# On the Spacetime Mean Curvature of Surfaces in General Relativity

**Dissertation**
der Mathematisch-Naturwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt von

Markus Wolff
aus Bietigheim-Bissingen

Tübingen
2023

Gedruckt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Eberhard Karls Universität Tübingen.

| | |
|---|---|
| Tag der mündlichen Qualifikation: | 19.10.2023 |
| Dekan: | Prof. Dr. Thilo Stehle |
| 1. Berichterstatter\-in: | Prof. Dr. Carla Cederbaum |
| 2. Berichterstatter\-in: | Prof. Dr. Gerhard Huisken |
| 3. Berichterstatter\-in: | Assoc. Prof. Dr. Theodora Bourni |

# Contents

## Acknowledgements

It is a daunting task to express my deep thanks in a short and concise way, when filling pages upon pages would still not be enough. Yet, I shall try to let these words measure up to my gratitude not in length but in sincerity.

My deepest gratitude goes towards my first and second advisors Carla Cederbaum and Gerhard Huisken who supported me in so much more than just this thesis, be it mathematics or otherwise. Their guidance, in particular in these trying times, went beyond anything I could have ever asked for. It has been a pleasure and a privilege to work with them. I can never repay their generosity.

I would also like to thank my colleagues and friends Albachiara, Andoni, Anna, Armando, Melanie, Oliver, Olivia, Sara, Sascha, Sophia and Stephen, who I listed in alphabetical order as to avoid playing favourites, for their camaraderie and support not only during my PhD, but also during a global pandemic. I will forever treasure the bonds we forged.

Further, I want to thank Annachiara Piubello and Rodrigo Avalos for pointing me towards some extremely helpful references regarding the contents of Section 8.

Last, but certainly not least, my eternal gratitude belongs to my family and close friends who stuck with me all this time despite probably not always understanding why someone would willingly torture themselves like this, and even worse with mathematics. I especially want to thank my partner Timo and closest friend Penelope who so intimately shared all the joys and pains of doing a PhD with me. I can only hope that you will be with me during the next part of my journey.

# 1 Summary

## 1.1 Summary in English

The *spacetime mean curvature* $\mathcal{H}^2$ of a codimension-2 surface $\Sigma$ in an ambient $n+1$-dimensional spacetime $(\mathfrak{M}, \mathfrak{g})$ is defined as the Lorentzian length of the mean curvature vector $\vec{\mathcal{H}}$. More precisely,

$$\mathcal{H}^2 = \mathfrak{g}(\vec{\mathcal{H}}, \vec{\mathcal{H}}).$$

If $\Sigma$ is contained within a spacelike hypersurface $(M, g)$ with second fundamental form $K$ with respect to a timelike unit normal $\vec{n}$, then

$$\mathcal{H}^2 = H^2 - P^2,$$

where $H$ denotes the mean curvature of $\Sigma$ as a hypersurface in $(M, g)$ and $P := \operatorname{tr}_\Sigma K$. Thus, the notion of spacetime mean curvature can be formally extended to initial data sets $(M, g, K)$. In a recent paper Cederbaum–Sakovich [24] constructed an asymptotic foliation of surfaces of constant spacetime mean curvature in asymptotically flat initial data sets which yields a new notion of center of mass. In time symmetry, this foliation agrees with the asymptotic foliation by constant mean curvature surfaces first proposed by Huisken–Yau [54] to define a notion of center of mass. Explicit examples [23, 24] suggest that this new definition of center of mass remedies some of the deficiencies of the Huisken–Yau center of mass in general initial data sets. Although $\mathcal{H}^2$ is indeed positive for each leaf in the construction of Cederbaum–Sakovich [24], by definition $\mathcal{H}^2$ is allowed to be at least locally negative in general, and trappend surfaces, where $\mathcal{H}^2 < 0$ globally, arise naturally in the context of General Relativity. This, together with the above observation and the Lorentz invariance of the spacetime mean curvature suggest it as an appealing geometric quantity to study for hypersurfaces in an initial data set.

Of course, we can also study the spacetime mean curvature of codimension-2 surfaces directly in the ambient spacetime, in particular for surfaces restriced to a null hypersurface. Appealing to this viewpoint, we will always consider the extrinsic curvature of a spacelike cross section of a null hypersurface as a codimension-2 surface in the ambient spacetime. Hence, a basis of the normal space will be given by a frame of two null vector fields $\{\underline{L}, L\}$. As the two null second fundamental forms depend on the choice of null frame along the null hypersurface, the spacetime mean curvature indeed arises naturally as a frame-independent geometric quantity for spacelike cross sections of a null hypersurface.

In this thesis, we will consider the spacetime mean curvature in three different contexts: In asymptotically flat initial data sets we study the evolution of hypersurfaces along a geometric flow related to their spacetime mean curvature. In hyperboloidal, totally umbilic warped product graphs in a class of static spacetimes, we give a characterization of surfaces of constant spacetime mean curvature under the null energy condition. Furthermore, we study the geometry of spacelike cross sections of the standard Minkowski lightcone related to their spacetime mean curvature.

In the case of an asymptotically flat initial data set $(M, g, K)$, we study the evolution of hypersurfaces along their inverse spacetime mean curvature. We briefly consider smooth solutions $F\colon [0, T) \times \Sigma \to M$ of *inverse space-time mean curvature flow* (STIMCF) given by the parabolic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}F = \frac{1}{\sqrt{\mathcal{H}^2}} = \frac{1}{\sqrt{H^2 - P^2}},$$

before developing a notion of weak solutions that satisfy a comparison principle. This comparison principle allows for solutions to form jumps such that the flow exists for all times without developing a singularity. This is based on joint work with Gerhard Huisken [53].

Such a notion of weak solutions was first considered by Huisken–Ilmanen [50, 51] for inverse mean curvature flow. They further showed that the monotonicity of the Hawking mass under smooth solutions to inverse mean curvature flow first observed by Geroch [43] and Jang–Wald [82] extends also to weak solutions in 3 dimensions. This allowed Huisken–Ilmanen [50] to give a proof of the Riemannian Penrose Inequality for connected apparent horizons in asymptotically flat Riemannian manifolds, as apparent horizons are given by minimal surfaces in time symmetry. In the general case, apparent horizons are given by marginally trapped surfaces, so the proof does not directly extend to non-time symmetric initial data sets $(M, g, K)$, but motivates to generalize the flow in the case of non-trivial extrinsic curvature $K$. This was first proposed by Moore in [63] where she studied weak solutions of inverse null mean curvature flow. Here, we propose inverse space-time mean curvature flow as another such generalization motivated by the work of Cederbaum–Sakovich [24].

Similar to inverse null mean curvature flow, a weak solution of inverse space-time mean curvature flow is a pair $(u, \nu)$ of a locally Lipschitz function $u$ and measurable unit vector field $\nu$. The main result of this section is the existence of weak solutions in maximal, asymptotically flat initial data sets. The construction closely follows the strategy of Huisken–Ilmanen [50] and Moore [63]. In particular, the locally Lipschitz function $u$ is obtained as the sublimit of solutions $u_\varepsilon$ to an elliptic regularisation. The solutions of the elliptic regularisation to inverse space-time mean curvature flow have first been constructed in the master thesis of the author [87]. Here, the restriction to maximal initial data sets is necessary to construct a lower barrier in the interior region. As in [63], the concept of unit normal has to be extended along jump regions where $u$ remains constant, and is obtained by studying the limiting behavior of the level-sets of the smooth functions $U_\varepsilon(x, z) := u_\varepsilon(x) - \varepsilon z$ in $M \times \mathbb{R}$, where $u_\varepsilon$ are the solutions of the elliptic regularisation. To extract a notion of unit normal even across jump regions, we heavily exploit that the level sets form a smooth, translating solution of inverse space-time mean curvature flow and use a regularity theorem from geometric measure theory. As we apply the regularity theory in the cylinder $M \times \mathbb{R}$, this restricts our construction to initial data sets of dimensions $n \leq 6$.

In the case of hyperboloidal, totally umbilic warped product graphs in a class of static spacetimes, we give a characterization of constant spacetime mean curvature surfaces under the null energy condition using an Alexandrov Theorem by Brendle [12]. The characterization is independent of the asymptotic behaviour, but heavily relies on the rigid structure of the ambient spacetime and the fact that the warped product graphs under consideration are totally umbilic with constant umbilicity factor.

The class of spacetimes under consideration has been extensively studied in spherical symmetry, see e.g. [21, 83, 85], but also contains the family of Birmingham–Kottler metrics [9, 56]. In the context of spacetime extensions similar to the Kruskal–Szekeres extension this class of spacetimes has further been considered in a joint work of Cederbaum and the author [25], and independently by Brill–Hayward [16] and Schindler–Aguirre [71]. In fact, we employ the results in [25] to extend the warped product graphs past the Killing horizon up to the minimal inner boundary to apply Brendle's Alexandrov Theorem. We note that under the assumptions made by Brendle in [12] that allow for a characterization of constant mean curvature surfaces for time symmetric slices, the spacetime satisfies the null energy condition and admits a spacetime extension as constructed in [25]. Assuming that the null energy condition is satisfied on the whole of the spacetime extension, we show that Brendle's Alexandrov Theorem is applicable also for totally umbilic warped product graphs and that the characterization of constant mean curvature surfaces leads to a characterization of constant spacetime mean curvature surfaces. As constant spacetime mean curvature surfaces agree with constant mean curvature surfaces in time symmetry, this can be seen as a generalization of Brendle's Alexandrov Theorem to general totally umbilic warped product graphs.

In the null case, we give two results for the standard lightcone in the $3+1$-dimensional Minkowski spacetime. In this case, all surfaces under consideration are spacelike cross sections of the lightcone which are conformal to the round 2-sphere. Moreover, the Gauss equation gives that the spacetime mean curvature $\mathcal{H}^2$ is proportional to the scalar curvature of the spacelike cross section. Due to this relation all surfaces of constant spacetime mean curvature on the lightcone have constant scalar curvature and are thus round spheres up to a conformal diffeomorphism in the Möbius group. By the isomorphism between the Möbius group and the restricted Lorentz group, all surfaces of constant spacetime mean curvature are hence given by round spheres up to a Lorentz transformation of the ambient Minkowski spacetime that leave the lightcone invariant. As the spacetime mean curvature fully determines the intrinsic curvature of the spacelike cross section, all results that we present here can be also be stated from an intrinsic viewpoint for conformally round 2-surfaces. However, by adopting the extrinsic viewpoint we gain additional geometric information on the spacelike cross sections. In particular, we will define a frame-independent, symmetric $(0,2)$-tensor $A$ which we call the *scalar second fundamental form* as a tensor-representative for the full vector-valued second fundamental form $\vec{\mathrm{II}}$ carrying the same geometric information for spacelike cross sections of the lightcone.

As a first result, we show that the Gauss equation further yields an equivalence between $2d$-Ricci flow for surfaces of genus 0 and an extrinsic curvature flow, which we will call *null mean curvature flow* here, for spacelike cross sections along the Minkowski lightcone. Null mean curvature flow was first studied by Roesch–Scheuer [67] along general null hypersurfaces to detect marginally outer trapped surfaces. As no such surfaces exist in the Minkowski lightcone, the flow develops singularities in finite time. Using the above equivalence to $2d$-Ricci flow, and a classical result first proven by Hamilton [47] we give a full characterization of all singularities as the flow rescaled by volume will converge to a surface of constant spacetime mean curvature. In the conformally round case, this result for $2d$-Ricci flow was initially proven by Hamilton [47] under the assumption of strictly positive scalar curvature. This assumption was eventually removed by Chow [30]. Their strategy relies on a Harnack inequality and an entropy bound, and yields a proof for the uniformization theorem, cf. Chen–Lu–Tian [28]. Later, independent proofs in the conformally round case were given by directly utilizing the uniformization theorem, cf. Bartz–Struwe–Ye, Struwe, and Andrews–Bryan [1, 7, 77]. Using the equivalence to null mean curvature flow we obtain yet a new proof of Hamilton's classical result. Although we only prove it under Hamilton's initial assumption, which equivalently states that we assume the mean curvature vector of the initial spacelike cross section to be spacelike, the proof does not rely on a choice of coordinates and only uses the maximum principle by studying the evolution of the scalar second fundamental form $A$. This is based on recently published single author work [89].

For the second result, we give a quantitative estimate of the fact that the tracefree part of the scalar second fundamental form $\mathring{A}$ vanishes if and only if the spacelike cross section of the lightcone is a surface of constant spacetime mean curvature. More precisely, we show that if $\mathcal{H}^2 \geq 0$ and the $L^2$ norm of $\mathring{A}$ is sufficiently small with respect to some a-priori bound on the spacelike cross section, then the conformal factor is $W^{2,2}$-close to the conformal factor of a surface of constant spacetime mean curvature. This statement can be understood as the analogue result to the work of De Lellis–Müller [36] in $\mathbb{R}^3$. Similar to the work of De Lellis–Müller, the proof of the statement consists of two parts:

In the first part, we prove a geometric estimate to show that the difference between $\mathcal{H}^2$ and its mean value in $L^2$ is uniformly controlled by the $L^2$ norm of $\mathring{A}$ if $\mathcal{H}^2 \geq 0$. We prove such an estimate in two ways: The first proof follows as an application of null mean curvature flow and the previously established full characterization of its singularities. The second proof is modelled on the proof of an almost Schur-lemma by De Lellis–Topping [37] using the Bochner formula. Using the estimate obtained by either of those methods, the second part of the prove then uses elliptic theory to obtain the desired $W^{2,2}$-estimates from the $L^2$ norm of the difference between $\mathcal{H}^2$ and its mean value under a suitable balancing condition. Here, the balancing condition is related to an associated timelike, future-pointing 4-vector of the spacelike cross section in the ambient Minkowski spacetime. This associated 4-vector is closely related to a notion of center in asymptotically hyperbolic manifolds defined by Cederbaum–Cortier–Sakovich [20]. It moreover transforms equivariantly under Lorentz transformations of the ambient Minkowski spacetime, which is precisely what allows us to state the $W^{2,2}$-estimate with respect to the uniquely determined surface of constant spacetime mean curvature that has the same associated 4-vector as the spacelike cross section under consideration.

As we have seen in all the above cases considered in this thesis, the concept of (constant) spacetime mean curvature naturally extends considerations for (constant) mean curvature in the context of General Relativity, and we have done so by explicitly extending several results to the setting of spacetime mean curvature. The observations and results here moreover suggest that the concept of (constant) spacetime mean curvature presents itself in the null case as the direct analogue to the concept of (constant) mean curvature in the Riemannian setting.

## 1.2 Zusammenfassung auf Deutsch (Summary in German)

Die *Raumzeit-mittlere Krümmung* $\mathcal{H}^2$ einer Fläche mit Kodimension 2 in einer umgebenden $n+1$-dimensionalen Raumzeit $(\mathfrak{M}, \mathfrak{g})$ ist definiert als die Lorentzsche Länge des mittleren Krümmungsvektors $\vec{\mathcal{H}}$. Dass heißt, es gilt

$$\mathcal{H}^2 = \mathfrak{g}(\vec{\mathcal{H}}, \vec{\mathcal{H}}).$$

In dem Fall, dass $\Sigma$ innerhalb einer raumartigen Hyperfläche $(M, g)$ mit zweiter Fundamentalform $K$ bezüglich einer zeitarigen Einheitsnormalen $\vec{n}$ liegt, gilt

$$\mathcal{H}^2 = H^2 - P^2,$$

wobei $H$ die mittlere Krümmung von $\Sigma$ in $(M, g)$ bezeichnet und es gilt $P := \operatorname{tr}_\Sigma K$. Insbesondere kann der Begriff der Raumzeit-mittleren Krümmung auch formal auf relativistische Anfangsdaten $(M, g, K)$ ausgeweitet werden. In einer kürzlich erschienenen Arbeit haben Cederbaum–Sakovich [24] eine asymptotische Blätterung von Flächen konstanter Raumzeit-mittlerer Krümmung in asymptotisch flachen relativistische Anfangsdatenn konstruiert und nutzen diese um eine neue Definition von Massenschwerpunkt zu geben. Im zeitsymmetrischen Fall stimmt diese Blätterung exakt mit der Blätterung von Flächen konstanter mittlerer Krümmung überein. Eine solche Blätterung wurde erstmals von Huisken–Yau [54] konstruiert um eine Definition von Massenschwerpunkt zu geben. Explizite Beispiele [23, 24] suggerieren, dass diese neue Defintion des Masseschwerpunkts einige Mängel des Huisken–Yau-Massenschwerpunkts in allgemeinen relativistischen Anfangsdaten korrigiert. Obwohl $\mathcal{H}^2$ tatsächlich positiv entlang den von Cederbaum–Sakovich [24] konstruierten Blättern ist, kann $\mathcal{H}^2$ per Definition im Allgemeinen zumindest lokal negativ sein. Gefangene Flächen, auf denen $\mathcal{H}^2$ überall negativ ist, treten auf natürliche Weise im Kontext der Allgemeinen Relativitätstheorie auf. Diese Tatsache, zusammen mit der obigen Beobachtung, sowie der Lorentzinvarianz der Raumzeit-mittleren Krümmung motivieren es diese ansprechend erscheinende geometrische Größe genauer in relativistischen Anfangsdaten zu studieren.

Unter Anderem kann man jedoch auch direkt Flächen der Kodimension 2 und deren Raumzeit-mittlere Krümmung in einer umgebenden Raumzeit betrachten, insbesondere solche, die in einer lichtartigen Hyperfläche enthalten sind. Von diesem Standpunkt aus betrachtet werden wir für einen raumartigen Querschnitt einer lichtartigen Hyperfläche stets die äußere Krümmung bezüglich der umgebenden Raumzeit betrachten, die den raumartigen Querschnitt als Fläche der Kodimension 2 enthält. Dabei betrachten wir stets einen Rahmen des Normalenraums, der aus zwei lichtartigen Vektorfeldern $\{\underline{L}, L\}$ besteht. Da die daraus resultierenden zwei lichtartigen, zweiten Fundamentalformen von der Wahl dieses Rahmens abhängen, präsentiert sich die Raumzeit-mittlere Krümmung auch für raumartige Querschnitte einer lichtartigen Hyperfläche auf natürliche Weise als eine rahmenunabhängige geometrische Größe.

In dieser Arbeit betrachten wir die Raumzeit-mittlere Krümmung in drei unterschiedlichen Fällen: In asymptotisch flachen relativistischen Anfangsdaten untersuchen wir die Evolution von Hyperflächen entlang eines geometrischen Flusses der in direktem Zusammenhang mit deren Raumzeit-mittlerer Krümmung steht. Im Fall von hyperboloidalen Graphen, die sich als verzerrtes Produkt in einer Klasse von statischen Raumzeiten ergeben, geben wir eine Charakterisierung von Flächen mit konstanter Raumzeit-mittlerer Krümmung, falls die umgebende Raumzeit die Null-Energiebedingung erfüllt. Des Weiteren betrachten wie die Geometrie raumartiger Querschnitte des Standardlichtkegels in der Minkowski-Raumzeit in Bezug auf deren Raumzeit-mittlere Krümmung.

Im Fall von asymptotisch flachen relativistischen Anfangsdaten $(M, g, K)$ untersuchen wir die Evolution einer Hyperfläche entlang deren inversen Raumzeit-mittleren Krümmung. Dabei beschäftigen wir uns knapp mit glatten Lösungen $F\colon [0,T)\times\Sigma\to M$ von *„inverse space-time mean curvature flow“* (STIMCF), welcher durch die parabolische Gleichung

$$\frac{\mathrm{d}}{\mathrm{d}t}F = \frac{1}{\sqrt{\mathcal{H}^2}} = \frac{1}{\sqrt{H^2 - P^2}},$$

beschrieben wird. Danach wenden wir uns der Entwicklung eines schwachen Lösungskonzepts zu, welches über ein Vergleichsprinzip definiert ist. Dieses Vergleichsprinzip erlaubt es Lösungen zu springen, so dass der Fluss für alle Zeiten existiert ohne eine Singularität zu bilden. Dies basiert auf gemeinsamer Arbeit mit Gerhard Huisken [53].

Ein solcher schwacher Lösungsbegriff wurde zuerst von Huisken–Ilmanen [50, 51] im Fall vom inversen mittleren Krümmungsfluss formuliert. Dabei zeigten sie weiterhin, dass die Monotonie der Hawkingmasse in 3 Dimensionen, die für den glatten Fluss zuerst von Geroch [43] und Jang–Wald [82] nachgewiesen wurde, auch für schwache Lösungen erfüllt ist. Dies führte zu einem Beweis der Riemannschen Penrose Ungleichung durch Huisken–Ilmanen [50] für zusammenhängende scheinbare Horizonte in asymptotisch flachen Riemannschen Mannigfaltigkeiten, da diese im zeitsymmetrischen Fall durch Minimalflächen gegegeben sind. Im allgemeinen Fall sind scheinbare Horizonte durch marginal gefangene Flächen gegeben, so dass der Beweis nicht direkt auf nicht-zeitsymmetrische relativistische Anfangsdaten $(M, g, K)$ übertragen werden kann. Stattdessen motiviert dies den Fluss an den Fall nicht trivialer äußerer Krümmung $K$ anzupassen. Eine solche Verallgemeinerung wurde zuerst von Moore [63] betrachtet, die ein schwaches Lösungskonzept für „inverse null mean curvature flow“ entwickelte. In dieser Arbeit betrachten wir, motiviert von der Arbeit von Cederbaum–Sakovich [24], „inverse space-time mean curvature flow“ als eine solche Verallgemeinerung.

Ähnlich zu dem schwachen Lösungskonzept für „inverse null mean curvature flow“, besteht eine schwache Lösung von „inverse space-time mean curvature flow“ aus einem Paar $(u, \nu)$

einer lokal-Lipschitz Funktion und einem messbaren Einheitsvektorfeld $\nu$. Das Hauptergebnis dieses Abschnitts ist die Existentz schwacher Lösungen in maximalen, asymptotisch flachen relativistischen Anfangsdaten. Dabei folgt die Konstruktion der schwachen Lösungen den Strategien von Huisken–Ilmanen [50] und Moore [63]. Insbesondere wird die lokal-Lipschitz Funktion $u$ als Häufungspunkt glatter Lösungen $u_\varepsilon$ einer elliptischen Regularisierung erhalten. Die Lösungen dieser ellitpischen Regularisierung von „inverse space-time mean curvature flow“ wurden zuerst in der Masterarbeit des Autors [87] konstruiert. Die Einschränkung auf maximale relativistische Anfangsdaten ist an dieser Stelle notwendig um die Existenz einer unteren Barriere in der inneren Region zu garantieren. Genau wie in [63] muss das Konzept einer äußeren Einheitsnormalen auch über Sprünge hinweg ausgeweitet werden, bei denen $u$ auf einer offenen Umgebung konstant bleibt. Wir erhalten eine solche Normale, indem wir das Grenzwertverhalten der Niveauflächen der glatten Funktionen $U_\varepsilon(x,z) := u_\varepsilon(x) - \varepsilon z$ im Zylinder $M \times \mathbb{R}$ untersuchen, wobei $u_\varepsilon$ die Lösungen der glatten Regularisierung bezeichne. Um zu zeigen, dass diese Konstruktion auch über Sprünge hinweg gelingt, nutzen wir die Tatsache, dass die Niveauflächen eine translatierende, glatte Lösung des „inverse space-time mean curvature flows“ bilden und nutzen ein Regularitätsresultat aus der geometrischen Maßtheorie. Da wir dieses Regularitätsresultat im Zylinder $M \times \mathbb{R}$ anwenden, schränkt dies unsere Konstruktion auf Dimensionen $n \leq 6$ ein.

Im Fall von hyperboloidalen, umbilischen Graphen, die sich als verzerrtes Produkt in einer Klasse von statischen Raumzeiten ergeben, charakterisieren wir Flächen von konstanter Raumzeit-mittlerer Krümmung unter der Null-Energiebedingung, indem wir ein Alexandrov Theorem nach Brendle [12] anwenden. Die Charakterisierung ist unabhängig von der asymptotischen Struktur, beruht jedoch auf der besonderen Form der umgebenden Raumzeit und der Tatsache, dass die umbilischen Graphen, die als solche verzerrten Produkte gegeben sind, konstanten Umbilizitätsfaktor besitzen.

Die hier betrachtete Klasse von Raumzeiten wurde bereits vielfach in sphärischer Symmetrie untersucht, siehe zum Beispiel [21, 85, 83], beinhaltet jedoch auch die Familie der Birmingham–Kottler Metriken [9, 56]. Zudem wurden in dieser Klasse Raumzeiterweiterungen, die in ihrer Struktur der Kruskal–Szekeres Erweiterung ähneln, von Cederbaum und dem Autor in gemeinsamer Arbeit [25] konstruiert, sowie in unabhängiger Arbeit von Brill–Hayward [16] und Schindler–Aguirre [71]. In der Tat benutzen wir die Ergebnisse aus [25], um die oben betrachteten Graphen über den Horizont hinaus bis zu einem minimalen inneren Rand fortzusetzen, so dass wir Brendle's Alexandrov Theorem anweden können. Wir wollen an dieser Stelle darauf hinweisen, dass die Voraussetzungen von Brendle in [12], unter denen eine Charakterisierung von Flächen konstanter mittlerer Krümmung in einem zeitsymmetrischen Schnitt möglich ist, implizieren, dass die umgebende Raumzeit, die in der obigen Klasse enthalten ist, die Null-Energiebedingung erfüllt und eine Raumzeiterweiterung, wie in [25] beschrieben, besitzt. Unter der Annahme, dass die Null-Energiebedingung auf der gesamten Raumzeiterweiterung erfüllt ist, zeigen wir, dass Brendle's Alexandrov Theorem auf den umbilischen Graphen, die sich als verzerrtes Produkt ergeben, anwendbar ist und dass eine Charakterisierung der Flächen konstanter mittlerer Krümmung gleichsam alle Flächen konstanter Raumzeit-mittlerer Krümmung charakterisiert. Da Flächen konstanterm Raumzeit-mittlerer Krümmung im zeitsymmetrischen Fall mit Flächen konstanter mittlerer Krümmung übereinstimmen, kann man dieses Ergebnis als eine Verallgemeinerung von Brendle's Alexandrov Theorem auf allgemeine umbilische Graphen interpretieren, die sich als verzerrtes Produkt ergeben.

Im lichtartigen Fall beweisen wir zwei Ergebnisse auf dem Standardlichtkegel in der $3+1$-dimensionalen Minkowski-Raumzeit. In diesem Fall sind alle Flächen, die wir betrachten, als raumartige Querschnitte des Lichtkegels gegeben und jeder dieser raumartigen Querschnitte ist zudem konform zur runden 2-Sphäre. Des Weiteren ergibt die Gauss-Gleichung, dass die Raumzeit-mittlere Krümmung der raumartigen Querschnitte direkt proportional zu deren Skalarkrümmung ist. Aufgrund dieser Beziehung besitzen alle Flächen konstanter Raumzeit-mittlerer Krümmung auf dem Lichtkegel konstante Skalarkrümmung und sind daher bis auf einen konformen Diffeomorphismus in der Möbius-Gruppe als runde Sphären gegeben. In Bezug auf den Gruppenisomorphismus zwischen der Möbius-Gruppe und der eigentlichen orthochronen Lorentz-Gruppe ergeben sich also alle Flächen konstanter Raumzeit-mittlerer Krümmung bis auf eine Lorentz-Transformation in der umgebenden Minkowski-Raumzeit, die den Lichtkegel invariant lassen, als eine runde Sphäre. Da die Raumzeit-mittlere Krümmung nach der obigen Beobachtung bereits die gesamte intrinsische Krümmung der raumartigen Querschnitte bestimmt, können die hier präsentierten Ergebnisse ebenso von einem intrinsichen Blickwinkel für konform runde 2-Flächen formuliert werden. Indem wir jedoch den extrinsischen Blickwinkel beibehalten, erhalten wir zusätzliche geometrische Informationen über den raumartigen Querschnitt. Insbesondere definieren wir einen rahmenunabhängigen, symmetrischen $(0,2)$-Tensor $A$, den wir die *skalare zweite Fundamentalform* nennen und der repräsentativ für die vektorwertige zweite Fundamentalform $\vec{\mathrm{II}}$ die gleiche geometrische Information über den raumartigen Querschnitt des Lichtkegels enthält.

Als ein erstes Ergebnis zeigen wir, dass die Gauss-Gleichung ebenfalls eine Äquivalenz zwischen dem $2d$-Ricci Fluss auf Flächen vom Geschlecht 0 und einem extrinsischem Krümmungsfluss, den wir hier *„null mean curvature flow"* nennen, für raumartige Querschnitte des Minkowski-Lichtkegels impliziert. „Null mean curvature flow" wurde zuerst von Roesch–Scheuer [67] entlang allgemeiner lichtartiger Hyperflächen untersucht um marginal nach außen gefangene Flächen zu detektieren. Da solche Flächen in der Minkowski-Raumzeit nicht existieren, entwickelt der Fluss eine Singularität in endlicher Zeit. Indem wir die obige Äquivalenz zum $2d$-Ricci Fluss mit einem klassichen Resultat, welches zuerst von Hamilton [46] bewiesen wurde, kombinieren, erhalten wir eine komplette Charakterisierungen aller Singularitäten, da der reskalierte, volumenerhaltende Fluss demnach zu einer Fläche konstanter

Raumzeit-mittlerer Krümmung konvergiert. Im konform runden Fall wurde dieses Resultat für den $2d$-Ricci Fluss von Hamilton zunächst nur unter der Voraussetzung strikt positiver Skalarkrümmung bewiesen. Das Resultat wurde später von Chow [30] auch auf den allgemeinen Fall erweitert. In beiden Fällen beruht der Beweis auf einer Harnack-Ungleichung und einer Entropieabschätzung und führt desweiteren zu einem neuen Beweis des Uniformisierungssatzes, siehe Chen–Lu–Tian [28]. Direkte Anwendung des Uniformisierungssatzes führte zu weiteren Beweisen dieses Resultats im konform runden Fall, siehe Bartz–Struwe–Ye, Struwe und Andrews–Bryan [1, 7, 77]. Indem wir die Äquivalenz zu „null mean curvature flow“ ausnutzen, erhalten wir einen weiteren Beweis dieses klassichen Resultats. Obwohl wir das Resultat hier nur unter den ursprünglichen Voraussetzungen von Hamilton beweisen, was bedeutet, dass wir annehmen, dass der initiale raumartige Querschnitt raumartigen mittleren Krümmungsvektor besitzt, ist der Beweis unabhängig von der Wahl der Koordinaten und beruht ausschließlich auf dem Maximumsprinzip, indem wir zusätzlich die Evolution der skalaren zweiten Fundamentalform $A$ betrachten. Dies basiert auf eigener, bereits veröffentlicher Arbeit, siehe [89].

Als zweites Ergebnis formulieren wir die Tatsache, dass der spurfreie Anteil $\mathring{A}$ der skalaren zweiten Fundamentalform eines raumartigen Querschnitts des Lichtkegels genau dann verschwindet, wenn der raumartigen Querschnitt eine Fläche konstanter Raumzeit-mittlerer Krümmung ist, in eine quantitative Abschätzung um. Genauer gesagt zeigen wir, dass der konforme Faktor eines raumartigen Querschnitts bereits $W^{2,2}$-nahe am konformen Faktor einer Fläche konstanter Raumzeit-mittlerer Krümmung sein muss, falls $\mathcal{H}^2 \geq 0$ und die $L^2$-Norm des spurfreien Anteils $\mathring{A}$ hinreichend klein ist. Dieses Ergebnis kann als das analoge Resultat zur Arbeit von De Lellis–Müller [36] in $\mathbb{R}^3$ betrachtet werden. Ähnlich zur Strategie von De Lellis–Müller besteht der Beweis aus zwei Schritten:

Im ersten Schritt beweisen wir eine geometrische Ungleichung, die zeigt, dass die Differenz zwischen $\mathcal{H}^2$ und seinem Mittelwert in $L^2$ uniform durch die $L^2$-Norm von $\mathring{A}$ kontrolliert ist, falls $\mathcal{H}^2 \geq 0$. Wir beweisen eine solche Ungleichung auf zwei unterschiedliche Weisen: Der erste Beweis folgt als eine Anwendung von „null mean curvature flow“und der obigen Charakterisierung aller Singularitäten. Der zweite Beweis ist von einem nahezu-Schur-Lemma von De Lellis–Topping [37] inspiriert und nutzt die Bochner-Formel. Der zweite Schritt besteht daraus elliptische Abschätzungen zu nutzen, um die gewünschte $W^{2,2}$-Abschätzung unter einer geeigneten Ausgleichsbedingung aus der $L^2$-Norm der Differenz zwischen $\mathcal{H}^2$ und seinem Mittelwert zu folgern, so dass das gewünschte Resultat durch Anwendung einer der obigen Ungleichungen folgt. Die hier formulierte Ausgleichsbedingung steht in Bezug zu einem zeitartigen, zukunftsgerichteten 4-Vektor in der umgebenden Minkowski-Raumzeit, der einem raumartigen Querschnitt zugeorndet ist. Dieser zugeordnete 4-Vektor steht zudem in direktem Bezug zu einem von Cederbaum–Cortier–Sakovich [20] definiertem Zentrumsbegriff in asymptotisch hyperbolischen Mannigfaltigkeiten. Er transformiert sich des Weiteren äquivariant unter Lorentztransformationen der umgebenden Minkowski-Raumzeit. Dies erlaubt es uns schließlich die $W^{2,2}$-Abschätzung bezüglich der Fläche konstanter Raumzeit-mittlerer Krümmung zu formulieren, die den gleichen zugeordneten 4-Vektor wie der ursprüngliche raumartige Querschnitt besitzt.

Die unterschiedlichen, in dieser Arbeit betrachteten Fälle weisen alle darauf hin, dass das Konzept der (konstanten) Raumzeit-mittleren Krümmung eine natürliche Verallgemeinerung der (konstanten) mittleren Krümmung im Hinblick auf die Allgemeine Relativitätstheorie darstellt. Die hier gemachten Beobachtungen und Ergebnisse suggerieren sogar, dass die (konstante) Raumzeit-mittlere Krümmung im lichtartigen Fall direkt analog zur (konstanten) mittleren Krümmung im Riemannschen Fall ist.

# 2 Overview and Contributions

Section 3 contains a brief listing of notation and conventions.

Section 4 contains a brief overview of the relevant background. In particular, we discuss General Relativity, the geometry of submanifolds, and evolution equations. All the material herein is well-known and we refer to the specific subsections for a list of references.

In Section 5, we introduce a notion of weak solutions of inverse space-time mean curvature flow and prove an existence result for maximal, asymptotically flat initial data sets. This is based on joint work with Gerhard Huisken [53]. As it is usual in mathematics, the authors of the manuscript [53] are listed alphabetically. The contents of Subsections 5.1 and 5.2 apart from a maximal existence result for smooth solutions (Theorem 5.2) contributed by Gerhard Huisken are contained in the master thesis of the author [87] supervised by Gerhard Huisken at the University of Tübingen, submitted in August, 2019. Besides the aforementioned maximal existence result for smooth solutions, Gerhard Huisken and I jointly discussed all of the statements and proof methods. Literature research and the technical calculations for the jointly discussed parts were done by me; the paper-writing is estimated to be 90% by me and 10% by Gerhard Huisken. The figures in [53] and Section 5 were generated by Axel Fehrenbach and Olivia Vičánek Martínez.

In Section 6 we prove a characterization of constant spacetime mean curvature surfaces on hyperboloidal, totally umbilic warped product graphs in a class of static spacetimes if the spacetime satisfies the null energy condition. This is based on single author work with a preprint available on arXiv [88]. All parts of the work are my own, but I am indebted to Carla Cederbaum for her suggestions to this problem and for helpful discussions.

In Section 7, we establish the equivalence between $2d$-Ricci flow for surfaces of genus 0 and null mean curvature flow in the standard lightcone of the Minkowski spacetime. This yields a full characterization of the singularity models for null mean curvature flow in the Minkowski lightcone, as well as a new proof of a classical result first proven by Hamilton under the assumption that the mean curvature vector of the initial spacelike cross section is spacelike. This is based on published single author work accepted in Calculus of Variations and Partial Differential Equations in January, 2023 [89].

In Section 8, we establish a De Lellis–Müller type estimate for spacelike cross sections of the standard Minkowski lightcone with $\mathcal{H}^2 \geq 0$ and subject to a-priori bounds. The desired $W^{2,2}$-estimate between the spacelike cross section under consideration and the constant spacetime mean curvature surface of reference relies on a notion of associated 4-vector. This is based on unpublished work by myself. Regarding the contents of Sections 7 and 8, all parts of the work are my own, but I am indebted to Carla Cederbaum for proposing to study the spacetime mean curvature of cross sections of the lightcone, and I am further indebted to both Carla Cederbaum and Gerhard Huisken for helpful suggestions and discussions.

# 3 Basic notation

Unless otherwise stated, we will follow these general conventions throughout the thesis:

Pairs $(\mathfrak{M}, \mathfrak{g})$, $(M, g)$, and $(\Sigma, \gamma)$ will always denote semi-Riemannian manifolds. In most cases $(\mathfrak{M}, \mathfrak{g})$ will denote a Lorentzian manifold, while $(\Sigma, \gamma)$ will always denote a Riemannian manifold.

Let $T^k_l M$ denote the $(k, l)$-tensorbundle on $M$ with sections $\Gamma(T^k_l M)$. As usual, we denote the vector bundle as $TM := T^1_0 M$.

For a semi-Riemannian manifold $(M, g)$, we denote the Levi-Civita connection by ${}^g\nabla$, and the Christoffel symbols in local coordinates $\{x^\alpha\}$ by ${}^g\Gamma^\mu_{\alpha\beta}$. We define the Riemann curvature tensor $\mathrm{Rm}_g$, the Ricci curvature $\mathrm{Ric}_g$, and the scalar curvature $\mathrm{R}_g$ as

$$\begin{aligned}
\mathrm{Rm}_g(X, Y, Z, W) &= g({}^g\nabla_X {}^g\nabla_Y W - {}^g\nabla_Y {}^g\nabla_X W - {}^g\nabla_{[X,Y]} W, Z), \\
\mathrm{Ric}_g(X, Y) &= \mathrm{tr}_g \, \mathrm{Rm}_g(X, \cdot, Y, \cdot), \\
\mathrm{R}_g &= \mathrm{tr}_g \, \mathrm{Ric}_g,
\end{aligned}$$

for vectorfields $X, Y, Z, W \in \Gamma(TM)$, where $\mathrm{tr}_g$ denotes the metric trace with respect to $g$. If clear from the context, we will usually omit the subscript $g$.

For a tensor $T$, we will denote the $k$-th tensor derivative by $\nabla^k T$. By a slight abuse of notation, we will further denote the gradient of a function $f \in C^\infty(M)$ by $\nabla f$, and we denote their Hessian either by $\mathrm{Hess} f = \mathrm{Hess}_g f$ or $\nabla^2 f$.

For a (symmetric) $(0, 2)$-tensor $T$, we define its trace-free part $\mathring{T}$ as

$$\mathring{T} := T - \frac{1}{n} \mathrm{tr}_g \, T g,$$

where $n$ denotes the dimension of $(M, g)$. We say $T$ is pure trace, if $\mathring{T} \equiv 0$.

We denote all possible inner products between tensors or tensor norms (induced by a semi-Riemannian metrik $g$) by $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_g$ and $|\cdot| = |\cdot|_g$, respectively.

Einstein summation convention is used whenever it is convenient and partial derivatives are denoted by $f_{,i} := \partial_i f$.

# 4 Preliminaries

## 4.1 An introduction to General Relativity

In 1915, Albert Einstein introduced General Relativity as a theory of gravitation that describes the universe in both space and time. We refer to [81] for a detailed introduction to the subject and introduce only some of the most relevant features of General Relativity. Mathematically, the models of this theory are described as spacetimes $(\mathfrak{M}, \mathfrak{g})$, given as Lorentzian manifolds, that satisfy the famous *Einstein Equations*

$$\mathfrak{Ric} - \frac{1}{2}\mathfrak{R}\mathfrak{g} = 8\pi\mathfrak{T}, \tag{1}$$

where $\mathfrak{T}$ is the so-called *stress-energy-(momentum) tensor* describing the matter model at hand, and $\mathfrak{Ric}$, and $\mathfrak{R}$ denote the Ricci-and scalar curvature of $(\mathfrak{M}, \mathfrak{g})$, respectively. Note that we chose geometric units such that the Gravitational constant $G$ and the speed of light $c$ equal 1. As we perceive the universe to posses 3 spatial dimensions and 1 dimension in time, $3+1$-dimensional spacetimes are often considered as the most relevant models, but the general $n+1$-dimensional case for $n \geq 3$ is also frequently studied, and we will do so in Sections 5 and 6.

As the underlying models are given by spacetimes, we may now differentiate between spacelike and timelike directions. More precisely, for any point $p \in \mathfrak{M}$ we call a vector $v \in T_p\mathfrak{M} \setminus \{0\}$

$$\begin{aligned}
&\text{spacelike} && \text{if } \mathfrak{g}_p(v,v) > 0,\\
&\text{timelike} && \text{if } \mathfrak{g}_p(v,v) < 0,\\
&\text{lightlike or null} && \text{if } \mathfrak{g}_p(v,v) = 0.
\end{aligned}$$

Following the most commonly used notation, we define the 0-vector in any tangent space to be spacelike. This notion is readily extended to vector fields on $\mathfrak{M}$ and along curves $\gamma$ in $\mathfrak{M}$. For example, a curve $\gamma$ in $\mathfrak{M}$ is called timelike if $\dot{\gamma}$ is timelike everywhere along $\gamma$. Such timelike curves, also referred to as *observers*, model the path of particles trough the universe that move with a speed strictly below the speed of light. Such particles are considered free-falling, if the curve is a geodesic. Similarly, lightlike curves model the path of lightrays. If $(\mathfrak{M}, \mathfrak{g})$ admits a global, timelike vector field $X$, then $X$ induces a *time-orientation* on $(\mathfrak{M}, \mathfrak{g})$ and we call $(\mathfrak{M}, \mathfrak{g})$ *time-oriented*. Hence, there exists a notion of future- and past-pointing for any causal (timelike or null) vector(-field) in the following way: We call a causal vector $v \in T_p\mathfrak{M}$ future-pointing if $\mathfrak{g}_p(X_p, v) < 0$, and past-pointing if $\mathfrak{g}_p(X_p, v) > 0$.

As the Einstein Equations (1) are hyperbolic in nature, they are in general hard to solve and there is an abundance of solutions even in Vacuum, i.e., $\mathfrak{T} = 0$. Note that an $n+1$-dimensional spacetime $(\mathfrak{M}, \mathfrak{g})$ with $n \geq 3$ satisfies the Einstein Equations (1) in Vacuum if and only if

$$\mathfrak{Ric} = 0.$$

Prominent examples of solutions to the Einstein Equations in Vacuum are the Minkowski and Schwarzschild spacetime, discussed below in Subsections 4.2 and 4.3, as well as the Kerr spacetime, see [81, Chapter 12.3]. Further, all of the above are examples of *isolated systems*, which are models of a universe that is empty outside of a spatially compact region that may contain any number of e.g. stars, black holes and galaxies. Heuristically, one expects that the gravitational effects of these celestial objects weaken as an observer moves farther and farther away from the spacially compact region, and the spacetime should approach the flat Minkowski spacetime, see Subsection 4.2. This is made rigorous in Subsection 4.5. On the other hand, one can also consider *cosmological models* to study the large scale structure of the universe. In this case, one looks at a modified version of the Einstein Equations

$$\mathfrak{Ric} - \frac{1}{2}\mathfrak{R}\mathfrak{g} + \Lambda\mathfrak{g} = 8\pi\mathfrak{T}, \tag{2}$$

where $\Lambda$ is the *cosmological constant*. In particular, $\Lambda = 0$ recovers (1).

However, instead of looking at models for a specific stress-energy tensor $\mathfrak{T}$, models $(\mathfrak{M}, \mathfrak{g})$ are often considered more broadly under some additional physically motivated energy assumptions on $\mathfrak{T}$, where one usually considers $\mathfrak{T}$ now to be defined by the left-hand side of (2). Here, we mention two prominent such energy conditions, that will be of relevance to Sections 5 and 6: We say that a (time-oriented) spacetime $(\mathfrak{M}, \mathfrak{g})$ satisfies the *dominant energy condition* (DEC), if $-\mathfrak{T}^{\alpha}_{\beta}X^{\beta}$ is causal, future-pointing for every timelike, future-pointing vectorfield $X$. We say that $(\mathfrak{M}, \mathfrak{g})$ satisfies the *null energy condition* (NEC), also referred to as the null convergence condition, if

$$\mathfrak{T}(X, X) = \mathfrak{Ric}(X, X) \geq 0 \tag{3}$$

for all null vector fields $X$. Note that the (DEC) implies the (NEC).

## 4.2 Special Relativity and the Minkowski spacetime

The theory of Special Relativity leads to the study of the properties of the Minkowski spacetime, in particular its isometry group. It was first considered by Einstein in 1905, before he later introduced the phenomenon of gravitation in the models via the Einstein Equations (1). We briefly recall some well-known facts of the Minkowski spacetime and its isometry group. See e.g. [62].

For any $n \geq 1$, the $(n+1)$-dimensional Minkowski spacetime $(\mathbb{R}^{n,1}, \eta)$ is given by $\mathbb{R}^{n,1} = \mathbb{R}^{n+1}$ equipped with the Minkowski metric $\eta$ given by

$$\eta = -\,\mathrm{d}t \otimes \mathrm{d}t + \sum_{i=1}^{n} dx^i \otimes \mathrm{d}x^i,$$

where $(t, x^1, \ldots, x^n)$ denote standard Cartesian coordinates on $\mathbb{R}^{n+1}$. For our purposes, we will only consider $n = 3$, but most observations in $3+1$-dimensions extend directly to the higher dimensional case. In Cartesian coordinates, it is easy to see that the Minkowski spacetime is flat, i.e., $\mathfrak{Rm} \equiv 0$, and in particular a solution of the Einstein Equations in Vacuum (with $\Lambda = 0$). Note that $\partial_t$ is a global timelike vectorfield, so $(\mathbb{R}^{3,1}, \eta)$ is time-oriented. Although it is a rather simple model of General Relativity, relativistic effects such as the twin paradox can already be observed in the Minkowski spacetime.

As $\eta$ is independent of the base point of any tangent space, and any tangent space can be canonically identified with $\mathbb{R}^{3,1}$, we may evaluate $\eta$ for any two points $p, q \in \mathbb{R}^{3,1}$ without ambiguity. In particular, we define the *standard lightcone* (centered at the origin) in the $3+1$-Minkowski spacetime as

$$C := \{p \in \mathbb{R}^{3,1} \colon \eta(p,p) = 0\},$$

with

$$C = C_+ \,\dot\cup\, \{0\} \,\dot\cup\, C_-,$$

where $C_+ := C \cap \{t > 0\}$, $C_- := C \cap \{t < 0\}$ contain all points $p$ that are null and future- and past-pointing, respectively. We will call $C_+$ the *future-pointing lightcone* and $C_-$ the *past-pointing lightcone*, respectively. Note further that $C_+$, $C_-$ are smooth null hypersurfaces, see Subsection 4.7 below.

The Lorentz group $O(3,1)$ is the group of matrizes $L$ in $\mathbb{R}^{4\times 4}$ such that

$$L^T \eta L = \eta$$

for $\eta$ in the above Cartesian coordinates. It is easy to see that any $L \in O(3,1)$ is an isometry of $(\mathbb{R}^{3,1}, \eta)$. More precisely, the Lorentz group is the subgroup of isometries of the Poincaré group that map the origin to itself and leave the lightcone invariant, i.e., $L(C) = C$, where the Poincaré group is the full isometry group of $\mathbb{R}^{3,1}$. Note that the Poincaré group is in fact the semi-direct product of the translations in $\mathbb{R}^{3,1}$ and the Lorentz group. The full Lorentz group is a 6-dimensional non-compact non-connected Lie group, but for the purpose of this thesis we will restrict our attention to the *restricted Lorentz group* $SO^+(3,1)$, which denotes the identity component of $O(3,1)$. Note that $SO^+(3,1)$ consists of all boosts and rotations that preserve the time-orientation (with respect to $\partial_t$), see Proposition 4.2 below. Thus $\Lambda(C_\pm) = C_\pm$ for all $\Lambda \in SO^+(3,1)$.

**Example 4.1.** Examples of isometries in $SO^+(3,1)$ are the rotations in the spatial coordinates and special Lorentz boosts.

(a) Here, we identify a rotation $D$ in $SO^+(3,1)$ of the spatial coordinates with the matrix

$$D = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix},$$

where $R \in SO(3)$ is a rotation in $\mathbb{R}^3$.

(b) A *special Lorentz boost* $\Lambda_a$ is given by the matrix

$$\Lambda_a = \begin{pmatrix} b & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & 0 & 0 & b \end{pmatrix}$$

for $a \in \mathbb{R}$ and $b := \sqrt{1+a^2}$.

More generally, for any $\vec{a} \in \mathbb{R}^3$ we may consider a *Lorentz boost* $\Lambda_{\vec{a}}$ such that

$$\Lambda_{\vec{a}}(\partial_t) = \left(\sqrt{1+|\vec{a}|^2}, \vec{a}\right).$$

This relation indeed uniquely determines $\Lambda_{\vec{a}}$ upon the convention that we do not further rotate the spatial directions perpendicular to $\vec{a}$, and we observe that $\Lambda_{(0,0,a)} = \Lambda_a$ as defined in Example 4.1 (b). Moreover, under this assumption it is direct to check that $\Lambda_{\vec{a}}$ can be decomposed as

$$\Lambda_{\vec{a}} = D_{\vec{a}} \circ \Lambda_{|\vec{a}|} \circ D_{\vec{a}}^{-1},$$

where $D_{\vec{a}}$ is the unique rotation that maps $\partial_3$ to $\frac{\vec{a}}{|\vec{a}|}$ without any further rotation of the $x^1$ and $x^2$ axis perpendicular to $\vec{a}$ for $\vec{a} \neq 0$, and we choose $D_{\vec{a}} = \mathrm{Id}$ if $\vec{a} = 0$. Hence, $D_{\vec{a}}$ is already uniquely determined by $\frac{\vec{a}}{|\vec{a}|}$ for $\vec{a} \neq 0$, and we may take the above decomposition also as a definition of $\Lambda_{\vec{a}}$.

For a general Lorentz transformation $\Lambda \in SO^+(3,1)$ we prove the following decomposition.

**Proposition 4.2.** *Let $\Lambda \in SO^+(3,1)$. Then there exists $\vec{a} \in \mathbb{R}^3$ and a rotation $D$ defined as above such that*

$$\Lambda = \Lambda_{\vec{a}} \circ D.$$

**Remark 4.3.** Note that $\vec{a}$ and $D$ are uniquely determined up to a choice of Cartesian coordinates, i.e., up to a choice of a positively oriented orthonormal basis on $\mathbb{R}^3$. As two such orthonormal frames are related by a uniquely determined rotation, $\vec{a}$ and $D$ transform under this rotation upon a change of the orthonormal frame. Hence, for fixed $\Lambda$ we may choose the orthonormal frame such that $D = \mathrm{Id}$.

*Proof.* Let $x^1$, $x^2$, $x^3$ denote Cartesian coordinates on $\mathbb{R}^3$. As $\Lambda \in SO^+(3,1)$, there exists an $\vec{a} \in \mathbb{R}^3$ such that

$$\Lambda(\partial_t) = \left(\sqrt{1 + |\vec{a}|^2}, \vec{a}\right).$$

Hence, $L := \Lambda_{\vec{a}}^{-1} \circ \Lambda$ is a linear isometry of the Minkowski spacetime with

$$L(\partial_t) = \partial_t$$

Further $L(\partial_i) = (b_i, \vec{b}_i)$ for some $b_i \in \mathbb{R}$, $\vec{b}_i \in \mathbb{R}^3$, $i = 1, 2, 3$. By Linearity,

$$L(\partial_i - b_i \partial_t) = (0, \vec{b}_i),$$

and as $L$ is an isometry, this implies that

$$1 - b_i^2 = |\vec{b}_i|^2 = 1 + b_i^2.$$

Thus $b_i = 0$ for all $i = 1, 2, 3$ and $L$ is of the form

$$L = \begin{pmatrix} 1 & 0 \\ 0 & R, \end{pmatrix}$$

for some $R \in \mathbb{R}^{3\times 3}$. In particular, $R$ is a linear isometry on $\mathbb{R}^3$ and hence $L$ is either a reflection in the spacial directions or a rotation $D$ as defined above. As the restricted Lorentz group is the identity component of the Lorentz group, it does not contain reflections. This concludes the proof. $\square$

## 4.3 A class of static spacetimes

We consider a class of $(n+1)$-dimensional static, warped product spacetimes defined in the following way: Let $h : (0, \infty) \to \mathbb{R}$ be smooth, unless otherwise stated, with finitely many, positive zeroes $r_0 := 0 < r_1 < \ldots < r_N < \infty =: r_{N+1}$ for some $N \geq 0$, and let $(\mathcal{N}, g_{\mathcal{N}})$ be a $(n-1)$-dimensional Riemannian manifold, $n \geq 3$. We say that a spacetime $(\mathfrak{M}, \mathfrak{g})$ is of *class* $\mathfrak{H}$ *with metric coefficient* $h$ *and fibre* $\mathcal{N}$ if

$$\mathfrak{M} = \mathbb{R} \times (r_i, r_{i+1}) \times \mathcal{N},$$
$$\mathfrak{g} = -h \, \mathrm{d}t^2 + \frac{1}{h} \, \mathrm{d}r^2 + r^2 g_{\mathcal{N}}$$

for some $0 \leq i \leq N$. It is easy to check that $\partial_t$ is a global Killing vector field. Thus any spacetime of class $\mathfrak{H}$ is static if $h > 0$ on the corresponding real interval $(r_i, r_{i+1})$.

Despite the strong symmetry assumptions, class $\mathfrak{H}$ contains an abundance of physically relevant modes both in the case of isolated systems as well as models considered in cosmology. In particular, class $\mathfrak{H}$ contains the spherically symmetric case, i.e., $(\mathcal{N}, g_{\mathcal{N}}) = (\mathbb{S}^{n-1}, \mathrm{d}\Omega^{n-1})$ where $\mathbb{S}^{n-1}$ denotes the standard round sphere equipped with the standard round metric $\mathrm{d}\Omega^{n-1}$. This case corresponds to the spacetimes of class $\mathcal{S}$ as considered by Cederbaum–Galloway [21], and contains many prominent examples such as the Minkowski spacetime, the Schwarzschild- and Reisner–Nordström spacetime, and the anti-de Sitter and Schwarzschild anti-de Sitter spacetime, which solve the Einstein Equations (2) in Vacuum with negative cosmological constant. If more generally $(\mathcal{N}, g_{\mathcal{N}})$ is a Riemannian manifold with constant sectional curvature, then spacetimes of class $\mathfrak{H}$ correspond to the family of Birmingham–Kottler metrics, cf. [9, 56].

**Example 4.4.** The $3+1$-dimensional (exterior) Schwarzschild spacetime of positive mass $m > 0$ (in Schwarzschild coordinates) is given by

$$\begin{aligned} \mathfrak{M}_{Schw} &= \mathbb{R} \times (2m, \infty) \times \mathbb{S}^2, \\ \mathfrak{g}_{Schw} &= -\left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 + \frac{1}{1 - \frac{2m}{r}} \mathrm{d}r^2 + r^2 \,\mathrm{d}\Omega^2. \end{aligned}$$

Similar, we can also consider the interior $3+1$-dimensional Schwarzschild spacetime corresponding to the interval $(0, 2m)$.

For a given $h$, the boundaries $\{r = r_i\}$ corresponding to the positive zeros $r_i$ are Killing horizons, and it appears natural to think of each spacetime of class $\mathfrak{H}$ corresponding to an intervall $(r_i, r_{i+1})$ as a different region of the same, larger spacetime. For the Schwarzschild spacetime, this was made rigorous by the construction of the *Kruskal–Szekeres spacetime*, see [57, 78]. In fact, the Kruskal–Szekeres spacetime is the maximal continuous extension of the Schwarzschild spacetime [70]. The question whether a general spacetime of class $\mathfrak{H}$ admits a similar spacetime extension across a Killing horizon $\{r = r_i\}$ was independently addressed by Brill–Hayward [16], Schindler–Aguirre [71] and Cederbaum and the author [25], showing that a spacetime of class $\mathfrak{H}$ can be extended past a Killing horizon if and only if the Killing horizon is non-degenerate, i.e., has non-vanishing surface gravity $\kappa$.

Recall that the *surface gravity* $\kappa$ of a Killing horizon is defined via the equation

$$\nabla_X X = \kappa X$$

evaluated at the Killing horizon, where $X$ is the Killing vector field corresponding to the Killing horizon. Hence, $\kappa$ describes the failure of the integral curves of $X$ to be affine null geodesic at the horizon. We call the Killing horizon *non-degenerate* if $\kappa \neq 0$. Circumventing some subtleties regarding the scaling of $X$, we will always choose $X = \partial_t$ for a spacetime of class $\mathfrak{H}$. Then one can directly compute the following:

**Lemma 4.5.** [81, Equation (12.5.16)]

$$\kappa_i = \pm \frac{h'(r_i)}{2},$$

*for the choice* $X = \partial_t$.

## 4.4 Hypersurface geometry and initial data sets

Let us for now consider an $(n+1)$-dimensional semi-Riemannian manifold $(\mathfrak{M}, \mathfrak{g})$ and let $M$ be a hypersurface in $\mathfrak{M}$. Let $g$ denote the induced metric on $M$. Unless otherwise stated, we will from now on always assume that $g$ in non-degenerate, $(M, g)$ is orientable, and there exists a smooth unit normal vector field $\nu$ on $M$. Note that in the following, all definitions depend on our choice of $\nu$. However, we will omit this dependence here, as we will usually refer directly to a unique choice of unit normal in the following sections. For a detailed introduction, we refer to [19, Chapter 3][1].

**Definition 4.6.** Let $(\mathfrak{M}, \mathfrak{g})$ be a semi-Riemannian manifold, $(M, g)$ a hypersurface with unit normal $\nu$. We then define the *second fundamental form* $h$ of $(M, g)$ as

$$h(X, Y) := -\mathfrak{g}({}^{\mathfrak{g}}\nabla_X Y, \nu)$$

for tangent vector fields $X, Y \in \Gamma(TM)$, where ${}^{\mathfrak{g}}\nabla$ denotes the Levi-Civita connection of $(\mathfrak{M}, \mathfrak{g})$. We further define the *mean curvature* $H$ of $(M, g)$ as

$$H := \operatorname{tr}_g h = g^{ij} h_{ij}$$

**Remark 4.7.** It is a well known fact that $h$ is a symmetric $(0, 2)$-tensor field. In particular, at each point $p \in M$ there exists an ON-frame of eigenvectors $\{v_p^i\}_{i=1}^n$ of $h_p$ with real eigenvalues $\lambda_1(p) \leq \ldots \leq \lambda_n(p)$, and

$$H(p) = \sum_{i=1}^{n} \lambda_i(p).$$

We call a hypersurface *totally umbilic*, if $h = \lambda g$ for some smooth function $\lambda \in C^\infty(M)$. In this case, $\lambda$ is called the *umbilicity factor of* $M$. Then, $\mathring{h} = 0$ and $H = n\lambda$.

[1]Note the different sign conventions of the Riemann tensor Rm.

Recall that $h$ satisfies the Gauss formula

$$^{\mathfrak{g}}\nabla_X Y = {}^g\nabla_X Y - \sigma h(X,Y)\nu$$

for all tangent vector fields $X, Y \in \Gamma(TM)$, where $\sigma := \mathfrak{g}(\nu,\nu)$ and ${}^g\nabla$ denotes the Levi-Civita connection of $(M,g)$.

Using the Gauss formula, one can derive the well-known Gauss- and Codazzi equations.

**Proposition 4.8** (Gauss Equations I)**.**

$$\begin{aligned}
\mathfrak{Rm}(X,Y,Z,W) &= \mathrm{Rm}_g(X,Y,Z,W) - \sigma h(X,Z)h(Y,W) + \sigma h(X,W)h(Y,Z), \\
\mathfrak{Ric}(X,Y) &= \mathrm{Ric}_g(X,Y) + \sigma\mathfrak{Rm}(X,\nu,Y,\nu) - \sigma H h(X,Y) + \sigma h^2(X,Y), \\
\mathfrak{R} - 2\sigma\mathfrak{Ric}(\nu,\nu) &= R_g - \sigma H^2 + g|K|_g^2
\end{aligned}$$

*for all tangent vector fields $X, Y, Z, W \in \Gamma(TM)$, where $\mathfrak{Rm}$, $\mathfrak{Ric}$ and $\mathfrak{R}$ denote the Riemann curvature tensor, Ricci tensor and scalar curvature of $(\mathfrak{M},\mathfrak{g})$, respectively, and $\sigma := \mathfrak{g}(\nu,\nu)$.*

**Proposition 4.9** (Codazzi Equation I)**.**

$$\mathfrak{Rm}(X,Y,Z,\nu) = ({}^g\nabla h)(X,Y,Z) - ({}^g\nabla h)(Y,X,Z)$$

*for all tangent vector fields $X, Y, Z \in \Gamma(TM)$, where $\mathfrak{Rm}$ is the Riemann curvature tensor of $(M,g)$.*

Now let $(\mathfrak{M},\mathfrak{g})$ be a spacetime. We call $(M,g)$ spacelike if $g$ is a Riemannian metric, and $(M,g)$ timelike if $g$ is a Lorentzian metric. Note that $(M,g)$ is spacelike if and only if $\nu$ is timelike, and $(M,g)$ is timelike if and only if $\nu$ is spacelike. If $(M,g)$ is a spacelike hypersurface in an ambient spacetime, we denote its second fundamental form by $K$ and its mean curvature by $\mathrm{tr}_g K$. If $(\mathfrak{M},\mathfrak{g})$ is time-orientable, we will always choose $\nu$ such that $\nu$ is future-pointing. In this context, one can derive the *constraint equations* from the Gauss and Codazzi equations.

**Corollary 4.10** (Constraint Equations)**.** Let $(\mathfrak{M},\mathfrak{g})$ be a time-orientable spacetime satisfying the Einstein Equations (2) (with cosmological constant $\Lambda$) with stress-energy tensor $\mathfrak{T}$. Let $(M,g)$ be a spacelike hypersurface with future-pointing unit normal $\nu$. Then $(M,g)$ satisfies the constraint equations

$$R_g + (\mathrm{tr}_g K)^2 - |K|_g^2 - 2\Lambda = 16\pi\boldsymbol{\mu}, \tag{4}$$

$$\mathrm{div}_g(K - (\mathrm{tr}_g K)g) = 8\pi\mathbf{J}, \tag{5}$$

where $\boldsymbol{\mu} := \mathfrak{T}(\nu,\nu)$ is called the *energy density* and $\mathbf{J} := \mathfrak{T}(\nu, \cdot|_{TM})$ is called the *momentum density.*

This motivates the following formal definition.

**Definition 4.11.** Let $(M, g)$ be a Riemannian manifold, $K$ a symmetric $(0, 2)$-tensor on $M$. We call the triple $(M, g, K)$ an *initial data set*, if the constraint equations

$$R_g + (\operatorname{tr}_g K)^2 - |K|_g^2 - 2\Lambda = 16\pi\boldsymbol{\mu}, \tag{6}$$

$$\operatorname{div}_g(K - (\operatorname{tr}_g K)g) = 8\pi\mathbf{J}, \tag{7}$$

are satisfied for some smooth function $\boldsymbol{\mu}$ and one-form $\mathbf{J}$. We call $\boldsymbol{\mu}$ the energy density and $\mathbf{J}$ the momentum density, respectively, and refer to Equations (6) and (7) as the *Hamiltonian and momentum constraint*, respectively.

**Remark 4.12.** By Corollary 4.10 it is easy to see that any spacelike hypersurface $M$ gives rise to an initial data set $(M, g, K)$. Conversely, it is natural to ask whether any initial data set $(M, g, K)$ can be realized as a spacelike hypersurface with induced metric $g$ and second fundamental form $K$ with respect to a timelike unit normal $\nu$ within an ambient spacetime $(\mathfrak{M}, \mathfrak{g})$, such that $\boldsymbol{\mu} := \mathfrak{T}(\nu, \nu)$ and $\mathbf{J} := \mathfrak{T}(\nu, \cdot|_{TM})$. In this case, we regard the Einstein Equations (2) as an evolutionary system for $\mathfrak{g}$ with initial data $(M, g, K)$, where we understand $g$ as the initial value restricted to tangent vectors of $M$, and we can interpret $K$ as the initial velocity prescribed on $M$. Under suitable conditions, this evolutionary system indeed admits a solution. This was first proven in the nominal work of Choquet-Bruhat–Geroch [29].

If $(M, g, K)$ indeed embeds into an ambient spacetime $(\mathfrak{M}, \mathfrak{g})$, and the respective energy-stress tensor $\mathfrak{T}$ satisfies the (DEC), then

$$\boldsymbol{\mu} \geq |\mathbf{J}|_g\,, \tag{8}$$

and we say that an initial data set $(M, g, K)$ satisfies the dominant energy condition (DEC), if $\boldsymbol{\mu}$ and $\mathbf{J}$ satisfy (8).

**Definition 4.13.** Let $(M, g, K)$ be an initial data set. We say $(M, g, K)$ is

- *time-symmetric* if $K \equiv 0$,
- *maximal* if $\operatorname{tr}_g K \equiv 0$.

Similar to minimal surfaces in Riemannian geometry, maximal initial data sets can be interpreted as critical points of the area functional. However, due to the fact that the unit normal is timelike in this setting and that one can usually decrease the area by the approximation of null directions, one generally seeks to maximise area. Note further that for any maximal initial data set $(M, g, K)$ satisfying the (DEC), $(M, g)$ has non-negative scalar curvature $R_g \geq 0$.

## 4.5 Asymptotically flat initial data sets

We briefly mentioned in Subsection 4.1 that isolated systems should be heuristically modelled by spacetimes that approach the Minkowski spacetime outside of a spatially compact set. Hence, any appropriately chosen spacelike hypersurface of such a model should approach Euclidean space outside of a compact set. This is made precise with the following definition, cf. [24] for $n = 3$.

**Definition 4.14.** Let $(M, g, K)$ be an $n$-dimensional initial data set. We say $(M, g, K)$ is *asymptotically flat* if $\Lambda = 0$, $\boldsymbol{\mu}$, $|\mathbf{J}|_g \in L^1(M)$, and there exists a compact set $\Omega \subseteq\subseteq M$ and a diffeomorphism $\Phi\colon M \setminus \Omega \to \mathbb{R}^n \setminus \overline{B_1(0)}$ such that

$$\begin{aligned} |g_{ij} - \delta_{ij}| + r\,|\partial_l g_{ij}| + r^2\,|\partial_l \partial_m g_{ij}| &\leq C\,|x|^{-\tau}\,, \\ |K_{ij}| + r\,|\partial_l K_{ij}| &\leq C|x|^{-\tau-1}, \end{aligned}$$

in Cartesian coordinates $x^i$ on $\mathbb{R}^n$ for some constant $C > 0$ and $\tau > \frac{n-2}{2}$, where $|x|$ denotes the Euclidean norm of $x$, and $g$ and $K$ are identified with their respective push-forwards on $\mathbb{R}^n \setminus \overline{B_1(0)}$.

For isolated systems in the above sense, there is a notion of total energy, momentum and mass going back to Arnowitt, Deser and Misner, cf. [3]. More precisely, we define the *ADM energy* $E_{ADM}$ and *ADM momentum* $\boldsymbol{P}_{ADM}$ of an asymptotically flat initial data set $(M, g, K)$ as

$$E_{ADM} := \frac{1}{2(n-1)\omega_{n-1}} \lim_{\rho\to\infty} \int_{S_\rho} (g_{ij,i} - g_{ii,j})\nu^j \, dV_{S_\rho}, \tag{9}$$

$$(\boldsymbol{P}_{ADM})_i := -\frac{1}{(n-1)\omega_{n-1}} \lim_{\rho\to\infty} \int_{S_\rho} (K_{ij} - (\operatorname{tr}_g K) g_{ij})\nu^j \, dV_{S_\rho}, \tag{10}$$

for $i = 1, ..., n$, where the integrals are taken over the round spheres $S_\rho$ in $\mathbb{R}^n$ centered at the origin and $\nu$ is the standard unit normal to $S_\rho$, cf. [24]. If $E_{ADM} \geq |\boldsymbol{P}_{ADM}|$ (where $|\cdot|$ denotes the Euclidean norm), we further define the *ADM mass* $m_{ADM}$ as

$$m_{ADM} = \sqrt{E_{ADM}^2 - |\boldsymbol{P}_{ADM}|^2}. \tag{11}$$

Note that under the assumptions of Definition 4.14, the ADM energy, momentum and mass are finite and independent of the choice of asymptotic chart, cf. [5, 33]. Further, it was first proven by Schoen–Yau in $n = 3$ that $E_{ADM} \geq |\boldsymbol{P}_{ADM}|$, if $(M, g, K)$ is geodesically complete and satisfies the (DEC). This is known as the celebrated *positive mass theorem*, cf. [72, 73]. See also [66, 86].

**Example 4.15.** Recall the Schwarzschild spacetime of positive mass $m > 0$ as in Example 4.4, and consider the constant time-slice $\{t = 0\}$, which is a time-symmetric spacelike hypersurface. Performing a change of coordinates, we see that the subset $\{t = 0\}$ is isometric to the *Schwarzschild manifold* $(M_{Schw}, g_{Schw})$ *in isotropic coordinates* given as

$$M_{Schw} = \mathbb{R}^3 \setminus \overline{B_{\frac{m}{2}}(0)}, \; g_{Schw} = \left(1 + \frac{m}{2\,|x|}\right)^4 \delta.$$

It is easy to check that the Schwarzschild manifold is asymptotically flat and a direct computation yields

$$m_{ADM} = E_{ADM} = m.$$

Note that $(M_{Schw}, g_{Schw})$ is not geodesically complete, but can be smoothly extended to all of $\mathbb{R}^3 \setminus \{0\}$ with the same metric $g_{Schw}$, and the extension is geodesically complete. In fact, the extension corresponds to two copies of $(M_{Schw}, g_{Schw})$ smoothly glued at the minimal surface $\{|x| = \frac{m}{2}\}$.

## 4.6 Asymptotically hyperbolic initial data sets

The anti-de Sitter spacetime is an important model in cosmology that solves the Einstein Equation (2) with negative cosmological constant $\Lambda$. More precisely, if we choose $\Lambda = \frac{n(n-1)}{2}$, then the anti-de Sitter spacetime is given by the spherically symmetric spacetime of class $\mathfrak{H}$ with metric coefficient $h(r) := 1 + r^2$. In particular, the constant time-slice $\{t = 0\}$ is a spacelike, time-symmetric hypersurface that is isometric to the $n$-dimensional hyperbolic space $(\mathbb{H}, g_{\mathbb{H}})$.

Similar to the definition of asymptotically flat initial data sets $(M, g, K)$, one can define a notion of *asymptotically hyperbolic* initial data sets for negative cosmological constant $\Lambda < 0$. In this context, we say that a Riemannian manifold $(M, g)$ is *asymptotically hyperbolic* if $g \to g_{\mathbb{H}}$ at an appropriate rate at infinity. See e.g. [84]. Notice that one can also isometrically embed the hyperbolic space $(\mathbb{H}, g_{\mathbb{H}})$ into the $(n + 1)$-dimensional Minkowski spacetime as either of the connected components of the set $\{p \in R^{n,1} \,|\, \eta(p, p) = -1\}$. More generally, we can consider the (*two sheeted*) *hyperboloids* $\{p \in R^{n,1} \,|\, \eta(p, p) = -r\}$ for some $r > 0$. Then the two connected components are spacelike, totally umbilic hypersurfaces with umbilicity factor $\lambda = \pm\frac{1}{r}$ with respect to the future pointing normal, where the sign of $\lambda$ depends on the choice of connected component. Further, each connected component asymptotes to a connected component $C_+$, $C_-$ of the lightcone $C$, see Subsection 4.2. Now, we call a general initial data set $(M, g, K)$ *asymptotically hyperboloidal* if $(M, g, K)$ formally asymptotes to a connected component of the hyperboloid $\{p \in R^{n,1} \,|\, \eta(p, p) = -1\}$, i.e., $(M, g)$ is a asymptotically hyperbolic manifold and $K \to g_{\mathbb{H}}$ at an appropriate rate. See e.g.

[68]. In this setting, there are also notions of total energy, mass and momentum, but the definitions are more subtle compared to the asymptotically flat case and we have no need of them here.

Here, we will not give a precise definition of asymptotically hyperbolic or asymptotically hyperboloidal initial data sets, as we will only encounter them in the following way in Section 6. We say a Riemannian manifold $(M, g)$ of the form

$$M = (r_0, \infty) \times \mathbb{S}^2, \tag{12}$$

$$g = \frac{1}{f(r) + \lambda^2 r^2} \tag{13}$$

is asymptotically hyperbolic, if $f \to 1$ as $r \to \infty$. As we will realize the above manifolds in Section 6 precisely as spacelike, totally umbilic graphs with $K = \lambda g$, we will similarly call them asymptotically hyperboloidal, if $f \to 1$ as $r \to \infty$. Note that these notions of asymptotically hyperbolic and asymptotically hyperboloidal indeed agree with the classical definitions (up to scaling) if suitable decay conditions are imposed on $f$ and its derivatives. One prominent example is the Schwarzschild anti-de Sitter family corresponding to $f = 1 - \frac{2m}{r}$ which we will discuss in more detail in Section 6.

## 4.7 Null hypersurfaces

We give a brief introduction into the definition and properties of null hypersurfaces. For details we refer the interested reader to [59, 67, 69]. Let $(\mathfrak{M}, \mathfrak{g})$ be an (ambient) spacetime. We say an orientable hypersurface $\mathcal{N}$ in $\mathfrak{M}$ is *null* or a *null hypersurface* if its induced metric $g$ is degenerate. In particular, there exists a null vector field $\underline{L} \in \Gamma(T\mathcal{N})$ such that

$$\underline{L}_p^\perp = T_p\mathcal{N}$$

for all $p \in \mathcal{N}$, where $\underline{L}_p^\perp = \{v_p \in T_p\mathfrak{M} \,|\, \mathfrak{g}_p(v_p, \underline{L}_p) = 0\}$. Observe that $\underline{L}$ is both tangent and normal to the null hypersurface $\mathcal{N}$. Note that for any $p \in \mathcal{N}$ all tangent vectors in $T_p\mathcal{N}$ are either spacelike or a multiple of $\underline{L}$, as $\mathfrak{g}_p(X_p, \underline{L}_p) \neq 0$ for any causal vector $X_p \in T_p\mathfrak{M}$ such that $X_p \neq c\underline{L}_p$.

Let $p \in \mathcal{N}$, as $\mathcal{N}$ is a hypersurface there exists an open neighbourhood $\mathcal{U} \subseteq \mathfrak{M}$ of $p$ and a function $v \in C^\infty(\mathcal{U})$ such that

$$\mathcal{U} \cap \mathcal{N} = \{v = 0\}$$

and $Dv := {}^{\mathfrak{g}}\nabla v \neq 0$ on $\mathcal{U}$, with $(Dv)_q^\perp = T_q\mathcal{N}$ for all $q \in \mathcal{U} \cap \mathcal{N}$. Hence, as $\mathfrak{g}$ is non-degenerate, we find that

$$Dv_q = a_q \underline{L}_q$$

for all $q \in \mathcal{U} \cap \mathcal{N}$ with $a_q \neq 0$. In particular, $Dv$ is null and tangent along $\mathcal{U} \cap \mathcal{N}$. Using that

$$\mathfrak{g}\left({}^{\mathfrak{g}}\nabla_{Dv}Dv, \cdot\right) = \frac{1}{2}\,\mathrm{d}\left(\mathfrak{g}(Dv, Dv)\right),$$

which can be directly verified for any semi-Riemannian manifold and (locally defined) smooth function by a computation in normal coordinates, we conclude that

$$\mathfrak{g}_q\left(\left({}^{\mathfrak{g}}\nabla_{Dv}Dv\right)_q, X_q\right) = 0 \text{ for all } X_q \in T_q\mathcal{N}$$

for all $q \in \mathcal{U} \cap \mathcal{N}$. Hence

$$\left({}^{\mathfrak{g}}\nabla_{Dv}Dv\right)_q = b_q Dv_q$$

for some constant $b_q \in \mathbb{R}$ for all $q \in \mathcal{U} \cap \mathcal{N}$. As all objects in the above considerations are smooth tensors, we can conclude that there exists a smooth function $\kappa \in C^\infty(\mathcal{N})$ such that

$$ {}^{\mathfrak{g}}\nabla_{\underline{L}}\underline{L} = \kappa \underline{L} \tag{14}$$

along $\mathcal{N}$. In particular, the integral curves of $\underline{L}$ can be reparametrized as null geodesics, and $\mathcal{N}$ is in fact ruled by null geodesics. We call $\underline{L}$ a *null generator* of $\mathcal{N}$ and if $(\mathfrak{M}, \mathfrak{g})$ is time-oriented we will assume from now on that $\underline{L}$ is past-pointing, unless otherwise stated.

Assume further that there exists an embedded, connected, spacelike surface $S_0 \subset \mathcal{N}$, such that any integral curve of $\underline{L}$ intersects $\Sigma_0$ exactly once, i.e., there exists a well-defined projection

$$\begin{aligned} \pi \colon \mathcal{N} &\to S_0, \\ p &\mapsto \gamma_p^{\underline{L}}(\,\widetilde{\lambda}\,) \in S_0, \end{aligned}$$

where $\gamma_p^{\underline{L}}$ denotes the integral curve of $\underline{L}$ starting at a point $p \in \mathcal{N}$ and where $\widetilde{\lambda}$ is the unique time such that $\gamma_p^{\underline{L}}$ intersects $S_0$. Given $\underline{L}$, $S_0$ and $s_0 \in \mathbb{R}$, we can define a unique scalar function $s \in C^\infty(\mathcal{N})$, such that $\underline{L}(s) = 1$, and $s(p) = s_0$ if and only if $p \in S_0$. For a point $p \in S_0$, we define $(s_-(p), s_+(p))$ as the maximal existence interval of the integral curves $\gamma_p^{\underline{L}}$ of $\underline{L}$ passing through $p$ at parameter time $\lambda = s_0$, and we assume that

$$\left(R_- := \sup_{S_0} s_-(p), R_+ := \inf_{S_0} s_-(p)\right) \neq \emptyset.$$

Since $\nabla s \neq 0$, the mapping

$$\Phi_{S_0} \colon (R_-, R_+) \times S_0 \to \mathcal{N}, (\lambda, p) \mapsto \gamma_p^{\underline{L}}(\lambda)$$

is a smooth embedding that maps into an open subset of $\mathcal{N}$. For convenience, we will from now on always assume that $\Phi_{S_0}$ is a diffeomorphism onto $\mathcal{N}$, and define the map $\pi \colon \mathcal{N} \to S_0$

to be the projection of $\Phi_{S_0}^{-1}$ onto $S_0$. As a consequence, the level sets $\Sigma_r := \{s = r\}$ form a smooth foliation of $\mathcal{N}$, where $\Sigma_{s_0} = S_0$, and for any point on $\Sigma_r$ there exist local coordinates $x^i$ on $\Sigma_r$ such that $[\underline{L}, \partial_i] = 0$, which we can identify locally with coordinates on $S_0$ pushed forward by $\Phi_{S_0}$. We call the family $(\Sigma_r)_{r\in(R_-,R_+)}$ a *background foliation of $\mathcal{N}$ with respect to $\underline{L}$*, and additionally consider the null vector field $L$ (with respect to our choice of background foliation) along $\mathcal{N}$ such that

$$\mathfrak{g}(\underline{L}, L) = 2$$

and $\{\underline{L}_p, L_p\}$ forms a null frame of the normal bundle $T_p^{\perp}\Sigma_r$ for all $p \in \Sigma_r$ and for all $r \in (R_-, R_+)$. Note that $L$ is uniquely determined and future-pointing by our above choice.

Notice that for any non-vanishing function $a \in C^\infty(\mathcal{N})$, $a\underline{L}$ is a null generator of $\mathcal{N}$, and we can consider the background foliation $(\Sigma_r^a)_r$ (with respect to the same choice of $S_0$). Recall from (14) that we can choose $a$ such that the integral curves of $a\underline{L}$ are geodesic. In this case, we call the resulting foliation a *geodesic background foliation*. Here, we will only consider $a > 0$ to preserve the time-orientation of $\underline{L}$.

We define the *null second fundamental form $\underline{\chi}$ of $\mathcal{N}$ with respect to $\underline{L}$* as

$$\underline{\chi}(X, Y) = \mathfrak{g}\left({}^{\mathfrak{g}}\nabla_X \underline{L}, Y\right)$$

for tangent vector fields $X, Y \in \Gamma(T\mathcal{N})$. Note that the induced metric $g$ and null second fundamental form $\underline{\chi}$ are symmetric $(0,2)$-tensors on $\mathcal{N}$ with

$$\begin{aligned} g(X + \underline{L}, Y + \underline{L}) &= g(X, Y), \\ \underline{\chi}(X + \underline{L}, Y + \underline{L}) &= \underline{\chi}(X, Y) \end{aligned}$$

for all tangent vector fields $X, Y \in \Gamma(T\mathcal{N})$. In particular, $g$ and $\underline{\chi}$ are fully determined by their restriction to a given background foliation and we write $(\gamma_r)_p := g_p\left(\cdot|_{\Sigma_r}, \cdot|_{\Sigma_r}\right)$ for $p \in \Sigma_r$.

Before discussing codimension-2 surfaces and their extrinsic curvature in a more general and detailed setting in the next subsection, consider now an embedded, spacelike, smooth cross section $\Sigma$ of $\mathcal{N}$, with embedding $\iota\colon \Sigma \to \mathcal{N}$. We will from now on always assume that any integral curve of $\underline{L}$ intersects $\Sigma$ exactly once. Then there exists a unique diffeomorphism $\Psi\colon \Sigma \to \Psi(\Sigma) \subseteq S_0$ that satisfies $\Psi(p) = (\pi \circ \iota)(p)$ for all $p \in \Sigma$, and moreover consider the function $F : \Sigma \to \mathbb{R}, p \mapsto s(\iota(p))$. For convenience, we will always assume that $\Psi$ is a diffeomorphism onto $S_0$. Conversely, for any such $\Psi$ and $F$, we can define the embedding

$$\iota\colon \Sigma \to \mathcal{N}, p \mapsto \gamma_{\Psi(p)}^{\underline{L}}(F(p)).$$

In particular, we can regard any such surface $\Sigma$ as a graph over $S_0$ with respect to the function $\omega := F \circ \Psi^{-1}$. Below, we will usually omit the explicit reference to the embedding $\iota$

and identify the functions $F \in C^\infty(\Sigma)$ and $\omega \in C^\infty(S_0)$ for convenience without ambiguity, as $F$ satisfies $F = \omega \circ \pi$. Hence, we can identify any such surface under consideration simply via the function $\omega$ that uniquely determines $\Sigma$ with respect to a fixed choice of background foliation. We will thus write $\Sigma = \Sigma_\omega$. For technical purposes, we will extend $F$ constantly along the integral curves of $\underline{L}$ to a neighbourhood of $\Sigma$ in $\mathcal{N}$, such that $\underline{L}(F) = 0$.

**Proposition 4.16.** *Let $p \in \Sigma = \Sigma_\omega$, then the map*

$$T_{F,p} \colon T_p\Sigma_{r=F(p)} \to T_p\Sigma, X \mapsto X' := X + X(F)\underline{L},$$

*is a well-defined isomorphism. Denoting the induced metric on $\Sigma$ by $\gamma_\Sigma$, we find that*

$$\begin{aligned}(\gamma_\Sigma)_p(X', Y') =& (\gamma_{F(p)})_p(X, Y),\\ \underline{\chi}_p(X', Y') =& \underline{\chi}_p(X, Y)\end{aligned}$$

*for all tangent vectors $X', Y' \in T_p\Sigma$. Moreover, the null vector field $L_\omega$ with $\mathfrak{g}(L_\omega, \underline{L}) = 2$ and $\mathfrak{g}_p((L_\omega)_p, X') = 0$ for all $p \in \Sigma$ and $X' \in T_p\Sigma$ is given by*

$$(L_\omega)_p = L_p - \left|{}^{\gamma_{F(p)}}\nabla F\right|^2 \underline{L}_p - 2\, {}^{\gamma_{F(p)}}\nabla F_p,$$

*where ${}^{\gamma_{F(p)}}\nabla F$ denotes the gradient of $F$ on $\Sigma_{r=F(p)}$.*

From now on and throughout this work a spacelike cross section will always refer to a surface $\Sigma \subseteq \mathcal{N}$ with the above properties, i.e., any integral curve of $\underline{L}$ intersects $\Sigma$ exactly once and that $\Phi \colon \Sigma \to S_0$ is a diffeomorphism for a given background foliation.

**Remark 4.17.**

(i) We can restrict $\underline{\chi}$ to a symmetric $(0, 2)$-tensor $\underline{\chi}_\Sigma$ on any spacelike cross section $\Sigma$. Note that Proposition 4.16 shows that $(\underline{\chi}_\Sigma)_p = (\underline{\chi}_{\Sigma'})_p$ for any two cross sections $\Sigma$, $\Sigma'$ with $p \in \Sigma \cap \Sigma'$ up to the above isomorphism of tangent spaces. Hence, we will simply denote the restriction also with $\underline{\chi}$ without ambiguity. In particular, the function

$$\underline{\theta} \colon p \mapsto \underline{\theta}(p) := \operatorname{tr}_\Sigma \underline{\chi}(p)$$

on $\Sigma$ can be extended to a well-defined smooth function $\underline{\theta}$ on all of $\mathcal{N}$ independent of the choice of $\Sigma$. More precisely, it is straightforward to see that the extension to all of $\mathcal{N}$ is given by $\underline{\theta}(p) := \operatorname{tr}_{\gamma_{s(p)}} \underline{\chi}(p)$. Note however, that $\underline{\chi}$, $\underline{\theta}$ depend on the choice of background foliation, see Remark 4.19 (ii), (iii).

(ii) As stated above, for each $\Sigma = \Sigma_\omega$ the function $\omega$ on $S_0$ is identified with $F$ by $F = \omega \circ \pi$. As $\pi$ induces a smooth family of isomorphisms between the tangent spaces of the spacelike cross sections $\Sigma_r$ and $S_0$, by Proposition 4.16 there exists an isomorphism

$$T_{\omega,p} \colon T_{\pi(p)} S_0 \to T_p\Sigma, X \mapsto \widetilde{X} := X + X(\omega)\underline{L}$$

for each $p \in \Sigma$. We may further use $\pi$ restricted to the spacelike cross sections $\Sigma_r$ to identify the metrics $\gamma_r$ and null second fundamental forms $\underline{\chi}_{\Sigma_r}$ with smooth families $(\pi_*\gamma_r)_{r\in(R_-,R_+)}$, $(\pi_*\underline{\chi}_{\Sigma_r})_{r\in(R_-,R_+)}$ of symmetric $(0,2)$-tensors on $S_0$. In particular, these smooth families allow us to identify the induced metric $\gamma_\Sigma$ and null second fundamental form $\underline{\chi}_\Sigma$ with a metric $\gamma_\omega$ and a symmetric $(0,2)$-tensor $\underline{\chi}_\omega$ on $S_0$.

*Proof.* The statement of Proposition 4.16 follows directly from Proposition 1 and Equation (9) in [59]. Note the different sign convention $k = -\underline{L}$. □

## 4.8 The Geometry of codimension-2 surfaces

Let us now consider embedded, spacelike codimension-2 surfaces $(\Sigma, \gamma)$ in an ambient (time-oriented) spacetime $(\mathfrak{M}, \mathfrak{g})$. Unless otherwise stated, we will always assume that $(\Sigma, \gamma)$ is orientable and that there exists a smooth orthonormal frame of the normal bundle $T^\perp\Sigma$. For additional information, we refer to [58, 64, 67].

**Definition 4.18.** We define the *vector-valued second fundamental form* of $(\Sigma, \gamma)$ in $(\mathfrak{M}, \mathfrak{g})$ as

$$\vec{\mathrm{II}} \colon \Gamma(T\Sigma) \times \Gamma(T\Sigma) \to \Gamma(T^\perp\Sigma), (X, Y) \mapsto ({}^{\mathfrak{g}}\nabla_X Y)^\perp,$$

where for all $p \in \Sigma$ and $\mathfrak{X} \in T_p\mathfrak{M}$, $\mathfrak{X}^\perp$ denotes the projection onto $T_p^\perp\Sigma$. We further define the (*codimension*-2) *mean curvature vector* $\vec{\mathcal{H}}$ of $\Sigma$ as

$$\vec{\mathcal{H}} := \operatorname{tr}_\Sigma \vec{\mathrm{II}} \in \Gamma(T^\perp\Sigma),$$

and the *spacetime mean curvature* $\mathcal{H}^2$ of $\Sigma$ as the Lorentzian length of $\vec{\mathcal{H}}$ in $(\mathfrak{M}, \mathfrak{g})$, i.e.,

$$\mathcal{H}^2 := \mathfrak{g}(\vec{\mathcal{H}}, \vec{\mathcal{H}}).$$

We say $(\Sigma, \gamma)$ is a *surface of constant spacetime mean curvature* or an *STCMC surface* if $\mathcal{H}^2$ is constant along $\Sigma$.

Similar to the second fundamental form of a hypersurface as defined in Subsection 4.4 $\vec{\mathrm{II}}$ is symmetric and satisfies a Gauss formula by definition, i.e.,

$${}^{\mathfrak{g}}\nabla_X Y = {}^{\gamma}\nabla_X Y + \vec{\mathrm{II}}(X, Y)$$

for any tangent vector fields $X, Y \in \Gamma(T\Sigma)$. If $\{e_0, e_1\}$ denotes an orthonormal frame of $T^\perp\Sigma$ with $e_0$ timelike and $e_1$ spacelike, we define the *second fundamental form $h_i$ of $\Sigma$ with respect to $e_i$* as

$$h_i(X,Y) := \mathfrak{g}({}^{\mathfrak{g}}\nabla_X e_i, Y) = -\mathfrak{g}(e_i, \vec{\mathrm{II}}(X,Y))$$

for tangent vector fields $X, Y \in \Gamma(T\Sigma)$. Similar to the properties of the second fundamental form $h$ of a hypersurface as stated in Remark 4.7, each $h_i$ is a symmetric $(0,2)$-tensor on $(\Sigma, \gamma)$. In particular, if we consider the case when $(\Sigma, \gamma)$ is contained in a spacelike hypersurface $(M, g)$ with second fundamental form $K$ with respect to a future timelike unit normal $\vec{n}$, then $(\Sigma, \gamma)$ is a hypersurface in $(M, g)$. In this case, $h$ will always denote the second fundamental form of $(\Sigma, \gamma)$ in $(M, g)$ with respect to some unit normal vectorfield $\nu$ tangent to $M$, and we find that $h_1 = h$, $h_0 = K(\cdot|_{T\Sigma}, \cdot|_{T\Sigma})$ with respect to the orthonormal frame $\{\vec{n}, \nu\}$ of $T^\perp\Sigma$. In particular,

$$\vec{\mathrm{II}} = -h\nu + K(\cdot|_{T\Sigma}, \cdot|_{T\Sigma})\vec{n}, \tag{15}$$

$$\vec{\mathcal{H}} = -H\nu + P\vec{n}, \tag{16}$$

and

$$\mathcal{H}^2 = H^2 - P^2, \tag{17}$$

where $H$ denotes the mean curvature of $(\Sigma, \gamma)$ in $(M, g)$ and $P := \operatorname{tr}_\Sigma K = \operatorname{tr}_M K - K(\nu, \nu)$. In particular, we can formally extend the definition of the spacetime mean curvature $\mathcal{H}^2$ to any hypersurface $(\Sigma, \gamma)$ of an initial data set $(M, g, K)$. In this context, Cederbaum–Sakovich [24] considered an asymptotic foliation of STCMC surfaces for asymptotically flat initial data sets as a notion of center-of-mass.

Equivalently, we may also consider a null frame $\{\underline{L}, L\}$ of $T^\perp\Sigma$ such that $\mathfrak{g}(\underline{L}, L) = 2$ and $\underline{L}$ is pastpointing. We then define the *null second fundamental forms* $\underline{\chi}$ and $\chi$ of $(\Sigma, \gamma)$ (with respect to $\underline{L}$ and $L$) as

$$\underline{\chi}(X,Y) := \mathfrak{g}({}^{\mathfrak{g}}\nabla_X \underline{L}, Y) = -\mathfrak{g}(\underline{L}, \vec{\mathrm{II}}(X,Y)),$$
$$\chi(X,Y) := \mathfrak{g}({}^{\mathfrak{g}}\nabla_X L, Y) = -\mathfrak{g}(L, \vec{\mathrm{II}}(X,Y)),$$

and the *null expansion* $\underline{\theta}$ and $\theta$ of $(\Sigma, \gamma)$ (with respect to $\underline{L}$ and $L$) as

$$\underline{\theta} := \operatorname{tr}_\Sigma \underline{\chi} = -\mathfrak{g}(\underline{L}, \vec{\mathcal{H}}),$$
$$\theta := \operatorname{tr}_\Sigma \chi = -\mathfrak{g}(L, \vec{\mathcal{H}}).$$

Again, we similarly note that

$$\vec{\mathrm{II}} = -\frac{1}{2}\chi\underline{L} - \frac{1}{2}\underline{\chi}L \tag{18}$$

$$\vec{\mathcal{H}} = -\frac{1}{2}\theta\underline{L} - \frac{1}{2}\underline{\theta}L, \tag{19}$$

and

$$\mathcal{H}^2 = \underline{\theta}\theta. \tag{20}$$

**Remark 4.19.**

(i) If $(\Sigma, \gamma)$ is contained in a null hypersurface $\mathcal{N}$ such that $\underline{L}$ is the null generator of $\mathcal{N}$ then we also defined a notion of null second fundamental form $\underline{\chi}$ of $\mathcal{N}$ and a smooth function $\underline{\theta}$ on $\mathcal{N}$. However, by Remark 4.17 those two definitions coincide and we may use both without ambiguity.

(ii) For any smooth, positive function $a \in C^\infty(\Sigma)$ we observe that $\left\{\underline{L}_a := a\underline{L}, L_a := \frac{1}{a}L\right\}$ is a null frame of $T^\perp\Sigma$ that retains the time orientation of $\underline{L}$ and $L$. Considering the function $\varphi$ such that $a = e^\varphi$, and the orthonormal frames $e_0 := \frac{1}{2}(L-\underline{L})$, $e_1 := \frac{1}{2}(L+\underline{L})$, $e_0^a := \frac{1}{2}(L_a - \underline{L}_a)$, $e_1^a := \frac{1}{2}(L_a + \underline{L}_a)$, we find

$$e_0^a = \cosh(\varphi)e_0 - \sinh(\varphi)e_1,$$
$$e_1^a = \cosh(\varphi)e_1 - \sinh(\varphi)e_0.$$

In this way, we can identify $a$ at each point $p \in \Sigma$ with a unqiue Lorentz boost in $\mathbb{R}^{1,1}$. Thus, we will call $a$ a *boost of the null frame* and $\{\underline{L}_a, L_a\}$ a *boosted null frame*. If again $(\Sigma, \gamma)$ is contained in a null hypersurface $\mathcal{N}$ such that $\underline{L}$ is the null generator of $\mathcal{N}$, and we now consider a non-vanishing function $a \in C^\infty(\mathcal{N})$ as in Subsection 4.7, then for any cross section $\Sigma = \Sigma_\omega$ we similarly find that $\left\{\underline{L}_a := a\underline{L}, L_a := \frac{1}{a}L_\omega\right\}$ is a null frame of $T^\perp\Sigma$, where $L_\omega$ is defined as in Proposition 4.16. Note however, that the function $\omega$ is defined with respect to the original background foliation with respect $\underline{L}$.

(iii) It is easy to check that the null second fundamental forms $\underline{\chi}$, $\chi$ and null expansions $\underline{\theta}$, $\theta$ for a given null frame $\{\underline{L}, L\}$ transform under a boost $a$ as

$$\underline{\chi}_a = a\underline{\chi} \qquad \chi_a = \frac{1}{a}\chi,$$
$$\underline{\theta}_a = a\underline{\theta} \qquad \theta_a = \frac{1}{a}\theta,$$

where $\underline{\chi}_a$, $\chi_a$, and $\underline{\theta}_a$, $\theta_a$ denote the null second fundamental forms and null expansions with respect to the boosted null frame $\{\underline{L}_a, L_a\}$, respectively.

On the other hand, for any given orthonormal frame $\{e_0, e_1\}$ the null vector fields $L := e_1 + e_0$, $\underline{L} := e_1 - e_0$ form a null frame with the above properties. Note that if $(\Sigma, \gamma)$ is contained in a spacelike hypersurface $(M, g)$ as above, then

$$\theta = H + P, \tag{21}$$

$$\underline{\theta} = H - P, \tag{22}$$

and we can thus formally define the expansions of a hypersurface $(\Sigma, \gamma)$ in an initial data set $(M, g, K)$ via the above relations. Similar to the recent work of Cederbaum–Sakovich [24], asymptotic foliations of constant expansion have been constructed by Metzger [61] in asymptotically flat initial data sets. Both foliations will generically only agree in time-symmetry, where they both agree with an asymptotic foliation of constant mean curvature surfaces first constructed by Huisken–Yau [54].

However, note that if we consider a surface $\Sigma$ that is the smooth intersection of two spacelike hypersurfaces in an ambient spacetime, then by Remark 4.19 (ii) the expansions $\underline{\theta}$, $\theta$ defined on the spacelike hypersurfaces will in general not agree with each other. However, as the Lorentzian length of the mean curvature vector $\mathcal{H}^2$ of $\Sigma$ is a Lorentz invariance, it will not depend on the choice of hypersurface and will in fact not depend on the causal character of any hypersurface containing $\Sigma$, compare also Equation (20). Due to this frame independence, $\mathcal{H}^2$ is an appealing property to study both from a mathematical and physical perspective. In particular, one is interested in the properties of surfaces of constant spacetime mean curvature.

**Remark 4.20.** We would like to emphasize again that despite the suggestive power of 2 as an exponent, $\mathcal{H}^2$ denotes the Lorentzian length of the mean curvature vector and may thus be (locally) negative. In General Relativity, surfaces with $\mathcal{H}^2 < 0$ are called *trapped* as the area of the surfaces decreases along all future causal directions if $\vec{\mathcal{H}}$ is future pointing. Heuristically, one thus expects such surfaces to shrink along the movement of future directions and remain contained within a spatially compact region. However, if we consider surfaces sufficiently close to a coordinate sphere in an asymptotically flat initial data set $(M, g, K)$, then $\mathcal{H}^2 > 0$ sufficiently far out in the asymptotic region. Examples of such surfaces are the leaves of the foliations of Cederbaum–Sakovich [24] and Metzger [61].

In the special case that $H \pm P = 0$, we call a surface a *marginally outer trapped* or *marginally inner trapped surface*, or MOTS or MITS for short, respectively. Such surfaces are exactly the critical points of the area functional in the null directions $L$ and $\underline{L}$, respectively, cf. Proposition 4.27 (i), and they are the models of apparent horizons for both initial data sets and null hypersurfaces and thus studied exhaustively in particular in the context of the Penrose conjecture, see for example [10, 58, 63] and [11, 60, 67, 69]. Further, we call a surface in an initial data set a *generalized apparent horizon* if $\mathcal{H}^2 = 0$ along the surface. By Equation (20), we note that any MOTS or MITS is a generalized apparent horizon, but the converse is not true and a Penrose inequality formulated for generalized apparent horizons seems to be false in general, see Carrasco–Mars [18].

Note that for a codimension-2 surface, $\vec{\mathrm{II}}$ does not fully encode all extrinsic geometric information, as one has to consider how any choice of frame of the normal space changes along the surface in tangential directions. To this end, we define the *connection* 1*-form* or *torsion* $\zeta$ for a given null frame $\{\underline{L}, L\}$ defined as

$$\zeta(V) := \frac{1}{2}\mathfrak{g}({}^{\mathfrak{g}}\nabla_V \underline{L}, L). \tag{23}$$

**Remark 4.21.** Note that up to possibly a sign, $\zeta$ agrees with the usual notion of torsion defined via a timelike unit normal $e_0$ and a spacelike unit normal $e_1$, where the relation between the two basis of the normal space is exactly as in Remark 4.19 (ii).

Returning our considerations to the case that $\Sigma$ is contained in a given null hypersurface, we know that $\Sigma$ can be written as a graph with respect to a given background foliation, cf. Subsection 4.7. Similar to Proposition 4.16 indeed all extrinsic curvature quantities can be computed with respect to this background foliation. Here, we will state everything with respect to the function $\omega$ and the respective tensors pushed forward onto $S_0$ via the projection $\pi$ as in Remark 4.17 (ii).

**Proposition 4.22.** *Let $(\Sigma, \gamma)$ be a spacelike cross section of a given null hypersurface $\mathcal{N}$ with null generator $\underline{L}$ as in Subsection 4.7. Let $\{\Sigma_r\}$ be a given background foliation with respect to $\underline{L}$, and let $\chi_r$ and $\zeta_r$ denote the push forwards onto $S_0$ via $\pi$ of the null second fundamental form and connection 1-form of $\Sigma_r$ with respect to the uniquely determined null normal vector field $L_r$ of $\Sigma_r$ such that $\mathfrak{g}(\underline{L}, L_r) = 2$ along $\Sigma_r$, respectively. Then $\Sigma$ can be written as a graph $\Sigma_\omega$ with respect to a smooth function $\omega$ on $S_0$, and for all $p \in \Sigma$*

$$\begin{aligned}
\chi_p(\widetilde{X}, \widetilde{Y}) =& (\chi_{\omega(q)})_q(X, Y) - 2\left(\mathrm{Hess}_{\gamma_\omega}\omega\right)_q(X, Y) + \left|{}^{\gamma_\omega}\nabla\omega\right|^2(q)(\underline{\chi}_\omega)_q(X, Y) \\
&- 2\kappa(p)X(\omega)Y(\omega) - 2X(\omega)(\zeta_{\omega(q)})_q(Y) - 2Y(\omega)(\zeta_{\omega(q)})_q(X), \\
\zeta_p(\widetilde{X}) =& (\zeta_{\omega(q)})_q(X) - (\underline{\chi}_\omega)_q(X, {}^{\gamma_\omega}\nabla\omega(q)) + \kappa(p)X(\omega),
\end{aligned}$$

*for tangent vectors $\widetilde{X}, \widetilde{Y} \in T_p\Sigma$, where $q = \pi(p) \in S_0$, and $X, Y \in T_qS_0$ are defined via the isomorphism in Remark 4.17 (ii), $\kappa$ is given by Equation* (14)*, and ${}^{\gamma_\omega}\nabla$,* $\mathrm{Hess}_{\gamma_\omega}$ *denote the gradient and Hessian with respect to the metric $\gamma_\omega$ on $S_0$, respectively. Similar to Remark 4.17 (ii), we will identify $\chi$, $\zeta$ with tensors $\chi_\omega$, $\zeta_\omega$ on $S_0$ via the above relations.*

The Proposition follows directly from Proposition 1 in [59]. Note the different sign conventions $k = -\underline{L}$, $s_l = -\zeta$. Note that together with Proposition 4.16, Proposition 4.22 shows that all geometric information of a spacelike cross section can be directly computed from a given background foliation.

We close this section by proving the Gauss- and Codazzi Equations for a codimension-2 surface $(\Sigma, \gamma)$ in an ambient spacetime $(\mathfrak{M}, \mathfrak{g})$. Here, we will express the equations with respect to a null frame of the given surface. A straightforward computation now yields the following Lemma:

**Lemma 4.23.** *Let $(x^i)$ denote local coordinates of $(\Sigma, \gamma)$. Then*

$$
\begin{aligned}
{}^{\mathfrak{g}}\nabla_{\partial_i}\partial_j &= {}^{\gamma}\nabla_{\partial_i}\partial_j - \frac{1}{2}\underline{\chi}_{ij}L - \frac{1}{2}\chi_{ij}\underline{L}, \\
{}^{\mathfrak{g}}\nabla_{\partial_i}\underline{L} &= \underline{\chi}_i^j\partial_j + \zeta(\partial_i)\underline{L}, \\
{}^{\mathfrak{g}}\nabla_{\partial_i}L &= \chi_i^j\partial_j - \zeta(\partial_i)L.
\end{aligned}
$$

With this, we obtain the following Gauss- and Codazzi Equations:

**Proposition 4.24** (Gauss Equations II)**.** *Let $(x^i)$ denote local coordinates of $(\Sigma, \gamma)$. Then*

$$
\begin{aligned}
\mathfrak{Rm}_{ijkl} &= \mathrm{Rm}_{ijkl} - \frac{1}{2}\chi_{jl}\underline{\chi}_{ik} - \frac{1}{2}\underline{\chi}_{jl}\chi_{ik} + \frac{1}{2}\chi_{jk}\underline{\chi}_{il} + \frac{1}{2}\underline{\chi}_{jk}\chi_{il}, \\
\mathfrak{Ric}_{ik} - \frac{1}{2}\mathfrak{Rm}_{i\underline{L}kL} - \frac{1}{2}\mathfrak{Rm}_{iLk\underline{L}} &= \mathrm{Ric}_{ik} - \frac{1}{2}\theta\underline{\chi}_{ik} - \frac{1}{2}\underline{\theta}\chi_{ik} + \frac{1}{2}(\chi\cdot\underline{\chi})_{ik} + \frac{1}{2}(\underline{\chi}\cdot\chi)_{ik}, \\
\mathfrak{R} - 2\mathfrak{Ric}(L,\underline{L}) + \frac{1}{2}\mathfrak{Rm}(\underline{L},L,L,\underline{L}) &= \mathrm{R} - \mathcal{H}^2 + |\vec{\mathrm{II}}|^2.
\end{aligned}
$$

*Proof.* Note that once the first equality is established, the others follow by taking a trace over the $j, l$ and $i, k$ entries, respectively. Using the identities in Lemma 4.23, we get

$$
\begin{aligned}
\mathfrak{Rm}_{ijkl} &= \mathfrak{g}\left({}^{\mathfrak{g}}\nabla_{\partial_i}{}^{\mathfrak{g}}\nabla_{\partial_j}\partial_l - {}^{\mathfrak{g}}\nabla_{\partial_j}{}^{\mathfrak{g}}\nabla_{\partial_i}\partial_l, \partial_k\right) \\
&= \mathfrak{g}\left({}^{\mathfrak{g}}\nabla_{\partial_i}\left(\nabla_{\partial_j}\partial_l - \frac{1}{2}\underline{\chi}_{jl}L - \frac{1}{2}\chi_{jl}\underline{L}\right) - {}^{\mathfrak{g}}\nabla_{\partial_j}\left(\nabla_{\partial_i}\partial_l - \frac{1}{2}\underline{\chi}_{il}L - \frac{1}{2}\chi_{il}\underline{L}\right), \partial_k\right) \\
&= \mathrm{Rm}_{ijkl} + \mathfrak{g}\left({}^{\mathfrak{g}}\nabla_{\partial_j}\left(\frac{1}{2}\underline{\chi}_{il}L + \frac{1}{2}\chi_{il}\underline{L}\right) - {}^{\mathfrak{g}}\nabla_{\partial_i}\left(\frac{1}{2}\underline{\chi}_{jl}L + \frac{1}{2}\chi_{jl}\underline{L}\right), \partial_k\right) \\
&= \mathrm{Rm}_{ijkl} - \frac{1}{2}\chi_{jl}\underline{\chi}_{ik} - \frac{1}{2}\underline{\chi}_{jl}\chi_{ik} + \frac{1}{2}\chi_{jk}\underline{\chi}_{il} + \frac{1}{2}\underline{\chi}_{jk}\chi_{il}.
\end{aligned}
$$

□

**Proposition 4.25** (Codazzi Equations II)**.** *Let $(x^i)$ denote local coordinates of $(\Sigma, \gamma)$. Then*

$$
\begin{aligned}
\nabla_i\underline{\chi}_{jk} - \nabla_j\underline{\chi}_{ik} &= \mathfrak{Rm}_{ijk\underline{L}} - \zeta_j\underline{\chi}_{ik} + \zeta_i\underline{\chi}_{jk}, \\
\nabla_i\chi_{jk} - \nabla_j\chi_{ik} &= \mathfrak{Rm}_{ijkL} + \zeta_j\chi_{ik} - \zeta_i\chi_{jk}.
\end{aligned}
$$

*Proof.* Using Lemma 4.23, we see that

$$\begin{aligned}
\mathfrak{Rm}_{ijk\underline{L}} &= \mathfrak{g}\left({}^{\mathfrak{g}}\nabla_{\partial_i}{}^{\mathfrak{g}}\nabla_{\partial_j}\underline{L} - {}^{\mathfrak{g}}\nabla_{\partial_i}{}^{\mathfrak{g}}\nabla_{\partial_j}\underline{L}, \partial_k\right) \\
&= \mathfrak{g}\left({}^{\mathfrak{g}}\nabla_{\partial_i}\left(\underline{\chi}_j^l\partial_l + \zeta_j\underline{L}\right) - {}^{\mathfrak{g}}\nabla_{\partial_j}\left(\underline{\chi}_i^l\partial_l + \zeta_i\underline{L}\right), \partial_k\right) \\
&= \nabla_i\underline{\chi}_{jk} - \nabla_j\underline{\chi}_{ik} + \zeta_j\underline{\chi}_{ik} - \zeta_i\underline{\chi}_{jk}.
\end{aligned}$$

A rearrangement yields the first identity. The second follows by an analogue computation. □

## 4.9 Evolution Equations

We will now state the evolution equations for a Riemannian manifold $(\Sigma, \gamma)$ along a normal direction both as a hypersurface in an ambient Riemannian manifold, and as a codimension-2 surface along a null hypersurface in an ambient spacetime.

First, let us consider the case, when $(\Sigma, \gamma)$ is an $n$-dimensional Riemannian manifold and $F_0\colon \Sigma \to M$ is a smooth embedding into a $n+1$-dimensional Riemannian manifold $(M, g)$. We now assume that there is a smooth 1-parameter family of embeddings $F\colon \Sigma\times[0,T) \to M$ such that

$$\begin{cases} F(\cdot, 0) & \equiv F_0, \\ \frac{\mathrm{d}}{\mathrm{d}t}F(x,t) & = -f(x,t)\nu(x,t), \end{cases} \tag{24}$$

for some smooth function $f\colon \Sigma \times [0,T) \to \mathbb{R}$, where $\nu(x,t)$ denotes the unit normal of $\Sigma_t := F(\Sigma, t)$ at $F(x,t)$ in $M$. We then get the following evolution equations:

**Proposition 4.26.** *Let $F : \Sigma \times [0,T) \to (M, g)$ be a smooth solution to* (24)*.*

(i) $\frac{\mathrm{d}}{\mathrm{d}t}\gamma_{ij} = -2fh_{ij}$, $\frac{\mathrm{d}}{\mathrm{d}t}\gamma^{ij} = 2fh^{ij}$, $\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{d}\mu = -fH\,\mathrm{d}\mu$,

(ii) $\frac{\mathrm{d}}{\mathrm{d}t}\nu = \nabla f$,

(iii) $\frac{\mathrm{d}}{\mathrm{d}t}h_{ij} = \operatorname{Hess} f_{ij} - fh_i^k h_{kj} + f\operatorname{Riem}_g(\partial_j, \nu, \partial_i, \nu)$

(iv) $\frac{\mathrm{d}}{\mathrm{d}t}H = \Delta f + f\left(\operatorname{Ric}_g(\nu,\nu) + |h|^2\right)$,

*where $\gamma$, $h$, and $H$ denote the induced metric, second fundamental form and mean curvature of $\Sigma_t$ in $M$, respectively,* $\mathrm{d}\mu$ *denotes the volume form of $\Sigma_t$, and $\nabla$,* Hess*, $\Delta$ denote the gradient, Hessian and Laplace–Beltrami operator of $\Sigma_t$, respectively.*

For a prove, we refer to the lecture notes of Huisken–Polden [52].

Let us now consider the case when $(\Sigma, \gamma)$ is 2-dimensional Riemannian manifold, $\mathcal{N}$ a null hypersurface in an ambient 4-dimensional spacetime $(\mathfrak{M}, \mathfrak{g})$, and there exists a smooth embedding $F_0 \colon \Sigma \to \mathcal{N}$. We now assume that there exists a smooth 1-parameter family of embeddings $F \colon \Sigma \times [0, T) \to \mathcal{N}$ such that

$$\begin{cases} F(\cdot, 0) & \equiv F_0, \\ \frac{\mathrm{d}}{\mathrm{d}t} F(x,t) & = \varphi(x,t)\underline{L}(x,t), \end{cases} \tag{25}$$

for some smooth function $\varphi \colon \Sigma \times [0, T) \to \mathbb{R}$, where $\underline{L}(x,t)$ is the null generator of $\mathcal{N}$ at $F(x,t)$. We then get the following evolution equations, also known as the Raychaudhuri optical equations:

**Proposition 4.27.** *Let $F : \Sigma \times [0, T) \to (M, g)$ be a smooth solution to* (25)*.*

(i) $\frac{\mathrm{d}}{\mathrm{d}t}\gamma_{ij} = 2\varphi\underline{\chi}_{ij}$, $\frac{\mathrm{d}}{\mathrm{d}t}\gamma^{ij} = -2\varphi\underline{\chi}^{ij}$, $\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{d}\mu = \varphi\underline{\theta}\,\mathrm{d}\mu$,

(ii) $\frac{\mathrm{d}}{\mathrm{d}t}\underline{\chi}_{ij} = \varphi\kappa\underline{\chi}_{ij} + \varphi \cdot (\underline{\chi})^2_{ij} - \varphi\mathfrak{Rm}(\underline{L}, \partial_i, \underline{L}, \partial_j)$,

(iii) $\frac{\mathrm{d}}{\mathrm{d}t}L = -2\nabla\varphi - 2\varphi\zeta^k\partial_k - \varphi\kappa L$,

(iv) $$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\chi_{ij} = &-2\mathrm{Hess}_{ij}\varphi - 2(\,\mathrm{d}\varphi_i \otimes \zeta_j + \,\mathrm{d}\varphi_j \otimes \zeta_i) \\ &-\varphi\left(\kappa\chi_{ij} - (\chi\underline{\chi})_{ij} + 2\nabla_i\zeta_j + 2\zeta_i \otimes \zeta_j + \mathfrak{Rm}(\underline{L}, \partial_i, L, \partial_j)\right), \end{aligned}$$

(v) $\frac{\mathrm{d}}{\mathrm{d}t}\underline{\theta} = \kappa\varphi\underline{\theta} - \varphi(|\underline{\chi}|^2 + \mathfrak{Ric}(\underline{L}, \underline{L}))$,

(vi) $\frac{\mathrm{d}}{\mathrm{d}t}\theta = -2\Delta\varphi - 4\gamma(\nabla\varphi, \zeta) - \varphi\kappa\theta - \varphi\left(|\vec{\mathrm{II}}|^2 + \mathfrak{Ric}(\underline{L}, L) - \frac{1}{2}\mathfrak{Rm}(\underline{L}, L, L, \underline{L}) + 2\,\mathrm{div}\,\zeta + 2\,|\zeta|^2\right)$,

*where $L$ is the unique normal null vector field of $\Sigma_t$ such that $\mathfrak{g}(\underline{L}, L) = 2$, $\kappa$ as in Equation* (14)*, $\gamma$, $\underline{\chi}$, $\chi$, $\underline{\theta}$, $\theta$ and $\zeta$ denote the induced metric, null second fundamental forms, null expansions, and connection* 1*-form of $\Sigma_t$ with respect to $\{\underline{L}, L\}$, respectively, and $\nabla$,* Hess *and $\Delta$ denote the Gradient, Hessian, and Laplace–Beltrami operator on $\Sigma_t$, respectively.*

**Remark 4.28.** Note that for arbitrary $\varphi$, the evolution equations corresponding to (25) will in general depend on the non-unique choice of null generator $\underline{L}$ of $\mathcal{N}$. In particular, if we want to study geometric flows along a null hypersurface, the speed $\varphi$ of the flow should be defined such that (25) is independent of the choice of null generator. Note that what we will call null mean curvature flow below in Section 7 is an example of a geometric flow that is indeed independent of the choice of null generator, cf. [67].

*Proof.*

(i) We compute

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\gamma_{ij} &= {}^{\mathfrak{g}}\nabla_{\varphi\underline{L}}\mathfrak{g}(\partial_i,\partial_j)\\
&= \mathfrak{g}({}^{\mathfrak{g}}\nabla_{\varphi\underline{L}}\partial_i,\partial_j) + \mathfrak{g}(\partial_i,{}^{\mathfrak{g}}\nabla_{\varphi\underline{L}}\partial_j)\\
&= \mathfrak{g}({}^{\mathfrak{g}}\nabla_{\partial_i}\varphi\underline{L},\partial_j) + \mathfrak{g}(\partial_i,{}^{\mathfrak{g}}\nabla_{\partial_j}\varphi\underline{L})\\
&= 2\varphi\underline{\chi}_{ij}.
\end{aligned}
$$

The evolution for the inverse of the induced metric follows by the general identity

$$
\frac{\mathrm{d}}{\mathrm{d}t}\gamma^{ij} = -\gamma^{ik}\gamma^{jl}\frac{\mathrm{d}}{\mathrm{d}t}\gamma_{kl}.
$$

Finally, note that $\vec{\mathcal{H}} = -\frac{1}{2}\theta\underline{L} - \frac{1}{2}\underline{\theta}L$, so by the first variation of area

$$
\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{d}\mu = -\mathfrak{g}(\vec{\mathcal{H}},\varphi\underline{L})\,\mathrm{d}\mu = \varphi\underline{\theta}\,\mathrm{d}\mu.
$$

(ii) Direct computation gives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\underline{\chi}_{ij} &= -{}^{\mathfrak{g}}\nabla_{\varphi\underline{L}}\mathfrak{g}({}^{\mathfrak{g}}\nabla_{\partial_i}\partial_j,\underline{L})\\
&= -\varphi\mathfrak{g}({}^{\mathfrak{g}}\nabla_{\partial_i}\partial_j,{}^{\mathfrak{g}}\nabla_{\underline{L}}\underline{L}) - \mathfrak{g}({}^{\mathfrak{g}}\nabla_{\varphi\underline{L}}{}^{\mathfrak{g}}\nabla_{\partial_i}\partial_j,\underline{L})\\
&= \varphi\kappa\underline{\chi}_{ij} - \mathfrak{g}({}^{\mathfrak{g}}\nabla_{\partial_i}{}^{\mathfrak{g}}\nabla_{\partial_j}(\varphi\underline{L}),\underline{L}) - \varphi\mathfrak{Rm}(\underline{L},\partial_i,\underline{L},\partial_L)\\
&= \varphi\kappa\underline{\chi}_{ij} + \mathfrak{g}({}^{\mathfrak{g}}\nabla_{\partial_j}(\varphi\underline{L}),{}^{\mathfrak{g}}\nabla_{\partial_i}\underline{L}) - \varphi\mathfrak{Rm}(\underline{L},\partial_i,\underline{L},\partial_L)\\
&= \varphi\kappa\underline{\chi}_{ij} + \varphi(\underline{\chi})^2_{ij} - \varphi\mathfrak{Rm}(\underline{L},\partial_i,\underline{L},\partial_L),
\end{aligned}
$$

where we used that $[\varphi\underline{L},\partial_i] = 0$ in the third line as $F$ induces a (local) background foliation on $\mathcal{N}$. Taking a trace, (ii) together with (i) yields (v).

(iii) Since we have

$$
\begin{aligned}
\mathfrak{g}(\frac{\mathrm{d}}{\mathrm{d}t}L, L) &= 0,\\
\mathfrak{g}(\frac{\mathrm{d}}{\mathrm{d}t}L, \underline{L}) &= \varphi\mathfrak{g}({}^{\mathfrak{g}}\nabla_{\underline{L}}L, \underline{L})\\
&= -\varphi\mathfrak{g}(L, {}^{\mathfrak{g}}\nabla_{\underline{L}}\underline{L})\\
&= -2\varphi\kappa,\\
\mathfrak{g}(\frac{\mathrm{d}}{\mathrm{d}t}L, \partial_i) &= \mathfrak{g}({}^{\mathfrak{g}}\nabla_{\varphi\underline{L}}L, \partial_i)\\
&= -\mathfrak{g}(L, {}^{\mathfrak{g}}\nabla_{\partial_i}\varphi\underline{L})\\
&= -2\partial_i\varphi - 2\varphi\zeta(\partial_i),
\end{aligned}
$$

we can conclude (iii).

(iv) Note that $\mathrm{Hess}\varphi_{ij} = \gamma(\nabla_{\partial_i}\nabla\varphi, \partial_j)$, so from (iii), we can compute that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\chi_{ij} =&\mathfrak{g}({}^{\mathfrak{g}}\nabla_{\varphi\underline{L}}{}^{\mathfrak{g}}\nabla_{\partial_i}L, \partial_j) + \mathfrak{g}({}^{\mathfrak{g}}\nabla_{\partial_i}L, {}^{\mathfrak{g}}\nabla_{\varphi\underline{L}}\partial_j)\\
=&\mathfrak{g}({}^{\mathfrak{g}}\nabla_{\partial_i}{}^{\mathfrak{g}}\nabla_{\varphi\underline{L}}L, \partial_j) + \mathfrak{Rm}(\varphi\underline{L}, \partial_i, \partial_j, L) + \mathfrak{g}({}^{\mathfrak{g}}\nabla_{\partial_i}L, {}^{\mathfrak{g}}\nabla_{\partial_j}\varphi\underline{L})\\
=&\mathfrak{g}\left({}^{\mathfrak{g}}\nabla_{\partial_i}\left(-2\nabla\varphi - 2\varphi\zeta^l\partial_l - \varphi\kappa L\right), \partial_j\right) - \varphi\mathfrak{Rm}(\underline{L}, \partial_i, L, \partial_j)\\
&+ \mathfrak{g}(\underline{\chi}_i^k\partial_k - \zeta_i L, \partial_j\varphi\underline{L} + \varphi\underline{\chi}_j^l\partial_l + \zeta_j\underline{L})\\
=&-2\mathrm{Hess}\varphi_{ij} - 2(\mathrm{d}\varphi_i\otimes\zeta_j + \mathrm{d}\varphi_j\otimes\zeta_i) - \varphi\kappa\chi_{ij}\\
&- \varphi\left(2\nabla_i\zeta_j + 2\zeta_i\otimes\zeta_j + \mathfrak{Rm}(\underline{L}, \partial_i, L, \partial_j) - (\chi\underline{\chi})_{ij}\right),
\end{aligned}
$$

where we again used that $[\varphi\underline{L}, \partial_i] = 0$ in the second line. Taking a trace, (i) and (iv) yield (vi).

□

**Remark 4.29.** Note that we only consider the evolution in directions normal to the surface. However, if we consider an additional tangential component in (24) and (25) this exactly corresponds to a change of coordinates on $\Sigma$ in time. Thus, by commuting $F$ with an appropriate smooth 1-parameter family of diffeomorphisms $\Phi_t\colon \Sigma \to \Sigma$, one can always assume that (24) and (25) are satisfied as above.

## 4.10 Spherical harmonics

We briefly recall the *spherical harmonics*, which are the eigenfunctions of the Laplace operator on the standard round sphere $(\mathbb{S}^2, \mathrm{d}\Omega^2)$. For detailed information and to the interested reader, we refer to [38]. In particular Chapters 1 and 4 in [38].

Considering a spherical harmonic $Y\colon \mathbb{S}^2 \to \mathbb{R}$, it will satisfy the equation

$$\Delta_{\mathbb{S}^2} Y + \lambda Y = 0$$

for some $\lambda \in \mathbb{R}$ by definition. First note that as rotations $R$ on $\mathbb{R}^3$ act as isometries on $\mathbb{S}^2$ one can check that $Y \circ R|_{\mathbb{S}^2}$ is also a spherical harmonic with respect to the same eigenvalue $\lambda$. Observe further that for $\lambda = 0$, the solutions are exactly the harmonic functions on $\mathbb{S}^2$ and hence the spherical harmonics for $\lambda = 0$ are the constant functions on $\mathbb{S}^2$. In general, one can check that this equation is intricately connected to the general Legendre equation for which the solutions are the Legendre Polynomials. In particular, there exists $l \in \mathbb{N}_0$ such that $\lambda = \lambda_l := -l(l+1)$, and the space of eigenfunctions to the eigenvalue $\lambda_l$ has dimension $2l+1$. Let $Y_l^k$ for $k = -l \ldots, l$ denote an orthonormal basis of the space of spherical harmonics with eigenvalue $\lambda_l$ with respect to the $L^2$-norm on $\mathbb{S}^2$. Moreover, one can check that

$$\int_{\mathbb{S}^2} Y_l^k Y_{l'}^{k'} \,\mathrm{d}\mu = 0$$

for $l \neq l'$. Hence, the functions $Y_l^k$ are $L^2$-orthogonal and in fact form an orthonormal basis of the smooth functions on $\mathbb{S}^2$. Thus, for any function $f$ on $\mathbb{S}^2$ one can write

$$f = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} a_{l,k} Y_l^k,$$

where the constants $a_{l,k} \in \mathbb{R}$ are given by

$$a_{l,k} := \int_{\mathbb{S}^2} f Y_l^k \,\mathrm{d}\mu.$$

As $Y_0^0 = \frac{1}{4\pi}$, $f_0 := f_{0,0}$ is the mean value of $f$ on $\mathbb{S}^2$, i.e.,

$$f_0 = \fint_{\mathbb{S}^2} f \,\mathrm{d}\mu = \frac{1}{4\pi} \int_{\mathbb{S}^2} f \,\mathrm{d}\mu.$$

Moreover, one can further check that all spherical harmonics with respect to the eigenvalue $\lambda_l$ can be obtained by restricting the homogenous polynomials of degree $l$ on $\mathbb{R}^3$ to $\mathbb{S}^2$. In particular, the Cartesian coordinate functions $x^i$ for $i = 1, 2, 3$ restrict to spherical harmonics

$f_i = x^i|_{\mathbb{S}^2}$ on $\mathbb{S}^2$ with respect to the eigenvalue $-2$. Here, we call the functions $f_i$ the *first spherical harmonics*, which form an $L^2$-orthogonal basis of the spherical harmonics with respect to the eigenvalue $-2$, and direct computation gives $||f_i||^2_{L^2(\mathbb{S}^2)} = \frac{4\pi}{3}$. If we consider spherical coordinates $r$, $\theta$, $\varphi$ on $\mathbb{R}^3$ with $(r, \theta, \varphi) \in (0, \infty) \times (0, \pi) \times (0, 2\pi)$ such that

$$\begin{aligned} x^1 =& r \sin\theta \sin\varphi, \\ x^2 =& r \sin\theta \cos\varphi, \\ x^3 =& r \cos\theta, \end{aligned} \tag{26}$$

then we note that

$$\begin{aligned} f_1 &= \sin\varphi \sin\theta, \\ f_2 &= \cos\varphi \sin\theta, \\ f_3 &= \cos\theta, \end{aligned} \tag{27}$$

where $\theta$, $\varphi$ now denote coordinates on $\mathbb{S}^2$. Lastly, we remark that under isometries of $\mathbb{S}^2$, which are exactly the rotations of the ambient space $\mathbb{R}^3$, the first spherical harmonics $f_i$ are transformed into spherical harmonics $\widetilde{f}_i$, which arise exactly as the first spherical harmonics with respect to the rotated Cartesian coordinates $\widetilde{x}^i$.

# 5 Inverse space-time mean curvature flow in asymptotically flat initial data sets

In this section we study inverse space-time mean curvature flow in asymptotically flat initial data sets as a generalization to inverse mean curvature flow in the time–symmetric case. We say a family of hypersurfaces $F\colon \Sigma \times [0,T) \to M$ in an initial data set $(M,g,K)$ is a (smooth) solution to *inverse space-time mean curvature flow (STIMCF)*, if

$$\frac{\mathrm{d}}{\mathrm{d}t}F = \frac{1}{\sqrt{H^2 - P^2}}\nu, \tag{28}$$

where $H$ is the mean curvature of $\Sigma_t = F(t,\Sigma)$ and $P = \operatorname{tr}_M K - K(\nu,\nu)$ is as in Subsection 4.8. Hence, the inverse of the speed of the flow is exactly given as the squareroot of the spacetime mean curvature $\mathcal{H}^2$, and the flow reduces to inverse mean curvature flow in the time-symmetric case.

In time-symmetry, Huisken–Ilmanen [50] used a weak notion of inverse mean curvature flow to give a proof of the Riemannian Penrose Inequality for connected apparent horizons. A generalization of inverse mean curvature flow has also been pursued by Moore [63], and the strategy presented here closely follows her analysis of inverse null mean curvature flow, as well as the techniques developed in [50]. Unlike inverse mean curvature flow, weak solutions of both inverse null mean curvature flow studied by Moore and inverse space-time mean curvature flow presented here satisfy an anisotropic comparison principle requiring a notion of unit normal even across jump regions, where the time-of-arrival function remains constant. The main result of this section is an existence theorem for weak solutions to inverse space-time mean curvature flow in maximal, asymptotically flat initial data sets (Theorem 5.33).

This section is based on joint work with Gerhard Huisken [53]. Additionally, Subsections 5.1 and 5.2 are also contained in the master thesis of the author [87], apart from a maximal existence result for the smooth flow (Theorem 5.2) in Subsection 5.1 due to Huisken. We refer to Overview and Contributions for a precise listing of the individual contributions.

As a consequence, the properties of smooth solutions discussed in Subsection 5.1 and the concept of elliptic regularisation introduced in Subsection 5.2 are only listed in brief detail for the convenience of the reader and the statements in these subsections are given without proof, where we refer to [53, 87] for detailed proofs.

The rest of the section is structured as follows: In Subsection 5.3 we analyse the limiting behaviour of the translating graphs defined via the elliptic regularisation and construct a notion of unit normal across jump regions. We define the concept of weak solutions in Subsection 5.4 and prove the main result in Subsection 5.5. Finally, we discuss the formation of jump regions in Subsection 5.6 and discuss the asymptotics of solutions in 5.7.

## 5.1 The smooth flow

We first concentrate on the properties of smooth solutions of (28).

Assuming that $\sqrt{H^2-P^2}\Big|_{\Sigma_0} > 0$, the flow is parabolic and the surfaces $\Sigma_t$ expand smoothly as long as their speed remains bounded, see Theorem 5.2. Additionally, we provide an upper bound on the inverse speed restricted to a ball with sufficiently small radius depending on the geometry of $(M, g, K)$, see Theorem 5.3. This will later serve as an interior gradient estimate in the level-set formulation of the flow.

**Lemma 5.1.** *Smooth solutions of* (28) *with* $\Phi := \sqrt{H^2-P^2} > 0$ *and* $\Psi := \frac{1}{\Phi}$ *satisfy the following evolution equations:*

*(i)* $\frac{\mathrm{d}}{\mathrm{d}t}\gamma_{ij} = 2\Psi h_{ij}$, $\frac{\mathrm{d}}{\mathrm{d}t}\gamma^{ij} = -2\Psi h^{ij}$, $\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{d}\mu_\gamma = \frac{H}{\sqrt{H^2-P^2}}\,\mathrm{d}\mu_\gamma$,

*(ii)* $\frac{\mathrm{d}}{\mathrm{d}t}\nu = -\nabla\Psi$,

*(iii)* $\frac{\mathrm{d}}{\mathrm{d}t}h_{ij} = -\operatorname{Hess}\Psi_{ij} + \Psi h_i^k h_{kj} - \Psi\operatorname{Riem}_g(\partial_i, \nu, \partial_j, \nu)$,

*(iv)* $\frac{\mathrm{d}}{\mathrm{d}t}H = -\Delta\Psi - \Psi\left(\operatorname{Ric}_g(\nu,\nu) + |h|^2\right)$,

*(v)* $\frac{\mathrm{d}}{\mathrm{d}t}P = \Psi\operatorname{tr}_\gamma D_\nu K + 2K_{\nu i}\nabla^i\Psi$,

*(vi)* $\frac{\mathrm{d}}{\mathrm{d}t}\Phi = \Phi^{-2}\left(\frac{H}{\Phi}\Delta\Phi - \frac{2H}{\Phi^2}|\nabla\Phi|^2 - H\left(\operatorname{Ric}_g(\nu,\nu) + |h|^2\right) - P\operatorname{tr}_\gamma D_\nu K + \frac{2P}{\Phi}K_{\nu i}\nabla^i\Phi\right)$,

*where $D$ denotes the Levi-Civita connection on $M$, and we otherwise use the same conventions as in Proposition 4.26.*

Throughout this section, we will always denote the Levi–Civita connection and gradient, Hessian and Laplace–Beltrami operator as $\nabla$, Hess, and $\Delta$, respectively, for any hypersurface under consideration, and will always denote the Levi–Civita connection and gradient of an ambient manifold as $D$. For a reader familiar with [53], we would like to emphasize that this is exactly the opposite convention as in [53], but as we otherwise always adhere to the above convention for the objects on surfaces $\Sigma$, we will also do so here for the sake of consistency.

While the main aim of this paper is the construction of a global weak solution to STIMCF, we begin by proving that for smooth closed initial hypersurfaces $F_0$ with $\Phi > 0$ a smooth solution to the flow exists as long as $\Phi > 0$ remains true. A corresponding result for inverse mean curvature flow in Euclidean space was shown in [51, Corollary 2.3], for general ambient manifolds the result appears to be new also for inverse mean curvature flow.

**Theorem 5.2.** *Suppose the initial hypersurface $F_0\colon \Sigma \to M$ is a smooth immersion satisfying $\Phi > \delta_0 > 0$. Then there exists $T > 0$ depending on $\delta_0$ and the regularity of $F_0, (M, g, K)$ with a unique smooth solution $F\colon \Sigma \times [0, T) \to M$ to* (28) *on $[0, T)$. If the space-time mean curvature $\Phi$ remains bounded from below by a constant $\delta_1 > 0$ for all $t \in [0, T)$, then the solution can be extended beyond $T$. In particular, if $[0, T_{max})$ is the maximal time interval of existence for a smooth solution of* (28) *with $T_{max} < \infty$, then the speed $\Psi = 1/\Phi$ is unbounded for $t \to T_{max}$.*

We further establish an interior upper bound on $\Phi$ which will be crucial in our later construction.

**Theorem 5.3.** *Let $x \in M^{n+1}$, let $d_x$ denote the distance to $x$, and let $R > 0$ be such that $B_R(x) \subset\subset M^{n+1}$, $\mathrm{Ric}_g \geq -\frac{1}{100(n+1)R^2}$ in $B_R(x)$, and there exists a function $p \in C^2(B_R(x))$ such that*

$$p(x) = 0,\ \frac{3}{2} d_x^2 \geq p \geq d_x^2 \text{ on } B_R(x),\ \ |\nabla p|_g \leq 3 d_x,\ \text{and } \nabla^2 p \leq 3g \text{ on } B_R(x).$$

*Assuming furthermore that $H_{\Sigma_s} > 0$ for $s \in [0, t]$, there exists a constant $C(n) > 0$ depending only on the dimension, such that*

$$\sup_{F(\Sigma,[0,t])\cap B_{R/2}(x)} \Phi \leq \max\left( \max_{\Sigma_0 \cap B_R(x)} \Phi, C(n)\left( \frac{1}{R} + \sup_{\overline{B}_R(x)} |K|_g + \sup_{\overline{B}_R(x)} |DK|_g^{\frac{1}{2}} \right)\right), \tag{29}$$

*where $\Phi = \sqrt{H^2 - P^2}$.*

**Remark 5.4.** We choose $p(y) := |x - y|^2$ in flat space with $R = \infty$, but in general $R > 0$ will depend on the injectivity radius and Ricci curvature. However, as argued in the Remark to [50, Definition 3.3], each $x \in M^{n+1}$ admits a positive radius $R$, such that the assumptions are satisfied.

## 5.2 Level-set description and elliptic regularisation

To reformulate STIMCF as a level-set flow, we assume that the smooth family of hypersurfaces $\{\Sigma_t\}$ evolving by STIMCF is given as level-sets

$$\Sigma_t = \partial\{x \in M | u(x) < t\}$$

of a smooth scalar function $u\colon M \to \mathbb{R}$ with $Du \neq 0$. Note that it is always possible to construct such a function $u$ as long as the flow remains smooth and parabolic. Then

$u(y) = t$ if and only if there exists $x \in \Sigma$, such that $F(x,t) = y$, and we call $u$ the *time-of-arrival function*. Since for fixed $x \in \Sigma$, $u \circ F(x,\cdot)$ is the identity map on the existence interval of $F$ we conclude that

$$|Du| = \sqrt{H^2 - P^2}. \tag{30}$$

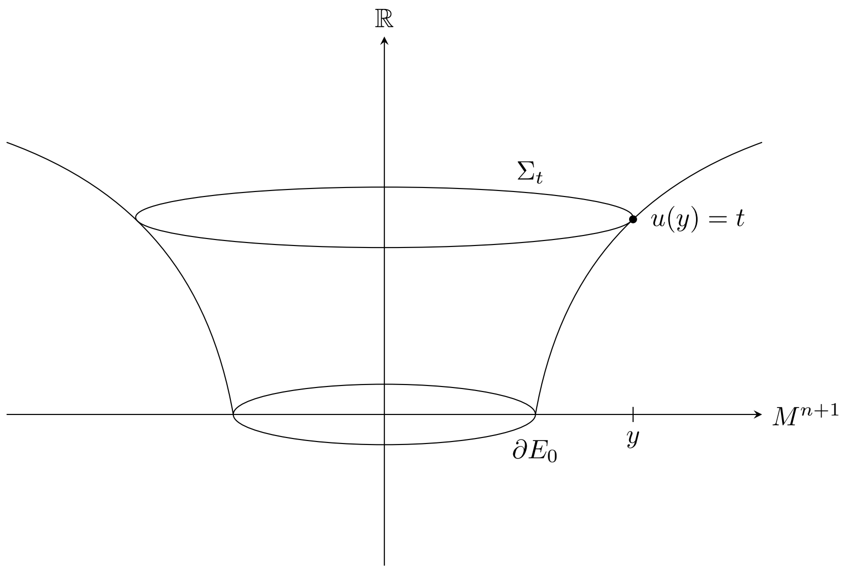


Figure 1: Time-of-arrival function $u$.

In this smooth setting we have

$$H = \operatorname{div}_g\left(\frac{Du}{|Du|_g}\right) \text{ and } P = \left(g^{\alpha\beta} - \frac{D^\alpha u D^\beta u}{|Du|_g^2}\right) K_{\alpha\beta}$$

such that (30) can be rearranged as

$$\operatorname{div}_g\left(\frac{Du}{|Du|_g}\right) = +\sqrt{|Du|_g^2 + \left(\Big(g^{\alpha\beta} - \frac{D^\alpha u D^\beta u}{|Du|_g^2}\Big) K_{\alpha\beta}\right)^2}.$$

The sign on the RHS is chosen such that STIMCF is consistent with inverse mean curvature flow in the time symmetric case and is further necessary to apply Theorem 5.3 below.

If we now assume that $\Sigma_0 = \partial E_0$, where $E_0$ is a precompact $C^2$-domain in $M$, we are led to the degenerate elliptic boundary value problem

$$\begin{cases} \operatorname{div}_g \left( \frac{Du}{|Du|_g} \right) - \sqrt{|Du|_g^2 + \left( \left( g^{\alpha\beta} - \frac{D^\alpha u D^\beta u}{|Du|_g^2} \right) K_{\alpha\beta} \right)^2} = 0, \\ u|_{\partial E_0} = 0. \end{cases} \tag{31}$$

We want to find weak solutions of (31) by elliptic regularisation in a broad class of asymptotically flat exterior regions and begin with the construction of subsolutions, motivated by the smooth spherical solutions $u(x) = n \ln \left( \frac{|x|}{R_0} \right)$ in the flat initial data set $(\mathbb{R}^{n+1}, \delta, 0)$.

**Lemma 5.5.** *Let $0 < \alpha < n$ and let $(M, g, K)$ be an asymptotically flat initial data set with asymptotic coordinate system $x : M \setminus \overline{\mathcal{B}} \to \mathbb{R}^{n+1} \setminus \overline{B_1(0)}$ as in Definition 4.14. Then, using the notation $\mathcal{O}_{R_0} = x^{-1} \left( \mathbb{R}^{n+1} \setminus \overline{B}_{R_0}(0) \right)$, there exists $R_0 = R_0(\alpha, g, K) > 1$, such that*

$$v \colon \mathcal{O}_{R_0} \to \mathbb{R},\ v(y) = \alpha \ln(|x(y)|) - \alpha \ln(R_0),$$

*is a smooth strict subsolution of (31) with $v|_{\partial \mathcal{O}_{R_0}} = 0$.*

**Remark 5.6.** Note that $v$ will remain a subsolution of the elliptic regularisation (32) defined below on compact regions for sufficiently small $\varepsilon > 0$ with respect to the same $R_0$.
Further, we would like to point out that here we use a slightly different notion of asymptotic flatness regarding the decay assumptions at infinity, cf. [53, Definition 2.1]. However, as we in fact only require the rather mild decay assumptions (64) as below in Subsection 5.7 to establish our results, we may ignore this subtlety.

Similar to the behaviour of inverse mean curvature flow, we expect solutions to form jump regions, where $Du = 0$ and (31) is not well-defined. To address this problem we use the method of *elliptic regularisation* and approximate weak solutions to inverse space-time mean curvature flow by smooth solutions of strictly elliptic equations. Let $\varepsilon > 0$ and consider the following strictly elliptic quasilinear PDE, writing now $|Du| = |Du|_g$ for simplicity,

$$\operatorname{div}_g \left( \frac{Du_\varepsilon}{\sqrt{\varepsilon^2 + |Du_\varepsilon|^2}} \right) - \sqrt{\varepsilon^2 + |Du_\varepsilon|^2 + \left( \left( g^{\alpha\beta} - \frac{D^\alpha u_\varepsilon D^\beta u_\varepsilon}{\varepsilon^2 + |Du_\varepsilon|^2} \right) K_{\alpha\beta} \right)^2} = 0. \tag{32}$$

Rescaling (32) via $\widehat{u}_\varepsilon := \frac{u_\varepsilon}{\varepsilon}$ gives

$$\operatorname{div}_g \left( \frac{D\widehat{u}_\varepsilon}{\sqrt{1 + |D\widehat{u}_\varepsilon|^2}} \right) - \sqrt{\varepsilon^2 + \varepsilon^2 |D\widehat{u}_\varepsilon|^2 + \left( \left( g^{\alpha\beta} - \frac{D^\alpha \widehat{u}_\varepsilon D^\beta \widehat{u}_\varepsilon}{1 + |D\widehat{u}_\varepsilon|^2} \right) K_{\alpha\beta} \right)^2} = 0, \tag{33}$$

which is equivalent to

$$\sqrt{\operatorname{div}_g\left(\frac{D\widehat{u}_\varepsilon}{\sqrt{1+|D\widehat{u}_\varepsilon|^2}}\right)^2-\left(\Big(g^{\alpha\beta}-\frac{D^\alpha\widehat{u}_\varepsilon D^\beta\widehat{u}_\varepsilon}{1+|D\widehat{u}_\varepsilon|^2}\Big)K_{\alpha\beta}\right)^2}=\varepsilon\sqrt{1+|D\widehat{u}_\varepsilon|^2}. \tag{34}$$

Note that the left-hand side in (34) corresponds to the square root of the spacetime mean curvature $\sqrt{\widehat{H}_\varepsilon^2-\widehat{P}^2}$ of the hypersurfaces $\widehat{\Sigma}_t^\varepsilon := \operatorname{graph}\left(\widehat{u}_\varepsilon-\frac{t}{\varepsilon}\right)$ in the initial data set $(M\times\mathbb{R}, g+\mathrm{d}z^2, \widetilde{K})$, where we extent $K$ onto $M\times\mathbb{R}$ by $\widetilde{K}_{ij} := K_{ij}$, $\widetilde{K}_{iz}=\widetilde{K}_{zz} := 0$. Hence, the downward translating graphs $\widehat{\Sigma}_t^\varepsilon$ solve (31) in $M\times\mathbb{R}$ with $U_\varepsilon(y,z)=u_\varepsilon(y)-\varepsilon z$, since $\widehat{\Sigma}_t^\varepsilon=\{U_\varepsilon(y,z)=t\}$. Equivalently, given smooth solutions $u_\varepsilon$ to (32), the hypersurfaces $\widehat{\Sigma}_t^\varepsilon$ are smooth translating solutions of STIMCF in $M\times\mathbb{R}$ with $\sqrt{\widehat{H}_\varepsilon-\widehat{P}^2}\Big|_{\widehat{\Sigma}_t^\varepsilon}>0$ along the hypersurfaces.

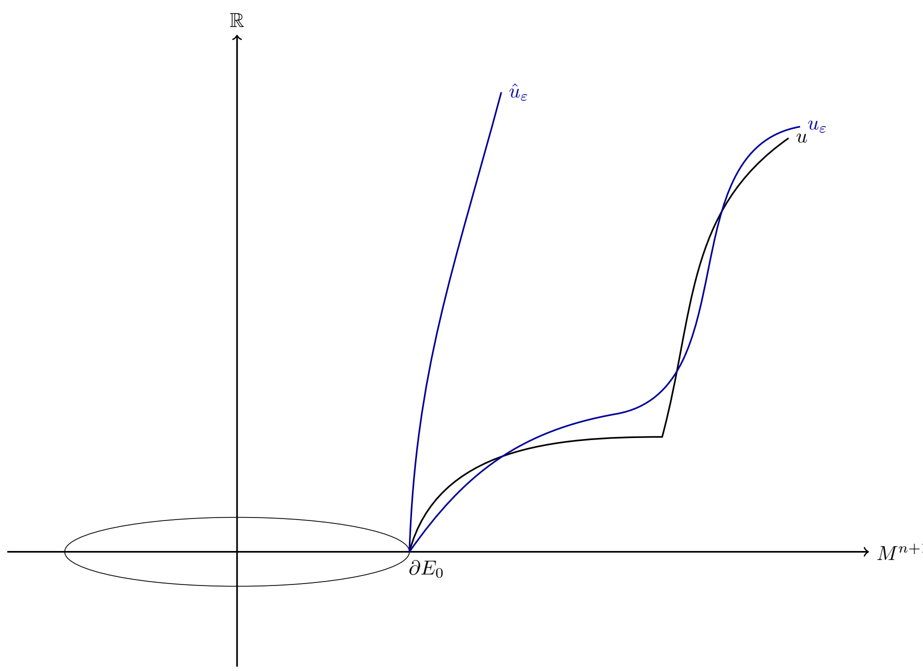


Figure 2: The time-of-arrival function $u$, the elliptic regularisation $u_\varepsilon$,and the rescaling $\hat{u}_\varepsilon$ .

We will dedicate the rest of this subsection to the existence of smooth solutions of the elliptic regularisation (32). Suppose that the initial hypersurface $\Sigma_0=\partial E_0$ is given as boundary of a precompact domain $E_0\subset M$ and let $F\subset(M\setminus E_0)$ be another precompact domain. Then a solution $u_\varepsilon$ of the regularisation (32) can only exist on $F$ if $\varepsilon > 0$ is sufficiently small, since the rescaled solution $\widehat{u}_\varepsilon$ will satisfy

$$\varepsilon\,|F| \le \int\limits_F \varepsilon\sqrt{1+|D\widehat{u}_\varepsilon|^2}\,\mathrm{d}y = \int\limits_F \operatorname{div}_g\left(\frac{D\widehat{u}_\varepsilon}{\sqrt{1+|D\widehat{u}_\varepsilon|^2}}\right)\mathrm{d}y = \int\limits_{\partial F} \frac{g(D\widehat{u}_\varepsilon,\eta)}{\sqrt{1+|D\widehat{u}_\varepsilon|^2}}\,\mathrm{d}\sigma \le |\partial F|\,,$$

where $\eta$ is the unit normal to $\partial U$. Therefore we need to specify boundary data for (32) on precompact domains $\Omega_L$ exhausting $M \setminus E_0$ as $L \to \infty$: We use the subsolution $v$ as in Lemma 5.5 to define the domains $F_L = \{x \in M : x \in M \setminus \mathcal{O}_{R_0} \text{ or } v(x) < L\}$ for all $0 \le L < \infty$. On $\Omega_L := F_L \setminus \overline{E}_0$ we consider the boundary value problem

$$\begin{cases} E^\varepsilon u_\varepsilon = 0 & \text{on } \Omega_L,\\ u_\varepsilon = 0 & \text{on } \partial E_0,\\ u_\varepsilon = L-2 & \text{on } \partial F_L, \end{cases} \tag{35}$$

where $E^\varepsilon u_\varepsilon$ is the ellipitic regularisation on the LHS of (32). As we only consider initial data sets with one asymptotically flat end and $v(x) \to \infty$ as $|x| \to \infty$, the domain $\Omega_L := F_L \setminus \overline{E}_0$ is precompact. If $M$ contains multiple asymptotic ends, we can also proceed as in the following under the additional assumption, that $E_0$ contains all but one end.

**Remark 5.7.** We prove existence of smooth solutions $u_\varepsilon$ of the elliptic regularisation (35) by using the method of continuity. For $s \in [0,1]$, we consider the boundary value problem

$$\begin{cases} E^{\varepsilon,s} u_{\varepsilon,s} = 0 & \text{on } \Omega_L,\\ u_{\varepsilon,s} = 0 & \text{on } \partial E_0,\\ u_{\varepsilon,s} = s(L-2) & \text{on } \partial F_L, \end{cases} \tag{36}$$

where the operator $E^{\varepsilon,s}u_{\varepsilon,s}$ is defined as

$$\operatorname{div}_g\left(\frac{Du_{\varepsilon,s}}{\sqrt{\varepsilon^2+|Du_{\varepsilon,s}|^2}}\right) - \sqrt{\varepsilon^2+|Du_{\varepsilon,s}|^2 + s\left(\left(g^{\alpha\beta} - \frac{D^\alpha u_{\varepsilon,s} D^\beta u_{\varepsilon,s}}{\varepsilon^2+|Du_{\varepsilon,s}|^2}\right)K_{\alpha\beta}\right)^2},$$

and aim to show that $S_\varepsilon := \{s \in [0,1] : (36) \text{ admits a unique solution in } C^{2,\alpha}(\overline{\Omega}_L)\} = [0,1]$ for sufficiently small $\varepsilon > 0$.

From now on, we also impose the additional condition that the initial data set $(M,g,K)$ is maximal, i.e., $\operatorname{tr}_M K = 0$, to ensure the existence of a subsolution barrier in the compact region. Therefore the quasilinear operator $E^{\varepsilon,s}$ becomes

$$E^{\varepsilon,s} = \operatorname{div}_g\left(\frac{Du_{\varepsilon,s}}{\sqrt{\varepsilon^2+|Du_{\varepsilon,s}|^2}}\right) - \sqrt{\varepsilon^2+|Du_{\varepsilon,s}|^2 + s\left(\frac{D^\alpha u_{\varepsilon,s} D^\beta u_{\varepsilon,s}}{\varepsilon^2+|Du_{\varepsilon,s}|^2}K_{\alpha\beta}\right)^2}. \tag{37}$$

In view of the work of Cederbaum–Sakovich [24] we can also interpret the method of continuity as modifying the underlying initial data set $(M, g, K)$ via $(M_s, g_s, K_s) := (M, g, \sqrt{s}K)$ instead of modifying the operator $E^\varepsilon$.

**Theorem 5.8.** *Let $(M, g, K)$ be an asymptotically flat, maximal initial data set and $E_0 \subset M$ a precompact $C^{2,\alpha}$ domain. Then, for every $L > 2$, satisfying $E_0 \subset F_L$ and $d(\partial E_0, \partial F_L) > 2$, there exists an $\varepsilon(L) > 0$, such that for $0 < \varepsilon < \varepsilon(L)$ and $s \in [0, 1]$ a smooth solution $u_{\varepsilon,s}$ to* (36) *satisfies the following a-priori estimates:*

(i) $u_{\varepsilon,s} \geq -\varepsilon$ *in* $\overline{\Omega}_L$, $u_{\varepsilon,s} \geq v + (s-1)(L-1) - 2$ *in* $\overline{F}_L \setminus F_0$,

(ii) $u_{\varepsilon,s} \leq s(L-2)$ *in* $\overline{\Omega}_L$,

(iii) $|Du_{\varepsilon,s}| \leq H_+ + \varepsilon$ *on* $\partial E_0$, $|Du_{\varepsilon,s}| \leq C(L)$ *on* $\partial F_L$,

(iv) $|Du_{\varepsilon,s}|(y) \leq \max\limits_{B_r(x)\cap\partial E_0} |Du| + \varepsilon + \frac{C(n)}{r} + C(n, ||K||, ||DK||)$ *for all* $y \in \overline{\Omega}_L$,

(v) $|u_{\varepsilon,s}|_{C^{2,\alpha}(\overline{\Omega}_L)} \leq C(\varepsilon, L, n, g, ||K||, ||DK||, \partial E_0)$,

*where $v$ is a subsolution of* (31) *as in Lemma 5.5, $H_+$ is the positive part of the mean curvature of $\partial E_0$, and $r > 0$ such that the conditions of Theorem 5.3 are satisfied at $y$ with $r$.*

**Remark 5.9.** Note that $E^{\varepsilon,s}$ is uniformly elliptic for any smooth function on the compact domain $\Omega_L$ and the assumptions on $L$ ensure that the boundary data of (5.7) are realized by a $C^{2,\alpha}$-function $\varphi$, such that $||\varphi||_{C^{2,\alpha}(\overline{\Omega_L})} \leq C(\partial E_0, L, g, n, ||K||)$. For $C^2$-domains, solutions in $C^{2,\alpha}(\Omega_L) \cap C^{1,\alpha}(\overline{\Omega}_L)$ still satisfying (i)-(iv) exist by approximation.

**Theorem 5.10.** *Let $(M, g, K)$ be an asympotically flat, maximal initial data set and $E_0 \subseteq M$ a precompact $C^{2,\alpha}$ domain. Then, for every $L > 2$, satisfying $E_0 \subset F_L$ and $d(\partial E_0, F_L) > 2$, there exists an $\varepsilon_0(L) \leq \varepsilon(L)$, such that a smooth solution $u_\varepsilon$ of* (35) *exists for all $\varepsilon < \varepsilon_0(L)$.*

Since the $C^1$ a-priori estimates in Theorem 5.8 are independent on $\varepsilon$, we can pass via Arzelà–Ascoli to a subsequence $u_{\varepsilon_k}$, such that $u_{\varepsilon_k} \to u$ locally uniformly, where $u$ is locally Lipschitz.

**Corollary 5.11.** Let $(M, g, K)$ be an asymptotically flat, maximal initial data set and $E_0 \subseteq M$ a precompact $C^2$ domain. Then there exists a locally Lipschitz function $u\colon M \setminus E_0 \to \mathbb{R}$ such that

(i) there exists a sequence $L_k \to \infty$, and a sequence $\varepsilon_k < \varepsilon_0(L_k)$ with $\varepsilon_k \to 0$, such that $u_{\varepsilon_k} \to u$ locally uniformly,

(ii) $0 \leq u \leq u^{(IMCF)}$, where $u^{(IMCF)}$ is the unique[2] weak solution to inverse mean curvature flow on $M \setminus E_0$,

(iii) $u \to \infty$ as $|x| \to \infty$,

(iv) $||u||_{C^{0,1}(\overline{B}_R(x))} \leq \frac{C(n)}{R} + C(n, ||K||, ||DK||)$, whenever $R < d(\partial E_0, x)$ and $R$ satisfies the assumptions of Theorem 5.3.

## 5.3 The limiting behavior of the translating graphs

We concluded in Corollary 5.11, that there exists a locally Lipschitz function $u$ on $M \setminus E_0$, and a subsequence $(\varepsilon_i)$ such that $u_i := u_{\varepsilon_i}$ converges to $u$ locally uniformly. We will show in this subsection that for $n < 6$ in addition to the function $u$ we can extract a limiting vectorfield $\nu$ on $(M \setminus E_0) \times \mathbb{R}$ that will be crucial in defining a weak solution to the anisotropic equation (28).

First note that the functions $U_i := u_i - \varepsilon z$ defined on $\Omega_L \times \mathbb{R}$ converge locally uniformly to the locally Lipschitz function $U(y, z) := u(y)$. In this and the following subsection, we will concentrate on the objects on the cylinder $M \times \mathbb{R}$ and study the limiting behaviour of the hypersurfaces $\widetilde{\Sigma}_t^\varepsilon := \{U_\varepsilon = t\}$ in $M \times \mathbb{R}$. In Subsection 5.4, we will define weak solutions as minimizers to a parametric variation principle and want to argue that the sublimits $U$ and $u$ are indeed minimizers on $(M \setminus \overline{E}_0) \times \mathbb{R}$ and $(M \setminus \overline{E}_0)$ respectively. As it is the case for inverse null mean curvature flow first studied by Moore in [63], the introduced bulk term energy requires a notion of a unit vector field $\nu$ across jump regions. Following their strategy, we first introduce a variational principle $\mathcal{J}_{U,\nu}$ for Caccioppoli sets on a compact subset $A$ inside a domain $\Omega$ defined as

$$\mathcal{J}_{U,\nu}^A(F) := |\partial^* F \cap A| - \int\limits_{F \cap A} \sqrt{|DU|^2 + P_\nu^2}, \tag{38}$$

where $P_\nu := (g^{ij} - \nu^i \nu^j) K_{ij}$ which here reduces to $P_\nu = -\nu^i \nu^j K_{ij}$ as we always impose that $\mathrm{tr}_M K \equiv 0$.

[2]with precompact level-sets as constructed in [50]

We say that $E$ minimizes (38) in a set $\Omega$ (form the inside/ outside respectively), if

$$\mathcal{J}^A_{U,\nu}(E) \leq \mathcal{J}^A_{U,\nu}(F) \tag{39}$$

for all $F$ ($F \subseteq E$, $E \subseteq F$ respectively), with $E \triangle F \subset\subset A \subset\subset \Omega$, where $\triangle$ denotes the symmetric difference. Since this does not depend on the particular choice of the compact set $A$ such that $E \triangle F \subset\subset A \subset\subset \Omega$, we will often omit the subscript $A$ in the following. Note that the well-known inequality

$$|\partial^*(E_1 \cup E_2)| + |\partial^*(E_1 \cap E_2)| \leq |\partial^* E_1| + |\partial^* E_2|, \tag{40}$$

implies

$$\mathcal{J}^A_{U,\nu}(E_1 \cup E_2) + \mathcal{J}^A_{U,\nu}(E_1 \cap E_2) \leq \mathcal{J}^A_{U,\nu}(E_1) + \mathcal{J}^A_{U,\nu}(E_2), \tag{41}$$

for Caccioppoli sets $E_1$ and $E_2$ satisfying $E_1 \triangle E_2 \subset\subset A$. In particular, $E$ minimizes $\mathcal{J}_{U,\nu}$, if and only if $E$ minimizes $J_{u,\nu}$ from the outside and the inside.

As already discussed, the bulk term $\sqrt{|DU|^2 + P_\nu^2}$ requires a notion of unit normal $\nu$ on all of $\Omega$, since the canonical choice $\nu = \frac{DU}{|DU|}$ fails across jump regions. Hence, the main task of this subsection will be to foliate the interior of jump regions of $U$ in $(M \setminus E_0) \times \mathbb{R}$, thus defining a notion of unit vector field $\nu$. In fact $\nu$ will turn out to be translation invariant and in particular gives rise to a well-defined vector field $\nu_M := \pi_{TM}\nu$ on $M$.

To establish the existence of this foliation, we will draw heavily upon regularity theory for obstable problems (42) below. In particular, if a Caccioppoli set E minimizes (38) and $|DU|$ admits an upper bound, $E$ is almost minimizing in the sense that

$$|\partial^* E \cap B_R| \leq |\partial^* F \cap B_R| + C(n, ||DU||_\infty, ||K||_{C^0}) R^{n+2},$$

for $E \triangle F \subset\subset B_R \subset M \times \mathbb{R}$. In particular, this allows us to apply regularity results of geometric measure theory to obtain higher regularity for $\partial^* E$. The following $C^{1,\alpha}$ result can be obtained by modifying the proof given in [79] for $\mathbb{R}^n$ to general Riemannian manifolds. We refer to the comments preceding [50, Regularity Theorem 1.3] for a broad overview of references.

**Theorem 5.12.** (Regularity Theorem)
*Let $f$ be a bounded, measurable function on a domain $\Omega \subset \widetilde{M}^m$ of a smooth Riemannian manifold $(\widetilde{M}^m, \widetilde{g})$ of dimension $m < 8$. Suppose $E \subset \Omega$ contains an open set $\mathfrak{U}$ and minimizes the functional*

$$|\partial^* F| + \int_F f \tag{42}$$

*with respect to competitors $F$, such that $\mathfrak{U} \subseteq F$ and $F \triangle E \subset\subset \Omega$. If $\partial\mathfrak{U}$ is $C^{1,\alpha}$, $0 < \alpha < \frac{1}{2}$, then $\partial E$ is a $C^{1,\alpha}$-submanifold of $\Omega$ with $C^{1,\alpha}$ estimates only depending on the distance to $\partial\Omega$, $\operatorname{ess\,sup}|f|$, $C^{1,\alpha}$-bounds for $\partial\mathfrak{U}$ and $C^1$-bounds on the metric $\widetilde{g}$.*

*When $m \geq 8$, this remains true away from a closed singular set $Z$ of dimension at most $m-8$, that is disjoint from $\overline{\mathfrak{U}}$.*

**Remark 5.13.** From here onwards we will always assume that $n < 6$ so the previous theorem applies in $M \times \mathbb{R}$ with $m = n+2$. If $n \geq 6$, the limit $u_\varepsilon \to u$ will lead to weak solutions of (28) with similar regularity properties away from the singular set.

Another essential tool that will be used in this section is the following Compactness Theorem for Caccioppoli sets minimizing (38).

**Theorem 5.14.** (Compactness Theorem)
*Let $(\widetilde{M}, \widetilde{g}, \widetilde{K})$ be an initial data set, let $\Omega \subseteq \widetilde{M}$, and let $E_i \subseteq \Omega$ be a sequence of sets with $C^{1,\alpha}_{loc}$ boundary such that $\partial E_i \to \partial E$ locally in $C^{1,\alpha}$. Let $(\nu_i)$ be a sequence of unit vector fields on $T\Omega$ satisfying $\nu_i|_{\partial E_i} = \nu_{\partial E_i}$, such that there exists a unit vector field $\nu \in T\Omega$ with $\nu_i \to \nu$ a.e. locally uniformly and $\nu|_{\partial E} = \nu_{\partial E}$. Further, let $U_i \in C^{0,1}_{loc}(\Omega)$, such that $U_i \to U$ locally uniformly for an $U \in C^{0,1}_{loc}(\Omega)$ and*

$$|DU_i| \to |DU| \ \ in\ L^1_{loc}(\Omega).$$

*Then, if $E_i$ minimizes $\mathcal{J}_{U_i,\nu_i}$ in $\Omega$, $E$ minimizes $\mathcal{J}_{U,\nu}$ in $\Omega$.*

*Proof.* As argued above, it suffices to show that $E$ minimizes $\mathcal{J}_{U,\nu}$ in $\Omega$ from the outside and from the inside. As both directions are similar in spirit, we merely show that $E$ minimizes $\mathcal{J}_{U,\nu}$ in $\Omega$ from the outside.

So let $E \subseteq F$ such that $F \setminus E \subset\subset \Omega$ and let $G \subset\subset \Omega$ such that $F \setminus E \subset\subset G$. We further consider a compact set $\widetilde{G} \subset\subset \Omega$ with smooth boundary, such that $G \subset int(\widetilde{G})$,

$$\left|\partial^*(F \cup E_i) \cap \partial\widetilde{G}\right| = \left|\partial^*(F \cap E_i) \cap \partial\widetilde{G}\right| = \left|\partial^* E_i \cap \partial\widetilde{G}\right| = 0$$

for $i$ large enough, and $\int\limits_{\partial\widetilde{G}} \left|\chi^-_{F\cup E_i} - \chi^+_{E_i}\right| \,\mathrm{d}\mathcal{H}^n \to 0$ as $i \to \infty$, which is possible as $F \cup E_i \to E$, $F \cap E_i \to E$, and $E_i \to E$ in $L^1_{loc}(\Omega \setminus G)$. Setting $F_i := E_i \cup (F \cap \widetilde{G})$, we see that

$$|\partial^* F_i \cap \Omega| = \left|\partial^* E_i \cap \partial(\Omega \setminus \widetilde{G})\right| + \left|\partial^*(F \cup E_i) \cap \widetilde{G}\right| + \int\limits_{\partial\widetilde{G}} \left|\chi^-_{F\cup E_i} - \chi^+_{E_i}\right| \,\mathrm{d}\mathcal{H}^n.$$

Furthermore $F_i \triangle E_i \subset\subset \Omega$, so $\mathcal{J}^{\widetilde{G}}_{U_i,\nu_i}(E_i) \leq \mathcal{J}^{\widetilde{G}}_{U_i,\nu_i}(F_i)$. With the above identity, we can conclude

$$\mathcal{J}^{\widetilde{G}}_{U_i,\nu_i}(E_i) \leq \mathcal{J}^{\widetilde{G}}_{U_i,\nu_i}(F \cup E_i) + \int\limits_{\partial \widetilde{G}} \left| \chi^-_{F\cup E_i} - \chi^+_{E_i} \right| \,\mathrm{d}\mathcal{H}^n.$$

Together with inequality (41), this implies that

$$\mathcal{J}^{\widetilde{G}}_{U_i,\nu_i}(F \cap E_i) - \int\limits_{\partial \widetilde{G}} \left| \chi^-_{F\cup E_i} - \chi^+_{E_i} \right| \,\mathrm{d}\mathcal{H}^n \leq \mathcal{J}^{\widetilde{G}}_{U_i,\nu_i}(F).$$

Using that $E_i \cap F \to E$ in $\widetilde{G}$, $U_i \to U$ and $\nu_i \to \nu$ a.e. locally uniformly, $|\nabla U_i| \to |\nabla U|$ in $L^1_{loc}$ and $\int\limits_{\partial \widetilde{G}} \left| \chi^-_{F\cup E_i} - \chi^+_{E_i} \right| \,\mathrm{d}\mathcal{H}^n \to 0$ as $i \to \infty$, we can conclude that

$$\mathcal{J}^{\widetilde{G}}_{U,\nu}(E) \leq \mathcal{J}^{\widetilde{G}}_{U,v}(F),$$

so $E$ minimizes $\mathcal{J}_{U,\nu}$ to the outside. □

We will see in Subsection 5.4, Lemmas 5.23 and 5.24, that the sublevelsets of smooth solutions of STIMCF satisfy the variational principle (39). Applying this to the smooth approximating solutions $U_\varepsilon$ constructed in Subsection 5.2, the Regularity Theorem 5.12 and the interior gradient estimate Theorem 5.3 yield the following:

**Corollary 5.15.** The downward translating graphs $\Sigma^\varepsilon_t = \{U_\varepsilon = t\}$ are locally uniformly bounded in $C^{1,\alpha}$ for sufficiently small $\varepsilon > 0$.

*Proof.* By Lemmas 5.23 and 5.24, we know that the sets $E^\varepsilon_t := \{U_\varepsilon < t\}$ minimize $\mathcal{J}_{U_\varepsilon,\nu_\varepsilon}$ on $E_b \setminus E_a$ for all $a \leq t < b$, where $\nu_\varepsilon = \frac{DU_\varepsilon}{|DU_\varepsilon|}$.

Let $(y,z) \in (M^{n+1} \setminus E_0) \times \mathbb{R}$ and define $d := dist((y,z), \partial E_0 \times \mathbb{R}) = dist(y, \partial E_0)$. We now take $L$ large enough, such that $B^M_{2r} \subset\subset F_L$, $2r < d$, and $r$ satisfies the assumptions of Theorem 5.3. So for $\varepsilon < \varepsilon(L)$ we have an upper bound of $\sqrt{|DU_\varepsilon|^2 + P^2_{\nu_\varepsilon}}$ on $B^{M\times\mathbb{R}}_r((y,z))$. Then the Regularity Theorem 5.12 implies that the hypersurfaces $\Sigma^\varepsilon_t \cap B^{M\times\mathbb{R}}_r((y,z))$ are uniformly bounded in $C^{1,\alpha}$. □

Using the locally uniform bounds on the downward translating graphs we are able to construct limiting hypersurfaces in a jump region of $U$ in $(M \setminus E_0) \times \mathbb{R}$.

**Proposition 5.16.** *Let $\mathcal{K}_{t_0}$ denote the interior of a jump region $\{U = t_0\}$, at a jump time $t_0$. Then each point $X_0 = (y_0, z_0) \in \mathcal{K}_{t_0}$ lies in a complete hypersurface $\Sigma_{X_0} \subseteq \overline{\mathcal{K}}_{t_0}$ that is the limit of a sequence $\widetilde{\Sigma}^{\varepsilon_{i_j}}_{t_{i_j}}$ and locally uniformly bounded in $C^{1,\alpha}$.*

*Proof.* We take a pointwise approach similar to Heidusch [49] and Moore [63]. We fix a target point $X_0 = (y_0, z_0) \in \mathcal{K}_{t_0}$. Taking the sequence $(\varepsilon_i) \to 0$, such that the solutions to the elliptic regularisation $u_{\varepsilon_i}$ converge to $u$, we consider the corresponding sequence of times $(t_i)$ defined by $t_i := U_{\varepsilon_i}(X_0) \in (-\infty, \infty)$, i.e., $X_0 \in \widetilde{\Sigma}^{\varepsilon_i}_{t_i}$. Note that $t_i \to t_0$, since $U_i \to U$ locally uniformly. Let $\iota(X_0)$ denote the injectivity radius of $X_0$ in $(M \setminus E_0) \times \mathbb{R}$ and set

$$d = d(X_0) := \min(\iota(X_0), r(X_0), dist(X_0, \partial\mathcal{K}_{t_0})),$$

where $r(X_0)$ is chosen as in the proof of Corollary 5.15. Therefore it exists $\varepsilon' > 0$, such that for all $i$ and $\varepsilon \le \varepsilon'$, the surface pieces $\widetilde{\Sigma}^{\varepsilon_i}_{t_i} \cap B^{M\times\mathbb{R}}_d(X_0)$ are $C^{1,\alpha}$ bounded uniformly in $i$ and $\varepsilon$. We now consider the exponential map

$$\exp_{X_0} = (\exp_{y_0}, \mathrm{id}_{\mathbb{R}})\colon T_{X_0}(M \times \mathbb{R}) \cap B^{n+2}_d(0, z_0) \to B^{M\times\mathbb{R}}_d(X_0),$$

and set $\widehat{\Sigma}^{\varepsilon_i}_{t_i} := \exp^{-1}_{X_0}\left(\widetilde{\Sigma}^{\varepsilon_i}_{t_i} \cap B^{M\times\mathbb{R}}_d(X_0)\right) \subseteq T_{X_0}(M \times \mathbb{R}) \cong \mathbb{R}^{n+2}$. Then the surfaces $\widehat{\Sigma}^{\varepsilon_i}_{t_i}$ are $C^{1,\alpha}$ bounded uniformly in $i$ and $\varepsilon$. In particular, we have uniform $C^{0,\alpha}$ bounds on the unit normal $\widehat{v}_i(\hat{X}_0)$ of $\widehat{\Sigma}^{\varepsilon_i}_{t_i}$ at $\hat{X}_0 = (0, z_0)$. By sequence compactness, there exists limit $\widehat{v}(\hat{X}_0)$, such that $\widehat{v}_i(\hat{X}_0) \to \widehat{v}(\hat{X}_0)$ uniformly (up to taking a subsequence). Then $v(\hat{X}_0)$ uniquely determines a hyperplane $\hat{T}$ centered at $\hat{X}_0$. By the uniform convergence of $\widehat{v}_i(\hat{X}_0)$ to $v(\hat{X}_0)$ and the uniform $C^{1,\alpha}$ bounds of the hypersurfaces $\widehat{\Sigma}^{\varepsilon_i}_{t_i}$, there exists $R \le d$, such that for $i >> 1$ large enough, we can write each $\widehat{\Sigma}^{\varepsilon_i}_{t_i}$ locally as a graph over $\hat{T} \cap B_R(\hat{X}_0)$. So there exists a $C^{1,\alpha}$ function $\hat{\omega}_i$ on $\hat{T} \cap B_R(\hat{X}_0)$, such that $\widehat{\Sigma}^{\varepsilon_i}_{t_i} \cap B_R(\hat{X}_0) = \mathrm{graph}(\hat{\omega}_i)$, and the functions $\hat{\omega}_i$ are uniformly $C^{1,\alpha}$ bounded. Using Arzelà–Ascoli and denoting the new Hölder exponent $0 < \beta < \alpha$ again by $\alpha$ for convenience, there is a further subsequence $\hat{\omega}_{i_j}$ and a $C^{1,\alpha}$ function $\hat{\omega}\colon \hat{T} \cap B_R(\hat{X}_0) \to \mathbb{R}$, such that

$$\hat{\omega}_{i_j} \to \hat{\omega} \text{ in } C^{1,\alpha}\left(\hat{T} \cap B_R(\hat{X}_0)\right),$$

$\hat{\omega}$ satisfies the same $C^{1,\alpha}$ bounds, and $\hat{\omega}$ is locally the graph of a hypersurface $\widehat{\Sigma}_{\hat{X}_0}$ around $\hat{X}_0$ with $\hat{T} = T_{\hat{X}_0}\widehat{\Sigma}_{\hat{X}_0}$. Thus the hypersurface $\exp_{X_0}(\widehat{\Sigma}_{\hat{X}_0})$ in $M^{n+1}\times\mathbb{R}$ is uniformly $C^{1,\alpha}$ bounded. By successively taking subsequences, the hypersurfaces $\widetilde{\Sigma}^{\varepsilon_i}_{t_i}$ converge in $C^{1,\alpha}_{loc}$ to a complete hypersurface that we will henceforth denote by $\widetilde{\Sigma}_{X_0}$ satisfying $\widetilde{\Sigma}_{X_0} \cap B^{M\times\mathbb{R}}_R(X_0) = \exp_{X_0}(\widehat{\Sigma}_{\hat{X}_0})$.

We want to conclude the proof by showing that $\widetilde{\Sigma}_{X_0} \subseteq \overline{\mathcal{K}}_{t_0}$. Consider a point $Y \in \widetilde{\Sigma}_{\hat{X}_0}$. Then there exists a sequence $(Y_i) \to Y$ with $Y_i \in \widetilde{\Sigma}^{\varepsilon_i}_{t_i}$, and

$$|U_{\varepsilon_i}(Y_i) - U(Y)| \le |U_{\varepsilon_i}(Y_i) - U(Y_i)| + |U(Y_i) - U(Y)| \to 0,$$

by the locally uniform convergence of $U_i$ to $U$, so $U(Y) = \lim_{i\to\infty} U_{\varepsilon_i}(Y_i) = \lim_{i\to\infty} t_i = t_0$. Hence, $Y \in \{U = t_0\}$. □

Using the $C^{1,\alpha}$ bounds on the limiting surfaces, we will dedicate the rest of this subsection on improving the above result. In particular, we will show that the hypersurfaces $\widetilde{\Sigma}_{X_0}$ indeed foliate $\mathcal{K}_{t_0}$ and are $C^{2,\alpha}_{loc}$ generalized apparent horizons that bound Caccioppoli sets that satisfy the comparison principle (38), see Theorem 5.19 and 5.20 below. To this end, we want to verify all assumptions to apply the Compactness Theorem 5.14 on $\mathcal{K}_{t_0}$. First, we argue that the limiting surfaces are already sufficient to define a notion of unit normal on $\mathcal{K}_{t_0}$.

**Proposition 5.17.** *Let $\mathcal{K}_{t_0}$ be the interior of a jump region, the sequence $(\varepsilon_i)$ as in Proposition 5.16 before, and $\nu_i$ the unit normal to the translating graphs $\widetilde{\Sigma}^{\varepsilon_i}_t$. Then there exists a Hölder continuous unit vector field $\nu$ on $\mathcal{K}_{t_0}$, and a subsequence $(i_j)$, such that $\nu_{i_j} \to \nu$ locally uniformly on $\mathcal{K}_{t_0}$. Moreover $\nu$ is translation invariant and everywhere normal to the hypersurfaces $\widetilde{\Sigma}_X$ as constructed in Proposition 5.16.*

*Proof.* Throughout the proof, let $\widetilde{\Sigma}_X$ denote a hypersurface through $X \in \mathcal{K}_{t_0}$ constructed in Proposition 5.16 as a sublimit of the level-sets $\{U_{\varepsilon_i} = t_i\}$ such that $U_{\varepsilon_i}(X) = t_i$. Note that the hypersurfaces $\widetilde{\Sigma}_X$ are not a-priori unique, as the construction relies on taking (subsequent) subsequences. However, comparing a point $X_0 = (x_0, z_0) \in \mathcal{K}_{t_0}$ with a vertical translate $X_\alpha = (x_0, z_0 + \alpha)$, and assuming that for a subsequence $i_j$

$$X_0 \in \widetilde{\Sigma}^{\varepsilon_{i_j}}_{t_{i_j}} = \operatorname{graph}\left(\frac{u_{i_j}}{\varepsilon_{i_j}} - \frac{t_{i_j}}{\varepsilon_{i_j}}\right) \to \widetilde{\Sigma}_{X_0},$$

we have $X_\alpha \in \widetilde{\Sigma}^{\varepsilon_{i_j}}_{t_{i_j} - \alpha\varepsilon_{i_j}}$, and

$$\widetilde{\Sigma}^{\varepsilon_{i_j}}_{t_{i_j} - \alpha\varepsilon_{i_j}} = \operatorname{graph}\left(\frac{u_{i_j}}{\varepsilon_{i_j}} - \frac{t_{i_j} - \alpha\varepsilon_{i_j}}{\varepsilon_{i_j}}\right) = \operatorname{graph}\left(\frac{u_{i_j}}{\varepsilon_{i_j}} - \frac{t_{i_j}}{\varepsilon_{i_j}}\right) + \alpha e_{n+2} \to \widetilde{\Sigma}_{X_0 + \alpha e_{n+2}},$$

so we have convergence for all vertical translates with respect to the same subsequence. Thus, it suffices to construct a unit vector field $\nu \in T\mathcal{K}_{t_0}$ along $\mathfrak{K}_{t_0} := int\{u = t_0\} = \mathcal{K}_{t_0} \cap (M \times \{0\})$ that is normal to the hypersurfaces $\widetilde{\Sigma}_X$ for $X \in \mathfrak{K}_{t_0}$, which is then trivially extended in the $z$-direction and satisfies all the desired properties.

Let $\zeta := \{X_k\}$ be a dense, countable subset of $\mathfrak{K}_{t_0}$. Then, by taking subsequent subsequences we can choose the hypersurfaces $\widetilde{\Sigma}_{X_k}$, such that

$$\widetilde{\Sigma}^{\varepsilon_{i_j}}_{t^k_{i_j}} \to \widetilde{\Sigma}_{X_k}$$

locally uniformly in $C^{1,\alpha}$ with respect to the same subsequence $(\varepsilon_{i_j})$ via a diagonal sequence argument. In particular, the hypersurfaces $\widetilde{\Sigma}_{X_k}$ are locally uniform limits of the level-sets

$\{U_{\varepsilon_{i_j}} = t^k_{i_j}\}$, which implies that if two surfaces touch each other, they have to do so tangentially and without intersecting anywhere. This yields a well-defined unit vector field on $\zeta$, which lies dense in $\mathfrak{K}_{t_0}$. In view of the (locally) uniform $C^{1,\alpha}$ estimates established in Corollary 5.15 and since the limiting hypersurfaces $\widetilde{\Sigma}_{X_k}$, $X_k \in \zeta$, do not intersect transversally, there is $r_0$ such that for $X_k, X_m \in \zeta$ with $|X_k - X_m| < r_0$ all $\widetilde{\Sigma}^{\varepsilon_{i_j}}_{t_{i_j}} \cap B_{2r_0}(X_k, 0)$ for $\varepsilon$ sufficiently small can be written as normal graphs over $\widetilde{S}_{X_k}$ with uniformly bounded $C^{1,\alpha}$ norm and

$$\left|\nu_{\varepsilon_{i_j}}(X_k) - \nu_{\varepsilon_{i_l}}(X_m)\right| \leq C\,|X_k - X_m|^{\alpha}$$

for all $j, l$ sufficiently large. Therefore, $\nu_{\varepsilon_{i_j}} \to \nu$ locally uniformly in $\zeta$ and $\nu$ can be extended to a locally Hölder continuous unit vector field on $\mathfrak{K}_{t_0}$.

It remains to show that $\nu(X)$ is normal to the hypersurfaces $\widetilde{\Sigma}_X$ for all $X \in \mathfrak{K}_{t_0} \setminus \zeta$. To this end, we construct $\widetilde{\Sigma}_X$ as in Proposition 5.16 by taking a further subsequence $(i_{j_k})_{k\in\mathbb{N}}$. In particular $\nu_{\varepsilon_{i_{j_k}}}(X) \to \nu_{\widetilde{\Sigma}_X}(X)$, but we also necessarily have $\nu_{\varepsilon_{i_{j_k}}} \to \nu$, so

$$\nu_{\widetilde{\Sigma}_X}(X) = \nu(X).$$

$\square$

Besides the existence of a measurable unit vector field $\nu$, the compactness theorems Theorem 5.14 and Theorem 5.31 below also require that $|DU_i| \to |DU|$ in $L^1_{loc}$. The interior gradient estimate Theorem 5.3 implies that for any $L$ large enough, $\varepsilon_i < \varepsilon_0(L)$ and domain $\Omega \subseteq \Omega_L$, it holds that

$$\sup_A |DU_i| \leq C(A),$$

for all $A \subset\subset \Omega$, where $C(A)$ is a positive constant only depending on $A$. Then, the Compactness Theorem for BV functions implies the weak convergence and semilowercontinuity of the gradient, i.e.,

$$DU_i \to DU \text{ in } (C^0_0(\Omega))^*,\ \ |DU|_{L^1} \leq \liminf_{i\to\infty} |DU_i|_{L^1}.$$

In the interior of jump regions, the $L^1_{loc}$ convergence is readily established by the weak convergence, but will demand a more delicate analysis away from jumps (see Lemma 5.37). Before proving this, we want to point out that this improvement of the convergence is not required in the respective Compactness Theorems for inverse mean curvature flow [50, Theorem 2.1] and for inverse null mean curvature flow [63, Compactness Property 9], where the proofs merely rely on the semilowercontinuity.

**Lemma 5.18.**

$$|DU_i| \to |DU| \ \ \textit{in } L^1_{loc}(\mathcal{K}_{t_0}).$$

*Proof.* Let $X_0 \in \mathcal{K}_{t_0}$, and for $\varepsilon > 0$ small enough, we consider the geodesic ball $B_\varepsilon(X_0) \subseteq\subseteq \mathcal{K}_{t_0}$ such that $\nu_i \to \nu$ uniformly and $|DU_i|$ uniformly bounded in $B_\varepsilon(X_0)$. Let $\varphi \in C_0^0(B_\varepsilon(X_0))$, then

$$\int_{B_\varepsilon(X_0)} \varphi\, |DU_i| = \int_{B_\varepsilon(X_0)} \langle DU_i, \varphi\nu\rangle + \varphi \langle DU_i, \nu_i - \nu\rangle \to \int_{B_\varepsilon(X_0)} \langle DU, \varphi\nu\rangle = 0$$

by the weak$^*$ convergence, which is implied by the Compactness Theorem for BV functions. Since $|DU_i|$ is uniformly (pointwise) bounded, the claim follows since $\varphi$ can be choosen arbitrarily close to 1 in $L^1_{loc}$ and $|DU| = 0$ on $\mathcal{K}_{t_0}$. □

We are now in the position to use the Compactness Theorem 5.14, to show that the hypersurfaces $\widetilde{\Sigma}_{X_0}$ bound minimizing Caccioppoli sets, which will allow us to improve their regularity.

**Theorem 5.19.** *Each hypersurface $\widetilde{\Sigma}_{X_0}$ constructed in Proposition 5.16 is a $C^{2,\alpha}$ generalized apparent horizon that bounds a Caccioppoli set that minimizes $\mathcal{J}_{U,\nu}$ in $\mathcal{K}_{t_0}$.*

*Proof.* Employing the Compactness Theorem 5.14, Proposition 5.16 and Lemma 5.18 imply that $\widetilde{\Sigma}_{X_0}$ bounds a Caccioppoli set $\widetilde{E}_{X_0}$ such that $\partial\widetilde{E}_{X_0} = \widetilde{\Sigma}_{X_0}$, $\nu$ is the outward unit normal to $\partial\widetilde{E}_{X_0}$, and $\widetilde{E}_{X_0}$ minimizes $\mathcal{J}_{U,\nu}$ in $\mathcal{K}_{t_0}$. To complete the proof it remains to show, that the hypersurfaces $\widetilde{\Sigma}_{X_0}$ are $C^{2,\alpha}$ generalized apparent horizons.

We recall the relationship between a function $\omega \in BV_{loc}(\Omega)$ and its subgraph

$$W = \{(x,t) \in \Omega \times \mathbb{R} \colon t < \omega(x)\}.$$

If in particular $\chi_W$ denotes the characteristic function of the subgraph $W$ of $\omega$, by [45, Theorem 14.6] it holds that

$$|\partial^* W| = \int\limits_{\Omega\times\mathbb{R}} |D\chi_W| = \int\limits_{\Omega} \sqrt{1 + |D\omega|^2}. \tag{43}$$

As argued by the construction in Proposition 5.16, we now choose a ball $B_R(X_0)$ in $M \times \mathbb{R}$ around a point $X_0 \in \widetilde{\Sigma}_{X_0}$, such that $\widetilde{\Sigma}_{X_0} \cap B_R(X_0) = \operatorname{graph}(\omega)$ for a function $\omega \in C^{1,\alpha}(T_{X_0}\widetilde{\Sigma}_{X_0} \cap B_R(X_0))$. Then $\widetilde{E}_{X_0} \cap B_R(X_0) = W$ is the subgraph of $\omega$, where $\Omega = T_{X_0}\widetilde{\Sigma}_{X_0} \cap B_R(X_0)$. Since $W$ minimizes $\mathcal{J}_{U,\nu}$ in $\mathcal{K}_{t_0}$, $\omega$ minimizes the functional

$$\mathcal{J}'_\nu(\omega) := \int\limits_{\Omega} \sqrt{1 + |D\omega|^2}\, \mathrm{d}x - \int\limits_{\Omega} \int\limits_{0}^{\omega(x)} |P_\nu|\,(x,s)\, \mathrm{d}s\, \mathrm{d}x,$$

where we used the identity (43) and the fact that $|DU| = 0$ on $\mathcal{K}_{t_0}$. The corresponding Euler-Lagrange equation is

$$\operatorname{div}\left(\frac{D\omega}{\sqrt{1+|D\omega|^2}}\right) + |P_\nu| = 0.$$

Note that this is exactly the equation characterizing a generalized apparent horizon, since $H = \operatorname{div}(\nu)$ and by construction $\nu = \frac{(-D\omega,1)}{\sqrt{1+|D\omega|^2}}$. Since $\omega \in C^{1,\alpha}(T_{X_0}\widetilde{\Sigma}_{X_0} \cap B_R(X_0))$, $\omega$ in particular weakly solves the uniformly elliptic equation

$$a^{ij} D_i D_j \omega = f,$$

where

$$a^{ij} := \frac{1}{\sqrt{1+|D\omega|^2}^3}\left(g^{ij} - \frac{D\omega^i \nabla\omega^j}{1+|D\omega|^2}\right), \ f := -\left|K_{ij}\frac{D\omega^i D\omega^j}{1+|D\omega|^2}\right|,$$

are $C^{0,\alpha}$ functions on $T_{X_0}\widetilde{\Sigma}_{X_0} \cap B_R(X_0)$. Schauder Theory, [44, Theorem 6.13], then implies that for $R(X_0)$ sufficiently small $\omega$ is in fact the unqiue solution in $C^{2,\alpha}(T_{X_0}\widetilde{\Sigma}_{X_0} \cap B_R(X_0))$ which solves the equation in the strong sense. Therefore $\Sigma_{X_0}$ is a generalized apparent horizon. $\square$

We can now use the apparent horizon equation on the hypersurfaces $\widetilde{\Sigma}_{X_0}$ to improve the result of Proposition 5.16, which is the concluding statement of this subsection.

**Theorem 5.20.** *Let $\mathcal{K}_{t_0}$ denote the interior of a jump region $\{U = t_0\}$ at a jump time $t_0$. Then each point $X_0 = (y_0, z_0)$ lies in a complete $C^{2,\alpha}$-hypersurface $\widetilde{\Sigma}_{X_0}$, such that the hypersurfaces $\widetilde{\Sigma}_{X_0}$ are generalized apparent horizons foliating $\mathcal{K}_{t_0}$ with unit normal $\nu \in C^{1,\alpha}_{loc}(\mathcal{K}_{t_0})$. If $(M, g, K)$ further satisfies the dominant energy condition* (8)*, then away from a set of $\mathcal{H}^n$ measure zero, the hypersurfaces are either vertical cylinders or translating graphs.*

**Remark 5.21.** Note that, although a leaf of the foliation $\widetilde{\Sigma}_{X_0}$ is always either locally a vertical cylinder or a translating graph, here we are able to fully characterize the set when $\widetilde{\Sigma}_{X_0}$ changes character if the (DEC) holds. This set of $\mathcal{H}^n$ measure zero along the hypersurface is essentially given as the set where $|P|$ fails to be differentiable along the hypersurface (compare the precise definition of the set $S_{X_0}$ in the proof below). In fact, we expect that the Hausdorff dimension of the $\mathcal{H}^n$ zero measure set is at most $n-1$ and is (if non-empty) a rectifiable varifold with bounded variation.

*Proof.* By Proposition 5.17, we already know that the hypersurfaces $\widetilde{\Sigma}_X$ can only touch tangentially and locally remain sheeted at the same side with respect to each other. Thus, since the hypersurfaces are in fact generalized apparent horizons, the possibility of them touching tangentially is immediately ruled out by the strong maximum principle. To show that the unit normal $\nu$ is indeed in $C^{1,\alpha}_{loc}(\mathcal{K}_{t_0})$, we fix a point $X_0 \in \mathcal{K}_{t_0}$ and choose sufficiently small $r > 0$, such that the geodesic ball $B_r(X_0) \subseteq\subseteq \mathcal{K}_{t_0}$ satisfies all the restrictions assumed in the previous propositions, and additionally such that each leaf of the foliation $\widetilde{\Sigma}_X$ intersecting the ball can be written as the graph of a function $\omega^X$ over the tangent space of $T\left(\widetilde{\Sigma}_{X_0} \cap B_r(X_0)\right)$ (this follows from the $C^{1,\alpha}_{loc}$ a-priori estimates on $B_r$ independent of the hypersurface). Clearly, if $X \to X_0$, $\omega^X \to \omega^{X_0}$ uniformly on $\overline{B}_{\frac{3}{4}r}$ by virtue of construction. Then, the Schauder interior estimates [44, Corollary 6.3] yield $C^{2,\alpha}$ convergence on $B_{\frac{r}{2}}(X)$. In particular, $\nu \in C^{1,\alpha}_{loc}(\mathcal{K}_{t_0})$.

We now assume that $(M, g, K)$ satisfies the dominant energy condition (8). Then, $(M \times \mathbb{R}, g + \mathrm{d}z^2, K)$ also satisfies the dominant energy condition. Since $\nu \in C^{1,\alpha}_{loc}$ along the hypersurfaces, we have $P_\nu \in C^{1,\alpha}_{loc}$, and $|P_\nu|$ extends to $C^{1,\alpha}_{loc}$, if $P \neq 0$, or $P = 0$ and $DP = 0$. For a leaf of the foliation $\widetilde{\Sigma}_{X_0}$ we therefore consider the $\mathcal{H}^n$ zero measure set $S_{X_0} := \overline{\{X \in \widetilde{\Sigma}_{X_0} \colon P = 0, DP \neq 0\}}$. Thus $|P_\nu| \in C^{1,\alpha}_{loc}(\widetilde{\Sigma}_{X_0} \setminus S_{X_0})$, which in particular implies that the hypersurfaces $\widetilde{\Sigma}_{X_0}$ are $C^{3,\alpha}_{loc}$ away from $S_{X_0}$ (and smooth away from $\{P_\nu = 0\}$). Closely following the proof of [72, Proposition 2], we can establish a Harnack inequality for the function $g(\partial_z, \nu)$ along the connected components of $\widetilde{\Sigma}_{X_0} \setminus S_{X_0}$, which yields that $\widetilde{\Sigma}_{X_0}$ is either a vertical cylinder or a translating graph along these connected components. This concludes the proof. $\square$

## 5.4 Variational formulation for weak solutions

By freezing $\sqrt{|Du|^2 + P_\nu^2}$ and treating it as a bulk term energy, we can interpret (31) as the Euler-Lagrange equation to the functional

$$\mathcal{J}^A_{u,\nu}(v) := \int\limits_A |Dv| + v\sqrt{|Du|^2 + P_\nu^2}. \tag{44}$$

If for all $A \subset \Omega$ compact

$$\mathcal{J}^A_{u,\nu}(u) \leq \mathcal{J}^A_{u,\nu}(v), \tag{45}$$

for all $v \in C^{0,1}_{loc}(\Omega)$ ($v \leq u$, $v \geq u$ respectively), such that $\{v \neq u\} \subset\subset A \subset \Omega$, we call $(u, \nu)$ a *weak solution (subsolution, supersolution respectively)* of (44) in $\Omega$.

**Remark 5.22.** Appealing to the results of Subsection 5.3, we will later define the concept of weak solutions of STIMCF, cf. Definition 5.27, on the cylinder $M \times \mathbb{R}$. We will thus frequently use the notion of weak solutions for pairs $(U, \nu)$ on open subsets $\Omega \subseteq M \times \mathbb{R}$ for the functional $\mathcal{J}_{U,\nu}$ as defined by (44). It is easy to check that all the results and observations regarding weak solutions $(u, \nu)$ of (44) on $M$ similarly hold for weak solutions $(U, \nu)$ of (44) on $M \times \mathbb{R}$, and in fact remain true for variational principles with general, bounded bulk term energy, cf. [80].

The relation between translation invariant weak solutions of (44) on $M \times \mathbb{R}$ and weak solutions on $M$ is established in Lemma 5.30 below.

For any $v, w \in C^{0,1}_{loc}(\Omega)$ satisfying $\{v \neq w\} \subset\subset \Omega$, we find that

$$\mathcal{J}^A_{u,\nu}(\min(v, w)) + \mathcal{J}^A_{u,\nu}(\max(v, w)) = \mathcal{J}^A_{u,\nu}(v) + \mathcal{J}^A_{u,\nu}(w),$$

so by choosing $w = u$, we can conclude that $u$ is a solution, if and only if $u$ is a subsolution and supersolution. Furthermore, smooth solutions to the corresponing Euler-Lagrange equation (31) are in fact minimizers of the comparison priniciple $\mathcal{J}_{u,\nu}$ as the following Lemma establishes.

**Lemma 5.23** (Smooth flow Lemma)**.** *Let $u$ be a smooth solution of* (31)*, $E_t := \{u < t\}$. Then the sets $E_t$ minimize $\mathcal{J}_{u,\nu}$ with $\nu = \frac{Du}{|Du|}$ in $E_b \setminus E_a$ for all $a \leq t < b$.*

*Proof.* Follows exactly as in [50, Lemma 2.3], [63, Lemma 15] by replacing the respective bulk term energies with $\sqrt{|Du|^2 + P_\nu^2}$. $\square$

We also see that the respective comparisons principles (38) for Caccioppoli sets and (44) for locally Lipschitz functions are indeed closely related.

**Lemma 5.24.** *Let $u$ be a locally Lipschitz function on an open set $\Omega$, $\nu$ a measurable (unit) vector field. Then $(u, \nu)$ is a weak solution (subsolution, supersolution respectively) of* (44) *in $\Omega$, if and only if for each $t > 0$, the sets $E_t := \{u < t\}$ minimize $\mathcal{J}_{u,\nu}$ in $\Omega$ (on the outside, the inside respectively).*

*Proof.* Is proven in complete analogue to [50, Lemma 1.1] and [63, Lemma 12] replacing the respective bulk term energies by $B_{u,\nu} := \sqrt{|Du|^2 + |P_\nu|^2}$. $\square$

**Remark 5.25.** As it is the case for inverse mean curvature flow and inverse null mean curvature flow this equivalence also extends to the initial value problems

$$\begin{gathered} u \in C^{0,1}_{loc}(M),\ \nu \text{ a measurable vector field on } T(M \setminus E_0), \\ E_0 = \{u < 0\}, \text{ and } (u, \nu) \text{ is a weak solution of (44) in } M \setminus E_0, \end{gathered} \tag{46}$$

and

$$\begin{aligned} u \in C^{0,1}_{loc}(M),\ \nu \text{ a measurable vector field on } T(M \setminus E_0),\\ \text{and } \forall t > 0\ E_t := \{u < t\} \text{ minimizes (38) in } M \setminus E_0. \end{aligned} \tag{47}$$

This follows directly from Lemma 5.24. Lastly, by approximating $s \searrow t$, we see that (46) and (47) are moreover equivalent to

$$\begin{aligned} u \in C^{0,1}_{loc}(M),\ \nu \text{ a measurable vector field on } T(M \setminus E_0),\\ \text{and } \forall t \geq 0\ E_t^+ := \{u \leq t\} \text{ minimizes (38) in } M \setminus E_0, \end{aligned} \tag{48}$$

via the Compactness Theorem 5.14.

Further, we can identify the weak mean curvature of the hypersurfaces $\widetilde{\Sigma}_t = \{u < t\}$ for weak solutions of (44) outside of jump regions, where $u$ remains constant.

Recall that the first variation $\delta(\mu)$ of a $C^1$ hypersurface $\widetilde{N}$ in $M \times \mathbb{R}$ is defined as

$$\delta(\mu)(X) := \int_{\widetilde{N}} \operatorname{div}_{\widetilde{N}} X \,\mathrm{d}\mu\ \forall X \in C^\infty_C(T(M \times \mathbb{R})),$$

where $\mu$ is the induced volume form on $\widetilde{N}$. For the following, we refer to [76] for a detailed introduction. If $\widetilde{N}$ has bounded variation, the Riesz Representation theorem implies that we can identify $\delta(\mu)$ with a vector valued measure. If the measure $\delta(\mu)$ is furthermore absolutely continuous with respect to $\mu$, the weak mean curvature vector $\vec{H} = -H\nu$ is defined via the Lebesque differentiation theorem,

$$\vec{H} := -D_\mu \delta(\mu).$$

The weak mean curvature $H$ is thus characterized by the following identity

$$\int_{\widetilde{N}} \operatorname{div}_{\widetilde{N}} X \,\mathrm{d}\mu = \int_{\widetilde{N}} Hg\,(\nu, X)\ \mathrm{d}\mu\ \forall X \in C^\infty_C(T(M \times \mathbb{R})). \tag{49}$$

**Lemma 5.26.** *Let $\widetilde{\Sigma}_t = \{u < t\}$ minimize $J_{u,\nu}$ in $\widetilde{E}_b \setminus \overline{\widetilde{E}}_a$, where $u \in C^{0,1}_{loc}(\widetilde{E}_b \setminus \overline{\widetilde{E}}_a)$, and let $\Omega$ be an open set, such that $\Omega \cap \widetilde{E}_b \setminus \overline{\widetilde{E}}_a$ contains no jump regions. Then the surfaces $\widetilde{\Sigma}_t = \partial \widetilde{E}_t$ have weak mean curvature $H$ satisfying*

$$H = \sqrt{|DU|^2 + P_\nu^2}$$

*a.e. in $\Omega \cap \widetilde{\Sigma}_t$ for a.e. $t \in (a, b)$.*

*Proof.* Follows exactly as in [50, Section 1] and [63, Lemma 16]. □

As suggested by Moore in [63] for inverse null mean curvature flow, we will define weak solutions of STIMCF on $M$ one dimension higher in $M \times \mathbb{R}$, as pairs $(U, \nu)$ of translation invariant locally Lipschitz functions $U$ and measurable unit vector fields $\nu$, where $U$ minimizes $\mathcal{J}_{U,\nu}$ on $(M \setminus E_0) \times \mathbb{R}$. In analogue to [63, Definition 15], we will incorporate the results of Subsection 5.3 into the general definition of weak solutions.

**Definition 5.27.** Let $E_0 \subset M$ be a precompact, open set with $C^2$ boundary $\Sigma_0 = \partial E_0$. We call the pair $(U, \nu)$ a *weak solution of inverse space-time mean curvature flow with initial condition* $E_0$, if $U \in C^{0,1}_{loc}(M \times \mathbb{R})$ and $\nu$ is a measurable unit vector field which satisfy

(i) $U$ is translation invariant in the vertical direction. In particular, there exists a locally Lipschitz function $u : M \to \mathbb{R}$, such that $U(y, z) = u(y)$. Moreover $u$ satisfies

$$u(x) \geq 0 \ \forall x \in M \setminus E_0,$$
$$u|_{\partial E_0} = 0, \ u(x) < 0 \ \forall x \in E_0,$$
$$u(x) \to +\infty \text{ as } dist(x, E_0) \to \infty.$$

(ii) The set $\widetilde{E}_t = \{U < t\}$ minimizes $\mathcal{J}_{U,\nu}$ in $(M \setminus E_0) \times \mathbb{R}$ for each $t > 0$. At jump times $t_0$, each point $X_0 = (y_0, z_0)$ in the interior $\mathcal{K}_{t_0}$ of the jump region $\{U = t_0\}$ lies in a $C^{1,\alpha}$ hypersurface $\widetilde{\Sigma}_{X_0}$ which is the boundary of a Caccioppoli set $\widetilde{E}_{X_0}$ that minimizes $\mathcal{J}_{U,\nu}$ in $\mathcal{K}_{t_0}$.

(iii) $\nu$ is a translation invariant with

$$\nu(X + \alpha e_z) = \nu(X) \ \forall X \in (M \setminus E_0) \times \mathbb{R}, \ \alpha \in \mathbb{R};$$
$\nu(X)$ is in $C^0_{loc}$ away from jump times and is the unit normal vector to $\partial\widetilde{E}_t$ at each point $X \in \partial\widetilde{E}_t$;
$\nu(X)$ is in $C^{1,\alpha}_{loc}(\mathcal{K}_{t_0})$ and is the unit normal vector to $\partial\widetilde{E}_{X_0}$ at each point $X \in \partial\widetilde{E}_{X_0}$ at jump times $t_0$ and points $X_0 \in \mathcal{K}_{t_0}$.

**Remark 5.28.**

(i) As we require the variational principle (38) for $\mathcal{J}_{U,\nu}$ to be satisfied everywhere, in particular in the interior of jump regions, we can argue as in Theorem 5.19 to conclude that the interior of any jump region is foliated by $C^{2,\alpha}$ generalized apparent horizons.

(ii) By Lemma 5.24, we find that any weak solutions $(U, \nu)$ of inverse space-time mean curvature flow is a weak solution of (44) on $(M \setminus E_0) \times \mathbb{R}$. Additionally, we formulated further restrictions that arise naturally in our construction that couple our choice of unit vector field $\nu$ to the function $U$ in an intuitively geometric way. Without these restrictions, there are in general several (translation invariant) weak solutions of (44), but where the function $U$ and the unit vector field $\nu$ are not coupled in any meaningful way. To see this, note that under the restriction $\operatorname{tr}_M K = 0$ there exists a unit vector $\nu_p \in T_pM$ with $K_p(\nu_p, \nu_p) = 0$ for all $p \in M$ and under fairly generic conditions on $K$ we can make that choice in a (uniformly) continuous way. In particular, $P_\nu = 0$ for this choice of unit vector field[3] and hence $(U^{IMCF}, \nu)$ is a translation invariant weak solution of (44) where $U^{IMCF}$ denotes the translation invariant weak solution of inverse mean curvature flow as constructed by Huisken–Ilmanen [50]. In particular, we can no longer expect any statement of uniqueness that solely relies on the comparison principle (44) and the methods of Huisken–Ilmanen developed for inverse mean curvature flow, cf. [50, Uniquness Theorem 2.2], can not be extended to the anisotropic case. Instead, a completely different approach has to be developed that also takes the the geometric restrictions of the unit vector field $\nu$ into account to conclusively answer the question of uniqueness for the concept of weak solutions to inverse space-time mean curvature flow. These observations are similarly relevant for the concept of weak solutions to inverse null mean curvature flow introduced by Moore [63].

**Example 5.29.** We give an illustrative example for the above observation that also highlights that weak solutions of (44) in general lack the geometric properties of weak solutions of STIMCF. We choose $(M, g) = (\mathbb{R}^3, \delta)$ and $E_0 = B_1(0)$. In particular, the corresponding weak solution to inverse mean curvature flow on $M$ is the smooth expanding sphere solution $v = 3\ln(|x|)$. We further choose $K := \frac{6}{1+r^6}\left(\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2\right)$ in polar coordinates, so $(M, g, K)$ is a maximal initial data set with $K\left(\frac{1}{r}\partial_\varphi, \frac{1}{r}\partial_\varphi\right) = 0$.

Hence, $\left(V, \frac{1}{r}\partial_\varphi\right)$ is a weak solution of (44) on $(M \setminus \overline{E_0}) \times \mathbb{R}$ where $V(x, z) = v(x)$ and thus the solution does not exhibit any jumps. However

$$H|_{\partial E_0} = 2 < 3 = \left|P_{|\partial E_0}\right|,$$

so any weak solution $(U, \nu)$ constructed as in Theorem 5.33 has to immediately jump to a generalized apparent horizon $\partial E_0^+$, see Corollary 5.43 below. In particular, $V \neq U$ and since $\mathbb{R}^3$ does not allow for any closed minimal surfaces we further have $\nu \neq \frac{1}{r}\partial_\varphi$ on $E_0^+ \setminus \overline{E_0}$. Hence, we found translation invariant weak solutions $\left(V, \frac{1}{r}\partial_\varphi\right)$ and $(U, \nu)$ of (44) such that neither the choice of function nor the choice of unit vector field agree. □

[3] One could of course always choose $\nu = \partial_z$ on $M \times \mathbb{R}$, but we want to emphasize that this phenomenon also persists on a large class of initial data sets $(M, g, K)$ with $\nu$ tangent to $M$.

By projecting the translation invariant solution down to the original initial data set, we find that $(u, \nu_M)$ is a weak solution of (44) on $M \setminus E_0$, where $\nu_M := \nu|_{TM}$ is the restriction of the translation invariant vector field $\nu$ onto the tangential space of $M$. As elaborated above, we lose geometric information of the solution during the process. Nonetheless, as the level-sets of $u$ are precompact it will be more convenient from a technical perspective to formulate the formation of jumps of a weak solution $(U, \nu)$ in Subsection 5.6 and and the blow-down procedure in Subsection 5.7 below with respect to $(u, \nu_M)$.

**Lemma 5.30.** *Let $(U(y,z) := u(y), \nu)$ be a weak solution of inverse space-time mean curvature flow with initial condition $E_0$. Then the pair $(u, \nu_M := \nu|_{TM})$ is a weak solution of (44) on $M \setminus E_0$, and $E_t = \{u < t\}$ minimizes (38) for each $t > 0$.*

*Proof.* Since we extended the tensor $K$ trivially in the $z$-direction, we find

$$\widetilde{P}_\nu = (\widetilde{g}^{ij} - \nu^i \nu^j)\widetilde{K}_{ij} = (g^{ij} - \nu^i_M \nu^j_M) K_{ij} = P_{\nu_M}.$$

In particular, we can conclude that $\sqrt{|DU|^2 + P^2_\nu}(y,z) = \sqrt{|Du|^2 + P^2_{\nu_M}}(y)$ for all $(y,z)$ in $(M \setminus E_0) \times \mathbb{R}$. The rest of the proof then proceeds as in the proof of [50, Theorem 3.1] and [63, Lemma 14]. □

We close this subsection with a Compactness Theorem for weak solutions of inverse space-time mean curvature flow.

**Theorem 5.31.** *Let $((U_i, \nu_i))_i$ be a sequence of weak solutions of (44) on open sets $\widetilde{\Omega}_i$ in $M \times \mathbb{R}$ with $U_i \in C^{0,1}_{loc}(\widetilde{\Omega}_i)$, $\nu_i$ measurable, a.e. locally uniformly continuous unit vector fields, such that $\widetilde{\Omega}_i \to \Omega$, $U_i \to U$ in $C^{0,1}_{loc}$ and $\nu_i \to \nu$ a.e. locally uniformly for a pair $(U, \nu)$, and $|DU_i| \to |DU|$ in $L^1_{loc}$. Then $(U, \nu)$ is a weak solution of (44) on $\Omega$.*

*If in addition, $(U_i, \nu_i)$ is a sequence of weak solutions of inverse space-time mean curvature flow as in Definition 5.27, then $(U, \nu)$ is a weak solution of inverse space-time mean curvature flow.*

**Remark 5.32.** As in the Remark following [50, Theorem 2.1] the statement of Theorem 5.31 is still valid if we allow $(U_i, \nu_i)$ to be a weak solution of (44) with respect to metrics $g_i$ and symmetric $(0,2)$-tensors $K_i$, such that $g_i \to g$ and $K_i \to K$ in $C^1_{loc}$.

*Proof.* By the stronger assumptions on the convergence of $|DU_i|$, we can replace the inductive structure of the proof of [50, Theorem 2.1] by a more direct argument. Let $V$ be a locally Lipschitz function, such that $\{V \neq U\} \subset\subset \Omega$. Let $\Phi \in C^1_c(\Omega)$ with $\Phi = 1$ on $\{V \neq U\}$. Then $V_i := \Phi V + (1 - \Phi) U_i$ is a locally Lipschitz comparison function for $U_i$ if $i >> 1$ is large enough. For an appropriate open, precompact set $W \subset \Omega$, such that $\operatorname{supp} \Phi \subset\subset W$, we have

$$\mathcal{J}^W_{U_i, \nu_i}(U_i) \leq \mathcal{J}^W_{U_i, \nu_i}(V_i).$$

Therefore, by definition of $\mathcal{J}_{U_i,\nu_i}$, we find that

$$\begin{aligned}\int\limits_W |DU_i| + U_i B_{U_i,\nu_i} &\leq \int\limits_W |DV_i| + V_i B_{U_i,\nu_i} \\ &\leq \int\limits_W \Phi\,|DV| + (1-\Phi)\,|DV| + |D\Phi|\,|U_i - V_i| + (\Phi V + (1-\Phi)U_i)B_{U_i,\nu_i},\end{aligned}$$

for $B_{U_i,\nu_i} := \sqrt{|DU_i|^2 + P_{\nu_i}^2}$ which implies

$$\int\limits_W \Phi\left(|DU_i| + U_i B_{U_i,\nu_i}\right) \leq \int\limits_W \Phi\left(|DV| + V B_{U_i,\nu_i}\right) + \int\limits_W |D\Phi|\,|U_i - V_i|\,.$$

By the choice of $\Phi$, we see that $|D\Phi|\,|U_i - V_i| \to 0$ uniformly, and by letting $i \to \infty$ we obtain

$$\mathcal{J}^W_{U,\nu}(U) \leq \mathcal{J}_{U,\nu}(V).$$

If we assume the pairs $(U_i, \nu_i)$ to be weak solutions of STIMCF the translation invariance of $(U_i, \nu_i)$ implies the translation invariance of the pair $(U, \nu)$. Using Lemma 5.24 and arguing as in Subsection 5.3, we conclude that the pair $(U, \nu)$ satisfies all properties of Definition 5.27, so the pair is a weak solution to inverse space-time mean curvature flow. □

## 5.5 Existence of weak solutions

We will now proof our main result. For this, we use the unit vector field $\nu$ constructed in Subsection 5.3 to extend the unit normal vector $\frac{DU}{|DU|}$ across jump regions and employ the Compactness Theorem 5.31 to the locally Lipschitz sublimit $U$.

**Theorem 5.33.** *Let $(M^{n+1}, g, K)$ be an asymptotically flat maximal initial data set as in Definition 2.1.. Then for any nonempty, precompact, open set $E_0 \subset M^{n+1}$ with $C^2$ boundary, there exists a weak solution of inverse space-time mean curvature flow with initial condition $E_0$.*

**Remark 5.34.**

(i) In initial data sets with multiple ends or inner boundary components the result holds analogously for initial data $E_0$ containing all but one end and all inner boundary components.

(ii) Note that with the methods presented in this paper we can similarly establish the existence of weak solutions of

$$\frac{\mathrm{d}}{\mathrm{d}t}F = \frac{H}{H^2 - P^2}$$

under the same conditions imposed on $(M, g, K)$. The speed of this flow corresponds precisely to the tangential projection of the codimension-2 formulation of inverse mean curvature flow in an ambient spacetime first proposed by Frauendiener [41]. The level-set equation in this case takes the form

$$\operatorname{div}_g\left(\frac{Du}{|Du|_g}\right) = \frac{1}{2}|Du|_g + \frac{1}{2}\sqrt{|Du|_g^2 + 4\left(\left(g^{ij} - \frac{D^i u D^j u}{|Du|^2}\right)K_{ij}\right)^2} \tag{50}$$

and leads to the same singularities and asymptotic behavior as STIMCF. In particular, their jumping behavior is driven by the same outward optimization property. See Subsection 5.6 below.

The proof proceeds as outlined above, where we essentially follow the strategy as in Subsection 5.3 outside of jump regions and establish the corresponding results in the following lemmata.

**Lemma 5.35.** *For a.e. $t \geq 0$,*

$$|\nabla u| > 0 \ \mathcal{H}^n\text{-a.e. on } \Sigma_t.$$

*Proof.* Follows as in [50, Lemma 5.1] directly from the co-area formula. □

**Lemma 5.36.** *Let $t > 0$. If $t$ is not a jump time, then $\widetilde{\Sigma}_t := \partial\{U < t\}$ is a complete hypersurface that is locally uniformly bounded in $C^{1,\alpha}$. If $t$ is a jump time, then $\widetilde{\Sigma}_t$, $\widetilde{\Sigma}_t^+ := \partial\{U > t\}$ are complete hypersurfaces that are locally uniformly bounded in $C^{1,\alpha}$.*

*Proof.* Let $U_\varepsilon$ be the smooth solution of STIMCF on $\Omega_L \times \mathbb{R}$ defined via the smooth solutions of the elliptic regularisation $u_\varepsilon$, which exist under the above assumptions by Theorem 5.10, and let $U$ be the sublimit as in Corollary 5.11, where $L_i \to \infty$, $0 < \varepsilon_i < \varepsilon_0(L_i)$ as $\varepsilon_i \to 0$. We treat the two cases separately.

(i) In the case that $t$ is not a jump time, where $\widetilde{\Sigma}_t = \widetilde{\Sigma}_t^+ = \{U = t\}$, the surface $\widetilde{\Sigma}_t$ is approximated by fixing a point $X_0 = (y_0, z_0) \in \widetilde{\Sigma}_t$ and considering the sequence of times $t_i$, such that $X_0 \in \widetilde{\Sigma}_{t_i}^{\varepsilon_i}$ for each $i$. It then follows exactly as in the proof of Proposition 5.16, that $\widetilde{\Sigma}_{t_i}^{\varepsilon_i}$ converges locally uniformly to $\widetilde{\Sigma}_t$ in $C^{1,\alpha}$ for a subsequence, and $\widetilde{\Sigma}_t$ satisfies the same locally uniform $C^{1,\alpha}$ bounds, where we denote the new Hölder exponent $0 < \beta < \alpha$ here and in the following again by $\alpha$ for convenience. Since in this case $\widetilde{\Sigma}_t = \{U = t\}$ and $U_i \to U$ locally uniformly, the convergence holds for the whole sequence.

(ii) If $t$ is a jump time, we will use a slightly different pointwise approach to approximate $\widetilde{\Sigma}_t$ and $\widetilde{\Sigma}_t^+$, since $\widetilde{\Sigma}_t \neq \widetilde{\Sigma}_t^+$, and hence argue more carefully in this case. To this end, let $X_0 \in \widetilde{\Sigma}_{t_0}^+$ at a jump time $t_0$. Since there are only countable many such $t_0$, there exists a sequence of points $X_i \in \widetilde{\Sigma}_{t_i}$ with $t_i > t_0$, such that $\lim\limits_{i\to\infty} X_i = X_0$, $\lim\limits_{i\to\infty} t_i = t_0$, and for $i >> 1$ large enough $\widetilde{\Sigma}_{t_i} = \widetilde{\Sigma}_{t_i}^+$. As argued above, each surface piece $\widetilde{\Sigma}_{t_i} \cap B_R^{M\times\mathbb{R}}(X_i)$ can therefore be written via the exponential map as the graph of a $C^{1,\alpha}$ function $\hat{\omega}_i$ over $T_{X_i}\hat{\Sigma}_{t_i}$, where

$$\hat{\Sigma}_{t_i} := \exp_{X_i}^{-1}\left(\widetilde{\Sigma}_{t_i} \cap B_R^{M\times\mathbb{R}}(X_i)\right).$$

Now consider the sequence $\nu_i$ of normal vectors to $\hat{\Sigma}_{t_i}$ at $\hat{X}_i$. By the uniform $C^{0,\alpha}$ bounds on $\hat{\nu}_i$, there exists a subsequence $\hat{\nu}_{i_k}$ and a unit vector $\hat{\nu} \in T_{X_0}M$, such that $\hat{\nu}_{i_j} \to \hat{\nu}$ uniformly. Let $\hat{T}$ denote the affine hyperplane orthogonal to $\hat{\nu}$ centered at $\hat{X}_0$. For $i >> 1$ large enough, we can write each surface $\hat{\Sigma}_{t_i}$ locally as the graph of a $C^{1,\alpha}$ function $\hat{\omega}_i$ over $\hat{T} \cap B_R^{n+2}(\hat{X}_0)$. By Arzelà–Ascoli, there exists a further subsequence $\hat{\omega}_{i_j}$ and a $C^{1,\alpha}$ function $\hat{\omega}\colon \hat{T} \cap B_R^{n+1}(\hat{X}_0) \to \mathbb{R}$, such that

$$\hat{\omega}_{i_j} \to \omega \text{ in } C^1(\hat{T} \cap B_R^{n+1}(\hat{X}_i)),$$

where $\hat{X}_0 \in \operatorname{graph}(\hat{\omega})$ and $\hat{T} = T_{X_0}\operatorname{graph}(\hat{\omega})$. We then consider $\omega := \hat{\omega} \circ \exp_{X_0}^{-1}$. In order to recognize $\operatorname{graph}(\omega)$ as a piece of $\widetilde{\Sigma}_{t_0}^+$, we consider a point $Y \in \operatorname{graph}(\omega)$. By construction, there exists a sequence $Y_j \in \operatorname{graph}(\omega_{i_j}) \subset \widetilde{\Sigma}_{t_i}$, such that $Y_j \to Y$. Hence $U(Y_i) = t_i$, which implies that $U(Y) = t_0$, so $Y \in \widetilde{E}_{t_0}^+ = \{U \leq t_0\}$. Assume that $Y \in int(\widetilde{E}_{t_0}^+)$, then there exists a $\delta > 0$, such that $B_\delta^{M\times\mathbb{R}}(Y) \subset int(\widetilde{E}_{t_0}^+)$. In particular $Y_j \subset int(\widetilde{E}_{t_0}^+)$ for $j >> 1$ large enough, which is a contradiction, since $U(Y_j) > t_0$. Thus $\operatorname{graph}(\omega) \subset \widetilde{\Sigma}_t^+$. We make an analogous argument in the case that $X_0 \in \widetilde{\Sigma}_{t_0}$ for a sequence of points $X_i \in \widetilde{\Sigma}_{t_i}$, where $t_i \nearrow t_0$. Again the limit is independent of the choice of subsequence and hence the full sequence converges.

□

Since $\widetilde{\Sigma}_{t_i}^{\varepsilon_i} \to \{U = t\}$ for the whole sequence away from jump regions, we can now argue as in Proposition 5.17 that $\nu_i \to \nu$ locally uniformly away from jumps for the whole sequence, and we can thus construct a locally Hölder continuous unit normal $\nu$ away from jump regions. Since there are at most countable jump times, we consider the normal vector field $\nu$ constructed via Proposition 5.17 in the interior of each jump region and by taking successive subsequences once more, we obtained a measurable unit vector field $\nu$ on all of $(M \setminus E_0) \times \mathbb{R}$, such that $U_i \to U$ locally uniformly, and $\nu_i \to \nu$ a.e. locally uniformly with respect to a fixed sequence $(\varepsilon_i)_{i\in\mathbb{N}}$. More precisely, case (ii) in particular shows that $\nu$ is continuous on $M \setminus E_0$ away from the interior of jump regions. Moreover, we will show in the next section that the exterior boundary of any jump region $\widetilde{\Sigma}_t^+ \cap \widetilde{\Sigma}_t^C$ is itself a generalized apparent horizon, so arguing with the Schauder interior estimates as in Theorem 5.20, we see that $\nu$ is continuous across the exterior boundary of any jump region. Thus the continuity of $\nu$ only fails at the interior boundary $\widetilde{\Sigma}_t \cap \left(\widetilde{\Sigma}_t^+\right)^C$ of jump regions. To reconcile, we have constructed $\nu$ such that a.e.

$$\nu(X) := \begin{cases} \frac{DU}{|DU|}(X) & \text{if } X \in \widetilde{\Sigma}_t \text{ at regular times } t, \\ \nu(X) & \text{if } X \in \mathcal{K}_{t_0} \text{ at a jump time } t_0, \text{ where } \nu \text{ as in Proposition 5.17}, \\ \lim\limits_{l\to\infty} \frac{DU}{|DU|}(X_l) & \text{if } X \in \widetilde{\Sigma}_{t_0}, \text{ for } X_l \to X,\ t_l \nearrow t_0,\ t_0 \text{ jump time}, \\ \lim\limits_{l\to\infty} \frac{DU}{|DU|}(X_l) & \text{if } X \in \widetilde{\Sigma}_{t_0}^+, \text{ for } X_l \to X,\ t_l \searrow t_0,\ t_0 \text{ jump time}. \end{cases} \tag{51}$$

To employ the Compactness Theorem 5.31, it remains to show that the gradients converge in $L^1_{loc}$.

**Lemma 5.37.**

$$|DU_i| \to |DU| \ \ \textit{in } L^1_{loc}.$$

Note that, we will in fact prove convergence of $DU_i$ in $L^2_{loc}$, which will imply the proposition, since we are claiming convergence on compact regions. More precisely, by the fact that the sequence $|DU_i|$ is locally uniformly bounded, dominated convergence yields convergence in $L^p_{loc}$ for all $p \geq 1$.

*Proof.* Since there at most countable many jump regions with boundary given by the union of two $C^{1,\alpha}_{loc}$ hypersurfaces and since the claim is already proven inside jump regions, see Lemma 5.18, it suffices to prove it on precompact open sets strictly away from any jump region. Hence, it suffices to proof the claim for small geodesic balls. Note further that

$$\int_\Omega |DU - DU_i|^2 = \int_\Omega |DU|^2 - 2\int_\Omega DU \cdot DU_i + \int_\Omega |DU_i|^2 .$$

Arguing by the uniform convergence of the unit normals and the weak * convergence similar as in Lemma 5.18, it suffices to prove that

$$\int_\Omega |DU_i|^2 \to \int_\Omega |DU|^2 ,$$

where me may assume that the domain $\Omega$ is a small geodesic ball.

Now, let $X_0 = (y_0, z_0) \in (M^{n+1} \setminus \overline{E_0}) \times \mathbb{R}$ away from any jump region and let $d := \min(\iota(X_0), dist(X_0, E_0 \times \mathbb{R}), r(X_0))$, where $\iota(X_0)$ again denotes the injectivity radius, and $r(X_0)$ is chosen such that the geodesic ball of radius $2r(X_0)$ centered at $X_0$ satisfies the assumptions of Theorem 5.3 and is compactly contained in the complement of the jump regions. Let $\phi \in C_c^\infty(\mathbb{R})$ be a compactly supported function with $\operatorname{supp}\phi \subseteq [z_0 - 3d, z_0 + 3d]$, $\phi \equiv 1$ on $[z_0 - 2, z_0 + 2]$ and $|\partial_z \phi| \leq 2$, and consider $\Phi := \phi^3$. Therefore

$$B_d^{M\times\mathbb{R}}(X_0) \subseteq Z := (\overline{B}_d^M(y_0) \times \mathbb{R}) \cap (M^{n+1} \times \operatorname{supp}\Phi).$$

Since $t_{\varepsilon_i} = U_{\varepsilon_i}(X_0) \to U(X_0) = t_0$ locally uniformly, we can choose $L >> 1$ large enough and $\varepsilon' < \varepsilon(L)$, such that there exist a $\delta > 0$ such that

$$Z \subseteq \bigcup_{t\in[t_0-\delta,t_0+\delta]} \widetilde{\Sigma}_t^{\varepsilon_i} \cap (M^{n+1} \times \operatorname{supp}\Phi),$$

with $\widetilde{\Sigma}_t^{\varepsilon_i} \cap (M^{n+1} \times \operatorname{supp}\Phi)$ staying strictly away from any jump region and $\partial\widetilde{\Sigma}_t^{\varepsilon_i} \cap (M^{n+1} \times \operatorname{supp}\Phi) = \emptyset$ for all $t \in [t_0 - \delta, t_0 + \delta]$ and all $\varepsilon_i < \varepsilon'$. Furthermore, there exists $R(t_0) > r(t_0) > 0$, such that for all $t \in [t_0 - \delta, t_0 + \delta]$ and all $\varepsilon_i < \varepsilon'$

$$\widetilde{\Sigma}_t^{\varepsilon_i} \cap (M^{n+1} \times \operatorname{supp}\Phi) \subseteq S(t_0) := (G_{R(t_0)}(X_0) \setminus G_{r(t_0)}) \times [z_0 - 3d, z_0 + 3d],$$

where $G_r := \{y \in M \setminus E_0 \colon dist(E_0, y) < r\}$. In particular, since the sublevelsets $\widetilde{E}_t^{\varepsilon_i} = \{U_{\varepsilon_i} < t\}$ are minimizing $\mathcal{J}_{U_{\varepsilon_i},\nu_{\varepsilon_i}}$ from the outside, we can conclude for all $t \in [t_0 - \delta, t_0 + \delta]$ and all $\varepsilon_i < \varepsilon'$, that

$$\left|\widetilde{\Sigma}_t^{\varepsilon_i} \cap (M^{n+1} \times \operatorname{supp}\Phi)\right| \leq |\partial^* S(t_0)'| - \int\limits_{S(t_0)'\setminus\widetilde{E}_t^{\varepsilon_i}} \sqrt{|DU_{\varepsilon_i}|^2 + \left|P_{\nu_{\varepsilon_i}}\right|^2} \leq |\partial^* S(t_0)'| =: C(t_0)$$

for $S(t_0)' := G_{R(t_0)} \times [z_0 - 3d, z_0 + 3d]$, where we used $S(t_0)' \cup \widetilde{E}_i^{\varepsilon_i}$ as a competitor. Additionally, since $S(t_0)$ is compact, the upper bound on $|DU_{\varepsilon_i}|$ implies that

$$\sqrt{H_{\varepsilon_i}^2 - P_{\nu_{\varepsilon_i}}^2} \leq C(t_0, g, n, ||K||, ||DK||),$$

for all $t \in [t_0 - \delta, t_0 + \delta]$ and all $\varepsilon_i < \varepsilon'$. Let $p \geq 1$ be fixed. Following the strategy of the calculation of the monotonicity of the Hawking mass in [50] and the prove of [63, Lemma 23], we use that the functions $U_i$ induces a smooth graphical solution of STIMCF. We then calculate the evolution of

$$\int\limits_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi(z) \left(H^2 - P^2\right)^p$$

using the evolution equations in Lemma 5.1. After a straightforward but rather lengthy computation, which we will therefore omit, we see that we can make the favorable choice $p = 2$ which yields

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi(z) \left(H^2 - P^2\right)^2 = &- 4 \int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi \left|\nabla \left(\sqrt{H^2 - P^2}\right)\right|^2 \left(\frac{2H}{\sqrt{H^2 - P^2}} + \frac{\sqrt{H^2 - P^2}}{2H}\right) \\
&- 4 \int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi \frac{P}{H} \nabla P \nabla \left(\sqrt{H^2 - P^2}\right) + H \nabla \Phi \nabla \left(\sqrt{H^2 - P^2}\right) \\
&- 4 \int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi \sqrt{H^2 - P^2} \left(H \left(\mathrm{Ric}_g(\nu_{\varepsilon_i}, \nu_{\varepsilon_i}) + |h|^2\right) + P \operatorname{tr}_{\widetilde{\Sigma}_t^{\varepsilon_i}} D_{\nu_{\varepsilon_i}} K\right) \\
&+ 8 \int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi(z) K_{\nu_{\varepsilon_i} j} P \nabla \left(\sqrt{H^2 - P^2}\right)^j \\
&+ \int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \left(H^2 - P^2\right)^{\frac{3}{2}} \left(\frac{\partial \Phi}{\partial z} \nu_{\varepsilon_i} + \Phi H\right),
\end{aligned}
$$

where $\nabla$ denotes the gradient on the graphs $\widetilde{\Sigma}_t^{\varepsilon_i}$ by our convention. Since $|\mathrm{Ric}| \leq C(t_0)$ on $S(t_0)$ by compactness, the upper bounds on $\sqrt{H^2 - P^2}$ and on $\left|\widetilde{\Sigma}_t^{\varepsilon_i} \cap (M^{n+1} \times \operatorname{supp} \Phi)\right|$ as well as a global bound on $K$ following from its asymptotics imply

$$
\int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \left(H^2 - P^2\right)^{\frac{3}{2}} \left(\frac{\partial \Phi}{\partial z} \nu_{\varepsilon_i} + \Phi H\right) - 4 \Phi \sqrt{H^2 - P^2} \left(H \mathrm{Ric}_g(\nu_{\varepsilon_i}, \nu_{\varepsilon_i}) + P \operatorname{tr}_{\widetilde{\Sigma}_t^{\varepsilon_i}} D_{\nu_{\varepsilon_i}} K\right) \leq C(t_0, K, DK).
$$

Furthermore, we can use the Peter–Paul inequality to estimate

$$
-4 \int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi \left|\nabla \left(\sqrt{H^2 - P^2}\right)\right|^2 \left(\frac{2H}{\sqrt{H^2 - P^2}} + \frac{\sqrt{H^2 - P^2}}{2H}\right) \leq -8 \int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi \left|\nabla \left(\sqrt{H^2 - P^2}\right)\right|^2,
$$

and

$$8\int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi(z) K_{\nu_{\varepsilon_i} j} P \nabla \left(\sqrt{H^2-P^2}\right)^j \leq \int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi \left|\nabla\left(\sqrt{H^2-P^2}\right)\right|^2 + 16\int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi P^2 \left|K_{\nu_{\varepsilon_i} j}\right|^2,$$

$$-2\int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi \frac{P}{H} \nabla P \nabla\left(\sqrt{H^2-P^2}\right) \leq \int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi \left|\nabla\left(\sqrt{H^2-P^2}\right)\right|^2 + 2\int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi \frac{P^2}{H^2} \left|\nabla P\right|^2,$$

$$-4\int_{\widetilde{\Sigma}_t^{\varepsilon_i}} H D\Phi \nabla\left(\sqrt{H^2-P^2}\right) \leq \int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi \left|\nabla\left(\sqrt{H^2-P^2}\right)\right|^2 + 4\int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \frac{\left|D\Phi\right|^2}{\Phi} H^2.$$

Note that $\frac{P^2}{H^2} \leq 1$ since $\sqrt{H^2-P^2} > 0$, and $\frac{|D\Phi|^2}{\Phi} \leq 36\phi$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi(z) \left(H^2-P^2\right)^2 \leq -5\int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi \left|\nabla\left(\sqrt{H^2-P^2}\right)\right|^2 + C(t_0, K, \nabla K), \tag{52}$$

if $|\nabla P|$ remains locally uniformly bounded. As computed in [53, Equation (13)],

$$\nabla_i P = \nabla_i(\operatorname{tr}_M K) - (D_i K)(\nu,\nu) - 2K(\nu, \partial_j) h^i_j,$$

so the desired bound is satisfied if the second fundamental form $h$ remains locally uniformly bounded. This was proven by Heidusch [49] in the case of inverse mean curvature flow using only the locally uniform upper bounds on the mean curvature and the $C^{1,\alpha}$-norm of the translating graphs. It is possible to derive the desired estimate also in this case. Details will be provided in an extended version of [53]. Integrating (52) yields

$$\int_{t_0-\delta}^{t_0+\delta} \int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi \left|\nabla\left(\sqrt{H^2-P^2}\right)\right|^2 \leq C(t_0, K, \nabla K). \tag{53}$$

Applying Fatou's Lemma, there exists a subsequence $i_j$ (henceforth just denoted as $i$ for convenience), such that

$$\int_{\widetilde{\Sigma}_t^{\varepsilon_i}} \Phi \left|\nabla\left(\sqrt{H^2-P^2}\right)\right|^2 < \infty \tag{54}$$

for almost every $t \in [t_0-\delta, t_0+\delta]$, but since $U_i$ are translating solutions, we can arrange this to be the case for any sequence of times, in particular (54) indeed holds for any $t \in [t_0-\delta, t_0+\delta]$.

Hence, $|DU_i| = \sqrt{H^2 - P^2}$ is uniformly bounded in $W^{1,2}_{loc}(\widetilde{\Sigma}^{\varepsilon_i}_t)$ for any $t$, so by Rellich–Kondrachov (writing everything locally as a graph over a fixed tangent plane), there exists a further subsequence $i_j$ and a function $f_t \in L^\infty(\widetilde{\Sigma}_t)$ such that $|DU_{i_j}| \to f_t$ in $L^2_{loc}$ (up to composition with the local graphs over the fixed tangent plane), which also implies a similarly defined pointwise a.e. convergence.

Additionally, by the locally uniform $C^{1,\alpha}$ convergence and the locally uniform gradient bound Theorem 5.3, we know that the surfaces $\widetilde{\Sigma}_t = \{U = t\}$ have a weak mean curvature vector $\vec{H}_t = -H_t\nu$ for a locally bounded function $H_t$, such that $\vec{H}^{\varepsilon_i}_{t_i} \to \vec{H}_t$ weakly in $L^2$. Since $\nu_i \to \nu$ locally uniformly, this implies that

$$\sqrt{|DU_i|^2 + K(\nu_i,\nu_i)^2} \to H_t \text{ weakly in } L^2,$$

where we used that $U_i$ is a strong solution of (31). Since $|DU_{i_j}|$ moreover converges strongly to $f_t$, we see that we in fact have strong convergence for $\sqrt{|DU_{i_j}|^2 + K(\nu_{i_j},\nu_{i_j})^2}$ and by the uniqueness of weak limits we find $f_t = \sqrt{H_t^2 - K(\nu,\nu)^2}$. In particular, $f_t$ is independent of the choice of subsequence, so $|DU_i|$ and $\vec{H}^{\varepsilon_i}_{t_i}$ converge strongly in $L^2_{loc}$ to $f_t$ and $\vec{H}_t$ respectively for the full sequence.

We now locally define $f(X) := f_t(X)$ if $X \in \widetilde{\Sigma}_t$ and notice that

$$f(X) = \lim_{i\to\infty} |DU_i| \circ \Pi_i(X)$$

pointwise a.e., where $\Pi_i$ denotes the projection of a point $X \in \widetilde{\Sigma_t}$ to the corresponding point $X_i \in \widetilde{\Sigma}^{\varepsilon_i}_{t_i}$ such that $X_i \to X$ as $\widetilde{\Sigma}^{\varepsilon_i}_{t_i} \to \widetilde{\Sigma}_t$ locally in $C^{1,\alpha}$. Since $U_i \to U$ locally uniformly, $\Pi_i$ is Hölder continuous for $i$ sufficiently large and thus $f$ is a measurable function. It is immediate that $f$ is locally bounded and non-negative everywhere.

In fact, $f > 0$ a.e. away from jump regions, where it is constructed: Assume that the set $f^{-1}(0)$ has positive measure. Then there is a small geodesic ball $B$ strictly away from any jump region and $E_0$, such that $S_f := f^{-1}(0) \cap B$ has positive measure. Define

$$M := \int_{S_f} |DU|\,,$$

where $M > 0$ due to Lemma 5.35. Using the above convergence to $f$ up to the projection $\Pi_i$ and the locally uniform bounds on $|DU_i|$, there exist sets $S^i_f$ such that

$$\int_{S^i_f} |DU_i| \to \int_{S_f} f = 0.$$

Due to the locally uniform convergence of the projections $\Pi_i$ to the identity, we know that for any (sufficiently small) domain $\Omega$ containing $S_f$ we also have $S^i_f \subseteq \Omega$ for $i$ sufficiently large. Using again that $|DU_i|$ is locally uniformly bounded, we see that for any $\varepsilon > 0$, there exists $\delta > 0$, $i_0 \in \mathbb{N}$ and a domain $\Omega$ such that $S_f \subseteq \Omega$, $\mathrm{Vol}(\Omega \setminus S_f) \leq \delta$ and

$$\int_\Omega |DU_i| < \varepsilon$$

for all $i \geq i_0$. In particular, for any $M > \varepsilon > 0$, we have

$$\int_\Omega |DU| \geq M > \varepsilon \geq \int_\Omega |DU_i| \,,$$

which gives a contradiction as

$$\liminf_{i\to\infty} \int_\Omega |DU_i| \geq \int_\Omega |DU|$$

as implied by the compactness theorem for BV functions, cf. Subsection 5.3. By a slightly refined argument, we in fact have $f \geq |DU|$ a.e..

Note that by the locally uniform $C^{1,\alpha}$ convergence of the translating graphs, we know that for any geodesic ball $B$ sufficiently small such that it remains strictly away from $E_0$ and any jump region, there exists $c \in \mathbb{R}$, $\delta > 0$ and $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, we have that $B \cap \partial \widetilde{\Sigma}_t^{\varepsilon_i} = \emptyset$ for all $t \in [\delta - c, \delta + c]$ and

$$\int_{c-\delta}^{c+\delta} \int_{B\cap\widetilde{\Sigma}_t^{\varepsilon_i}} \frac{1}{|DU_i|} \,\mathrm{d}\mu_i \,\mathrm{d}t = \int_B 1 = \int_{c-\delta}^{c+\delta} \int_{B\cap\widetilde{\Sigma}_t} \frac{1}{|DU|} \,\mathrm{d}\mu \,\mathrm{d}t$$

by the co-area formula. Applying it again in a similar fashion yields

$$\int_B |DU_i|^2 \to \int_B f\,|DU| \,.$$

Using the co-area formula once more together with the above considerations gives

$$\begin{aligned}
\int_B f\,|DU| &= \int_{B\setminus f^{-1}(0)} f\,|DU| \\
&= \int_{c-\delta}^{c+\delta} \int_{(B\setminus f^{-1}(0))\cap\widetilde{\Sigma}_t^{\varepsilon_i}} \frac{f}{|DU_i|} \,|DU|\, \mathrm{d}\mu_i \,\mathrm{d}t \\
&\to \int_{c-\delta}^{c+\delta} \int_{(B\setminus f^{-1}(0))\cap\widetilde{\Sigma}_t} |DU|\, \mathrm{d}\mu \,\mathrm{d}t \\
&= \int_{B\setminus f^{-1}(0)} |DU|^2 = \int_B |DU|^2 \,,
\end{aligned}$$

and as the sequence is constant in $i$ by the first identity the two integrals agree. Hence,

$$\int_B |DU_i|^2 \to \int_B |DU|^2 \,,$$

which concludes the proof. $\square$

Theorem 5.33 now follows by using the Compactness Theorem 5.31 for weak solutions.

## 5.6 Outward optimization principle and jump formation

We already know that the interior of jump regions of a weak solution $(U, \nu)$ is foliated by $C^{2,\alpha}$ generalized apparent horizons in $(M \setminus E_0) \times \mathbb{R}$, but did not discuss when such jumps occur. In this section, we will study the selection of jump times of weak solutions of STIMCF via an outward optimization property.

Let $\Omega$ be an open set in $M^{n+1}$, $\nu$ a measurable vector field on $M$. We call the set $E$ outward optimizing in $\Omega$ with respect to $\nu$, if $E$ minimizes area minus bulk term energy $|P_\nu|$ on the outside in $\Omega$. That is, if

$$|\partial^* E \cap A| \leq |\partial^* F \cap A| - \int\limits_{F \setminus E} |P_\nu| \,, \tag{55}$$

for any set $F$ containing $E$, such that $F \setminus E \subset A \subset\subset \Omega$, where $P_\nu = (g^{ij} - \nu^i \nu^j)\, K_{ij}$ as usual. We further call the set $E$ strictly outward optimizing in $\Omega$, if equality in (55) implies that $F \cap \Omega = E \cap \Omega$ up to a set of measure zero. Further, we define the strictly outward optimizing hull (in $\Omega$) $E' = E'_\Omega$ of a measurable set $E \subset \Omega$ to be the intersection of Lebesque points of all strictly outward optimizing sets in $\Omega$ that contain $E$. Up to a set of measure zero, $E'$ may be realized as a countable intersection, so $E'$ is in particular strictly outward optimizing and open. Due to the asymptotic decay of $g$ and $K$, the existence of strictly outward optimizing sets follows from the isoperimetric inequality if we allow $\Omega$ to be sufficiently large. In particular, any set admits a precompact outward optimizing hull if the domain $\Omega$ is sufficiently large.

Note that, unlike the corresponding outward optimization property of inverse null mean curvature flow introduced by Moore [63, Section 6], the bulk term energy in (55) is non-positive everywhere. This suggests that in non time-symmetric initial data sets, the additional energy induced by the second fundamental form $K$ makes the evolving surfaces jump sooner than it is the case for inverse mean curvature flow. In fact, it is immediate, that (strictly) outward optimizing surfaces are (strictly) outward minimizing as defined by Huisken–Ilmanen [50]. This was to be expected as inverse mean curvature flow acts as an upper barrier, cf. Corollary 5.11. However, as we can not expect STIMCF to be a simple reparameterization of inverse mean curvature flow, this underlines the role of the unit normal and the anisotropy of the problem, as the level sets are not only outward minimizing with respect to area, but outward optimizing with respect to the unit normal and second fundamental form $K$. Analogue to the respective optimization properties of inverse mean curvature flow and inverse null mean curvature flow, we establish the following:

**Theorem 5.38** (Outward optimization property)**.** *Suppose that $(U,\nu)$ is a weak solution of inverse space-time mean curvature flow with initial condition $E_0$, such that the level-sets $E_t=\{u<t\}$ of the projection $(u,\nu_M)$ onto $M$ are precompact for all $t>0$. Suppose further that $M$ has no compact components.*
*Then $E_t:=\{u<t\}$ is outward optimizing in $M$ with respect to $\nu_M$ for $t>0$, and $E_t^+:=\{u\leq t\}$ is outward optimizing in $M$ with respect to $\nu_M$ for $t\geq 0$. Furthermore, we have that*

*(i)* *$E_t^+$ is strictly outward optimizing in $\Omega$ with respect to $\nu_M$ for all $t\geq 0$, where $\Omega$ is an open set containing $E_t^+$ such that $\Omega$ does not contain any jump regions $\mathcal{K}_{t'}$ for $t'>t$.*

*(ii)* $(E_t)'_\Omega = E_t^+$ ,

*(iii)*

$$\left|\partial^* E_t^+\right| = |\partial^* E_t| + \int\limits_{E_t^+\setminus E_t} |P_{\nu_M}|\,,$$

*for all $t>0$. This precisely extends to $E_0$, if $E_0$ is outward optimizing.*

**Remark 5.39.**

(i) Note that (iii) in Theorem 5.38 implies that $|\Sigma_t| + \int_{\{u\leq t\}\setminus E_0} |P_{\nu_M}|$ is continuous on $(0,\infty)$, and continuous in 0 precisely when $\Sigma_0$ is outward optimizing. Moreover, the quantity is monotone under the smooth flow with

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|\Sigma_t| + \int\limits_{\{u\leq t\}\setminus E_0} |P_{\nu_M}|\right) = \int\limits_{\Sigma_t}\sqrt{\frac{H+|P|}{H-|P|}} \geq |\Sigma_t|\,.$$

(ii) By the continuity of the solution, there exists an open set $\Omega$ with the desired properties $\forall t\geq 0$. In particular, if $t'$ denotes the next time the solution will jump, we can choose $\Omega:=\{u<t''\}$ for any $t<t''<t'$. If we drop this restriction, then the outward optimizing hull $E_t'$ will agree with $E_t^+$ up to a "cost free" union of disjoint, open sets that are confined to other jump regions. We will make this statement precise in the following Proposition 5.40.

(iii) Similarly, the sets $\{U < t\}$ and $\{U \leq t\}$ are outward optimizing in $(M \setminus E_0) \times \mathbb{R}$ with respect to $\nu$. Moreover, if there exists a family of smooth solutions $(U_i, \nu_i)$ such that $U_i \to U$ locally uniformly and $\nu_i \to \nu$ locally uniformly in jump times of $U$ (which in particular is satisfied by the weak solutions constructed in Theorem 5.33) then $\{U \leq t\}$ is strictly outward optimizing in $(M \setminus E_0) \times \mathbb{R}$.

*Proof.* The fact that $E_t$ is outward optimizing immediately follows from the results of Lemma 5.24 and Lemma 5.30, since $E_t$ in particular minimizes $\mathcal{J}_{u,\nu}$ from the outside for all $t > 0$, so for all sets $F$ such that $E_t \subset F$ and $F \setminus E \subset A \subset\subset M$, it holds that

$$|\partial^* E_t \cap A| \leq |\partial^* F \cap A| - \int\limits_{F \setminus E_t} \sqrt{|Du|^2 + P_{\nu_M}^2} \leq |\partial^* F \cap A| - \int\limits_{F \setminus E_t} |P_{\nu_M}| \,.$$

By Remark 5.25, we also have for all $t \geq 0$ that

$$\left|\partial^* E_t^+ \cap A\right| - \int\limits_{E_t^+ \cap A} \sqrt{|Du|^2 + P_{\nu_M}^2} \leq |\partial^* F \cap A| - \int\limits_{F \cap A} \sqrt{|Du|^2 + P_{\nu_M}^2}, \tag{56}$$

for $F$ such that $E_t^+ \triangle F \subset A \subset\subset M \setminus E_0$. In particular, for any $t \geq 0$, we find that $E_t^+$ is outward optimizing in $M$, i.e.

$$\left|\partial^* E_t^+ \cap A\right| \leq |\partial^* F \cap A| - \int\limits_{F \setminus E_t^+} |P_{\nu_M}| \,, \tag{57}$$

for any $E_t^+ \subset F$, $F \setminus E_t^+ \subset A \subset\subset M$. We now proof $(i) - (iii)$.

(i) Let $\Omega$ be as above, suppose there exists $F \subset A \subset\subset \Omega$ containing $E_t^+$, such that

$$\left|\partial^* E_t^+ \cap A\right| = |\partial^* F \cap A| - \int\limits_{F \setminus E_t^+} |P_{\nu_M}|$$

Assume that $F \neq E_t^+$ and is not a set of measure zero. With regards to (56), this implies that $|Du| = 0$ a.e. on $F \setminus E_t^+$ and $F$ is also outward optimizing. Since the Lebesque points of an outward optimizing set form an open outward optimizing set, we can assume by a modification of measure zero, that $F$ is open. Then $u$ is constant on every connected component of $F \setminus E_t^+$. A contradiction, as $\Omega$ contains no jump regions outside of $E_t^+$. Hence, $E_t^+$ is outward optimizing in $\Omega$.

(ii) Since $E_t^+$ is strictly outward optimizing in $\Omega$, we have $E_t' := (E_t)_\Omega' \subseteq E_t^+$ by definition. If

$$|\partial^* E_t' \cap A| = \left|\partial^* E_t^+ \cap A\right| - \int\limits_{E_t^+ \setminus E_t'} |P_{\nu_M}|\,,$$

the strict optimization property of $E_t'$ implies that $E_t' = E_t^+$. Otherwise

$$|\partial^* E_t' \cap A| < \left|\partial^* E_t^+ \cap A\right| - \int\limits_{E_t^+ \setminus E_t'} |P_{\nu_M}|\,,$$

but this contradicts (56), since $|Du| = 0$ on $E_t^+ \setminus E_t'$.

(iii) Since $E_t$ is outward optimizing, we can use $E_t^+$ as a competitor to obtain

$$|\partial^* E_t \cap A| \leq \left|\partial^* E_t^+ \cap A\right| - \int\limits_{E_t^+ \setminus E_t} |P_{\nu_M}|\,,$$

for $t > 0$, and for $t = 0$ if $E_0$ happens to be outward optimizing itself. Again, since $|Du| = 0$ on $E_t^+ \setminus E_t$, strict inequality would contradict (56).

□

**Proposition 5.40.** *Let $(U, \nu)$ be a weak solution of STIMCF, such that $(U, \nu)$ and $M$ satisfy all assumptions of Theorem 5.38. Let $\Omega$ be a domain in $M \setminus \overline{E_0}$ such that $E_t^+ = \{u \leq t\} \subset \Omega$, and assume that $E_t$ admits a precompact outward optimizing hull $E_t' = (E_t)_\Omega'$ in $\Omega$ with respect to $\nu_M$. Then $E_t^+ = \{u \leq t\} \subseteq (E_t)_\Omega'$ up to a set of measure zero.*
*If we further assume that any jump region intersecting $\Omega$ is compactly contained in $\Omega$, then up to a set of measure zero all connected components of $E_t' \setminus E_t^+$ are contained in jump regions $\{u = t'\}$ for times $t' > t$ and*

$$\left|\partial^*(E_t' \setminus E_t^+)\right| = \int_{E_t' \setminus E_t^+} |P_{\nu_M}| \tag{58}$$

*Moreover, if any connected component $F$ of $E_t' \setminus E_t^+$ satisfies (58) by itself and is contained in the interior of a jump region, then $\nu_{\partial^* F} = \nu_M = \nu$ a.e. along $\partial^* F$.*

**Remark 5.41.** Note that the additional assumptions of Proposition 5.40 are satisfied for solutions which jump only a finite number of times. In particular, $E_t' = E_t^+$ for smooth solutions of STIMCF. In the case of a weak solution exhibiting jumps, the statement of Proposition 5.40 suggests that the flow does not want to increase the number of connected components of the level sets, even if the outward optimization principle would prefer to jump farther. See also the corresponding connectedness lemma, Lemma 5.45, for weak solutions of STIMCF below.

*Proof.* We define $P_{\nu_M}$ and $B_{u,\nu_M} = \sqrt{|Du|^2 + P_{\nu_M}}$ as before and we will write $E_t' = (E_t)'_\Omega$ for convenience. By assumption $E_t'$ is precompact. To establish the above claim, we will largely exploit the formula (40), which allows us state (41) more generally, as we have

$$|\partial^*(E_1 \cup E_2)| - \int\limits_{E_1 \cup E_2} f + |\partial^*(E_1 \cap E_2)| - \int\limits_{E_1 \cap E_2} f \leq |\partial^* E_1| - \int\limits_{E_1} f + |\partial^* E_2| - \int\limits_{E_2} f, \tag{59}$$

for any (locally) integrable function $f$ and precompact Caccioppoli sets $E_1$, $E_2$. Here, we will always consider $f = |P_{\nu_M}|$. As $E_t^+$ minimizes $\mathcal{J}_{u,\nu_M}$, we have

$$\left|\partial^* E_t^+\right| - \int\limits_{E_t^+} B_{u,\nu_M} \leq \left|\partial^*(E_t^+ \cap E_t')\right| - \int\limits_{E_t^+ \cap E_t'} B_{u,\nu_M},$$

and since $|Du| = 0$ on $E_t^+ \setminus E_t$, we can conclude that in fact

$$\left|\partial^* E_t^+\right| - \int\limits_{E_t^+} |P_{\nu_M}| \leq \left|\partial^*(E_t^+ \cap E_t')\right| - \int\limits_{E_t^+ \cap E_t'} |P_{\nu_M}|. \tag{60}$$

Choosing $E_1 = E_t^+$, $E_2 = E_t'$ in (59), we can conclude using (60) that

$$\left|\partial^*(E_t^+ \cup E_t')\right| - \int\limits_{(E_t^+ \cup E_t') \setminus E_t'} |P_{\nu_M}| \leq |\partial^* E_t'|.$$

Hence $E_t' = E_t^+ \cup E_t'$ up to a set of measure zero, as $E_t'$ is strictly outward optimizing in $\Omega$. Therefore $E_t^+ \subset E_t'$ up to a set of measure zero.

We now assume in addition that all jump regions intersecting $\Omega$ are compactly contained in $\Omega$. First consider a Caccioppoli set $B$ such that $E_t^+ \subseteq B \subseteq \Omega$ with $B \setminus E_t^+ \subseteq\subseteq \Omega$. Considering $E_1 = E_t^+$, $E_2 = B \setminus E_t^+$, (59) yields that

$$\begin{aligned}|\partial^* B| - \int_B |P_{\nu_M}| &\leq \left|\partial^* E_t^+\right| - \int_{E_t^+} |P_{\nu_M}| + \left|\partial^*(B \setminus E_t^+)\right| - \int_{B \setminus E_t^+} |P_{\nu_M}| \\ &\leq |\partial^* B| - \int_B |P_{\nu_M}| + \left|\partial^*(B \setminus E_t^+)\right| - \int_{B \setminus E_t^+} |P_{\nu_M}|,\end{aligned}$$

where we used that $E_t^+$ is outward optimizing in $\Omega$. In particular,

$$0 \leq \left|\partial^*(B \setminus E_t^+)\right| - \int_{B \setminus E_t^+} |P_{\nu_M}| \tag{61}$$

for all Cacciopoli sets $B$ with $E_t^+ \subseteq B \subseteq\subseteq \Omega$. As in the proof of Theorem 5.38 (i), consider a Cacciopoli set $F$ such that $E_t^+ \subseteq F \subseteq\subseteq \Omega$ with

$$\left|\partial^* E_t^+\right| - \int_{E_t^+} |P_{\nu_M}| = |\partial^* F| - \int_F |P_{\nu_M}| \,. \tag{62}$$

As before, we can conclude that $F$ is outward optimizing in $\Omega$ and (56) implies that

$$\left|\partial^* E_t^+\right| \leq |\partial^* F| - \int_{F\setminus E_t^+} B_{u,\nu_M} \leq |\partial^* F| - \int_{F\setminus E_t^+} |P_{\nu_M}| = \left|\partial^* E_t^+\right| .$$

Hence, equality holds everywhere and $|Du| \equiv 0$ a.e. on $F \setminus E_t^+$. We conclude that either $F = E_t^+$ or each connected component of $F \setminus E_t^+$ lies in a jump region $\{u = t'\}$ for $t' > t$. In particular, $F \setminus E_t^+$ remains strictly away from $E_t^+$ with positive distance and thus

$$\left|\partial^*(F \setminus E_t^+)\right| = \int_{F\setminus E_t^+} |P_{\nu_M}| \,.$$

Now consider $F_1, F_2$ such that $E_t^+ \subseteq F_1, F_2 \subseteq\subseteq \Omega$ satisfying (62). Then $(F_1 \cap F_2) \setminus E_t^+$ has a strict positive distance to $E_t^+$. Using that $E_t^+$ is outward optimizing in $\Omega$ and (59), we find

$$\begin{aligned}
\left|\partial^* E_t^+\right| - \int_{E_t^+} |P_{\nu_M}| &\leq |\partial^*(F_1 \cup F_2)| - \int_{F_1\cup F_2} |P_{\nu_M}| \\
&\leq |\partial^* F_1| - \int_{F_1} |P_{\nu_M}| + |\partial^* F_2| - \int_{F_2} |P_{\nu_M}| - |\partial^*(F_1 \cap F_2)| + \int_{F_1\cap F_2} |P_{\nu_M}| \\
&= 2\left|\partial^* E_t^+\right| - 2\int_{E_t^+} |P_{\nu_M}| - |\partial^*(F_1 \cap F_2)| + \int_{F_1\cap F_2} |P_{\nu_M}| \\
&= \left|\partial^* E_t^+\right| - \int_{E_t^+} |P_{\nu_M}| - \left|\partial^*((F_1 \cap F_2) \setminus E_t^+)\right| + \int_{(F_1\cap F_2)\setminus E_t^+} |P_{\nu_M}| \,.
\end{aligned}$$

This implies

$$\left|\partial^*((F_1 \cap F_2) \setminus E_t^+)\right| \leq \int_{(F_1\cap F_2)\setminus E_t^+} |P_{\nu_M}| \,.$$

By (61), we have equality. In particular, $F_1 \cup F_2$ satisfies (62). Now let $F$ denote the union of all such sets satisfying (62). Then $F$ satisfies (62) and $E_t^+ \subseteq F \subseteq\subseteq \Omega$ as all jump regions intersecting $\Omega$ are compactly contained in $\Omega$. By construction, $F$ is strictly outward optimizing in $\Omega$, so $E_t^+ \subseteq E_t' \subseteq F$. In particular, $|Du| \equiv 0$ a.e. on $E_t' \setminus E_t^+$, and as $E_t'$ is outward optimizing in $\Omega$, we have

$$\begin{aligned}|\partial^* E_t'| - \int_{E_t'\setminus E_t^+} B_{u,\nu_M} &= |\partial^* E_t'| - \int_{E_t'\setminus E_t^+} |P_{\nu_M}| \\ &\leq |\partial^* F| - \int_{F\setminus E_t'} |P_{\nu_M}| - \int_{E_t'\setminus E_t^+} |P_{\nu_M}| \\ &= |\partial^* F| - \int_{F\setminus E_t^+} |P_{\nu_M}| \\ &= \left|\partial^* E_t^+\right|.\end{aligned}$$

Note that (56) implies equality, and thus

$$|\partial^* E_t'| = |\partial^* F| - \int_{F\setminus E_t'} |P_{\nu_M}|.$$

As $E_t'$ is strictly outward optimizing in $\Omega$, this implies $E_t' = F$, and by the properties of $F$ we find that $E_t'$ is as claimed.

It remains to show that for any connected component $F$ of $E_t'\setminus E_t^+$ such that $\overline{F} \subseteq int\{u = t'\}$ $(t' > t)$ and $F$ satisfies (58) by itself, it holds that $\nu_{\partial^* F} = \nu_M = \nu$ a.e. along $\partial^* F$. By Theorem 5.20, the interior $\mathcal{K}_{t'}$ of the jump region $\{U = t'\}$ is foliated by generalized apparent horizons with $\nu \in C^{1,\alpha}_{loc}(\mathcal{K}_{t_0})$. By the translation invariance, we find

$$|P_{\nu_M}| = |P_\nu| = \operatorname{div}_{M\times\mathbb{R}} \nu = \operatorname{div}_M \nu_M,$$

and the claim then follows using the divergence theorem. □

**Proposition 5.42.** *For $t \geq 0$, we find that*

*(i)* $H = |P_{\nu_M}|$ *on* $\partial E_t^+ \cap \partial E_t^C$,

*(ii)* $H \geq |P_{\nu_M}|$ *in the weak sense on* $\partial E_t^+ \cap \partial E_t$.

*Proof.*

(i) Since $E_t^+$ is outward optimizing, we know that $E_t^+$ minimizes the functional

$$|\partial^* E_t^* \cap A| - \int\limits_{E_t^+\cap A} |P_{\nu_M}| \tag{63}$$

against any competitor $F$, such that $E_t^+ \subseteq F$, $F \setminus E_t^+ \subset A \subset\subset \Omega$. Now let $y \in \partial E_t^+$ such that $y \notin \overline{E}_t$. Then locally around $y$ $\partial E_t$ and $\partial E_t^+$ are separated by a jump region, and since $\partial E_t^+$ is $C^{1,\alpha}$, there exists an $R > 0$, such that $B_R(y) \cap \partial E_t^+$ is given by the graph over a $C^{1,\alpha}$ function $\omega$ and $W := B_R(y) \cap int(E_t^+) \subset E_t^+ \setminus \overline{E}_t$ is the subgraph of $\omega$, and $|Du| = 0$ on $W$. Using that $E_t^+$ minimizes (38), in particular from the inside, we conclude that $E_t^+$ minimizes (63) from the inside and outside for $K = \overline{B}_R(y)$. We conclude that $W$ minimizes (63) and therefore, as in Theorem 5.19, we see that $\omega$ minimizes the functional

$$\mathcal{J}'_{\nu_M}(\omega) := \int\limits_A \sqrt{1 + |D\omega|^2} - \int\limits_A \int\limits_0^{\omega(x)} |P_{\nu_M}| \,\mathrm{d}s \,\mathrm{d}x.$$

In particular $\omega \in C^{2,\alpha}_{loc}$, and $\partial E_t^+ \cap \overline{E_t}^C$ satisfies $H = |P_{\nu_M}|$.

(ii) If $\partial E_t$ and $\partial E_t^+$ only agree on a set of measure zero, there is nothing to show. So we assume there exists an $y \in \partial E_t$, such that $E_t \cap B_R(y) \subset E_t^+$ for some $R > 0$ and w.l.o.g. that $\partial E_t \cap B_R(y)$ is given as the graph over a $C^{1,\alpha}$ function $\omega$. Then similar to before, we can conclude that $\omega$ is a supersolution for $\mathcal{J}'_{\nu_M}$, i.e.,

$$0 \leq \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathcal{J}'(\omega + \varepsilon\eta) = \int\limits_K \frac{D\eta \cdot D\omega}{\sqrt{1 + |D\omega|^2}} - \eta \, |P_\nu| \,,$$

for all $\eta \in C_c^\infty(\partial E_t \cap B_R(y))$, such that $\eta \geq 0$. As $\partial E_t$ admits a weak mean curvature $H$, we can conclude that

$$0 \leq \int\limits_K \eta \,(H - |P_\nu|)$$

for all $\eta \in C_c^\infty(\partial E_t \cap B_R(y))$, such that $\eta \geq 0$. This establishes the claim.

□

Choosing appropriate initial data $E_0$, the properties of jump formation allow us to formulate conditions that will force any weak solution to jump. Therefore the existence of weak solutions Theorem 5.33 yields a method of detecting generalized apparent horizons.

**Corollary 5.43.**

(i) If $E_0$ is not outward minimizing, then $E_0^+ \neq E_0$,

(ii) If $E_0$ satisfies $H_{\partial E_0} < |\mathrm{tr}_{\partial E_0} K|$, then $\partial E_0^+$ is a $C^{2,\alpha}$ generalized apparent horizon disjoint from $\partial E_0$.

**Remark 5.44.** The conclusion in (ii) is generally stronger than in (i), as we can not guarantee that $\partial E_0^+ \cap E_0 = \emptyset$ in the first case. However, the assumption in (i) is more flexible as it does not depend on the sign of the mean curvature and is completely unrelated to $K$. Note that (i) also follows from Corollary 5.11, as $E_0^{+,(IMCF)} \subseteq E_0^+$, so the solution has to jump immediately if inverse mean curvature flow jumps at $t = 0$.

Regarding (ii), note that hypersurfaces satisfying $H < |\mathrm{tr}_{\partial E_0} K|$ are often referred to as *trapped surfaces* in the context of General Relativity, and weakly trapped if the inquality is not assumed to be strict. Assuming the existence of a weakly trapped surface, Eichmair [39] can similarly provide the existence of a generalized apparent horizon using the Perron method. Moreover, Eichmair can in fact construct an outermost generalized apparent horizon. Our result suggests that this outermost generalized apparent horizon is outward minimizing with respect to area (at least if all conditions of Theorem 5.33 are statisfied).

## 5.7 Asymptotic behavior

Finally, we discuss the asymptotic behavior of weak solutions $(U, \nu)$ of STIMCF as constructed in Subsection 5.5 via the projection $(u, \nu_M)$ on $M$ as defined in Subsection 5.4. We use a blowdown argument similar to [50, Section 7] to see that the level-sets of $u$ become asymptotically round. Moreover, we can use the notion of unit normal $\nu_M$ to show that the surfaces become in fact uniformly starshaped outside of a compact set if that set contains all jump regions.

To begin, we establish the connectedness Lemma [50, Lemma 4.2] in the case of STIMCF.

**Lemma 5.45.**

*(i) If $(u, \nu_M)$ is a weak solution of* (44)*, then $u$ has no strict local maxima or minima.*

*(ii) Suppose $M$ is connected and simply connected*[4] *with a single, asymptotically flat end, and $(E_t)_{t>0}$ is a solution of* (38) *for all $t > 0$ with initial condition $E_0$. If $\partial E_0$ is connected, $\Sigma_t = \partial E_t$ is connected, as long as it remains compact.*

*Proof.* We obtain (i) be arguing in analogue to [63, Lemma 19]. Once (i) is established, (ii) follows exactly as in the proof of [50, Lemma 4.2]. □

Let $(U, \nu)$ be a weak solution of STIMCF on $(M, g, K)$ with initial condition $E_0 \subseteq M$ and assume that $M$ has exactly one asymptotically flat end. We recall the set $\mathcal{O}_{R_0} \subseteq M$ as defined in Lemma 5.5 and may assume without loss of generality that $E_0 \subseteq M \setminus \mathcal{O}_{R_0}$. Thus we can regard $(U, \nu)$ (up to identification via the asymptotic chart $\Phi$) as a weak solution to

[4] Note that we may relax this assumption and allow curves that loop around $E_0$.

STIMCF on $(\Omega, g, K)$ where $\Omega = \Phi(\mathcal{O}_{R_0})$ is an open subset of $\mathbb{R}^{n+1}$. For $\lambda > 0$ we define the blowdown objects

$$\Omega^\lambda := \lambda \cdot \Omega,\ g^\lambda(y) := g\left(\frac{y}{\lambda}\right),\ K^\lambda(y) := \frac{1}{\lambda} K\left(\frac{y}{\lambda}\right),\ u^\lambda(y) := u\left(\frac{y}{\lambda}\right),\ \nu^\lambda_\Omega(y) := \nu_\Omega\left(\frac{y}{\lambda}\right),$$

where $\lambda \cdot A := \{x\colon \frac{x}{\lambda} \in A\}$, and $E^\lambda_t := \lambda \cdot E_t = \{u^\lambda < t\}$. Then $(u^\lambda, \nu^\lambda_\Omega)$ is a weak solution of (44) on $(\Omega^\lambda, g^\lambda, K^\lambda)$.

**Proposition 5.46.** *Let $(M, g, K)$ be a triple, such that $(M, g)$ is a Riemannian manifold, $K$ a symmetric $(0, 2)$ tensor on $M$ with $\operatorname{tr}_M K = 0$, s.t. $M$ has exactly one asymptotically flat end with*

$$|g - \delta| = o(1),\ \ |\nabla g| = o\left(\frac{1}{|y|}\right),\ K = o\left(\frac{1}{|y|}\right),\ \ |\nabla K| = o\left(\frac{1}{|y|^2}\right). \tag{64}$$

*Let $E_0$ be a compact $C^2$ domain in $M$ and $(U, \nu)$ a weak solution of STIMCF with initial condition $E_0$ constructed as in Theorem 5.33. Then there exist constants $c_\lambda \to \infty$ such that*

$$u^\lambda - c_\lambda \to v := n \ln(|y|)$$

*locally uniformly on $R^{n+1} \setminus \{0\}$ as $\lambda \to 0$, where $v$ is the standard expanding sphere solution on of inverse mean curvature flow on $(R^{n+1} \setminus \{0\}, \delta, 0)$.*

*Proof.* Let everything be as above. The asymptotic conditions (64) and Remark 5.4 imply that there is an $R_1 \geq R_0$, such that

$$r \geq c\,|y|\,,\ dist(y, \partial E_0) \geq c\,|y|$$

for all $y \in \mathbb{R}^{n+1} \setminus B_{R_1}(0)$, where $r$ satisfies the assumptions of Theorem 5.3 for $y$. Arguing as in [50, Lemma 7.1], we can use Theorem 5.3 and the fact that there is a suitable subsolution $\left(B_{\exp(\alpha t)}(0)\right)_{t_1 \leq t < \infty}$ for $0 < \alpha < n$ in the asymptotic region even under weaker decay assumptions, cf. Remark 5.6, to establish gradient and eccentricity estimates that are preserved under the blowdown by scaling invariance. Defining the constants $c_\lambda := \max_{\mathbb{S}_1(0)} \left|u^\lambda\right|$ for $\lambda$ sufficiently small we can use Arzelà–Ascoli to conclude that there exists as subsequence $(\lambda_{k_l})$, again denoted by $(\lambda_k)$, and local Lipschitz function $v \in C^{0,1}_{loc}(\mathbb{R}^{n+1} \setminus \{0\})$, such that

$$\widetilde{u}^{\lambda_k} := u^{\lambda_k} - c_{\lambda_k} \to v \text{ locally uniformly in } R^{n+1} \setminus \{0\},$$

and there exists $y_0 \in \mathbb{S}_1(0)$ with $v(y_0) = 0$. Moreover the eccentricity estimates imply that the level-sets of $v$ are non-empty and compact for all $t \in \mathbb{R}$, cf. [50, Lemma 7.1]. We are now left to show that $v$ is a weak solution of (44) in $(\mathbb{R}^{n+1}\setminus\{0\},\delta,0)^5$, i.e., a solution for the corresponding comparison principle for inverse mean curvature flow in $(R^{n+1}\setminus\{0\},\delta)$. Then by [50, Proposition 7.2], $v(y)=n\ln(|y|)$ is the standard expanding sphere solution. Since this is then in particular true for a subsequence of any subsequence, it follows that the full sequence converges, proving the claim.

However to conclude this, we need to confirm the stronger assumptions for the Compactness Theorem 5.31, in particular the local $L^1$ convergence of the gradients. To this end, we consider the solutions of the elliptic regularisation $u_{\varepsilon_n}$ on $F_{L_n}\setminus\overline{E_0}$, where $L_n\to\infty$ s.t. $u_{\varepsilon_n}\to u$ locally uniformly on $\Omega_0$. Now choose a subsequence $(L_{n_k})$, such that $L_{n_k}\geq\alpha\ln(\lambda^{-2})-\alpha\ln(R_0)$ and a (possibly different) subsequence $(\varepsilon_{n_k})$, s.t. $\varepsilon_{n_k}<\varepsilon_0(L_{n_k})$ and

$$\left|\left|u_{\varepsilon_{n_k}}-u\right|\right|_{C^0(\overline{F_{L_{n_k}}\cap\mathcal{O}_{R_3}})}\leq\lambda_k.$$

Now we define the subsets $\Omega_k:=\lambda_k\cdot\left(F_{L_{n_k}}\cap\mathcal{O}_{R_3}\right)=B_{e^\alpha\lambda^{-1}}(0)\setminus\overline{B}_{R_3\lambda}(0)\to\mathbb{R}^{n+1}\setminus\{0\}$ and consider the functions $\widetilde{u}_k(y):=u_{\varepsilon_{n_k}}\left(\frac{y}{\lambda_k}\right)-c_{\lambda_k}$. Then $\widetilde{u}_k\to v$ locally uniformly and $\widetilde{u}_k$ solves (32) in $(\Omega_k,g^{\lambda_k},K^{\lambda_k})$. In particular $\widetilde{U}_k(y,z):=\widetilde{u}_k(y)-\varepsilon z\to V(y,z):=v(y)$ locally uniformly in $(\mathbb{R}^{n+1}\setminus\{0\})\times\mathbb{R}$ and $\widetilde{U}_k$ is a smooth solution to STIMCF on $(\widetilde{M}_k,g^{\lambda_k}+\mathrm{d}z^2,\widetilde{K}^k)$, where $\widetilde{M}_k:=\Omega_k\times\mathbb{R}$ and $\widetilde{K}^k_{ij}=K^{\lambda_k}_{ij}$, $\widetilde{K}^k_{iz}=\widetilde{K}^k_{zz}=0$. Since the local gradient estimate is also satisfied for $\widetilde{U}_k$ on $\widetilde{M}_k$, we are in the same situation as in Corollary 5.11. Then arguing as in Subsection 5.3 and Subsection 5.5, and applying a modified Compactness Theorem as in Remark 5.32, we conclude that there exists an a.e. locally uniformly continuous unit vector field $\eta$, such that $(V,\eta)$ is a weak solution of (44) in $((\mathbb{R}^{n+1}\setminus\{0\})\times\mathbb{R},\delta,0)$. By Lemma 5.24, $v$ solves (44) on $(R^{n+1},\delta,0)$, concluding the proof. In particular, the convergence holds for any choice $(\lambda_k,L_{n_k},\varepsilon_{n_k})$ as above. □

**Corollary 5.47.** Away from jump regions, the level-sets $\Sigma_t=\{u=t\}$ become uniformly starshaped as $t\to\infty$. More precisely, if there exists $R_{reg}>0$ such that $u$ has no jumps on $\{|x|\geq R_{reg}\}$ in the asymptotic chart, then for any $\delta<1$, there exists $R(\delta)\geq R_{reg}$, such that

$$\langle v(y),y\rangle\geq(1-\delta)\,|y|\;\text{ for all }\;|y|\geq R(\delta),\tag{65}$$

in the asymptotic chart.

*Proof.* Let $A:=\{1\leq|x|\leq 3\}$ and choose $\lambda_k=2^{-k}$. For $0<\alpha<n$ as above, we choose $L_{n_k}\geq\alpha\ln(\lambda^{-2})-\alpha\ln(R_0)$, such that $A\subset\lambda_k\Omega_k$, where $\Omega_k$ is defined as in the proof of Proposition 5.46, but we replace the constant $R_3$ by $R_4:=\max\{R_3,R_{reg}\}$. Since $g$ satisfies the decay assumptions (64), $\delta$ and $g$ are equivalent on $\Omega_k$ with constants independent of $k$

[5] As $K\equiv 0$ we do not need to specify a choice of $\nu$.

and the locally uniform convergence of $\nu_\varepsilon$ implies that we can choose a subsequence $(\varepsilon_{n_k})$ such that

$$\left|\left|\nu_{\varepsilon_{n_k}} - \nu\right|\right|_{(\Omega_k,\delta)} \leq \frac{\delta}{2}$$

for all $k \geq k_1$, where we now have uniform convergence on all of $\Omega_k$, as it contains no jump regions. Further, Proposition 5.46 implies that there exists a $k_2 \in \mathbb{N}$ such that

$$\left|\left|\nu^{\lambda_k}_{\varepsilon_{n_k}} - \frac{x}{|x|}\right|\right|_A \leq \frac{\delta}{2}$$

for all $k \geq k_2$. Let $k_0 := \max(k_1, k_2)$, then for all $x \in A$, for all $k \geq k_0$, we have

$$\begin{aligned}\left\langle \nu^{\lambda_k}, \frac{x}{|x|}\right\rangle &= \left\langle \nu^{\lambda_k} - \nu^{\lambda_k}_{\varepsilon_{n_k}} + \nu^{\lambda_k}_{\varepsilon_{n_k}} - \frac{x}{|x|} + \frac{x}{|x|}, \frac{x}{|x|}\right\rangle \\ &\geq 1 - \left|\nu^{\lambda_k} - \nu^{\lambda_k}_{\varepsilon_{n_k}}\right| - \left|\nu^{\lambda_k}_{\varepsilon_{n_k}} - \frac{x}{|x|}\right| \\ &\geq 1 - \left|\left|\nu_{\varepsilon_{n_k}-\nu}\right|\right|_{(\lambda_k\Omega_{L_{n_k}},\delta)} - \left|\left|\nu^{\lambda_k}_{\varepsilon_{n_k}} - \frac{x}{|x|}\right|\right|_A \\ &\geq 1 - \frac{\delta}{2} - \frac{\delta}{2} = 1-\delta.\end{aligned}$$

Hence

$$\left\langle \nu^{\lambda_k}, x\right\rangle \geq (1-\delta)\,|x| \text{ for all } x \in A, \text{ and all } k \geq k_0.$$

Since $\nu^{\lambda_k}(x) = \nu\left(\frac{x}{\lambda_k}\right)$, a rescaling by $\lambda_k$ gives

$$\langle \nu(x), x\rangle \geq (1-\delta)\,|x| \text{ for } \lambda_k^{-1} \leq |x| \leq 3\lambda_k^{-1},$$

where $k \geq k_0$. Note that $\lambda_{k+1}^{-1} < 3\lambda_k^{-1}$, so we conclude that

$$\langle \nu(x), x\rangle \geq (1-\delta)\,|x| \text{ for } |x| \geq R(\delta),$$

with $R(\delta) := \lambda_{k_0}^{-1}$. □

**Remark 5.48.**

(i) Similar as in [50], one can see that in the case of $n=2$ $||\mathring{A}||^2_{L^2} \to 0$ as $t \to \infty$, so the level-sets approach coordinate spheres in $W^{2,2}$, cf. [36].

(ii) We expect the solutions to be smooth outside some compact set based on arguments similar as in [51]. In particular, all jump regions are contained in a compact set such that the level-sets are uniformly starshaped outside of that set.

(iii) In fact, we expect the solution to be well adapted to a new concept of center of mass proposed by Cederbaum–Sakovich [24] as $t \to \infty$, as we expect the level-sets of our solutions to asymptotically approach the foliation of STCMC surfaces constructed by them in the asymptotic region. The center of mass proposed by Cederbaum–Sakovich remedies some of the deficiencies of the center of mass formulation via surfaces of constant mean curvature first proposed by Huisken–Yau [54]. In particular, we expect our flow to exhibit better asymptotic behavior in non time-symmetric initial data sets than inverse mean curvature flow.

# 6 Uniqueness of STCMC surfaces on hyperboloids in class $\mathfrak{H}$

In this section, we give a characterization for STCMC surfaces on totally umbilic warped product graphs in spacetimes of class $\mathfrak{H}$ provided the spacetime satisfies the null energy condition. This generalizes a characterization of CMC surfaces in case of a time–symmetric warped product slice by Brendle [15]. In particular, we show that any STCMC surface on an hyperboloid in the Schwarzschild spacetime of mass $m > 0$ is a round sphere of constant radius. This section is based on single author work with a preprint available on arXiv [88].

The proof relies heavily on the warped product structure of the ambient spacetime and the spacelike graphs under consideration by showing that the result of Brendle [15] is applicable even for general warped product graphs if the graph can be extended past any Killing horizon to a minimal boundary and a second order ordinary differential inequality is satisfied for some non-negative function uniquely determining the spacelike graph. The main result then follows from a full characterization of totally umbilic warped product graphs, which we will refer to as hyperboloids. In particular, we can verify that the above conditions are satisfied for sufficiently small umbilicity factor and such that the differential inequality is in fact satisfied as an ODE.

This section is structured as follows: In Subsection 6.1 we briefly introduce the result of Brendle [15] for the convenience of the reader, before translating them into conditions on a suitable spacetime of class $\mathfrak{H}$ in Subsection 6.2. In Subsection 6.3 we study warped product graphs and give a full characterization of all such graphs that are totally umbilic. In Subsection 6.4 we study the (NEC) along such graphs, and discuss their extendability across a non-degenerate Killing horizon in Subsection 6.5. We then establish the main result of this section in Subsection 6.6.

## 6.1 Brendle's Alexandrov Theorem

For $n \geq 3$, $\overline{r} > 0$, consider a Riemannian manifold $(M, g)$ of the form

$$\begin{aligned} M &= [0, \overline{r}) \times \mathcal{N}, \\ g &= \mathrm{d}r^2 + \omega(r)^2 g_{\mathcal{N}}, \end{aligned}$$

where $(\mathcal{N}, g_{\mathcal{N}})$ is an $(n-1)$-dimensional compact Riemannian manifold with

$$\mathrm{Ric}_{g_{\mathcal{N}}} \geq \rho g_{\mathcal{N}}$$

for some constant $\rho \in \mathbb{R}$. Moreover, $\omega\colon [0, \overline{r}) \to \mathbb{R}$ is a positive function satisfying

(H1) $\omega'(0) = 0$, $\omega''(0) > 0$,

(H2) $\omega'(r) > 0$ for all $r \in (0, \overline{r})$,

(H3)

$$2\frac{\omega''(r)}{\omega(r)} - (n-2)\frac{\rho - \omega'(r)^2}{\omega(r)^2} \text{ is non-decreasing for } r \in (0, \overline{r}),$$

(H4)

$$\frac{\omega''(r)}{\omega(r)} + \frac{\rho - \omega'(r)^2}{\omega(r)^2} > 0 \text{ for all } r \in (0, \overline{r}).$$

**Theorem 6.1** (Brendle, Theorem 1.1 [15])**.** *Suppose that $(M, g)$ is a warped product manifold satisfying conditions (H1)–(H3). Moreover, let $\Sigma$ be a closed, embedded, orientable hypersurface in $(M, g)$ with constant mean curvature. Then $\Sigma$ is umbilic. If, in addition, the condition (H4) holds, then $\Sigma$ is a slice $N \times \{r\}$ for some $r \in (0, \overline{r})$.*

**Remark 6.2.**

(i) We note that it suffices to assume that the left-hand side in (H4) is non-zero, cf. condition (H4') in [15], and it in fact suffices to assume that it is non-zero on a dense subset.

(ii) Note that under condition (H2), there exists a change of coordinates such that

$$\begin{aligned} M &= [r_H, \widetilde{r}) \times \mathcal{N}, \\ g &= \frac{1}{h(r)}\,\mathrm{d}r^2 + r^2 g_{\mathcal{N}}, \end{aligned}$$

where $h\colon [r_H, \widetilde{r}) \to \mathbb{R}$ is such that

- $h(r) > 0$ on $(r_H, \widetilde{r})$,
- $h(r_H) = 0$ and $h'(r_H) > 0$,
- $\dfrac{h'(r)}{r} - (n-2)\dfrac{\rho - h(r)}{r^2}$ is non-decreasing on $(r_H, \widetilde{r})$,
- $\dfrac{h'(r)}{2r} + \dfrac{\rho - h(r)}{r^2} > 0$ on $(r_H, \widetilde{r})$,

where the above are equivalent to (H1), (H3), (H4), and we can again weaken the last inequality as in (i) if desired. Compare [15, Section 5], where we stated everything in [15] for $h$ and $\omega$ with their roles reversed for notational convenience, and also use $r$ as a coordinate in both cases by a slight abuse of notation.

## 6.2 The null energy condition in class $\mathfrak{H}$

Throughout this section, we will always consider $n+1$-dimensional spacetimes of class $\mathfrak{H}$ with metric coefficient $h$ and fibre $\mathcal{N}$ as defined in Subsection 4.3. For now, we will only consider the exterior region and for simplicity will always assume that it corresponds to the interval $(r_N, \infty)$. Let $(M_0, g^0)$ denote the time-symmetric ($K \equiv 0$) time slice $\{t = 0\}$ with induced metric $g^0$. Note that the warped product manifolds considered by Brendle [15] are precisely of the form $(M_0, g^0)$ after a change of coordinates as in Remark 6.2 using (H2), and $h$ and $(\mathcal{N}, g_{\mathcal{N}})$ fully determine the respective spacetime of class $\mathfrak{H}$. By Remark 6.2 and Lemma 4.5, it is straightforward to see that condition (H1) is equivalent to the fact that $\{r = r_H\}$ is a non-degenerate Killing horizon in $(\mathfrak{M}, \mathfrak{g})$ with $r_H = r_N$. Moreover, Wang–Wang–Zhang [83] pointed out that condition (H3) of Brendle precisely translates to $(\mathfrak{M}, \mathfrak{g})$ satisfying the null energy condition. For the convenience of the reader, we collect the respective equivalences in the following Lemma:

**Lemma 6.3.** *Let $(\mathfrak{M}, \mathfrak{g})$ be a spacetime of class $\mathfrak{H}$ with metric coefficient $h$, and let $f := \sqrt{h}$ on $(r_N, \infty)$. Then the following are equivalent:*

*(i)* $(\mathfrak{M}, \mathfrak{g})$ *satisfies the (NEC).*

*(ii)* $\Delta_0 f g^0 - \mathrm{Hess}_0 f + f\mathrm{Ric}^0 \geq 0$ *on $M_0$, where $\Delta_0$, $\mathrm{Hess}_0$, and $\mathrm{Ric}^0$ denote the Laplacian, the Hessian and Ricci curvature with respect to $g^0$, respectively.*

*(iii)* *It holds*

$$\frac{1}{2}h'' + \frac{(n-3)}{2r}h' - \frac{n-2}{r^2}h + r^{-2}\mathrm{Ric}_{g_{\mathcal{N}}}(X, X) \geq 0$$

*on $M_0$ for all unit tangent vector fields $X \in \Gamma(T\mathcal{N})$ on $\mathcal{N}$, where $\mathrm{Ric}^{\mathcal{N}}$ denotes the Ricci curvature on $\mathcal{N}$.*

*(iv)* *The function $x = h - \alpha \colon (r_N, \infty) \to \mathbb{R}$ is a solution of the ordinary differential inequality*

$$\frac{1}{2}x'' + \frac{(n-3)}{2s}x' - \frac{(n-2)}{s^2}x \geq 0,$$

*where $(n-2)\alpha$ is the minimum of the smallest eigenvalue of $\mathrm{Ric}_{g_{\mathcal{N}}}$ on $\mathcal{N}$.*

For the equivalence of (i) to (iii), we refer to the respective results of Brendle and Wang–Wang–Zhang, see Proposition 2.1. in [15] and Lemma 3.8 in [83], where $\alpha$ is the same as the constant $\rho$ considered by Brendle [15] as above. The fourth equivalence is immediate, since we assume $\mathcal{N}$ to be compact. We moreover observe that

$$L_g^*(f) = -\Delta_0 f g^0 + \mathrm{Hess}_0 f - f\mathrm{Ric}^0,$$

where $L_g^*$ denotes the formal $L^2$ adjoint of the linearization of the scalar curvature operator, cf. [35] Lemma 2.2 . Thus, the (NEC) implies the existence of a non-trivial supersolution $f > 0$ of the formal $L^2$ adjoint of the linearization of the scalar curvature operator. As equality for the (NEC) implies a non-trivial kernel, we can conclude from [35] Lemma 2.3 that then the scalar curvature $R^0$ of the time-symmetric slices must necessarily be constant in this case.

Wang [85] further noticed that the (NEC) is in fact also related to an eigenvalue analysis of the Ricci curvature tensor $\mathrm{Ric}^0$ of the time-symmetric time slices. Namely, the (NEC) implies monotonocity for the difference between the eigenvalue $h\mathrm{Ric}^0_{rr}$ and any eigenvalue of $\mathrm{Ric}^0|_{N\times N}$. Compare Lemma 5.1 in [85]. Analogous to spherical symmetry, we can establish this monotonicity in the general case:

**Lemma 6.4.** *Let $(\mathfrak{M}, \mathfrak{g})$ be a spacetime of class $\mathfrak{H}$ satisfying the (NEC), and let $X$ be a unit tangent vector on $\mathcal{N}$. Then*

$$r^n\left(\frac{(n-2)}{2r}h' - \frac{(n-2)}{r^2}h + r^{-2}\mathrm{Ric}_{g_\mathcal{N}}(X,X)\right) \tag{66}$$

*is monotone non-decreasing in $r$.*

*Proof.*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}r}\left(r^n\left(\frac{(n-2)}{2r}h' - \frac{(n-2)}{r^2}h + r^{-2}\mathrm{Ric}_{g_\mathcal{N}}(X,X)\right)\right)\\
&\quad= \frac{(n-2)}{2}r^{n-1}h'' + \frac{(n-2)(n-3)}{2}r^{n-2}h' - (n-2)^2 r^{n-3}h + (n-2)r^{n-3}\mathrm{Ric}_{g_\mathcal{N}}(X,X)\\
&\quad= (n-2)r^{n-1}\left(\frac{1}{2}h'' + \frac{(n-3)}{2r}h' - \frac{(n-2)}{r^2}h + r^{-2}\mathrm{Ric}_{g_\mathcal{N}}(X,X)\right) \geq 0,
\end{aligned}$$

where we used Lemma 6.3 in the last line. $\square$

Thus, this holds true for any unit eigenvector $X \in \Gamma(T\mathcal{N})$ of $\mathrm{Ric}_{g_\mathcal{N}}$, in particular for the minimum $(n-2)\alpha$, and condition (H4) of Brendle [15] is equivalent to the monotone quantity (66) being strictly positive everywhere. Due to the monotonicity, it suffices to check this at the inner boundary, so condition (H4) is in particular implied by the boundary condition

$$h'(r_H)r_H + 2\alpha > 0,$$

and hence immediate for $\alpha \geq 0$. Compare Remark 5.2 in [85]. However, as stated in Remark 6.2 it is enough to assume that (66) is non-trivial on a dense subset. This more general assumption is in fact needed in our case, as we will extend the graphical slices past the original inner boundary, and condition (H4) will in general fail in the interior of the Killing horizon even in spherical symmetry, for example in the case of the subextremal Reissner–Nordström spacetime corresponding to

$$h(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}$$

for constants $m$, $q$ with $m > |q|$. Thus, we will usually refer to the more general assumption

$$\frac{(n-2)}{2r}h' - \frac{(n-2)}{r^2}h + r^{-2}\mathrm{Ric}^N(X,X) \neq 0 \tag{67}$$

for any unit eigenvector $X$ tangent to $N$ on a dense subset of $(0,\infty)$.

## 6.3 Graphical spacelike hypersurfaces

We consider spacelike warped product graphs over the canonical $\{t = 0\}$ time slice $M_0$ in spacetimes of Class $\mathfrak{H}$. More precisely, we look at hypersurfaces $M_T$ of the form

$$M_T = \{(T(s), s) \colon s \in (r_1, r_2)\} \times \mathcal{N}$$

for some smooth function $T : (r_1, r_2) \to \mathbb{R}$ with $r_H < r_1 < r_2 \leq \infty$. We will refer to $T$ as the radial height function. We further denote the induced metric and second fundamental form of $M_T$ as $g^T$ and $K^T$, respectively.

Note that any spacelike slice in a static spacetime can always be written as a graph, so the above assumption is only restrictive in the sense that we assume that $T$ is only depending on $r$. For general graphical initial data sets $(M_T, g^T, K^T)$ given as $M_T = \mathrm{graph}_{M_0} T$, the spacelike condition yields a restriction on the gradient of $T$, i.e. that $1 - h\left|\nabla_0 T\right|^2 > 0$, where $\nabla_0$ denotes the gradient on $M_0$. Using the computations of Cederbaum–Nerz [23] for graphs in general static spacetimes with coordinates $\{x^i\}$ on $M_0$, we get

$$\begin{aligned} \partial_i^T &= \partial_i + T_{,i}\partial_t, \\ g_{ij}^T &= g_i j - hT_{,i}T_{,j}, \end{aligned}$$

and the future timelike unit normal $\vec{n}$ is given by $\vec{n} = \frac{\partial_t + h\nabla_0 T}{f\sqrt{1-h|\nabla_0 T|^2}}$ with $f := \sqrt{h}$ as above. Moreover, the second fundamental form $K^T$ is given by[6]

$$K^T(\partial_i^T, \partial_j^T) = \frac{f\mathrm{Hess}_0 T(\partial_i, \partial_j) + T_{,i}f_{,j} + f_{,i}T_{,j} - h\left\langle \nabla_0 T, \nabla_0 f\right\rangle T_{,i}T_{,j}}{\sqrt{1 - h\left|\nabla_0 T\right|^2}}.$$

[6]Note that there is a slight mistake in the formula in [23] which has been corrected in [24]

If $M_T = \{(T(s), s, x^I)\}$ embeds in Class $\mathfrak{H}$ with coordinates $\{s, x^I\}$, where $x^I$ denote (local) coordinates of $\mathcal{N}$, this yields

$$\begin{aligned} g^T &= \frac{1}{h_T}\,\mathrm{d}s^2 + s^2 g_{\mathcal{N}}, \\ K^T &= a_T\,\mathrm{d}s^2 + b_T s^2 g_{\mathcal{N}}, \end{aligned} \tag{68}$$

where

$$h_T = \frac{h}{1 - h\,|\nabla_0 T|^2}, \tag{69}$$

$$a_T = \frac{fT'' + f'T'(3 - h\,|\nabla_0 T|^2)}{\sqrt{1 - h\,|\nabla_0 T|^2}}, \tag{70}$$

$$b_T = \frac{f^3 T'}{s\sqrt{1 - h\,|\nabla_0 T|^2}}, \tag{71}$$

with the tangent vector fields $\partial_s = \partial_r + T'\partial_t$, $\partial_I$ and the future unit normal

$$\vec{n}^T = \frac{\partial_t + h\nabla_0 T}{f\sqrt{1 - h\,|\nabla_0 T|^2}}.$$

In particular, we see from (68) that $(M_T, g^T)$ is again a warped product manifold as considered by Brendle [15] with $h_T \geq h$, and that $K$ also satisfies a similar block diagonal form. The main observation in this subsection is to see that both the intrinsic and extrinsic curvature for such spacelike warped product graphs are fully determined by the difference $h_T - h$ in Class $\mathfrak{H}$. This essentially follows from the following Lemma:

**Lemma 6.5.**

$$\begin{aligned} b_T^2 &= \frac{h_T - h}{s^2}, \\ h_T a_T b_T &= \frac{h_T' - h'}{2s}. \end{aligned}$$

*Proof.* By (69) and using that $|\nabla_0 T|^2 = h \cdot (T')^2$, we see that

$$h_T - h = \frac{h}{1 - h\,|\nabla_0 T|^2} - h = \frac{h^3 \cdot (T')^2}{1 - h\,|\nabla_0 T|^2}. \tag{72}$$

Thus, the first identity follows by taking a square of (71). Taking a derivative of (72) the second identity follows from straightforward computation. □

**Remark 6.6.** Further, the difference $h_T - h$ also uniquely determines the radial height function $T$ up to a choice of sign of the derivative and a constant of integration, as

$$|T'| = \frac{1}{h}\sqrt{\frac{h_T - h}{h_T}}.$$

More precisely $(M_T, g^T, K^T)$ is fully determined by the choice of function $b_T$ with $h_T = h + r^2 b_T^2$ and

$$T' = \frac{r b_T}{h\sqrt{h + r^2 b_T^2}}.$$

As a consequence, this rigid structure yields a full characterization of totally umbilic spacelike warped product graphs in Class $\mathfrak{H}$:

**Corollary 6.7.** Let $(M_T, g^T, K^T)$ be a spacelike warped product graph as above, and we further assume that $K^T = \lambda_T g^T$ for some smooth function $\lambda_T$. Then $\lambda_T$ is constant, and $(M_T, g^T, K^T)$ is fully determined by the choice $b_T = \lambda_T$.

In particular, $h_T = h + \lambda_T^2 s^2$, and motivated by this we refer to such spacelike totally umbilic warped product graphs as *hyperboloids* in Class $\mathfrak{H}$.

*Proof.* Since $K^T = \lambda_T g^T$, we have

$$h_T a_T = \lambda_T = b_T,$$

in particular

$$h_T a_T b_T = b_T^2.$$

Using Lemma 6.5, we see that

$$(h_T - h)' = \frac{2}{s}(h_T - h),$$

and solving the ODE gives $h_T - h = Cs^2$, where necessarily $C \geq 0$ since $h_T \geq h$. Lastly, this yields

$$\lambda_T^2 = b_T^2 = C,$$

completing the proof. $\square$

**Remark 6.8.** Similarly, we can characterize all rotationally symmetric graphs with $\operatorname{tr}_{M_T} K^T \equiv C$ via

$$h_T = h + \left(\frac{C}{n}s + \frac{c_1}{s^{n-1}}\right)^2$$

for some real constant $c_1 \in \mathbb{R}$. The choice $c_1 = 0$ corresponds to the totally umbilic case and we recover the hyperboloids. For $C = 0$, we obtain a 1-parameter family of maximal hypersurfaces. These CMC graphs have been considered by Bartnik–Simon [6] as barries in the Minkowski spacetime. See also [4, 42].

**Remark 6.9.** Although the assumption of being a graph in the above sense is much more restrictive in the timelike case, we can establish a similar warped product structure for timelike graphs with radial height function $T = T(s)$ where we now require that $h\left|\nabla_0 T\right|^2 - 1 > 0$. More precisely, we find that

$$\begin{aligned}
g_T &= -\frac{1}{h_T}\,\mathrm{d}s^2 + s^2 g_{\mathcal{N}},\\
K_T &= a_T\,\mathrm{d}s^2 + b_T s^2 g_{\mathcal{N}},
\end{aligned}$$

with

$$\begin{aligned}
h_T &= \frac{h}{h\left|\nabla_0 T\right|^2 - 1},\\
a_T &= \frac{fT'' + f'T'(3 - h\left|\nabla_0 T\right|^2)}{\sqrt{h\left|\nabla_0 T - 1\right|^2}},\\
b_T &= \frac{f^3 T'}{s\sqrt{h\left|\nabla_0 T\right|^2 - 1}},
\end{aligned}$$

and find the relations

$$\begin{aligned}
b_T^2 &= \frac{h_T + h}{s^2},\\
h_T a_T b_T &= -\frac{(h_T + h)'}{2s}.
\end{aligned}$$

In the totally umbilic case, this leads to the same ODE system characterizing rotationally symmetric photon surfaces in class $\mathcal{S}$ derived by Cederbaum–Galloway [21][7]. In [22], Cederbaum–Jahns–Vičánek Martínez fully characterize the behavior of solutions to this ODE, in particular showing that rotationally symmetric photon surfaces are either photon spheres or warped product graphs of the above sense away from singular radii.

In particular $h_T = \lambda_T^2 s^2 - h$ with $\lambda_T \neq 0$ constant, so up to dividing $h_T$ by $s^2$, the function determining the induced metric is the effective potential studied by Cederbaum–Jahns–Vičánek Martínez [22] in order the characterize solutions of the ODE system (away from singular radii where these coordinates break down).

[7]This has been extended to Class $\mathfrak{H}$ by Cederbaum and the author in [25].

## 6.4 The (NEC) on spacelike warped product graphs

We first show that we can rewrite the (NEC) along any spacelike warped product graph $(M_T, g^T, K^T)$ as a tensor inequality adapted to the slice. Recall that the (NEC) along $(M_T, g^T, K^T)$ equivalently implies that for any unit vector $V_T$ on $(M_T, g^T, K^T)$

$$
\begin{aligned}
0 \leq & \mu_T - \frac{1}{2}\mathfrak{R} \pm 2J_T(V_T) + \mathfrak{Ric}(V_T, V_T) \\
= & \mu_T - \frac{1}{2}\mathfrak{R} \pm 2J_T(V_T) + (\operatorname{tr}_T K^T K^T - (K^T)^2)(V_T, V_T) \\
& + \operatorname{Ric}^T(V_T, V_T) - \mathfrak{Rm}(V_T, \vec{n}^T, V_T, \vec{n}^T),
\end{aligned} \tag{73}
$$

where we used the once contracted Gauss equation in the last line. Here $\mu_T$, $J_T$ denote the energy and momentum density of $(M_T, g^T, K^T)$, respectively, and $\operatorname{tr}_T$ denotes the trace with respect to $g^T$. We will similarly denote the respective quantities on the $\{t = 0\}$ time slice $M_0$ with a subscript 0. A direct computation yields the following well-known curvature identities in class $\mathfrak{H}$:

**Lemma 6.10.** *For any warped product graph $(M_T, g^T)$, we have that*

$$
\begin{aligned}
\operatorname{Ric}^T_{ss} &= -\frac{(n-1)}{2s}\frac{h_T'}{h_T}, \\
\operatorname{Ric}^T_{IJ} &= (\operatorname{Ric}_{\mathcal{N}})_{IJ} - \left((n-2)h_T + \frac{1}{2}sh_T'\right)(g_{\mathcal{N}})_{IJ}, \\
\mathrm{R}^T &= s^{-2}\mathrm{R}_{\mathcal{N}} - \frac{(n-1)}{s^2}\left((n-2)(h_T) + sh_T'\right), \\
\frac{\operatorname{Hess}_T f_T}{f_T} &= \frac{1}{2}\frac{h_T''}{h_T}\,\mathrm{d}s^2 + \frac{1}{2}rh_T' g_{\mathcal{N}}, \\
\frac{\Delta_T f_T}{f_T} &= \frac{1}{2}h_T'' + \frac{(n-1)}{2s}h_T',
\end{aligned}
$$

*where $\operatorname{Ric}^T$, $R^T$, $\operatorname{Hess}_T$, and $\Delta_T$ denote the Ricci curvature, scalar curvature, Hessian and Laplacian along $(M_T, g^T)$ respectively, and $f_T$ is defined as $f_T := \sqrt{h_T}$ on $I$.*

**Lemma 6.11.** *Let $(\mathfrak{M}, \mathfrak{g})$ be a spacetime of Class $\mathfrak{H}$. Then*

$$\begin{aligned}
\mathfrak{Rm}(\cdot, \partial_t, \cdot, \partial_t) &= f_0 \mathrm{Hess}_0 f_0(\cdot, \cdot) \\
\mathfrak{Rm}(\cdot, \partial_r, \cdot, \partial_r) &= f_0 \mathrm{Hess}_0 f_0(\partial_r, \partial_r)\, \mathrm{d}t^2 - f^{-3} \mathrm{Hess}_0 f_0(\partial_I, \partial_J)\, \mathrm{d}x^I\, \mathrm{d}x^J \\
\mathfrak{Rm}(\cdot, \partial_r, \cdot, \partial_t) &= -\frac{1}{2} h''\, \mathrm{d}t\, \mathrm{d}r \\
\mathfrak{Rm}(\cdot, \partial_t, \cdot, \partial_r) &= -\frac{1}{2} h''\, \mathrm{d}r\, \mathrm{d}t \\
\mathfrak{R} &= -h'' - \frac{(n-1)}{s^2} \left((n-2)h + 2sh'\right) + s^{-2} \mathrm{R}_{\mathcal{N}},
\end{aligned}$$

*where $\mathrm{R}_{\mathcal{N}}$ denotes the scalar curvature on $\mathcal{N}$, and $f_0 := \sqrt{h}$.*

We now define an isomorphism between the tangent bundles of $M_T$ and $M_0$ in the following way: Let $V_T = c_1 f_T(s)\partial_s + \frac{c_2}{s} X$ be a tangent vector field along $M_T$, where $X$ is a unit vector field tangent to $\mathcal{N}$. We define the vector field $V_0$ tangent to $M_0$ as $V_0 := c_1 f(s)\partial_r + \frac{c_2}{s} X$. Note that this isomorphism is not induced by an isometry between $M_T$ and $M_0$ unless $T = \text{const.}$. Using Lemmas 6.5, 6.10, 6.11, we establish the following:

**Lemma 6.12.** *For $f_0 := f = \sqrt{h}$ as above, we find*

(i) $\mu_T = \mu_0 = \frac{1}{2} R_0 = \frac{1}{2}\mathfrak{R} + \frac{\Delta_0 f_0}{f_0}$,

(ii) $J_T \equiv 0$,

(iii) $\mathrm{Ric}^T(V_T, V_T) + (\mathrm{tr}_T K^T K^T - (K^T)^2)(V_T, V_T) = \mathrm{Ric}^0(V_0, V_0)$,

(iv) $\mathfrak{Rm}(V_T, \vec{n}^T, V_T, \vec{n}^T) = \mathfrak{Rm}(V_0, \partial_t, V_0, \partial_t) = \frac{\mathrm{Hess}_0 f_0(V_0, V_0)}{f_0}$

**Remark 6.13.** As the spacetime is static, it is unsurprising that with these identities (73) directly reduces to Lemma 6.3 (ii). However, as we aim to employ the result of Brendle [12] directly on the slice, we will instead write (73) as a tensor inequality involving $f_T := \sqrt{h_T}$.

We also want to emphasize the vanishing momentum constraint $J_T \equiv 0$, as the converse is also true in the following sense: If $(\widetilde{M}, \widetilde{g}, \widetilde{K})$ is a rotationally symmetric initial data set as above determined by the functions $\widetilde{h}$, $\widetilde{a}$, $\widetilde{b}$, then $\widetilde{J} \equiv 0$ if and only if $(\widetilde{M}, \widetilde{g}, \widetilde{K})$ embeds as a warped product graph into a spacetime of Class $\mathfrak{H}$ with $h := \widetilde{h} - s^2\widetilde{b}^2$. In the context of the construction of such rotationally symmetric initial data sets, this yields the maximal future development under the additional assumption $J \equiv 0$, cf. Cabrera Pacheco–Wolff [17].

*Proof of Lemma 6.12.* In view of the Remark, we only prove (ii), as the derivation of (i), (iii), (iv) follows by direct computation from Lemmas 6.10, 6.11. In coordinates $J_T$ is given as

$$(J_T)_i = (g^T)^{jk} \left( K_{ki,j} - \Gamma^l_{jk} K_{li} - \Gamma^l_{ij} K_{kl} \right) - \operatorname{tr}_T K_{,i},$$

for indices $i, j, k, l \in \{s, I, J, K, L\}$. Using the block diagonal structure of $K$ and the well-known identities for the Christoffel symbols in class $\mathfrak{H}$, a direct computation shows that

$$J^T = \frac{2}{s} \left( h_T a_T - b_T - s b'_T \right) \mathrm{d}s.$$

Thus, it remains to show that $h_T a_T - b_T - s b'_T$ vanishes along $M_T$. As $b_T$ is in particular continuous, there exists a closed set $\mathcal{X} \subset I_T := (r_1, r_2)$ of measure zero, such that for all $s \in I_T \setminus \mathcal{X}$ there exists an open neighborhood $U_s$ of $s$, such that either $b_T \neq 0$ or $b_T$ vanishes identically on $U_s$. In the first case, multiplying the equation by $b_T$ yields

$$b_T \left( h_T a_T - b_T - s b'_T \right) = h_T a_T - b_T^2 - s b'_T b_T = 0,$$

which vanishes by the identities of Lemma 6.5 and using

$$2 b'_T b_T = (b_T^2)' = \frac{h'_T - h'}{s^2} - 2 \frac{h_T - h}{s^3}.$$

Since we assumed $b_T \neq 0$ , we have $h_T a_T - b_T - s b'_T = 0$. On the other hand, if $b_T$ vanishes identically on a neighborhood, then $T = \text{const.}$ by Remark 6.6, so $h_T = h$ and $a = 0$. In particular $h_T a_T - b_T - s b'_T = 0$. Therefore $h_T a_T - b_T - s b'_T$ vanishes on $I_T \setminus \mathcal{X}$. By continuity, it has to vanish on all of $I_T$, which concludes the proof of (2). $\square$

We now establish the relevant tensor inequality adapted to the warped product graphs $(M_T, g^T, K^T)$.

**Proposition 6.14.** *Let $(\mathfrak{M}, \mathfrak{g})$ be a spacetime of Class $\mathfrak{H}$ that satisfies the (NEC), and let $(M_T, g^T, K^T)$ be a warped product graph in $(\mathfrak{M}, \mathfrak{g})$. Then, for all unit vector fields $V_T$ on $(M_T, g^T, K^T)$ we have*

$$B^T(V_T, V_T) \leq \frac{\Delta_T f_T}{f_T} - \frac{\operatorname{Hess}_T f_T(V_T, V_T)}{f_T} + \operatorname{Ric}^T(V_T, V_T), \tag{74}$$

*where*

$$B^T = \left( \frac{1}{2}(h_T - h)'' + \frac{(n-3)}{2s}(h_T - h)' - \frac{(n-2)}{s^2}(h_T - h) \right) s^2 g_{\mathcal{N}}.$$

*Proof.* Note that

$$\begin{aligned}(K^T)^2 &= h_T a_T^2 \,\mathrm{d}s^2 + b_T^2 s^2 g_{\mathcal{N}},\\ \operatorname{tr}_{M_T} K^T &= h_T a_T + (n-1) b_T, \\ |K^T|^2 &= h_T^2 a_T^2 + (n-1) b_T^2,\end{aligned} \tag{75}$$

and using Lemma 6.5, direct computation shows

$$\operatorname{tr}_{M_T} K^T K^T - (K^T)^2 = \frac{(n-1)}{2} \frac{h_T' - h'}{s h_T} \,\mathrm{d}s^2 + \left( \frac{h_T' - h'}{2s} + \frac{(n-2)}{s^2} (h_T - h) \right) s^2 g_{\mathcal{N}}. \tag{76}$$

For a unit vector $V_T = c_1 f_T \partial_s + \frac{c_2}{s} X$ on $M_T$, by Lemma 6.12 the (NEC) gives

$$0 \leq \frac{\Delta_0 f_0}{f_0} - \frac{\mathrm{Hess}_0 f_0}{f_0}(V_0, V_0) + \mathrm{Ric}^T(V_T, V_T) + \left( \operatorname{tr}_T K^T K^T - (K^T)^2 \right)(V_T, V_T).$$

Note that a direct computation yields

$$\begin{aligned}&\left( \frac{\Delta_T f_T}{f_T} - \frac{\Delta_0 f_0}{f_0} \right) - \left( \frac{\mathrm{Hess}_T f(V_T, V_T)}{f_T} - \frac{\mathrm{Hess}_0 f_T(V_0, V_0)}{f_0} \right) - \left( \operatorname{tr}_{M_T} K^T K^T - K^2 \right)(V_T, V_T)\\ &= c_2^2 \left( \frac{1}{2}(h_T - h)'' + \frac{(n-3)}{2s}(h_T - h)' - \frac{(n-2)}{s^2}(h_T - h) \right)\\ &= B^T(V_T, V_T).\end{aligned}$$

Inserting this into the tensor inequality above yields the claim. □

In particular, the (NEC) on $(\mathfrak{M}, \mathfrak{g})$ implies the same tensor inequality

$$0 \leq \frac{\Delta_T f_T}{f_T} - \frac{\mathrm{Hess}_T f_T(V, V)}{f_T} + \mathrm{Ric}^T(V, V) \tag{77}$$

on general warped product initial data sets $(M_T, g^T, K^T)$, provided that $B^T$ is positive semi-definite. Moreover, $B^T \geq 0$ if and only if $x = h_T - h$ is a non-negative solution of the linear ordinary differential inequality

$$\frac{1}{2} x'' + \frac{(n-3)}{2s} x' - \frac{(n-2)}{s^2} x \geq 0. \tag{78}$$

Note that this is the same differential inequality as in Lemma 6.3 (iv) for the function $h - \alpha$, which is equivalent to the (NEC) on $(\mathfrak{M}, \mathfrak{g})$. By linearity, we have that $h_T - \alpha$ solves the above differential inequality (78), which by Lemma 6.3 implies that the spacetime of Class $\mathfrak{H}$ with metric coefficient $h_T$ satisfies the (NEC).

**Remark 6.15.** The exact solutions of (78) as an ODE are given by a 2-parameter family of solutions of the form

$$x = \frac{C_1}{s^{-n+2}} + C_2 s^2.$$

In spherical symmetry, $h = 1+x$ correspond to the Schwarzschild de Sitter and Schwarzschild anti-de Sitter family depending on the sign of $C_2$, which describe the static, spherically symmetric Vacuum solutions of the Einstein Equations (with cosmological constant depending on $C_2$). These are precisely the spacetimes of class $\mathcal{S}$ such that the time-symmetric slices have constant scalar curvature. Compare [35, Lemma 2.3].

In this sense, if $B^T \geq 0$, the spacetime of Class $\mathfrak{H}$ with metric coefficient $h_T$ inherits the (NEC) from the spacetime of Class $\mathfrak{H}$ with metric coefficient $h$. This makes apparent that we can use the (NEC) on $(\mathfrak{M}, \mathfrak{g})$ to classify CMC surfaces on a large class of general warped product graphs $(M_T, g^T, K^T)$ provided we can extend the tensor inequality in an appropriate way until the first zero $r_T$ of $h_T$ to verify Brendle's condition (H1) in [15]. As $h_T \geq h$ on $I$, a positive minimal, inner boundary of $(M_T, g^T)$ will in general be hidden behind a Killing horizon of $(\mathfrak{M}, \mathfrak{g})$.

## 6.5 Extending the graph past the Killing horizon to the minimal boundary

As mentioned in Subsection 4.3, Killing horizons arise in spacetimes of Class $\mathfrak{H}$ as zeros of $h$. As in Subsection 4.3 we assume that $h$ finitely many positive, simple zeros $0 < r_1 < \ldots < r_N = r_H$, such that all arising Killing horizons are non-degenerate. As (H2) is equivalent to the fact that the outermost Killing horizon $\{r = r_H\}$ is non-degenerate, this is rather a necessary than a restrictive assumption in lieu of applying the results of Brendle [15].

However, as $r \to r_N$ the $(t, r)$-coordinate system breaks down, and in general $M_T$ can no longer be described as a graph of a radial function $T$, since by Remark 6.6

$$|T'| = \frac{1}{h}\sqrt{\frac{h_T - h}{h_T}}.$$

We want to argue that we can extend the graph of $T$ past any Killing horizon $\{r = r_l\}$ for all $1 \leq q \leq N$ in different coordinates in the case that $h_T(r_l) > 0$ and extend the notion of the (NEC) as an ordinary differential inequality on a suitable spacetime extension. As we only need this up to the inner boundary of $M_T$, i.e $\{s = r_T\}$ for the biggest zero $r_T$ of $h_T$, we always have that $h_T(r_l) > h(r_l) = 0$ for all $1 \leq l \leq N$ with $r_l > r_T$.

Using the above assumptions that each Killing horizon is non-degenerate, Brill–Hayward [16], Schindler–Aguirre [71], and Cederbaum and the author [25] showed independently that a spacetime $(\mathfrak{M}, \mathfrak{g})$ of Class $\mathfrak{H}$ admits a spacetime extension called the *generalized Kruskal–Szekeres extension*, that extends the radial coordinate $r$ to $(0,\infty)$ away from the zeros of $h$. Throughout this subsection we will use the conventions of [25]. In each coordinate chart, the spacetime extension is given by a warped product manifold $(\mathbb{P}_l \times \mathcal{N}, \widetilde{g}_l)$, where

$$\begin{aligned}
&\mathbb{P}_l := \{(u,v) \in \mathbb{R}^2 : uv \in \operatorname{Im}(f_l)\},\\
&\widetilde{g}_l = (F_l \circ \rho)(\,\mathrm{d}u \otimes \,\mathrm{d}v + \,\mathrm{d}v \otimes \,\mathrm{d}u) + \rho^2 g_{\mathcal{N}},
\end{aligned}$$

with $\rho = f_l^{-1}(uv)$, $F_l = \frac{2K_l}{f_l'}$, where $f_l$ is the unique strictly increasing solution of

$$\frac{f_l}{f_l'} = C_l h, \tag{79}$$

on $(r_{l-1}, r_{l+1})$ with $f'(r_l) = 1$, $C_l := \frac{1}{h'(r_l)}$, $1 \le l \le N$, where we recall from Subsection 4.3 that $r_0 = 0$, $r_{N+1} = \infty$. Note that the original spacetime $(\mathfrak{M}, \mathfrak{g})$ corresponds to $\{u, v > 0\}$ in $\mathbb{P}_N \times \mathcal{N}$. Moreover, we have the explicit coordinate transformations between $(u,v)$ and $(t,r)$ coordinates

$$\begin{aligned}
v(t,r) &= \sqrt{f_l(r)} \exp\left(\frac{1}{2C_l} t\right),\\
u(t,r) &= \sqrt{f_l(r)} \exp\left(-\frac{1}{2C_l} t\right)
\end{aligned}$$

on each coordinate patch $\mathbb{R} \times (r_j, r_{j+1}) \times \mathcal{N}$ ($j \in \{l-1, l\}$), where $(t,r)$ coordinates are defined. A direct computation in the $(u,v)$ coordinates using Equation (79) gives

$$\begin{aligned}
\mathfrak{Ric}_{uv} &= -\frac{F_l}{2}\left(h'' + \frac{(n-1)}{\rho} h'\right),\\
\mathfrak{Ric}_{IJ} &= (\mathrm{Ric}_{g_{\mathcal{N}}})_{IJ} - \left((n-2)h + \rho h'\right)(g_{\mathcal{N}})_{IJ}.
\end{aligned}$$

Let $L = a\partial_u + b\partial_v + \frac{c}{\rho} X$ be a null vector field, where $X$ is a unit vector on $\mathcal{N}$. Then

$$0 = 2F_l ab + c^2.$$

Using this identity, we see that

$$\begin{aligned}
0 &\le \mathfrak{Ric}(L,L)\\
&= -F_l ab \left(h'' + \frac{(n-1)}{\rho} h'\right) + \frac{c^2}{s^2}\left(\mathrm{Ric}_{g_{\mathcal{N}}}(X,X) - (n-2)h + \rho h'\right)\\
&= c^2 \left(\frac{1}{2}(h-1)'' + \frac{(n-3)}{2\rho}(h-1)' - \frac{(n-2)}{\rho^2}(h) + \mathrm{Ric}_{g_{\mathcal{N}}}(X,X)\right),
\end{aligned}$$

which is again equivalent to the linear ordinary differential inequality (78) for $x = h - \alpha$ by the same arguments as in the proof of Lemma 6.3. Therefore the spacetime extension satisfies the (NEC), if and only if $x = h - \alpha$ satisfies (78) on $(0, \infty)$.

A crucial observation in [25] is that any solution $f_l$ of (79) is of the form

$$f_l = h \exp\left(\frac{R_l}{K_l}\right),$$

where $R_l$ is a smooth function on $(r_{l-1}, r_{l+1})$ uniquely determined up to a constant. Although the construction circumvents the need to do so, this yields a tortoise function $R^*$, i.e., a primitive of $\frac{1}{h}$, a-posteriori on each of the intervals $(r_{l-1}, r_l)$ and $(r_l, r_{l+1})$ by defining $R^* := K_l \ln(|f_l|)$.

Now, we notice that by Remark 6.6 $T$ is a primitive of the function $\frac{1}{\psi_T}$, where $\Psi_T = \pm h\sqrt{\frac{h_T}{h_T - h}}$ is well-defined and $\Psi_T'(r_l) = h'(r_l) \neq 0$ for any $r_l > r_T$. In particular, the construction of Cederbaum and the author [25] yields that $T$ satisfies

$$T = \pm\left(C_l \ln(|h|) + C_l \frac{1}{2} \ln\left(\frac{h_T}{h_T - h}\right) + \widetilde{R}_T\right)$$

for some smooth function $\widetilde{R}_T$ on $(r_{l-1}, r_{l+1})$. Using this explicit behavior of $T$, we see that for the $+$-case

$$\begin{aligned} v(s) &= h\sqrt[4]{\frac{h_T - h}{h_T}} \exp\left(\frac{1}{2K}(R + \widetilde{R}_T)\right), \\ u(s) &= \sqrt[4]{\frac{h_T}{h_T - h}} \exp\left(\frac{1}{2K}(R - \widetilde{R}_T)\right), \end{aligned}$$

so $M_T$ extends smoothly across the horizon, and crosses the horizon at $v = 0$, $u = u(r_T)$, and similarly at $v = v(r_T)$, $u = 0$ in the $-$-case. Therefore we can extend any warped product graph across any non-degenerate Killing horizon up to its minimal, inner boundary.

## 6.6 Uniqueness of STCMC surfaces on hyperboloids

Combining the result of the previous subsections with the results of Brendle [15], and Cederbaum and the author [25], we acquire the following:

**Theorem 6.16.** *Let $h : (0,\infty) \to \mathbb{R}$ be a smooth function with finitely many, positive simple zeros $r_1, \ldots, r_N$. Let $(\mathfrak{M}, \mathfrak{g})$ be the corresponding spacetime of Class $\mathfrak{H}$ with metric coefficient $h$, and assume that the generalized Kruskal–Szekeres extension of $(\mathfrak{M}, \mathfrak{g})$ satisfies the (NEC) condition.*
*Let $x$ be some non-negative solution of the ordinary differential inequality* (78)*, and consider the warped product graph $M_T$, where $T$ is such that $h_T = h + x$. Then $M_T$ extends into the generalized Kruskal–Szekeres extension until its minimal inner boundary, corresponding to the first zero $r_T$ of $h_T$.*
*Additionally, if $r_T > 0$ with $h'(r_T) > 0$ and $h_T$ satisfies* (67) *on a dense set of $(r_T, \infty)$, then all compact CMC surfaces in $M_T$ are leaves $\{s\} \times \mathcal{N}$.*

*Proof.* By [25], $(\mathfrak{M}, \mathfrak{g})$ extends onto all positive radii, and the generalized Kruskal–Szekeres extension is covered by a countable, smooth Atlas. As observed in the previous subsection the (NEC) implies that the ordinary differential inequality (78) holds for $h - \alpha$ on all of $(0, \infty)$, and the graph $M_T$ of $T$ with $h_T = h + x$ is well defined across any non-degenerate Killing horizon up until the first zero $r_T$ of $h_T$.

Since $x$ is a non-negative solution of (78), we have the tensor inequality (77), and by Remark 6.6 there exists a graph $T$ such that $h_T = h + x$, where $T$ is uniquely determined by $x$ up to a choice of sign of $T'$ and a constant of integration. Now assume that the first zero $r_T$ of $h_T$ satisfies $h_T'(r_T) > 0$, and $h_T$ satisfies (67) on a dense set of $(r_T, \infty)$. In particular, conditions (H1)-(H3) in [12, Theorem 1.1] are satisfied. Thus, any compact CMC surface in $M_T$ is totally umbilic. Moreover, since (67) holds on a dense subset, we can conclude as in [12, p. 18] that any compact CMC surface is in fact a leaf of the canonical foliation $\{s\} \times \mathcal{N}$. $\square$

Note that this result is independent of the extrinsic curvature $K$. Therefore it also suffices to apply Brendle's result directly to the totally geodesic slices in the spacetime of Class $\mathfrak{H}$ with metric coefficient $h_T$, as this spacetime will satisfy the (NEC) by Remark 6.15. Note that this observation is consistent with the duality of constant, positive mean curvature slices in spacetimes with zero cosmological constant and maximal slices in spacetimes with negative cosmological constant, cf. [34].

We now want to incorporate the extrinsic curvature $K$ into our result. Due to the difficulty in adapting Brendle's method in the presence of $P = \operatorname{tr}_\Sigma K$ and its evolution, we restrict ourselves to the special case of totally umbilic warped product graphs, which we have fully characterized in Corollary 6.7. Note that on a hyperboloid, $P = \operatorname{tr}_\Sigma K = (n-1)\lambda$

is constant and the same for any embedded surface $\Sigma$, so the evolution of $P$ along any deformation is trivial. Hence, any surface $\Sigma$ in $(M_T, g^T, K^T)$ has constant spacetime mean curvature $\mathcal{H}^2$, and constant expansion $\theta_\pm$, if and only if it is a CMC surface. Moreover, by Remark 6.15 $Cs^2$ is an exact solution of (78), so it in fact solves (78) as an ODE. Using Theorem 6.16, we acquire our main result for totally umbilic warped product graphs in Class $\mathfrak{H}$.

**Theorem 6.17.** *Let $h : (0,\infty) \to \mathbb{R}$ be a smooth function with finitely many, positive, simple zeros $r_1, \dots, r_N$. Let $(\mathfrak{M}, \mathfrak{g})$ be the corresponding spacetime with metric coefficient $h$. Assume that the generalized Kruskal–Szekeres extension satisfies the (NEC) condition. Assume further, that $h$ satisfies* (67) *on a dense subset of* $(0,\infty)$.
*Then there exists a constant $C_0 = C_0(h, h') \in (0,\infty]$, such that for any hyperboloid $(M_T, g^T, K^T)$ with umbilicity factor $\lambda_T^2 < C_0$, we have: If $\Sigma \subset M_T$ is an orientable, closed, embedded hypersurface with constant spacetime mean curvature, then $\Sigma$ is a slice $\{s\} \times \mathcal{N}$.*

*Proof.* For $C = \lambda_T^2$ small enough, $h_T$ has at least one positive zero $r_T$ with $r_T \to r_N$ as $C \to 0$. By continuity, we have $h'(q_T) > 0$ for small enough $C$. We define $C_0$ as the supremum over all $C$, such that these conditions are still satisfied.

Further, we have

$$\frac{(n-2)}{2s}h_T' - \frac{(n-2)}{s^2}h_T + s^{-2}\mathrm{Ric}_{g_{\mathcal{N}}}(X,X) = \frac{(n-2)}{2r}h' - \frac{(n-2)}{r^2}h + r^{-2}\mathrm{Ric}_{g_{\mathcal{N}}}(X,X) \neq 0$$

on a dense subset of $(r_T, \infty)$ by assumption. Thus, Theorem 6.16 applies to $(M_T, g^T, K^T)$, and any CMC surface $\Sigma$ is a slice $\{s\} \times \mathcal{N}$. Since $(M_T, g^T, K^T)$ has constant umbilicity factor, any STCMC surface is a slice $\{s\} \times \mathcal{N}$. □

Note that all assumptions are in particular satisfied for any constant $C \geq 0$ in the Kruskal–Szekeres extension of the Schwarzschild spacetime with positive mass.

**Corollary 6.18.** Let $(\mathfrak{M}, \mathfrak{g})$ be the Schwarzschild spacetime with positive mass. Then any closed, embedded STCMC surface $\Sigma$ in an hyperboloid $(M_T, g^T, K^T)$ is a slice $\{s\} \times \mathbb{S}^2$.

**Remark 6.19.** Note that a direct computation yields that

$$\mathcal{H}^2 = \frac{(n-1)h(s)}{s^2}$$

for a spherical slice $\{s\} \times \mathbb{S}^{n-1}$ for any warped product graph $(M_T, g^T, K^T)$. As we have to extend any hyperboloid with $\lambda_T \neq 0$ across the horizon where $h(r_H) = 0$ into a region where $h < 0$, both the case of generalized apparent horizons $\mathcal{H}^2 = 0$, and STCMC surfaces with $\mathcal{H}^2 < 0$ naturally occur in hyperboloids. In particular, in the latter case they are trapped in the sense of both Remark 4.20 and Remark 5.44.

# 7 $2d$-Ricci flow on the standard Minkowski lightcone

In this section, we study the evolution of spacelike cross sections of the standard lightcone in the $3+1$-Minkowski spacetime under a geometric flow. Here, a spacelike cross section is always assumed to satisfy the assumptions made in Subsection 4.7. By observing that any spacelike cross section of the lightcone can be uniquely identified with a metric in the conformal class of the round 2-sphere, we establish an equivalence between $2d$-Ricci flow in the conformal class of the sphere and an extrinsic curvature flow along the lightcone. This extrinsic curvature flow, which we will call *null mean curvature flow* here, was first studied by Roesch–Scheuer [67] to detect MOTS in null hypersurfaces. As no MOTS exists on the Minkowski lightcone, the flow develops singularities which we can fully characterize by drawing on the equivalence to Ricci flow and a classical result first proven by Hamilton [47]. Moreover, by studying null mean curvature flow along the lightcone we obtain an independent proof of Hamilton's classical result for spacelike cross sections with spacelike mean curvature vector, which corresponds to Hamilton's initial assumption of strictly positive scalar curvature for surfaces of genus 0. This proof relies solely on the maximum principle by exploiting a new notion of scalar second fundamental form. This section is based on published single author work in [89].

This section is structured as follows: In Subsection 7.1 we briefly review $2d$-Ricci flow. In Subsection 7.2 we collect all relevant properties of spacelike cross sections of the Minkowski lightcone, and define a notion of scalar second fundamental form in Subsection 7.3. In Subsection 7.4 we establish the equivalence between null mean curvature flow on the lightcone and $2d$-Ricci flow on surfaces of genus 0, and obtain a full classification of singularities of null mean curvature flow on the lightcone. In Subsection 7.5 we give a new proof of Hamilton's classical result using null mean curvature flow.

## 7.1 $2d$-Ricci flow

A family of metrics $\{g(t)\}$ is said to evolve under Ricci flow, if

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = -2Ric_{g(t)}, \tag{80}$$

which is a non-linear parabolic system. As $\mathrm{Ric}_g = \frac{1}{2}\,\mathrm{R}_g\, g$ in 2-dimensions, the flow agrees with the Yamabe flow in 2-dimensions, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = -\,\mathrm{R}_{g(t)}\, g(t). \tag{81}$$

In particular, it is easy to see that $2d$-Ricci flow preserves the conformal class of the metrics under the flow. However, this also causes some analytical difficulties compared to the 3-dimensional case, as the evolution of $|\mathring{\mathrm{Ric}}_g|^2$ can not be exploited along the flow. Nonetheless, a classical result first proven by Hamilton [47] gives a full characterization of all singularity models:

**Theorem 7.1** (H. '88 [47], C. '91 [30], B.–S.–Y. '94 [7], S. 2002 [77], A.–B. 2009 [1])**.**
*If $g$ is any metric on a Riemann surface, then under Hamilton's Ricci flow, $g$ converges to a metric of constant curvature.*

**Remark 7.2.**

(i) Here, Hamilton's Ricci flow denotes Ricci flow renormalized by volume, which can in fact always be realized as a rescaling of the unnormalized Ricci flow, cf. Equation (95) below. The following thus gives a full characterization of the singularity models for Ricci flow in the sense that whenever the unnormalized Ricci flow approaches a singularity, the flow smoothly converges to a surface of constant scalar curvature upon rescaling.

(ii) In the case of surfaces of genus 0 this was initially proven by Hamilton [47] for metrics of strictly positive scalar curvature. This assumption was eventually removed by Chow [30]. Their techniques rely on establishing a Harnack inequality and an entropy bound. Their approach also gives an independent proof of the uniformization theorem, c.f. [28]. Due to the difficulty of this approach, the conformally round case was later revisited and several new proofs were given utilizing the uniformization theorem to study the evolution of the conformal factor, see [1, 7, 77].

## 7.2 The standard lightcone in the $3+1$-dimensional Minkowski spacetime

For our purpose, it is most convenient to introduce the null coordinates $v := r+t$, $u := r-t$ on $\mathbb{R}^{3,1}$. Then $\eta$ can be written as

$$\eta = \frac{1}{2}\left(\,\mathrm{d}u\;\mathrm{d}v + \,\mathrm{d}v\;\mathrm{d}u\right) + r^2\;\mathrm{d}\Omega^2$$

with $r = r(u,v) = \frac{u+v}{2}$. Now, all past-pointing standard lightcones in the Minkowski spacetime are given by the sets $\{v = \text{const.}\}$ (and similarly all future-pointing lightcones are given by $\{u = \text{const.}\}$). From now on, we will work on the null hypersurface $\mathcal{N} = \{v = 0\} = C_-$, i.e., the past-pointing standard lightcone centered at the origin, cf. Subsection 4.2. However, all identities derived for $\mathcal{N}$ will also analogously hold on all level sets of $v$ and $u$ respectively, in particular on $C_+ = \{u = 0\}$. Note that $\mathcal{N}$ has the induced degenerate metric

$$r^2\;\mathrm{d}\Omega^2,$$

and is generated by the geodesic integral curves of $\underline{L} := 2\partial_u$. Note that $\underline{L}$ is past-pointing and its integral curves are geodesics. Recall that the null generator $\underline{L}$ of a null hypersurface is both tangential and normal to $\mathcal{N}$, and by choice of $\underline{L}$ we have $\underline{L}(r) = 1$. Thus, $r$ restricts to an affine parameter along $\mathcal{N}$. In particular, we can represent any spacelike cross section $\Sigma$ of $\mathcal{N}$ (which intersects any integral curve of $\underline{L}$ exactly once) as a graph over $\mathbb{S}^2$, i.e., $\Sigma = \Sigma_\omega = \{\omega = r\} \subseteq \mathcal{N}$. In particular, $\Sigma$ has the induced metric

$$\gamma = \omega^2 \, \mathrm{d}\Omega^2,$$

so $(\Sigma, \gamma)$ is conformally round. Conversely, for any conformally round metric $\gamma_\omega = \omega^2 \, \mathrm{d}\Omega^2$ there exists a unique spacelike cross section $\Sigma_\omega$ such that $(\Sigma_\omega, \gamma_\omega)$ embeds into $\mathcal{N}$, where we will omit the subscript $\omega$ in the following when it is clear from the context. This observation is similar to an idea developed by Fefferman–Graham [40], and their construction indeed yields the standard lightcone in the $3+1$-Minkowski spacetime in the case of the round 2-sphere.

We now want to represent the extrinsic curvature of a spacelike cross section $(\Sigma, \gamma)$ of $\mathcal{N}$ as a codimension-2 surface with respect to a particular null frame. Recall that the null generator $\underline{L}$ is both tangent and normal to $\mathcal{N}$, in particular $\underline{L}$ is normal to any spacelike cross section $(\Sigma, \gamma)$. We further consider a normal null vector field $L$ along $\Sigma$ such that $\eta(\underline{L}, L) = 2$. This uniquely determines $L$ such that $\{\underline{L}, L\}$ form a frame of the normal bundle $T^\perp\Sigma$ of $\Sigma$. Note that $L$ is future-pointing.

We now remark that the standard round spheres $\{\Sigma_s\}_{s\in(0,\infty)}$ form a level-set foliation with respect to the integral curves of the null generator $\underline{L}$. It is easy to verify that for any leaf $\Sigma_s$, we have $L_{\Sigma_s} = 2\partial_v$ and find

$$\begin{aligned}
\underline{\chi}_s &= \chi_s = s \, \mathrm{d}\Omega^2, \\
\underline{\theta}_s &= \theta_s = \frac{2}{s}, \\
\zeta_s &= 0.
\end{aligned}$$

From this background foliation, we can explicitly compute all extrinsic curvature quantities for any spacelike cross section $\Sigma$ with respect to the null frame $\{\underline{L}, L\}$ as defined in Subsection 4.8 by Propositions 4.16 and 4.22.

**Proposition 7.3.** *For any spacelike cross section* $(\Sigma, \gamma)$ *of* $\mathcal{N}$*, we find*

(i) $\gamma = \omega^2 \, \mathrm{d}\Omega^2$,

(ii) $\underline{\chi} = \frac{1}{\omega}\gamma$,

(iii) $\underline{\theta} = \frac{2}{\omega}$,

(iv) $\chi = \frac{1}{\omega}(1 + |\nabla\omega|^2)\gamma - 2\,\mathrm{Hess}\,\omega$

(v) $\theta = 2\left(\frac{1}{\omega} + \frac{|\nabla\omega|^2}{\omega} - \Delta\omega\right)$,

(vi) $\zeta = -\frac{\mathrm{d}\,\omega}{\omega}$,

*where* Hess *and* $\Delta$ *denote the Hessian and Laplace–Beltrami operator on* $(\Sigma, \gamma)$*, respectively.*

**Remark 7.4.** The fact that the null second fundamental form $\underline{\chi}$ depends only pointwise on $\omega$ together with the background foliation of round spheres gives

$$|\vec{\mathrm{II}}|^2 = \left\langle \underline{\chi}, \chi \right\rangle = \frac{1}{2}\mathcal{H}^2,$$

and thus

$$\mathrm{R} = \frac{1}{2}\mathcal{H}^2 \tag{82}$$

by the twice contracted Gauss equation, Proposition 4.24, which can also be directly verified from (iii) and (v) in Proposition 7.3. Since $\Sigma$ is 2-dimensional, we can therefore express the Riemann tensor of the surface as

$$\mathrm{Rm}_{ijkl} = \frac{1}{4}\mathcal{H}^2 \left(\gamma_{ik}\gamma_{jl} - \gamma_{jk}\gamma_{il}\right). \tag{83}$$

We would like to emphasize here that $\mathcal{H}^2$ refers by definition to the signed Lorentzian length of the mean curvature tensor and can therefore be (locally) negative despite the suggestive power of 2 as an exponent.

In particular, we always have

$$\mathring{\vec{\mathrm{II}}} = -\frac{1}{2}\mathring{\chi}\underline{L}, \tag{84}$$

so $|\mathring{\vec{\mathrm{II}}}|^2 = 0$ although $\mathring{\vec{\mathrm{II}}} \neq 0$, and the property of $\vec{\mathrm{II}}$ being pure trace is instead more accurately captured by $|\mathring{\chi}|^2 = 0$. Along $\mathcal{N}$, this is made precise by the following proposition.

**Proposition 7.5.** *Let $(\Sigma, \gamma)$ be a spacelike cross section of $\mathcal{N}$ with $\overset{\circ}{\chi} = 0$. Then $\mathcal{H}^2$ is constant and strictly positive along $\Sigma$. In particular, $\gamma$ is a metric of constant scalar curvature.*

**Remark 7.6.** Note that

$$\overset{\circ}{\chi} = 2\omega^2 \overset{\circ}{\mathrm{Hess}}_{\mathbb{S}^2}\left(\frac{1}{\omega}\right), \tag{85}$$

where $\mathrm{Hess}_{\mathbb{S}^2}$ denotes the Hessian on $(\mathbb{S}^2, \mathrm{d}\Omega^2)$. One can also verify by computation in coordinates that $\overset{\circ}{\mathrm{Hess}}_{\mathbb{S}^2}\left(\frac{1}{\omega}\right) = 0$ if and only if $\omega$ is of the form

$$\omega(\vec{x}) = \frac{\rho}{\sqrt{1 + ||\vec{a}||^2} - \vec{a} \cdot \vec{x}}, \tag{86}$$

for a constant $\rho > 0$ and a fixed vector $\vec{a} \in \mathbb{R}^3$, which are exactly the metrics of constant scalar curvature on $\mathbb{S}^2$. Hence, the converse statement of Proposition 7.5 is also true. It is a well-known fact that all such metrics can be obtained from the round metric by a suitable Möbius transformation, cf. [59, Proposition 6], [85, Section 5.2]. Moreover, the metrics (86) describe exactly the images of round spheres (as cross sections of $\mathcal{N}$) after a suitable Lorentz transformation in $\mathrm{SO}^+(3,1)$ in the Minkowski spacetime, which leave the lightcone $\mathcal{N}$ invariant. These observations illustrate once again the well-known fact that the Möbius group is isomorphic to the restricted Lorentz group $\mathrm{SO}^+(3,1)$.

*Proof of Proposition 7.5.* Combining the Codazzi equation Proposition 4.25 for $\chi$ with the explicit form of $\zeta$ from Proposition 7.3, we find

$$\nabla_i \chi_{jk} - \frac{\mathrm{d}\,\omega_i}{\omega}\chi_{jk} = \nabla_j \chi_{ik} - \frac{\mathrm{d}\,\omega_j}{\omega}\chi_{ik}. \tag{87}$$

Multiplying the equation by $\underline{\theta} = \frac{2}{\omega} > 0$, we get

$$\nabla_i \left(\underline{\theta}\chi\right)_{jk} = \nabla_j \left(\underline{\theta}\chi\right)_{jk}. \tag{88}$$

Hence $\nabla\left(\underline{\theta}\chi\right)$ is totally symmetric and since $\mathrm{tr}_\gamma\, \underline{\theta}\chi = \mathcal{H}^2$, we find

$$\nabla_i \mathcal{H}^2 = \mathrm{div}\left(\underline{\theta}\chi\right)_i = \frac{1}{2}\nabla_i \mathcal{H}^2 + \mathrm{div}\left(\underline{\theta}\overset{\circ}{\chi}\right)_i = \frac{1}{2}\nabla_i \mathcal{H}^2$$

by assumption. Therefore $\mathcal{H}^2$ is constant, in particular $\gamma$ is a metric of constant scalar curvature by the Gauss equation (82). Finally, the Gauss–Bonnet Theorem ensures the positivity of R and hence $\mathcal{H}^2$. $\square$

## 7.3 A scalar second fundamental form

Motivated by the proof of Proposition 7.5, we regard the symmetric $(0,2)$-tensor $A := \underline{\theta}\chi$ as a scalar representation of the vector valued second fundamental form $\vec{\mathrm{II}}$, and call $A$ the *scalar second fundamental form.* This can also be regarded as a choice of gauge. Rephrasing Proposition 7.5 in terms of $A$, we see that we can prove the following identity in complete analogy to the properties of the second fundamental form $h$ of an embedded, orientable surface in $\mathbb{R}^3$, cf. [2, Lemma 8.15].

**Proposition 7.7.** *Let $(\Sigma, \gamma)$ be a spacelike cross section of $\mathcal{N}$. Then $\nabla A$ is totally symmetric, i.e.,*

$$\nabla_i A_{jk} = \nabla_j A_{ik}. \tag{89}$$

*In particular, we find*

$$|\nabla A|^2 \geq \frac{3}{4}\left|\nabla \mathcal{H}^2\right|^2, \tag{90}$$

*and $\mathring{A} = 0$ if and only if $\mathcal{H}^2$ is a strictly positive constant.*

We further derive the propagation equations for the geometric objects $A$ and $\mathcal{H}^2$ from Proposition 4.27:

**Lemma 7.8.**

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,t} A_{ij} &= -\,2 \operatorname{Hess}_{ij}(\underline{\theta}\varphi),\\
\frac{\mathrm{d}}{\mathrm{d}\,t} \mathcal{H}^2 &= -\,2\Delta(\underline{\theta}\varphi) - (\underline{\theta}\varphi)\mathcal{H}^2.
\end{aligned}$$

*Proof.* From Proposition 4.27 (iv) and (v), and $\underline{\chi} = \frac{1}{2}\underline{\theta}\gamma$, we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,t} A_{ij} =& -2\underline{\theta} \operatorname{Hess}_{ij}\varphi - 2\underline{\theta}(\,\mathrm{d}\,\varphi_i \otimes \zeta_j + \,\mathrm{d}\,\varphi_j \otimes \zeta_i)\\
& - \varphi\underline{\theta}\left(2\nabla_i \zeta_j + 2\zeta_i \otimes \zeta_j - \frac{1}{2}A_{ij}\right) - \frac{1}{2}\varphi\underline{\theta}A_{ij}\\
=& -2\underline{\theta} \operatorname{Hess}_{ij}\varphi - 2\underline{\theta}(\,\mathrm{d}\,\varphi_i \otimes \zeta_j + \,\mathrm{d}\,\varphi_j \otimes \zeta_i) - \varphi\underline{\theta}\left(2\nabla_i \zeta_j + 2\zeta_i \otimes \zeta_j\right).
\end{aligned}$$

We now observe that the remaining terms on the right hand side exactly combine into $-2\operatorname{Hess}_{ij}(\underline{\theta}\varphi)$ due to the explicit formulas for $\underline{\theta}$ and $\zeta$ as stated in Proposition 7.3. Taking a trace, where $\frac{\mathrm{d}}{\mathrm{d}\,t}\gamma^{ij} = -\varphi\underline{\theta}\gamma^{ij}$, completes the proof. □

We close this section by establishing a null version of the Simons' identity for $A$ in the $3+1$-Minkowski lightcone $\mathcal{N}$ which will be crucial for our later analysis.

**Lemma 7.9** (Null Simons' Identity)**.**

$$\Delta A_{ij} = \operatorname{Hess}_{ij}\mathcal{H}^2 + \frac{1}{2}\mathcal{H}^2 \mathring{A}_{ij}.$$

*Proof.* In the following lines, we will make frequent use of the Codazzi equation for A, Lemma 7.7, and the fact that for any symmetric $(0,2)$-tensor $T$, we have that

$$\nabla_k\nabla_l T_{ij} - \nabla_l\nabla_k T_{ij} = \operatorname{Rm}_{kljm} T_i^m + \operatorname{Rm}_{klim} T_j^m.$$

Thus, we compute

$$\begin{aligned}
\nabla_k\nabla_l A_{ij} =& \nabla_k\left(\nabla_i A_{lj}\right)\\
=& \nabla_i\nabla_k A_{jl} + \operatorname{Rm}_{kilm} A_j^m + \operatorname{Rm}_{kijm} A_l^m\\
=& \nabla_i(\nabla_j A_{kl}) + \operatorname{Rm}_{kilm} A_j^m + \operatorname{Rm}_{kijm} A_l^m\\
=& \nabla_i\nabla_j A_{kl} + \frac{1}{4}\mathcal{H}^2\left(\left(\gamma_{kl}\gamma_{im} - \gamma_{il}\gamma_{km}\right)A_j^m + \left(\gamma_{kj}\gamma_{im} - \gamma_{ij}\gamma_{km}\right)A_l^m\right)\\
=& \nabla_i\nabla_j A_{kl} + \frac{1}{4}\mathcal{H}^2\left(A_{ij}\gamma_{kl} + A_{il}\gamma_{kj} - A_{kj}\gamma_{il} - A_{kl}\gamma_{ij}\right),
\end{aligned}$$

where we have used (83) in the second to last line. Taking a trace with respect to the $kl$ entries yields the claim. □

## 7.4 Null mean curvature flow on the Minkowski lightcone

We will now investigate null mean curvature flow restricted to the (past-pointing) standard lightcone $\mathcal{N}$ in the $3+1$-Minkowski spacetime. Recall that along the standard lightcone in the $3+1$-Minkowski spacetime null mean curvature flow is defined as

$$\frac{\mathrm{d}}{\mathrm{d}\,t}x = \frac{1}{2}\eta\left(\vec{\mathcal{H}}, L\right)\underline{L} = -\frac{1}{2}\theta\underline{L},$$

as first studied by Roesch–Scheuer in a more general case [67]. Note that since $\underline{L}(r) = 1$, the above is equivalent to the following evolution equation for $\omega$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\omega = -\frac{1}{2}\theta. \tag{91}$$

Recalling the considerations on $2d$-Ricci flow in Subsection 7.1, we see that for surfaces of genus 0 the uniformization theorem yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}\gamma_{ij} = -2\,\mathrm{Ric}_{ij} \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\omega^2\ \mathrm{d}\Omega^2\right) &= -2\mathcal{K}\omega^2\ \mathrm{d}\Omega^2 \\ \Leftrightarrow \frac{2}{\omega}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\omega\right)\gamma_{ij} &= -2\mathcal{K}\gamma_{ij} \\ \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}\,t}\omega &= -\omega\mathcal{K},\end{aligned}$$

where $\mathcal{K}$ denotes the Gauss curvature. Note that by the twice contracted Gauss equation (82) and the explicit form of $\underline{\theta}$, we have that $\theta = 2\omega\mathcal{K}$. Therefore, 2-dimensional Ricci flow in the conformal class of the round sphere is equivalent to null mean curvature flow on the past-pointing standard lightcone in the $3+1$-Minkowski spacetime. Thus, Theorem 7.1 immediately yields the following:

**Theorem 7.10.** *Let $(\Sigma_0, \gamma_0)$ be a spacelike cross section of the past-pointing standard lightcone $\mathcal{N}$ in the $3+1$-Minkowski spacetime. Then the solution of null mean curvature flow starting from $\Sigma_0$ extinguishes in the tip of the cone in finite time and the renormalization by volume converges to a surface of constant spacetime mean curvature, which exactly arise as the image of a round sphere of a Lorentz transformation in $\mathrm{SO}^+(3,1)$ consisting of a Lorentz boost with boost vector*

$$z = \begin{pmatrix}\sqrt{1+||\vec{a}||} \\ \vec{a}\end{pmatrix}$$

*for a vector $\vec{a} \in \mathbb{R}^3$ and a rotation determined by the choice of coordinates on $\mathbb{S}^2$.*

**Remark 7.11.** Since the general structure of the standard lightcone derived in Subsection 7.2 extends directly to higher dimensions (up to some possibly dimension dependent constants), the geometric intuition developed in Subsection 7.2 also holds for the standard lightcone in the $n+1$ dimensional Minkowski spacetime, $n \geq 3$. In particular, the Gauss equation yields

$$\mathrm{R} = \frac{n-1}{n}\mathcal{H}^2. \tag{92}$$

From this, we can similarly establish that null mean curvature flow is proportional to the Yamabe flow [48] for the conformal class of the round metric on $\mathbb{S}^{n-1}$ in all dimensions $n-1 \geq 2$. More precisely, the metrics evolve under renormalized null mean curvature flow as

$$\frac{\mathrm{d}}{\mathrm{d}\,\tilde{t}}\tilde{g}(\,\tilde{t}\,) = -\frac{1}{n-1}\left(\tilde{R} - \fint \tilde{R}\right)\tilde{g}(\,\tilde{t}\,). \tag{93}$$

Since not all metrics on $\mathbb{S}^{n-1}$ are necessarily conformally round in higher dimension $n-1 \geq 3$ they can thus not be embedded isometrically into the standard Minkowski lightcone.

Similar to the 2-dimensional case, renormalized Yamabe flow has been subject to thorough investigation using various different methods. The case of the conformal class of the round sphere was first treated separately by Chow [31] under the additional assumption of positive Ricci curvature, which is preserved under the flow. A uniform approach for locally conformally flat metrics was later provided by Ye [90]. Schwetlick–Struwe [74] performed a precise blow-up analysis and showed that singularities arising in the blow-up procedure can by ruled out by employing the positive mass theorem (cf. [72]) if the initial energy is less than some uniform bound depending on the Yamabe invariant of the initial metric and the Yamabe energy of the round sphere. The general approach by Brendle [12, 14] leads to a short proof of the conformally round case [13]. We suspect that the techniques developed in the next subsection could be applied to gain a new proof of this result, possibly under similar restrictions as Chow [31].

## 7.5 A new proof of Hamilton's classical result

With this approach to $2d$-Ricci flow, we give a new proof of Hamilton's classical result:

**Theorem 7.12** (cf. [47])**.** *Let $(\Sigma_0, \gamma_0)$ be a surface with conformally round metric $\gamma_0$ and strictly positive scalar curvature. Then a solution of renormalized Ricci flow exists for all time and the metrics $\gamma_t$ converge to a smooth metric $\gamma_\infty$ of constant scalar curvature in $C^k$ for all $k \in \mathbb{N}$ as $t \to \infty$.*

Note that the assumption of strictly positive scalar curvature translates by the Gauss equation (82) to the assumption that the mean curvature vector $\vec{\mathcal{H}}$ is spacelike everywhere. Throughout this section, we will use the extrinsic objects $A$, $\mathcal{H}^2$ evolving under null mean curvature flow on the standard lightcone in the 3+1-Minkowski spacetime, but will frequently exploit its equivalence to $2d$-Ricci flow to switch freely between the frameworks of null mean curvature flow and $2d$-Ricci flow. A key tool in the proof will be to first study the evolution of $|\mathring{A}|^2$ along the unnormalized flow which will yield a crucial gradient estimate. Translating these to the renormalized flow will then yield the proof of Theorem 7.12.

**Remark 7.13.** Note that there does not seem to be a direct connection between $\mathring{A}$ and the auxiliary term $M = \mathring{\mathrm{Hess}} f$ in the modified renormalized flow in [7, 47], where $f$ solves

$$\Delta f = \left(\mathrm{R} - \fint_\Sigma \mathrm{R}\right).$$

To see this, consider any stationary point of the renormalized flow where $f$ is necessarily constant while $\mathring{A} = 0$ holds for all functions $\omega$ of the form (86) arising from Lorentz transformations.

We start by computing the relevant evolution equations for the unnormalized flow.

**Proposition 7.14.** *For a smooth solution to null mean curvature flow, we find*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}|A|^2 &= \Delta|A|^2 - 2|\nabla A|^2 + \frac{1}{2}\left(\mathcal{H}^2\right)^3,\\ \frac{\mathrm{d}}{\mathrm{d}\,t}\mathcal{H}^2 &= \Delta\mathcal{H}^2 + \frac{1}{2}\left(\mathcal{H}^2\right)^2.\end{aligned}$$

*Proof.* For $\varphi = -\frac{1}{2}\theta$, the evolution equation for $\mathcal{H}^2$ is immediate from Lemma 7.8. Combining the evolution equation for A from Lemma 7.8 with the null Simons' identity, Lemma 7.9, we have

$$\frac{\mathrm{d}}{\mathrm{d}\,t}A_{ij} = \Delta A_{ij} - \frac{1}{2}\mathcal{H}^2\mathring{A}_{ij}.$$

Hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}|A|^2 &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\gamma^{ik}\gamma^{jl}A_{ij}A_{jk}\right)\\ &= 2\gamma^{ik}\gamma^{jl}A_{ij}\frac{\mathrm{d}}{\mathrm{d}\,t}A_{kl} + 2\frac{\mathrm{d}}{\mathrm{d}\,t}\gamma^{ik}\gamma^{jl}A_{ij}A_{kl}\\ &= 2\left\langle A, \Delta A - \frac{1}{2}\mathcal{H}^2\mathring{A}\right\rangle + \mathcal{H}^2|A|^2\\ &= \Delta|A|^2 - 2|\nabla A|^2 - \mathcal{H}^2|\mathring{A}|^2 + \mathcal{H}^2|A|^2\\ &= \Delta|A|^2 - 2|\nabla A|^2 + \frac{1}{2}\left(\mathcal{H}^2\right)^3.\end{aligned}$$

□

Therefore, as we already know from Ricci flow, cf. [32, Corollary 2.11], the positivity of $\mathcal{H}^2$ is preserved under the flow by the parabolic maximum principle [32, Proposition 2.9]. In particular, the flow exists only for finite time, as $\mathcal{H}^2 \to \infty$ in finite time.

We would like to point out the similarity to the evolution of the corresponding quantities for mean curvature flow in Euclidean space, where the second fundamental form $h$ and mean curvature $H$ are replaced here by $A$ and $\mathcal{H}^2$ in the evolution equations. However, compared to the work of Huisken (cf. [2, Theorem 8.6]) for mean curvature flow the slight differences will allow for a much more direct approach using only the maximum principle without any pinching condition and without the need to employ a Stampacchia iteration. In fact, here we give a more direct proof of an improved gradient estimate as in the original work by the author [89], cf. Proposition 5.4 and Theorems 5.6 and 5.7 in [89].

We state the gradient estimate in the language of $2d$-Ricci flow.

**Theorem 7.15.** *Let $\{\Sigma_t\}_{t\in[0,T_{max})}$ be a family of closed, topological 2-spheres with strictly positive scalar curvature $R > 0$ evolving under Ricci flow. For any $p > \frac{1}{2}$, $\eta > 0$, there exists $C_\eta > 0$ only depending on $\eta$, $p$ and $\Sigma_0$, such that*

$$|\nabla \mathrm{R}| \leq \eta^2 \mathrm{R}^p + C_\eta.$$

*for all $t \in [0, T_{max})$.*

**Remark 7.16.** As $\mathcal{H}^2 = 2\,\mathrm{R}$ by the Gauss Equation (82), it suffices to proof

$$\left|\nabla \mathcal{H}^2\right| \leq \eta^2 \left(\mathcal{H}^2\right)^p + C_\eta.$$

Choosing $p = \frac{3}{2}$, we get the crucial gradient estimate

$$|\nabla \mathrm{R}| \leq \eta^2 \mathrm{R}_{max}^{\frac{3}{2}}$$

for any $\eta > 0$ and $t$ sufficiently close to $T_{max}$, as $\mathrm{R}_{max} \to \infty$. This estimate allowed Hamilton to conclude that the ratio of $\mathrm{R}_{min}$ and $\mathrm{R}_{max}$ converges to 1 in the 3-dimensional case using the Theorem of Bonnet–Myers, cf. [32, Lemma 3.22]. We now established this estimate for 2-dimensional Ricci flow and can argue analogously. Compare also the corresponding result by Huisken for mean curvature flow of convex surfaces, cf. [2, Corollary 8.16].

**Corollary 7.17.** As $t \to T_{max}$,

$$\frac{\mathcal{H}^2_{max}}{\mathcal{H}^2_{min}} = \frac{\mathrm{R}_{max}}{\mathrm{R}_{min}} \to 1,$$
$$\mathrm{diam}(\Sigma_t) \to 0.$$

*Proof of Theorem 7.15.* A direct computation using Proposition 7.14 yields

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathring{A}|^2 = \Delta|\mathring{A}|^2 - 2\left(|\nabla A|^2 - \frac{1}{2}\left|\nabla \mathcal{H}^2\right|^2\right) \leq \Delta|\mathring{A}|^2 - \frac{1}{2}\left|\nabla \mathcal{H}^2\right|^2,$$

where we additionally used Proposition 7.7, Equation (90). As $\left|\nabla \mathcal{H}^2\right|^2 = \gamma^{ij}\partial_i\mathcal{H}^2\partial_j\mathcal{H}^2$, a straightforward computation gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left|\nabla \mathcal{H}^2\right|^2 &= 2\left\langle \nabla \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^2, \nabla \mathcal{H}^2\right\rangle + \frac{1}{2}\mathcal{H}^2\left|\nabla \mathcal{H}^2\right|^2 \\ &= 2\left\langle \nabla\Delta \mathcal{H}^2, \nabla \mathcal{H}^2\right\rangle + \frac{5}{2}\mathcal{H}^2.\end{aligned}$$

Using the Bochner formula

$$\Delta\left|\nabla f\right|^2 = 2\left\langle\nabla\Delta f, \nabla f\right\rangle + 2\left|\nabla^2 f\right|^2 + 2\mathrm{Ric}_\gamma(\nabla f, \nabla f) \tag{94}$$

for any smooth function $f$, where $\nabla^2$ denotes the Hessian of $f$, and the fact that $\mathrm{Ric}_\gamma = \frac{1}{4}\mathcal{H}^2\gamma$ by the Gauss formula (82), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\nabla\mathcal{H}^2\right|^2 = \Delta\left|\nabla\mathcal{H}^2\right|^2 - 2\left|\nabla^2\mathcal{H}^2\right|^2 + 2\mathcal{H}^2\left|\nabla\mathcal{H}^2\right|^2.$$

We note that this is consistent with the evolution of the gradient of the scalar curvature under $2d$-Ricci flow, as we would expect. Then, direct computation gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{|\nabla\mathcal{H}^2|^2}{\mathcal{H}^2} &= \Delta\frac{|\nabla\mathcal{H}^2|^2}{\mathcal{H}^2} + 2\frac{\left\langle\nabla\left|\nabla\mathcal{H}^2\right|^2, \nabla\mathcal{H}^2\right\rangle}{(\mathcal{H}^2)^2} - 2\frac{|\nabla\mathcal{H}^2|^4}{(\mathcal{H}^2)^3} - 2\frac{|\nabla^2\mathcal{H}^2|^2}{\mathcal{H}^2} + \frac{3}{2}\left|\nabla\mathcal{H}^2\right|^2 \\ &\leq \Delta\frac{|\nabla\mathcal{H}^2|^2}{\mathcal{H}^2} + \frac{3}{2}\left|\nabla\mathcal{H}^2\right|^2\end{aligned}$$

using the Cauchy–Schwarz and Young's inequality. Hence

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\right)\left(\frac{|\nabla\mathcal{H}^2|^2}{\mathcal{H}^2} + 3|\mathring{A}|^2\right) \leq 0$$

and therefore any initial bound is preserved for the sum. Thus

$$\frac{|\nabla\mathcal{H}^2|^2}{\mathcal{H}^2} \leq \frac{|\nabla\mathcal{H}^2|^2}{\mathcal{H}^2} + 3|\mathring{A}|^2 \leq C(\Sigma_0).$$

The above estimate follows from multiplying by $\mathcal{H}^2$ and using Young's inequality. □

We briefly recall some properties of $n$-dimensional Ricci flow renormalized by volume (cf. [32, Chapter 3.6]), i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}\,\widetilde{t}}\widetilde{\gamma}(\,\widetilde{t}\,) = -2\widetilde{\mathrm{Ric}}(\,\widetilde{t}\,) + \frac{2}{n}\widetilde{r}\widetilde{\gamma}(\,\widetilde{t}\,), \tag{95}$$

where $\widetilde{r} = \fint\widetilde{\mathrm{R}}$, such that along any solution we have that

$$\mathrm{Vol}(\widetilde{\gamma}(\,\widetilde{t}\,)) = \mathrm{Vol}(\widetilde{\gamma}(0)) = \mathrm{const.}.$$

Given a solution of Ricci flow $\gamma(t)$, $t \in [0, T)$, the metrics $\widetilde{\gamma}(\,\widetilde{t}\,) := c(t)\gamma(t)$, with

$$c(t) := \exp\left(\frac{2}{n}\int_0^t r(\tau)\,\mathrm{d}\,\tau\right), \ \widetilde{t}(t) := \int_0^t c(\tau)\,\mathrm{d}\,\tau,$$

satisfy (95) with initial condition $\widetilde{\gamma}(0) = \gamma(0)$, so we can always renormalize a given solution of Ricci flow. Moreover, we have the following transformation laws for evolution equations by Hamilton:

**Lemma 7.18** (Hamilton, see [32, Lemma 3.26])**.** *If an expression $X = X(\gamma)$ formed algebraically from the metric and the Riemann curvature tensor has degree $k$, i.e., $X(c\gamma) = c^k X(\gamma)$, and if under the Ricci flow*

$$\frac{\mathrm{d}}{\mathrm{d}t} X = \Delta X + Y,$$

*then the degree of $Y$ is $k-1$ and the evolution under the normalized Ricci flow $\frac{\mathrm{d}}{\mathrm{d}\widetilde{t}}\widetilde{\gamma}_{ij} = -2\widetilde{\mathrm{Ric}}_{ij} + \frac{2}{n}\widetilde{r}\widetilde{\gamma}_{ij}$ of $\widetilde{X} := X(\widetilde{\gamma})$ is given by*

$$\frac{\mathrm{d}}{\mathrm{d}\widetilde{t}}\widetilde{X} = \widetilde{\Delta}\widetilde{X} + \widetilde{Y} + \frac{2k}{n}\widetilde{X}.$$

Recall that by [32, Remark 3.27], this lemma also extends to the corresponding partial differential inequalities if $Y$ is of degree $k-1$, and is further also applicable to arbitrary tensor derivatives of such expressions as used frequently throughout [32, Chapter 3].

**Remark 7.19.** In this section, as before, we will only look at the case when $n = 2$ and $\gamma(t)$ is conformal to the round sphere for each $t$. Thus $\widetilde{\gamma}(\widetilde{t})$ is conformally round, and by the Gauss–Bonnet Theorem

$$\widetilde{r}(t) = \frac{8\pi}{\mathrm{Vol}(\widetilde{\gamma}(\widetilde{t}))} = \frac{8\pi}{\mathrm{Vol}(\widetilde{\gamma}(0))}$$

is positive and remains constant along the flow.

From now on, will assume without loss of generality that

$$\frac{1}{2}\,\mathrm{R}_{max}(t) \leq \mathrm{R}(x,t) \tag{96}$$

for all $t \in [0, T_{max}), x \in \Sigma_t$ (this is ultimately satisfied for $t$ sufficiently close to $T_{max}$ due to Corollary 7.17). Note that due to the relation between $\mathcal{H}^2$ and R via the Gauss equation (82), combining Proposition 7.14 with the fact that $\mathrm{R}_{max} \to \infty$ as $t \to T$, we find that $\mathrm{R}_{max} \geq (T-t)^{-1}$. In particular,

$$\mathrm{R}(t,x) \geq \frac{1}{2(T-t)}. \tag{97}$$

In the following, we will switch freely between the framework of (renormalized) 2-d Ricci flow and null mean curvature flow along the past-pointing standard lightcone. Recall that most importantly, bounds for $\mathcal{H}^2$ and its derivatives correspond directly to bounds on the scalar curvature and its derivatives via the Gauss equation (82). In our analysis of the renormalized flow we will closely follow the outline of Hamilton's strategy presented in [32, Chapter 3] for 3-dimensional Ricci flow, and include the proofs for the sake of completeness. We start by establishing the following lemma:

**Lemma 7.20.** *For the renormalized flow* (95), *we have that*

(i) $\widetilde{T} = \infty$,

(ii) $\lim\limits_{\widetilde{t}\to\infty} \frac{\widetilde{\mathcal{H}}^2_{min}}{\widetilde{\mathcal{H}}^2_{max}} = 1$

(iii) *There exists* $C_1 > 0$ *such that* $\frac{1}{C_1} \leq \widetilde{\mathcal{H}}^2_{min}(\,\widetilde{t}\,) \leq \widetilde{\mathcal{H}}^2_{max}(\,\widetilde{t}\,) \leq C_1$ *for all* $\widetilde{t}$,

(iv) $\operatorname{diam}(\widetilde{\gamma}(\,\widetilde{t}\,)) \leq C_2$

(v) $|\mathring{\widetilde{A}}|^2 \leq C_3 e^{-\delta \widetilde{t}}$,

(vi) $\left|\widetilde{\nabla}\widetilde{\mathcal{H}}^2\right|^2 \leq C_4 e^{-\delta \widetilde{t}}$,

(vii) $\widetilde{\mathcal{H}}^2_{max} - \widetilde{\mathcal{H}}^2_{min} \leq C_5 e^{-\delta \widetilde{t}}$,

*for constants* $C_1, C_2, C_3, C_4, C_5$ *independent of* $t$.

*Proof.* In the following, $C, \widetilde{C}$ will always denote positive constants independent of $t$ that may vary from line to line.

(i) By substitution rule, we find that

$$\int_0^{\widetilde{t}(t_0)} \widetilde{r}(\widetilde{\tau})\, \mathrm{d}\,\widetilde{\tau} = \int_0^{t_0} r(\tau)\, \mathrm{d}\,\tau.$$

Combining this formula for $t \to T$ with (97), we can conclude (i) since $\widetilde{r}$ is constant.

(ii) Follows immediately from Corollary 7.17.

(iii) By the Bishop–Gromov volume comparison, we have that

$$\operatorname{Vol}(\widetilde{\gamma}(0)) = \operatorname{Vol}(\widetilde{\gamma}(\,\widetilde{t}\,)) \leq C \operatorname{diam}(\widetilde{\gamma}(\,\widetilde{t}\,))^2,$$

and recall that by the Bonnet–Myers Theorem

$$\operatorname{diam}(\widetilde{\gamma}(\widetilde{t})) \leq C \left(\widetilde{\mathcal{H}}^2_{max}\right)^{-\frac{1}{2}}. \tag{98}$$

Thus, $\widetilde{\mathcal{H}}^2_{max}$ is uniformly bounded from above. Now note that $\widetilde{\Sigma}_{\widetilde{t}}$ is a topological sphere, in particular simply connected. Hence, Klingenberg's injectivity radius estimate yields that

$$\operatorname{inj}(\widetilde{\gamma}(\widetilde{t})) \geq C \left(\widetilde{\mathcal{H}}^2_{max}\right)^{-\frac{1}{2}},$$

and therefore

$$\operatorname{Vol}(\widetilde{\gamma}(0)) = \operatorname{Vol}(\widetilde{\gamma}(\widetilde{t})) \geq C \left(\widetilde{\mathcal{H}}^2_{max}\right)^{-1},$$

so $\widetilde{\mathcal{H}}^2_{max}$ is also uniformly bounded from below. Since the inequality (96) is preserved under rescaling, we have that $\widetilde{\mathcal{H}}^2$ is uniformly bounded by the Gauss Equation (82) and we can therefore pick some constant $C > 0$ such that (iii) is satisfied.

(iv) Follows directly from (iii) via (98).

We prove (v) and (vi) simultaneously. Consider $\Psi := \frac{\left|\nabla \mathcal{H}^2\right|^2}{\mathcal{H}^2} + 3|\mathring{A}|^2$. Then $\Psi$ is of degree $-2$ and we saw in the proof of Theorem 7.15 that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\right) \Psi \leq 0$$

Thus, by Lemma 7.20, we find that

$$\left(\frac{\mathrm{d}}{\mathrm{d}\widetilde{t}} - \widetilde{\Delta}\right) \widetilde{\Psi} \leq -\widetilde{C}\widetilde{\Psi}.$$

By the maximum principle, we can now conclude that

$$\frac{\left|\widetilde{\nabla}\widetilde{\mathcal{H}}^2\right|^2}{\widetilde{\mathcal{H}}^2} + K|\widetilde{\mathring{A}}|^2 \leq Ce^{-\delta\widetilde{t}},$$

so (v) and (vi) follow since $\widetilde{\mathcal{H}}^2$ is uniformly bounded. Lastly (vii) follows form (iv) and (vi). $\square$

In particular $\left|\widetilde{\mathrm{R}} - \widetilde{r}\right| \leq Ce^{-\delta\widetilde{t}}$, so we know that the evolution speed of the renormalized flow (95) is integrable. We thus acquire uniform bounds and $C^0$ convergence of the metric due to a Lemma by Hamilton (cf. [32, Lemma 6.10]):

**Corollary 7.21.** Let $\gamma(t)$ be a solution of $2d$-Ricci flow with $\mathrm{R} > 0$. Then the renormalized flow (95) exists for all time, and there exists a constant $C > 0$ such that

$$\frac{1}{C}\widetilde{\gamma}(0) \leq \widetilde{\gamma}(\,\widetilde{t}\,) \leq C\widetilde{\gamma}(0),$$

and $\widetilde{\gamma}(\widetilde{t})$ converges uniformly to a limiting metric $\widetilde{\gamma}(\infty)$ on compact sets as $\widetilde{t} \to \infty$.

**Remark 7.22.** Since the renormalized metrics are also conformally round, i.e., $\widetilde{\gamma}(\,\widetilde{t}\,) = \widetilde{\omega}^2(\,\widetilde{t}\,)\,\mathrm{d}\,\Omega^2$, Corollary 7.21 in particular yields a uniform bound on the conformal factors $\widetilde{\omega}(t)$ depending only on $\widetilde{\omega}(0)$.

To complete the proof of Theorem 7.12, it remains to show that $\widetilde{\gamma}(\infty)$ is in fact smooth and that the renormalized flow converges in $C^k$ for any $k$. In particular, due to Lemma 7.20 (ii), $\widetilde{\gamma}(\infty)$ is then a metric of constant scalar curvature. We thus require bounds for the derivatives of the renormalized metrics, and by a standard argument it suffices to bound the derivatives of the Riemann tensor. However, for $n = 2$, the Riemann tensor and its derivatives are fully determined by the scalar curvature and its derivatives. By the Gauss equation (82), it thus suffices to find appropriate bounds for $\widetilde{\mathcal{H}}^2$ and its derivatives.

**Lemma 7.23.** *For all $k \in \mathbb{N}$, there exist $C_k, \delta_k > 0$, such that*

$$\left|\widetilde{\nabla}^k\widetilde{\mathcal{H}^2}\right|^2 \leq C_k e^{-\delta_k\widetilde{t}}$$

*Proof.* Since $n = 2$, there exists a fixed constant $C$, such that $\left|\nabla^k\mathcal{H}^2\right|^2 = C\left|\nabla^k\operatorname{Rm}\right|^2$, and thus the evolution of $\left|\nabla^k\mathcal{H}^2\right|^2$ along the unnormalized flow can be estimated by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left|\nabla^k\mathcal{H}^2\right|^2 \leq{}& \Delta\left|\nabla^k\mathcal{H}^2\right|^2 - 2\left|\nabla^{k+1}\mathcal{H}^2\right|^2\\ &+ C(k)\sum_{l=0}^{k}\left|\nabla^l\mathcal{H}^2\right|\left|\nabla^{k-l}\mathcal{H}^2\right|\left|\nabla^k\mathcal{H}^2\right|,\end{aligned} \tag{99}$$

where $C(k)$ denotes a constant only depending on $k$, cf. [32, Chapter 3].

We will proof the statement by strong induction, where in the following $C_k, C_{k+1}$ will be constants only depending on $k$ which may vary from line to line. The statement is true for $k = 1$ as proven in Lemma 7.20 (vi).

We now assume that the statement is true for all $1 \leq l \leq k$, and proceed from $k$ to $k+1$. We define $f := \left|\nabla^{k+1}\mathcal{H}^2\right|^2 + K\mathcal{H}^2\left|\nabla^k\mathcal{H}^2\right|^2$, where $K$ is a positive constant to be determined later. Then $f$ is of degree $-k-3$ and according to (99) its evolution under Ricci flow is given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f \leq & \Delta \left|\nabla^{k+1}\mathcal{H}^2\right|^2 - 2\left|\nabla^{k+2}\mathcal{H}^2\right|^2 + C_{k+1}\sum_{l=0}^{k+1}\left|\nabla^l\mathcal{H}^2\right|\left|\nabla^{k+1-l}\mathcal{H}^2\right|\left|\nabla^{k+1}\mathcal{H}^2\right| \\
& + K\left|\nabla^k\mathcal{H}^2\right|^2\Delta\mathcal{H}^2 + \frac{K}{2}\left(\mathcal{H}^2\right)^2\left|\nabla^k\mathcal{H}^2\right|^2 \\
& + K\mathcal{H}^2\Delta\left|\nabla^k\mathcal{H}^2\right|^2 - 2K\mathcal{H}^2\left|\nabla^{k+1}\mathcal{H}^2\right|^2 + K\mathcal{H}^2 C_k\sum_{l=0}^{k}\left|\nabla^l\mathcal{H}^2\right|\left|\nabla^{k-l}\mathcal{H}^2\right|\left|\nabla^k\mathcal{H}^2\right| \\
\leq & \Delta f - 2K\left\langle\nabla\mathcal{H}^2, \nabla\left|\nabla^k\mathcal{H}^2\right|^2\right\rangle - 2K\mathcal{H}^2\left|\nabla^{k+1}\mathcal{H}^2\right|^2 \\
& + C_{k+1}\sum_{l=0}^{k+1}\left|\nabla^l\mathcal{H}^2\right|\left|\nabla^{k+1-l}\mathcal{H}^2\right|\left|\nabla^{k+1}\mathcal{H}^2\right| + K\mathcal{H}^2 C_k\sum_{l=0}^{k}\left|\nabla^l\mathcal{H}^2\right|\left|\nabla^{k-l}\mathcal{H}^2\right|\left|\nabla^k\mathcal{H}^2\right| \\
\leq & \Delta f + C_k(K, \mathcal{H}^2, \nabla^{1\leq l\leq k}\mathcal{H}^2) + (C_{k+1} - K)\mathcal{H}^2\left|\nabla^{k+1}\mathcal{H}^2\right|^2,
\end{aligned}$$

where we have used Young's inequality in the last line and collected all remaining terms in $C_k(K, \mathcal{H}^2, \nabla^{1\leq l\leq k}\mathcal{H}^2)$. In particular, we used the estimate

$$-2K\left\langle\nabla\mathcal{H}^2, \nabla\left|\nabla^k\mathcal{H}^2\right|^2\right\rangle \leq 4K\frac{\left|\nabla\mathcal{H}^2\right|^2\left|\nabla^k\mathcal{H}^2\right|^2}{\mathcal{H}^2} + K\mathcal{H}^2\left|\nabla^{k+1}\mathcal{H}^2\right|^2.$$

We now choose $K := C_{k+1}$, and thus we find that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \Delta\right) f \leq C_k(\mathcal{H}^2, \nabla^{1\leq l\leq k}\mathcal{H}^2),$$

where $C_k(\mathcal{H}^2, \nabla^{1\leq l\leq k}\mathcal{H}^2)$ is in fact a sum of products of derivatives with at least order 1 and at most order $k$ such that the factors only depend on $k$, and possibly $\mathcal{H}^2$, and $C_k(\mathcal{H}^2, \nabla^{1\leq l\leq k}\mathcal{H}^2)$ is of degree $-k-4$. Hence, the evolution of $\widetilde{f}$ along the renormalized flow is given by

$$\left(\frac{\mathrm{d}}{\mathrm{d}\widetilde{t}} - \widetilde{\Delta}\right)\widetilde{f} \leq C_k(\widetilde{\mathcal{H}^2}, \widetilde{\nabla}^{1\leq l\leq k}\widetilde{\mathcal{H}^2}) - (k+3)\widetilde{f} \leq \widetilde{C}e^{-\widetilde{\delta}\widetilde{t}} - (k+3)\widetilde{f}$$

for some $\widetilde{C}, \widetilde{\delta} > 0$ by induction, as $\widetilde{\mathcal{H}^2}$ is uniformly bounded by Lemma 7.20. Now choosing $\delta < \min(\widetilde{\delta}, k+3)$, we find that

$$(\partial_{\widetilde{t}} - \widetilde{\Delta})\left(e^{\delta\widetilde{t}}\widetilde{f} - \widetilde{C}\widetilde{t}\right) \leq 0.$$

So by the maximum principle, there exists $C_0 > 0$ such that

$$e^{\delta\widetilde{t}}\widetilde{f} - \widetilde{C}\widetilde{t} \leq C_0 \Leftrightarrow \widetilde{f} \leq e^{-\delta\widetilde{t}}(C_0 + \widetilde{C}\widetilde{t}).$$

Since exponential decay wins over linear growth, there exists an appropriate constant $C_k > 0$ for any choice $0 < \delta_k < \delta$ such that

$$\widetilde{f} \leq C_k e^{-\delta_k\widetilde{t}}.$$

This concludes the proof. □

From this, we can conclude the uniform convergence in $C^k$ for any $k \in \mathbb{N}$ and Theorem 7.12 is proven.

# 8 A De Lellis–Müller type estimate on the lightcone

A well-known result by De Lellis–Müller [36] states that any closed surface $\Sigma$ in $\mathbb{R}^3$ with $\mathring{h}$ sufficiently small in $L^2$ is $W^{2,2}$-close to a round sphere $\mathbb{S}_r(\vec{a})$, where $r$ and $\vec{a}$ denote the area radius and the Euclidean center of mass of $\Sigma$, respectively. This can be regarded as a quantitative refinement of the fact that any closed surface $\Sigma$ in $\mathbb{R}^3$ with $\mathring{h} = 0$ is necessarily a round surface by the Codazzi Equation, cf. Proposition 4.9.

In Section 7, we have proven an analogous statement that any spacelike cross section $\Sigma$ of the standard Minkowski lightcone with $\mathring{A} = 0$ is a surface of constant spacetime mean curvature, where $A$ is the scalar second fundamental form as defined in Subsection 7.3, cf. Proposition 7.7. As before, a spacelike cross section is always assumed to satisfy the assumptions made in Subsection 4.7. Similar to the result of De Lellis–Müller, we will prove in this section that any spacelike cross section $\Sigma$ with $\mathcal{H}^2 \geq 0$ and $\mathring{A}$ sufficiently small in $L^2$ is $W^{2,2}$-close to a surface of constant spacetime mean curvature if some uniform bounds on the conformal factor are satisfied. The STCMC surface of reference is fully determined by a timelike, future-pointing 4-vector $\mathbf{Z}$ in the ambient Minkowski spacetime associated to the cross section which transforms equivariantly under Lorentz transformations in $SO^+(3,1)$.

The proof consists of two steps: In the first step, we establish a geometric, scaling invariant estimate of the form

$$|\Sigma| \int_\Sigma \left| A - \frac{⨍ \mathcal{H}^2}{2} \gamma \right|^2 \leq C\, |\Sigma| \int_\Sigma |\mathring{A}|^2. \tag{100}$$

for any spacelike cross section $\Sigma$ of the standard lightcone with $\mathcal{H}^2 \geq 0$, where $C > 0$ is a uniform constant. We will in fact prove this estimate using two different methods and obtain it for two explicit, but different constants $C$. The first proof is given as an application of null mean curvature flow studied in Section 7. The second proof is modelled on the proof of an almost Schur-lemma by De Lellis–Topping [37] using the Bochner formula.

Smallness on the left-hand side of the equation implies that the Gauss curvature of the rescaled surface is close to 1 in $L^2$, and elliptic theory will yield a $W^{2,2}$ estimate for the conformal factor under a suitable balancing condition. This balancing condition is intricately connected to the equivariance of $\mathbf{Z}$ under Lorentz transformations. This will allow us to conclude the desired result, see Theorem 8.23.

Alternatively, we can interpret the result intrinsically in the class of conformally round surfaces, see Corollary 8.22.

Throughout this section, we will always consider the future-pointing lightcone $\{t = r\} = \{u = 0\}$, cf. Subsections 4.2 and 7.2. Note that all curvature identities for spacelike cross sections established on the past-pointing lightcone, and all properties of null mean curvature flow along the past-pointing lightcone proven in Section 7 analogously hold on the future-pointing standard lightcone. Similarly, all results of this section can be directly translated to the past-pointing lightcone, where we choose to phrase everything with respect to the future-pointing lightcone purely for convenience.

This section is structured as follows: In Subsection 8.1 we define an associated 4-vector for any spacelike cross section of the future-pointing standard lightcone, and show that it equivariantly transforms under a restricted Lorentz transformation of the ambient spacetime. We will further recall that the Gauss curvature is invariant under restricted Lorentz transformations up to a change of coordinates on $\mathbb{S}^2$. In Subsection 8.2 we give two proofs of the geometric estimate (100). In Subsection 8.3 we establish a $W^{2,2}$-estimate for surfaces with uniformly bounded conformal factor and Gauss curvature sufficiently close to 1 in $L^2$ under a suitable balancing condition. In Subsection 8.4 we combine the results of the previous subsections to obtain Theorem 8.23.

## 8.1 An associated 4-vector in the Minkowski spacetime

Let $(\Sigma,\gamma)$ be a spacelike cross section of the future-pointing lightcone. Then, the *associated 4-vector $\boldsymbol{Z}$ of $\Sigma$ in the ambient Minkowski spacetime* is defined as

$$\begin{aligned}\mathbf{Z}(\Sigma)^0 &:= \frac{1}{|\Sigma|}\int_\Sigma t\,\mathrm{d}\mu_\Sigma,\\ \mathbf{Z}(\Sigma)^i &:= \frac{1}{|\Sigma|}\int_\Sigma x^i\,\mathrm{d}\mu_\Sigma,\end{aligned} \tag{101}$$

where $x^i$ denotes the restriction of the standard Euclidean coordinates on $\mathbb{R}^3$ to $\Sigma$. In particular, the spacial coordinates are similarly defined as the Euclidean center of mass for a surface in $\mathbb{R}^3$, see e.g. [23, 24, 61]. In fact, up to rescaling to unit length and thus projecting to the hyperboloid $\{p\in\mathbb{R}^{3,1}\colon \eta(p,p)=-1\}$, this associated 4-vector is directly equivalent to a notion of hyperbolic center defined by Cederbaum–Cortier–Sakovich for surfaces in asymptotically hyperbolic initial data sets [20]. While they argue abstractly that this notion of center transforms equivariantly for surfaces on a hypberboloid in the Minkowski spacetime, the approach to prove this for spacelike cross sections of the standard lightcone presented here will be more explicit and computational in nature.

To this end, we recall that any spacelike cross sections $(\Sigma,\gamma)$ is conformally round with conformal factor $\omega$ and $\Sigma=\mathrm{graph}_{\mathbb{S}^2}\omega$. Thus, for any spacelike cross section along the future-pointing lightcone, we have $t|_\Sigma=r|_\Sigma=\omega$, $x^i|_\Sigma=\omega f_i$, where $f_i$ denote the first spherical harmonics as defined in Subsection 4.10. Hence,

$$\begin{aligned}\mathbf{Z}(\Sigma)^0 &:= \frac{1}{|\Sigma|}\int_{\mathbb{S}^2}\omega^3\,\mathrm{d}\mu_{\mathbb{S}^2},\\ \mathbf{Z}(\Sigma)^i &:= \frac{1}{|\Sigma|}\int_{\mathbb{S}^2} f_i\omega^3\,\mathrm{d}\mu_{\mathbb{S}^2}.\end{aligned}\tag{102}$$

In this context, we further note that the spatial coordinates closely resemble a notion of center of mass in conformal geometry, cf. [55, Definition 2.8], up to replacing $\omega^3$ by $\omega^2$ in the integrals. See also [26, 27]. By a topological argument, it is a well-known fact that for any conformally round surface there exists a conformal transformation in the Möbius group such that one can achieve

$$\frac{1}{|\Sigma|}\int_{\mathbb{S}^2} f_i\omega^2\,\mathrm{d}\mu_{\mathbb{S}^2} = 0 \text{ for all } i,$$

see [26, 27, 65]. As we will show that the associated 4-vector transforms equivariantly under Lorentz boosts in $SO^+(3,1)$, see Proposition 8.4 below, we can achieve a similar balancing condition without relying on an implicit argument and uniquely identify the respective Lorentz transformation with respect to $\mathbf{Z}$ up to isometries on $\mathbb{S}^2$.

As the future-pointing lightcone is mapped onto itself under any Lorentz transformation $\Lambda$ in $SO^+(3,1)$, the image of any smooth spacelike cross section $\Sigma$ of the lightcone is itself a smooth, spacelike cross section $\Lambda(\Sigma)$.

**Lemma 8.1.** *Let $\Sigma$ be a spacelike cross section with conformal factor $\omega$, and consider $\Lambda \in SO^+(3,1)$. Then the conformal factor $\omega_\Lambda$ of $\Lambda(\Sigma)$ is given by*

$$\omega_\Lambda = \frac{\omega\circ\Phi}{\sqrt{1+|\vec{a}|^2} - \vec{a}^i f_i},$$

*where $\Phi\colon \mathbb{S}^2 \to \mathbb{S}^2$ is a diffeomorphism, and $\Phi$ and $\vec{a}\in\mathbb{R}^3$ are uniquely determined by $\Lambda$.*

A straightforward computation then yields the following as a corollary:

**Corollary 8.2.** Let $\Sigma$ be a spacelike cross section with Gauss curvature $\mathcal{K}$, and consider $\Lambda \in SO^+(3,1)$. Then the Gauss curvature $\mathcal{K}_\Lambda$ of $\Lambda(\Sigma)$ satisfies

$$\mathcal{K}_\Lambda = \mathcal{K}\circ\Phi.$$

**Remark 8.3.**

(i) In particular, $\Phi$ is in fact a Möbius transformation on $\mathbb{S}^2$ uniquely determined for each $\Lambda \in SO^+(3,1)$ by the isomorphism between the Möbius group and the restricted Lorentz group, cf. Subsection 4.2. In the context of Möbius transformations and the conformal geometry of $\mathbb{S}^2$, both the invariance of the Gauss curvature and the precise transformation of the conformal factor under a Möbius transformation as stated in Lemma 8.1 are well-known facts, see e.g. [55] for a short summary and further references. Here, we give a different proof of the statement in Lemma 8.1 as we approach it from an extrinsic viewpoint with the Lorentz transformations acting as isometries on the ambient Minkowski spacetime.

(ii) For $\Sigma_\rho$, i.e., $\omega = \rho > 0$ for some positive constant, we recover the well-known formula for STCMC surfaces discussed in Section 7, cf. Remark 7.6 Equation (86). Direct computation gives

$$\mathbf{Z}(\Lambda(\Sigma_\rho)) = \rho(\sqrt{1+|\vec{a}|^2}, \vec{a}).$$

Hence, $\mathbf{Z}(\Lambda(\Sigma_\rho)) = \Lambda(\mathbf{Z}(\Sigma))$.

Moreover, for any timelike, future-pointing vector $z \in \mathbb{R}^{3,1}$, one can check by direct computation that the spacelike cross section corresponding to the conformal factor

$$\omega_z \colon = \frac{-\eta(z,z)}{z^0 - z^i f_i} \tag{103}$$

is an STCMC surface with $\mathbf{Z}(\Sigma_{\omega_z}) = z$. In this way, there is a one-to-one correspondence between timelike, future-pointing vectors and STCMC surfaces via $\mathbf{Z}$.

*Proof of Lemma 8.1.* We first prove the statement for special Lorentz boosts $\Lambda_a$ for some $a \in \mathbb{R}$ defined as in Subsection 4.2. In particular, the standard coordinates transform as

$$\begin{aligned} t' &= bt + az, \\ x' &= x, \\ y' &= y, \\ z' &= at + bz, \end{aligned}$$

where $t', x', y', z'$ denote different Cartesian coordinates. We now consider spherical coordinates $t, r, \theta, \varphi$, and $t', r', \theta', \varphi'$ given via (26) as considered in Subsection 4.10. As the lightcone is invariant under $\Lambda$, we know that $\omega_\Lambda = t' = r'$, and by the above identities it is easy to check that

$$r = r'(b - a\cos\theta').$$

Hence,

$$\omega_\Lambda(\theta',\varphi') = \frac{\omega(\theta,\varphi)}{b - a\cos\theta'}. \tag{104}$$

Moreover, we see that

$$\begin{aligned}
\sin\theta\sin\varphi &= \frac{\sin\theta'}{b - a\cos\theta'}\sin\varphi',\\
\sin\theta\cos\varphi &= \frac{\sin\theta'}{b - a\cos\theta'}\cos\varphi',\\
\cos\theta &= \frac{b\cos\theta' - a}{b - a\cos\theta'}.
\end{aligned} \tag{105}$$

In particular, we obtain $\varphi = \varphi'$ and

$$\theta = \arccos\left(\frac{b\cos\theta' - a}{b - a\cos\theta'}\right) \tag{106}$$

A direct computation gives $\theta(\theta') \to 0$ as $\theta' \to 0$ and $\theta(\theta') \to \pi$ as $\theta' \to \pi$, and

$$\frac{\mathrm{d}\theta}{\mathrm{d}\theta'} = \frac{1}{b - a\cos\theta'} > 0. \tag{107}$$

Thus, as $b > |a|$, we note that

$$\Phi_a\colon (0,\pi)\times(0,2\pi) \to (0,\pi)\times(0,2\pi)\colon (\theta',\varphi') \mapsto \left(\arccos\left(\frac{b\cos\theta' - a}{b - a\cos\theta'}\right), \varphi'\right)$$

extends smoothly to a diffeomorphism $\Phi_a\colon \mathbb{S}^2 \to \mathbb{S}^2$. This establishes the claim in the case of a special Lorentz boost.

For a general $\Lambda \in SO^+(3,1)$, we may assume wlog by Proposition 4.2 and Remark 4.3 that

$$\Lambda = \Lambda_{\vec{a}} = D_{\vec{a}} \circ \Lambda_{|\vec{a}|} \circ D_{\vec{a}}^{-1},$$

as rotations act as isometries on $\mathbb{S}^2$ and the standard lightcone, and the spherical harmonics transform naturally under rotations, cf. Subsection 4.10.

For a rotation $D$, let $\Phi_D\colon \mathbb{S}^2 \to \mathbb{S}^2, \vec{x} \mapsto R(\vec{x})$ denote the corresponding diffeomorphism on $\mathbb{S}^2$. In fact, $\Phi_D$ acts as an isometry on the round sphere. It is easy to check that

$$\omega_D(\vec{x}) = \omega \circ \Phi_D^{-1}(\vec{x}).$$

In view of the above decomposition, the claim directly follows by observing that for $D = D_{\vec{a}}$

$$\begin{aligned}\left(\sqrt{1+|\vec{a}|^2} - |\vec{a}|\cos\theta\right)(\Phi_D^{-1})(\vec{x}) &= \sqrt{1+|\vec{a}|^2} - (|\vec{a}|\partial_3)\cdot\Phi_D^{-1}(\vec{x}) \\ &= \sqrt{1+|\vec{a}|^2} - \Phi_D\left(|\vec{a}|\partial_3\right)\cdot\vec{x} \\ &= \sqrt{1+|\vec{a}|^2} - \vec{a}\cdot\vec{x} \\ &= \sqrt{1+|\vec{a}|^2} - \vec{a}^i f_i,\end{aligned}$$

where we used that $\Phi_D$ is a linear isometry on the round sphere. Note further that by the above decomposition

$$\Phi_\Lambda = \Phi_D \circ \Phi_{|\vec{a}|} \circ \Phi_D^{-1}.$$

□

From this, the following proposition is easily established:

**Proposition 8.4.** *Let $\Sigma$ be a spacelike cross section with assiciated 4-vector $\boldsymbol{Z}$, and consider $\Lambda \in SO^+(3,1)$. Then $|\Lambda(\Sigma)| = |\Sigma|$, and $\boldsymbol{Z}$ is a future-timelike vector, such that*

$$\Lambda(\boldsymbol{Z}(\Sigma)) = \boldsymbol{Z}(\Lambda(\Sigma)).$$

**Remark 8.5.** Closely following the proof of the fact that $|\Lambda(\Sigma)| = |\Sigma|$ presented below, it is easy to check that in fact

$$\int_{\Lambda(\Sigma)} f \circ \Phi = \int_\Sigma f$$

for any continuous function $f$ on $\mathbb{S}^2$. Hence, by Corollary 8.2, we find

$$||\mathcal{K}_\Lambda - 1||^2_{L^2(\Lambda(\Sigma))} = ||\mathcal{K} - 1||^2_{L^2(\Sigma)}\,. \tag{108}$$

*Proof of Proposition 8.4.* Similar to the arguments in Lemma 8.1 the claims is readily established for rotations in view of (102) as integrals on the round sphere naturally transform under rotations which act as isometries on the round sphere. Hence, considering a decomposition of $\Lambda$ as before, it only remains to prove the claim for a special Lorentz boost $\Lambda_a$.

Now let $\Lambda_a$ be a special Lorentz boost for $a \in \mathbb{R}$. Recall from the proof of Lemma 8.1 that

$$\omega_\Lambda(\theta', \varphi') = \frac{\omega(\theta(\theta'), \varphi')}{b - a\cos\theta'},$$

with $\theta$ given by (106). From this, we can explicitly compute that

$$\begin{aligned}
\sin\theta &= \frac{\sin\theta'}{b-a\cos\theta'}, \\
b+a\cos\theta &= \frac{1}{b-a\cos\theta'}, \\
b\cos\theta + a &= \frac{\cos\theta'}{b-a\cos\theta'}.
\end{aligned}$$

Then, direct computation using (107) gives

$$|\Lambda(\Sigma)| = \int_0^{2\pi}\int_0^{\pi} \frac{\omega(\theta(\theta'),\varphi(\varphi'))^2}{(b-a\cos\theta')^2}\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\varphi' = \int_0^{2\pi}\int_0^{\pi} \omega(\theta(\theta'),\varphi)^2\sin\theta(\theta')\frac{\mathrm{d}\theta}{\mathrm{d}\theta'}\,\mathrm{d}\theta'\,\mathrm{d}\varphi = |\Sigma|\,,$$

and

$$\begin{aligned}
\int_{\mathbb{S}^2}\omega_\Lambda^3\,\mathrm{d}\mu &= \int_0^{2\pi}\int_0^{\pi} \frac{\omega(\theta(\theta'),\varphi(\varphi'))^3}{(b-a\cos\theta')^3}\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\varphi' \\
&= b\int_0^{2\pi}\int_0^{\pi}\omega(\theta(\theta'),\varphi(\varphi'))^3\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}\theta'}\,\mathrm{d}\theta'\,\mathrm{d}\varphi' + a\int_0^{2\pi}\int_0^{\pi}\omega(\theta(\theta'),\varphi(\varphi'))^3\cos\theta\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}\theta'}\,\mathrm{d}\theta'\,\mathrm{d}\varphi' \\
&= b\int_0^{2\pi}\int_0^{\pi}\omega(\theta,\varphi)^3\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi + a\int_0^{2\pi}\int_0^{\pi}\omega(\theta,\varphi)^3\cos\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= b\int_{\mathbb{S}^2}\omega^3\,\mathrm{d}\mu + a\int_{\mathbb{S}^2}\omega^3 f_3\,\mathrm{d}\mu.
\end{aligned}$$

Similarly

$$\begin{aligned}
\int_{\mathbb{S}^2}\omega_\Lambda^3 f_1\,\mathrm{d}\mu &= \int_{\mathbb{S}^2}\omega^3 f_1\,\mathrm{d}\mu, \\
\int_{\mathbb{S}^2}\omega_\Lambda^3 f_2\,\mathrm{d}\mu &= \int_{\mathbb{S}^2}\omega^3 f_2\,\mathrm{d}\mu, \\
\int_{\mathbb{S}^2}\omega_\Lambda^3 f_3\,\mathrm{d}\mu &= b\int_{\mathbb{S}^2}\omega^3 f_3\,\mathrm{d}\mu + a\int_{\mathbb{S}^2}\omega^3\,\mathrm{d}\mu.
\end{aligned}$$

By (102) it follows that

$$\mathbf{Z}(\Lambda_a(\Sigma)) = \Lambda_a(\mathbf{Z}(\Sigma)),$$

and it remains to show that $\mathbf{Z}(\Sigma)$ is timelike, future-pointing. By the above, we may assume without loss of generality that

$$\mathbf{Z} = \begin{pmatrix} \mathbf{Z}^0 \\ 0 \\ 0 \\ \mathbf{Z}^3 \end{pmatrix},$$

after a suitable rotation of the spatial coordinates in the ambient Minkowski spacetime. Thus

$$|\Sigma| \left( \mathbf{Z}^0 - |\mathbf{Z}^3| \right) \geq \int_{\mathbb{S}^2} \omega^3 (1 - |f_3|) > 0,$$

which implies that $\mathbf{Z}$ is timelike, future-pointing. $\square$

Using the one-to-one correspondence between STCMC surfaces and timelike, future-pointing vectors induced by the definition of $\mathbf{Z}$, we define the following a-priori class.

**Definition 8.6.** Let $\kappa > 0$. We say a spacelike cross section $\Sigma$ is *$\kappa$-bounded*, if

$$(1+\kappa)^{-1} \omega_{\mathbf{Z}} \leq \omega \leq (1+\kappa) \omega_{\mathbf{Z}},$$

where $\mathbf{Z} = \mathbf{Z}(\Sigma)$.

**Remark 8.7.** As we only look at smooth spacelike cross sections any spacelike cross section is of course $\kappa$-bounded for some appropriate $\kappa = \kappa(\omega) > 0$. As all the estimates derived below assume $\kappa$ to be fixed a-priori, we may rephrase them as depending an a suitable sup-bound on $\omega$ with constants explicitly depending on this sup-bound.

We close this subsection with some lemmas to be used later.

**Lemma 8.8.** *$\kappa$-boundedness is preserved under rescaling and Lorentz transformations $\Lambda \in SO^+(3,1)$.*

*Proof.* As $\mathbf{Z}(\Sigma_{\omega_{\mathbf{Z}}}) = \mathbf{Z}$ we can conclude that

$$\Lambda(\Sigma_{\omega_{\mathbf{Z}}}) = \Sigma_{\omega_{\Lambda(\mathbf{Z})}}$$

using Proposition 8.4 and the one-to-one correspondence between STCMC surfaces on the future-pointing standard lightcone and timelike, future-pointing vectors as in Remark 8.3 (ii). Furthermore, notice that $\mathbf{Z}(\Sigma_{\rho\omega}) = \rho \mathbf{Z}(\Sigma_\omega)$ and $\omega_{\rho z} = \rho \omega_z$ for any spacelike cross section $\Sigma_\omega$, constant $\rho > 0$ and timelike, future-pointing vector $z$. Under these considerations, it readily follows from the above considerations that $\kappa$-boundedness is preserved under both rescaling and Lorentz transformations. $\square$

**Lemma 8.9.** *Let $\Sigma$ be $\kappa$-bounded. Then up to isometries on $\mathbb{S}^2$ there exists a unique Lorentz-transformation $\Lambda \in SO^+(3,1)$ such that $\omega_\Lambda$ satisfies*

$$\int_{\mathbb{S}^2} f_i \omega_\Lambda^3 \, \mathrm{d}\mu = 0 \text{ for all } i = 1,2,3$$

*and*

$$(1+\kappa)^{-2} r \le \omega_\Lambda \le (1+\kappa)^2 r,$$

*where $r$ is the area radius of $\Sigma$.*

*Proof.* Integrating the inequality in Definition 8.6 immediately yields that

$$(1+\kappa)^{-1} r_{\mathbf{Z}} \le r \le (1+\kappa) r_{\mathbf{Z}}, \tag{109}$$

where $r_{\mathbf{Z}} := \sqrt{-\eta(\mathbf{Z}(\Sigma), \mathbf{Z}(\Sigma))}$. Let $\Lambda \in SO^+(3,1)$ be the Lorentz boost (unique up to rotations) such that

$$\Lambda(\mathbf{Z}(\Sigma)) = (r_{\mathbf{Z}}, 0, 0, 0)$$

By Proposition 8.4 it follows that

$$\int_{\mathbb{S}^2} f_i \omega_\Lambda^3 \, \mathrm{d}\mu = 0.$$

In particular, $\omega_{\Lambda(\mathbf{Z})} = r_{\mathbf{Z}}$. Hence

$$(1+\kappa)^{-1} r_{\mathbf{Z}} \le \omega_\Lambda \le (1+\kappa) r_{\mathbf{Z}},$$

as $\kappa$-boundedness is preserved by Lemma 8.8. The claim then follows by Equation (109). $\square$

**Remark 8.10.** In fact, it holds that

$$(1+\kappa)^{-1} r \le \omega_\Lambda \le (1+\kappa)^2 r$$

as $r_{\mathbf{Z}} := \sqrt{-\eta(\mathbf{Z}, \mathbf{Z})} \ge r$ with equality if and only if $\omega = \omega_{\mathbf{Z}}$. To see this, we can assume wlog that $\mathbf{Z} = \left(r_{\mathbf{Z}}, \vec{0}\right)$ using Proposition 8.4 as above. Thus

$$|\Sigma_{\omega_{\mathbf{Z}}}| = \frac{1}{|\Sigma|^2} \int_{\mathbb{S}^2} \left( \int \omega^3 \right)^2 = 4\pi \frac{\left(\int \omega^3\right)^2}{\left(\int \omega^2\right)^2} \ge |\Sigma| \, ,$$

which follows from applying the Hölder inequality twice.

## 8.2 A geometric estimate

We now prove the desired scaling invariant, geometric estimate (100). First, we establish the estimate using null mean curvature flow.

**Theorem 8.11.** *Let $\{\Sigma_t\}_t$ be a family of spacelike cross sections evolving under null mean curvature flow with $\mathcal{H}^2 \geq 0$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|\Sigma| \int \frac{1}{2}(\mathcal{H}^2)^2 - 2|\mathring{A}|^2\right) \geq 0.$$

*As a consequence, we have*

$$|\Sigma| \int \frac{1}{2}(\mathcal{H}^2)^2 - 2|\mathring{A}|^2 \leq 128\pi^2 = \frac{1}{2}\left(\int \mathcal{H}^2\right)^2,$$

*with equality if and only if $\{\Sigma_t\}$ is a family of shrinking STCMC surfaces.*

**Remark 8.12.**

(i) By the strong maximum principle we have $\mathcal{H}^2 > 0$ for any $t > 0$ if $\mathcal{H}^2 \geq 0$ for $\Sigma_0$, as we can rule out $\mathcal{H}^2 \equiv 0$ by Gauss–Bonnet. Thus, we may entirely rely on Theorem 7.12, obtained by studying null mean curvature flow, to prove the above estimate, and do not need to evoke the equivalence to $2d$-Ricci flow and employ the more general result, cf. Theorem 7.1. See Section 7 for details regarding these results.

(ii) The strategy presented here is motivated by a short, unpublished prove by Huisken of the inequality

$$\left|\left|h - \frac{\fint H}{2}\gamma\right|\right|_{L^2(\Sigma)} \leq 2||\mathring{h}||_{L^2(\Sigma)}$$

for strictly starshaped surfaces $\Sigma$ in $\mathbb{R}^3$ with $H > 0$ using *inverse mean curvature flow*, where 2 is indeed the optimal constant.

(ii) Along a null hypersurface, inverse mean curvature flow is defined as the projection of codimension-2 inverse mean curvature flow in an ambient spacetime $(\mathfrak{M}, \mathfrak{g})$ to the null hypersurface, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}x = -\frac{1}{2}\frac{\mathfrak{g}(\vec{\mathfrak{H}}, L)}{\mathcal{H}^2}\underline{L} = -\frac{1}{\underline{\theta}}\underline{L},$$

which is well-defined as long as $\underline{\theta} \neq 0$ along the null hypersurface. Recall that $\underline{\theta}$ is indeed a smooth function on the null hypersurface, cf. Subsection 4.7 Remark 4.17.

Hence, inverse mean curvature flow is given as an ordinary differential equation rather than a parabolic system along a null hypersurface. More explicitly, for any spacelike cross section $\Sigma_\omega$ along the standard lightcone in the $3+1$-Minkowski spacetime, the solution of inverse mean curvature flow is given by the smooth family $\{\Sigma_{\omega(t)}\}$ with

$$\omega(t) = \omega e^{\frac{t}{2}}.$$

In particular, any scaling invariant quantity remains unchanged under the flow.

Before proving Theorem 8.11, we establish the following auxiliary lemma following from the Hölder inequality:

**Lemma 8.13.** *Let $(X, \mu)$ be a finite measure space, and $f$ a bounded, non-negative, measurable function with $\int f \,\mathrm{d}\mu > 0$. Then*

$$\int f \int f^2 \leq \mu(X) \int f^3$$

*with equality if and only if $f$ is constant.*

*Proof.* Using the Hölder inequality, we see that

$$\int f \int f^2 \leq \frac{\mu(X)}{\int f} \left( \int f^2 \right)^2.$$

In particular, it suffices to prove that

$$\left( \int f^2 \right)^2 \leq \int f \int f^3.$$

However, this directly follows by applying the Hölder inequality again for $f$ using the finite measure space $(X, f\mu)$. □

*Proof of Theorem 8.11.* Recall that we have proven in Section 7 that under null mean curvature flow

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^2 &= \Delta\mathcal{H}^2 + \frac{1}{2}\left(\mathcal{H}^2\right)^2, \\
\frac{\mathrm{d}}{\mathrm{d}t}|\mathring{A}|^2 &= \Delta|\mathring{A}|^2 - 2\left(|\nabla A|^2 - \frac{1}{2}\left|\nabla\mathcal{H}^2\right|^2\right) \leq \Delta|\mathring{A}|^2 - \frac{1}{2}\left|\nabla\mathcal{H}^2\right|^2.
\end{aligned}$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{d}\mu_\gamma = -\frac{1}{2}\mathcal{H}^2\,\mathrm{d}\mu_\gamma$$

by Proposition 4.27. A direct computation yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\Sigma|\int\frac{1}{2}(\mathcal{H}^2)^2-2|\mathring{A}|^2\right)\geq&\int\mathcal{H}^2\int|\mathring{A}|^2-\frac{1}{4}\int\mathcal{H}^2\int(\mathcal{H}^2)^2\\&+|\Sigma|\int\frac{1}{2}\Delta(\mathcal{H}^2)^2-\left|\nabla\mathcal{H}^2\right|^2+\frac{1}{2}(\mathcal{H}^2)^3-2\Delta|\mathring{A}|^2+\left|\nabla\mathcal{H}^2\right|^2\\&-|\Sigma|\int\frac{1}{2}\mathcal{H}^2\left(\frac{1}{2}(\mathcal{H}^2)^3-2|\mathring{A}|^2\right)\\=&\int\mathcal{H}^2\int|\mathring{A}|^2+|\Sigma|\int\mathcal{H}^2|\mathring{A}|^2\\&+\frac{1}{2}\left(|\Sigma|\int(\mathcal{H}^2)^3-\int\mathcal{H}^2\int(\mathcal{H}^2)^2\right)\\\geq&0,\end{aligned}$$

where we used Lemma 8.13 in the last line, as all assumptions are satisfied by Gauss–Bonnet. Thus, we have proven the monotonicity.

Note that $|\Sigma|\int\left(\frac{1}{2}(\mathcal{H}^2)^2-2|\mathring{A}|^2\right)$ is scaling invariant, thus the monotonicity is also satisfied for any conformally equivalent flow, in particular for volume preserving Ricci flow, cf. Subsection 7.5. Using Theorem 7.12, the convergence to a round limit with $\mathcal{H}^2=\text{const.}$ and $|\mathring{A}|^2=0$, gives

$$|\Sigma|\int\frac{1}{2}(\mathcal{H}^2)^2-2|\mathring{A}|^2\leq128\pi^2=\frac{1}{2}\left(\int\mathcal{H}^2\right)^2,$$

where the second identity holds by Gauss–Bonnet. Lastly, equality is achieved if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|\Sigma|\int\frac{1}{2}(\mathcal{H}^2)^2-2|\mathring{A}|^2\right)=0,$$

so by Proposition 7.7 and Lemma 8.13 if $\{\Sigma_t\}$ is a family of shrinking STCMC surfaces. □

From this, we obtain the desired estimate:

**Theorem 8.14.** *Let $(\Sigma,\gamma)$ be a spacelike cross section of the standard lightcone in the $3+1$-dimensional Minkowski spacetime with $\mathcal{H}^2\geq0$. Then, we have that*

$$|\Sigma|\int_\Sigma\left|A-\frac{2}{r^2}\gamma\right|^2\leq3\,|\Sigma|\int_\Sigma|\mathring{A}|^2,$$

where $r=\sqrt{\frac{|\Sigma|}{4\pi}}$ denotes the area radius of $\Sigma$. Moreover, equality holds if and only if $\Sigma$ is a surface of constant spacetime mean curvature.

**Remark 8.15.**

(i) Using the Gauss equation (82), we see that in fact

$$\frac{2}{r^2} = \frac{1}{2} \fint \mathcal{H}^2$$

by Gauss–Bonnet. We have thus proven equivalently that

$$|\Sigma| \int_\Sigma \left| A - \frac{1}{2} \fint \mathcal{H}^2 \gamma \right|^2 \leq 3\, |\Sigma| \int_\Sigma |\mathring{A}|^2. \tag{110}$$

(ii) It turns out that the constant $C = 3$ is not optimal. See Theorem 8.17 below.

*Proof of Theorem 8.14.* We first rewrite the left-hand side as

$$\begin{aligned} |\Sigma| \int_\Sigma \left| A - \frac{2}{r^2}\gamma \right|^2 &= |\Sigma| \int_\Sigma \left| A - \frac{\fint \mathcal{H}^2}{2}\gamma \right|^2 \\ &= |\Sigma| \int_\Sigma \left( |A|^2 - \left( \fint \mathcal{H}^2 \right) \mathcal{H}^2 + \frac{1}{2} \left( \fint \mathcal{H}^2 \right)^2 \right) \\ &= |\Sigma| \int_\Sigma |A|^2 - \frac{1}{2} \left( \int_\Sigma \mathcal{H}^2 \right)^2 \\ &= |\Sigma| \int_\Sigma |A|^2 - 128\pi^2 \end{aligned}$$

As $\mathcal{H}^2 \geq 0$ on $\Sigma$, this is preserved under the solution of null mean curvature flow with initial data $\Sigma_0 = \Sigma$ by the parabolic maximum principle. In fact, the strong maximum principle yields that $\mathcal{H}^2 > 0$ for all positive times $t > 0$. Hence, the claim follows from Theorem 8.11. □

**Corollary 8.16.** If $\mathcal{H}^2 \geq 0$,

$$|\Sigma| \left\| \mathcal{H}^2 - \fint \mathcal{H}^2 \right\|^2_{L^2(\Sigma)} \leq 4\, |\Sigma|\, ||\mathring{A}||^2_{L^2(\Sigma)}.$$

We note that the estimates in Theorem 8.14 and Corollary 8.16 are equivalent. We now give a different proof of Corollary 8.16, which will in fact yield an improved estimate.

**Theorem 8.17.** *If* $\mathcal{H}^2 \geq 0$,

$$|\Sigma| \left\| \mathcal{H}^2 - \fint \mathcal{H}^2 \right\|^2_{L^2(\Sigma)} \leq 2\, |\Sigma|\, ||\mathring{A}||^2_{L^2(\Sigma)}.$$

**Remark 8.18.**

(i) The proof is motivated by the work of DeLellis–Topping [37] on an almost-Schur lemma. In fact, one may view the above as a generalization of these inequalities to $n = 2$ in the case of surfaces of genus 0.

(ii) Equivalently, it holds that

$$|\Sigma| \int_\Sigma \left| A - \frac{2}{r^2}\gamma \right|^2 \leq 2\,|\Sigma| \int_\Sigma |\mathring{A}|^2 \tag{111}$$

for spacelike cross sections with $\mathcal{H}^2 \geq 0$. We expect that one can adapt the arguments of DeLellis–Topping [37] to show that 2 is indeed the optimal constant and that the claim is in general false if the assumption $\mathcal{H}^2 \geq 0$ is dropped.

(iii) This approach directly extends to the standard $n+1$ Minkowski lightcone for arbitrary dimension $n \geq 3$, yielding a similar estimate under the assumption that $\mathrm{Ric} \geq 0$ for the spacelike cross section. In fact, the estimate directly follows from the estimate of DeLellis–Topping as

$$\mathring{A} = \frac{2(n-1)}{n-3} \mathring{\mathrm{Ric}}_\gamma, \;\; \mathcal{H}^2 = \frac{n-1}{n-2} R.$$

(iv) Similar to Theorem 8.11, it is easy to check that equality holds if and only if $\Sigma$ is an STCMC surface.

*Proof.* Recall that by Proposition 7.7, we have

$$\nabla_k A_{ij} = \nabla_i A_{kj}.$$

Taking a trace yields

$$\operatorname{div} \mathring{A} = \frac{1}{2}\, \mathrm{d}\mathcal{H}^2. \tag{112}$$

Now consider the elliptic equation

$$\Delta f = \mathcal{H}^2 - \fint \mathcal{H}^2. \tag{113}$$

Then (113) has a unique solution $f$ such that $\int f = 0$, cf. [69, Chapter 2.3]. Integration by parts using (112) and (113) gives

$$\begin{aligned}
\int \left(\mathcal{H}^2 - \fint \mathcal{H}^2\right)^2 &= -\int \left\langle \nabla \mathcal{H}^2, \nabla f \right\rangle \\
&= -2 \int \operatorname{div} \mathring{A}(\nabla f) \\
&= 2 \int \left\langle \mathring{A}, \mathrm{Hess} f \right\rangle \\
&= 2 \int \left\langle \mathring{A}, \mathring{\mathrm{Hess}} f \right\rangle \\
&\leq 2\, ||\mathring{A}||_{L^2} \cdot ||\mathring{\mathrm{Hess}} f||_{L^2}
\end{aligned}$$

Using the Bochner formula (94) and recalling the fact that $\mathrm{Ric}_\gamma = \frac{1}{4}\mathcal{H}^2\gamma$, we find that

$$||\mathring{\mathrm{Hess}} f||_{L^2}^2 = \int |\mathrm{Hess} f|^2 - \frac{1}{2}(\Delta f)^2 = \int \frac{1}{2}(\Delta f)^2 - \frac{1}{4}\mathcal{H}^2 \, |\nabla f|^2 \leq \frac{1}{2} \int \left(\mathcal{H}^2 - \fint \mathcal{H}^2\right)^2 .$$

Combining both inequalities implies the claim. □

## 8.3 Elliptic estimates under a balancing condition

We dedicate most of this subsection to prove the following proposition:

**Proposition 8.19.** *Let $(\Sigma, \gamma)$ be a conformally round surface with conformal factor $\omega$ such that $|\Sigma| = 4\pi$ and $C_0^{-1} \leq \omega \leq C_0$, $C_0 > 1$. We further assume that $\Sigma$ satisfies the balancing condition*

$$\int_{\mathbb{S}^2} f_i \omega^3 = 0 \text{ for all } i = 1, 2, 3, \tag{114}$$

*where $f_i$ denote the first spherical harmonics. Then there exists an $\varepsilon > 0$ only depending on $C_0$, such that if $||\mathcal{K} - 1||_{L^2(\mathbb{S}^2)} \leq \varepsilon$, we have*

$$||\omega - 1||_{W^{2,2}(\mathbb{S}^2)} \leq C \cdot ||\mathcal{K} - 1||_{L^2(\mathbb{S}^2)} ,$$

*where $C$ is a constant only depending on $C_0$.*

Estimates in similar spirit have been proven in [26, 55, 75]. In fact, we will closely follow the strategy outlined by Shi–Wang–Wu [75], replacing the sup-estimates with the application of the Hölder inequality. Compared to [75] we assume additional uniform sup-bounds on $\omega$, and we note the different balancing condition (114) with respect to the usual balancing condition found in the literature, see e.g. [26, 27, 55, 65, 75], which here is adapted to the definition of the associated 4-vector, cf. Subsection 8.1.

We later want to apply Proposition 8.19 for spacelike cross sections in the following way: Observe that for the rescaled conformal factor $\widetilde{\omega} := \frac{\omega}{r}$, where $r$ denotes again the area radius of $\Sigma$, we find $\left|\widetilde{\Sigma}\right| = 4\pi$ and

$$\left|\left|\widetilde{\mathcal{K}} - 1\right|\right|^2_{L^2(\mathbb{S}^2)} \leq \frac{1}{\inf \widetilde{\omega}^2} \frac{1}{64\pi} |\Sigma| \left|\left|\mathcal{H}^2 - \fint \mathcal{H}^2\right|\right|^2_{L^2(\Sigma)}, \tag{115}$$

using the Gauss equation (82). Moreover, the balancing condition (114) can always be achieved by Lemma 8.9.

We first establish several auxiliary lemmas for the smooth function $u := \ln(\omega)$ on $\mathbb{S}^2$.

**Lemma 8.20.** *Under the assumptions of Proposition 8.19 and for any $\eta > 0$, there exists a $\delta > 0$ such that if*

$$||\mathcal{K} - 1||_{L^2(\mathbb{S}^2)} \leq \delta,$$

*then $|u|_\infty \leq \eta$.*

*Proof.* Assume that this is false. Then there exists $\eta_0 > 0$ and a sequence $(u_k)_{k\in\mathbb{N}}$ of smooth functions corresponding to conformally round surfaces $\Sigma_k$ with $|\Sigma_k| = 4\pi$, $w_k = e^{u_k}$ satisfies (114), and $0 < \eta_0 \leq |u_k|_\infty \leq C_1$, $C_1 := \ln(C_0)$ for all $k$ with

$$||\mathcal{K}_k - 1||_{L^2(\mathbb{S}^2)} \to 0$$

as $k \to \infty$. Note that with respect to $u_k$ the Gauss curvature $\mathcal{K}_k$ satisfies

$$\Delta_{\mathbb{S}^2} u_k = 1 - e^{2u_k}\mathcal{K}_k,$$

and in particular, by the above properties on the sequence,

$$||\Delta_{\mathbb{S}^2} u_k||_{L^2(\mathbb{S}^2)} \leq C$$

for some constant independent of $k$. By the uniform estimate on $|u_k|_\infty$, standard elliptic estimates yield that $||u_k||_{W^{2,2}(\mathbb{S}^2)}$ is bounded independent of $k$, cf. [8, Appendix H]. As $W^{2,2}$ embeds compactly into $W^{1,2}$ and $C^{0,\alpha}$ for some $\alpha \in (0,1)$ on $\mathbb{S}^2$, cf. [8, Appendix C], there exists a subsequence $(u_k)$ such that $u_k \to u_\infty \in W^{1,2} \cap C^0$ (after possibly taking successive subsequences) with convergence both in $C^0$ and $W^{1,2}$. In particular, $u_\infty$ satisfies $\eta_0 \leq |u_\infty|_\infty \leq C_1$,

$$\int_{\mathbb{S}^2} f_i e^{3u_\infty} = 0,$$

and weakly solves the equation

$$\Delta_{\mathbb{S}^2} u_\infty = 1 - e^{2u_\infty}.$$

By appealing to standard arguments in regularity theory, we see that $u_\infty$ is indeed smooth and solves the above equation in the strong sense. In particular $(\mathbb{S}^2, e^{2u_\infty})$ is a smooth conformally round surface with area $4\pi$ and constant Gauss curvature everywhere equal to 1. However, this gives an immediate contradiction as $u_\infty = 0$ is the unique solution under the balancing condition (114). $\square$

**Lemma 8.21.** *For any bounded, measurable function $u$ on $\mathbb{S}^2$ with $|u| \leq \ln(C_0)$, $C_0 > 1$ and $k \in \mathbb{N}$, we have*

$$\left|\left|1 + ku - e^{ku}\right|\right|_{L^2(\mathbb{S}^2)} \leq C\,|u|_\infty\,||u||_{L^2(\mathbb{S}^2)}$$

*for some constant $C$ only depending on $C_0$ and $k$.*

*Proof.* By the mean value theorem, we know that for each $x \in \mathbb{R} \setminus \{0\}$, there exists $x' \in \mathbb{R}$ with $|x'| \in (0, |x|)$ such that

$$e^x - 1 = x \cdot e^{x'}.$$

In particular, for any $p \in \mathbb{S}^2$ with $u(p) \neq 0$, we find that

$$\begin{aligned}
\left|1 + ku(p) - e^{ku(p)}\right| &= \left|ku(p) - ku(p)e^{k(u(p))'}\right| \\
&= k\,|u(p)|\left|1 - e^{(ku(p))'}\right| \\
&= k\,|u(p)|\,|(ku(p))'|\,e^{(ku(p))''} \leq C\,|u(p)|^2,
\end{aligned}$$

and the inequality is trivially satsified whenever $u(p) = 0$. Hence

$$\left|\left|1 + ku - e^{ku}\right|\right|_{L^2(\mathbb{S}^2)} \leq C\,||u||^2_{L^4(\mathbb{S}^2)} \leq C\,|u|_\infty\,||u||_{L^2(\mathbb{S}^2)}$$

$\square$

We now prove Proposition 8.19:

*Proof of Proposition 8.19.* We decompose $u$ into $u = u_0 + u_1 + u_2$ with respect to the $L^2$-orthonormal basis of spherical harmonics on $\mathbb{S}^2$, cf. Subsection 4.10, where

$$u_0 := \frac{1}{4\pi}\int_{\mathbb{S}^2} u,$$

$u_1 := a^1 f_1 + a^2 f_2 + a^3 f_3$ with

$$a^i := \frac{3}{4\pi}\int_{\mathbb{S}^2} u \cdot f_i.$$

Note that $u_2 := u - u_0 - u_1$ is perpendicular to $u_0, u_1$ in $L^2$. We first aim to bound the $L^2$-norm of $u$. As

$$||u||_{L^2(\mathbb{S}^2)} \leq \left(|u_0| + \sum_{i=1}^{3}|a^i| + ||u_2||_{L^2(\mathbb{S}^2)}\right),$$

it suffices to bound the individual terms. In the following, $C$ will always denote a constant that may change from line to line only depending on $C_0$, unless otherwise stated. Direct estimation gives

$$\begin{aligned}
|u_0| = \frac{1}{8\pi}\left|\int 2u\right| &= \frac{1}{8\pi}\left|\int 2u + 1 - \mathcal{K}e^{2u}\right| \\
&= \frac{1}{8\pi}\left|\int (2u + 1 - e^{2u}) + (1-\mathcal{K})e^{2u}\right| \\
&\leq C\left(\left|\left|1 + 2u - e^{2u}\right|\right|_{L^2(\mathbb{S}^2)} + ||1-\mathcal{K}||_{L^2(\mathbb{S}^2)}\right),
\end{aligned}$$

and

$$4\pi|a^i| = \left|\int 3uf_i\right| = \left|\int (3u + 1 - e^{3u})f_i\right| \leq C\left|\left|1 + 3u - e^{3u}\right|\right|_{L^2(\mathbb{S}^2)},$$

where we used the Gauss–Bonnet theorem and the balancing condition (114) in the first and second computation, respectively, and the Hölder inequality in both cases in the last line. It remains to bound $||u_2||_{L^2(\mathbb{S}^2)}$.
We note that by our choice of decomposition, we find that for the operator $L$ defined as

$$L(v) := \Delta v + 2v$$

$u_2$ satisfies

$$L(u_2) = L(u) - 2u_0 = (1 + 2u - e^{2u}) + (1-\mathcal{K})e^{2u} - 2u_0,$$

where we used the explicit formula for the Gauss curvature. Multiplying the equation by $u_2$, integration by parts yields

$$\int |\nabla u_2|^2 - 2u_2^2 = -\int u_2 L(u_2) \leq C\,||u_2||_{L^2(\mathbb{S}^2)}\left(\left|\left|1 + 2u - e^{2u}\right|\right|_{L^2(\mathbb{S}^2)} + ||1-\mathcal{K}||_{L^2(\mathbb{S}^2)}\right),$$

where we used the Hölder inequality and the already established bound on $u_0$. Moreover, we also know that by our choice of decomposition

$$u_2 = \sum_{l=2}^{\infty} \sum_{k=-l}^{l} a_{l,k} Y_l^k,$$

where $Y_l^k$ denote spherical harmonics with eigenvalues $\lambda_l = -l(l+1) \leq -6$. Hence, partial integration yields

$$\int |\nabla u_2|^2 = -\int \sum_{l=2}^{\infty} \sum_{k=-l}^{l} \lambda_l a_{l,k}^2 (Y_l^k)^2 \geq 6 \, ||u_2||^2_{L^2(\mathbb{S}^2)} \, ,$$

and we conclude that

$$4 \, ||u_2||^2_{L^2(\mathbb{S}^2)} \leq C \, ||u_2||_{L^2(\mathbb{S}^2)} \left( \left|\left| 1 + 2u - e^{2u} \right|\right|_{L^2(\mathbb{S}^2)} + ||1 - \mathcal{K}||_{L^2(\mathbb{S}^2)} \right).$$

Now, using the Peter–Paul inequality, we may conclude that

$$||u_2||_{L(\mathbb{S}^2)} \leq C \left( \left|\left| 1 + 2u - e^{2u} \right|\right|_{L^2(\mathbb{S}^2)} + ||1 - \mathcal{K}||_{L^2(\mathbb{S}^2)} \right)$$

after taking a square root. Combining the above estimates, we find that

$$\begin{aligned} ||u||_{L^2(\mathbb{S}^2)} &\leq C \left( ||1 - \mathcal{K}||_{L^2(\mathbb{S}^2)} + \left|\left| 1 + 2u - e^{2u} \right|\right|_{L^2(\mathbb{S}^2)} + \left|\left| 1 + 3u - e^{3u} \right|\right|_{L^2(\mathbb{S}^2)} \right) \\ &\leq C \left( ||1 - \mathcal{K}||_{L^2(\mathbb{S}^2)} + |u|_\infty \, ||u||_{L^2(\mathbb{S}^2)} \right), \end{aligned}$$

using Lemma 8.21. Let $\eta > 0$ such that $\eta C \leq \frac{1}{2}$. Then by Lemma 8.20 there exists $\delta(\eta)$ (only depending on $C_0$) such that

$$||u||_{L(\mathbb{S}^2)} \leq 2C \, ||1 - \mathcal{K}||_{L^2(\mathbb{S}^2)}$$

if $||1 - \mathcal{K}||_{L^2(\mathbb{S}^2)} \leq \delta(\eta)$. As

$$||\Delta u||_{L^2(\mathbb{S}^2)} = \left|\left| 1 - \mathcal{K} e^{2u} \right|\right|_{L^2(\mathbb{S}^2)} \leq C \left( 2 \, ||u||_{L^2(\mathbb{S}^2)} + ||1 - \mathcal{K}||_{L^2(\mathbb{S}^2)} \right),$$

it follows by standard elliptic estimates, cf. [8, Appendix H], that

$$||u||_{W^{2,2}(\mathbb{S}^2)} \leq C \, ||1 - \mathcal{K}||_{L^2(\mathbb{S}^2)}$$

if $||1 - \mathcal{K}||_{L^2(\mathbb{S}^2)} \leq \delta(\eta)$. The claim then follows with $\varepsilon := \delta(\eta)$ and noting that $||\omega - 1||_{W^{2,2}(\mathbb{S}^2)} \leq C \, ||u||_{W^{2,2}(\mathbb{S}^2)}$. □

Before stating the main result in the next subsection, we give a partial result formulated as an intrinsic result on conformally round surfaces.

**Corollary 8.22.** Let $(\Sigma, \gamma)$ be a conformally round surface with non-negative scalar curvature $\mathrm{R} \geq 0$ and area radius $r$. Assume that $C_0^{-1} r \leq \omega \leq C_0 r$ for some constant $C_0$, and that the balancing condition (114) holds. Then there exits $\varepsilon > 0$ only depending on $C_0$ such that if

$$|\Sigma| \cdot ||\overset{\circ}{\mathrm{Hess}}_{\mathbb{S}^2} v||^2_{L^2(\mathbb{S}^2)} \leq \varepsilon$$

for $v := \frac{1}{\omega}$, then

$$||\omega - r||_{W^{2,2}(\mathbb{S}^2)} \leq C\,|\Sigma| \cdot ||\overset{\circ}{\mathrm{Hess}}_{\mathbb{S}^2} v||_{L^2(\mathbb{S}^2)}$$

for a constant $C$ only depending on $C_0$.

*Proof.* By the Gauss equation, $\Sigma$ has $\mathcal{H}^2 \geq 0$ as a spacelike cross section of the standard lightcone. Note further that

$$||\mathring{A}||_{L^2(\Sigma)} = 4||\overset{\circ}{\mathrm{Hess}}_{\mathbb{S}^2} \frac{1}{\omega}||^2_{L^2(\mathbb{S}^2)},$$

cf. Remark 7.6. Then the result immediately follows from Theorem 8.17 and Proposition 8.19 for the rescaled surface with conformal factor $\frac{\omega}{r}$. Multiplying the estimate by $r$ then yields the result. $\square$

## 8.4 A De Lellis–Müller type estimate on the Minkowski lightcone

We now state the main result of this section.

**Theorem 8.23.** *Let $(\Sigma, \gamma)$ be a spacelike cross section of the future-pointing standard lightcone with $\mathcal{H}^2 \geq 0$, area radius $r$, and associated 4-vector $\boldsymbol{Z} = \boldsymbol{Z}(\Sigma)$. Assume further that $\Sigma$ is $\kappa$-bounded for some fixed $\kappa > 0$. Then there exists $\varepsilon > 0$ only depending on $\kappa$ such that if*

$$|\Sigma| \cdot ||\mathring{A}||^2_{L^2(\Sigma)} \leq \varepsilon$$

*then*

$$||\omega - \omega_{\boldsymbol{Z}}||_{W^{2,2}(\mathbb{S}^2)} \leq C(\kappa, \boldsymbol{Z}) \cdot |\Sigma|\,||\mathring{A}||_{L^2(\Sigma)},$$

*where $C(\kappa, \Lambda)$ is a constant only depending on $\kappa$ and $\boldsymbol{Z}$.*

*Proof.* By Lemma 8.9 there exists $\Lambda \in SO^+(3,1)$ (uniquely determined up to isometries on $\mathbb{S}^2$) such that $\omega_{\Lambda^{-1}(\mathbf{Z})} = r_{\mathbf{Z}}$, as this is equivalent to

$$\int_{\mathbb{S}^2} f_i \omega^3 = 0 \text{ for all } i = 1, 2, 3.$$

Note that the components of $\Lambda$ are uniquely determined by $\mathbf{Z}$ (up to a choice of rotation). By Remark 8.5, Lemma 8.9, Theorem 8.17, and Equation (115), we find for $\widetilde{\omega}_{\Lambda^{-1}} := r^{-1}\omega_{\Lambda^{-1}}$ that

$$\begin{aligned}
||\widetilde{\mathcal{K}}_{\Lambda^{-1}} - 1||^2_{L(\mathbb{S}^2)} &\leq C(\kappa) \left|\Lambda^{-1}(\Sigma)\right| \left|\left|\mathcal{H}^2_{\Lambda^{-1}} - \fint \mathcal{H}^2_{\Lambda^{-1}}\right|\right|^2_{L^2(\Lambda^{-1}(\Sigma))} \\
&= C(\kappa)\,|\Sigma| \left|\left|\mathcal{H}^2 - \fint \mathcal{H}^2\right|\right|^2_{L^2(\Sigma)} \\
&\leq 4C(\kappa)\,|\Sigma|\,||\mathring{A}||^2_{L^2(\Sigma)}.
\end{aligned}$$

Hence, for $\varepsilon$ sufficiently small, we may use Proposition 8.19 as $\widetilde{\omega}_{\Lambda^{-1}}$ has area $4\pi$ and satisfies the balancing condition (114). Multiplying by $r$, we find

$$||\omega_{\Lambda^{-1}} - r||_{W^{2,2}(\mathbb{S}^2)} \leq C(\kappa)\,|\Sigma|\,||\mathring{A}||_{L^2(\Sigma)}$$

Now notice that by the definition of $\omega_{\mathbf{Z}}$, $r_{\mathbf{Z}}$ and Lemma 8.1, we find that

$$\omega - \frac{r}{r_{\mathbf{Z}}}\omega_{\mathbf{Z}} = \frac{\omega_{\Lambda^{-1}} \circ \Phi_\Lambda - r}{\sqrt{1 + |\vec{a}^2|} - \vec{a}_i f_i},$$

where $\Phi_\Lambda$ and $\vec{a}$ are uniquely determined by $\mathbf{Z}$ up to a rotation by the above consideration. Hence

$$||\omega - \frac{r}{r_{\mathbf{Z}}}\omega_{\mathbf{Z}}||_{W^{2,2}(\mathbb{S}^2)} \leq C(\mathbf{Z})||\omega_{\Lambda^{-1}} - r||_{W^{2,2}(\mathbb{S}^2)} \leq C(\kappa, \mathbf{Z})\,|\Sigma|\,||\mathring{A}||_{L^2(\Sigma)},$$

where the constant $C(\mathbf{Z})$ does not depend on the choice of rotation as rotations act as isometries on $\mathbb{S}^2$. On the other hand, note that $W^{2,2}$ embedds compactly into $C^0$ on $\mathbb{S}^2$. Thus, we find that $|\omega_{\Lambda^{-1}} - r|_\infty \leq C(\kappa)\,|\Sigma|\,||\mathring{A}||_{L^2(\Sigma)}$. Notice further that

$$r_{\mathbf{Z}} = r_{\mathbf{Z}(\Lambda^{-1}(\Sigma))} = \frac{1}{|\Lambda^{-1}(\Sigma)|}\int_{\mathbb{S}^2} \omega^3_{\Lambda^{-1}} \leq r + |\omega_{\Lambda^{-1}} - r|_\infty\,.$$

By Remark 8.10, we conclude that

$$|r_{\mathbf{Z}} - r| \leq C(\kappa)\,|\Sigma|\,||\mathring{A}||_{L^2(\Sigma)}.$$

Hence,

$$||\frac{r}{r_{\mathbf{Z}}}\omega_{\mathbf{Z}} - \omega_{\mathbf{Z}}||_{W^{2,2}(\mathbb{S}^2)} = |r - r_{\mathbf{Z}}| \cdot \left|\left|\frac{\omega_{\mathbf{Z}}}{r_{\mathbf{Z}}}\right|\right|_{W^{2,2}(\mathbb{S}^2)} \leq C(\mathbf{Z})\,|r_{\mathbf{Z}} - r| \leq C(\kappa, \mathbf{Z})\,|\Sigma|\,||\mathring{A}||_{L^2(\Sigma)},$$

where the constant $C(\mathbf{Z})$ again does not depend on the choice of rotation. The claim then follows directly by the triangle inequality. □

Note that as an intermediate step, we have proven that

$$\left|\left|\frac{\omega}{r} - \frac{\omega_{\mathbf{Z}}}{r_{\mathbf{Z}}}\right|\right|^2_{W^{2,2}(\mathbb{S}^2)} \leq C(\kappa, \mathbf{Z})\, |\Sigma|\, ||\mathring{A}||^2_{L^2(\Sigma)}, \tag{116}$$

which allows for a direct comparison of the spacelike cross section $\Sigma$ and the STCMC surface of reference both rescaled to have area radius 1. Using this, we establish a further conclusion from Theorem 8.23 as a last result of this section.

**Theorem 8.24.** *Let $(\Sigma_k)_{k\in\mathbb{N}}$ be a sequence of spacelike cross sections with $\mathcal{H}^2_k \geq 0$ with conformal factor $\omega_k$, area radius $r_k$ and associated 4-vector $\boldsymbol{Z}_k$. Assume further that $\Sigma_k$ is $\kappa$-bounded for all $k$ with $\kappa$ independent of $k$, and there exists $\delta > 0$ such that*

$$\left(\boldsymbol{Z}^0_k\right)^2 \geq (1+\delta)\left(\left(\boldsymbol{Z}^1_k\right)^2 + \left(\boldsymbol{Z}^2_k\right)^2 + \left(\boldsymbol{Z}^3_k\right)^2\right), \tag{117}$$

*and*

$$|\Sigma_k| \cdot ||\mathring{A}_k||^2_{L^2(\Sigma_k)} \to 0 \tag{118}$$

*as $k \to \infty$. Then there exists a subsequence $(k_l)_{l\in\mathbb{N}}$ and a surface of constant spacetime mean curvature $\widetilde{\Sigma}$ with conformal factor $\widetilde{\omega}$ and area radius 1 such that*

$$\frac{\omega_{k_l}}{r_{k_l}} \to \widetilde{\omega} \text{ in } W^{2,2}(\mathbb{S}^2),$$

*and*

$$\frac{\boldsymbol{Z}_{k_l}}{r_{\boldsymbol{Z}_{k_l}}} \to \boldsymbol{Z}(\widetilde{\Sigma}),$$

*where $r_{\boldsymbol{Z}_{k_l}} = \sqrt{-\eta(\boldsymbol{Z}_{k_l}, \boldsymbol{Z}_{k_l})}$, and $\eta(\boldsymbol{Z}(\widetilde{\Sigma}), \boldsymbol{Z}(\widetilde{\Sigma})) = -1$.*

**Remark 8.25.** As $W^{2,2}$ embeds compactly into $C^0$ on $\mathbb{S}^2$, cf. [8, Appendix C], the conformal factors converge in $C^0$. Note that for any uniformly converging sequence of spacelike cross sections, we have convergence of the associated 4-vectors to the associated 4-vector of the limiting surface, which is easy to verify from the definition of $\mathbf{Z}$, cf. Subsection 8.1. In particular, the sequence is $\kappa$-bounded with $\kappa$ independent of $k$ and satisfies (117) for $k$ sufficiently large. Hence, these are necessary conditions for the conclusions of Theorem 8.24 to hold.

For example, the sequence of STCMC surfaces corresponding to the conformal factors

$$\omega_k = \frac{1}{\sqrt{1+k^2} - k\cos\theta}$$

has area radius 1, is $\kappa$-bounded for any $\kappa > 0$ independent of $k$, and $\mathring{A}_k = 0$ for all $k$. However, the sequence $\omega_k$ does not converge to a limit, even in $C^0$, due to the non-compactness of the restricted Lorentz group $SO^+(3,1)$. This issue is precisely avoided by (117), as the renormalized associated 4-vectors will be restricted to a compact subset of the hyperboloid $\{p \in \mathbb{R}^{3,1} \colon \eta(p,p) = -1,\, p^0 > 0\}$.

*Proof of Theoem 8.24.* We define the sequence

$$z_k := \frac{\mathbf{Z}_{k_l}}{r_{\mathbf{Z}_{k_l}}}.$$

In particular $\eta(z_k, z_k) = -1$ and as (117) is preserved under rescaling, we see that

$$(1+\delta)\left((z_k^1)^2 + (z_k^2)^2 + (z_k^3)^2\right) \leq (z_k^0)^2 = (z_k^1)^2 + (z_k^2)^2 + (z_k^3)^2 + 1.$$

Hence

$$z_k \in C_\delta := \left\{p \in \mathbb{R}^{3,1} \colon (p^1, p^2, p^3) \in \overline{B}_{\frac{1}{\delta}}(0),\, p^0 = \sqrt{1 + (p^1)^2 + (p^2)^2 + (p^3)^2}\right\}$$

for all $k$, where $\overline{B}_{\frac{1}{\delta}}(0)$ denotes the closed ball of radius $\frac{1}{\delta}$ in $\mathbb{R}^3$ centered around the origin. As the set is compact, there exists a subsequence $(k_l)_{l\in\mathbb{N}}$ such that

$$z_{k_l} \to z \in C_\delta.$$

In particular, $z$ is timelike, future-pointing with $\eta(z,z) = -1$. Let $\widetilde{\Sigma}$ be the STCMC surface with conformal factor $\widetilde{\omega} := \omega_z$ as defined in Remark 8.3 (ii). Then $\widetilde{\Sigma}$ has area radius $\widetilde{r} = r_z := \sqrt{-\eta(z,z)} = 1$. Moreover, as $z_{k_l} \to z$, we see by the explicit definition (103) that

$$\omega_{z_{k_l}} \to \widetilde{\omega} \tag{119}$$

in $C^2$ (in fact $C^k$ for any fixed $k$). It only remains to show the $W^{2,2}$ convergence.

To this end, recall that $\omega_{z_{k_l}} = \frac{1}{r_{\mathbf{Z}_{k_l}}} \omega_{\mathbf{Z}_{k_l}}$. It follows that

$$\left|\left|\frac{\omega_{k_l}}{r_{k_l}} - \omega_{z_{k_l}}\right|\right|^2_{W^{2,2}(\mathbb{S}^2)} \leq C(\kappa, \mathbf{Z}_{k_l})\, |\Sigma_{k_l}|\, ||\mathring{A}_{k_l}||^2_{L^2(\Sigma)}, \tag{120}$$

from (116), where the constant $C(\kappa, \mathbf{Z}_{k_l})$ does not depend on $r_{\mathbf{Z}}$. Hence $C(\kappa, \mathbf{Z}_{k_l}) = C(\kappa, z_{k_l})$, and by following the arguments in the proof of Theorem 8.23 we note that the constant can be chosen such that it continuously depends on $z_{k_l}$. Evoking once again that $z_{k_l} \to z$, there exists a suitable constant $C(\kappa, z)$ only depending on $\kappa$ and $z$ such that

$$C(\kappa, z_{k_l}) \leq C(\kappa, z)$$

fo all $l$. The claim then follows from (119) and (120) using assumption (118) and the triangle inequality. $\square$

# 9 References


[1] Ben Andrews and Paul Bryan. Curvature bounds by isoperimetric comparison for normalized Ricci flow on the two-sphere. *Calculus of Variations and Partial Differential Equations*, 39:419–428, 2009.

[2] Ben Andrews, Bennet Chow, Christine Guenther, and Mat Langford. *Extrinsic geometric flows*. American Mathematical Society, Providence, Rhode Island, 2020.

[3] Richard Arnowitt, Stanley Deser, and Charles Misner. The Dynamics of General Relativity. *arXiv:gr-qc/0405109*, 2004.

[4] Robert Bartnik. Existence of maximal surfaces in asymptotically flat spacetimes. *Comm. Math. Phys.*, 94:155–175, 1984.

[5] Robert Bartnik. The Mass of an Asymptotically Flat Manifold. *Communications on Pure and Applied Mathematics*, 39:661–693, 1986.

[6] Robert Bartnik and Leon Simon. Spacelike hypersurfaces with prescribed boundary values and mean curvature. *Comm. Math. Phys.*, 87(1):131–152, 1982–1983.

[7] Janet Bartz, Michael Struwe, and Rugang Ye. A new approach to the Ricci flow on $\mathbb{S}^2$. *Annali Della Scuola Normale Superiore Di Pisa-classe Di Scienze*, 21:475–482, 1994.

[8] Arthur Besse. *Einstein Manifolds*. Springer, Berlin–Heidelberg–New York, 2008.

[9] Danny Birmingham. Topological black holes in anti-de Sitter space. *Class. Quantum Grav.*, 16(4):1197, 1999.

[10] Hubert Bray and Marcus A. Khuri. PDEs which imply the Penrose Conjecture. *Asian J. Math.*, 15(4):557–610, 2011.

[11] Hubert Bray and Henri Roesch. Proof of a null geometry Penrose conjecture. *Notice of the AMS*, 65(2):156–162, 2018.

[12] Simon Brendle. Convergence of the Yamabe flow for arbitrary initial energy. *Journal of Differential Geometry*, 69(2):217–278, 2005.

[13] Simon Brendle. A short proof of the Yamabe flow on $S^n$. *Pure and Applied Mathematics Quarterly*, 3(2):499–512, 2007.

[14] Simon Brendle. Convergence of the Yamabe flow in dimension 6 and higher. *Inventiones mathematicae*, 170:541–576, 2007.

[15] Simon Brendle. Constant mean curvature surfaces in warped product manifolds. *Publ. math. IHES*, 117:247–269, 2013.

[16] Dieter R. Brill and Sean A. Hayward. Global structure of a black hole cosmos and its extremes. *Class. Quantum Grav.*, 11:359–370, 1993.

[17] Armando Cabrera Pacheco and Markus Wolff. Families of non time-symmetric initial data sets and Penrose-like energy inequalities. *arXiv:2310.13547*, 2023.

[18] Alberto Carrasco and Marc Mars. A counterexample to a recent version of the Penrose conjecture. *Class. Quantum Grav.*, 27, 2010.

[19] Carla Cederbaum. Mathematical Relativity. lecture notes, current version, Summer Semester 2022 at University of Tübingen, 2022.

[20] Carla Cederbaum, Julien Cortier, and Anna Sakovich. On the Center of Mass of Asymptotically Hyperbolic Initial Data Sets. *Ann. Herni Poincaré*, 17:1505–1528, 2016.

[21] Carla Cederbaum and Gregory J. Galloway. Photon surfaces with equipotential time slices. *J. Math. Phys.*, 62, 2021.

[22] Carla Cederbaum, Sophia Jahns, and Olivia Vičánek Martínez. On equipotential photon surfaces in electrostatic spacetimes of arbitrary dimension. *unpublished*, 2021.

[23] Carla Cederbaum and Christopher Nerz. Explicit Riemannian Manifolds with Unexpectedly Behaving Center of Mass. *Ann. Herni Poincaré*, 16:1609–1631, 2015.

[24] Carla Cederbaum and Anna Sakovich. On the center of mass and foliations by constant spacetime mean curvature surfaces for isolated systems in General Relativity. *Calculus of Variations and Partial Differential Equations*, 60(214), 2021.

[25] Carla Cederbaum and Markus Wolff. Some new perspectives on the Kruskal–Szekeres extension. *arXiv:2310.06946*, 2023.

[26] Sun-yung Alice Chang. The Moser-Trudinger inequality and applications to some problems in conformal geometry. *IAS/ Park City Mathematical series*, pages 65–125, 1996.

[27] Sun-yung Alice Chang and Paul C. Yang. Prescribing Gaussian curvature on $S^2$. *Acta Math.*, 159:215–259, 1987.

[28] Xiuxiong Chen, Peng Lu, and Gang Tian. A note on uniformization of Riemann surfaces by Ricci flow. *Proceedings of the American Mathematical Society*, 134(11):3391–3393, 2006.

[29] Yvonne Choquet-Bruhat and Robert Geroch. Global aspects of the Cauchy problem in general relativity. *Communications in Mathematical Physics*, 14:329–335, 1969.

[30] Bennet Chow. The Ricci flow on the 2-sphere. *Journal of Differential Geometry*, 33(2):325–334, 1991.

[31] Bennet Chow. The Yamabe flow on locally conformally flat metrics with positive Ricci curvature. *Communications on Pure and Applied Mathematics*, 45:1003–1014, 1992.

[32] Bennet Chow, Peng Lu, and Lei Ni. *Hamilton's Ricci flow.* American Mathematical Society, Providence, Rhode Island, 2010.

[33] Piotr T. Chruściel. Boundary conditions at spatial infinity from a Hamiltonian point of view. *Topological properties and global structure of space-time (Erice, 1985).* volume 138 of *Nato Adv. Sci. Inst. Ser. B Phys.*, pages 49–59. Plenum, New York, 1986.

[34] Piotr T. Chruściel and Paul Tod. An angular momentum bound at null infinity. *Adv. Theor. Math. Phys.*, 13:1317–1334, 2009.

[35] Justin Corvino. Scalar Curvature Deformation and a Glueing Construction for the Einstein Constraint Equations. *Comm. Math. Phys.*, 214:137–189, 2000.

[36] Camillo De Lellis and Stefan Müller. Optimal rigidity estimates for nearly umbilical surfaces. *J. Diff. Geom.*, pages 75–110, 2005.

[37] Camillo De Lellis and Peter Topping. Almost-Schur lemma. *Calculus of Variations and Partial Differential Equations*, 43:347–354, 2012.

[38] Costas Efthimiou and Christopher Frye. *Spherical harmonics in p dimensions.* World Scientific, Singapore, 2014.

[39] Michael Eichmair. Existence, regularity, and properties of generalized apparent horizons. *Comm. Math. Phys.*, 294:745–760, 2010.

[40] Charles Fefferman and Charles Robin Graham. Conformal invariants. In *Élie Cartan et les mathématiques d'aujourd'hui - Lyon, 25-29 juin 1984*, number S131 in Astérisque. Société mathématique de France, 1985.

[41] Jörg Frauendiener. On the Penrose Inequality. *Phys. Rev. Lett.*, 87:101101, 2001.

[42] Claus Gerhardt. H-surfaces in Lorentzian manifolds. *Comm. Math. Phys.*, 89:523–553, 1983.

[43] Robert Geroch. Energy extraction. *Ann. New York Acac. Sci.*, 224:108–117, 1973.

[44] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer, Berlin–Heidelberg–New York, 1998.

[45] Enrico Giusti. *Minimal Surfaces and Functions of Bounded Variation.* Birkhäuser Inc., Basel, Boston, Stuttgart, 1984.

[46] Richard S. Hamilton. Three-Manifolds with positive Ricci-curvature. *J. Diff. Geom.*, 17:255–306, 1982.

[47] Richard S. Hamilton. The Ricci flow on surfaces. *Math and General Relativity, Contemporary Mathematics*, 71:237–262, 1988.

[48] Richard S. Hamilton. Lectures on geometric flows. unpublished, 1989.

[49] Mirjam Heidusch. Zur Regularität des inversen mittleren Krümmungsflusses. *(Doctoral thesis)*, 2001.

[50] Gerhard Huisken and Tom Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *J. Diff. Geom.*, 59(3):353–437, 2001.

[51] Gerhard Huisken and Tom Ilmanen. Higher regularity for the inverse mean curvature flow. *J. Diff. Geom.*, 80(3):433–451, 2008.

[52] Gerhard Huisken and Alexander Polden. Geometric evolution equations for hypersurfaces. *Calculus of Variations and Geometric Evolution Problems, CIME Lectures at Cetraro of 1996 (S. Hildebrandt and M. Struwe, eds.)*, pages 45–84, 1999.

[53] Gerhard Huisken and Markus Wolff. On the evolution of hypersurfaces along their inverse spacetime mean curvature. *arXiv:2208.05709*, 2022.

[54] Gerhard Huisken and Shing-Tung Yau. Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant man curvature. *Invent. Math.*, 124:281–311, 1996.

[55] Sergiu Klainerman and Jeremie Szeftel. Effective results on uniformization and intrinsic GCM spheres in perturbations of Kerr. *arXiv:1912.12195*, 2019.

[56] Friedrich Kottler. Über die physikalischen Grundlagen der Einsteinschen Gravitationstheorie. *Annalen der Physik*, 56:401–462, 1918.

[57] Martin D. Kruskal. Maximal extension of schwarzschild metric. *Phys. Rev.*, 119:1743–1745, 1960.

[58] Marc Mars. Present status of the Penrose inequality. *Class. Quantum Grav.*, 26, 2009.

[59] Marc Mars and Alberto Soria. The asymptotic behavior of the Hawking energy along null asymptotically flat hypersurfaces. *Classical and Quantum Gravity*, 32(18):185020, 2015.

[60] Marc Mars and Alberto Soria. On the Penrose inequality along null hypersurfaces. *Class. Quantum Grav.*, 33:115019, 2016.

[61] Jan Metzger. Foliations of asymptotically flat 3-manifolds by 2-surfaces of prescribed mean curvature. *J. Diff. Geom.*, 77, 2007.

[62] Charles Misner, Kip Thorne, and John Wheeler. *Gravitation*. W.H Freeman and Company, United States of America, 1973.

[63] Kristen Moore. On the evolution of hypersurfaces by their inverse null mean curvature. *J. Diff. Geom.*, 98(3):425–466, 2014.

[64] Barret O'Neill. *Semi-Riemannian Geometry with applications to Relativity.* Academic Press, San Diego, California, 1983.

[65] Enrico Onofri. On the positivity of the effective action in a theory of random surfaces. *Comm. Math. Phys.*, 86(3):321–326, 1982.

[66] Thomas Parker and Clifford H. Taubes. On Witten's Proof of the Positive Energy Theorem. *Communications in Mathematical Physics*, 84:223–238, 1982.

[67] Henri Roesch and Julian Scheuer. Mean Curvature Flow in Null Hypersurfaces and the Detection of MOTS. *Communications in Mathematical Physics*, 390:1–25, 2022.

[68] Anna Sakovich. The Jang equation and the positive mass theorem in the asymptotically hyperbolic setting. *arXiv:2003.07762*, 2020.

[69] Johannes Sauter. Foliations of null hypersurfaces and the Penrose inequality. *Doctoral thesis, ETH Zürch*, 2008.

[70] Jan Sbierski. The $C_0$-inextendibility of the Schwarzschild spacetime and the spacelike diameter in Lorentzian geometry. *J. Diff. Geom.*, 108(2):319–378, 2018.

[71] J. C. Schindler and A. Aguirre. Algorithms for the explicit computation of Penrose diagrams. *Class. Quantum Grav.*, 35, 2018.

[72] Richard Schoen and Shing Tung Yau. On the proof of the positive mass conjecture in general relativity. *Communications in Mathematical Physics*, 65(1):45 – 76, 1979.

[73] Richard Schoen and Shing-Tung Yau. Proof of the positive mass theorem ii. *Comm. Math. Phys.*, 79:231–260, 1981.

[74] Hartmut Schwetlick and Michael Struwe. Convergence of the Yamabe flow for "large" energies. *Journal für die reine und angewandte Mathematik*, 562:59–100, 2003.

[75] Yuguang Shi, Guofang Wang, and Jie Wu. On the behavior of quasi-local mass at the infinity along nearly round surfaces. *Annals of Global Analysis and Geometry*, 36:419–441, 2009.

[76] Leon Simon. Introduction to geometric measure theory. *Tsinghua Lectures*, 2014.

[77] Michael Struwe. Curvature flows on surfaces. *Annali Della Scuola Normale Superiore di Pisa-classe Di Scienze*, Ser. 5, 1(2):247–274, 2002.

[78] György Szekeres. On the Singularities of a Riemannian Manifold. *General Relativity and Gravitation*, 34:2001–2016, 2002. reprinted from Publ. Math. Debrecen **7** (1960), 285–301.

[79] Italo Tamanini. Regularity results for almost minimal oriented hypersurfaces in $R^n$. *Quaderni del Dipartimento Matematica dell'Universita de Lecce*, 1984.

[80] Augusto Visintin. Nucleation and mean curvature flow. *Comm. Part. Diff. Equ.*, 23:17–53, 1998.

[81] Robert M. Wald. *General Relativity.* The University of Chicago Press, Chicago 60637, 1984.

[82] Robert M. Wald and Pong Soo Jang. The positive energy conjecture and the cosmic censor hypothesis. *J. Math. Phys.*, 18(41), 1977.

[83] M. Wang, Y. Wang, and Y. Zhang. Minkowski formulae and Alexandrov Theorems in spacetime. *J. Diff. Geom.*, 105:249–290, 2017.

[84] Xiaodong Wang. The Mass of Asymptotically Hyperbolic manifolds. *J. Differential Geom.*, 57(2):273–299, 2001.

[85] Ye-Kai Wang. A spacetime Alexandrov Theorem. *Doctoral thesis, Columbia University*, 2014.

[86] Edward Witten. A new proof of the positive energy theorem. *Communications in Mathematical Physics*, 80:381–402, 1981.

[87] Markus Wolff. On the evolution of hypersurfaces inside maximal initial data sets along their inverse spacetime mean curvature. *Master thesis, University of Tübingen*, 2019.

[88] Markus Wolff. On effects of the null energy condition on totally umbilic hypersurfaces in a class of static spacetimes. *arXiv:2310.11229*, 2023.

[89] Markus Wolff. Ricci flow on surfaces along the standard lightcone in the 3 + 1-Minkowski spacetime. *Calculus of Variations and Partial Differential Equations*, 62(90), 2023.

[90] Rugang Ye. Global existence and convergence of Yamabe flow. *Journal of Differential Geometry*, 39:35–50, 1994.