On Fano and Calabi-Yau varieties with hypersurface Cox rings

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INTRODUCTION

This thesis contributes to the explicit classification of Fano and Calabi-Yau varieties.

A *Fano variety* is a normal projective variety with an ample anticanonical divisor. These varieties gain attention due to their significant role in the Minimal Model Program, a systematic approach to the classification of projective varieties up to birational equivalence proposed by Mori [98,99]. The Fano varieties among the smooth surfaces are precisely the classically known del Pezzo surfaces: a product of two projective lines, the projective plane and its blow-ups in up to eight points in general position. The classification of smooth Fano threefolds by Iskovskikh [82,83] and Mori/Mukai [100,101] marked an important milestone in the study of Fano varieties. In higher dimensions the classification of smooth Fano varieties is still an open problem. Particular results such as the classification of smooth toric Fano varieties up to dimension nine [10,11,77,104,110] indicate that the amount of Fano varieties rapidly increases with each dimension step.

The rich diversity of singular Fano varieties is illustrated by log del Pezzo surfaces, i.e., two-dimensional Fano varieties with at most log terminal singularities. Here we find classification results for log del Pezzo surfaces of Gorenstein index at most three [6,58,103]. Furthermore, log del Pezzo surfaces with symmetry are an active field of research [37–39,48–50]. In particular for surfaces with a torus action we mention the complete classification of Gorenstein log del Pezzo surfaces [47,70,74,125], see also [4, Sec. 5.4.4].

Over the last two decades, extensive progress was made in the classification of $\mathbb{Q}$-factorial Fano threefolds of Picard number one with at most terminal singularities, often called $\mathbb{Q}$-*Fano threefolds* for short. Major contributions come from Brown, Prokhorov, Reid and Suzuki [5,25,26,111–115,124]. The toolbox of these authors contains the so-called *graded ring method*, initiated by Reid. Roughly speaking this refers to the two-staged study of Fano varieties in terms of the anticanonical ring. First, ingredients from geometry and computer aid allow to produce lists including all Hilbert series associated with the varieties in question. Then in a second step one checks which candidates for the Hilbert series actually do occur, e.g. by applying constructive methods. An accurate introduction to the graded ring approach can be found in [5]. The finite list of candidates for Hilbert series of $\mathbb{Q}$-Fano threefolds is documented in the Graded Ring Database [29].

In connection with the graded ring approach it is natural to consider embeddings into weighted projective spaces. More generally, embeddings into toric varieties are a widely used approach for classifying Fano varieties [35, 41, 80, 116, 117]. In the first chapter of this thesis, which presents joint work with J. Hausen and M. Wrobel, we investigate *non-degenerate toric complete intersections*, meaning complete intersections in projective toric varieties arising from a *non-degenerate system* of Laurent polynomials; see Definitions 1.3.6 and 1.4.1. This notion is originally due to Khovanskii in the smooth case [76]. We approach the singular case by means of the anticanonical complex, a generalization of the Fano polytope from toric geometry introduced in [17, 72]. Theorem 1.1.1 shows that non-degenerate toric complete intersections indeed admit an anticanonical complex. This leads to Bertini type statements on terminal and canonical singularities, namely that a non-degenerate toric complete intersection $X \subseteq Z$ inherits precisely the terminal (canonical) singularities from the minimal open toric subvariety of Z containing X.

Using this result we treat terminal Fano threefolds showing up as non-degenerate toric complete intersection in a fake weighted projective space. Here, by a *fake weighted projective space* we mean any normal $\mathbb{Q}$-factorial projective toric variety of Picard number one, thus generalizing the well-known weighted projective spaces. Toric terminal Fano threefolds have been classified by Kasprzyk [87]. We present results for the non-toric case.

Theorem 1. *Any non-toric terminal Fano general complete intersection threefold $X = X_1 \cap \cdots \cap X_s$ in a fake weighted projective space Z is a member of precisely one of the following families, specified by the generator degree matrix Q and the relation degree matrix μ having the classes of the torus invariant prime divisors $[D_i] \in \mathrm{Cl}(Z)$ resp. the classes $[X_i] \in \mathrm{Cl}(Z)$ as its columns. We also list $-\mathcal{K}$, $-\mathcal{K}^3$ and $h^0(-\mathcal{K})$,*

No.	Cl(Z)	Q	μ	$-\mathcal{K}$	$-\mathcal{K}^3$	$h^0(-\mathcal{K})$
1	$\mathbb{Z}$	$[1\ 1\ 1\ 1\ 1]$	2	3	54	30
2	$\mathbb{Z}$	$[1\ 1\ 1\ 1\ 1]$	3	2	24	15
3	$\mathbb{Z}$	$[1\ 1\ 1\ 1\ 1]$	4	1	4	5
4	$\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix}1&1&1&1&1\\\bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{2}\end{bmatrix}$	$\begin{bmatrix}3\\\bar{0}\end{bmatrix}$	$\begin{pmatrix}2\\\bar{1}\end{pmatrix}$	8	5
5	$\mathbb{Z}$	$[1\ 1\ 1\ 1\ 2]$	4	2	16	11
6	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}1&1&1&1&2\\\bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}4\\\bar{0}\end{bmatrix}$	$\begin{pmatrix}2\\\bar{1}\end{pmatrix}$	8	5
7	$\mathbb{Z}$	$[1\ 1\ 1\ 2\ 2]$	4	3	27	16
8	$\mathbb{Z}$	$[1\ 1\ 1\ 2\ 2]$	6	1	3/2	3
9	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}1&1&1&2&2\\\bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}4\\\bar{0}\end{bmatrix}$	$\begin{pmatrix}3\\\bar{1}\end{pmatrix}$	27/2	8
10	$\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix}1&1&1&2&2\\\bar{0}&\bar{1}&\bar{2}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\\bar{0}\end{bmatrix}$	$\begin{pmatrix}1\\\bar{1}\end{pmatrix}$	1/2	1
11	$\mathbb{Z}$	$[1\ 1\ 1\ 1\ 3]$	6	1	2	4
12	$\mathbb{Z}$	$[1\ 1\ 1\ 2\ 3]$	6	2	8	7

13	$\mathbb{Z}$	$[1\ 1\ 2\ 2\ 3]$	6	3	27/2	9
14	$\mathbb{Z}$	$[1\ 1\ 2\ 3\ 3]$	6	4	64/3	13
15	$\mathbb{Z}$	$[1\ 2\ 2\ 3\ 3]$	6	5	125/6	12
16	$\mathbb{Z}$	$[1\ 1\ 1\ 2\ 4]$	8	1	1	3
17	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 4 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 8 \\ \bar{0} \end{bmatrix}$	$\begin{pmatrix} 1 \\ \bar{1} \end{pmatrix}$	1/2	1
18	$\mathbb{Z}$	$[1\ 2\ 3\ 3\ 4]$	12	1	1/6	1
19	$\mathbb{Z}$	$[1\ 1\ 3\ 4\ 4]$	12	1	1/4	2
20	$\mathbb{Z}$	$[1\ 1\ 2\ 2\ 5]$	10	1	1/2	2
21	$\mathbb{Z}$	$[1\ 1\ 2\ 3\ 6]$	12	1	1/3	2
22	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 2 & 3 & 6 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 12 \\ \bar{0} \end{bmatrix}$	$\begin{pmatrix} 1 \\ \bar{1} \end{pmatrix}$	1/6	1
23	$\mathbb{Z}$	$[1\ 1\ 1\ 4\ 6]$	12	1	1/2	3
24	$\mathbb{Z}$	$[1\ 1\ 2\ 6\ 9]$	18	1	1/6	2
25	$\mathbb{Z}$	$[1\ 1\ 4\ 5\ 10]$	20	1	1/10	2
26	$\mathbb{Z}$	$[1\ 1\ 3\ 8\ 12]$	24	1	1/12	2
27	$\mathbb{Z}$	$[1\ 2\ 3\ 10\ 15]$	30	1	1/30	1
28	$\mathbb{Z}$	$[1\ 1\ 6\ 14\ 21]$	42	1	1/42	2
29 30	$\mathbb{Z}$	$[1\ 1\ 1\ 1\ 1\ 1]$	$[2\ 2]$ $[2\ 3]$	2 1	32 6	19 6
31	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 2 & 2 \\ \bar{0} & \bar{0} \end{bmatrix}$	$\begin{pmatrix} 2 \\ \bar{1} \end{pmatrix}$	16	9
32	$\mathbb{Z}$	$[1\ 1\ 1\ 2\ 2\ 2]$	$[4\ 4]$	1	2	3
33	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 & 4 \\ \bar{0} & \bar{0} \end{bmatrix}$	$\begin{pmatrix} 1 \\ \bar{1} \end{pmatrix}$	1	1
34	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 & 4 \\ \bar{0} & \bar{0} \end{bmatrix}$	$\begin{pmatrix} 1 \\ \bar{0} \end{pmatrix}$	1	2
35	$\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 & 4 \\ \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}$	$\begin{pmatrix} 1 \\ \bar{1} \\ \bar{1} \end{pmatrix}$	1/2	0
36	$\mathbb{Z}$	$[1\ 2\ 2\ 2\ 3\ 3]$	$[6\ 6]$	1	1/2	1
37	$\mathbb{Z}$	$[1\ 1\ 2\ 3\ 3\ 3]$	$[6\ 6]$	1	2/3	2
38	$\mathbb{Z}$	$[1\ 2\ 2\ 3\ 3\ 3]$	$[6\ 6]$	2	8/3	3
39	$\mathbb{Z}$	$[1\ 1\ 1\ 1\ 1\ 1\ 1]$	$[2\ 2\ 2]$	1	8	7
40	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{pmatrix} 1 \\ \bar{1} \end{pmatrix}$	4	3
41	$\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{pmatrix} 1 \\ \bar{1} \\ \bar{1} \end{pmatrix}$	2	1

42	$\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{pmatrix} 1 \\ \bar{1} \\ \bar{1} \\ \bar{1} \end{pmatrix}$	1	0

Moreover, each of these constellations defines non-degenerate toric complete intersections with at most terminal singularities in a fake weighted projective space.

Fano varieties with mild singularities are *Mori dream spaces* [19] in the sense that they are normal projective varieties X with finitely generated divisor class group $\mathrm{Cl}(X)$ and finitely generated *Cox ring*

$$\mathcal{R}(X) \coloneqq \bigoplus_{[D] \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)).$$

The Cox ring fixes a variety up to small quasimodifications. Together with an ample class a Mori dream space can be reconstructed from its Cox ring using geometric invariant theory. In particular, Fano varieties are entirely determined by their Cox ring. We refer to [4] for more background and the combinatorial treatment of this topic.

It turns out that, under suitable assumptions on the ambient toric variety, the Cox ring of a non-degenerate toric complete intersection is given by its defining equations in homogeneous coordinates; see Corollary 1.4.13. This applies in particular to the varieties from Theorem 1.

Corollary 2. *For any non-toric terminal Fano general complete intersection threefold $X = X_1 \cap \dots \cap X_s$ in a fake weighted projective space Z we have $\mathrm{Cl}(X) = \mathrm{Cl}(Z)$ and the Cox ring of X is given by*

$$\begin{aligned} \mathcal{R}(X) &= \mathbb{K}[T_1, \dots, T_{s+4}]/\langle g_1, \dots, g_s \rangle, \\ \deg(T_i) &= [D_i] \in \mathrm{Cl}(Z), \\ \deg(g_j) &= [X_j] \in \mathrm{Cl}(Z), \end{aligned}$$

where $\mathbb{K}[T_1, \dots, T_{s+4}] = \mathcal{R}(Z)$ is the Cox ring of the fake weighted projective space Z and $g_1, \dots, g_s \in \mathcal{R}(Z)$ are the defining $\mathrm{Cl}(Z)$-homogeneous polynomials for $X_1, \dots, X_s \subseteq Z$. Moreover, $T_1, \dots, T_{s+4}$ define a minimal system of prime generators for $\mathcal{R}(X)$.

We turn to smooth Fano varieties. Although there exists no comprehensive description of smooth Fano fourfolds so far, there are plenty of results providing a clear direction. Let us mention for example partial classifications in terms of the Fano index by Fujita [57], Mukai [102] and Wiśniewski [130] as well as detailed studies of the birational geometry including strong bounds on the Picard number by Casagrande [30–34]. As a result the uncharted territories mainly restrict to Fano fourfolds of index one. Tackling this class in Picard number one Küchle [92] and Przyjalkowski/Shramov [116] have classified all smooth Fano fourfolds showing up as general complete intersection in a weighted projective space. In terms of Cox rings, here the hypersuface case consists precisely of the smooth Fano fourfolds of Picard number one that have a *hypersuface Cox ring*, i.e., $\mathcal{R}(X)$ admits $\mathrm{Cl}(X)$-homogeneous generators such that the relation ideal is generated by a single equation. From the geometric point of view, a variety with a hypersurface Cox ring is an especially neatly embedded hypersurface of a toric variety [4, Sec. 3.2.5].

Note that the first examples of smooth Fano varieties with a hypersurface Cox ring show up in dimension three; see [51, Thms. 4.1 and 4.5], where, based on the classifications mentioned before, the Cox rings of the smooth Fano threefolds of Picard numbers one and two have been computed. While there are no smooth del Pezzo surfaces with a hypersurface Cox ring, the singular case provides many examples [48].

In Chapter 2, together with J. Hausen and A. Laface, we provide classification results on smooth Fano fourfolds of Picard number two with a hypersurface Cox ring. While other recent Cox ring based classifications of Fano varieties exploit special relations, e.g. quadrics [55, 71], trinomials [56] or quadronomials [63], we consider equations of a suitably general form. This leads to the notion of *general* and *spread hypersurface Cox rings*, which is made precise in Definitions 2.4.3 and 2.4.5. Note that the latter intends to make the somewhat nebulous term general more concrete and verifiable for explicit examples. Here we restrict to a non-technical statement of our result.

Theorem 3. *Every smooth Fano fourfold X of Picard number two that has a general hypersurface Cox ring is isomorphic to a member of one of the following families of smooth Fano fourfolds, specified by their Cox ring generator degrees $w_1, \ldots, w_7$, the relation degree μ and the anticanonical class $-\mathcal{K}$ in* $\mathrm{Cl}(X) = \mathbb{Z}^2$.

No.	$[w_1, \ldots, w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$
1	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$(1,1)$	$(3,2)$	432
2		$(2,1)$	$(2,2)$	256
3		$(3,1)$	$(1,2)$	80
4		$(1,2)$	$(3,1)$	270
5		$(2,2)$	$(2,1)$	112
6		$(3,2)$	$(1,1)$	26
7	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$(1,1)$	$(2,2)$	416
8		$(1,2)$	$(2,1)$	163
9		$(2,1)$	$(1,2)$	224
10		$(2,2)$	$(1,1)$	52
11	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$(1,1)$	$(1,2)$	464
12		$(1,2)$	$(1,1)$	98
13	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(1,2)$	$(3,2)$	352
14		$(2,3)$	$(2,1)$	65
15	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(1,3)$	$(2,1)$	83
16	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$(2,1)$	$(3,2)$	352
17		$(3,2)$	$(2,1)$	81

No.	$[w_1, \ldots, w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$
18	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$(3,1)$	$(1,1)$	38
19		$(2,1)$	$(2,1)$	192
20		$(1,1)$	$(3,1)$	432
21	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$(3,1)$	$(2,1)$	113
22	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,2)$	$(2,3)$	272
23		$(3,3)$	$(1,2)$	51
24	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 1 \end{bmatrix}$	$(4,4)$	$(1,2)$	34
25	$\begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 1 & 1 \end{bmatrix}$	$(6,6)$	$(1,2)$	17
26	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,2)$	$(1,3)$	216
27	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,4)$	$(1,2)$	64
28	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,6)$	$(1,1)$	8
29	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,2)$	$(2,2)$	192
30		$(3,3)$	$(1,1)$	18

No.	$[w_1,\dots,w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$
31	$\begin{bmatrix} 1&1&1&2&0&0&0 \\ 0&0&0&1&1&1&1 \end{bmatrix}$	$(4,2)$	$(1,2)$	48
32	$\begin{bmatrix} 1&1&1&2&0&0&0 \\ 0&0&0&2&1&1&1 \end{bmatrix}$	$(4,4)$	$(1,1)$	12
33	$\begin{bmatrix} 1&1&2&1&0&0&0 \\ 0&1&3&2&1&1&1 \end{bmatrix}$	$(4,6)$	$(1,3)$	50
34 35 36	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&1&1&1&1&1&1 \end{bmatrix}$	$(2,2)$ $(3,3)$ $(4,4)$	$(3,4)$ $(2,3)$ $(1,2)$	378 144 20
37	$\begin{bmatrix} 1&1&1&1&2&0&0 \\ 0&1&1&1&2&1&1 \end{bmatrix}$	$(4,4)$	$(2,3)$	96
38	$\begin{bmatrix} 1&1&1&1&3&0&0 \\ 0&1&1&1&3&1&1 \end{bmatrix}$	$(6,6)$	$(1,2)$	10
39	$\begin{bmatrix} 1&1&1&2&3&0&0 \\ 0&1&1&2&3&1&1 \end{bmatrix}$	$(6,6)$	$(2,3)$	48
40 41	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&1&1&1&1&1&1 \end{bmatrix}$	$(2,2)$ $(3,3)$	$(2,4)$ $(1,3)$	352 99
42 43	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&2&2&2&1&1&1 \end{bmatrix}$	$(2,4)$ $(3,6)$	$(2,5)$ $(1,3)$	304 54
44	$\begin{bmatrix} 1&1&1&2&0&0&0 \\ 0&1&1&2&1&1&1 \end{bmatrix}$	$(4,4)$	$(1,3)$	66
45	$\begin{bmatrix} 1&1&1&2&0&0&0 \\ 0&2&2&4&1&1&1 \end{bmatrix}$	$(4,8)$	$(1,3)$	36
46	$\begin{bmatrix} 1&1&2&3&0&0&0 \\ 0&1&2&3&1&1&1 \end{bmatrix}$	$(6,6)$	$(1,3)$	33
47	$\begin{bmatrix} 1&1&2&3&0&0&0 \\ 0&2&4&6&1&1&1 \end{bmatrix}$	$(6,12)$	$(1,3)$	18
48	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&1&1&1&2&1&1 \end{bmatrix}$	$(2,2)$	$(3,5)$	433
49	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&2&2&2&3&1&1 \end{bmatrix}$	$(3,6)$	$(2,5)$	145
50	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&1&1&2&1&1&1 \end{bmatrix}$	$(2,4)$	$(2,3)$	144

No.	$[w_1,\dots,w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$
51	$\begin{bmatrix} 1&1&1&2&0&0&0 \\ 0&1&1&3&1&1&1 \end{bmatrix}$	$(4,6)$	$(1,2)$	22
52	$\begin{bmatrix} 1&1&1&2&1&0&0 \\ 0&1&1&3&2&1&1 \end{bmatrix}$	$(4,6)$	$(2,3)$	65
53 54	$\begin{bmatrix} 1&1&1&1&1&1&0 \\ -1&0&0&0&0&1&1 \end{bmatrix}$	$(2,0)$ $(4,0)$	$(4,1)$ $(2,1)$	431 62
55	$\begin{bmatrix} 1&1&1&1&1&2&0 \\ -1&0&0&0&0&1&1 \end{bmatrix}$	$(3,0)$	$(4,1)$	376
56	$\begin{bmatrix} 1&1&1&1&1&3&0 \\ -1&0&0&0&0&1&1 \end{bmatrix}$	$(4,0)$	$(4,1)$	341
57	$\begin{bmatrix} 1&1&1&1&3&1&0 \\ -1&0&0&0&0&1&1 \end{bmatrix}$	$(6,0)$	$(2,1)$	31
58	$\begin{bmatrix} 1&1&1&1&3&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(6,0)$	$(1,2)$	16
59	$\begin{bmatrix} 1&1&1&2&3&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(6,0)$	$(2,2)$	64
60	$\begin{bmatrix} 1&1&1&2&3&1&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(6,0)$	$(3,2)$	80
61	$\begin{bmatrix} 1&1&1&1&2&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(4,0)$	$(2,2)$	128
62	$\begin{bmatrix} 1&1&1&1&2&1&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(4,0)$	$(3,2)$	160
63	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(3,0)$	$(2,2)$	192
64	$\begin{bmatrix} 1&1&1&1&1&1&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(3,0)$	$(3,2)$	240
65	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(2,0)$	$(3,2)$	432
66	$\begin{bmatrix} 1&1&1&1&1&1&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(2,0)$	$(4,2)$	480
67	$\begin{bmatrix} 1&1&1&1&1&2&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(2,0)$	$(5,2)$	624

Moreover, each of the items 1 to 67 defines a non-empty family of smooth Fano fourfolds of Picard number two and any two members from different families are not isomorphic.

In the third chapter of this dissertation we pursue the Cox ring approach in the world of *Calabi-Yau varieties*, meaning normal projective varieties X with trivial canonical class $\mathcal{K}_X$, at most canonical singularities and $h^i(X, \mathcal{O}_X) = 0$ for $i = 1, \dots, \dim(X) - 1$. Calabi-Yau varieties, especially threefolds, are intensively studied from various perspectives, also including diverse classification approaches such as [106–108, 128, 129] or more recently [45, 61, 64]. Similar to Fano varieties, they are considered a building block in the Minimal Model Program. Note that a smooth Calabi-Yau variety of dimension at most three is a Mori dream space if and only if its cone of effective divisors is rational polyhedral [97]. More general, Mori dream spaces of Calabi-Yau type are completely characterized via the singularities of their total coordinate space $\operatorname{Spec} \mathcal{R}(X)$ [88]. Furthermore, Calabi-Yau varieties are central to the interdisciplinary research field of mirror symmetry, connecting algebraic geometry and theoretical physics. Work by i.a. Batyrev [12–14] in this area emphasizes Calabi-Yau hypersurfaces and complete intersections in toric varieties as a rich source of explicit examples. The interest from physics also includes several large-scale classifications of Calabi-Yau varieties based on combinatorial approaches and computer aid [36, 60, 78, 122]. Besides, the graded ring method mentioned above applies to the study of Calabi-Yau varieties as well [22–24, 118].

From the view of classical algebraic geometry Calabi-Yau varieties can be seen as generalization of K3 surfaces, which are precisely the smooth Calabi-Yau varieties of dimension two. Cox rings of K3 surfaces have been studied in [3, 8, 109], in particular describing several classes of K3 surfaces with a hypersurface Cox ring. Our results concern smooth Calabi-Yau threefolds with a general hypersurface Cox ring. By and large, the case of Picard number one is covered by Oguiso's classification of smooth Calabi-Yau threefolds that are general complete intersections in some weighted projective space [105] providing all smooth Calabi-Yau threefolds with a general hypersurface Cox ring and freely generated Picard group. There is one additional family if one allows torsion in the Picard group; see Proposition 3.1.1. Note that Przyjalkowski and Shramov have established explicit bounds for smooth Calabi-Yau weighted complete intersections in any dimension [116].

The main result of Chapter 3 is the classification of all smooth Calabi-Yau threefolds of Picard number two that have a general hypersurface Cox ring.

Theorem 4. *Every smooth Calabi-Yau threefold X of Picard number two that has a general hypersurface Cox ring is isomorphic to a member of one of the following families of smooth Calabi-Yau threefolds, specified by their Cox ring generator degrees $w_1, \ldots, w_6$, the relation degree μ and an ample class u in $\operatorname{Cl}(X)$.*

No.	$\operatorname{Cl}(X)$	$[w_1, \ldots, w_6]$	μ	u
1	$\mathbb{Z}^2$	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2	$\mathbb{Z}^2 \times \mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{2} & \bar{0} & \bar{1} & \bar{2} \end{bmatrix}$	$\begin{bmatrix} 3 \\ 3 \\ \bar{0} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ \bar{0} \end{bmatrix}$
3	$\mathbb{Z}^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
4	$\mathbb{Z}^2$	$\begin{bmatrix} 1 & 1 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 6 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
5	$\mathbb{Z}^2$	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
6	$\mathbb{Z}^2$	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
7	$\mathbb{Z}^2 \times \mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{2} & \bar{1} & \bar{2} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ \bar{0} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ \bar{0} \end{bmatrix}$
8	$\mathbb{Z}^2$	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 2 & 3 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
9	$\mathbb{Z}^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
10	$\mathbb{Z}^2$	$\begin{bmatrix} 1 & 1 & 1 & 3 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 6 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

No.	Cl(X)	$[w_1, \dots, w_6]$	μ	u
11	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&1&1&0&0 \\ 0&0&0&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
12	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&1&1&0&0 \\ 0&0&0&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
13	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&1&3&0&0 \\ 0&0&0&2&1&1 \end{bmatrix}$	$\begin{bmatrix} 6 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
14	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&1&3&0&0 \\ -1&0&0&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 6 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
15	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&1&1&0&0 \\ -1&0&0&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
16	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&1&1&0&0 \\ -1&0&0&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
17	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&1&1&0&0 \\ -2&0&0&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
18	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&1&1&0&0 \\ -2&0&0&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
19	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&1&2&1&0 \\ 0&0&0&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 6 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
20	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&1&4&1&0 \\ 0&0&0&2&1&1 \end{bmatrix}$	$\begin{bmatrix} 8 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
21	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&1&5&2&0 \\ 0&0&0&2&1&1 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
22	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&2&5&1&0 \\ 0&0&0&2&1&1 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
23	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&2&7&3&0 \\ 0&0&0&2&1&1 \end{bmatrix}$	$\begin{bmatrix} 14 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$
24	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&1&2&1&0 \\ -2&0&0&0&1&1 \end{bmatrix}$	$\begin{bmatrix} 6 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
25	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&1&1&3&0 \\ -2&0&0&0&1&1 \end{bmatrix}$	$\begin{bmatrix} 7 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$
26	$\mathbb{Z}^2$	$\begin{bmatrix} 2&1&1&1&3&0 \\ -2&0&0&0&1&1 \end{bmatrix}$	$\begin{bmatrix} 8 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$
27	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&2&5&1&0 \\ 0&0&1&3&1&1 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
28	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&2&5&1&0 \\ 0&0&1&3&1&1 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$
29	$\mathbb{Z}^2$	$\begin{bmatrix} 1&1&1&4&1&0 \\ -1&0&0&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 8 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 1 \end{bmatrix}$
30	$\mathbb{Z}^2$	$\begin{bmatrix} 1&2&1&1&1&0 \\ -1&-1&0&0&1&1 \end{bmatrix}$	$\begin{bmatrix} 6 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Moreover, each of the items 1 to 30 defines a non-empty family of smooth Calabi-Yau threefolds of Picard number two and any two members from different families are not isomorphic.

CHAPTER

ONE

NON-DEGENERATE TORIC COMPLETE INTERSECTIONS

We classify the terminal Fano threefolds that are embedded into a fake weighted projective space via a general system of Laurent polynomials. Varieties arising in this way were originally studied by Khovanskii in the smooth case [76] and provide useful tools for constructing explicit examples of Mori dream spaces with prescribed properties. Our key tool for the combinatorial treatment of singularities is the anticanonical complex, which generalizes the Fano polytope from toric geometry and has been used to study Fano varieties with torus action so far [17, 72]. In this chapter, we enlarge the area of application to complete intersections in toric varieties defined by non-degenerate systems of Laurent polynomials. The results of this chapter are published in the joint work [67].

1.1 Results

The idea behind anticanonical complexes is to extend the features of the Fano polytopes from toric geometry to wider classes of varieties and thereby to provide combinatorial tools for the treatment of the singularities of the minimal model programme. If X is any $\mathbb{Q}$-Gorenstein variety, i.e. some positive multiple of a canonical divisor K_X is Cartier, then these singularities are defined in terms of *discrepancies* that means the coefficients $a(E)$ of the exceptional divisors E showing up in the ramification formula for a resolution $\pi\colon X' \to X$ of singularities:

$$K_{X'} \ = \ \pi^* K_X + \sum a(E)E.$$

The variety X has at most *terminal, canonical* or *log terminal* singularities if always $a(E) > 0$, $a(E) \geq 0$ or $a(E) > -1$. We briefly look at the toric case. For an n-dimensional toric Fano variety Z, one defines the *Fano polytope* to be the convex hull $A \subseteq \mathbb{Q}^n$ over the primitive ray generators of the describing fan of Z. For any toric resolution $\pi\colon Z' \to Z$ of singularities, the exceptional divisors E_ϱ are given by rays of the fan of Z' and one obtains the discrepancies as

$$a(E_\varrho) \ = \ \frac{\|v_\varrho\|}{\|v'_\varrho\|} - 1,$$

where $v_\varrho \in \varrho$ is the shortest non-zero lattice vector and $v'_\varrho \in \varrho$ is the intersection point of ϱ and the boundary ∂A. In particular, a toric Fano variety Z is always log terminal and Z has at most terminal (canonical) singularities if and only if its corresponding Fano polytope A contains no lattice points except the origin and its vertices (no lattice points in its interior except the origin). This allows the use of lattice polytope methods in the study of singular toric Fano varieties; see [21, 85, 86] for work in this direction.

This principle has been extended by replacing the Fano polytope with a suitable polyhedral complex, named *anticanonical complex* in the setting of varieties with a torus action of complexity one, which encodes discrepancies in full analogy to the toric Fano polytope; see [17]. The more recent work [72] provides an existence result of anticanonical complexes for torus actions of higher complexity subject to conditions on a rational quotient. Applications to the study of singularities and Fano varieties can be found in [7, 28, 73].

In the present chapter, we provide an anticanonical complex for subvarieties of toric varieties arising from non-degenerate systems of Laurent polynomials in the sense of Khovanskii [76]; see also Definition 1.3.6. Even in the hypersurface case, the subvarieties obtained this way form an interesting example class of varieties which is actively studied by several authors; see for instance [13, 53, 81].

We briefly indicate the setting; see Section 1.3 for the details. Let $F = (f_1, \ldots, f_s)$ be a non-degenerate system of Laurent polynomials in n variables and let Σ be any fan in $\mathbb{Z}^n$ refining the normal fan of the Minkowski sum $B_1 + \ldots + B_s$ of the Newton polytopes B_j of f_j. Moreover, denote by Z the toric variety associated with Σ. We are interested in the *non-degenerate toric complete intersection* defined by F and Σ, that means the variety

$$X \;=\; X_1 \cap \ldots \cap X_s \;\subseteq Z,$$

where $X_i \subseteq Z$ is the closure of $V(f_i) \subseteq \mathbb{T}^n$. By Theorem 1.3.12, the variety $X \subseteq Z$ is a locally complete intersection, equals the closure of $V(F) \subseteq \mathbb{T}^n$ and, in the Cox ring of Z, the defining homogeneous equations of X generate a complete intersection ideal. Theorem 1.4.4 shows that the union $Z_X \subseteq Z$ of all torus orbits intersecting X is open in Z and thus the corresponding cones form a subfan $\Sigma_X \subseteq \Sigma$. Moreover, the support of Σ_X equals the tropical variety of $V(F) \subseteq \mathbb{T}^n$.

We come to the first main result of this chapter. Suppose that Z_X is $\mathbb{Q}$-Gorenstein. Then, for every $\sigma \in \Sigma_X$, we have a linear form $u_\sigma \in \mathbb{Q}^n$ evaluating to -1 on every primitive ray generator v_ϱ, where ϱ is an extremal ray of σ. We set

$$A(\sigma) \;\coloneqq\; \{v \in \sigma;\; 0 \geq \langle u_\sigma, v\rangle \geq -1\} \;\subseteq\; \sigma.$$

Theorem 1.1.1. *Let $X \subseteq Z$ be an irreducible non-degenerate toric complete intersection. Then $X \subseteq Z$ admits ambient toric resolutions. Moreover, if Z_X is $\mathbb{Q}$-Gorenstein, then X is so and X has an anticanonical complex*

$$\mathcal{A}_X \;=\; \bigcup_{\sigma \in \Sigma_X} A(\sigma).$$

That means that for all ambient toric modifications $Z' \to Z$ the discrepancy of any exceptional divisor $E_{X'} \subseteq X'$ is given in terms of the defining ray $\varrho \in \Sigma'$ of its host $E_{Z'} \subseteq Z'$, the primitive generator $v_\varrho \in \varrho$ and the intersection point v'_ϱ of ϱ and the boundary $\partial \mathcal{A}_X$ as

$$a(E_{X'}) \;=\; \frac{\|v_\varrho\|}{\|v'_\varrho\|} - 1.$$

Observe that in the above setting, each vertex of $\mathcal{A}_X$ is a primitive ray generator of the fan Σ. Thus, in the non-degenerate complete toric intersection case, all vertices of the anticanonical complex are integral vectors; this does definitely not hold in other situations, see [17, 72]. The following consequence of Theorem 1.1.1 yields in particular Bertini type statements on terminal and canonical singularities.

Corollary 1.1.2. *Consider a subvariety $X \subseteq Z$ as in Theorem 1.1.1 and the associated anticanonical complex $\mathcal{A}_X$.*

(i) *X has at most log-terminal singularities.*
(ii) *X has at most terminal singularities if and only if $\mathcal{A}_X$ contains no lattice points except the origin and its vertices.*
(iii) *X has at most canonical singularities if and only if $\mathcal{A}_X$ contains no interior lattice points except the origin.*

Moreover, X has at most terminal (canonical) singularities if and only if its ambient toric variety Z_X has at most terminal (canonical) singularities.

As an application of the first main result, we classify the general non-toric terminal Fano non-degenerate complete intersection threefolds sitting in fake weighted projective spaces; for the meaning of "general" in this context, see Definition 1.4.12. According to [76], the general toric complete intersection is non-degenerate. Moreover, under suitable assumptions on the ambient toric variety, we obtain the divisor class group and the Cox ring for free in the general case; see Corollary 1.4.13. This, by the way, allows us to construct many Mori dream spaces with prescribed properties; see for instance Example 1.4.16.

We turn to the second main result. Recall that a fake weighted projective space is an n-dimensional toric variety arising from a complete fan with $n+1$ rays. Any fake weighted projective space Z is uniquely determined up to isomorphism by its degree matrix Q, having as its columns the divisor classes $[D_i] \in \mathrm{Cl}(Z)$ of the toric prime divisors $D_1, \ldots, D_{n+1}$ of Z.

Theorem 1.1.3. *Any non-toric terminal Fano general complete intersection threefold $X = X_1 \cap \ldots \cap X_s$ in a fake weighted projective space Z is a member of precisely one of the following families, specified by the generator degree matrix Q and the relation degree matrix μ with respect to the $\mathrm{Cl}(Z)$-grading. We also list $-\mathcal{K}$, $-\mathcal{K}^3$ and $h^0(-\mathcal{K})$,*

No.	$\mathrm{Cl}(Z)$	Q	μ	$-\mathcal{K}$	$-\mathcal{K}^3$	$h^0(-\mathcal{K})$
1			2	3	54	30
2	$\mathbb{Z}$	$[1\ 1\ 1\ 1\ 1]$	3	2	24	15
3			4	1	4	5
4	$\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix}1&1&1&1&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{2}\end{bmatrix}$	$\begin{bmatrix}3\\ \bar{0}\end{bmatrix}$	$\begin{pmatrix}2\\ \bar{1}\end{pmatrix}$	8	5
5	$\mathbb{Z}$	$[1\ 1\ 1\ 1\ 2]$	4	2	16	11
6	$\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}1&1&1&1&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{0}\end{bmatrix}$	$\begin{pmatrix}2\\ \bar{1}\end{pmatrix}$	8	5
7	$\mathbb{Z}$	$[1\ 1\ 1\ 2\ 2]$	4	3	27	16
8			6	1	3/2	3
9	$\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}1&1&1&2&2\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}4\\ \bar{0}\end{bmatrix}$	$\begin{pmatrix}3\\ \bar{1}\end{pmatrix}$	27/2	8
10	$\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix}1&1&1&2&2\\ \bar{0}&\bar{1}&\bar{2}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}6\\ \bar{0}\end{bmatrix}$	$\begin{pmatrix}1\\ \bar{1}\end{pmatrix}$	1/2	1
11	$\mathbb{Z}$	$[1\ 1\ 1\ 1\ 3]$	6	1	2	4
12	$\mathbb{Z}$	$[1\ 1\ 1\ 2\ 3]$	6	2	8	7
13	$\mathbb{Z}$	$[1\ 1\ 2\ 2\ 3]$	6	3	27/2	9
14	$\mathbb{Z}$	$[1\ 1\ 2\ 3\ 3]$	6	4	64/3	13
15	$\mathbb{Z}$	$[1\ 2\ 2\ 3\ 3]$	6	5	125/6	12
16	$\mathbb{Z}$	$[1\ 1\ 1\ 2\ 4]$	8	1	1	3
17	$\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}1&1&1&2&4\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}8\\ \bar{0}\end{bmatrix}$	$\begin{pmatrix}1\\ \bar{1}\end{pmatrix}$	1/2	1
18	$\mathbb{Z}$	$[1\ 2\ 3\ 3\ 4]$	12	1	1/6	1
19	$\mathbb{Z}$	$[1\ 1\ 3\ 4\ 4]$	12	1	1/4	2
20	$\mathbb{Z}$	$[1\ 1\ 2\ 2\ 5]$	10	1	1/2	2
21	$\mathbb{Z}$	$[1\ 1\ 2\ 3\ 6]$	12	1	1/3	2
22	$\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix}1&1&2&3&6\\ \bar{0}&\bar{1}&\bar{1}&\bar{0}&\bar{1}\end{bmatrix}$	$\begin{bmatrix}12\\ \bar{0}\end{bmatrix}$	$\begin{pmatrix}1\\ \bar{1}\end{pmatrix}$	1/6	1
23	$\mathbb{Z}$	$[1\ 1\ 1\ 4\ 6]$	12	1	1/2	3
24	$\mathbb{Z}$	$[1\ 1\ 2\ 6\ 9]$	18	1	1/6	2
25	$\mathbb{Z}$	$[1\ 1\ 4\ 5\ 10]$	20	1	1/10	2
26	$\mathbb{Z}$	$[1\ 1\ 3\ 8\ 12]$	24	1	1/12	2

27	$\mathbb{Z}$	$[1\ 2\ 3\ 10\ 15]$	30	1	1/30	1
28	$\mathbb{Z}$	$[1\ 1\ 6\ 14\ 21]$	42	1	1/42	2
29	$\mathbb{Z}$	$[1\ 1\ 1\ 1\ 1\ 1]$	$[2\ 2]$	2	32	19
30			$[2\ 3]$	1	6	6
31	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 2 & 2 \\ \bar{0} & \bar{0} \end{bmatrix}$	$\begin{pmatrix} 2 \\ \bar{1} \end{pmatrix}$	16	9
32	$\mathbb{Z}$	$[1\ 1\ 1\ 2\ 2\ 2]$	$[4\ 4]$	1	2	3
33	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 & 4 \\ \bar{0} & \bar{0} \end{bmatrix}$	$\begin{pmatrix} 1 \\ \bar{1} \end{pmatrix}$	1	1
34	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 & 4 \\ \bar{0} & \bar{0} \end{bmatrix}$	$\begin{pmatrix} 1 \\ \bar{0} \end{pmatrix}$	1	2
35	$\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{1} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 4 & 4 \\ \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}$	$\begin{pmatrix} 1 \\ \bar{1} \\ \bar{1} \end{pmatrix}$	1/2	0
36	$\mathbb{Z}$	$[1\ 2\ 2\ 2\ 3\ 3]$	$[6\ 6]$	1	1/2	1
37	$\mathbb{Z}$	$[1\ 1\ 2\ 3\ 3\ 3]$	$[6\ 6]$	1	2/3	2
38	$\mathbb{Z}$	$[1\ 2\ 2\ 3\ 3\ 3]$	$[6\ 6]$	2	8/3	3
39	$\mathbb{Z}$	$[1\ 1\ 1\ 1\ 1\ 1\ 1]$	$[2\ 2\ 2]$	1	8	7
40	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{pmatrix} 1 \\ \bar{1} \end{pmatrix}$	4	3
41	$\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{pmatrix} 1 \\ \bar{1} \\ \bar{1} \end{pmatrix}$	2	1
42	$\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^3$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 2 \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$	$\begin{pmatrix} 1 \\ \bar{1} \\ \bar{1} \\ \bar{1} \end{pmatrix}$	1	0

Moreover, each of these constellations defines non-degenerate toric complete intersections with at most terminal singularities in a fake weighted projective space. In addition, for the divisor class groups, we have $\mathrm{Cl}(X) = \mathrm{Cl}(Z)$ *and the Cox ring of* X *is given by*

$$\begin{aligned} \mathcal{R}(X) &= \mathbb{K}[T_1, \ldots, T_{s+4}]/\langle g_1, \ldots, g_s \rangle, \\ \deg(T_i) &= [D_i] \in \mathrm{Cl}(Z), \\ \deg(g_j) &= [X_j] \in \mathrm{Cl}(Z), \end{aligned}$$

where $\mathbb{K}[T_1, \ldots, T_{s+4}] = \mathcal{R}(Z)$ *is the Cox ring of the fake weighted projective space* Z *and* $g_1, \ldots, g_s \in \mathcal{R}(Z)$ *are the defining* $\mathrm{Cl}(Z)$*-homogeneous polynomials for* $X_1, \ldots, X_s \subseteq Z$*. Moreover,* $T_1, \ldots, T_{s+4}$ *define a minimal system of prime generators for* $\mathcal{R}(X)$*.*

We note some observations around this classification and link to the existing literature.

Remark 1.1.4. The toric terminal Fano complete intersection threefolds in a fake weighted projective space are precisely the three-dimensional terminal fake weighted projective spaces; up to isomorphy, there are eight of them [87].

Recall that the *Fano index* of X is the maximal positive integer q_X such that $\mathcal{K} = q_X D$ with a Weil divisor D on X.

Remark 1.1.5. For the X of Theorem 1.1.3 with $\mathrm{Cl}(Z)$ torsion free, we have $q_X = -\mathcal{K}$, regarding $-\mathcal{K} \in \mathrm{Cl}(Z) = \mathbb{Z}$ as an integer. In the remaining cases, q_X is given by

No.	4	6	9	10	17	22	31	33	34	35	40	41	42
q_X	2	1	3	1	1	1	1	1	1	1	1	1	1

Remark 1.1.6. Embeddings into weighted projective spaces have been intensely studied by several authors. Here is how Theorem 1.1.3 relates to well-known classifications in this case.

(i) Numbers 1, 2, 3, 5, 11, 12, 29, 30 and 39 from Theorem 1.1.3 are smooth and thus appear in the classification of smooth Fano threefolds of Picard number one [84, § 12.2].

(ii) Every variety X from Theorem 1.1.3 with Fano index $q_X = 1$ defined by at most two equations in a weighted projective space Z occurs in [80, Lists 16.6, 16.7].

(iii) The items from [80, Lists 16.6, 16.7] which don't show up in Theorem 1.1.3 are not realizable as general complete intersections in a fake weighted projective space.

Recall that the *Gorenstein index* of a $\mathbb{Q}$-Gorenstein variety X is the minimal positive integer $\imath_X$ such that $\imath_X K_X$ is a Cartier divisor. So, $\imath_X = 1$ means that X is Gorenstein.

Remark 1.1.7. The Gorenstein varieties in Theorem 1.1.3 are precisely the smooth ones. This is a direct application of Corollary 1.4.5 showing that Z_X is the union of all torus orbits of dimension at least three and Proposition 1.4.9 which ensures that X and Z_X have the same Gorenstein index.

Remark 1.1.8. The anticanonical self intersection $-\mathcal{K}^3$ together with the first coefficients of the Hilbert series of X from Theorem 1.1.3 with $\mathrm{Cl}(Z)$ having torsion occur in the Graded Ring Database [5, 29]. Here are the corresponding IDs:

No.	4	6	9	10	17	22	31	33	34	35	40	41	42
ID	40245	23386	41176	2122	3508	1249	32755	4231	5720	237	14885	4733	258

We observe that Numbers 17 and 36 from Theorem 1.1.3 both realise the numerical data from ID 3508 in the Graded Ring Database but the general members of the respective families are non-isomorphic.

Remark 1.1.9. For Numbers 35 and 42 from Theorem 1.1.3 the linear system $|-K_X|$ is empty. In particular these Fano threefolds X do not admit an elephant, that means a member of $|-K_X|$ with at most canonical singularities. There appear to be only few known examples for this phenomenon, compare [80, 16.7] and [120, Sec. 4].

Remark 1.1.10. Numbers 4, 6, 9, 10, 22, 33, 34, 35, 40, 41 and 42 from Theorem 1.1.3 do not show up in any reference known to the authors apart from the candidate list for terminal Fano threefolds provided by the Graded Ring Database.

1.2 Background on toric varieties

In this section, we gather the necessary concepts and results from toric geometry and thereby fix our notation. We briefly touch some of the fundamental definitions but nevertheless assume the reader to be familiar with the foundations of the theory of toric varieties. We refer to [43, 46, 59] as introductory texts.

Our ground field $\mathbb{K}$ is algebraically closed and of characteristic zero. We write $\mathbb{T}^n$ for the standard n-torus, that means the n-fold direct product of the multiplicative group $\mathbb{K}^*$. By a torus we mean an affine algebraic group $\mathbb{T}$ isomorphic to some $\mathbb{T}^n$. A toric variety is a normal algebraic variety Z containing a torus $\mathbb{T}$ as a dense open subset such that the multiplication on $\mathbb{T}$ extends to an action of $\mathbb{T}$ on Z.

Toric varieties are in covariant categorical equivalence with lattice fans. In this context, a lattice is a free $\mathbb{Z}$-module of finite dimension. Moreover, a quasifan (a fan) in a lattice N is a finite collection Σ of (pointed) convex polyhedral cones σ in the rational vector space $N_\mathbb{Q} = \mathbb{Q} \otimes_\mathbb{Z} N$ such that given $\sigma \in \Sigma$, we have $\tau \in \Sigma$ for all faces $\tau \preccurlyeq \sigma$ and for any two $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of both, σ and σ'. The toric variety Z and its acting torus $\mathbb{T}$ associated with a fan Σ in N are constructed as follows:

$$\mathbb{T} \;:=\; \operatorname{Spec} \mathbb{K}[M], \qquad Z \;:=\; \bigcup_{\sigma \in \Sigma} Z_\sigma, \qquad Z_\sigma \;:=\; \operatorname{Spec} \mathbb{K}[\sigma^\vee \cap M],$$

where M is the dual lattice of N and $\sigma^\vee \subseteq M_\mathbb{Q}$ is the dual cone of $\sigma \subseteq N_\mathbb{Q}$. The inclusion $\mathbb{T} \subseteq Z$ of the acting torus is given by the inclusion of semigroup algebras arising from the inclusions $\sigma^\vee \cap M \subseteq M$ of additive semigroups. In practice, we will mostly deal with $N = \mathbb{Z}^n = M$, where $\mathbb{Z}^n$ is identified with its dual via the standard bilinear form $\langle u, v \rangle = u_1 v_1 + \ldots + u_n v_n$. In this setting, we have $N_\mathbb{Q} = \mathbb{Q}^n = M_\mathbb{Q}$. Moreover, given a lattice homomorphism $F\colon N \to N'$, we write as well $F\colon N_\mathbb{Q} \to N'_\mathbb{Q}$ for the associated vector space homomorphism.

We briefly recall Cox's quotient construction $p\colon \hat{Z} \to Z$ of a toric variety Z given by a fan Σ in $\mathbb{Z}^n$ from [44]. We denote by $v_1, \ldots, v_r \in \mathbb{Z}^n$ the primitive generators of Σ, that means the shortest non-zero integral vectors of the rays $\varrho_1, \ldots, \varrho_r \in \Sigma$. We will always assume that $v_1, \ldots, v_r$ span $\mathbb{Q}^n$ as a vector space; geometrically this means that Z has no torus factor. By $D_i \subseteq Z$ we denote the toric prime divisor corresponding to $\varrho_i \in \Sigma$. Throughout the chapter, we will make free use of the notation introduced around Cox's quotient presentation.

Construction 1.2.1. Let Σ be a fan in $\mathbb{Z}^n$ and Z the associated toric variety. Consider the linear map $P\colon \mathbb{Z}^r \to \mathbb{Z}^n$ sending the i-th canonical basis vector $e_i \in \mathbb{Z}^r$ to the i-th primitive generator $v_i \in \mathbb{Z}^n$ of Σ, denote by $\delta = \mathbb{Q}^r_{\geq 0}$ the positive orthant and define a fan $\hat{\Sigma}$ in $\mathbb{Z}^r$ by

$$\hat{\Sigma} \;:=\; \{\delta_0 \preccurlyeq \delta;\; P(\delta_0) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

As $\hat{\Sigma}$ consists of faces of the orthant δ, the toric variety $\hat{Z}$ defined by $\hat{\Sigma}$ is an open $\mathbb{T}^r$-invariant subset of $\bar{Z} = \mathbb{K}^r$. We also regard the linear map $P\colon \mathbb{Z}^r \to \mathbb{Z}^n$ as an $n \times r$ matrix $P = (p_{ij})$ and then speak about the generator matrix of Σ. The generator matrix P defines a homomorphism of tori:

$$p\colon \mathbb{T}^r \;\to\; \mathbb{T}^n, \qquad t \;\mapsto\; (t_1^{p_{11}} \cdots t_r^{p_{1r}}, \ldots, t_1^{p_{n1}} \cdots t_r^{p_{nr}}).$$

This homomorphism extends to a morphism $p\colon \hat{Z} \to Z$ of toric varieties, which in fact is a good quotient for the action of the quasitorus $H = \ker(p)$ on $\hat{Z}$. Let P^* be the transpose of P, set $K := \mathbb{Z}^r/\mathrm{im}(P^*)$ and let $Q\colon \mathbb{Z}^r \to K$ be the projection. Then $\deg(T_i) := Q(e_i) \in K$ defines a K-graded polynomial ring

$$\mathcal{R}(Z) \;:=\; \bigoplus_{w \in K} \mathcal{R}(Z)_w \;:=\; \bigoplus_{w \in K} \mathbb{K}[T_1, \ldots, T_r]_w \;=\; \mathbb{K}[T_1, \ldots, T_r].$$

There is an isomorphism $K \to \mathrm{Cl}(Z)$ from the grading group K onto the divisor class group $\mathrm{Cl}(Z)$ sending $Q(e_i) \in K$ to the class $[D_i] \in \mathrm{Cl}(Z)$ of the toric prime divisor $D_i \subseteq Z$ defined by the ray ϱ_i through v_i. Moreover, the K-graded polynomial ring $\mathcal{R}(Z)$ is the Cox ring of Z; see [4, Sec. 2.1.3].

We now explain the correspondence between effective Weil divisors on a toric variety Z and the K-homogeneous elements in the polynomial ring $\mathcal{R}(Z)$. For any variety X, we denote by $X_{\mathrm{reg}} \subseteq X$ the open subset of its smooth points and by $\mathrm{WDiv}(X)$ its group of Weil divisors. We need the following pull back construction of Weil divisors with respect to morphisms $\varphi\colon X \to Y$: Given a Weil divisor D having $\varphi(X)$ not inside its support, restrict D to a Cartier divisor on Y_{reg}, apply the usual pull back and turn the result into a Weil divisor on X by replacing its prime components with their closures in X.

Definition 1.2.2. Consider a toric variety Z and its quotient presentation $p\colon \hat{Z} \to Z$. A *describing polynomial* of an effective divisor $D \in \mathrm{WDiv}(Z)$ is a K-homogeneous polynomial $g \in \mathcal{R}(Z)$ with $\mathrm{div}(g) = p^*D \in \mathrm{WDiv}(\bar{Z})$.

Example 1.2.3. An effective toric divisor $a_1D_1 + \ldots + a_rD_r$ on Z has the monomial $T_1^{a_1} \cdots T_r^{a_r} \in \mathcal{R}(Z)$ as a describing polynomial. Moreover, in $K = \mathrm{Cl}(X)$, we have

$$\deg(T_1^{a_1} \cdots T_r^{a_r}) \;=\; Q(a_1, \ldots, a_r) \;=\; [a_1D_1 + \ldots + a_rD_r].$$

We list the basic properties of describing polynomials, which in fact hold in the much more general framework of Cox rings; see [4, Prop. 1.6.2.1 and Cor 1.6.4.6].

Proposition 1.2.4. *Let Z be a toric variety with quotient presentation $p\colon \hat{Z} \to Z$ as in Construction 1.2.1 and let D be any effective Weil divisor on Z.*

(i) *There exist describing polynomials for D and any two of them differ by a non-zero scalar factor.*
(ii) *If g is a describing polynomial for D, then, identifying K and $\mathrm{Cl}(Z)$ under the isomorphism presented in Construction 1.2.1, we have*

$$p_*(\mathrm{div}(g)) \;=\; D, \qquad \deg(g) \;=\; [D] \;\in\; \mathrm{Cl}(Z) \;=\; K.$$

(iii) *For every K-homogeneous element $g \in \mathcal{R}(Z)$, the divisor $p_*(\mathrm{div}(g))$ is effective and has g as a describing polynomial.*

Let us see how base points of effective divisors on toric varieties are detected in terms of fans and homogeneous polynomials. Recall that each cone $\sigma \in \Sigma$ defines a distinguished point $z_\sigma \in Z$ and the toric variety Z is the disjoint union over the orbits $\mathbb{T}^n \cdot z_\sigma$, where $\sigma \in \Sigma$.

Proposition 1.2.5. *Let Z be the toric variety arising from a fan Σ in $\mathbb{Z}^n$ and D an effective Weil divisor on Z. Then the base locus of D is $\mathbb{T}^n$-invariant. Moreover, a point $z_\sigma \in Z$ is not a base point of D if and only if D is linearly equivalent to an effective toric divisor $a_1D_1 + \cdots + a_rD_r$ with $a_i = 0$ whenever $v_i \in \sigma$.*

In the later construction and study of non-degenerate subvarieties of toric varieties, we make essential use of the normal fan of a lattice polytope and the correspondence between polytopes and divisors for toric varieties. Let us briefly recall the necessary background and notation.

Reminder 1.2.6. Consider a polytope $B \subseteq \mathbb{Q}^n$. We write $B' \preccurlyeq B$ for the faces of B. One obtains a quasifan $\Sigma(B)$ in $\mathbb{Z}^n$ by

$$\Sigma(B) \ := \ \{\sigma(B');\ B' \preccurlyeq B\}, \qquad \sigma(B') \ := \ \mathrm{cone}(u - u';\ u \in B,\ u' \in B')^\vee,$$

called the *normal fan* of B. The assignment $B' \mapsto \sigma(B')$ sets up an inclusion-reversing bijection between the faces of B and the cones of $\Sigma(B)$.

Note the slight abuse of notation: the normal fan $\Sigma(B)$ is a fan in the strict sense only if the polytope B is of full dimension n, otherwise $\Sigma(B)$ is a quasifan. Given quasifans Σ and Σ' in $\mathbb{Z}^n$, we speak of a *refinement* $\Sigma' \to \Sigma$ if Σ and Σ' have the same support and every cone of Σ' is contained in a cone of Σ.

Reminder 1.2.7. Let $B = B_1 + \cdots + B_s$ be the Minkowski sum of polytopes $B_1, \ldots, B_s \subseteq \mathbb{Q}^n$. Each face $B' \preccurlyeq B$ has a unique presentation

$$B' \ = \ B_1' + \cdots + B_s', \qquad B_1' \preccurlyeq B_1, \ldots, B_s' \preccurlyeq B_s.$$

The normal fan $\Sigma(B)$ of B is the coarsest common refinement of the normal fans $\Sigma(B_i)$ of the B_i. The cones of $\Sigma(B)$ are given as

$$\sigma(B') \ = \ \sigma(B_1') \cap \cdots \cap \sigma(B_s'),$$

where $B' \preccurlyeq B$ and $B' = B_1' + \ldots + B_s'$ is the above decomposition. In particular, $\sigma(B_i') \in \Sigma(B_i)$ is the minimal cone containing $\sigma(B') \in \Sigma(B')$.

Reminder 1.2.8. Let $B \subseteq \mathbb{Q}^n$ be an n-dimensional polytope with integral vertices and let Σ be any complete fan in $\mathbb{Z}^n$ with generator matrix $P = [v_1, \ldots, v_r]$. Define a vector $a \in \mathbb{Z}^r$ by

$$a \ := \ (a_1, \ldots, a_r), \qquad a_i \ := \ -\min_{u \in B} \langle u, v_i \rangle.$$

Observe that the a_i are indeed integers, because B has integral vertices. For $u \in B$ set $a(u) := P^* u + a$ and let $B(u) \preccurlyeq B$ be the minimal face containing u. Then the entries of the vector $a(u) \in \mathbb{Q}^r$ satisfy

$$a(u)_i \geq 0, \text{ for } i = 1, \ldots, r, \qquad a(u)_i = 0 \ \Leftrightarrow \ v_i \in \sigma(B(u)).$$

Proposition 1.2.9. *Let $B \subseteq \mathbb{Q}^n$ be a lattice polytope and Σ any complete fan in $\mathbb{Z}^n$ with generator matrix $P = [v_1, \ldots, v_r]$. With $a \in \mathbb{Z}^r$ from Reminder 1.2.8, we define a divisor on the toric variety Z arising from Σ by*

$$D \ := \ a_1 D_1 + \cdots + a_r D_r \ \in \ \mathrm{WDiv}(Z).$$

Moreover, for every vector $u \in B \cap \mathbb{Z}^n$, we have $a(u) \in \mathbb{Z}^r$ as in Reminder 1.2.8 and obtain effective divisors $D(u)$ on Z, all of the same class as D by

$$D(u) \ := \ a(u)_1 D_1 + \cdots + a(u)_r D_r \ \in \ \mathrm{WDiv}(Z).$$

If Σ refines the normal fan $\Sigma(B)$, then D and all $D(u)$ are base point free. If Σ equals the normal fan $\Sigma(B)$, then the divisors D and $D(u)$ are even ample.

1.3 Laurent systems and their Newton polytopes

We consider systems F of Laurent polynomials in n variables. Any such system F defines a Newton polytope B in $\mathbb{Q}^n$. The objects of interest are completions $X \subseteq Z$ of the zero set $V(F) \subseteq \mathbb{T}^n$ in the toric varieties Z associated with refinements of the normal fan of B. In Proposition 1.3.10, we interpret Khovanskii's non-degeneracy condition [76] in terms of Cox's quotient presentation of Z. Theorem 1.3.12 gathers complete intersection properties of the embedded varieties $X \subseteq Z$ given by non-degenerate systems of Laurent polynomials.

We begin with recalling the basic notions around Laurent polynomials and Newton polytopes. Laurent polynomials are the elements of the Laurent polynomial algebra for which we will use the short notation

$$\mathrm{LP}(n) \ := \ \mathbb{K}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}].$$

Definition 1.3.1. Take any Laurent polynomial $f = \sum_{\nu \in \mathbb{Z}^n} \alpha_\nu T^\nu \in \mathrm{LP}(n)$. The *Newton polytope* of f is

$$B(f) \ := \ \mathrm{conv}(\nu \in \mathbb{Z}^n; \ \alpha_\nu \neq 0) \ \subseteq \ \mathbb{Q}^n.$$

Given a face $B \preccurlyeq B(f)$ of the Newton polytope, the associated *face polynomial* is defined as

$$f_B \ = \ \sum_{\nu \in B \cap \mathbb{Z}^n} \alpha_\nu T^\nu \ \in \ \mathrm{LP}(n).$$

Construction 1.3.2. Consider a Laurent polynomial $f \in \mathrm{LP}(n)$ and a fan Σ in $\mathbb{Z}^n$. The pullback of f with respect to the homomorphism $p\colon \mathbb{T}^r \to \mathbb{T}^n$ defined by the generator matrix $P = (p_{ij})$ of Σ has a unique presentation as

$$p^* f(T_1, \ldots, T_r) \;=\; f(T_1^{p_{11}} \cdots T_r^{p_{1r}}, \ldots, T_1^{p_{n1}} \cdots T_r^{p_{nr}}) \;=\; T^\nu g(T_1, \ldots, T_r)$$

with a Laurent monomial $T^\nu = T_1^{\nu_1} \cdots T_r^{\nu_r} \in \mathrm{LP}(r)$ and a K-homogeneous polynomial $g \in \mathbb{K}[T_1, \ldots, T_r]$ being coprime to each of the variables $T_1, \ldots, T_r$. We call g the *Σ-homogenization* of f.

Lemma 1.3.3. *Consider a Laurent polynomial $f \in \mathrm{LP}(n)$ with Newton polytope $B(f)$ and a fan Σ in $\mathbb{Z}^n$ with generator matrix $P \coloneqq [v_1, \ldots, v_r]$ and associated toric variety Z. Let $a \coloneqq (a_1, \ldots, a_r)$ be as in Reminder 1.2.8 and $D \in \mathrm{WDiv}(Z)$ the push forward of $\mathrm{div}(f) \in \mathrm{WDiv}(\mathbb{T}^n)$.*

(i) *The Σ-homogenization g of f is a describing polynomial of D and with the homomorphism $p\colon \mathbb{T}^r \to \mathbb{T}^n$ given by P, we have*

$$g \;=\; T^a p^* f \;\in\; \mathcal{R}(Z), \qquad T^a \;\coloneqq\; T^{a_1} \cdots T^{a_r}.$$

(ii) *The Newton polytope of g equals the image of the Newton polytope of f under the injection $\mathbb{Q}^n \to \mathbb{Q}^r$ sending u to $a(u) \coloneqq P^* u + a$, in other words*

$$B(g) \;=\; P^* B(f) + a \;=\; \{a(u);\ u \in B(f)\}.$$

(iii) *Consider a face $B \preccurlyeq B(f)$ and the associated face polynomial f_B. Then the corresponding face $P^*B + a \preccurlyeq B(g)$ has the face polynomial*

$$g_{P^*B+a} \;=\; g(\tilde{T}_1, \ldots, \tilde{T}_r), \qquad \tilde{T}_i \;\coloneqq\; \begin{cases} 0 & v_i \in \sigma(B), \\ T_i & v_i \notin \sigma(B). \end{cases}$$

Moreover, for each monomial T^ν of $g - g_{P^*B+a}$ there is a proper face $\sigma \prec \sigma(B)$ such that every variable T_i with $v_i \in \sigma(B) \setminus \sigma$ divides T^ν.

(iv) *The degree $\deg(g) \in K$ of the Σ-homogenization g of f and the divisor class $[D] \in \mathrm{Cl}(Z)$ of $D \in \mathrm{WDiv}(Z)$ are given by*

$$\deg(g) \;=\; Q(a) \;=\; [a_1 D_1 + \ldots + a_r D_r] \;=\; [D].$$

(v) *If Σ is a refinement of the normal fan of $B(f)$, then the divisor $D \in \mathrm{WDiv}(Z)$ is base point free on Z.*

Proof. Assertions (i) to (iii) are direct consequences of Reminder 1.2.8. Assertion (iv) is clear by Proposition 1.2.4 and (v) follows from Proposition 1.2.9. □

Remark 1.3.4. Situation as in Lemma 1.3.3. If $\mathrm{Cl}(Z)$ is torsion-free, then every polynomial g' with $B(g') = B(g) = B_\mu$ is homogeneous of degree $\mu = \deg(g) = Q(a)$. This becomes false when $\mathrm{Cl}(Z)$ is not torsion-free. In this case there can be interior points of B_μ which are not of the form $P^*u + a$ where $u \in B$. Hence one finds a non-homogeneous polynomial having B_μ as its Newton polytope.

Example 1.3.5. Consider $B = \mathrm{conv}((0,1),(0,2),(3,0)) \subseteq \mathbb{Q}^2$. The normal fan Σ of B has the generator matrix P and the associated toric variety is the fake weighted projective plane Z with $\mathrm{Cl}(Z) = \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and degree matrix Q as follows

$$P = \begin{bmatrix} -2 & 1 & 1 \\ -3 & 3 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{2} \end{bmatrix}, \qquad a = (6,-3,0).$$

We compare the lattice polytopes B and $B_\mu = P^*B + a$. First observe

$$\mu = Q(a) = (3, \bar{0}) \in \mathrm{Cl}(Z), \qquad B_\mu = \mathrm{conv}((3,0,0),(0,3,0),(0,0,3)) \subseteq \mathbb{Q}^3.$$

It turns out that B contains four lattice points whereas B_μ contains ten lattice points. For convenience we list them explicitly,

$$\begin{aligned} B \cap \mathbb{Z}^2 &= \{(0,1),(0,2),(3,0),(1,1)\}, \\ B_\mu \cap \mathbb{Z}^3 &= \{(3,0,0),(0,3,0),(0,0,3),(1,1,1),(0,2,1), \\ &\qquad (2,1,0),(1,2,0),(1,0,2),(0,1,2,),(2,0,1)\}. \end{aligned}$$

The first four points listed in $B_\mu \cap \mathbb{Z}^3$ are precisely those stemming from lattice points in B. In other words, these are precisely the exponent vectors of monomials of $\mathrm{Cl}(Z)$-degree $\mu = (3, \bar{0})$. The remaining lattice points in B_μ correspond to monomials having 3 as $\mathbb{Z}$-part of their $\mathrm{Cl}(Z)$-degree as well yet with a torsion component different from $\bar{0}$.

Here are the basic notions around systems of Laurent polynomials; observe that item (iii) is precisely Khovanskii's non-degeneracy condition stated in [76, Sec. 2.1].

Definition 1.3.6. Let $f_1, \ldots, f_s \in \mathrm{LP}(n)$ be Laurent polynomials with Newton polytopes $B_j := B(f_j) \subseteq \mathbb{Q}^n$.

(i) We speak of $F = (f_1, \ldots, f_s)$ as a *system* in $\mathrm{LP}(n)$ and define the Newton polytope of F to be the Minkowski sum

$$B \;:=\; B(F) \;=\; B_1 + \ldots + B_s \;\subseteq\; \mathbb{Q}^n.$$

(ii) The *face system* F' of F associated with a face $B' \preccurlyeq B$ of the Newton polytope is the Laurent system

$$F' \;=\; F_{B'} \;=\; (f'_1, \ldots, f'_s),$$

where $f'_j = f_{B'_j}$ are the face polynomials associated with the faces $B'_j \preccurlyeq B_j$ from the presentation $B' = B'_1 + \ldots + B'_s$.

(iii) We call F *non-degenerate* if for every face $B' \preccurlyeq B$, the differential $\mathcal{D}F'(z)$ is of rank s for all $z \in V(F') \subseteq \mathbb{T}^n$.

(iv) Let Σ be a fan in $\mathbb{Z}^n$. The Σ-*homogenization* of $F = (f_1, \ldots, f_s)$ is the system $G = (g_1, \ldots, g_s)$, where g_j is the Σ-homogenization of f_j.

(v) By an F-*fan* we mean a fan Σ in $\mathbb{Z}^n$ that refines the normal fan $\Sigma(B)$ of the Newton polytope B of F.

Note that Condition 1.3.6 (iii) is fullfilled for suitably general choices of F; see also Section 1.5. Even more, it is a concrete condition in the sense that for every explicitly given Laurent system F, we can explicitly check non-degeneracy.

Remark 1.3.7. Every system in $\mathrm{LP}(n)$ can be turned into a system of polynomials by multiplication with a suitable monomial without affecting non-degeneracy. Moreover, a system $F = (f_1, \ldots, f_s)$ of polynomials is non-degenerate if and only if for every face $B' \preccurlyeq B$ the associated face system $F' = (f'_1, \ldots, f'_s)$ satisfies

$$T_1 \cdots T_n \in \sqrt{\langle f'_1, \ldots, f'_s, m'_1, \ldots, m'_d \rangle}$$

where $m'_1, \ldots, m'_d$ denote the $(s \times s)$-minors of $\mathcal{D}F'$. This condition can be checked by Gröbner basis computations for instance.

Construction 1.3.8. Consider a system $F = (f_1, \ldots, f_s)$ in $\mathrm{LP}(n)$, a fan Σ in $\mathbb{Z}^n$ and the Σ-homogenization G of F. Define subvarieties

$$\bar{X} := V(G) := V(g_1, \ldots, g_s) \subseteq \bar{Z}, \qquad X := \overline{V(f_1)} \cap \ldots \cap \overline{V(f_s)} \subseteq Z,$$

where Z is the toric variety associated with Σ and $\bar{Z} = \mathbb{K}^r$. The quotient presentation $p\colon \hat{Z} \to Z$ gives rise to a commutative diagram

$$\begin{array}{ccc} \hat{X} & \subseteq & \hat{Z} \\ {\scriptstyle /\!\!/ H} \big\downarrow {\scriptstyle p} & & {\scriptstyle p} \big\downarrow {\scriptstyle /\!\!/ H} \\ X & \subseteq & Z \end{array}$$

where $\hat{X} := \bar{X} \cap \hat{Z} \subseteq \hat{Z}$ as well as $X \subseteq Z$ are closed subvarieties and $p\colon \hat{X} \to X$ is a good quotient for the induced H-action on $\hat{X}$. In particular, $X = p(\hat{X})$.

In our study of $\bar{X}$, $\hat{X}$ and X, the decompositions induced from the respective ambient toric orbit decompositions will play an important role. We work with distinguished points $z_\sigma \in Z$. In terms of Cox's quotient presentation, $z_\sigma \in Z$ becomes explicit as $z_\sigma = p(z_{\hat{\sigma}})$, where $\hat{\sigma} = \mathrm{cone}(e_i;\ v_i \in \sigma) \in \hat{\Sigma}$ and the coordinates of the distinguished point $z_{\hat{\sigma}} \in \hat{Z}$ are $z_{\hat{\sigma},i} = 0$ if $v_i \in \sigma$ and $z_{\hat{\sigma},i} = 1$ otherwise.

Construction 1.3.9. Consider a system $F = (f_1, \ldots, f_s)$ in $\mathrm{LP}(n)$, a fan Σ in $\mathbb{Z}^n$ and the Σ-homogenization $G = (g_1, \ldots, g_s)$ of F. For every cone $\sigma \in \Sigma$ define

$$g_j^\sigma \ := \ g_j(T_1^\sigma, \ldots, T_r^\sigma), \qquad T_i^\sigma \ := \ \begin{cases} 0 & v_i \in \sigma, \\ T_i & v_i \notin \sigma. \end{cases}$$

This gives us a system $G^\sigma := (g_1^\sigma, \ldots, g_s^\sigma)$ of polynomials in $\mathbb{K}[T_i;\ v_i \notin \sigma]$. In the coordinate subspace $\bar{Z}(\sigma) = V(T_i;\ v_i \in \sigma)$ of $\mathbb{K}^r$, we have

$$\bar{X}(\sigma) \ := \ \bar{X} \cap \bar{Z}(\sigma) \ = \ V(G^\sigma) \ \subseteq \ \bar{Z}(\sigma).$$

Note that $\bar{Z}(\sigma)$ equals the closure of the toric orbit $\mathbb{T}^r \cdot z_{\hat{\sigma}} \subseteq \mathbb{K}^r$. Consider as well the toric orbit $\mathbb{T}^n \cdot z_\sigma \subseteq Z$ and define locally closed subsets

$$\hat{X}(\sigma) \ \coloneqq \ \hat{X} \cap \mathbb{T}^r \cdot z_{\hat{\sigma}} \ \subseteq \ \hat{X}, \qquad X(\sigma) \ \coloneqq \ X \cap \mathbb{T}^n \cdot z_\sigma \ \subseteq \ X.$$

Then we have $X(\sigma) = p(\hat{X}(\sigma))$ and $X \subseteq Z$ is the disjoint union of the subsets $X(\sigma)$, where $\sigma \in \Sigma$.

The key step for our investigation of varieties $X \subseteq Z$ defined by Laurent systems is to interpret the non-degeneracy condition of a system F in terms of its Σ-homogenization G.

Proposition 1.3.10. *Let $F = (f_1, \ldots, f_s)$ be a non-degenerate system in* $\mathrm{LP}(n)$ *and let Σ be an F-fan in $\mathbb{Z}^n$.*

(i) *The differential $\mathcal{D}G(\hat{z})$ of the Σ-homogenization G of F is of full rank s at every point $\hat{z} \in \hat{X}$.*

(ii) *For each cone $\sigma \in \Sigma$, the differential $\mathcal{D}G^\sigma(\hat{z})$ of the system G^σ is of full rank s at every point $\hat{z} \in \hat{X}(\sigma)$.*

(iii) *For every $\sigma \in \Sigma$, the scheme $\hat{X}(\sigma) \coloneqq \hat{X} \cap \mathbb{T}^r \cdot z_{\hat{\sigma}}$, provided it is non-empty, is a closed subvariety of pure codimension s in $\mathbb{T}^r \cdot z_{\hat{\sigma}}$.*

Proof. We care about (i) and on the way also prove (ii). Since $g_1, \ldots, g_s$ are H-homogeneous, the set of points $\hat{z} \in \hat{Z}$ with $\mathcal{D}G(\hat{z})$ of rank strictly less than s is H-invariant and closed in $\hat{Z}$. Thus, as $p\colon \hat{Z} \to Z$ is a good quotient for the H-action, it suffices to show that for the points $\hat{z} \in \hat{X}$ with a closed H-orbit in $\hat{Z}$, the differential $\mathcal{D}G(\hat{z})$ is of rank s. That means that we only have to deal with the points $\hat{z} \in \hat{X} \cap \mathbb{T}^r \cdot z_{\hat{\sigma}}$, where $\sigma \in \Sigma$.

So, consider a point $\hat{z} \in \hat{X} \cap \mathbb{T}^r \cdot z_{\hat{\sigma}}$, let $\sigma' \in \Sigma(B)$ be the minimal cone with $\sigma \subseteq \sigma'$ and let $B' \preccurlyeq B$ be the face corresponding to $\sigma' \in \Sigma(B)$. Then we have the Minkowski decomposition

$$B' \ = \ B'_1 + \ldots + B'_s, \qquad B'_j \preccurlyeq B_j = B(f_j).$$

From Reminder 1.2.7 we infer that $\sigma'_j = \sigma(B'_j)$ is the minimal cone of the normal fan $\Sigma(B'_j)$ with $\sigma \subseteq \sigma'_j$. Let F' be the face system of F given by $B' \subseteq B$. Define $G' = (g'_1, \ldots, g'_s)$, where g'_j is the face polynomial of g_j defined by

$$P^*B'_j + a_j \ \preccurlyeq \ P^*B_j + a_j \ = \ B(g_j), \qquad g_j \ = \ T^{a_j} p^* f_j.$$

According to Lemma 1.3.3 (iii), the polynomials g'_j only depend on the variables T_i with $v_i \notin \sigma(B'_j)$. Moreover, we have

$$g'_j \ = \ g_j^\sigma, \quad j = 1, \ldots, s,$$

because due to the minimality of $\sigma'_j = \sigma(B'_j)$ each monomial of $g_j - g'_j$ is a multiple of some T_i with $v_i \in \sigma$. Thus, $G' = G^\sigma$. Using the fact that $\hat{z}_i = 0$ if and only if $v_i \in \sigma$, we observe

$$g_j^\sigma(\hat{z}) = g_j(\hat{z}) = 0, \ j = 1, \ldots, s, \qquad \mathrm{rank}\ \mathcal{D}G^\sigma(\hat{z}) = s \ \Rightarrow \ \mathrm{rank}\ \mathcal{D}G(\hat{z}) = s.$$

This reduces the proof of (i) to showing that $\mathcal{D}G^\sigma(\hat{z})$ is of full rank s, and the latter proves (ii). Choose $\tilde{z} \in \mathbb{T}^r$ such that $\tilde{z}_i = \hat{z}_i$ for all i with $v_i \notin \sigma$. Using again that the polynomials g_i' only depend on T_i with $v_i \notin \sigma$, we see

$$g_j^\sigma(\tilde{z}) = g_j^\sigma(\hat{z}) = 0, \; j = 1, \ldots, s, \qquad \mathcal{D}G^\sigma(\hat{z}) = \mathcal{D}G^\sigma(\tilde{z}).$$

We conclude that $F'(p(\tilde{z})) = 0$ holds. Thus, the non-degeneracy condition on the Laurent system F ensures that $\mathcal{D}F'(p(\tilde{z}))$ is of full rank s. Moreover, we have

$$\mathcal{D}G^\sigma(\hat{z}) \;=\; \mathcal{D}G^\sigma(\tilde{z}) \;=\; (T^{a_1}, \ldots, T^{a_s})(\tilde{z}) \cdot \mathcal{D}F'(p(\tilde{z})) \circ \mathcal{D}p(\tilde{z}).$$

Since $T^{a_j}(\tilde{z}) \neq 0$ holds for $j = 1, \ldots, s$ and $p\colon \mathbb{T}^r \to \mathbb{T}^n$ is a submersion, we finally obtain that $\mathcal{D}G^\sigma(\hat{z})$ is of full rank s, which proves (i) and (ii). Assertion (iii) follows from (ii) and the Jacobian criterion for complete intersections. ☐

Remark 1.3.11. Given a system F in $\mathrm{LP}(n)$ and an F-fan Σ in $\mathbb{Z}^n$, let G be the Σ-homogenization of F. The proof of Proposition 1.3.10 shows that the following statements are equivalent:

(i) F is non-degenerate,
(ii) all $\mathcal{D}G^\sigma(\hat{z})$, where $\sigma \in \Sigma$ and $\hat{z} \in \hat{X}(\sigma)$, are of full rank,
(iii) all $\mathcal{D}G^\sigma(\hat{z})$, where $\sigma, \sigma' \in \Sigma$ with $\sigma \preccurlyeq \sigma'$ and $\hat{z} \in \hat{X}(\sigma')$, are of full rank.

A first application gathers complete intersection properties for the varieties defined by a non-degenerate Laurent system. Note that the codimension condition imposed on $\bar{X} \setminus \hat{X}$ in the fourth assertion below allows computational verification for explicitly given systems of Laurent polynomials.

Theorem 1.3.12. *Consider a non-degenerate system $F = (f_1, \ldots, f_s)$ in $\mathrm{LP}(n)$, an F-fan Σ in $\mathbb{Z}^n$ and the Σ-homogenization $G = (g_1, \ldots, g_s)$ of F.*

(i) *The variety $\bar{X} = V(G)$ in $\bar{Z} = \mathbb{K}^r$ is a complete intersection of pure dimension $r - s$ with vanishing ideal*

$$I(\bar{X}) \;=\; \langle g_1, \ldots, g_r \rangle \;\subseteq\; \mathbb{K}[T_1, \ldots, T_r].$$

(ii) *With the zero sets $V(F) \subseteq \mathbb{T}^n$ and $V(G) \subseteq \mathbb{K}^r$ and the notation of Construction 1.3.8, we have*

$$\hat{X} \;=\; \overline{V(G) \cap \mathbb{T}^r} \;\subseteq\; \hat{Z}, \qquad\qquad X \;=\; \overline{V(F)} \;\subseteq\; Z.$$

In particular, the irreducible components of $X \subseteq Z$ are the closures of the irreducible components of $V(F) \subseteq \mathbb{T}^n$.

(iii) *The closed hypersurfaces $X_j = \overline{V(f_j)} \subseteq Z$, where $j = 1, \ldots, s$, represent X as a scheme-theoretic locally complete intersection*

$$X \;=\; X_1 \cap \ldots \cap X_s \;\subseteq\; Z.$$

(iv) *If $\bar{X} \setminus \hat{X}$ is of codimension at least two in $\bar{X}$, then $\bar{X}$ is irreducible and normal and, moreover, X is irreducible.*

Proof. Assertion (i) is clear by Proposition 1.3.10 (i) and the Jacobian criterion for complete intersections. For (ii), we infer from Proposition 1.3.10 (ii) that, provided it is non-empty, the intersection $\hat{X} \cap \mathbb{T}^r \cdot z_{\hat{\sigma}}$ is of dimension $r - s - \dim(\hat{\sigma})$. In particular no irreducible component of $V(G)$ is contained in $\hat{X} \setminus \mathbb{T}^r$. The assertions follow.

We prove (iii). Each f_j defines a divisor on Z having support X_j and according to Lemma 1.3.3 (v) this divisor is base point free on Z. Thus, for every $\sigma \in \Sigma$, we find a monomial $h_{\sigma,j}$ of the same K-degree as g_j without zeroes on the affine chart $\hat{Z}_{\hat{\sigma}} \subseteq \hat{Z}$ defined by $\hat{\sigma} \in \hat{\Sigma}$. We conclude that the invariant functions $g_1/h_{\sigma,1}, \ldots, g_s/h_{\sigma,s}$ generate the vanishing ideal of X on the affine toric chart $Z_\sigma \subseteq Z$.

We turn to (iv). Proposition 1.3.10 and the assumption that $\bar{X} \setminus \hat{X}$ is of codimension at least two in $\bar{X}$ allow us to apply Serre's criterion and we obtain that $\bar{X}$ is normal. In order to see that $\bar{X}$ is irreducible, note that H acts on $\bar{Z}$ with attractive fixed point $0 \in \bar{Z}$. This implies $0 \in \bar{X}$, Hence $\bar{X}$ is connected and thus, by normality, irreducible. □

Remark 1.3.13. If in the setting of Theorem 1.3.12, the dimension of $\bar{Z} \setminus \hat{Z}$ is at most $r - s - 2$, for instance if Z is a fake weighted projective space, then the assumption of Statement (iv) is satisfied.

The statements (i) and (iv) of Theorem 1.3.12 extend in the following way to the pieces cut out from $\bar{X}$ by the closures of the $\mathbb{T}^r$-orbits of $\bar{Z} = \mathbb{K}^r$.

Proposition 1.3.14. *Consider a non-degenerate system $F = (f_1, \ldots, f_s)$ in* LP(n)*, an F-fan Σ in $\mathbb{Z}^n$, the Σ-homogenization $G = (g_1, \ldots, g_s)$ of F, a cone $\sigma \in \Sigma$ and*

$$\bar{Z}(\sigma) \;=\; V(T_i;\ v_i \in \sigma), \qquad \bar{X}(\sigma) \;=\; \bar{X} \cap \bar{Z}(\sigma).$$

If $\bar{X}(\sigma) \setminus \hat{X}(\sigma)$ is of codimension least one in $\bar{X}(\sigma)$, then $\bar{X}(\sigma) = \bar{X} \cap \bar{Z}(\sigma)$ is a subvariety of pure codimension s in $\bar{Z}(\sigma)$ with vanishing ideal

$$I(\bar{X}(\sigma)) \;=\; \langle g_1^\sigma, \ldots, g_s^\sigma \rangle \;\subseteq\; \mathbb{K}[T_i;\ v_i \notin \sigma].$$

If $\bar{X}(\sigma) \setminus \hat{X}$ is of codimension at least two in $\bar{X}(\sigma)$, then the variety $\bar{X}(\sigma)$ is irreducible and normal.

Proof. If $\bar{X}(\sigma) \setminus \hat{X}(\sigma)$ is of codimension least one in $\bar{X}(\sigma)$, then Proposition 1.3.10 and the Jacobian criterion ensure that $\bar{X}(\sigma)$ is a complete intersection in $\mathbb{K}^r$ with the equations $g_j = 0$, $j = 1, \ldots, s$, and $T_i = 0$, $v_i \in \sigma$. This gives the first statement. If $\bar{X}(\sigma) \setminus \hat{X}(\sigma)$ is of codimension at least two in $\bar{X}(\sigma)$, then we obtain irreducibility and normality as in the proof of (iv) of Theorem 1.3.12, replacing $\bar{X}$ with $\bar{X}(\sigma)$. □

1.4 Non-degenerate toric complete intersections

We take a closer look at the geometry of the varieties $X \subseteq Z$ arising from non-degenerate Laurent systems. The main statements of the section are Theorem 1.4.2, showing that $X \subseteq Z$ is always quasismooth and Theorem 1.4.4 giving details on how X sits inside Z.

Using these, we can prove Theorem 1.1.1 which describes the anticanonical complex. First we give a name to our varieties $X \subseteq Z$, motivated by Theorem 1.4.4. Finally, we see that for a general choice of the defining Laurent system and an easy-to-check assumption on the ambient toric variety Z, we obtain divisor class group and Cox ring of X for free, see Corollary 1.4.13.

Definition 1.4.1. By a *non-degenerate toric complete intersection* we mean a variety $X \subseteq Z$ defined by a non-degenerate system F in $\mathrm{LP}(n)$ and an F-fan Σ in $\mathbb{Z}^n$.

An immediate but important property of non-degenerate toric complete intersections is quasismoothness; see also [2] for further results in this direction. The second statement in the theorem below is Khovanskii's resolution of singularities [76, Thm. 2.2]. Observe that our proof works without any ingredients from the theory of holomorphic functions.

Theorem 1.4.2. *Let F be a non-degenerate system in $\mathrm{LP}(n)$ and Σ an F-fan in $\mathbb{Z}^n$. Then the variety X is normal and quasismooth in the sense that $\hat{X}$ is smooth. Moreover, $X \cap Z_{\mathrm{reg}} \subseteq X_{\mathrm{reg}}$. In particular, if Z is smooth, then X is smooth.*

Proof. By Proposition 1.3.10 (i), the variety $\hat{X}$ is smooth. As smooth varieties are normal and the good quotient $p\colon \hat{X} \to X$ preserves normality, we see that X is normal. Moreover, if Z is smooth, then the quasitorus $H = \ker(p)$ acts freely on $p^{-1}(Z_{\mathrm{reg}})$, hence on $\hat{X} \cap p^{-1}(Z_{\mathrm{reg}})$ and thus the quotient map $p\colon \hat{X} \to X$ preserves smoothness over $X \cap Z_{\mathrm{reg}}$. □

The next aim is to provide details on the position of X inside the toric variety Z. The considerations elaborate the transversality statement on X and the torus orbits of Z made in [76] for the smooth case.

Definition 1.4.3. Let Z be the toric variety arising from a fan Σ in $\mathbb{Z}^n$. Given a closed subvariety $X \subseteq Z$, we set

$$\Sigma_X \ := \ \{\sigma \in \Sigma;\ X(\sigma) \neq \emptyset\}, \qquad X(\sigma) \ = \ X \cap \mathbb{T}^n \cdot z_\sigma.$$

Theorem 1.4.4. *Consider a non-degenerate system $F = (f_1, \ldots, f_s)$ in $\mathrm{LP}(n)$, an F-fan Σ in $\mathbb{Z}^n$ and the associated toric complete intersection $X \subseteq Z$.*

(i) *For every $\sigma \in \Sigma_X$, the scheme $X(\sigma) \cap \mathbb{T}^n \cdot z_\sigma$ is a closed subvariety of pure codimension s in $\mathbb{T}^n \cdot z_\sigma$.*
(ii) *The subset $\Sigma_X \subseteq \Sigma$ is a subfan and the subset $Z_X := \mathbb{T}^n \cdot X \subseteq Z$ is an open toric subvariety.*
(iii) *All maximal cones of Σ_X are of dimension $n - s$ and the support of Σ_X equals the tropical variety of $V(F) \subseteq \mathbb{T}^n$.*

Proof. We prove (i). Given a cone $\sigma \in \Sigma_X$ consider $\hat{\sigma} \in \hat{\Sigma}$ and the corresponding affine toric charts and the restricted quotient map:

$$\begin{array}{ccccccc}
\bar{X} \cap \hat{Z}_{\hat{\sigma}} & = & \hat{X}_{\hat{\sigma}} & \subseteq & \hat{Z}_{\hat{\sigma}} & \subseteq & p^{-1}(Z_\sigma) \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\
X \cap Z_\sigma & = & X_\sigma & \subseteq & Z_\sigma & &
\end{array}$$

From Proposition 1.3.10 we infer that $\hat{X}(\hat{\sigma}) = \mathbb{T}^r \cdot z_{\hat{\sigma}} \cap \hat{X}$ is a reduced subscheme of pure codimension s in $\mathbb{T}^r \cdot z_{\hat{\sigma}}$. The involved vanishing ideals on Z_σ and $\hat{Z}_{\hat{\sigma}}$ satisfy

$$I(X_\sigma) + I(\mathbb{T}^n \cdot z_\sigma) \ = \ I(\hat{X}_{\hat{\sigma}})^H + I(\mathbb{T}^r \cdot z_{\hat{\sigma}})^H \ = \ \Big(I(\hat{X}_{\hat{\sigma}}) + I(\mathbb{T}^r \cdot z_{\hat{\sigma}})\Big)^H.$$

We conclude that the left hand side ideal is radical. In order to see that $X(\sigma)$ is of codimension s in $\mathbb{T}^n \cdot z_\sigma$, look at the restriction

$$p\colon \mathbb{T}^r \cdot z_{\hat{\sigma}} \ \to \ \mathbb{T}^n \cdot z_\sigma.$$

This is a geometric quotient for the H-action, it maps $\hat{X}(\hat{\sigma})$ onto $X(\sigma)$ and, as $\hat{X}(\hat{\sigma})$ is H-invariant, it preserves codimensions.

We prove (ii) and (iii). First note that, due to (i), for any $\sigma \in \Sigma_X$ we have $\dim(\sigma) \le n - s$. We compare Σ_X with $\mathrm{trop}(X)$. Tevelev's criterion [126] tells us that a cone $\sigma \in \Sigma$ belongs to Σ_X if and only if $\sigma^\circ \cap \mathrm{trop}(X) \neq \emptyset$ holds. As Σ is complete, we conclude that $\mathrm{trop}(X)$ is covered by the cones of Σ_X.

We show that the support of every cone of Σ_X is contained in $\mathrm{trop}(X)$. The tropical structure theorem provides us with a balanced fan structure Δ on $\mathrm{trop}(X)$ such that all maximal cones are of dimension $n - s$; see [94, Thm. 3.3.6]. Together with Tevelev's criterion, the latter yields that all maximal cones of Σ_X are of dimension $n - s$. The balancy condition implies that every cone $\delta_0 \in \Delta$ of dimension $n - s - 1$ is a facet of at least two maximal cones of Δ. We conclude that every cone $\sigma \in \Sigma_X$ of dimension $n - s$ must be covered by maximal cones of Δ.

Knowing that $\mathrm{trop}(X)$ is precisely the union of the cones of Σ_X, we directly see that Σ_X is a fan: Given $\sigma \in \Sigma_X$, every face $\tau \preccurlyeq \sigma$ is contained in $\mathrm{trop}(X)$. In particular, τ° intersects $\mathrm{trop}(X)$. Using once more Tevelev's criterion, we obtain $\tau \in \Sigma_X$. □

Corollary 1.4.5. *Let $X \subseteq Z$ be a non-degenerate toric complete intersection given by $F = (f_1, \ldots, f_s)$ in $\mathrm{LP}(n)$ and a simplicial F-fan Σ. If $\bar{X} \setminus \hat{X}$ is of dimension strictly less than $r - n$, then we have*

$$\Sigma_X \ = \ \{\sigma \in \Sigma;\ \dim(\sigma) \le n - s\}.$$

Proof. Assume that $\sigma \in \Sigma$ is of dimension $n - s$ but does not belong to Σ_X. Then $X(\sigma) = \emptyset$ and hence $\hat{X}(\sigma) = \emptyset$. This implies

$$\bar{X}(\sigma) \ = \ V(g_1, \ldots, g_s) \cap V(T_i;\ v_i \in \sigma) \ = \ \bigcup_{\hat{\sigma} \prec \tau} \bar{X} \cap \mathbb{T}^r \cdot z_\tau.$$

As Σ is simplicial, P defines a bijection from $\hat{\Sigma}$ onto Σ. Moreover, $\hat{\sigma}$ and σ both have $n - s$ rays and we can estimate the dimension of $\bar{X}(\sigma)$ as

$$\dim(\bar{X}(\sigma)) \ \ge \ r - s - (n - s) \ = \ r - n.$$

Due to $\dim(\bar{X} \setminus \hat{X}) < r - n$, we have $\bar{X} \cap \mathbb{T}^r \cdot z_\tau \subseteq \hat{X}$ for some $\hat{\sigma} \prec \tau \in \hat{\Sigma}$. Thus, σ is a proper face of $P(\tau) \in \Sigma_X$. This contradicts to Theorem 1.4.4 (iii). □

Example 1.4.6. Let $f = S_1 + S_2 + 1 \in \mathbb{K}[S_1, S_2, S_3]$ and Σ the fan in $\mathbb{Z}^3$ given via its generator matrix $P = [v_1, \ldots, v_5]$ and maximal cones $\sigma_{ijk} = \mathrm{cone}(v_i, v_j, v_k)$:

$$P = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix}, \qquad \Sigma^{\max} = \{\sigma_{124}, \sigma_{134}, \sigma_{234}, \sigma_{125}, \sigma_{135}, \sigma_{235}\}.$$

Then f is non-degenerate in $\mathrm{LP}(3)$ and Σ is an F-fan. Thus, we obtain a nondegenerate toric hypersurface $X \in Z$. The Σ-homogenization of f is

$$g = T_1^2 + T_2^2 + T_3^2.$$

The minimal ambient toric variety $Z_X \subseteq Z$ is the open toric subvariety given by the fan Σ_X with the maximal cones $\sigma_{ij} = \mathrm{cone}(v_i, v_j)$ given as follows

$$\Sigma_X^{\max} = \{\sigma_{14}, \sigma_{15}, \sigma_{24}, \sigma_{25}, \sigma_{34}, \sigma_{35}\}.$$

In particular, the fan Σ_X is a proper subset of the set of all cones of dimension at most two of the fan Σ.

Remark 1.4.7. The variety X from Example 1.4.6 is a rational $\mathbb{K}^*$-surface as constructed in [4, Sec. 5.4]. More generally, every weakly tropical general arrangement variety in the sense of [63, Sec. 5] is an example of a non-degenerate complete toric intersection.

We approach the proof of Theorem 1.1.1. The following pull back construction relates divisors of Z to divisors on X.

Remark 1.4.8. Let $X \subseteq Z$ be an irreducible non-degenerate toric complete intersection. Denote by $\imath\colon X \cap Z_{\mathrm{reg}} \to X$ and $\jmath\colon X \cap Z_{\mathrm{reg}} \to Z_{\mathrm{reg}}$ the inclusions. Then Theorems 1.4.2 and 1.4.4 (ii) yield a well defined pull back homomorphism

$$\mathrm{WDiv}^{\mathbb{T}}(Z) = \mathrm{WDiv}^{\mathbb{T}}(Z_{\mathrm{reg}}) \to \mathrm{WDiv}(X), \qquad D \mapsto D|_X := \imath_* \jmath^* D,$$

where we set $\mathbb{T} = \mathbb{T}^n$ for short. By Theorem 1.4.4 (i), this pull back sends any invariant prime divisor on Z to a sum of distinct prime divisors on X. Moreover, we obtain a well defined induced pullback homomorphism for divisor classes

$$\mathrm{Cl}(Z) \to \mathrm{Cl}(X), \qquad [D] \mapsto [D]|_X.$$

The remaining ingredients are the adjunction formula given in Proposition 1.4.9 and Proposition 1.4.10 providing canonical divisors which are suitable for the ramification formula.

Proposition 1.4.9. *Let $X \subseteq Z$ be an irreducible non-degenerate toric complete intersection given by a system $F = (f_1, \ldots, f_s)$ in* $\mathrm{LP}(n)$.

(i) *Let $C_j \in \mathrm{WDiv}(Z)$ be the push forward of $\mathrm{div}(f_j)$ and K_Z an invariant canonical divisor on Z. Then the canonical class of X is given by*

$$[K_X] = [K_Z + C_1 + \ldots + C_s]|_X \in \mathrm{Cl}(X).$$

(ii) *The variety X is $\mathbb{Q}$-Gorenstein if and only if Z_X is $\mathbb{Q}$-Gorenstein. If one of these statements holds, then X and Z_X have the same Gorenstein index.*

Proof. Due to Theorem 1.4.2 and Theorem 1.4.4 (ii) it suffices to have the desired canonical divisor on $Z_{\mathrm{reg}} \cap X \subseteq X_{\mathrm{reg}}$. By Theorem 1.3.12, the classical adjunction formula applies, proving (i). For (ii), note that the divisors C_j on Z are base point free by Lemma 1.3.3 (v) and hence Cartier. The assertions of (ii) follow. □

Proposition 1.4.10. *Consider an irreducible non-degenerate system F in $\mathrm{LP}(n)$, a refinement $\Sigma' \to \Sigma$ of F-fans and the associated modifications $\pi\colon Z' \to Z$ and $\pi\colon X' \to X$. Then, for every $\sigma \in \Sigma_X$, there are canonical divisors $K_X(\sigma)$ on X and $K_{X'}(\sigma)$ on X' such that*

(i) *$K_{X'}(\sigma) = \pi^* K_X(\sigma)$ holds on $X' \setminus Y'$, where $Y' \subseteq Z'$ is the exceptional locus of the toric modification $\pi\colon Z' \to Z$,*
(ii) *$K_{X'}(\sigma) - \pi^* K_X(\sigma) = K_{Z'}|_{X'} - \pi^* K_Z|_{X'}$ holds on $\pi^{-1}(Z_\sigma) \cap X'$, where $Z_\sigma \subseteq Z_X$ is the affine toric chart defined by $\sigma \in \Sigma_X$.*

Proof. Fix $\sigma \in \Sigma_X$. Then there is a vertex $u \in B$ of the Newton polytope $B = B(F)$ such that the maximal cone $\sigma(u) \in \Sigma(B)$ contains σ. Write $u = u_1 + \ldots + u_s$ with vertices $u_j \in B(f_j)$. With the corresponding vertices $a(u_j) = P^* u_j + a_j$ of the Newton polytopes $B(g_j)$, we define

$$D(\sigma, j) \;\coloneqq\; a(u_j)_1 D_1 + \ldots + a(u_j)_r D_r \;\in\; \mathrm{WDiv}(Z).$$

Let $C_j \in \mathrm{WDiv}(Z)$ be the push forward of $\mathrm{div}(f_j)$. Propositions 1.2.5 and 1.2.9 together with Lemma 1.3.3 (v) tell us

$$[D(\sigma, j)] = [C_j] = \deg(g_j) \in K = \mathrm{Cl}(Z), \qquad \mathrm{supp}(D(\sigma, j)) \cap Z_\sigma = \emptyset.$$

Also for the Σ'-homogenization G' of F, the vertices $u_j \in B(f_j)$ yield corresponding vertices $a'(u_j) \in B(g'_j)$ and define divisors

$$D'(\sigma, j) \;\coloneqq\; a'(u_j)_1 D_1 + \ldots + a'(u_j)_{r+l} D_{r+l} \;\in\; \mathrm{WDiv}(Z').$$

As above we have the push forwards $C'_j \in \mathrm{WDiv}(Z')$ of $\mathrm{div}(f_j)$ and, by the same arguments, we obtain

$$[D'(\sigma, j)] = \deg(g'_j) \in K' = \mathrm{Cl}(Z'), \qquad \mathrm{supp}(D'(\sigma, j)) \cap \pi^{-1}(Z_\sigma) = \emptyset.$$

Take the invariant canonical divisors K_Z on Z and $K_{Z'}$ in Z' with multiplicity -1 along all invariant prime divisors and set

$$K_X(\sigma) \;\coloneqq\; (K_Z + \sum_{j=1}^{s} D(\sigma, j))|_X, \qquad K_{X'}(\sigma) \;\coloneqq\; (K_{Z'} + \sum_{j=1}^{s} D'(\sigma, j))|_{X'}.$$

According to Proposition 1.4.9, these are canonical divisors on X and X' respectively. Properties (i) and (ii) are then clear by construction. □

Proof of Theorem 1.1.1. First observe that $\mathcal{A}_X$ is an anticanonical complex for the toric variety Z_X. Now, choose any regular refinement $\Sigma' \to \Sigma$ of the defining F-fan Σ of the irreducible non-degenerate toric complete intersection $X \subseteq Z$. This gives us modifications $\pi\colon Z' \to Z$ and $\pi\colon X' \to X$. Standard toric geometry and Theorem 1.4.2 yield that both are resolutions of singularities.

Proposition 1.4.10 provides us with canonical divisors on X' and X. We use them to compute discrepancies. Over each $X \cap Z_\sigma$, where $\sigma \in \Sigma_X$, we obtain the discrepancy divisor as

$$K_{X'}(\sigma) - \pi^* K_X(\sigma) \ = \ K_{Z'}|_X - \pi^* K_{Z_X}|_X.$$

By Theorem 1.4.4 (i), every exceptional prime divisor $E'_X \subseteq X'$ admits a unique exceptional prime divisor $E'_Z \subseteq Z'$ with $E'_X \subseteq E'_Z$. Remark 1.4.8 guarantees that the discrepancy of E'_X with respect to $\pi\colon X' \to X$ and that of E'_Z with respect to $\pi\colon Z' \to Z_X$ coincide. □

We conclude the section by discussing the divisor class group and the Cox ring of a non-degenerate complete toric intersection and the effect of a general choice of the defining Laurent system.

Proposition 1.4.11. *Consider a non-degenerate system $F = (f_1, \ldots, f_s)$ in* LP(n)*, an F-fan Σ in $\mathbb{Z}^n$ and the associated toric complete intersection $X \subseteq Z$. Assume that $\bar{X} \setminus \hat{X}$ is of codimension at least two in $\bar{X}$. If the pullback* $\mathrm{Cl}(Z) \to \mathrm{Cl}(X)$ *is an isomorphism, then the Cox ring of X is given by*

$$\mathcal{R}(X) \ = \ \mathbb{K}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_s\rangle, \qquad \deg(T_i) \ = \ [X_i] \ \in \ \mathrm{Cl}(X),$$

where $G = (g_1, \ldots, g_s)$ is the Σ-homogenization of F. In this situation, we have moreover the following statements.

(i) *If $\bar{X} \cap V(T_i) \setminus \hat{X}$ is of codimension at least two in $\bar{X} \cap V(T_i)$, then T_i defines a prime element in $\mathcal{R}(X)$.*

(ii) *If $\deg(g_j) \neq \deg(T_i)$ holds for all i, j, then the variables $T_1, \ldots, T_r$ define a minimal system of generators for $\mathcal{R}(X)$.*

Proof. According to Theorem 1.3.12 (iv) ensures that $\bar{X}$ is normal. This allows us to apply [4, Cor. 4.1.1.5], which shows that the Cox ring $\mathcal{R}(X)$ is as claimed. The supplementary assertion (i) is a consequence of Proposition 1.3.14 and (ii) is clear. □

Definition 1.4.12. Let $B_1, \ldots, B_s \subseteq \mathbb{Q}^n$ be integral polytopes. The *Laurent space* associated with $B_1, \ldots, B_s$ is the finite-dimensional vector space

$$V(B_1, \ldots, B_s) \ := \ \bigoplus_{j=1}^{s} \mathbb{K}[T^\nu;\, \nu \in B_j \cap \mathbb{Z}^n].$$

Given a non-empty open set $U \subseteq V(B_1, \ldots, B_s)$, we refer to the elements $F \in U$ and also to the possible associated toric complete intersections as *U-general.*

Following common (ab)use, we say "the general Laurent system $F = (f_1, \dots, f_s)$ in $\mathrm{LP}(n)$ satisfies ..." if we mean "there is a $U \subseteq V(B_1, \dots, B_s)$ such that every U-general F satisfies ...", where B_j denotes the Newton polytope of f_j for $j = 1, \dots, s$. By [76, Thm. 2.2], the general Laurent system is non-degenerate.

Corollary 1.4.13. *Let $F = (f_1, \dots, f_s)$ be a general Laurent system in $\mathrm{LP}(n)$ and Σ an F-fan in $\mathbb{Z}^n$. For the associated toric complete intersection $X = X_1 \cap \dots \cap X_s \subseteq Z$ assume that each X_i, regarded as push-forward of $\mathrm{div}(f_i) \in \mathrm{WDiv}(\mathbb{T}^n)$, is ample for Z and*

$$\dim(\bar{Z} \setminus \hat{Z}) \ \leq \ r - s - 2, \qquad \dim(X) \ \geq \ 3.$$

Then the variety X is irreducible and normal, the pullback $\mathrm{Cl}(Z) \to \mathrm{Cl}(X)$ is an isomorphism and the Cox ring of X is given as

$$\mathcal{R}(X) = \mathbb{K}[T_1, \dots, T_r]/\langle g_1, \dots, g_s \rangle, \qquad \deg(T_i) = [D_i] \ \in \ \mathrm{Cl}(X) = \mathrm{Cl}(Z),$$

where $G = (g_1, \dots, g_s)$ is the Σ-homogenization of $F = (f_1, \dots, f_s)$ and $D_i \subseteq Z$ the toric prime divisor corresponding to $T_i \in \mathcal{R}(Z) = \mathbb{K}[T_1, \dots, T_r]$.

The proof of this Corollary is covered by the subsequent two remarks, which we formulate separately as they touch aspects of independent interest.

Remark 1.4.14. Let $F = (f_1, \dots, f_s)$ be a Laurent system in $\mathrm{LP}(n)$ and Σ an F-fan in $\mathbb{Z}^n$. Then Lemma 1.3.3 (ii) tells us that F is general if and only if its Σ-homogenization G is general.

The second remark shows in particular that the easy-to-check assumption $\dim(\bar{Z} \setminus \hat{Z}) \leq r - s - 2$ might even be weakened and that it suffices to assume that suitable restrictions of the X_i are ample.

Remark 1.4.15. Consider a toric variety Z and a non-degenerate toric complete intersection $X = X_1 \cap \dots \cap X_s$ in Z of dimension at least three and assume that each X_i is ample on Z. Then X is constructed by passing stepwise to hypersurfaces:

$$X'_0 \ \coloneqq \ Z, \qquad X'_j \ \coloneqq \ X'_{j-1} \cap X_j \ \subseteq \ Z, \qquad j = 1, \dots, s.$$

Then $X = X'_s$ and each X'_j is a non-degenerate toric complete intersection in Z. In each step, $X_j|_{X'_{j-1}}$ is a base point free and ample divisor on X'_{j-1}; see Lemma 1.3.3. The Grothendieck-Lefschetz Theorem from [119] provides us with a pullback isomorphism

$$\mathrm{Cl}(X'_{j-1}) \ \to \ \mathrm{Cl}(X'_j)$$

for a general choice of $X_j|_{X'_{j-1}}$ with respect to its linear system. In the initial step, the linear system of X_1 is just the projective space over the corresponding homogeneous component of the Cox ring, that means that we have

$$|X_1| \ = \ \mathbb{P}\mathcal{R}(Z)_{[X_1]}.$$

Now consider a general $X = X_1 \cap \cdots \cap X_s \subseteq Z$ and suppose that $\bar{X}_j' \setminus \hat{X}_j'$ is of codimension at least two in $\bar{X}_j'$ in each step. Then we may apply Proposition 1.4.11 stepwise, where in each step we observe

$$\mathcal{R}(X_{j-1}')_{\left[X_j|_{X_{j-1}'}\right]} \;=\; \mathcal{R}(Z)_{[X_j]}/\langle g_1, \ldots, g_{j-1}\rangle_{[X_j]}.$$

Thus the general choice of $X_1 \cap \cdots \cap X_s \subseteq Z$ induces a general choice of the divisor $X_j|_{X_{j-1}'}$ on X_{j-1}' in each step. In particular, we obtain $\mathrm{Cl}(Z) = \mathrm{Cl}(X)$ and see that the statements of Proposition 1.4.11 apply to $X \subseteq Z$.

Example 1.4.16. Corollary 1.4.13 enables us to produce Mori dream spaces with prescribed properties. For instance, consider general toric hypersurfaces

$$X \;=\; V(f) \;\subseteq\; \mathbb{P}_{1,1,2} \times \mathbb{P}_{1,1,2} \;=\; Z,$$

where f is $\mathbb{Z}^2$-homogeneous of bidegree (d_1, d_2) with $d_1, d_2 \in \mathbb{Z}_{\geq 1}$. Corollary 1.4.13 directly yields $\mathrm{Cl}(X) = \mathbb{Z}^2$ and delivers the Cox ring as

$$\mathcal{R}(X) \;=\; \mathbb{K}[T_0, T_1, T_2, S_0, S_1, S_2]/\langle f\rangle, \qquad \begin{array}{ll} w_0 = w_1 = (1,0), & w_2 = (2,0), \\ u_0 = u_1 = (0,1), & u_2 = (0,2), \end{array}$$

where $w_i = \deg(T_i)$ and $u_i = \deg(S_i)$. Corollary 1.1.2 tells us that X has worst canonical singularities. Moreover, if for instance $d_1 = d_2 = d$, then in the cases

$$d > 4, \qquad d = 4, \qquad d < 4,$$

the Mori dream space X is of general type, satisfies $\mathcal{K}_X = 0$ or is Fano, accordingly; use Proposition 1.4.9.

1.5 The non-degeneracy condition

The purpose of this section is to prove that the non-degenerate Laurent systems with given Newton polytopes form a non-empty open subset of the Laurent space. This involves a proof for non-degeneracy of a general Laurent system as stated by Khovanskii as part of [76, Thm. 2.2] but without the use of complex spaces.

Proposition 1.5.1. *The general Laurent system in* $\mathrm{LP}(n)$ *is non-degenerate.*

Proof. Consider integral polytopes $B_1, \ldots, B_s \subseteq \mathbb{Q}^n$ and let $B_{j1}, \ldots, B_{jm_j}$ be the faces of B_j. Given a choice of vertices $b_1, \ldots, b_s$, where $b_j \in B_j$, we consider the index tuples

$$\kappa = (k_1, \ldots, k_s), \qquad 1 \leq k_j \leq m_j, \qquad b_j \in B_{jk_j}.$$

Write $V = V(B_1, \ldots, B_s)$ for the Laurent space. For $F \in V$, we denote by F_κ the face system of F given by $(B_{1k_1}, \ldots, B_{sk_s})$. Then we have morphisms

$$\Phi_\kappa \colon V \times \mathbb{T}^n \;\to\; V \times \mathbb{K}^s, \qquad (F, z) \;\mapsto\; (F, F_\kappa(z)).$$

For a system $F \in V$ and a vector $y \in \mathbb{K}^s$, the fiber of the morphism Φ_κ over (F, y) is given as

$$\Phi_\kappa^{-1}(F, y) \;=\; \{(F, z);\; z \in \mathbb{T}^n,\; F_\kappa(z) = y\} \;\cong\; F_\kappa^{-1}(y).$$

In particular, the fiber over $0 \in \mathbb{K}^s$ equals the solution set of $F_\kappa = 0$ in the torus $\mathbb{T}^n$. The differential of Φ_κ at any point $(F, z) \in \Phi_\kappa^{-1}(F, 0)$ is of the form

$$\mathcal{D}\Phi_\kappa(F, z) \;=\; \begin{bmatrix} \mathrm{id}_V & 0 \\ * & \mathcal{D}F_\kappa(z) \end{bmatrix}.$$

Let K denote the set of all relevant κ, that means those with $F_\kappa = 0$ having non-empty solution set for the general F. Then, for $\kappa \in K$ and F being general, $\Phi_\kappa^{-1}(F, 0)$ is of dimension $n - s$. Thus, by semicontinuity of fiber dimension, almost all fibers $\Phi_\kappa^{-1}(F, y)$ are of dimension $n - s$ for $\kappa \in K$. Consequently, Φ_κ is dominant whenever $\kappa \in K$.

Assume $b_1 = \ldots = b_s = 0$ for the moment. Then, for all face polynomials $F_\kappa = (f_{1\kappa}, \ldots, f_{s\kappa})$ of the general F, each $f_{j\kappa}$ has a constant term c_j, only depending on F and j. For every $\kappa \in K$, Sard's Theorem [123, Lemma 2.4] yields that the set $Y_\kappa \subseteq V \times \mathbb{K}^s$ of regular values of Φ_κ is non-empty and open in $V \times \mathbb{K}^s$. Set

$$Y \;:=\; \bigcap_{\kappa \in K} Y_\kappa \;\subseteq\; V \times \mathbb{K}^s.$$

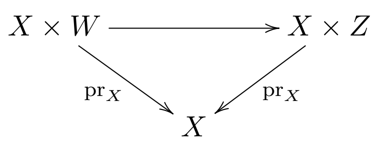
Then for every $y \in Y$, scaling c_j by $1 - y_j/c_j$, we turn any general F into a Laurent system such that $(F, 0)$ is a regular value of Φ_κ for all $\kappa \in K$. We conclude that in the case $b_1 = \ldots = b_s = 0$ all morphisms Φ_κ, where $\kappa \in K$, have $(F, 0)$ as a regular value for the general system $F \in V$.

Now look at arbitrary vertices $b_1, \ldots, b_s$, where $b_j \in B_j$. From the previous consideration, we know that for the general system $F' = (T^{-b_1} f_{1\kappa}, \ldots, T^{-b_s} f_{s\kappa})$, all Φ_κ have $(F', 0)$ as a regular value. Multiplying componentwise with the monomials $T^{-b_1}, \ldots, T^{-b_s}$, we see that also for arbitrary $b_1, \ldots, b_s$, all morphisms Φ_κ have $(F, 0)$ as a regular value for the general system $F \in V$.

Finally, by finiteness of the number of possible choices, we see that for the general $F \in V$, for all choices of $b_1, \ldots, b_s$ and all the associated tuples κ, the morphisms Φ_κ have $(F, 0)$ as a regular value. By the nature of the differential of Φ_κ, we conclude that the general $F \in V$ is non-degenerate. □

Lemma 1.5.2. *Let a reductive algebraic group G act on a prevariety W, and let $p\colon W \to Z$ be a good quotient onto a complete variety Z. Then for any prevariety X and closed subset $A \subseteq X \times W$ the projection $\mathrm{pr}_X(A) \subseteq X$ is closed whenever A is invariant under the G-action on $X \times W$ where G acts trivially on X.*

Proof. The projections onto the first factor fit into the following commutative diagram



where the horizontal arrow is the good quotient for G, acting trivially on X. Since $A \subseteq X \times W$ is invariant under the G-action, the image of A in $X \times Z$ is closed. Since Z is complete, the image $\mathrm{pr}_X(A)$ is closed in X. □

We introduce the relation space as homogeneous analog to the Laurent space. This enables us to use techniques from toric geometry to investigate the Laurent space.

Definition 1.5.3. Let $B_1, \dots, B_s \subseteq \mathbb{Q}^n$ be integral polytopes, $B = B_1 + \cdots + B_s$ the Minkowski sum and Σ a fan in $\mathbb{Z}^n$. The *relation space* associated with $B_1, \dots, B_s$ and Σ is the finite-dimensional vector space

$$V_\Sigma(B_1, \dots, B_s) \coloneqq \bigoplus_{j=1}^{s} \mathbb{K}[T_1, \dots, T_r]_{\mu_j}, \qquad \mu_j = Q(a_j) \in \mathrm{Cl}(Z)$$

where a_j is as in Reminder 1.2.8 for each polytope B_j.

Proposition 1.5.4. *Let $B_1, \dots, B_s \subseteq \mathbb{Q}^n$ be integral polytopes, $B = B_1 + \cdots + B_s$ the Minkowski sum, and Σ a fan in $\mathbb{Z}^n$ refining the normal fan of B. Then we have an isomorphism of vector spaces*

$$\begin{aligned} \Phi\colon V(B_1, \dots, B_s) &\to V_\Sigma(B_1, \dots, B_s), \\ (f_1, \dots, f_s) &\mapsto (T^a p^* f_1, \dots, T^a p^* f_s). \end{aligned}$$

In particular, Σ-homogenization establishes a one-to-one correspondence between Laurent systems $(f_1, \dots, f_s)$ in $\mathrm{LP}(n)$ having $B_1, \dots, B_s$ as Newton polytopes and systems $(g_1, \dots, g_s)$ of homogeneous polynomials with Newton polytopes $B_{\mu_1}, \dots, B_{\mu_s}$.

Proof. Observe that Φ is a well-defined injection since P^* is injective and satisfies $\ker(Q) = P^*(\mathbb{Z}^n)$. As Φ is defined componentwise, it suffices to verify that Φ is surjective in the case $s = 1$, i.e., $B_1 = B$. Let $a = (a_1, \dots, a_r)$ be as in Reminder 1.2.8 and consider the divisorial polytope

$$B(D) = \{u \in \mathbb{Q}^n;\ \langle u, v_i \rangle \geq -a_i\} \subseteq \mathbb{Q}^n$$

associated with the toric divisor $D = a_1 D_1 + \cdots + a_r D_r$ arising from B. Gale duality yields that all monomials of degree $Q(a) = [D]$ stem from a lattice point of $B(D)$, i.e.,

$$Q^{-1}(Q(a)) \cap \mathbb{Z}^r_{\geq 0} = P^*(B(D) \cap \mathbb{Z}^n) + a.$$

Since Σ refines the normal fan of B, the divisorial polytope $B(D)$ equals B. We conclude that each monomial of degree $Q(a)$ is of the form $T^{a+P^*(u)} = T^a p^* T^u$ for some $u \in B \cap \mathbb{Z}^n$, hence Φ is surjective. The supplement is clear by Lemma 1.3.3 (i). □

Remark 1.5.5. The linear map from Proposition 1.5.4 fails to be surjective in general if one drops the assumption that Σ refines the normal fan of B.

For instance, consider the describing fan Σ of $\mathbb{P}_2$ with generator matrix P together with the lattice polytope $B \subseteq \mathbb{Q}^2$,

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \qquad B = \operatorname{conv}(0, e_1, e_2, e_1 + e_2).$$

Here we have $a = (0, 0, 2)$, thus $D = 2D_3$ and $[D] = Q(a) = 2 \in \mathbb{Z} = \operatorname{Cl}(\mathbb{P}_2)$. There are precisely six monomials of degree 2 in T_1, T_2, T_3, which correspond to the lattice points of the divisorial polytope $B(D) = \operatorname{conv}(0, 2e_1, 2e_2)$. However, B is a proper subset of $B(D)$ only having four lattice points. This means that the Laurent space $V(B)$ is fourdimensional but the relation space $V_\Sigma(B)$ is of dimension six.

Lemma 1.5.6. *Consider a simplicial fan Σ refining $\Sigma(B)$. The $G = (g_1, \ldots, g_s) \in V_\Sigma(B_1, \ldots, B_s)$ arising as Σ-homogenization of a non-degenerate system F in* $\operatorname{LP}(n)$ *and satisfying $B(g_j) = B_{\mu_j}$ for all $1 \leq j \leq s$ form an open subset of $V_\Sigma(B_1, \ldots, B_s)$.*

Proof. Throughout the proof we denote $V = V_\Sigma(B_1, \ldots, B_s)$ for short. Observe that $B(g_i) = B_{\mu_j}$ simply means that the monomials T^ν corresponding to the vertices ν of B_{μ_j} occur in g_i with non-zero coefficient. Thus the set U of all $G \in V$ with $B(g_i) = B_{\mu_j}$ for any $1 \leq j \leq s$ is open in V. According to Proposition 1.5.4 each such G appears as Σ-homogenization of a system $(f_1, \ldots, f_s)$ in $\operatorname{LP}(n)$ with $B(f_j) = B_j$ for all $1 \leq j \leq s$. Note that the latter property ensures that Σ is an F-fan. This allows us to apply Remark 1.3.11 and thus check non-degeneracy of F in terms of the Σ-homogenization G.

Next, we show that the $G \in V$ not satisfying condition (iii) from Remark 1.3.11 form a closed subset of V. This implies in particular that those $G \in U$ arising from a non-degenerate F form an open subset of U. Now fix $\sigma \in \Sigma$ and set

$$W_\sigma := \bigcup_{\sigma \preccurlyeq \tau} \mathbb{T}^r \cdot z_{\hat{\tau}} = \overline{\mathbb{T}^r \cdot z_{\hat{\sigma}}} \subseteq \hat{Z}.$$

Then $p(W_\sigma) = \overline{\mathbb{T}^n \cdot z_\sigma} \subseteq Z$, in particular $p(W_\sigma)$ is complete. Moreover W_σ is an H-invariant closed subset of $\hat{Z}$, thus the restriction $p : W_\sigma \to p(W_\sigma)$ is a good quotient for the H-action on W_σ. Now consider the morphism

$$\varphi_\sigma : V \times W_\sigma \to \mathbb{K}^s \times \mathbb{K}^{s \times r}, \quad (G, \hat{z}) \mapsto (G(\hat{z}), \mathcal{D}G^\sigma(\hat{z})).$$

The preimage A_σ of $\{0\} \times M$ where M denotes the set of matrices of rank strictly less than s is a closed H-invariant subset of $V \times W_\sigma$. Lemma 1.5.2 shows that $\operatorname{pr}_V(A_\sigma) \subseteq V$ is closed. We finish the proof by observing that $\operatorname{pr}_V(A_\sigma)$ is precisely the set of $G \in V$ which do not satisfy condition (iii) from Remark 1.3.11 for fixed σ. ☐

Proposition 1.5.7. *Let $B_1, \ldots, B_s \subseteq \mathbb{Q}^n$ be lattice polytopes. The non-degenerate Laurent systems $(f_1, \ldots, f_s)$ in* $\operatorname{LP}(n)$ *with $B(f_j) = B_j$ for $j = 1, \ldots, s$ form a non-empty open subset of the Laurent space $V(B_1, \ldots, B_s)$.*

Proof. Consider the Minkowski sum $B = B_1 + \cdots + B_s$. Then there exists a simplicial fan Σ in $\mathbb{Z}^n$ that refines the normal fan $\Sigma(B)$ of B; see for instance [43, Prop. 11.1.7]. Now Lemma 1.5.6 applies to $B_1, \ldots, B_s$ and Σ. From the correspondence in Proposition 1.5.4 we infer that the non-degenerate Laurent systems $F = (f_1, \ldots, f_s)$ with $B(f_j) = B_j$ for all $1 \leq j \leq s$ form an open subset of U of $V(B_1, \ldots, B_s)$. Finally, Proposition 1.5.1 ensures that U is non-empty. $\square$

1.6 Computing intersection numbers

This section is devoted to a simple algorithm for computing intersection numbers on projective $\mathbb{Q}$-factorial toric varieties in terms of combinatorial data in the rational divisor class group. In particular this enables us to compute intersection numbers on Mori dream spaces that have a complete intersection Cox ring; see Remark 1.6.6.

Setting 1.6.1. Consider an n-dimensional projective $\mathbb{Q}$-factorial toric variety Z with divisor class group K together with an ample class $u \in K_{\mathbb{Q}}$ and the induced K-grading on the polynomial algebra

$$S \coloneqq \mathbb{K}[T_1, \ldots, T_r], \quad \deg(T_i) \coloneqq w_i = [D_i],$$

where $D_1, \ldots, D_r$ are the torus invariant prime divisors of Z. Let $v_1, \ldots, v_t \in K_{\mathbb{Q}}$ be the primitive lattice vectors lying on the rays of the GIT-fan $\Lambda(S)$ and N_i the number of generator degrees w_j lying on the ray $\mathrm{cone}(v_i)$. Our goal is to explicitly compute the intersection number $u_1 \cdots u_n \in \mathbb{Q}$ for given $u_1, \ldots, u_n \in K_{\mathbb{Q}}$.

We state the key formula for computing intersection numbers provided by [4, Prop. 2.4.2.11].

Proposition 1.6.2. *Situation as in Setting 1.6.1. Consider pairwise different generator degrees $w_{i_1}, \ldots, w_{i_n}$ and the complementary degrees $w_{j_1}, \ldots, w_{j_{r-n}}$. Then*

$$w_{i_1} \cdots w_{i_n} = \begin{cases} 1/[K : \langle w_{j_1}, \ldots, w_{r-n} \rangle], & \text{if } u \in \mathrm{cone}(w_{i_1}, \ldots, w_{i_r})^{\circ}, \\ 0 & \text{else }. \end{cases}$$

Remark 1.6.3. In the situation of Setting 1.6.1 we can compute an intersection product of the form $v_1^{l_1} \cdots v_t^{l_t}$ in the following cases.

Case 1: $l_i \leq N_i$ for all $i = 1, \ldots, t$. Here, for each v_i we find pairwise different generator degrees $w_{i_1}, \ldots, w_{i l_i}$ each of them being of the form $w_{i_j} = c_j v_i$ with some $c_j \in \mathbb{Z}$. Successively replacing $v_i^{l_i}$ by $c_1 \cdots c_{l_i} w_{i_1} \cdots w_{i_j}$ in $v_1^{l_1} \cdots v_t^{l_t}$ leads eventually to

$$v_1^{l_1} \cdots v_t^{l_t} = c w_{i_1} \cdots w_{i_n}$$

for some $c \in \mathbb{Z}$ and pairwise different generator degrees $w_{i_1}, \ldots, w_{i_n}$. Explicit computation is now done by applying Proposition 1.6.2.

Case 2: $u \notin \mathrm{cone}(v_i;\ l_i < N_i)$. Here, the according divisors do not meet, hence $v_1^{l_1} \cdots v_t^{l_t}$ vanishes.

Remark 1.6.4. Consider the situation of Setting 1.6.1. Since the toric variety Z defined by u is $\mathbb{Q}$-factorial, i.e., u lives in the relative interior of some full-dimensional GIT-cone, for any subset $I \subseteq \{1, \ldots, t\}$ the cone $\mathrm{cone}(v_i;\ i \in I)$ is full-dimensional whenever it contains u.

In the situation of Setting 1.6.1 the following algorithm computes intersection numbers by gradually reducing the problem to the feasible cases discussed in Remark 1.6.3. To keep a record of this process the algorithm treats $u_1 \cdots u_n$ as a formal polynomial in $v_1, \dots, v_t$ with rational coefficients.

Algorithm 1.6.5 (Computing intersection numbers)**.**

Input: the degree matrix Q, an ample class $u \in K_{\mathbb{Q}}$ for Z, $u_1, \dots, u_n \in K_{\mathbb{Q}}$
Output: the intersection number $u_1 \cdots u_n \in \mathbb{Q}$
foreach $1 \le i \le n$ **do**
  compute $c_{ij} \in \mathbb{Q}$ such that $u_i = c_{i1}v_1 + \dots + c_{it}v_t$ in $K_{\mathbb{Q}}$;
end
set $f \coloneqq \prod_{i=1}^n (c_{i1}v_1 + \ldots + c_{it}v_t) \in \mathbb{Q}[v_1, \dots, v_t]$;
repeat
  foreach *term* $cv_1^{l_1} \cdots v_t^{l_t}$ *of* f **do**
    if $l_i \le N_i$ *for all* $i = 1, \dots, t$ **then**
      compute $d \coloneqq v_1^{l_1} \cdots v_t^{l_t}$ by Remark 1.6.3;
      set $f \coloneqq f - cv_1^{l_1} \cdots v_t^{l_t} + cd$;
    else
      if $u \notin \operatorname{cone}(v_i;\ l_i < N_i)$ **then**
        set $f \coloneqq f - cv_1^{l_1} \cdots v_t^{l_t}$;
      else
        choose some $1 \le j \le t$ with $l_j > N_j$;
        compute $c_i \in \mathbb{Q}$ such that $v_j = \sum_{l_i < N_i} c_i v_i$;
        set

$$f \coloneqq f + c \left(\prod_{i \neq j} v_i^{l_i} \cdot v_j^{l_j - 1} \cdot \left(\sum_{l_i < N_i} c_i v_i \right) - v_1^{l_1} \cdots v_t^{l_t} \right)$$

      end
    end
  end
until f *is constant*;
return f;

Proof. The only step which is not obviously doable is the computation of the rational numbers c_i in the innermost else-branch. Here, Remark 1.6.4 ensures that the demanded presentation of v_j as linear combination over all v_i with $l_i < N_i$ exists.

We show that the algorithm terminates. For any monomial $v^l = v_1^{l_1} \cdots v_t^{l_t}$ of f consider the sum $L(v^l)$ of all exponents l_j with $l_j > N_j$. The maximum of all $L(v^l)$ where v^l runs over all monomials of f is strictly decreased in each step of the repeat structure until it equals zero. Thus, after a finite number of steps all monomials of f satisfy $l_i \le N_i$ for all $i = 1, \dots, t$. In the following step all terms of f will be substituted by rationals numbers, hence f becomes constant and the algorithm terminates.

According to Remark 1.6.3 the output is the desired intersection number. $\square$

Mori dream spaces with complete intersection Cox rings inherit intersection theory from ambient toric varieties thus fit into Setting 1.6.1 when it comes to computing intersection numbers; see [4, Sec. 3.3.3].

Remark 1.6.6. Consider an irreducible $\mathbb{Q}$-factorial projective variety X with finitely generated divisor class group $K = \mathrm{Cl}(X)$ and a complete intersection Cox ring, i.e., there is a graded presentation

$$\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_s \rangle, \quad s = r - \dim(X) - \dim(K_{\mathbb{Q}}),$$

such that $T_1, \ldots, T_r$ define pairwise non-associated K-primes in $\mathcal{R}(X)$. Moreover, let Z be a $\mathbb{Q}$-factorial completion of the canonical ambient toric variety of X given by the above presentation of $\mathcal{R}(X)$ and $u \in K_{\mathbb{Q}}$ and ample class for Z.

For any $u_1, \ldots, u_{\dim(X)} \in K_{\mathbb{Q}}$ the intersection product $u_1 \cdots u_{\dim(X)}$ on X equals $\deg(g_1) \cdots \deg(g_s) \cdot u_1 \cdots u_{\dim(X)}$ on Z. Thus the desired intersection number can be computed by Algorithm 1.6.5 with $n := \dim(X) + s$ and the following input data

$$Q = [\deg(T_1), \ldots, \deg(T_r)], \quad u, \quad u_i = \begin{cases} u_i & 1 \le i \le \dim(X), \\ \deg(g_{i-n}) & \dim(X) + 1 \le i \le n. \end{cases}$$

1.7 Proof of Theorem 1.1.3

Here we prove Theorem 1.1.3. The first and major part uses the whole theory developed in this chapter to establish suitable upper bounds on the specifying data. Having reduced the problem to working out a manageable number of cases, we proceed computationally, which involves besides a huge number of divisibility checks the search for lattice points inside polytopes tracing back to the terminality criterion provided in Corollary 1.1.2. A second and minor part concerns verifying and distinguishing items listed in Theorem 1.1.3, where we succeed with Corollary 1.4.13 and the computation of suitable invariants.

We fix the notation around a non-degenerate complete intersection X in an n-dimensional fake weighted projective space Z. The defining fan of Σ in $\mathbb{Z}^n$ is simplicial, complete and we denote its primitive generators by $v_0, \ldots, v_n$. The divisor class group $\mathrm{Cl}(Z)$ is of the form

$$\mathrm{Cl}(Z) \ = \ \mathbb{Z} \times \mathbb{Z}/t_1\mathbb{Z} \times \cdots \times \mathbb{Z}/t_q\mathbb{Z}.$$

By $w_i = (x_i, \eta_{i1}, \ldots, \eta_{iq}) \in \mathrm{Cl}(Z)$ we denote the classes of the torus invariant prime divisors D_i on Z. Recall that any n of $w_0, \ldots, w_n$ generate $\mathrm{Cl}(Z)$. Moreover, as the $\mathrm{Cl}(Z)$-grading on $\mathcal{R}(Z)$ is pointed, we may assume

$$0 \ < \ x_0 \ \le \ \ldots \ \le \ x_n.$$

As before, $X \subseteq Z$ arises from a Laurent system F in $\mathrm{LP}(n)$ and Σ is an F-fan. We denote by $G = (g_1, \ldots, g_s)$ the Σ-homogenization of $F = (f_1, \ldots, f_s)$. Recall that the $\mathrm{Cl}(Z)$-degree $\mu_j = (u_j, \zeta_{j1}, \ldots, \zeta_{jq})$ of g_j is base point free.

Lemma 1.7.1. *A divisor class* $[D] \in \mathrm{Cl}(Z)$ *is base point free if and only if for any* $i = 0, \ldots, n$ *there exists an* $l_i \in \mathbb{Z}_{\geq 1}$ *with* $[D] = l_i w_i \in \mathrm{Cl}(Z)$.

Proof. This is a direct consequence of Proposition 1.2.5 and the fact that the maximal cones of Σ are given by $\mathrm{cone}(v_j;\ j \neq i)$ for $i = 0, \ldots, n$. □

The following lemma provides effective bounds on the orders $t_1, \ldots, t_q$ of the finite cyclic components of $\mathrm{Cl}(Z)$ in terms of the $\mathbb{Z}$-parts x_i of the generator degrees $w_0, \ldots, w_n$ and u_j of the relation degrees $\mu_1, \ldots, \mu_s$ for any toric complete intersection X in a weighted projective space Z with $x_0 = 1$.

Lemma 1.7.2. *Assume* $x_0 = 1$. *Moreover, let* $\mu = (u, \zeta_1, \ldots, \zeta_q) \in \mathrm{Cl}(Z)$ *be a base point free divisor class. Then, for any* $k = 1, \ldots, q$ *and* $j = 1, \ldots, n$, *we have*

$$t_k \ \mid \ \mathrm{lcm}\left(\frac{u}{x_i};\ i = 1, \ldots, n,\ i \neq j\right).$$

In particular all t_k *divide* u. *Moreover, for the* $\mathbb{Z}$*-parts* u_j *of the relation degrees* μ_j, *we see that each of* $t_1, \ldots, t_q$ *divides* $\gcd(u_1, \ldots, u_s)$.

Proof. Due to $x_0 = 1$, we may assume $\eta_{01} = \ldots = \eta_{0q} = 0$. Lemma 1.7.1 delivers $l_i \in \mathbb{Z}_{\geq 1}$ with $\mu = l_i w_i$. For $i = 0, \ldots, n$ that means

$$(l_0, 0, \ldots, 0) \ = \ l_0 w_0 \ = \ \mu \ = \ l_i w_i \ = \ (l_i x_i, l_i \eta_{i1}, \ldots, l_i \eta_{iq}).$$

Thus, we always have $u = l_i x_i$ and $l_i \eta_{ik} = 0$. Now, fix $1 \leq j \leq n$. As any n of the w_i generate $\mathrm{Cl}(Z)$, we find $\alpha \in \mathbb{Z}^{n+1}$ with $\alpha_j = 0$ and

$$\alpha_0 w_0 + \ldots + \alpha_n w_n \ = \ (1, \bar{1}, \ldots, \bar{1}).$$

Scalar multiplication of both sides with $\mathrm{lcm}(l_i;\ 1 \leq i \leq n,\ i \neq j)$ gives the first claim. The second one is clear. □

The next bounding lemma uses terminality. Given $\sigma \in \Sigma$, let $I(\sigma)$ be the set of indices such that the v_i with $i \in I(\sigma)$ are precisely the primitive ray generators of σ and $u_\sigma \in \mathbb{Q}^n$ a linear form evaluating to -1 on each v_i with $i \in I(\sigma)$. As before, we look at

$$A(\sigma) \ := \ \{v \in \sigma;\ 0 \geq \langle u_\sigma, v \rangle \geq -1\} \ \subseteq \ \sigma.$$

The point $z_\sigma \in Z$ is at most a terminal singularity of Z if and only if 0 and the v_i with $i \in I(\sigma)$ are the only lattice points in $A(\sigma)$. According to Theorem 1.1.1, the analogous statement holds for the points $x \in X$ with $x \in \mathbb{T}^n \cdot z_\sigma$.

Lemma 1.7.3. *Consider $\sigma \in \Sigma$ such that $z_\sigma \in Z$ is at most a terminal singularity of Z.*
(i) *If σ is of dimension two, then σ is a regular cone and $\mathrm{Cl}(Z)$ is generated by the w_i with $i \notin I(\sigma)$. In particular, $\gcd(x_i;\ i \notin I(\sigma)) = 1$ holds.*
(ii) *If σ is of dimension at least two, then $\gcd(x_i;\ i \notin I(\sigma))$ is strictly less than the sum over all x_i with $i \in I(\sigma)$.*

Proof. The first assertion can easily be verified directly. We turn to the second one. Using $x_i \in \mathbb{Z}_{\geq 1}$ and $x_0 v_0 + \cdots + x_n v_n = 0$, we obtain

$$v' \ :=\ -\sum_{i \notin I(\sigma)} x_i v_i \ =\ \sum_{i \in I(\sigma)} x_i v_i \ \in\ \sigma^\circ \cap \mathbb{Z}^n.$$

Write $v' = \gcd(x_i;\ i \notin I(\sigma)) v$ with $v \in \sigma^\circ \cap \mathbb{Z}^n$. Due to $\dim(\sigma) \geq 2$, the vector v does not occur among $v_0, \ldots, v_n$. Evaluating u_σ yields

$$0 \ \geq\ \langle u_\sigma, v \rangle \ =\ \gcd(x_i;\ i \notin I(\sigma))^{-1} \langle u_\sigma, v' \rangle \ =\ -\gcd(x_i;\ i \notin I(\sigma))^{-1} \sum_{i \in I(\sigma)} x_i.$$

By assumption, we have $v \notin A(\sigma)$. Consequently, the right hand side term is strictly less than -1. This gives us the desired estimate. □

We turn to bounds involving the Fano property of a toric complete intersection threefold X in a fake weighted projective space Z. A tuple $\xi = (x_0, \ldots, x_n)$ of positive integers is ordered if $x_0 \leq \ldots \leq x_n$ holds and well-formed if any n of its entries are coprime. For an ordered tuple ξ, we define

$$m(\xi) \ :=\ \mathrm{lcm}(x_0, \ldots, x_n), \qquad\qquad M(\xi) \ :=\ \begin{cases} 2m(\xi), & x_n = m(\xi), \\ m(\xi), & x_n \neq m(\xi). \end{cases}$$

We deal with well-formed ordered tuples $\xi = (x_0, \ldots, x_n)$ with $n \geq 4$. As we will see, the Fano property forces the inequality

$$(n-3)M(\xi) \ <\ x_0 + \cdots + x_n. \tag{1.1}$$

Lemma 1.7.4. *Consider an ordered $\xi = (x_0, \ldots, x_4)$ such that any three of $x_0, \ldots, x_4$ are coprime and condition (1.1) is satisfied. Then $x_4 \leq 41$ holds or we have $1 \leq x_0, x_1, x_2 \leq 2$ and $x_3 = x_4$.*

Proof. We first settle the case $x_4 = m(\xi)$. Then x_4 is divided by each of $x_0, \ldots, x_3$. This implies

$$\gcd(x_i, x_j) \ =\ \gcd(x_i, x_j, x_4) \ =\ 1, \qquad 0 \leq i < j \leq 3.$$

Consequently, $x_0 \cdots x_3$ divides x_4. Subtracting x_4 from both sides of the inequality (1.1) leads to

$$x_0 \cdots x_3 \ \leq\ x_4 \ <\ x_0 + \ldots + x_3.$$

Using $1 \le x_0 \le \cdots \le x_3$ and pairwise coprimeness of $x_0, \dots, x_3$, we conclude that the tuple (x_0, x_1, x_2, x_3) is one of

$$(1,1,2,3), \qquad\qquad (1,1,1,x_3).$$

In the first case, we arrive at $x_4 < x_0 + \cdots + x_3 = 7$. In the second one, $x_4 = dx_3$ holds with $d \in \mathbb{Z}_{\ge 1}$. Observe

$$dx_3 \ = \ x_4 \ < \ x_0 + \cdots + x_3 \ = \ 3 + x_3.$$

Thus, we have to deal with $d = 1, 2, 3$. For $d = 1$, we arrive at $x_3 = x_4$ and the cases $d = 2, 3$ lead to $x_3 \le 2$ which means $x_4 < 5$.

Now we consider the case $x_4 < m(\xi)$. Then $m(\xi) = lx_4$ with $l \in \mathbb{Z}_{\ge 2}$. From inequality (1.1) we infer $l \le 4$ as follows:

$$lx_4 \ = \ m(\xi) \ < \ x_0 + \cdots + x_4 \ \le \ 5x_4.$$

We first treat the case $x_3 = x_4$. Using the assumption that any three of $x_0, \dots, x_4$ are coprime, we obtain

$$\gcd(x_i, x_4) \ = \ \gcd(x_i, x_3, x_4) \ = \ 1, \qquad i = 0, 1, 2.$$

Consequently, $x_2 x_4 \le m(\xi) = lx_4$ and $x_2 \le l \le 4$. For $l = 2$ this means $1 \le x_0, x_1, x_2 \le 2$. For $l = 3, 4$, we use again (1.1) and obtain

$$x_4 \ < \ \frac{1}{l-2}(x_0 + x_1 + x_2) \ \le \ 12.$$

Now we turn to the case $x_3 < x_4$. Set for short $d_i := \gcd(x_i, x_4)$. Then, for all $0 \le i < j \le 3$, we observe

$$\gcd(d_i, d_j) \ = \ \gcd(x_i, x_j, x_4) \ = \ 1.$$

Consequently $d_0 \cdots d_3 \mid x_4$. For $i = 0, \dots, 3$, write $x_i = f_i d_i$ with $f_i \in \mathbb{Z}_{\ge 1}$. Then f_i divides lx_4 and hence l. Fix $i_0, \dots, i_3$ pairwise distinct with $d_{i_0} \le \cdots \le d_{i_3}$. Using (1.1), we obtain

$$(l-1) d_{i_0} \cdots d_{i_3} \ \le \ (l-1) x_4 \ < \ f_{i_0} d_{i_0} + \cdots + f_{i_3} d_{i_3} \ \le \ (2 + 2l) d_{i_3}.$$

For the last estimate, observe that due to $l = 2, 3, 4$, all $f_i \ne 1$ have a common factor 2 or 3. Thus, as any three of $x_0, \dots, x_3$ are coprime, we have $f_i = 1$ for at least two i. We further conclude

$$d_{i_0} d_{i_1} d_{i_2} \ < \ \frac{(2+2l)}{l-1} \ \le \ 6.$$

This implies $d_{i_0} = d_{i_1} = 1$ and $d_{i_2} \le 5$. We discuss the case $f_{i_3} = 1$. There, we have $x_{i_3} = d_{i_3}$, hence $x_{i_3} \mid x_4$. By assumption, $x_0 \le \cdots \le x_3 < x_4$ and thus $x_{i_3} < x_4$. We conclude $d_{i_3} = x_{i_3} \le x_4/2$. From above we infer

$$(l-1)x_4 \ < \ f_{i_0} d_{i_0} + \cdots + f_{i_3} d_{i_3} \ \le \ l(2 + d_{i_2}) + \frac{x_4}{2}.$$

Together with $l = 2, 3, 4$ and $d_{i_2} \leq 5$ as observed before, this enables us to estimate x_4 as follows:

$$x_4 \ < \ 2l\frac{2 + d_{i_2}}{2l - 3} \ \leq \ 28.$$

Now let $f_{i_3} > 1$. Then $2d_{i_3} \leq f_{i_3}d_{i_3} = x_{i_3} < x_4$ holds. This gives $d_{i_3} < x_4/2$. Using $d_{i_3} \mid x_4$ we conclude $d_{i_3} \leq x_4/3$. Similarly as before, we proceed by

$$(l-1)x_4 \ < \ f_{i_0}d_{i_0} + \cdots + f_{i_3}d_{i_3} \ \leq \ l(2 + d_{i_2}) + ld_{i_3} \ \leq \ l(2 + d_{i_2}) + l\frac{x_4}{3}.$$

Again, inserting $l = 2, 3, 4$ and the bound $d_{i_2} \leq 5$ finally leads to the desired estimate

$$x_4 \ < \ 3l\frac{2 + d_{i_2}}{2l - 3} \ \leq \ 42.$$

□

Lemma 1.7.5. *Consider a well-formed ordered $\xi = (x_0, \ldots, x_5)$ satisfying (1.1). Then $x_5 \leq 21$ holds or we have $1 \leq x_0, x_1 \leq 2$ and $x_2 = x_3 = x_4 = x_5$.*

Proof. Let $x_5 \geq 22$. We have $M(\xi) = lx_5$ with $l \geq 2$. From (1.1) we infer $2lx_5 < 6x_5$, hence $l = 2$. Thus, we can reformulate (1.1) as

$$3x_5 \ < \ x_0 + \cdots + x_4.$$

Moreover, $M(\xi) = 2x_5$ implies $a_i x_i = 2x_5$ with suitable $a_i \in \mathbb{Z}_{\geq 2}$ for $i = 0, \ldots, 4$. In particular, the possible values of $x_0, \ldots, x_4$ are given as

$$x_5, \quad \frac{2}{3}x_5, \quad \frac{1}{2}x_5, \quad \frac{2}{5}x_5, \quad \frac{1}{3}x_5, \quad \frac{2}{7}x_5, \quad \ldots .$$

We show $x_4 = x_5$. Suppose $x_4 < x_5$. Then $x_4 \leq 2x_5/3$. We have $x_1 \geq 2x_5/3$, because otherwise $x_1 \leq x_5/2$ and thus

$$3x_5 \ < \ x_0 + \ldots + x_4 \ \leq \ \frac{1}{2}x_5 + \frac{1}{2}x_5 + \frac{2}{3}x_5 + \frac{2}{3}x_5 + \frac{2}{3}x_5 \ = \ 3x_5,$$

a contradiction. We conclude $x_1 = \cdots = x_4 = 2x_5/3$. By well-formedness, the integers $x_1, \ldots, x_5$ are coprime. Combining this with

$$3x_1 \ = \ \cdots \ = \ 3x_4 \ = \ 2x_5$$

yields $x_5 = 3$, contradicting $x_5 \geq 22$. Thus, $x_4 = x_5$, and we can update the previous reformulation of (1.1) as

$$2x_5 \ < \ x_0 + \cdots + x_3.$$

We show $x_3 = x_5$. Suppose $x_3 < x_5$. Then, by the limited stock of possible values for the x_i, the displayed inequality forces $x_3 = 2x_5/3$ and one of the following

$$x_2 \ = \ \frac{2}{3}x_5, \quad x_1 \ = \ \frac{2}{3}x_5, \frac{1}{2}x_5, \ \frac{2}{5}x_5, \qquad\qquad x_2 \ = \ \frac{1}{2}x_5, \quad x_1 \ = \ \frac{1}{2}x_5.$$

By well-formedness, $x_1, \ldots, x_5$ are coprime. Depending on the constellation, this leads to $x_5 = 3, 6, 15$, contradicting $x_5 \geq 22$. Thus, $x_3 = x_5$. Observe

$$x_5 \ < \ x_0 + x_1 + x_2, \qquad \gcd(x_i, x_j, x_5) = 1, \quad 0 \leq i < j \leq 2.$$

We show $x_2 = x_5$ by excluding all values $x_2 < x_5$. First note $x_2 > x_5/3$. Assume $x_2 = 2x_5/5$. Then, by the above inequality, $x_1 = 2x_5/5$. We obtain

$$5x_1 \ = \ 5x_2 \ = \ 2x_5.$$

thus $\gcd(x_1, x_2, x_5) = 1$ implies $x_5 = 5$, a contradiction to $x_5 \geq 22$. Next assume $x_2 = x_5/2$. The inequality leaves us with

$$x_1 \ = \ \frac{1}{2}x_5, \ \frac{2}{5}x_5, \ \frac{1}{3}x_5, \ \frac{2}{7}x_5.$$

Thus, using $\gcd(x_1, x_2, x_5) = 1$ we arrive at $x_5 = 2, 10, 6, 14$ respectively, contradicting $x_5 \geq 22$. Finally, let $x_2 = 2x_5/3$. Then we have to deal with

$$x_1 \ = \ \frac{2}{3}x_5, \ \frac{1}{2}x_5, \ \frac{2}{5}x_5, \ \frac{1}{3}x_5, \ \frac{2}{7}x_5, \ \frac{1}{4}x_5, \ \frac{2}{9}x_5, \ \frac{1}{5}x_5, \ \frac{2}{11}x_5.$$

Using $\gcd(x_1, x_2, x_5) = 1$ gives $x_5 = 3, 6, 15, 3, 21, 12, 9, 15$ in the first eight cases, excluding those. Thus, we are left with the three cases

$$x_2 \ = \ \frac{2}{3}x_5, \qquad x_1 \ = \ \frac{2}{11}x_5, \qquad x_0 \ = \ \frac{2}{11}x_5, \ \frac{1}{6}x_5, \ \frac{2}{13}x_5.$$

In the first one, coprimeness of x_0, x_1, x_5 gives $x_5 = 11$ and in the second one coprimeness of x_0, x_2, x_5 implies $x_5 = 6$. The third case is excluded by

$$\gcd(x_1, x_2, x_5) = 1 \ \Rightarrow \ x_5 = 33, \qquad\qquad \gcd(x_0, x_2, x_5) = 1 \ \Rightarrow \ x_5 = 39.$$

Thus, $x_2 = x_5$. We care about x_0 and x_1. Well-formedness and $x_2 = \ldots = x_5$ yield that x_0, x_5 as well as x_1, x_5 are coprime. Thus, we infer $1 \leq x_0, x_1 \leq 2$ from

$$a_0 x_0 \ = \ 2x_5, \qquad\qquad a_1 x_1 \ = \ 2x_5.$$

□

Lemma 1.7.6. *There exist only two ordered well-formed septuples* $(x_0, \ldots, x_6)$ *satisfying (1.1), namely* $(1, 1, 1, 1, 1, 1, 1)$ *and* $(2, 2, 3, 3, 3, 3, 3)$.

Proof. The case $x_6 = 1$ gives the first tuple. Let $x_6 > 1$. Then $M(\xi) = l x_6$ holds with $l \geq 2$. Using (1.1), we see $3 l x_6 < 7 x_6$ which means $l = 2$. We obtain

$$5x_6 \ < \ x_0 + \cdots + x_5$$

by adapting the inequality (1.1) to the present setting. Similar to the preceding proof, $M(\xi) = 2x_6$ leads to presentations

$$x_i \;=\; \frac{2}{a_i}x_6, \qquad a_i \in \mathbb{Z}_{\geq 2}, \quad i = 0, \ldots, 5.$$

Now, pick the unique j with $x_0 \leq \cdots \leq x_{j-1} < x_j = \cdots = x_6$. Well-formedness implies $j \geq 2$. Moreover $x_{j-1} \leq 2x_6/3$ holds and thus

$$5x_6 \;<\; \frac{2}{3}jx_6 + (6-j)x_6 \;=\; \frac{18-j}{3}x_6.$$

This implies $j < 3$. Thus $j = 2$, which means $x_0 \leq x_1 < x_2 = \cdots = x_6$. Adapting the inequality (1.1) accordingly gives

$$x_6 \;<\; x_0 + x_1.$$

Moreover, by well-formedness, x_0, x_6 as well as x_1, x_6 are coprime. Consequently, we can deduce $1 \leq x_0 \leq x_1 \leq 2$ from

$$a_0x_0 \;=\; 2x_6, \qquad\qquad a_1x_1 \;=\; 2x_6.$$

Now, $x_6 > 1$ excludes $x_1 = 1$. Next, $x_0 = 1$ would force $x_6 = 2 = x_1$, contradicting the choice of j. Thus, we arrive at $x_0 = x_1 = 2$ and $x_2 = \cdots = x_6 = 3$. □

The last tool package for the proof of Theorem 1.1.3 supports the verification of candidates in the sense that it allows us to show that each of the specifying data in the list do indeed stem from a toric complete intersection.

Reminder 1.7.7. Consider any complete toric variety Z arising from a lattice fan Σ in $\mathbb{Z}^n$. With every invariant Weil divisor $C = a_1D_1 + \ldots + a_rD_r$ on Z one associates its *divisorial polytope*

$$B(C) \;=\; \{u \in \mathbb{Q}^n;\ \langle u, v_i\rangle \geq -a_i,\ i = 1, \ldots, r\} \;\subseteq\; \mathbb{Q}^n.$$

If C is base point free, then $B(C)$ has integral vertices and Σ refines the normal fan of $B(C)$. If in addition C is ample, then $B(C)$ is a full-dimensional lattice polytope having Σ as its normal fan.

Given base point free classes $\mu_1, \ldots, \mu_s$ on a toric variety Z, the question is whether or not these are the relation degrees of a (general) toric complete intersection. The following criterion relies on Corollary 1.4.13.

Remark 1.7.8. Consider a complete toric variety Z given by a fan Σ in $\mathbb{Z}^n$ and let $\mu_1, \ldots, \mu_s \in \mathrm{Cl}(Z)$ such that each μ_j admits a base point free representative $C_j \in \mathrm{WDiv}^{\mathbb{T}}(Z)$. Being integral, the $B(C_j)$ can be realized as Newton polytopes:

$$B(C_j) \;=\; B(f_j), \qquad f_j \;\in\; \mathrm{LP}(n), \qquad j \;=\; 1, \ldots, s.$$

Consider the system $F = (f_1, \ldots, f_s)$ in $\mathrm{LP}(n)$. The fan Σ refines the normal fan of the Minkowski sum $B(C_1) + \cdots + B(C_s)$ and hence is an F-fan. For the Σ-homogenization $G = (g_1, \ldots, g_s)$ of F we have $\deg(g_j) = \mu_j \in \mathrm{Cl}(Z)$.

(i) If F is non-degenerate, then the associated variety $X \subseteq Z$ is a toric complete intersection.
(ii) If F is non-degenerate, then there is a non-empty open $U \subseteq V_F$ such that every $G' \in U$ defines a non-degenerate $F' \in \mathrm{LP}(n)$.
(iii) If F is general, $C_1, \ldots, C_s \in \mathrm{WDiv}(Z)$ are ample and we have $\dim(\bar{Z} \setminus \hat{Z}) \leq r-s-2$, then $\mathrm{Cl}(X) = \mathrm{Cl}(Z)$ holds and the Cox ring of X is given as

$$\mathcal{R}(X) \ = \ \mathbb{K}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_s \rangle,$$

where $\deg(T_i) \in \mathrm{Cl}(X)$ is the class $[D_i] \in \mathrm{Cl}(Z)$ of the invariant prime divisor $D_i \subseteq Z$ corresponding to T_i.

Proof of Theorem 1.1.3. Let Z be a fake weighted projective space arising from a fan Σ in $\mathbb{Z}^n$ and let $X = X_1 \cap \cdots \cap X_s \subseteq Z$ be a general (non-degenerate) terminal Fano complete intersection threefold. Write $G = (g_1, \ldots, g_s)$ for the Σ-homogenization of the defining Laurent system $F = (f_1, \ldots, f_s)$ of $X \subseteq Z$. We have

$$\mathrm{Cl}(Z) \ = \ \mathbb{Z} \times \mathbb{Z}/t_1\mathbb{Z} \times \cdots \times \mathbb{Z}/t_q\mathbb{Z}$$

for the divisor class group of Z. As before, the generator degrees $w_i = \deg(T_i)$ and the relation degrees $\mu_j = \deg(g_j)$ in $\mathrm{Cl}(Z)$ are given as

$$w_i \ = \ [D_i] \ = \ (x_i, \eta_{i1}, \ldots, \eta_{iq}), \qquad \mu_j = [X_j] \ = \ (u_j, \zeta_{j1}, \ldots, \zeta_{jq}).$$

We assume that the presentation $X \subseteq Z$ is irredundant in the sense that no g_i has a monomial T_i; otherwise, as the $\mathrm{Cl}(Z)$-grading is pointed, we may write $g_j = T_i + h_j$ with h_j not depending on T_i and, eliminating T_i, we realize X in a smaller fake weighted projective space. Moreover, suitably renumbering, we achieve

$$x_0 \leq \cdots \leq x_n, \qquad\qquad u_1 \leq \cdots \leq u_s.$$

According to the generality condition, we may assume that every monomial of degree μ_j shows up in the relation g_j, where $j = 1, \ldots, s$. In particular, as Lemma 1.7.1 shows $\mu_j = l_{ji} w_i$ with $l_{ji} \in \mathbb{Z}_{\geq 1}$, we see that each power $T_i^{l_{ji}}$ is a monomial of g_j. By irredundance of the presentation, we have $l_{ji} \geq 2$ for all i and j.

We will now establish effective bounds on the w_i and μ_j that finally allow a computational treatment of the remaining cases. The following first constraints are caused by terminality. By Corollary 1.4.5, all two-dimensional cones of Σ belong to Σ_X and by Corollary 1.1.2, the toric orbits corresponding to these cones host at most terminal singularities of Z. Thus, Lemma 1.7.3 (i) tells us that $\mathrm{Cl}(Z)$ is generated by any $n-1$ of $w_0, \ldots, w_n$. In particular, any $n-1$ of $x_0, \ldots, x_n$ are coprime and, choosing suitable generators for $\mathrm{Cl}(Z)$, we achieve

$$\mathrm{Cl}(Z) \ = \ \mathbb{Z} \times \mathbb{Z}/t_1\mathbb{Z} \times \cdots \times \mathbb{Z}/t_q\mathbb{Z}, \qquad q \ \leq \ n-1.$$

Next, we see how the Fano property of X contributes to bounding conditions. Generality and Corollary 1.4.13 ensure that X inherits its divisor class group from the ambient fake weighted projective space Z. Moreover, by Proposition 1.4.9, the anticanonical class $-\mathcal{K}_X$ of X is given in terms of the generator degrees $w_i = \deg(T_i)$, the relation degrees $\mu_j = \deg(g_j)$ and $n = s + 3$ as

$$-\mathcal{K}_X \;=\; w_0 + \cdots + w_n - \mu_1 - \cdots - \mu_s \;\in\; \mathrm{Cl}(Z) \;=\; \mathrm{Cl}(X).$$

Now, consider the tuples $\xi = (x_0, \ldots, x_n)$ and $(u_1, \ldots, u_s)$ of $\mathbb{Z}$-parts of the generator and relation degrees. As seen above, we have $u_j = l_{ji} x_i$ with $l_{ji} \in \mathbb{Z}_{\geq 2}$ for all i and j. Thus, $m(\xi) = \mathrm{lcm}(x_0, \ldots, x_n)$ divides all u_j, in particular $m(\xi) \leq u_j$. Moreover, if $m(\xi) \neq x_n$, then we even have $2m(\xi) \leq u_j$. Altogether, with $M(\xi) \coloneqq 2m(\xi)$ if $m(\xi) = x_n$ and $M(\xi) \coloneqq m(\xi)$ else, we arrive in particular at the inequality (1.1):

$$(n-3)M(\xi) \;=\; sM(\xi) \;\leq\; u_1 + \cdots + u_s \;<\; x_0 + \cdots + x_n.$$

This allows us to conclude that the number s of defining equations for our $X \subseteq Z$ is at most three. Indeed, inserting $2x_n \leq u_j$ and $x_i \leq x_n$, we see that $2sx_n$ is strictly less than $(n+1)x_n = (s+4)x_n$. We go through the cases $s = 1, 2, 3$ and provide upper bounds on the generator degrees $x_0, \ldots, x_n$.

Let $s = 1$. Then $n = 4$. We will show $x_4 \leq 41$. As noted above any three of $x_0, \ldots, x_4$ are coprime. Thus, Lemma 1.7.4 applies, showing that we have $x_4 \leq 41$ or the tuple $(x_0, \ldots, x_4)$ is one of

$$(1, 1, 1, x_4, x_4), \quad (1, 1, 2, x_4, x_4), \quad (1, 2, 2, x_4, x_4).$$

In the latter case, consider $\sigma = \mathrm{cone}(v_0, v_1, v_2) \in \Sigma$. Corollary 1.4.5 ensures $\sigma \in \Sigma_X$. Due to by Corollary 1.1.2, we may apply Lemma 1.7.3 (ii), telling us

$$x_4 \;=\; \gcd(x_3, x_4) \;<\; x_0 + x_1 + x_2 \;\leq\; 5.$$

Let $s = 2$. Then $n = 5$. We will show $x_5 \leq 21$. According to Lemma 1.7.5, we only have to treat the case $x_2 = \ldots = x_5$. As noted above, we have

$$x_5 \;=\; \gcd(x_2, \ldots, x_5) \;=\; 1.$$

Let $s = 3$. Then $n = 6$. Lemma 1.7.6 leaves us with $(x_0, \ldots, x_6)$ being one of the tuples $(1, 1, 1, 1, 1, 1)$ and $(2, 2, 3, 3, 3, 3, 3)$. As before, we can exclude the second configuration.

Next, we perform a computational step. Subject to the bounds just found, we determine all ordered, well formed tuples $\xi = (x_0, \ldots, x_n)$, where $n = s+3$ and $s = 1, 2, 3$, that admit an ordered tuple $(u_1, \ldots, u_s)$ such that

$$u_1 + \cdots + u_s < x_0 + \cdots + x_n, \qquad l_{ji} \coloneqq \frac{u_j}{x_i} \in \mathbb{Z}_{\geq 2}, \; j = 1, \ldots, s, \; i = 0, \ldots, n$$

holds and any $n-1$ of $x_0, \ldots, x_n$ are coprime. This is an elementary computation leaving us with about a hundred tuples $\xi = (x_0, \ldots, x_n)$, each of which satisfies $x_0 = 1$.

As consequence, we can bound the data of the divisor class group $\mathrm{Cl}(Z)$. As noted, we have $q \leq n - 1$ and Lemma 1.7.2 now provides upper bounds on the orders t_k of the finite cyclic factors. This allows us to compute a list of specifying data $(Q, \mu_1, \ldots, \mu_s)$ of candidates for $X \subseteq Z$ by building up degree maps

$$Q\colon \mathbb{Z}^{n+2} \ \to \ \mathrm{Cl}(Z) = \mathbb{Z} \times \mathbb{Z}/t_1\mathbb{Z} \times \cdots \times \mathbb{Z}/t_q\mathbb{Z}, \qquad e_i \ \mapsto \ w_i$$

and pick out those that satisfy the constraints established so far. In a further step, we check the candidates for terminality using the criterion provided Corollary 1.1.2; computationally, this amounts to a search of lattice points in integral polytopes. The affirmatively tested candidates form the list of Theorem 1.1.3. All the computations have been performed with the Magma programs available at [68].

Remark 1.7.8 shows that each specifying data (Q, μ) in the list of Theorem 1.1.3 stems indeed from a general toric complete intersection X in the fake weighted projective space Z. Finally, Corollary 1.4.13 ensures that the Cox ring of all listed X is as claimed. In particular, none of the X is toric. Most of the listed families can be distinguished via the divisor class group $\mathrm{Cl}(X)$, the anticanonical self intersection $-\mathcal{K}_X^3$ and $h^0(-\mathcal{K}_X)$. For Numbers 12 and 39, observe that their Cox rings have non-isomorphic configurations of generator degrees, which also distinguishes the members of these families. □

CHAPTER
TWO

SMOOTH FANO FOURFOLDS OF PICARD NUMBER TWO

In this chapter we classify the smooth Fano fourfolds of Picard number two that have a general hypersurface Cox ring.

This chapter is organized as follows. First, we present our classification results; see Theorem 2.1.1. In the following Sections 2.2 and 2.3 we build up the necessary background on factorially graded algebras and Mori dream spaces for proving our results. Section 2.4 discusses the concept of a general hypersurface Cox ring and provides tools for explicitly constructing examples. Sections 2.5 to 2.9 are then devoted to the classification procedure. Afterwards we investigate different aspects of the varieties from Theorem 2.1.1. Section 2.10 is dedicated to a geometric description in terms of elementary contractions, in Section 2.11 we determine Hodge numbers and, finally, Section 2.12 discusses the connection between automorphisms and deformations of these varieties. The results of this chapter are published in the joint work [66].

2.1 Classification results

By a *Fano variety*, we mean a normal projective complex variety with an ample anticanonical divisor. The Cox ring of a smooth Fano variety X is known to be a finitely generated $\mathbb{C}$-algebra [19]. We restrict our attention to simply structured Cox rings: We say that a variety X with divisor class group $\mathrm{Cl}(X) = K$ has a *hypersurface Cox ring* if we have a K-graded presentation

$$\mathcal{R}(X) \;=\; R_g \;=\; \mathbb{C}[T_1, \ldots, T_r]/\langle g \rangle,$$

where g is homogeneous of degree $\mu \in K$ and $T_1, \ldots, T_r$ define a minimal system of K-homogeneous generators. In this situation, we call $\mathcal{R}(X)$ *spread* if each monomial of degree μ is a convex combination of monomials of g. Moreover, we call $\mathcal{R}(X)$ *general (smooth, Fano)* if g admits an open neighbourhood U in the vector space of all μ-homogeneous polynomials such that every $h \in U$ yields a hypersurface Cox ring R_h of a normal (smooth, Fano) variety X_h with divisor class group K; see also Definition 2.4.5.

We approach our main result, concerning Fano fourfolds of Picard number two. The notation is as follows. For any hypersurface Cox ring $\mathcal{R}(X) = R_g$ graded by $\mathrm{Cl}(X) = K$, we write $w_i = \deg(T_i) \in K$ for the generator degrees and $\mu = \deg(g) \in K$ for the degree of the relation. Moreover, in this setting, the anticanonical class of X is given by

$$-\mathcal{K} \; = \; w_1 + \ldots + w_r - \mu \; \in \; \mathrm{Cl}(X) \; = \; K.$$

If R_g is the Cox ring of a Fano variety X, then X can be reconstructed as the GIT quotient of the set of $(-\mathcal{K})$-semistable points of $\operatorname{Spec} R_g$ by the quasitorus $\operatorname{Spec} \mathbb{C}[K]$. In this setting, we refer to the Cox ring generator degrees $w_1, \ldots, w_r \in K$ and the relation degree $\mu \in K$ as the *specifying data of the Fano variety* X. In the case of a smooth Fano fourfold X of Picard number two, $\mathrm{Cl}(X)$ equals $\mathbb{Z}^2$ and thus $\operatorname{Spec} \mathbb{C}[K]$ is a two-dimensional torus. Hence the hypersurface Cox ring R_g is of dimension six and has seven generators.

Theorem 2.1.1. *The following table lists the specifying data $w_1, \ldots, w_7$ and μ in $\mathrm{Cl}(X) = \mathbb{Z}^2$, the anticanonical class $-\mathcal{K}$ and $\mathcal{K}^4$ for all smooth Fano fourfolds of Picard number two with a spread hypersurface Cox ring.*

No.	$[w_1, \ldots, w_7]$	$\deg(g)$	$-\mathcal{K}$	$\mathcal{K}^4$
1	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$(1,1)$	$(3,2)$	432
2		$(2,1)$	$(2,2)$	256
3		$(3,1)$	$(1,2)$	80
4		$(1,2)$	$(3,1)$	270
5		$(2,2)$	$(2,1)$	112
6		$(3,2)$	$(1,1)$	26
7	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$(1,1)$	$(2,2)$	416
8		$(1,2)$	$(2,1)$	163
9		$(2,1)$	$(1,2)$	224
10		$(2,2)$	$(1,1)$	52
11	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$(1,1)$	$(1,2)$	464
12		$(1,2)$	$(1,1)$	98
13	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(1,2)$	$(3,2)$	352
14		$(2,3)$	$(2,1)$	65
15	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(1,3)$	$(2,1)$	83
16	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$(2,1)$	$(3,2)$	352
17		$(3,2)$	$(2,1)$	81
18	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$(3,1)$	$(1,1)$	38
19		$(2,1)$	$(2,1)$	192
20		$(1,1)$	$(3,1)$	432
21	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$	$(3,1)$	$(2,1)$	113
22	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,2)$	$(2,3)$	272
23		$(3,3)$	$(1,2)$	51

No.	$[w_1, \ldots, w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$
24	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 1 \end{bmatrix}$	$(4,4)$	$(1,2)$	34
25	$\begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 1 & 1 \end{bmatrix}$	$(6,6)$	$(1,2)$	17
26	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,2)$	$(1,3)$	216
27	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,4)$	$(1,2)$	64
28	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,6)$	$(1,1)$	8
29	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,2)$	$(2,2)$	192
30		$(3,3)$	$(1,1)$	18
31	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(4,2)$	$(1,2)$	48
32	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 & 1 \end{bmatrix}$	$(4,4)$	$(1,1)$	12
33	$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 1 & 1 \end{bmatrix}$	$(4,6)$	$(1,3)$	50
34	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,2)$	$(3,4)$	378
35		$(3,3)$	$(2,3)$	144
36		$(4,4)$	$(1,2)$	20
37	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}$	$(4,4)$	$(2,3)$	96
38	$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(6,6)$	$(1,2)$	10
39	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 2 & 3 & 1 & 1 \end{bmatrix}$	$(6,6)$	$(2,3)$	48

No.	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$
40 41	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&1&1&1&1&1&1 \end{bmatrix}$	$(2,2)$ $(3,3)$	$(2,4)$ $(1,3)$	352 99
42 43	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&2&2&2&1&1&1 \end{bmatrix}$	$(2,4)$ $(3,6)$	$(2,5)$ $(1,3)$	304 54
44	$\begin{bmatrix} 1&1&1&2&0&0&0 \\ 0&1&1&2&1&1&1 \end{bmatrix}$	$(4,4)$	$(1,3)$	66
45	$\begin{bmatrix} 1&1&1&2&0&0&0 \\ 0&2&2&4&1&1&1 \end{bmatrix}$	$(4,8)$	$(1,3)$	36
46	$\begin{bmatrix} 1&1&2&3&0&0&0 \\ 0&1&2&3&1&1&1 \end{bmatrix}$	$(6,6)$	$(1,3)$	33
47	$\begin{bmatrix} 1&1&2&3&0&0&0 \\ 0&2&4&6&1&1&1 \end{bmatrix}$	$(6,12)$	$(1,3)$	18
48	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&1&1&1&2&1&1 \end{bmatrix}$	$(2,2)$	$(3,5)$	433
49	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&2&2&2&3&1&1 \end{bmatrix}$	$(3,6)$	$(2,5)$	145
50	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&1&1&2&1&1&1 \end{bmatrix}$	$(2,4)$	$(2,3)$	144
51	$\begin{bmatrix} 1&1&1&2&0&0&0 \\ 0&1&1&3&1&1&1 \end{bmatrix}$	$(4,6)$	$(1,2)$	22
52	$\begin{bmatrix} 1&1&1&2&1&0&0 \\ 0&1&1&3&2&1&1 \end{bmatrix}$	$(4,6)$	$(2,3)$	65
53 54	$\begin{bmatrix} 1&1&1&1&1&1&0 \\ -1&0&0&0&0&1&1 \end{bmatrix}$	$(2,0)$ $(4,0)$	$(4,1)$ $(2,1)$	431 62

No.	$[w_1, \dots, w_7]$	μ	$-\mathcal{K}$	$\mathcal{K}^4$
55	$\begin{bmatrix} 1&1&1&1&1&2&0 \\ -1&0&0&0&0&1&1 \end{bmatrix}$	$(3,0)$	$(4,1)$	376
56	$\begin{bmatrix} 1&1&1&1&1&3&0 \\ -1&0&0&0&0&1&1 \end{bmatrix}$	$(4,0)$	$(4,1)$	341
57	$\begin{bmatrix} 1&1&1&1&3&1&0 \\ -1&0&0&0&0&1&1 \end{bmatrix}$	$(6,0)$	$(2,1)$	31
58	$\begin{bmatrix} 1&1&1&1&3&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(6,0)$	$(1,2)$	16
59	$\begin{bmatrix} 1&1&1&2&3&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(6,0)$	$(2,2)$	64
60	$\begin{bmatrix} 1&1&1&2&3&1&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(6,0)$	$(3,2)$	80
61	$\begin{bmatrix} 1&1&1&1&2&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(4,0)$	$(2,2)$	128
62	$\begin{bmatrix} 1&1&1&1&2&1&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(4,0)$	$(3,2)$	160
63	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(3,0)$	$(2,2)$	192
64	$\begin{bmatrix} 1&1&1&1&1&1&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(3,0)$	$(3,2)$	240
65	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(2,0)$	$(3,2)$	432
66	$\begin{bmatrix} 1&1&1&1&1&1&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(2,0)$	$(4,2)$	480
67	$\begin{bmatrix} 1&1&1&1&1&2&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$	$(2,0)$	$(5,2)$	624

Any two smooth Fano fourfolds of Picard number two with specifying data from distinct items of the table are not isomorphic to each other. Moreover, each of the items 1 to 67 even defines a general smooth Fano hypersurface Cox ring and thus provides the specifying data for a whole family of smooth Fano fourfolds.

Let us compare the result with existing classifications. Wiśniewski classified in [130] the smooth Fano fourfolds of Picard number and Fano index at least two, where the Fano index is the largest integer ι such that $-\mathcal{K} = \iota H$ holds with an ample divisor H.

Remark 2.1.2. In eight cases, the families listed in Theorem 2.1.1 consist of varieties of Fano index two and in all other cases, the varieties are of Fano index one. The conversion between Theorem 2.1.1 and Wiśniewski's results as presented in the table [84, 12.7] is as follows:

Thm. 2.1.1	2	7	29	40	59	61	63	66
[84, 12.7]	5	12	4	10	1	2	3	13

Theorem 2.1.1 has no overlap with Batyrev's classification [11] of smooth toric Fano fourfolds. Indeed, toric varieties have polynomial rings as Cox rings which are by definition no hypersurface Cox rings. However, there is some interaction with the case of torus actions of complexity one.

Remark 2.1.3. Eleven of the families of Theorem 2.1.1 admit small degenerations to smooth Fano fourfolds with an effective action of a three-dimensional torus. Here are these families and the corresponding varieties from [56, Thm. 1.2].

Thm. 2.1.1	[56, Thm. 1.2]
1	4.A: $m = 1$, $c = 0$
4	4.C: $m = 1$
7	2
13	5: $m = 1$
20	4.A: $m = 1$, $c = -1$
34	1
48	10: $m = 2$
53	7: $m = 1$
65	12: $m = 2$, $a = b = c = 0$
66	11: $m = 2$, $a_2 = 1$
67	11: $m = 2$, $a_2 = 2$

Moreover, observe that for the families 1, 20, 48, 53, 65, 66 and 67 of Theorem 2.1.1 the degeneration process gives a Fano smooth intrinsic quadric; compare [55, Thm. 1.3].

Remark 2.1.4. Coates, Kasprzyk and Prince classified in [41] the smooth Fano fourfolds that arise as complete intersections of ample divisors in smooth toric Fano varieties of dimension at most eight. Comparing anticanonical self-intersection numbers as well as the first six coefficients of the Hilbert series yields that at least the 17 families 14, 15, 24, 25, 28, 30, 32, 33, 38, 44, 45, 46, 47, 51, 52, 57 and 58 of Theorem 2.1.1 do not show up in [41].

2.2 Factorial gradings

Here we provide the first part of the algebraic and combinatorial tools used in our classification. We recall the basic concepts on factorially graded algebras and, as a new result, prove Proposition 2.2.4, locating the relation degrees of a factorially graded complete intersection algebra. Moreover, we recall and discuss the GIT-fan of the quasitorus action associated with a graded affine algebra.

For the moment, $\mathbb{K}$ is any field. Let R be a *K-graded algebra*, which, in this chapter, means that K is a finitely generated abelian group and R is a $\mathbb{K}$-algebra coming with a direct sum decomposition into $\mathbb{K}$-vector subspaces

$$R \;=\; \bigoplus_{w \in K} R_w$$

such that $R_w R_{w'} \subseteq R_{w+w'}$ holds for all $w, w' \in R$. An element $f \in R$ is *homogeneous* if $f \in R_w$ holds for some $w \in K$; in that case, w is the *degree* of f, written $w = \deg(f)$. We say that R is *K-integral* if it has no homogeneous zero divisors.

Consider the rational vector space $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$ associated with K. The *effective cone* of R is the convex cone generated by all degrees admitting a non-zero homogeneous element:

$$\mathrm{Eff}(R) \ := \ \mathrm{cone}(w \in K;\ R_w \neq 0) \ \subseteq \ K_{\mathbb{Q}}.$$

The K-grading of R is called *pointed* if $R_0 = \mathbb{K}$ holds and the effective cone $\mathrm{Eff}(R)$ contains no line. Note that $\mathrm{Eff}(R)$ is polyhedral, if the $\mathbb{K}$-algebra R is finitely generated.

Lemma 2.2.1. *Let R be a K-graded algebra. Assume that R is K-integral and every homogeneous unit of R is of degree zero.*

(i) *If $R_0 = \mathbb{K}$ holds, then the K-grading is pointed and for every non-zero torsion element $w \in K$, we have $R_w = 0$.*

(ii) *The K-grading is pointed if and only if there is a homomorphism $\kappa\colon K \to \mathbb{Z}$ defining a pointed $\mathbb{Z}$-grading with effective cone $\mathbb{Q}_{\geq 0}$.*

Proof. We prove (i). It suffices to show that there is no non-zero $w \in K$ with $R_w \neq 0$ and $R_{-w} \neq 0$. Consider $f \in R_w$ and $f' \in R_{-w}$, both being non-zero. Then ff' is a non-zero element of R_0 and hence constant. Thus, f and f' are both units. By assumption, we have $w = 0$.

We prove (ii). If the K-grading is pointed, then we find a hyperplane $U \subseteq K_{\mathbb{Q}}$ intersecting $\mathrm{Eff}(X)$ precisely in the origin. Let $K_U \subseteq K$ be the subgroup consisting of all elements $w \in K$ with $w \otimes 1 \in U$. Then $K/K_U \cong \mathbb{Z}$ holds and we may assume that the projection $\kappa\colon K \to \mathbb{Z}$ sends the effective cone to the positive ray. Using (i), we see that for the induced $\mathbb{Z}$-grading all homogeneous elements of degree zero are constant. The reverse implication is clear according to (i). $\square$

Let R be a K-integral algebra. A homogeneous non-zero non-unit $f \in R$ is *K-irreducible*, if admits no decomposition $f = f'f''$ with homogeneous non-zero non-units $f', f'' \in R$. A homogeneous non-zero non-unit $f \in R$ is *K-prime*, if for any two homogeneous $f', f'' \in R$ we have that $f \mid f'f''$ implies $f \mid f'$ or $f \mid f''$. Every K-prime element is K-irreducible. The algebra R is called *K-factorial*, or the K-grading just *factorial*, if R is K-integral and every homogeneous non-zero non-unit is a product of K-primes. In a K-factorial algebra, the K-prime elements are exactly the K-irreducible ones.

An ideal $\mathfrak{a} \subseteq R$ is *homogeneous* if it is generated by homogeneous elements. Moreover, an ideal $\mathfrak{a} \subseteq R$ is *K-prime* if for any two homogeneous $f, f' \in R$ we have that $ff' \in \mathfrak{a}$ implies $f \in \mathfrak{a}$ or $f' \in \mathfrak{a}$. A homogeneous ideal $\mathfrak{a} \subseteq R$ is K-prime if and only if $R/\mathfrak{a}$ is K-integral. We say that homogeneous elements $g_1, \ldots, g_s \in R$ *minimally generate* the K-homogeneous ideal $\mathfrak{a} \subseteq R$ if they generate $\mathfrak{a}$ and no proper subcollection of $g_1, \ldots, g_s$ does so.

Lemma 2.2.2. *Let R be a K-graded algebra such that the grading is pointed, factorial and every homogeneous unit is of degree zero. If $g_1, \ldots, g_s \in R$ minimally generate a K-prime ideal of R, then each g_i is a K-prime element of R.*

Proof. Assume that g_1 is not K-prime. Then g_1 is not K-irreducible and we can write $g_1 = g_1'g_1''$ with homogeneous non-zero non-units $g_1', g_1'' \in R$. As the ideal $\langle g_1, \ldots, g_s \rangle \subseteq R$

is K-prime, it contains one of g_1' and g_1'', say g_1'. That means that

$$g_1' \;=\; h_1 g_1 + \ldots + h_s g_s$$

holds with homogeneous elements $h_i \in R$. Take a coarsening $K \to \mathbb{Z}$ of the K-grading as provided by Lemma 2.2.1 (ii). Then the above representation of g_1' yields

$$\deg_{\mathbb{Z}}(g_1') \;=\; \deg_{\mathbb{Z}}(h_1) + \deg_{\mathbb{Z}}(g_1) \;=\; \ldots \;=\; \deg_{\mathbb{Z}}(h_s) + \deg_{\mathbb{Z}}(g_s).$$

Consequently, $\deg_{\mathbb{Z}}(g_1') \geq \deg_{\mathbb{Z}}(g_1)$ or $h_1 = 0$. Since the $\mathbb{Z}$-grading of R is pointed, we have $\deg_{\mathbb{Z}}(g_1') < \deg_{\mathbb{Z}}(g_1') + \deg_{\mathbb{Z}}(g_1'') = \deg_{\mathbb{Z}}(g_1)$. Thus, $h_1 = 0$ holds. This implies $g_1 = g_1' g_1'' \in \langle g_2, \ldots, g_s \rangle$. A contradiction. □

Given a finitely generated abelian group K and $w_1, \ldots, w_r \in K$, there is a unique K-grading on the polynomial algebra $\mathbb{K}[T_1, \ldots, T_r]$ satisfying $\deg(T_i) = w_i$ for $i = 1, \ldots, r$. We call such grading a *linear* grading of $\mathbb{K}[T_1, \ldots, T_r]$.

Lemma 2.2.3. *Consider a linear K-grading on $\mathbb{K}[T_1, \ldots, T_r]$ and a K-homogeneous $g \in \mathbb{K}[T_1, \ldots, T_r]$. Moreover, let $1 \leq i_1, \ldots, i_q \leq r$ be pairwise distinct. Assume that T_{i_1} is not a monomial of g and that $g, T_{i_2}, \ldots, T_{i_q}$ minimally generate a K-prime ideal in $\mathbb{K}[T_1, \ldots, T_r]$. Then we have a presentation*

$$\deg(g) \;=\; \sum a_j \deg(T_j), \quad j \neq i_1, \ldots, i_q, \; a_j \in \mathbb{Z}_{\geq 0}.$$

Proof. Suppose that $\deg(g)$ allows no representation as a positive combination over the $\deg(T_j)$ with $j \notin \{i_1, \ldots, i_q\}$. Then each monomial of g must have a factor T_{i_j} for some $j = 1, \ldots, q$. Write

$$g \;=\; g_1 T_{i_1} + g_2 T_{i_2} + \ldots + g_q T_{i_q} \;=\; g_1 T_{i_1} + h$$

with polynomials $g_j \in \mathbb{K}[T_1, \ldots, T_r]$ such that g_1 depends on none of $T_{i_2}, \ldots, T_{i_q}$. By assumption, $g_1 T_{i_1}$ is non-zero and we have a K-integral factor ring

$$\mathbb{K}[T_1, \ldots, T_r]/\langle g, T_{i_2}, \ldots, T_{i_q} \rangle \;\cong\; \mathbb{K}[T_j;\; j \neq i_2, \ldots, i_r]/\langle g_1 T_{i_1} \rangle.$$

Consequently, $g_1 T_{i_1}$ is a K-prime polynomial. This implies $g_1 = c \in \mathbb{K}^*$ and thus we arrive at $g = cT_{i_1} + h$; a contradiction to the assumption that T_{i_1} is not a monomial of g. □

If R is a finitely generated K-graded algebra, then R admits homogeneous generators $f_1, \ldots, f_r$. Turning the polynomial ring $\mathbb{K}[T_1, \ldots, T_r]$ into a K-graded algebra via $\deg(T_i) := \deg(f_i)$, we obtain an epimorphism of K-graded algebras:

$$\pi \colon \mathbb{K}[T_1, \ldots, T_r] \;\to\; R, \qquad T_i \;\mapsto\; f_i.$$

Together with a choice of K-homogeneous generators $g_1, \ldots, g_s$ for the ideal $\ker(\pi)$, we arrive at *K-graded presentation* of R by homogeneous generators and relations:

$$R \;=\; \mathbb{K}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_s \rangle.$$

We call such presentation *irredundant* if $\ker(\pi)$ contains no elements of the form $T_i - h_i$ with $h_i \in \mathbb{K}[T_1, \ldots, T_r]$ not depending on T_i.

Proposition 2.2.4. *Let R a finitely generated K-graded algebra such that the grading is pointed, factorial and every homogeneous unit is of degree zero. Let*

$$R \;=\; \mathbb{K}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_s \rangle$$

be an irredundant K-graded presentation with $\dim(R) = r - s$ *such that $T_1, \ldots, T_r$ define K-prime elements in R. Then, for every $l = 1, \ldots, s$, we have*

$$\deg(g_l) \;\in\; \bigcap_{1 \le i < j \le r} \mathrm{cone}(\deg(T_k);\; k \ne i,\; k \ne j) \;\subseteq\; K_{\mathbb{Q}}.$$

Proof. It suffices to show that for any two $1 \le i < j \le r$, we can represent each $\deg(g_l)$ as a positive combination over the $\deg(T_k)$, where $k \ne i, j$. For $l = 1, \ldots, s$, set

$$g_{l,j} \;:=\; g_l(T_1, \ldots, T_{j-1}, 0, T_{j+1}, \ldots, T_r) \;\in\; \mathbb{K}[T_1, \ldots, T_r].$$

Since T_j defines a K-prime element in R, the ideal $\langle T_j \rangle \subseteq R$ is K-prime and $\langle T_j \rangle$ lifts to a K-prime ideal

$$I_j \;:=\; \langle g_1, \ldots, g_s, T_j \rangle \;=\; \langle g_{1,j}, \ldots, g_{s,j}, T_j \rangle \;\subseteq\; \mathbb{K}[T_1, \ldots, T_r].$$

Then $\mathbb{K}[T_1, \ldots, T_r]/I_j$ is isomorphic to $R/\langle T_j \rangle$. The latter algebra is of dimension $r-s-1$ due to our assumptions. Thus, $g_{1,j}, \ldots, g_{s,j}, T_j$ minimally generate I_j. By Lemma 2.2.2, each $g_{l,j}$ is K-prime and hence defines a K-integral factor algebra

$$\mathbb{K}[T_m;\; m \ne j]/\langle g_{l,j} \rangle \;\cong\; \mathbb{K}[T_1, \ldots, T_r]/\langle g_l, T_j \rangle.$$

We conclude that g_l, T_j minimally generate a K-prime ideal in $\mathbb{K}[T_1, \ldots, T_r]$. Thus, we may apply Lemma 2.2.3 and obtain the assertion. □

We turn to the geometric point of view. So, $\mathbb{K}$ is now algebraically closed of characteristic zero and R an affine K-graded algebra, where affine means that R is finitely generated over $\mathbb{K}$ and has no nilpotent elements. Then we have the affine variety $\bar{X}$ with R as its algebra of global functions and the quasitorus H with K as its character group:

$$\bar{X} \;=\; \mathrm{Spec}\, R, \qquad\qquad H \;=\; \mathrm{Spec}\, \mathbb{K}[K].$$

The K-grading of R defines an action of H on $\bar{X}$, which is uniquely determined by the property that each $f \in R_w$ satisfies $f(h \cdot x) = \chi^w(h) f(x)$ for all $x \in \bar{X}$ and $h \in H$, where χ^w is the character corresponding to $w \in K$. We take a look at the geometric invariant theory of the H-action on $\bar{X}$; see [4, 15]. The *orbit cone* $\omega_x \subseteq K_{\mathbb{Q}}$ associated with $x \in \bar{X}$ and the *GIT-cone* $\lambda_w \subseteq K_{\mathbb{Q}}$ associated with $w \in \mathrm{Eff}(R)$ are defined as

$$\omega_x \;=\; \mathrm{cone}(w \in K;\; f(x) \ne 0 \text{ for some } f \in R_w), \qquad \lambda_w \;:=\; \bigcap_{x \in \bar{X}, w \in \omega_x} \omega_x.$$

Orbit cones as well as GIT-cones are convex polyhedral cones and there are only finitely many of them. The basic observation is that the GIT-cones form a fan $\Lambda(R)$ in $K_{\mathbb{Q}}$, the *GIT-fan*, having the effective cone $\mathrm{Eff}(R)$ as its support.

Remark 2.2.5. Let K be a finitely generated abelian group and R a K-integral affine algebra. Fix a K-graded presentation

$$R \;=\; \mathbb{K}[T_1,\dots,T_r]/\langle g_1,\dots,g_s\rangle.$$

This yields an H-equivariant closed embedding $\bar{X} = V(g_1,\dots,g_s) \subseteq \mathbb{K}^r$ of affine varieties. Moreover, we have a homomorphism

$$Q\colon \mathbb{Z}^r \;\to\; K, \qquad \nu \;\mapsto\; \nu_1\deg(T_1) + \ldots + \nu_r\deg(T_r).$$

An $\bar{X}$*-face* is a face $\gamma_0 \preceq \gamma$ of the orthant $\gamma := \mathbb{Q}^r_{\geq 0}$ admitting a point $x \in \bar{X}$ such that one has

$$x_i \neq 0 \;\;\Longleftrightarrow\;\; e_i \in \gamma_0$$

for the coordinates $x_1,\dots,x_r$ of x and the canonical basis vectors $e_1,\dots,e_r \in \mathbb{Z}^r$. Write $\mathfrak{S}(\bar{X})$ for the set of all $\bar{X}$-faces of $\gamma \subseteq \mathbb{Q}^r$. Then we have

$$\{Q(\gamma_0);\; \gamma_0 \in \mathfrak{S}(\bar{X})\} \;=\; \{\omega_x;\; x \in \bar{X}\}.$$

That means that the projected $\bar{X}$-faces are exactly the orbit cones. The $\bar{X}$-faces define a decomposition into locally closed subsets

$$\bar{X} \;=\; \bigcup_{\gamma_0 \in \mathfrak{S}(X)} \bar{X}(\gamma_0), \qquad \bar{X}(\gamma_0) \;:=\; \{x \in \bar{X};\; x_i \neq 0 \Leftrightarrow e_i \in \gamma_0\} \;\subseteq\; \bar{X}.$$

Definition 2.2.6. Let $I = \{i_1,\dots,i_k\}$ be a subset of $\{1,\dots,r\}$. Then the face γ_I of the orthant $\gamma = \mathbb{Q}^r_{\geq 0}$ associated with I is defined as

$$\gamma_I \;:=\; \gamma_{i_1,\dots,i_k} \;:=\; \operatorname{cone}(e_{i_1},\dots,e_{i_k}).$$

Moreover, for a polynomial $g \in \mathbb{K}[T_1,\dots,T_r]$, the polynomial g_I associated with I is defined as

$$g_I \;:=\; g(\tilde{T}_1,\dots,\tilde{T}_r), \qquad \tilde{T}_i \;:=\; \begin{cases} T_i, & i \in I, \\ 0, & i \notin I. \end{cases}$$

Remark 2.2.7. In the setting of Remark 2.2.5, let $I = \{i_1,\dots,i_k\}$ be a subset of $\{1,\dots,r\}$.

(i) γ_I is an $\bar{X}$-face if and only if $\langle g_{1,I},\dots,g_{s,I}\rangle$ contains no monomial.

(ii) If $\deg(g_j) \notin \operatorname{cone}(w_i;\; i \in I)$ holds for $j = 1,\dots,s$, then γ_I is an $\bar{X}$-face.

(iii) If $(w_i;\; i \in I)$ is linearly independent in K, then γ_I is an $\bar{X}$-face if and only if none of $g_1,\dots,g_s$ has a monomial $T_{i_1}^{l_1}\cdots T_{i_k}^{l_k}$ with $l_1,\dots,l_k \in \mathbb{Z}_{\geq 0}$.

Proposition 2.2.8. *Let K be a finitely generated abelian group and R an affine algebra with a pointed K-grading. Consider a K-graded presentation*

$$R \;=\; \mathbb{K}[T_1,\dots,T_r]/\langle g_1,\dots,g_s\rangle$$

such that $T_1,\dots,T_r$ define non-constant elements in R. Assume that there are a GIT-cone $\lambda \in \Lambda(R)$ of dimension at least two and an index i with $\deg(T_i) \in \lambda^\circ$.

(i) *There exists a j such that g_j has a monomial $T_i^{l_i}$ with $l_i \in \mathbb{Z}_{\geq 0}$.*
(ii) *There exists a j such that $\deg(g_j) = l_i \deg(T_i)$ holds with $l_i \in \mathbb{Z}_{\geq 0}$.*
(iii) *If $s = 1$ holds, then, $\deg(T_k)$ generates a ray of $\Lambda(R)$ whenever $k \neq i$.*

Proof. Because of $\deg(T_i) \in \lambda^\circ$, the ray τ generated by $\deg(T_i)$ is not an orbit cone. Thus, $\mathbb{Q}_{\geq 0} e_i$ is not an $\bar{X}$-face. This means that some g_j has a monomial $T_i^{l_i}$, which in particular proves (i) and (ii). To obtain (iii), first observe that $\deg(T_k) \in K_\mathbb{Q}$ is non-zero and thus lies in the relative interior of some GIT-cone $\varrho \in \Lambda(R)$ of positive dimension. Suppose that ϱ is not a ray. Then (i) yields that besides $T_i^{l_i}$ also $T_k^{l_k}$ is a monomial of the relation g_1. We conclude that $\gamma_{i,k}$ is an $\bar{X}$-face. Thus, $\deg(T_i)$ and $\deg(T_k)$ lie on a ray of $\Lambda(R)$. A contradiction. □

2.3 Mori dream spaces

Mori dream spaces, introduced in [79], behave optimally with respect to the minimal model programme and are characterized as the normal projective varieties with finitely generated Cox ring. Well known example classes are the projective toric or spherical varieties and, most important for the present chapter, the smooth Fano varieties. In this section, we provide a brief summary of the combinatorial approach [4, 16, 62] to Mori dream spaces, adapted to our needs. Moreover, as a new observation, we present Proposition 2.3.6, locating the relation degrees of a Cox ring inside the effective cone of a quasismooth Mori dream space.

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, R be a K-graded affine $\mathbb{K}$-algebra and consider the action of $H = \operatorname{Spec} \mathbb{K}[K]$ on variety $\bar{X} = \operatorname{Spec} R$. Mori dream spaces are obtained as quotients of the H-action. We briefly recall the general framework. Each cone $\lambda \in \Lambda(R)$ of the GIT-fan defines an H-invariant open set of *semistable points* and a *good quotient*:

$$\bar{X}^{ss}(\lambda) \; = \; \{x \in \bar{X};\ \lambda \subseteq \omega_x\} \; \subseteq \; \bar{X}, \qquad\qquad \bar{X}^{ss}(\lambda) \; \to \; \bar{X}^{ss}(\lambda) /\!\!/ H,$$

where $\omega_x \subseteq K_\mathbb{Q}$ denotes the orbit cone of $x \in \bar{X}$. Each of the quotient varieties $\bar{X}^{ss}(\lambda) /\!\!/ H$ is projective over $\operatorname{Spec} R_0$ and whenever $\lambda' \subseteq \lambda$ holds for two GIT-cones, then we have $\bar{X}^{ss}(\lambda) \subseteq \bar{X}^{ss}(\lambda')$ and thus an induced projective morphism $\bar{X}^{ss}(\lambda) /\!\!/ H \to \bar{X}^{ss}(\lambda') /\!\!/ H$ of the quotient spaces.

The K-grading of R is *almost free* if the (open) set $\bar{X}_0 \subseteq \bar{X}$ of points $x \in \bar{X}$ with trivial isotropy group $H_x \subseteq H$ has complement of codimension at least two in $\bar{X}$. Moreover, the *moving cone* of R is the convex cone $\operatorname{Mov}(R) \subseteq K_\mathbb{Q}$ obtained as the union over all $\lambda \in \Lambda(R)$, where $\bar{X}^{ss}(\lambda)$ has a complement of codimension at least two in $\bar{X}$.

Remark 2.3.1. Let R be a K-graded affine algebra such that the grading is factorial and any homogeneous unit is constant. Then R admits a system $f_1, \ldots, f_r$ of pairwise non-associated K-prime generators. Moreover, if $f_1, \ldots, f_r$ is such a system of generators for R, then the following holds.

(i) The K-grading is almost free if and only if any $r-1$ of $\deg(f_1), \ldots, \deg(f_r)$ generate K as a group.

(ii) If the K-grading is almost free, then the orbit cones ω_x, where $x \in \bar{X}$, and the moving cone are given by

$$\begin{aligned} \omega_x &= \operatorname{cone}(\deg(f_i);\ f_i(x) \neq 0), \\ \operatorname{Mov}(R) &= \bigcap_{i=1}^{r} \operatorname{cone}(\deg(f_j);\ j \neq i). \end{aligned}$$

We say that a K-graded affine $\mathbb{K}$-algebra R is an *abstract Cox ring* if it is integral, normal, has only constant homogeneous units, the K-grading is almost free, pointed, factorial and the moving cone $\operatorname{Mov}(R)$ is of full dimension in $K_{\mathbb{Q}}$.

Construction 2.3.2. Let R be an abstract Cox ring and consider the action of the quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$ on the affine variety $\bar{X} = \operatorname{Spec} R$. For every GIT-cone $\lambda \in \Lambda(R)$ with $\lambda^\circ \subseteq \operatorname{Mov}(R)^\circ$, we set

$$X(\lambda) \ := \ \bar{X}^{ss}(\lambda) /\!\!/ H.$$

The following proposition tells us in particular that Construction 2.3.2 delivers Mori dream spaces; see [4, Thm. 3.2.14, Prop. 3.3.2.9 and Rem. 3.3.4.2].

Proposition 2.3.3. *Let $X = X(\lambda)$ arise from Construction 2.3.2. Then X is normal, projective and of dimension* $\dim(R) - \dim(K_{\mathbb{Q}})$*. The divisor class group and the Cox ring of X are given as*

$$\operatorname{Cl}(X) \ = \ K, \qquad \mathcal{R}(X) \ = \ \bigoplus_{\operatorname{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)) \ = \ \bigoplus_{K} R_w \ = \ R.$$

Moreover, the cones of effective, movable, semiample and ample divisor classes of X are given in $\operatorname{Cl}_{\mathbb{Q}}(X) = K_{\mathbb{Q}}$ *as*

$$\operatorname{Eff}(X) \ = \ \operatorname{Eff}(R), \qquad \operatorname{Mov}(X) \ = \ \operatorname{Mov}(R),$$

$$\operatorname{SAmple}(X) = \lambda, \qquad \operatorname{Ample}(X) = \lambda^\circ.$$

By [4, Cor. 3.2.1.11], all Mori dream space arise from Construction 2.3.2. For the subsequent work, we have to get more concrete, meaning that we will work in terms of generators and relations.

Construction 2.3.4. Let R be an abstract Cox ring and $X = X(\lambda)$ be as in Construction 2.3.2. Fix a K-graded presentation

$$R \ = \ \mathbb{K}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_s \rangle$$

such that the variables $T_1, \ldots, T_r$ define pairwise non-associated K-primes in R. Consider the orthant $\gamma = \mathbb{Q}^r_{\geq 0}$ and the projection

$$Q \colon \mathbb{Z}^r \ \to \ K, \qquad e_i \ \mapsto \ w_i := \deg(T_i).$$

An X-*face* is an $\bar{X}$-face $\gamma_0 \preceq \gamma$ with $\lambda^\circ \subseteq Q(\gamma_0)^\circ$. Let $\mathrm{rlv}(X)$ be the set of all X-faces and $\pi\colon \bar{X}^{ss}(\lambda) \to X$ the quotient map. Then we have a decomposition

$$X \;=\; \bigcup_{\gamma_0 \in \mathrm{rlv}(X)} X(\gamma_0)$$

into pairwise disjoint locally closed sets $X(\gamma_0) := \pi(\bar{X}(\gamma_0))$, which we also call the *pieces* of X.

Recall that X is $\mathbb{Q}$-*factorial* if for every Weil divisor on X some non-zero multiple is locally principal. Moreover, X is *locally factorial* if every stalk $\mathcal{O}_x$, where $x \in X$ is a (closed) point, is a unique factorization domain. Finally, X is *quasismooth* if the open set $\bar{X}^{ss}(\lambda) \subseteq \bar{X}$ of semistable points is a smooth variety.

Proposition 2.3.5. *Consider the situation of Construction 2.3.4.*

(i) *The variety X is $\mathbb{Q}$-factorial if and only if* $\dim(\lambda) = \dim(K_{\mathbb{Q}})$ *holds for* $\lambda = \mathrm{SAmple}(X)$.

(ii) *The variety X is locally factorial if and only if for every X-face $\gamma_0 \preceq \gamma$, the group K is generated by $Q(\gamma_0 \cap \mathbb{Z}^r)$.*

(iii) *The variety X is quasismooth if and only if every $\bar{X}(\gamma_0)$ consists of smooth points of $\bar{X}$ for every X-face $\gamma_0 \preceq \gamma$.*

(iv) *The variety X is smooth if and only if X is locally factorial and quasismooth.*

We refer to [4, Cor. 1.6.2.6, Cor. 3.3.1.8, Cor. 3.3.1.9] for the above statements. Next we describe the impact of quasismoothness on the position of the relation degrees.

Proposition 2.3.6. *In the situation of Construction 2.3.4, assume* $\dim(R) = r - s$ *and let X be quasismooth. Then, for every $j = 1, \ldots, s$, we have*

$$\deg(g_j) \;\in\; \bigcap_{\gamma_0 \in \mathrm{rlv}(X)} \left(Q(\gamma_0 \cap \mathbb{Z}^r) \cup \bigcup_{i=1}^{r} w_i + Q(\gamma_0 \cap \mathbb{Z}^r) \right).$$

Proof. Consider any X-face γ_I, where $I \subseteq \{1, \ldots, r\}$, and choose a point $x \in \bar{X}(\gamma_I)$. Then $x_i \neq 0$ holds if and only if $i \in I$. For any monomial T^ν, we have

$$\frac{\partial T^\nu}{\partial T_k}(x) \neq 0 \;\Rightarrow\; \nu \;\in\; \gamma_I \cup \gamma_I + e_k \;\Rightarrow\; \deg(T^\nu) = Q(\nu) \in Q(\gamma_I) \cup Q(\gamma_I) + w_k.$$

Now, since X is quasismooth, we have $\mathrm{grad}_{g_j}(x) \neq 0$ for all $j = 1, \ldots, s$. Thus, every g_j must have a monomial T^{ν_j} with non-vanishing gradient at x. □

Finally, in case of a complete intersection Cox ring, we have an explicit description of the anticanonical class; see [4, Prop. 3.3.3.2].

Proposition 2.3.7. *In the situation of Construction 2.3.4, assume that* $\dim(R) = r - s$ *holds. Then the anticanonical class of X is given in* $K = \mathrm{Cl}(X)$ *as*

$$-\mathcal{K}_X \;=\; \deg(T_1) + \cdots + \deg(T_r) - \deg(g_1) - \cdots - \deg(g_s).$$

2.4 General hypersurface Cox rings

First, we make our concept of a general hypersurface Cox ring precise. Then we present the toolbox to be used in the proof of Theorem 2.1.1 for verifying that given specifying data, that means a collection of the generator degrees and a relation degree, allow indeed a smooth general hypersurface Cox ring. We will have to deal with the following setting.

Construction 2.4.1. Consider a linear, pointed, almost free K-grading on the polynomial ring $S := \mathbb{K}[T_1, \ldots, T_r]$ and the quasitorus action $H \times \bar{Z} \to \bar{Z}$, where

$$H \ := \ \operatorname{Spec} \mathbb{K}[K], \qquad \bar{Z} \ := \ \operatorname{Spec} S \ = \ \mathbb{K}^r.$$

As earlier, we write $Q\colon \mathbb{Z}^r \to K$, $e_i \mapsto w_i := \deg(T_i)$ for the degree map. Assume that $\operatorname{Mov}(S) \subseteq K_{\mathbb{Q}}$ is of full dimension and fix $\tau \in \Lambda(S)$ with $\tau^\circ \subseteq \operatorname{Mov}(S)^\circ$. Set

$$\hat{Z} \ := \ \bar{Z}^{ss}(\tau), \qquad Z \ := \ \hat{Z} /\!\!/ H.$$

Then Z is a projective toric variety with divisor class group $\operatorname{Cl}(Z) = K$ and Cox ring $\mathcal{R}(Z) = S$. Moreover, fix $0 \neq \mu \in K$, and for $g \in S_\mu$ set

$$R_g := S/\langle g \rangle, \qquad \bar{X}_g := V(g) \subseteq \bar{Z}, \qquad \hat{X}_g := \bar{X}_g \cap \hat{Z}, \qquad X_g := \hat{X}_g /\!\!/ H \subseteq Z.$$

Then the factor algebra R_g inherits a K-grading from S and the quotient $X_g \subseteq Z$ is a closed subvariety. Moreover, we have

$$X_g \ \subseteq \ Z_g \ \subseteq \ Z$$

where $Z_g \subseteq Z$ is the minimal ambient toric variety of X_g, that means the (unique) minimal open toric subvariety containing X_g.

Remark 2.4.2. In the situation of Construction 2.4.1, there is a (unique) GIT-cone $\lambda \in \Lambda(R_g)$ such that we have

$$\hat{X}_g \ = \ \bar{X}_g^{ss}(\lambda), \qquad X_g \ = \ \bar{X}_g^{ss}(\lambda) /\!\!/ H.$$

Thus, if R_g is an abstract Cox ring and $T_1, \ldots, T_r$ define pairwise non-associated K-primes in R_g, then X_g is as in Construction 2.3.4. In particular

$$\operatorname{Cl}(X) \ = \ K, \qquad \mathcal{R}(X_g) \ = \ R_g$$

hold for the divisor class group and the Cox ring of X_g. Moreover, in $K_{\mathbb{Q}}$ we have the following

$$\tau^\circ \ = \ \operatorname{Ample}(Z) \ \subseteq \ \operatorname{Ample}(Z_g) \ = \ \operatorname{Ample}(X_g) \ = \ \lambda^\circ.$$

We are ready to formulate the precise definitions for our notions around hypersurface Cox rings.

Definition 2.4.3. Consider the situation of Construction 2.4.1.

(i) We call R_g a *hypersurface Cox ring* if $T_1, \dots, T_r$ define a minimal system of K-homogeneous generators for R_g.
(ii) We say that R_g is *spread* if every monomial $T^\nu \in \mathbb{K}[T_1, \dots, T_r]$ of degree $\mu = \deg(g) \in K$ is a convex combination of monomials of g.

Here, we tacitly identify a monomial $T^\nu = T_1^{\nu_1} \cdots T_r^{\nu_r}$ with its exponent vector $\nu = (\nu_1, \dots, \nu_r) \in \mathbb{Q}^r$ when we speak about convex combinations of monomials.

Remark 2.4.4. In the setting of Construction 2.4.1, assume that R_g is a hypersurface Cox ring.
(i) Since $T_1, \dots, T_r$ define a minimal system of K-homogeneous generators, R_g is not a polynomial ring.
(ii) As the K-grading is pointed, the T_i define pairwise non-associated K-prime elements in R_g.
(iii) R_g is spread if and only if the Newton polytope of g equals the convex hull over all monomials of degree $\mu = \deg(g) \in K$.

Definition 2.4.5. Consider the situation of Construction 2.4.1 and denote by $S_\mu \subseteq S = \mathbb{K}[T_1, \dots, T_r]$ the homogeneous component of degree $\mu \in K$.
(i) A *general hypersurface Cox ring* is a family R_g, where $g \in U$ with a non-empty open $U \subseteq S_\mu$, such that each R_g is a hypersurface Cox ring.
(ii) We say that a general hypersurface Cox ring R_g is *spread* if each R_g, where $g \in U$, is spread.
(iii) We say that a general hypersurface Cox ring R_g is *smooth (Fano)* if for some $\tau \in \Lambda(S)$ all the resulting X_g, where $g \in U$, are smooth (Fano).

Intrinsic quadrics provide first simple examples for general hypersurface Cox rings.

Example 2.4.6. Let $r \geq 5$. We run Construction 2.4.1 for $S = \mathbb{K}[T_1, \dots, T_r]$ with the $\mathbb{Z}$-grading given by $\deg(T_i) := 1$. The choice of $\tau = \mathbb{Q}_{\geq 0}$ and $\mu = 2$ leads to a quadric hypersurface

$$X = V(g) \subseteq \mathbb{P}_{r-1} = Z.$$

The quadratic polynomials $g \in U_\mu$ such that the according quadric $V(g) \subseteq \mathbb{P}_{r-1}$ is of full rank form a non-empty open subset U of S_μ. For any $g \in U$ the ring R_g is factorial; see e.g. [121, Satz 60.12]. So R_g is an abstract Cox ring. For any T_i we have

$$R_g/\langle T_i \rangle \cong \mathbb{K}[T_1, \dots, T_r]/\langle g, T_i \rangle \cong \mathbb{K}[T_1, \dots, T_{r-1}]/\langle T_1^2 + \dots + T_k^2 \rangle$$

for some $k \geq 3$. Thus, $T_i \in R_g$ is prime. Remark 2.4.9 ensures that $T_1, \dots, T_r \in R_g$ form a minimal generator system for R_g. Altogether, U defines a smooth Fano general hypersurface Cox ring with specifying data $Q = [1, \dots, 1]$ and $\mu = 2$.

We turn to the toolbox for verifying that given specifying data $w_1, \dots, w_r \in K$ and $\mu \in K$ as in Construction 2.4.1 lead to a smooth Fano general hypersurface Cox ring R_g in the above sense.

Remark 2.4.7. In the notation of Construction 2.4.1, a general hypersurface Cox ring R_g is Fano if and only if the generator and relation degrees satisfy

$$-\mathcal{K} \ = \ w_1 + \cdots + w_r - \mu \ \in \ \mathrm{Mov}(R_g)^\circ.$$

In this case, the unique cone $\tau \in \Lambda(S)$ with $-\mathcal{K} \in \tau^\circ$ defines Fano varieties X_g for all $g \in U$; see Proposition 2.3.7 and Remark 2.4.2.

In the notation of Construction 2.4.1, we denote by $U_\mu \subseteq S_\mu$ the non-empty open set of polynomials $f \in S$ of degree $\mu \in K$ such that each monomial of S_μ is a convex combination of monomials of f.

Remark 2.4.8. If R_g, where $g \in U$, is a general hypersurface Cox ring, then R_g, where $g \in U \cap U_\mu$, is a spread general hypersurface Cox ring. In particular, we can always assume a general hypersurface Cox ring to be spread.

Remark 2.4.9. In the situation of Construction 2.4.1, consider the rings R_g for $g \in U_\mu$. Then the following statements are equivalent.

(i) The variables $T_1, \ldots, T_r$ form a minimal system of generators for all R_g, where $g \in U_\mu$.
(ii) The variables $T_1, \ldots, T_r$ form a minimal system of generators for one R_g with $g \in U_\mu$.
(iii) We have $\mu \neq w_i$ for $i = 1, \ldots, r$.
(iv) The polynomial $g \in U_\mu$ is not of the form $g = T_i + h$ with $h \in S_\mu$ not depending on T_i.

Lemma 2.4.10. *Consider a linear, pointed K-grading on $S := \mathbb{K}[T_1, \ldots, T_r]$. Then, for any $0 \neq \mu \in K$ the irreducible polynomials $g \in S_\mu$ form an open subset of S_μ.*

Proof. Lemma 2.2.1 (ii) provides us with a coarsening homomorphism $\kappa\colon K \to \mathbb{Z}$ that turns S into a pointed $\mathbb{Z}$-graded algebra. Then S_μ is a vector subspace of the (finite dimensional) vector space $S_{\kappa(\mu)}$ of $\kappa(\mu)$-homogeneous polynomials and we may assume $K = \mathbb{Z}$ for the proof. Since the K-grading of S is pointed, we have $S^* = S_0 \setminus \{0\}$. Thus, a polynomial $g \in S_\mu$ is reducible if and only if it is a product of homogeneous polynomials of non-zero K-degree.

Now, let $u, v \in \mathbb{Z}$ with $u + v = \mu$ and $S_u \neq \{0\} \neq S_v$. Then the set of μ-homogeneous polynomials g admitting a factorization $g = fh$ with $f \in S_u$, $h \in S_u$ is exactly the affine cone over the image of the projectivized multiplication map

$$\mathbb{P}(S_u) \times \mathbb{P}(S_v) \ \to \ \mathbb{P}(S_\mu), \qquad\qquad ([f],[h]) \ \mapsto \ [fh]$$

and thus is a closed subset of S_μ. As there are only finitely many such presentations $u + v = \mu$, the reducible $g \in S_\mu$ form a closed subset of S_μ. □

Proposition 2.4.11. *Consider the setting of Construction 2.4.1. For $1 \leq i \leq r$ denote by $U_i \subseteq S_\mu$ the set of all $g \in S_\mu$ such that g is prime in S and T_i is prime in R_g. Then $U_i \subseteq S_\mu$ is open. Moreover, U_i is non-empty if and only if there is a μ-homogeneous prime polynomial not depending on T_i.*

Proof. By Lemma 2.4.10, the $g \in S_\mu$ being prime in S form an open subset $U \subseteq S_\mu$. For any $g \in U$, the variable T_i defines a prime in R_g if and only if the polynomial $g_i := g(T_1, \ldots, T_{i-1}, 0, T_{i+1}, \ldots, T_n)$ is prime in $\mathbb{K}[T_j;\ j \neq i]$. Thus, using again Lemma 2.4.10, we see that the $g \in U$ with $T_i \in R_g$ prime form the desired open subset $U_i \subseteq U$. The supplement is clear. □

Remark 2.4.12. An easy way to check the criterion from Proposition 2.4.11 is to look for μ-homogeneous prime binomials. Recall that a binomial $T^\kappa - T^\nu$ is prime if and only if T^κ, T^ν are not divisible by a common variable and $\gcd(\kappa_1 - \nu_1, \ldots, \kappa_r - \nu_r) = 1$ holds.

Checking the normality and K-factoriality of R_g amounts, in our situation, to proving factoriality. We will use Dolgachev's criterion, see [53, Thm. 1.2] and [54], which tells us that a polynomial $g = \sum a_\nu T^\nu$ in $\mathbb{K}[T_1, \ldots, T_r]$ defines a unique factorization domain if the Newton polytope $\Delta \subseteq \mathbb{Q}^r$ of g satisfies the following conditions:
(i) $\dim(\Delta) \geq 4$,
(ii) each coordinate hyperplane of $\mathbb{Q}^r$ intersects Δ non-trivially,
(iii) the dual cone of $\mathrm{cone}(\Delta_0 - u;\ u \in \Delta_0)$ is regular for each one-dimensional face $\Delta_0 \preceq \Delta$,
(iv) for each face $\Delta_0 \preceq \Delta$ the zero locus of $\sum_{\nu \in \Delta_0} a_\nu T^\nu$ is smooth along the torus $\mathbb{T}^r = (\mathbb{K}^*)^r$.
We will call for short a convex polytope $\Delta \subseteq \mathbb{Q}^r_{\geq 0}$ with properties (i)–(iii) from above a *Dolgachev polytope*.

Proposition 2.4.13. *In the situation of Construction 2.4.1, suppose that one of the following conditions is fulfilled:*
(i) *K is of rank at most $r-4$ and torsion free, there is a $g \in S_\mu$ such that $T_1, \ldots, T_r$ define primes in R_g, we have $\mu \in \tau^\circ$ and μ is base point free on Z.*
(ii) *The set $\mathrm{conv}(\nu \in \mathbb{Z}^r_{\geq 0};\ Q(\nu) = \mu)$ is a Dolgachev polytope.*
Then there is a non-empty open subset of polynomials $g \in S_\mu$ such that the ring R_g is factorial.

Proof. Assume that (i) is satisfied. If $\mu = \deg(T_i)$ holds for some i, then, as the grading is pointed, we have a non-empty open set of polynomials $g = T_i + h$ in S_μ with h not depending on T_i. The corresponding R_g are all factorial. Now assume $\mu \neq \deg(T_i)$ for all i. By Proposition 2.4.11, the set $U \subseteq S_\mu$ of all prime $g \in S_\mu$ such that $T_1, \ldots, T_r$ define primes in R_g is open and, by assumption, $U \subseteq S_\mu$ is non-empty. Remark 2.4.9 yields that $T_1, \ldots, T_r$ form a minimal system of generators for R_g. We conclude that for all $f \in U$, the complement of $\hat{X}_g$ in $\bar{X}_g$ is of codimension at least two. Since μ is base point free and ample on Z, we can apply [9, Cor. 2.3], telling us that after suitably shrinking, U is still non-empty and R_g is the Cox ring of X_g for all $g \in U$. In particular, R_g is K-factorial. Since K is torsion free, R_g is a unique factorization domain.

Assume that (ii) holds. As $\Delta := \mathrm{conv}(\nu \in \mathbb{Z}^r_{\geq 0};\ Q(\nu) = \mu)$ is a Dolgachev polytope, we infer from [76, §2, Thm. 2] that there is a non-empty open subset of polynomials $g \in S_\mu$ with Newton polytope Δ satisfying the above conditions (i) to (iv). Thus, Dolgachev's criterion shows that R_g is a factorial ring. □

Remark 2.4.14. In the situation of Construction 2.4.1, assume that Z is a fake weighted projective space, i.e., Z is $\mathbb{Q}$-factorial and $\mathrm{Cl}(Z) = K$ is of rank one. Then $\mu \in \mathrm{Cl}(Z)$ is base point free if and only if there is an $l_i \in \mathbb{Z}_{\geq 1}$ with $\mu = l_i w_i$ for all $1 \leq i \leq r$.

According to Remark 1.7.8 general base point free hypersurfaces in fake weighted projective spaces always stem from Cox ring embeddings. This fact provides us with the following criterion for general hypersurface Cox rings of Picard number one.

Proposition 2.4.15. *In the situation of Construction 2.4.1, suppose that K is of rank one, $r \geq 5$ holds and that for any $i = 1, \ldots, r$ there is an $l_i \in \mathbb{Z}_{\geq 1}$ with $\mu = l_i w_i$. Then there is a non-empty open subset of polynomials $g \in S_\mu$ such that the ring R_g is normal and K-factorial, and $T_1, \ldots, T_r \in R_g$ are prime. In particular, there is a general hypersurface Cox ring with specifying data $w_1, \ldots, w_r$ and μ.*

We use the concept of algebraic modifications [4, Sec. 4.1.2] to provide further factoriality criteria for graded hypersurface rings. These will apply to several cases where the relation degree lies on the boundary of the moving cone.

Let us briefly recall the notion of polynomials arising from Laurent polynomials by homogenization with respect to a lattice fan from Sections 1.2, 1.3 and 1.5.

Remark 2.4.16. Let Σ be a complete lattice fan in $\mathbb{Z}^n$ and $v_1, \ldots, v_r$ the primitive lattice vectors generating the rays of Σ. Consider the following mutually dual exact sequences

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^r \xrightarrow[e_i \mapsto v_i]{P} \mathbb{Z}^n$$

$$0 \longleftarrow K \xleftarrow{Q} \mathbb{Z}^r \xleftarrow{P^*} \mathbb{Z}^n \longleftarrow 0$$

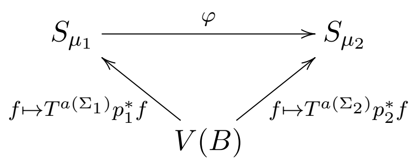
This induces a pointed K-grading on the polynomial algebra $S = \mathbb{K}[T_1, \ldots, T_r]$ via $\deg(T_i) := Q(e_i) \in K$. For any $w \in K$ we denote $S_w \subseteq S$ for the finite-dimensional vector space of homogeneous polynomials of degree w.

Moreover, fix a lattice polytope $B \subseteq \mathbb{Q}^n$ and set

$$a(\Sigma) := (a_1, \ldots, a_r) \in \mathbb{Z}^r, \qquad a_i := -\min_{u \in B} \langle u, v_i \rangle.$$

We call $\mu = Q(a(\Sigma)) \in K$ the Σ-*degree* of B. Besides $\mu \in K = \mathrm{Cl}(Z)$ regarded as a divisor class is base point free if Σ refines the normal fan of B. The Σ-*homogenization* of a Laurent polynomial $f \in \mathbb{K}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$ with Newton polytope $B(f)$ equal to B is the μ-homogeneous polynomial $g = T^{a(\Sigma)} p^* f \in S$ where $p\colon \mathbb{T}^r \to \mathbb{T}^n$ is the homomorphism of tori associated with P. Each spread polynomial $g \in S_\mu$ arises as Σ-homogenization of a Laurent polynomial f with $B(f) = B$ provided that Σ refines $\Sigma(B)$.

Let Σ_1, Σ_2 be lattice fans refining the normal fan $\Sigma(B)$ of B. The vector space $V(B)$ of all Laurent polynomials of the form $\sum_{\nu \in B \cap \mathbb{Z}^r} a_\nu T^\nu$ fits into the following commutative diagram of vector space isomorphisms



Moreover, if $g \in S_{\mu_1}$ is spread, then $\varphi(g) \in S_{\mu_2}$ is spread as well and g, $\varphi(g)$ are homogenizations of a common Laurent polynomial with respect to different fans Σ_i.

We state an adapted version of [4, Thm. 4.1.2.2]; see also [4, Prop. 4.1.2.4].

Theorem 2.4.17. *Let $f \in \mathrm{LP}(n)$ be a Laurent polynomial and $\Sigma_2 \preceq \Sigma_1$ a refinement of fans in $\mathbb{Z}^n$. Moreover, let $g_i \in \mathbb{K}[T_1, \ldots, T_{r_i}]$ be the respective Σ_i-homogenization of f and consider the K_i-graded algebra*

$$R_{g_i} = \mathbb{K}[T_1, \ldots, T_{r_i}]/\langle g_i \rangle.$$

Assume that g_1, g_2 are prime polynomials, $T_1, \ldots, T_{r_1}$ define K_1-primes in R_{g_1} and $T_1, \ldots, T_{r_2}$ define K_2-primes in R_{g_2}. Then the following statements are equivalent.

(i) *The algebra R_{g_1} is factorially K_1-graded.*
(ii) *The algebra R_{g_2} is factorially K_2-graded.*

Now let us bring this theorem in the context of general hypersurface rings. We show that factoriality is inherited between general hypersurface rings with relation degrees stemming from a common lattice polytope.

Proposition 2.4.18. *Let $B \subseteq \mathbb{Q}^n$ be a lattice polytope, $\Sigma_2 \preceq \Sigma_1 \preceq \Sigma(B)$ a refinement of fans in $\mathbb{Z}^n$, and $\mu_i \in K_i$ the respective Σ_i-degree. Assume that for $i = 1, 2$ there is a μ_i-homogeneous prime polynomial g_i and a non-empty open subset $U_i \subseteq S_{\mu_i}$ such that for all $g_i \in U_i$ the variables $T_1, \ldots, T_{r_i}$ define K_i-primes in the K_i-graded algebra*

$$R_{g_i} = \mathbb{K}[T_1, \ldots, T_{r_i}]/\langle g_i \rangle.$$

Then the following statements are equivalent.

(i) *There is a non-empty open subset of polynomials $g_1 \in S_{\mu_1}$ such that R_{g_1} is K_1-factorial.*
(ii) *There is a non-empty open subset of polynomials $g_2 \in S_{\mu_2}$ such that R_{g_2} is K_2-factorial.*

Proof. We know that the subset $U_{\mu_i} \subseteq S_{\mu_i}$ of spread μ_i-homogeneous polynomials is open and non-empty. According to Remark 2.4.16 there is an isomorphism $\varphi : S_{\mu_1} \to S_{\mu_2}$ of vector spaces such that g and $\varphi(g)$ arise as Σ_i-homogenization of the same Laurent polynomial whenever $g \in U_{\mu_1}$. Besides, by Lemma 2.4.10 the μ_i-homogeneous prime polynomials form an open subset of S_{μ_i}, which is non-empty by assumption. Therefore, by suitably shrinking U_1 and U_2 we achieve

- $\varphi(U_1) = U_2$,
- g_1 and $g_2 \coloneqq \varphi(g_1)$ are respective Σ_i-homogenizations of a common Laurent polynomial whenever $g_1 \in U_1$,
- for every $g_1 \in U_1$ the ring R_{g_1} is integral and $T_1, \ldots, T_{r_1} \in R_{g_1}$ are K_1-prime,
- for every $g_2 \in U_2$ the ring R_{g_2} is integral and $T_1, \ldots, T_{r_2} \in R_{g_2}$ are K_2-prime.

In this situation Theorem 2.4.17 tells us that for any $g_1 \in U_1$ and $g_2 \coloneqq \varphi(g_1)$ we have

$$R_{g_1} \text{ is } K_1\text{-factorial} \iff R_{g_2} \text{ is } K_2\text{-factorial.}$$

Now let $V_1 \subseteq S_{\mu_1}$ be a non-empty open subset such that R_{g_1} is factorially graded for each $g_1 \in V_1$. Then $V_2 \coloneqq \varphi(U_1 \cap V_1)$ is a non-empty open subset of S_{μ_2} and R_{g_2} is K_2-factorial for all $g_2 \in V_2$. This proves "(i) ⇒ (ii)". The inverse implication is shown analogously. □

Corollary 2.4.19. *Let $B \subseteq \mathbb{Q}^n$ be an integral n-simplex, Σ a fan in $\mathbb{Z}^n$ refining the normal fan of B, and $\mu \in K$ the Σ-degree of B. Assume that there is a μ-homogeneous prime polynomial g and a non-empty open subset $U \subseteq S_\mu$ such that for all $g \in U$ the variables $T_1, \ldots, T_r$ define K-primes in the K-graded algebra*

$$R_g = \mathbb{K}[T_1, \ldots, T_r]/\langle g \rangle.$$

Then there is a non-empty open subset of polynomials $g \in S_\mu$ such that R_g is K-factorial.

Proof. Since B a is simplex, the toric variety associated with $\Sigma(B)$ is $\mathbb{Q}$-factorial and of Picard number one. Now we apply Proposition 2.4.18 to the refinement $\Sigma \preceq \Sigma(B)$ and the suitable open subset of polynomials provided by Proposition 2.4.15. □

In many situations we encounter it can be read of straight from the specifying data whether the conditions from Corollary 2.4.19 are met.

Corollary 2.4.20. *Situation as in Construction 2.4.1. Assume that we have $r \geq 5$, $K = \mathbb{Z}^2$ and the degree matrix is of the form*

$$Q = [w_1, \ldots, w_{r+1}] = \begin{bmatrix} x_1 & \ldots & x_r & 0 \\ -d_1 & \ldots & -d_r & 1 \end{bmatrix}, \qquad x_i \in \mathbb{Z}_{\geq 1},\ d_i \in \mathbb{Z}_{\geq 0}.$$

Then for any $\mu = (\mu_1, \mu_2) \in K = \mathbb{Z}^2$ satisfying the subsequent conditions there is a non-empty open subset of polynomials $g \in S_\mu$ such that R_g is factorial:

(i) *for each i there exists some $l_i \in \mathbb{Z}_{\geq 1}$ with $\mu = l_i x_i$,*
(ii) *$\mu_2 = -\min_\nu \nu_1 d_1 + \cdots + d_r \nu_r$ where the minimum runs over all lattice points $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{Z}^r_{\geq 0}$ with $\nu_1 x_1 + \cdots + \nu_r x_r = \mu_1$,*
(iii) *there is some $g \in S_\mu$ such that $T_1, \ldots, T_{r+1}$ define primes in R_g.*

Proof. Observe that each $r-1$ of $x_1, \ldots, x_r$ generate $\mathbb{Z}$ as a group since the first coordinate of w_{r+1} vanishes and the $\mathbb{Z}^2$-grading associated with Q is almost free according to the assumptions made in Construction 2.4.1. Consider the weighted projective space

$$Z' := \mathbb{P}(x_1, \ldots, x_r).$$

Condition (i) ensures that $\mu_1 \in \mathbb{Z} = \mathrm{Cl}(Z')$ regarded as a divisor class on Z' is ample and base point free. Choose some representative $D \in \mathrm{WDiv}(Z')$ of μ_1. The associated divisorial polytope $B := B(D) \subseteq \mathbb{Q}^{r-1}$ is a full-dimensional integral simplex.

The normal fan Σ' of B is a lattice fan in $\mathbb{Z}^{r-1}$ corresponding with Z'. Write $v_1, \ldots, v_r \in \mathbb{Z}^{r-1}$ for the primitive ray generators of Σ'. Observe that the maps

$$P' \colon \mathbb{Z}^r \to \mathbb{Z}^{r-1},\ e_i \mapsto v_i, \qquad Q' \colon \mathbb{Z}^r \to \mathbb{Z},\ e_i \mapsto x_i$$

fit into a mutually dual pair of exact sequences as shown in Remark 2.4.16. Now set

$$v_{r+1} := d_1 v_1 + \cdots + d_r v_r \in \mathbb{Z}^{r-1}, \qquad d := (d_1, \ldots, d_r) \in \mathbb{Z}^r.$$

The second row of Q encodes the relation satisfied by $v_1, \ldots, v_{r+1}$ thus the following maps constitute a pair of mutually dual sequences as well

$$P \colon \mathbb{Z}^{r+1} \to \mathbb{Z}^n,\ e_i \mapsto v_i, \qquad Q \colon \mathbb{Z}^{r+1} \to \mathbb{Z}^2,\ e_i \mapsto w_i.$$

Since the first r columns of Q generate $\mathbb{Z}^2$, the vector $v_{r+1} \in \mathbb{Z}^{r-1}$ is primitive; see [4, Lemma 2.1.4.1]. This allows us to consider the stellar subdivision Σ of Σ' along v_{r+1}.

We show that $\mu \in \mathbb{Z}^2$ is the Σ-degree μ_B of B. First note that μ_1 is the Σ'-degree of B by construction. Consider

$$a' := a(\Sigma') = (a'_1, \ldots, a'_r), \qquad a := a(\Sigma) = (a_1, \ldots, a_{r+1})$$

from Remark 2.4.16. Since Σ arises from Σ' by introducing an $(r+1)$-th ray, we have $a_i = a'_i$ for $i = 1, \ldots, r$. From this we infer

$$\mu_1 = Q'(a') = a_1 x_1 + \cdots + a_r x_r, \qquad \mu_B = Q(a) = a_1 w_1 + \cdots + a_{r+1} w_{r+1}.$$

As the first coordinate of w_{r+1} vanishes, we conclude that the first coordinate of μ_B equals μ_1. It remains to investigate the second coordinate of μ_B. We have

$$a_{r+1} = -\min_{u \in B} \langle u,\, v_{r+1} \rangle = -\min_{u \in B} \langle u,\, P'(d) \rangle = -\min_{u \in B} \langle (P')^* u,\, d \rangle.$$

Using this presentation of a_{r+1}, the second coordinate of μ_B is given as

$$a_{r+1} - \sum_{i=1}^{r} a_i d_i = -\min_{u \in B} \langle (P')^* u,\, d \rangle - \langle a',\, d \rangle = -\min_{u \in B} \langle (P')^* u + a',\, d \rangle.$$

From condition (ii) and the fact that the lattice points $\nu \in \mathbb{Z}^r_{\geq 0}$ with $Q(\nu) = \mu_1$ are precisely those of the form $\nu = (P')^* u + a'$ for some lattice point $u \in B$ follows that the second coordinate of μ_B equals μ_2. Altogether we have verified $\mu = \mu_B$.

The above discussion combined with condition (iii) ensures that we may apply Corollary 2.4.19 to Q and μ which finishes the proof. □

Proposition 2.4.21. *In the setting of Construction 2.4.1, assume that Z_g and $\hat{X}_g$ both are smooth. Then X_g is smooth.*

Proof. Consider the quotient map $p\colon \hat{Z} \to Z$. Since Z_g is smooth, H acts freely on $p^{-1}(Z_g)$. Thus, X_g inherits smoothness from $\hat{X}_g = p^{-1}(X_g)$. □

Lemma 2.4.22. *Consider a linear, pointed K-grading on $S := \mathbb{K}[T_1, \ldots, T_r]$. Let $\lambda \in \Lambda(S)$ and set $W := (\mathbb{K}^r)^{ss}(\lambda)$. Then, for any $\mu \in K$, the polynomials $g \in S_\mu$ such that* $\mathrm{grad}(g)$ *has no zeroes in W form an open subset of S_μ.*

Proof. Consider the morphism $\varphi\colon S_\mu \times W \to \mathbb{K}^r$ sending (g, z) to $\mathrm{grad}_z(g)$ and the projection $\mathrm{pr}_1\colon S_\mu \times W \to S_\mu$ onto the first factor. Then our task is to show that $S_\mu \setminus \mathrm{pr}_1(\varphi^{-1}(0))$ is open in S_μ. We make use of the action of $H = \mathrm{Spec}\,\mathbb{K}[K]$ on W given by the K-grading and the commutative diagram

$$\begin{array}{ccc} S_\mu \times W & \longrightarrow & S_\mu \times W /\!\!/ H \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & S_\mu & \end{array}$$

where the horizontal arrow is the good quotient for H, acting trivially on S_μ and on W as indicated above. Since $\varphi^{-1}(0) \subseteq S_\mu \times W$ is invariant under the H-action, the image of $\varphi^{-1}(0)$ in $S_\mu \times W /\!\!/ H$ is closed. Since $W /\!\!/ H$ is projective, the image $\mathrm{pr}_1(\varphi^{-1}(0))$ is closed in S_μ. □

Proposition 2.4.23. *Consider the situation of Construction 2.4.1. Then the polynomials $g \in S_\mu$ such that $g \in S$ is prime and $\hat{X}_g$ is smooth form an open subset $U \subseteq S_\mu$. Moreover, U is non-empty if and only if there are $g_1, g_2 \in S_\mu$ such that $g_1 \in S$ is prime and* $\mathrm{grad}(g_2)$ *has no zeroes in $\hat{Z}$.*

Proof. By Lemma 2.4.10, the set V_1 of all prime polynomials of S_μ is open. Moreover, by Lemma 2.4.22, the set of all polynomials of S_μ such that $\mathrm{grad}(g)$ has no zeroes in $\hat{Z}$ is open. The assertion follows from $U = V_1 \cap V_2$. □

Corollary 2.4.24. *Let X be a variety with a general hypersurface Cox ring R. If X is smooth, then R is a smooth general hypersurface Cox ring.*

Proposition 2.4.25. *Consider the situation of Construction 2.4.1. If $\mu \in \mathrm{Cl}(Z)$ is base point free, then there is a non-empty open subset of $g \in S_\mu$ such that $X_g \cap Z^{\mathrm{reg}}$ is smooth.*

Proof. Observe that $\mathbb{P}(S_\mu)$ is the complete linear system associated with the divisor class $\mu \in \mathrm{Cl}(Z)$. If μ is a base point free class on Z, we can apply Bertini's first theorem [89, Thm. 4.1] stating that there is a non-empty open subset $U \subseteq S_\mu$ such that for each $g \in U$ the singular locus of X_g is precisely $X_g \cap Z^{\mathrm{sing}}$. In particular, $X_g \cap Z^{\mathrm{reg}}$ is smooth for all $g \in U$. □

Remark 2.4.26. In the situation of Construction 2.4.1, let $N(g)$ be the Newton polytope of g. For $I \subseteq \{1, \dots, r\}$, let $\gamma_I \preccurlyeq \gamma$ and $g_I \in \mathbb{K}[T_1, \dots, T_r]$ be as in Definition 2.2.6. Then [4, Prop. 3.1.1.12] yields the equivalence of the following statements.

(i) We have $\bar{X}_g \cap \bar{Z}(\gamma_I) \neq \emptyset$.

(ii) The polynomial g_I is not a monomial.

(iii) The number of vertices of $N(g)$ contained in γ_I differs from one.

If, in addition, $Z(\gamma_I) \neq \emptyset$ holds, then (i)–(iii) are equivalent to

(iv) We have $X_g \cap Z(\gamma_I) \neq \emptyset$.

In particular, for the non-empty open subset $U_\mu \subseteq S_\mu$ of polynomials $f \in S$ of degree $\mu = \deg(g) \in K$ such that each monomial of S_μ is a convex combination of monomials of f, we obtain $Z_g = Z_{g'}$ for all $g, g' \in U_\mu$.

Definition 2.4.27. In the setting of Remark 2.4.26, we call $Z_\mu := Z_g$, where $g \in U_\mu$, the *μ-minimal ambient toric variety.*

Proposition 2.4.28. *In the situation of Construction 2.4.1 the following statements are equivalent.*

(i) *The μ-minimal ambient toric variety Z_μ is smooth.*

(ii) *For each $\gamma_I \preceq \gamma$ with $\tau^\circ \in Q(\gamma_I)^\circ$ and $|Q^{-1}(\mu) \cap \gamma_I| \neq 1$ the group K is generated by $Q(\gamma_I \cap \mathbb{Z}^r)$.*

Proof. First recall that a toric variety is smooth if its closed orbits are smooth. For any spread $g \in S_\mu$ all closed orbits of Z_μ intersect X_g non-trivially by construction of the minimal ambient toric variety. Thus Z_μ is smooth if and only if all orbits $Z(\gamma_I)$ intersecting X_g non-trivially are smooth. Observe that the number of vertices of $N(g)$ contained in $\gamma_I \preceq \gamma$ equals $|Q^{-1}(\mu) \cap \gamma_I|$. Hence, according to Remark 2.4.26, a toric orbit $Z(\gamma_I)$ intersects X_g non-trivially if and only if $|Q^{-1}(\mu) \cap \gamma_0| \neq 1$ holds. Finally, Proposition 2.3.5 tells us that $Z(\gamma_I)$ is smooth if and only if $Q(\gamma_I \cap \mathbb{Z}^r)$ spans K. □

Corollary 2.4.29. *In the setting of in Construction 2.4.1, assume* $\operatorname{rank}(K) = 2$ *and that $Z_\mu \subseteq Z$ is smooth. If $\mu \in \tau$ holds, then μ is base point free. Moreover, then there is a non-empty open subset of polynomials $g \in S_\mu$ such that X_g is smooth.*

Proof. According to [4, Prop. 3.3.2.8], the class $\mu \in \operatorname{Cl}(Z)$ is base point free on Z if and only if the following holds:

$$\mu \ \in \bigcap_{\gamma_0 \in \operatorname{rlv}(Z)} Q(\gamma_0 \cap \mathbb{Z}^r).$$

To check the latter, let $\gamma_0 \in \operatorname{rlv}(Z)$. As $K_{\mathbb{Q}}$ is two-dimensional, we find $1 \leq i, j \leq r$ with $e_i, e_j \in \gamma_0$ and $\lambda^\circ \subseteq \operatorname{cone}(w_i, w_j)^\circ$. If w_i, w_j generate K as a group, then K is torsion-free, w_i, w_j form a Hilbert basis for $\operatorname{cone}(w_i, w_j)$ and thus μ is a positive combination of w_i, w_j. Otherwise, the toric orbit $Z(\gamma_{i,j})$ is not smooth, hence not contained in Z_μ. The latter means $V(g) \cap \bar{Z}(\gamma_{i,j}) = \emptyset$, which in turn shows that g has a monomial of the form $T_i^{l_i} T_j^{l_j}$ where $l_i + l_j > 0$. Thus, μ is a positive combination of w_i and w_j.

Knowing that μ is base point free, we obtain the supplement as a direct consequence of smoothness of Z_μ and Proposition 2.4.25. □

2.5 Proof of Theorem 2.1.1: Constraints on hypersurface Cox rings

We prepare the setting for the proof of Theorem 2.1.1. We work in the combinatorial framework for Mori dream spaces provided in the preceding sections. The ground field is now $\mathbb{K} = \mathbb{C}$, due to the references we use; see Remark 2.5.12. The major part of proving Theorem 2.1.1, is to figure out the candidates for specifying data of smooth general hypersurface Cox rings of Fano fourfolds of Picard number two. Having found the candidates, the remaining task is to verify them, that means to show that the given specifying data indeed define a smooth general hypersurface Cox ring of a Fano fourfold.

Setting 2.5.1. Consider a K-graded algebra R and $X = X(\lambda)$, where $\lambda \in \Lambda(R)$ with $\lambda^\circ \subseteq \mathrm{Mov}(R)^\circ$, as in Construction 2.3.2. Assume that $\dim(K_\mathbb{Q}) = 2$ holds and that we have an irredundant K-graded presentation

$$R \;=\; R_g \;=\; \mathbb{C}[T_1, \ldots, T_r]/\langle g\rangle$$

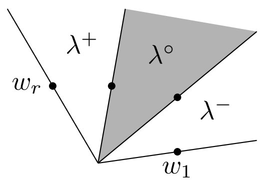
such that the T_i define pairwise nonassociated K-primes in R. Write $w_i \coloneqq \deg(T_i)$, $\mu \coloneqq \deg(g)$ for the degrees in K, also when regarded in $K_\mathbb{Q}$. Suitably numbering $w_1, \ldots, w_r$, we ensure counter-clockwise ordering, that means that we always have

$$i \le j \implies \det(w_i, w_j) \ge 0.$$

Note that each ray of $\Lambda(R)$ is of the form $\varrho_i = \mathrm{cone}(w_i)$, but not vice versa. We assume X to be $\mathbb{Q}$-factorial. According to Proposition 2.3.5 this means $\dim(\lambda) = 2$. Then the effective cone of X is uniquely decomposed into three convex sets,

$$\mathrm{Eff}(X) \;=\; \lambda^- \cup \lambda^\circ \cup \lambda^+,$$

where λ^- and λ^+ are convex polyhedral cones not intersecting $\lambda^\circ = \mathrm{Ample}(X)$ and $\lambda^- \cap \lambda^+$ consists of the origin. By Remark 2.3.1 and Proposition 2.3.3, each of λ^- and λ^+ contains at least two of the degrees $w_1, \ldots, w_r$.



Note that λ^- as well as λ^+ might be one-dimensional. As a GIT-cone in $K_\mathbb{Q} \cong \mathbb{Q}^2$, the closure $\lambda = \mathrm{SAmple}(X)$ of $\lambda^\circ = \mathrm{Ample}(X)$ is the intersection of two projected $\bar{X}$-faces and thus we find at least one of the w_i on each of its bounding rays.

Remark 2.5.2. Setting 2.5.1 is respected by orientation preserving automorphisms of K. If we apply an orientation reversing automorphism of K, then we regain Setting 2.5.1 by reversing the numeration of $w_1, \ldots, w_r$. Moreover, we may interchange the numeration of T_i and T_j if w_i and w_j share a common ray without affecting Setting 2.5.1. We call these operations *admissible coordinate changes.*

Working in Setting 2.5.1 enables us to obtain constraints on the generator degrees as well as the relation degree from their geometric constellation. For this the following lemmas are crucial.

Lemma 2.5.3. *Consider a locally factorial $X = X(\lambda)$ arising from Construction 2.3.4 with only one relation, i.e., $s = 1$. Let i, j with $\lambda \subseteq \operatorname{cone}(w_i, w_j)$. Then either w_i, w_j generate K as a group, or g_1 has precisely one monomial of the form $T_i^{l_i} T_j^{l_j}$, where $l_i + l_j > 0$.*

Proof. If $\gamma_{i,j}$ is an X-face, then Proposition 2.3.5 (ii) tells us that w_i and w_j generate K as a group. Now consider the case that $\gamma_{i,j}$ is not an X-face. Then we must have $\lambda^\circ \not\subseteq Q(\gamma_{i,j})^\circ$ or $\gamma_{i,j}$ is not an $\bar{X}$-face. Proposition 2.3.5 (i) excludes the first possibility. Thus, the second one holds, which in turn means that g_1 has precisely one monomial of the form $T_i^{l_i} T_j^{l_j}$, where $l_i + l_j > 0$. □

Lemma 2.5.4. *Let $X = X(\lambda)$ be as in Setting 2.5.1 and let $1 \le i < j < k \le r$. If X is locally factorial, then w_i, w_j, w_k generate K as a group provided that one of the following holds:*

(i) *$w_i, w_j \in \lambda^-$, $w_k \in \lambda^+$ and g has no monomial of the form $T_k^{l_k}$,*
(ii) *$w_i \in \lambda^-$, $w_j, w_k \in \lambda^+$ and g has no monomial of the form $T_i^{l_i}$,*
(iii) *$w_i \in \lambda^-$, $w_j \in \lambda^\circ$, $w_k \in \lambda^+$.*

Moreover, if (iii) holds, then g has a monomial of the form $T_j^{l_j}$ where l_j is divisible by the order of the factor group $K/\langle w_i, w_k \rangle$. In particular, if K is torsion-free, then l_j is a multiple of $\det(w_i, w_k)$.

Proof. Assume that (i) holds. If K is generated by w_i, w_k or by w_j, w_k, then we are done. Consider the case that none of the pairs w_i, w_k and w_j, w_k generates K. Applying Lemma 2.5.3 to each of the pairs shows that g has precisely one monomial of the form $T_i^{l_i} T_k^{l_k}$ with $l_i + l_k > 0$ and precisely one monomial of the form $T_j^{l_j} T_k^{l'_k}$ with $l_j + l'_k > 0$. By assumption, we must have $l_i, l_j > 0$. We conclude that $\gamma_{i,j,k}$ is an X-face. Since X is locally factorial, Proposition 2.3.5 (ii) yields that w_i, w_j, w_k generate K. If (ii) holds, then a suitable admissible coordinate change leads to (i).

Assume that (iii) holds. If K is generated by w_i, w_k there is nothing to show. We consider the case where w_i, w_k do not generate K. Lemma 2.5.3 yields that g has a monomial of the form $T_i^{l_i} T_k^{l_k}$ with $l_i + l_k > 0$. From Proposition 2.2.8 we infer that g has a monomial of the form $T_j^{l_j}$ with $l_j > 0$ as well. Remark 2.2.5 says that $\gamma_{i,j,k}$ is an X-face. Since X is locally factorial, Proposition 2.3.5 (ii) yields that w_i, w_j, w_k generate K.

We turn to the supplement. Consider $G = K/\langle w_i, w_k \rangle$. Since w_i, w_j, w_k generate K, the class of w_j generates G. From $l_j w_j = \mu = l_i w_i + l_k w_k$ we infer $l_j w_j = 0 \in G$, thus l_j is divisible by $\operatorname{ord}(w_j) = \operatorname{ord}(G)$. □

Lemma 2.5.5. *Assume u, w_1, w_2 generate the abelian group $\mathbb{Z}^2$. If $w_i = a_i w$ holds with a primitive $w \in \mathbb{Z}^2$ and $a_i \in \mathbb{Z}$, then (u, w) is a basis for $\mathbb{Z}^2$ and u is primitive.*

Lemma 2.5.6. *Let $w_1, \dots, w_4 \in \mathbb{Z}^2$ such that* $\det(w_1, w_3)$*,* $\det(w_1, w_4)$*,* $\det(w_2, w_3)$ *and* $\det(w_2, w_4)$ *all equal one. Then $w_1 = w_2$ or $w_3 = w_4$ holds.*

It turns out that for locally factorial $X(\lambda)$ and a suitably general relation g the GIT-fan of R can be read from the geometric constellation of the Cox ring generator degrees $w_1, \dots, w_r$ and the relation degree μ even without explicit knowledge of their coordinates.

Lemma 2.5.7. *In Setting 2.5.1, assume that $X = X(\lambda)$ is locally factorial and R_g a spread hypersurface Cox ring.*

(i) *If w_i lies on the ray through μ, then g has a monomial of the form $T_i^{l_i}$ where $l_i \geq 2$.*

(ii) *If w_i, w_j, where $i \neq j$, lie on the ray through μ, then $\varrho_i = \varrho_j \in \Lambda(R_g)$ holds.*

Proof. We show (i). Suppose that g has no monomial of the form $T_i^{l_i}$ where $l_i \geq 2$. As R_g is a hypersurface Cox ring, also T_i is not a monomial of g. Then, on one of the extremal rays of $\mathrm{Eff}(R)$, we find a w_j such that $\gamma_{i,j}$ is a X-face; see Remark 2.2.7 (i). Proposition 2.3.5 (ii) yields that w_i, w_j generate $\mathbb{Z}^2$ as a group. In particular, w_i is primitive. Hence $\mu = kw_i$ holds for some $k \in \mathbb{Z}_{\geq 1}$. As R_g is spread, T_i^k must be a monomial of g. In addition, we obtain $k \geq 2$. A contradiction.

We prove (ii). Assertion (i) just proven and Remark 2.2.7 (i) tell us that $\gamma_{i,j}$ is an $\bar{X}$-face. Thus, being a ray, $Q(\gamma_{i,j}) = \varrho_i = \varrho_j$ belongs to the GIT-fan $\Lambda(R_g)$. □

Proposition 2.5.8. *Situation as in Setting 2.5.1. Assume that $X(\lambda)$ is locally factorial and R is a spread hypersurface Cox ring. The ray ϱ_i is a GIT-cone if and only if one of the following conditions hold:*

(i) *μ is not contained in ϱ_i,*

(ii) *$\mu \in \varrho_i$ and $w_j \in \varrho_i$ holds for some $i \neq j$.*

Proof. The if-part is a direct consequence of Remark 2.2.5 (ii) and Lemma 2.5.7 (ii). We turn to the only-if-part. So assume $\varrho_i \in \Lambda(R)$. If $\mu \notin \varrho_i$ holds, we are done. We consider the case $\mu \in \varrho_i$. Being a GIT-cone ϱ_i is the intersection of some projected $\bar{X}$-faces. Due to $\mathrm{rank}(K) = 2$ every projected $\bar{X}$-face is of the form $\mathrm{cone}(w_k, w_l)$ with $k \leq l$. We conclude that there is some $\bar{X}$-face $\gamma_I \preceq \gamma$ such that $Q(\gamma_I) = \mathrm{cone}(w_i, w_k)$ holds. After suitably renumbering $w_1, \dots, w_r$, we may assume $i \in I$. According to Lemma 2.5.7 the polynomial g owns a monomial of the form $T_i^{l_i}$. Since γ_I is an $\bar{X}$-face, Remark 2.2.5 (i) ensures that g_I, see Definition 2.2.6, has monomial T^ν not equal to $T_i^{l_i}$. Note that μ lies on an extremal ray of $Q(\gamma_I)$. We conclude that I contains some $i \neq j$ with $w_j \in \varrho_i$. □

Remark 2.5.9. In Setting 2.5.1 all full-dimensional cones of the GIT-fan $\Lambda(R)$ are of the form $\mathrm{cone}(w_i, w_j)$. Moreover, taking counter-clockwise ordering of $w_1, \dots, w_r$ into account we observe that $\eta = \mathrm{cone}(w_i, w_j)$ is a two-dimensional GIT-cone if and only if

(i) both ϱ_i and ϱ_j are distinct GIT-cones, and

(ii) for any $w_k \in \eta^\circ$ the ray ϱ_k is not a GIT-cone.

Corollary 2.5.10. *Situation as in Setting 2.5.1. Assume that $X(\lambda)$ is locally factorial and R is a spread hypersurface Cox ring. Then the full-dimensional cones of $\Lambda(R)$ are* *precisely the cones* $\eta = \mathrm{cone}(w_i, w_j)$ *where* $\varrho_i \neq \varrho_j$ *and one of the following conditions is satisfied:*

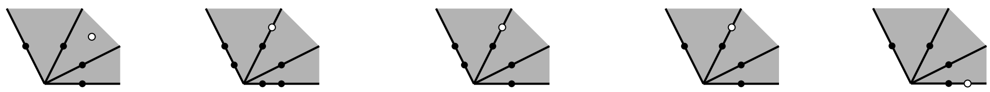
(i) $\mu \in \varrho_i$ *holds,* ϱ_i *contains at least two generator degrees and* η° *contains no generator degree,*
(ii) $\mu \in \varrho_j$ *holds,* ϱ_j *contains at least two generator degrees and* η° *contains no generator degree,*
(iii) $\mu \in \eta^\circ$ *holds and there is at most one* $w_k \in \eta^\circ$*, which must lay on the ray through* μ*,*
(iv) $\mu \notin \eta$ *holds and* η° *contains no generator degrees.*

Now we finalize the arrangements for the proof of Theorem 2.1.1.

Remark 2.5.11. In Setting 2.5.1, consider the rays $\varrho_i := \mathrm{cone}(w_i) \subseteq \mathbb{Q}^2$, where $i = 1, \ldots, r$, and the degree $\mu = \deg(g)$ of the relation. Set

$$\Gamma \ := \ \varrho_1 \cup \cdots \cup \varrho_r, \qquad\qquad \Gamma^\circ \ := \ \Gamma \cap \mathrm{Eff}(R)^\circ.$$

Then a suitable admissible coordinate change turns the setting into one of the following



(I) $\mu \notin \Gamma$ — (IIa) $\mu \in \Gamma^\circ$, $\varrho_1 = \varrho_2$, $\varrho_{r-1} = \varrho_r$ — (IIb) $\mu \in \Gamma^\circ$, $\varrho_1 \neq \varrho_2$, $\varrho_{r-1} = \varrho_r$ — (IIc) $\mu \in \Gamma^\circ$, $\varrho_1 \neq \varrho_2$, $\varrho_{r-1} \neq \varrho_r$ — (III) $\mu \in \varrho_1$

where the figures exemplarily sketch the case $r = 6$, the black dots indicate the generator degrees and the white dot stands for the relation degree.

Our proof of Theorem 2.1.1 will be split into Parts I, IIa, IIb, IIc and III according to the constellations of Remark 2.5.11. The reason why we restrict Theorem 2.1.1 to the ground field $\mathbb{K} = \mathbb{C}$ is that we use the following references on complex Fano varieties.

Remark 2.5.12. Let X be a smooth complex Fano variety. Then the divisor class group $\mathrm{Cl}(X)$ of X is torsion free; see for instance [84, Prop. 2.1.2]. Moreover, if $\dim(X) = 4$ holds, then [31, Rem. 3.6] tells us that any $\mathbb{Q}$-factorial projective variety being isomorphic in codimension one to X is smooth as well. In terms of Construction 2.3.2, the latter means that all varieties $X(\eta)$ are smooth, where $\eta \in \Lambda(R)$ is full-dimensional with $\eta^\circ \subseteq \mathrm{Mov}(R)^\circ$.

2.6 Proof of Theorem 2.1.1: Collecting candidates I

We treat Case I from Remark 2.5.11. Here the degree of the defining relation is not proportional to any of the Cox ring generator degrees. Here are first constraints on the possible specifying data in this situation.

Proposition 2.6.1. *Situation as in Setting 2.5.1. Assume that* $r = 7$, $K \cong \mathbb{Z}^2$ *holds, every two-dimensional* $\lambda \in \Lambda(R)$ *with* $\lambda^\circ \subseteq \mathrm{Mov}(R)^\circ$ *defines a locally factorial* $X(\lambda)$

and μ doesn't lie on any of the rays $\varrho_1, \ldots, \varrho_7$. Then, after a suitable admissible coordinate change, we have $\mu \in \operatorname{cone}(w_4, w_5)^\circ$ and one of the following holds:

(i) $w_1 = w_2$ *and* $w_5 = w_6$,	(iv) $w_2 = w_3$ *and* $w_6 = w_7$,
(ii) $w_1 = w_2$ *and* $w_6 = w_7$,	(v) $w_3 = w_4$ *and* $w_5 = w_6$,
(iii) $w_2 = w_3$ *and* $w_5 = w_6$,	(vi) $w_3 = w_4$ *and* $w_6 = w_7$.

Proof of Proposition 2.6.1. The assumption $\mu \notin \varrho_i$ implies $\varrho_i \in \Lambda(R)$ for $i = 1, \ldots, 7$, see Remark 2.2.7 (ii). Proposition 2.2.4 gives $\mu \in \operatorname{cone}(w_3, w_5)$. The latter cone is the union of $\operatorname{cone}(w_3, w_4)$ and $\operatorname{cone}(w_4, w_5)$; both are GIT-cones, one of them is two-dimensional and hosts μ in its relative interior. A suitable admissible coordinate change yields $\mu \in \operatorname{cone}(w_4, w_5)^\circ$.

First we show that if $w_i \in \varrho_j$ holds for some $1 \le i < j \le 4$, then two of w_5, w_6, w_7 coincide. Consider the case $w_5, w_6 \in \varrho_5$. By assumption $X(\lambda)$ is locally factorial for $\lambda = \operatorname{cone}(w_4, w_5)$. Thus, we can apply Lemma 2.5.4 to w_i, w_j, w_5 and also to w_i, w_j, w_6 and obtain that each of the triples generates K as a group. Lemma 2.5.5 yields that w_5 and w_6 are primitive and hence, lying on a common ray, coincide. Now, assume $w_6 \notin \varrho_5$. Then we consider $X = X(\lambda)$ for $\lambda = \operatorname{cone}(w_5, w_6)$. Using Lemma 2.5.4 as before, see that w_i, w_j, w_6 as well as w_i, w_j, w_7 generate K as a group. For the primitive generator w of $\varrho_i = \varrho_j$, we infer $\det(w, w_6) = 1$ and $\det(w, w_7) = 1$ from Lemma 2.5.5. Moreover, $\gamma_{5,6}$ and $\gamma_{5,7}$ are X-faces due to Remark 2.2.7 (ii). Thus, Proposition 2.3.5 (ii) yields $\det(w_5, w_6) = 1$ and $\det(w_5, w_7) = 1$. Lemma 2.5.6 yields $w_6 = w_7$.

We conclude the proof by showing that at least two of $w_1, \ldots, w_4$ coincide. Consider the case $w_2 \in \varrho_3$. Then, by the first step, there are $5 \le i < j \le 7$ with $w_i = w_j$. Taking $X(\lambda)$ for $\lambda = \operatorname{cone}(w_4, w_5)$ and applying Lemma 2.5.4 to w_2, w_i, w_j as well as to w_3, w_i, w_j, we obtain that each of these triples generates K. Because of $w_i = w_j$, we directly see that w_2 and w_3, each being part of a $\mathbb{Z}$-basis, are primitive and hence coincide. We are left with the case that $\lambda' = \operatorname{cone}(w_2, w_3)$ is of dimension two. By assumption, the variety X' defined by λ' is locally factorial. Moreover, Remark 2.2.7 (ii) provides us with the X'-faces $\gamma_{1,3}, \gamma_{2,3}, \gamma_{1,4}$ and $\gamma_{2,4}$. By Proposition 2.3.5 (ii), all corresponding determinants $\det(w_k, w_m)$ equal one. Lemma 2.5.6 shows that at least two of $w_1, \ldots, w_4$ coincide. □

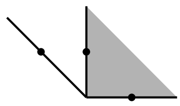
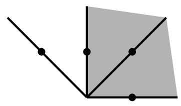
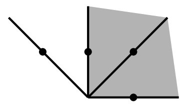
We are ready to enter Part I of the proof of Theorem 2.1.1. The task is to work out further the degree constellations left by Proposition 2.6.1. This leads to major multistage case distinctions.

Proof of Theorem 2.1.1: Part I. This part of the proof treats the case that $\mu = \deg(g)$ doesn't lie on any of the rays $\varrho_i = \operatorname{cone}(w_i)$. In particular, by Remark 2.2.7 (ii), all rays $\varrho_1, \ldots, \varrho_7$ belong to the GIT-fan $\Lambda(R)$. By Remark 2.5.12, every two-dimensional $\eta \in \Lambda(R)$ with $\eta^\circ \subseteq \operatorname{Mov}(R)^\circ$ produces a smooth variety $X(\eta)$. Thus, we can apply Proposition 2.6.1, which leaves us with $\mu \in \operatorname{cone}(w_4, w_5)^\circ$ and the six possible constellations for $w_1, \ldots, w_7$ given there. Again by Remark 2.5.12, the divisor class group of X is torsion free, that means that we have $K = \mathbb{Z}^2$.

Constellation 2.6.1 (i). We have $w_1 = w_2$ and $w_5 = w_6$. Lemma 2.5.4 applied to w_1, w_2, w_5 shows that w_1, w_5 form a basis of $\mathbb{Z}^2$. Thus, a suitable admissible coordinate change gives $w_1 = (1, 0)$ and $w_6 = (0, 1)$. Applying Lemma 2.5.4 also to w_1, w_2, w_7 and w_i, w_5, w_6 where $i = 1, \ldots, 4$ yields the first coordinate of $w_1, \ldots, w_4$ and the second coordinate of w_7 equal one. Thus, the degree matrix has the form

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -a_7 \\ 0 & 0 & b_3 & b_4 & 1 & 1 & 1 \end{bmatrix}, \qquad b_3, b_4, a_7 \in \mathbb{Z}_{\geq 0}.$$

We determine the possible values of b_3 and b_4. If $b_3 > 0$ holds, then $\eta = \mathrm{cone}(w_2, w_3)$ is two-dimensional and satisfies $\eta^\circ \subseteq \mathrm{Mov}(R)^\circ$. Because of $\mu \in \mathrm{cone}(w_4, w_5)^\circ$, none of the monomials of g is of the form $T_1^{l_1} T_j^{l_j}$ with $j = 3, 4$. Lemma 2.5.3 applied to $X(\eta)$ gives $b_j = \det(w_1, w_j) = 1$ for $j = 3, 4$. If $b_3 = 0$ and $b_4 > 0$ hold, we argue similarly with $\eta = \mathrm{cone}(w_2, w_4)$ and obtain $b_4 = 1$. Altogether, we arrive at the three cases

2.6.1 (i-a): $b_3 = b_4 = 0$, 2.6.1 (i-b): $b_3 = 0$, $b_4 = 1$, 2.6.1 (i-c): $b_3 = b_4 = 1$.

Case 2.6.1 (i-a). Here, the semiample cone λ of $X = X(\lambda)$ must be the positive orthant. Thus, X being Fano just means that both coordinates of the anticanonical class $-\mathcal{K}_X \in K = \mathbb{Z}^2$ are strictly positive. According to Proposition 2.3.7, we have

$$-\mathcal{K}_X \ = \ (4 - a_7 - \mu_1, \, 3 - \mu_2).$$

We conclude $1 \leq \mu_2 \leq 2$ and $1 \leq \mu_1 < 4 - a_7$ which implies in particular $0 \leq a_7 \leq 2$. Thus, the weights $w_1, \ldots, w_7$ and the degree μ must be as in Theorem 2.1.1, Numbers 1 to 12.

Case 2.6.1 (i-b). Here, either $\lambda = \mathrm{cone}(w_3, w_4)$ or $\lambda = \mathrm{cone}(w_4, w_5)$ holds. In any case, the anticanonical class is given as

$$-\mathcal{K}_X \ = \ (4 - a_7 - \mu_1, \, 4 - \mu_2).$$

First assume that $\lambda = \mathrm{cone}(w_3, w_4)$ holds. Then, X being Fano, we have $-\mathcal{K}_X \in \lambda^\circ$. The latter is equivalent to the inequalities

$$4 - \mu_2 \ > \ 0, \qquad \mu_2 - \mu_1 - a_7 \ > \ 0.$$

Using $\mu \in \mathrm{cone}(w_4, w_5)^\circ$, we conclude $1 \leq \mu_1 < \mu_2 \leq 3$ and $0 \leq a_7 \leq 1$. Thus, we end up with

$$a_7 = 0 \text{ and } \mu = (1, 2), (1, 3), (2, 3), \qquad a_7 = 1 \text{ and } \mu = (1, 3).$$

Note that in all cases, $\gamma_{1,2,3,4}$ is an X-face according to Remark 2.2.7 (ii). Since X is quasismooth, Proposition 2.3.6 yields

$$\mu \ \in \ Q(\gamma_{1,2,3,4}) \cup w_7 + Q(\gamma_{1,2,3,4}).$$

This excludes $a_7 = 0$ and $\mu = (1, 3)$. The remaining three cases are Numbers 13 to 15 of Theorem 2.1.1.

Now, assume that $\lambda = \mathrm{cone}(w_4, w_5)$ holds. The condition that $X = X(\lambda)$ is Fano means $-\mathcal{K}_X \in \lambda^\circ$, which translates into the inequalities $0 < 4 - a_7 - \mu_1 < 4 - \mu_2$. Moreover, $\mu \in \lambda^\circ$ implies $\mu_1 < \mu_2$ and we conclude

$$1 \ \le\ \mu_1 \ <\ \mu_2 \ <\ \mu_1 + a_7 \ \le\ 3.$$

This is only possible for $a_7 = 2$ and $\mu = (1, 2)$. Then we have $w_4 = (1, 1)$ and $w_7 = (-2, 1)$. In particular, g admits no monomial of the form $T_4^{l_4} T_7^{l_7}$. Lemma 2.5.3 tells us that w_4 and w_7 generate $K = \mathbb{Z}^2$ as a group. A contradiction.

Case 2.6.1 (i-c). Applying Remark 2.5.12 and Lemma 2.5.4 to $X(\eta)$ with $\eta = \mathrm{cone}(w_4, w_5)$ and w_3, w_4, w_7 yields $\det(w_4, w_7) = 1$. From this we infer $a_7 = 0$. Thus, either $\lambda = \mathrm{cone}(w_2, w_3)$ or $\lambda = \mathrm{cone}(w_4, w_5)$ holds. In any case, the anticanonical class is

$$-\mathcal{K}_X \ =\ (4 - \mu_1,\ 5 - \mu_2).$$

Assume $\lambda = \mathrm{cone}(w_2, w_3)$. Then the Fano condition $-\mathcal{K}_X \in \lambda^\circ$ implies $\mu_1 + 1 < \mu_2$. Remark 2.2.7 (ii) says that $\gamma_{1,2,3,4}$ is an X-face. As before, Proposition 2.3.6 gives

$$\mu \ \in\ Q(\gamma_{1,2,3,4}) \cup w_7 + Q(\gamma_{1,2,3,4}).$$

We conclude $\mu_1 + 1 \ge \mu_2$. A contradiction. Now, assume $\lambda = \mathrm{cone}(w_4, w_5)$. Then $-\mathcal{K}_X \in \lambda^\circ$ yields $\mu_1 \ge \mu_2$. But we have $\mu \in \mathrm{cone}(w_4, w_5)^\circ$, hence $\mu_1 < \mu_2$. A contradiction.

Constellation 2.6.1 (ii). We have $w_1 = w_2$ and $w_6 = w_7$. Lemma 2.5.4 applied to w_1, w_6, w_7 shows that w_1, w_7 generate $\mathbb{Z}^2$. Hence, a suitable admissible coordinate change yields $w_1 = w_2 = (1, 0)$ and $w_6 = w_7 = (0, 1)$. Applying Lemma 2.5.4 to w_3, w_6, w_7 and w_4, w_6, w_7, we obtain that the first coordinates of w_3 and w_4 both equal one. Thus, the degree matrix has the form

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & a_5 & 0 & 0 \\ 0 & 0 & b_3 & b_4 & 1 & 1 & 1 \end{bmatrix}, \qquad\qquad a_5, b_3, b_4 \in \mathbb{Z}_{\ge 0}.$$

By assumption w_4 and w_5 don't lie on a common ray. Consequently, $b_4 = 0$ or $a_5 = 0$ holds. If $a_5 = 0$ holds, then we are in Constellation 2.6.1 (i) just treated. So, assume $a_5 > 0$. Then $b_3 = b_4 = 0$ holds. Taking $X(\eta)$ for $\eta = \mathrm{cone}(w_5, w_6)$ and applying Lemma 2.5.3 to w_5, w_6 yields $a_5 = 1$. We arrive at the degree matrix

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Observe that either $\lambda = \mathrm{cone}(w_4, w_5)$ or $\lambda = \mathrm{cone}(w_5, w_6)$ holds. In any case, the anticanonical class of $X = X(\lambda)$ is given as

$$-\mathcal{K}_X \ =\ (5 - \mu_1,\ 3 - \mu_2).$$

First, assume $\lambda = \mathrm{cone}(w_4, w_5)$. Then X being Fano means $0 < 3 - \mu_2 < 5 - \mu_1$. We conclude $\mu_2 \le 2$ and $\mu_1 \le \mu_2 + 1$. Moreover, $\mu \in \mathrm{cone}(w_4, w_5)^\circ$ gives $0 < \mu_2 < \mu_1$. Thus, we have $\mu_1 = \mu_2 + 1$ and arrive at the possibilities $\mu = (2,1), (3,2)$, which are Numbers 16 and 17 in Theorem 2.1.1. Now, let $\lambda = \mathrm{cone}(w_5, w_6)$. Then X being Fano gives $0 < 5 - \mu_1 < 3 - \mu_2$. We conclude $\mu = (4,1)$. Remark 2.2.7 (ii) provides us with the X-face $\gamma_{5,6,7}$. Proposition 2.3.6 says that μ should lie in $Q(\gamma_{5,6,7})$ or in $w_1 + Q(\gamma_{5,6,7})$. A contradiction.

Constellation 2.6.1 (iii). We have $w_2 = w_3$ and $w_5 = w_6$. Lemma 2.5.4 applied to w_2, w_5, w_6 shows that w_2, w_5 form a basis of $\mathbb{Z}^2$. A suitable admissible coordinate change leads to $w_2 = w_3 = (1,0)$ and $w_5 = w_6 = (0,1)$. Again Lemma 2.5.4, this time applied to w_1, w_5, w_6, to w_4, w_5, w_6 and to w_2, w_3, w_7, shows $w_1 = (1, -a_1)$, $w_4 = (1, b_4)$ and $w_7 = (-a_7, 1)$ where $a_1, a_7, b_4 \in \mathbb{Z}_{\ge 0}$. From $\det(w_1, w_7) > 0$ we infer $a_1 = 0$ or $a_7 = 0$. Since the case $w_1 = w_2$ is already covered by Constellation 2.6.1 (i), we may assume $w_1 \neq w_2$ i.e. $a_1 > 0$ and $a_7 = 0$. Hence, the degree matrix is

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -a_1 & 0 & 0 & b_4 & 1 & 1 & 1 \end{bmatrix}, \quad a_1 \in \mathbb{Z}_{\ge 1},\ b_4 \in \mathbb{Z}_{\ge 0}.$$

We claim $b_4 = 0$. Suppose $b_4 > 0$. Then, $\eta = \mathrm{cone}(w_3, w_4)$ is a two-dimensional GIT-chamber with $\eta^\circ \subseteq \mathrm{Mov}(R)^\circ$. Consider the associated variety $X(\eta)$. Due to $\mu \in \mathrm{cone}(w_4, w_5)^\circ$, none of the monomials of g is of the form $T_3^{l_3} T_4^{l_4}$. From Lemma 2.5.3 we infer $a_1 + b_4 = \det(w_1, w_4) = 1$, hence $a_1 = 0$ or $b_4 = 0$. A contradiction.

So, we have $b_4 = 0$. By Proposition 2.3.7 the anticanonical class $-\mathcal{K}_X$ is given as

$$-\mathcal{K}_X = (4 - \mu_1,\ 3 - a_1 - \mu_2).$$

Here, we have $\lambda = \mathrm{cone}(w_3, w_5)$, which is the positive orthant. Thus, $X = X(\lambda)$ being Fano means that both coordinates of $-\mathcal{K}_X$ are positive. We directly obtain $\mu_1 \le 3$. Moreover, from $3 - a_1 - \mu_2 > 0$ we deduce $a_1 = 1$ and $\mu_2 = 1$. We end up with specifying data as in Numbers 18 to 20 from Theorem 2.1.1.

Constellation 2.6.1 (iv). We have $w_2 = w_3$ and $w_6 = w_7$. Lemma 2.5.4 applied to w_2, w_6, w_7 shows that w_2, w_6 generate the group $\mathbb{Z}^2$. A suitable admissible coordinate change leads to $w_2 = w_3 = (1,0)$ and $w_6 = w_7 = (0,1)$. Applying Lemma 2.5.4 to w_1, w_6, w_7 and w_4, w_6, w_7 shows that the first coordinate of both w_1 and w_4 equals one. Lemma 2.5.4 applied to w_2, w_3, w_5 yields that the second coordinate of w_5 equals one. Since the case $w_5 = w_6$ has already been treated in Constellation 2.6.1 (iii), we may assume $w_5 \neq (0,1)$. Thereby, $\det(w_4, w_5) > 0$ gives $w_4 = (1,0)$. The degree matrix is

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -a_1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad a_1, \in \mathbb{Z}_{\ge 1}.$$

By Proposition 2.3.7 the anticanonical class $-\mathcal{K}_X$ of X is given by

$$-\mathcal{K}_X = (5 - \mu_1,\ 3 - a_1 - \mu_2).$$

From $X = X(\lambda)$ being Fano we infer $-\mathcal{K}_X \in \mathrm{Mov}(R)^\circ$ i.e. both coordinates of $-\mathcal{K}_X$ are positive. We directly obtain $\mu_1 \leq 4$. From $3 - a_1 - \mu_2 > 0$ we deduce $a_1 = 1$ and $\mu_2 = 1$. Now, we have $\det(w_1, w_5) = 2$, thus Lemma 2.5.3 tells us that g must have a monomial of the form $T_1^{l_1} T_5^{l_5}$ where $l_1 + l_5 > 0$. From $\mu \in \mathrm{cone}(w_4, w_5)^\circ$ we infer $l_1, l_5 > 0$. Moreover, $\mu_2 = 1$ means $l_5 = l_1 + 1$. Using $l_1 + l_5 = \mu_1 \leq 4$ amounts to $l_1 = 1$, $l_5 = 2$ and thus $\mu = (3, 1)$. We have arrived at Number 21 from Theorem 2.1.1.

Constellation 2.6.1 (v). We have $w_3 = w_4$ and $w_5 = w_6$. Lemma 2.5.4 applied to w_3, w_4, w_5 shows that w_4, w_5 generate the group $\mathbb{Z}^2$. A suitable admissible coordinate change leads to $w_3 = w_4 = (1, 0)$ and $w_5 = w_6 = (0, 1)$. Applying Lemma 2.5.4 to w_1, w_5, w_6 and w_2, w_5, w_6 shows that w_1, w_2 are primitive. Thus, for $i = 1, 2$ either $w_i = w_3$ holds or $\eta = \mathrm{cone}(w_2, w_3)$ gives rise to a smooth variety $X(\eta)$. According to Lemma 2.5.3, the latter implies $w_i = (1, -1)$. This amounts to $w_1 = w_2$ or $w_2 = w_3$. Hence, this constellation is completely covered by Constellations 2.6.1 (i), (iii).

Constellation 2.6.1 (vi). We have $w_3 = w_4$ and $w_6 = w_7$. Lemma 2.5.4 applied to w_3, w_4, w_6 shows that w_4, w_6 generate the group $\mathbb{Z}^2$. A suitable admissible coordinate change leads to $w_3 = w_4 = (1, 0)$ and $w_6 = w_7 = (0, 1)$. Applying Lemma 2.5.4 to w_1, w_6, w_7 and w_2, w_6, w_7 shows that w_1, w_2 are primitive. Thus, for $i = 1, 2$ either $w_i = w_3$ holds or $\eta = \mathrm{cone}(w_2, w_3)$ gives rise to a smooth variety $X(\eta)$. According to Lemma 2.5.3, the latter implies $w_i = (1, -1)$. This amounts to $w_1 = w_2$ or $w_2 = w_3$. Hence, this constellation is completely covered by Constellations 2.6.1 (ii), (iv). □

2.7 Proof of Theorem 2.1.1: Collecting candidates II

In the present part of the proof the Fano property will often lead to numerical constraints which are presented directly as or can be rephrased as an inequation of the form

$$x_1 \cdots x_n \leq x_1 + \cdots + x_n, \quad x_1, \ldots, x_n \in \mathbb{Z}_{\geq 1}. \tag{2.1}$$

Lemma 2.7.1. *Let $n \in \mathbb{Z}_{\geq 2}$ and consider positive integers $x_1 \leq \cdots \leq x_n$ satisfying Eq.* (2.1)*. Then $x_1, \ldots, x_{n-1}$ all equal one or $x_n \leq n^2 - n$ holds.*

Proof. Observe $x_1 \cdots x_n \leq n x_n$. From this we infer $x_i \leq n$ for all $i = 1, \ldots, n-1$. If $x_i > 1$ holds for some $1 \leq i \leq n-1$, i.e. $x_1 \cdots x_{n-1} > 1$, then we obtain

$$x_n \leq \frac{x_1 + \cdots + x_{n-1}}{x_1 \cdots x_{n-1} - 1} \leq n(n-1) = n^2 - n.$$

□

This allows us to explicitly present the solutions of Eq. (2.1) in the cases we will face in the subsequent parts of the proof of Theorem 2.1.1.

Remark 2.7.2. The following table describes the solutions of Eq. (2.1) for $n = 3, 4, 5$ where $x_1, \ldots, x_n$ are in ascending order. Here, $*$ stands for an arbitrary positive integer.

n	x_1	x_2	x_3	x_4
3	1	1	$*$	—
	1	2	2	—
	1	2	3	—
4	1	1	1	$*$
	1	1	2	3
	1	1	2	4

n	x_1	x_2	x_3	x_4	x_5
5	1	1	1	1	$*$
	1	1	1	2	2
	1	1	1	2	3
	1	1	1	2	4
	1	1	1	2	5
	1	1	1	3	3

Part IIa • We discuss Case IIa from Remark 2.5.11, i.e., the degree of the relation lies in the interior of the effective cone, is proportional to some Cox ring generator degree and $\varrho_1 = \varrho_2$ as well as $\varrho_{r-1} = \varrho_r$ hold.

Lemma 2.7.3. *In Setting 2.5.1 assume that* $\mathrm{Mov}(R) = \mathrm{Eff}(R)$ *and* $\mu \in \mathrm{Eff}(R)^\circ$ *hold. Let* Ω *denote the set of two-dimensional cones* $\eta \in \Lambda(R)$ *with* $\eta^\circ \subseteq \mathrm{Mov}(R)^\circ$*.*

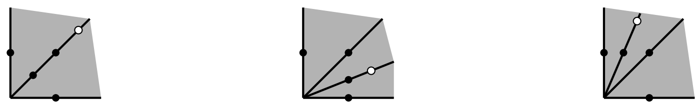
(i) *If* $X(\eta)$ *is locally factorial for some* $\eta \in \Omega$*, then* $\mathrm{Eff}(R)$ *is a regular cone and every* w_i *on the boundary of* $\mathrm{Eff}(R)$ *is primitive.*
(ii) *If* $X(\eta)$ *is locally factorial for all* $\eta \in \Omega$*, then, for any* $w_i \in \mathrm{Eff}(R)^\circ$*, we have* $w_i = w_1 + w_r$ *in* $K_{\mathbb{Q}}$ *or* g *has a monomial of the form* $T_i^{l_i}$*.*

Proof. We show (i). Let $w_i \in \varrho_r$. Due to $\mu \in \mathrm{Eff}(R)^\circ$, the relation g has no monomial of the form $T_i^{l_i}$. Thus, Lemmas 2.5.4 and 2.5.5 applied to the triple w_1, w_2, w_i show that w_i is primitive. Analogously, we see that any $w_i \in \varrho_1$ is primitive. In particular, we have $w_1 = w_2$. Thus, applying Lemma 2.5.4 to w_1, w_2, w_r, we obtain that $\mathrm{Eff}(R)$ is a regular cone.

We turn to (ii). Throughout this paragraph we regard $w_1, \ldots, w_r$ as elements of $K_{\mathbb{Q}} = \mathbb{Q}^2$. By (i), we may assume $w_1 = w_2 = (1, 0)$ and $w_{r-1} = w_r = (0, 1)$. Consider $w_i \in \mathrm{Eff}(R)^\circ$ such that $T_i^{l_i}$ is not a monomial of g. Then we find GIT-cones $\eta_1 \subseteq \mathrm{cone}(w_1, w_i)$ and $\eta_2 \subseteq \mathrm{cone}(w_i, w_r)$ defining locally factorial varieties $X(\eta_1)$ and $X(\eta_2)$ respectively. Lemma 2.5.4, applied to w_1, w_2, w_i together with $X(\eta_1)$ and to w_i, w_{r-1}, w_r together with $X(\eta_2)$ shows $w_i = (1, 1) = w_1 + w_r$. □

Proof of Theorem 2.1.1: Part IIa. We deal with the specifying data of a smooth general hypersurface Cox ring R as in Remark 2.5.11 IIa defining a smooth Fano fourfold $X = X(\lambda)$. By Proposition 2.2.4, the relation degree μ lies on ϱ_3, ϱ_4 or ϱ_5. We claim that we can't have $\varrho_3 = \varrho_4 = \varrho_5$. Otherwise Corollary 2.5.10 shows $\eta = \mathrm{cone}(w_1, w_3) \in \Lambda(R)$. Since $X(\eta)$ is smooth by Remark 2.5.12, we may apply Lemma 2.5.4 to the triple w_1, w_3, w_4. According to Lemma 2.5.5 we obtain $\det(w_1, v) = 1$ where v denotes the primitive generator of the ray ϱ_3. Analogous arguments yield $\det(v, w_7) = 1$. Using both determinantal equations we conclude that v and $w_1 + w_7$ are collinear. In particular $w_1 + w_7$ generates $\varrho_3 = \varrho_4 = \varrho_5$. Lemma 2.7.3 (i) tells us $w_1 = w_2$ and $w_6 = w_7$. As a result, Proposition 2.3.7 gives $-\mathcal{K}_X \in \varrho_3$. Moreover, Lemma 2.5.7 (ii) tells us $\varrho_3 \in \Lambda(R_g)$

and thus $\lambda = \varrho_3$, which contradicts $\mathbb{Q}$-factoriality, see Proposition 2.3.5 (i). A suitable admissible coordinate change yields $\mu \notin \varrho_5$ and we are left with the following three constellations:



(i) $\varrho_3 = \varrho_4$, $\mu \in \varrho_3$ (ii) $\varrho_3 \neq \varrho_4$, $\mu \in \varrho_3$ (iii) $\varrho_3 \neq \varrho_4$, $\mu \in \varrho_4$

By Lemma 2.7.3 (i), we can assume $w_1 = w_2 = (1,0)$ and $w_6 = w_7 = (0,1)$. We show $w_5 = (0,1)$. Otherwise, by Lemma 2.7.3 (ii), we must have $w_5 = (1,1)$. Consider $\lambda' = \mathrm{cone}(w_5, w_6)$. Then $\mu \notin \lambda'$ holds. Remark 2.2.7 (ii) tells us that $\gamma_{5,6}$ is an X'-face and hence λ' is a GIT-cone. The associated variety X' is smooth according to Remark 2.5.12. Thus, Proposition 2.3.6 yields $\mu \in w_i + \lambda'$ for some $1 \leq i \leq 7$. By the geometry of the possible degree constellations, only $i = 1, 2$ come into consideration. We conclude $\mu = (e+1, e+f)$ with $e, f \in \mathbb{Z}_{\geq 0}$. Positive orientation of (μ, w_5) gives $f = 0$. Hence, μ is primitive. By Lemma 2.5.7 (i), this contradicts R_g being a spread hypersurface ring.

Constellation (i). Let $v = (v_1, v_2)$ be the primitive generator of $\varrho_3 = \varrho_4$. Due to Lemma 2.5.7 (ii), we have $\varrho_3 \in \Lambda(R_g)$ and thus also $\lambda' = \mathrm{cone}(w_3, w_7)$ is a GIT-cone. The associated variety X' is smooth by Remark 2.5.12. Applying Lemmas 2.5.4 and 2.5.5 to the triple w_3, w_4, w_7 yields $v_1 = 1$ and that the first coordinates of w_3, w_4 are coprime. Arguing similarly with w_1, w_3, w_4 gives $v_2 = 1$. So, the degree matrix has the form

$$Q = \begin{bmatrix} 1 & 1 & a & b & 0 & 0 & 0 \\ 0 & 0 & a & b & 1 & 1 & 1 \end{bmatrix}, \qquad a, b \in \mathbb{Z}_{\geq 1}, \quad \gcd(a,b) = 1.$$

We may assume $a \leq b$. By Lemma 2.5.7 (i), the relation g has monomials of the form $T_3^{l_3}$ and $T_4^{l_4}$. Since $\gcd(a,b) = 1$ holds, we conclude $\mu_1 = \mu_2 = dab$ with $d \in \mathbb{Z}_{\geq 1}$. In particular $\mu_1 \geq ab$ holds. By Proposition 2.3.7, the anticanonical class is given as

$$-\mathcal{K}_X \ = \ (2 + a + b - \mu_1,\ 3 + a + b - \mu_2).$$

From X being Fano we deduce $-\mathcal{K}_X \in \mathrm{Eff}(R)^\circ$, that means that each coordinate of $-\mathcal{K}_X$ is positive. Thus, we obtain

$$2 + a + b \ > \ dab \ \geq \ ab.$$

This implies $a = 1$ or $a = 2, b = 3$. Consider the case $a = 1$. Here we have $\mu = dw_4$, thus R_g being spread and irredundant ensures $d \geq 2$. Now using the inequality again leads to $3 + (1-d)b > 0$ and we end up with possibilities

$$b = 1,\ d = 2, 3, \qquad\qquad b = 2,\ d = 2,$$

leading to the specifying data of Numbers 22 to 24 of Theorem 2.1.1. The constellation $a = 2, b = 3$ immediately implies $d = 1$, which gives the specifying data of Number 25 of Theorem 2.1.1.

Constellation (ii). Here we obtain $w_4 = (0,1)$ by the same arguments used for showing $w_5 = (0,1)$. Write $w_3 = (a_3, b_3)$ and let k be the unique positive integer with $\mu = kw_3$. Then $k \geq 2$ as R_g is spread and $T_1, \ldots, T_7$ form a minimal system of generators. By Proposition 2.3.7, the anticanonical class of $X = X(\lambda)$ is given as

$$-\mathcal{K}_X \;=\; (2 + (1-k)a_3,\; 4 + (1-k)b_3).$$

Moreover, we have $\varrho_3 \notin \Lambda(R_g)$ due to Lemma 2.5.7 (i) and Remark 2.2.7 (i), the defining GIT-cone λ of X is the positive orthant. Thus the Fano condition $-\mathcal{K}_X \in \lambda^\circ$ simply means that both coordinates of $-\mathcal{K}_X$ are positive. This leads to $a_3 = 1$, $k = 2$ and $b_3 \leq 3$. These are Numbers 26 to 28 of Theorem 2.1.1.

Constellation (iii). We obtain $w_3 = (1,0)$ by analogous arguments as used for showing $w_5 = (0,1)$ before. The degree $w_4 = (a_4, b_4)$ has to be determined. A suitable admissible coordinate change yields $a_4 \geq b_4$. By Proposition 2.3.7 the anticanonical class of $X = X(\lambda)$ is given as

$$-\mathcal{K}_X = (3 + (1-k)a_4,\; 3 + (1-k)b_4),$$

where $k \in \mathbb{Z}_{\geq 0}$ is defined via $\mu = kw_3$. As in the preceding constellation, we see that λ is the positive orthant. Thus, $X(\lambda)$ being Fano just means that both coordinates of $-\mathcal{K}_X$ are positive. We end up with the specifying data from Numbers 29 to 32 of Theorem 2.1.1. □

Part IIb • This part deals with Case IIb from Remark 2.5.11: Precisely one extremal ray of $\mathrm{Eff}(R)$ contains more than one Cox ring generator degree, the relation degree lies in the relative interior of $\mathrm{Eff}(R)$ and is proportional to some generator degree.

Lemma 2.7.4. *Situation as in Setting 2.5.1. Assume that R_g is a spread hypersurface Cox ring. If $\mu \in \mathrm{Eff}(R)^\circ$ holds and every two-dimensional $\eta \in \Lambda(R)$ with $\eta^\circ \subseteq \mathrm{Mov}(R)^\circ$ defines a locally factorial $X(\eta)$, then there is at most one ray ϱ_i which is not contained in the boundary of $\mathrm{Eff}(R)$ and contains more than one Cox ring generator degree w_i.*

Proof. Suppose there are indices $1 < i < j < r$ where ϱ_i, ϱ_j, are different rays not contained in the boundary of $\mathrm{Eff}(R)$ such that $w_{i_1}, w_{i_2} \in \varrho_i$, $w_{j_1}, w_{j_2} \in \varrho_j$ holds for some $1 < i_1 < i_2 < j_1 < j_2 < r$. From Lemma 2.5.7 (ii) we infer $\varrho_i, \varrho_j \in \Lambda(R)$. As a result, there must be some full-dimensional $\eta \in \Lambda(R)$ with $\eta^\circ \subseteq \mathrm{Mov}(R)^\circ$ and $\eta \subseteq \mathrm{cone}(w_{i_1}, w_{j_1})$. By assumption, $X(\eta)$ is locally factorial. From Lemma 2.5.4 we deduce that three of $w_{i_1}, w_{i_2}, w_{j_1}, w_{j_2}$ generate K as group. Using Lemma 2.5.5 we may assume $\varrho_i = \mathbb{Q}_{\geq 0} e_1$ and $\varrho_j = \mathbb{Q}_{\geq 0} e_2$. Applying Lemma 2.5.4 to w_1, w_{j_1}, w_{j_2} shows that $w_1 = (1, b_1) \in K_{\mathbb{Q}}$ holds for some $b_1 < 0$. Analogously we obtain $w_r = (a_r, 1) \in K_{\mathbb{Q}}$ with some $a_r < 0$. However, this leads to $\det(w_1, w_r) = 1 - b_1 a_r \leq 0$. A contradiction. □

Lemma 2.7.5. *Consider the situation of Setting 2.5.1. If we have $w_2 = w_3$ and $\mu \in \varrho_2$, then $w_4 \in \varrho_2$ holds.*

Proof. Suppose $w_4 \notin \varrho_2$. Then every monomial of g not being divisible by T_1 is of the form $T_2^{l_2} T_3^{l_3}$ where $l_2 + l_3 > 0$. Since g is prime, thus not divisible by T_1, at least one such monomial occurs with non-zero coefficient in g. From $w_2 = w_3$ we deduce that $g_1 := g(0, T_2, \dots, T_r)$ is a classical homogeneous polynomial in T_2, T_3, thus admits a presentation $g_1 = \ell_1 \cdots \ell_m$ where $\ell_1, \dots, \ell_m$ are linear forms in T_2 and T_3. Here $w_2 = w_3$ ensures that $\ell_1, \dots, \ell_m$ are homogeneous w.r.t. the K-grading. Observe $m > 1$ as the presentation of $R = R_g$ is irredundant; see Setting 2.5.1. We conclude that g_1 is not K-prime, hence $T_1 \in R$ is not K-prime either. A contradiction. □

Proof of Theorem 2.1.1: Part IIb. As in the previous parts of the proof we work in Setting 2.5.1. In this part we consider the constellation IIb of Remark 2.5.11. Let us establish first constraints. Lemma 2.5.4 shows that w_1, w_6, w_7 generate $\mathbb{Z}^2$ as group. According to Lemma 2.5.5, by applying a suitable admissible coordinate we achieve $w_1 = (1, 0)$ and that the vector $e_2 = (0, 1)$ generates the ray $\varrho_6 = \varrho_7$. The following discussion splits into two major cases

$$\varrho_2 \neq \varrho_3 \quad \text{and} \quad \varrho_2 = \varrho_3.$$

We start with $\varrho_2 \neq \varrho_3$. Here Proposition 2.2.4 ensures $\mu \notin \varrho_2$. This enables us to apply Lemma 2.5.4 to w_2, w_6, w_7. We deduce that the first coordinate of w_2 equals one. So far, the degree matrix is of the form

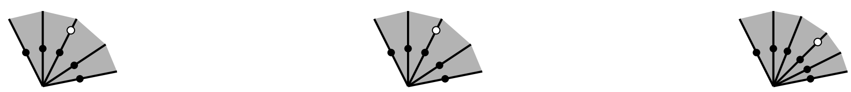
$$Q = \begin{bmatrix} 1 & 1 & a_3 & a_4 & a_5 & 0 & 0 \\ 0 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \end{bmatrix}, \quad a_4, a_5 \in \mathbb{Z}_{\geq 0},\ a_3, b_2, \dots, b_7 \in \mathbb{Z}_{\geq 1}.$$

We claim $\mu \in \varrho_3$. Otherwise, $\eta = \operatorname{cone}(w_2, w_3) \in \Lambda(R)$ holds; see Corollary 2.5.10. The associated variety $X(\eta)$ is smooth by Remark 2.5.12. Applying Lemma 2.5.3 to w_1, w_3 yields $b_3 = \det(w_1, w_3) = 1$. Again by Lemma 2.5.3 we obtain

$$1 - a_3 b_2 = b_3 - a_3 b_2 = \det(w_2, w_3) = 1.$$

This leads to $a_3 = 0$ or $b_2 = 0$. If $a_3 = 0$ holds, then Proposition 2.2.4 shows that μ lies in the boundary of $\operatorname{Eff}(R)$. Besides $b_2 = 0$ means $\varrho_1 = \varrho_2$. Both cases contradict the assumptions of constellation IIb. So we must have $\mu \in \varrho_3$.

The relative positions of $\varrho_3, \varrho_4, \varrho_5$ establish three subcases:



(i) $\varrho_3 = \varrho_4$ (ii) $\varrho_4 = \varrho_5$ (iii) $\varrho_3 \neq \varrho_4 \neq \varrho_5$

Constellation IIb (i). Let $v = (v_1, v_2) \in \mathbb{Z}^2$ be a primitive lattice vector on $\varrho_3 = \varrho_4$. According to Corollary 2.5.10 we have $\eta = \operatorname{cone}(w_2, w_3) \in \Lambda(R)$. Moreover $X(\eta)$ is smooth by Remark 2.5.12. Applying Lemmas 2.5.4 and 2.5.5 to the triples w_1, w_3, w_4 and w_2, w_3, w_4 yields $v_2 = 1$ and $\det(w_2, v) = 1$. Hence $1 - v_1 b_2 = 1$. This implies $v_1 = 0$ or $b_2 = 0$, hence $\varrho_3 = \varrho_7$ or $\varrho_1 = \varrho_2$. As seen before, this is a contradiction.

Constellation IIb (ii). Let $v = (v_1, v_2) \in \mathbb{Z}^2$ be a primitive lattice vector on $\varrho_4 = \varrho_5$. According to Corollary 2.5.10 we have $\eta = \text{cone}(w_2, w_4) \in \Lambda(R)$. Moreover $X(\eta)$ is smooth by Remark 2.5.12. Applying Lemmas 2.5.4 and 2.5.5 to the triples w_1, w_4, w_5 and w_2, w_4, w_5 yields $v_2 = 1$ and $\det(w_2, v) = 1$. Hence $1 - v_1 b_2 = 1$. This implies $v_1 = 0$ or $b_2 = 0$. As seen before, this is a contradiction.

Constellation IIb (iii). Corollary 2.5.10 provides us with GIT-cones

$$\eta_1 = \text{cone}(w_2, w_4), \quad \eta_2 = \text{cone}(w_5, w_6).$$

The associated varieties $X(\eta_1)$ and $X(\eta_2)$ both are smooth by Remark 2.5.12. Let us consider $X(\eta_2)$. We may apply Lemma 2.5.3 to both pairs w_4, w_6 and w_4, w_7. From this we infer that w_6 and w_7 are primitive, thus $w_6 = w_7 = (0, 1)$. In addition, we obtain $w_4 = (1, b_4)$ with some $b_4 \in \mathbb{Z}_{\geq 1}$. Note that $\det(w_2, w_4) > 0$ implies $b_4 \geq 2$.

To keep up an overview, we give the degree matrix with the entries known so far

$$Q = \begin{bmatrix} 1 & 1 & a_3 & 1 & a_5 & 0 & 0 \\ 0 & b_2 & b_3 & b_4 & b_5 & 1 & 1 \end{bmatrix}, \quad a_5 \in \mathbb{Z}_{\geq 0},\ a_3, b_2, b_3, b_5 \in \mathbb{Z}_{\geq 1},\ b_4 \in \mathbb{Z}_{\geq 2}.$$

Moreover, Lemma 2.5.7 (i) provides us with some $k \in \mathbb{Z}_{\geq 2}$ such that $\mu = k w_3$ holds. To continue we have to distinguish between $\varrho_5 = \varrho_6$ and $\varrho_5 \neq \varrho_6$.

We consider the case $\varrho_5 = \varrho_6$ first. Here, the first coordinate of w_5 vanishes. Applying Lemma 2.5.3 to w_4, w_5 gives $w_5 = (0, 1)$. By Proposition 2.3.7 the anticanonical class $-\mathcal{K}_X$ of X is given by

$$-\mathcal{K}_X = (3 + (1-k)a_3,\ 3 + b_2 + b_4 + (1-k)b_3).$$

From X being Fano we infer $-\mathcal{K}_X \in \text{Eff}(R)^\circ$, in particular $3 + (1-k)a_3 > 0$. Hence, we must have one of the following configurations

$$k = 2 \text{ and } a_3 = 1, 2, \qquad k = 3 \text{ and } a_3 = 1.$$

Now consider the smooth variety $X(\eta_1)$. As we have $\det(w_1, w_4) = b_4 \geq 2$, Lemma 2.5.3 yields that g has a monomial of the form $T_1^{l_1} T_4^{l_4}$. Thus b_4 divides $\mu_2 = k b_3$. Applying Lemma 2.5.4 to w_1, w_3, w_4 yields $\gcd(b_3, b_4) = 1$. So b_4 must divide k. From $k = 2, 3$ and $b_4 \geq 2$ we conclude $b_4 = k$. Now, we can examine all three possible configurations.

- For $k = 2$ and $a_3 = 1$ we obtain $w_2 = (1, 1)$, $w_3 = (1, b_3)$, $w_4 = (1, 2)$. Due to the counter-clockwise order of w_2, w_3, w_4, this amounts to $\varrho_2 = \varrho_3$ or $\varrho_3 = \varrho_4$. A contradiction.
- For $k = 2$ and $a_3 = 2$ the condition $\det(w_3, w_4) > 0$ gives $b_3 < 4$. Moreover, $\det(w_2, w_3) > 0$ shows $b_3 > 2$. Hence, $b_3 = 3$. We end up with specifying data as in Number 33 from Theorem 2.1.1.

- For $k = 3$ and $a_3 = 1$ we arrive at

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 1 & 1 \end{bmatrix}, \quad \mu = (3, 6).$$

  Here we have $-\mathcal{K}_X = (1, 3)$, which lies in the relative interior of the one-dimensional GIT-cone ϱ_4. According to Proposition 2.3.5 (i), this contradicts X being $\mathbb{Q}$-factorial.

It remains to consider the case $\varrho_5 \neq \varrho_6$. Corollary 2.5.10 provides us with GIT-cones

$$\eta_1 = \operatorname{cone}(w_2, w_4), \quad \eta_2 = \operatorname{cone}(w_4, w_5), \quad \eta_3 = \operatorname{cone}(w_5, w_6).$$

For all $i = 1, 2, 3$ the associated variety $X(\eta_i)$ is smooth by Remark 2.5.12. Lemma 2.5.3 applied to $X(\eta_3)$ and w_5, w_6 shows that the first coordinate of w_5 equals one. Applying Lemma 2.5.3 to $X(\eta_2)$ and w_4, w_5 gives $w_5 = (1, b_4 + 1)$. We give an intermediate result on the degree matrix

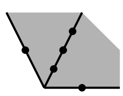
$$Q = \begin{bmatrix} 1 & 1 & a_3 & 1 & 1 & 0 & 0 \\ 0 & b_2 & b_3 & b_4 & b_4 + 1 & 1 & 1 \end{bmatrix}, \quad a_3, b_2, b_3, b_5 \in \mathbb{Z}_{\geq 1},\ b_4 \in \mathbb{Z}_{\geq 2}.$$

For $i = 4, 5$ applying Lemma 2.5.3 to $X(\eta_1)$ and w_1, w_i yields that g must have a monomial of the form $T_1^{l_1} T_i^{l_i}$, thus b_i divides $\mu_2 = kb_3$. For both triples w_1, w_3, w_4 and w_1, w_3, w_5 we may apply Lemma 2.5.4 to $X(\eta_1)$. From this we deduce $\gcd(b_3, b_4) = 1$ and $\gcd(b_3, b_4 + 1) = 1$. Together we obtain that $b_4(b_4 + 1)$ divides k, in particular $k \geq 6$ holds. Proposition 2.3.7 says the anticanonical class $-\mathcal{K}_X$ of X is given by

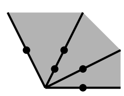
$$-\mathcal{K}_X = (4 + (1 - k)a_3,\ 3 + b_2 + 2b_4 - (1 - k)b_3).$$

In the present case X being Fano implies that both coordinates of $-\mathcal{K}_X$ are positive. From $4 + (1 - k)a_3 > 0$ we deduce $k \leq 4$. A contradiction.

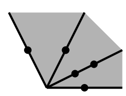
We turn to the second major case of Part IIb and investigate $\varrho_2 = \varrho_3$. Applying Lemmas 2.5.4 and 2.5.5 to the triples w_2, w_3, w_6 and w_2, w_3, w_7 shows that w_6, w_7 are primitive. Consequently $w_6 = w_7 = (0, 1)$ holds. We have to work out the subsequent degree constellations.



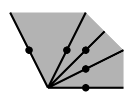
(iv) $\varrho_3 = \varrho_4 = \varrho_5$



(v) $\varrho_3 \neq \varrho_4, \varrho_4 = \varrho_5$



(vi) $\varrho_3 = \varrho_4, \varrho_4 \neq \varrho_5$



(vii) $\varrho_3 \neq \varrho_4 \neq \varrho_5$

Constellation IIb (iv). Let $v = (v_1, v_2) \in \mathbb{Z}^2$ be the primitive lattice vector lying on $\varrho_2 = \cdots = \varrho_5$. Then for each $i = 2, \ldots, 5$ we find a presentation $w_i = d_i v$ with some $d_i \in \mathbb{Z}_{\geq 1}$. Similarly $\mu = kv$ holds for some $k \in \mathbb{Z}_{\geq 2}$. Observe $\lambda = \operatorname{cone}(w_2, w_6)$.

Lemma 2.5.4 yields that w_2, w_3, w_7 generate $\mathbb{Z}^2$ as a group. Now Lemma 2.5.5 says $v_1 = \det(v, w_7) = 1$. This gives the degree matrix the following look

$$Q = \begin{bmatrix} 1 & d_2 & d_3 & d_4 & d_5 & 0 & 0 \\ 0 & d_2v_2 & d_3v_2 & d_4v_2 & d_5v_2 & 1 & 1 \end{bmatrix},$$

where we may assume $d_2 \leq \cdots \leq d_5$. Lemma 2.5.7 (i) guarantees that for any $2 \leq i \leq 5$ the relation g has a monomial of the form $T_i^{l_i}$ with $l_i \in \mathbb{Z}_{\geq 2}$, hence we obtain

$$l_i d_i v = l_i w_i = \mu = kv.$$

From this we conclude that k is divisible by each of $d_2, \ldots, d_5$. Moreover Lemma 2.5.4 yields that w_i, w_j, w_7 generate $\mathbb{Z}^2$ as a group for any $2 \leq i < j \leq 5$. In particular $d_2, \ldots, d_5$ are pairwise coprime. Thus $d_2 \cdots d_5$ divides k as well. By Proposition 2.3.7 the anticanonical class $-\mathcal{K}_X$ of X is given by

$$-\mathcal{K}_X = (1 + d_2 + \cdots + d_5 - k,\ 2 + (d_2 + \cdots + d_5 - k)v_2).$$

The Fano condition $-\mathcal{K}_X \in \lambda^\circ$ is equivalent to the inequations

$$\begin{aligned} 2 + (d_2 + \cdots + d_5 - k)v_2 &> (1 + d_2 + \cdots + d_5 - k)v_2, \\ d_2 + \cdots + d_5 + 1 &> k. \end{aligned}$$

The first inequation directly gives $v_2 = 1$ and the second implies $d_2 + \cdots + d_5 \geq d_2 \cdots d_5$. According to Remark 2.7.2 we have one of the following two configurations:

$$d_2 = d_3 = d_4 = 1, \qquad d_2 = d_3 = 1,\ d_4 = 2, \text{ and } d_5 = 3.$$

We treat the first configuration. Recall that $k = l_5 d_5$ holds for some $l_5 \in \mathbb{Z}_{\geq 2}$. Inserting into the second Fano inequation yields $4 - (1 - l_5)d_5 > 0$. We arrive at one of

$$l_5 = 2 \text{ and } d_5 = 1, 2, 3, \qquad l_5 = 3, 4 \text{ and } d_5 = 1.$$

This immediately leads to specifying data as in Numbers 34 to 38 from Theorem 2.1.1.

Dealing with the second configuration from above means to determine μ or equivalently k. The second Fano inequation tells us $k \leq 7$. Since k is divisible by $d_2 \cdots d_5 = 6$, we end up with $k = 6$. This amounts to Number 39 from Theorem 2.1.1.

Constellation IIb (v). The present constellation assumes that $\varrho_2 = \varrho_3$ is a not an extremal ray of $\mathrm{Eff}(R)$ and that w_4, w_5 share a common ray. Remark 2.5.12 allows us to apply Lemma 2.7.4. From this we infer that $\varrho_4 = \varrho_5$ must be an extremal ray of $\mathrm{Eff}(R)$, hence coincide with $\varrho_6 = \varrho_7$. In particular μ is not contained in $\varrho_4 = \varrho_5$. Thus we may apply Lemma 2.5.4 to both triples w_2, w_3, w_4 and w_2, w_3, w_5. This shows that w_4, w_5 both are primitive, hence $w_4 = w_5 = w_6 = (0, 1)$.

Let $v = (v_1, v_2) \in \mathbb{Z}^2$ be the primitive generator of the ray $\varrho_2 = \varrho_3$. We have $w_2 = d_2 v$ and $w_3 = d_3 v$ for some $d_2, d_3 \in \mathbb{Z}_{\geq 1}$. Here we may assume $d_2 \leq d_3$. Lemmas 2.5.4

and 2.5.5 applied to the triple w_2, w_3, w_7 yield $v_1 = \det(v, w_7) = 1$. Additionally we obtain $\gcd(d_2, d_3) = 1$. So far, the degree matrix is given by

$$Q = \begin{bmatrix} 1 & d_2 & d_3 & 0 & 0 & 0 & 0 \\ 0 & d_2v_2 & d_3v_2 & 1 & 1 & 1 & 1 \end{bmatrix}$$

According to the assumptions of Part IIb the relation degree μ lives in the relative interior of $\mathrm{Eff}(R)$ and is contained in some ϱ_i. This amounts to $\mu \in \varrho_2$. Consequently, we find some $k \in \mathbb{Z}_{\geq 2}$ such that $\mu = kv$. Lemma 2.5.7 (i) shows that k is divisible by both of the coprime integers d_2, d_3, in particular $d_2d_3 \mid k$ holds. According to Proposition 2.3.7 the anticanonical class $-\mathcal{K}_X$ of X is given by

$$-\mathcal{K}_X = (1 + d_2 + d_3 - k,\ 4 + (d_2 + d_3 - k)v_2).$$

Since $\lambda = \mathrm{cone}(w_2, w_4)$ is contained in the positive orthant, the Fano property of X implies that the first coordinate of $-\mathcal{K}_X$ is positive. This leads to $d_2+d_3 \geq d_2d_3$. As d_2, d_3 are coprime, we conclude $d_2 = 1$. Suppose $d_3 = 1$. Then $w_2 = w_3$ holds. Lemma 2.7.5 says $w_4 \in \varrho_2$. A contradiction to $\varrho_2 \neq \varrho_4$. We are left with $d_3 > 1$. From $d_3 | k \leq d_3 + 1$ we conclude $k = d_3$. We arrive at $\mu = w_3$. This is not possible, since R_g is spread and comes with an irredundant presentation; see also Setting 2.5.1.

Constellation IIb (vi). Let $v = (v_1, v_2) \in \mathbb{Z}^2$ denote the primitive generator of the ray $\varrho_2 = \varrho_3 = \varrho_4$. We have $w_i = d_iv$ with some $d_i \in \mathbb{Z}_{\geq 1}$ for $i = 2, 3, 4$. For any $2 \leq i < j \leq 4$ Lemma 2.5.4 shows that w_i, w_j, w_7 generate the group $\mathbb{Z}^2$. Using Lemma 2.5.5 yields that d_2, d_3, d_4 are pairwise coprime. In addition we infer $v_1 = \det(v, w_7) = 1$. Up to now, the degree matrix has the form

$$Q = \begin{bmatrix} 1 & d_2 & d_3 & d_4 & a_5 & 0 & 0 \\ 0 & d_2v_2 & d_3v_2 & d_4v_2 & b_5 & 1 & 1 \end{bmatrix}, \quad a_5 \in \mathbb{Z}_{\geq 0},\ b_5 \in \mathbb{Z}_{\geq 1}.$$

Let $k \in \mathbb{Z}_{\geq 2}$ such that $\mu = kv$. Lemma 2.5.7 (i) shows that k is divisible by each of the pairwise coprime integers d_2, d_3, d_4, in particular $d_2d_3d_4$ divides k. For the further discussion we have to distinguish between $\varrho_5 = \varrho_6$ and $\varrho_5 \neq \varrho_6$.

First, we consider the case $\varrho_5 = \varrho_6$. Here the first coordinate a_5 of w_5 vanishes. Moreover, μ is not contained in ϱ_5 because of $\mu \in \mathrm{Eff}(R)^\circ$. Applying Lemmas 2.5.4 and 2.5.5 to the triple w_2, w_3, w_5 yields that w_5 is primitive, hence $w_5 = (0, 1)$. By Proposition 2.3.7 the anticanonical class $-\mathcal{K}_X$ of X is given by

$$-\mathcal{K}_X = (1 + d_2 + d_3 + d_4 - k,\ 3 + (d_2 + d_3 + d_4 - k)v_2).$$

Observe $\lambda = \mathrm{cone}(w_2, w_5)$. Thus $X = X(\lambda)$ being Fano is equivalent to the inequations

$$\begin{aligned} 3 + (d_2 + d_3 + d_4 - k)v_2 &> (1 + d_2 + d_3 + d_4 - k)v_2, \\ 1 + d_2 + d_3 + d_4 - k &> 0. \end{aligned}$$

Subtracting $(d_2 + d_3 + d_4 - k)v_2$ from both sides in the first inequation yields $v_2 \leq 2$. Plugging $d_2d_3d_4 \mid k$ into the second inequation shows $d_2 + d_3 + d_4 \geq d_2d_3d_4$. By Remark 2.7.2 this leads to $d_2 = 1$ and one of the following two configurations

$$d_3 = 1, \qquad d_3 = 2 \text{ and } d_4 = 3.$$

We study the first configuration i.e. $d_3 = 1$. Recall that $k = md_4$ holds for some $m \in \mathbb{Z}_{\geq 1}$. Inserting into the second Fano inequation gives $3 + (1 - m)d_4 > 0$. We arrive at

$$d_4 = 1 \text{ and } m = 2, 3, \qquad d_4 = 2 \text{ and } m = 2.$$

Altogether we end up with specifying data as in Numbers 40 to 45.

Dealing with the second configuration from above, $d_3 = 2$, means to determine μ or equivalently k. The second Fano inequation gives $k \leq 6$. Since $d_2d_3d_4 = 6$ divides k, we conclude $k = 6$. This amounts to Numbers 46 and 47 from Theorem 2.1.1.

The next step is to consider the case $\varrho_5 \neq \varrho_6$. According to Proposition 2.2.4 we have either $\mu \in \varrho_2$ or $\mu \in \varrho_5$. We deal with $\mu \in \varrho_2$ first and show afterwards that the latter does not occur. Here, Corollary 2.5.10 provides us with two GIT-cones

$$\eta_1 = \operatorname{cone}(w_2, w_5), \quad \eta_2 = \operatorname{cone}(w_5, w_6).$$

According to Remark 2.5.12 the associated varieties $X(\eta_1)$ and $X(\eta_2)$ both are smooth. Consider $X(\eta_2)$. Applying Lemma 2.5.3 to w_5, w_6 yields that the first coordinate of w_5 equals one. Lemmas 2.5.4 and 2.5.5 applied to $X(\eta_2)$ and w_2, w_3, w_5 gives $\det(v, w_5) = 1$. Due to $v_1 = 1$, we obtain $w_5 = (1, v_2 + 1)$. So far, the degree matrix is of the form

$$Q = \begin{bmatrix} 1 & d_2 & d_3 & d_4 & 1 & 0 & 0 \\ 0 & d_2v_2 & d_3v_2 & d_4v_2 & v_2 + 1 & 1 & 1 \end{bmatrix}.$$

By Proposition 2.3.7 the anticanonical class $-\mathcal{K}_X$ of X is given by

$$-\mathcal{K}_X = (2 + d_2 + d_3 + d_4 - k,\ 3 + (d_2 + d_3 + d_4 + 1 - k)v_2).$$

Since X is Fano, we have $-\mathcal{K}_X \in \operatorname{Mov}(R)^\circ$. Note $\operatorname{Mov}(R) = \operatorname{cone}(v, w_6)$. From this we infer $\det(v, -\mathcal{K}_X) > 0$, i.e.,

$$3 + (1 + d_2 + d_3 + d_4 - k)v_2 > (2 + d_2 + d_3 + d_4 - k)v_2.$$

One directly obtains $v_2 \leq 2$. Remark 2.2.5 ensures that $\gamma_{5,6}$ is an $X(\eta_2)$-face. This allows us to apply Proposition 2.3.6 telling us that $\mu \in w_i + \operatorname{Pos}_{\mathbb{Z}}(w_5, w_6)$ holds for some $1 \leq i \leq 7$. Only $i = 1$ comes into consideration because of $\mu \in \varrho_2$ and the geometric constellation of $w_1, \ldots, w_7$. So we have a presentation

$$kv = \mu = w_1 + ew_5 + fw_6, \quad e, f \in \mathbb{Z}_{\geq 0}.$$

Vanishing of the first coordinate of w_6 shows $e = k - 1$. Considering the second coordinate of μ we obtain $kv_2 = (k - 1)(v_2 + 1) + f$. Term manipulation gives $v_2 + 1 = k + f$, thus

$k \le v_2 + 1$. We conclude $k \le 3$. Lemma 2.5.7 (i) ensures that k is a proper multiple of each of d_2, d_3, d_4. Thus we must have $d_2 = d_3 = d_4 = 1$. We have arrived at specifying data of the following form

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & v_2 & v_2 & v_2 & v_2+1 & 1 & 1 \end{bmatrix}, \quad \mu = (k, kv_2),$$
$$v_2 \le 2, \quad k \le v_2 + 1.$$

We show that the configuration $v_2 = 2$ and $k = 2$ does not show up. Here, $w_5 = (1,3)$ and $\mu = (2,4)$ holds. Consequently g has no monomial of the form $T_1^{l_1} T_5^{l_5}$. Thus Lemma 2.5.3 yields $\det(w_1, w_5) = 1$. A contradiction. The remaining configurations amount to specifying data as in Numbers 48 and 49 from Theorem 2.1.1.

To conclude the discussion of Constellation IIb (vi), suppose $\mu \in \varrho_5$. Here we have $\lambda = \mathrm{cone}(w_2, w_6)$. Lemma 2.5.4 applied to w_i, w_6, w_7 yields that the first coordinate of w_i equals one for $i = 2,3,4$. In particular w_2, w_3, w_4 are primitive, hence coincide with the primitive generator $v = (1, v_2)$ of the ray $\varrho_2 = \varrho_3 = \varrho_4$. Up to now, the degree matrix is of the form

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & a_5 & 0 & 0 \\ 0 & v_2 & v_2 & v_2 & b_5 & 1 & 1 \end{bmatrix}, \quad a_5, b_5 \in \mathbb{Z}_{\ge 1}$$

By Proposition 2.3.7 the anticanonical class $-\mathcal{K}_X$ of X is given by

$$-\mathcal{K}_X = (4 + (1-k)a_5,\ 3v_2 + 2 + (1-k)b_5).$$

From X being Fano we infer that $-\mathcal{K}_X$ lives in the relative interior of $\lambda = \mathrm{cone}(v, w_6)$. This leads to the inequation

$$3v_2 + 2 + (1-k)b_5 > (4 + (1-k)a_5)v_2,$$

This inequation can be simplified by subtracting $3v_2$ from both sides. Moreover, combining $\varrho_2 \ne \varrho_5$ with counter-clockwise ordering of the generator degrees gives $\det(v, w_5) > 0$, hence $b_5 > a_5 v_2$. Altogether we obtain

$$1 + (1-k)b_5 \ge v_2 + (1-k)a_5 v_2 > v_2 + (1-k)b_5.$$

This forces $v_2 = 0$. A contradiction to $\varrho_1 \ne \varrho_2$.

Constellation IIb (vii). Let $v = (v_1, v_2) \in \mathbb{Z}^2$ be the primitive lattice vector lying on the ray $\varrho_2 = \varrho_3$. For $i = 2,3$ we have $w_i = d_i v$, with some $d_i \in \mathbb{Z}_{\ge 1}$. Lemmas 2.5.4 and 2.5.5 applied to w_2, w_3, w_6 yield $v_1 = \det(v, w_6) = 1$ and $\gcd(d_2, d_3) = 1$ in addition. For the further discussion we have to distinguish between $\varrho_5 = \varrho_6$ and $\varrho_5 \ne \varrho_6$.

First we consider $\varrho_5 = \varrho_6$. Here, the first coordinate of w_5 vanishes. Applying Lemmas 2.5.4 and 2.5.5 to the triple w_2, w_3, w_5 shows $w_5 = (0,1)$. According to Proposition 2.2.4 we face one of the following cases

$$\mu \in \varrho_2 = \varrho_3, \qquad \mu \in \varrho_4.$$

Assume $\mu \in \varrho_4$. Here Lemma 2.5.7 (i) provides us with some $k \in \mathbb{Z}_{\geq 2}$ such that $\mu = kw_4$. Moreover Lemmas 2.5.4 and 2.5.5, this time applied to w_i, w_6, w_7 for $i = 2, 3$, give that w_2, w_3 are primitive, thus both equal v. So far, the degree matrix has the form

$$Q = [w_1, \ldots, w_7] = \begin{bmatrix} 1 & 1 & 1 & a_4 & 0 & 0 & 0 \\ 0 & v_2 & v_2 & b_4 & 1 & 1 & 1 \end{bmatrix}, \quad a_4, b_4 \in \mathbb{Z}_{\geq 1}.$$

The anticanonical class $-\mathcal{K}_X$ of X is given by Proposition 2.3.7 as

$$-\mathcal{K}_X = (3 + (1-k)a_4,\ 3 + 2v_2 + (1-k)b_4).$$

Here, we have $\lambda = \mathrm{cone}(w_3, w_5)$; see also Corollary 2.5.10. As a result the Fano condition $-\mathcal{K}_X \in \lambda^\circ$ is equivalent to the inequations

$$3 + 2b + (1-k)b_4 > (3 + (1-k)a_4)v_2, \qquad 3 + (1-k)a_4 > 0.$$

The second inequation implies that we must have one of the following configurations

$$k = 2 \text{ and } a_4 = 1, 2, \qquad k = 3 \text{ and } a_4 = 1.$$

Inserting values for k and a_4 into the first inequation from above provides us with further restrictions on v_2, b_4 in the respective configuration. We examine this for all three configurations of k and a_4.

- For $k = 2$ and $a_4 = 1$ we get $b_4 < 3$. From $b_4 - v_2 = \det(w_3, w_4) > 0$ follows $v_2 = 1$ and $b_4 = 2$. We arrive at specifying data Number 50 from Theorem 2.1.1.
- For $k = 2$ and $a_4 = 2$ we obtain $3 - (b_4 - 2v_2) > v_2$. Due to anti-clockwise ordering of w_1, v, w_4 we have $v_2 > 0$ and $b_4 - 2v_2 > 0$. Together we obtain $b_4 - 2v_2 = 1$ and $v_2 = 1$. Therefore $b_4 = 3$. We end up with specifying data as in Number 51 from Theorem 2.1.1.
- For $k = 3$ and $a_4 = 1$ we get $3 - 2(b_4 - v_2) > v_2$. Taking $b_4 - v_2 = \det(w_3, w_4) > 0$ into account yields $v_2 < 1$. A contradiction.

We turn to $\mu \in \varrho_2$. As v is the primitive lattice vector generating ϱ_2, we have $\mu = kv$ for some $k \in \mathbb{Z}_{\geq 1}$. Moreover, Corollary 2.5.10 provides us with GIT-cones

$$\eta_1 = \mathrm{cone}(w_3, w_4), \quad \eta_2 = \mathrm{cone}(w_4, w_5).$$

Each of them give rise to smooth a variety $X(\eta_i)$; see Remark 2.5.12. Now we want to determine $w_4 = (a_4, b_4)$. We apply Lemma 2.5.4 to $X(\eta_2)$ and w_4, w_6, w_7. Thus we obtain $a_4 = \det(w_4, w_6) = 1$. Applying Lemmas 2.5.4 and 2.5.5 again, this time to $X(\eta_1)$ and w_2, w_3, w_4, shows $\det(v, w_4) = 1$. We have $v_1 = a_4 = 1$, thus we obtain $b_4 = v_2 + 1$. So far the degree matrix looks like

$$Q = \begin{bmatrix} 1 & d_2 & d_3 & 1 & 0 & 0 & 0 \\ 0 & d_2 v_2 & d_3 v_2 & v_2 + 1 & 1 & 1 & 1 \end{bmatrix}.$$

Proposition 2.3.7 tells us that the anticanonical class $-\mathcal{K}_X$ of X is given by

$$-\mathcal{K}_X = (2 + d_2 + d_3 - k,\ 4 + (1 + d_2 + d_3 - k)v_2).$$

From X being Fano we deduce $-\mathcal{K}_X \in \mathrm{Mov}(R)^\circ$. This yields $\det(v, -\mathcal{K}_X) > 0$, i.e.,

$$4 + (1 + d_2 + d_3 - k)v_2 > (2 + d_2 + d_3 - k)v_2.$$

From this inequation one directly obtains $v_2 \le 3$. Remark 2.2.7 (ii) guarantees that $\gamma_{4,5}$ is an $X(\eta_2)$-face. Applying Proposition 2.3.6 yields $\mu \in w_i + \mathrm{Pos}_{\mathbb{Z}}(w_4, w_5)$ for some $1 \le i \le 7$. Only $i = 1$ comes into consideration due to $\mu \in \varrho_2 = \varrho_3$ and the geometric constellation of $w_1, \ldots, w_7$. So, we have a presentation

$$kv = \mu = w_1 + ew_4 + fw_5, \quad e, f \in \mathbb{Z}_{\ge 0}.$$

As the first coordinate of w_5 vanishes, we must have $e = k - 1$. Considering the second coordinate of μ gives $kv_2 = (k-1)(v_2+1)+f$. Term manipulation leads to $v_2+1 = k+f$. This implies $k \le v_2 + 1$, in particular $k \le 4$. Lemma 2.5.7 (i) ensures that k is a proper multiple of both d_2 and d_3. Thus, d_2 and d_3 being coprime, we must have $d_2 = 1$ and $d_3 = 1, 2$. Note that $d_2 = d_3 = 1$ implies $w_2 = w_3$. Lemma 2.7.5 shows that this case does not occur as we have $\varrho_1 \ne \varrho_2$ and $\varrho_3 \ne \varrho_4$. We are left with $d_3 = 2$. This forces $k = 4$ and thus $v_2 = 3$. The resulting specifying data is

$$Q = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 6 & 4 & 1 & 1 & 1 \end{bmatrix}, \quad \mu = (4, 12).$$

Here we have $-\mathcal{K}_X = (1, 4)$, which lies in the relative interior of the one-dimensional GIT-cone ϱ_4. According to Proposition 2.3.5 (i), this contradicts X being $\mathbb{Q}$-factorial.

The next step is to consider $\varrho_5 \ne \varrho_6$. According to Proposition 2.3.5 one of the pairwise different rays ϱ_3, ϱ_4, ϱ_5 contains μ. We claim that only $\mu \in \varrho_4$ is possible.

Suppose $\mu \in \varrho_3$. Here Corollary 2.5.10 ensures $\eta = \mathrm{cone}(w_4, w_5) \in \Lambda(R)$. The associated variety $X(\eta)$ is smooth by Remark 2.5.12. Applying Lemma 2.5.3 to the pairs w_4, w_5 and w_4, w_6 shows that $\det(w_4, w_5)$, $\det(w_4, w_6)$ both equal one. Lemmas 2.5.4 and 2.5.5 applied to w_2, w_3, w_5 yields $\det(v, w_5) = 1$. In the same way we obtain $\det(v, w_6) = 1$. Altogether, Lemma 2.5.6 says $\varrho_3 = \varrho_4$ or $\varrho_5 = \varrho_6$. A contradiction.

Suppose $\mu \in \varrho_5$. Corollary 2.5.10 provides us with GIT-cones

$$\eta_1 = \mathrm{cone}(w_3, w_4), \quad \eta_2 = \mathrm{cone}(w_4, w_6).$$

According to Remark 2.5.12 they give rise to smooth varieties $X(\eta_1)$ and $X(\eta_2)$. Let us determine $w_4 = (a_4, b_4)$. Applying Lemma 2.5.3 to $X(\eta_1)$ and w_1, w_4 gives $b_4 = 1$. Lemma 2.5.4 applied to $X(\eta_2)$ and w_4, w_6, w_7 shows $a_4 = 1$. We obtain

$$\det(v, w_4) = 1 - v_2 \le 0.$$

As v and w_3 are proportional and the generator degrees are in counter-clockwise order, this contradicts $\varrho_3 \ne \varrho_4$.

We must have $\mu \in \varrho_4$. According to Corollary 2.5.10 the following GIT-cones are in the game

$$\eta_1 = \operatorname{cone}(w_2, w_5), \quad \eta_2 = \operatorname{cone}(w_5, w_6).$$

The associated varieties $X(\eta_1)$ and $X(\eta_2)$ both are smooth; see Remark 2.5.12. Lemma 2.5.3 applied to $X(\eta_2)$ and w_5, w_6 yields that the first coordinate of w_5 equals one. Recall that $v = (1, v_2)$ is the primitive generator of the ray $\varrho_2 = \varrho_3$. Applying Lemmas 2.5.4 and 2.5.5 to $X(\eta_1)$ and w_2, w_3, w_5 gives $w_5 = (1, v_2 + 1)$. For $i = 2, 3$ applying Lemmas 2.5.4 and 2.5.5 to w_i, w_6, w_7 shows that w_i is primitive. Hence w_2 and w_3 equal v. At this point the degree matrix is of the form

$$Q = \begin{bmatrix} 1 & 1 & 1 & a_4 & 1 & 0 & 0 \\ 0 & v_2 & v_2 & b_4 & v_2+1 & 1 & 1 \end{bmatrix}, \quad a_4, b_4 \in \mathbb{Z}_{\geq 1}.$$

Besides, Lemma 2.5.7 (i) tells us $\mu = kw_4$ for some $k \in \mathbb{Z}_{\geq 2}$. By Proposition 2.3.7 the anticanonical class $-\mathcal{K}_X$ of X is

$$-\mathcal{K}_X = (4 + (1-k)a_4,\ 3 + 3v_2 + (1-k)b_4).$$

Since X is Fano, $-\mathcal{K}_X \in \operatorname{Mov}(R)^\circ$ holds. This is equivalent to the inequations

$$3 + 3v_2 + (1-k)b_4 > (4 + (1-k)a_4)v_2, \qquad 4 + (1-k)a_4 > 0.$$

Note that counter-clockwise ordering of w_3, w_4, w_5 implies $a_4 \geq 2$. For this reason the second inequation yields $k = 2$ and $a_4 = 2, 3$. We have to determine v_2, b_4 for both configurations. Inserting $k = 2$ into the first inequation leads to $3 - (b_4 - a_4v_2) > v_2$. Using $b_4 - a_4v_2 = \det(w_3, w_4) > 0$ shows $a_4v_2 - b_4 = 1$. This implies $v_2 = 1$ and thus $b_4 = a_4 + 1$. For $a_4 = 2$ this leads to Number 52 from Theorem 2.1.1. For $a_4 = 3$ the resulting specifying data is

$$Q = \begin{bmatrix} 1 & 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 4 & 2 & 1 & 1 \end{bmatrix}, \quad \mu = (6, 8).$$

Here we have $-\mathcal{K}_X = (1, 2)$, which lies in the relative interior of the one-dimensional GIT-cone ϱ_5. According to Proposition 2.3.5 (i), this contradicts X being $\mathbb{Q}$-factorial. □

Part IIc • We elaborate Case IIc from Remark 2.5.11. This means that each extremal ray of $\operatorname{Eff}(R)$ contains precisely one Cox ring generator degree and the relation degree shares a common ray with some generator degree living in the relative interior of $\operatorname{Eff}(R)$.

Proof of Theorem 2.1.1, Part IIc. In the present part of the proof, we have that $\varrho_2, \dots, \varrho_6$ are not contained in the boundary of $\operatorname{Eff}(R)$. Observe that $\varrho_2, \dots, \varrho_6$ do not coincide. Otherwise, $\operatorname{Mov}(R)$ is one-dimensional; a contradiction to the assumptions made in Setting 2.5.1. Remark 2.5.12 allows us to use Lemma 2.7.4. From this we infer that either $\varrho_2, \dots, \varrho_6$ are pairwise different or that there is a unique ray ϱ_j, where $2 \leq j \leq 5$, such that ϱ_j contains between two and four of $w_2, \ldots, w_6$. We end up with four subcases where N denotes the number of indices i such that $w_i \in \varrho_j$ holds.

(i) $N = 4$ (ii) $N = 3$ (iii) $N = 2$ (iv) $\varrho_i \neq \varrho_j$ for $i \neq j$

Constellation (IIc-i). A suitable admissible coordinate change leads to

$$\varrho_2 = \varrho_3 = \varrho_4 = \varrho_5, \quad \varrho_5 \neq \varrho_6.$$

Let $v \in \mathbb{Z}^2$ be the primitive generator of the ray ϱ_2. Proposition 2.2.4 tells us $\mu \in \varrho_2$. We may apply Lemmas 2.5.4 and 2.5.5 to w_2, w_3, w_7. This shows that v, w_7 generate $\mathbb{Z}^2$. Thus, a further admissible coordinate change gives $v = (1, 0)$ and $w_7 = (0, 1)$. Now, Lemma 2.5.4 applied to w_2, w_3, w_6 yields that the second coordinate of w_6 equals one. So far, the degree matrix is given by

$$Q = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ -b_1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad a_1, \ldots, a_6, b_1 \in \mathbb{Z}_{\geq 1}.$$

We may assume $a_2 \leq \cdots \leq a_5$. For each $1 \leq i < j \leq 5$ we apply Lemma 2.5.4 to the triple w_i, w_j, w_7 and obtain $\gcd(a_i, a_j) = 1$. Besides, $\mu \in \varrho_2$ means that the second coordinate μ_2 of the relation degree μ vanishes. By Proposition 2.3.7 the anticanonical class $-\mathcal{K}_X$ of X is thus given as

$$-\mathcal{K}_X = (a_1 + \cdots + a_6 - \mu_1,\ 2 - b_1).$$

From X being Fano we infer $-\mathcal{K}_X \in \mathrm{Mov}(R)^\circ$. As $\mathrm{Mov}(R)$ is contained in the positive orthant, we immediately obtain $b_1 = 1$. Observe $\det(w_1, w_6) = a_1 + a_6 > 1$. By Lemma 2.5.3 the polynomial g must have a monomial of the form $T_1^{l_1} T_6^{l_6}$. Since μ_2 vanishes, we must have $l_1 = l_6$. We conclude that $a_1 + a_6$ divides μ_1. Besides, Lemma 2.5.4 tells us that w_1, w_i, w_6 generate $\mathbb{Z}^2$ for any $2 \leq i \leq 5$. We conclude that the group $\mathbb{Z}^2$ is also spanned by

$$w_1 + w_6 = (a_1 + a_6, 0), \quad w_i = (a_i, 0), \quad w_6 = (a_6, 1).$$

From this we deduce $\gcd(a_1 + a_6, a_i) = 1$. Using $-\mathcal{K}_X \in \mathrm{Mov}(R)^\circ$ again we obtain $\det(-\mathcal{K}_X, w_6) > 0$ or equivalently

$$a_1 + \cdots + a_5 > \mu_1. \tag{2.2}$$

Lemma 2.5.3 applied to w_1, w_7 provides us with some $d_1 \in \mathbb{Z}_{\geq 1}$ such that $d_1 a_1 = \mu_1$ holds. Moreover, Lemma 2.5.7 tells us that for any $2 \leq i \leq 5$ there is some $d_i \in \mathbb{Z}_{\geq 2}$ such that $\mu = d_i w_i$ holds. In particular, the first coordinate μ_1 of μ is divisible by each of the pairwise coprime integers $a_1, \ldots, a_5$. We arrive at the following inequation

$$a_1 + \cdots + a_5 > a_1 \cdots a_5.$$

Note that $a_1, \dots, a_5$ are not necessarily in ascending order. Applying Remark 2.7.2 to a suitable permuation of $a_1, \dots, a_5$ yields that we have to deal with the following configurations:
(1) $a_1 = \dots = a_4 = 1$ and $a_5 \geq 1$,
(2) $a_1 \geq 2$ and $a_2 = \dots = a_5 = 1$,
(3) $a_1 = a_2 = a_3 = 1$, $a_4 = 2$ and $a_5 = 3$,
(4) $a_1 = 2$, $a_2 = a_3 = a_4 = 1$, $a_5 = 3$,
(5) $a_1 = 3$, $a_2 = a_3 = a_4 = 1$, $a_5 = 2$.

Constellation (IIc-i-1). We have $\mu_1 = d_5 a_5$ where $d_5 \geq 2$. Inserting into the Fano condition Eq. (2.2) yields $4 + a_5 > d_5 a_5$. Then again, this leads to

$$a_5 = 1 \text{ and } d_5 = 2, 3, 4, \qquad a_5 = 2, 3 \text{ and } d_5 = 2.$$

We examine these configurations explicitly.

- For $a_5 = 1$ and $d_5 = 2$ we have $\mu_1 = 2$. From μ_1 being divisible by $a_6 + 1$ we infer $a_6 = 1$. This amounts to Number 53 from Theorem 2.1.1.
- For $a_5 = 1$ and $d_5 = 3$ we have $\mu_1 = 3$. From μ_1 being divisible by $a_6 + 1$ we infer $a_6 = 2$. This amounts to Number 54 from Theorem 2.1.1.
- For $a_5 = 1$ and $d_5 = 4$ we have $\mu_1 = 4$. From μ_1 being divisible by $a_6 + 1$ we infer $a_6 = 1, 3$. This amounts to Numbers 55 and 56 from Theorem 2.1.1.
- For $a_5 = 2$ and $d_5 = 2$ we have $\mu_1 = 4$. From μ_1 being divisible by $a_6 + 1$ we infer $a_1 + a_6 = 2, 4$. This contradicts $\gcd(a_1 + a_6, 2) = 1$.
- For $a_5 = 3$ and $d_5 = 2$ we have $\mu_1 = 6$. From μ_1 being divisible by $a_6 + 1$ we infer $a_6 = 1, 2, 5$. We obtain $a_6 = 1$ due to $\gcd(a_1 + a_6, 3) = 1$. This amounts to Number 57 from Theorem 2.1.1.

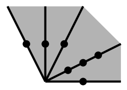
Constellation (IIc-i-2). We have $d_1 a_1 = \mu_1 = l_1 a_1 + l_6 a_6$ where $l_1, l_6 > 0$. This implies $d_1 \geq 2$. Here, inserting into Eq. (2.2) gives $4 + a_1 > d_1 a_1$. This leaves us with $d_1 = 2$ and $a_1 = 2, 3$. Suppose $a_1 = 2$. Then we have $\mu_1 = 4$. From $a_1 + a_6 | \mu_1$ we infer $a_6 = 2$. This contradicts $\gcd(a_1, a_6) = 1$. Suppose $a_1 = 3$. Then we have $\mu_1 = 6$. From $a_1 + a_6 | \mu_1$ we infer $a_6 = 3$. This contradicts $\gcd(a_1, a_6) = 1$.

Constellation (IIc-i-3). We have $\mu_1 = 3d_5$. Inserting into Eq. (2.2) gives $8 > 3d_5$, hence $d_5 = 2$ and $\mu_1 = 6$. From $a_1 + a_6 | \mu_1$ we infer $a_6 = 1, 2, 5$. In each of these cases the choice of a_6 contradicts $\gcd(a_6 + 1, 2) = 1$ or $\gcd(a_6 + 1, 3) = 1$.

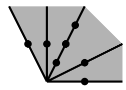
Constellation (IIc-i-4). We have $\mu_1 = 3d_5$. Inserting into Eq. (2.2) gives $8 > 3d_5$, hence $d_5 = 2$ and $\mu_1 = 6$. From $a_1 + a_6 | \mu$ we infer $a_6 = 3$. This contradicts $\gcd(a_5, a_6) = 1$.

Constellation (IIc-i-5). We have $\mu_1 = 2d_5$. Inserting into Eq. (2.2) gives $8 > 2d_5$, hence $d_5 = 2, 3$. Suppose $d_5 = 2$. Then we have $\mu_1 = 4$. From $3 + a_6 | \mu_1$ we infer $a_6 = 1$. This contradicts $\gcd(a_1 + a_6, 2) = 1$. Suppose $d_5 = 3$. Then we have $\mu_1 = 6$. From $3 + a_6 | \mu_1$ we infer $a_6 = 3$. This contradicts $\gcd(a_1 + a_6, 2) = 1$.

Constellation (IIc-ii). After a suitable admissible coordinate change we have $\varrho_3 = \varrho_4$ and one of the following constellations.



(1) $\varrho_2 = \varrho_3$, $\varrho_4 \neq \varrho_5$, $\varrho_5 \neq \varrho_6$



(2) $\varrho_2 \neq \varrho_3$, $\varrho_4 = \varrho_5$, $\varrho_5 \neq \varrho_6$

Constellation (IIc-ii-1). After a suitable admissible coordinate change we may assume that $e_1 = (1,0)$ generates the ray $\varrho_2 = \varrho_3 = \varrho_4$. Proposition 2.2.4 says $\mu \in \varrho_2$ or $\mu \in \varrho_5$. Thus we may apply Lemma 2.5.4 to both triples w_2, w_3, w_6 and w_2, w_3, w_7. From this we infer that the first coordinates of w_6, w_7 equal one. By applying a suitable admissible coordinate change we achieve $w_7 = (0,1)$. Up to now, the degree matrix is of the form

$$Q = [w_1, \ldots, w_7] = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ -b_1 & 0 & 0 & 0 & b_5 & 1 & 1 \end{bmatrix}, \quad a_i, b_i \in \mathbb{Z}_{\geq 1}.$$

We narrow the position of the relation degree. Suppose $\mu \in \varrho_2$. Then $\eta = \mathrm{cone}(w_5, w_6)$ is a GIT-cone leading to a smooth variety $X(\eta)$; see Corollary 2.5.10 and Remark 2.5.12. Applying Lemma 2.5.3 to $X(\eta)$ and w_5, w_6 gives $a_5 = \det(w_5, w_7) = 1$. Again Lemma 2.5.3, this time applied to w_5, w_6, shows $1 - a_6 b_5 = \det(w_5, w_6) = 1$. Hence $a_6 = 0$ or $b_5 = 0$. However, the current geometric constellation of the generator degrees ensures $a_6, b_5 > 0$. A contradiction. So we must have $\mu \in \varrho_5$.

Lemma 2.5.7 tells us that $\mu = k w_5$ holds for some $k \in \mathbb{Z}_{\geq 2}$. By Proposition 2.3.7 the anticanonical class $-\mathcal{K}$ of X is given by

$$-\mathcal{K}_X = (a_1 + a_2 + a_3 + a_4 + a_6 + (1-k)a_5,\ -b_1 + 2 + (1-k)b_5).$$

From X being Fano we infer $-\mathcal{K} \in \mathrm{Mov}(R)^\circ$, in particular $-b_1 + (1-k)b_5 + 2 > 0$. As we have $b_1, b_5 > 0$ and $k \geq 2$, this inequation does not hold. We conclude that the present constellation does not admit any candidate for a smooth Fano variety.

Constellation (IIc-ii-2). By applying a suitable admissible coordinate change we achieve that $e_1 = (1,0)$ generates the ray $\varrho_3 = \varrho_4 = \varrho_5$. Proposition 2.2.4 gives $\mu \in \varrho_3$. Corollary 2.5.10 provides us with GIT-cones

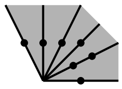
$$\eta_1 = \mathrm{cone}(w_2, w_3), \quad \eta_2 = \mathrm{cone}(w_3, w_6).$$

Remark 2.5.12 ensures smoothness of the associated varieties $X(\eta_1)$, $X(\eta_2)$. We may apply Lemma 2.5.4 to $X(\eta_1)$ and both triples w_1, w_3, w_4 and w_2, w_3, w_4. From this we obtain $w_i = (a_i, -1)$ for $i = 1, 2$. Applying Lemma 2.5.4 to $X(\eta_2)$ and w_3, w_4, w_j yields that the second coordinate of w_6, w_7 equals one. We arrive at the following degree matrix

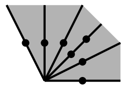
$$Q = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad a_1, \ldots, a_6 \in \mathbb{Z}_{\geq 1},\ a_7 \in \mathbb{Z}_{\geq 0}.$$

Note that $\mu \in \varrho_3$ implies that the second coordinate of μ vanishes. Proposition 2.3.7 yields that the second coordinate of $-\mathcal{K}_X$ vanishes as well i. e. $-\mathcal{K}_X \in \varrho_3$. Since ϱ_3 is a GIT-cone of dimension one, this contradicts $\mathbb{Q}$-factoriality of X; see Proposition 2.3.5.

Constellation (IIc-iii). Here, there is exactly one index $2 \leq j \leq 5$ such that $\varrho_j = \varrho_{j+1}$ holds. A suitable admissible coordinate change reduces the situation to the following cases.



(1) $j = 2$



(2) $j = 3$

Constellation (IIc-iii-1). We apply a suitable admissible coordinate change such that $e_1 = (1, 0)$ spans $\varrho_2 = \varrho_3$. Proposition 2.2.4 says that μ is contained in one of $\varrho_3, \varrho_4, \varrho_5$.

First, we treat the case $\mu \in \mu_3$. Corollary 2.5.10 provides us with GIT-cones

$$\eta_1 = \operatorname{cone}(w_3, w_4), \quad \eta_2 = \operatorname{cone}(w_5, w_6).$$

According to Remark 2.5.12 the varieties $X(\eta_1)$, $X(\eta_2)$ both are smooth. Lemma 2.5.4 applied to $X(\eta_1)$ and w_2, w_3, w_j shows $w_j = (a_j, 1)$ for $j = 5, 6, 7$. A suitable admissible coordinate change leads to $w_7 = (0, 1)$. From Lemma 2.5.3 applied to $X(\eta_2)$ and w_5, w_7 we infer $a_5 = \det(w_5, w_7) = 1$. Applying Lemma 2.5.3 once more, this time to w_5, w_6, yields $1 - a_6 = \det(w_5, w_6) = 1$. We conclude $a_6 = 0$, thus $w_6 = (0, 1) = w_7$. A contradiction to $\varrho_6 \neq \varrho_7$.

Repeating the above arguments with $\eta_1 = \operatorname{cone}(w_3, w_5)$ instead of $\eta_1 = \operatorname{cone}(w_3, w_4)$ yields that the case $\mu \in \varrho_4$ neither shows up.

Now we turn to $\mu \in \varrho_5$. We have $\eta = \operatorname{cone}(w_3, w_4) \in \Lambda(R)$; see Corollary 2.5.10 Remark 2.5.12 says that $X(\eta)$ is smooth. Lemma 2.5.4 applied to w_2, w_3, w_j shows $w_j = (a_j, 1)$ for $j = 4, 6$. A suitable admissible coordinate change leads to $w_6 = (0, 1)$. From $\varrho_1 \neq \varrho_2$ we deduce $w_1 = (a_1, -b_1)$ for some $a_1, b_1 \in \mathbb{Z}_{\geq 1}$. Moreover, observe $a_4 > 0$, which is due to $\varrho_4 \neq \varrho_6$. This amounts to $\det(w_1, w_4) = a_1 + a_4 b_1 > 1$. However, Lemma 2.5.3 says $\det(w_1, w_4) = 1$. A contradiction.

Constellation (IIc-iii-2). We apply a suitable admissible coordinate change such that $e_1 = (1, 0)$ generates $\varrho_3 = \varrho_4$. Proposition 2.2.4 gives $\mu \in \varrho_3$ or $\mu \in \varrho_5$.

Suppose $\mu \in \varrho_3$. Then $\eta = \operatorname{cone}(w_5, w_6) \in \Lambda(R)$ defines a smooth $X(\eta)$; see Corollary 2.5.10 and Remark 2.5.12. Applying Lemma 2.5.4 to w_3, w_4, w_j shows $w_j = (a_j, 1)$ for $j = 5, 6, 7$. A suitable admissible coordinate leads to $w_7 = (0, 1)$. Applying Lemma 2.5.3 to both pairs w_5, w_7 and w_6, w_7 yields $w_5 = (1, 1) = w_6$. A contradiction to $\varrho_5 \neq \varrho_6$.

We deal with $\mu \in \varrho_5$. Corollary 2.5.10 provides us with GIT-cones

$$\eta_1 = \operatorname{cone}(w_2, w_3), \quad \eta_2 = \operatorname{cone}(w_3, w_6).$$

According to Remark 2.5.12 the associated varieties $X(\eta_1)$ and $X(\eta_2)$ both are smooth. Applying Lemma 2.5.4 to $X(\eta_2)$ and the triples w_3, w_4, w_6 and w_3, w_4, w_7 shows that the first coordinate of w_6 and w_7 equals one. We apply Lemma 2.5.3 to $X(\eta_1)$ and w_1, w_3

respectively w_2, w_3. As a result, the second coordinates of w_1 and w_2 both equal minus one. So far the degree matrix is of the form

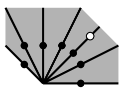
$$Q = \begin{bmatrix} a_1 & a_2 & 1 & 1 & a_5 & a_6 & 0 \\ -1 & -1 & 0 & 0 & b_5 & 1 & 1 \end{bmatrix}, \quad a_i, b_5 \in \mathbb{Z}_{\geq 1}.$$

Remark 2.2.5 ensures that we may apply Proposition 2.3.6 to $X(\eta_1)$ and $\gamma_{1,3}$. From this we infer that μ admits a presentation

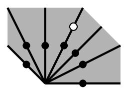
$$\mu = w_j + l_1 w_1 + l_3 w_3, \quad l_1, l_3 \in \mathbb{Z}_{\geq 0}.$$

Because of $\mu \in \varrho_5$ and the geometry of the present degree constellation, only $j = 6, 7$ come into consideration. This amounts to $\mu_2 = 1 - l_1$. If $l_1 = 0$ holds, μ is primitive. Since R_g is spread, this implies that T_5 is a monomial of g. This contradicts the choice of an irredundant presentation for R. If $l_1 > 0$ holds, μ_2 is negative, hence μ cannot lie on ϱ_5; a contradiction.

Constellation (IIc-iv). Proposition 2.2.4 says μ lies in one of ϱ_3, ϱ_4, ϱ_5. If $\mu \in \varrho_5$ holds, a suitable admissible coordinate change leads to $\mu \in \varrho_3$. This reduces the situation to



(1) $\mu \in \varrho_3$



(2) $\mu \in \varrho_4$

Constellation (IIc-iv-1). We apply a suitable admissible coordinate change such that $e_1 = (1, 0)$ spans the ray ϱ_4. Corollary 2.5.10 provides us with GIT-cones

$$\eta_1 = \mathrm{cone}(w_4, w_5), \quad \eta_2 = \mathrm{cone}(w_5, w_6)$$

Then $X(\eta_1)$ and $X(\eta_2)$ both are smooth by Remark 2.5.12. For $j = 5, 6, 7$ Lemma 2.5.3 applies to $X(\eta_1)$ and w_4, w_j. Thus the second coordinate of w_5, w_6, w_7 equals one. A suitable suitable admissible coordinate leads to $w_7 = (0, 1)$. Applying Lemma 2.5.3 again, this time to $X(\eta_2)$ and w_5, w_6 respectively w_5, w_7, yields $w_5 = (1, 1) = w_6$. This contradicts $\varrho_5 \neq \varrho_6$.

Constellation (IIc-iv-2). Corollary 2.5.10 provides us with GIT-chambers

$$\eta_1 = \mathrm{cone}(w_2, w_3), \quad \eta_2 = \mathrm{cone}(w_3, w_5), \quad \eta_3 = \mathrm{cone}(w_5, w_6).$$

According to Remark 2.5.12 every associated variety $X(\eta_i)$ is smooth. Consider $X(\eta_1)$. Lemma 2.5.3 applied to w_1, w_3 and w_2, w_3 yields $\det(w_1, w_3) = 1$ and $\det(w_2, w_3) = 1$. By applying a suitable admissible coordinate change we achieve

$$Q = \begin{bmatrix} 1 & a_3 + 1 & a_3 & a_4 & \dots & a_7 \\ 0 & 1 & 1 & b_4 & \dots & b_7 \end{bmatrix}, \quad a_i, b_i \in \mathbb{Z}_{\geq 1}.$$

According to Lemma 2.5.7 (i) we have $\mu = kw_4$ for some $k \in \mathbb{Z}_{\geq 2}$. In particular μ_2 is divisible by b_4. Applying Lemma 2.5.3 to the pair w_1, w_j for $j = 5, 6, 7$ shows that μ_2 is divisible by each of b_5, b_6, b_7. From Lemma 2.5.4 applied to any triple w_1, w_i, w_j where $4 \leq i < j \leq 7$ follows that $b_4, \ldots, b_7$ are pairwise coprime. Together we obtain $b_4 \cdots b_7 \mid \mu_2$. By Proposition 2.3.7 the anticanonical class $-\mathcal{K}$ of X is given by

$$-\mathcal{K} = (2 + a_3 + \cdots + a_7 - \mu_1,\ 2 + b_4 + \cdots b_7 - \mu_2).$$

Since X is Fano, $-\mathcal{K} \in \mathrm{Mov}(R)^\circ$ holds. In particular, the second coordinate of $-\mathcal{K}$ is positive. This leads to the following inequation:

$$b_4 \cdots b_7 \leq \mu_2 < b_4 + \cdots + b_7 + 2. \tag{2.3}$$

Let $4 \leq j_1, \ldots, j_4 \leq 7$ be pairwise different indices such that $b_{j_1} \leq \cdots \leq b_{j_4}$ holds. We can use Remark 2.7.2 to get constraints on $b_{j_1}, \ldots, b_{j_4}$. In doing so we interpret $b_{j_1} \cdots b_{j_4} \leq b_{j_1} + \cdots b_{j_4} + 1$ as an inequation in five variables where we only consider solutions where at least one variable equals one and the remaining variables are pairwise coprime. This leads to $b_{j_1} = b_{j_2} = 1$ and one of the following configurations

$$b_{j_3} = 1, \qquad b_{j_3} = 2 \text{ and } b_{j_4} = 3, 5.$$

We study the case $b_{j_3} = 1$. First of all we show that $b_5 = b_6 = b_7 = 1$ does not occur. Suppose $b_5 = b_6 = b_7 = 1$. Lemma 2.5.3 applied two $X(\eta_3)$ yields $\det(w_5, w_6) = 1$ and $\det(w_5, w_7) = 1$. This amounts to $a_6 = a_5 - 1 = a_7$ i.e. $w_6 = w_7$. A contradiction. We conclude $b_{j_4} \geq 2$ and $b_4 = 1$. The latter implies $\mu_2 = k$. Since $b_{j_1}, b_{j_2}, b_{j_3}$ all equal one, we may assume $j_1 < j_2 < j_3$ in what follows. We know that the second coordinate of $w_3, w_{j_1}, w_{j_2}, w_{j_3}$ equals one. Thus counter-clockwise orientation of these four generator degrees implies

$$\det(w_3, w_{j_3}) = a_3 - a_{j_3} \geq 3.$$

Note $w_4 \in \eta_2^\circ$. We apply Lemma 2.5.4 to $X(\eta_2)$ and w_2, w_4, w_{j_3} as well as w_3, w_4, w_{j_3}. From this we infer that $k = \mu_2$ is divisible by both $\det(w_2, w_{j_3})$ and $\det(w_3, w_{j_3})$. Observe

$$\det(w_2, w_{j_3}) = \det(w_3, w_{j_3}) + 1.$$

Thus μ_2 has a factor of the form $n(n+1)$ where $n \geq 3$, in particular $\mu_2 \geq 12$. Furthermore, we have $\mu_2 = db_{j_4}$ for some $d \in \mathbb{Z}_{\geq 1}$. Suppose $d = 1$. Then $\mu_2 = b_{j_4}$ is true, hence g has no monomial of the form $T_3^{l_3} T_{j_4}^{l_{j_4}}$ where $l_3 + l_{j_4} > 0$. Lemma 2.5.3 yields $\det(w_3, w_{j_4}) = 1$. Since $\mu \in \mathrm{cone}(w_3, w_{j_4})$ holds, μ is an integral positive linear combination over w_3, w_{j_4}. A contradiction. We are left with $d > 1$. Inserting into (2.3) yields $0 < (d-1)b_{j_4} < 5$. This forces $\mu_2 = db_{j_4} \leq 8$. A contradiction.

We turn to $b_{j_3} = 2$ and $b_{j_4} = 3$. Inserting into (2.3) gives $kb_4 = \mu_2 < 9$. Applying Lemma 2.5.4 to $X(\eta_2)$ to w_2, w_4, w_j as as well w_3, w_4, w_j yields that k is divisible by both $\det(w_2, w_j)$ and $\det(w_3, w_j)$ for all $j > 4$. Due to $b_{j_1} = b_{j_2}$ we may assume $j_1 < j_2$ from here on. We have that the second coordinates of w_3, w_{j_1}, w_{j_2} all equal one. Thus from w_3, w_{j_1}, w_{j_2} being oriented counter-clockwise we deduce

$$\det(w_3, w_{j_2}) = a_3 - a_{j_2} \geq 2.$$

Note $j_2 > 4$ and $\det(w_2, w_{j_2}) = \det(w_3, w_{j_2}) + 1$. We conclude that k has a factor of the form $n(n+1)$ where $n \geq 2$. Now $\mu_2 = kb_4 < 9$ forces $\mu_2 = k = 6$ and $b_4 = 1$ i.e. $j_1 = 4$. Additionally, we obtain $a_3 - a_{j_2} = 2$. Furthermore, we have

$$2a_3 - a_{j_3} = \det(w_2, w_{j_3}) \mid k = 6, \qquad 2a_3 - a_{j_3} + 2 = \det(w_3, w_{j_3}) \mid k = 6.$$

The only positive integer n such that 6 is divisible by n as well as $n+2$ is $n = 1$. Thus $2a_3 - a_{j_3} = 1$. Similar arguments give $3a_3 - a_{j_4} = 3$. We have

$$w_{j_2} = (a_3 - 2,\, 1), \qquad w_{j_3} = (2a_3 - 1,\, 2), \qquad w_{j_4} = (3a_3 - 3,\, 3).$$

Checking the orientation of w_{j_2}, w_{j_3}, w_{j_4} shows $j_2 = 7$, $j_3 = 5$, and $j_4 = 6$. Hence we may apply Lemma 2.5.3 to $X(\eta_3)$ and $w_{j_3} = w_5$, $w_{j_2} = w_7$. We obtain $\det(w_5, w_7) = 1$, yet this is a contradiction, since

$$\det(w_{j_3}, w_{j_2}) = 2a_3 - 1 - 2(a_3 - 2) = 3.$$

Finally we deal with $b_{j_3} = 2$ and $b_{j_4} = 5$. Inserting into (2.3) gives $kb_4 = \mu_2 < 11$. As $b_4 \cdots b_7 = 10$ divides μ_2, we arrive at $\mu_2 = 10$. Due to $b_{j_1} = b_{j_2}$ we assume $j_1 < j_2$ from here on. We have that the second coordinates of w_3, w_{j_1}, w_{j_2} all equal one. Thus from w_3, w_{j_1}, w_{j_2} being oriented counter-clockwise we deduce

$$\det(w_3, w_{j_2}) = a_3 - a_{j_2} \geq 2.$$

Note $j_2 > 4$ and $\det(w_2, w_{j_2}) = \det(w_3, w_{j_2}) + 1$. Applying Lemma 2.5.4 to $X(\eta_2)$ and w_2, w_4, w_{j_2} as well as w_3, w_4, w_{j_2} yields that k is divisible by both $\det(w_2, w_{j_2})$ and $\det(w_3, w_{j_2})$. We conclude that k and thus $\mu_2 = 10$ has a factor of the form $n(n+1)$ for some $n \geq 2$. A contradiction. □

2.8 Proof of Theorem 2.1.1: Collecting candidates III

We treat Case III from Remark 2.5.11, i.e., the degree μ of the relation lies in the bounding ray ϱ_1 of the effective cone.

Lemma 2.8.1. *Let $X = X(\lambda)$ be as in Setting 2.5.1 and $1 \leq i < j \leq r$ such that g neither depends on T_i nor on T_j. If X is quasismooth, then w_i, w_j lie either both in λ^- or both in λ^+.*

Proof. Otherwise, we may assume $w_i \in \lambda^-$ and $w_j \in \lambda^+$. Then $\gamma_{i,j}$ is an X-face and $\bar{X}(\gamma_{i,j})$ is a singular point of $\bar{X}$. According to Proposition 2.3.5 (iv), this contradicts quasismoothness of X. □

Proof of Theorem 2.1.1: Part III. We may assume that the ray ϱ_1 is generated by the vector $(1, 0)$. Let m be the number with $w_1, \ldots, w_m \in \varrho_1$ and $w_{m+1}, \ldots, w_7 \notin \varrho_1$. Observe that due to $\mu \in \varrho_1$, the relation g only depends on $T_1, \ldots, T_m$.

The first step is to show that only for $m = 5$, the specifying data $w_1, \ldots, w_7$ and μ in $K = \mathbb{Z}^2$ allow a hypersurface Cox ring. Since $\mu \in \varrho_1$, Proposition 2.2.4 yields $m \geq 3$. As

Mov(X) is of dimension two, we must have $m \le 5$; see Setting 2.5.1. Lemma 2.8.1 shows $w_{m+1}, \ldots, w_r \in \lambda^+$. Applying Lemma 2.5.4 to triples w_1, w_2, w_i for $i \ge m+1$, we obtain

$$\mu = (\mu_1, 0), \qquad w_i = (a_i, 0),\ i = 1, \ldots, m, \qquad w_i = (a_i, 1),\ i = m+1, \ldots, 7,$$

where, for any two $1 \le i < j \le m$, the numbers a_i and a_j are coprime and we may assume $a_7 = 0$. Moreover, we must have $a_{m+1} = \ldots = a_6$, because otherwise we obtain a GIT cone $\lambda \neq \eta \in \Lambda(R)$ with $\eta^\circ \in \mathrm{Mov}(R)^\circ$ and the associated variety $X(\eta)$ is not quasismooth by Lemma 2.8.1, contradicting Remark 2.5.12. Proposition 2.3.7 and the fact that X is Fano give us

$$(a_1 + \ldots + a_6 - \mu_1,\ 7 - m) \ = \ -\mathcal{K}_X \ \in \ \lambda^\circ \ = \ \mathrm{cone}((1,0),\ (a_{m+1}, 1))^\circ.$$

Since $a_1, \ldots, a_m$ are pairwise coprime, the component μ_1 of the degree of the relation g is greater or equal to $a_1 \cdots a_m$. Using moreover $a_{m+1} = \ldots = a_6$, we derive from the above Fano condition

$$a_1 \cdots a_m \ \le \ \mu_1 \ < \ a_1 + \cdots + a_m - a_{m+1},$$

where we may assume $a_1 \le \ldots \le a_m$. We exclude $m = 3$: here, $g = g(T_1, T_2, T_3)$, the above inequality forces $a_1 = a_2 = 1$, hence $g(T_1, T_2, 0)$ is classically homogeneous and T_3 is not prime in R, a contradiction. Let us discuss $m = 4$. The above inequality and pairwise coprimeness of the a_i leave us with

$$a_1 = a_2 = a_3 = 1, \qquad\qquad a_1 = a_2 = 1,\ a_3 = 2,\ a_4 = 3.$$

In the case $a_3 = 1$, we must have $\mu_1 = ka_4$ with some $k \in \mathbb{Z}_{\ge 2}$, because otherwise, the relation would be redundant or, seen similarly as above, one of T_1, T_2, T_3 would not be prime in R. The inequality gives $(k-1)a_4 < 3 - a_{m+1}$. We arrive at the following possibilities:

$$a_{m+1} = a_4 = 1,\ k = 2, \quad a_{m+1} = 0,\ a_1 = 1,\ k = 2, 3, \quad a_{m+1} = 0,\ a_1 = k = 2.$$

The first constellation implies that R is not factorial and hence is excluded. In the each of remaining ones, X is a product of $\mathbb{P}_2$ and a surface Y which must be smooth as X is so. Moreover, for the Picard numbers, we have

$$\rho(X) \ = \ \rho(\mathbb{P}_2) + \rho(Y).$$

Thus, $\rho(Y) = 1$. Finally, being a Mori fiber, Y is a del Pezzo surface. We arrive at $Y = \mathbb{P}_2$ and hence X is toric. A contradiction to X having a hypersurface Cox ring. We conclude that $m = 5$ is the only possibility. In this case, $\lambda = \mathrm{cone}(w_1, w_6)$ holds and our degree matrix is of the form

$$Q = \begin{bmatrix} a_1 & \ldots & a_5 & a_6 & 0 \\ 0 & \ldots & 0 & 1 & 1 \end{bmatrix} \qquad 1 \le a_1 \le \ldots \le a_5, \quad 0 \le a_6.$$

As mentioned before, g neither depends on T_6 nor on T_7. Consequently, we can write R as a polynomial ring over a K-graded subalgebra $R' \subseteq R$ as follows:

$$R \;=\; R'[T_6, T_7], \qquad R' \;:=\; \mathbb{C}[T_1, \ldots, T_5]/\langle g \rangle.$$

Moreover, R' is $\mathbb{Z}$-graded via $\deg(T_i) := a_i$. We claim that the $\mathbb{Z}$-graded algebra R' is a smooth Fano hypersurface Cox ring. if the K-graded algebra R is so. First observe that R' inherits the properties of an abstract Cox ring from R. Moreover, with $\bar{X}' = V(g) \subseteq \mathbb{C}^5$, we have $\bar{X} = \bar{X}' \times \mathbb{C}^2$. Now, the action of the one-dimensional torus $H' = \operatorname{Spec} \mathbb{C}[\mathbb{Z}]$ on $\bar{X}'$ admits a unique projective quotient in the sense of Construction 2.3.2, namely

$$X' \;=\; \hat{X}' /\!\!/ H', \qquad\qquad \hat{X}' \;=\; \bar{X}' \setminus \{0\}.$$

Propositions 2.3.3 and 2.3.7 show that X' is a Fano variety. Observe that each X'-face of $\gamma_0' \preccurlyeq \gamma'$ of the orthant $\gamma' \subseteq \mathbb{Q}^5$ defines and X-face $\gamma_0 = \gamma_0' + \operatorname{cone}(e_6, e_7)$. In particular, using Proposition 2.3.5 (ii) and (iv), we see that X' is smooth if X is so. Moreover, R' is a smooth hypersurface Cox ring if R is so. The smooth Fano threefolds with hypersurface Cox ring are listed in [51, Thm. 4.1], which gives us the possible values of $a_1, \ldots, a_5$ and from the Fano condition on X, we infer $a_6 + \mu_1 < a_1 + \ldots + a_5$. So, we end up with the specifying data as in Theorem 2.1.1 Numbers 58 to 67. ☐

2.9 Proof of Theorem 2.1.1: Verification

The next to last step in the proof of Theorem 2.1.1 is to make sure that specifying data as in Numbers 1 to 67 indeed lead to smooth general hypersurface Cox rings.

Let (Q, μ) be specifying data from Theorem 2.1.1. Consider the linear $\mathbb{Z}^2$-grading on $S = \mathbb{K}[T_1, \ldots, T_7]$ given by $Q : \mathbb{Z}^7 \to \mathbb{Z}^2$. We run Construction 2.4.1 with $\tau \in \Lambda(S)$ such that $-\mathcal{K} \in \tau^\circ$ holds. In each case one easily verifies $-\mathcal{K} \in \operatorname{Mov}(S)^\circ$, so the desired τ always exists. We start constructing a non-empty open subset $U \subseteq U_\mu$ as in Definition 2.4.5, thereby obtaining a Fano general hypersurface Cox ring. Afterwards we shrink U suitably to achieve smoothness.

Table 2.1 on page 106 lists for each $1 \leq i \leq 7$ a μ-homogeneous prime binomial $T^\kappa - T^\nu \in S$ not depending on T_i. Thus, Proposition 2.4.11 provides us with a non-empty open subset $U \subseteq S_\mu$ such that $T_1, \ldots, T_r$ define primes in R_g for all $g \in U$. Since $\mu \neq w_i$ holds for all i, Remark 2.4.9 ensures that $T_1, \ldots, T_7$ are a minimal system of generators for R_g, whenever $g \in U$. The next step is to make sure that each R_g admits unique factorization. Here we encounter three different classes of candidates.

Numbers 1–21, 26–41, 44, 46, 48, and 50–67. One directly checks that the convex hull over the $\nu \in \mathbb{Z}^7$ with $Q(\nu) = \mu$ is Dolgachev polytope; we have used the Magma program from Intrinsic A.3.5 for this purpose. Proposition 2.4.13 (ii) ensures that R_g is factorial after suitably shrinking U.

Numbers 22–25. Here, the cone $\tau' = \operatorname{cone}(w_3) \in \Lambda(S)$ satisfies $(\tau')^\circ \subseteq \operatorname{Mov}(S)^\circ$. Thus, Construction 2.4.1 gives raise to a toric variety Z'. We have $\mu \in (\tau')^\circ$ and one directly verifies that μ is base point free for Z'. Hence, Proposition 2.4.13 (i) shows that after shrinking U suitably, R_g admits unique factorization in R_g for all $g \in U$.

Numbers 42, 43, 45, 47, 49. After applying a suitable coordinate change, the specifying data (Q, μ) is as in the following table.

No.	Q	μ
42 43	$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{bmatrix}$	$(4, -2)$ $(6, -3)$
45	$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 4 & 0 \\ -1 & -1 & -1 & -1 & -1 & -2 & 1 \end{bmatrix}$	$(8, -4)$
47	$\begin{bmatrix} 1 & 1 & 1 & 2 & 4 & 6 & 0 \\ -1 & -1 & -1 & -1 & -2 & -3 & 1 \end{bmatrix}$	$(12, -6)$
49	$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 0 \\ -1 & -1 & -1 & -1 & -1 & -2 & 1 \end{bmatrix}$	$(6, -3)$

We apply Corollary 2.4.20. In each case, conditions (i) and (ii) from Corollary 2.4.20 can be directly checked. Condition (iii) is not affected by coordinate changes, hence is fullfilled according to the preceding considerations of U. As a result, we may shrink U such that each R_g is a factorial ring.

Cutting U down to U_μ leads to a general Fano hypersurface Cox ring. The final step is to attain X_g being smooth. Applying the Magma program from Intrinsic A.2.5 shows that Z_μ is smooth in all 67 cases. Observe that we have $\mu \in \tau$ except for Numbers 13, 14, 15, and 33. These four cases will be treated separately. For the other case we can verify base point freeness of μ on Z and hence may apply Corollary 2.4.29 allowing us to shrink U once more such that X_g is smooth for all $g \in U$. We turn to the four exceptional cases. For smoothness of X_g, it suffices to show that $\hat{X}_g$ is smooth; see Proposition 2.4.21. By Proposition 2.4.23, it suffices to find some $g \in S_\mu$ such that $\operatorname{grad}(g)$ has no zeroes in $\hat{Z}$, then shrinking U suitably yields that $\hat{X}_g$ is smooth for all $g \in U$. We just chose a random g of degree μ and verified this using [69]. The subsequent polynomials $g_{13}, \ldots, g_{33}$ do the job for the respective numbers of Theorem 2.1.1:

$$
\begin{aligned}
g_{13} = {} & 8T_1T_5^2 + 7T_1T_5T_6 + 7T_1T_5T_7 + 6T_1T_6^2 + 4T_1T_6T_7 + T_1T_7^2 + 7T_2T_5^2 + 7T_2T_5T_6 + 3T_2T_5T_7 \\
& + 8T_2T_6^2 + 5T_2T_6T_7 + 8T_2T_7^2 + 5T_3T_5^2 + 4T_3T_5T_6 + 9T_3T_5T_7 + 2T_3T_6^2 + 9T_3T_6T_7 + T_3T_7^2 \\
& + 8T_4T_5 + 3T_4T_6 + 6T_4T_7,
\end{aligned}
$$

$$
\begin{aligned}
g_{14} = {} & T_1^2T_5^3 + T_1^2T_5^2T_6 + T_1^2T_5^2T_7 + 4T_1^2T_5T_6^2 + T_1^2T_5T_6T_7 + T_1^2T_5T_7^2 + 6T_1^2T_6^3 + T_1^2T_6^2T_7 \\
& + T_1^2T_6T_7^2 + 3T_1^2T_7^3 + T_1T_2T_5^3 + T_1T_2T_5^2T_6 + 6T_1T_2T_5^2T_7 + T_1T_2T_5T_6^2 + T_2T_5T_6T_7 \\
& + T_1T_2T_5T_7^2 + T_1T_2T_6^3 + 2T_1T_2T_6^2T_7 + T_1T_2T_6T_7^2 + T_1T_2T_7^3 + 8T_1T_3T_5^3 + T_1T_3T_5^2T_6 \\
& + 8T_1T_3T_5^2T_7 + T_1T_3T_5T_6^2 + T_1T_3T_5T_6T_7 + T_1T_3T_5T_7^2 + T_1T_3T_6^3 + 5T_1T_3T_6^2T_7 \\
& + T_1T_3T_6T_7^2 + T_1T_3T_7^3 + 3T_1T_4T_5^2 + T_1T_4T_5T_6 + 4T_1T_4T_5T_7 + 9T_1T_4T_6^2 + T_1T_4T_6T_7 \\
& + T_1T_4T_7^2 + 5T_2^2T_5^3 + 3T_2^2T_5^2T_6 +^2 T_5^2T_7 + 5T_2^2T_5T_6^2 +^2 T_5T_6T_7 + T_2^2T_5T_7^2 + 5T_2^2T_6^3 \\
& + T_2^2T_6^2T_7 + T_2^2T_6T_7^2 9T_2^2T_7^3 + T_2T_3T_5^3 + 4T_2T_3T_5^2T_6 + T_2T_3T_5^2T_7 + T_2T_3T_5T_6^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 2T_2T_3T_5T_6T_7 + T_2T_3T_5T_7^2 + 9T_2T_3T_6^3 + T_2T_3T_6^2T_7 + T_2T_3T_6T_7^2 + 9T_2T_3T_7^3 + T_2T_4T_5^2 \\
&+ 6T_2T_4T_6^2 + T_2T_4T_6T_7 + T_2T_4T_7^2 + 5T_3^2T_5^3 + T_3^2T_5^2T_6 + 8T_3^2T_5^2T_7 + T_3^2T_5T_6^2 + T_3^2T_5T_6T_7 \\
&+ 4T_3^2T_5T_7^2 + T_3^2T_6^3 + 7T_3^2T_6^2T_7 + T_3^2T_6T_7^2 + 8T_3^2T_7^3 + T_3T_4T_5^2 + T_3T_4T_5T_6 + 5T_3T_4T_5T_7 \\
&+ T_3T_4T_6^2 + T_3T_4T_6T_7 + 3T_3T_4T_7^2 + T_4^2T_5 + T_4^2T_6 + T_4^2T_7,
\end{aligned}
$$

$$
\begin{aligned}
g_{15} = {} & T_1^4T_7^3 + 6T_1^3T_2T_7^3 + 9T_1^3T_3T_7^3 + 4T_1^3T_5T_7^2 + 3T_1^3T_6T_7^2 + 4T_1^2T_2^2T_7^3 + T_1^2T_2T_3T_7^3 \\
&+ 3T_1^2T_2T_5T_7^2 + 5T_1^2T_2T_6T_7^2 + 8T_1^2T_3^2T_7^3 + 5T_1^2T_3T_5T_7^2 + 5T_1^2T_3T_6T_7^2 + 7T_1^2T_4T_7^2 \\
&+ 2T_1^2T_5^2T_7 + 5T_1^2T_5T_6T_7 + 3T_1^2T_6^2T_7 + 8T_1T_2^3T_7^3 + 7T_1T_2^2T_3T_7^3 + 7T_1T_2^2T_5T_7^2 \\
&+ 6T_1T_2^2T_6T_7^2 + 6T_1T_2T_3^2T_7^3 + 2T_1T_2T_3T_5T_7^2 + 5T_1T_2T_3T_6T_7^2 + 2T_1T_2T_4T_7^2 + 2T_1T_2T_5^2T_7 \\
&+ 6T_1T_2T_5T_6T_7 + 2T_1T_2T_6^2T_7 + 5T_1T_3^3T_7^3 + 4T_1T_3^2T_5T_7^2 + 7T_1T_3^2T_6T_7^2 + 4T_1T_3T_4T_7^2 \\
&+ 3T_1T_3T_5^2T_7 + 9T_1T_3T_5T_6T_7 + 6T_1T_3T_6^2T_7 + 8T_1T_4T_5T_7 + T_1T_4T_6T_7 + 3T_1T_5^3 + 8T_1T_5^2T_6 \\
&+ 4T_1T_5T_6^2 + 4T_1T_6^3 + 4T_2^4T_7^3 + 3T_2^3T_3T_7^3 + 6T_2^3T_5T_7^2 + 9T_2^3T_6T_7^2 + 9T_2^2T_3^2T_7^3 \\
&+ 8T_2^2T_3T_5T_7^2 + 9T_2^2T_3T_6T_7^2 + 4T_2^2T_4T_7^2 + 2T_2^2T_5^2T_7 + 2T_2^2T_5T_6T_7 + 7T_2^2T_6^2T_7 + 6T_2T_3^3T_7^3 \\
&+ 7T_2T_3^2T_5T_7^2 + 5T_2T_3^2T_6T_7^2 + 9T_2T_3T_4T_7^2 + 7T_2T_3T_5^2T_7 + 3T_2T_3T_5T_6T_7 + 8T_2T_3T_6^2T_7 \\
&+ 7T_2T_4T_5T_7 + 9T_2T_4T_6T_7 + 3T_2T_5^3 + 4T_2T_5^2T_6 + 8T_2T_5T_6^2 + 4T_2T_6^3 + 8T_3^4T_7^3 + 8T_3^3T_5T_7^2 \\
&+ 4T_3^3T_6T_7^2 + 8T_3^2T_4T_7^2 + 6T_3^2T_5^2T_7 + 2T_3^2T_5T_6T_7 + 3T_3^2T_6^2T_7 + T_3T_4T_5T_7 + 9T_3T_4T_6T_7 \\
&+ 5T_3T_5^3 + T_3T_5^2T_6 + 4T_3T_5T_6^2 + 4T_3T_6^3 + 8T_4^2T_7 + T_4T_5^2 + 6T_4T_5T_6 + 4T_4T_6^2,
\end{aligned}
$$

$$
\begin{aligned}
g_{33} = {} & 2T_1^4T_5^6 + 7T_1^4T_5^5T_6 + 3T_1^4T_5^5T_7 + 9T_1^4T_5^4T_6^2 + 9T_1^4T_5^4T_6T_7 + 6T_1^4T_5^4T_7^2 + 8T_1^4T_5^3T_6^3 \\
&+ 8T_1^4T_5^3T_6^2T_7 + 8T_1^4T_5^3T_6T_7^2 + 8T_1^4T_5^3T_7^3 + 6T_1^4T_5^2T_6^4 + 9T_1^4T_5^2T_6^3T_7 + 8T_1^4T_5^2T_6^2T_7^2 \\
&+ 9T_1^4T_5^2T_6T_7^3 + 6T_1^4T_5^2T_7^4 + 5T_1^4T_5T_6^5 + T_1^4T_5T_6^4T_7 + 5T_1^4T_5T_6^3T_7^2 + 5T_1^4T_5T_6^2T_7^3 \\
&+ 6T_1^4T_5T_6T_7^4 + T_1^4T_5T_7^5 + 4T_1^4T_6^6 + 8T_1^4T_6^5T_7 + 2T_1^4T_6^4T_7^2 + 5T_1^4T_6^3T_7^3 + 5T_1^4T_6^2T_7^4 \\
&+ 5T_1^4T_6T_7^5 + 7T_1^4T_7^6 + 4T_1^3T_2T_5^5 + 6T_1^3T_2T_5^4T_6 + 5T_1^3T_2T_5^4T_7 + 4T_1^3T_2T_5^3T_6^2 \\
&+ 9T_1^3T_2T_5^3T_6T_7 + 9T_1^3T_2T_5^3T_7^2 + 7T_1^3T_2T_5^2T_6^3 + 8T_1^3T_2T_5^2T_6^2T_7 + 6T_1^3T_2T_5^2T_6T_7^2 \\
&+ 9T_1^3T_2T_5^2T_7^3 + 3T_1^3T_2T_5T_6^4 + 2T_1^3T_2T_5T_6^3T_7 + 8T_1^3T_2T_5T_6^2T_7^2 + 9T_1^3T_2T_5T_6T_7^3 \\
&+ 8T_1^3T_2T_5T_7^4 + 5T_1^3T_2T_6^5 + 4T_1^3T_2T_6^4T_7 + 8T_1^3T_2T_6^3T_7^2 + T_1^3T_2T_6^2T_7^3 + 6T_1^3T_2T_6T_7^4 \\
&+ 3T_1^3T_2T_7^5 + 6T_1^3T_4T_5^4 + 5T_1^3T_4T_5^3T_6 + 5T_1^3T_4T_5^3T_7 + 4T_1^3T_4T_5^2T_6^2 + T_1^3T_4T_5^2T_6T_7 \\
&+ 4T_1^3T_4T_5^2T_7^2 + 8T_1^3T_4T_5T_6^3 + T_1^3T_4T_5T_6^2T_7 + 2T_1^3T_4T_5T_6T_7^2 + 8T_1^3T_4T_5T_7^3 + 8T_1^3T_4T_6^4 \\
&+ 3T_1^3T_4T_6^3T_7 + 5T_1^3T_4T_6^2T_7^2 + 3T_1^3T_4T_6T_7^3 + 8T_1^3T_4T_7^4 + 4T_1^2T_2^2T_5^4 + 7T_1^2T_2^2T_5^3T_6 \\
&+ 6T_1^2T_2^2T_5^3T_7 + 3T_1^2T_2^2T_5^2T_6^2 + 4T_1^2T_2^2T_5^2T_6T_7 + 2T_1^2T_2^2T_5^2T_7^2 + 7T_1^2T_2^2T_5T_6^3 \\
&+ 3T_1^2T_2^2T_5T_6^2T_7 + T_1^2T_2^2T_5T_6T_7^2 + T_1^2T_2^2T_5T_7^3 + 7T_1^2T_2^2T_6^4 + 3T_1^2T_2^2T_6^3T_7 + 7T_1^2T_2^2T_6^2T_7^2 \\
&+ 6T_1^2T_2^2T_6T_7^3 + 2T_1^2T_2^2T_7^4 + 7T_1^2T_2T_4T_5^3 + T_1^2T_2T_4T_5^2T_6 + 8T_1^2T_2T_4T_5^2T_7 + 4T_1^2T_2T_4T_5T_6^2 \\
&+ 3T_1^2T_2T_4T_5T_6T_7 + 3T_1^2T_2T_4T_5T_7^2 + T_1^2T_2T_4T_6^3 + 8T_1^2T_2T_4T_6^2T_7 + T_1^2T_2T_4T_6T_7^2 \\
&+ 2T_1^2T_2T_4T_7^3 + 6T_1^2T_3T_5^3 + 8T_1^2T_3T_5^2T_6 + 3T_1^2T_3T_5^2T_7 + 5T_1^2T_3T_5T_6^2 + 6T_1^2T_3T_5T_6T_7 \\
&+ 8T_1^2T_3T_5T_7^2 + 9T_1^2T_3T_6^3 + 9T_1^2T_3T_6^2T_7 + 9T_1^2T_3T_6T_7^2 + 3T_1^2T_3T_7^3 + 8T_1^2T_4^2T_5^2 \\
&+ 9T_1^2T_4^2T_5T_6 + 3T_1^2T_4^2T_5T_7 + 4T_1^2T_4^2T_6^2 + 3T_1^2T_4^2T_6T_7 + 4T_1^2T_4^2T_7^2 + 9T_1T_2^3T_5^3
\end{aligned}
$$

$$+ 6T_1T_2^3T_5^2T_6 + T_1T_2^3T_5^2T_7 + 2T_1T_2^3T_5T_6^2 + 5T_1T_2^3T_5T_6T_7 + 8T_1T_2^3T_5T_7^2 + 4T_1T_2^3T_6^3$$
$$+ 6T_1T_2^3T_6^2T_7 + 2T_1T_2^3T_6T_7^2 + 9T_1T_2^3T_7^3 + 6T_1T_2^2T_4T_5^2 + 7T_1T_2^2T_4T_5T_6 + 8T_1T_2^2T_4T_5T_7$$
$$+ T_1T_2^2T_4T_6^2 + 9T_1T_2^2T_4T_6T_7 + 9T_1T_2^2T_4T_7^2 + 6T_1T_2T_3T_5^2 + 9T_1T_2T_3T_5T_6 + 6T_1T_2T_3T_5T_7$$
$$+ 2T_1T_2T_3T_6^2 + 2T_1T_2T_3T_6T_7 + 4T_1T_2T_3T_7^2 + 5T_1T_2T_4^2T_5 + T_1T_2T_4^2T_6 + 5T_1T_2T_4^2T_7$$
$$+ 6T_1T_3T_4T_5 + 4T_1T_3T_4T_6 + 7T_1T_3T_4T_7 + 3T_1T_4^3 + 8T_2^4T_5^2 + T_2^4T_5T_6 + T_2^4T_5T_7 + T_2^4T_6^2$$
$$+ 8T_2^4T_6T_7 + 5T_2^4T_7^2 + 6T_2^3T_4T_5 + 9T_2^3T_4T_6 + 7T_2^3T_4T_7 + 9T_2^2T_3T_5 + 3T_2^2T_3T_6 + 9T_2^2T_3T_7$$
$$+ 4T_2^2T_4^2 + 6T_2T_3T_4 + 8T_3^2.$$

We finish the proof by showing that each two varieties belonging to different families from Theorem 2.1.1 are non-isomorphic. If two such varieties are isomorphic, then their Cox rings are isomorphic as graded rings. Let us highlight two invariants in connection with graded rings.

An important invariant of a graded ring R is the *set of generator degrees* $\Omega_R \subseteq K$, which in the situation of Setting 2.5.1 is given as

$$\Omega_R = \{w_1, \ldots, w_r\} \subseteq K.$$

The set of generator degrees is unique and does not depend on a graded presentation of R. From this emerges another invariant: Choose pairwise different $u_1, \ldots, u_m \in K$ such that $\Omega_R = \{u_1, \ldots, u_m\}$ and set $d_i \coloneqq \dim_{\mathbb{K}} R_{u_i}$. By suitably reordering $u_1, \ldots, u_m$ we achieve $d_1 \leq \ldots \leq d_m$. We call $(d_1, \ldots, d_m)$ the *generator degree dimension tuple* of R. If two graded rings are isomorphic, then their generator degree dimension tuples coincide.

With the help of the Magma function Intrinsic A.4.2 we have computed the anticanonical self-intersection numbers of the families from Theorem 2.1.1. Among all 67 families there are 56 families with a unique anticanonical self-intersection number $\mathcal{K}^4$. It remains to deal with those cases where families from Theorem 2.1.1 cannot be distinguished by their anticanonical self-intersection number. There are precisely nine such cases.

Case 1: Numbers 1, 20, 65. Any member X of these families satisfies $\mathcal{K}_X^4 = 432$. However, the subsequent table shows that the generator degree dimension tuples $(d_1, \ldots, d_l)$ associated with each family are pairwise different.

No.	l	d_1	d_2	d_3
1	2	3	4	–
20	3	1	3	6
65	2	2	5	–

Case 2: Numbers 13, 16, 40. Any member X of these families satisfies $\mathcal{K}_X^4 = 352$. However, the subsequent table shows that the generator degree dimension tuples $(d_1, \dots, d_l)$ associated with each family are pairwise different.

No.	l	d_1	d_2	d_3
13	3	3	3	10
16	3	2	4	9
40	3	1	3	6

Case 3: Numbers 19, 29, 63. Any member X of these families satisfies $\mathcal{K}_X^4 = 192$. However, the subsequent table shows that the generator degree dimension tuples $(d_1, \dots, d_l)$ associated with each family are pairwise different.

No.	l	d_1	d_2	d_3
19	3	1	3	6
29	3	3	3	10
63	3	2	5	–

Case 4: Numbers 30, 47. Let X_{30} be a member of Number 30 and X_{47} a member of Number 47. The anticanonical self-intersection number of both X_{30} and X_{47} equals 18. On the other side, the Cox ring of X_{30} has three generator degrees whereas the Cox ring of X_{47} has five generator degrees. Consequently, X_{30} and X_{47} are non-isomorphic.

Case 5: Numbers 31, 39. Let X_{31} be a member of Number 31 and X_{39} a member of Number 39. The anticanonical self-intersection number of both X_{31} and X_{39} equals 48. On the other side, the Cox ring of X_{31} has three generator degrees whereas the Cox ring of X_{39} has five generator degrees. Consequently, X_{31} and X_{39} are non-isomorphic.

Case 6: Numbers 35, 50. Let X_{35} be a member of Number 35 and X_{50} a member of Number 50. The anticanonical self-intersection number of both X_{35} and X_{50} equals 144. On the other side, the Cox ring of X_{35} has three generator degrees whereas the Cox ring of X_{50} has four generator degrees. Consequently, X_{35} and X_{50} are non-isomorphic.

Case 7: Numbers 27, 59. Let X_{27} be a member of Number 27 and X_{59} a member of Number 59. The anticanonical self-intersection number of both X_{27} and X_{59} equals 64. On the other side, the Cox ring of X_{27} has three generator degrees whereas the Cox ring of X_{59} has four generator degrees. Consequently, X_{27} and X_{59} are non-isomorphic.

Case 8: Numbers 14, 52. Let X_{14} be a member of Number 14 and X_{52} a member of Number 52. The anticanonical self-intersection number of both X_{14} and X_{52} equals 65. On the other side, the Cox ring of X_{14} has three generator degrees whereas the Cox ring of X_{52} has five generator degrees. Consequently, X_{14} and X_{52} are non-isomorphic.

Case 9: Numbers 3, 60. Let X_3 be a member of Number 3 and X_{60} a member of Number 60. The anticanonical self-intersection number of both X_3 and X_{60} equals 80. On the other side, the Cox ring of X_3 has two generator degrees whereas the Cox ring of X_{60} has five generator degrees. Consequently, X_3 and X_{60} are non-isomorphic.

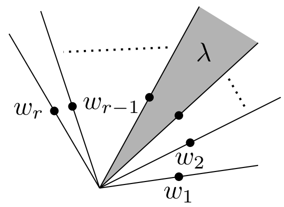
Table 2.1: Binomials used to ensure primeness of $T_1, \dots, T_7 \in R_g$ in the verification part of the proof of Theorem 2.1.1

No.	T_1	T_2	T_3	T_4	T_5	T_6	T_7
1	$T_2T_7 - T_3T_6$	$T_1T_7 - T_3T_6$	$T_1T_6 - T_2T_7$	$T_2T_7 - T_3T_6$	$T_2T_6 - T_3T_7$	$T_2T_5 - T_3T_7$	$T_2T_6 - T_3T_5$
2	$T_2^2T_6 - T_3T_4T_5$	$T_1^2T_6 - T_3T_4T_5$	$T_1T_4T_5 - T_2^2T_6$	$T_1T_3T_5 - T_2^2T_6$	$T_1T_4T_7 - T_3^2T_6$	$T_1T_4T_7 - T_3^2T_5$	$T_1T_4T_5 - T_3^2T_6$
3	$T_2^3T_6 - T_3^3T_7$	$T_1^3T_6 - T_3^3T_7$	$T_1^3T_7 - T_2^3T_6$	$T_2^3T_6 - T_3^3T_7$	$T_1T_2^2T_6 - T_4^3T_7$	$T_1T_2^2T_5 - T_4^3T_7$	$T_1T_2^2T_6 - T_4^3T_5$
4	$T_2T_5T_6 - T_3T_7^2$	$T_1T_5T_6 - T_3T_7^2$	$T_1T_7^2 - T_2T_5T_6$	$T_2T_5T_6 - T_3T_7^2$	$T_1T_7^2 - T_4T_6^2$	$T_1T_7^2 - T_4T_5^2$	$T_1T_5^2 - T_4T_6^2$
5	$T_2^2T_5T_7 - T_3^2T_6^2$	$T_1^2T_5T_7 - T_3^2T_6^2$	$T_1^2T_6^2 - T_2^2T_5T_7$	$T_2^2T_5T_7 - T_3^2T_6^2$	$T_1T_4T_7^2 - T_2T_3T_6^2$	$T_1T_4T_7^2 - T_2T_3T_5^2$	$T_1T_4T_5^2 - T_2T_3T_6^2$
6	$T_2T_4^2T_5T_7 - T_3^3T_6^2$	$T_1T_4^2T_5T_7 - T_3^3T_6^2$	$T_1^3T_6^2 - T_2T_4^2T_5T_7$	$T_1^2T_2T_5T_7 - T_3^3T_6^2$	$T_1T_3^2T_6^2 - T_2^2T_4T_7^2$	$T_1T_3^2T_5^2 - T_2^2T_4T_7^2$	$T_1T_3^2T_6^2 - T_2^2T_4T_5^2$
7	$T_2T_6 - T_3T_5$	$T_1T_6 - T_3T_5$	$T_1T_5 - T_2T_6$	$T_2T_6 - T_3T_5$	$T_1T_6 - T_3^2T_7$	$T_1T_5 - T_3^2T_7$	$T_2T_5 - T_3T_6$
8	$T_2^2T_4T_7^2 - T_3T_6^2$	$T_1^2T_4T_7^2 - T_3T_6^2$	$T_1T_6^2 - T_2^2T_4T_7^2$	$T_1T_2^2T_7^2 - T_3T_6^2$	$T_1T_2T_4T_7^2 - T_3T_6^2$	$T_1T_2T_4T_7^2 - T_3T_5^2$	$T_1T_6^2 - T_4T_5^2$
9	$T_2^3T_7 - T_4^2T_6$	$T_1^3T_7 - T_4^2T_6$	$T_2^3T_7 - T_4^2T_6$	$T_1^2T_6 - T_2^3T_7$	$T_1T_3^2T_7 - T_4^2T_6$	$T_1T_3^2T_7 - T_4^2T_5$	$T_1T_4T_6 - T_3^2T_5$
10	$T_2^2T_4T_5T_7 - T_3^2T_6^2$	$T_1^2T_4T_5T_7 - T_3^2T_6^2$	$T_1^2T_6^2 - T_2^2T_4T_5T_7$	$T_1T_2^2T_5T_7 - T_3^2T_6^2$	$T_2^4T_7^2 - T_3T_4T_6^2$	$T_2^4T_7^2 - T_3T_4T_5^2$	$T_1T_4T_6^2 - T_2T_3T_5^2$
11	$T_2T_6 - T_4T_5$	$T_1T_6 - T_4T_5$	$T_2T_6 - T_4T_5$	$T_1T_5 - T_2T_6$	$T_2T_3^2T_7 - T_4T_6$	$T_2T_3^2T_7 - T_4T_5$	$T_2T_5 - T_3T_6$
12	$T_2^4T_4T_7^2 - T_3T_5^2$	$T_1^4T_4T_7^2 - T_3T_5^2$	$T_1T_5^2 - T_2^4T_4T_7^2$	$T_1T_2^4T_7^2 - T_3T_5^2$	$T_1T_6^2 - T_3^5T_7^2$	$T_1T_5^2 - T_3^5T_7^2$	$T_1T_6^2 - T_4T_5^2$
13	$T_2T_5T_7 - T_3T_6^2$	$T_1T_5T_7 - T_3T_6^2$	$T_1T_6^2 - T_2T_5T_7$	$T_1T_5T_6 - T_2T_7^2$	$T_1T_6^2 - T_2T_7^2$	$T_1T_5^2 - T_2T_7^2$	$T_1T_6^2 - T_2T_5^2$
14	$T_3^2T_6^2T_7 - T_4^2T_5$	$T_3^2T_6^2T_7 - T_4^2T_5$	$T_1^2T_6^2T_7 - T_4^2T_5$	$T_1T_3T_6^3 - T_2^2T_7^3$	$T_1^2T_6^3 - T_2T_3T_7^3$	$T_1^2T_5^3 - T_2T_3T_7^3$	$T_1^2T_6^3 - T_2T_3T_5^3$
15	$T_2^4T_7^3 - T_3T_5^3$	$T_1^4T_7^3 - T_3T_5^3$	$T_1T_5^3 - T_2^4T_7^3$	$T_1T_5T_6^2 - T_2^4T_7^3$	$T_1T_6^3 - T_2^3T_3T_7^3$	$T_1T_5^3 - T_2^3T_3T_7^3$	$T_1T_5^3 - T_4T_6^2$
16	$T_2T_3T_6 - T_4T_5$	$T_1T_3T_6 - T_4T_5$	$T_1T_2T_6 - T_4T_5$	$T_1T_5 - T_2T_3T_6$	$T_1T_4T_7 - T_3^2T_6$	$T_1T_5 - T_3^2T_7$	$T_1T_5 - T_3^2T_6$
17	$T_2T_5^2 - T_3^3T_6^2$	$T_1T_5^2 - T_3^3T_6^2$	$T_1^3T_6^2 - T_2T_5^2$	$T_2T_5^2 - T_3^3T_6^2$	$T_1T_3^2T_6^2 - T_2^2T_4T_7^2$	$T_1T_2T_4T_7^2 - T_3T_5^2$	$T_1T_2T_4T_6^2 - T_3T_5^2$
18	$T_2^3T_6 - T_3^3T_7$	$T_1^2T_3T_5^2T_6 - T_4^3T_7$	$T_1^2T_2T_5^2T_6 - T_4^3T_7$	$T_1^2T_3T_5^2T_6 - T_2^3T_7$	$T_1T_2T_3T_7^2 - T_4^3T_6$	$T_1T_2T_3T_7^2 - T_4^3T_5$	$T_1T_2T_3T_5^2 - T_4^3T_6$
19	$T_2^2T_6 - T_3T_4T_5$	$T_1^2T_7^3 - T_3^2T_5$	$T_1^2T_7^3 - T_2^2T_5$	$T_1^2T_7^3 - T_3^2T_5$	$T_1T_3T_6^2 - T_4^2T_7$	$T_1T_3T_5^2 - T_4^2T_7$	$T_1T_3T_6^2 - T_4^2T_5$
20	$T_2T_7 - T_3T_6$	$T_1T_6^2 - T_4T_5$	$T_1T_6^2 - T_4T_5$	$T_1T_6^2 - T_2T_5$	$T_2T_6 - T_3T_7$	$T_2T_5 - T_3T_7$	$T_2T_6 - T_3T_5$
21	$T_2^3T_7 - T_3^3T_6$	$T_1^2T_5T_6^2 - T_3T_4^2T_7$	$T_1^2T_5T_6^2 - T_2T_4^2T_7$	$T_1^2T_5T_6^2 - T_2^2T_3T_7$	$T_1T_2T_3T_7^2 - T_4^3T_6$	$T_2^2T_3T_7 - T_4^2T_5$	$T_2^2T_3T_6 - T_4^2T_5$
22	$T_2^2T_5T_6 - T_3^2$	$T_1^2T_5T_6 - T_3^2$	$T_1^2T_5^2 - T_2^2T_6T_7$	$T_1^2T_5^2 - T_2^2T_6T_7$	$T_1^2T_6T_7 - T_4^2$	$T_1^2T_5T_7 - T_4^2$	$T_1^2T_5T_6 - T_4^2$
23	$T_2^2T_4T_5T_6 - T_3^3$	$T_1^2T_4T_5T_6 - T_3^3$	$T_1^3T_5^2T_6 - T_2^2T_4T_7^2$	$T_1^3T_5^2T_6 - T_2^2T_3T_7^2$	$T_1^2T_2T_7^3 - T_4^3$	$T_1^2T_2T_7^3 - T_4^3$	$T_1^2T_2T_5^3 - T_4^3$
24	$T_2^4T_6^3T_7 - T_3^4$	$T_1^4T_6^3T_7 - T_3^4$	$T_1^4T_7^4 - T_2^4T_5^3T_6$	$T_1^4T_6T_7^3 - T_2T_3^3T_5$	$T_1^3T_3T_6T_7^2 - T_4^2$	$T_1^3T_3T_5T_7^2 - T_4^2$	$T_1^3T_3T_5^2T_6 - T_4^2$
25	$T_2^6T_5T_6^4T_7 - T_3^3$	$T_1^6T_5T_6^4T_7 - T_3^3$	$T_1^3T_2^3T_6^5T_7 - T_4^2$	$T_1T_2^5T_5T_6^2T_7^3 - T_3^3$	$T_1T_2^3T_3T_6^4 - T_4^2$	$T_1T_2^3T_3T_5^4 - T_4^2$	$T_1T_2^3T_3T_6^4 - T_4^2$
26	$T_2^2T_4T_5 - T_3^2$	$T_1^2T_4T_5 - T_3^2$	$T_1^2T_5T_6 - T_2^2T_7^2$	$T_1^2T_5^2 - T_2^2T_6T_7$	$T_1^2T_4^2 - T_2^2T_6T_7$	$T_1^2T_5^2 - T_2^2T_4T_7$	$T_1^2T_5^2 - T_2^2T_4T_6$
27	$T_2^2T_4T_6^2T_7 - T_3^2$	$T_1^2T_4T_6^2T_7 - T_3^2$	$T_1^2T_5^4 - T_2^2T_4T_6^3$	$T_1T_2T_6^2T_7^2 - T_3^2$	$T_1T_2T_6^2T_7^2 - T_3^2$	$T_1T_2T_4^2T_7^2 - T_3^2$	$T_1T_2T_4^2T_6^2 - T_3^2$
28	$T_2^2T_5^2T_6^3T_7 - T_3^2$	$T_1^2T_5^2T_6^3T_7 - T_3^2$	$T_1^2T_5^6 - T_2^2T_4T_6^2T_7^3$	$T_1T_3T_5^2T_6 - T_2^2T_7^6$	$T_1T_3T_4^2T_6 - T_2^2T_7^6$	$T_1T_3T_4T_5^2 - T_2^2T_7^6$	$T_1T_3T_5^2T_6 - T_2^2T_4^6$
29	$T_2^2T_5^2 - T_3^2T_6T_7$	$T_1^2T_5^2 - T_3^2T_6T_7$	$T_1^2T_6T_7 - T_2^2T_5^2$	$T_1^2T_5T_7 - T_2^2T_6^2$	$T_2^2T_6T_7 - T_4^2$	$T_2^2T_5T_7 - T_4^2$	$T_2^2T_5T_6 - T_4^2$
30	$T_2^3T_5^2T_6 - T_3^2T_4T_7^2$	$T_1^3T_5^2T_6 - T_3^2T_4T_7^2$	$T_1^2T_4T_7^2 - T_2^3T_5^2T_6$	$T_1^3T_5T_6^2 - T_2^3T_7^3$	$T_1T_3^2T_6^3 - T_2^3T_7^3$	$T_1T_3^2T_5^3 - T_2^3T_7^3$	$T_1T_3^2T_6^3 - T_2^3T_5^3$
31	$T_2T_3^3T_7^2 - T_4^2$	$T_1T_3^3T_7^2 - T_4^2$	$T_1^3T_2T_7^2 - T_4^2$	$T_1^3T_3T_7^2 - T_2^4T_5T_6$	$T_1T_2T_3^2T_6T_7 - T_4^2$	$T_1T_2T_3^2T_5T_7 - T_4^2$	$T_1T_2T_3^2T_5T_6 - T_4^2$
32	$T_2^4T_7^4 - T_3^4T_5^3T_6$	$T_1^4T_7^4 - T_3^4T_5^3T_6$	$T_1^4T_5^3T_6 - T_2^4T_7^4$	$T_1^4T_5^2T_6^2 - T_2T_3^3T_7^4$	$T_1T_2^3T_6^4 - T_3^2T_4T_7^2$	$T_1T_2^3T_5^4 - T_3^2T_4T_7^2$	$T_1T_2^3T_6^4 - T_3^2T_4T_5^2$

33	$T_2^4T_6T_7 - T_3^2$	$T_1^4T_5^3T_6^2T_7 - T_3^2$	$T_1^3T_4T_5^2T_7^2 - T_2^4T_6^2$	$T_1^4T_6^5T_7 - T_2^4T_5^2$	$T_1^4T_6^3T_7^3 - T_2T_3T_4$	$T_1^4T_5^3T_7^3 - T_2T_3T_4$	$T_1^4T_5^3T_6^3 - T_2T_3T_4$
34	$T_3^2 - T_4T_5$	$T_1T_3T_7 - T_4T_5$	$T_1T_2T_7 - T_4T_5$	$T_1T_3T_7 - T_2T_5$	$T_1T_3T_7 - T_2T_4$	$T_1T_5T_7 - T_3^2$	$T_1T_5T_6 - T_3^2$
35	$T_3T_5^2 - T_4^3$	$T_1T_3T_5T_7 - T_4^3$	$T_1T_2T_5T_7 - T_4^3$	$T_1T_3T_5T_7 - T_2^3$	$T_1T_2T_3T_7 - T_4^3$	$T_1T_4T_5T_7 - T_2^3$	$T_1T_4T_5T_6 - T_2^3$
36	$T_2^2T_4^2 - T_3T_5^3$	$T_1^2T_3T_4T_6^2 - T_5^4$	$T_1^2T_2T_4T_6^2 - T_5^4$	$T_1^2T_2T_3T_6^2 - T_5^4$	$T_1^2T_3T_4T_6^2 - T_2^4$	$T_2T_3T_5^2 - T_4^4$	$T_2T_3T_5^2 - T_4^4$
37	$T_3^2T_5 - T_4^4$	$T_1T_3T_4^2T_7 - T_5^2$	$T_1T_2T_4^2T_7 - T_5^2$	$T_1T_2^2T_3T_7 - T_5^2$	$T_1^2T_2T_3T_6^2 - T_4^4$	$T_1^4T_7^4 - T_2^2T_5$	$T_1^4T_6^4 - T_2^2T_5$
38	$T_3^3T_4^3 - T_5^2$	$T_1^3T_3^3T_6T_7^2 - T_4^6$	$T_1^3T_2^3T_6T_7^2 - T_4^6$	$T_1^3T_3^3T_6T_7^2 - T_2^6$	$T_1^5T_3T_6^4T_7 - T_2T_4^5$	$T_1^3T_4^3T_7^3 - T_2T_3^2T_5$	$T_1^3T_4^3T_6^3 - T_2T_3^2T_5$
39	$T_2T_3T_4^2 - T_5^2$	$T_1^4T_4T_6T_7^3 - T_3^6$	$T_1^4T_4T_6T_7^3 - T_2^6$	$T_1^3T_2^3T_6T_7^2 - T_3^6$	$T_1^4T_4T_7^4 - T_2T_3^5$	$T_1^3T_5T_7^3 - T_3^4T_4$	$T_1^3T_5T_6^3 - T_3^4T_4$
40	$T_2T_3 - T_4^2$	$T_1^2T_5T_6 - T_3^2$	$T_1^2T_5T_6 - T_2^2$	$T_1^2T_5T_6 - T_3^2$	$T_1T_2T_7 - T_3T_4$	$T_1T_2T_7 - T_3T_4$	$T_1T_2T_5 - T_3T_4$
41	$T_2^2T_3 - T_4^3$	$T_1^2T_4T_5T_6 - T_3^3$	$T_1^2T_4T_5T_6 - T_2^3$	$T_1^2T_2T_5T_6 - T_3^3$	$T_1T_2T_4T_7 - T_3^3$	$T_1T_2T_4T_7 - T_3^3$	$T_1T_2T_4T_5 - T_3^3$
42	$T_2T_3 - T_4^2$	$T_1^2T_5T_6^2T_7 - T_3T_4$	$T_1^2T_5T_6^2T_7 - T_2T_4$	$T_1^2T_5T_6^2T_7 - T_2T_3$	$T_1^2T_6^4 - T_2T_3$	$T_1^2T_5^4 - T_2T_3$	$T_1^2T_6^4 - T_2T_3$
43	$T_2^2T_3 - T_4^3$	$T_1^3T_5^5T_7 - T_3T_4^2$	$T_1^3T_5^5T_7 - T_2T_4^2$	$T_1^3T_5^5T_7 - T_2^2T_3$	$T_1^2T_4T_6^2T_7^2 - T_2^3$	$T_1^2T_4T_5^2T_7^2 - T_2^3$	$T_1^2T_4T_5^2T_6^2 - T_2^3$
44	$T_2^3T_3 - T_4^2$	$T_1^4T_6^3T_7 - T_3^4$	$T_1^4T_6^3T_7 - T_2^4$	$T_1^3T_2T_5^2T_7 - T_3^4$	$T_1T_2T_3^2T_7 - T_4^2$	$T_1T_2T_3^2T_7 - T_4^2$	$T_1T_2T_3^2T_5 - T_4^2$
45	$T_2^3T_3 - T_4^2$	$T_1^4T_5^2T_6^2T_7^4 - T_3^2T_4$	$T_1^4T_5^2T_6^2T_7^4 - T_2^2T_4$	$T_1^4T_5^5T_7^3 - T_2^2T_3^2$	$T_1^4T_6^4T_7^4 - T_2T_3T_4$	$T_1^4T_5^4T_7^4 - T_2T_3T_4$	$T_1^4T_5^4T_6^4 - T_2T_3T_4$
46	$T_3^3 - T_4^2$	$T_1^6T_5T_6^4T_7 - T_3^3$	$T_1^5T_2T_5^4T_7 - T_4^2$	$T_1^3T_2^3T_5T_7^2 - T_3^3$	$T_1^4T_3T_6T_7^3 - T_2^6$	$T_1^4T_3T_5T_7^3 - T_2^6$	$T_1^4T_3T_5^3T_6 - T_2^6$
47	$T_3^3 - T_4^2$	$T_1^6T_5^6T_6^3T_7^3 - T_4^2$	$T_1^6T_5^8T_6^4 - T_2^3T_4$	$T_1^6T_6^{11}T_7 - T_2^6$	$T_1T_2T_3^2T_7^2 - T_4^2$	$T_1T_2T_3^2T_7^2 - T_4^2$	$T_1T_2T_3^2T_5^2 - T_4^2$
48	$T_2T_3 - T_4^2$	$T_1T_5 - T_3^2$	$T_1T_5 - T_2^2$	$T_1T_5 - T_3^2$	$T_1T_2T_7 - T_3T_4$	$T_1T_4T_7 - T_3^2$	$T_1T_4T_6 - T_3^2$
49	$T_2^2T_3 - T_4^3$	$T_1^3T_6^5T_7 - T_3T_4^2$	$T_1^3T_6^5T_7 - T_2T_4^2$	$T_1^3T_6^5T_7 - T_2^2T_3$	$T_1^2T_4T_6^2T_7^2 - T_2^3$	$T_1^2T_2T_7^4 - T_4^3$	$T_1^2T_2T_6^4 - T_4^3$
50	$T_2^2T_5^2 - T_3^2T_6T_7$	$T_1^2T_7^4 - T_3T_4T_6$	$T_1^2T_7^4 - T_2T_4T_6$	$T_1T_2T_5^3 - T_3^2T_6T_7$	$T_3^2T_6T_7 - T_4^2$	$T_3^2T_5T_7 - T_4^2$	$T_3^2T_5T_6 - T_4^2$
51	$T_2T_3^3T_7^2 - T_4^2$	$T_1^4T_6^5T_7 - T_3^4T_5^2$	$T_1^4T_6^5T_7 - T_2^4T_5^2$	$T_1T_2^3T_5^3 - T_3^4T_7^2$	$T_2^2T_4T_7 - T_3^4T_6^2$	$T_2^2T_4T_7 - T_3^4T_5^2$	$T_2^2T_4T_5 - T_3^4T_6^2$
52	$T_2^2T_4T_6 - T_3^3T_5T_7$	$T_1^4T_6^3T_7^3 - T_3T_4T_5$	$T_1^4T_6^3T_7^3 - T_2T_4T_5$	$T_1^3T_3T_6^2T_7^3 - T_2^2T_5^2$	$T_2^2T_4T_7 - T_3^4T_6^2$	$T_2T_3T_5^2 - T_4^2$	$T_2T_3T_5^2 - T_4^2$
53	$T_3^2 - T_4T_5$	$T_1T_5T_7 - T_4^2$	$T_1T_5T_7 - T_4^2$	$T_1T_5T_7 - T_2^2$	$T_1T_2T_7 - T_4^2$	$T_1T_5T_7 - T_3^2$	$T_1T_6 - T_2T_4$
54	$T_2^2T_4^2 - T_3T_5^3$	$T_1T_4^2T_6 - T_5^4$	$T_1T_4^2T_6 - T_5^4$	$T_1T_2^2T_6 - T_5^4$	$T_1T_4^2T_6 - T_2^4$	$T_2T_3T_5^2 - T_4^4$	$T_1^2T_6^2 - T_2T_3^2T_4$
55	$T_3T_5^2 - T_4^3$	$T_1^2T_3T_7^2 - T_4^3$	$T_1^2T_2T_7^2 - T_4^3$	$T_1^2T_3T_7^2 - T_2^3$	$T_1^2T_3T_7^2 - T_4^3$	$T_1T_4T_5T_7 - T_2^3$	$T_2^2T_4 - T_3T_5^2$
56	$T_2^2T_4^2 - T_3T_5^3$	$T_1^4T_7^4 - T_4^3T_5$	$T_1^4T_7^4 - T_4^3T_5$	$T_1^4T_7^4 - T_2^3T_5$	$T_1^4T_7^4 - T_2T_4^3$	$T_2T_3T_5^2 - T_4^4$	$T_2T_3^3 - T_4^3T_5$
57	$T_3^3T_4^3 - T_5^2$	$T_1^5T_6T_7^4 - T_3T_4^2T_5$	$T_1^5T_6T_7^4 - T_2T_4^2T_5$	$T_1^5T_6T_7^4 - T_2^2T_3T_5$	$T_1^3T_4^2T_6T_7^2 - T_2^5T_3$	$T_1^3T_4^3T_7^3 - T_2T_3^2T_5$	$T_2T_3^3T_4^2 - T_5^2$
58	$T_3^3T_4^3 - T_5^2$	$T_3^3T_4^3 - T_5^2$	$T_1^3T_4^3 - T_5^2$	$T_1^3T_3^3 - T_5^2$	$T_1^2T_4^4 - T_2T_3^5$	$T_1T_2^4T_4 - T_3^6$	$T_1T_2^4T_4 - T_3^6$
59	$T_2T_3T_4^2 - T_5^2$	$T_1T_3T_4^2 - T_5^2$	$T_1T_2T_4^2 - T_5^2$	$T_2^3T_3^3 - T_5^2$	$T_1^3T_2^2T_3 - T_4^3$	$T_1^3T_2T_3^2 - T_4^3$	$T_1^3T_2T_3^2 - T_4^3$
60	$T_2T_3T_4^2 - T_5^2$	$T_1T_3T_4^2 - T_5^2$	$T_1T_2T_4^2 - T_5^2$	$T_2^3T_3^3 - T_5^2$	$T_1^3T_2^2T_3 - T_4^3$	$T_1^3T_2T_3^2 - T_4^3$	$T_1^3T_2T_3^2 - T_4^3$
61	$T_3^2T_5 - T_4^4$	$T_3^2T_5 - T_4^4$	$T_1^2T_5 - T_4^4$	$T_1^4 - T_3^2T_5$	$T_1^2T_3^2 - T_2T_4^3$	$T_1^3T_3 - T_4^4$	$T_1^3T_3 - T_4^4$
62	$T_3^2T_5 - T_4^4$	$T_3^2T_5 - T_4^4$	$T_1^2T_5 - T_4^4$	$T_1^4 - T_3^2T_5$	$T_1^2T_3^2 - T_2T_4^3$	$T_1^3T_3 - T_4^4$	$T_1^3T_3 - T_4^4$
63	$T_3T_5^2 - T_4^3$	$T_3T_5^2 - T_4^3$	$T_1T_5^2 - T_4^3$	$T_1^3 - T_3T_5^2$	$T_1^2T_3 - T_4^3$	$T_1^2T_2 - T_5^3$	$T_1^2T_2 - T_5^3$
64	$T_3T_5^2 - T_4^3$	$T_3T_5^2 - T_4^3$	$T_1T_5^2 - T_4^3$	$T_1^3 - T_3T_5^2$	$T_1^2T_3 - T_4^3$	$T_1^2T_2 - T_5^3$	$T_1^2T_2 - T_5^3$
65	$T_3^2 - T_4T_5$	$T_3^2 - T_4T_5$	$T_1^2 - T_4T_5$	$T_1T_5 - T_3^2$	$T_1T_4 - T_3^2$	$T_1T_5 - T_3^2$	$T_1T_5 - T_3^2$
66	$T_3^2 - T_4T_5$	$T_3^2 - T_4T_5$	$T_1^2 - T_4T_5$	$T_1T_5 - T_3^2$	$T_1T_4 - T_3^2$	$T_1T_5 - T_3^2$	$T_1T_5 - T_3^2$
67	$T_3^2 - T_4T_5$	$T_3^2 - T_4T_5$	$T_1^2 - T_4T_5$	$T_1T_5 - T_3^2$	$T_1T_4 - T_3^2$	$T_1T_5 - T_3^2$	$T_1T_5 - T_3^2$

2.10 Birational geometry

We begin with a look at the birational geometry of the Fano fourfolds from Theorem 2.1.1. Let us briefly recall the necessary background. Consider any $\mathbb{Q}$-factorial Mori dream space $X = X(\lambda)$ arising from an abstract Cox ring $R = \oplus_K R_w$ as in Construction 2.3.2. Assume that $K_{\mathbb{Q}} = \mathrm{Cl}_{\mathbb{Q}}(X)$ is of dimension two. Then the GIT-fan $\Lambda(R)$ looks as follows:



where, as in Setting 2.5.1, we order the generator degrees $w_1, \dots, w_r \in K$ of R counter-clockwise. The moving cone $\mathrm{Mov}(X)$ is spanned by w_2 and w_{r-1}. If $w_2 \in \lambda$ holds, then with $\tau = \mathrm{cone}(w_2)$ we have

$$\bar{X}^{ss}(\lambda) \ \subseteq\ \bar{X}^{ss}(\tau),$$

which induces a morphism $\pi\colon X \to Y$ from $X = \bar{X}^{ss}(\lambda)/\!\!/H$ onto $Y = \bar{X}^{ss}(\tau)/\!\!/H$. Recall that π is an elementary contraction in the sense of [31]. In particular, we have the following two possibilities:

- If $w_2 \notin \mathrm{cone}(w_1)$ holds, then $\pi\colon X \to Y$ is birational and contracts the prime divisor $D_1 \subseteq X$ corresponding to the ray through w_1. In this case, we write $X \sim Y$ for the morphism π and denote by $C \subseteq Y$ the center of the contraction.
- $\pi\colon X \to Y$ is a proper fibration with $\dim(Y) < \dim(X)$. In this case, we write $X \to Y$ for the morphism π and denote by $F \subseteq X$ the general fiber.

Similarly, if $w_{r-2} \in \mathrm{Mov}(X)$ holds, we use the same notation. In general, λ need not to have common rays with $\mathrm{Mov}(X)$. However, given a ray $\varrho \subseteq \mathrm{Mov}(X)$, we find a small quasimodification $X \dashrightarrow X'$, where X' stems from a chamber $\lambda' \in \Lambda(R)$ sharing the ray ϱ with $\mathrm{Mov}(X)$. We then write $X' \sim Y$ or $X' \to Y$ etc. accordingly.

Remark 2.10.1. If X is as in Theorem 2.1.1, then X admits at least one elementary contraction and at most one small quasimodification $X \dashrightarrow X'$. If there is one, then X' is smooth due to Remark 2.5.12.

Now assume in addition that X has a hypersurface Cox ring and consider the toric embedding $X = X_g \subseteq Z$ from Construction 2.4.1. Given an elementary contraction of $\pi\colon X_g \to Y$, a suitable choice of the cone τ in Construction 2.4.1 leads to a commutative diagram

$$\begin{array}{ccc} X & \subseteq & Z \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_Z} \\ Y & \subseteq & W \end{array}$$

where $\pi_Z\colon Z \to W$ is an elementary contraction of the ambient toric variety Z. In particular, we have in this setting that for every point $y \in Y$, the fiber $\pi^{-1}(y) \subseteq X$ is contained in the fiber $\pi_Z^{-1}(y) \subseteq Z$. This gives in particular a description for the general fiber $F \subseteq X$ as a subvariety of the general fiber $F_Z \subseteq Z$.

Let us fix the necessary notation to formulate the result. By $Y_{d;a_1^{k_1},\dots,a_n^{k_n}}$ we denote a (not necessarily general) hypersurface of degree d in the weighted projective space $\mathbb{P}_{a_1^{k_1},\dots,a_n^{k_n}}$, where, as usual, $a_i^{k_i}$ means that $a_i \in \mathbb{Z}_{\geq 1}$ is repeated k_i times. For a hypersurface of degree d in the classical projective space $\mathbb{P}_n$ we just write $Y_{d;n}$. In our situation, this notation applies to the target spaces $Y \subseteq W$ in case of a birational elementary contraction and to the general fiber $F \subseteq F_Z$ in case of a fibration.

Proposition 2.10.2. *The subsequent table lists the possible elementary contractions for X as in Theorem 2.1.1, where X is not a cartesian product; the notation Y^* in the context of a birational contraction indicates that the target space is singular.*

No.	Contraction 1	Contraction 2
1	$X \to \mathbb{P}_3$ $F = Y_{1;2}$	$X \to \mathbb{P}_2$ $F = Y_{1;3}$
2	$X \to \mathbb{P}_3$ $F = Y_{1;2}$	$X \to \mathbb{P}_2$ $F = Y_{2;3}$
3	$X \to \mathbb{P}_3$ $F = Y_{1;2}$	$X \to \mathbb{P}_2$ $F = Y_{3;3}$
4	$X \to \mathbb{P}_3$ $F = Y_{2;2}$	$X \to \mathbb{P}_2$ $F = Y_{1;3}$
5	$X \to \mathbb{P}_3$ $F = Y_{2;2}$	$X \to \mathbb{P}_2$ $F = Y_{2;3}$
6	$X \to \mathbb{P}_3$ $F = Y_{2;2}$	$X \to \mathbb{P}_2$ $F = Y_{3;3}$
7	$X \to \mathbb{P}_3$ $F = Y_{1;2}$	$X \sim Y_{2;5}$ $C = \mathbb{P}_1$
8	$X \to \mathbb{P}_3$ $F = Y_{2;2}$	$X \sim Y_{3;5}$ $C = \mathbb{P}_1$
9	$X \to \mathbb{P}_3$ $F = Y_{1;2}$	$X \sim Y^*_{3;5}$ $C = \mathbb{P}_1$
10	$X \to \mathbb{P}_3$ $F = Y_{2;2}$	$X \sim Y^*_{4;5}$ $C = \mathbb{P}_1$
11	$X \to \mathbb{P}_3$ $F = Y_{1;2}$	$X \sim Y^*_{3;1^4,2^2}$ $C = \mathbb{P}_1$
12	$X \to \mathbb{P}_3$ $F = Y_{2;2}$	$X \sim Y^*_{5;1^4,2^2}$ $C = \mathbb{P}_1$
13	$X \to \mathbb{P}_2$ $F = Y_{2;3}$	$X' \to \mathbb{P}_2$ $F = Y_{1;3}$

No.	Contraction 1	Contraction 2
14	$X \to \mathbb{P}_2$ $F = Y_{3;3}$	$X' \to \mathbb{P}_2$ $F = Y_{2;3}$
15	$X \to \mathbb{P}_2$ $F = Y_{3;2}$	$X' \sim Y_{4;1^5,2}$ $C = \mathbb{P}_1$
16	$X \to \mathbb{P}_3$ $F = Y_{1;2}$	$X' \to \mathbb{P}_1$ $F = Y_{2;4}$
17	$X \to \mathbb{P}_3$ $F = Y_{2;2}$	$X' \to \mathbb{P}_1$ $F = Y_{3;4}$
18	$X \sim Y_{4;5}$ $C = \mathbb{P}_2$	$X \to \mathbb{P}_2$ $F = Y_{3;3}$
19	$X \sim Y_{3;5}$ $C = \mathbb{P}_2$	$X \to \mathbb{P}_2$ $F = Y_{2;3}$
20	$X \sim Y_{2;5}$ $C = \mathbb{P}_2$	$X \to \mathbb{P}_2$ $F = Y_{1;3}$
21	$X \sim Y_{4;1^5,2}$ $C = \mathbb{P}_2$	$X \to \mathbb{P}_1$ $F = Y_{3;4}$
22	$X' \to \mathbb{P}_1$ $F = Y_{2;4}$	$X \to \mathbb{P}_2$ $F = Y_{2;3}$
23	$X' \to \mathbb{P}_1$ $F = Y_{3;4}$	$X \to \mathbb{P}_2$ $F = Y_{3;3}$
24	$X' \to \mathbb{P}_1$ $F = Y_{4;1^4,2}$	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$
25	$X' \to \mathbb{P}_1$ $F = Y_{6;1^3,2,3}$	$X \to \mathbb{P}_2$ $F = Y_{6;1^2,2,3}$
26	$X \to \mathbb{P}_1$ $F = Y_{2;4}$	$X \to \mathbb{P}_3$ $F = Y_{2;2}$

No.	Contraction 1	Contraction 2
27	$X \to \mathbb{P}_1$ $F = Y_{4;1^4,2}$	$X \to \mathbb{P}_3$ $F = Y_{2;2}$
28	$X \to \mathbb{P}_1$ $F = Y_{6;1^4,3}$	$X \to \mathbb{P}_3$ $F = Y_{2;2}$
29	$X \to \mathbb{P}_2$ $F = Y_{2;3}$	$X \to \mathbb{P}_2$ $F = Y_{2;3}$
30	$X \to \mathbb{P}_2$ $F = Y_{3;3}$	$X \to \mathbb{P}_2$ $F = Y_{3;3}$
31	$X \to \mathbb{P}_2$ $F = Y_{2;3}$	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$
32	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$
33	$X' \sim Y^*_{6;1^4,2,3}$ $C = \{\mathrm{pt}\}$	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$
34	$X \sim Y_{2;5}$ $C = \mathbb{P}_1 \times \mathbb{P}_1$	$X \to \mathbb{P}_1$ $F = Y_{2;4}$
35	$X \sim Y_{3;5}$ $C = Y_{3;3}$	$X \to \mathbb{P}_1$ $F = Y_{3;4}$
36	$X \sim Y_{4;5}$ $C = Y_{4;3}$	$X \to \mathbb{P}_1$ $F = Y_{4;4}$
37	$X \sim Y_{4;1^5,2}$ $C = Y_{4;1^3,2}$	$X \to \mathbb{P}_1$ $F = Y_{4;1^4,2}$
38	$X \sim Y_{6;1^5,3}$ $C = Y_{6;1^3,3}$	$X \to \mathbb{P}_1$ $F = Y_{6;1^4,3}$
39	$X \sim Y_{6;1^4,2,3}$ $C = Y_{6;1^2,2,3}$	$X \to \mathbb{P}_1$ $F = Y_{6;1^3,2,3}$
40	$X \sim Y_{2;5}$ $C = \mathbb{P}_1$	$X \to \mathbb{P}_2$ $F = Y_{2;3}$
41	$X \sim Y_{3;5}$ $C = Y_{3;2}$	$X \to \mathbb{P}_2$ $F = Y_{3;3}$
42	$X \sim Y^*_{4;1^3,2^3}$ $C = \mathbb{P}_1$	$X \to \mathbb{P}_2$ $F = Y_{2;3}$
43	$X \sim Y^*_{6;1^2,2^3}$ $C = Y_{3;2}$	$X \to \mathbb{P}_2$ $F = Y_{3;3}$
44	$X \sim Y_{4;1^5,2}$ $C = Y_{4;1^2,2}$	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$

No.	Contraction 1	Contraction 2
45	$X \sim Y^*_{8;1^3,2^2,4}$ $C = Y_{4;1^2,2}$	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$
46	$X \sim Y_{6;1^4,2,3}$ $C = Y_{6;1,2,3}$	$X \to \mathbb{P}_2$ $F = Y_{6;1^2,2,3}$
47	$X \sim Y^*_{12;1^3,2,4,6}$ $C = Y_{6;1,2,3}$	$X \to \mathbb{P}_2$ $F = Y_{6;1^2,2,3}$
48	$X \sim \mathbb{P}_4$ $C = \mathbb{P}_1$	$X' \to \mathbb{P}_1$ $F = Y_{2;4}$
49	$X \sim Y^*_{6;1^2,2^3,3}$ $C = Y_{6;2^3,4}$	$X' \to \mathbb{P}_1$ $F = Y_{3;4}$
50	$X \sim Y^*_{4;1^5,2}$ $C = \mathbb{P}_1$	$X \to \mathbb{P}_2$ $F = Y_{2;3}$
51	$X \sim Y^*_{6;1^5,3}$ $C = \mathbb{P}_1$	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$
52	$X \sim Y^*_{6;1^4,2,3}$ $C = \mathbb{P}_1$	$X \to \mathbb{P}_1$ $F = Y_{4;1^4,2}$
53	$X \sim \mathbb{P}_4$ $C = \mathbb{P}_1 \times \mathbb{P}_1$	$X \sim Q_4$ $C = \{\mathrm{pt}\}$
54	$X \sim Y_{4;1^5,2}$ $C = Y_{4;3}$	$X \sim Y_{4;5}$ $C = \{\mathrm{pt}\}$
55	$X \sim \mathbb{P}_4$ $C = Y_{3;3}$	$X \sim Y^*_{3;1^5,2}$ $C = \{\mathrm{pt}\}$
56	$X \sim \mathbb{P}_4$ $C = Y_{4;3}$	$X \sim Y_{3;1^5,3}$ $C = \{\mathrm{pt}\}$
57	$X \sim Y_{6;1^4,2,3}$ $C = Y_{4;1^3,3}$	$X \sim Y_{6;1^5,3}$ $C = \{\mathrm{pt}\}$
60	$X \to Y_{6;1^3,2,3}$ $F = \mathbb{P}_1$	$X \sim Y^*_{6;1^4,2,3}$ $C = \{\mathrm{pt}\}$
62	$X \to Y_{4;1^4,2}$ $F = \mathbb{P}_1$	$X \sim Y^*_{4;1^5,2}$ $C = \{\mathrm{pt}\}$
64	$X \to Y_{3;4}$ $F = \mathbb{P}_1$	$X \sim Y^*_{3;5}$ $C = \{\mathrm{pt}\}$
66	$X \to Y_{2;4}$ $F = \mathbb{P}_1$	$X \sim Y^*_{2;5}$ $C = \{\mathrm{pt}\}$
67	$X \to Y_{2;4}$ $F = \mathbb{P}_1$	$X \sim Y^*_{2;1^5,2}$ $C = \{\mathrm{pt}\}$

The remaining families of Theorem 2.1.1 consist of cartesian products $Y \times \mathbb{P}_1$ where the first factor Y is a smooth threedimensional Fano hypersurface of Picard number one as displayed in the following table.

No.	58	59	61	63	65
Y	$Y_{6;1^4,3}$	$Y_{6;1^3,2,3}$	$Y_{4;1^4,2}$	$Y_{3;4}$	$Y_{2;4}$

The proof of this proposition is basically a case by case analysis of the contraction maps in coordinates. We restrict ourselves to perform this in the subsequent remark for one case, where we even go a bit deeper into the matter and specify also the singular fibers of the fibration.

Remark 2.10.3. We take a closer look at the varieties X from No. 9 of Theorem 2.1.1. In this case the specifying data, that means the degree matrix Q and the degree μ of the relation g, are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (2,1).$$

Due to $-\mathcal{K} = (1,2)$, we have $\lambda = \operatorname{cone}(w_1, w_5)$. Observe that $\operatorname{Mov}(R)$ and λ share the rays ϱ_1 and ϱ_5. Thus X admits two elementary contractions $\pi_1 \colon X \to Y_1$ and $\pi_2 \colon X \to Y_2$ associated to ϱ_1 resp. ϱ_5. To study π_1 and π_2 we make use of the toric embedding $X = X_g \subseteq Z$ from Construction 2.4.1.

First, we discuss π_1. Since $w_2 \in \varrho_1$ holds, the morphism π_1 is a fibration. Moreover, π_1 is the restriction of the corresponding ambient toric elementary contraction $\pi_{1,Z}$ of Z, which in turn is explicitly given as follows:

$$\begin{array}{ccccc} \bar{X} & \subseteq & \mathbb{K}^7 & \xrightarrow{(z_1,\ldots,z_7)\mapsto(z_1,\ldots,z_4)} & \mathbb{K}^4 \\ \downarrow & & \downarrow & & \downarrow \\ X & \subseteq & Z & \xrightarrow{\quad \pi_{1,Z} \quad} & \mathbb{P}_3 \end{array}$$

Suitably sorting the terms of g yields a presentation $g = q_1T_5 + q_2T_6 + fT_7$ where $q_1, q_2 \in \mathbb{K}[T_1, \ldots, T_4]$ both are quadrics and $f \in \mathbb{K}[T_1, \ldots, T_4]$ is a cubic, each of which is general. Note that $V(g) \subseteq \mathbb{K}^7$ projects onto $\mathbb{K}^4$ thus $Y_1 = \mathbb{P}_3$. For any point $y = [y_1, \ldots, y_4] \in \mathbb{P}_3$ the fiber $\pi_{1,Z}^{-1}(y)$ of the ambient toric variety is given by the equations

$$y_2T_1 - y_1T_2 \;=\; y_3T_2 - y_2T_3 \;=\; y_4T_3 - y_3T_4 \;=\; 0.$$

Besides we have $y_i \neq 0$ for some i. Taking this into account one directly checks $\pi_{1,Z}^{-1}(y) \cong \mathbb{P}_2$. Being homogeneous g is compatible with this isomorphism, thereby we obtain

$$\pi_1^{-1}(y) \;\cong\; V(y_iq_1(y)T_0 + y_iq_2(y)T_1 + f(y)T_2) \;\subseteq\; \mathbb{P}_2.$$

We conclude that the general fiber $\pi_1^{-1}(y)$ is isomorphic to $\mathbb{P}_1$. In addition, $V(q_1, q_2, f) \subseteq \mathbb{P}_3$ consists of precisely 12 points $p_1, \ldots, p_{12}$, each of which has fiber $\pi_1^{-1}(p_i) \cong \mathbb{P}_2$.

We turn to π_2. From $w_7 \notin \varrho_5$ follows that π_2 is a birational morphism contracting the prime divisor $V(T_7) \subseteq X$. The according elementary contraction $\pi_{2,Z}$ of the ambient toric variety Z is the blow-up of $\mathbb{P}_5$ along $C = V_{\mathbb{P}_5}(T_0, \dots, T_3) \cong \mathbb{P}_1$. The situation is as in the subsequent diagram:

$$
\begin{array}{ccccc}
\bar{X} & \subseteq & \mathbb{K}^7 & \xrightarrow{(z_1,\dots,z_7)\mapsto(z_1z_7,\dots,z_4z_7,z_5,z_6)} & \mathbb{K}^6 \\
\downarrow & & \downarrow & & \downarrow \\
X & \subseteq & Z & \xrightarrow{\pi_{2,Z}} & \mathbb{P}_5
\end{array}
$$

The target variety $Y_2 \subseteq \mathbb{P}_5$ of π_2 is $V(g') \subseteq \mathbb{P}_5$ where $g' = g(T_0, \dots, T_6, 1)$. From this we infer $C \subseteq Y_2$, so C is the center of π_2 as well. In particular π_2 is the blow-up of Y_2 along C. Moreover, the polynomial g' is an irreducible cubic living in $\langle T_0, \dots, T_3 \rangle^2$. Consequently, Y_2 is singular at every point of C.

2.11 Hodge numbers

Here we determine the Hodge numbers of the Fano fourfolds from Theorem 2.1.1. First, we note the following simple observation.

Proposition 2.11.1. *Let X be a smooth projective Fano fourfold of Picard rank 2. Then the Hodge diamond of X is the following.*

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 2 & & 0 & & \\
 & 0 & & h^{1,2} & & h^{2,1} & & 0 & \\
0 & & h^{1,3} & & h^{2,2} & & h^{3,1} & & 0 \\
 & 0 & & h^{3,2} & & h^{2,3} & & 0 & \\
 & & 0 & & 2 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

Proof. Ampleness of $-K_X$ and Kawamata-Viehweg vanishing give $h^{p,0}(X) = 0$ for any $p > 0$. Moreover, plugging $H^i(X, \mathcal{O}) = 0$ for $i = 1, 2$ into the cohomology sequence associated with the exponential sequence yields $H^2(X, \mathbb{C}) \cong \mathbb{C}^2$. The Hodge decomposition together with $h^{1,0}(X) = h^{0,1}(X) = 0$ shows $h^{1,1}(X) = 2$. □

By symmetry, we are left with computing the Hodge numbers $h^{2,1}$, $h^{3,1}$ and $h^{2,2}$. Here comes our result.

Proposition 2.11.2. *The subsequent table lists the Hodge numbers $h^{2,1}$, $h^{3,1}$ and $h^{2,2}$ for X as in Theorem 2.1.1.*

No.	$h^{2,1}$	$h^{3,1}$	$h^{2,2}$	*No.*	$h^{2,1}$	$h^{3,1}$	$h^{2,2}$	*No.*	$h^{2,1}$	$h^{3,1}$	$h^{2,2}$
1	0	0	3	23	0	13	103	45	1	50	288
2	0	0	10	24	0	35	218	46	1	24	163
3	0	0	29	25	0	114	591	47	1	159	793
4	0	0	3	26	0	0	10	48	0	0	3
5	0	3	40	27	0	20	138	49	1	2	31
6	0	30	185	28	0	112	570	50	0	3	40
7	0	0	4	29	0	1	22	51	0	65	356
8	0	1	23	30	0	45	255	52	0	20	139
9	0	0	14	31	0	10	94	53	0	0	3
10	0	18	126	32	0	100	508	54	0	6	72
11	0	0	5	33	0	24	162	55	0	0	8
12	0	12	95	34	0	0	4	56	0	1	21
13	0	0	4	35	0	1	28	57	0	25	181
14	0	6	65	36	0	22	162	58	52	0	2
15	0	5	55	37	0	5	60	59	21	0	2
16	0	0	6	38	0	71	402	60	21	0	2
17	0	9	77	39	0	24	170	61	10	0	2
18	0	21	143	40	0	0	4	62	10	0	2
19	0	1	22	41	1	1	23	63	5	0	2
20	0	0	3	42	0	0	10	64	5	0	2
21	0	5	53	43	1	19	131	65	0	0	2
22	0	0	10	44	1	5	54	66	0	0	2
								67	0	0	2

Proof. We consider the toric embedding $X = X_g \subseteq Z_g$ as provided by Construction 2.4.1. The five-dimensional toric ambient variety Z_g is smooth and the decomposition

$$X \;=\; \bigcup_{\gamma_0 \in \mathrm{rlv}(X)} X(\gamma_0)$$

from Construction 2.3.4 is obtained by cutting down the toric orbit decomposition of Z_g. Now the idea is to compute the Hodge numbers in question via the Hodge-Deligne polynomial, being defined for any variety Y as

$$e(Y) \;:=\; \sum_{p,q} e^{p,q}(Y) x^p \bar{x}^q \;\in\; \mathbb{Z}[x, \bar{x}],$$

with $e^{p,q}(Y)$ as in [52, p. 280]. We also write $e^{p,q}$ instead of $e^{p,q}(Y)$. Recall that $e^{p,q} = e^{q,p}$ holds. Moreover, in case that Y is smooth and projective, the $e^{p,q}$ are related to the Hodge numbers as follows:

$$e^{p,q}(Y) \;=\; (-1)^{p+q} h^{p,q}(Y).$$

The Hodge-Deligne polynomial is additive on disjoint unions, multiplicative on cartesian products. We list the necessary steps for computing it in low dimensions. On $Y = \mathbb{C}^*$, it evaluates to $x\bar{x} - 1$. For a hypersurface $Y \subseteq (\mathbb{C}^*)^n$ with no torus factors, one has the Lefschetz type formula

$$e^{p,q}(Y) \;=\; e^{p+1,q+1}((\mathbb{C}^*)^n), \qquad \text{for } p + q > n - 1,$$

see [52, p. 290]. Moreover, according to [52, p. 291], with the Newton polytope Δ of the defining equation of Y, one has the following identity

$$\sum_{q\geq 0} e^{p,q}(Y) \;=\; (-1)^{p+n-1}\binom{n}{p+1} + (-1)^{n-1}\varphi_{n-p}(\Delta),$$

where, denoting by $l^*(B)$ the number of interior points of a polytope B, the function φ_i is defined as

$$\varphi_0(\Delta) \;:=\; 0, \qquad \varphi_i(\Delta) \;:=\; \sum_{j=1}^{i}(-1)^{i+j}\binom{n+1}{i-j} l^*(j\Delta),$$

This leads to an explicit formula for all $e^{p,0}(Y)$. Moreover, for $\dim(Y) \leq 3$, all the numbers $e^{p,q}$ are directly calculated using the above formulas. For $\dim(Y) = 4$, the values of $e^{1,1} + e^{1,2} + e^{1,3}$ and $e^{2,1} + e^{2,2}$ and $e^{3,1}$ can be directly computed using the above formulas. By the symmetry $e^{p,q} = e^{q,p}$ these sums involve just four numbers which thus can be expressed in terms of one of them, say $e^{1,2}$, plus known quantities. To determine the value of $e^{1,2}$ one passes to a smooth compactification Y' of Y for which

$$e^{1,2}(Y') \;=\; -h^{1,2}(Y') \;=\; -h^{3,2}(Y') \;=\; e^{3,2}(Y')$$

holds by Serre's duality and then observes that $e^{3,2}$ can be computed for all the strata via the Lefschetz formula. Now, we apply these principles to the strata $Y = X(\gamma_0)$ that have no torus factor and compute the desired $e^{p,q}$. If $Y = X(\gamma_0)$ has a torus factor, then we use multiplicativity of the Hodge-Deligne polynomial and again the above principles. □

Finally, we extend the discussion of the varieties X from Number 9 of Theorem 2.1.1 started in Remark 2.10.3 by some topological aspects.

Remark 2.11.3. Let X be as in Theorem 2.1.1, No. 9. Recall that we have a fibration $X \to \mathbb{P}_3$ with general fiber $F = \mathbb{P}_1$ and precisely 12 special fibers $F_1, \ldots, F_{12}$, lying over $p_1, \ldots, p_{12} \in \mathbb{P}_3$, each of the F_i being isomorphic to $\mathbb{P}_2$. We claim

$$F_i^2 \;=\; 1 \quad \text{for } i = 1, \ldots 12, \qquad\qquad F_i \cdot F_j = 0 \quad \text{for } 1 \leq i < j \leq 12.$$

The second part is clear because of F_i and F_j do not intersect for $i < j$. In order to establish the first part, we show $F_1^2 = 1$, where we may assume $p_1 = [1, 0, 0, 0]$. Consider the zero sets $L_1, L_2 \subseteq X$ of two general polynomials in the variables T_2, T_3, T_4. By definition $L_1 \cap L_2 = F$ and $L_1 \sim L_2$, that is the two surfaces are rationally equivalent. Thus $L_i \sim F + S_i$ for some surface S_i. Observe that we have

$$F \cdot L_i \;=\; 0, \qquad\qquad S_i \cdot L_i \;=\; 0$$

because L_i is rationally equivalent to a complete intersection of two general polynomials in $T_1, \ldots, T_4$, which has empty intersection with L_i. We deduce

$$F^2 \;=\; -F \cdot S_1 \;=\; S_1 \cdot S_1 \;=\; S_1 \cdot S_2,$$

using $S_1 \sim S_2$ in the last step. For computing the last intersection number, we may assume $L_1 = V(T_2, T_3, g)$ and $L_2 = V(T_2, T_4, g)$. Then $S_1 = V(T_2, T_3, h_1)$ with

$$h_1 \ = \ T_4^{-1}(q_1(T_1, 0, 0, T_4)T_5 + q_2(T_1, 0, 0, T_4)T_6 + f(T_1, 0, 0, T_4)T_7),$$

where the division by T_4 can be performed because by hypothesis q_1, q_2 and f do not contain a pure power of T_1. Similarly $S_2 = V(T_2, T_4, h_2)$, where

$$h_2 \ = \ T_3^{-1}(q_1(T_1, 0, T_3, 0)T_5 + q_2(T_1, 0, T_3, 0)T_6 + f(T_1, 0, T_3, 0)T_7).$$

It follows that

$$\begin{aligned} S_1 \cap S_2 &= V(T_2, T_3, T_4, \alpha_1 T_1 T_5 + \alpha_2 T_1 T_6 + \alpha_3 T_1^2 T_7, \beta_1 T_1 T_5 + \beta_2 T_1 T_6 + \beta_3 T_1^2 T_7) \\ &= V(T_2, T_3, T_4, \alpha_1 T_5 + \alpha_2 T_6 + \alpha_3 T_1 T_7, \beta_1 T_5 + \beta_2 T_6 + \beta_3 T_1 T_7). \end{aligned}$$

Now one directly checks that $S_1 \cap S_2$ is a point and the intersection is transverse. Thus, we arrive at $S_1 \cdot S_2 = 1$, proving the $F_1^2 = 1$. Now, fix two general linear forms $\ell_1, \ell_2 \in \mathbb{C}[T_1, \ldots, T_4]$ and set

$$E \ := \ V(T_6, T_7, g) \ \subseteq \ X, \qquad\qquad L \ := \ V(\ell_1, \ell_2, g) \ \subseteq \ X.$$

We claim that the classes of $E, L, F_1, \ldots, F_{12}$ in $H^{2,2}(X) \cap H^4(X, \mathbb{Q})$ are linearly independent. First observe that $F_1, \ldots, F_{12}$ are linearly independent: passing to the self-intersection, $\sum_i a_i F_i \sim 0$ turns into $\sum_i a_i^2 = 0$ and thus, being rational numbers, all a_i vanish. Now, by definition of L one has $L^2 = L \cdot F_i = 0$ for any i, in particular the class of L cannot be in the linear span of the classes of the 12 fibers. The statement then follows from $E \cdot L = 2$, which in turn holds due to

$$E \cap L \ = \ V(\ell_1, \ell_2, T_6, T_7, g) \ = \ V(\ell_1, \ell_2, T_6, T_7, q_1 T_5) \ = \ V(\ell_1, \ell_2, T_6, T_7, q_1).$$

Combining linear independence of $E, L, F_1, \ldots, F_{12} \in H^{2,2}(X) \cap H^4(X, \mathbb{Q})$ with $h^{2,2}(X) = 14$ as provided by Proposition 2.11.2, we retrieve that the varieties X from Number 9 of Theorem 2.1.1 satisfy the Hodge Conjecture; which, in this case, is known to hold also by [42] and [127, Proof of Lemma 15.2].

2.12 Deformations and automorphisms

We take a look at the deformations of the varieties from Theorem 2.1.1. For any variety X, we denote by $\mathcal{T}_X$ its tangent sheaf. If X is Fano, then it is unobstructed and thus its versal deformation space is of dimension $h^1(X, \mathcal{T}_X)$. The following observation makes precise how the problem of determining $h^1(X, \mathcal{T}_X)$ is connected with determining the automorphisms in our setting.

Proposition 2.12.1. *Let X be a smooth Fano variety X with a general hypersurface Cox ring $\mathcal{R}(X) = \mathbb{C}[T_1, \ldots, T_r]/\langle g \rangle$ and associated minimal toric embedding $X \subseteq Z$.*

Assume that $\mu = \deg(g) \in \mathrm{Cl}(Z)$ *is base point free and no* $w_i = \deg(T_i) \in \mathrm{Cl}(Z)$ *lies in* $\mu + \mathbb{Z}_{\geq 0}w_1 + \cdots + \mathbb{Z}_{\geq 0}w_r$*. Then we have*

$$\begin{aligned} h^1(X, \mathcal{T}_X) &= \dim(\mathcal{R}(Z)_\mu) - 1 + \mathrm{rank}(\mathrm{Cl}(Z)) - \sum_{i=1}^r \dim(\mathcal{R}(Z)_{w_i}) + h^0(X, \mathcal{T}_X) \\ &= -1 + \dim(\mathcal{R}(Z)_\mu) - \dim(\mathrm{Aut}(Z)) + \dim(\mathrm{Aut}(X)). \end{aligned}$$

Proof. First look at $0 \to \mathcal{T}_X \to \imath^*\mathcal{T}_Z \to \mathcal{N}_X \to 0$, the normal sheaf sequence for the inclusion $\imath\colon X \subseteq Z$. By assumption, $\mu - \mathcal{K}_X$ is ample and thus we obtain

$$h^1(X, \mathcal{T}_X) - h^0(X, \mathcal{T}_X) \;=\; -h^0(X, \imath^*\mathcal{T}_Z) + h^0(X, \mathcal{N}_X) + h^1(X, \imath^*\mathcal{T}_Z),$$

according to the Kawamata-Viehweg vanishing theorem. The task is to evaluate the right hand side. First, note that we have

$$h^0(X, \mathcal{N}_X) \;=\; \dim(\mathcal{R}(X)_\mu) \;=\; \dim(\mathcal{R}(Z)_\mu) - 1.$$

For the remaining two terms, we use the Euler sequence of Z restricted to X which in our setting is given by

$$0 \longrightarrow \mathcal{O}_X \otimes \mathrm{Cl}(Z) \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_X(D_i) \longrightarrow \imath^*\mathcal{T}_Z \longrightarrow 0,$$

where $D_i \subseteq X$ denotes the prime divisor defined by the Cox ring generator T_i. Since X is Fano, $h^i(X, \mathcal{O}_X)$ vanishes for all $i > 0$. As first consequence, we obtain

$$h^0(X, \imath^*\mathcal{T}_Z) \;=\; \sum_{i=1}^r \dim(\mathcal{R}(X)_{w_i}) - \mathrm{rank}(\mathrm{Cl}(Z)) \;=\; \sum_{i=1}^r \dim(\mathcal{R}(Z)_{w_i}) - \mathrm{rank}(\mathrm{Cl}(Z)),$$

using $\mathcal{R}(X)_{w_i} \cong H^0(X, D_i)$ and $\mathcal{R}(X)_{w_i} = \mathcal{R}(Z)_{w_i}$, where the latter holds by assumption. Moreover, we can conclude

$$h^1(X, \imath^*\mathcal{T}_Z) \;=\; \sum_{i=1}^r h^1(X, D_i).$$

We evaluate the right hand side. Since X has a general hypersurface Cox ring, Z is smooth [4, Prop. 3.3.1.12] and μ is base point free, we can infer smoothness of

$$D_i \;=\; V(g) \cap V(T_i) \;\subseteq\; Z$$

from Bertini's theorem. Now choose $\varepsilon > 0$ such that $\varepsilon D_i - \mathcal{K}_X$ is nef and big. Then, using once more the Kawamata-Viehweg vanishing theorem, we obtain

$$h^1(X, D_i) \;=\; h^1(X, \mathcal{K}_X + (\varepsilon D_i - \mathcal{K}_X) + (1-\varepsilon)D_i) \;=\; 0.$$

Consequently, $h^1(X, \imath^*\mathcal{T}_Z)$ vanishes. This gives the first equality of the assertion. The second one follow from [44, Thm. 4.2] and [93, Lemma 3.4]. □

Observe that Proposition 2.12.1 applies in particular to all smooth Fano non-degenerate toric hypersurfaces in the sense of Khovanskii [76] and Definition 1.4.1 of this thesis, where Lemma 1.3.3 (v) guarantees base point freeness of $\mu \in \mathrm{Cl}(Z)$. Concerning the varieties from Theorem 2.1.1, we can say the following.

Corollary 2.12.2. *For each of the Fano varietes X listed in Theorem 2.1.1, except possibly numbers 13, 14, 15, 33 and 67, we have*

$$h^1(X, \mathcal{T}_X) \;=\; -1 + \dim(\mathcal{R}(Z)_\mu) - \dim(\mathrm{Aut}(Z)) + \dim(\mathrm{Aut}(X)).$$

Proof. Using [4, Prop. 3.3.2.8] one directly checks that $\mu \in \mathrm{Cl}(X)$ and hence also $\mu \in \mathrm{Cl}(Z)$ are base point free in all cases except the Numbers 13, 14, 15 and 33. Number 67 violates the assumption on the generator degrees. □

The only serious task left open by Proposition 2.12.1 for explicitly computing $h^1(X, \mathcal{T}_X)$ is to determine the dimension of $\mathrm{Aut}(X)$. As general tools we mention [65, Thm. 4.4], the algorithms presented thereafter and their implementation provided by [75]. The subsequent example discussions indicate how one might proceed in concrete cases.

Example 2.12.3. The variety X from No. 65 is a product of the smooth projective quadric $Q_4 \subseteq \mathbb{P}_4$ and a projective line. So, X is known to be infinitesimally rigid. Via Proposition 2.12.1, this is seen as follows:

$$\begin{aligned} h^1(X, \mathcal{T}_X) &= -1 + \dim(\mathcal{R}(Z)_\mu) - \dim(\mathrm{Aut}(Z)) + \dim(\mathrm{Aut}(X)) \\ &= -1 + 15 - 27 + 13 \\ &= 0. \end{aligned}$$

All ingredients are classical: First, by [20, Cor. I.2] the unit component of the automorphism group of a product is the product of the unit components of the respective automorphism groups. Second, $\mathrm{Aut}(Q_n) = \mathrm{O}(n)$ is of dimension $n(n-1)/2$.

Example 2.12.4. For the varieties X from No. 1, the algorithm [75] is feasible and tells us that $\mathrm{Aut}(X)$ is of dimension 12. In particular, we see that also these varieties are infinitesimally rigid:

$$\begin{aligned} h^1(X, \mathcal{T}_X) &= -1 + \dim(\mathcal{R}(Z)_\mu) - \dim(\mathrm{Aut}(Z)) + \dim(\mathrm{Aut}(X)) \\ &= -1 + 12 - 23 + 12 \\ &= 0. \end{aligned}$$

In suitable linear coordinates respecting the grading, $g = T_1T_5 + T_2T_6 + T_3T_7$ holds and the automorphisms on X are induced by the five-dimensional diagonally acting torus respecting g and the group $\mathrm{GL}(3)$ acting on $\mathcal{R}(X)_{w_1} \oplus \mathcal{R}(X)_{w_5}$ via

$$A \cdot (T_1, T_2, T_3, T_4; T_5, T_6, T_7) \;:=\; (A \cdot (T_1, T_2, T_3), T_4; (A^{-1})^t \cdot (T_5, T_6, T_7)).$$

The two previous examples fit into the class of *intrinsic quadrics*, that means varieties having a hypersurface Cox ring with a quadric as defining relation. The ideas just observed lead to the following general observation.

Corollary 2.12.5. *Let X be a variety satisfying all the assumptions of Proposition 2.12.1 and assume that $\mathrm{Aut}_H(\bar{Z})$ acts almost transitively on $\mathcal{R}(Z)_\mu$.*

(i) *The variety X is infinitesimally rigid and the dimension of its automorphism group is given by*

$$\dim(\mathrm{Aut}(X)) \;=\; \dim(\mathrm{Aut}_H(\bar{Z})) - (\dim(\mathcal{R}(Z)_\mu) - 1) - \mathrm{rank}(\mathrm{Cl}(Z)).$$

(ii) *If X is an intrinsic quadric, then $\mathrm{Aut}_H(\bar{Z})$ acts almost transitively on $\mathcal{R}(Z)_\mu$ and thus the statements from (i) hold for X.*

Proof. We take $X \subseteq Z$ as in Construction 2.4.1. According to [65, Thm. 4.4 (iv)], the unit component $\mathrm{Aut}(X)^0$ equals the stabilizer $\mathrm{Aut}(Z)^0_X$ of $X \subseteq Z$ under the action of $\mathrm{Aut}(Z)^0$ on Z. Thus, using [4, Thm. 4.2.4.1], we obtain

$$\begin{aligned}
\dim(\mathrm{Aut}(X)) &= \dim(\mathrm{Aut}(Z)^0_X) \\
&= \dim(\mathrm{Aut}_H(\bar{Z})^0_{\bar{X}}) - \dim(H) \\
&= \dim(\mathrm{Aut}_H(\bar{Z})^0) - (\dim(\mathcal{R}(Z)_\mu) - 1) - \mathrm{rank}(\mathrm{Cl}(Z)),
\end{aligned}$$

where $\mathcal{R}(Z)_\mu$ is the space of defining equations and "-1" pops up as we are looking for only the zero sets of these equations. Thus, Proposition 2.12.1 gives the first statement. For the second one, note that $\mathrm{Aut}_H(\bar{Z})$ acts almost transitively on $\mathcal{R}(Z)_\mu$ due to [55, Prop. 2.1]. □

Let us take up once more the geometric discussion of the varieties from No. 9 of Theorem 2.1.1 started in Remarks 2.10.3 and 2.11.3. Using geometric properties observed so far, we see $\mathrm{Aut}(X)$ is trivial.

Remark 2.12.6. Let X be as in Theorem 2.1.1, No. 9. We claim that $\mathrm{Aut}(X)$ is finite in this case. As a consequence, we obtain

$$\begin{aligned}
h^1(X, \mathcal{T}_X) &= \dim(\mathcal{R}(Z)_\mu) - 1 + \mathrm{rank}(\mathrm{Cl}(Z)) - \sum_{i=1}^{r} \dim(\mathcal{R}(Z)_{w_i}) \\
&= 40 - 1 + 2 - 29 \\
&= 12.
\end{aligned}$$

Look at the fibration $\pi_1 \colon X \to Y_1 = \mathbb{P}_3$ from Remark 2.10.3. By [20, Prop. I.1], there is an induced action of the unit component $\mathrm{Aut}(X)^0$ on Y_1 turning π_1 into an equivariant map. This means in particular that the induced action permutes the image points of the 12 singular fibers of π_1. By the generality assumption, these 12 points don't lie in a common hyperplane and thus the induced action of $\mathrm{Aut}(X)^0$ on Y_1 must be trivial. Recall that any point of the fiber π_1 over $[y] = [y_1, \ldots, y_4]$ has Cox coordinates

$$[y, x, z] \;=\; [y_1, \ldots, y_4, x_1, x_2, z], \qquad \text{where} \quad q_1(y)x_1 + q_2(y)x_2 + f(y)z = 0,$$

with general quadrics q_1, q_2 and a general cubic f in the first four variables. Let us see in these terms what it means that the π_1-fibers are invariant under $\mathrm{Aut}(X)^0$. Consider the action of the characteristic quasitorus $H = \mathrm{Spec}\,\mathbb{C}[\mathrm{Cl}(Z)]$ on $\bar{Z} = \mathbb{C}^r$ given by the $\mathrm{Cl}(Z)$-grading of $\mathbb{C}[T_1, \ldots, T_r]$. The group $\mathrm{Aut}_H(\bar{Z})$ of H-equivariant automorphisms is concretely given as

$$G \;=\; \mathrm{GL}(4) \times \mathrm{GL}(2) \times \mathbb{K}^*.$$

According to [65, Thm. 4.4], we obtain $\mathrm{Aut}(X)^0$ as a factor group of the unit component of the subgroup $\mathrm{Aut}_H(\bar{X})$ of $\mathrm{Aut}_H(\bar{Z})$ stabilizing $\bar{X} \subseteq \bar{Z}$. We take a closer look at the action of an element $\gamma = \mathrm{diag}(A_1, A_2, \alpha_3)$ of $\mathrm{Aut}_H(\bar{X})$ on $\hat{X} \subseteq \bar{X}$. Given general $y \in \mathbb{C}^4$ and $x \in \mathbb{C}^2$, we find $z \in \mathbb{C}$ such that $[y, x, z]$ is a point of $\hat{X}$. In particular, $\gamma \cdot [y, x, z]$ belongs to the fiber of π_1 over $[y]$. The latter implies $A_1 \cdot y = \eta y$ with $\eta \in \mathbb{K}^*$ and for the matrix $A_2 = (a_{ij})$ it gives

$$\begin{aligned} 0 &= q_1(y)(a_{11}x_1 + a_{12}x_2) + q_2(y)(a_{21}x_1 + a_{22}x_2) + \alpha_3 f(y)z \\ &= q_1(y)((a_{11} - \alpha_3)x_1 + a_{12}x_2) + q_2(y)(a_{21}x_1 + (a_{22} - \alpha_3)x_2). \end{aligned}$$

Recall that this holds for any general choice of y and x. As a consequence, we arrive at $a_{11} - \alpha_3 = 0 = a_{12}$, because otherwise $q_1 q_2^{-1} \in \mathbb{C}(T_1, T_2)$ holds in $\mathbb{C}(X)$ which is impossible due to the general choice of q_1 and q_2. By the same argument, we see $a_{22} - \alpha_3 = 0 = a_{21}$. Thus, γ acts trivially on each fiber of π_1 and we conclude that $\mathrm{Aut}(X)$ is of dimension zero.

Proposition 2.12.1 suggests that the infinitesimal deformations of X can be obtained by varying the coefficients of the defining equation in the Cox ring. As a possible approach to turn this impression into a precise statement, we mention the comparison theorem of Christophersen and Kleppe [40, Thm. 6.2] which relates in particular deformations of a variety to deformations of its Cox ring.

CHAPTER

THREE

SMOOTH CALABI-YAU THREEFOLDS OF SMALL PICARD NUMBER

We classify the smooth Calabi-Yau threefolds of Picard number one and two that have a general hypersurface Cox ring. The main result of this chapter is Theorem 3.1.3. On our way to the classification we use and complement the techniques developed in the previous chapter. Parts of this chapter are published in [96].

3.1 Results

A *Calabi-Yau variety* is an irreducible normal projective variety X such that the canonical class $\mathcal{K}_X$ of X is trivial, X has at most canonical singularities and $H^i(X, \mathcal{O}_X)$ vanishes for all $i = 1, \ldots, \dim(X) - 1$.

Let us recall the notions on hypersurfaces Cox ring that are necessary to state our results; cf. Section 2.4. We say that a normal irreducible projective variety X with finitely generated divisor class group $\mathrm{Cl}(X) = K$ has a *hypersurface Cox ring* if its Cox ring $\mathcal{R}(X)$ admits a K-graded presentation

$$\mathcal{R}(X) = R_g = \mathbb{K}[T_1, \ldots, T_r]/\langle g \rangle$$

with a homogeneous polynomial g of degree $\mu \in K$ such that $T_1, \ldots, T_r$ form a minimal system of K-prime generators for R_g. Note that the number of generator degrees is

$$r = \mathrm{rank}(K) + \dim(X) + 1.$$

We say that R_g resp. g is *spread* if each monomial of degree μ is a convex combination over those monomials showing up in g with non-zero coefficient. Besides, we call R_g *general (smooth, Calabi-Yau)* if g admits an open neighbourhood U in the finite dimensional vector space of all μ-homogeneous polynomials such that every $h \in U$ yields a hypersurface Cox ring R_h of a normal (smooth, Calabi-Yau) variety X_h with divisor class group K.

Any projective variety X with Cox ring R_g is encoded by R_g and an ample class $u \in K$ in the sense that X occurs as the GIT quotient of the set of u-semistable points of

$\operatorname{Spec} R_g$ by the quasitorus $\operatorname{Spec} \mathbb{K}[K]$. In this setting, we write $w_i = \deg(T_i)$ and refer to the Cox ring generator degrees $w_1, \ldots, w_r \in K$, the relation degree $\mu \in K$ and an ample class $u \in K$ as *specifying data* of the variety X. The class u can be omitted whenever $\operatorname{rank}(K) = 1$ holds since two varieties with the same Cox ring are isomorphic in this case.

Proposition 3.1.1. *The following table lists specifying data, $w_1, \ldots, w_5$ and $\mu = \deg(g)$ in $\operatorname{Cl}(X)$ for all smooth Calabi-Yau threefolds X of Picard number one that have a spread hypersurface Cox ring.*

No.	$\operatorname{Cl}(X)$	$[w_1, \ldots, w_5]$	μ
1	$\mathbb{Z}$	$\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}$	5
2	$\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$	$\begin{bmatrix}1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4}\end{bmatrix}$	$(5, \bar{0})$
3	$\mathbb{Z}$	$\begin{bmatrix}1 & 1 & 1 & 1 & 2\end{bmatrix}$	6
4	$\mathbb{Z}$	$\begin{bmatrix}1 & 1 & 1 & 1 & 4\end{bmatrix}$	8
5	$\mathbb{Z}$	$\begin{bmatrix}1 & 1 & 1 & 2 & 5\end{bmatrix}$	10

Moreover, each of the items 1 to 5 even defines a general smooth Calabi-Yau hypersurface Cox ring and thus provides the specifying data for a whole family of smooth Calabi-Yau threefolds. Any two smooth Calabi-Yau threefolds of Picard number one with specifying data from distinct items of the table are not isomorphic to each other.

Remark 3.1.2. Number 1, 3, 4 and 5 from Proposition 3.1.1 are covered by Oguiso's classification of smooth Calabi-Yau threefolds showing up as general complete intersection in some weighted projective space [105, Thm 4.1]. Moreover Number 2 from Proposition 3.1.1 does not appear in [105, Thm 4.1] since its Picard group is not cyclic.

The main result of this chapter is concerned with Calabi-Yau threefolds of Picard number two over the field of complex numbers.

Theorem 3.1.3. *The following table lists specifying data, $w_1, \ldots, w_6$, μ and u in $\operatorname{Cl}(X)$ for all complex smooth Calabi-Yau threefolds X of Picard number two that have a spread hypersurface Cox ring.*

No.	$\operatorname{Cl}(X)$	$[w_1, \ldots, w_6]$	μ	u
1	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&0&0&0 \\ 0&0&0&1&1&1\end{bmatrix}$	$\begin{bmatrix}3 \\ 3\end{bmatrix}$	$\begin{bmatrix}1 \\ 1\end{bmatrix}$
2	$\mathbb{Z}^2 \times \mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix}1&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ \bar{0}&\bar{1}&\bar{2}&\bar{0}&\bar{1}&\bar{2}\end{bmatrix}$	$\begin{bmatrix}3 \\ 3 \\ \bar{0}\end{bmatrix}$	$\begin{bmatrix}1 \\ 1 \\ \bar{0}\end{bmatrix}$
3	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&1&0&0 \\ 0&0&1&1&1&1\end{bmatrix}$	$\begin{bmatrix}4 \\ 4\end{bmatrix}$	$\begin{bmatrix}2 \\ 1\end{bmatrix}$
4	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&3&0&0 \\ 0&0&1&3&1&1\end{bmatrix}$	$\begin{bmatrix}6 \\ 6\end{bmatrix}$	$\begin{bmatrix}2 \\ 1\end{bmatrix}$

No.	Cl(X)	$[w_1,\ldots,w_6]$	μ	u
5	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&0&0&-1\\0&0&0&1&1&1\end{bmatrix}$	$\begin{bmatrix}2\\3\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$
6	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&0&0&-2\\0&0&0&1&1&1\end{bmatrix}$	$\begin{bmatrix}1\\3\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$
7	$\mathbb{Z}^2 \times \mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix}1&1&1&0&0&-3\\0&0&0&1&1&1\\\bar{0}&\bar{1}&\bar{2}&\bar{1}&\bar{2}&\bar{0}\end{bmatrix}$	$\begin{bmatrix}0\\3\\\bar{0}\end{bmatrix}$	$\begin{bmatrix}1\\1\\\bar{0}\end{bmatrix}$
8	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&0&0&-3\\0&0&0&2&3&1\end{bmatrix}$	$\begin{bmatrix}0\\6\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$
9	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&1&0&0\\-2&0&0&0&1&1\end{bmatrix}$	$\begin{bmatrix}4\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$
10	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&3&0&0\\-2&0&0&0&1&1\end{bmatrix}$	$\begin{bmatrix}6\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$
11	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&1&0&0\\0&0&0&1&1&1\end{bmatrix}$	$\begin{bmatrix}4\\3\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$
12	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&1&0&0\\0&0&0&1&1&1\end{bmatrix}$	$\begin{bmatrix}4\\3\end{bmatrix}$	$\begin{bmatrix}1\\2\end{bmatrix}$
13	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&3&0&0\\0&0&0&2&1&1\end{bmatrix}$	$\begin{bmatrix}6\\4\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$
14	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&3&0&0\\-1&0&0&1&1&1\end{bmatrix}$	$\begin{bmatrix}6\\2\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$
15	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&1&0&0\\-1&0&0&1&1&1\end{bmatrix}$	$\begin{bmatrix}4\\2\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$
16	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&1&0&0\\-1&0&0&1&1&1\end{bmatrix}$	$\begin{bmatrix}4\\2\end{bmatrix}$	$\begin{bmatrix}1\\2\end{bmatrix}$
17	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&1&0&0\\-2&0&0&1&1&1\end{bmatrix}$	$\begin{bmatrix}4\\1\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$

No.	Cl(X)	$[w_1,\ldots,w_6]$	μ	u
18	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&1&0&0\\-2&0&0&1&1&1\end{bmatrix}$	$\begin{bmatrix}4\\1\end{bmatrix}$	$\begin{bmatrix}1\\2\end{bmatrix}$
19	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&2&1&0\\0&0&0&1&1&1\end{bmatrix}$	$\begin{bmatrix}6\\3\end{bmatrix}$	$\begin{bmatrix}3\\1\end{bmatrix}$
20	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&4&1&0\\0&0&0&2&1&1\end{bmatrix}$	$\begin{bmatrix}8\\4\end{bmatrix}$	$\begin{bmatrix}3\\1\end{bmatrix}$
21	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&5&2&0\\0&0&0&2&1&1\end{bmatrix}$	$\begin{bmatrix}10\\4\end{bmatrix}$	$\begin{bmatrix}3\\1\end{bmatrix}$
22	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&2&5&1&0\\0&0&0&2&1&1\end{bmatrix}$	$\begin{bmatrix}10\\4\end{bmatrix}$	$\begin{bmatrix}3\\1\end{bmatrix}$
23	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&2&7&3&0\\0&0&0&2&1&1\end{bmatrix}$	$\begin{bmatrix}14\\4\end{bmatrix}$	$\begin{bmatrix}4\\1\end{bmatrix}$
24	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&2&1&0\\-2&0&0&0&1&1\end{bmatrix}$	$\begin{bmatrix}6\\0\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$
25	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&1&3&0\\-2&0&0&0&1&1\end{bmatrix}$	$\begin{bmatrix}7\\0\end{bmatrix}$	$\begin{bmatrix}4\\1\end{bmatrix}$
26	$\mathbb{Z}^2$	$\begin{bmatrix}2&1&1&1&3&0\\-2&0&0&0&1&1\end{bmatrix}$	$\begin{bmatrix}8\\0\end{bmatrix}$	$\begin{bmatrix}4\\1\end{bmatrix}$
27	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&2&5&1&0\\0&0&1&3&1&1\end{bmatrix}$	$\begin{bmatrix}10\\6\end{bmatrix}$	$\begin{bmatrix}3\\1\end{bmatrix}$
28	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&2&5&1&0\\0&0&1&3&1&1\end{bmatrix}$	$\begin{bmatrix}10\\6\end{bmatrix}$	$\begin{bmatrix}3\\2\end{bmatrix}$
29	$\mathbb{Z}^2$	$\begin{bmatrix}1&1&1&4&1&0\\-1&0&0&1&1&1\end{bmatrix}$	$\begin{bmatrix}8\\2\end{bmatrix}$	$\begin{bmatrix}5\\1\end{bmatrix}$
30	$\mathbb{Z}^2$	$\begin{bmatrix}1&2&1&1&1&0\\-1&-1&0&0&1&1\end{bmatrix}$	$\begin{bmatrix}6\\0\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$

Moreover, each of the items 1 to 30 even defines a general smooth Calabi-Yau hypersurface Cox ring and thus provides the specifying data for a whole family of smooth Calabi-Yau threefolds. Any two smooth Calabi-Yau threefolds of Picard number two with specifying data from distinct items of the table are not isomorphic to each other.

Remark 3.1.4. Each spread hypersurface Cox ring R with specifying data as in Number 3, 4, and 30 from Theorem 3.1.3 provides actually two full-dimensional GIT cones λ_1, λ_2 with $\lambda_i^\circ \subseteq \mathrm{Mov}(R)^\circ$. In any of these constellations the resulting varieties $X(\lambda_1)$ and $X(\lambda_2)$ are isomorphic.

Hypersurfaces in toric Fano varieties form a rich source of examples for Calabi-Yau varieties, e.g. [1,12,13]. Theorem 3.1.3 comprises several varieties of this type.

Remark 3.1.5. Any Mori dream space X can be embedded into a projective toric variety by choosing a graded presentation of its Cox ring $\mathcal{R}(X)$; see [4, Sec. 3.2.5] for details. The following table shows for which varieties X from Theorem 3.1.3 the presentation $\mathcal{R}(X) = R_g$ gives rise to an embedding into a (possibly singular) toric Fano variety. Observe that in our situation this simply means $\mu \in \mathrm{Ample}(X)$.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
✓	✓	✗	✗	✓	✓	✗	✗	✗	✗	✓	✗	✓	✓	✓

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
✗	✓	✗	✓	✓	✓	✓	✓	✗	✗	✗	✗	✓	✓	✗

3.2 Mori dream spaces with hypersurface Cox rings

In this section we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. By a *Mori dream space* we mean an irreducible normal projective variety X with a finitely generated divisor class group $\mathrm{Cl}(X)$ and a finitely generated Cox ring $\mathcal{R}(X)$. We basically use the same tools as shown in Sections 2.2 to 2.4 of this thesis. For convenience, we gather the basic facts on the combinatorial description of Mori dream spaces with a hypersurface Cox ring and how to construct families of them with prescribed properties in this section.

Recall that an *abstract Cox ring* is an integral normal affine $\mathbb{K}$-algebra with a grading by a finitely generated abelian group K such that R has only constant homogeneous units and the grading is almost free, pointed, factorial and the moving cone $\mathrm{Mov}(R)$ is of full dimension in $K_{\mathbb{Q}}$. Abstract Cox rings are the basic ingredient for the combinatorial description of Mori dream spaces since all of them arise from the following construction.

Construction 3.2.1. Let R be an abstract Cox ring and consider the action of the quasitorus $H = \mathrm{Spec}\,\mathbb{K}[K]$ on the affine variety $\bar{X} = \mathrm{Spec}\,R$. For every GIT-cone $\lambda \in \Lambda(R)$ with $\lambda^\circ \subseteq \mathrm{Mov}(R)^\circ$, we set

$$X(\lambda) \ := \ \bar{X}^{ss}(\lambda) /\!\!/ H.$$

Then X is normal, projective and of dimension $\dim(R) - \dim(K_{\mathbb{Q}})$. The divisor class group and the Cox ring of X are given as

$$\mathrm{Cl}(X) \ = \ K, \qquad\qquad \mathcal{R}(X) \ = \ \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)) \ = \ \bigoplus_{K} R_w \ = \ R.$$

Moreover, the cones of effective, movable, semiample and ample divisor classes of X are given in $\mathrm{Cl}_{\mathbb{Q}}(X) = K_{\mathbb{Q}}$ as

$$\mathrm{Eff}(X) \ = \ \mathrm{Eff}(R), \qquad \mathrm{Mov}(X) \ = \ \mathrm{Mov}(R),$$

$$\mathrm{SAmple}(X) = \lambda, \qquad \mathrm{Ample}(X) = \lambda^\circ.$$

Choosing homogeneous generators for an abstract Cox ring gives rise to a closed embedding into a projective toric variety.

Construction 3.2.2. In the situation of Construction 3.2.1, consider a K-graded presentation

$$R = \mathbb{K}[T_1, \ldots, T_r]/\mathfrak{a}$$

where $T_1, \ldots, T_r$ define pairwise non-associated K-primes in R and $\mathfrak{a} \subseteq S = \mathbb{K}[T_1, \ldots, T_r]$ is a homogeneous ideal. The GIT-fan $\Lambda(S)$ w.r.t. the diagonal H-action on $\mathbb{K}^r = \operatorname{Spec} S$ refines the GIT-fan $\Lambda(R)$. Let $\tau \in \Lambda(S)$ with $\lambda^\circ \subseteq \tau^\circ$. Running Construction 3.2.1 for S and τ yields a projective toric variety Z fitting in the following diagram

$$\begin{array}{ccc} \bar{X}^{\mathrm{ss}}(\lambda) & \longrightarrow & (\mathbb{K}^r)^{\mathrm{ss}}(\tau) \\ {\scriptstyle /\!\!/ H} \downarrow & & \downarrow {\scriptstyle /\!\!/ H} \\ X & \xrightarrow{\imath} & Z \end{array}$$

The embedding $\imath : X \to Z$ is neat, i.e., it is a closed embedding, the torus invariant prime divisors on Z restrict to pairwise different prime divisors on X and the induced pullback of divisor class groups $\imath^* : \operatorname{Cl}(Z) \to \operatorname{Cl}(X)$ is an isomorphism.

Now we specialize to the case where R is a hypersurface ring and explain how geometrical properties of X interact with the combinatorial data behind.

Construction 3.2.3. In the situation of Construction 3.2.1, assume that R admits a K-graded presentation

$$R \ = \ \mathbb{K}[T_1, \ldots, T_r]/\langle g \rangle$$

such that the variables $T_1, \ldots, T_r$ define pairwise non-associated K-primes in R. Consider the positive orthant $\gamma = \mathbb{Q}^r_{\geq 0}$ and the degree homomorphism

$$Q \colon \mathbb{Z}^r \ \to \ K, \qquad e_i \ \mapsto \ w_i := \deg(T_i).$$

An $\bar{X}$*-face* is a face $\gamma_0 \preceq \gamma$ admitting a point $x \in \bar{X}$ such that one has

$$x_i \neq 0 \quad \Longleftrightarrow \quad e_i \in \gamma_0$$

for the coordinates $x_1, \ldots, x_r$ of x and the canonical basis vectors $e_1, \ldots, e_r \in \mathbb{Z}^r$. Moreover, an X*-face* is an $\bar{X}$-face $\gamma_0 \preceq \gamma$ with $\lambda^\circ \subseteq Q(\gamma_0)^\circ$. Let $\operatorname{rlv}(X)$ be the set of all X-faces and $\pi \colon \bar{X}^{ss}(\lambda) \to X$ the quotient map. Then we have a decomposition

$$X \ = \ \bigcup_{\gamma_0 \in \operatorname{rlv}(X)} X(\gamma_0)$$

into pairwise disjoint locally closed sets $X(\gamma_0) := \pi(\bar{X}(\gamma_0))$.

Remark 3.2.4. We consider the situation of Construction 3.2.3. Any subset $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, r\}$ defines a face γ_I of the orthant by

$$\gamma_I := \gamma_{i_1,\ldots,i_k} := \operatorname{cone}(e_{i_1}, \ldots, e_{i_k}) \preceq \gamma.$$

Moreover, the polynomial $g_I \in \mathbb{K}[T_1, \ldots, T_r]$ associated with I is defined as

$$g_I := g(\tilde{T}_1, \ldots, \tilde{T}_r), \qquad \tilde{T}_i := \begin{cases} T_i, & i \in I, \\ 0, & i \notin I. \end{cases}$$

Then γ_I is an $\bar{X}$-face if and only if g_I is no monomial.

Proposition 3.2.5. *Consider the situation of Construction 3.2.3.*

(i) *The variety X is $\mathbb{Q}$-factorial if and only if* $\dim(\lambda) = \dim(K_{\mathbb{Q}})$ *holds for* $\lambda = \mathrm{SAmple}(X)$.

(ii) *The variety X is locally factorial if and only if for every X-face $\gamma_0 \preceq \gamma$, the group K is generated by $Q(\gamma_0 \cap \mathbb{Z}^r)$.*

(iii) *X is smooth if and only if $\bar{X}^{\mathrm{ss}}$ is smooth and $X \subseteq Z^{\mathrm{reg}}$ holds.*

Furthermore, for hypersurface Cox rings, we have an explicit description of the anticanonical class.

Proposition 3.2.6. *In the situation of Construction 3.2.3, the anticanonical class of X is given in $K = \mathrm{Cl}(X)$ as*

$$-\mathcal{K}_X \;=\; \deg(T_1) + \cdots + \deg(T_r) - \deg(g).$$

We call an irreducible normal variety X *weakly Calabi-Yau* if its canonical class $\mathcal{K}_X$ vanishes. For varieties with hypersurface Cox ring this notion only depends on the generator degrees and the relation degree. Moreover, it turns out that smooth weakly Calabi-Yau hypersurfaces are Calabi-Yau varieties in the strong sense.

Remark 3.2.7. Consider the situation of Construction 3.2.3.

(i) From Proposition 3.2.6 we deduce that X is weakly Calabi-Yau if and only if $\mu = w_1 + \cdots + w_r$ holds. In particular, μ lies in the relative interior of $\mathrm{Eff}(R)$ whenever X is weakly Calabi-Yau.

(ii) If X is weakly Calabi-Yau, then Proposition 3.2.6 shows that X is an anticanonical hypersurface of a projective toric variety Z as in Construction 3.2.2. If, in addition, X is smooth, then Proposition 3.2.5 allows us to apply [2, Prop. 6.1]. From this we infer $h^i(X, \mathcal{O}_X) = 0$ for all $0 < i < \dim(X)$, hence X is Calabi-Yau.

In what follows we describe in outline the toolbox for producing general hypersurface Cox rings with given data established in Section 2.4; proofs and more details can be found at the same place.

Construction 3.2.8. Consider a linear, pointed, almost free K-grading on the polynomial ring $S := \mathbb{K}[T_1, \ldots, T_r]$ and the quasitorus action $H \times \bar{Z} \to \bar{Z}$, where

$$H \ := \ \operatorname{Spec} \mathbb{K}[K], \qquad \bar{Z} \ := \ \operatorname{Spec} S \ = \ \mathbb{K}^r.$$

We write $Q\colon \mathbb{Z}^r \to K$, $e_i \mapsto w_i := \deg(T_i)$ for the degree map. Assume that $\operatorname{Mov}(S) \subseteq K_{\mathbb{Q}}$ is of full dimension and fix $\tau \in \Lambda(S)$ with $\tau^\circ \subseteq \operatorname{Mov}(S)^\circ$. Set

$$\hat{Z} \ := \ \bar{Z}^{ss}(\tau), \qquad Z \ := \ \hat{Z} /\!\!/ H.$$

Then Z is a projective toric variety with divisor class group $\operatorname{Cl}(Z) = K$ and Cox ring $\mathcal{R}(Z) = S$. Moreover, fix $0 \neq \mu \in K$, and for $g \in S_\mu$ set

$$R_g := S/\langle g \rangle, \qquad \bar{X}_g := V(g) \subseteq \bar{Z}, \qquad \hat{X}_g := \bar{X}_g \cap \hat{Z}, \qquad X_g := \hat{X}_g /\!\!/ H \subseteq Z.$$

Then the factor algebra R_g inherits a K-grading from S and the quotient $X_g \subseteq Z$ is a closed subvariety. Moreover, we have

$$X_g \ \subseteq \ Z_g \ \subseteq \ Z$$

where $Z_g \subseteq Z$ is the minimal ambient toric variety of X_g, that means the (unique) minimal open toric subvariety containing X_g.

Remark 3.2.9. In the situation of Construction 3.2.8 assume that R_g is normal, factorially graded and $T_1, \ldots, T_r$ define pairwise non-associated K-primes in R_g. Then R_g is an abstract Cox ring and we find a GIT-cone $\lambda \in \Lambda(R_g)$ with $\tau^\circ \subseteq \lambda^\circ$ and $\hat{X}_g = \bar{X}^{\mathrm{ss}}(\lambda)$. This brings us into the situation of Constructions 3.2.1 and 3.2.2, so we have

$$\operatorname{Cl}(X_g) \ = \ K, \qquad \mathcal{R}(X_g) \ = \ R_g, \qquad \tau^\circ \subseteq \operatorname{Ample}(X_g).$$

Moreover, for any $g \in U_\mu$ the variables $T_1, \ldots, T_r$ form a minimal system of generators for all R_g if and only if we have $\mu \neq w_i$ for $i = 1, \ldots, r$.

Constructing a general hypersurface Cox ring with prescribed specifying data essentially means to to find a suitable open subset $U \subseteq S_\mu$ such that R_g, where $g \in U$, satisfies the conditions from the above remark. In the subsequent text we present several criteria to check these conditions.

Proposition 3.2.10. *Consider the setting of Construction 3.2.8. For $1 \le i \le r$ denote by $U_i \subseteq S_\mu$ the set of all $g \in S_\mu$ such that g is prime in S and T_i is prime in R_g. Then $U_i \subseteq S_\mu$ is open. Moreover, U_i is non-empty if and only if there is a μ-homogeneous prime polynomial not depending on T_i.*

Proposition 3.2.11. *In the situation of Construction 3.2.8, suppose that K is of rank one, $r \ge 5$ holds and that for any $i = 1, \ldots, r$ there is an $l_i \in \mathbb{Z}_{\ge 1}$ with $\mu = l_i w_i$. Then there is a non-empty open subset of polynomials $g \in S_\mu$ such that the ring R_g is normal and K-factorial, and $T_1, \ldots, T_r \in R_g$ are prime. In particular, there is a general hypersurface Cox ring with specifying data $w_1, \ldots, w_r$ and μ.*

By a *Dolgachev polytope* we mean a convex polytope $\Delta \subseteq \mathbb{Q}^r_{\geq 0}$ of dimension at least four such that each coordinate hyperplane of $\mathbb{Q}^r$ intersects Δ non-trivially and the dual cone of $\operatorname{cone}(\Delta_0 - u;\ u \in \Delta_0)$ is regular for each one-dimensional face $\Delta_0 \preceq \Delta$.

Proposition 3.2.12. *In the situation of Construction 3.2.8, there is a non-empty open subset of polynomials $g \in S_\mu$ such that the ring R_g is factorial provided that one of the following conditions is fulfilled:*

(i) *K is of rank at most $r-4$ and torsion free, there is a $g \in S_\mu$ such that $T_1, \dots, T_r$ define primes in R_g, we have $\mu \in \tau^\circ$ and μ is base point free on Z.*

(ii) *The set $\operatorname{conv}(\nu \in \mathbb{Z}^r_{\geq 0};\ Q(\nu) = \mu)$ is a Dolgachev polytope.*

(iii) *$r \geq 5$, $K = \mathbb{Z}^2$, there is some $g \in S_\mu$ such that $T_1, \dots, T_r$ define primes in R_g, and the degree matrix is of the form*

$$Q = \begin{bmatrix} x_1 & \dots & x_{r-1} & 0 \\ -d_1 & \dots & -d_{r-1} & 1 \end{bmatrix}, \qquad x_i \in \mathbb{Z}_{\geq 1},\ d_i \in \mathbb{Z}_{\geq 0},$$

such that the first coordinate μ_1 of $\mu \in \mathbb{Z}^2$ is a multiple of each of $x_1, \dots, x_{r-1}$ and the second coordinate μ_2 of μ satisfies

$$\mu_2 = -\min_\nu \nu_1 d_1 + \dots + d_{r-1}\nu_{r-1}$$

where the minimum runs over all $\nu \in \mathbb{Z}^{r-1}_{\geq 0}$ with $\nu_1 x_1 + \dots + \nu_{r-1}x_{r-1} = \mu_1$.

We give another easy to check factoriality criterion for homogeneous polynomials with degree arising from a lattice polytope in the following sense.

Remark 3.2.13. Let Σ be a complete lattice fan in $\mathbb{Z}^n$ and $v_1, \dots, v_r$ the primitive lattice vectors generating the rays of Σ. Consider the mutually dual exact sequences

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^r \xrightarrow[e_i \mapsto v_i]{P} \mathbb{Z}^n$$

$$0 \longleftarrow K \xleftarrow{\ Q\ } \mathbb{Z}^r \xleftarrow{\ P^*\ } \mathbb{Z}^n \longleftarrow 0$$

The Σ-*degree* of a lattice polytope $B \subseteq \mathbb{Q}^n$ is $Q(a(\Sigma)) \in K$ where

$$a(\Sigma) := (a_1, \dots, a_r) \in \mathbb{Z}^r, \qquad a_i := -\min_{u \in B} \langle u, v_i \rangle.$$

Proposition 3.2.14. *Let $B \subseteq \mathbb{Q}^n$ be an integral n-simplex, Σ a fan in $\mathbb{Z}^n$ refining the normal fan of B, and $\mu \in K$ the Σ-degree of B. Assume that there is a μ-homogeneous prime polynomial and a non-empty open subset $U \subseteq S_\mu$ such that for all $g \in U$ the variables $T_1, \dots, T_r$ define K-primes in the K-graded algebra*

$$R_g = \mathbb{K}[T_1, \dots, T_r]/\langle g \rangle.$$

Then there is a non-empty open subset of polynomials $g \in S_\mu$ such that R_g is K-factorial.

The spread μ-homogeneous polynomials form an open subset $U_\mu \subseteq S_\mu$. Moreover all polynomials $g \in U_\mu$ share the same minimal ambient toric variety Z_g. We call $Z_\mu := Z_g$, where $g \in U_\mu$, the *μ-minimal ambient toric variety.* The following propositions enable us to verify smoothness of Z_μ and the general X_g in a purely combinatorial manner.

Proposition 3.2.15. *In the situation of Construction 3.2.8 the following statements are equivalent.*

(i) *The μ-minimal ambient toric variety Z_μ is smooth.*

(ii) *For each $\gamma_0 \preceq \gamma$ with $\tau^\circ \in Q(\gamma_0)^\circ$ and $|Q^{-1}(\mu) \cap \gamma_0| \neq 1$ the group K is generated by $Q(\gamma_0 \cap \mathbb{Z}^r)$.*

Proposition 3.2.16. *In the setting of Construction 3.2.8, assume* $\operatorname{rank}(K) = 2$ *and that $Z_\mu \subseteq Z$ is smooth. If $\mu \in \tau$ holds, then μ is base point free. Moreover, then there is a non-empty open subset of polynomials $g \in S_\mu$ such that X_g is smooth.*

A further ingredient needed in the proofs of Proposition 3.1.1 and Theorem 3.1.3 are invariants in connection with hypersurface Cox rings that distinguish varieties with different specifying data. Let us highlight generator and relation degrees of a graded algebra; for a detailed discussion of this topic we refer to [65, Sec. 2].

Remark 3.2.17. Let $R = \bigoplus_{w \in K} R_w$ be an integral pointed K-graded algebra. We denote $S(R) = \{w \in K;\ R_w \neq 0\}$. An important invariant of R is the *set of generator degrees generator degrees*

$$\Omega_R \coloneqq \{w \in S(R);\ R_w \nsubseteq R_{<w}\} \subseteq K$$

where $R_{<w}$ denotes the subalgebra of R spanned by all homogeneous components $R_{w'}$ such that $w = w' + w_0$ holds for some $0 \neq w_0 \in S(R)$. In the situation of Setting 3.5.1 the set of generator degrees is given as

$$\Omega_R = \{w_1, \ldots, w_r\} \subseteq K.$$

The set of generator degrees is unique and does not depend on a graded presentation of R. From this emerges another invariant: Choose pairwise different $u_1, \ldots, u_m \in K$ such that $\Omega_R = \{u_1, \ldots, u_m\}$ and set $d_i \coloneqq \dim_{\mathbb{K}} R_{u_i}$. By suitably reordering $u_1, \ldots, u_m$ we achieve $d_1 \leq \ldots \leq d_m$. We call $(d_1, \ldots, d_m)$ the *generator degree dimension tuple* of R. If two graded algebras are isomorphic, then they share the same generator degree dimension tuples.

Moreover, if R admits an irredundant graded presentation $R = \mathbb{K}[T_1, \ldots, T_r]/\langle g \rangle$, then the *relation degree* $\mu = \deg(g) \in K$ is unique and does not depend on the choice of the minimal graded presentation.

3.3 Proof of Proposition 3.1.1

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero. The proof of Proposition 3.1.1 can be seen as a lightweight version of the proof of Theorem 3.1.3. They are similar in their structure yet the first one does not involve detailed elaboration of combinatorial configurations. Indeed, the combinatorial input restricts to the following remark.

Remark 3.3.1. The following tables describes the solutions of the inequation

$$x_1 \cdots x_n \leq x_1 + \cdots + x_n, \quad x_1, \ldots, x_n \in \mathbb{Z}_{\geq 1}$$

for $n = 3, 4, 5$ where $x_1, \ldots, x_n$ are in ascending order. Here, $*$ stands for an arbitrary positive integer.

n	x_1	x_2	x_3	x_4
3	1	1	$*$	—
	1	2	2	—
	1	2	3	—
4	1	1	1	$*$
	1	1	2	3
	1	1	2	4

n	x_1	x_2	x_3	x_4	x_5
5	1	1	1	1	$*$
	1	1	1	2	2
	1	1	1	2	3
	1	1	1	2	4
	1	1	1	2	5
	1	1	1	3	3

Proof of Proposition 3.1.1. Let X be a smooth Calabi-Yau threefold of Picard number one with a spread hypersurface Cox ring. Fix a graded presentation

$$\mathcal{R}(X) = R_g = \mathbb{K}[T_1, \ldots, T_5]/\langle g \rangle$$

and observe that we are in the situation of Construction 3.2.3. Our major task is to verify that $Q = [w_1, \ldots, w_5]$, where $w_i = \deg(T_i)$, is as in one of the items 1 to 5 from Proposition 3.1.1.

We claim that for $i = 1, \ldots, 5$ a power $T_i^{l_i}$ shows up amongst the monomials of g . Suppose that g has no monomial of the form $T_i^{l_i}$ for some $1 \leq i \leq 5$. Then $\gamma_i \preceq \gamma$ is an X-face by Remark 3.2.4. As X is locally factorial, Proposition 3.2.5 (ii) says that w_i is a generator for K, in particular $\mu = l_i w_i$ holds for some $l_i \in \mathbb{Z}_{\geq 1}$. The respective monomial $T_i^{l_i}$ is of degree μ and admits no presentation as convex combination over other monomials of the same degree. Since g is spread, $T_i^{l_i}$ has a non-zero coefficient in g. A contradiction.

Now Remark 3.2.4 shows that that any twodimensional face $\gamma_{i,j} \preceq \gamma$ is an X-face. Thus any two w_i, w_j form a generating set for K; see Proposition 3.2.5 (ii). Being an abelian group of rank one generated by two elements K is of the form $\mathbb{Z} \times \mathbb{Z}/t\mathbb{Z}$ for some $t \in \mathbb{Z}_{\geq 1}$. According to this presentation of K we denote

$$w_i = (a_i, \zeta_i), \ \mu = (\alpha, \theta), \quad a_i, \alpha \in \mathbb{Z}, \ \zeta_i, \theta \in \mathbb{Z}/t\mathbb{Z}.$$

By applying a suitable automorphism of K and reordering $T_1, \ldots, T_5$ we achieve

$$1 \leq a_1 \leq \cdots \leq a_5.$$

Our next task is to figure out all possible configurations of $(a_1, \ldots, a_5)$. Observe that $a_1, \ldots, a_5$ are pairwise coprime. As the existence of monomials $T_i^{l_i}$ of degree μ ensures that α is divisible by each of $a_1, \ldots, a_5$, we obtain $a_1 \cdots a_5 \mid \alpha$. From Proposition 3.2.6 and X being Calabi-Yau we infer $\mu = w_1 + \cdots + w_5$. This leads to

$$a_1 \cdots a_5 \mid a_1 + \cdots + a_5. \tag{3.1}$$

Remark 3.3.1 yields $a_1 = a_2 = a_3 = 1$ and we are left with one of the following configurations

$$a_4 = 1, \qquad a_4 = 2 \text{ and } a_5 = 3, 5.$$

Inserting $a_4 = 1$ into Eq. (3.1) amounts to $a_5 \mid 4$, hence $a_5 = 1, 2, 4$. Besides, $a_4 = 2$ and $a_5 = 3$ does not satisfy Eq. (3.1). To sum it up, $(a_1, \ldots, a_5)$ must be one of the following

$$(1,1,1,1,1), \quad (1,1,1,1,2), \quad (1,1,1,1,4), \quad (1,1,1,2,5).$$

When K is torsion-free, these configurations lead to Numbers 1, 3, 4, and 5 from Proposition 3.1.1.

The next step is to study the torsion subgroup of K. We produce upper bounds on the order t of the torsion subgroup of K for each of the above configurations of $(a_1, \ldots, a_5)$. Since any of these configurations satisfies $a_1 = 1$ we achieve $\zeta_1 = 0$ by applying a suitable automorphism of K. Recall that $T_i^{l_i}$ shows up as a monomial of g for all $i = 1, \ldots, 5$. From this we infer $\theta = l_1\zeta_1 = 0$. Moreover w_1, w_j form a generating set for K for any $j > 1$. Thus each ζ_j is a generator for $\mathbb{Z}/t\mathbb{Z}$. Then again $l_j\zeta_j = \theta = 0$ forces $t \mid l_j$ for $j = 2, \ldots, 5$. Using the presentation $l_i = \alpha/a_i$ we obtain that t is a divisor of

$$d := \gcd\left(\frac{\alpha}{a_2}, \ldots, \frac{\alpha}{a_5}\right).$$

The following table lists the data in question explicitly.

$(a_1, \ldots, a_5)$	α	d
$(1,1,1,1,1)$	5	5
$(1,1,1,1,2)$	6	3
$(1,1,1,1,4)$	8	2
$(1,1,1,2,5)$	10	1

One directly sees that K admits no torsion in the last case. We deal with the three remaining cases. Since we are interested in the case where K is not torsion-free we assume $t > 1$. From d being a prime multiple of t we deduce $t = d$. Consider indices $i \neq j$ with $a_i = a_j = 1$. From w_i, w_j spanning K as a group we infer $\zeta_i \neq \zeta_j$. Thus in the first case, $a_5 = 1$, the elements $\zeta_1, \ldots, \zeta_5$ are pairwise different. After suitably reordering $T_1, \ldots, T_5$, we end up with specifying data as in Number 2 from Proposition 3.1.1. For the two remaining cases, $a_5 = 2, 4$, we obtain that $\zeta_1, \ldots, \zeta_4$ are pairwise different. Note that the cyclic factor $\mathbb{Z}/t\mathbb{Z}$ of K has order at most three in these cases; a contradiction. Hence these configurations of $(a_1, \ldots, a_5)$ do not admit torsion in K.

The next to last step is to make sure that each item from Proposition 3.1.1 stems from a general smooth Calabi-Yau hypersurface Cox ring. Fix specifying data (Q, μ) as listed in Proposition 3.1.1 and run Construction 3.2.8. Proposition 3.2.11 guarantees that for the given data there exists indeed a general hypersurface Cox ring $\mathcal{R}(X) = R_g$. Observe that X is quasismooth since a power of each variable T_i shows up in g with non-zero coefficient, hence $\overline{X}$ is smooth apart from the origin. Furthermore, one directly checks that X is locally factorial using Proposition 3.2.5 (ii). Altogether X is smooth according to Proposition 3.2.5 (iii). Moreover, Remark 3.2.7 (i) says that X is weakly Calabi-Yau, thus, being smooth, Calabi-Yau by Remark 3.2.7 (ii).

Finally we have to verify that each two smooth Calabi-Yau threefolds with specifying data from different items from Proposition 3.1.1 are non-isomorphic. Varieties from Number 2 have a unique divisor class group among the varieties from Proposition 3.1.1. Thus it suffices to consider families 1, 3, 4, and 5. Dealing with pointed $\mathbb{Z}$-gradings, the assumption $a_1, \ldots, a_5 \geq 1$ makes the set of Cox ring generator degrees $\{w_1, \ldots, w_5\} = \{a_1, \ldots, a_5\}$ unique. We conclude the discussion by observing that Numbers 1, 3, 4 and 5 have pairwise different set of generator degrees. □

3.4 A Flop Lemma

The aim of this section is to give a direct proof that small birational modifications of Mori dream spaces with Picard number two that have a trivial canonical class are connected by flops; see Proposition 3.4.2.

Let us briefly recall the notion of flops [90, 91] as well as some surrounding terminology. A proper birational morphism $\varphi : X \to Y$ of normal varieties is called *extremal*, if X is $\mathbb{Q}$-factorial and for each two Cartier divisors D_1, D_2 on X there are $a_1, a_2 \in \mathbb{Z}$ where at least one of a_1, a_2 is non-zero and $a_1 D_1 - a_2 D_2$ is linearly equivalent to the pullback $\varphi^* C$ of some Cartier divisor C on Y. This is essentially a condition on the Picard numbers of X and Y.

Lemma 3.4.1. *Let $\varphi : X \to Y$ be a dominant morphism of normal varieties with finitely generated Picard groups. Then the following statements are equivalent.*

(i) *For each two Cartier divisors D_1, D_2 on X there are $a_1, a_2 \in \mathbb{Z}$ where at least one of a_1, a_2 is non-zero and $a_1 D_1 - a_2 D_2$ is linearly equivalent to the pullback $\varphi^* C$ of some Cartier divisor C on Y.*

(ii) *We have $\rho(X) - \rho(Y) \leq 1$.*

Proof. Being a dominant morphism, φ induces an injective pull-back homomorphism of Picard groups $\varphi^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$. In particular $\varphi^*\mathrm{Pic}(Y)$ is of rank $\rho(Y) = \mathrm{rank}\,\mathrm{Pic}(Y)$. Consider the factor group $G := \mathrm{Pic}(X)/\varphi^*\mathrm{Pic}(Y)$. We have

$$\dim G_{\mathbb{Q}} = \mathrm{rank}\,\mathrm{Pic}(X) - \mathrm{rank}\,\varphi^*\mathrm{Pic}(Y) = \rho(X) - \rho(Y).$$

Now observe that (i) means that each two elements from G lay on a common ray in the rational vector space $G_{\mathbb{Q}}$ i.e. $\dim G_{\mathbb{Q}} \leq 1$. □

A Weil divisor D on a variety X is said to be *relatively ample* w.r.t a morphism $\varphi : X \to Y$ of varieties, or just φ-*ample*, if there is an open affine covering $Y = \bigcup V_i$ such that D restricts to an ample divisor on each $\varphi^{-1}(V_i)$. A birational map $\psi : X^- \dashrightarrow X^+$ of $\mathbb{Q}$-factorial weakly Calabi-Yau varieties is a *flop* if it fits into a commutative diagram

$$\begin{array}{ccc} X^- & \overset{\psi}{\dashrightarrow} & X^+ \\ & {\scriptstyle \varphi^-}\searrow \quad \swarrow{\scriptstyle \varphi^+} & \\ & Y & \end{array}$$

where $\varphi^- : X^- \to Y$ and $\varphi^+ : X^- \to Y$ are small proper birational morphisms, φ^- is extremal and there is a Weil divisor D on X^- such that $-D$ is φ^--ample and the proper transform of D on X^+ is φ^+-ample.

Proposition 3.4.2. *Let R be an abstract Cox ring with grading group K of rank two and $\lambda, \eta \in \Lambda(R)$ full-dimensional cones with $\lambda^\circ, \eta^\circ \subseteq \mathrm{Mov}(R)^\circ$. Consider the varieties $X(\lambda)$ and $X(\eta)$ arising from Construction 3.2.1. If the canonical class of $X(\lambda)$ is trivial, then there is a sequence of flops*

$$X(\lambda) \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_k \dashrightarrow X(\eta).$$

We study the toric setting first. Consider $S = \mathbb{K}[T_1, \ldots, T_r]$ with a linear, pointed, almost free grading of an abelian group K of rank two and the associated action of the quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$ on $\mathbb{K}^r$. Let us recall some facts about toric varieties arising from GIT-cones as treated e.g. in [4, Chap. 2–3]. The degree homomorphism $Q : \mathbb{Z}^r \to K$, $e_i \mapsto w_i := \deg(T_i)$ gives rise to a pair of mutually dual exact sequences:

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^r \xrightarrow{\ P\ } \mathbb{Z}^n$$

$$0 \longleftarrow K \xleftarrow{\ Q\ } \mathbb{Z}^r \xleftarrow{\ P^*\ } \mathbb{Z}^n \longleftarrow 0$$

Given a GIT-cone $\tau \in \Lambda(S)$ with $\tau^\circ \subseteq \mathrm{Mov}(S)^\circ$, the associated toric variety $Z = (\mathbb{K}^r)^{\mathrm{ss}}(\tau) /\!\!/ H$ has the describing fan $\Sigma(\tau)$ given by

$$\Sigma(\tau) = \{P(\gamma_0^*);\ \gamma_0 \in \mathrm{rlv}(\tau)\}, \qquad \mathrm{rlv}(\tau) = \{\gamma_0 \preceq \gamma;\ \tau^\circ \subseteq Q(\gamma_0)^\circ\}$$

In particular all such fans share the same one-skeleton consisting of the pairwise different rays generated by $v_1, \ldots, v_r$ where $v_i := P(e_i) \in \mathbb{Z}^n$. Moreover, we denote Z_{γ_0} for the affine toric variety associated with the lattice cone $P(\gamma_0^*) \subseteq \mathbb{Q}^n$. The covering of Z by affine toric charts then formulates as

$$Z = \bigcup_{\gamma_0 \in \mathrm{rlv}(\tau)} Z_{\gamma_0}.$$

Lemma 3.4.3. *Let $\tau_1, \tau_2 \in \Lambda(S)$ with $\tau_i^\circ \subseteq \mathrm{Mov}(S)^\circ$. Then for any $\gamma_1 \in \mathrm{rlv}(\tau_1)$, $\gamma_2 \in \mathrm{rlv}(\tau_2)$ we have*

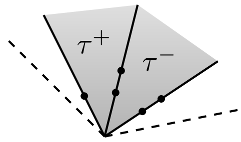
$$P(\gamma_2^*) \subseteq P(\gamma_1^*) \Longleftrightarrow \gamma_1 \subseteq \gamma_2.$$

Proof. The implication "$\Leftarrow$" is clear. We show "$\Rightarrow$". Note that the cones $P(\gamma_1^*) \in \Sigma(\tau_1)$ and $P(\gamma_2^*) \in \Sigma(\tau_2)$ both live in lattice fans having precisely $v_1, \ldots, v_r$ as primitive ray generators. Thus for $j = 1, 2$ and any v_i we have

$$v_i \in P(\gamma_j^*) \iff \mathbb{Q}_{\geq 0}\, v_i \text{ is an extremal ray of } P(\gamma_j^*) \iff e_i \in \gamma_j^*.$$

From this we infer that $P(\gamma_2^*) \subseteq P(\gamma_1^*)$ implies $\gamma_2^* \subseteq \gamma_1^*$. This in turn means $\gamma_1 \subseteq \gamma_2$. $\square$

Let $\tau^-, \tau^+ \subseteq \mathbb{Q}^2 = K_\mathbb{Q}$ be full-dimensional GIT-cones with $(\tau^-)^\circ, (\tau^+)^\circ \subseteq \operatorname{Mov}(S)^\circ$ intersecting in a common ray $\tau^0 := \tau^- \cap \tau^+$.



Consider the projective toric varieties Z^0, Z^-, Z^+ associated with τ^0, τ^- and τ^+ and denote $\Sigma^0 = \Sigma(\tau^0)$, $\Sigma^- = \Sigma(\tau^-)$ and $\Sigma^+ = \Sigma(\tau^+)$ for the describing fans. Moreover the inclusions of the respective semistable points induce proper birational toric morphims $\varphi^- \colon Z^- \to Z^0$, $\varphi^+ \colon Z^+ \to Z^0$ described by the refinements of fans $\Sigma^- \preceq \Sigma^0$ and $\Sigma^+ \preceq \Sigma^0$ respectively. This yields a small birational map $\psi \colon Z^- \dashrightarrow Z^+$ as shown in the diagram

$$\begin{array}{ccccc}
(\mathbb{K}^r)^{\mathrm{ss}}(\tau^-) & \subseteq & (\mathbb{K}^r)^{\mathrm{ss}}(\tau^0) & \supseteq & (\mathbb{K}^r)^{\mathrm{ss}}(\tau^+) \\
{\scriptstyle /\!\!/ H}\downarrow & & {\scriptstyle /\!\!/ H}\downarrow & & {\scriptstyle /\!\!/ H}\downarrow \\
Z^- & \xrightarrow{\varphi^-} & Z^0 & \xleftarrow{\varphi^+} & Z^+ \\
& & \underset{\psi}{\dashrightarrow} & &
\end{array}$$

Lemma 3.4.4. *Let $-D$ be an ample divisor on Z^-, then D regarded as a divisor on Z^+ is φ^+-ample.*

Proof. By suitably applying an automorphism of K and relabeling $w_1, \ldots, w_r \in K$ we achieve counter-clockwise ordering i.e.

$$i \leq j \Longrightarrow \det(w_i, w_j) \geq 0$$

and $\det(w^-, w^+) \geq 0$ for all $w^- \in \tau^-$, $w^+ \in \tau^+$. Moreover, we name the indices of the weights that approximate τ^0 from the outside

$$i^- := \max(i; w_i \in \tau^-), \qquad i^+ := \min(i; w_i \in \tau^+).$$

The geometric constellation of $w_1, \ldots, w_r$ in $\mathbb{Q}^2$ directly yields that the set of minimal cones of $\operatorname{rlv}(\tau^0)$ is

$$\{\gamma_i;\ i^- < i < i^+\} \cup \{\gamma_{i,j};\ i \leq i^-,\ j \geq i^+\},$$

where $\gamma_{i_1,\ldots,i_r} = \operatorname{cone}(e_{i_1}, \ldots, e_{i_r}) \preceq \gamma$. The corresponding cones $P(\gamma_0)^*$ are precisely the maximal cones of Σ^0, in particular the associated toric charts Z_{γ_0} form an open affine covering of Z^0. We show that D is ample on each open subset $(\varphi^+)^{-1}(Z_{\gamma_0})$ of Z^+.

First, note that φ^+ is an isomorphism over the affine toric charts of Z^0 associated with the common minimal cones of $\mathrm{rlv}(\tau^0)$ and $\mathrm{rlv}(\tau^+)$, namely all $Z_{\gamma_{i,j}}$ where $i \le i^+$ and $j \ge i^+$. In particular each preimage $(\varphi^+)^{-1}(Z_{\gamma_{i,j}})$ is affine. Since Z^+ is $\mathbb{Q}$-factorial by Proposition 3.2.5 (i), the divisor D is $\mathbb{Q}$-Cartier thus restricts to an ample divisor on any open affine subvariety of Z^+.

It remains to consider the charts of Z^0 defined by the faces of the form γ_j. Let us fix some $i^- < j < i^+$. The minimal cones $\gamma_0 \in \mathrm{rlv}(\tau^+)$ with $\gamma_j \subseteq \gamma_0$ are precisely those of the form $\gamma_{j,i}$ where $i \ge j$. As the toric morphism φ^+ is described by the refinement $\Sigma^+ \preceq \Sigma^0$, Lemma 3.4.3 yields

$$U := (\varphi^+)^{-1}\left(Z_{\gamma_j}\right) = \bigcup_{i \ge i^+} Z_{\gamma_{j,i}} \subseteq Z^+.$$

Note that $U \subseteq Z^+$ is an open toric subset and the maximal cones of the associated subfan Σ' of Σ^+ are precisely the cones $P(\gamma_{j,i}^*)$ where $i \ge i^+$. This shows that the rays of Σ' are the rays of Σ^+ minus ϱ_j. Thus the divisor class group of U is given by $\mathrm{Cl}(U) = K/\langle w_j \rangle$ and the projection corresponds to the restriction of divisor classes

$$\begin{array}{ccc} \mathrm{Cl}(Z^+) & \xrightarrow{\imath^*} & \mathrm{Cl}(U) \\ \cong \downarrow & & \downarrow \cong \\ K & \longrightarrow & K/\langle w_j \rangle \end{array}$$

Taking $\operatorname{rank} K = 2$ into account, we may choose suitable coordinates leading to an isomorphism $\mathrm{Cl}(U)_{\mathbb{Q}} \cong \mathbb{Q}$ such that for any $w \in \mathrm{Cl}(Z^+)$ the restriction $\imath^*(w)$ to $\mathrm{Cl}(U)$ and $\det(w_j, w)$ have the same sign. Graphically this means that the sign of $\imath^*(w) \in \mathrm{Cl}(U)$ is positive if w lies above the ray τ^0 and negative if w lies below τ^0.

Since we know the maximal cones of Σ' we may compute the ample cone of U as

$$\mathrm{Ample}(U) = \bigcap_{i \ge i^+} (\imath^* \circ Q(\gamma_{j,i}))^\circ = \mathbb{Q}_{>0} \subseteq \mathbb{Q} = \mathrm{Cl}(U)_{\mathbb{Q}}.$$

Note that $[-D] \in \mathrm{Ample}(Z^-) = \tau^-$ lies below τ, thus the class of $-D$ (regarded on Z^+) restricted to U is negative, hence $\imath^*[D] \in \mathrm{Ample}(U)$. In other words, D is ample on U. Altogether, we conclude that D is φ^+-ample. □

Proof of Proposition 3.4.2. First, we deal with the case that λ and η intersect in a common ray $\varrho := \lambda \cap \eta$. Consider a K-graded presentation

$$R = \mathbb{K}[T_1, \ldots, T_r]/\mathfrak{a}$$

where $T_1, \ldots, T_r$ define pairwise non-associated K-primes in R and $\mathfrak{a} \subseteq S = \mathbb{K}[T_1, \ldots, T_r]$ is a homogeneous ideal. The GIT-fan $\Lambda(S)$ w.r.t. the H-action on S refines the GIT-fan $\Lambda(R)$. We may choose $\tau^+, \tau^- \in \Lambda(S)$ such that

$$\left(\tau^-\right)^\circ \subseteq \lambda^\circ \quad \left(\tau^+\right)^\circ \subseteq \eta^\circ, \quad \tau^- \cap \tau^+ = \varrho.$$

The toric morphisms φ_Z^-, φ_Z^+ arising from the face relations $\varrho \preceq \tau^-, \tau^+$ of GIT-cones are compatible with the toric morphisms φ^-, φ^+ arising from $\varrho \preceq \lambda, \eta$ as shown in the following diagram where the vertical arrows are neat embeddings as in Construction 3.2.2

$$\begin{array}{ccccc} Z(\tau^-) & \xrightarrow{\varphi_Z^-} & Z(\varrho) & \xleftarrow{\varphi_Z^+} & Z(\tau^+) \\ \uparrow & & \uparrow & & \uparrow \\ X(\lambda) & \xrightarrow{\varphi^-} & X(\varrho) & \xleftarrow{\varphi^+} & X(\eta) \end{array}$$

We claim that the resulting birational map $\psi : X(\lambda) \dashrightarrow X(\eta)$ is a flop. First observe that X^- is $\mathbb{Q}$-factorial by Proposition 3.2.5 (i) and φ^-, φ^+ are small birational morphisms; see [4, Rem. 3.3.3.4]. Lemma 3.4.1 ensures that φ^- is extremal.

Let D_Z be a torus invariant divisor on $Z(\tau^-)$ such that $-D_Z$ is ample for $Z(\tau^-)$. Since $X(\lambda) \subseteq Z(\tau^-)$ is neatly embedded, we may restrict D_Z to a divisor D_X on $X(\lambda)$. Note that $-D_X$ is ample since $-D_Z$ is so. In particular $-D_X$ is φ^--ample. Lemma 3.4.4 yields that D_Z is φ_Z^+-ample. Let $U \subseteq Z(\varrho)$ be an affine open subset such that D_Z is ample on

$$V := \left(\varphi_Z^+\right)^{-1}(U) \subseteq Z(\tau^+).$$

The further restriction of D_Z from V to $V \cap X(\eta)$ is still ample. In other words, D_X restricted to $(\varphi^+)^{-1}(X(\tau) \cap U)$ is ample. We conclude that D_X is φ^+-ample.

Altogether $\psi : X(\lambda) \dashrightarrow X(\eta)$ is a flop.

In the general case we find full-dimensional GIT-cones $\lambda = \eta_1, \ldots, \eta_k = \eta$ where $\eta_i^\circ \subseteq \mathrm{Mov}(R)^\circ$ holds for all i and each intersection $\eta_i \cap \eta_{i+1}$ is a ray of $\Lambda(R)$. According to the preceding discussion, we may successively construct the desired sequence of flops. □

3.5 Combinatorial constraints on smooth hypersurface Cox rings

The proof of Theorem 3.1.3 basically uses the combinatorial framework for the classification of smooth Mori dream spaces of Picard number two with hypersurface Cox ring established in Section 2.5. Let us recall the notation from there and slightly extend it to address the torsion subgroup of the grading group explicitly. We also present the accompanying toolkit. Moreover we add some new tools for dealing with torsion.

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

Setting 3.5.1. Consider $K = \mathbb{Z}^2 \times \Gamma$ where Γ is some finite abelian group of order t, a K-graded algebra R and $X = X(\lambda)$, where $\lambda \in \Lambda(R)$ with $\lambda^\circ \subseteq \mathrm{Mov}(R)^\circ$, as in Construction 3.2.1. Assume that we have an irredundant K-graded presentation

$$R \;=\; R_g \;=\; \mathbb{K}[T_1, \ldots, T_r]/\langle g \rangle$$

such that the T_i define pairwise nonassociated K-primes in R. Write $w_i := \deg(T_i)$, $\mu := \deg(g)$ for the degrees in K. According to the presentation $K = \mathbb{Z}^2 \times \Gamma$ we denote

$$w_i = (u_i, \zeta_i), \quad \mu = (\alpha, \theta), \qquad u_i, \alpha \in \mathbb{Z}^2, \quad \zeta_i, \theta \in \Gamma.$$

Similarly the degree matrix $Q = [w_1, \ldots, w_r]$ is divided into a free part Q^0 and a torsion part Q^{tor}, i.e., we set

$$Q^0 = \begin{bmatrix} u_1 & \ldots & u_r \end{bmatrix}, \quad Q^{\mathrm{tor}} = \begin{bmatrix} \zeta_1 & \ldots & \zeta_r \end{bmatrix}.$$

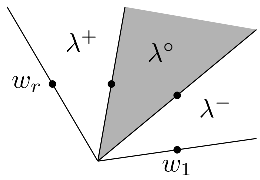
Regarded as elements of $K_{\mathbb{Q}}$ we identify w_i with u_i and μ with α. Suitably numbering $w_1, \ldots, w_r$, we ensure counter-clockwise ordering, that means that we always have

$$i \le j \implies \det(w_i, w_j) = \det(u_i, u_j) \ge 0.$$

Note that each ray of $\Lambda(R)$ is of the form $\varrho_i = \mathrm{cone}(w_i)$, but not vice versa. We assume X to be $\mathbb{Q}$-factorial. According to Proposition 3.2.5 this means $\dim(\lambda) = 2$. Then the effective cone of X is uniquely decomposed into three convex sets,

$$\mathrm{Eff}(X) \;=\; \lambda^- \cup \lambda^\circ \cup \lambda^+,$$

where λ^- and λ^+ are convex polyhedral cones not intersecting $\lambda^\circ = \mathrm{Ample}(X)$ and $\lambda^- \cap \lambda^+$ consists of the origin.



Remark 3.5.2. Setting 3.5.1 is respected by orientation preserving automorphisms of K. If we apply an orientation reversing automorphism of K, then we regain Setting 3.5.1 by reversing the numeration of $w_1, \ldots, w_r$. Moreover, we may interchange the numeration of T_i and T_j if w_i and w_j share a common ray without affecting Setting 3.5.1. We call these operations *admissible coordinate changes*. Note that any automorphism of $\mathbb{Z}^2$ naturally extends to an automorphism of $K = \mathbb{Z}^2 \times \Gamma$ acting as the identity on Γ.

We state an adapted version of Proposition 2.2.4 locating the relation degree.

Proposition 3.5.3. *In the situation of Setting 3.5.1 we have* $\mu \in \mathrm{cone}(w_3, w_{r-2}) \subseteq K_{\mathbb{Q}}$.

A further important observation is that the GIT-fan structure of R_g can be read of from the geometric constellation of $w_1, \ldots, w_r$ and μ.

Proposition 3.5.4. *Situation as in Setting 3.5.1. Assume that $X(\lambda)$ is locally factorial and R is a spread hypersurface Cox ring. Then the full-dimensional cones of $\Lambda(R)$ are precisely the cones $\eta = \mathrm{cone}(w_i, w_j)$ where $\varrho_i \neq \varrho_j$ and one of the following conditions is satisfied:*

(i) *$\mu \in \varrho_i$ holds, ϱ_i contains at least two generator degrees and η° contains no generator degree,*

(ii) *$\mu \in \varrho_j$ holds, ϱ_j contains at least two generator degrees and η° contains no generator degree,*

(iii) *$\mu \in \eta^\circ$ holds and there is at most one $w_k \in \eta^\circ$, which must lay on the ray through μ,*
(iv) *$\mu \notin \eta$ holds and η° contains no generator degrees.*

The following lemmas are a crucial in gaining constraints on the specifying data.

Lemma 3.5.5. *Situation as in Setting 3.5.1. Let i, j with $\lambda \subseteq \operatorname{cone}(w_i, w_j)$. If $X = X(\lambda)$ is locally factorial, then either w_i, w_j generate K as a group, or g has precisely one monomial of the form $T_i^{l_i} T_j^{l_j}$, where $l_i + l_j > 0$.*

Lemma 3.5.6. *Let $X = X(\lambda)$ be as in Setting 3.5.1 and let $1 \leq i < j < k \leq r$. If X is locally factorial, then w_i, w_j, w_k generate K as a group provided that one of the following holds:*
(i) *$w_i, w_j \in \lambda^-$, $w_k \in \lambda^+$ and g has no monomial of the form $T_k^{l_k}$,*
(ii) *$w_i \in \lambda^-$, $w_j, w_k \in \lambda^+$ and g has no monomial of the form $T_i^{l_i}$,*
(iii) *$w_i \in \lambda^-$, $w_j \in \lambda^\circ$, $w_k \in \lambda^+$.*

Moreover, if (iii) holds, then g has a monomial of the form $T_j^{l_j}$ where l_j is divisible by the order of the factor group $K/\langle w_i, w_k \rangle$. In particular l_j is a multiple of $\det(u_i, u_k)$.

Lemma 3.5.7. *Assume u, w_1, w_2 generate the abelian group $\mathbb{Z}^2$. If $w_i = a_i w$ holds with a primitive $w \in \mathbb{Z}^2$ and $a_i \in \mathbb{Z}$, then (u, w) is a basis for $\mathbb{Z}^2$ and u is primitive.*

Now we present some structural observations which prove useful at different places inside the proof of Theorem 3.1.3 when we deal with specific configurations of generator and relation degrees.

Lemma 3.5.8. *Let $u_1, \ldots, u_4 \in \mathbb{Z}^2$ such that $\det(u_1, u_3)$, $\det(u_1, u_4)$, $\det(u_2, u_3)$ and $\det(u_2, u_4)$ all equal one. Then $u_1 = u_2$ or $u_3 = u_4$ holds.*

Lemma 3.5.9. *In Setting 3.5.1, assume that $X = X(\lambda)$ is locally factorial and R_g a spread hypersurface Cox ring. If w_i lies on the ray through μ, then g has a monomial of the form $T_i^{l_i}$ where $l_i \geq 2$.*

Lemma 3.5.10. *In Setting 3.5.1 assume that $\operatorname{Mov}(R) = \operatorname{Eff}(R)$ and $\mu \in \operatorname{Eff}(R)^\circ$ hold. Let Ω denote the set of two-dimensional cones $\eta \in \Lambda(R)$ with $\eta^\circ \subseteq \operatorname{Mov}(R)^\circ$.*
(i) *If $X(\eta)$ is locally factorial for some $\eta \in \Omega$, then $\operatorname{Eff}(R)$ is a regular cone and every u_i on the boundary of $\operatorname{Eff}(R)$ is primitive.*
(ii) *If $X(\eta)$ is locally factorial for all $\eta \in \Omega$, then, for any $w_i \in \operatorname{Eff}(R)^\circ$, we have $u_i = u_1 + u_r$ or g has a monomial of the form $T_i^{l_i}$.*

Lemma 3.5.11. *Situation as in Setting 3.5.1. Assume that R_g is a spread hypersurface Cox ring. If $\mu \in \operatorname{Eff}(R)^\circ$ holds and every two-dimensional $\eta \in \Lambda(R)$ with $\eta^\circ \subseteq \operatorname{Mov}(R)^\circ$ defines a locally factorial $X(\eta)$, then there is at most one ray ϱ_i which is not contained in the boundary of $\operatorname{Eff}(R)$ and contains more than one w_i.*

Lemma 3.5.12. *Situation as in Setting 3.5.1. If we have $w_2 = w_3$ and $\mu \in \varrho_2$, then $w_4 \in \varrho_2$ holds.*

We have to bear in mind that the divisor class group $K = \mathrm{Cl}(X)$ of a smooth Calabi-Yau threefold X is not necessarily torsion-free. The following lemmas show that in the case of a hypersurface Cox ring the order of the torsion subgroup is bounded in terms of monomials of the relation degree. A first important constraint is that the torsion subgroup of K is cyclic.

Lemma 3.5.13. *Situation as in Setting 3.5.1. If $X = X(\lambda)$ is locally factorial and $\mu \notin \lambda$, then $K \cong \mathbb{Z}^2$ holds.*

Proof. We have $\lambda = \mathrm{cone}(w_i, w_j)$ for some generator degrees w_i, w_j lying on the boundary of λ. Due to $\mu \notin \lambda$, there is no monomial $T_i^{l_i} T_j^{l_j}$ of degree μ. Lemma 3.5.5 yields that K is generated by w_i, w_j. Since $\mathrm{rank}(K) = 2$, this implies $K \cong \mathbb{Z}^2$. □

Lemma 3.5.14. *Situation as in Setting 3.5.1. If X is locally factorial, then $K \cong \mathbb{Z}^2 \times \mathbb{Z}/t\mathbb{Z}$ holds.*

Proof. Both λ^- and λ^+ contain at least two Cox ring generator degrees. This allows us to choose w_i, w_j, w_k such that Lemma 3.5.6 applies. This ensures that K is generated by three elements. By Setting 3.5.1 we have $\mathrm{rank}(K) = 2$, thus K is as claimed. □

Lemma 3.5.15. *Situation as in Setting 3.5.1. Let $1 \le i, j \le n$ with $\mathrm{cone}(w_i, w_j) \cap \lambda^\circ \neq \emptyset$. If $X = X(\lambda)$ is locally factorial and $\mu \in \lambda$ holds, then there is a monomial $T_i^{l_i} T_j^{l_j}$ of degree μ where $l_i + l_j > 0$.*

Proof. Since g is μ-homogeneous, we are done when g has a monomial of the form $T_i^{l_i} T_j^{l_j}$.

We assume that g has no monomial of the form $T_i^{l_i} T_j^{l_j}$. Then ϱ_i and ϱ_j both are GIT-rays, thus none of w_i, w_j lies in λ°. This forces $\lambda \subseteq \mathrm{cone}(w_i, w_j)$. Then Lemma 3.5.5 tells us that w_i, w_j generate K as a group. Using $\mu \in \lambda \subseteq \mathrm{cone}(w_i, w_j)$ we deduce that μ is an positive integral combination over w_i, w_j, i.e., there exists a monomial as desired. □

Lemma 3.5.16. *Situation as in Setting 3.5.1. Let $1 \le i, j, k \le r$ such that w_i, w_j, w_k generate K as a group, $\det(u_i, u_j) = 1$ and $\mathrm{cone}(w_i, w_j) \cap \lambda^\circ \neq \emptyset$. If X is locally factorial, then $t \mid l_k$ holds for any monomial $T_i^{l_i} T_k^{l_k}$ of degree μ.*

Proof. Using $\det(u_i, u_j) = 1$ enables us to apply a suitable admissible coordinate change such that $\zeta_i = \zeta_j = 0$. Moreover we may assume $\lambda \in \mu$; otherwise Lemma 3.5.13 yields $t = 1$ and there is nothing left to show. This allows us to use Lemma 3.5.15. From this we infer that $\mu = (\alpha, \theta)$ is an integral positive combination over w_i, w_j, thus $\theta = 0$. Since w_i, w_j, w_k generate K as a group, ζ_k is a generator for Γ. Using $\zeta_i = 0$ we obtain $l_k \zeta_k = \theta = 0$ whenever $T_i^{l_i} T_k^{l_k}$ is of degree μ. This implies $t \mid l_k$. □

Lemma 3.5.17. *Situation as in Setting 3.5.1. Assume that $X = X(\lambda)$ is locally factorial. If $\det(u_1, u_r) = 1$ and $\alpha = l_k u_k$ holds, then $t \mid l_k$.*

Proof. Lemma 3.5.6 yields that w_1, w_k, w_r generate K as a group. Besides $T_k^{l_k}$ is of degree μ by Lemma 3.5.9 (i). Now Lemma 3.5.16 tells us $t \mid l_k$. □

Lemma 3.5.18. *Let $w_i = (u_i, \zeta_i) \in \mathbb{Z}^2 \times \mathbb{Z}/t\mathbb{Z}$ for $1 \le i \le 3$. If $u_1 = u_2$ holds and w_1, w_2, w_3 span $\mathbb{Z}^2 \times \mathbb{Z}/t\mathbb{Z}$ as a group, then $\zeta_1 - \zeta_2$ is a generator for $\mathbb{Z}/t\mathbb{Z}$.*

Proof. Choose $a_i \in \mathbb{Z}$ such that $\zeta_i = \overline{a_i} \in \mathbb{Z}/t\mathbb{Z}$. Then $\mathbb{Z}^3$ is the linear hull of the columns of the following matrix, which we modify by subtracting the second from the first column

$$\begin{bmatrix} u_1 & u_2 & u_3 & 0 \\ a_1 & a_2 & a_3 & t \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & u_2 & u_3 & 0 \\ a_1 - a_2 & a_2 & a_3 & t \end{bmatrix}$$

Observe that u_2, u_3 form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$, thus a suitable unimodular row operation gives

$$\begin{bmatrix} 0 & u_2 & u_3 & 0 \\ a_1 - a_2 & a_2 & a_3 & t \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & u_2 & u_3 & 0 \\ a_1 - a_2 & 0 & 0 & t \end{bmatrix}$$

The columns of the right-hand side matrix still form a generating system for $\mathbb{Z}^3$. This implies that $\overline{a_1 - a_2} = \zeta_1 - \zeta_2$ generates the group $\mathbb{Z}/t\mathbb{Z}$. □

Lemma 3.5.19. *Situation as in Setting 3.5.1. If X is locally factorial,* $\det(u_1, u_r) = 1$, *and $u_i = u_j$ holds for some $1 < i < j < r$, then $\zeta_1 - \zeta_2$ is a generator for Γ. In particular K is torsion-free or $t \neq 2, 4$ holds.*

Proof. First note that w_i, w_j share a common ray in $K_{\mathbb{Q}}$, thus do not lie in the relative interior of the GIT-cone λ; see Proposition 3.5.4. So we have $w_i \in \lambda^-$ or $w_i \in \lambda^+$. By applying an orientation reversing coordinate change if necessary we achieve $w_i \in \lambda^-$.

We have $K = \mathbb{Z}^2 \times \mathbb{Z}/t\mathbb{Z}$; see Lemma 3.5.14. Using $\det(u_1, u_r) = 1$ enables us to apply a suitable admissible coordinate change such that $\zeta_1 = \zeta_r = \overline{0}$. Remark 3.2.7 ensures that g has no monomial of the form $T_r^{l_r}$. Hence Lemma 3.5.6 yields that both triples w_1, w_i, w_r and w_1, w_j, w_r generate K as a group. In particular ζ_i, ζ_j both are generators for $\mathbb{Z}/t\mathbb{Z}$. Moreover Lemma 3.5.6 tells us that w_i, w_j, w_r form a generating set for K. Lemma 3.5.18 yields that $\zeta_i - \zeta_j$ is a generator for $\mathbb{Z}/t\mathbb{Z}$. The proof is finished by the fact that the difference of two generators for $\mathbb{Z}/2\mathbb{Z}$ resp. $\mathbb{Z}/4\mathbb{Z}$ is never a generator for the respective group. □

3.6 Proof of Theorem 3.1.3: Collecting candidates

The first and major task in the proof of Theorem 3.1.3 is to show that we find specifying data for any given smooth Calabi-Yau threefold X with spread hypersurface Cox ring among the items displayed in Theorem 3.1.3. This is done by a case-by-case analysis of the geometric constellation of the Cox ring generator degrees.

Now the ground field is $\mathbb{K} = \mathbb{C}$. The sole reason for this is the reference involved in the proof of the following proposition.

Proposition 3.6.1. *Consider the situation of Setting 3.5.1. If $X(\lambda)$ is a smooth weakly Calabi-Yau threefold, then any variety $X(\eta)$ arising from a full-dimensional GIT-cone η satisfying $\eta^\circ \subseteq$* $\mathrm{Mov}(R)^\circ$ *is smooth.*

Proof. Proposition 3.4.2 provides us with a sequence of flops

$$X(\lambda) = X_1 \dashrightarrow \cdots \dashrightarrow X_k = X(\eta).$$

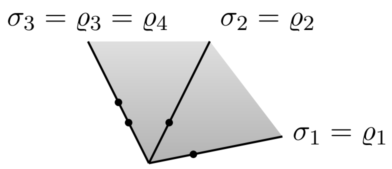
According to [91, Thm. 6.15], see also [90], flops of threefolds preserve smoothness. So we successively obtain smoothness for all varieties in the above sequence, especially for $X(\eta)$. □

Given a positive integer n, a sum of the form $n_1 + \cdots + n_k = n$ where $n_1, \ldots, n_k \in \mathbb{Z}_{\geq 1}$ is called an *integer partition* of n. If one wants to emphasize the order of the summands, one calls such a sum an *integer composition* of n. For instance, $1+1+2=4$ and $1+2+1=4$ are two different integer compositions of 4 but they are equal as integer partitions.

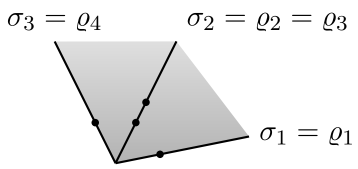
Remark 3.6.2. In Setting 3.5.1 the geometric constellation of $w_1, \ldots, w_r$ is described by an integer composition of r in the following sense: First, we take into account that some of the rays $\varrho_i = \operatorname{cone}(w_i)$ may coincide and label the actual rays properly. Let $1 \leq j_1 < \cdots < j_s \leq r$ such that $\varrho_{j_k} \neq \varrho_{j_l}$ holds for $j_k \neq j_l$ and each ϱ_i equals some ϱ_{j_k}. Set $\sigma_k \coloneqq \varrho_{j_k}$. We denote N_k for the number of Cox ring generator degrees w_i lying on σ_k. Then the distribution of the degrees w_i on the rays σ_k is encoded by the composition

$$N_1 + \cdots + N_s = r.$$

For example, when $r = 4$ holds, the integer compositions $1+1+2=4$ and $1+2+1=4$ correspond to the constellations of $w_1, \ldots, w_4$ illustrated below.



$1+1+2=4$



$1+2+1=4$

Proposition 3.6.3. *Situation as in Setting 3.5.1. If X is a weakly Calabi-Yau threefold, then $r = 6$ holds and the constellation of $w_1, \ldots, w_6$ corresponds to one of the following integer partitions $N_1 + \ldots + N_s = 6$ in the sense of Remark 3.6.2.*

	s	N_1	N_2	N_3	N_4	N_5	N_6
I	2	3	3	—	—	—	—
II	3	2	2	2	—	—	—
III	3	1	2	3	—	—	—
IV	4	1	1	2	2	—	—
V	4	1	1	1	3	—	—
VI	5	1	1	1	1	2	—
VII	6	1	1	1	1	1	1

Proof. Observe $r = \dim(X) + \dim(K_{\mathbb{Q}}) + 1 = 6$. The subsequent table shows all integer partitions $N_1 + \cdots + N_s = 6$.

	s	N_1	N_2	N_3	N_4	N_5	N_6
	1	6	—	—	—	—	—
	2	1	5	—	—	—	—
	2	2	4	—	—	—	—
I	2	3	3	—	—	—	—
II	3	2	2	2	—	—	—
III	3	1	2	3	—	—	—
	3	1	1	4	—	—	—
IV	4	1	1	2	2	—	—
V	4	1	1	1	3	—	—
VI	5	1	1	1	1	2	—
VII	6	1	1	1	1	1	1

Our task is to show that in the situation of Setting 3.5.1 those partitions without roman label do not admit a composition corresponding to the constellation of $w_1, \ldots, w_6$ in $K_{\mathbb{Q}}$.

Observe that in the cases $s = 1$ and $s = 2$ where $N_1 = 1$, $N_2 = 5$ the moving cone $\mathrm{Mov}(R)$ of R must be one-dimensional; a contradiction. From Proposition 3.5.3 we deduce that any constellation given by $N_1 + N_2 = 2 + 4 = 6$ forces μ to live in the boundary of $\mathrm{Eff}(R)$. This contradicts Remark 3.2.7. Furthermore, the partition $N_1 + N_2 + N_3 = 1 + 1 + 4$ comprises precisely two compositions, that is to say

$$N_1 + N_2 + N_3 = 1 + 4 + 1 \quad \text{and} \quad N_1 + N_2 + N_3 = 1 + 1 + 4.$$

The first of them implies that $\mathrm{Mov}(R)$ is one-dimensional; a contradiction. Considering the latter, Proposition 3.5.3 shows that μ lies on the boundary of $\mathrm{Eff}(R)$; a contradiction to Remark 3.2.7. □

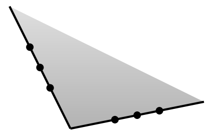
We work in Setting 3.5.1 for the proof of Theorem 3.1.3. According to Remark 3.2.7 (i) it suffices to determine the degree matrix $Q = [w_1, \ldots, w_6]$ in order to figure out candidates for specifying data of X since the relation degree μ is given by

$$\mu = w_1 + \cdots + w_6.$$

When Q and μ are fixed, we cover all possibilities (up to isomorphism) by picking an interior point u of each full-dimensional GIT-chamber λ with $\lambda^\circ \subseteq \mathrm{Mov}(R)^\circ$.

Our proof of Theorem 3.1.3 will be split into Parts I, …, VII discussing the constellations of $w_1, \ldots, w_6$ in the sense of Remark 3.6.2 given by the accordingly labeled integer partition of six from Proposition 3.6.3.

Part I • We consider $3 + 3 = 6$ i.e. the generator degrees w_i are evenly distributed on two rays σ_1, σ_2. So $w_1, \ldots, w_6$ lie all in the boundary of $\mathrm{Eff}(R)$.



Lemma 3.5.10 (i) tells us that each w_i is primitive and $\mathrm{Eff}(R)$ is regular. In particular $u_1 = u_2 = u_3$ and $u_4 = u_5 = u_6$. A suitable admissible coordinate change leads to

$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

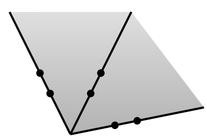
If K is torsion-free, this leads to specifying data as in Number 1 from Theorem 3.1.3.

We assume that K admits torsion. Remark 3.2.7 (i) implies $\alpha = u_1 + \cdots + u_6 = (3, 3)$. Lemma 3.5.15 guarantees that $T_2^3 T_4^3$ is of degree μ. Lemma 3.5.6 tells us that w_1, w_2, w_4 generate K as a group and we have $\det(u_1, u_4) = 1$. Thus we may apply Lemma 3.5.16. From this we infer $t \mid 3$, hence $t = 3$ i.e. $K = \mathbb{Z}^2 \times \mathbb{Z}/3\mathbb{Z}$; see also Lemma 3.5.14. Furthermore Lemma 3.5.6 yields that K is generated by each of the triples

$$(w_1, w_2, w_4), \quad (w_1, w_3, w_4), \quad (w_2, w_3, w_4).$$

Since $u_1 = u_2 = u_3$, we conclude that η_1, η_2, η_3 are pairwise different. Otherwise two of w_1, w_2, w_3 coincide, hence K is generated by two elements; a contradiction. In the same manner we obtain that η_4, η_5, η_6 are pairwise different. After suitably reordering $T_1, \ldots, T_6$ we arrive at specifying data as in Number 2 from Theorem 3.1.3.

Part II • We discuss the degree constellation determined by $2 + 2 + 2 = 6$. Here the generator degrees w_i are evenly distributed on three rays $\sigma_1, \sigma_2, \sigma_3$.



We have $\mu \in \sigma_2$ by Proposition 3.5.3. Proposition 3.5.4 provides us with two GIT-cones

$$\eta_1 = \mathrm{cone}(w_1, w_3), \qquad \eta_2 = \mathrm{cone}(w_3, w_5).$$

According to Proposition 3.6.1 the associated varieties $X(\eta_1)$, $X(\eta_2)$ both are smooth. Lemma 3.5.10 (i) yields $u_1 = u_2$, $u_5 = u_6$ and $\det(u_1, u_5) = 1$. After applying a suitable admissible coordinate change the degree matrix is of the form

$$Q^0 = \begin{bmatrix} 1 & 1 & a_3 & a_4 & 0 & 0 \\ 0 & 0 & b_3 & b_4 & 1 & 1 \end{bmatrix}, \qquad a_3, a_4 \in \mathbb{Z}_{\geq 1}.$$

We may assume $a_3 \leq a_4$. Let $v = (v_1, v_2) \in \mathbb{Z}^2$ be the primitive vector lying on σ_2. Applying Lemma 3.5.6 to $X(\eta_2)$ and the triple w_3, w_4, w_5 shows $\gcd(a_3, a_4) = 1$. In addition, we obtain $v_1 = 1$ from Lemma 2.5.5. Lemma 3.5.6 again, this time applied to $X(\eta_1)$ and w_1, w_2, w_3, gives $v_2 = 1$. From $v_1 = v_2$ we deduce $a_3 = b_3$ and $a_4 = b_4$. Lemma 3.5.9 ensures that μ_1 is divisible by both a_3 and a_4, thus $a_3 a_4 \mid \mu_1$. Remark 3.2.7 (i) says $\mu = w_1 + \ldots + w_6$. We conclude

$$a_3 a_4 \mid \mu_1 = a_3 + a_4 + 2.$$

First we deduce $a_4 \mid a_3 + 2$. Moreover we obtain $a_3 \leq 4$ due to $a_3 \leq a_4$. Altogether the integers a_3, a_4 are bounded, so we just have to examine the possible configurations.

- $a_3 = 1$: From $a_4 \mid a_3 + 2 = 3$ we infer $a_4 = 1, 3$. Now we show that K is torsion-free. For $a_4 = 1$ we have

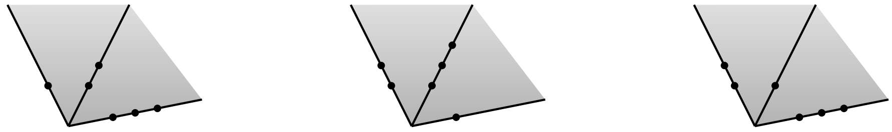
$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \alpha = (4, 4).$$

  Observe $\mu^0 = 4u_3$. Lemma 3.5.13 tells us $t \mid 4$, thus K is torsion-free according to Lemma 3.5.19. Similarly, for $a_4 = 3$ we have

$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 1 \end{bmatrix}, \qquad \alpha = (6, 6).$$

  Observe $\alpha = 2u_4$. Lemma 3.5.13 tells us $t \mid 2$, thus K is torsion-free according to Lemma 3.5.19. We arrive at specifying data as in Numbers 3 and 4 from Theorem 3.1.3. Observe $X(\eta_1) \cong X(\eta_2)$ in both cases due to the symmetry of the geometric constellation of $w_1, \ldots, w_6, \mu$. Thus it suffices to list an ample class for $X(\eta_1)$ only.
- $a_3 = 2$: From $a_4 \mid a_3 + 2 = 4$ and $a_3 \leq a_4$ we infer $a_4 = 2$, 4. This contradicts $\gcd(a_3, a_4) = 1$.
- $a_3 = 3$: From $a_4 \mid a_3 + 2 = 5$ and $a_3 \leq a_4$ we infer $a_4 = 5$. This leads to $\mu_1 = a_3 + a_4 + 2 = 10$. A contradiction to $a_3 \mid \mu_1$.
- $a_3 = 4$: From $a_4 \mid a_3 + 2 = 6$ and $a_3 \leq a_4$ we infer $a_4 = 6$. This contradicts $\gcd(a_3, a_4) = 1$.

Part III • In this part we consider the arrangements of $w_1, \ldots, w_6$ associated with the integer partition $1 + 2 + 3 = 6$. Here we have precisely three rays $\sigma_1, \sigma_2, \sigma_3$ each of which contains a different number of Cox ring generator degrees. A suitable admissible coordinate change turns the setting into one of the following:



III-i III-ii III-iii

Case III-i. Here we have $\lambda = \operatorname{cone}(w_1, w_4)$. Let $v \in \mathbb{Z}^2$ be a primitive vector on σ_2. Proposition 3.5.3 and Remark 3.2.7 (i) tell us $\mu \in \lambda^\circ \cup \sigma_2$. This allows us to apply Lemma 3.5.6 to w_i, w_4, w_5 for $i = 1, 2, 3$. From this we infer $\det(u_i, v) = 1$ for $i = 1, 2, 3$. In particular u_1, u_2, u_3 are primitive, thus $u_1 = u_2 = u_3$. Applying Lemma 3.5.6 to the triple w_1, w_2, w_6 shows $\det(u_1, u_6) = 1$. A suitable admissible coordinate change amounts to $v = (0, 1)$ and

$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -a_6 \\ 0 & 0 & 0 & b_4 & b_5 & 1 \end{bmatrix}, \qquad a_6, b_4, b_5 \in \mathbb{Z}_{\geq 1}$$

We may assume $b_4 \leq b_5$. To proceed we have to take the position of μ into account.

Assume $\mu \in \lambda^\circ$. Then we may apply Lemmas 2.5.5 and 3.5.6 to the two triples w_1, w_2, w_4 and w_1, w_2, w_5. We obtain that u_4 and u_5 both are primitive, hence

$$u_4 = u_5 = v = (0, 1).$$

From Remark 3.2.7 (i) we infer $\alpha = (3 - a_6, 3)$. Since μ lives in the relative interior of λ, which is the positive orthant, we end up with $a_6 = 1, 2$. We show that K is torsion free in both cases.

- $a_6 = 1$. The free parts of the specifying data are given as

$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \alpha = (2, 3).$$

  Lemma 3.5.15 ensures that $T_1^2 T_4^3$ is of degree μ. Moreover Lemma 3.5.6 shows that both triples w_1, w_4, w_5 and w_1, w_2, w_4 generate K as a group. Applying Lemma 3.5.16 to w_1, w_4, w_5 and $T_1^2 T_4^3$ yields $t \mid 3$. Again Lemma 3.5.16, this time applied to w_1, w_2, w_4 and $T_1^2 T_3^3$ shows $t \mid 2$. Altogether $t = 1$, thus K is torsion-free.
- $a_6 = 2$. The free parts of the specifying data are given as

$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \alpha = (1, 3).$$

  Lemma 3.5.15 ensures that $T_1^1 T_4^3$ of degree μ. Moreover Lemma 3.5.6 shows that w_1, w_2, w_4 generate K as a group. Applying Lemma 3.5.16 to w_1, w_2, w_4 and $T_1 T_4^3$ shows $t = 1$ i.e. K is torsion-free.

Eventually this leads to specifying data as in Numbers 5 and 6 from Theorem 3.1.3.

Assume $\mu \in \sigma_2$. Recall that $v = (0, 1)$ spans the ray σ_2. So here we have $\alpha_1 = 0$. From Remark 3.2.7 (i) we obtain $a_6 = 3$ and $\alpha_2 = b_4 + b_5 + 1$. Lemma 3.5.9 yields $b_4, b_5 \mid \alpha_2$. Applying Lemma 3.5.6 to w_1, w_4, w_5 shows $\gcd(b_4, b_5) = 1$. We conclude

$$b_4 b_5 \mid \alpha_2 = b_4 + b_5 + 1.$$

This implies $b_5 \mid b_4 + 1$. Moreover we deduce $b_4 \leq 3$. We discuss the resulting cases:

- $b_4 = 1$: From $b_5 \mid b_4 + 1 = 2$ we deduce $b_5 = 1, 2$. For $b_5 = 1$ we have
$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad \alpha = (0,3).$$
Suppose that K is torsion-free. Then $w_4 = w_5$ holds. Reversing the order of the variables by applying a suitable admissible coordinate change enables us to use Lemma 3.5.12. This forces two of the rays σ_i to coincide; a contradiction. So K has torsion. From $\alpha = 3u_4$ and Lemma 3.5.17 we obtain $t \mid 3$, hence $t = 3$. So we have $K = \mathbb{Z}^2 \times \mathbb{Z}/3\mathbb{Z}$. Using $\det(u_1, u_6) = 1$ enables us to apply a suitable admissible coordinate change such that $\zeta_1 = \zeta_6 = 0$. Now Lemma 3.5.6 shows that both triples w_1, w_2, w_6 and w_1, w_3, w_6 generate K as a group. From this we infer that ζ_2 and ζ_3 both are generators for $\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Lemma 3.5.6 yields that w_2, w_3, w_6 form a generating set for K as well. This forces $\zeta_2 \neq \zeta_3$. Otherwise $w_2 = w_3$ holds, thus K is spanned by two elements; a contradiction. Similarly, Lemma 3.5.6 applied to w_1, w_4, w_6 and w_1, w_5, w_6 yields that ζ_4 and ζ_5 both are generators for $\mathbb{Z}/3\mathbb{Z}$. Moreover applying Lemma 3.5.6 to w_1, w_4, w_5 ensures $\zeta_4 \neq \zeta_5$. After suitably reordering T_2, T_3 and T_4, T_5 we arrive at Number 7 from Theorem 3.1.3.
We turn to $b_5 = 2$. Here the free parts of degree matrix and relation degree are given by
$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix}, \quad \alpha = (0,4).$$
Note $\alpha = 2u_5$. From Lemma 3.5.17 we infer $t \mid 2$, hence K is torsion-free according to Lemma 3.5.19. Moreover every μ-homogeneous polynomial not depending on T_6 is a linear combination over the monomials T_4^4, $T_4^2T_5$, T_5^2, thus reducible. This implies that $T_6 \in R$ is not prime. A contradiction.
- $b_4 = 2$: From $b_5 \mid b_4 + 1 = 3$ and $b_4 \leq b_5$ follows $b_5 = 3$. This leads to
$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 2 & 3 & 1 \end{bmatrix}, \quad \alpha = (0,6).$$
Observe $\alpha = 3u_2 = 2u_3$. Lemma 3.5.17 yields $t \mid 2$ and $t \mid 3$, hence $t = 1$. So K is torsion-free. We end up with Number 8 from Theorem 3.1.3.
- $b_4 = 3$: From $b_5 \mid b_4 + 1 = 4$ and $b_4 \leq b_5$ we infer $b_5 = 4$. This implies $\alpha_2 = 8$; a contradiction to $b_4 \mid \alpha_2$.

Case III-ii. Here, we have $\lambda = \operatorname{cone}(w_2, w_5)$. Proposition 3.5.3 says $\mu \in \sigma_2$. Let $v \in \mathbb{Z}^2$ be a primitive vector on σ_2. Applying Lemma 3.5.6 to w_2, w_3, w_5 as well as w_2, w_3, w_6 shows $\det(v, u_5) = 1$ and $\det(v, u_6) = 1$. In particular u_5, u_6 are primitive and lie on the same ray hence coincide. Again by Lemma 3.5.6, now applied to w_1, w_5, w_6, we obtain $\det(u_1, u_5) = 1$. A suitable admissible coordinate change amounts to $v = (1,0)$ and $u_5 = (0,1)$. As a result the free part Q^0 of the degree matrix Q is of the form
$$Q^0 = \begin{bmatrix} 1 & a_2 & a_3 & a_4 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad a_2, a_3, a_4 \in \mathbb{Z}_{\geq 1}.$$

Note that the second coordinate of u_1 is determined by $\alpha_2 = 0$ and Remark 3.2.7 (i). Furthermore, we may assume $a_2 \leq a_3 \leq a_4$. Lemma 3.5.9 shows that α_1 is divisible by each of a_2, a_3, a_4. From applying Lemma 3.5.6 to all triples w_i, w_j, w_6 where $2 \leq i < j \leq 4$ we infer that a_2, a_3, a_4 are pairwise coprime. This leads to

$$a_2 a_3 a_4 \mid a_2 + a_3 + a_4 + 1.$$

According to Remark 3.3.1 we have $a_2 = 1$ and one of the following two configurations

$$a_3 = 1, \quad a_3 = 2 \text{ and } a_4 = 3.$$

Note that $a_3 = 2$ and $a_4 = 3$ amounts to $\alpha_1 = 7$; a contradiction to $a_3 \mid \alpha_1$. So we have $a_3 = 1$. Then $a_4 \mid \alpha_1 = 3 + a_4$ holds. We conclude $a_4 \mid 3$ i.e. $a_4 = 1, 3$. We show that K is torsion-free in both cases:

- $a_4 = 1$. Here we have

$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \alpha = (4, 0).$$

  Note $\alpha = 4u_2$, thus $t \mid 4$ by Lemma 3.5.17. Now Lemma 3.5.19 ensures that K is torsion-free.

- $a_4 = 2$. Here we have

$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 3 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \alpha = (6, 0).$$

  Note $\alpha = 2u_4$, thus $t \mid 2$ by Lemma 3.5.17. Now Lemma 3.5.19 ensures that K is torsion-free.

We have arrived at Numbers 9 and 10 from Theorem 3.1.3.

Case III-iii. From Lemma 3.5.10 (i) we obtain

$$u_1 = u_2 = u_3, \qquad u_5 = u_6, \qquad \det(u_1, u_6) = 1.$$

A suitable admissible coordinate change brings the degree matrix into the following form

$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & a_4 & 0 & 0 \\ 0 & 0 & 0 & b_4 & 1 & 1 \end{bmatrix}, \qquad a_4, b_4 \in \mathbb{Z}_{\geq 1}.$$

Moreover, Proposition 3.5.3 tells us $\mu \in \mathrm{cone}(w_1, w_4)^\circ$ or $\mu \in \varrho_4$. Let us first assume $\mu \in \mathrm{cone}(w_1, w_4)^\circ$. According to Proposition 3.5.4 we have GIT-cones

$$\eta_1 = \mathrm{cone}(w_1, w_4), \qquad \eta_2 = \mathrm{cone}(w_4, w_5),$$

both of them giving rise to a smooth variety $X(\eta_i)$; see Proposition 3.6.1. We obtain that K is torsion-free by applying Lemma 3.5.13 to $X(\eta_2)$. Applying Lemma 3.5.10 (ii) gives $u_4 = u_1 + u_6 = (1, 1)$. We have arrived at Numbers 11 and 12 from Theorem 3.1.3.

The next step is to consider $\mu \in \varrho_4$. Lemma 3.5.9 provides us with some $k \in \mathbb{Z}_{\geq 2}$ such that $\mu = kw_4$ holds. Using Remark 3.2.7 (i) gives $kb_4 = \alpha_2 = b_4 + 2$. We conclude $b_4 \mid 2$. This leads to one of the following two configurations

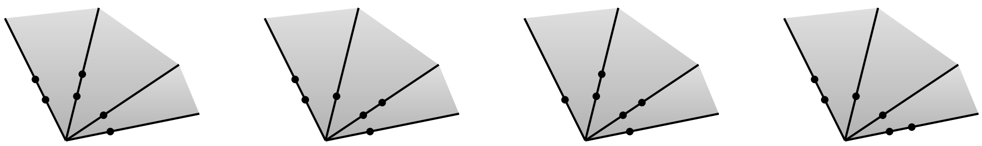
$$k = 3 \text{ and } b_4 = 1, \qquad k = 2 \text{ and } b_4 = 2.$$

Suppose $k = 3$. Using Remark 3.2.7 (i) again shows $3a_4 = 3 + a_4$, equivalently $2a_4 = 3$. A contradiction. We must have $k = 2$ and $b_4 = 2$. Here Remark 3.2.7 (i) implies $2a_4 = 3 + a_4$, thus $a_4 = 3$. We have

$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \end{bmatrix}, \qquad \alpha = (6, 4).$$

From $\alpha = 2u_4$ we infer $t \mid 2$ by Lemma 3.5.17. Thus Lemma 3.5.19 yields that K is torsion-free. This amounts to Number 13 from Theorem 3.1.3.

Part IV • This parts deals with the case of $w_1, \ldots, w_6$ being disposed on four rays according to the integer partition $1 + 1 + 2 + 2 = 6$. A suitable admissible coordinate change leads to one of the subsequent constellations:



IV-i IV-ii IV-iii IV-iv

Case IV-i. Here, Proposition 3.5.3 tells us $\mu \in \sigma_3$. As a result, Proposition 3.5.4 provides us with two GIT-cones

$$\eta_1 = \text{cone}(w_2, w_3), \qquad \eta_2 = \text{cone}(w_3, w_5).$$

Proposition 3.6.1 ensures that the associated varieties $X(\eta_1)$ and $X(\eta_2)$ both are smooth. Let $v \in \mathbb{Z}^2$ denote the primitive lattice vector lying on σ_3. Consider $X(\eta_2)$. Applying Lemmas 2.5.5 and 3.5.6 to the triples w_3, w_4, w_5 and w_3, w_4, w_6 yields $u_5 = u_6$ and $\det(v, u_5) = 1$. Thus we may apply a suitable admissible coordinate change such that $v = (1, 0)$ and $u_5 = u_6 = (0, 1)$. We apply Lemma 3.5.6 again, this time to w_1, w_5, w_6 and w_2, w_5, w_6. This shows that the first coordinate of both u_1 and u_2 equals one. Now, consider $X(\eta_1)$. We apply Lemma 3.5.6 to w_1, w_3, w_4, hence obtain $u_1 = (1, -1)$. Analogously, we obtain $u_2 = (1, -1)$, thus $u_1 = u_2$. This contradicts $\sigma_1 \neq \sigma_2$.

Case IV-ii. Proposition 3.5.3 says $\mu \in \text{cone}(w_2, w_4)$. First, we assume $\mu \in \varrho_4 = \sigma_3$. Then Proposition 3.5.4 ensures $\lambda = \text{cone}(w_2, w_5)$. Let $v \in \mathbb{Z}^2$ be the primitive lattice vector on σ_2. Applying Lemmas 2.5.5 and 3.5.6 to all four triples

$$(w_2, w_3, w_5), \quad (w_2, w_3, w_6), \quad (w_2, w_5, w_6), \quad (w_3, w_5, w_6)$$

shows that u_2, u_3, u_5, u_6 are primitive, thus $u_2 = u_3$ and $u_5 = u_6$. Additionally we obtain $\det(u_2, u_5) = 1$. Lemma 3.5.6 again, this time applied to w_1, w_5, w_6, tells us $\det(u_1, u_5) = 1$. A suitable admissible coordinate change eventually amounts to

$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & a_4 & 0 & 0 \\ -b_1 & 0 & 0 & b_4 & 1 & 1 \end{bmatrix}, \qquad a_4, b_1, b_4 \in \mathbb{Z}_{\geq 1}.$$

From Remark 3.2.7 (i) we infer $\alpha = (a_4 + 3,\, b_4 - b_1 + 2)$. Lemma 3.5.9 provides us with some $k \in \mathbb{Z}_{\geq 2}$ such that $\mu = kw_4$. In particular $a_4 \mid \alpha_1 = a_4 + 3$. This implies $a_4 = 1, 3$. Suppose $a_4 = 1$. Then $k = 4$ holds. This leads to $4b_4 = \alpha_2 = b_4 - b_1 + 2$, hence $3b_4 = 2 - b_1$. A contradiction to $b_1, b_4 \geq 1$. We are left with $a_4 = 3$ and $k = 2$. Inserting into $\alpha = ku_3$ gives $2b_4 = b_4 - b_1 + 2$, thus $b_4 = 2 - b_1$. This forces $b_1 = 1$ and $b_4 = 1$ due to $b_1, b_4 \geq 1$. Moreover $k = 2$ implies $t \mid 2$ by Lemma 3.5.17. Thus K is torsion-free according to Lemma 3.5.19. We have arrived at Number 14 from Theorem 3.1.3.

We turn to the case $\mu \notin \sigma_3$. Here Proposition 3.5.4 provides us with two GIT-cones

$$\eta_1 = \operatorname{cone}(w_2, w_4), \qquad \eta_2 = \operatorname{cone}(w_4, w_5).$$

According to Proposition 3.6.1 the according varieties $X(\eta_1)$ and $X(\eta_2)$ both are smooth. Consider $X(\eta_2)$. Lemma 3.5.5 applied to w_4, w_5 and w_4, w_6 yields $\det(u_4, u_5) = 1$ as well as $\det(u_4, u_6) = 1$. Besides, Lemmas 2.5.5 and 3.5.6 applied to w_1, w_5, w_6 give us $\det(u_1, u_5) = 1$. Now consider $X(\eta_1)$. Let $v \in \mathbb{Z}^2$ be the primitive vector contained in σ_2. Applying Lemmas 2.5.5 and 3.5.6 to w_2, w_3, w_4 and w_2, w_3, w_5 shows $\det(v, u_4) = 1$ and $\det(v, u_5) = 1$. Now we apply an admissible coordinate change such that $v = (1, 0)$ and $u_5 = (0, 1)$ holds. Taking the determinantal equations from above into account amounts to the following degree matrix

$$Q^0 = \begin{bmatrix} 1 & a_2 & a_3 & 1 & 0 & 0 \\ -b_1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad a_2, a_3, b_1 \in \mathbb{Z}_{\geq 1}.$$

We may assume $a_2 \leq a_3$. From Remark 3.2.7 (i) follows $\alpha_2 = 3 - b_1$. Proposition 3.5.3 guarantees that μ lives in the positive orthant, hence $b_1 \leq 3$. Furthermore, Lemma 3.5.5 applied w.r.t $X(\eta_1)$ and the pairs w_2, w_5 and w_3, w_5 shows $a_2, a_3 \mid \alpha_1$. Applying Lemma 3.5.6 to w_2, w_3, w_5 shows $\gcd(a_2, a_3) = 1$. Consequently $a_2 a_3 \mid \alpha_1 = a_2 + a_3 + 2$ holds. We end up with $a_2 = 1$ and $a_3 = 1, 3$.

Let us discuss the case $a_3 = 1$. Here specifying data looks as follows

$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ -b_1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad \alpha = (4,\, 3 - b_1), \qquad b_1 \in \{1, 2, 3\}.$$

Suppose $b_1 = 3$. This implies $\alpha = (4, 0) = 4u_2$. Lemmas 3.5.17 and 3.5.19 yield that K is torsion-free. So $w_2 = w_3$ holds. Note that $\alpha_2 = 0$ means $\mu \in \varrho_2$. In this situation Lemma 3.5.12 says $w_4 \in \varrho_2$. A contradiction to $\sigma_2 \neq \sigma_3$. So we have $b_1 = 1, 2$. Observe $\mu \in \eta_1^\circ$. Applying Lemma 3.5.13 to $X(\eta_2)$ guarantees that K is torsion-free. We end up with Numbers 15 to 18 from Theorem 3.1.3.

We turn to $a_3 = 3$. Here we have $\alpha_1 = 6$. According to Lemma 3.5.5 applied to $X(\eta_1)$ and w_3, w_4, there must be some monomial $T_3^{l_3}T_4^{l_4}$ of degree μ because of $\det(u_3, u_4) = 3$. As the second coordinate of u_3 vanishes, $l_4 = \alpha_2 = 3 - b_1$ holds. Inserting this into the equation $\alpha_1 = l_3a_3 + l_4a_4$ yields $3l_3 + 3 - b_1 = \alpha_1 = 6$. This forces b_1 to be divisible by 3, hence $b_1 = 3$. We arrive at the following data

$$Q^0 = \begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ -3 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \alpha = (6, 0).$$

Observe that this grading does not admit any monomial of the form $T_1^{l_1}T_4^{l_4}$ of degree μ. Thus $\det(u_1, u_4) = 1$ by Lemma 3.5.5 applied to $X(\eta_1)$ and w_1, w_4. A contradiction.

Case IV-iii. Proposition 3.5.4 ensures $\lambda = \mathrm{cone}(w_3, w_5)$. Let $v, v' \in \mathbb{Z}^2$ be the primitive ray generators of σ_2, σ_3. We may apply Lemmas 2.5.5 and 3.5.6 to at least one of the triples w_2, w_3, w_4 and w_2, w_4, w_5. From this we infer $\det(v, v') = 1$. A suitable admissible coordinate change leads to $v = (1, 0)$ and $v' = (0, 1)$. Applying Lemma 3.5.6 to w_2, w_3, w_6 yields $w_6 = (-a_6, 1)$ for some $a_6 \in \mathbb{Z}_{\geq 1}$. Similarly, one obtains $w_1 = (1, -b_1)$ with $b_1 \in \mathbb{Z}_{\geq 1}$. Counter-clockwise orientation yields $\det(w_1, w_6) = 1 - a_6b_1 \leq 0$. We conclude $b_1 = a_6 = 1$, hence $w_1 = -w_6$. This contradicts $\mathrm{Eff}(R)$ being pointed.

Case IV-iv. We have $\mu \in \mathrm{cone}(w_3, w_4)$ by Proposition 3.5.3. Suppose $\mu \in \mathrm{cone}(w_3, w_4)^\circ$. Proposition 3.6.1 allows us to apply Lemma 3.5.10 (ii). From this we infer $u_3 = u_4$, thus $\sigma_2 = \sigma_3$; a contradiction. So we have $\mu \in \sigma_2 \cup \sigma_3$. Taking the symmetry in the geometric constellation of $w_1, \ldots, w_6$ into account a suitable admissible coordinate change amounts to $\mu \in \sigma_2$. Lemma 3.5.10 yields $u_1 = u_2$, $u_5 = u_6$ and $u_4 = u_1 + u_6$. Furthermore we obtain $\det(u_1, u_6) = 1$. After applying another suitable admissible coordinate change the degree matrix is of the following form

$$Q^0 = \begin{bmatrix} 1 & 1 & a_3 & 1 & 0 & 0 \\ 0 & 0 & b_3 & 1 & 1 & 1 \end{bmatrix}, \quad a_3, b_3 \in \mathbb{Z}_{\geq 1}.$$

From X being Calabi-Yau we infer $\alpha = (a_3 + 3,\ b_3 + 3)$; see Remark 3.2.7. Besides Lemma 3.5.9 provides us with some $k \in \mathbb{Z}_{\geq 2}$ such that $\mu = kw_3$ holds. Altogether we obtain $(k-1)a_3 = 3 = (k-1)b_3$, hence $a_3 = b_3$. This contradicts $\sigma_2 \neq \sigma_3$.

Part V • In this part we study the case of $w_1, \ldots, w_6$ being disposed on four rays according to the integer partition $1 + 1 + 1 + 3 = 6$. After applying a suitable admissible coordinate change, we face one of the two constellations below.

V-i V-ii

Case V-i. According to Proposition 3.5.3 and Remark 3.2.7 (i) either $\mu \in \mathrm{cone}(w_3, w_4)^\circ$ or $\mu \in \sigma_2$ holds. First, we assume $\mu \in \mathrm{cone}(w_3, w_4)^\circ$. Here Proposition 3.5.4 provides us with two GIT-cones

$$\eta_1 = \mathrm{cone}(w_3, w_4), \qquad \eta_2 = \mathrm{cone}(w_4, w_5).$$

Proposition 3.6.1 tells us that each η_i defines a smooth variety $X(\eta_i)$. Let $v \in \mathbb{Z}^2$ be the primitive vector lying on σ_1. Consider $X(\eta_1)$. Applying Lemmas 3.5.6 and 3.5.7 to w_1, w_2, w_5 and w_1, w_2, w_6 shows that $\det(v, u_5)$ and $\det(v, u_6)$ both equal one. Now consider $X(\eta_2)$. From Lemma 3.5.5 applied to both pairs w_4, w_5 and w_4, w_6 we infer $\det(u_4, u_5) = 1$ and $\det(u_4, u_6) = 1$. We are in the situation of Lemma 3.5.8, hence $v = u_4$ or $u_5 = u_6$ i.e. $\sigma_1 = \sigma_2$ or $\sigma_3 = \sigma_4$. A contradiction.

We turn to $\mu \in \sigma_2$. Here $\lambda = \mathrm{cone}(w_3, w_5)$ holds due to Proposition 3.5.4. Let $v \in \mathbb{Z}^2$ be the primitive lattice vector in σ_1. Applying Lemma 3.5.6 to w_1, w_2, w_5 and w_1, w_2, w_6 shows $\det(v, u_5) = 1$ and $\det(v, u_6) = 1$. We find a suitable admissible coordinate change that amounts to $v = (1, 0)$ and

$$Q^0 = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ 0 & 0 & 0 & b_4 & 1 & 1 \end{bmatrix}, \quad a_1, \ldots, a_5, b_4 \in \mathbb{Z}_{\geq 1}.$$

We may assume $a_1 \leq a_2 \leq a_3$. Lemma 3.5.9 says $\mu = kw_4$ holds for some $k \in \mathbb{Z}_{\geq 2}$, in particular $b_4 \mid \alpha_2$. From Remark 3.2.7 (i) follows $\alpha_2 = b_4 + 2$, hence $b_4 \mid 2$. This leads to one of the configurations

$$b_4 = 1 \text{ and } k = 3, \qquad b_4 = 2 \text{ and } k = 2.$$

Furthermore, Lemma 3.5.5 shows that k is divisible by a_i for all $i = 1, 2, 3$. Lemma 3.5.6 applied to all triples w_i, w_j, w_6 where $1 \leq i < j \leq 4$ guarantees that $a_1, \ldots, a_4$ are pairwise coprime. As as result we obtain $a_1 a_2 a_3 \mid k$. Since $k \leq 3$ holds, this forces

$$a_1 = a_2 = 1 \text{ and } a_3 = 1, k.$$

Remark 3.2.7 (i) says $\mu = w_1 + \ldots + w_6$. We combine this with $\mu = kw_4$, consider the first coordinate of α and eventually obtain that a_5 is determined by

$$a_5 = (k - 1)a_4 - a_3 - 2.$$

Observe $b_4(k - 1) = 2$ for both configurations of b_4, k in question. Besides, note that $\det(u_4, u_5) > 0$ means $b_4 a_5 < a_4$. Plugging the above presentation of a_5 into this inequation yields

$$a_4 < b_4(2 + a_3).$$

At this point we have found upper bounds on all entries of Q^0. Let us make things explicit:

- $b_4 = 1$, $k = 3$ and $a_3 = 1$: We have $0 < a_5 < a_4 < 3$ i.e. $a_4 = 2$ and $a_5 = 1$. This amounts to
$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad \alpha = (6, 3).$$
Given that K is torsion-free this leads to specifying data as in Number 19 from Theorem 3.1.3.
Now assume that the torsion subgroup of K is non-trivial. From $\alpha = 3u_4$ and Lemma 3.5.17 we infer $t = 3$ i.e. $K = \mathbb{Z}^2 \times \mathbb{Z}/3\mathbb{Z}$. Using $\det(u_5, u_6) = 1$ enables us to apply a suitable admissible coordinate change such that $\zeta_5 = \zeta_6 = 0$. Moreover, Lemma 3.5.6 tells us that K is generated by each of the triples
$$(w_1, w_2, w_6), \quad (w_1, w_3, w_6), \quad (w_2, w_3, w_6).$$
Combining this with $u_1 = u_2 = u_3$ we deduce that ζ_1, ζ_2, ζ_3 are pairwise different; otherwise K would be spanned by only two elements. After suitably reordering T_1, T_2, T_3 we may assume $\zeta_1 = 0$. Again Lemma 3.5.6 shows that w_1, w_5, w_6 form a generating set for K as well. This contradicts $\zeta_1 = \zeta_5 = \zeta_6 = 0$. As a consequence, K must be torsion-free.
- $b_4 = 1$, $k = 3$ and $a_3 = 3$: We have $a_4 < 5$. We exclude $a_4 = 2$ because this choice of a_4 implies $a_5 = -1$; a contradiction. Due to $\gcd(a_3, a_4) = 1$ the case $a_4 = 3$ does not show up either. The remaining case $a_4 = 4$ leads to $a_5 = 3$. However, Lemma 3.5.6 applied to w_3, w_5, w_6 states $\gcd(a_3, a_5) = 1$. A contradiction.
- $b_4 = 2$, $k = 2$ and $a_3 = 1$: We have $a_4 < 6$. From $b_4 a_5 < a_4$ we deduce $a_4 \geq 3$. From $a_4 = 3$ follows $a_5 = 0$; a contradiction. So we end up with
$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & a_4 & a_5 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \end{bmatrix}, \quad (a_4, a_5) = (4, 1), (5, 2).$$
From $k = 2$ and Lemma 3.5.17 we infer $t \mid 2$. Now Lemma 3.5.19 ensures that K is torsion-free. So we arrive at Numbers 20 and 21 from Theorem 3.1.3.
- $b_4 = 2$, $k = 2$ and $a_3 = 2$: We have $a_4 < 8$. From $a_5 = (k-1)a_4 - a_3 - 2 = a_4 - 4$ we deduce $a_4 \geq 5$. The case $a_4 = 6$ is excluded by $\gcd(a_3, a_4) = 1$. The remaining cases are $(a_4, a_5) = (5, 1), (7, 3)$.
With $(a_4, a_5) = (5, 1)$ we obtain
$$Q^0 = \begin{bmatrix} 1 & 1 & 2 & 5 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \end{bmatrix}, \quad \alpha = (10, 4).$$
From $\alpha = 2u_4$ we infer $t \mid 2$ by Lemma 3.5.17. Furthermore, Lemma 3.5.15 ensures that $T_3^3 T_5^4$ is of degree μ. Applying Lemma 3.5.16 to w_1, w_3, w_5 and $T_3^3 T_5^4$ yields $t \mid 3$. Altogether $t = 1$ i.e. K is torsion-free. This leads to Number 22 from Theorem 3.1.3.
Finally consider $(a_4, a_5) = (7, 3)$. Here we have
$$Q^0 = \begin{bmatrix} 1 & 1 & 2 & 7 & 3 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \end{bmatrix}, \quad \alpha = (14, 4).$$

Lemma 3.5.15 ensures that $T_3T_5^4$ is of degree μ. Applying Lemma 3.5.16 to w_1, w_3, w_5 and $T_3^3T_5^4$ yields $t = 1$. i.e. K is torsion-free. We end up with specifying data as in Number 23 from Theorem 3.1.3.

Case V-ii. Proposition 3.5.3 says $\mu \in \sigma_2$. We have $\lambda = \mathrm{cone}(w_2, w_5)$ by Proposition 3.5.4. Let $v \in \mathbb{Z}^2$ denote the primitive generator of the ray σ_2. Applying Lemmas 3.5.6 and 3.5.7 to the triples w_2, w_3, w_5 and w_2, w_3, w_6 shows $\det(v, u_5) = 1$ and $\det(v, u_6) = 1$. A suitable admissible coordinate change leads to

$$Q^0 = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ -b_1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad a_1, \ldots, a_5, b_1 \in \mathbb{Z}_{\geq 1},$$

where $a_2 \leq a_3 \leq a_4$. Remark 3.2.7 (i) yields $\alpha_2 = 2 - b_1$. Since $\mu \in \varrho_2$ means $\alpha_2 = 0$, we conclude $b_1 = 2$. Applying Lemma 3.5.6 to w_i, w_j, w_6 for any $1 \leq i < j \leq 4$ shows that $a_1, \ldots, a_4$ are pairwise coprime. According to Lemma 3.5.9 the first coordinate α_1 of α is divisible by a_2, a_3, a_4. For $i = 2, 3, 4$ applying Lemma 3.5.6 to w_1, w_i, w_5 gives $\gcd(a_1 + 2a_5, a_i) = 1$ as well as $\det(u_1, u_5) = a_1 + 2a_5 \mid \alpha_1$. Altogether we have

$$a_1a_2a_3a_4 + 2a_2a_3a_4a_5 = (a_1 + 2a_5)a_2a_3a_4 \mid \alpha_1 = a_1 + \cdots + a_5. \tag{3.2}$$

In particular, the left-hand side is less than or equal to the right-hand side. Plugging $a_2, a_3 \leq a_4$, $a_1a_2a_2a_4 \geq a_1$ and $a_2a_3a_4a_5 \geq a_5$ into Eq. (3.2) leads to $a_2a_3a_4a_5 \leq 3a_4$, thus $a_2a_3a_5 \leq 3$. Then $a_2 \leq a_3$ enforces $a_2 = 1$. We end up with the following configurations

$$a_3 = 1 \text{ and } a_5 = 1, 2, 3, \qquad a_3 = 2, 3 \text{ and } a_5 = 1.$$

We gain bounds on a_4 in terms of a_3, a_5 by combining $a_1a_2a_3a_4 \geq a_1$ and Eq. (3.2) again:

$$(2a_3a_5 - 1)a_4 \leq a_3 + a_5 + 1. \tag{3.3}$$

As a result we obtain $a_3 = 1$; otherwise we must have $a_3 = 2, 3$ and $a_5 = 1$ thus Eq. (3.3) gives $3a_4 \leq 5$ forcing $a_4 = 1$, which is a contradiction to $a_3 \leq a_4$. From Lemma 3.5.5 applied to w_1, w_6 we deduce that a_1 divides $\alpha_1 = a_1 + a_4 + a_5 + 2$, thus

$$a_1 \mid a_4 + a_5 + 2. \tag{3.4}$$

Finally, we make things explicit and go through the cases $a_5 = 1, 2, 3$:

- $a_5 = 1$: We have $a_4 \leq 3$ due to Eq. (3.3). When $a_4 = 1$ we end up with $a_1 = 1, 2, 4$ due to Eq. (3.4). None of these configurations satisfies Eq. (3.2). When $a_4 = 2$ holds, Eq. (3.4) gives $a_1 = 1, 5$. The first value, $a_1 = 1$, leads to

$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \alpha = (6, 0).$$

  From $\alpha = 3u_4$ and Lemma 3.5.17 we infer $t \mid 3$. Moreover, Lemma 3.5.15 ensures that $T_1^2T_5^4$ is of degree μ. Applying Lemma 3.5.16 to w_1, w_2, w_5 and $T_1^2T_5^4$ shows $t \mid 2$. Altogether $t = 1$ i.e. K is torsion-free. We arrive at Number 24 from Theorem 3.1.3. The second value, $a_1 = 5$, does not satisfy Eq. (3.2). Finally, with $a_4 = 4$ we obtain $a_1 = 1, 7$ from Eq. (3.4). None of these configurations satisfies Eq. (3.2).

- $a_5 = 2$: Inserting into Eq. (3.3) yields $3a_4 \leq 4$, hence $a_4 = 1$. Now Eq. (3.4) leads to $a_1 = 1, 5$. None of these configurations satisfies Eq. (3.2).
- $a_5 = 3$: Inserting into Eq. (3.3) yields $5a_4 \leq 5$, hence $a_4 = 1$. Now Eq. (3.4) amounts to $a_1 = 1, 2, 3, 6$. Applying Lemma 3.5.6 to the triple w_1, w_5, w_6 shows $\gcd(a_1, a_5) = 1$. This reduces the situation to $a_1 = 1, 2$.
  With $a_1 = 1$ we obtain

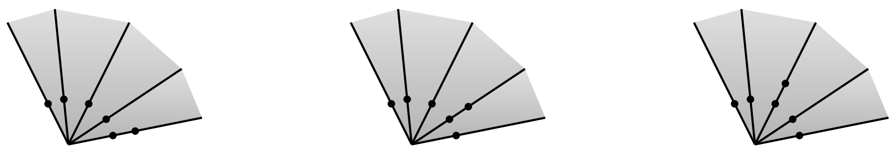
$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \alpha = (7, 0).$$

  Lemma 3.5.15 ensures that $T_1 T_5^2$ is of degree μ. Applying Lemma 3.5.16 to w_1, w_2, w_5 and $T_1 T_5^2$ shows $t = 1$ i.e. K is torsion-free. We arrive at specifying data as in Number 25 from Theorem 3.1.3.
  For $a_1 = 2$ we have the following data

$$Q^0 = \begin{bmatrix} 2 & 1 & 1 & 1 & 3 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \alpha = (8, 0).$$

  Again Lemma 3.5.15 ensures that $T_1 T_5^2$ is of degree μ. Applying Lemma 3.5.16 to w_1, w_2, w_5 and $T_1 T_5^2$ shows $t = 1$ i.e. K is torsion-free. We arrive at specifying data as in Number 26 from Theorem 3.1.3.

Part VI • Here we treat the case where the generator degrees $w_1, \ldots, w_6$ lie on five different rays. After a suitable admissible coordinate change, we are in the situation of one of the constellations illustrated below.



VI-i VI-ii VI-iii

Case VI-i. We claim $\mu \in \sigma_3$. Otherwise Proposition 3.5.3 tells us $\mu \in \sigma_2 \cup \mathrm{cone}(w_3, w_4)^\circ$. Then $\eta = \mathrm{cone}(w_4, w_5)$ is a GIT-cone by Proposition 3.5.4. Consider the associated variety $X(\eta)$. Proposition 3.6.1 ensures that $X(\eta)$ is smooth. Let $v \in \mathbb{Z}^2$ denote the primitive generator of the ray σ_1. Lemmas 3.5.6 and 3.5.7 apply to w_1, w_2, w_5 and w_1, w_2, w_6, therefore $\det(v, u_5) = 1$ and $\det(v, u_6) = 1$. Moreover, Lemma 3.5.5 applied to both pairs w_4, w_5 and w_4, w_6 gives $\det(u_4, u_5) = 1$ and $\det(u_4, u_6) = 1$. We use Lemma 2.5.6 and obtain $v = u_4$ or $u_5 = u_6$, thus $\sigma_1 = \sigma_3$ or $\sigma_4 = \sigma_5$. A contradiction.

So we have $\mu \in \sigma_3$. Proposition 3.5.4 provides us with GIT-cones

$$\eta_1 = \mathrm{cone}(w_1, w_3), \qquad \eta_2 = \mathrm{cone}(w_3, w_5).$$

According to Proposition 3.6.1 the associated varieties $X(\eta_1)$, $X(\eta_2)$ both are smooth. Let us consider $X(\eta_1)$. Applying Lemma 3.5.5 to the pairs w_1, w_3 and w_2, w_3 yields

$$\det(u_1, u_3) = 1 \quad \text{and} \quad \det(u_2, u_3) = 1.$$

From this we also infer that u_1 and u_2 both are primitive, hence $u_1 = u_2$. Now consider $X(\eta_2)$. Lemma 3.5.6 applied to both triples w_1, w_2, w_5 and w_1, w_2, w_6 gives

$$\det(u_1, u_5) = 1 \quad \text{and} \quad \det(u_1, u_6) = 1.$$

After applying a suitable admissible coordinate change the degree matrix is as follows

$$Q^0 = \begin{bmatrix} 1 & 1 & a_3 & a_4 & a_5 & 0 \\ 0 & 0 & 1 & b_4 & 1 & 1 \end{bmatrix}, \quad a_3, a_4, a_5, b_4 \in \mathbb{Z}_{\geq 1}.$$

Lemma 3.5.9 provides us with some $k \in \mathbb{Z}_{\geq 2}$ such that $kw_4 = \mu$. Remark 3.2.7 (i) says $\alpha_2 = b_4 + 3$. Together we obtain $b_4 \mid 3$, thus $b_4 = 1, 3$. This implies $k = 1 + 3/b_4$. Moreover, applying Lemma 3.5.6 to w_3, w_4, w_6 yields

$$a_3 = \det(w_3, w_6) \mid k = 1 + \frac{3}{b_4}. \tag{3.5}$$

Again by Remark 3.2.7 (i) we have $ka_4 = \alpha_1 = a_3 + a_4 + a_5 + 2$. This determines a_4 by

$$a_4 = \frac{a_3 + a_5 + 2}{(k-1)} = \frac{b_4(a_3 + a_5 + 2)}{3}. \tag{3.6}$$

Consider $b_4 = 1$. From w_3, w_4, w_5 being oriented counter-clockwise we infer $a_3 > a_4 > a_5$, in particular $a_3 \geq 3$. Then Eq. (3.5) enforces $a_3 = 4$. This implies $a_5 \leq 2$. For $a_5 = 1, 2$ one directly checks that Eq. (3.6) does not yield an integer value for a_4. A contradiction.

Consider $b_4 = 3$. Equation (3.5) says $a_3 = 1, 2$. From $a_3 > a_5 > 0$ we deduce $a_3 = 2$ and $a_5 = 1$. Now inserting into Eq. (3.6) leads to $a_4 = 5$, hence

$$Q^0 = \begin{bmatrix} 1 & 1 & 2 & 5 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 & 1 \end{bmatrix}, \quad \alpha = (10, 6).$$

Lemma 3.5.15 makes sure that $T_3^5 T_6$ is of degree μ. Consider $X(\eta_2)$. Applying Lemma 3.5.16 to w_3, w_5, w_6 and $T_3^4 T_6$ yields $t = 1$ i.e. K is torsion-free. This amounts to Numbers 27 and 28 from Theorem 3.1.3.

Case VI-ii. Repeating the arguments from Case VI-i shows $\mu \in \sigma_3$. By Proposition 3.5.4 we have $\lambda = \operatorname{cone}(w_3, w_5)$. Let $v \in \mathbb{Z}^2$ denote the primitive lattice in σ_2. Applying Lemmas 3.5.6 and 3.5.7 to w_2, w_3, w_5 as well as w_2, w_3, w_6 shows $\det(v, u_5) = 1$ and $\det(v, u_6) = 1$. By a suitable admissible coordinate thus we achieve

$$Q^0 = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ -b_1 & 0 & 0 & b_4 & 1 & 1 \end{bmatrix}, \quad a_1, \ldots, a_5, b_1, b_4 \in \mathbb{Z}_{\geq 1},$$

where $a_2 \le a_3$. By Lemma 3.5.9 there is some $k \in \mathbb{Z}_{\ge 2}$ with $kw_4 = \mu$. According to Remark 3.2.7 (i) we obtain $kb_4 = \alpha_2 = b_4 - b_1 + 2$. This leads to $(k-1)b_4 = 2 - b_1$. Since the left-hand side is positive, we conclude $b_1 = 1$ and consequently $b_4 = 1$, $k = 2$. Moreover, applying Lemma 3.5.6 to w_1, w_4, w_5 yields

$$a_1 + a_5 = \det(u_1, u_5) \mid k = 2.$$

From this we infer $a_1 = 1$ and $a_5 = 1$. Analogously we obtain $a_2, a_3 \mid 2$. Applying Lemma 3.5.6 to w_2, w_3, w_6 shows $\gcd(a_2, a_3) = 1$. Besides we have $a_2 \le a_3$. Altogether we arrive at $a_2 = 1$ and $a_3 = 1, 2$. From Remark 3.2.7 (i) we infer $a_4 = a_3 + 3$.

The case $a_3 = 1$ amounts to

$$Q^0 = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \alpha = (6, 2).$$

Note $\alpha = 3u_4$, thus $t \mid 3$ by Lemma 3.5.17. Furthermore, Lemma 3.5.15 guarantees that $T_3^6 T_6^2$ is of degree μ. Lemma 3.5.6 says that w_3, w_5, w_6 form a generating system for K. Thus we may apply Lemma 3.5.16 to w_3, w_5, w_6 and $T_3^6 T_6^2$. From this we infer $t \mid 2$. Altogether $t = 1$, hence K is torsion-free. We have arrived at Number 29 from Theorem 3.1.3.

To conclude Case VI-ii suppose $a_3 = 2$. Then $a_4 = 5$ holds. On the other side, Lemma 3.5.6 applied to w_3, w_4, w_5 yields that a_3 and $a_4 - a_5 = 4$ are coprime. A contradiction.

Case VI-iii. Proposition 3.5.3 says $\mu \in \sigma_3$. By Proposition 3.5.4 we find GIT-chambers

$$\eta_1 = \operatorname{cone}(w_2, w_3), \qquad \eta_2 = \operatorname{cone}(w_4, w_5)$$

each of which defines a smooth variety $X(\eta_i)$; see Proposition 3.6.1. Let $v \in \mathbb{Z}^2$ be the primitive lattice vector on the ray σ_3. Consider $X(\eta_1)$. Applying Lemma 3.5.6 to w_1, w_3, w_4 as well as w_2, w_3, w_4 yields $\det(u_1, v) = 1$ and $\det(u_2, v) = 1$. Analogously we obtain $\det(v, u_5) = 1$ and $\det(v, u_6) = 1$ when considering $X(\eta_2)$. Performing a suitable admissible coordinate change leads to

$$Q^0 = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad a_1, \ldots, a_5 \in \mathbb{Z}_{\ge 1}$$

where $a_1 < a_2$ and $\gcd(a_3, a_4) = 1$. Observe that u_1, u_5 do not span $\mathbb{Z}^2$, in particular w_1, w_5 do not span K. Thus Lemma 3.5.5 gives us a monomial of the form $T_1^{l_1} T_5^{l_5}$ and degree μ. Recall that $\mu \in \sigma_3$ means that the second coordinate of α vanishes. From this we conclude $l_1 = l_5$ hence $a_1 + a_5 \mid \alpha_1$. In the same way we obtain $a_2 + a_5 \mid \alpha_1$. Furthermore, we infer $a_3, a_4 \mid \alpha_1$ from Lemma 3.5.9. Applying Lemma 3.5.6 to w_i, w_j, w_5 for all $1 \le i < j \le 4$ shows that the four integers $a_1 + a_5$, $a_2 + a_5$, a_3, a_4 are pairwise coprime. We conclude

$$(a_1 + a_5)(a_2 + a_5) a_3 a_4 \mid \alpha_1.$$

Expanding the left-hand side and inserting the description for μ provided by Remark 3.2.7 (i) while weaking the divisibility condition to an estimation leads to

$$a_1a_2a_3a_4 + a_1a_3a_4a_5 + a_2a_3a_4a_5 + a_3a_4a_5^2 \leq a_1 + \cdots + a_5.$$

One quickly checks that only $a_1 = 1, a_2 = 2, a_3 = a_4 = a_5 = 1$ satisfies this inequation by suitably estimating the single terms. For example, suppose $a_3 > 1$. Then

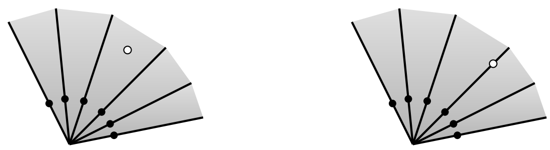
$$a_1a_2a_3a_4 + a_2a_3a_4a_5 + a_2a_3a_4a_5 + a_3a_4a_5^2 \geq 2a_2 + a_3 + a_4 + a_5 > a_1 + \cdots + a_5.$$

Note that second estimation is due to $a_2 > a_1$. Anyways, this is a contradiction. We give the free parts of the final degree matrix and resulting relation degree explicitly

$$Q^0 = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \alpha = (6, 0).$$

Lemma 3.5.15 ensures that both $T_1^3T_5^3$ and $T_2^2T_5^2$ are of degree μ. Moreover Lemma 3.5.6 yields that both triples w_1, w_5, w_6 and w_2, w_3, w_5 form a generating system for K. Applying Lemma 3.5.16 to w_1, w_5, w_6 and $T_1^3T_5^3$ gives $t \mid 3$. Lemma 3.5.16 again, this time applied to w_2, w_3, w_5 and $T_2^2T_5^2$, yields $t \mid 2$. Altogether $t = 1$ i.e. K is torsion-free. Furthermore, the symmetry in the geometric constellation of $w_1, \ldots, w_6, \mu$ reveals $X(\eta_1) \cong X(\eta_2)$. This becomes even clearer if one applies an admissible coordinate change, namely adding the first row of Q to the second row. Consequently, it suffices to list $X(\eta_1)$. We have arrived at specifying data as in Number 30 from Theorem 3.1.3.

Part VII • We work out the the constellation where the Cox ring generator degrees $w_1, \ldots, w_6$ lie on pairwise different rays i.e. we have $\sigma_i = \varrho_i$ for all $i = 1, \ldots, 6$. Proposition 3.5.3 says $\mu \in \operatorname{cone}(w_3, w_4)$. After applying a suitable admissible coordinate change we have either $\mu \in \operatorname{cone}(w_3, w_4)^\circ$ or $\mu \in \varrho_3$.



VII-a: $\mu \in \operatorname{cone}(w_3, w_4)^\circ$ VII-b: $\mu \in \varrho_3$

Case VII-a. Here, we assume $\mu \in \operatorname{cone}(w_3, w_4)^\circ$. According to Proposition 3.5.4 the cones

$$\eta_1 = \operatorname{cone}(w_2, w_3), \qquad \eta_2 = \operatorname{cone}(w_3, w_4), \qquad \eta_3 = \operatorname{cone}(w_4, w_5)$$

are GIT-cones leading to smooth varieties $X(\eta_i)$; see also Proposition 3.6.1. Let us consider $X(\eta_1)$. Lemma 3.5.5 applied to w_1, w_3 and w_2, w_3 yields $\det(u_1, u_3) = 1$ and $\det(u_2, u_3) = 1$. Thus a suitable admissible coordinate change leads to

$$Q^0 = \begin{bmatrix} 1 & 1 & 0 & -a_4 & -a_5 & -a_6 \\ -b_1 & 0 & 1 & b_4 & b_5 & b_6 \end{bmatrix}, \quad a_i, b_i \in \mathbb{Z}_{\geq 1}.$$

Consider $X(\eta_3)$. Applying Lemma 3.5.5 to w_4, w_5 and w_4, w_6 gives $\det(u_4, u_5) = 1$ and $\det(u_4, u_6) = 1$. Since $a_4 \neq 0$, this is equivalent to

$$b_5 = \frac{a_5 b_4 - 1}{a_4}, \qquad b_6 = \frac{a_6 b_4 - 1}{a_4}. \tag{3.7}$$

Now consider $X(\eta_2)$. Applying Lemma 3.5.5 to the pair w_3, w_i for $i = 4, 5, 6$ shows that α_1 is divisible by each of a_4, a_5, a_6. Moreover, Lemma 3.5.6 applied to w_2, w_i, w_j where $4 \leq i < j \leq 6$ ensures that a_4, a_5, a_6 are pairwise coprime. Together with Remark 3.2.7 (i) we obtain

$$a_4 a_5 a_6 \mid \alpha_1 = a_4 + a_5 + a_6 - 2.$$

One quickly checks that this forces two of a_4, a_5, a_6 to equal one. Suppose $a_5 = a_6 = 1$. Then Eq. (3.7) implies $b_5 = b_6$, thus $u_5 = u_6$. A contradiction. So we must have $a_4 = 1$, in particular

$$b_5 = a_5 b_4 - 1, \qquad b_6 = a_6 b_4 - 1. \tag{3.8}$$

Furthermore, Lemma 3.5.5 applied to w_2, w_j gives $b_j \mid \alpha_2$ for $j = 4, 5, 6$. In addition, applying Lemma 3.5.6 to all triples w_2, w_i, w_j where $4 \leq i < j \leq 6$ shows that b_4, b_5, b_6 are pairwise coprime. Once again by Remark 3.2.7 (i) we obtain

$$b_4 b_5 b_6 \mid \alpha_2 = b_4 + b_5 + b_6 + 1 - b_1. \tag{3.9}$$

Note that the right-hand side is positive due to the position of μ. From this we deduce $b_4 b_5 b_6 \leq b_4 + b_5 + b_6$. According to Remark 3.3.1 this inequation implies that either two of b_4, b_5, b_6 equal one or $\{b_4, b_5, b_6\} = \{1, 2, 3\}$.

We exclude the first option. Here we have $b_5 \neq b_6$ by Eq. (3.8), thus $b_4 = 1$. However, we also have $a_i = 1$ for some $i \in \{5, 6\}$. Then again Eq. (3.8) implies $b_i = a_i - 1 = 0$. A contradiction. So we have $\{b_4, b_5, b_6\} = \{1, 2, 3\}$.

Inserting into Eq. (3.9) amounts to $b_1 = 1$. Currently the degree matrix has the form

$$Q^0 = \begin{bmatrix} 1 & 1 & 0 & -1 & -a_5 & -a_6 \\ -1 & 0 & 1 & b_4 & b_5 & b_6 \end{bmatrix}.$$

Recall that $a_5 = 1$ or $a_6 = 1$ holds. So we have $b_4 > b_5$ or $b_4 > b_6$ due to the counter-clockwise orientation of w_4, w_5, w_6. From this we infer $b_4 \neq 1$, hence $b_i = 1$ for some $i \in \{5, 6\}$. We are left with the cases $b_4 = 2, 3$. With $b_4 = 3$, inserting into Eq. (3.8) gives $3a_i - 1 = b_i = 1$. A contradiction to $a_i \in \mathbb{Z}_{\geq 1}$. With $b_4 = 2$ we deduce $a_i = 1$ from Eq. (3.8). In particular $w_i = (-1, 1) = -w_1$ holds. A contradiction to $\mathrm{Eff}(R)$ being pointed.

Case VII-b. Here, we assume $\mu \in \varrho_3$. Proposition 3.5.4 provides us with two GIT-cones

$$\eta_1 = \mathrm{cone}(w_2, w_4), \qquad \eta_2 = \mathrm{cone}(w_4, w_5).$$

Both of them give rise to a smooth variety $X(\eta_i)$; see Proposition 3.6.1. Consider $X(\eta_2)$. Applying Lemma 3.5.5 to both pairs w_4, w_5 and w_4, w_6 yields $\det(u_4, u_5) = 1$ and $\det(u_4, u_6) = 1$. A suitable admissible coordinate change leads to

$$Q^0 = \begin{bmatrix} a_1 & a_2 & a_3 & 1 & a_5 & 0 \\ -b_1 & -b_2 & -b_3 & 0 & 1 & 1 \end{bmatrix}, \quad a_i, b_i \in \mathbb{Z}_{\geq 1}.$$

Lemma 3.5.9 provides us with some $k \in \mathbb{Z}_{\geq 2}$ such that $\mu = kw_3$ holds. In particular, we have $\alpha_1 = ka_3$. Now consider $X(\eta_1)$. Lemma 3.5.6 applied to the triples w_1, w_3, w_5 and w_2, w_3, w_5 shows that $a_1 + b_1 a_5$ and $a_2 + b_2 a_5$ both divide k. Moreover, applying Lemma 3.5.6 to w_1, w_2, w_5 yields $\gcd(a_1 + b_1 a_5, a_2 + b_2 a_5) = 1$. Together with Remark 3.2.7 (i) we obtain

$$(a_1 + b_1 a_5)(a_2 + b_2 a_5)a_3 \mid \alpha_1 = a_1 + a_2 + a_3 + a_5 + 1.$$

We expand the left-hand side and give a rough estimation:

$$\begin{aligned}(a_1 + b_1 a_5)(a_2 + b_2 a_5)a_3 &= a_1 a_2 a_3 + a_1 a_3 a_5 b_2 + a_2 a_3 a_5 b_1 + a_3 a_5^2 b_1 b_2 \\ &\geq a_1 + a_2 + a_3 + a_5\end{aligned}$$

Since $\gcd(n, n+1) = 1$ is true for every integer n, this inequation shows that equality holds in the above divisibility condition. From this we infer

$$a_2(a_1 a_3 - 1) + a_1(a_3 a_5 b_2 - 1) + a_3(a_2 a_5 b_1 - 1) + a_5(a_3 a_5 b_1 b_2 - 1) = 1.$$

Observe that every summand on the left-hand side is non-negative, hence precisely one of them equals one while the other vanish. Since a_1, a_2, a_3, a_5 are non-zero, the factor in the parenthesis vanishes whenever the whole summand vanishes. There are two summands where b_1 shows up in the second factor. At least one of those parenthesis must vanish, hence $b_1 = 1$. Repeating this argument yields $b_2 = 1$ as well as $a_3 = 1$. Similarly, we obtain $a_1 = 1$ or $a_2 = 1$. Altogether we have $u_3 = (1, -b_3)$ and $u_i = (1, -1)$ where $i \in \{1, 2\}$. This implies $\det(u_i, u_3) = 1 - b_3 \leq 0$; a contradiction to our assumption that $w_1, \ldots, w_6$ are in counter-clockwise order.

3.7 Proof of Theorem 3.1.3: Verification

The second mission in the proof of Theorem 3.1.3 is to ensure that the list of specifying data given there does not contain any superfluous items. So we have to verify that all items from Theorem 3.1.3 are realized by pairwise non-isomorphic smooth Calabi-Yau threefolds having a (general) hypersurface Cox ring. We make extensive use of the toolbox from Section 3.2.

Lemma 3.7.1. *Consider n-dimensional varieties X_1, X_2 with hypersurface Cox rings having relation degree μ_1 resp. μ_2. If X_1 and X_2 are isomorphic, then $\mu_1^n = \mu_2^n$ where μ_i^n is the self-intersection number of μ_i regarded as a divisor class on X_i.*

Proof. Let $\varphi : X_1 \to X_2$ be an isomorphism. Then the induced pull-back maps

$$\varphi^* : \mathcal{R}(X_2) \to \mathcal{R}(X_1), \qquad \tilde{\varphi}^* : \mathrm{Cl}(X_2) \to \mathrm{Cl}(X_1).$$

form an isomorphism $(\varphi^*, \tilde{\varphi}^*)$ of $\mathrm{Cl}(X_i)$-graded algebras. From this we deduce that the pull-back $\tilde{\varphi}^*(\mu_2) \in \mathrm{Cl}(X_2)$ of the relation degree $\mu_2 \in \mathrm{Cl}(X_2)$ of $\mathcal{R}(X_2)$ is the unique relation degree $\mu_1 \in \mathrm{Cl}(X_1)$ of $\mathcal{R}(X_1)$; see also Remark 3.2.17. Hence $\mu_1^n = \tilde{\varphi}^*(\mu_2)^n = \mu_2^n$. □

Proof of Theorem 3.1.3: Verification. We show that each item from Theorem 3.1.3 indeed stems from a smooth Calabi-Yau threefold with a general hypersurface Cox ring.

Let (Q, μ, u) be specifying data as presented in Theorem 3.1.3. Consider the linear K-grading on $S = \mathbb{K}[T_1, \ldots, T_6]$ given by $Q : \mathbb{Z}^6 \to K$. We run Construction 3.2.8 with the unique GIT-chamber $\tau \in \Lambda(S)$ containing u in its relative interior τ°. In doing so $u \in \mathrm{Mov}(S)^\circ$ guarantees $\tau^\circ \subseteq \mathrm{Mov}(S)^\circ$. In what follows we construct a non-empty open subset $U \subseteq U_\mu$ of polynomials satisfying the conditions from Remark 2.4.2, thereby obtaining a smooth general Calabi-Yau hypersurface Cox ring. This is done by starting with $U := U_\mu$ and shrinking U successively.

Since $\mu \neq w_i$ holds for all i, Remark 3.2.9 ensures that $T_1, \ldots, T_6$ form a minimal system of generators for R_g, whenever $g \in U_\mu$. We want to achieve K-primeness of $T_1, \ldots, T_6 \in R$. Here Numbers 2 and 7 have to be treated separately. For all remaining items from Theorem 3.1.3 and any $1 \leq i \leq 6$ we find in Table 3.1 on page 164 a μ-homogeneous prime binomial $T^\kappa - T^\nu \in S$ not depending on T_i. Thus, Proposition 3.2.10 allows us to shrink U such that $T_1, \ldots, T_6$ define primes in R_g for all $g \in U$.

Number 2. For Number 2 observe that all the generator degrees $w_i = \deg(T_i)$ are indecomposable in the weight monoid

$$S(R) = \{u \in K;\ R_u \neq 0\} = \mathrm{Pos}_{\mathbb{Z}}(w_1, \ldots, w_6) \subseteq K.$$

Thus every $T_i \in R_g$ is K-irreducible. As soon as we know that R_g is K-factorial, we may conclude that T_i is K-prime.

Number 7. Table 3.1 on page 164 shows μ-homogeneous prime binomials $T^\kappa - T^\nu \in S$ not depending on T_i for $i = 1, \ldots, 5$. Thus, Proposition 3.2.10 allows us to shrink U such that $T_1, \ldots, T_5$ define primes in R_g for all $g \in U$.

Observe that T_6 defines a K-prime in R_g if and only if $h := g(T_1, \ldots, T_5, 0) \in S$ is K-prime. Since S is a UFD, thus K-factorial, the latter is equivalent to $h \in S$ being K-irreducible. The only monomials of degree μ not depending on T_6 are T_4^3 and T_5^3, hence $h = aT_4^3 - bT_5^3$. Note that T_4^3, T_5^3 are vertices of the polytope

$$\mathrm{conv}\left(\nu \in \mathbb{Z}^6_{\geq 0};\ \deg(T^\nu) = \nu\right).$$

From g being spread we infer $a, b \in \mathbb{K}^*$. For degree reasons, any non-trivial factorization of h has a linear form $\ell = a'T_4 + b'T_5$ with $a', b' \in \mathbb{K}^*$ among its factors. From $w_4 \neq w_5$ we deduce that such ℓ is not homogeneous w.r.t the K-grading. We conclude that h

admits no non-trivial presentation as product of homogeneous elements, i.e., $h \in S$ is K-irreducible. This implies that $T_6 \in R_g$ is K-prime.

We take the next step, that is to make sure that each R_g is normal and factorially graded. For example this holds when R_g admits unique factorization. Whenever K is torsion-free the converse is also true. Here we encounter different classes of candidates.

Numbers 1, 2, 5, 6, 10 – 22, and 26 – 28. One directly checks that the convex hull over the $\nu \in \mathbb{Z}^6_{\geq 0}$ with $Q(\nu) = \mu$ is Dolgachev polytope; we have used the Magma program from Intrinsic A.3.5 for this task. Proposition 3.2.12 (ii) ensures that R_g is factorial after suitably shrinking U.

Numbers 3, 4, and 30. Here, the cone $\tau' = \operatorname{cone}(w_3) \in \Lambda(S)$ satisfies $(\tau')^\circ \subseteq \operatorname{Mov}(S)^\circ$. Thus, Construction 2.4.1 gives raise to a toric variety Z'. We have $\mu \in (\tau')^\circ$ and one directly verifies that μ is base point free for Z'. Hence Proposition 3.2.12 (i) shows that after shrinking U suitably, R_g admits unique factorization for all $g \in U$.

Number 7. We are aiming to apply Proposition 3.2.14. For this purpose we have to verify that μ occurs as degree associated with a simplex in the sense of Remark 3.2.13. The following polytope does the job:

$$B = \operatorname{conv}((0,0,0,0),\ (0,0,0,3)\ (0,0,9,-3),\ (3,0,3,-1),\ (3,3,3,-2)) \subseteq \mathbb{Q}^4.$$

The rays of its normal fan $\Sigma(B)$ are given as the columns of the following matrix

$$P_1 = \begin{bmatrix} -2 & 0 & -1 & 0 & 1 \\ -1 & 1 & 0 & 1 & -1 \\ -2 & 1 & 1 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 \end{bmatrix}.$$

Now consider the stellar subdivision Σ_2 of $\Sigma(B)$ along $(-1,0,0,0)$. The associated data of Σ_2 is $K_2 = \mathbb{Z}^2 \times \mathbb{Z}/3\mathbb{Z}$ and

$$P_2 = \begin{bmatrix} -2 & 0 & -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 & -1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \overline{0} & \overline{1} & \overline{2} & \overline{1} & \overline{2} & \overline{0} \end{bmatrix}.$$

We compute the Σ_2-degree μ_2 of B. Observe $a(\Sigma_2) = (9,0,0,0,3)$. From this we infer $\mu_2 = Q_2(a(\Sigma_2)) = (0,3,\overline{0})$. Note that (Q_2, μ_2) coincides with the specifying data (Q, μ) for which we run the verification process. In the previous step of this process we have ensured that $U \subseteq S_\mu$ is a non-empty open subset of prime polynomials such that $T_1, \ldots, T_6$ define K-primes in R_g whenever $g \in U$. According to Proposition 3.2.14 we may shrink U such that R_g is K-factorial for each $g \in U$.

Finally, Bechtold's criterion [18, Cor. 0.6; 63, Prop. 4.1] directly implies that R_g is normal since each five of $w_1, \ldots, w_6$ generate K as a group.

Numbers 8, 9, 10, and 24, 25 . By applying a suitable coordinate change we achieve that the degree matrix Q and the relation degree μ are as in the following table.

No.	Q	μ
8	$\begin{bmatrix} 1 & 1 & 1 & 6 & 9 & 0 \\ -1 & -1 & -1 & -4 & -6 & 1 \end{bmatrix}$	$(18, -12)$
9	$\begin{bmatrix} 1 & 1 & 1 & 6 & 9 & 0 \\ -1 & -1 & -1 & -4 & -6 & 1 \end{bmatrix}$	$(8, -4)$
10	$\begin{bmatrix} 0 & 2 & 2 & 2 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}$	$(8, -4)$
24	$\begin{bmatrix} 0 & 2 & 2 & 4 & 3 & 1 \\ 1 & -1 & -1 & -2 & -2 & -1 \end{bmatrix}$	$(12, -6)$
25	$\begin{bmatrix} 0 & 2 & 2 & 2 & 7 & 1 \\ 1 & -1 & -1 & -1 & -4 & -1 \end{bmatrix}$	$(14, -7)$

We apply Proposition 3.2.12 (iii). In the last three cases it is necessary to reorder the variables such that Q has precisely the shape requested by Proposition 3.2.12 (iii). Now the conditions from there can be directly checked. As a result, we may shrink U such that each R_g is a factorial ring.

Number 26. Again we want to use Proposition 3.2.14 thus we have to present μ as degree associated with a simplex in the sense of Remark 3.2.13. Consider

$$B = \mathrm{conv}((0,0,0,0),\ (0,0,0,8)\ (0,8,0,0),\ (0,0,4,0),\ (2,2,1,2)) \subseteq \mathbb{Q}^4.$$

Its normal fan $\Sigma_1 = \Sigma(B)$ has the rays given by the columns of the matrix

$$P_1 = \begin{bmatrix} 0 & 0 & 1 & -1 & 3 \\ 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & -1 & 1 \end{bmatrix}.$$

Now consider the stellar subdivision Σ_2 of $\Sigma(B)$ along $(1,0,0,0)$. Here associated data of Σ_2 is given by $K_2 = \mathbb{Z}^2$ and

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 3 & -1 & 1 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 3 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

We compute the Σ_2-degrees μ_2 of B. Observe $a(\Sigma_2) = (0,8,0,0,0)$. From this we infer $\mu_2 = Q_2(a(\Sigma_2)) = (8,0)$. Here (Q_2, μ_2) equals (Q, μ) from the specifying data for which we run the verification process. In the previous step of this process we have ensured that $U \subseteq S_\mu$ is a non-empty open subset such that $T_1, \ldots, T_6$ define primes in R_g whenever $g \in U$. Now Proposition 2.4.18 shows that we may shrink U such that R_g is factorial for each $g \in U$.

At this point we have that U defines a general hypersurface Cox ring. Note that Proposition 2.3.7 immediately yields that the corresponding varieties X_g are weakly

Calabi-Yau. The next step is to attain X_g being smooth. Checking the condition from Proposition 3.2.15 with the help of the Magma program from Intrinsic A.2.5 shows that Z_μ is smooth in all 30 cases. Observe that we have $\mu \in \tau$ except for Numbers 12, 16, 18, and 27. Whenever $\mu \in \tau$ holds we may apply Corollary 2.4.29 allowing us to shrink U once more such that X_g is smooth for all $g \in U$. The four exceptional cases turn out to be small quasimodifications of smooth weakly Calabi-Yau threefolds, hence are smooth by Proposition 3.6.1. Eventually Remark 3.2.7 (ii) ensures that X_g is Calabi-Yau.

The last task in the proof of Theorem 3.1.3 is to make sure that two varieties from different families from Theorem 3.1.3 are non-isomorphic. Note that if two varieties from Theorem 3.1.3 are isomorphic, then their Cox rings are isomorphic as graded rings. For each family from Theorem 3.1.3 we give the number l of generator degrees, the entries of the generator degree dimension tuple $(d_1, \ldots, d_l)$ and the self-intersection number μ^3 of the relation degree in the following table.

No.	l	d_1	d_2	d_3	d_4	d_5	d_6	μ^3
1	2	3	3	–	–	–	–	486
2	6	1	1	1	1	1	1	162
3	3	2	2	6	–	–	–	512
4	4	2	2	5	31	–	–	864
5	3	1	3	5	–	–	–	513
6	6	1	1	1	1	2	3	243
7	3	1	3	8	–	–	–	594
8	4	1	3	29	66	–	–	1944
9	3	1	2	6	–	–	–	512
10	4	1	2	5	31	–	–	864
11	3	2	3	7	–	–	–	513
12	3	2	3	7	–	–	–	512
13	3	2	3	31	–	–	–	864
14	4	1	2	4	31	–	–	864
15	4	1	2	4	8	–	–	520

No.	l	d_1	d_2	d_3	d_4	d_5	μ^3
16	4	1	2	4	8	–	512
17	4	1	2	5	9	–	539
18	4	1	2	5	9	–	512
19	4	1	3	4	10	–	567
20	4	1	3	4	32	–	896
21	4	1	3	7	35	–	992
22	5	1	2	3	4	28	784
23	5	1	2	4	7	32	912
24	5	1	1	3	4	8	432
25	4	1	1	4	21	–	686
26	4	1	1	3	14	–	512
27	5	1	2	3	6	31	864
28	5	1	2	3	6	31	872
29	5	1	1	3	4	29	808
30	5	1	1	3	4	4	432

Most of the varieties from Theorem 3.1.3 are distinguished by the generator degree dimension tuple. Note that the pairs having the same generator dimension degree tuple are precisely Numbers 11 & 12, 15 & 16, 17 & 18 and 27 & 28 as they share the same Cox ring. These pairs can be distinguished by the relation degree self-intersection number. □

Table 3.1: Binomials used to ensure primeness of $T_1, \ldots, T_6 \in R_g$ in the proof of Theorem 3.1.3

No.	T_1	T_2	T_3	T_4	T_5	T_6
1	$-T_2^3T_5T_6^2+T_3^3T_4^3$	$-T_1^3T_5T_6^2+T_3^3T_4^3$	$T_1^3T_4^3-T_2^3T_5T_6^2$	$T_1^3T_5^3-T_2T_3^2T_6^3$	$T_1^3T_4^3-T_2T_3^2T_6^3$	$T_1^3T_5^3-T_2T_3^2T_4^3$
2	—	—	—	—	—	—
3	$-T_2^3T_4T_5^3+T_3^4$	$-T_1^3T_4T_5^3+T_3^4$	$T_1^3T_2T_5^2T_6^2-T_4^4$	$T_1^3T_2T_5^2T_6^2-T_3^4$	$-T_1^3T_2T_6^4+T_3^4$	$-T_1^3T_2T_5^4+T_3^4$
4	$-T_2^6T_5^5T_6+T_4^2$	$-T_1^6T_5^5T_6+T_4^2$	$-T_1^3T_2^3T_5^6+T_4^2$	$T_1T_2^5T_5^5T_6-T_3^6$	$-T_2^6T_6^6+T_3^3T_4$	$-T_2^6T_5^6+T_3^3T_4$
5	$-T_2^2T_5^3+T_3^5T_6^3$	$-T_1^2T_5^3+T_3^5T_6^3$	$T_1^5T_6^3-T_2^2T_5^3$	$-T_1^5T_6^3+T_2^2T_5^3$	$-T_1^5T_6^3+T_2^2T_4^3$	$T_2^2T_5^3-T_3^2T_4^3$
6	$-T_2^5T_5T_6^2+T_3T_4^3$	$-T_1^5T_5T_6^2+T_3T_4^3$	$T_1T_4^3-T_2^5T_5T_6^2$	$-T_1^4T_3^3T_6^3+T_2T_5^3$	$-T_1^4T_3^3T_6^3+T_2T_4^3$	$T_2T_5^3-T_3T_4^3$
7	$-T_2^4T_3^2T_4T_6^2+T_5^3$	$-T_1^2T_3^4T_4T_6^2+T_5^3$	$-T_1^4T_2^2T_4T_6^2+T_5^3$	$T_1^2T_2^5T_3^2T_6^3-T_5^3$	$T_1^2T_2^5T_3^2T_6^3-T_4^3$	—
8	$-T_2^{11}T_3^7T_6^6+T_5^2$	$-T_1^{11}T_3^7T_6^6+T_5^2$	$-T_1^7T_2^{11}T_6^6+T_5^2$	$-T_1^{15}T_2^2T_3T_6^6+T_5^2$	$T_1^2T_2T_3^{15}T_6^6-T_4^3$	$T_4^3-T_5^2$
9	$-T_2T_3^3+T_4^4$	$T_1^3T_3T_5^5T_6-T_4^4$	$T_1^3T_2T_5^5T_6-T_4^4$	$T_1^3T_3T_5^5T_6-T_2^4$	$-T_1^2T_3^2T_6^4+T_2^3T_4$	$-T_1^2T_3^2T_5^4+T_2^3T_4$
10	$T_2^5T_3-T_4^2$	$-T_1^5T_3T_5^5T_6^5+T_4^2$	$-T_1^5T_2T_5^5T_6^5+T_4^2$	$T_1^6T_5^{11}T_6-T_2^2T_3^4$	$T_2^5T_3-T_4^2$	$T_2^5T_3-T_4^2$
11	$-T_2^4T_5^3+T_3^3T_4T_6^2$	$-T_1^4T_5^3+T_3^3T_4T_6^2$	$T_1^3T_4T_6^2-T_2^4T_5^3$	$T_1^4T_5^3-T_2^2T_3^2T_6^3$	$T_1^2T_2^2T_6^3-T_3T_4^3$	$T_1^2T_2^2T_5^3-T_3T_4^3$
12	$-T_2^4T_5^3+T_3^3T_4T_6^2$	$-T_1^4T_5^3+T_3^3T_4T_6^2$	$T_1^3T_4T_6^2-T_2^4T_5^3$	$T_1^4T_5^3-T_2^2T_3^2T_6^3$	$T_1^2T_2^2T_6^3-T_3T_4^3$	$T_1^2T_2^2T_5^3-T_3T_4^3$
13	$-T_2T_3^5T_5^4+T_4^2$	$-T_1T_3^5T_5^4+T_4^2$	$-T_1^5T_2T_5^4+T_4^2$	$T_1^5T_2T_5^4-T_3^6T_6^4$	$-T_1T_2^2T_3^3T_6^4+T_4^2$	$-T_1T_2^2T_3^3T_5^4+T_4^2$
14	$T_2^3T_3^3T_5T_6-T_4^2$	$-T_1^4T_3^2T_5T_6^5+T_4^2$	$-T_1^4T_2^2T_5T_6^5+T_4^2$	$T_1^5T_3T_5^7-T_2^6T_6^2$	$-T_1^3T_3^3T_6^5+T_4^2$	$-T_1^3T_3^3T_5^5+T_4^2$
15	$-T_2^3T_4T_6+T_3^4T_5^2$	$T_1T_4^3-T_3^4T_6^2$	$T_1T_4^3-T_2^4T_6^2$	$-T_1^4T_5^6+T_2T_3^3T_6^2$	$T_1^4T_6^6-T_2T_3T_4^2$	$T_1^4T_5^6-T_2T_3T_4^2$
16	$-T_2^3T_4T_6+T_3^4T_5^2$	$T_1T_4^3-T_3^4T_6^2$	$T_1T_4^3-T_2^4T_6^2$	$-T_1^4T_5^6+T_2T_3^3T_6^2$	$T_1^4T_6^6-T_2T_3T_4^2$	$T_1^4T_5^6-T_2T_3T_4^2$
17	$T_2^4T_5-T_3^3T_4$	$-T_1^2T_4^2T_5^3+T_3^4T_6$	$-T_1^2T_4^2T_5^3+T_2^4T_6$	$-T_1^4T_6^9+T_3^4T_5$	$-T_1^4T_6^9+T_3^3T_4$	$-T_1^4T_5^9+T_3^3T_4$
18	$T_2^4T_5-T_3^3T_4$	$-T_1^2T_4^2T_5^3+T_3^4T_6$	$-T_1^2T_4^2T_5^3+T_2^4T_6$	$-T_1^4T_6^9+T_3^4T_5$	$-T_1^4T_6^9+T_3^3T_4$	$-T_1^4T_5^9+T_3^3T_4$
19	$-T_3^4T_5^2T_6+T_4^3$	$-T_3^4T_5^2T_6+T_4^3$	$-T_1^4T_5^2T_6+T_4^3$	$T_1^4T_3^2T_6^3-T_2^3T_5^3$	$-T_2^2T_3^4T_6^3+T_4^3$	$-T_2T_3^2T_5^3+T_4^3$
20	$-T_2^4T_3T_5^3T_6+T_4^2$	$-T_1^4T_3T_5^3T_6+T_4^2$	$-T_1T_2^4T_5^3T_6+T_4^2$	$T_1^8T_6^4-T_2^3T_3T_5^4$	$T_2^3T_3^5T_6^4-T_4^2$	$T_2^3T_3T_5^4-T_4^2$
21	$T_2^7T_3^3T_6^4-T_4^2$	$T_1^7T_3^3T_6^4-T_4^2$	$T_1^3T_2^7T_6^4-T_4^2$	$-T_1T_2^9T_6^4+T_3^2T_5^4$	$T_1^4T_2T_3^5T_6^4-T_4^2$	$-T_1T_2T_5^4+T_4^2$
22	$T_2^5T_3^2T_5T_6^3-T_4^2$	$T_1^5T_3^2T_5T_6^3-T_4^2$	$T_1^3T_4T_5^2-T_2^{10}T_6^4$	$T_1^6T_5^4-T_2^4T_3^3T_6^4$	$-T_1^3T_2^5T_3T_6^4+T_4^2$	$-T_1^3T_2^3T_5^4+T_4^2$
23	$T_2T_3^2T_5^3T_6-T_4^2$	$T_1T_3^2T_5^3T_6-T_4^2$	$-T_1^3T_2^5T_5^2T_6^2+T_4^2$	$-T_1^{10}T_2^4T_6^4+T_3T_5^4$	$-T_1^9T_2^5T_6^4+T_4^2$	$-T_3T_5^4+T_4^2$
24	$-T_2^6+T_3^4T_4$	$-T_1^3T_4T_5T_6^5+T_3^6$	$-T_1^3T_4T_5T_6^5+T_2^6$	$-T_1^2T_5^4+T_2^5T_3$	$-T_1^4T_4T_6^8+T_3^6$	$-T_1T_3^3T_5^2+T_4^3$
25	$-T_2^4T_3^3+T_4^7$	$-T_1^7T_6^{14}+T_3^5T_4^2$	$-T_1^7T_6^{14}+T_2^5T_4^2$	$-T_1^7T_6^{14}+T_2^2T_3^5$	$-T_1^2T_4^5T_6^4+T_3^7$	$T_1T_5^2-T_2T_3^4T_4^2$
26	$-T_2^7T_3+T_4^8$	$-T_1T_3^6T_6^2+T_4^8$	$-T_1T_2^6T_6^2+T_4^8$	$-T_1T_3^6T_6^2+T_2^8$	$-T_1^4T_6^8+T_2T_3T_4^6$	$-T_1T_5^2+T_2^6T_3^2$
27	$-T_2^3T_3^3T_5T_6^2+T_4^2$	$-T_1^3T_3^3T_5T_6^2+T_4^2$	$T_1^7T_5^3T_6^3-T_4^2$	$-T_1^2T_2^2T_5^6+T_3^5T_6$	$-T_1^5T_2^3T_3T_6^5+T_4^2$	$-T_1T_2T_3^2T_5^4+T_4^2$
28	$-T_2^3T_3^3T_5T_6^2+T_4^2$	$-T_1^3T_3^3T_5T_6^2+T_4^2$	$T_1^7T_5^3T_6^3-T_4^2$	$-T_1^2T_2^2T_5^6+T_3^5T_6$	$-T_1^5T_2^3T_3T_6^5+T_4^2$	$-T_1T_2T_3^2T_5^4+T_4^2$
29	$-T_3^7T_5T_6+T_4^2$	$-T_3^7T_5T_6+T_4^2$	$-T_2^7T_5T_6+T_4^2$	$T_1T_3^7T_6^3-T_2^6T_5^2$	$-T_1T_2^6T_3T_6^3+T_4^2$	$-T_1^3T_5^5+T_4^2$
30	$T_2^2T_5^2-T_3^5T_4$	$-T_1T_4^4T_5+T_3^6$	$-T_1^3T_5^3+T_2^2T_4^2T_6^2$	$-T_1^3T_5^3+T_2^2T_3^2T_6^2$	$-T_1^2T_2^2T_6^4+T_3^3T_4^3$	$-T_2^2T_5^2+T_3T_4^5$

APPENDIX

A

MAGMA PROGRAMS

Throughout this thesis we have performed computations with the support of the computer algebra system MAGMA [27]. The according functions have been bundled together into a MAGMA package which is available at [95]. In this chapter we describe the essential intrinsics of this package used for producing our results and provide examples how to use them. The involved data is mainly stored in elementary structures from the MAGMA language, such as sequences, allowing the users to easily modify the examples given in this chapter.

A.1 Elementary algebraic and combinatoric intrinsics

We represent an element $a = (x, \overline{y_1}, \ldots, \overline{y_q})$ of a finitely generated abelian group

$$K = \mathbb{Z}^n \times \mathbb{Z}/t_1\mathbb{Z} \times \cdots \times \mathbb{Z}/t_q\mathbb{Z}$$

by an integer sequence $[x_1, \ldots, x_n, y_1, \ldots, y_q]$ of the length $n + q$ together with a second integer sequence $\texttt{T} = [t_1, \ldots, t_q]$ containing the orders of the finite cyclic factors of K. Moreover a group homomorphism $Q : \mathbb{Z}^r \to K$ identified with the matrix

$$Q = \begin{bmatrix} a_1 & \ldots & a_r \end{bmatrix} = \begin{bmatrix} x_1 & \ldots & x_r \\ \overline{y_{11}} & \ldots & \overline{y_{r1}} \\ \vdots & & \vdots \\ \overline{y_{1q}} & \ldots & \overline{y_{rq}} \end{bmatrix}$$

is represented by the sequence of its rows regarded as integer sequences together with the accompanying sequence $\texttt{T} = [t_1, \ldots, t_q]$.

Intrinsic A.1.1 (`IsZZGenerating`)**.** Check if given elements of K form a generating set.

Input: $u_1, \ldots, u_s \in K$.

Parameters: if K has torsion, the torsion sequence $\texttt{T} = [t_1, \ldots, t_q]$.

Output: returns true if and only if $u_1, \ldots, u_s$ form a generating set for K as a group.

Example. First, we check that the group $\mathbb{Z}^3$ is generated by

$$u_1 = (1, 0, 0), \quad u_2 = (0, 1, 0), \quad u_3 = (3, 3, 3), \quad u_4 = (4, 4, 4).$$

Then we verify that $w_1 = (1, 2, \overline{1})$, $w_2 = (1, -1, \overline{1})$ do not span $\mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z}$ as a group.

```
> u1 := [1,0,0];
> u2 := [0,1,0];
> u3 := [3,3,3];
> u4 := [4,4,4];
> IsZZGenerating([u1,u2,u3,u4]);
true

> w1 := [1,2,1];
> w2 := [1,-1,1];
> IsZZGenerating([w1, w2] : T := [2]);
false
```

Intrinsic A.1.2 (`FiberPoints`)**.** Compute the intersection of a fiber of a group homomorphism with the positive orthant.

Input: a homomorphism $Q : \mathbb{Z}^r \to K$ such that $Q(\mathbb{Q}^r_{\geq 0}) \subseteq \mathbb{Q}^n = K_{\mathbb{Q}}$ is a pointed cone, and $w \in K$.

Parameters: if K has torsion, the torsion sequence $\mathtt{T} = [t_1, \ldots, t_q]$.

Output: a sequence of all lattice points $\nu \in \mathbb{Z}^r_{\geq 0}$ with $Q(\nu) = w$.

Example. We consider $K = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and compute all $\mu \in \mathbb{Z}^4_{\geq 0}$ with $Q(\mu) = w$ where

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \overline{0} & \overline{0} & \overline{1} & \overline{1} \end{bmatrix}, \quad w = (2, \overline{1}).$$

```
> Q := [[1,1,1,1], [0,0,1,1]];
> w := [2, 1];
> FiberPoints(Q, w : T := [2]);
[
    [ 0, 1, 0, 1 ],
    [ 1, 0, 0, 1 ],
    [ 0, 1, 1, 0 ],
    [ 1, 0, 1, 0 ]
]
```

Intrinsic A.1.3 (`IsHomPermutation`)**.** Check if there is a permution that fixes the columns of a given matrix and translates a given subsets of $\mathbb{Z}^n$ into another give one.

Input: an integral $(m \times n)$-matrix Q, finite subsets $E, F \subseteq \mathbb{Z}^n$.

Output: returns true if and only if there is a permutation $\sigma \in S_n$ such that

- the i-th column of Q equals the $\sigma(i)$-th column of Q,
- $E = \{\sigma(v);\ v \in F\}$ where $\sigma(v) := (v_{\sigma(1)}, \ldots, v_{\sigma(n)})$.

Example. Consider the $\mathbb{Z}^2$-grading on $\mathbb{K}[T_1, \dots, T_6]$ given by $\deg(T_i) := w_i$ with the columns w_i of the following matrix Q and polynomials g_1, g_2

$$Q = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad g_1 = T_1T_4 + T_2T_5 + T_3T_6, \quad g_2 = T_1T_6 + T_2T_4 + T_3T_5.$$

We use `IsHomPermutation` to check whether there is a permutation $\sigma \in S_6$ giving rise to a graded automorphism $\varphi : T_i \mapsto T_{\sigma(i)}$ on $\mathbb{K}[T_1, \dots, T_6]$ such that $\varphi(g_1) = g_2$.

```
> S := PolynomialAlgebra(Rationals(), 6);
> Q := [[1,1,1,0,0,0], [0,0,0,1,1,1]];
> g1 := S.1*S.4 + S.2*S.5 + S.3*S.6;
> g2 := S.1*S.6 + S.2*S.4 + S.3*S.5;
> E := [Exponent(f) : f in Monomials(g1)];
> F := [Exponents(f) : f in Monomials(g2)];
> IsHomPermutation(Q, E, F);
true [ 1, 2, 3, 6, 4, 5 ]
```

A.2 Tools for hypersurface rings

We present computational tools for dealing with questions arising in the context of Construction 2.4.1. For convenience, let us recall the notation around the central objects from there. We consider the polynomial algebra $S = \mathbb{K}[T_1, \dots, T_r]$ together with a pointed linear K-grading described by the degree map $Q : \mathbb{Z}^r \to K$, $e_i \mapsto \deg(T_i)$. For any homogeneous polynomial $g \in S$ of degree $\mu \in K$ we set

$$R := R_g := \mathbb{K}[T_1, \dots, T_r]/\langle g \rangle.$$

Moreover, any GIT-cone $\tau \in \Lambda(S)$ with $\tau^\circ \subseteq \mathrm{Mov}(S)^\circ$ gives rise to a projective toric variety Z with a closed subvariety X_g as shown in the following diagram:

$$\begin{array}{ccc} V(g) & \subseteq & \mathbb{K}^r \\ \downarrow & & \downarrow \\ X_g & \subseteq & Z \end{array}$$

Intrinsic A.2.1 (`SearchPrimeBinomial`)**.** Check if $T_i \in R_g$ is prime for general g by looking for prime binomials. Implements Proposition 2.4.11/Remark 2.4.12.

Input: Q, μ, an in index $1 \le i \le r$.

Parameters: if K has torsion, the torsion sequence `T` $= [t_1, \dots, t_q]$.

Output: Returns true if there is a μ-homogeneous prime binomial not depending on T_i and false otherwise. If true, also returns exponents of such a prime binomial.

Example. We perform the test for T_1 and T_6 with data as in Number 7 from Theorem 3.1.3:

$$K = \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, \quad Q = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \overline{0} & \overline{1} & \overline{2} & \overline{1} & \overline{2} & \overline{0} \end{bmatrix}, \quad \mu = (0, 3, \overline{0}).$$

In the first case the output additionally provides us with the binomial $T_5^3 - T_2^4T_3^2T_4T_6^2$. In the second case, the output can be easily verified by hand since the only monomials of degree μ not depending on T_6 are T_4^3 and T_5^3.

```
> Q := [[1,1,1,0,0,-3], [0,0,0,1,1,1], [0,1,2,1,2,0]];
> mu := [0,3,0];
> SearchPrimeBinomial(Q, mu, 1 : T := [3]);
true [
    [ 0, 0, 0, 0, 3, 0 ],
    [ 0, 4, 2, 1, 0, 2 ]
]
> SearchPrimeBinomial(Q, mu, 6 : T := [3]);
false
```

Intrinsic A.2.2 (`DimHomComp`)**.** Compute the dimension of homogeneous components.

Input: Q, μ, $w \in K$.

Parameters: if K has torsion, the torsion sequence `T` $= [t_1, \dots, t_q]$.

Output: the vector space dimension of R_w.

Example. We compute the dimension of R_w for the data

$$Q = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad \mu = (1,1), \quad w = (2,2).$$

```
> Q := [[1,1,1,0,0,0], [0,0,0,1,1,1]];
> mu := [1,1];
> w := [2,2];
> DimHomComp(Q, mu, w);
27
```

Intrinsic A.2.3 (`GeneratorDegreeDimensionTuple`)**.** Computes the generator degree dimension tuple of R.

Input: Q, μ.

Parameters: if K has torsion, the torsion sequence `T` $= [t_1, \dots, t_q]$.

Output: the generator degree dimension tuple of R.

Example. We compute the generator degree dimension tuple with data as in Number 3 from Theorem 3.1.3:

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (4,4).$$

```
> Q := [[1,1,1,1,0,0], [0,0,1,1,1,1]];
> mu := [4,4];
> GeneratorDegreeDimensionTuple(Q, mu);
[ 2, 2, 6 ]
```

Intrinsic A.2.4 (`HilbertCoeffs`)**.** Compute the first coefficients of the Hilbert series of R.

Input: Q, μ, $n \in \mathbb{Z}_{\geq 2}$.

Parameters: if K has torsion, the torsion sequence $\mathtt{T} = [t_1, \dots, t_q]$.

Output: the first n coefficients of the Hilbert series of R.

Example. We compute the first six coefficients of the Hilbert series of Number 1 from Theorem 2.1.1.

```
> Q := [[1,1,1,1,0,0,0], [0,0,0,0,1,1,1]];
> mu := [1,1];
> HilbertCoeffs(Q, mu, 6);
[ 1, 90, 700, 2695, 7371, 16456 ]
```

Intrinsic A.2.5 (`IsMuAmbientSmooth`)**.** Check if the μ-minimal ambient toric variety Z_μ is smooth. Implements Proposition 2.4.28. Assumes $\mathrm{rank}(K) \leq 2$.

Input: Q, μ, an ample class $u \in \tau^\circ$ for Z.

Parameters: if K has torsion, the torsion sequence $\mathtt{T} = [t_1, \dots, t_q]$.

Output: returns true if and only if Z_μ is smooth.

Example. We perform the test for Number 33 from Theorem 2.1.1 i.e. with the data

$$K = \mathbb{Z}^2, \quad Q = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 1 & 1 \end{bmatrix}, \quad \mu = (4,6), \quad u = (1,3).$$

```
> Q := [[1,1,2,1,0,0,0], [0,1,3,2,1,1,1]];
> mu := [4, 6];
> u := [1, 3];
> SD := SpecifyingData(Q, [mu], u);
> IsMuAmbientSmooth(SD);
true
```

Intrinsic A.2.6 (`QuasismoothTest`)**.** Check if the location of $\mu \in K_\mathbb{Q}$ is compatible with X_g being quasismooth. Implements Proposition 2.3.6. Assumes $\mathrm{rank}(K) \leq 2$.

Input: Q, μ, an ample class $u \in \tau^\circ$ for Z.

Parameters: if K has torsion, the torsion sequence $\mathtt{T} = [t_1, \dots, t_q]$.

Output: Returns true if and only if

$$\mu \in \bigcap_{\gamma_I \in \mathrm{rlv}(X)} \left(Q(\gamma_I \cap \mathbb{Z}^r) \cup \bigcup_{i=1}^{r} w_i + Q(\gamma_I \cap \mathbb{Z}^r) \right).$$

If false, also returns $I \subseteq \{1, \dots, r\}$ such that $\mu \notin (Q(\gamma_I \cap \mathbb{Z}^r) \cup \bigcup_{i=1}^r w_i + Q(\gamma_I \cap \mathbb{Z}^r))$

Example. We show that the following data does not lead to a quasismooth X_g:

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (1,3), \qquad u = (2,1).$$

```
> Q := [[1,1,1,1,0,0], [0,0,1,1,1,1]];
> mu := [1, 3];
> u := [2, 1];
> QuasismoothTest(Q, mu, u);
false [ 1, 3 ]
```

A.3 Newton polytopes and non-degenerate systems

The following intrinsics offer explicit treatment of the concepts introduced for systems of (Laurent) polynomials in Section 1.3. We use the Magma categories `RngMPol` for polynomials and `TorPol` for convex polytopes.

Intrinsic A.3.1 (`NewtonPolytope`)**.** Returns the Newton polytope of a polynomial. This is a wrapper function to apply the existing intrinsic `NewtonPolytope` to the data type `RngMPolElt`.

Input: a polynomial $f \in \mathbb{K}[T_1, \ldots, T_r]$.

Output: the Newton polytope $B(f) \subseteq \mathbb{Q}^r$ of f.

Intrinsic A.3.2 (`FacePolynomial`)**.** Compute the face polynomial of a given polynomial.

Input: a polynomial $f \in \mathbb{K}[T_1, \ldots, T_r]$ and a face $B' \preceq B(f)$ of its Newton polytope

Output: the face polynomial f' associated with B'.

Intrinsic A.3.3 (`FaceSystem`)**.** Compute the face system of a given system of polynomials.

Input: a system F of polynomials and a face $B' \preceq B(F)$ of its Newton polytope.

Output: the face system F' associated with B'.

Intrinsic A.3.4 (`IsNondegenerate`)**.** Check if a system of polynomials is non-degenerate in the sense of Definition 1.3.6.

Input: a system F of polynomials.

Output: returns true if and only if F is non-degenerate. If F is not non-degenerate, also returns a face $B' \preceq B(F)$ which does not satisfy Definition 1.3.6 (iii).

Example. We investigate the system F consisting of the single polynomial

$$f = (T_1 - T_2)(T_3 - T_4) + T_5^2.$$

Since f fails to be non-degenerate, we also compute a critical face polynomial f'.

```
> S<[T]> := PolynomialAlgebra(Rationals(), 5);
> f := (T[1] - T[2])*(T[3] - T[4]) + T[5]^2;
> ver, B0 := IsNondegenerate([f]);
> ver;
false
> B0;
2-dimensional polytope B0 with 4 vertices:
    (1, 0, 1, 0, 0),
    (1, 0, 0, 1, 0),
    (0, 1, 1, 0, 0),
    (0, 1, 0, 1, 0)
> FacePolynomial(f, B0);
T[1]*T[3] - T[1]*T[4] - T[2]*T[3] + T[2]*T[4]
```

Intrinsic A.3.5 (`IsDolgachevPolytope`)**.** Check if a polytope is Dolgachev. This is used in connection with Dolgachev's factoriality criterion; see also Proposition 2.4.13 (ii).

Input: a polytope $P \subseteq \mathbb{Q}^r$.

Output: true if P is a Dolgachev polytope, false otherwise.

Example. We consider $Q : \mathbb{Z}^7 \to \mathbb{Z}^2$, $\mu \in \mathbb{Z}^2$ as in Number 1 from Theorem 2.1.1 and verify that $\mathrm{conv}(\nu \in \mathbb{Z}^7_{\geq 0};\, Q(\nu) = \mu) \subseteq \mathbb{Q}^7$ is a Dolgachev polytope.

```
> Q := [[1,1,1,1,0,0,0], [0,0,0,0,1,1,1]];
> mu := [1,1];
> P := Polytope(FiberPoints(Q, mu));
> P;
5-dimensional polytope P with 12 generators
> Vertices(P);
[
    (1, 0, 0, 0, 0, 1, 0),
    (0, 0, 0, 1, 1, 0, 0),
    (0, 0, 1, 0, 0, 0, 1),
    (1, 0, 0, 0, 1, 0, 0),
    (0, 0, 1, 0, 0, 1, 0),
    (0, 1, 0, 0, 0, 0, 1),
    (0, 0, 1, 0, 1, 0, 0),
    (0, 0, 0, 1, 0, 0, 1),
    (0, 1, 0, 0, 0, 1, 0),
    (1, 0, 0, 0, 0, 0, 1),
    (0, 1, 0, 0, 1, 0, 0),
    (0, 0, 0, 1, 0, 1, 0)
]
> IsDolgachevPolytope(P);
true
```

A.4 Intersection numbers

Intrinsic A.4.1 (`ToricIntersectionProduct`)**.** Compute intersection numbers on an n-dimensional $\mathbb{Q}$-factorial projective toric variety Z. Implements Algorithm 1.6.5.

Input: degree map $Q : \mathbb{Z}^r \to K$, ample class $u \in K_{\mathbb{Q}}$ for Z, $u_1, \ldots, u_n \in K_{\mathbb{Q}}$.

Parameters: if K has torsion, the torsion sequence `T` $= [t_1, \ldots, t_q]$.

Output: the intersection number $u_1 \cdots u_n \in \mathbb{Q}$.

Example 1. We compute the anticanonical self-intersection number of the toric complete intersection Number 41 from Theorem 1.1.3. We have $K = \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$ and input data

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{bmatrix}, \qquad u = 1,$$

$$u_1 = u_2 = u_3 = -\mathcal{K} = 1, \quad u_4 = u_5 = u_6 = \mu_i = 2.$$

```
> Q := [ [1,1,1,1,1,1,1],
         [0,0,0,0,1,1,1],
         [0,0,1,1,0,0,1]];
> mu := [ [2], [2], [2]];
> u := [1];
> K := [1];
> D := mu cat [K, K, K];
> ToricIntersectionProduct(Q, u, D : T := [2,2]);
2
```

Example 2. We compute the anticanonical self-intersection number of a variety X with hypersurface Cox ring and specifying data as in Number 2 from Theorem 2.1.1; see also Remark 1.6.6. Here the anticanonical class of X is also ample for a $\mathbb{Q}$-factorial ambient toric variety $X \subseteq Z$. So the input data is

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad u = (2,2),$$

$$u_1 = \cdots = u_4 = -\mathcal{K} = (2,2), \quad u_5 = \mu = (2,1).$$

```
> Q := [ [1,1,1,1,0,0,0],
>        [0,0,0,0,1,1,1] ];
> mu := [2, 1];
> u := [2,2];
> K := [2,2];
> D := [K, K, K, K, mu];
> ToricIntersectionProduct(Q, u, D);
256
```

Intrinsic A.4.2 (`FanoDegree`)**.** Compute the anticanonical self-intersection number of a $\mathbb{Q}$-factorial projective Fano variety X with complete intersection Cox ring and Picard number at most two.

Input: specifying data Q and $[\mu_1, \ldots, \mu_s]$ for X.

Parameters: if K has torsion, the torsion sequence `T` $= [t_1, \ldots, t_q]$.

Output: the anticanonical self-intersection number of X.

Example. We compute the anticanonical self-intersection number of Number 27 from Theorem 2.1.1 i.e. for the following data

$$Q = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (2,4).$$

```
> Q := [[1,1,1,0,0,0,0],
>       [0,0,2,1,1,1,1] ];
> mu := [2, 4];
> FanoDegree(Q, [mu]);
64
```

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