w_j in order to kill all cocycles of word length two, we obtain a minimal Sullivan algebra of the form $(\Lambda(v_1, \ldots, w_1, \ldots), d_1)$. It is the minimal Sullivan model of the CDGA $(H, 0)$, where $H^0 = \mathbb{Q}$ and $H^+ = \oplus_i \mathbb{Q}[v_i]$. Its homotopy Lie algebra L is generated by elements x_i, y_j and it holds $L = A \oplus [L, L]$, with A being generated by the elements x_i.

So far, the examples were mainly dealing with minimal Sullivan algebras that are well-known to us, so it was already clear which class of spaces they belong to. In the following, we will take a look at some cases where this is less obvious and see if we can find a realization during the course of this section.

Example 5.1.11. Suppose $|v_1|, |v_2|$ are odd and consider the minimal Sullivan algebra given by $(\Lambda(v_1, v_2, w), d_1(w) = v_1 \wedge v_2)$. The corresponding homotopy Lie algebra has basis $\{x_1, x_2, y\}$ with $|x_i|$ even and proposition 5.1.5 implies that $[x_1, x_2] = y$, so we have $L = \langle x_1, x_2, [x_1, x_2]\rangle_{\mathbb{Q}}$, the graded $\mathbb{Q}$-vector space generated by these elements. In comparison to example 5.1.9, while now $[x_2, x_1] \neq 0$, we have the relations $[x_1, [x_1, x_2]]$ and $[x_2, [x_1, x_2]]$ and L is not free.
Suppose next we add two generators w_1, w_2 with $d_1(w_i) = v_i \wedge w$, so the minimal Sullivan algebra we are looking at is $(\Lambda(v_1, v_2, w, w_1, w_2), d_1)$. A basis for L is now $\{x_1, x_2, y, y_1, y_2\}$, where

$$y = -[x_2, x_1] = [x_1, x_2], \quad y_1 = -[y, x_1] = [x_1, [x_1, x_2]], \quad y_2 = -[y, x_2] = [x_2, [x_1, x_2]],$$

so $L = \langle x_1, x_2, [x_1, x_2], [x_1, [x_1, x_2]], [x_2, [x_1, x_2]]\rangle_{\mathbb{Q}}$. The first relations that appear here are of bracket length four. One can repeat this process, expanding d_1 and creating brackets in L until it becomes $\mathbb{L}_{(x_1, x_2)}$, the free Lie algebra in two generators.

Example 5.1.12. Assume we are given a minimal Sullivan algebra $(\Lambda(v_1, v_2, w), d_1(w) = v_1 \wedge v_2)$ with $|v_1|, |v_2|$ being even. Again, proposition 5.1.5 yields $y = [x_2, x_1]$ and the homotopy Lie algebra is $L = \langle x_1, x_2, [x_1, x_2]\rangle_{\mathbb{Q}}$ with $|x_i|$ odd. However, this time we have already relations of bracket length two, namely the expressions $[x_1, x_1], [x_2, x_2]$ which would be non-zero in the free graded Lie algebra $\mathbb{L}_{(x_1, x_2)}$.
Now, given two additional generators w_1, w_2 with $d_1(w_1) := v_1^2$ and $d_1(w_2) := v_2^2$, we arrive at the minimal Sullivan algebra $(\Lambda(v_1, v_2, w, w_1, w_2), d_1)$. We see that $[x_1, x_1] = \frac{1}{2}y_1$ and $[x_2, x_2] = \frac{1}{2}y_2$, so these brackets now exist in L. On the other hand, there are no brackets of length ≥ 3 in L, which poses relations. However, it can be quite difficult to distinguish relations of bracket length ≥ 3 from brackets that vanish due to the Jacobi identity.
To clarify, we add another two generators w_1', w_2' by setting

$$d_1(w_1') := v_1 \wedge w - v_2 \wedge w_1, \quad d_1(w_2') := v_2 \wedge w - v_1 \wedge w_2.$$

For the dual basis elements y_1', y_2' in L it then holds $[y, x_1] = y_1' = [y_1, x_2]$ and $[y, x_2] = y_2' = [y_2, x_1]$. This means we now have some non-vanishing brackets of length three, but by our calculations they are related through the following identities

$$[[x_2, x_1], x_1] - \frac{1}{2}[[x_1, x_1], x_2] = 0, \quad [[x_2, x_1], x_2] - \frac{1}{2}[[x_2, x_2], x_1] = 0$$

that are essentially due to the Jacobi identity in L or, equivalently, the condition $d_1^2 = 0$ in $(\Lambda V, d_1)$.

These examples illustrate that, in some sense, the more complicated the differential d_1 of the Sullivan algebra is, the closer is the homotopy Lie algebra L to being free. Moreover, examples 5.1.11 and 5.1.12 show that it is difficult to determine a criterion that, without explicitly calculating L from $(\Lambda V, d_1)$, tells us exactly which brackets are relations in L. This essentially boils down to the fact that providing a basis for free graded Lie algebras is anything but easy, as on elements of bracket length ≥ 3 the Jacobi identity makes things complicated. We will come across this problem again soon, when we want to determine a free Lie model for a given homotopy Lie algebra L.

Example 5.1.13. Finally, consider the separable minimal Sullivan algebra $(\Lambda(v_1, v_2, v_3, w), d_1)$ with $|v_1| = |v_3|$, $|v_i| \geq 2$ and $d_1(w) = v_1 \wedge v_2 + v_2 \wedge v_3$. It follows that $[x_3, x_1] = 0$ and $(-1)^{|v_1|}[x_2, x_1] = y = (-1)^{|v_2|}[x_3, x_2]$ in L.

5.2 A Free Model for the Homotopy Lie Algebra

Assume L is a connected chain Lie algebra and $A \subseteq L$ is any subspace such that $L = A \oplus [L, L]$. Using the universal property of the free graded Lie algebra, the inclusion $A \hookrightarrow L$ extends to a morphism $\sigma\colon \mathbb{L}_A \to L$. In proposition 1.4.12 we showed that σ is surjective and called elements in $\ker(\sigma)$ relations of L. We may apply this to the case that L is the homotopy Lie algebra of $(\Lambda V, d_1)$ where we, as before, specify A to be the space generated by the elements x_i, $i \in I$. View σ as surjective DGL morphism

$$\psi\colon (\mathbb{L}_A, 0) \to (L, 0),$$

which can now be improved to a quasi-isomorphism by sequentially erasing the kernel of the induced map $H(\psi)$ in each level. For this, assume by induction $\psi\colon (\mathbb{L}_{A\oplus B_{\leq k}}, \partial) \to (L, 0)$ to be already constructed such that $H_l(\psi)$ is an isomorphism for $l < k$. Choose a basis (γ_i) of $\ker(H_k(\psi))$ and define B_{k+1} to be the vector space generated by elements (b_i), with $|b_i| := k+1$. Extend ∂ through setting $\partial(b_i) := z_i$, where $z_i \in \mathbb{L}_{A\oplus B_{\leq k}}$ is a cycle representing γ_i. Finally, as the differential in L is zero, we may extend ψ on B_{k+1} by setting $\psi(b_i) := 0$. A more detailed description of this construction is provided in 4.3.2. We arrive at a free Lie model of the form

$$\psi\colon (\mathbb{L}_{A\oplus B}, \partial) \xrightarrow{\simeq} (L, 0)$$

which implies the existence of a DGA quasi-isomorphism

$$C^*(\psi)\colon (\Lambda V, d_1) = C^*(L, 0) \xrightarrow{\simeq} C^*(\mathbb{L}_{A\oplus B}, \partial).$$

A space X for which $(\mathbb{L}_{A\oplus B}, \partial)$ is a cellular Lie model will then be a space which has $(\Lambda V, d_1)$ as its minimal Sullivan model. However, before we come to this, we will first make a connection between $(\Lambda V, d_1)$ and the vector space $A \oplus B$ together with ∂, since these essentially provide a description of X.

Lemma 5.2.1. *The free Lie algebra $(\mathbb{L}_{A\oplus B}, \partial)$ constructed above is a minimal free connected chain Lie algebra.*

Proof. Only the minimality remains to show. Write $\partial = \partial_0 + \partial_1 + \dots$, where $\partial_i\colon A \oplus B \to \mathbb{L}^{(i+1)}_{A\oplus B}$ raises bracket length by i. In particular,

$$\partial_0\colon A \oplus B \to \mathbb{L}^{(1)}_{A\oplus B} = A \oplus B,$$

and we claim that $\partial_0 = 0$. It is clear that $\partial|_A = 0$, so let $b \in B_{k+1}$ be some generator. Write $\partial(b) = x + y$ with $x \in A \oplus B_k$ and $y \in \mathbb{L}^{(\geq 2)}_{A\oplus B_{\leq k-1}}$, as $|y| = k$. It follows that

$$\partial(x + y) = 0 \quad \text{and} \quad H_k(\psi)([x + y]) = 0.$$

Since the differential in L is zero, this implies $\psi(x) = -\psi(y)$, but $\psi|_A = \mathrm{id}_A$ and $\psi|_B = 0$. Thus $\psi(x) \in A$ and $\psi(y) \in \mathbb{L}^{(\geq 2)}_{A\oplus B}$, as ψ is a DGL morphism. It follows that $\psi(x) = 0 = \psi(y)$, so in particular $x \in B_k$. Now, if $x \neq 0$, then by the prior construction step, $\partial(x) = z$ represents a non-trivial element in $\ker(H_{k-1}(\psi)) \subseteq H_{k-1}(\mathbb{L}_{A\oplus B_{\leq k-1}}, \partial)$, but now $\partial(x) = -\partial(y)$ implies that $[z] = 0$, since $y \in \mathbb{L}_{A\oplus B_{\leq k-1}}$. ☐

What follows is the existence of a quasi-isomorphism

$$\theta\colon C_*(\mathbb{L}_{A\oplus B}, \partial) \xrightarrow{\simeq} (sA \oplus sB \oplus \mathbb{Q}, 0)$$

of complexes by proposition 4.3.7. Together with ψ as above, it yields an isomorphism

$$\Phi := (H(\theta)^*)^{-1} \circ H(C^*(\psi))\colon H^+(\Lambda V, d_1) \xrightarrow{\cong} sA^* \oplus sB^*$$

of graded vector spaces. We have already shown that the elements $v_i \in V$ are in one-to-one correspondence to the generators $x_i \in A$ via the duality of V and $sL = V^*$. We retrieve this correspondence in the isomorphism above, in the following sense. To begin with, note that the elements $([v_i])_{i\in I}$ in $H^+(\Lambda V, d_1)$ are linear independent, since the differential is minimal. However, the same does not hold for the products $[v_k \wedge v_l] = [v_k][v_l]$, but we will come back to this later. It follows that we may complete $([v_i])_{i\in I}$ to a basis $\{[v_i], \beta_r\}$ of $H^+(\Lambda V, d_1)$, where $r \in R$ for a suitable index set R. Note that due to $V = V^{\geq 2}$, it holds $|\beta_r| \geq 4$.

Lemma 5.2.2. *The family $(\Phi([v_i]))_{i\in I}$ is a basis for sA^*, while $(\Phi(\beta_r))_{r\in R}$ is a basis for sB^*.*

Proof. The space sA is generated by elements sx_k, $k \in I$ with $\langle v_i; sx_k\rangle = \delta_{ik}$ and $\langle w_j; sx_k\rangle = 0$. Since $\psi|_A$ is the identity and $\psi|_B = 0$, we get

$$\langle v_i; C_*(\psi)(sx_k)\rangle = \delta_{ik} \quad \text{and} \quad \langle v_i; C_*(\psi)(sb)\rangle = 0,$$

which means that $H(C^*(\psi))([v_i]) = [sx_i]^* \in H(\Lambda s\mathbb{L}_{A\oplus B})^*$. On the other hand, $\theta(sx_i) = sx_i$, since sx_i has bracket length one. It follows that $H(\theta)([sx_i]) = sx_i$. Denote by $sx_i^* \in sA^*$ the dual basis of sx_i, then $H(\theta)^*(sx_i) = [sx_i]^*$ and thus $\Phi([v_i]) = sx_i^*$. □

While it is not new that a basis $(v_i)_{i\in I}$ of $\ker(d_1)$ provides a basis for A via the dual basis of $sL = V^*$, this now shows that a choice for the remaining basis elements $\beta_r \in H^+(\Lambda V, d_1)$ can be related to generators of B. In particular, since all appearing graded vector spaces are of finite type, the number of elements β_r of a fixed degree $|\beta_r| = k + 1$ tell us how many generators B_k has, i.e. how many generators will be added in the $(k-1)$-th step of the construction. What remains is to specify this relationship and to calculate the differential of generators in B.

Proposition 5.2.3. *Let $(\Lambda V, d_1)$ with basis $\{v_i, w_j\}$ as before, where $i \in I$ and $j \in J$. The classes $[v_i]$ may be complemented to a vector space basis $\{[v_i], \beta_r\}_{i\in I, r\in R}$ of $H^+(\Lambda V, d_1)$. Let L be the homotopy Lie algebra of $(\Lambda V, d_1)$ and $(\mathbb{L}_{A\oplus B}, \partial)$ the free DGL of the constructed free minimal model for $(L, 0)$. Then it holds:*

(i) There exists a basis $(x_i)_{i\in I}$ of A, where $|x_i| = |v_i| - 1$ and a basis $(b_r)_{r\in R}$ of B with $|b_r| = |\beta_r| - 1$. In particular, since A and B are of finite type, this determines their dimension in each grade.

(ii) Let $k_1 + 1 = \min\{|\beta_r| : r \in R\}$, then $B = B_{\geq k_1}$ and for all $b \in B_{k_1}$ it holds that $\partial(b) \in [A, A]$.

Proof. The first claim is an immediate consequence of the isomorphism Φ from lemma 5.2.2. It also implies that for k_1 as defined, we find $B = B_{\geq k_1}$. For the second claim, let us take a closer look at the initial step of the construction of $(\mathbb{L}_{A\oplus B}, \partial)$. For the morphism $\psi\colon (\mathbb{L}_A, 0) \to (L, 0)$ it necessarily holds that

$$\min\{k \in \mathbb{N} \mid \ker(\psi_k) \neq 0\} = k_1 - 1,$$

since otherwise, B would have generators of degree lower than k_1. Then $\ker(\psi_{k_1-1}) \subseteq [\mathbb{L}_A, \mathbb{L}_A]_{k_1-1}$ are the lowest-degree relations of L, for which we pick a basis (z_i). By definition, B_{k_1} is then generated by elements (b_i) with $|b_i| := k_1$ and a differential is defined on $(\mathbb{L}_{A\oplus B_{k_1}})$ by $\partial(a) := 0$ for all $a \in A$ and $\partial(b_i) = z_i$. Now, since $z_i \in [\mathbb{L}_A, \mathbb{L}_A]_{k_1-1}$, it is generated by elements $[x, y]$ with $x, y \in \mathbb{L}_A = A \oplus [\mathbb{L}_A, \mathbb{L}_A]$. If x or y are not in A, we may again express them as a bracket, until eventually z_i is a linear combination of elements of the form

$$[a_1, \ldots, [a_{m-1}, a_m] \ldots]$$

of varying bracket length m and fixed degree $k_1 - 1$, where $a_1, \ldots, a_m \in A$. It follows that $\partial(b_i) \in [A, A]$. □

Remark 5.2.4. Remember that if $(\mathbb{L}_{A\oplus B}, \partial)$ is a cellular Lie model for X, the cells in X correspond to a basis of $A \oplus B$, while the attaching maps are determined by the class of their differential. While the differential on $x_i \in A$ is trivial, it can be complicated for elements in B. Proposition 5.2.3 associates the cells in X with a basis of $H^+(\Lambda V, d_1)$ and makes a first attempt in expressing the differential of elements in B. We will see later on that the property $\partial(b) \in [A, A]$ is beneficial for a further description of the related attaching maps. One should note however that for elements in $B_{>k_1}$, this statement is in general not true. While the differential is still a linear combination of brackets, these brackets can now contain elements from B of lower degree, which will make it difficult to calculate the associated attaching map.

In the remainder of this section, we will see that in some cases, the differential of elements in B be calculated directly from $(\Lambda V, d_1)$. That being said, we will also see that in general, the calculation of $\partial(b)$ straight from the minimal Sullivan algebra $(\Lambda V, d_1)$ has its limits, especially for the generators $b \in B$ of higher degree.

When we look back at the construction of $(\mathbb{L}_{A\oplus B}, \partial)$, the differentials of the generators in B were used to kill unwanted cocycles in $\ker(H(\psi))$. For the initial construction step, we related the first generators of B that were added to the lowest-degree relations of L. In a general construction step however, it proves to be difficult, if even possible, to obtain a full expression of $\ker(H_k(\psi))$ in terms of $\ker(\sigma_k)$, the relations of degree k in L. Nevertheless, the following discussion shows it is quite easy to see that relations naturally yield elements in the kernel of $H(\psi)$.

Remark 5.2.5. Let us see how the connection between generators of B and relations in L can be improved. By construction, for each $k \geq 1$ we obtain a commutative diagram

$$\begin{tikzcd} (\mathbb{L}_{A\oplus B_{\leq k}}, \partial) \arrow[rr, "\psi"] & & (L, 0) \\ & (\mathbb{L}_A, 0) \arrow[ul, "\lambda"] \arrow[ur] & \end{tikzcd}$$

where λ is induced by the inclusion $A \hookrightarrow A \oplus B_{\leq k}$ of graded vector spaces. On homology level, this in turn induces

$$\begin{tikzcd} H(\mathbb{L}_{A\oplus B_{\leq k}}, \partial) \arrow[rr, "H(\psi)"] & & L \\ & \mathbb{L}_A \arrow[ul, "H(\lambda)"] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

which implies $H(\lambda)(\ker(\sigma)) \subseteq \ker(H(\psi))$. Therefore,

$$H_k(\lambda)\colon \ker(\sigma_k) \to \ker(H_k(\psi))$$

is well-defined.

Remember that $L \cong \mathbb{L}_A/\ker(\sigma)$ and that the kernel $J := \ker(\sigma) \subseteq [\mathbb{L}_A, \mathbb{L}_A]$ is an ideal. As such, it holds $[J, \mathbb{L}_A] \subseteq J$, so we can write $J = R \oplus [J, \mathbb{L}_A]$ for a suitable subspace $R \subseteq J$. A basis (r_i) of R is a minimal set of the generators for J, hence we refer to the elements r_i as *generators of the relations.* As before, let $k \geq 1$ and suppose $\mathbb{L}_{A\oplus B_{\leq k}}$ is already constructed.

Remark 5.2.6. The elements $H_k(\lambda)(r_i)$ are linear independent, and thus can be complemented to form a basis of $\ker(H_k(\psi))$. In fact, we show that $H(\lambda)$ only vanishes on $[J, \mathbb{L}_A]$.

Let $y \in \ker(\sigma_k) \subseteq [\mathbb{L}_A, \mathbb{L}_A]$ be non-trivial and assume $H_k(\lambda)(y) = 0$. This means we find $x \in \mathbb{L}_{A\oplus B_{\leq k}}$ with $|x| = k+1$ and $\partial(x) = \lambda(y)$. The only elements in $\mathbb{L}_{A\oplus B_{\leq k}}$ of degree $k+1$ that have word length one lie in A_{k+1} and are thus cycles, so we may deduce that $x \in [\mathbb{L}_{A\oplus B_{\leq k}}, \mathbb{L}_{A\oplus B_{\leq k}}]$. This means x is generated by elements $[z_1, z_2]$, with $z_i \in \mathbb{L}_{A\oplus B_{\leq k}}$ and $\partial([z_1, z_2]) = [\partial(z_1), z_2] - (-1)^{|z_1|}[z_1, \partial(z_2)] \in [\mathbb{L}_A, \mathbb{L}_A]$. But the elements $\partial(z_1), \partial(z_2) \in \mathbb{L}_A$ lie in the kernel of σ, so we have $\partial([z_1, z_2]) \in [J, \mathbb{L}_A]$ and therefore $y \in [J, \mathbb{L}_A]$. It follows that $\ker(H(\lambda)) \subseteq [J, \mathbb{L}_A]$, and thus $\ker(H(\lambda)) = [J, \mathbb{L}_A]$.

In other words, we may regard the space of the generators for the relations of L_k as a subspace of $\ker(H_k(\psi))$. In particular, for each such element $r_i \in \ker(\sigma_k)$ there exists a corresponding element γ_i in a basis of $\ker(H_k(\psi))$, for which in turn a generator $b_i \in B_{k+1}$ is introduced to kill it.
That being said, in most cases calculating the elements r_i for given L is a futile effort, since finding a vector space basis for graded Lie algebras can be quite complicated. Later, we will provide an outlook how the situation changes for the better if we restrict to odd generators in the Sullivan algebra $(\Lambda V, d_1)$, which implies that the homotopy Lie algebra L is concentrated in even degrees. For now, however, we have to restrict ourselves to elements of bracket length two, which simplifies things by a lot.

Example 5.2.7. Let $(x_i)_{i\in I}$ be an ordered basis of A and pick two elements x_i, x_j from this basis. We assume that $i \leq j$ and that $|x_i|$ is odd if $i = j$, such that $[x_j, x_i]$ is non-trivial in the free Lie algebra $\mathbb{L}_A$. Suppose $[x_j, x_i] = 0$ in L, meaning $[x_j, x_i] \in \ker(\sigma)$, and $(\mathbb{L}_{A\oplus B_{\leq k}}, \partial)$ is already constructed, where $k := |x_i| + |x_j|$. Then $[x_j, x_i]$ is a generator for the relations in L, and thus $[[x_j], [x_i]]$ provides a basis element of $\ker(H_k(\psi))$. By construction, a generator $b \in B_{k+1}$ will be added such that $\partial(b) = [x_j, x_i]$.
Note that for basis elements $y \notin A$, the same argumentation can not be applied, as they can be brackets themselves. For example, there are some cases where $y = [x, x]$, with $x \in A$ being some basis element. Then $[x, y] = 0$ in L, but this relation is due to the Jacobi identity, and thus does not provide a generator for the relations. We have seen this already in example 5.1.8. For that reason, the bracket $[x, y]$ does not add a generator to B.
Furthermore, a relation $[x_j, x_i]$ implies that elements of the form $[x, [x_j, x_i]]$ with $x \in A$ are relations as well. However, they do not provide generators for the relations. In fact, we see that $\partial((-1)^{|x|+1}[x, b]) = [x, [x_j, x_i]]$ in $(\mathbb{L}_{A\oplus B_{\leq k+1}}, \partial)$, since $\partial|_{\mathbb{L}_A} = 0$. Thus, the cycle $[x, [x_j, x_i]]$ is already killed in $(\mathbb{L}_{A\oplus B_{\leq k+1}}, \partial)$, without the need of an additional generator.

As already hinted, when we restrict ourselves to graded Lie algebras L concentrated in even degrees, the detection of relations becomes much easier. For now, let us translate the above into the language of Sullivan algebras.

Remark 5.2.8.

(a) Remember that in the situation that L is the homotopy Lie algebra of $(\Lambda V, d_1)$, the elements $x_i \in L$ from the dual basis of $v_i \in V$ generate a space A such that $L = A \oplus [L, L]$. As we have seen, a product $u_k \wedge u_l$ with vanishing coefficients $\lambda^j_{k,l}$ with regard to $d_1(w_j)$ causes the corresponding bracket $[z_l, z_k]$ to vanish. Now, in the special case that $u_k = v_k$ and $u_l = v_l$, we have that $z_k = x_k$ and $z_l = x_l$ are basis elements in A. But then $[x_l, x_k] = 0$ is a generator for the relations, so we need to introduce $b \in B$ with $\partial(b) = [x_l, x_k]$ and $|b| = |x_k| + |x_l| + 1 = |v_k| + |v_l| - 1 = |w_j|$.

(b) Similarly, fix $j \in J$ and suppose that

$$d_1(w_j) = \sum_{k,l} \lambda^j_{k,l} v_k \wedge v_l$$

for a number of $m \geq 2$ distinct basis elements $v_k \wedge v_l \in \Lambda^2 V$, and that $\lambda^{j'}_{k,l} = 0$ for all $j \neq j' \in J$, which is always the case if the considered Sullivan algebra is separated. Note that then, in particular, the elements $v_k \wedge v_l$ have the same degree. Then $y_j = \mu^j_{k,l}[x_l, x_k]$ due to proposition 5.1.5 and with the prefactors defined in remark 5.1.6. In particular, for any two distinct elements $v_k \wedge v_l \neq v_p \wedge v_q$ in this basis of $\Lambda^2 V$ we obtain

$$\mu^j_{k,l}[x_l, x_k] - \mu^j_{p,q}[x_q, x_p] = 0.$$

There are $\binom{m}{2}$ such pairs, of which at most $m-1$ are linear independent and thus provide generators for the relations. These can for instance be obtained by fixing one index, say (p,q), and let (k,l) vary over the remainder of the index set. Note that in fact,

$$\mu^j_{k,l}[x_l, x_k] - \mu^j_{k',l'}[x_{l'}, x_{k'}] = \mu^j_{k,l}[x_l, x_k] - \mu^j_{p,q}[x_q, x_p] - (\mu^j_{k',l'}[x_{l'}, x_{k'}] - \mu^j_{p,q}[x_q, x_p])$$

for an element $\mu^j_{k,l}[x_l, x_k] - \mu^j_{k',l'}[x_{l'}, x_{k'}]$. This leads to the addition of $m-1$ generators in B, with differentials given by

$$\partial(b) = \mu^j_{k,l}[x_l, x_k] - \mu^j_{p,q}[x_q, x_p].$$

The arguments above can not be applied for arbitrary products $u_k \wedge u_l$, since in these cases the corresponding brackets need not be generators for the relations.

Proposition 5.2.9. *We can describe certain steps in the construction of* $(\mathbb{L}_{A\oplus B}, \partial)$ *in greater detail.*

(i) A basis element $v_k \wedge v_l \in \Lambda^2 V$ *for which the coefficient* $\lambda^j_{k,l}$ *in* $d_1(w_j)$ *is zero for all* $j \in J$ *implies that there exists a generator* $b \in B_{|v_k|+|v_l|-1}$ *with* $\partial(b) = [x_l, x_k]$.

(ii) Assume that, for $m \geq 2$ *distinct basis elements* $v_k \wedge v_l \in \Lambda^2 V$*, there exists exactly one* $j \in J$ *such that the coefficients* $\lambda^j_{k,l}$ *in* $d_1(w_j)$ *are non-zero,* $k, l \in I$*. Fix one element* $v_p \wedge v_q$*, then for each of the remaining* $m-1$ *elements* $v_k \wedge v_l$ *a generator* $b \in B_{|w_j|}$ *is introduced such that*

$$\partial(b) = \mu^j_{k,l}[x_l, x_k] - \mu^j_{p,q}[x_q, x_p].$$

Proof. This is proven in remark 5.2.8 above. □

Remark 5.2.10. To put these statements into the context of proposition 5.2.3, consider the products $[v_k \wedge v_l] = [v_k][v_l] \in H^+(\Lambda V, d_1)$. By assumption, there exist only a finite amount of these in each degree. Again, we will first fix one element $v_k \wedge v_l$ from a basis of $\Lambda^2 V$ and assume that $\lambda^j_{k,l} = 0$ for all $j \in J$. This is equivalent to

$$0 \neq [v_k][v_l] \notin H^+(\Lambda V, d_1) \setminus \mathbb{Q}([v_k][v_l]),$$

meaning $[v_k][v_l]$ is not a linear combination of the remaining elements in $H^+(\Lambda V, d_1)$. In particular, there exists a basis of $H^+(\Lambda V, d_1)$ containing $[v_k][v_l]$. This element thus corresponds to a generator $b \in B$, and by remark 5.2.8 it holds $\partial(b) = [x_l, x_k]$.
Similarly, suppose $m \geq 2$ distinct basis elements $v_k \wedge v_l \in \Lambda^2 V$ appear solely in the basis expression of $d_1(w_j)$ for exactly one $j \in J$ and fix one element $v_p \wedge v_q$. This can be equivalently expressed by the linear dependence

$$\lambda^j_{p,q}[v_q][v_q] = \sum_{k,l} \lambda^j_{k,l}[v_k][v_l],$$

where the $m-1$ products $[v_k][v_l]$ on the right-hand side are linear independent. These then correspond to $m-1$ generators in B, and by remark 5.2.8 their differentials calculate as $\mu^j_{k,l}[x_l, x_k] - \mu^j_{p,q}[x_q, x_p]$.

As we can see, the differential d_1 determines how the products $[v_k][v_l]$ are related and thus to what extent they can occur in a basis of $H^+(\Lambda V, d_1)$. In turn, the elements β_r in a basis of $H^+(\Lambda V, d_1)$ that are not generated by $([v_i])_{i\in I}$ specify the amount of generators of B in each degree. In a way, proposition 5.2.9 states that the differential of generators in B which correspond to products $[v_k][v_l]$ appearing in a basis of $H^+(\Lambda V, d_1)$ in the sense of proposition 5.2.9 can be calculated. Prior to this, we only knew that the differential of the lowest-degree generators in B lies in $[A, A]$.

Example 5.2.11. The homotopy Lie algebra of the minimal Sullivan algebra $(\Lambda(v), 0)$ with $|v|$ odd is the free graded Lie algebra in one generator, $L = \mathbb{L}_{(x)}$ with $|x| = |v| - 1$. For the case that $|v|$ is even, however, we get $v^2 \in \Lambda^2 V$ but no differential, so $[x, x] = 0$ in L which defines a relation. We therefore introduce a generator $b_1 \in B$ with $\partial(b_1) = [x, x]$, as stated in proposition 5.2.9. However, the construction of the free Lie model for L is far from being finished with $(\mathbb{L}_{(x,b_1)}, \partial)$. For example $[x, b_1]$ now is a cycle, so we need another generator $b_2 \in B$ with $\partial(b_2) = [x, b_1]$, which in turn results in the formation of new cycles.
Moreover, $H^+(\Lambda(v), 0) = \Lambda(v)$ is generated by products v^n of degree $n|v|$, thus by proposition 5.2.3 we know that B is generated by $(b_n)_{n\geq 1}$ with $|b_n| = (n+1)|v| - 1$. We get a free Lie model of the form $\psi\colon (\mathbb{L}_{\mathbb{Q}x\oplus B}, \partial) \xrightarrow{\simeq} L$ for which $\psi(x) = x$ and $\psi|_B = 0$. We can generalize the calculation of the differential to $\partial(b_{n+1}) = [x, b_n]$ for $n \geq 2$, as follows from

$$\partial([x, b_n]) = [x, [x, b_{n-1}]] = 0$$

by the Jacobi identity. The class $[[x, b_n]]$ then generates $\ker(H_{(n+2)|v|-2}(\psi))$, which leads to the introduction of b_{n+2} with degree $(n+2)|v| - 1$ and differential $[x, b_n]$. Note however that we have only established a formalism for the differential of b_1 and had to calculate the differentials of the other generators b_n manually.

Example 5.2.12. For the minimal Sullivan algebra $(\Lambda(v, w), d_1(w) = v^2)$, where $|v| \geq 2$ is even, we have already seen in example 5.1.8 that $L = \mathbb{L}_{(x)}$ is a free graded Lie algebra.

Example 5.2.13. For a minimal Sullivan algebra $(\Lambda(v_1, v_2), 0)$ with $|v_1|, |v_2| \geq 2$ it immediately follows that $x_1, x_2 \in A$ and $b \in B$ with $\partial(b) = [x_2, x_1]$ for the product $v_1 \wedge v_2$, since the differential is trivial. This concludes the construction of the free Lie model if both v_1 and v_2 have odd degree. If $|v_i|$ is even such that $0 \neq v_i^2 \in \Lambda^2 V$, we get two additional generators $b_i \in B$ for these products such that $\partial(b_i) = [x_i, x_i]$. Further, similar to the previous example, in this case generators of increasing degree will be needed in the construction of $(\mathbb{L}_{A\oplus B}, \partial)$. This can be averted by introducing generators w_i with $d_1(w_i) = v_i^2$, such that $[x_i, x_i] \neq 0$ in L. The cohomology of the resulting Sullivan algebra $(\Lambda(v_1, v_2, w_1, w_2), d_1)$ is generated by $\{[v_1], [v_2], [v_1][v_2]\}$ in positive degrees, thus we arrive at the free Lie algebra $(\mathbb{L}_{A\oplus B}, \partial)$ with A generated by x_1, x_2 and B having a single generator b with $\partial(b) = [x_2, x_1]$, just as in the odd case before. Finally, for the remaining case that $|v_1|$ is odd and $|v_2|$ is even, the minimal Sullivan algebra $(\Lambda(v_1, v_2, w_2), d_1(w_2) = v_2^2)$ leads to the same free graded Lie algebra.

Example 5.2.14. In example 5.1.10 we introduced a minimal Sullivan algebra of the form

$$(\Lambda V, d_1) := (\Lambda(v_1, \ldots, w_1, \ldots), d_1)$$

with homology $H^+(\Lambda V, d_1) = \oplus_i \mathbb{Q}[v_i]$. For the homotopy Lie algebra $L = A \oplus [L, L]$, proposition 5.2.3 directly implies that $L = \mathbb{L}_A$ with A generated by $(x_i)_i$.

The following examples show that a lot of the information regarding the construction of $(\mathbb{L}_{A\oplus B}, \partial)$ from $(\Lambda V, d_1)$ is not covered by proposition 5.2.9. While there are cases in which we can obtain the free Lie model manually, in a general setting we fail to give an exact description.

Example 5.2.15. Consider the minimal Sullivan algebra $(\Lambda(v_1, v_2, w), d_1(w) = v_1 \wedge v_2)$, which yields the homotopy Lie algebra $L = \langle x_1, x_2, [x_1, x_2]\rangle_{\mathbb{Q}}$ from example 5.1.11, with $|v_i|$ odd and therefore $|x_i|$ even. Then A is generated by x_1, x_2 and the only non-trivial bracket shall be $[x_1, x_2]$. In particular, since $[x_1, [x_1, x_2]]$ and $[x_2, [x_1, x_2]]$ do not vanish in $\mathbb{L}_A$, there will be two elements b_1 and b_2 added in the construction of B with $\partial(b_1) = [x_1, [x_1, x_2]]$ and $\partial(b_2) = [x_2, [x_1, x_2]]$. However, this for example leads to $[x_1, b_2] - [x_2, b_1]$ being a cycle due to the Jacobi identity, and this element lies in the kernel of $H(\psi)$. So we need to add another generator b_3 to B and define

$\partial(b_3) := [x_1, b_2] - [x_2, b_1]$.
This actually completes the construction, as follows from proposition 5.2.3 and the fact that

$$\{[v_1], [v_2], [v_1 \wedge w], [v_2 \wedge w], [v_1 \wedge v_2 \wedge w]\}$$

is a basis for $H^+(\Lambda(v_1, v_2, w), d_1)$. We get a free Lie model

$$\psi\colon (\mathbb{L}_{A \oplus B}, \partial) \to (L, 0), \quad \psi(x_i) = x_i, \quad \psi(b_r) = 0,$$

where A is generated by x_1, x_2 and B is generated by b_1, b_2, b_3 and with differentials as provided above.

Note that here, the cycles $[x_1, [x_1, x_2]]$ and $[x_2, [x_1, x_2]]$ are essentially due to the cocycles $v_1 \wedge w$ and $v_2 \wedge w$, but as we have pointed out before, it is not generally true that the brackets $[x_1, y]$ and $[x_2, y]$ provide generators for the relations. We may apply proposition 5.1.5 to read off $[x_1, y] = 0$ and $[x_2, y] = 0$ from the Sullivan model, but have to check if they provide generators for the relations by hand. We will revisit and expand this example later.

Example 5.2.16. When looking at $(\Lambda(v_1, v_2, w), d_1(w) = v_1 \wedge v_2)$ with $|v_1|, |v_2|$ being even, as usual, things get more complicated. We employ proposition 5.2.9 and see that $x_1, x_2 \in A$ and $b_1, b_2 \in B$ with $\partial(b_i) = [x_i, x_i]$. This actually takes care of the brackets $[x_1, [x_1, x_2]]$ and $[x_2, [x_1, x_2]]$ as well, since we do have the identities

$$[x_1, [x_1, x_2]] = -\frac{1}{2}[x_2, [x_1, x_1]] \quad \text{and} \quad [x_2, [x_1, x_2]] = -\frac{1}{2}[x_1, [x_2, x_2]],$$

leading to $\partial(-\frac{1}{2}[x_2, b_1]) = [x_1, [x_1, x_2]]$ and $\partial(-\frac{1}{2}[x_1, b_2]) = [x_2, [x_1, x_2]]$. In similar fashion to example 5.2.11 we can continue the calculation by hand and see that for each $n \geq 1$ we get generators $b_n^1, b_n^2 \in B$ with

$$\partial(b_{n+1}^1) = [x_1, b_n^1] \quad \text{and} \quad \partial(b_{n+1}^2) = [x_2, b_n^2],$$

where $b_1^1 := b_1$ and $b_1^2 := b_2$. These correspond to the basis $\{[v_1]^n, [v_2]^n \mid n \geq 1\}$ of the cohomology algebra $H^+(\Lambda(v_1, v_2, w), d_1)$ using proposition 5.2.3.
Through adding two generators w_1, w_2 with $d_1(w_1) := v_1^2$ and $d_1(w_2) := v_2^2$ we end up with $(\Lambda(v_1, v_2, w, w_1, w_2), d_1)$. We can no longer make use of proposition 5.2.9, but we see that the homotopy Lie algebra no longer has relations of bracket length two. On the other hand, we now need generators $b \in B$ to kill cycles of bracket length three. For example, the cocycles

$$v_1 \wedge w - v_2 \wedge w_1 \quad \text{and} \quad v_2 \wedge w - v_1 \wedge w_2$$

define non-trivial, linear independent elements in the cohomology of the Sullivan algebra and thus lead to two generators in B of degree $2(|v_1| - 1) + |v_2|$ and $2(|v_2| - 1) + |v_1|$.

Example 5.2.17. Finally, in the minimal Sullivan algebra $(\Lambda(v_1, v_2, v_3, w), d_1)$ with $|v_1| = |v_3|$, $|v_i| \geq 2$ and $d_1(w) = v_1 \wedge v_2 + v_2 \wedge v_3$, the product $v_1 \wedge v_3$ of cocycles yields b with $\partial(b) = [x_3, x_1]$. The pair $(v_1 \wedge v_2, v_2 \wedge v_3)$ is responsible for b' with $\partial(b') = (-1)^{|v_1|}[x_2, x_1] - (-1)^{|v_2|}[x_3, x_2]$. Here, $|b| = 2|v_1| - 1$ and $|b'| = |v_1| + |v_2| - 1$.

These examples show that along the process, we might need to add generators to B which are not directly traceable back to a generating relation in L. They are added due to unwanted cycles that appear in the free Lie model along its construction and are difficult to link to the underlying minimal Sullivan model. For instance, in example 5.2.15 a generator b_3 is added due to the cycle $[x_1, b_2] - [x_2, b_1]$, which appears due to the symmetry and Jacobi relations. We have linked this generator to the class of the element $v_1 \wedge v_2 \wedge w = d_1(w) \wedge w$, which is essentially a cocycle due to symmetry and the condition $d_1^2 = 0$. These two directly correlate to the symmetry and Jacobi-identity in the homotopy Lie algebra L. However, how we might directly read off the differential

$$\partial(b_3) = [x_1, b_2] - [x_2, b_1]$$

directly from the Sullivan algebra is unclear at this point.

5.3 Calculating Attaching Maps via Whitehead Products

Let us briefly recall where we are. Starting with a minimal Sullivan algebra $(\Lambda V, d_1)$ and a choice of a basis $(v_i)_{i\in I}$ for $\ker(d_1)$, we complemented this to a basis of V, denoting the additional basis elements by w_j, $j \in J$. This defines a unique basis on the homotopy Lie algebra L of the minimal Sullivan algebra, the dual basis. We denoted by $sx_i \in sL$ those elements in this basis which vanish everywhere except on v_i. The elements $(x_i)_{i\in I}$ then generate a subspace $A \subseteq L$ for which it holds $L = A \oplus [L, L]$. With that, there exists a surjection $\mathbb{L}_A \to L$ which can be used to obtain a free Lie model $(\mathbb{L}_{A\oplus B}, \partial) \to (L, 0)$ of L. Moreover, we have discussed how certain algebraic data in $(\Lambda V, d_1)$ is responsible for generators in A and B. What is left is the transgression into the world of CW complexes.

For this, remember that we can describe a CW complex X such that $(\mathbb{L}_{A\oplus B}, \partial)$ is a Lie model for X. Here, for $n \geq 1$

- the $(n + 1)$-cells $\mathbb{B}_i^{n+1}$ are in one-to-one correspondence to a basis (c_i) of $A_n \oplus B_n$,
- $(\mathbb{L}_{A_{\leq n}\oplus B_{\leq n}}, \partial)$ is a Lie model for X^{n+1},
- the class $\gamma_i \in \pi_n(X^n)$ of the attaching map of a cell $\mathbb{B}_i^{n+1}$ is given by the homology class $[\partial(c_i)]) \in H(\mathbb{L}_{A_{<n}\oplus B_{<n}}, \partial)$ under the isomorphism τ,
- X^0 is the space consisting of one point x_0.

By the isomorphism $\Phi\colon H^+(\Lambda V, d_1) \xrightarrow{\cong} sA^* \oplus sB^*$ and the finite type assumption, a basis of $H^{n+1}(\Lambda V, d_1)$ translates into a basis of $A_n \oplus B_n$. We get the following consequence.

Proposition 5.3.1. *A minimal Sullivan algebra $(\Lambda V, d_1)$ with $V = V^{\geq 2}$ of finite type provides a minimal Sullivan model for a CW complex X for which the number of cells in X^n is equal to the dimension of $H^n(\Lambda V, d_1)$.*

Proof. As for any vector space of finite type, we get $A^* \cong A$ and $B^* \cong B$, thus Φ provides isomorphisms $H^{n+1}(\Lambda V, d_1) \cong A_n \oplus B_n$. Since the cellular Lie model establishes a direct connection between the cells of dimension $n + 1$ and a basis of $A_n \oplus B_n$, the claim follows. $\square$

Remark 5.3.2. We can specify this a little. The elements v_i of degree $|v_i| =: k_i + 1$ correspond to x_i in a basis of A with $|x_i| = k_i$. Since $\partial|_A = 0$, it follows that the attaching map of the corresponding cell $\mathbb{B}^{k_i+1}$ is trivial, resulting in a sphere $\mathbb{S}^{k_i+1}$ that is attached to X^{k_i} at the base point. We can deduce inductively that, given a fix $n \geq 1$, the skeleton X^{n+1} has a subspace

$$Y^{n+1} := \bigvee_{i\in I_n} \mathbb{S}_i^{k_i+1} \subseteq X^{n+1},$$

where by $I_n \subseteq I$ we mean the index set that belongs to the elements v_i with $|v_i| \leq n + 1$. Note that, in particular, $(\mathbb{L}_{A_{\leq n}}, 0)$ is a cellular Lie model for Y^{n+1}.

As for the elements $\beta_r \in H^+(\Lambda V, d_1)$ that correspond to a basis of B, the situation is more complicated. For these, we have established a few things.

(a) When $k_1 + 1 = \min\{|\beta_r| : r \in R\}$, then $B = B_{\geq k_1}$ and $\partial(b) \in [A, A]$ for all $b \in B_{k_1}$.

(b) Assume $v_k \wedge v_l$ is a product in the basis of $\Lambda^2 V$ for which the associated coefficient $\lambda_{k,l}^j$ in the basis representation of $d_1(w_j)$ vanishes for all $j \in J$. Then $v_k \wedge v_l$ corresponds to a basis element $b \in B_{|v_k|+|v_l|-1}$ with $\partial(b) = [x_l, x_k] \in [A, A]$. Moreover, $[v_k][v_l]$ is not a linear combination of elements in $H^+(\Lambda V, d_1) \setminus \mathbb{Q}([v_k][v_l])$.

(c) If $d_1(w_j) = \sum_{k,l} \lambda_{k,l}^j v_k \wedge v_l$ and if $\lambda_{k,l}^{j'} = 0$ for all $j \neq j' \in J$, then fixing one element $v_p \wedge v_q$, we get a generator $b \in B_{|w_j|}$ with $\partial(b) = \mu_{k,l}^j [x_l, x_k] - \mu_{p,q}^j [x_q, x_p] \in [A, A]$ for each remaining element $v_k \wedge v_l$ in the above linear combination. Moreover, $[v_p][v_q]$ is a linear combination of the remaining products $[v_k][v_l]$, whereas these are linear independent.

According to the cellular Lie model, for each b of a basis of B_n a cell is attached by a representative of the class

$$\tau(s[\partial(b)]) \in \pi_n(X^n) \otimes \mathbb{Q}.$$

Here, we have to choose b such that $\tau(s[\partial(b)])$ can be represented by a map $\mathbb{S}^n \to X^n$. Moreover, since the cases above share the fact that $\partial(b) \in [A, A]$, our goal now is to evaluate τ on cycles in $s[A, A]$.

The following arguments actually work for any free chain Lie algebra $(\mathbb{L}_{A\oplus B}, \partial)$ of finite type, where $\partial(a_i) = 0$, (a_i) being a basis of A. We can therefore apply this to our case, with specific basis $(x_i)_{i\in I}$ provided by V. Let X denote the CW complex that has $(\mathbb{L}_{A\oplus B}, \partial)$ as cellular Lie model, the skeleton X^{n+1} being modelled by $(\mathbb{L}_{A_{\leq n}\oplus B_{\leq n}}, \partial)$. The space $Y^{n+1} \subseteq X^{n+1}$ is then a bouquet of spheres $\mathbb{S}_i^{k_i+1}$, one for each element a_i, $k_i := |a_i|$. As noted before, $(\mathbb{L}_{A_{\leq n}}, 0)$ is a cellular Lie model for Y^{n+1}.

Lemma 5.3.3. *Let $n \geq 1$ and denote by $\lambda\colon \mathbb{L}_{A_{\leq n}} \to \mathbb{L}_{A_{\leq n}\oplus B_{\leq n}}$ and $j\colon Y^{n+1} \to X^{n+1}$ the inclusions. Further, let*

$$\tau\colon sH(\mathbb{L}_{A_{\leq n}\oplus B_{\leq n}}, \partial) \xrightarrow{\cong} \pi_*(X^{n+1}) \otimes \mathbb{Q}$$
$$\tau'\colon sH(\mathbb{L}_{A_{\leq n}}, 0) = s\mathbb{L}_{A_{\leq n}} \xrightarrow{\cong} \pi_*(Y^{n+1}) \otimes \mathbb{Q}$$

be the isomorphisms associated with the Lie models. Then $\tau \circ sH(\lambda) = (\pi_*(j) \otimes \mathbb{Q}) \circ \tau'$, meaning the following diagram commutes:

$$\begin{tikzcd}
\pi_*(X^{n+1}) \otimes \mathbb{Q} & sH(\mathbb{L}_{A_{\leq n}\oplus B_{\leq n}}, \partial) \arrow[l, "\tau", "\cong"'] \\
\pi_*(Y^{n+1}) \otimes \mathbb{Q} \arrow[u, "\pi_*(j)\otimes\mathbb{Q}"] & s\mathbb{L}_{A_{\leq n}} \arrow[l, "\tau'", "\cong"'] \arrow[u, "sH(\lambda)"']
\end{tikzcd}$$

Proof. The Lie models yield quasi-isomorphisms

$$m\colon C^*(\mathbb{L}_{A_{\leq n}\oplus B_{\leq n}}, \partial) \xrightarrow{\simeq} A_{PL}(X^{n+1}),$$
$$m'\colon C^*(\mathbb{L}_{A_{\leq n}}, 0) \xrightarrow{\simeq} A_{PL}(Y^{n+1}).$$

Remember that the minimal Sullivan model for a space is a minimal Sullivan model for the Lie model of the space and vice versa. Thus, denoting by $(\Lambda V_{X^{n+1}}, d)$ and $(\Lambda V_{Y^{n+1}}, d)$ the corresponding minimal Sullivan models, we can extend the homotopy commutative diagram from theorem 4.5.2 to obtain

$$\begin{tikzcd}
 & C^*(\mathbb{L}_{A_{\leq n}}, 0) \arrow[d, "m'"', "\simeq"] & C^*(\mathbb{L}_{A_{\leq n}\oplus B_{\leq n}}, \partial) \arrow[l, "C^*(\lambda)"'] \arrow[d, "\simeq"', "m"] & \\
(\Lambda V_{Y^{n+1}}, d) \arrow[ur, bend left, "\varphi'", "\simeq"'] \arrow[r, "\simeq"] & A_{PL}(Y^{n+1}) & A_{PL}(X^{n+1}) \arrow[l, "A_{PL}(j)"'] & (\Lambda V_{X^{n+1}}, d) \arrow[l, "\simeq"'] \arrow[ul, bend right, "\varphi"', "\simeq"] \arrow[lll, bend left, "\varphi_j"]
\end{tikzcd}$$

where φ and φ' are lifts and φ_j is a Sullivan representative of j. It follows that $C^*(\lambda) \circ \varphi \simeq \varphi' \circ \varphi_j$. Now, due to proposition 4.2.9, $C^*(\mathbb{L}_{A_{\leq n}}, 0)$ is the Sullivan algebra $(\Lambda(s\mathbb{L}_{A_{\leq n}})^*, d = d_0 + d_1)$. Since d_0 is dual to the derivative in $\mathbb{L}_{A_{\leq n}}$ which is zero, it too vanishes. It follows that $C^*(\mathbb{L}_{A_{\leq n}}, 0)$ is a minimal Sullivan algebra and we may apply proposition 3.4.11 to deduce that

$$Q(C^*(\lambda)) \circ Q(\varphi) = Q(C^*(\lambda) \circ \varphi) = Q(\varphi' \circ \varphi_j) = Q(\varphi') \circ Q(j),$$

where as usual we write $Q(j)$ for $Q(\varphi_j)$. This yields the following commutative diagram

$$\begin{tikzcd}
(\pi_*(X^{n+1}) \otimes \mathbb{Q})^* \arrow[d, "(\pi_*(j)\otimes\mathbb{Q})^*"'] & V_{X^{n+1}} \arrow[l, "\cong"', "\nu_{X^{n+1}}"] \arrow[r, "Q(\varphi)"] \arrow[d, "Q(j)"'] & (s\mathbb{L}_{A_{\leq n}\oplus B_{\leq n}})^* \arrow[d, "Q(C^*(\lambda))"] \\
(\pi_*(Y^{n+1}) \otimes \mathbb{Q})^* & V_{Y^{n+1}} \arrow[l, "\cong"', "\nu_{Y^{n+1}}"] \arrow[r, "Q(\varphi')"] & (s\mathbb{L}_{A_{\leq n}})^*,
\end{tikzcd}$$

where the left-hand square is due to remark 3.6.7 with the horizontal isomorphisms being induced by the dual pairings $V_{X^{n+1}} \times \pi_*(X^{n+1})$ and $V_{Y^{n+1}} \times \pi_*(Y^{n+1})$. Now, as the linear part of a CDGA morphism commutes with the linear part of the differential, we may view the right-hand square as a commutative diagram of complexes, the differential in $V_{X^{n+1}}$ and $V_{Y^{n+1}}$ being trivial due to the minimality of the Sullivan models. The induced maps between homology groups commute, so we have

$$H(Q(C^*(\lambda))) \circ H(Q(\varphi)) = H(Q(\varphi')) \circ Q(j).$$

Now by proposition 4.2.13, the maps $H(Q(\varphi))$ and $H(Q(\varphi'))$ dualize to isomorphisms

$$\begin{aligned}
H(Q(\varphi))^* \colon sH(\mathbb{L}_{A_{\leq n}\oplus B_{\leq n}}, \partial) &\xrightarrow{\cong} sL_{X^{n+1}}, \\
H(Q(\varphi'))^* \colon sH(\mathbb{L}_{A_{\leq n}}, 0) &\xrightarrow{\cong} sL_{Y^{n+1}},
\end{aligned}$$

where $L_{X^{n+1}}$ and $L_{Y^{n+1}}$ denote the homotopy Lie algebras associated with the minimal Sullivan algebras $(\Lambda V_{X^{n+1}}, d)$ and $(\Lambda V_{Y^{n+1}}, d)$. Hence, by dualizing the diagram above, we get the following commutative diagram

$$\begin{tikzcd}
\pi_*(X^{n+1}) \otimes \mathbb{Q} \arrow[r, "\cong", "\mu"'] & sL_{X^{n+1}} & sH(\mathbb{L}_{A_{\leq n}\oplus B_{\leq n}}, \partial) \arrow[l, "H(Q(\varphi))^*"', "\cong"] \\
\pi_*(Y^{n+1}) \otimes \mathbb{Q} \arrow[u, "\pi_*(j)\otimes\mathbb{Q}"] \arrow[r, "\cong", "\mu'"'] & sL_{Y^{n+1}} \arrow[u, "Q(j)^*"] & sH(\mathbb{L}_{A_{\leq n}}, 0). \arrow[l, "H(Q(\varphi'))^*"', "\cong"] \arrow[u, "H(Q(C^*(\lambda)))^*"']
\end{tikzcd}$$

The isomorphisms μ and μ' are the same as in theorem 4.1.10 and by definition coincide, up to sign, with the dual isomorphisms $(\nu_{X^{n+1}})^*$ and $(\nu_{Y^{n+1}})^*$. Finally, remember that $\tau = \mu^{-1} \circ H(Q(\varphi))^*$ and $\tau' = \mu'^{-1} \circ H(Q(\varphi'))^*$, which are exactly the horizontal parts of the diagram from right to left. Finally, for the morphism $H(Q(C^*(\lambda)))^*$, use the isomorphism described in proposition 4.2.9 to identify $C^*(\lambda) = (C_*(\lambda))^* = (\Lambda\overline{\lambda})^*$ with $\Lambda\overline{\lambda}^*$. It follows that $Q(C^*(\lambda)) = \overline{\lambda}^*$ and thus $H(Q(C^*(\lambda)))^* = H(\overline{\lambda}) = sH(\lambda)$, as $\overline{\lambda}(sx) = s\lambda(x)$ for all $x \in \mathbb{L}_{A_{\leq n}}$. □

Preparation 5.3.4. Let us proceed by investigating the isomorphism τ' a little bit further. We know by theorem 4.5.2 that $(\mathbb{L}_{A_{\leq n}}, 0)$ is a cellular Lie model for Y^{n+1}, which implies that there is a one-to-one correspondence between basis elements $a_i \in A_{\leq n}$ of degree k_i and spheres $\mathbb{S}_i^{k_i+1}$ in Y^{n+1}. However, we have yet to specify this correspondence, which we will do now.
Again, τ' is a composition of two isomorphisms $s\mathbb{L}_{A_{\leq n}} \cong sL_{Y^{n+1}}$ and $sL_{Y^{n+1}} \cong \pi_*(Y^{n+1}) \otimes \mathbb{Q}$. The first one is provided by a minimal Sullivan model for $C^*(\mathbb{L}_{A_{\leq n}}, 0)$ and the second one comes from the duality of $V_{Y^{n+1}}$ and $\pi_*(Y^{n+1}) \otimes \mathbb{Q}$ which is induced by a minimal Sullivan model for Y^{n+1}. In both cases, the isomorphism only depends on the unique isomorphy class of the model. Now remember that $C^*(\mathbb{L}_{A_{\leq n}}, 0) = (\Lambda(s\mathbb{L}_{A_{\leq n}})^*, d_1)$ is itself a minimal Sullivan algebra. We may therefore choose φ' in the proof of lemma 5.3.3 to be the identity. This means the Lie model m' is also a minimal Sullivan model

$$m' \colon (\Lambda V_{Y^{n+1}}, d_1) \xrightarrow{\simeq} A_{PL}(Y^{n+1})$$

for the space Y^{n+1}. It follows that the first part of τ', the isomorphism $H(Q(\varphi'))^*$, is the identity on $sL_{Y^{n+1}} = s\mathbb{L}_{A_{\leq n}}$. For the second part, as Y^{n+1} is a bouquet of spheres, note that $H^*(Y^{n+1};\mathbb{Q}) = \oplus_i H^*(\mathbb{S}_i^{k_i+1};\mathbb{Q})$ generated by the elements $[\mathbb{S}_i^{k_i+1}]^*$ dual to the fundamental classes. We may therefore choose m' to be the minimal Sullivan model that sends the basis element in $V_{Y^{n+1}}$ that is provided by a_i to the corresponding representatives of $[\mathbb{S}_i^{k_i+1}]^*$. That means, supposing $\mathbb{L}_{A_{\leq n}}$ is provided, a basis element $a_i \in A_{\leq n}$ defines a dual basis element $v_i \in V_{Y^{n+1}}$ with $d(v_i) = 0$. Vice versa, if we start with $V_{Y^{n+1}}$, a basis $(v_i)_{i\in I}$ of $\ker(d_1)$ as discussed yields a dual basis $(x_i)_{i\in I}$ of $A_{\leq n}$. Then, for every such basis element $v_i \in V_{Y^{n+1}}$ with $|v_i| =: k_i + 1$, choose a corresponding generator $[\mathbb{S}_i^{k_i+1}]^* \in \oplus_i H^*(\mathbb{S}_i^{k_i+1};\mathbb{Q})$ and take m' to be the morphism for which $H(m')([v_i]) = [\mathbb{S}_i^{k_i+1}]^*$.

Remember that following the notation of remark 2.2.2, we may understand appearing homotopy classes as elements in the classical or rational homotopy groups, depending on the context.

Proposition 5.3.5. *For each basis element $a_i \in A_{\leq n}$, let $|a_i| := k_i \leq n$ and denote by $[a_i]$ the corresponding class in $(\mathbb{L}_{A_{\leq n}\oplus B_{\leq n}}, \partial)$. Let $\iota_i\colon \mathbb{S}_i^{k_i+1} \hookrightarrow Y^{n+1}$ be the inclusion of the corresponding sphere, with class $\alpha_i := [\iota_i]$. It then holds $\tau(s[a_i]) = \pi_*(j)(\alpha_i)$.*

Proof. By lemma 5.3.3, it remains to show that $\tau'(sa_i) = \alpha_i$. We already identified τ' with the inverse of the dual isomorphism

$$(\nu_{Y^{n+1}})^*\colon \pi_*(Y^{n+1}) \otimes \mathbb{Q} \xrightarrow{\cong} (V_{Y^{n+1}})^* = s\mathbb{L}_{A_{\leq n}}, \quad \alpha \mapsto \langle -;\alpha\rangle,$$

with the dual pairing from construction 3.6.4. This definition extends to elements $\alpha \otimes \lambda$ using linearity. Remember that $\langle v;\alpha_i\rangle \in \mathbb{Q}$ for $v \in V_{Y^{n+1}}^{k_i+1}$ was defined to be the prefactor of $Q(\iota_i)(v) = Q(\varphi_{\iota_i})(v) \in \mathbb{Q}e$, where e is the generator of degree $k_i + 1$ from the minimal Sullivan model of the sphere $\mathbb{S}_i^{k_i+1}$ and φ_{ι_i} is a Sullivan representative of ι_i. The corresponding homotopy commutative diagram is

$$\begin{tikzcd} A_{PL}(\mathbb{S}_i^{k_i+1}) & A_{PL}(Y^{n+1}) \arrow[l, "A_{PL}(\iota_i)"'] \\ (\Lambda(e,\dots),d) \arrow[u, "\simeq"] & (\Lambda V_{Y^{n+1}}, d). \arrow[l, "\varphi_{\iota_i}"'] \arrow[u, "\simeq"', "m'"] \end{tikzcd}$$

with m' as in preparation 5.3.4 above. As usual, we identify the morphisms $H(A_{PL}(\iota_i)) = H^*(\iota_i,\mathbb{Q})$ and cohomologies $H(A_{PL}(Y^{n+1})) = H^*(Y^{n+1};\mathbb{Q}) = \oplus_i H^*(\mathbb{S}_i^{k_i+1};\mathbb{Q})$. Then $H^*(\iota_i,\mathbb{Q})$ is the identity on $H^*(\mathbb{S}_i^{k_i+1};\mathbb{Q})$ and vanishes on the other terms. We chose m' such that for each basis element v_j that is a cocycle, $m'(v_j)$ represents the generator $[\mathbb{S}_j^{k_j+1}]^*$ of $H^*(\mathbb{S}_j^{k_j+1};\mathbb{Q})$. It follows that $\varphi_{\iota_i}(v_i) = e$ and $\varphi_{\iota_i}(v_j) = 0$ for $j \neq i$, which implies that $\langle v_j;\alpha_i\rangle = \delta_{ij}$. In other words, $\alpha_i \in \pi_{k_i+1}(Y^{n+1}) \otimes \mathbb{Q}$ is the unique basis element dual to $v_i \in V_{Y^{n+1}}^{k_i+1}$, and therefore $(\nu_{Y^{n+1}})^*(\alpha_i) = sa_i$ or respectively $\tau'(sa_i) = [\iota_i]$. □

Corollary 5.3.6. *For basis elements $a_1,\dots,a_k \in A_{\leq n}$ it holds*

$$\tau(s[[a_k,\dots[a_2,a_1]\dots]]) = (-1)^{|a_k|\dots|a_2|}\pi_*(j)([\alpha_k,\dots,[\alpha_2,\alpha_1]_W\dots]_W),$$

where α_i is the class of the inclusion $\iota_i\colon \mathbb{S}_i^{k_i+1} \hookrightarrow Y^{n+1}$ and, as before, $j\colon Y^{n+1} \hookrightarrow X^{n+1}$

Proof. Assume first that $k = 2$. Since $a_1, a_2 \in A$ are cycles, it holds $[[a_2,a_1]] = [[a_2],[a_1]] \in H(\mathbb{L}_{A\oplus B},\partial)$. From remark 4.4.6 we see that $\tau(s[[a_2],[a_1]]) = (-1)^{|a_2|}[\tau(s[a_2]),\tau(s[a_1])]_W$, and by proposition 5.3.5 we see that $\tau(s[a_i]) = j_*(\alpha_i) = [j\circ\iota_i]$. Finally,

$$\begin{aligned}[j\circ\iota_2, j\circ\iota_1]_W &= (j\circ\iota_2)\vee(j\circ\iota_1)\circ a_{k_2+1,k_1+1} = j\circ(\iota_2\vee\iota_1)\circ a_{k_2+1,k_1+1}\\ &= j\circ[\iota_2,\iota_1]_W\end{aligned}$$

implies that $[[j\circ\iota_2],[j\circ\iota_1]] = \pi_*(j)([\alpha_2,\alpha_1])$. By induction, this easily extends to the general case $k \geq 2$. □

Let us return to the minimal Sullivan algebra $(\Lambda V, d_1)$. In the beginning of this section, we specified some conditions which lead to basis elements $b \in B$ with differential $\partial(b) \in s[A, A]$. Corollary 5.3.6 now allows us to express the attaching maps of the respective cells of X in terms of the Whitehead product.

Theorem 5.3.7. *Let $(\Lambda V, d_1)$ as before, with basis $\{v_i, w_j\}_{i\in I, j\in J}$ and let $\{[v_i], \beta_r\}_{i\in I, r\in R}$ be a basis of $H^+(\Lambda V, d_1)$. Then $(\Lambda V, d_1)$ is a minimal Sullivan model for a CW complex X, for which holds:*

(i) The number of cells in X^n corresponds to the dimension of $H^n(\Lambda V, d_1)$. More precisely, for each $i \in I$ a cell of dimension $|v_i|$ is attached to the base point, and for each $r \in R$ a cell of dimension $|\beta_r|$ may be attached non-trivially.

(ii) Let $k_1 + 1 = \min\{|\beta_r| : r \in R\}$ and $R_1 \subseteq R$ such that $r \in R_1$ if $|\beta_r| = k_1 + 1$. Then X has a subspace

$$Y := \vee_{i\in I} \mathbb{S}^{|v_i|} \cup_f \dot{\bigcup}_{r\in R_1} \mathbb{B}^{k_1+1} \subseteq X,$$

where f is a family of Whitehead products of inclusions $\iota_k \colon \mathbb{S}^{|v_k|} \hookrightarrow \vee_{i\in I} \mathbb{S}^{|v_i|}$.

In particular, if $R_1 = R$, no further cells are attached and we get $Y = X$. In this case, $(\Lambda V, d_1)$ is the minimal Sullivan model of a wedge of spheres, with cells of dimension $k_1 + 1$ attached by Whitehead products in $\pi_{k_1}(\vee_{i\in I} \mathbb{S}^{|v_i|})$.

Proof. As follows from the cellular Lie model, the cells in X^n are in one-to-one correspondence to a basis of $A_{n-1} \oplus B_{n-1}$. By proposition 5.2.3, there exists a basis $(x_i)_{i\in I}$ of A with $|x_i| = |v_i| - 1$ and a basis $(b_r)_{r\in R}$ of B with $|b_r| = |\beta_r| - 1$. This proves the first claim, since $\partial|_A = 0$ yields $\partial(x_i) = 0$ for all $i \in I$ and zero is represented by the constant map.
For the second claim, we again employ proposition 5.2.3 to obtain a basis $(b_r)_{r\in R_1}$ of B_{k_1}, for whose elements $\partial(b_r) \in [A, A]$ holds. As A is generated by $(x_i)_{i\in I}$, each $\partial(b_r)$ is a linear combination of elements of the form

$$[x_{i_1}, \ldots, [x_{i_{k-1}}, x_{i_k}] \ldots],$$

where $k \geq 2$. By corollary 5.3.6, these evaluate to Whitehead brackets in the classes of the respective inclusions. □

In many cases, even for degrees higher than $k_1 + 1$ the attaching maps of some cells can be expressed in terms of the Whitehead product. Moreover, in some cases, we can directly calculate these expressions from the minimal Sullivan algebra $(\Lambda V, d_1)$.

Theorem 5.3.8. *Let X be the CW complex constructed from $(\Lambda V, d_1)$ and fix $n \geq 1$. As before, $Y^n \subseteq X^n$ denotes the bouquet of spheres up to dimension n, with $j \colon Y^n \hookrightarrow X^n$ and classes α_i represented by the inclusions $\iota_i \colon \mathbb{S}^{|v_i|} \hookrightarrow Y^n$. It holds:*

(i) A basis element $v_k \wedge v_l \in (\Lambda^2 V)^{n+1}$ for which the coefficient $\lambda^j_{k,l}$ in $d_1(w_j)$ is zero for all $j \in J$ implies that a cell $\mathbb{B}^{n+1}$ is attached to X^n by the class

$$-(-1)^{|v_k|}\pi_*(j)([\alpha_l, \alpha_k]_W) \in \pi_n(X^n).$$

Then $[v_k][v_l]$ is an element in a basis of $H^+(\Lambda V, d_1)$.

(ii) Assume that, for $m \geq 2$ distinct basis elements $v_k \wedge v_l \in (\Lambda^2 V)^{n+1}$, there exists exactly one $j \in J$ such that the coefficients $\lambda^j_{k,l}$ in $d_1(w_j)$ are non-zero, $k, l \in I$. Fix one element $v_p \wedge v_q$, then for each of the remaining $m - 1$ elements $v_k \wedge v_l$ a cell $\mathbb{B}^{n+1}$ is attached to X^n by the class

$$(-1)^{|v_p|}\mu^j_{p,q}\pi_*(j)([\alpha_q, \alpha_p]_W) - (-1)^{|v_k|}\mu^j_{k,l}\pi_*(j)([\alpha_l, \alpha_k]_W) \in \pi_n(X^n).$$

Then the $m - 1$ products $[v_k][v_l]$ are elements in a basis of $H^+(\Lambda V, d_1)$.

Moreover, if the products $[v_k][v_l]$ above in combination with the elements $([v_i])_{i\in I}$ provide a full basis of $H^+(\Lambda V, d_1)$, this concludes the description of X.

Proof. In both cases, proposition 5.2.9 implies the existence of generators in B and provides a calculation of their differentials. Using corollary 5.3.6 we evaluate

$$\tau(s[[x_l, x_k]]) = -(-1)^{|v_k|}\pi_*(j)([\alpha_l, \alpha_k]_W),$$

as claimed. Finally, remark 5.2.10 explains the resulting linear independence of products $[v_k][v_l]$, such that they can be integrated into a basis of $H^+(\Lambda V, d_1)$. As the dimension of $H^n(\Lambda V, d_1)$ equals to the number of cells in X^n, no further cells are attached if this amounts to a full basis of $H^+(\Lambda V, d_1)$. □

Example 5.3.9. Let us start again with the Sullivan algebra $(\Lambda(v), 0)$ in a single generator v. In case $|v| = 2n+1$ is odd, there are no products due to degree reasons, and thus Y is a space with a base point and an attached $(2n+1)$-cell, hence $Y = \mathbb{S}^{2n+1}$. Then $H^+(\Lambda(v), 0) = \mathbb{Q}v$ implies that there are no other cells and therefore $Y = X$. Note that we could also read this off the homotopy Lie algebra, which is already the free graded Lie algebra in one generator of degree $2n$, and is a cellular Lie model for $\mathbb{S}^{2n+1}$.
Suppose otherwise, $|v| = 2n$ is even. Again, we start with a sphere $\mathbb{S}^{2n}$, but this time $v^2 \neq 0$. Since $d_1 = 0$, theorem 5.3.8 states that we then need to attach a $4n$-cell to $\mathbb{S}^{2n}$ using a representative f_n of the class

$$[f_n] := -[[\mathrm{id}_{\mathbb{S}^{2n}}], [\mathrm{id}_{\mathbb{S}^{2n}}]]_W \in \pi_{4n-1}(\mathbb{S}^{2n})$$

For the case $n = 1$, it holds $[[\mathrm{id}_{\mathbb{S}^{2n}}], [\mathrm{id}_{\mathbb{S}^{2n}}]]_W = 2\eta \in \pi_3(\mathbb{S}^2)$, where η is the class of the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$ which is a generator of $\pi_3(\mathbb{S}^2) = \mathbb{Z}$. In particular, $[f_1] \neq 0$ in $\pi_3(\mathbb{S}^2) \otimes \mathbb{Q}$. Remember that $\mathbb{C}P^2$ is obtained by attaching a 4-cell to $\mathbb{C}P^1 = \mathbb{S}^2$ via η. So what we are doing here is the rational analogue of this construction, as multiplication with two is an isomorphism in $\pi_*(\mathbb{S}^2_{\mathbb{Q}})$. Nevertheless, the construction of X is far from finished, as theorem 5.3.7 states that there is a n-cell in X for every product v^n. In example 5.2.11 it was shown that the free Lie model has generators x and $(b_n)_{n\geq 1}$ with $\partial(b_{n+1}) = [x, b_n]$ and $\partial(b_1) = [x, x]$. However, we lack the tools to calculate the corresponding attaching maps for all generators but b_1, since we have not evaluated τ on cycles $[[x, b_n]]$.
It should be mentioned that in the case of $|v| = 2$ it is already well known that $(\Lambda(v), 0)$ is the Sullivan minimal model of $\mathbb{C}P^\infty$, as it is an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. This was also carried out in example 3.5.14.

Example 5.3.10. Take $(\Lambda(v, w), d(w) = v^2)$ with $|v| = 2n$, $|w| = 4n-1$. We again get $Y = \mathbb{S}^{2n}$ for the generator v, but the product v^2 is in the image of d_1. Since the remaining basis element w is a coboundary, theorem 5.3.8 does not make any statements regarding the attachment of further cells. In this case too, theorem 5.3.7 yields that $X = Y$.

Example 5.3.11. Let $(\Lambda(v_1, v_2, \dots), d_1)$ with $|v_1| = k$, $|v_2| = n$, where we distinguish between even and odd grades, and a generator w_i with $d_1(w_i) = v_i^2$ exists if and only if $|v_i|$ is even. This summarizes the cases discussed in example 5.2.13, and it follows that $\{[v_1], [v_2], [v_1][v_2]\}$ is a basis for the cohomology. Without loss of generality, we may assume $k \leq n$, such that $X^n = \mathbb{S}^k \vee \mathbb{S}^n$. Due to the product $v_1 \wedge v_2$, a cell $\mathbb{B}^{k+n}$ will be attached to $\mathbb{S}^k \vee \mathbb{S}^n$ using the Whitehead product $[\alpha_1, \alpha_2]_W \in \pi_{k+n-1}(X^n)$, where $\alpha_1 = [i_k]$ and $\alpha_2 = [i_n]$ and i_k, i_n are the inclusions of the k- and n-spheres. Since the representative $[i_k, i_n]_W = a_{k,n}$ is the universal (k,n)-Whitehead product, we obtain $X^n \cup_{a_{k,n}} \mathbb{B}^{k+n} = \mathbb{S}^k \times \mathbb{S}^n$. Again, a look at the basis of the cohomology algebra above shows that this finishes the construction of X, so $(\Lambda(v_1, v_2, \dots), d_1)$ is the minimal Sullivan model of $\mathbb{S}^k \times \mathbb{S}^n$. Note that we obtain the same result by calculating the tensor product of the minimal Sullivan models of the spheres $\mathbb{S}^k$ and $\mathbb{S}^n$, since the model behaves well with product spaces.

Example 5.3.12. Remember the minimal Sullivan algebra $(\Lambda(v_1,\dots,w_1,\dots),d_1)$ of finite type, $|v_i| := k_i$, that is the minimal Sullivan model of $(H,0)$ with $H^0=\mathbb{Q}$ and $H^+=\oplus_i\mathbb{Q}[v_i]$ introduced in example 5.1.10. By example 5.2.14 its homotopy Lie algebra is $\mathbb{L}_A$, where A is generated by $(x_i)_i$, so $(\Lambda(v_1,\dots,w_1,\dots),d_1)$ is the minimal Sullivan model of a wedge of spheres $\vee_i\mathbb{S}^{k_i}$.

As in the previous sections, we can apply these constructions to minimal Sullivan algebras for which finding a suitable representative in the corresponding rational homotopy class is not so obvious.

Example 5.3.13. Consider $(\Lambda(v_1,v_2,w),d_1(w)=v_1\wedge v_2)$, where we first restrict ourselves to the easier case that $|v_1|=k$ and $|v_2|=n$ are odd. We again start our construction with the bouquet $\mathbb{S}^k\vee\mathbb{S}^n$, but in this case, theorem 5.3.8 makes no statement concerning the attachment of further cells. However, in example 5.2.15, we showed that a free Lie model for the homotopy Lie algebra L is given by $(\mathbb{L}_{A\oplus B},\partial)$, with A being generated by x_1,x_2 and B by b_1,b_2,b_3. Using corollary 5.3.6, the classes of the attaching maps f_1,f_2 of the cells belonging to b_1 and b_2 evaluate to $[\alpha_1,[\alpha_1,\alpha_2]_W]_W$ and $[\alpha_2,[\alpha_1,\alpha_2]_W]_W$, with $\alpha_1=[i_1]$ and $\alpha_2=[i_2]$ the classes of the inclusions of the spheres. This leads to

$$Y := (\mathbb{S}^k\vee\mathbb{S}^n)\cup_f(\mathbb{B}^{3k-1}\dot\cup\mathbb{B}^{3n-1}).$$

Unfortunately, we can not explicitly calculate the class of the attaching map of the cell in X that corresponds to $b_3\in B$, we just know it is some class in $\pi_{2(n+k)-2}(Y)$, for which we choose a representative g. All in all, this brings us to

$$X = Y\cup_g\mathbb{B}^{2(n+k)-1}$$

and we may conclude that $(\Lambda(v_1,v_2,w),d_1)$ is the minimal Sullivan model of X. A different argumentation for this example is included in chapter 15 of [7].

Example 5.3.14. Switching to the case that the elements v_i in the minimal Sullivan algebra $(\Lambda(v_1,v_2,w),d_1(w)=v_1\wedge v_2)$ have even degree, we have calculated the corresponding free Lie model in example 5.2.16. We get a cell for each of the generators $[v_1]^n$ and $[v_2]^n$ of $H^*(\Lambda(v_1,v_2,w),d_1)$, but we can only calculate the attaching maps for $n\le 2$. That is, two cells $\mathbb{B}^{2|v_i|}$ will be attached to $\mathbb{S}^{|v_1|}\vee\mathbb{S}^{|v_2|}$ by the classes $[\alpha_i,\alpha_i]_W$.

Example 5.3.15. Returning to the second part of example 5.2.16, we considered a minimal Sullivan algebra $(\Lambda(v_1,v_2,w,w_1,w_2),d_1)$ with v_1,v_2 of even degree and differential $d_1(w)=v_1\wedge v_2$, $d_1(w_1)=v_1^2$, $d_1(w_2)=v_2^2$. Let us specify $|v_1|=|v_2|=2$ and describe a corresponding CW complex without reference to previous examples. The cohomology algebra in positive degrees has basis

$$[v_1],\quad [v_2],\quad [v_1\wedge w-v_2\wedge w_1],\quad [v_2\wedge w-v_1\wedge w_2],\quad [v_1\wedge v_2\wedge w-v_2^2\wedge w_1].$$

By theorem 5.3.7, the construction of X starts with $\mathbb{S}^2_1\vee\mathbb{S}^2_2$ to which we attach two cells $\mathbb{B}^5_1,\mathbb{B}^5_2$ via Whitehead products in $\pi_4(\mathbb{S}^2_1\vee\mathbb{S}^2_2)$. We can not directly calculate them from our theorem, but we see hands on that the homotopy Lie algebra L has the relations $[x_1,[x_2,x_2]]$ and $[x_2,[x_1,x_1]]$. Again, with α_i being the class of the inclusion $\mathbb{S}^2_i\hookrightarrow\mathbb{S}^2_1\vee\mathbb{S}^2_2$, the attaching maps are $[\alpha_1,[\alpha_2,\alpha_2]_W]_W$ and $[\alpha_2,[\alpha_1,\alpha_1]_W]_W$. Lastly, for the remaining basis element of $H^+(\Lambda V,d_1)$ we obtain a cell $\mathbb{B}^7$ attached by some map

$$f\colon\mathbb{S}^6\to\mathbb{S}^2_1\vee\mathbb{S}^2_2\cup\dot{\bigcup}_{i\in\{1,2\}}\mathbb{B}^5_i.$$

Example 5.3.16. Lastly, let us consider the minimal Sullivan algebra $(\Lambda(v_1,v_2,v_3,w),d_1)$ with $|v_1|=|v_3|$, $|v_i|\ge 2$ and $d_1(w)=v_1\wedge v_2+v_2\wedge v_3$. We obtain the skeleton $Y=\mathbb{S}^{|v_1|}\vee\mathbb{S}^{|v_2|}\vee\mathbb{S}^{|v_3|}$ and attach two cells, $\mathbb{B}^{2|v_1|}$ for the product $v_1\wedge v_3$ and $\mathbb{B}^{|v_1|+|v_2|}=\mathbb{B}^{|w|+1}$ for the pair $(v_1\wedge v_2,v_2\wedge v_3)$. the attaching maps are $-(-1)^{|v_1|}[\alpha_3,\alpha_1]_W$ and $[\alpha_3,\alpha_2]_W-[\alpha_2,\alpha_1]_W$.

As a final remark, note that spaces allowing for minimal Sullivan models of the form $(\Lambda V, d_1)$ are sometimes called *coformal spaces* in literature. This is essentially equivalent to having a Lie model $(L, 0)$, where $L \cong L_X$ is the homotopy Lie algebra, since $C^*(L, 0) = (\Lambda V, d_1)$. Thus, the rational homotopy type of a coformal space is a formal equivalence of its homotopy Lie algebra $L_X = \pi_*(\Omega X) \otimes \mathbb{Q}$. One may note the similarities to the definition of a formal space, and in fact, these notions are dual to each other. In [2] it is shown that spaces which are both formal and coformal can be characterized in terms of Koszul algebras.
By our results, a coformal space with rational cohomology of finite type can, up to rational homotopy type, be thought of as a wedge of spheres to which the first cells are attached to by Whitehead products. Moreover, the dimension of the cohomology in each grade tells us the amount of cells in each dimension.

5.4 Outlook: Minimal Sullivan Algebras with Odd Generators

We already showed in theorem 5.3.7 that any realization X obtained as explained in the very beginning of this chapter has a subspace of the form $Y := \vee_i \mathbb{S}^{|v_i|}$, to with cells are attached via Whitehead brackets in $\pi_*(Y) \otimes \mathbb{Q}$. We could calculate some of these brackets directly from $(\Lambda V, d_1)$, but in many examples we had to do the calculation manually. In this outlook, we want to describe a formalism that allows a wider variety of brackets, such as the ones appearing in example 5.3.13, to be obtained directly from $(\Lambda V, d_1)$ In order to do so, we have to make some additional assumptions concerning the grade of V.
As has become clear in the preceding examples, the situation is always easier when the vector space V of a Sullivan algebra $(\Lambda V, d_1)$ is concentrated in odd degrees. Therefore, it is not too surprising that we can improve the results a little bit at the cost of the additional requirement that V is concentrated in odd degrees. In essence, the immediate advantage is that the homotopy Lie algebra L of such V is then concentrated in even degrees, such that anti-symmetry and Jacobi identity reads

$$[x, y] = -[y, x] \quad \text{and} \quad [x, [y, z]] = [[x, y], z] + [y, [x, z]] \quad \text{for } x, y, z \in L,$$

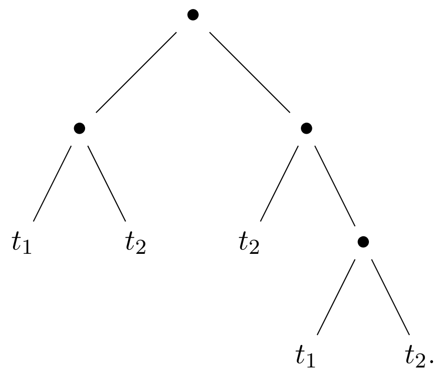
allowing L to be regarded as a Lie algebra in the non-graded sense. As we shall see, this makes it possible to provide a relatively straightforward description of a basis for the free Lie algebra $\mathbb{L}_A$, where $A \subseteq L$ such that $L = A \oplus [L, L]$. This will make it easier to find elements of a basis for $\ker(\sigma)$, where $\sigma\colon \mathbb{L}_A \to L$ is the associated surjection. For each such, in turn a generator of B is added in the first step of the construction of $(\mathbb{L}_{A \oplus B}, \partial)$.

Preparation 5.4.1. Let W be any $\mathbb{Q}$-vector space with basis T. By a *tree* we mean a formal expression on T that is recursively defined by the following:

- each element $t \in T$ is a tree,
- if t_1, t_2 are trees, then so is $t = (t_1, t_2)$.

We denote the set of all trees by $M(T)$ and observe that the bracket $(,)$ defines a binary operation on $M(T)$. Note that the naming is not arbitrary, as $M(T)$ is just the set of bracketed words in the elements of T, or in other words the *free magma* on the set T. Each element in $M(T)$ can be depicted by a binary, rooted tree whose leaves are elements of T.
Note that for an arbitrary element $t \in M(T)$ it either holds $t \in T$ or $t = (t_1, t_2)$. In the latter case, we call t_1, t_2 the *left, right immediate subtree of* t. The elements in T are called *leaves.* A tree is therefore a bracketed expression in its leaves. By $|t|$ we denote the *length* of the tree t, which is the bracket length with respect to $(,)$. Note that there exists a natural map $g\colon M(T) \to \mathbb{L}_T = \mathbb{L}_W$ defined by $g(t) = t$ if $t \in T$ and $g(t) = [g(t_1), g(t_2)]$ if $t = (t_1, t_2)$. In other words, $g(t) \in \mathbb{L}_W$ is obtained by replacing the brackets $(,)$ in the expression t with the Lie bracketing $[\ ,\ ]$. Given the elements in T are graded, W is a graded $\mathbb{Q}$ vector space and therefore, $\mathbb{L}_W$ is a graded free Lie algebra.

Example 5.4.2. For $T = \{t_1, t_2\}$, a tree $t \in M(T)$ is given by $((t_1, t_2), (t_2, (t_1, t_2)))$. It has length $|t| = 5$, right immediate subtree (t_1, t_2) and left immediate subtree $(t_2, (t_1, t_2))$. Moreover, $g(t) = [[t_1, t_2], [t_2, [t_1, t_2]]] \in \mathbb{L}_W$, where $W = \mathbb{Q}t_1 \oplus \mathbb{Q}t_2$ and t_1, t_2 is given any non-negative degree. We may visualize t as



As one might already have guessed, we will later set $W := A$ and $T := (x_i)_{i \in I}$. For now, let us stay in this general setting for a little while longer. The goal is to describe a subset $H \subseteq M(T)$ such that $g(H)$ is a basis of $\mathbb{L}_W$. In the following, assume that T has been given a total order. We say a total order $<$ on $M(T)$ is *compatible with the tree length* if for $s, t \in M(T)$ we have $s < t$ whenever $|s| < |t|$. Note that such order exists.

Definition 5.4.3. *(Hall sets)*
Consider any total order on $M(T)$ that is compatible with the tree length. A subset $H \subseteq M(T)$ can be recursively defined as follows. We start by including all trees of length one, such that $T \subseteq H$. For a tree $t \in M(T)$ of length greater than one, $t = (t_1, t_2)$, it then holds $t \in H$ if

(a) $t_1, t_2 \in H$ with $t_1 < t_2$ and

(b) either $t_2 \in T$ or $t_2 = (t_1^2, t_2^2)$ with $t_1^2 \leq t_1$,

where t_1^2, t_2^2 are the left and right immediate subtrees of t_2. The set H constructed this way is called *Hall set*, the elements $t \in H$ will be referred to as *Hall trees*.

Remark 5.4.4. Note that if $t = (t_1, t_2) \in H$ with $t_2 \in T$ we necessarily have $t_1 \in T$, since $t_1 < t_2$ and the order is compatible with the tree length. It should be pointed out that there is more than one way to introduce Hall sets. In some texts, for example, the order is reversed, which leads to Hall trees that are mirrored versions of the trees that we will be looking at. For the description of a basis of free Lie algebras, other definitions of Hall sets work as well. One should be aware, however, that the obtained basis elements can change depending on the definition used.

Note that the tree length defines a canonical grading on $M(T)$ and therefore H. This is preserved by the map $g\colon M(T) \to \mathbb{L}_W$, when we regard the grading on $\mathbb{L}_W$ defined by the bracket length. For any $n \geq 1$, we denote the subspace of H consisting of trees of length n by H_n.

Example 5.4.5. Starting with $T = \{t_1, t_2\}$, the Hall trees up to length five are given by

$$\begin{aligned}
H_1 :&\quad t_1, t_2\\
H_2 :&\quad (t_1, t_2)\\
H_3 :&\quad (t_1, (t_1, t_2)), (t_2, (t_1, t_2))\\
H_4 :&\quad (t_1, (t_1, (t_1, t_2))), (t_2, (t_1, (t_1, t_2))), (t_2, (t_2, (t_1, t_2)))\\
H_5 :&\quad (t_1, (t_1, (t_1, (t_1, t_2)))), (t_2, (t_1, (t_1, (t_1, t_2)))), (t_2, (t_2, (t_1, (t_1, t_2)))),\\
&\quad (t_2, (t_2, (t_2, (t_1, t_2)))), ((t_1, t_2), (t_1, (t_1, t_2))), ((t_1, t_2), (t_2, (t_1, t_2))).
\end{aligned}$$

These can be pictured in the sense of example 5.4.2. In fact, the tree there belongs to H_5.

As we can see, the definition of a Hall set $H \subseteq M(T)$ allows a systematical description of each of its trees $t \in H$. However, it is the following result to which Hall sets owe their importance. It states that the problem of finding a basis for the free Lie algebra $\mathbb{L}_W$ essentially reduces to finding a Hall set $H \subseteq M(T)$. In this way, Hall trees provide an algorithm to determine a basis of free Lie algebras.

Proposition 5.4.6. *Let $H \subseteq M(T)$ be a Hall set. The elements $g(t)$, $t \in H$ form a $\mathbb{Q}$-vector space basis of $\mathbb{L}_W$.*

Proof. For the proof of this statement, we refer to [21]. □

Remark 5.4.7. This allows to find bases of free graded Lie algebras in even generators. In case that there is a similar statement for general $\mathbb{L}_W$, the following could be improved in the sense that the restriction on the grade of V could be dropped.

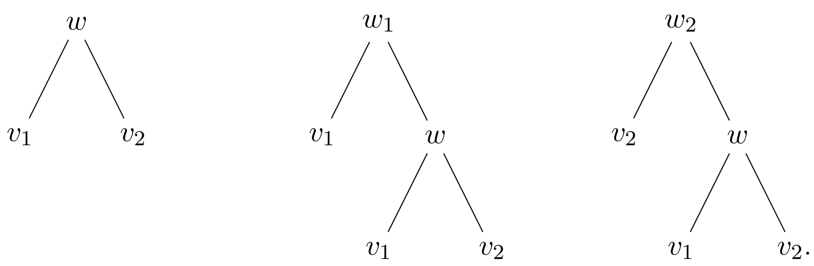
Assume now that $(\Lambda V, d_1)$ is a minimal Sullivan algebra, with $V = V^{\geq 2}$ of finite type, basis $(v_i, w_j)_{i,j}$ as before and homotopy Lie algebra L. As usual, let $L = A \oplus [L, L]$ with $x_i \in A$ and $y_j \in [L, L]$. Our goal is now to describe a way in which we can express the basis elements y_j in terms of brackets of the x_i. We will then compare these brackets with a Hall basis of $\mathbb{L}_A$ in order to detect generators for the relations in L. More precisely, each Hall basis element of $\mathbb{L}_A$ that evaluates to zero in L provides a generating relation in L. As we already know, these in turn are responsible for the addition of generators in a first step of the construction of a free Lie model for L.
Note however that the bracket expressions we obtain for the elements y_j may differ from the Hall basis elements $g(t)$. Nevertheless, as elements in $\mathbb{L}_A$ they might still be the same, since we can rearrange brackets using the symmetry and Jacobi relations. Furthermore, in case of the coefficients $\lambda^j_{p,q}$ in $d_1(w_j)$ being non-trivial for multiple basis elements $u_p \wedge u_q \in \Lambda^2 V$, we will obtain multiple expressions of y_j as seen in proposition 5.1.5. Thus, there may be further calculations necessary in order to spot generators for the relations of L. Still, a basis provided by Hall sets makes these computations more tangible and, for instance, tells us the amount of basis elements for each bracket length.

Remark 5.4.8. Let $(\Lambda V, d_1)$ and L as above and consider $T_V := (v_i)_{i \in I}$. Now, using d_1, each element w_j provides possibly multiple trees in $M(T_V)$ in the following way. Write $d_1(w_j) = \sum_{p,q} \lambda^j_{p,q} u_p \wedge u_q$ with unique coefficients $\lambda^j_{p,q} \neq 0$ using the basis $(u_p \wedge u_q)$ of $\Lambda^2 V$. Write down the bracket (u_p, u_q) for each basis element $u_p \wedge u_q$ appearing in the linear combination. In case that $u_p, u_q \in T_V$, the product is of the form $v_p \wedge v_q$ and we have that $(v_p, v_q) \in M(T_V)$ is a tree. If not, u_p or u_q are elements of the family $(w_j)_{j \in J}$. By our convention, $p \leq q$ with index set $I + J$, thus we arrive at either $u_p \in T_V$ and $u_q \in (w_j)_{j \in J}$ or both $u_p, u_q \in (w_j)_{j \in J}$. Without loss of generality, assume $u_p = v_p$ and $u_q = w_q$, where $p \in I$ and $q \in J$. Repeat the process, expressing $d_1(w_q) = \sum_{p',q'} \lambda^q_{p',q'} u_{p'} \wedge u_{q'}$, leading to brackets $(v_p, (u_{p'}, u_{q'}))$. If we continue in this fashion, eventually all elements in the bracket are in T_V. This way, each element w_j provides possibly many trees in $M(T_V)$. If in each step, the appearing linear combinations consist of only one element, we get a single tree from w_j.

Example 5.4.9.

(a) Consider again the minimal Sullivan algebra $(\Lambda(v_1, v_2, w), d_1(w) = v_1 \wedge v_2)$ with $|v_i|$ odd. We directly obtain the tree $(v_1, v_2) \in M(v_1, v_2)$ from w using d_1.

(b) Let us pursue this example a little bit further. First, observe that $v_1 \wedge w$ and $v_2 \wedge w$ are cocycles, so we may introduce generators w_1, w_2 with $d_1(w_i) = v_i \wedge w$ to arrive at $(\Lambda(v_1, v_2, w, w_1, w_2), d_1)$. Translating this into trees with leaves $\{v_1, v_2\}$, for w_1 and w_2 we first get (v_1, w) and (v_2, w), then continue by replacing w to obtain $(v_1, (v_1, v_2)), (v_2, (v_1, v_2)) \in M(v_1, v_2)$. Note that these are in particular Hall trees. The pictures are



(c) We continue in this fashion, introducing generators $w_{1,1}, w_{2,2}, w_{1,2}$ with

$$d_1(w_{1,1}) = v_1 \wedge w_1, \quad d_1(w_{2,2}) = v_2 \wedge w_2, \quad d_1(w_{1,2}) = v_1 \wedge w_2 - v_2 \wedge w_1.$$

The elements $w_{1,1}$ and $w_{2,2}$ lead to the Hall trees $(v_1, (v_1, (v_1, v_2)))$ and $(v_2, (v_2, (v_1, v_2)))$ respectively, for $w_{1,2}$ we have to choose between the two summands, leading to two choices of trees, namely $(v_1, (v_2, (v_1, v_2)))$ and $(v_2, (v_1, (v_1, v_2)))$. Only the latter is a Hall tree.

(d) Lastly, let us add two more generators w_1' and w_2' by setting

$$d_1(w_1') = w \wedge w_1 - v_2 \wedge w_{1,1}, \qquad d_1(w_2') = w \wedge w_2 - v_1 \wedge w_{2,2}.$$

For each element w_1', w_2' we again get two trees. Starting with w_1', the first summand gives $((v_1, v_2), (v_1, (v_1, v_2)))$, and for the second one we get the Hall tree $(v_2, (v_1, (v_1, (v_1, v_2))))$ of length five. The element w_2' provides a second Hall tree $((v_1, v_2), (v_2, (v_1, v_2)))$ of length five and another tree $(v_1, (v_2, (v_2, (v_1, v_2))))$ for the second summand in its differential.

We now switch to the set $T = (x_i)_{i \in I}$. There exists an obvious bijection $M(T_V) \to M(T)$ induced by the one-to-one correspondence of the elements v_i and x_i, $T_V \cong T$ (we just renamed the leaves). Thus, any tree obtained from w_j as described above in remark 5.4.8 can be viewed as an element in $M(T)$. Remember, there exists a natural map $g\colon M(T) \to \mathbb{L}_{(x_i)_{i \in I}} = \mathbb{L}_A$, which replaces the parenthesis with Lie brackets.

Proposition 5.4.10. *For each element w_j, let $t_j \in M(T_V)$ be any tree obtained by successively replacing the basis elements that are not coboundaries with products that appear in the linear combination of their image under d_1, as described in remark 5.4.8, then renaming the leaves. It holds that $\mu_j g(t_j) = y_j$ for some $\mu_j \neq 0$.*

Proof. Write $d_1(w_j) = \sum_{p,q} \lambda^j_{p,q} u_p \wedge u_q$. By proposition 5.1.5 it follows $y_j = \mu^j_{p,q}[z_p, z_q]$ for all p, q appearing in the sum, where $\mu^j_{p,q} \neq 0$. Now $t_j = (u_p, u_q)$ for some p, q, where we further replace u_p or u_q if they have a non-vanishing differential. Thereby it holds $u_p = v_p$ if and only if $z_p = x_p$ (and the same holds for u_q, z_q), so whenever we replace u_p or u_q by some other bracket $(u_{p'}, u_{q'})$, we may apply proposition 5.1.5 again to replace z_p respectively z_q by the bracket $\mu^p_{p',q'}[z_{p'}, z_{q'}]$ or $\mu^q_{p',q'}[z_{p'}, z_{q'}]$ respectively. □

We have now established a way how to express the basis elements y_j in brackets of the x_i. Furthermore, we have learned of a way to describe a specific basis of a free Lie algebra. Applied to our case, this means we can now more easily spot missing brackets in the homotopy Lie algebra L. To be more clear, we have a way to describe the first construction step of the free Lie algebra model $(\mathbb{L}_{A \oplus B}, \partial)$ of L. Remember that $\psi\colon (\mathbb{L}_A, 0) \to (L, 0)$ is a surjection that should be improved to a DGL quasi-isomorphism by sequentially killing the kernel. By the definition of ψ its kernel are exactly the missing brackets, so a basis for $\ker(\psi)$ is provided by the Hall trees that are missing in L. While doing this, we worked over the grading of the Lie algebras, which in general is different from the grading given by the bracket length. However, using Hall trees, we can describe some steps in this construction more precisely.

- Let k_1 such that $B = b_{\geq k_1}$ with notation from theorem 5.3.7. This means the first appearing relations of L are of degree $k_1 - 1$ and $\partial(b) \in [A, A]$ for all $b \in B_{k_1}$. We thus find Hall trees $t \in H$ such that $\psi(g(t)) = 0$. By proposition 5.4.6, the elements $g(t)$ provide a basis for $\ker(\psi_{k_1-1})$.
- In the construction of $(\mathbb{L}_{A\oplus B}, \partial)$ this means that for each such t a generator b will be added such that $\partial(b) = g(t)$. It follows that each such t belongs to a cell in the realization X of $(\Lambda V, d_1)$ that is attached to the bouquet by a Whitehead bracket. This bracket is directly determined by t.

Note that in addition to the minimal grade $k_1 - 1$ of the relations, we can also argue over the minimal bracket length. That is, let $n \geq 2$ be the least bracket length of the relations of L. Then all $t \in H_n$ with $\psi(g(t)) = 0$ contribute to the addition of generators in B, since the differential ∂ can only increase the bracket length.

Example 5.4.11. (Continuation of example 5.4.9)

(a) Let $(\Lambda(v_1, v_2, w), d_1(w) = v_1 \wedge v_2)$ with $|v_i|$ odd. The only tree we obtain is (v_1, v_2), so for the basis element y dual to w it holds $y = \mu[x_1, x_2]$. It follows that in L the Hall bases of H_3, $[x_1, [x_1, x_2]], [x_2, [x_1, x_2]]$ are missing. This leads to the attachment of two cells $\mathbb{B}_1^{2|v_1|+|v_2|-1}$, $\mathbb{B}_2^{2|v_2|+|v_1|-1}$ by the classes $[\alpha_1, [\alpha_1, \alpha_2]_W]_W$ and $[\alpha_2, [\alpha_1, \alpha_2]_W]_W$ to $Y = \mathbb{S}^{|v_1|} \vee \mathbb{S}^{|v_2|}$. We have already established this in example 5.3.13.

(b) For $(\Lambda(v_1, v_2, w, w_1, w_2), d_1)$ the elements w_1, w_2 evaluate to the Hall trees $(v_1, (v_1, v_2))$ and $(v_2, (v_1, v_2))$, so the dual basis elements y_j essentially belong to a Hall basis,

$$y_1 = \mu_1[x_1, [x_1, x_2]], \qquad y_2 = \mu_2[x_2, [x_1, x_2]].$$

In turn, now all brackets $g(t)$ for $t \in H_4$ vanish in L, which leads to three cells being attached to Y via Whitehead brackets

$$[\alpha_1, [\alpha_1, [\alpha_1, \alpha_2]_W]_W]_W, \quad [\alpha_2, [\alpha_1, [\alpha_1, \alpha_2]_W]_W]_W, \quad [\alpha_2, [\alpha_2, [\alpha_1, \alpha_2]_W]_W]_W.$$

(c) The case of $(\Lambda(v_1, v_2, w, w_1, w_2, w_{1,1}, w_{2,2}, w_{1,2}), d_1)$ is similar to the one above, with H_5 instead of H_4. Note that for $w_{1,2}$ the two trees lead to the bracket expressions

$$y_{1,2} = \mu[x_1, [x_2, [x_1, x_2]]] = \mu[x_2, [x_1, [x_1, x_2]]],$$

in accordance to the Jacobi identity. Only the latter is a Hall tree.

(d) Lastly, let us examine $(\Lambda(v_1, v_2, w, w_1, w_2, w_{1,1}, w_{2,2}, w_{1,2}, w_1', w_2'), d_1)$. We have already translated the information on the basis elements into trees using d_1. We obtain all Hall trees up to bracket length four, and two of bracket length five. Comparing this to example 5.4.5, we see that four basis elements of H_5 are missing, so we know that there will be four cells attached to Y using Whitehead brackets that we can read off the missing Hall trees. One such is for example $[[\alpha_1, \alpha_2]_W, [\alpha_1, [\alpha_1, \alpha_2]_W]_W]_W$.

Remark 5.4.12. Note that the condition of V being concentrated in even degrees is only necessary if one wants to work with Hall trees. The general strategy of translating the elements w_j into trees, providing (possibly multiple) bracket expressions of the $y_j \in L$ works this way for any Sullivan algebra $(\Lambda V, d_1)$. What makes the general case more complicated is the location of missing basis elements in L when compared to $\mathbb{L}_A$.

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